Assignment: Mathematical Writing Practice II

Alexander Elzenaar
Upper Hutt College

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1 Task

A tank contains $1000\,\mathrm{L}$ of pure water. Brine that contains $0.05\,\mathrm{kg}$ of salt per litre of water enters the tank at a rate of $5\,\mathrm{L\,min^{-1}}$. Brine that contains $0.04\,\mathrm{kg}$ of salt per litre of water enters the tank at a rate of $10\,\mathrm{L\,min^{-1}}$. The solution is kept thoroughly mixed and drains from the tank at a rate of $15\,\mathrm{L\,min^{-1}}$. How much salt is in the tank after one hour?

Ensure that you write 'properly'. That means using complete sentences, justifying all logic, and aiming for clarity!

2 Hints

A list of things to think about:

- What information are you given?
- How can you model the amount of salt remaining?
- What will be your strategy for working out an answer?

3 Example Answer

This problem is very similar to a problem that appeared in the 2015 scholarship paper.

We begin by attempting to write a DE to model the situation. We will use t to denote the time since initial conditions ($0\,\mathrm{kg}$ of salt in the tank), S to denote the mass of salt in the tank, and C to denote the concentration of salt in the tank. Immediately, we see that $C = \frac{S}{1000}$.

We notice that we have two inputs of salt: an input of $0.05\,\mathrm{kg\,L^{-1}} \times 5\,\mathrm{L\,min^{-1}} = 0.25\,\mathrm{kg\,min^{-1}}$, and an input of $0.04\,\mathrm{kg\,L^{-1}} \times 10\,\mathrm{L\,min^{-1}} = 0.4\,\mathrm{kg\,min^{-1}}$. We have a single output of $C\,\mathrm{kg\,L^{-1}} \times 15\,\mathrm{L\,min^{-1}} = 15C\,\mathrm{kg\,min^{-1}}$.

Since $\frac{\mathrm{d}S}{\mathrm{d}t} = \text{rate in} - \text{rate out}$, we have that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0.25 + 0.4 - 15C = 0.65 - 15C.$$

From the result above, we can substitute in order to obtain a separable DE in S,

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0.65 - \frac{15S}{1000}.$$

Solving this for S, we find that

$$\begin{aligned}
& \int \frac{\mathrm{d}S}{0.65 - \frac{15S}{1000}} = \int \mathrm{d}t \\
\Rightarrow \quad & -\frac{1000}{15} \ln\left|0.65 - \frac{15S}{1000}\right| = t + K && \text{(where } K \text{ is an arbitrary constant)} \\
\Rightarrow \quad & \ln\left|0.65 - \frac{15S}{1000}\right| = -\frac{15t}{1000} + K' && \text{(where } K' = -\frac{15K}{1000}\text{)} \\
\Rightarrow \quad & S = \left|Re^{-\frac{15t}{1000}} - \frac{650}{15}\right| && \text{(where } R = \frac{1000}{15}e^{K'}\text{)}
\end{aligned}$$

Remembering that our initial condition was that $S(0) = 0\,\mathrm{kg}$, we see that $R = \frac{650}{15}$. Hence, after $60\,\mathrm{min}$, we have $S(60) = \left|\frac{650}{15}e^{-\frac{15\times 60}{1000}} - \frac{650}{15}\right| = 25.715\,\mathrm{kg}$.