Scholarship Calculus

There are five questions, worth a total of 28 marks.
Attempt ALL questions, showing all working.
Read questions carefully before attempting them.
Marks are available for partial answers.
The amount of time expected to be spent per question may not necessarily correlate "nicely" to the number of marks.
Diagrams may be used to support answers.
Candidates who do not provide diagrams for some questions may be disadvantaged.
Some marks are given for clarity and neatness of solutions or proofs.

Time Allowed: Three Hours
Scholarship: 35 marks
Outstanding: 56 marks (including at least 10 marks in each question)

Question:	Aircraft	Zeno's Limits	Trig Integrals	Functions	Accounting	Total
Points:	4	0	10	14	0	28
Score:						

Available Grades: *No Scholarship* *Scholarship* *Outstanding*

Question 1: Aircraft

(a) Consider an aircraft coming in to land at an airport. The approach path of the aircraft satisfies three conditions: (156/ap)

> 1. The cruising altitude is h when descent starts, at a horizontal distance ℓ from touch-down at the origin.
> 2. The pilot must maintain a constant horizontal speed of v throughout descent.
> 3. The absolute value of the vertical acceleration should not exceed a constant k, which is much less than the acceleration due to gravity.

i. Find a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ that satisfies the first condition, by imposing suitable restrictions on $P(x)$ and $P'(x)$ at the start of descent and at touchdown. (4)

ii. Use the second and third conditions to show that

$$\frac{6hv^2}{\ell^2} \leq k$$

iii. Suppose that an airline decides not to allow the vertical acceleration of a plane to exceed $k = 1385\,\mathrm{km/h^2}$. If the cruising altitude of a plane is $11\,000\,\mathrm{m}^2$ and the speed is $480\,\mathrm{km\,h^{-1}}$, how far away from the airport should the pilot start descent?

(b) It so happens that the vertical flight profile of the aircraft during the whole of its flight can be modelled by the equation

$$y = 6 - (0.01x)^2 - (0.01x)^4 + \sin(0.09x)$$

where $-1.40721 \leq x \leq 1.44154$.

i. Find the maximum altitude of the aircraft.

ii. Approximate the total distance travelled by the aircraft using Simpson's rule with $n = 10$.

> If $f(x)$ and $f'(x)$ are both continuous over the interval $a \leq x \leq b$, the arc length of the curve $y = f(x)$ over that interval is given by
>
> $$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,\mathrm{d}x$$

Question 2: Zeno's Limits

(a) Zeno, an ancient Greek philosopher, examined the following scenario in around 450 BC. Use your knowledge of limits to explain why it is possible for Homer to walk to the end of the path.

> **Zeno's Dichotomy Paradox:** Suppose Homer wishes to walk to the end of a path. Before he can get there, he must get halfway there. Before he can get halfway there, he must get a quarter of the way there. Before travelling a quarter, he must travel one-eighth; before an eighth, one-sixteenth; and so on. This requires Homer to complete an infinite number of tasks – an impossibility.
>
> In fact, Homer cannot even begin to walk - since any possible (finite) first distance could be divided in half, and hence would not be first after all. Hence travel over any finite distance can neither be completed nor begun.

(b) Prove that $\frac{\mathrm{d}}{\mathrm{d}x} \sin x + \cos x = \cos x - \sin x$. You may assume the following limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

(c) Evaluate $\lim_{x \to -\infty} \frac{e^{2x} + e^x + 4}{7e^{2x} + e^x - 4}$.

Question 3: Trig Integrals

(a) Evaluate $\int \cos^3 x \sin x \,\mathrm{d}x$. (3)

(b) Differentiate $2\theta \sin\theta$, and hence find $\int \cos\sqrt{x}\,\mathrm{d}x$. (3)

(c) Prove, by integration, that the area of a circle of radius r is $A = \pi r^2$. You may find the substitution $x = r\sin\theta$ useful. (4)

(d) A useful result is that $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$. (Cam03)

i. Given that $x + a > 0$ and $x + b > 0$, and that $b > a$, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x+a}{x+b}\right) = \frac{\sqrt{b-a}}{(x+b)\sqrt{a+b+2x}}$$

ii. Hence, or otherwise, find the following integral (for $x > -1$):

$$\int \frac{1}{(x+3)\sqrt{x+1}}\,\mathrm{d}x$$

Question 4: Functions

(a) For what values of $k > 0$ does the function f defined by

$$f(x) = \frac{\ln x}{k} - \frac{kx}{x+1}$$

have local extrema? For each such k locate and classify the extrema, and explain the reasons for your conclusions carefully.

(b) Prove (without explicit calculation) that $e^\pi > \pi^e$. (10)

Hint: Begin by taking the natural log of both sides, and try to define a suitable function that has the essential properties that yield the inequality.

(c) Let $f(x) = x^5 + 3x^3 + x - 10$. Find $a \in \mathbb{Z}$ such that $\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(48) = \frac{1}{a}$. (4)

Question 5: Accounting

A company produces three items, A, B, and C. The company has three factories, each of which produces the three items in the quantities per hour indicated in the following table:

		Plant		
		I	*II*	*III*
Item	*A*	1	2	3
	B	2	1	4
	C	3	1	1

It costs \$1000 per hour to operate plant I, \$400 per hour to operate plant II, and \$2400 per hour to operate plant III.

(a) An order is placed with the company for three units of item A, five of item B, and six of item C. Determine the number of hours each plant should be operated to produce at least the required number of items for the order at minimum cost.

(b) The accounting department wishes to assign values to each item produced as a measure of their respective contributions to company profits. If we let x_1 be the value per unit of item A, x_2 the value per unit of B, and x_3 the value per unit of C, then what we are trying to do is determine values of x_1, x_2, x_3 that will maximize $R = 3x_1 + 5x_2 + 6x_3$ (recalling that the order was three units of A, two of B, and six of C).

i. Set up and solve the linear programming problem to maximize R.

ii. If we allow our cost of operating plant II to increase to \$1000 per hour, what effect will this have on R?

Some problems taken, either directly or with modification, from: Stewart's *Calculus* (7th Ed); University of Cambridge entrance examinations; Erdman's *Exercises and Problems in Calculus*; Rammaha's *Challenging Problems for Calculus Students*; Whipkey's *The Power of Mathematics*.