NCEA Level 3 Calculus (Integration)
18. Substitution

Recall that the *chain rule* is:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$$

Since integration is (in some sense) the inverse of differentiation, we can write:

$$\int f'(g(x))g'(x)\,\mathrm{d}x = f(g(x)) + C.$$

For a mnemonic, we can let $u = g(x)$. Then $\mathrm{d}u = g'(x)\,\mathrm{d}g^*$, and then (by substitution) we have

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(u)\,\mathrm{d}u = f(u) + C = f(g(x)) + C.$$

Example. Suppose we wish to find $\int \sin x \cos x\,\mathrm{d}x$. Then let $u = \sin x$, so $\mathrm{d}u = \cos x\,\mathrm{d}x$ and

$$\int \sin x \cos x\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C.$$

In this case, we also could have used a trigonometric identity.

Example. Suppose we wish to find $\int xe^{x^2}\,\mathrm{d}x$. We can let $u = x^2$, and then $du = 2x\,\mathrm{d}x \Rightarrow \mathrm{d}x = \frac{du}{2x}$. Hence:

$$\int xe^{x^2}\,\mathrm{d}x = \int \frac{1}{2}e^u\,\mathrm{d}u = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

Example. Suppose we wish to find $\int \frac{4}{x}(\ln x)^3\,\mathrm{d}x$. We let $u = \ln x$, and then $\mathrm{d}u = \frac{\mathrm{d}x}{x}$. Hence:

$$\int \frac{4}{x}(\ln x)^3\,\mathrm{d}x = 4\int u^3\,\mathrm{d}u = u^4 + C = (\ln u)^4 + C.$$

Questions

1. $\boxed{\text{A}}$ Find the following indefinite integrals:
   (a) $\int \sin 2x\,\mathrm{d}x$
   (b) $\int (4x - 44)^{2019}\,\mathrm{d}x$
   (c) $\int 4x\sqrt{x^2+3}\,\mathrm{d}x$
   (d) $\int (3t-4)^2\,\mathrm{d}x$
   (e) $\int \frac{x}{x^2+1}\,\mathrm{d}x$
   (f) $\int \frac{2}{4x+3}\,\mathrm{d}x$
   (g) $\int e^{2x+1}\,\mathrm{d}x$
   (h) $\int \sec 4x \tan 4x\,\mathrm{d}x$
   (i) $\int 2\cos x + \sin 2x\,\mathrm{d}x$
   (j) $\int -2x\csc^2(3x^2)\,\mathrm{d}x$
   (k) $\int \frac{3}{x^3} - \frac{4}{x+1}\,\mathrm{d}x$

*this is not rigorous, but it 'works'

(l) $\int e^{x/2} + \frac{2}{x}\,\mathrm{d}x$

(m) $\int x^2 \sec^2 x^3 + 9\,\mathrm{d}x$

(n) $\int -\csc(\tan x)\cot(\tan x)\sec^2 x\,\mathrm{d}x$

(o) $\int \frac{\cos x - \sin x}{\cos x + \sin x}\,\mathrm{d}x$

(p) $\int \frac{2017}{x \ln x}\,\mathrm{d}x$

2. $\boxed{\texttt{M}}$ By using the substitution $x = \sin\theta$, find

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

3. $\boxed{\texttt{M}}$ Compute the following definite integrals:

(a) $\int_0^1 xe^{-x^2}\,\mathrm{d}x$

(b) $\int_{-\pi/3}^{\pi/3} x^4 \sin x\,\mathrm{d}x$ (hint: no substitution is required)

(c) $\int_0^1 \cos(\pi t/2)\,\mathrm{d}t$

(d) $\int_0^1 (3t-1)^{50}\,\mathrm{d}t$

(e) $\int_0^1 \sqrt[3]{1+7x}\,\mathrm{d}x$

(f) $\int_0^1 \frac{\mathrm{d}x}{1+\sqrt{x}}\,\mathrm{d}x$

(g) $\int_{-1}^2 x(x-1)^3\,\mathrm{d}x$

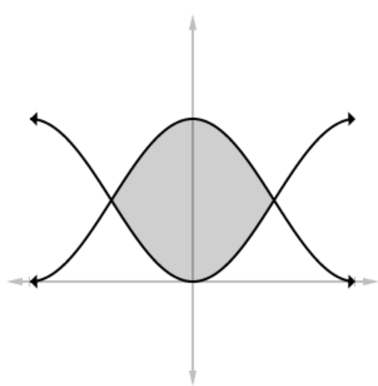
(h) $\int_0^3 x\sqrt{1+x^2}\,\mathrm{d}x$

4. $\boxed{\texttt{M}}$ Find the area enclosed by the curve $y = 4\sin 3x \cos x$ and the $x-$axis from $x = 0$ to $x = \frac{\pi}{3}$.

5. $\boxed{\texttt{E}}$ Find k such that $\int_0^k e^{2x}\,\mathrm{d}x = 40$.

6. $\boxed{\texttt{E}}$ Calculate the area enclosed by the curve $y = \frac{3x-2}{x+4}$ and the lines $y = 0$, $x = 1$, and $x = 5$.

7. $\boxed{\texttt{E}}$ Find the area between the curves $y = \sin^2 kx$ and $y = \cos^2 kx$ shaded below.



8. $\boxed{\texttt{M}}$ Find $\int \tan\theta\,\mathrm{d}\theta$ and $\int \cot\theta\,\mathrm{d}\theta$.

9. $\boxed{\texttt{A}}$ Complete the following working:

$$\begin{aligned}\int \sec x \, \mathrm{d}x &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, \mathrm{d}x \\ &= \int \frac{\ldots}{\sec x + \tan x} \, \mathrm{d}x \\ \text{Let } u = \ldots & \\ &= \int \frac{1}{\ldots} \, \mathrm{d}u \\ &= \ldots\end{aligned}$$

10. $\boxed{\texttt{M}}$ Show that

$$\int x^4 \sin x \, \mathrm{d}x = 4x(x^2 - 6x)\sin x - (x^4 - 12x^2 + 24)\cos x + C.$$

11. $\boxed{\texttt{M}}$ If $y = x\sqrt{\sin x^3 + \cos x^3}$, find $\pi \int_0^1 y^2 \, \mathrm{d}x$.

12. $\boxed{\texttt{M}}$ The velocity of a particle at time t is given by $v = \frac{\cos(\sqrt{2t+1})}{\sqrt{2t+1}}$. What is the position of the particle at time $t = 5$, given that $x(0.5) = 0$?

13. $\boxed{\texttt{S}}$ Evaluate $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, \mathrm{d}x$ [*Hint: use the substitution* $x = \frac{\pi}{2} - u$ *and add the result to the original integral.*]

14. $\boxed{\texttt{S}}$ Scholarship 1999:
    (a) Evaluate $\int \cos^5 x \, \mathrm{d}x$ using the substitution $t = \sin x$.
    (b) i. If $f(x) = \cos^5 x$, what are $f(0)$, $f'(0)$, and $f''(0)$?
        ii. Hence evaluate a, b, and c in the approximation $\cos^5 x \approx a + bx + cx^2$.
        iii. Use this to give an approximation for $\int \cos^5 x \, \mathrm{d}x$.
    (c) Evaluate $\int_0^{0.6} \cos^5 x \, \mathrm{d}x$ to three significant figures, using:
        i. The exact integration in (a).
        ii. The expression in (b)(iii).
        iii. Simpson's rule.