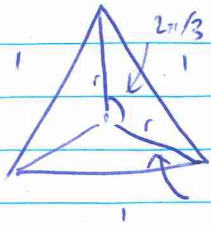
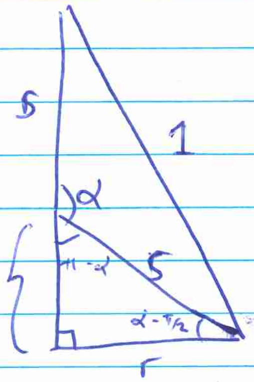
Schol trig problem set 5.

1. Assume the tetrahedron has side length 1 for simplicity.



top-down view.

$$1 = 2r^2 - 2r^2\cos\tfrac{2\pi}{3}$$
$$1 = 2r^2\left(1-\cos\tfrac{2\pi}{3}\right)$$
$$= 2r^2\left(1+\tfrac{1}{2}\right)$$
$$r^2 = \tfrac{1}{3}.$$



$1^2 =$

$$\sqrt{1^2-r^2} - s$$
$$= \sqrt{1-\tfrac13} - s$$
$$= \sqrt{\tfrac23} - s.$$

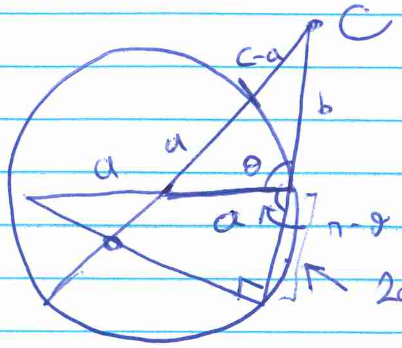
$$s^2 = r^2 + \left(\sqrt{\tfrac23}-s\right)^2 = r^2 + \tfrac23 - 2\sqrt{\tfrac23}\,s + s^2$$

$$s = \frac{r^2 + 2/3}{2\sqrt{2/3}} = \frac{\sqrt3}{2\sqrt2}. \qquad s^2 = \frac38.$$

$$1^2 = 2s^2 - 2s^2\cos\alpha$$
$$= 2\cdot\tfrac38\,(1-\cos\alpha)$$
$$\cancel{\tfrac{8}{64} = 1-\cos\alpha} \qquad \cancel{\cos\alpha = 1 - \tfrac{8}{24} = \tfrac{16}{24}.}$$
$$\frac86 = 1-\cos\alpha \qquad \cos\alpha = 1-\frac86 = -\frac26 = -\frac13.$$

$$\alpha = \arccos\left(-\tfrac13\right) \approx \underline{109.47^\circ}$$

2. The proof is the same, but now the chords intersect at the point C outside the circle:



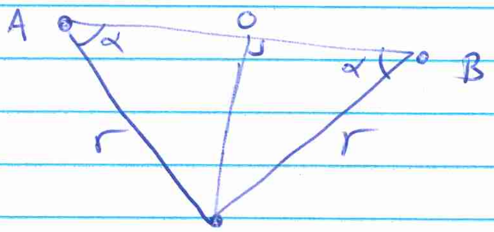
$2a\cos(\pi-\theta) = -2a\cos\theta$

$$(a+c).(c-a) = (b+2a\cos\theta)\,b$$

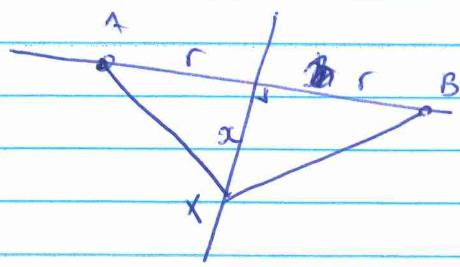
$$c^2-a^2 = b^2-2ab\cos\theta$$

$$c^2 = a^2+b^2-2ab\cos\theta.$$

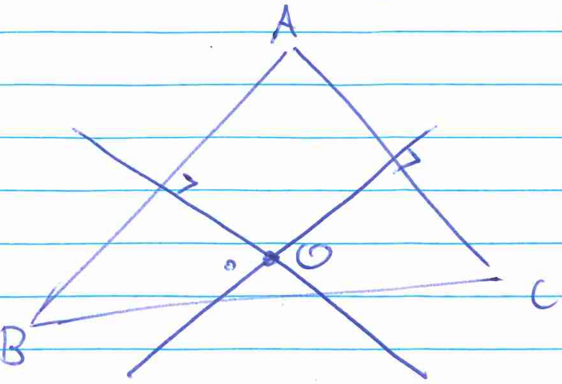
3a. Suppose X is equidistant from A and B. Then ABX is isosceles. Drop the perpendicular from X to AB at O. Then $|AO| = r\cos\alpha = |BO|$; so OX is the perp. bisec. of AB.



Conversely, if X lies on the perpendicular bisector of AB, then $|AX| = \sqrt{x^2+r^2} = |BX|$.



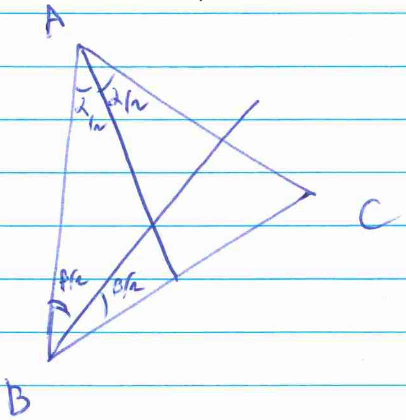
3b. Let ABC be a triangle. Let L_C be the perp bisector of AB, and let L_B be the perp bisector of AC. Then the intersection O of L_B and L_C is equidistant from A and B (since it lies on L_C) and equidistant from A and C (since it lies on L_B).



So $|OB| = |OA| = |OC|$.

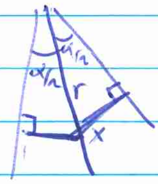
Thus O also lies on the perp bisec. of BC, and hence the three bisectors do intersect at a single point. Further, since $|OA| = |OB| = |OC|$, ABC lie on a circle centred at O with radius $|OA|$.

3c.

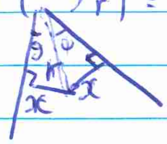


Let L_A be the angle bisector at A.

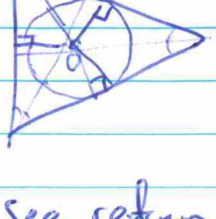
Then if X is on L_A,
$d(X, AC) = r\sin\frac{\alpha}{2}$
$= d(X, AB)$.



Conversely, if X is equidistant from AB and AC,
then $\theta = \sin^{-1}(x/r) = \phi$.



Hence if O is the intersection pt of L_A and L_B,
then $d(O, AB) = d(O, AC) = \cancel{d}$
and $d(O, AB) = d(O, BC)$
$d(O, AC) = d(O, BC) \Rightarrow O$ lies on L_C.



Hence O is equidistant from all three sides and is thus the centre of the incircle.

Qn 4: See return on prob sheet.