NCEA Level 2 Mathematics
6. Simultaneous Equations

Let's go back to last week, where we had the following example of a quadratic model.
Example. The following table gives the instantaneous rates of reaction for the a particular chemical reaction.

Time (min)	**Concentration of reactant** (M)	**Instantaneous reaction rate** (M/min)
0	0.0054	
10	0.0044	8.0×10^{-5}
26	0.0034	5.0×10^{-5}
44	0.0027	3.1×10^{-5}
70	0.0020	1.8×10^{-5}
120	0.0014	8.0×10^{-6}

It is known that the reaction rate is modelled by a quadratic function of the reaction concentration. Let's call the reaction rate at a particular concentration $R(C)$; so

$$R(C) = XC^2 + YC + Z.$$

By using the values in the table above, we have that

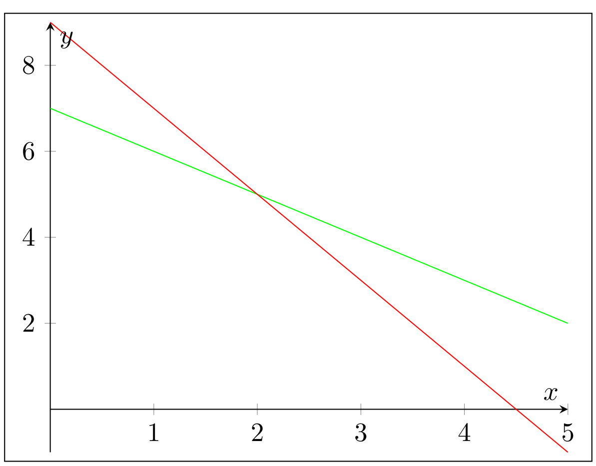
$$\begin{aligned} 8.0 \times 10^{-5} &= X \cdot (0.0044)^2 + Y(0.0044) + Z \\ 5.0 \times 10^{-5} &= X \cdot (0.0034)^2 + Y(0.0034) + Z \\ 3.1 \times 10^{-5} &= X \cdot (0.0027)^2 + Y(0.0027) + Z. \end{aligned}$$

I'll leave off the end, because the point is we used a computer to solve this system of equations. This week, we'll learn a *systematic* method for solving such systems: not because it's easier than using a computer, but because it's interesting to see what's going on geometrically. (In my Y13 notes, there's a lot more detail — and proofs, which I'll skip this year.)

The main idea to get your head around this year is that a system of equations is *geometric*.
Example. Consider the system of equations

$$\begin{cases} x + y = 7 \\ 5x + 10y = 45. \end{cases}$$

If we plot all the values (x, y) which satisfy these equations, we obtain:



The key point here is that the point which satisfies both solutions is simply the geometric point of intersection.

In order to find this point algebraically, we can solve the first equation for y: $y = 7 - x$. We can then substitute this into the other: $5x + 10(7 - x) = 45$. Simplifying, we have

$$
\begin{aligned}
5x + 70 - 10x &= 45 \\
25 &= 5x \\
5 &= x.
\end{aligned}
$$

Let's think about this a little more explicitly: since the same pair of values (x, y) makes both equations true, then if we can find y in terms of x using one equation then we can substitute it straight into the other one *because the symbol y represents the same thing in both.*

In another situation, we might have a line and a parabola.
Example (Extract from the sample L2 assessment for this topic.)**.** The student committee is planning the upcoming Performing Arts Showcase at your school this year. They are trying to determine how much they should make the price of adult tickets this year.

Determine the price they should set the adult tickets in 2012 if they want to make a profit of \$3210 from ticket sales, given that:

- The cost of a child ticket is planned to be \$5 and the cost of an adult ticket is not determined yet.
- The relationship between the expected number of child tickets and adult tickets to be sold can be modelled by $200x + y^2 = 80000$, where x represents the number of child tickets and y represents the number of adult tickets.
- The planning committee wants to make \$3210 from ticket sales.
- There needs to be only one possibility for the number of child and adult tickets to be sold.

