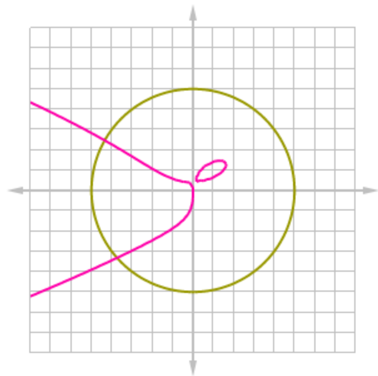
NCEA Level 3 Calculus (Differentiation)
11. Implicit Differentiation

This week we continue the study of more interesting curves which we began last week. Consider the curves

$$x^2 + y^2 = 25 \text{ and } x^3 + y^4 = 5xy - 2x$$

graphed here:



We can solve for the first y, as $y = \pm\sqrt{25 - x^2}$; however, the second is much harder to solve and so we cannot find its derivative using the techniques we have studied so far. These equations are examples of *implicit functions* of x. Note that neither is a 'real' function since they both fail the vertical-line test.

The key observation here is that **differentiation is an operation**, similar to addition. Just like we can add 3 to both sides of the true equation $2 + 4 = 6$ to obtain another true equation $2 + 3 + 4 = 3 + 6$, we can differentiate both sides of an equation to obtain another true equation. The only catch is that we must remember that y is a function of x and so we must employ the chain rule.

Example. If $x^2 + y^2 = 25$, by differentiating both sides with respect to x we obtain $2x + \frac{\mathrm{d}y}{\mathrm{d}x}2y = 0$ and therefore we have $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. Note that this depends on both x and y which makes sense: at $x = 0$, for example, we have two gradients (both of which are zero).

Example. If $x^3 + y^4 = 5xy - 2x$, then by differentiating both sides with respect to x we obtain $3x^2 + \frac{\mathrm{d}y}{\mathrm{d}x}4y^3 = 5y + 5x\frac{\mathrm{d}y}{\mathrm{d}x} - 2$ (being careful to use the product and chain rules in differentiating). Hence we have that the derivative is:

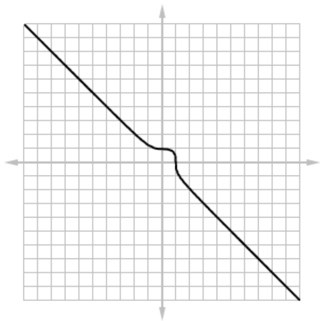
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y - 3x^2 - 2}{4y^3 - 5x}$$

Be careful to always specify which is the variable which you are differentiating with respect to.

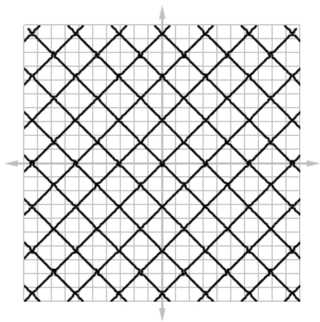
Questions

1. $\boxed{\texttt{M}}$ In each case, look at the cool pictures and find y':

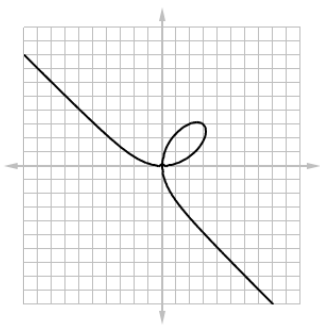
(a) $x^3 + y^3 = 1$



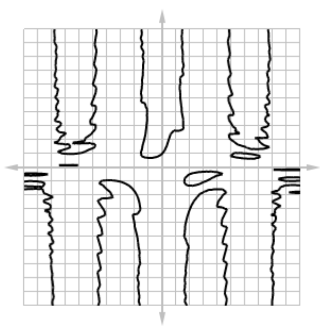
(b) $\sin^2 y + \cos^2 x = 1$



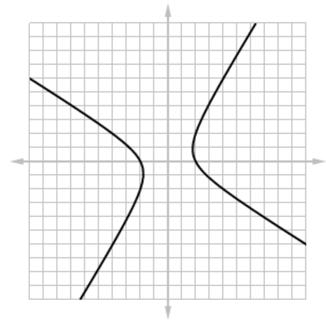
(c) $x^3 + y^3 = 6xy$ (the folium of Descartes)



