Level Three Calculus

Second Edition

ALEX ELZENAAR

https://github.com/aelzenaar/ncea-notes

Preface for the navigator

These notes are my second attempt at a coherent introduction to calculus at the level of NCEA Level 3 and NZ Scholarship.

I have made a few philosophical changes from the first edition:-

- I treat anti-differentiation at the same time as differentiation. (I do introduce the $\int$ notation and the term 'indefinite integral' here, though I would rather not.)

- The notes are split into "topic" chapters: *The basics* (the basic formal manipulations of derivatives and anti-derivatives), *Geometry of curves* (studying curves via differentiation), *Geometry of spaces* (definite integrals, the fundamental theorem of calculus, and arc lengths, surface area, and volumes), and *Motion and change* (differential equations).

- I have dropped many of the proofs; my justification for this change is threefold(!). Firstly, the students that 'need' the proofs will see them in a Stage I university course. Secondly, many of the proofs in elementary calculus often obscure a nugget of geometry behind the formal manipulation of limits, and so I would rather include intuitive geometric *justifications* for results in the space the proofs formerly were. Finally, most students at Level 3 are simply not ready for proofs: either they don't understand why proof is required, or their level of mathematical sophistication means that the proofs seem esoteric. I have included copious references to textbooks where proofs can be found.

I feel the need also to point out that these notes are incredibly geometric. *If you don't like teaching geometry, these are not the notes for you.*

Prerequisite material

Firstly, a hard fact: for a student to be successful in L3 calculus, they should have a good understanding of the material at L2 and earlier (I would generally expect that students with less than a merit in the level 2 algebra standard will struggle).

In these notes, I will use material from algebra and geometry at L2 or earlier liberally; I try to point it out when I use some of the more obsure results. I do not use any material from any of the level three standards, except trigonometry.

So, in general, the prerequisites and expectations for these notes are:-

- A good understanding of L2 coordinate geometry and algebra.

- A decent understanding of L3 trigonometry, *including the manipulation of identities*.

For some of the sections, knowledge of a little physics (L1 and/or L2) would be nice. I cover the material in the L2 calculus standard quickly so this is not formally a prerequisite, but a student who doesn't understand the material there well will struggle with these notes. Roughly speaking, the differentiation material there is more important.

I would strongly recommend revising the material on functions (section 4 of my own level 2 notes).

Recommended textbooks

I have used the following textbooks when writing these notes, in roughly increasing order of sophistication:

- *Calculus made easy*, by Sivanus P. Thompson and Martin Gardner. This book is perhaps at the correct level mathematically speaking for a Y12/13 student, but it is not very geometric. It is certainly worth looking at, though.
- *Calculus*, by James Stewart. This is one of the standard first-year computational calculus books. It has many examples and many exercises, but lacks soul.
- *Calculus*, by Michael Spivak. This is often called the 'One True Calculus Book',[1] but is more properly an introduction to real analysis. As such, it is too diffcult for all but the most motivated high school students.
- (For the sake of completeness,) *Advanced Calculus*, by L. H. Loomis and S. Sternberg. This is the author's favourite calculus book, but is eminently unsuitable for high school students of any motivation. It is incredibly geometric, wonderfully well written, and almost impenetrable. This book inspired these notes, philosophically speaking.

[1] For example, by the *Chicago undergraduate mathematics bibliography*: `https://www.ocf.berkeley.edu/~abhishek/chicmath.htm` (somewhat useful, if taken with a grain of salt).

Preface for the student

I don't have much to say, really. I could spend time explaining why calculus is useful, but I won't do that here because we'll see a lot of examples of calculus 'in the wild' as we progress (just as a taster, we'll look at some physics, some biology, some economics, and maybe even some statistics). I could equally well spend time trying to explain exactly what calculus *is* exactly, but this page is not large enough to contain such an exposition.

Instead, I will give some study advice.

Firstly, you must read the notes. You must sleep with them beneath your pillow. You must work through the examples yourself. You must do all the problems. You must ask questions.

