NCEA Level 3 Calculus (Differentiation)
2. Limits (Homework)

Reading

You may be wondering why we bother introducing the concept of limits: after all, we are simply replacing one handwavy picture-based definition (that of the derivative) with another! I will give the answer in two parts:

1. Limits are a more general and hence more useful concept; and
2. It is much easier to formally define a limit than a derivative.

Limits are more general

The obvious use of limit notation this year is to 'plug gaps' in functions; however, we can also (as you have seen) take limits of things towards infinity. This allows us to formalise things like infinite sums: we define the value of an infinite sum to be a special kind of limit.

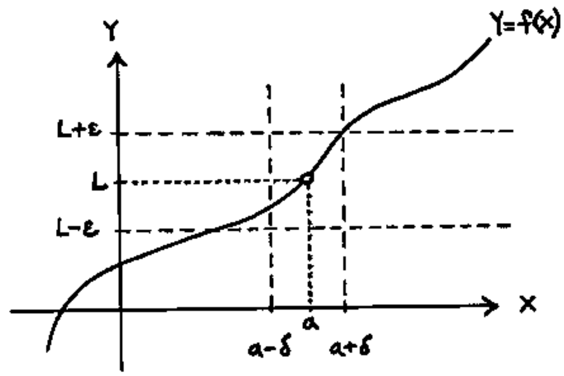
Limiting situations come up surprisingly often in physics and chemistry as well, if we want to look at the behaviour of a system in the long term: say the concentration of a particular compound in solution can be modelled by $C(t) = \frac{k}{t^2}$; then, if we wait a long time (i.e. let $t \to \infty$), we predict that the concentration becomes negligible.

It is easier to formally define a limit

Suppose that we have some function f such that

$$\lim_{x \to a} f(x) = L.$$

All we are saying here is (intuitively) that we can make the value of f *as close as we like* to being L, by taking x to be sufficiently close to a. I will not state the formal definition here (it is easy enough to find), except to state that it is a little stricter than this intuitive statement suggests (i.e. for all $x \neq a$ within the interval $(a - \delta, a + \delta)$ we must have $f(x)$ be within $(L - \varepsilon, L + \varepsilon)$).



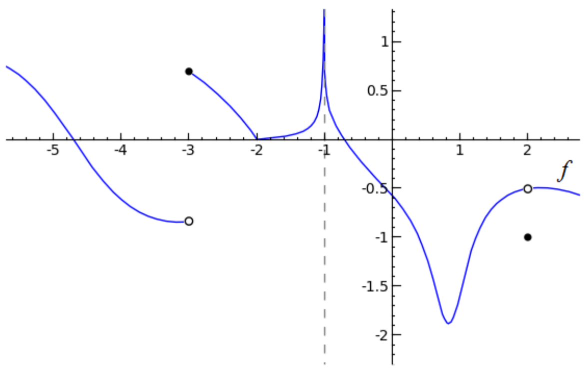
Questions

Derivatives and limits allow us to classify functions and their behaviour. Consider the following:

Properties of Functions

- A function is **increasing** if its derivative is positive.
- A function is **decreasing** if its derivative is negative.
- A function is **concave down** if its derivative is decreasing.
- A function is **concave up** if its derivative is increasing.
- A function f is **continuous** at a point a if $\lim_{x\to a} f(x) = f(a)$.

1. Describe all the function properties given above geometrically, and give an example of each.
2. Consider the function graphed below.



   (a) Find $\lim_{x\to -2} f(x)$ and $\lim_{x\to 2} f(x)$.

   (b) Does $\lim_{x\to -3} f(x)$ exist? Why/why not?

   (c) Does $\lim_{x\to 0} f(x)$ exist? Why/why not?

   (d) On what intervals is $f(x)$ continuous?

   (e) At what points is $f(x)$ not differentiable?

3. On an axis, sketch a graph of some function f that has the following features:
   - Is continuous for $0 < x < 5$ and $5 < x < 9$ and is discontinuous when $x = 5$
   - Is concave down $(f''(x) < 0)$ for $0 < x < 5$
   - Has $f'(x) = 0$ at $(3, 8)$
   - Has $\lim_{x\to 5} f(x) = 6$.
   - Is not differentiable at $(7, 3)$.