Solutions to L3 Calculus Differentiation Exam 1

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Question One

Part (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 1 - \frac{1}{x^2} - \frac{2}{x^3}$$

(1 mark)

Part (b)

We have $\frac{\mathrm{d}r}{\mathrm{d}t} = 0.5$. Since $A = \pi r^2$, we can write $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t} \times 2\pi r = \pi r$. After 10 seconds, the radius will be 5 metres and so $\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right|_{t=10} = 5\pi$. (3 marks)

Part (c)

Noting that an equilateral triangle has angles of $\frac{\pi}{3}$, the height of the rectangle must be $\frac{L-x}{2} \tan\frac{\pi}{3} = \frac{\sqrt{3}}{2}(L-x)$. Hence the area of the rectangle is

$$A = \frac{\sqrt{3}}{2}(L-x)x = \frac{\sqrt{3}}{2}(Lx - x^2)$$

and, taking the derivative,

$$A' = \frac{L\sqrt{3}}{2} - x\sqrt{3}$$

The area will be maximised when $A' = 0$, which is exactly when $x = L/2$. (4 marks)

Question Two

Part (a)

(i)

$$f'(x) = -2\sin(2x) + \frac{1}{2}e^{x/2}$$
$$f''(x) = -4\cos(2x) + \frac{1}{4}e^{x/2}$$

(2 marks)

(ii) We have $f''(0) = -4 + \frac{1}{4} < 0$, so the function is concave down and the point is a minimum. (1 mark)

Part (b)

(i) $g(3) = 9$ (1 mark)

(ii) The function approaches a value near $y = 5.5$ on the left, but a value near $y = 9$ from the right. Hence the left and right limits are different, and the limit of the function at $x = 3$ does not exist. (2 marks)

(iii) $\lim_{x \to 5} = 7$ (1 mark)

Part (c)

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}(x^2 - 5x) &= \lim_{h \to 0} \frac{[(x+h)^2 - 5(x+h)] - (x^2 - 5x)}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 5x - 5h - x^2 + 5x}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 - 5h}{h} \\
&= \lim_{h \to 0} 2x - 5 + h \\
&= 2x - 5.
\end{aligned}$$

(3 marks)

Question Three

Part (a)

$$\begin{aligned}
\tfrac{\mathrm{d}x}{\mathrm{d}t} &= \frac{(1 + \tan t)(\sec t)' - (\sec t)(1 + \tan t)'}{(1 + \tan t)^2} \\
&= \frac{(1 + \tan t)(\sec t \tan t) - (\sec t)(\sec^2 t)}{1 + 2\tan t + \tan^2 t} \\
&= \frac{\sec t \tan t + \sec t \tan^2 t - \sec^3 t}{1 + 2\tan t + \tan^2 t}
\end{aligned}$$

(2 marks)

Part (b)

(i) We have $y'(t) = 4t$ and $x'(t) = 4$. Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = t = \frac{x}{4}$. (2 marks)

(ii) Suppose that $B = (x, y)$. Then the distance we wish to minimise is

$$D = \sqrt{(x+8)^2 + (y-2)^2} = \sqrt{(4t+8)^2 + (2t^2 - 2)^2} = \sqrt{4t^4 + 8t^2 + 64t + 68}$$

Then we can take the derivative and set it to zero:

$$\begin{aligned}
\tfrac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{2(4t^4 + 8t^2 + 64t + 68)} \times 16t^3 + 16t + 64 \\
0 &= 16t^3 + 16t + 64 \\
&= t^3 + t + 4 \\
t &\approx -1.3788.
\end{aligned}$$

(4 marks)

Part (c)

Taking the derivative:

$$\begin{aligned} y &= \frac{1}{m}[\sec(m\ln\theta)]^2 \\ \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{1}{m}\cdot 2\cdot\sec(m\ln\theta)\cdot\sec(m\ln\theta)\tan(m\ln\theta)\cdot\frac{m}{\theta} \\ &= \frac{2}{\theta}\cdot\sec^2(m\ln\theta)\cdot\tan(m\ln\theta). \end{aligned}$$

Plugging in $\theta = 1$, we have $\frac{2}{1}\cdot\sec^2(m\ln 1)\cdot\tan(m\ln 1) = 2\cdot 1\cdot 0 = 0$. (2 marks)