NCEA Level 3 Calculus (Integration)
19. Differential Equations (Homework)

Reading

So far, we have looked at equations where the unknown is either a number or a point in n-dimensional space (that is, a sequence of n numbers). In order to generate these equations, we took various combinations of the basic arithmetical operations and applied them to our unknowns.

Here, for comparison, are two well-known differential equations, the first "ordinary" and the second "partial":

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + k^2 x = 0,$$
$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right).$$

The first is the equation for simple harmonic motion, which has the general solution $x(t) = A \sin kt + B \cos kt$; the second is the heat equation which describes the way that the distribution of heat in a physical medium changes with time.

For many reasons, differential equations represent a jump in sophistication. One is that the unknowns are *functions*, which are much more complicated objects than numbers or n-dimensional points. (For example, the first equation above asks what function x of t has the property that if you differentiate it twice then you get $-k^2$ times the original function.) A second is that the basic operations one performs on functions include differentiation and integration, which are considerably less "basic" than addition and multiplication. A third is that differential equations which can be solved in "closed form," that is, by means of a formula for the unknown function f, are the exception rather than the rule, even when the equations are natural and important.

From 'The Princeton Companion to Mathematics', I.4 §1.5

Questions

1. Solve the following equations for $y(t)$:
   (a) $e^{y-t} \frac{\mathrm{d}y}{\mathrm{d}t} = 1$
   (b) $\frac{\mathrm{d}y}{\mathrm{d}t} = ty^2$
   (c) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\sec^2 y}$
   (d) $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{t}{\sec t \sin y}$ (*Hint: first show that* $\frac{\mathrm{d}}{\mathrm{d}x}[\cos x - x \sin x] = -x \cos x$)
2. A copper ball with temperature 100 °C is dropped into a basin of water with constant temperature 30 °C. After 3 minutes the temperature of the ball has decreased to 70 °C. When will it reach a temperature of 31 °C?
3. Consider a tank of water. The rate of flow of water into the tank is a constant $3\,\mathrm{L\,s^{-1}}$; the flow out is directly proportional to the volume of water in the tank. Initially, the volume of water in the tank is 100 L; if the volume were to increase to $120\,\mathrm{L\,s^{-1}}$, the rate of water flowing out would exactly balance the rate of water flowing in.
   (a) Form a differential equation and find the volume of water after ten minutes.
   (b) Does the outward rate of flow ever become greater than the incoming rate of flow?