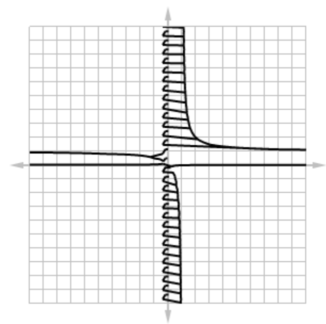
(d) $y \cos x = 1 + \sin(xy)$



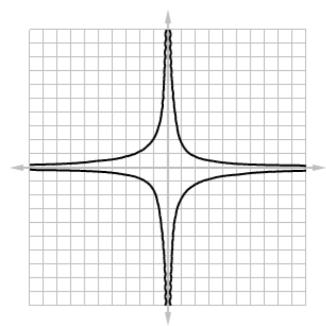
(e) $x^2 + xy - y^2 = 4$



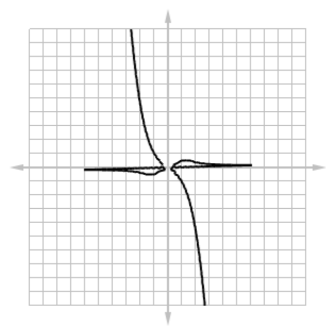
(f) $\frac{1}{x} + \frac{1}{y} = 1$



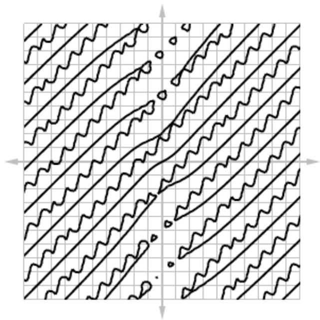
(g) $x^2y^2 + x \sin y = 4$



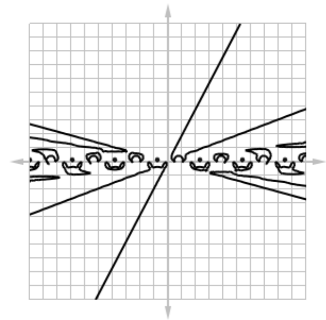
(h) $x^4y^2 - x^3y + 2xy^3 = 0$



(i) $\tan(x-y) = \frac{y}{1+x^2}$



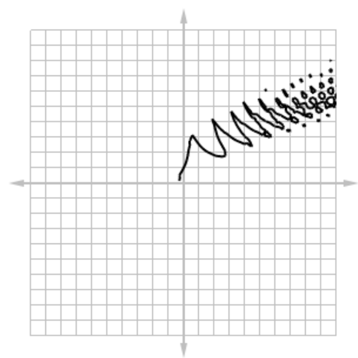
(j) $\sin\left(\frac{x}{y}\right) = \frac{1}{2}$



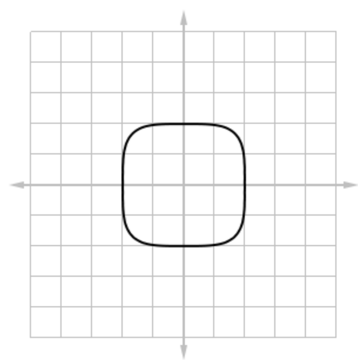
2. $\boxed{\mathtt{M}}$ Consider the circle $x^2 + y^2 = 1$. Find the equation of the tangent to the curve at $(\sqrt{2}, \sqrt{2})$.

3. $\boxed{\mathtt{S}}$ The ellipse $x^2 + 3y^2 = 36$ has two tangent lines passing through the point $(12, 3)$. Find both. *This question is similar to one from the 2015 Scholarship paper.*

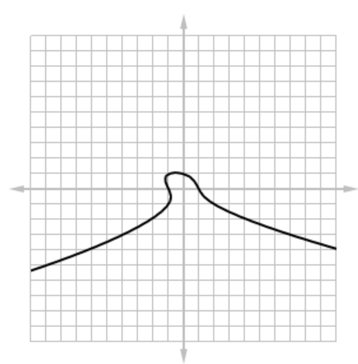
4. $\boxed{\mathtt{M}}$ Find x' and y' if $\ln(y) = \sin(xy) + \frac{x}{y}$.



