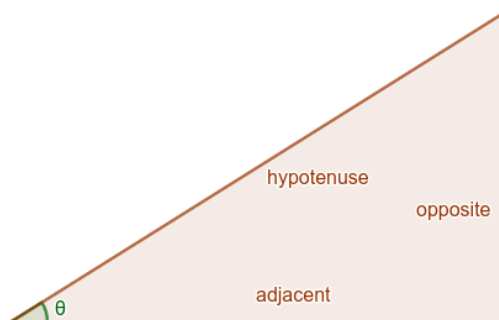


NCEA Level 2 Mathematics

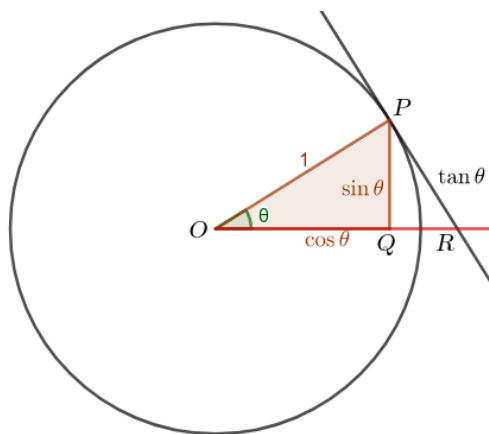
3. Trigonometry



We are now going to look at triangles inside circles. Now, last year we learned that any triangles with two equal angles are similar; in particular, if we take ratios of sides, we obtain the same value. This means that if we have any right-angled triangle with angle θ like the one above, then the ratios $\frac{\text{opposite}}{\text{hypotenuse}}$, $\frac{\text{adjacent}}{\text{hypotenuse}}$, and $\frac{\text{opposite}}{\text{adjacent}}$ all depend only on the angle θ ; we call them the sine, cosine, and tangent of the angle respectively:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin \theta}{\cos \theta}.$$

In particular, if we draw our triangle inside a unit circle then we can draw the following:



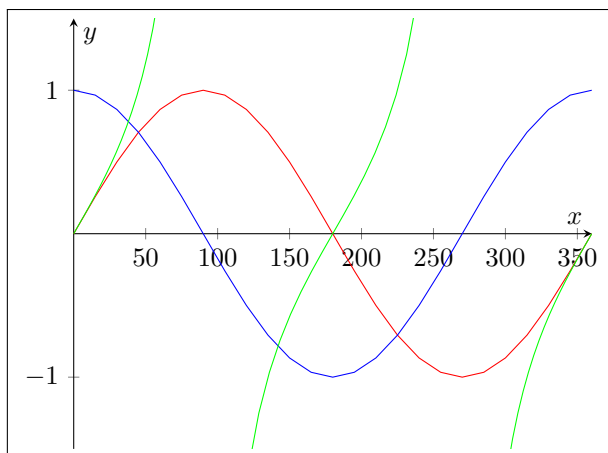
In fact, we can take this as our definition of sin and cos. To show that $\tan \theta$ is indeed the line segment marked, first notice that since the triangle OPR is right-angled, the angle at the intersection of the horizontal line and the tangent line is $90^\circ - \theta$; so the other non-right-angle in the triangle PQR is θ . Hence the hypotenuse of PQR is $\frac{\text{adjacent}}{\cos \theta} = \frac{\sin \theta}{\cos \theta}$, as proposed.

Note also that, from this diagram, we have

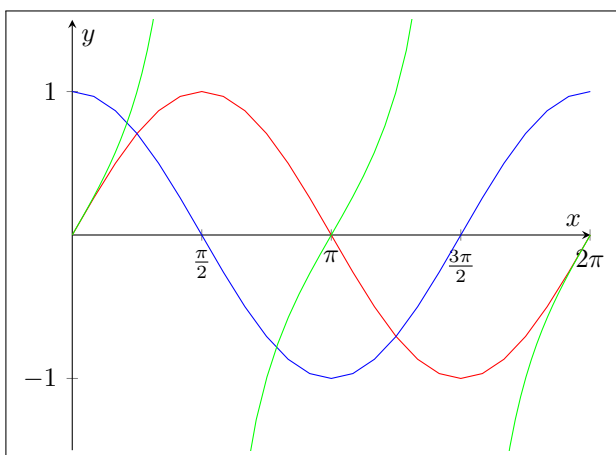
$$\sin^2 \theta + \cos^2 \theta = 1$$

for every angle θ .

