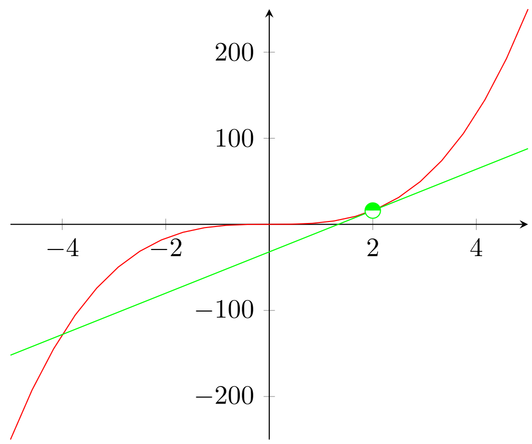
NCEA Level 3 Calculus (Differentiation)
6. Tangent and Normal Lines

Last year, we defined the tangent line of a function at a point to be the unique line passing through that point such that the line has the same slope as the function at that point. More formally, if we have some function f which is differentiable at a point (x_0, y_0) then the tangent line to the function at that point is

$$(y - y_0) = f'(x_0)(x - x_0).$$

Example. Consider $f : x \mapsto 2x^3$. Then $f' : x \mapsto 6x^2$, and the tangent line to the function at $(2, 16)$ is $(y - 16) = 24(x - 2)$, or in slope-intercept form $y = 24x - 32$.

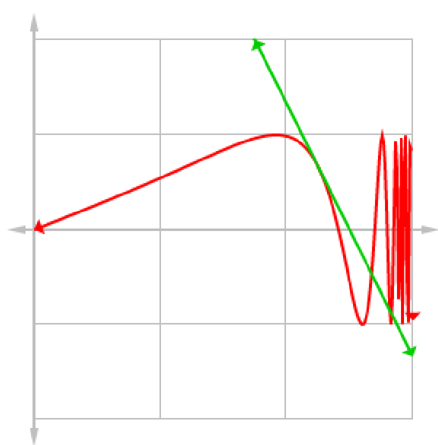


The tangent line of a function at a point is the best linear approximation to the function at that point.

We further defined the normal line of a function at a point to be the unique line passing through the point at right angles to the tangent line. If the gradient of the tangent line is m, then the gradient of the normal line is $m^\perp = -\frac{1}{m}$.*

Example. The slope of the normal line of the function f defined above at the given point is $-\frac{1}{24}$.

Example. Consider the function $y = \sin \tan x$. The derivative of this function is $y' = \sec^2 x \cos \tan x$; at the point $P(\frac{3\pi}{8}, 0.6649)$, the slope is -5.1.



The tangent line at P is $y = \sin(\sqrt{2} + 1) + \cos(\sqrt{2} + 1)((\sqrt{2} + 1)^2 + 1)(x - \frac{3\pi}{8})$, or (approximately) $y = -5.1(x - 1.178) + 0.6649$.

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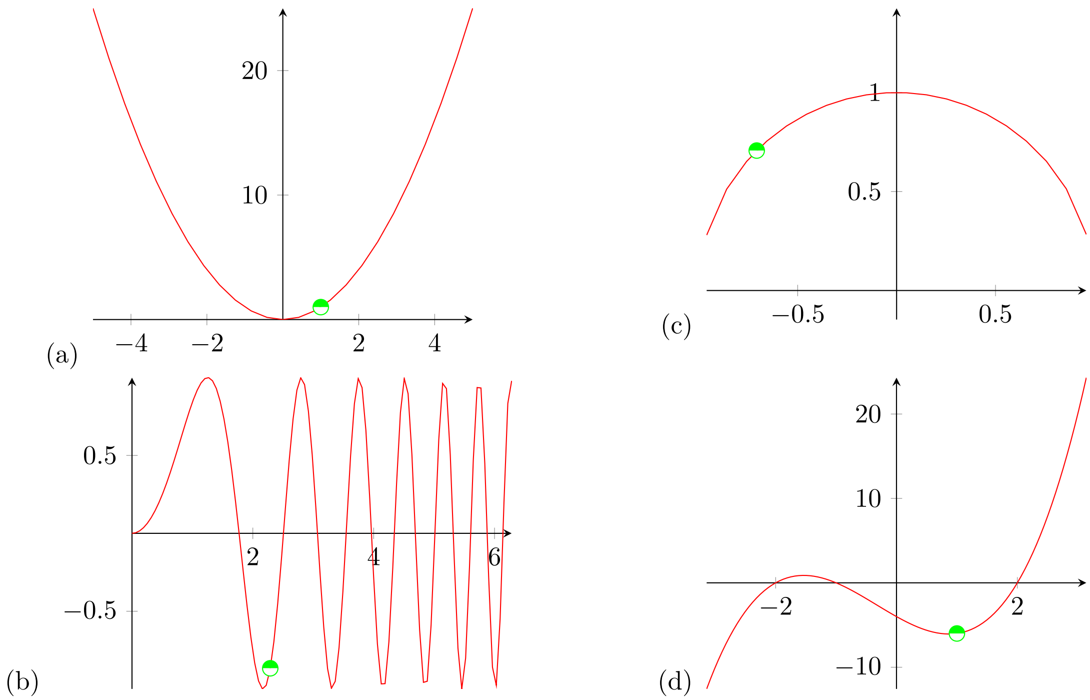
*See supplementary proof sheet.