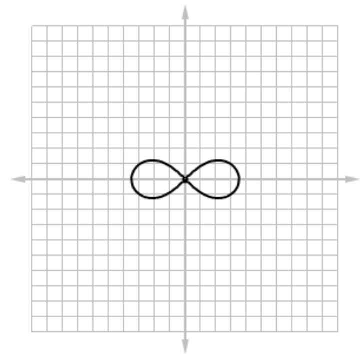
5. $\boxed{\mathtt{M}}$ Find y'' if $x^4 + y^4 = 16$.



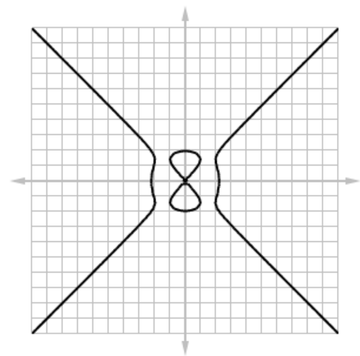
6. $\boxed{\mathtt{M}}$ If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.



7. $\boxed{\texttt{M}}$ Find a tangent line to the curve $2(x^2+y^2)^2 = 25(x^2-y^2)$ at the point $(3,1)$. This curve is known as a lemniscate.



8. $\boxed{\texttt{M}}$ Find a tangent line to the curve $y^2(y^2-4) = x^2(x^2-5)$ at the point $(0,-2)$. This curve is known as a devil's curve.



9. Consider the ellipse $x^2 - xy + y^2 = 3$.
   (a) $\boxed{\texttt{A}}$ Find the points where the ellipse crosses the x-axis.
   (b) $\boxed{\texttt{M}}$ Show that the tangent lines of the curve at these points are parallel.
   (c) $\boxed{\texttt{E}}$ Find the maximum and minimum points of the curve.

10. $\boxed{\texttt{E}}$ Consider a circle C that is tangent to $3x+4y-12=0$ at $(0,3)$ and contains $(2,-1)$. Set up equations that would determine the centre (h,k) and radius r of C.

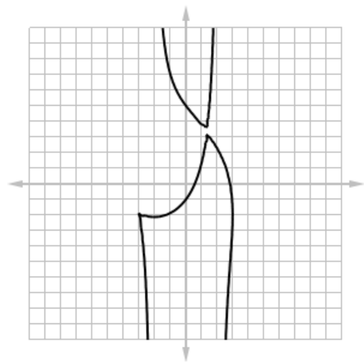
11. $\boxed{\texttt{S}}$ The Bessel function of order 0, $y = J(x)$, satisfies the differential equation

$$xy'' + y' + xy = 2$$

for all values of x. The value of the function at 0 is $J(0) = 1$.
   (a) Find $J'(0)$.
   (b) Use implicit differentiation to find $J''(0)$.

12. $\boxed{\text{S}}$ Consider the following family of curves, known as Durer's shell curves (shown here for $a = 2$, $b = 3$):

$$(x^2 + xy + ax - b^2)^2 = (b^2 - x^2)(x - y + a)^2.$$

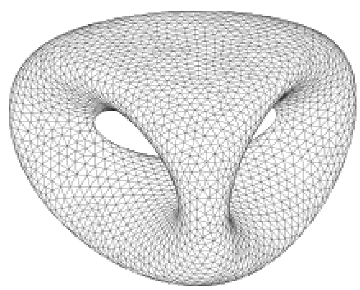


(a) For which value(s) of b does the curve become a straight line?

(b) Suppose that we restrict $a = \frac{b}{2}$. Find all non-differentiable points on the curve.

13. $\boxed{\text{O}}$ Moving into three dimensions, let us consider the surface described by

$$2y(y^2 - 3x^2)(1 - z^2) + (x^2 + y^2)^2 - (9z^2 - 1)(1 - z^2) = 0.$$



(a) Verify that the point $\left(1, 1, \frac{1}{\sqrt{3}}\right)$ is on the surface.

(b) Find the values of $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$ at this point (holding y and x constant, respectively). What do these derivatives represent?

(c) Write down the equations of the tangent lines to the surface in the y and x directions.

(d) Find an equation for the unique plane containing both tangent lines. Describe what this plane represents geometrically.