Schol Trig problem sheet 6.

1a. Let $x = \sin y$. Then:

$$2\arctan\left(\frac{x}{1+\sqrt{1-x^2}}\right) = 2\arctan\left(\frac{\sin y}{1+\sqrt{1-\sin^2 y}}\right)$$

$$= 2\arctan\left(\frac{\sin y}{1+\cos y}\right)$$

$$= 2\arctan\left(\frac{\pm\sqrt{1-\cos^2 y}}{1+\cos y}\right)$$ (where $\pm$ is the sign of y)

$$= 2\arctan\left(\frac{\pm\sqrt{(1-\cos y)(1+\cos y)}}{1+\cos y}\right)$$

$$= 2\arctan\left(\pm\sqrt{\frac{1-\cos y}{1+\cos y}}\right).$$

But $\tan\frac{y}{2} = \pm\sqrt{\frac{1-\cos y}{1+\cos y}}$ (where $\pm$ is the sign of y)

So $2\arctan\left(\frac{x}{1+\sqrt{1-x^2}}\right) = 2\cdot\frac{y}{2} = y = \arcsin x$.

b. Let $y = \operatorname{versin} x = 1-\cos x$. Then:

$$\arctan\left(\frac{\sqrt{2y-y^2}}{1-y}\right) = \arctan\left(\frac{\pm\sqrt{2-2\cos x-(1-\cos x)^2}}{\cos x}\right)$$

$$= \arctan\left(\frac{\pm\sqrt{1-\cos^2 x}}{\cos x}\right)$$

$$= \arctan\left(\frac{\sin x}{\cos x}\right) = x = \operatorname{arcversin} y.$$

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2a. $\tan(\arctan a + \arctan b) = \dfrac{\tan\arctan a + \tan\arctan b}{1 - \tan\arctan a \tan\arctan b}$

$$= \frac{a+b}{1-ab}$$

So $\arctan a + \arctan b = \arctan\left(\frac{a+b}{1-ab}\right)$.

b. $\arctan 1 + \arctan 2 = \arctan\left(\frac{3}{-1}\right) = \arctan(-3)$.

$\arctan(-3) + \arctan 3 = \arctan\left(\frac{3-3}{1-3\cdot 3}\right) = 0$.

However, note that the formula in (a) only tells us that $\tan(\arctan 1 + \arctan 2 + \arctan 3) = 0$. Hence, $\arctan 1 + \arctan 2 + \arctan 3 = 0 + n\pi$ for some integer n.
A rough calculation shows that

$$0 < \arctan 1 + \arctan 2 + \arctan 3 < 4$$

So we must have $n = 1$.

c. $\arctan \tau + \arctan \frac{1}{\tau} = \arctan\left(\frac{\tau + 1/\tau}{1 - \tau/\tau}\right)$

$$= \arctan(\infty) = \pi/2.$$

If $\tau < 0$, $\arctan \tau + \arctan 1/\tau$

$$= -\arctan(-\tau) - \arctan(-1/\tau)$$
$$= -\pi/2.$$

3a. $p = \frac{b}{a}$, $q = \frac{y}{x}$ $\Rightarrow$ $p \otimes q = \frac{ay+bx}{ax-by} = \frac{ya+xb}{xa-yb} = q \otimes p$.

b. Suppose $g = \frac{m}{n}$ is such that $g \otimes k = g$ for all $k = \frac{b}{a}$ rational.

Then $\frac{m}{n} = \frac{m}{n} \otimes \frac{b}{a} = \frac{nb+ma}{na-mb}$

$\Rightarrow$ $m(na-mb) = n(nb+ma)$

$\Rightarrow$ $mna - m^2 b = n^2 b + mna$

$n^2 b = -m^2 b$.

~~If $b \neq 0$ $n = \pm m$ works, so $g = \pm 1$. But~~

~~$\frac{-1}{1} \otimes \frac{b}{a} = \frac{b+a}{}$~~

For us to pick n, m for all b/a, in particular it must work for $b = 1$. But then $n^2 = -m^2$, which is impossible unless $m = n = 0$ and no such rational exists.

c. Try $F = 1.0/1$. Then if $k = \frac{b}{a}$ then

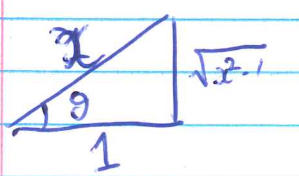
$F \otimes \frac{b}{a} = \frac{1 \cdot b + 0 \cdot a}{1 \cdot a - 0 \cdot b} = b/a$.

So F works.

d. $\frac{b}{a} \otimes \frac{-b}{a} = 0 = F$, so if k is rational then $k' = -k$ works.

e. $\arctan\left(\frac{b}{a}\right) + \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{\frac{b}{a} + \frac{y}{x}}{1 - \frac{by}{ax}}\right)$

$= \arctan\left(\frac{bx-ay}{ax-by}\right)$ ✓

4. 
$\sec x = \theta$, $\tan \theta = \sqrt{x^2+1}$.

5. $\arcsin a = 2\arccos a \Rightarrow a = \sin(2\arccos a)$

$= 2\sin \arccos a \cdot \sin \arcsin a$

$= 2 \cdot \sqrt{1-a^2} \cdot a$

$\frac{1}{2} = \sqrt{1-a^2} \Rightarrow a = \sqrt{3}/2$ works.