Questions

1. $\boxed{\texttt{A}}$ Draw the tangent and normal lines to each function at the indicated points.



2. $\boxed{\texttt{A}}$ When is the tangent line a good approximation to a function?

3. $\boxed{\texttt{M}}$ Find the tangent and normal lines to the following functions at the given points.
   (a) $x \mapsto \sin x$ at $(0, 0)$
   (b) $x \mapsto \sin x$ at $(\pi, 0)$
   (c) $x \mapsto e^x$ at $(0, 1)$
   (d) $x \mapsto \sec x$ at $(0, 0)$
   (e) $x \mapsto x^2$ at $(1, 1)$
   (f) $x \mapsto \sqrt{x}$ at $(1, 1)$
   (g) $x \mapsto (x^4 - 3x^2 + 5)^3$ at $(0, 125)$
   (h) $x \mapsto \cos \tan x$ at $(\pi, 1)$
   (i) $x \mapsto \left(x + \frac{1}{x^2}\right)^{\sqrt{7}}$ at $(1, 2^{\sqrt{7}})$.

4. $\boxed{\texttt{S}}$ Consider the surface $S : (x, y) \mapsto z = x^2 + 2\sin(y^2 + 1) + 2$.
   (a) Differentiate S with respect to x, and find the tangent line in the x direction at $(0, 0)$.
   (b) Differentiate S with respect to y, and find the tangent line in the y direction at $(0, 0)$.
   (c) Hence describe the tangent *plane* of the surface S at $(0, 0)$.

5. $\boxed{\texttt{M}}$ Find the equation of the tangent line to $y = x + \tan x$ at (π, π).

6. $\boxed{\texttt{M}}$ Find an equation for the normal line to the curve $y = \frac{1}{\sqrt{x^2 - x}}$ at $(2, \frac{1}{\sqrt{2}})$.

7. Consider the curve $y = \tan(2 \sin x)$.

(a) $\boxed{\texttt{A}}$ Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x \sec^2(2\sin x)$.

(b) $\boxed{\texttt{M}}$ Find the equation of:

i. The tangent to the curve at $(\pi, 0)$

ii. The normal to the curve at $(0, 0)$

8. $\boxed{\texttt{M}}$ Find the best linear approximation to $y = 3x^3 + 2x + 4$ around $x = 2$.

9. $\boxed{\texttt{M}}$ Find the point(s) on the graph of the function $y = x^2$ such that the slope of the normal to the curve at that point is $m^{\perp} = -1$.

10. $\boxed{\texttt{E}}$ The tangents to the curve $y = \frac{1}{4}(x-2)^2$ at points P and $Q = (6, 4)$ are perpendicular. What is the x-ordinate of P?

11. Consider the function $y = \sin x$.

(a) $\boxed{\texttt{M}}$ Find the best linear approximation to this function around $(0, 0)$.

(b) $\boxed{\texttt{M}}$ Find the percentage error of this approximation to 1dp when $x = \pi$.

(c) $\boxed{\texttt{E}}$ At what point on the curve ($x \geq 0$) does the percentage error of the approximation rise above 100%?

12. (a) $\boxed{\texttt{M}}$ Find two points on the graph of $y = 1/x$ that share a common normal line.

(b) $\boxed{\texttt{M}}$ Show that there are no more such points.

(c) $\boxed{\texttt{M}}$ Show that there are no two points on the graph that share a common tangent line.

(d) $\boxed{\texttt{E}}$ Repeat parts (a)-(c) for the general hyperbola-like curve $y = x^{-n}$ (where n is a positive integer).

(e) $\boxed{\texttt{S}}$ What is the situation for the even more general case of the curve $y = x^r$, where r is any real number?

13. Consider the quartic polynomial $p(x) = 2x^4 - 4x^3 - 23x^2 + 84x - 61$.

(a) $\boxed{\texttt{M}}$ Find the best linear approximation to p around the point $(2, 15)$.

(b) $\boxed{\texttt{S}}$ Find the unique quadratic polynomial $q(x)$ such that $q(2) = p(2)$, $q'(2) = p'(2)$, and $q''(2) = p''(2)$. This is the best quadratic approximation to p at the point $(2, 15)$.

(c) $\boxed{\texttt{O}}$ Show that the best quadratic approximation to a function f at the point (x_0, y_0) is given by

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2.$$