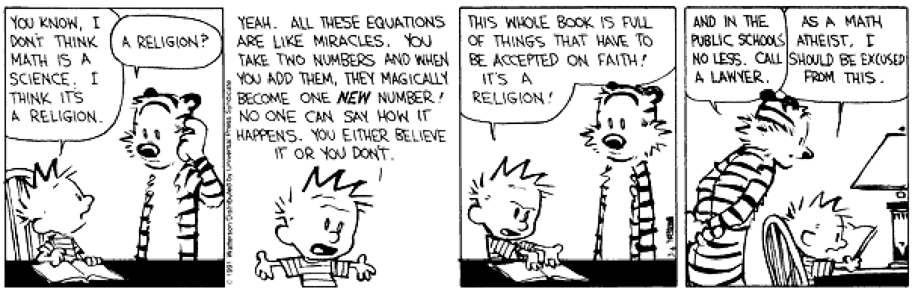
Because no student ever follows my first piece of advice, I will give you a second, easier option. For each topic, there are a few homework problems set. *At a minimum*, you should do all these problems. (But beware, if you *only* do these problems, you will be woefully underprepared for any situation you need calculus for.)

Secondly, draw pictures. I do my best to include lots of diagrams (some even in colour!), but one can never have too many pictures. (As a young girl called Alice once perceptively remarked, "What is the use of a book without pictures or conversations?"[2])

Thirdly, and I cannot stress this enough, *your exam grades do not matter.*[*citation needed*] It is perfectly possible to pass calculus exams without understanding the material, but if you do that (by, for example, trying to memorise everything in leu of understanding it), you are cheating yourself out of an education. If you understand the material, you will be prepared for every subject you may wish to take next year (and, as a bonus, you'll pass the exam).

Let us begin.

[2] *Alice in Wonderland*, by Lewis Carroll.



Contents

Chapter I

The basics

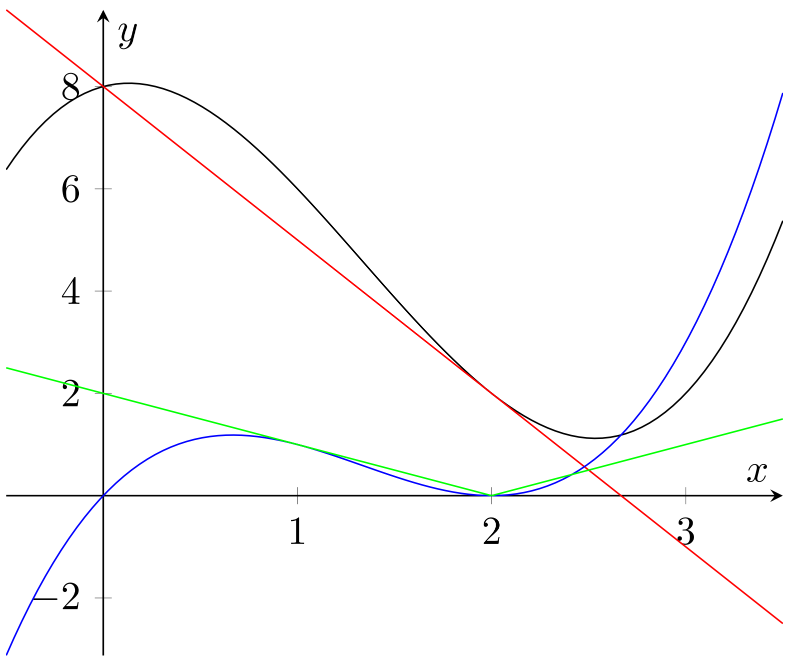


Figure 1: The graph of a function $f(x)$ (black), its tangent line $\tilde{f}(x)$ at 2 (red), $f(x) - \tilde{f}(x)$ (blue), and $y = |x - 2|$ (green).

I.1 Approximations

Our original definition of the derivative was motivated in part by finding the 'best' linear approximation to a curve at a given point. Indeed, if f is a function then, for all h such that $f(x_0 + h)$ is defined, we have

$$f(x_0 + h) - f(x_0) = hf'(x_0) + \vartheta(h)$$

where $\lim_{h\to 0} \vartheta(h)/h = 0$; in particular, for any x where f is defined (moreover, we need f to be defined everywhere *between* x and x_0) we have

