NCEA Level 3 Calculus (Differentiation)
5. The Product and Quotient Rules

Two weeks ago we saw that the derivative of a product is not simply the product of the derivatives; for example, take $(x)(x)$. If we differentiate each term and multiply, we obtain 1; however, the derivative of x^2 is (of course) $2x$. Ensure you understand why this is a counterexample to the naive rule $(fg)' = f'g'$ before continuing.

Suppose f and g are functions; then the *real* product rule is

$$(fg)' = gf' + fg'.$$

Example. Consider $y = 2t \sin t$. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = 2\sin t + 2t\cos t$.

We can also write a rule for the derivative of a quotient of functions. You will be asked to prove it as an exercise, using the product rule.

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Example. Consider $f(s) = \sqrt{\frac{s^2+1}{s^2+4}}$. Then

$$\begin{aligned} f'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s^1+1}{s^2+4}\right] \cdot \frac{1}{2\sqrt{\frac{s^2+1}{s^2+4}}} \\ &= \frac{(s^2+4)(2s)-(s^2+1)(2s)}{(s^2+4)^2} \cdot \frac{\sqrt{s^2+4}}{2\sqrt{s^2+1}} \\ &= \frac{6s}{(s^2+4)^{\frac{3}{2}}(s^2+1)^{\frac{1}{2}}} \end{aligned}$$

Questions

1. $\boxed{\texttt{A}}$ In each case, find $\frac{\mathrm{d}y}{\mathrm{d}t}$.

   (a) $y = \left(3+2t^2\right)^4$

   (b) $y = \frac{t^3}{\ln t}$

   (c) $y = t\sqrt{t}$

   (d) $y = 2t\sin t - (t^2-2)\cos t$

   (e) $y = \frac{t}{\sqrt{a^2-t^2}}$ (a constant)

   (f) $y = \frac{1}{8}t^8\left(1-t^2\right)^{-4}$

   (g) $y = e^t \ln t$

   (h) $y = \log\left[1 + \frac{t^2+3t+17}{t^{17}}\right]$

   (i) $y = \sin\left[e^{\tan t}\ln\tan t\right]$

   (j) $y = \frac{3t-2}{\sqrt{2t+1}}$

   (k) $y = \frac{\sec 2t}{1+\tan 2t}$

   (l) $y = \frac{(t-1)(t-4)}{(t-2)(t-3)}$

   (m) $y = t\sin^2(\cos\sqrt{\sin \pi t})$

   (n) $y = \sqrt[5]{t\tan t}$

   (o) $y = \dfrac{(t+\lambda)^4}{t^4+\lambda^4}$

2. $\boxed{\texttt{A}}$ If $f(x) = e^{-x}$, find $f(0) + xf'(0)$.

3. $\boxed{\texttt{M}}$ Suppose f and g are functions (g not the zero function). Write $\frac{f}{g} = fg^{-1}$ and prove the quotient rule from the product rule.

4. $\boxed{\texttt{M}}$ Show that if f, g, and h are functions then $(fgh)' = f'gh + fg'h + fgh'$.

5. $\boxed{\texttt{E}}$ Suppose $f(x) = f(-x)$ for all x in the domain of f. Prove that $f'(x) = -f'(-x)$ for all x in the domain of $f'(x)$.

6. $\boxed{\texttt{E}}$ Consider the function defined by $f(x) = x^x$.
   (a) Rewrite f in the form $f(x) = e^{x \ln x}$, and hence find $f'(x)$.
   (b) Find $\frac{\mathrm{d}y}{\mathrm{d}t}$ if $y = (t^2+3)^{(t^2+3)}$.

7. $\boxed{\texttt{E}}$ A circle that closely fits points on a local section of a curve can be drawn for any continuous curve. The radius of curvature of the curve is defined as the radius of the approximating circle, which changes as we move around the curve.

$$\text{radius of curvature} = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|}$$

   Find the radius of curvature of the curve $y = e^{-x}\sin x$ at the point $(0, 0)$.

8. $\boxed{\texttt{M}}$ Show that $y = xe^{-x}$ satisfies the differential equation $xy' = (1-x)y$.

9. $\boxed{\texttt{M}}$ If $y = \ln \frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}$, find y''.

10. $\boxed{\texttt{M}}$ Find the equation of the tangent line to the graph of $y = \ln \cos \frac{x-1}{x}$ at the point $(1, 0)$.

11. $\boxed{\texttt{M}}$ Show that $y = (1 + x + \ln x)^{-1}$ satisfies the differential equation $xy' = y(y \ln x - 1)$.

12. $\boxed{\texttt{E}}$ Find the angle at which $y = x^2 \ln[(x-2)^2]$ cuts the x-axis at the point $(0, 0)$.

13. $\boxed{\texttt{M}}$ When $x = 0$, is the curve $y = (x+20)^2(2x^2-3)^6 - \ln\sin(x - \frac{\pi}{2})$ concave up or concave down?

14. $\boxed{\texttt{M}}$ If $y = \frac{e^x}{\sin x}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 - \cot x)$.

15. $\boxed{\texttt{E}}$ Find $f'(x+3)$ if $f(x+3) = (x+5)^7$.

16. $\boxed{\texttt{E}}$ The number a is called a **double root** of some polynomial function f if $f(x) = (x-a)^2 g(x)$ for some polynomial g. Prove that a is a double root of f if and only if a is a root of both f and f'.

17. $\boxed{\texttt{E}}$ Show that there is no function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0 + \frac{b_1}{x} + \cdots + \frac{b_m}{x^m}$$

   such that $f'(x) = 1/x$.

18. $\boxed{\texttt{S}}$ Let's try to generalise the product rule for nth derivatives of products. Recall that the notation $f^{(n)}(x)$ means the nth derivative of f with respect to x. Assume in each case that the relevant derivatives exist.
   (a) Find a formula for $(fg)'' = (fg)^{(2)}$ in terms of the derivatives of f and g.
   (b) What about $(fg)^{(3)}$?
   (c) If you can, prove Leibniz' formula for the nth derivative of a product using *induction*:

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$$