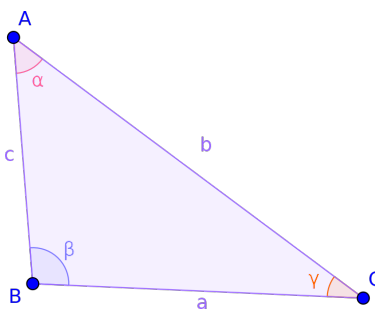
Since the sin of an angle is just the height of the point above the x -axis in the diagram above, we have that $-1 \leq \sin \theta \leq 1$; similarly, $-1 \leq \cos \theta \leq 1$. Note that when $\theta = 90^\circ$, the tangent line becomes horizontal and so never intersects the x -axis: so $\tan 90^\circ$ is undefined. We can even graph $\sin \theta$ (red), $\cos \theta$ (blue), and $\tan \theta$ (green):



If we graph them in radians, only the labels on the x -axis change:



Let us now begin to look at more general triangles:



Theorem (Sine rule). *In any triangle, with the angles and sides labelled as above, we have*

$$\frac{\alpha}{\sin a} = \frac{\beta}{\sin b} = \frac{\gamma}{\sin c}.$$

Proof. Drop an altitude from B to AC , creating two new right-angled triangles. Then the length of this line can be calculated using both of the resulting right-angled triangles: so $c \sin \alpha = a \sin \gamma$ and $\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$. This proves the theorem. \square

Theorem (Cosine rule). *In any triangle, with the angles and sides labelled as above, we have*

$$a^2 = b^2 + c^2 - bc \cos \alpha.$$

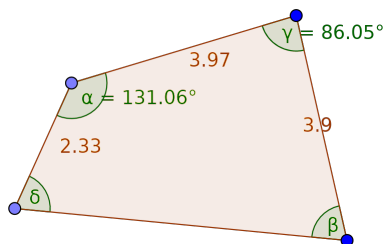
Proof. Drop an altitude from B to AC , creating two new right-angled triangles. Then the length b can be split into two lengths, $c \cos \alpha$ and $b - c \cos \alpha$; the length of the altitude is $c \sin \alpha$. Now, apply the Pythagorean theorem to the triangle including the angle γ :

$$a^2 = (b - c \cos \alpha)^2 + c^2 \sin^2 \alpha = b^2 - 2bc \cos \alpha + c^2 \cos^2 \alpha + c^2 \sin^2 \alpha = b^2 + c^2 - 2bc \cos \alpha.$$

