# NCEA Level 2 Mathematics
11. Slopes and Differentiation

The main theme this year so far has been geometry and modelling the real world. This week, we move from modelling the stationary world of fields, fences, and compound interest to the world of continuity and motion.

We have already seen that if we model a changing quantity with a linear equation of the form $y = mx + c$, then it is possible to talk about the rate of change of that quantity by means of the slope of the graph of the equation.

Example. Suppose that the amount of sand in a pile (in kilograms, perhaps) is modelled by the equation mass $= 3(\text{time}) + 1$. Then the slope of the graph of mass versus time is 3, and so it makes sense to say that the rate of change of mass is three kilograms per unit of time.

Our goal this week is to describe the slopes of functions which are not necessarily lines. Let us look at some arbitrary function f, and let us draw the graph $y = f(x)$. Then, recalling that we originally defined slope to be $m = \frac{\text{change in } y}{\text{change in } x}$, we can look at the slope between $y_0 = f(x_0)$ and $y_1 = f(x_1)$:
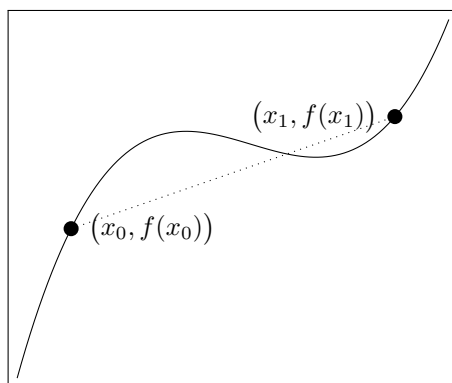
The average slope betwen $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is just the slope of the dotted line, which is simply

$$\frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

If we move the point (x_0, y_0) closer to (x_1, y_1) along the curve, then it makes sense that the slope of the dotted line will match the slope of $y = f(x)$ at the point (x_1, y_1) better and better (the dotted lines in the diagram below). If we imagine moving the point (x_0, y_0) until it coincides with the point (x_1, y_1), then the 'average' slope (indicated by the solid line below) will in face be the actual slope of the curve at x_1.
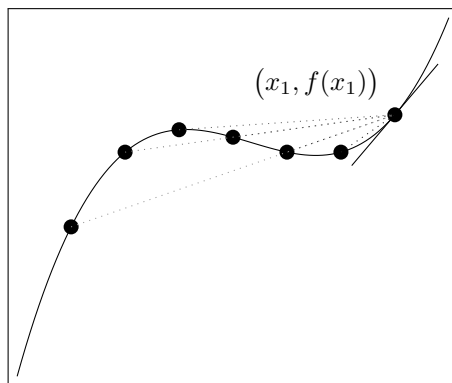
Example. Suppose we look at the curve $y = x^2$. The two points $(0, 0)$ and $(2, 4)$ are on this curve; the average slope of the curve between $x = 0$ and $x = 2$ is therefore $\frac{4-0}{2-0} = 2$.

Now, suppose we want to find the actual slope of the parabola at $x = 2$. We will do this in a slightly tricky way: finding the average slope between $x = 2$ and $x = 2 + h$, and then setting h to zero. The coordinates of the point on the parabola at $x = 2 + h$ will be $(2 + h, (2 + h)^2) = (2 + h, 4 + 4h + h^2)$ and so the average slope that we are looking for is

$$\frac{\text{rise}}{\text{run}} = \frac{(4 + 4h + h^2) - 4}{(2 + h) - 2} = \frac{4h + h^2}{h} = 4 + h;$$

hence the average slope between $x = 2$ and $x = 2 + h$ when $h = 0$ (i.e. when the two points are the same), and thus the actual slope of $y = x^2$ at $x = 2$, is 4.

