NCEA Level 3 Calculus (Integration)
24. Kinematics

Calculus was independently developed by Sir Isaac Newton to describe mechanical motion in physics. This use is known as **kinematics** (from the Greek *kinein*, 'to move'). Suppose a particle moves from position x_0 to position x_1 over a time Δt. We call the ratio

$$\frac{x_1 - x_0}{\Delta t}$$

the **average velocity** of the particle; if we let $x_1 \to x_2$ (or let $\Delta t \to 0$), we obtain the derivative $\frac{\mathrm{d}x}{\mathrm{d}t} = v$, the **instantaneous velocity** of the particle at the point x (usually just abbreviated to velocity).

Similarly, the rate of change of velocity is called *acceleration*. We have $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$. (Out of interest, the third derivative of displacement is known as *jerk*, and the fourth is *jounce*.)

Now, suppose we know the velocity of a particle at each instant over a given time interval. Suppose we split the interval up into small intervals, each of length Δt. Then the total distance travelled is approximated by $\sum v\Delta t$, where the sum is taken for each small interval. If we make the intervals smaller, then clearly our approximation becomes better; and to obtain the true answer, we need only take an integral.

Displacement, s		$\int_{t_0}^{t_1} v \,\mathrm{d}t$
Velocity, v	$\frac{\mathrm{d}s}{\mathrm{d}t}$	$\int_{t_0}^{t_1} a \,\mathrm{d}t$
Acceleration, a	$\frac{\mathrm{d}v}{\mathrm{d}t}$	

We can prove the following **kinematic equations** if acceleration is kept constant over a time period Δt. These equations should be familar to all of those that took level 2 physics, and they are derived by finding areas underneath a velocity-time graph: in short, via calculus.

$$\begin{aligned} v_f &= v_i + a\Delta t \\ s &= v_i \Delta t + \frac{1}{2} a \Delta t^2 \\ v_f^2 &= v_i^2 + 2as \\ s &= \frac{v_f + v_i}{2}\Delta t \end{aligned}$$

Questions

All distances are given in m, and all times in s, unless otherwise stated.

1. A particle moves from $x = 2\,\mathrm{m}$ to $x = 3\,\mathrm{m}$ over $3\,\mathrm{s}$. What is its average velocity over that time? A

2. A particle moves from $(3, 4)$ to $(12, -3)$ over a time $10\,\mathrm{s}$. If the displacements are measured in metres, what is the magnitude of its average velocity during that period? A

3. An object A has a positive acceleration a, and a second object B has a negative acceleration $-a$. Both are moving in the same direction. Which of the following is *not* true? A
   (a) Object B is slowing down compared to object A.
   (b) Object B has a lower velocity than object A.
   (c) At some point, object B will reach a velocity of zero and then start moving in the opposite direction.
   (d) If object B is behind object A, the two will never cross paths.

4. Suppose a particle has a constant velocity of $34\,\mathrm{m\,s^{-1}}$. How long does it take for the particle to travel $150\,\mathrm{m}$? A

5. Derive the kinematic equations, by considering the integrals of a velocity function $v(t)$ with constant derivative a. M

6. The velocity v of an object t seconds after it moves from the origin is given by M

$$v(t) = 3t^2 - 6t - 24.$$

   (a) Write down the formula for the acceleration of the particle after t seconds.
   (b) Work out the initial velocity and acceleration.
   (c) When is the object at rest momentarily?
   (d) When did the object return to the origin?
   (e) What was the total distance travelled by the object before it returned to the origin?

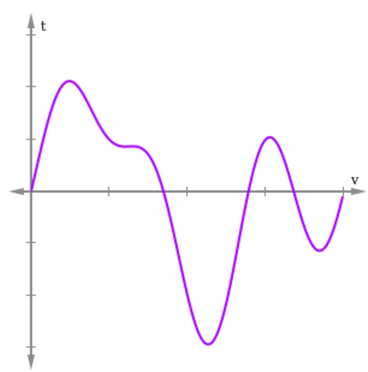
7. A well-wrapped food parcel is dropped from an aeroplane flying at a height of 500 m above the ground. The constant acceleration due to gravity is $-9.81\,\mathrm{m\,s^{-2}}$. Air resistance is negligible. M

   (a) How long does it take for the food parcel to hit the ground?
   (b) How fast is the food parcel moving when it hits the ground?

8. A racing car travelling at $210\,\mathrm{km\,h^{-1}}$ skids for a distance of 150 m after its brakes are applied. The brakes provide a constant deceleration. M

   (a) What is the deceleration in $\mathrm{m\,s^{-2}}$?
   (b) How long does it take for the car to stop?

9. The following is a graph of the instantaneous velocity of an object moving in one dimension over time. M



   (a) Draw the acceleration of the object over time.
   (b) Draw the position of the object over time, if it was originally located at $x = 0$.

10. The velocity of an Olympic sprinter is modelled by M

$$v_x = a(1 - e^{-bt}),$$

where $a = 11.81\,\mathrm{m\,s^{-1}}$ and $b = 0.6887\,\mathrm{s^{-1}}$. Find an expression for the distance travelled after time t.

11. The displacement of an object moving in a straight line on either side of a fixed origin is given by E

$$s(t) = 2t^3 - 12t^2 + 18t + 3.$$

   (a) Find the minimum velocity of the object. Carefully prove that you have found a minimum.
   (b) What is the distance between the origin and the object when its velocity is at a minimum?

12. The acceleration of a rocket propelled washing machine is given by $\frac{\mathrm{d}v}{\mathrm{d}t} = 9t^3 - t^4 + t^{-3/2}$, where $0 \leq t \leq 10$. Find the distance which it has travelled after 10 seconds if its initial velocity (at $t = 0$) was $90\,\mathrm{m\,s^{-1}}$. M

13. The acceleration of an object is given by $a(t) = 0.2t + 0.3\sqrt{t}$ for $0 \leq t \leq 10$, where a is the acceleration of the object in $\mathrm{m\,s^{-2}}$ and t is the time in seconds from the instant that movement began. The object was moving with a velocity of $5\,\mathrm{m\,s^{-1}}$ when $t = 4$. How far was the object from its starting point after nine seconds? M

14. A ball is thrown straight up from the edge of the roof of a building, with initial velocity v_0. A second ball is dropped from the roof 1.00 s later. Both feel a constant acceleration due to gravity, $g = -9.81\,\mathrm{m\,s^{-2}}$. E
    (a) Suppose the height of the building is 20.00 m. What must be the initial speed v_0 if both balls are to hit the ground at the same time?
    (b) Consider a second building of unknown height h; if the first ball is thrown upwards with initial velocity v_1, and both balls hit the ground at the same time, give an expression for h in terms of v_1.