

NCEA Level 3 Calculus (Differentiation)

8. Optimisation

Recall from Level 2 that a *local maximum* of a function f is some point $(x, f(x))$ such that, for a sufficiently small interval I around x , for all $y \in I$ $f(y) \leq f(x)$; a *local minimum* is defined in a similar way (see exercises).

Example. The function $x \mapsto x^2$ has a local minimum at $(0, 0)$.

Example. The function $x \mapsto 2x^3 + 15x^2 + 36x + 2$ has a local maximum at $(-3, -25)$ and a local minimum at $(-2, -26)$.

Example. The function $x \mapsto \sin x$ has a local maximum at $(2n\pi + \frac{\pi}{2}, 1)$ for every $n \in \mathbb{Z}$, and a local minimum at $(2n\pi - \frac{\pi}{2}, -1)$ for every $n \in \mathbb{Z}$.

Local extrema are also sometimes called *relative extrema*.

It is possible to prove that if f is defined around some value x , if f has a relative extremum at $(x, f(x))$, and if f' is defined at x , then $f'(x) = 0$. Because of this, we define a *critical point* of a function f to be some value x such that either $f'(x) = 0$, or $f'(x)$ is undefined. In the first case, we also call the value a *stationary point*.

All local extrema occur at critical points, but not all critical points occur at extrema.

Example. The function $x \mapsto 2x^3 + 15x^2 + 36x + 2$ above has critical points $x = -2$ and $x = -3$. Both of these are local extrema.

Example. The function $x \mapsto x^3$ above has a critical point at $x = 0$, but does not have a local extrema there.

Example. The function $x \mapsto \frac{1}{x}$ does not have a critical point at $x = 0$, because it is not defined there.

We can use the first derivative to classify extrema as either maxima or minima.

The First Derivative Test

1. Determine all critical points of f .
2. Determine the sign of $f'(x)$ to the left and right of each critical point x_0 :
 - If $f'(x)$ changes from positive to negative as we move from left to right across x_0 , then $f(x)$ has a local maximum at x_0 .
 - If $f'(x)$ changes from negative to positive as we move from left to right across x_0 , then $f(x)$ has a local minimum at x_0 .
 - If $f'(x)$ does not change sign across x_0 , then $f(x)$ does not have a relative extremum at x_0 (e.g. $y = x^3$).

Recall last week's material; using the second derivative, we can come up with a second test:

The Second Derivative Test

1. Compute $f'(x)$ and $f''(x)$.
2. Find all the stationary points of f by finding all the points x_0 such that $f'(x_0) = 0$.
3. Determine the sign of $f''(x)$ for each stationary point x_0 :
 - If $f''(x_0) < 0$, then $f(x)$ has a relative maximum at x_0 .
 - If $f''(x_0) > 0$, then $f(x)$ has a relative minimum at x_0 .
 - If $f''(x_0) = 0$, then $f(x)$ could have a relative maximum, a relative minimum, or neither.

Example. Find and classify the critical points of $y = x^3 - 3x^2 + 6$.

Solution. We have $\frac{dy}{dx} = 3x^2 - 6x$ and $\frac{d^2y}{dx^2} = 6x - 6$. Hence the critical points are $x = 0$ and $x = 2$. At the former point, $\frac{d^2y}{dx^2} < 0$, and so the point is a maximum; at the latter point, $\frac{d^2y}{dx^2} > 0$ and so the point is a minimum.

Example. Find two numbers whose difference is 100 and whose product is a minimum.

Solution. Let the two numbers be x and $x + 100$. We wish to minimise $y = x(x + 100)$; clearly $y' = 2x + 100$, and so $x = -50$ is a critical point. To the left of $x = -50$, the derivative is negative; to the right, the derivative is positive. Hence $x = -50$ is indeed a minimum. The two required numbers are therefore -50 and 50.

Example. Find and classify the critical points of $y = (x - 1)^2 + \ln x$.

