NCEA Level 3 Calculus (Integration)
19. Differential Equations

Many physical problems can be expressed by writing different rates of change in terms of each other. For example, for a spring pulled a distance x away from its equilibrium point we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

for some constant k; and for a falling stone at distance r from the centre of the Earth, we have

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -g\frac{r_0^2}{r^2}$$

where r_0 is the radius of the Earth and g is a constant. These kinds of equations are known as **differential equations**.

Suppose $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$. It would be nice if we could find y in terms of x only. This is in fact possible, using substitution:

$$\begin{aligned} \tfrac{\mathrm{d}y}{\mathrm{d}x} &= f(x)g(y) \\ \Rightarrow \frac{1}{g(y)}\tfrac{\mathrm{d}y}{\mathrm{d}x} &= f(x) \\ \Rightarrow \int \frac{1}{g(y)}\tfrac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int f(x)\,\mathrm{d}x. \end{aligned}$$

Now, let $G(y)$ be an antiderivative of $\frac{1}{g(y)}$ (with respect to y). By the chain rule, then,

$$\frac{\mathrm{d}}{\mathrm{d}x}G(y) = \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}$$

and so

$$\int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = G(y) = \int \frac{1}{g(y)}\,\mathrm{d}y.$$

Hence we have

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

This way of solving differential equations is called **separation of variables**.

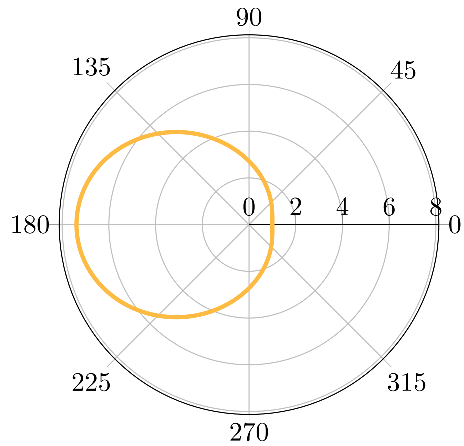
Example. Suppose we know that $y\frac{\mathrm{d}y}{\mathrm{d}x} = e^x$. Then we can separate the variables:

$$\begin{aligned} \int y\,\mathrm{d}y &= \int e^x\,\mathrm{d}x \\ \Rightarrow \frac{1}{2}y^2 &= e^x + C \\ \Rightarrow y^2 &= 2e^x + C. \end{aligned}$$

If we know that the curve passes through $(0, 0)$, then $0 = 2e^0 + C$ and $C = -2$, so $y^2 = 2e^x - 2$.

To check our answer, let us now use implicit differentiation to differentiate this curve. We have $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2e^x$ so and $y\frac{\mathrm{d}y}{\mathrm{d}x} = e^x$ as expected: our solution is correct.

Example. Suppose that $r(\theta)$ is implicitly defined by $\frac{\mathrm{d}r}{\mathrm{d}\theta} = r\sin\theta$ with the condition $r(\pi) = e$. Separating variables, we have $\int \frac{\mathrm{d}r}{r} = \int \sin\theta\,\mathrm{d}\theta$; so $\ln|r| = -\cos\theta + C$ and therefore $r = Ke^{-\cos\theta}$ for some constant K. But $e = Ke^{-\cos\pi} = Ke^0 = K$; so $r(\theta) = ee^{-\cos\theta} = e^{1-\cos\theta}$. Graphing this:



Questions

1. Find y in terms of x in each case, if each curve passes through $(1, 1)$: $\boxed{\texttt{M}}$

   (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = yx$

   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} + x = yx$

   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} + y = yx$

   (d) $\sqrt{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$

   (e) $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+2)^2$

   (f) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2+1}{2y}e^x$

   (g) $\frac{\mathrm{d}y}{\mathrm{d}x} = x\cos^2 y$

   (h) $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \tan y$

   (i) $2y\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + 2x + 1$

   (j) $\sin y \frac{\mathrm{d}y}{\mathrm{d}x} = 3x$

   (k) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x(e^{x^2}+2)}{6y^2}$

2. (a) Show that one antiderivative of $f(x) = x\sin x\,\mathrm{d}x$ is $F(x) = \sin x - x\cos x$. $\boxed{\texttt{E}}$

   (b) Find $y(\pi)$ if $y(0) = \pi$ and
$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \theta y \sin\theta.$$

3. Newton's law of cooling states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings; in other words, if the temperature of the body at time t is T and the ambient temperature is T_∞ then $\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_\infty)$ (where k is some constant.) $\boxed{\texttt{E}}$

   (a) A loaf of bread is taken from the oven at a temperature of 400 °C and is set down on a bench in an area with an ambient temperature of 20 °C. It is found that it takes ten minutes for the loaf to cool to half its initial temperature. How many minutes will it take for the loaf to cool to 30 °C?

