# NCEA Level 3 Calculus (Differentiation)
# 13. Inverse Functions

A function is one-to-one (or injective) if $f(x) = f(y)$ implies that $x = y$ (i.e. two different inputs can never give the same output/the function passes the horizontal-line test).
**Definition.** Let $f$ be a function that is one-to-one. Then the **inverse** of $f$ is the (unique) function $f^{-1}$ such that

$$f(x) = y \iff f^{-1}(y) = x.$$

In other words, $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$.
**Example.** Here are some functions with their inverses:

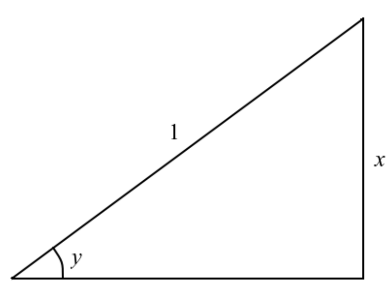
| Function | Inverse | Notes |
|---|---|---|
| $e^x$ | $\ln x$ | Note that $\ln x$ is defined only when $x > 0$ since $e^x > 0$ for all real $x$. |
| $\sin x$ | $\sin^{-1} x$ | Note that $\sin^{-1} x$ is only defined when $-\pi < x \leq \pi$ since otherwise $\sin x$ is not one-to-one. |
| $\cos x$ | $\cos^{-1} x$ | Note that $\cos^{-1} x$ is only defined when $-\pi < x \leq \pi$ since otherwise $\cos x$ is not one-to-one. |
| $\tan x$ | $\tan^{-1} x$ | Note that $\tan^{-1} x$ is defined for all $x$ (why?), and so $\tan^{-1} x \neq \frac{\sin^{-1} x}{\cos^{-1} x}$. |
| $x^2$ | $\sqrt{x}$ | When $x$ is positive. |

The graph of the inverse of a function is the reflection of the graph of the original function around the line $x = y$ (essentially, we swap the $x$ and $y$ axes).

Let us now find the derivative of $y = \sin^{-1} x$.

$$\begin{aligned} y &= \sin^{-1} x \\ \sin y &= x \\ \frac{\mathrm{d}y}{\mathrm{d}x} \cos y &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos y} = \frac{1}{\cos \sin^{-1} x} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

The identity $\cos \sin^{-1} x = \sqrt{1 - x^2}$ comes from the following triangle:




**Theorem** (Inverse Trigonometric Derivatives)**.** *The following table gives the derivatives of inverses of the three primary trigonometric functions and their reciprocals.*

| *Function* | *Derivative* | *Function* | *Derivative* |
|---|---|---|---|
| $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ | $\csc^{-1} x$ | $-\frac{1}{x\sqrt{x^2-1}}$ |
| $\cos^{-1} x$ | $-\frac{1}{\sqrt{1-x^2}}$ | $\sec^{-1} x$ | $\frac{1}{x\sqrt{x^2-1}}$ |
| $\tan^{-1} x$ | $\frac{1}{1+x^2}$ | $\cot^{-1} x$ | $-\frac{1}{x^2+1}$ |

## Questions

1. $\boxed{\texttt{M}}$ Prove or disprove the statement that $f : x \mapsto x^2 + x + 1$ is one-to-one (where $x$ is real).
2. $\boxed{\texttt{M}}$ Prove or disprove the statement that $f : x \mapsto 2^x$ is one-to-one (where $x$ is a positive real).
3. $\boxed{\texttt{M}}$ Determine whether the following functions have inverses on the given interval:
   (a) $x \mapsto x^3$ (on $\mathbb{R}$)
   (b) $y \mapsto y^4$ (on $\mathbb{R}$)
   (c) $y \mapsto y^4$ (for $y \geq 0$)
   (d) $y \mapsto y^4$ (for $y > 0$)
   (e) $\theta \mapsto \cos^{-1}\theta$ (on $\mathbb{R}$)
   (f) $\theta \mapsto \cos^{-1}\theta$ (for $-1 \leq \theta \leq 1$)
4. $\boxed{\texttt{A}}$ True or false:
   (a) $\cos^{-1} x = \frac{1}{\cos x}$
   (b) If $x > 0$ then $(\ln x)^6 = 6 \ln x$
   (c) $\tan^{-1}(-1) = \frac{3\pi}{4}$ (think about which arm of $\tan x$ we're talking about)
   (d) The inverse of $y = e^{3x}$ is $y = \frac{1}{3} \ln y$.
5. $\boxed{\texttt{A}}$ Find the derivative of $y = \ln(e^x)$ in two different ways.
6. $\boxed{\texttt{A}}$ Find $y'$ if:
   (a) $y = \sin^{-1} 2x$
   (b) $x = \sin^2 y$
   (c) $y = x + \tan^{-1} y$
   (d) $y = \ln \sin x - \frac{1}{2} \sin^2 x$
   (e) $y = 24 \arctan x + \arcsin \sqrt{x}$
   (f) $y = \sqrt{\sec^{-1} 2x}$
7. $\boxed{\texttt{E}}$ Find the local extrema, areas of concavity, and inflection points of the following functions; hence sketch their graphs.
   (a) $y = e^x \sin x$ for $-\pi < x < \pi$
   (b) $y = x + \ln(x^2 + 1)$
   (c) $y = \sin^{-1}(1/x)$
8. $\boxed{\texttt{E}}$ If $f'(x) = \tan^{-1} x$, find $(f^{-1})'(x)$.
9. $\boxed{\texttt{E}}$ Suppose $f$ is a function, and $f^{-1}$ is the inverse of $f$. Prove that $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$.
10. $\boxed{\texttt{S}}$ Prove the formulae for the derivatives of $\cos^{-1}$ and $\tan^{-1}$, using a similar method to that for $\sin^{-1} x$.
11. $\boxed{\texttt{S}}$ Scholarship 2012: Consider the equation $x^n = \tan(ny)$, where $n$ is a constant. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$.