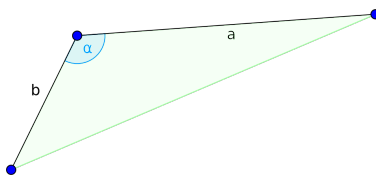
□

Questions

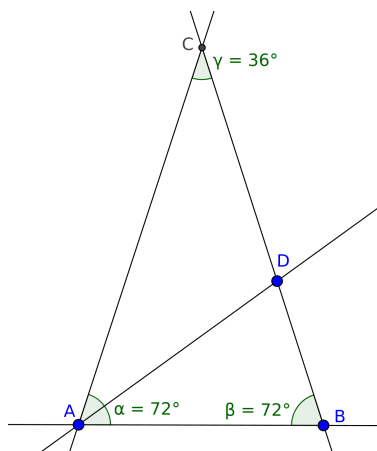
- A beam of gamma rays is to be used to treat a tumor known to be 5.7 cm beneath the patient's skin. To avoid damaging a vital organ, the radiologist moves the source over by 8.3 cm.
 - At what angle to the patient's skin must the radiologist aim the gamma ray source to hit the tumor?
 - How far will the beam have to travel through the patient's body before reaching the tumor?
- A surveyor measures the three sides of a triangular field and finds that they are 117, 165, and 257 metres long.
 - What is the measure of the largest angle of the field?
 - What is the area of the field?
- A field has the shape of a quadrilateral (four-sided shape) that is *not* a rectangle. Three sides measure 2.33, 3.97, and 3.9 kilometres, and two angles measure $\alpha = 131.06^\circ$ and $\gamma = 86.05^\circ$ (as in the figure).



- By dividing the quadrilateral into two triangles, find its area.
 - Find the length of the fourth side.
 - Find the measures of the other two angles, β and δ .
- A surveyor is standing on top of a peak. She can see two prominent peaks ahead of her, and from previous measurements she knows that one of them is 8 km away from her and the other is 11 km away. She measures the angle between them to be 120° . How far apart are the two peaks (measured *along the ground*):
 - If that they have same height?
 - If the surveyor and the closer peak are at the same height, but the peak which is further away is 200 m higher?
 - Consider a triangle with sides of 5, 7, and 10 kilometres.
 - Find the measure of the largest angle of this triangle.
 - Find the area of the triangle.
 - Find the area of the triangle below.



7. For each item below, decide whether or not such a triangle exists. If at least one does, how many exist?
- Exactly one angle greater than 90° .
 - Two angles greater than $\pi/2$.
 - Two sides of length 200,000.
 - Three sides of length 200,000.
 - Sides of length 90, 30, and 30.
8. Prove that, if a quadrilateral has equal diagonals, then it is a rectangle. (We used this fact last week!)
9. Let ABC be a triangle.
- Let $X = m(A, B)$, $Y = m(B, C)$, and $Z = m(C, A)$ be the midpoints of the sides; then the lines CX , AY , and BZ are called the *medians* of the triangle. Show that the three medians always intersect at a single point N (the *centroid*).
 - Let k be the perpendicular bisector of AB , ℓ be the perpendicular bisector of BC , and m be the perpendicular bisector of CA . Show that k , ℓ , and m intersect at a single point O (the *circumcentre*). Show that O is the centre of the circle passing through A , B , and C .
 - Let λ be the line passing through A that bisects the angle of the triangle at A . Define μ and ν similarly as the angle bisectors at B and C . Show that λ , μ , and ν intersect at a single point P (the *incentre*). Show that P is the centre of the circle which is tangent to the three sides of the triangle.
 - Let ρ be the line through A perpendicular to BC ; define σ and τ similarly to be lines through B and C . These lines are known as the *altitudes* of the triangle. Show that ρ , σ , and τ intersect at a single point (the *orthocentre*).
10. This question requires you to find exact values for trig functions *without* using a calculator. [Schol 1999]
- Sketch an equilateral triangle of side 2 units. Hence, or otherwise, show that $\cos \frac{\pi}{3} = \frac{1}{2}$, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, and $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$.
 - Sketch a right angled isosceles triangle whose equal sides enclose the right angle and are of length 1. Hence, or otherwise, show that $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$.
11. Consider the 75-75-36 triangle ABC given in the figure. The angle α has been bisected into two angles, and the resulting line meets the triangle at D .



- Show that ABC and ABD are similar triangles.
- Hence, or otherwise, show that $\frac{AB}{BD} = \frac{AB+BD}{AB}$.
- Show that the ratio of the long side of the triangle to the short side of the triangle is $\frac{AB}{BD} = \frac{1+\sqrt{5}}{2} = \phi$.
- Show that $\cos 72^\circ = \frac{1}{2\phi}$.
- Find $\sin 36^\circ$ and $\sin 72^\circ$.