More generally, suppose we are to find the slope of the parabola at (x_0, x_0^2). To do this, we will find the average slope between this point and the point $(x_0 + h, (x_0 + h)^2) = (x_0 + h, x_0^2 + 2x_0h + h^2)$. The calculation runs as follows:

$$\frac{\text{rise}}{\text{run}} = \frac{(x_0^2 + 2x_0h + h^2) - x_0^2}{(x_0 + h) - x_0} = \frac{2x_0h + h^2}{h} = 2x_0.$$

Does this function give us the slope we expect? For $x < 0$, the parabola is downward-sloping (if we move to the right, we move down) and so we would expect the slope to be negative; and for $x > 0$, the parabola is upwards-sloping and so we would expect the slope to be positive. The function $x \mapsto 2x$ satisfies both these criteria, and so this is reassuring: our algebraic slope-finding seems to have given a reasonable answer.

Hence, given the function $f : x \mapsto x^2$, we can write down another function $f' : x \mapsto 2x$ that gives the slope of the graph $y = f(x)$ at any point we choose.

It is, in fact, possible with most functions f we have met to write down another function f' such that the second function (called the *derivative* of f) gives the slope of $y = f(x)$ at any point we choose.

Example. Let's try $f(x) = x^3$ now; we'll go a bit quicker this time.

$$\frac{\text{rise}}{\text{run}} = \frac{(x + h)^3 - x^3}{(x + h) - h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = 3x^2 + 3xh + h^2$$

and so (setting h to zero) it seems that the derivative of f is $f' : x \mapsto 3x^2$.

Does this make sense? Well, $y = x^3$ is always sloping upwards, and $3x^2$ is never negative, so yes — this does make sense.

With the basic examples out of the way, I will state (without proof, but the proof is similar to the ideas above) the following theorem.

Theorem (Power rule)**.** *If f is a function defined by $f(x) = ax^r$, where r is any real number, then the derivative of f is the function defined by $f'(x) = rax^{r-1}$.*

This matches our example above: for x^2, we obtained $2x = 2x^{2-1}$ and for x^3 we obtained $3x^2 = 3x^{3-1}$. Note though that this theorem holds when r is any real number. Hence:

Example.

1. The derivative of $3x^{2018}$ is $6054x^{2017}$.

2. The derivative of $9x^\pi$ is $9\pi x^{\pi-1}$.

3. The derivative of 97 is 0.

4. The derivative of $\sqrt{x}$ is $\frac{1}{2}x^{-1/2}$.

5. The derivative of $1/x$ is $-\frac{1}{x^2}$.

We also have the following theorem which allows us to combine derivatives:

Theorem. *If f and g are functions, and λ is any number, then*

1. $(\lambda)' = 0$ — *the derivative of a flat line/constant is zero.*

2. $(f + g)' = f' + g'$ — *the derivative of the sum of two functions is the sum of the derivatives.*

3. $(\lambda f)' = \lambda f'$ — *the derivative of a number times a function is the number times the derivative.*

Hence,

Example.

1. The derivative of $2x^2 + 3x$ is $4x + 3$.

2. The derivative of $19x^{\sqrt{2}} - 3$ is $19\sqrt{2}x^{\sqrt{2}-1}$.

Finally, we can obviously take derivatives with respect to variables that are not x; obviously the two functions $x \mapsto f(x)$ and $y \mapsto f(y)$ are the same function, and the derivative of $3y^2$ with respect to y is $6y$.

If we want to make the variable of differentiation clear, we can use the alternative Leibniz notation: if $y = f(x)$, then the derivative is

$$\text{either} \qquad f' \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x}.$$

If we take the derivative of the derivative, the result is called the second derivative and is notated by

$$\text{either} \qquad f'' \qquad \text{or} \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

In general, the nth derivative is notated by

$$\text{either} \qquad f^{(n)} \qquad \text{or} \qquad \frac{\mathrm{d}^n y}{\mathrm{d}x^n}.$$


Maria Agnesi (1718–1799), an Italian mathematician, was the author of one of the first influential textbooks on calculus, and was also the first woman appointed as a mathematics professor at a university.

