NCEA Level 3 Calculus (Integration)
24. Formal Definitions of $\exp$ and $\ln$

Today we will write down formal definitions of the natural exponential and logarithm functions.

The Natural Logarithm

We can integrate x^3, x^2, x^{-2017}, and x^0 easily using the reverse power rule (respectively, they become $x^4/4$, $x^3/3$, $-x^{-2016}/2016$, and 0). However, we cannot integrate x^{-1}:

$$\int x^{-1}\,\mathrm{d}x = \frac{1}{-1+1}x^{-1+1} = \frac{1}{0}x^0 = ???$$

But we know that the indefinite integral of x^{-1} must exist, since there is obviously a finite area beneath the curve $y = x^{-1}$ over (for example) the interval $[1, 2]$. We therefore make the following **definition**:

$$\ln x := \int_1^x \frac{1}{t}\,\mathrm{d}t.$$

How does one pronounce '*ln*'? `log`.

We can prove the log rules using this definition; as a sampler, we prove that $\ln x + \ln y = \ln xy$:

$$\begin{aligned}
\ln x + \ln y &= \int_1^x \frac{1}{t}\,\mathrm{d}t + \int_1^y \frac{1}{t}\,\mathrm{d}t \\
&= \int_1^x \frac{1}{t}\,\mathrm{d}t + \int_x^{xy} \frac{1}{u/x}\frac{1}{x}\,\mathrm{d}u \quad (u = tx) \\
&= \int_1^x \frac{1}{t}\,\mathrm{d}t + \int_x^{xy} \frac{1}{u}\,\mathrm{d}u \\
&= \int_1^x \frac{1}{t}\,\mathrm{d}t + \int_x^{xy} \frac{1}{t}\,\mathrm{d}t \\
&= \int_1^{xy} \frac{1}{t}\,\mathrm{d}t \\
&= \ln xy.
\end{aligned}$$

The Exponential Function

We define the exponential function to be the function implicitly defined by $\exp(x) = y \iff \ln(y) = x$. Consider the following (where $y = \exp(x)$):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{1/y} = y$$

So $\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = \exp(x)$.

We can also write $\exp(x) \equiv e^x$.

Note that e is the unique number such that $\ln(e) = 1$:

$$\ln(e) = \ln(e^1) = \ln(\exp(1)) = 1.$$

Questions

1. $\boxed{\text{S}}$ Prove the following:
   (a) $\ln 1 = 0$
   (b) $\ln x > 0$ for all $x > 1$.
   (c) $\ln x < 0$ for all $0 < x < 1$.
   (d) $a \ln x = \ln(x^a)$ (Remember, multiplication is just repeated addition.)
   (e) $\ln x - \ln y = \ln(x/y)$
   (f) $e^{x+y} = e^x e^y$
   (g) $e^{x-y} = e^x / e^y$
   (h) $e^0 = 1$
   (i) $(e^x)^y = e^{xy}$