NCEA Level 3 Calculus (Integration)
19. Differential Equations

Many physical problems can be expressed by writing different rates of change in terms of each other. For example, for a spring pulled a distance x away from its equilibrium point we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

for some constant k; and for a falling stone at distance r from the centre of the Earth, we have

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -g\frac{r_0^2}{r^2}$$

where r_0 is the radius of the Earth and g is a constant. These kinds of equations are known as **differential equations**.

Suppose $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$. It would be nice if we could find y in terms of x only. This is in fact possible, using substitution:

$$\begin{aligned}\tfrac{\mathrm{d}y}{\mathrm{d}x} &= f(x)g(y)\\ \Rightarrow \frac{1}{g(y)}\tfrac{\mathrm{d}y}{\mathrm{d}x} &= f(x)\\ \Rightarrow \int \frac{1}{g(y)}\tfrac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int f(x)\,\mathrm{d}x.\end{aligned}$$

Now, let $G(y)$ be an antiderivative of $\frac{1}{g(y)}$ (with respect to y). By the chain rule, then,

$$\frac{\mathrm{d}}{\mathrm{d}x}G(y) = \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}$$

and so

$$\int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = G(y) = \int \frac{1}{g(y)}\,\mathrm{d}y.$$

Hence we have

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

This way of solving differential equations is called **separation of variables**.

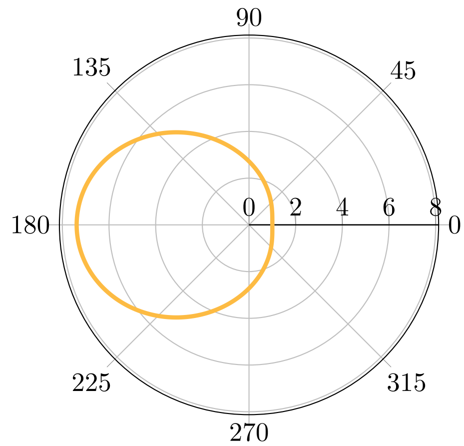
Example. Suppose we know that $y\frac{\mathrm{d}y}{\mathrm{d}x} = e^x$. Then we can separate the variables:

$$\begin{aligned}\int y\,\mathrm{d}y &= \int e^x\,\mathrm{d}x\\ \Rightarrow \frac{1}{2}y^2 &= e^x + C\\ \Rightarrow y^2 &= 2e^x + C.\end{aligned}$$

If we know that the curve passes through $(0, 0)$, then $0 = 2e^0 + C$ and $C = -2$, so $y^2 = 2e^x - 2$.

To check our answer, let us now use implicit differentiation to differentiate this curve. We have $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2e^x$ so and $y\frac{\mathrm{d}y}{\mathrm{d}x} = e^x$ as expected: our solution is correct.

Example. Suppose that $r(\theta)$ is implicitly defined by $\frac{\mathrm{d}r}{\mathrm{d}\theta} = r\sin\theta$ with the condition $r(\pi) = e$. Separating variables, we have $\int \frac{\mathrm{d}r}{r} = \int \sin\theta\,\mathrm{d}\theta$; so $\ln|r| = -\cos\theta + C$ and therefore $r = Ke^{-\cos\theta}$ for some constant K. But $e = Ke^{-\cos\pi} = Ke^0 = K$; so $r(\theta) = ee^{-\cos\theta} = e^{1-\cos\theta}$. Graphing this:



Questions

1. Find y in terms of x in each case, if each curve passes through $(1,1)$: M

   (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = yx$

   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} + x = yx$

   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} + y = yx$

   (d) $\sqrt{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$

   (e) $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+2)^2$

   (f) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2+1}{2y}e^x$

   (g) $\frac{\mathrm{d}y}{\mathrm{d}x} = x\cos^2 y$

   (h) $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \tan y$

   (i) $2y\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + 2x + 1$

   (j) $\sin y \frac{\mathrm{d}y}{\mathrm{d}x} = 3x$

   (k) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x(e^{x^2}+2)}{6y^2}$

2. (a) Show that one antiderivative of $f(x) = x\sin x\,\mathrm{d}x$ is $F(x) = \sin x - x\cos x$. E

   (b) Find $y(\pi)$ if $y(0) = \pi$ and
   $$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \theta y \sin\theta.$$

3. Newton's law of cooling states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings; in other words, if the temperature of the body at time t is T and the ambient temperature is T_∞ then $\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_\infty)$ (where k is some constant.) E

   (a) A loaf of bread is taken from the oven at a temperature of 400 °C and is set down on a bench in an area with an ambient temperature of 20 °C. It is found that it takes ten minutes for the loaf to cool to half its initial temperature. How many minutes will it take for the loaf to cool to 30 °C?

