NCEA Level 3 Calculus (Integration)
20. Partial Fractions

This is a Scholarship topic! The algebraic computations required can get quite messy.
Definition (Rational Function)**.** A **rational function** is a function f which can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

for suitable polynomials p and q (where $q \neq 0$).

We can already integrate some rational functions; in particular, those of the form $f(x) = p'(x)/p(x)$:

$$\int \frac{p'(x)}{p(x)} \, \mathrm{d}x = \ln\left|p(x)\right| .$$

This week we will learn a technique that, in theory, allows us to integrate *all* rational functions. To understand the idea, note that we can easily integrate all functions of the form

$$f(x) = \frac{A}{(ax+b)^n}$$

for real constants A, a, b, and n.

Our task is simply to 'deconstruct' arbitrary fractions into this form.
Example.

$$\frac{2}{x-4} + \frac{3}{x+1} = \frac{5x-10}{x^2-3x-4}$$

so

$$\int \frac{5x-10}{x^2-3x-4} \, \mathrm{d}x = \int \frac{2}{x-4} + \frac{3}{x+1} \, \mathrm{d}x = 2\ln|x-4| + 2\ln|x-4| .$$

In effect, the technique of partial fractions is the reverse of this: we *decompose* the more complex rational function into two or more functions which are easier to integrate.

Let $f(x) = \dfrac{p(x)}{q(x)}$. Then we have four cases

1. $q(x)$ is the product of distinct linear factors.
2. $q(x)$ is the product of linear factors, some of which are repeated.
3. $q(x)$ is the product of distinct factors, some of which are irreducible quadratics.
4. $q(x)$ contains a repeated irreducible quadratic factor.

The degree of p must be less than the degree of q, so you may need to use long division before applying the technique of partial fractions.

We consider only the first two cases here. See Stewart §7.4 for the others.

Type I: Distinct linear factors

Suppose that $q(x) = (\alpha_1 x + \beta_1) + \cdots + (\alpha_n x + \beta_n)$. Then the partial fraction decomposition is of the form

$$\sum_{i=1}^{n} \frac{A_i}{\alpha_i x + \beta_i}.$$

Example.

$$\frac{11x-2}{6x^2+x-1} = \frac{11x-2}{(2x+1)(3x-1)} = \frac{A}{2x+1} + \frac{B}{3x-1}.$$

So $11x - 2 = A(3x-1) + B(2x+1)$. Let $x = 1/3$, so $B = \frac{11/3-2}{2/3+1} = 1$; then let $x = -1/2$, so $A = \frac{11/2+2}{3/2+1} = 3$.

Hence

$$\int \frac{11x-2}{6x^2+x-1}\,\mathrm{d}x = \int \frac{3}{2x+1} + \frac{1}{3x-1}\,\mathrm{d}x = \frac{3}{2}\ln|2x+1| + \frac{1}{2}\ln|3x-1| + C.$$

Type II: Repeated linear factors

Suppose some factor $(\alpha_i x + \beta_i)^r$ appears in the factorisation of $Q(x)$. Then the partial fraction decomposition will include

$$\sum_{j=1}^{r} \frac{A_{i_j}}{(\alpha_i x + \beta_i)^j}.$$

Example. Consider $\int \frac{2x+4}{x^3-2x^2}\,\mathrm{d}x$. We wish to find a partial fraction expansion:

$$\begin{aligned} \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{2x+4}{x^3-2x^2} &\iff 2x+4 = Ax(x-2) + B(x-2) + Cx^2 \\ &\iff 2x+4 = (A+C)x^2 + (B-2A)x - 2B \end{aligned}$$

Matching coefficients, we find $B = -2$, $A = -2$, and $C = 2$. Then:

$$\begin{aligned} \int \frac{2x+4}{x^3-2x^2}\,\mathrm{d}x &= \int \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}\,\mathrm{d}x \\ &= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C \\ &= 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C. \end{aligned}$$

Questions

1. $\boxed{S}$ Find $\displaystyle\int \frac{A}{(ax+b)^n}\,\mathrm{d}x$ if A, a, b, and n are real constants.

