NCEA Level 3 Calculus (Differentiation)
5. The Product and Quotient Rules

Goal for this week

To be able to take derivatives of things multiplied by other things, and to be able to decide which differentiation rules to use for a problem.

Two sections ago we saw that the derivative of a product is not simply the product of the derivatives; for example, take $(x)(x)$. If we differentiate each term and multiply, we obtain 1; however, the derivative of x^2 is (of course) $2x$. Ensure you understand why this is a counterexample to the naive rule $(fg)' = f'g'$ before continuing.

Suppose f and g are functions; then the *real* product rule is

$$(fg)' = gf' + fg'.$$

Proof. We simply apply the limit laws to the definition of the derivative and use a little trick (note that the expression highlighted in purple is exactly zero).

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x)}{h} \\
&= f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f(x)g'(x) + g(x)f'(x).
\end{aligned}$$

$\square$

Example. Consider $y = 2t\sin t$. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = 2\sin t + 2t\cos t$.

We can also write a rule for the derivative of a quotient of functions. You will be asked to prove it as an exercise, using the product rule.

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Example. Consider $f(s) = \sqrt{\frac{s^2+1}{s^2+4}}$. Then

$$f'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s^1+1}{s^2+4}\right] \cdot \frac{1}{2\sqrt{\frac{s^2+1}{s^2+4}}} = \frac{(s^2+4)(2s) - (s^2+1)(2s)}{(s^2+4)^2} \cdot \frac{\sqrt{s^2+4}}{2\sqrt{s^2+1}} = \frac{6s}{(s^2+4)^{\frac{3}{2}}(s^2+1)^{\frac{1}{2}}}.$$

With our rules (sum, chain, and product, together with our basic derivatives), we can now differentiate almost any combination of functions that we are currently aware of. The process of differentiation is entirely mechanical, and can be easily performed by a computer. As such, learning to differentiate more complicated combinations of functions is very similar to learning how to add, multiply, and perform long division, and is only a matter of practice.

Like basic arithmetic, differentiating functions is not interesting on its own; but being able to differentiate will allow us to talk more clearly about the geometry and behaviour of functions.

Questions

1. In each case, find $\frac{\mathrm{d}y}{\mathrm{d}t}$. A

   (a) $y = \left(3+2t^2\right)^4$

   (b) $y = \dfrac{t^3}{\ln t}$

   (c) $y = t\sqrt{t}$

   (d) $y = 2t\sin t - (t^2-2)\cos t$

   (e) $y = \dfrac{t}{\sqrt{a^2-t^2}}$ (a constant)

   (f) $y = \dfrac{1}{8}t^8\left(1-t^2\right)^{-4}$

   (g) $y = e^t \ln t$

   (h) $y = \log\left[1+\dfrac{t^2+3t+17}{t^{17}}\right]$

   (i) $y = \sin\left[e^{\tan t}\ln\tan t\right]$

   (j) $y = \dfrac{3t-2}{\sqrt{2t+1}}$

   (k) $y = \dfrac{\sec 2t}{1+\tan 2t}$

   (l) $y = \dfrac{(t-1)(t-4)}{(t-2)(t-3)}$

   (m) $y = t\sin^2(\cos\sqrt{\sin \pi t})$

   (n) $y = \sqrt[5]{t\tan t}$

   (o) $y = \dfrac{(t+\lambda)^4}{t^4+\lambda^4}$ (λ constant)

2. If $f(x) = e^{-x}$, find $f(0) + xf'(0)$. A

3. Show that $\frac{\mathrm{d}}{\mathrm{d}x}e^{\tan x}e^{-\cot x} = \left(\frac{\mathrm{d}}{\mathrm{d}x}e^{\tan x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}e^{-\cot x}\right)$. Reconcile this with our statement above that the naive product rule does not work in general. M

4. Suppose f and g are functions (g not the zero function). Write $\frac{f}{g} = fg^{-1}$ and prove the quotient rule from the product rule. M

5. Show that $y = xe^{-x}$ satisfies the differential equation $xy' = (1-x)y$. M

6. If $y = \ln\frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}$, find y''. M

7. Find the equation of the tangent line to the graph of $y = \ln\cos\frac{x-1}{x}$ at the point $(1, 0)$. M

8. Show that $y = (1+x+\ln x)^{-1}$ satisfies the differential equation $xy' = y(y\ln x - 1)$. M

9. Find the angle at which $y = x^2\ln[(x-2)^2]$ cuts the x-axis at the point $(0, 0)$. E

10. When $x = 0$, is the curve $y = (x+20)^2(2x^2-3)^6 - \ln\sin(x-\frac{\pi}{2})$ concave up or concave down? M

11. If $y = \frac{e^x}{\sin x}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y(1-\cot x)$. M

12. Show that if f, g, and h are functions then $(fgh)' = f'gh + fg'h + fgh'$. M

13. Suppose $f(x) = f(-x)$ for all x in the domain of f. Prove that $f'(x) = -f'(-x)$ for all x in the domain of $f'(x)$. E

14. Consider the function defined by $f(x) = x^x$. E

   (a) Rewrite f in the form $f(x) = e^{x\ln x}$, and hence find $f'(x)$.

   (b) Find $\frac{\mathrm{d}y}{\mathrm{d}t}$ if $y = (t^2+3)^{(t^2+3)}$.

15. A circle that closely fits points on a local section of a curve can be drawn for any continuous curve. The radius of curvature of the curve is defined as the radius of the approximating circle, which changes as we move around the curve. E

$$\text{radius of curvature} = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|}$$

Find the radius of curvature of the curve $y = e^{-x}\sin x$ at the point $(0, 0)$.

16. Find $f'(x+3)$ if $f(x+3) = (x+5)^7$. E

17. The number a is called a **double root** of some polynomial function f if $f(x) = (x-a)^2 g(x)$ for some polynomial g. Prove that a is a double root of f if and only if a is a root of both f and f'. E

18. Show that there is no function of the form E

$$f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0 + \frac{b_1}{x} + \cdots + \frac{b_m}{x^m}$$

such that $f'(x) = 1/x$.

19. Let's try to generalise the product rule for nth derivatives of products. Recall that the notation $f^{(n)}(x)$ means the nth derivative of f with respect to x. Assume in each case that the relevant derivatives exist. S
    (a) Find a formula for $(fg)'' = (fg)^{(2)}$ in terms of the derivatives of f and g.
    (b) What about $(fg)^{(3)}$?
    (c) If you can, prove Leibniz' formula for the nth derivative of a product using *induction*:

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$$