   (b) A detective is called to the scene of a crime where a dead body has just been found. She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and is found to be 24 °C. The detective checks the programmable thermostat and finds that the room has been kept at a constant 20 °C for the past 3 days.

   After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 22 °C. This last temperature reading was taken exactly one hour after the first one. Assuming that the victim's body temperature was normal (37.5 °C) prior to death, at what time did the victim die?

4. We will apply calculus concepts to chemical rates of reaction. E

   (a) A *first-order reaction* is one whose rate depends linearly on the concentration of one reactant A; in other words, $-\frac{\mathrm{d}[A]}{\mathrm{d}t} = k[A]$.

   One example of a first-order reaction is the decomposition of hydrogen peroxide:

   $$2\,H_2O_2(aq) \longrightarrow 2\,H_2O + O_2(g)$$

   What percentage of hydrogen peroxide will have decomposed after 600 seconds, if the reaction constant is $k = 6.40 \times 10^{-5}\,\mathrm{s}^{-1}$?

   (b) If a reaction depends linearly on the concentrations of two reactants, A and B (as in $\mathrm{A} + \mathrm{B} \longrightarrow \mathrm{C}$) then the rate of reaction is given by

   $$-\frac{\mathrm{d}[A]}{\mathrm{d}t} = -\frac{\mathrm{d}[B]}{\mathrm{d}t} = \frac{\mathrm{d}[C]}{\mathrm{d}t} = k[A][B].$$

   If we consider the reaction $NO_2 + CO \longrightarrow CO_2 + NO$, the rate is experimentally found to be *second-order* in the reactant NO_2 and independent of the concentration of carbon monoxide. It follows that the rate of reaction is

   $$\frac{\mathrm{d}[NO_2]}{\mathrm{d}t} = -k[NO_2]^2$$

   where k is some constant.

   Initially, the concentration of NO_2 is $2.0\,\mathrm{mol\,L^{-1}}$; after ten minutes, the concentration has decreased to $1.0\,\mathrm{mol\,L^{-1}}$. How long will it take for the concentration to become $0.5\,\mathrm{mol\,L^{-1}}$?

5. It is known that the motion of a particle is described by the differential equation M

   $$v = \frac{4\sin(2t)}{x}.$$

   Initially, the particle is two metres away from the origin in the positive $x-$direction. Find the particle's position after ten seconds.

6. Suppose that $y'(x) = e^{x+2y}$, and $y(0) = 0$. Find $y(x)$ explicitly. M

7. Assume that the rate of reproduction of some population P is proportional to the number of pairs of individuals; so M

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2.$$

   Show that the size of the population becomes infinitely large in a finite time.

8. Recall from physics that **simple harmonic motion** occurs when the force F on an object is proportional to (and opposite in direction to) its displacement k from some zero point ($F = -kx$). Recall also that Newton's second law tells us that the acceleration felt by an object of mass m is given by $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{F}{m}$. We wish to find a formula for x, the displacement of the object, at time t. We have: M

   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{kx}{m}$$

   Show that $x = A\cos(\sqrt{k/m}\cdot t)$ is a solution of the given differential equation.

9. Consider the general wave equation, $y = A\sin(kx - \omega t)$ (where A, k, and ω are constant). We write $\frac{\partial y}{\partial x}$ for the derivative of y with respect to x holding t constant, and $\frac{\partial y}{\partial t}$ for the derivative of y with respect to t keeping x constant. S

   Show that $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ for some constant c.

10. Physics: Write down a differential equation modelling the charge on the capacitor in an RC circuit over time. Solve the equation. S

11. Scholarship 2000: The piriform is the curve defined by the equation $16y^2 = x^3(8-x)$ where $x \geq 0$. By solving the differential equation S

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2(6-x)}{8y}$$

($y = 0$ when $x = 0$), show the piriform is the solution.

12. Scholarship 2015: Determine all differentiable equations of the form $y = f(x)$ which have the properties: S

$$f'(x) = (f(x))^3 \text{ and } f(0) = 2$$

13. If you are more interested in nice geometry than applications, fill in the details here. Consider the family of curves consisting of all circles tangent to the y-axis at $(0,0)$: that is, all circles with equations of the form O

$$(x-r)^2 + y^2 = r^2.$$

   (a) Show that $2(x-r) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x-r}{y}$.

   (b) By substituting the original equation, eliminate the parameter r.

   (c) Thus we have found a differential equation satisfied by all of the circles in the family. We now want to find the set of curves such that each of our new curves is orthogonal to all of the curves in the original family: that is, the set of curves such that each intersects every circle at right angles. We thus need to solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy}{x^2-y^2}.$$

   Justify this.

   (d) Use the substitution $z = y/x$, and rewrite the differential equation in terms of z and x only. The result should be separable; solve it as usual, and graph the resulting family of functions to check that they are indeed the ones we are looking for.