Solution. The total profit that will be made is $5x + my$, where m is the cost of an adult ticket (yet to be determined); we therefore have three important pieces of information: $5x + my = 3210$, $200x + y^2 = 80000$, and the curves of these two equations intersect at only one point. Solving the first for x, we have $x = \frac{3210 - my}{5}$; substituting, we have

$$
\begin{aligned}
200\left(\frac{3210 - my}{5}\right) + y^2 &= 80000 \\
128400 - 40my + y^2 &= 80000 \\
48400 + 40my + y^2 &= 0
\end{aligned}
$$

When we learned about parabolae, we learned that we could transform them into vertex-intercept form. Let's do that here, because our goal is to find a value m such that the vertex of $48400 + 40my + y^2 = x$ is sitting exacty on the y-axis — our parabola is flipped sideways, but the idea is the same. In fact, to make it clearer let's look at the 'graphy' form:*

$$(y + 20m)^2 + 48400 - 400m^2 = x.$$

This parabola is sitting on the y-axis when $48400 - 400m^2$ is zero (because this is the x-shift); hence $m = 11$, and the committee should sell the adult tickets for \$11 each.

*The idea of switching from looking at something in an 'equationy' sense and turning something into a parameter is actually quite a powerful idea that's applicable in many different situations.

Questions

1. Graph and solve the system of equations

$$\begin{cases} 1 = x + y, \\ 2 = 2x - y. \end{cases}$$

2. Let us explore how many solutions we can obtain for simple systems. If

$$\begin{cases} a = bx + cy, \\ p = qx + ry \end{cases}$$

is a system of simultaneous equations in x and y then depending on the constants we choose there are three possible situations:

- We can have no pairs (x, y) that satisfy both equations.
- We can have precisely one pair (x, y) that satisfies both equations.
- We can have infinitely many pairs (x, y) that satisfy both equations.

(We will prove this next year.)

(a) Draw a diagram showing each situation geometrically.

(b) Give an example of a system for each case, taking care to show that your systems have the desired number of solutions.

3. Graph (using a computer or a calculator) and solve the system of equations

$$\begin{cases} y = 2x + 3, \\ x^2 + 2xy - 1 = 0. \end{cases}$$

4. [Extract from the sample L2 assessment for this topic.] For the performing arts showcase example above, you must also provide the planning committee with the change in the adult tickets sales in 2011 compared to 2010. You are given that:

- **2010 Performing Arts Showcase**
  - The total number of tickets sold was 400.
  - The relationship between the number of child tickets and adult tickets sold can be modelled by $x^2 + y = 22750$, where x represents the number of child tickets and y represents the number of adult tickets.
- **2011 Performing Arts Showcase**
  - The cost of a child ticket was \$5 and the cost of an adult ticket was \$10.
  - The relationship between the number of child tickets and adult tickets sold can be modelled by $xy = 1000 + 100x$, where x represents the number of child tickets and y represents the number of adult tickets.
  - More than 300 tickets were sold.
  - The money generated from ticket sales was \$2050.

5. For which values of c does the system of equations

$$\begin{cases} y^2 = 2x^2 + xy \\ y = cx - 2 \end{cases}$$

have precisely two solutions?

6. Find the parabola passing through the three points $(0,0)$, $(1,1)$, and $(3,0)$.

7. The graph of the equation $xy = a$ (where a is a number) is called a *hyperbola*.
   (a) Show that $xy = a$ and $xy = b$ never intersect if $a \neq b$.
   (b) Find the point of intersection between the curves

$$\begin{cases} x(y+1) = 4 \\ y = 2x - 3. \end{cases}$$

8. The line $y = 2x - 3$ intersects the circle $x^2 - 6x + y^2 = 0$ precisely twice.
   (a) Find the coordinates of the centre of the circle.
   (b) Find the points of intersection.
   (c) How could the constant term 3 of the line be changed such that the line becomes tangent with the circle?