$$f(x) - f(x_0) = (x - x_0)f'(x_0) + \vartheta(x - x_0)$$

where we have simply replaced h with $x - x_0$. We already defined the tangent line to f at x_0 to be the line given by $y - f(x_0) = (x - x_0)f'(x_0)$; so the above equation tells us that if (x, y) is on the tangent line then

$$f(x) - y = [(x - x_0)f'(x_0) + \vartheta(x - x_0) + f(x_0)] - [(x - x_0)f'(x_0) + f(x_0)] = \vartheta(x - x_0)$$

and so we have that

$$\frac{f(x) - y}{x - x_0} = \frac{\vartheta(x - x_0)}{x - x_0} \xrightarrow{x - x_0 \to 0} 0 :$$

in other words, as we move our point x of interest closer and closer to x_0, the 'error' in the tangent line approximation vanishes at a faster rate than our movement. Figure 1 illustrates these relationships; notice that at points x close to the point of tangency (in this case $x_0 = 2$), $\tilde{f}(x) - f(x)$ is approaching zero faster than the distance $|x - x_0|$.

Given that the tangent line is, in this sense, a good linear approximation, it is natural to ask the following question:

If f is differentiable at x_0, what is the best polynomial approximation to f around the point x_0?

As a reminder, a polynomial is a function p of the form $p(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$ where $p_n \neq 0$. The various p_k are called the *coefficients*, and n is the *degree* of p.

This question is an important one, because if we can answer it then we can always replace differentiable functions (which are difficult to calculate — what is sin(32.341)?) with polynomials (which only require a finite number of multiplications and additions to calculate) with only a small loss of information.

Our plan of attack will be, given a function f that is differentiable (at least) n times at some point x_0, to find a polynomial

$$p(x) = p_n(x-x_0)^n + p_{n-1}(x-x_0)^{n-1} + \cdots + p_1(x-x_0) + p_0$$

satisfying the following conditions:

$$\begin{aligned} p(x_0) &= f(x_0) \\ p'(x_0) &= f'(x_0) \\ &\vdots \\ p^{(n)}(x_0) &= f^{(n)}(x_0). \end{aligned}$$

(For convenience, the zeroth derivative of f is f itself.)

Our argument above shows that the linear polynomial approximation at x_0 is $p(x) = f'(x_0)(x - x_0) + f(x_0)$. We will need the following lemma:

Lemma. *Let* $p(x) = p_n(x-x_0)^n + p_{n-1}(x-x_0)^{n-1} + \cdots + p_1(x-x_0) + p_0$ *be a polynomial. Then* $p_k = \frac{p^{(k)}(x_0)}{k!}$ *for all* $0 \le k \le n$.

Proof. We have the following:

$$\begin{aligned} p(x) = p_n(x-x_0)^n + \cdots + p_3(x-x_0)^3 + p_2(x-x_0)^2 + p_1(x-x_0) + p_0 &\implies p(x_0) = p_0 \\ p'(x) = np_n(x-x_0)^{n-1} + \cdots + 3p_3(x-x_0)^2 + 2p_2(x-x_0) + p_1 &\implies p'(x_0) = p_1 \\ p''(x) = n(n-1)p_n(x-x_0)^{n-2} + \cdots + (3\cdot 2)p_2(x-x_0) + 2p_2 &\implies p''(x_0) = 2p_2 \\ p^{(3)}(x) = n(n-1)(n-2)p_n(x-x_0)^{n-3} + \cdots + (3\cdot 2)p_2 &\implies p^{(3)}(x_0) = (2\cdot 3)p_3 \end{aligned}$$

and in general, completing the pattern,

$$p^{(k)}(x_0) = (k!)p_k.$$

□

Thus, if we want to match f with a polynomial of degree n such that $f^{(k)}(x_0) = p^{(k)}(x_0)$ for every $0 \le k \le n$ we simply choose $p(x) = p_n(x-x_0)^n + p_{n-1}(x-x_0)^{n-1} + \cdots + p_1(x-x_0) + p_0$ such that

$$p_k = \frac{f^{(k)}(x_0)}{k!}.$$

This polynomial is called the nth *Taylor polynomial* of f at x_0; we might even write $\mathsf{T}_{n,x_0} f$ for this polynomial.

