

NCEA Level 3 Calculus (Integration)

19. Differential Equations (Homework)

Reading

So far, we have looked at equations where the unknown is either a number or a point in n -dimensional space (that is, a sequence of n numbers). In order to generate these equations, we took various combinations of the basic arithmetical operations and applied them to our unknowns.

Here, for comparison, are two well-known differential equations, the first “ordinary” and the second “partial”:

$$\frac{d^2x}{dt^2} + k^2x = 0,$$
$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right).$$

The first is the equation for simple harmonic motion, which has the general solution $x(t) = A \sin kt + B \cos kt$; the second is the heat equation which describes the way that the distribution of heat in a physical medium changes with time.

For many reasons, differential equations represent a jump in sophistication. One is that the unknowns are *functions*, which are much more complicated objects than numbers or n -dimensional points. (For example, the first equation above asks what function x of t has the property that if you differentiate it twice then you get $-k^2$ times the original function.) A second is that the basic operations one performs on functions include differentiation and integration, which are considerably less “basic” than addition and multiplication. A third is that differential equations which can be solved in “closed form,” that is, by means of a formula for the unknown function f , are the exception rather than the rule, even when the equations are natural and important.

From ‘The Princeton Companion to Mathematics’, I.4 §1.5

Questions

1. Solve the following equations for $y(t)$:

(a) $e^{y-t} \frac{dy}{dt} = 1$

(b) $\frac{dy}{dt} = ty^2$

(c) $\frac{dy}{dt} = \frac{1}{\sec^2 y}$

(d) $\frac{dy}{dt} = -\frac{t}{\sec t \sin y}$ (*Hint: first show that $\frac{d}{dx} [\cos x - x \sin x] = -x \cos x$*)

2. A copper ball with temperature 100°C is dropped into a basin of water with constant temperature 30°C . After 3 minutes the temperature of the ball has decreased to 70°C . When will it reach a temperature of 31°C ?
3. Consider a tank of water. The rate of flow of water into the tank is a constant 3 L s^{-1} ; the flow out is directly proportional to the volume of water in the tank. Initially, the volume of water in the tank is 100 L ; if the volume were to increase to 120 L s^{-1} , the rate of water flowing out would exactly balance the rate of water flowing in.
- (a) Form a differential equation and find the volume of water after ten minutes.
- (b) Does the outward rate of flow ever become greater than the incoming rate of flow?