

NCEA Level 3 Calculus (Integration)

21. Integration by Parts

The substitution rule is the inverse of the chain rule; similarly, there is an inverse of the product rule.

$$\begin{aligned}\frac{d}{dx} [f(x)g(x)] &= f'(x)g(x) + f(x)g'(x) \\ \Longleftrightarrow \int f'(x)g(x) + f(x)g'(x) dx &= f(x)g(x) \\ \Longleftrightarrow \int f(x)g'(x) dx &= f(x)g(x) - \int f'(x)g(x) dx\end{aligned}$$

Mnemonically,

$$\int u dv = uv - \int v du.$$

Example. Consider $\int x \sin x dx$, which does not yield to any obvious change of variable. Let $u = x$, and let $dv = \sin x dx$. So $du = dx$, and $v = -\cos x$. Hence:

$$\int x \sin x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C,$$

where C is an arbitrary constant. Check that $(-x \cos x + \sin x)' = x \sin x$.

The aim is to end up with an easier integral than the one that was started with. A good choice for u is usually (in descending order of usefulness):

1. Logarithms
2. Powers of x
3. Exponentials
4. Trig functions

Questions

1. S Compute the following indefinite integrals.

- (a) $\int x e^x dx$
- (b) $\int x^2 e^{2x} dx$
- (c) $\int \ln x dx$
- (d) $\int p^5 \ln p dp$
- (e) $\int t^3 e^{-t^2} dt$
- (f) $\int \sin \ln y dy$
- (g) $\int x \tan^2 x dx$

2. S Prove that

$$\int \cos^n x dx = \frac{1}{n} \sin x \cos^{(n-1)} x + \frac{n-1}{n} \int \cos^{(n-2)} x dx$$

3. S If $I_n = \int_0^n x^n e^x dx$, write down an explicit general formula for I_n .

4. [S] Evaluate $\int (\ln x)^2 dx$.
5. [S] Compute $\int_0^\lambda te^{-\lambda t} dt$.
6. [S] Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, and $f'(4) = 3$. Evaluate $\int_1^4 xf''(x) dx$.
7. [S] A particle moving in one dimension has a velocity function $v(t) = t^2e^{-t}$ (where t is in seconds). What is its displacement from its starting position after three minutes?
8. [S] Find the area bounded by $y = x^2 \ln x$ and $y = 4 \ln x$
9. Scholarship 2012:
- (a) [E] Find $\frac{d}{dx}[x \cos x]$ and use this result to find $\int x \sin x dx$.
- (b) [S] Hence find the value of $\int_0^{n\pi} x \sin x dx$ for integer values of n .
10. [O] Scholarship 2016:
- (a) A function $f(x)$, where x is a real number, is defined implicitly by the formula

$$f(x) = x - \int_0^{\pi/2} f(x) \sin(x) dx.$$

Find the explicit expression for $f(x)$ in simplest form.

- (b) A curve passing through the point $(1, 1)$ has the property that at each point (x, y) on the curve, the gradient of the curve is $x - 2y$; that is, $\frac{dy}{dx} = x - 2y$.
- i. Show that $\frac{d}{dx}e^{2x}y = xe^{2x}$.
- ii. Hence, or otherwise, find the equation of the curve.
11. [S] It is well known that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Using this result, show that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

12. [O] Find $I = \int e^x \cos x dx$.
13. [S] Recall that $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$. Find $\int \tan^{-1} x dx$.