

NCEA Level 3 Calculus (Integration)

18. Substitution

Recall that the **chain rule** for differentiation is given by

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

Since integration is (in some sense) the inverse of differentiation, we can write (by applying the fundamental theorem of calculus)

$$\int f'(g(x))g'(x) dx = f(g(x)) + C.$$

For a mnemonic, we can let $u = g(x)$. Then $du = g'(x) dx$ * and so, by the rule we just wrote down, we have

$$\int f'(g(x))g'(x) dx = \int f'(u) du = f(u) + C = f(g(x)) + C.$$

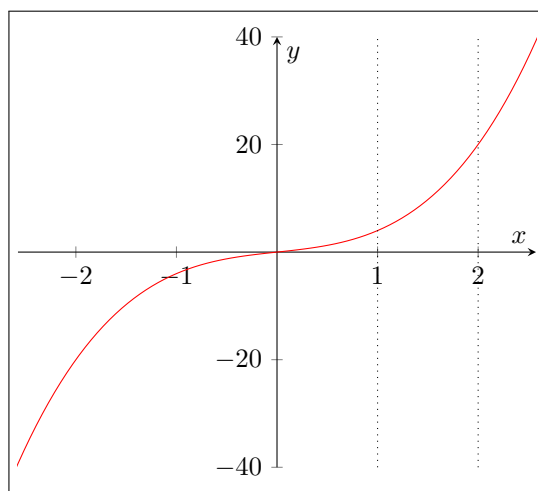
In Leibniz notation, we have

$$\int f'(g(x))g'(x) dx = \int \frac{df}{dg} \frac{dg}{dx} dx = \int \frac{df}{dg} dg = \int f'(g) dg = f(g) + C = f(g(x)) + C,$$

and so one can intuitively think about this (here we substitute g out) as the cancellation of differentials underneath an integral sign.

This rule, which gives us a kind of chain rule for integration, is called **substitution**, or the **inverse chain rule**. It can be thought of as a change in coordinate system from an x -based system to one based on u , and we have to ‘resize’ our area based on how much u stretches the coordinate system compared to x — and this ‘stretch factor’ is simply $\frac{du}{dx}$.

Example. For example, consider $\int_1^2 2x(x^2 + 1) dx$; we are finding the area shown here between the dotted lines.

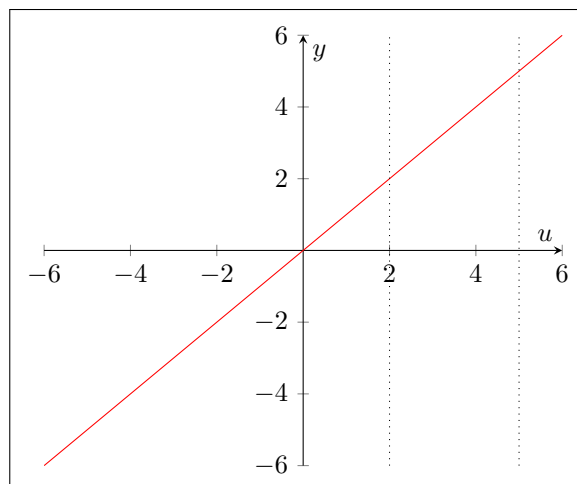


Let us make the substitution $u = x^2 + 1$, so $\frac{du}{dx} = 2x$ and our integral becomes

$$\int_1^2 2x(x^2 + 1) dx = \int_{u^{-1}(2)}^{u^{-1}(5)} \frac{du}{dx} u(x) dx = \int_2^5 u du.$$

We can graph our region of integration again.

*This rearrangement is not rigorous, but it works.



This new coordinate system, which is $2x$ times as large as the older one, is much simpler to integrate inside!

Examples.

1. Suppose we wish to find $\int \sin x \cos x \, dx$. Then let $u = \sin x$, so $du = \cos x \, dx$ and

$$\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C.$$

2. In this case, we also could have used a trigonometric identity. Suppose we wish to find $\int xe^{x^2} \, dx$. We can let $u = x^2$, and then $du = 2x \, dx \Rightarrow dx = \frac{du}{2x}$. Hence:

$$\int xe^{x^2} \, dx = \int \frac{1}{2}e^u \, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

3. Suppose we wish to find $\int \frac{4}{x}(\ln x)^3 \, dx$. We let $u = \ln x$, and then $du = \frac{dx}{x}$. Hence:

$$\int \frac{4}{x}(\ln x)^3 \, dx = 4 \int u^3 \, du = u^4 + C = (\ln u)^4 + C.$$