Questions

1. Since the sum rule for derivatives holds (i.e. $(f + g)' = f' + g'$), one might be tempted to guess that the product rule $(fg)' = f'g'$ also holds. Unfortunately, this is not the case.

    (a) Differentiate $f(x) = 4x^3$, $g(x) = 2x^2$, and $h(x) = 8x^5$.

    (b) Notice that $fg = h$, but $f'g' \neq h'$.

    (c) There is in fact a product rule for derivatives, but this year you won't need it. Find the derivative $\frac{dy}{dx}$ if $y = (3x + 2)(x^2 + 1)$.

2. Differentiate the function, with respect to x or the stated variable.

    (a) $a(x) = 186.5$

    (b) $b(x) = \sqrt{30}$

    (c) $c(x) = 5x - 1$

    (d) $z = -4x^{10}$

    (e) $e(x) = x^3 - 4x + 6$

    (f) $f(t) = \frac{1}{2}t^6 - 3t^4 + t$ with respect to t

    (g) $g(x) = x^2(1 - 2x)$

    (h) $h(x) = (x - 2)(2x + 3)$

    (i) $y = x^{-2/5}$

    (j) $B(y) = cy^{-6}$ with respect to y

3. The following part-questions require you to find higher derivatives of functions.

    (a) Find $\frac{d^2 x}{dt^2}$ if $x = 3t^2 + 4$.

    (b) Suppose $f(x) = 9x^2 + \frac{1}{x^3} + \sqrt{x}$. Find $f''(x)$.

    (c) Find $\frac{d^3 y}{dt^3}$ if $y = 3x^{-1}$.

    (d) Find $f''(x)$ if $f'(x) = 9x^2 + 3x^{-2}$.

    (e) Find $a(t) = \frac{d^2 s}{dt^2}$ if $s(t) = 3t - 4t^{-1}$.

4. If we 'zoom out' from the graph of any polynomial, and ignore the 'wiggly bits' in the middle, the result always 'looks like' either the graph of $y = \pm x^2$, or the graph of $y = \pm x^3$.

    (a) Check this by zooming out from $y = 3x^6 + 2x^3 - 17x^2 + 3$ and $y = -2x^9 + 30x^2 - x$ on a graphing calculator.

    (b) Intuitively justify why the derivative of an odd polynomial (one which has an odd highest power of x) is an even polynomial, and the derivative of an even polynomial is an odd polynomial.

5. Where is the graph of $y = \frac{x^2}{2} - \frac{x^3}{3}$ increasing?

6. Draw the graph of the slope function (derivative) for each of the following graphed functions.

(a)

(b)

(c)

(d)

