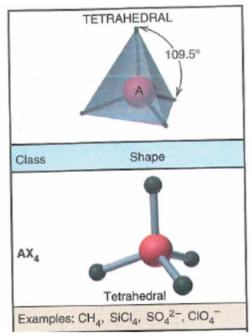
NCEA Level 3 Trigonometry (exercise set)
5. The Sine and Cosine Laws

Goal To consider the relations of arbitrary (i.e. non-right-angled) triangles and associated circles.

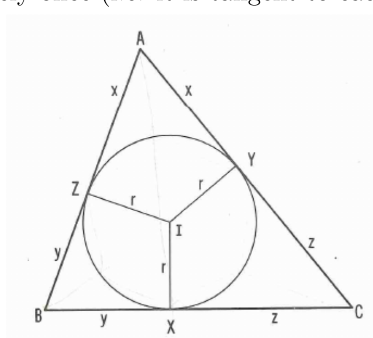
1. There exists a point O inside the regular tetrahedron $ABCD$ such that O is equidistant from the four vertices. (You may assume this without proof.) Compute the angle between OA and OB.



(Image credit: Silberberg, *Chemistry* 3e, figure 10.8.)

2. The proof of the cosine rule, as given in the notes, only works if $c \leq a$ (i.e. if the chords intersect inside the circle). Show that if $c > a$ then the same idea works (consider exercise 5.3).

3. (a) Let A and B be two points. Show that the set of all points equidistant from A and B is just the line perpendicular to the segment AB, that passes through the midpoint of A and B.
   (b) The circumcentre (centre of the circumcircle) of a triangle ABC is the point equidistant from all three vertices. Show that the circumcentre is the point of intersection of the perpendicular bisectors of the sides of the triangle (so in particular all three lines pass through a single point).
   (c) Suppose ABC is a triangle. Show that there exists a (unique) circle with centre inside ABC that touches each side of ABC precisely once (i.e. it is tangent to each side).



   (Image credit: Coxeter, *Geom. Revisited*, figure 1.4A.) Hints:
   i. Draw the triangle ABC.
   ii. Suppose the angle at A has measure α; draw the line through A that bisects the angle there (i.e. cuts it into two angles of measure $\alpha/2$). Likewise, draw angle bisector lines through B and C.
   iii. Show that, if P is on the perpendicular bisector at the vertex A, then P is equidistant from AB and AC. (The distance from a point P to a line ℓ is the length of the segment with one end P and perpendicular to ℓ at the other end.)
   iv. Use (iii) and a similar argument to (b) above to show that all three bisectors intersect at a single point O that is equidistant to all three sides.

4. Prove the *Steiner-Lehmus theorem*: if a triangle has two equal angle bisectors (see 2(c)(ii) above), then it is isoceles.

   "This theorem was sent to the great Swiss geometer Jacob Steiner in 1840 by C. L. Lehmus (whose name would otherwise have been forgotten long ago) with a respect for a pure geometrical proof. Steiner gave a fairly complicated proof which inspired many other people to look for simpler methods. Papers on the Steiner-Lehmus theorem appeared in various journals in 1842, 1844, 1848, almost every year from 1854 till 1864, and with a good deal of regularity during the next hundred years." (Coxeter and Greitzer, *Geometry Revisited*, §1.5. Random House (1967).)

   Remark: This may be the most difficult exercise in any of the exercise sets for this topic. However, I will not give any hints because there are multiple productive avenues of study for you. A number of solutions are given in the paper linked in the footnote, and a purely geometric one is given in the citation for the above quote.[1]

Additional reading Coxeter, *Intro. to Geometry*, sections 1.4 to 1.6 inclusive. Coxeter, *Geometry Revisited*, chapter 1.

[1] Róbert Oláh-G'al and József Sándor, *On Trigonometric Proofs of the Steiner-Lehmus Theorem.* Forum Geometricorum, Volume 9 (2009) 155160. `http://forumgeom.fau.edu/FG2009volume9/FG200914.pdf`. See also `https://cs.nyu.edu/pipermail/fom/2004-August/008394.html` for a philosophical reason that no 'easy' proof exists.