Example. 0, 1, 0, -1, 0, 1, Consider $f(x) = \sin x$. Then $f^{(0)}(0) = 0$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = -1$, and in general

$$f^{(n)}(0) = \begin{cases} 0 & n \text{ even} \\ (-1)^k & n = 2k+1 \text{ odd} \end{cases}$$

In particular, $\mathsf{T}_{2k+1,0} f = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + \frac{(-1)^k}{(2k+1)!}x^{2k+1}$; the Taylor polynomials for $k = 0$ (red), $k = 1$ (green), and $k = 2$ (blue) are graphed in figure 2.

We can even do the same error estimation that we performed above, although it is a little tedious; we find that

$$\lim_{x \to x_0} \frac{f(x) - \mathsf{T}_{n,x_0} f(x)}{(x-x_0)^n} = 0 \tag{I.1}$$

(so $\mathsf{T}_{n,x_0} \to f(x)$ faster than $(x-x_0)^n \to 0$).

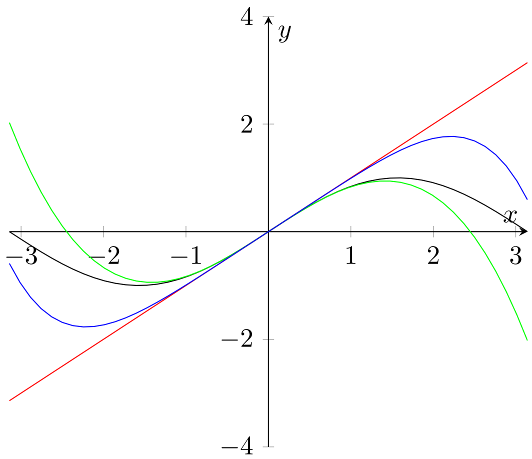


Figure 2: The first few Taylor polynomials of sine.

Note that this only tells us that the Taylor polynomials $\mathsf{T}_{n,x_0} f$ approximate f *very, very, very* close to the point x_0. It is not the case, in general, that the Taylor polynomials are a good approximation *around* the point x_0. For example, in one of the problems from the previous section we saw that

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is such that $f^{(n)}(0) = 0$ for all n, and thus $\mathsf{T}_{n,0} f(x) = 0$ for all n; so even for large n, the Taylor polynomials are a terrible approximation!

Exercises and Problems

1. Find the best quadratic approximation to $f(x) = 1/(1 + x^2)$ at zero.

2. Find the first, second, third, and nth Taylor polynomials of the following functions at zero.

   a) $x \mapsto e^x$

   b) $x \mapsto \cos x$

   c) $x \mapsto \frac{1}{1+x}$

3. a) According to problem set 6 of the trigonometry notes (remember those!), $\arctan a + \arctan b = \arctan\left[(a + b)/(1 - ab)\right]$ as long as the sum on the left is between $\pm\pi/2$. Show that $\pi/4 = 5 \arctan(1/5) - \arctan(1/239)$.

   b) Show that $\pi = 3.14159\ldots$.

4. a) Let $p(x)$ be a polynomial of degree n, and let x_0 be any point. Show that for all x, $p(x) = \mathsf{T}_{n,x_0} p(x)$.

   b) Write the function $p(x) = 22 - 49x + 35x^2 - 10x^3 + x^4$ as a polynomial in $(x - 3)$.

5. Suppose f and g both have n derivatives at x_0. Let λ be a real number. Calculate:

   a) $\mathsf{T}_{n,x_0}(\lambda f)$

   b) $\mathsf{T}_{n,x_0}(f + g)$

   c) $\mathsf{T}_{n,x_0}(fg)$

   d) $\mathsf{T}_{n-1,x_0}(f')$

6. Prove formula I.1 above.

References

An introduction to the approximation of sufficiently differentiable functions by polynomials can be found in Spivak, chapter 19. With a little extra work one can work out what degree of polynomial is needed at a point x_0 to reduce the error to less than a specified amount; to do this one needs integration (which we have not yet seen), and the details are in Spivak.

Homework problems

1. Find the best quartic (degree 4 polynomial) approximation to $f(x) = x^5 + x^3 + x$ about (a) 0 and (b) 1.

2. Square roots are difficult. Compute an approximation to $\sqrt{4.003}$ by hand. (Hint: expand $x \mapsto \sqrt{x}$ as a Taylor series about 4.)