(e)
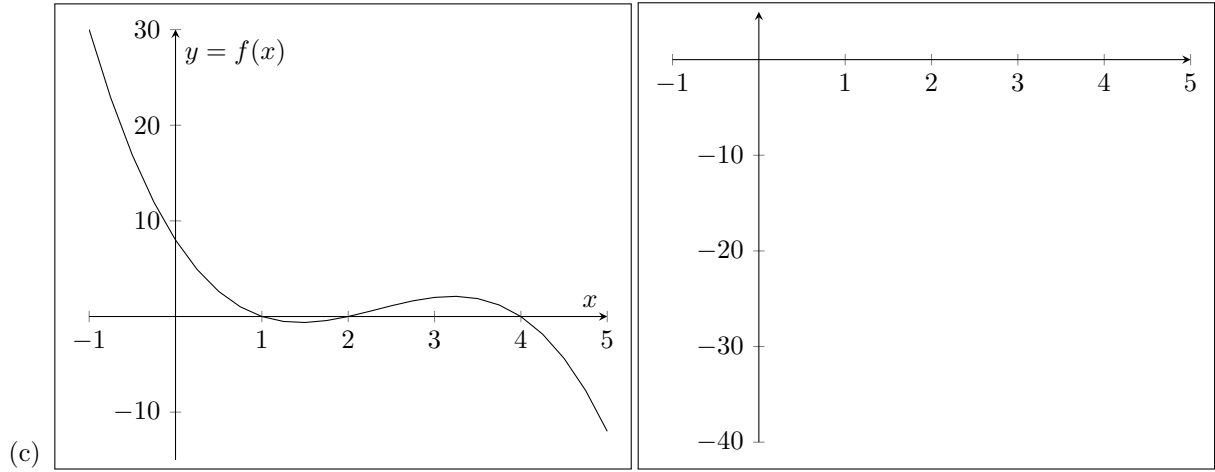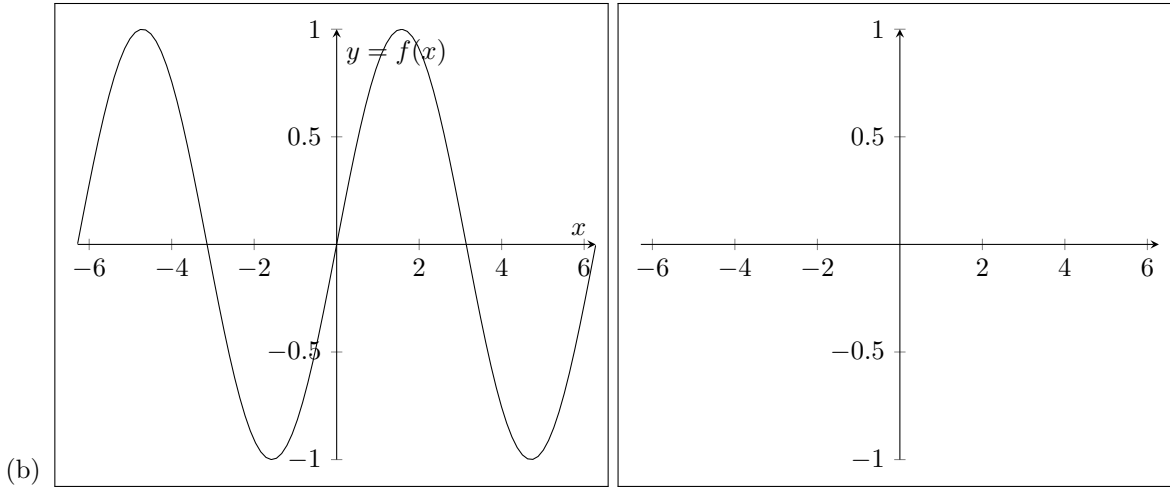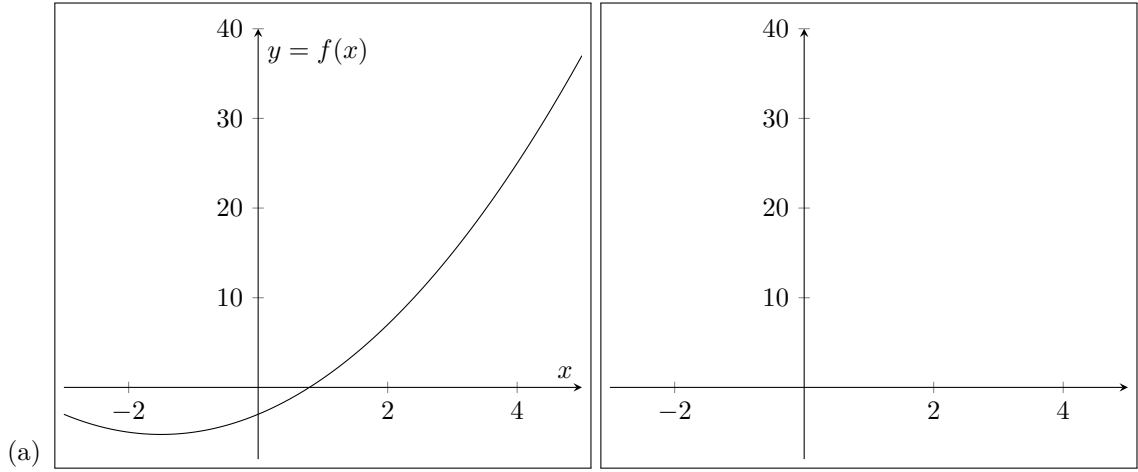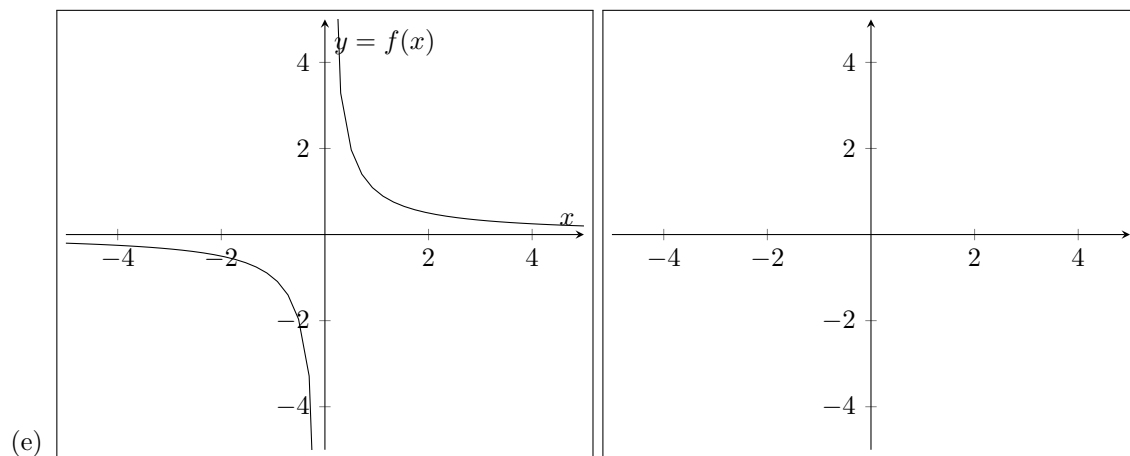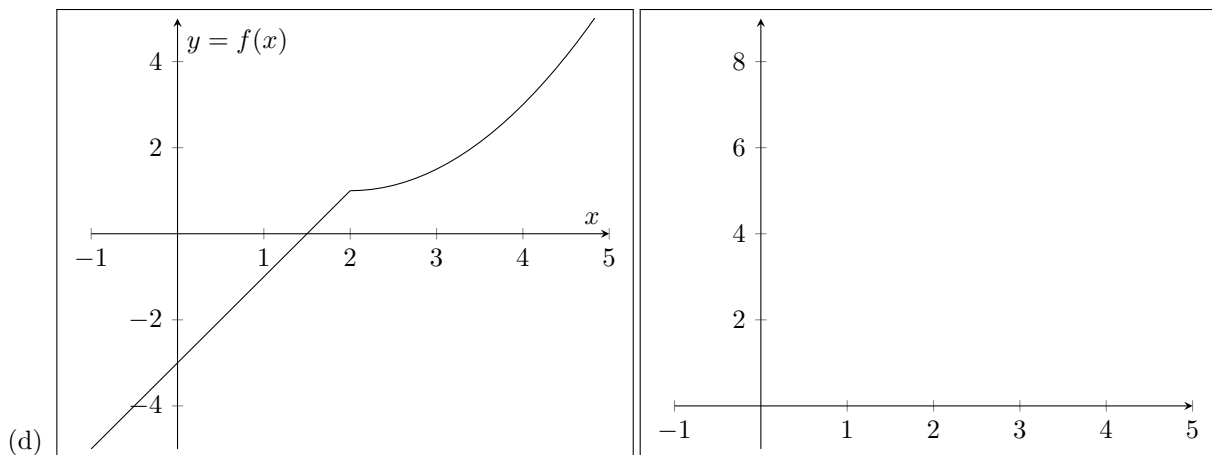
7. Car tires need to be inflated properly because overinflation or underinflation can cause premature treadwear. The graph shows tire life L (in thousands of kilometres) for a certain type of tire at various pressures P (in kPa), as well as a quadratic function that models the tire life.