2. $\boxed{S}$ Evaluate, using partial fractions:

   (a) $\displaystyle\int \frac{3x-1}{(x-3)(x+4)}$

   (b) $\displaystyle\int \frac{1}{x^2-3x-4}$

   (c) $\displaystyle\int \frac{1}{x^2-6x-7}$

   (d) $\displaystyle\int \frac{11x+17}{2x^2+7x-4}$

   (e) $\displaystyle\int \frac{5x-10}{x^2-3x-4}$

   (f) $\displaystyle\int \frac{x+7}{x^2-x-6}$

   (g) $\displaystyle\int \frac{1}{x^2+5x+6}$

   (h) $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}$

3. Some more interesting problems:

   (a) $\boxed{S}$ Rewrite in the form $\displaystyle\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$ and integrate:

   $$\int \frac{4x}{x^3-x^2-x+1}\,\mathrm{d}x.$$

   (b) $\boxed{S}$ Use the obvious substitution and divide through:

   $$\int \frac{\sqrt{x+1}}{x}\,\mathrm{d}x.$$

4. $\boxed{S}$ Use appropriate substitutions to evaluate:

   (a) $\displaystyle\int \frac{\cos\theta}{\sin^2\theta+4\sin\theta-5}\,\mathrm{d}\theta$

   (b) $\displaystyle\int \frac{e^{3x}}{e^{2x}+4}\,\mathrm{d}t$

   (c) $\displaystyle\int \frac{5+2\ln x}{x(1+\ln x)^2}$

5. $\boxed{\mathtt{E}}$ Don't use a sledgehammer to kill a fly, and compute the following:

   (a) $\displaystyle\int \frac{x^2(5x^2+4x-3)}{x^5+x^4-x^3+1}\,\mathrm{d}x$‘

   (b) $\displaystyle\int \frac{x^2+1}{x(x^2+3)}\,\mathrm{d}x$

6. $\boxed{\mathtt{O}}$ We have already computed $\int \sec x\,\mathrm{d}x$ via a bit of a trick, but we could also use partial fractions.

   (a) Show that $\sec x = \frac{\cos x}{1-\sin^2 x}$.

   (b) Hence, or otherwise, compute $\int \sec x\,\mathrm{d}x$.

7. $\boxed{\mathtt{S}}$ Solve the following differential equation for $y(x)$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-a^4}{x^2-a^2}$$

8. $\boxed{\mathtt{S}}$ Scholarship 2008:

   (a) $\dfrac{A}{x} + \dfrac{B}{P-x} = \dfrac{1}{x(P-x)}$ where x is a variable and P is a constant. Find A and B in terms of P.

   (b) When a rumour about a teacher is started at a school of size P students, it spreads at a rate (in students per day) that is proportional to the product of the number of students who know the rumour, N, and those who do not. Find an expression for the number of students N who know the rumour after t days.

   (c) For a particular rumour about a teacher, 0.5% of students know the rumour initially. The principal will need to act to stop the rumour once more than half the school's students know it. When $\dfrac{1}{5}$ of the students know the rumour, the number who know the rumour is increasing at a rate of $0.08P$ students per day. How long will it be before the principal must act?

9. $\boxed{\mathtt{O}}$ Scholarship 2015: The rate of spread of a rumour at a particular school is proportional to both the number of students who know a rumour, S, and the number of students who do not. If N is the total number of students in the school, then $\frac{\mathrm{d}S}{\mathrm{d}t} = kS(N-S)$. Initially, two students knew the rumour. Show that the number of students who know the rumour at time t is $S(t) = \dfrac{N}{1+\dfrac{1}{2}e^{-kNt}(N-2)}$.

10. $\boxed{\mathtt{O}}$ Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{x^2+1}$.

   (a) Find a partial expansion of the given rational function as follows:

$$\frac{x^2+x-2}{3x^3-x^2+3x-1} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

   (b) Hence (or otherwise) compute:

$$\int_{\pi/4}^{\pi/3} \frac{x^2+x-2}{3x^3-x^2+3x-1}\,\mathrm{d}x.$$

11. $\boxed{\mathtt{S}}$ (Revenge of the limits.) Compute the following series. [*Hint: this sheet is on partial fractions.*]

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n(n+1)}\right)$$