   (b) A detective is called to the scene of a crime where a dead body has just been found. She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and is found to be 24 °C. The detective checks the programmable thermostat and finds that the room has been kept at a constant 20 °C for the past 3 days.

   After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 22 °C. This last temperature reading was taken exactly one hour after the first one. Assuming that the victim's body temperature was normal (37.5 °C) prior to death, at what time did the victim die?

4. We will apply calculus concepts to chemical rates of reaction. E

   (a) A *first-order reaction* is one whose rate depends linearly on the concentration of one reactant A; in other words, $-\frac{\mathrm{d}[A]}{\mathrm{d}t} = k[A]$.

   One example of a first-order reaction is the decomposition of hydrogen peroxide:

   $$2\,\mathrm{H_2O_2(aq)} \longrightarrow 2\,\mathrm{H_2O} + \mathrm{O_2(g)}$$

   What percentage of hydrogen peroxide will have decomposed after 600 seconds, if the reaction constant is $k = 6.40 \times 10^{-5}\,\mathrm{s^{-1}}$?

   (b) If a reaction depends linearly on the concentrations of two reactants, A and B (as in $\mathrm{A + B \longrightarrow C}$) then the rate of reaction is given by

   $$-\frac{\mathrm{d}[A]}{\mathrm{d}t} = -\frac{\mathrm{d}[B]}{\mathrm{d}t} = \frac{\mathrm{d}[C]}{\mathrm{d}t} = k[A][B].$$

   If we consider the reaction $\mathrm{NO_2 + CO \longrightarrow CO_2 + NO}$, the rate is experimentally found to be *second-order* in the reactant $\mathrm{NO_2}$ and independent of the concentration of carbon monoxide. It follows that the rate of reaction is

   $$\frac{\mathrm{d}[\mathrm{NO_2}]}{\mathrm{d}t} = -k[\mathrm{NO_2}]^2$$

   where k is some constant.

   Initially, the concentration of $\mathrm{NO_2}$ is $2.0\,\mathrm{mol\,L^{-1}}$; after ten minutes, the concentration has decreased to $1.0\,\mathrm{mol\,L^{-1}}$. How long will it take for the concentration to become $0.5\,\mathrm{mol\,L^{-1}}$?

5. It is known that the motion of a particle is described by the differential equation M

   $$v = \frac{4\sin(2t)}{x}.$$

   Initially, the particle is two metres away from the origin in the positive $x-$direction. Find the particle's position after ten seconds.

6. Suppose that $y'(x) = e^{x+2y}$, and $y(0) = 0$. Find $y(x)$ explicitly. M

7. Assume that the rate of reproduction of some population P is proportional to the number of pairs of individuals; so M

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2.$$

   Show that the size of the population becomes infinitely large in a finite time.

8. Recall from physics that **simple harmonic motion** occurs when the force F on an object is proportional to (and opposite in direction to) its displacement k from some zero point ($F = -kx$). Recall also that Newton's second law tells us that the acceleration felt by an object of mass m is given by $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{F}{m}$. We wish to find a formula for x, the displacement of the object, at time t. We have: M

   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{kx}{m}$$

   Show that $x = A\cos(\sqrt{k/m}\cdot t)$ is a solution of the given differential equation.

9. Consider the general wave equation, $y = A\sin(kx - \omega t)$ (where A, k, and ω are constant). We write $\frac{\partial y}{\partial x}$ for the derivative of y with respect to x holding t constant, and $\frac{\partial y}{\partial t}$ for the derivative of y with respect to t keeping x constant. S

   Show that $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ for some constant c.

10. Physics: Write down a differential equation modelling the charge on the capacitor in an RC circuit over time. Solve the equation. S

11. Scholarship 2000: The piriform is the curve defined by the equation $16y^2 = x^3(8-x)$ where $x \geq 0$. By solving the differential equation S

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2(6-x)}{8y}$$

($y = 0$ when $x = 0$), show the piriform is the solution.

12. Scholarship 2015: Determine all differentiable equations of the form $y = f(x)$ which have the properties: S

$$f'(x) = (f(x))^3 \text{ and } f(0) = 2$$