Questions

1. Find the following indefinite integrals.

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|-----------------------------------|--|
| (a) $\int \sin 2x \, dx$ | (i) $\int 2 \cos x + \sin 2x \, dx$ |
| (b) $\int (4x - 44)^{2019} \, dx$ | (j) $\int -2x \csc^2(3x^2) \, dx$ |
| (c) $\int 4x\sqrt{x^2 + 3} \, dx$ | (k) $\int \frac{3}{x^3} - \frac{4}{x+1} \, dx$ |
| (d) $\int (3x - 4)^2 \, dx$ | (l) $\int e^{x/2} + \frac{2}{x} \, dx$ |
| (e) $\int \frac{x}{x^2+1} \, dx$ | (m) $\int x^2 \sec^2 x^3 + 9 \, dx$ |
| (f) $\int \frac{2}{4x+3} \, dx$ | (n) $\int -\csc(\tan x) \cot(\tan x) \sec^2 x \, dx$ |
| (g) $\int e^{2x+1} \, dx$ | (o) $\int \frac{\cos x - \sin x}{\cos x + \sin x} \, dx$ |
| (h) $\int \sec 4x \tan 4x \, dx$ | (p) $\int \frac{2017}{x \ln x} \, dx$ |

2. By using the substitution $x = \sin \theta$, find

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$$\int \frac{1}{\sqrt{1-x^2}} \, dx.$$

3. Compute the following definite integrals:

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- (a) $\int_0^1 x e^{-x^2} dx$
- (b) $\int_{-\pi/3}^{\pi/3} x^4 \sin x dx$ (hint: no substitution is required)
- (c) $\int_0^1 \cos(\pi t/2) dt$
- (d) $\int_0^1 (3t-1)^{50} dt$
- (e) $\int_0^1 \sqrt[3]{1+7x} dx$
- (f) $\int_0^1 \frac{dx}{1+\sqrt{x}}$
- (g) $\int_{-1}^2 x(x-1)^3 dx$
- (h) $\int_0^3 x\sqrt{1+x^2} dx$

4. Find the area enclosed by the curve $y = 4 \sin 3x \cos x$ and the x -axis from $x = 0$ to $x = \frac{\pi}{3}$.

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5. Find k such that $\int_0^k e^{2x} dx = 40$.

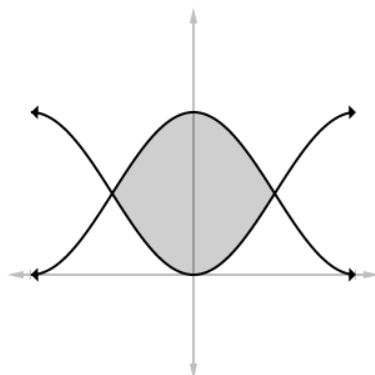
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6. Calculate the area enclosed by the curve $y = \frac{3x-2}{x+4}$ and the lines $y = 0$, $x = 1$, and $x = 5$.

E

7. Find the area between the curves $y = \sin^2 kx$ and $y = \cos^2 kx$ shaded below.

E



8. Find $\int \tan \theta d\theta$ and $\int \cot \theta d\theta$.

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9. Complete the following working:

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$$\begin{aligned} \int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\ &= \int \frac{\dots}{\sec x + \tan x} dx \\ \text{Let } u &= \dots \\ &= \int \frac{1}{\dots} du \\ &= \dots \end{aligned}$$

10. Show that

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$$\int x^4 \sin x dx = 4x(x^2 - 6x) \sin x - (x^4 - 12x^2 + 24) \cos x + C.$$

11. If $y = x\sqrt{\sin x^3 + \cos x^3}$, find $\pi \int_0^1 y^2 dx$. M
12. The velocity of a particle at time t is given by $v = \frac{\cos(\sqrt{2t+1})}{\sqrt{2t+1}}$. What is the position of the particle at time $t = 5$, given that $x(0.5) = 0$? (Recall that $v = \frac{dx}{dt}$.) M
13. Evaluate $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ [*Hint: use the substitution $x = \frac{\pi}{2} - u$ and add the result to the original integral.*] S
14. Scholarship 1999: S
- (a) Evaluate $\int \cos^5 x dx$ using the substitution $t = \sin x$.
 - (b)
 - i. If $f(x) = \cos^5 x$, what are $f(0)$, $f'(0)$, and $f''(0)$?
 - ii. Hence evaluate a , b , and c in the approximation $\cos^5 x \approx a + bx + cx^2$.
 - iii. Use this to give an approximation for $\int \cos^5 x dx$.
 - (c) Evaluate $\int_0^{0.6} \cos^5 x dx$ to three significant figures, using:
 - i. The exact integration in (a).
 - ii. The expression in (b)(iii).
 - iii. Simpson's rule.