Quadratic polynomial fit (tire pressure vs. life)

$L = -0.0092P^2 + 4.553P - 434.5891$
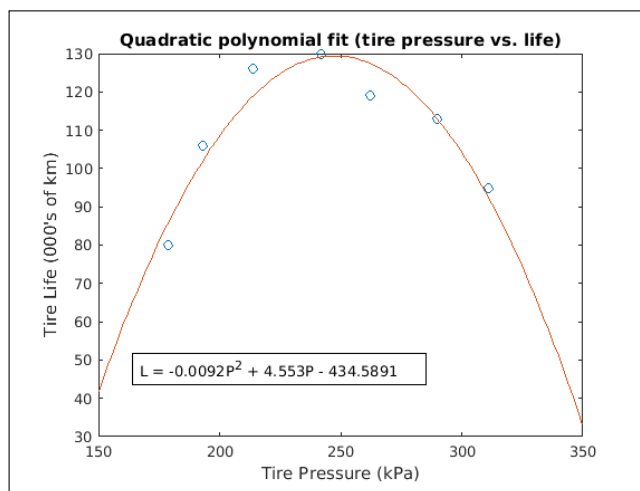
Use the model to estimate $\frac{\mathrm{d}L}{\mathrm{d}P}$ when $P = 200$ and when $P = 300$. What is the meaning of the derivative? What is the significance of the sign of the derivatives?

8. A function f is given by $f(x) = 2 - 4x + 4x^2 + ax^3$. The gradient of the graph at the point where $x = 1$ is 3. Find the value of a.

9. Let f be a function of x defined by $f(x) = 3x^2 + 6x + 6$. Show that f is a solution of the *differential equation* $f(x) - f'(x) = 3x^2$.

10. Show that the function f of y defined by $f(y) = \frac{3}{y^2} + 2y$ is not differentiable at $y = 0$.

11. It is natural to ask if there is any function such that the function is its own derivative.

    (a) An obvious candidate is the function K defined by $K(x) = 0$. Show that $\frac{\mathrm{d}K}{\mathrm{d}x} = K(x)$.

    (b) For a more interesting example, we will look at the exponential functions $y = a^x$. Draw the gradient function of $y = 2^x$; explain why looking at the exponential functions is probably a productive way to answer our question.

    (c) Suppose we try to use the same technique as we used with the parabola. Show that the average slope between (x, a^x) and $(x + h, a^{x+h})$ is given by

    $$a^x \left(\frac{a^h - 1}{h} \right).$$

    (d) For a^x to be its own slope, we want $\frac{a^h - 1}{h}$ to get closer and closer to 1 as h gets closer and closer to zero. Unfortunately we can't just substitute $h = 0$ straight in. Why?

    (e) Let $a = (1 + h)^{1/h}$ (where we take h very small), and show that $\frac{a^h - 1}{h} = 1$. This suggests that a works as a base for our exponential-which-is-its-own-derivative if we let h get closer to zero here.

    (f) Again, we can't let $h = 0$ in a — so let $n = 1/h$, and show that $a = \left(1 + \frac{1}{n}\right)^n$.

    (g) If $n \to 0$, then $h \to \infty$. Therefore, our base we want is just $\left(1 + \frac{1}{n}\right)^n$ if we let n get infinitely large: the notation we use is

    $$\lim_n \left(1 + \frac{1}{n}\right)^n,$$

    which you can read as the *limit* with respect to n. Calculate this base for some large value of n. Is the number you see familiar?