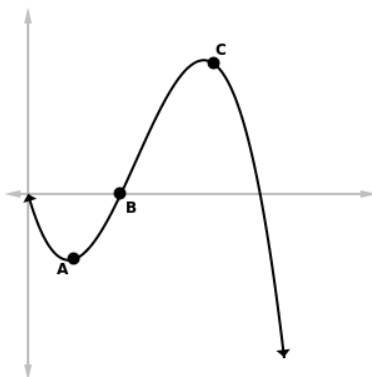
Solution. The derivative is $y' = 2x - 2 + \frac{1}{x}$. We therefore have one critical point at $x = 0$ (where y' is undefined); this is an asymptote. Setting $y' = 0$, we have $0 = 2x - 2 + \frac{1}{x} = 2x^2 - 2x + 1$ which has no real roots. Hence $x = 0$ is the only critical point, and the curve has no local extrema.

Example. A rectangular plot of land is to be fenced using two varieties of fence. Two opposite sides will use fences selling for \$3 per metre, while the other two sides will use cheaper fence selling for \$2 per metre. Given that the total budget is \$1200, what is the greatest area of land which can be fenced?

Solution. Let x be the length of one of the expensive sides; then the length of one of the cheaper sides is $\frac{1}{2}(1200 - 3x)$, and the total area is $A = \frac{1}{2}x(1200 - 3x) = \frac{1}{2}(1200x - 3x^2)$. Hence $\frac{dA}{dx} = 600 - 3x$. We wish to find the maximum area, so set $\frac{dA}{dx} = 0$; hence $3x = 600$ and $x = 200$. Note that the second derivative is always negative, so this stationary point must be a maximum as required. The length of the other side will be $\frac{1}{2}(1200 - 600) = 300$, and so the maximum area is $300 \times 200 = 60000$ square metres.

Questions

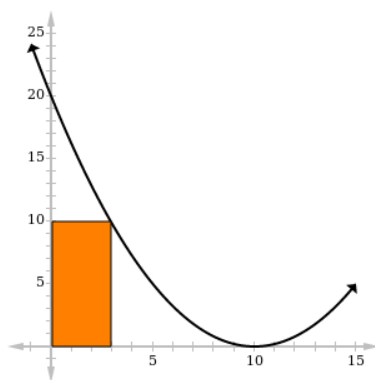
1. **A** Write down a definition of a local minimum similar to that given for a maximum.
2. **A** Show that $f(x) = x^4$ has $f''(0) = 0$ but not a point of inflection at $x = 0$ (in fact, it has a minimum at that point).
3. **A** Describe the advantages and disadvantages of the first and second derivative tests for local extrema.
4. **M** Describe the local extrema, concavity, and points of inflection of the function $f(x) = x^4 - 4x^3$.
5. **A** Consider the following graph:



Find the signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the three points A, B, and C.

6. **A** Without using calculus, find the extreme value(s) of $y = 3x^2 - 258x + 5598$.

7. M Find all the local extrema of the following curves in the given intervals, and classify them as maxima, minima, or neither.
- (a) $f(x) = \sin x - \cos x$ on the interval $0 < x < \pi$
- (b) $g(x) = x^3 - x^2 + x - 1$ on the interval $-\infty < x < \infty$
8. M The sum of two positive numbers x and y is 16. Find the smallest possible value for $S = x^2 + y^2$.
9. M A box with an open top is to be constructed from a square piece of cardboard with a side length of 3 m by cutting out a square from each of the four corners and bending up the sides. Find the dimensions of the resultant box of maximum volume.
10. M Find the dimensions of a rectangle with area 1000 m^2 such that the perimeter is minimised.
11. E A large orange rectangle is to be drawn with one corner sitting on the origin and the opposite corner lying on the curve $y = 0.2(x - 10)^2$. What is the maximum possible area of the rectangle?



12. M A window consisting of a rectangle topped with a semicircle is to have a fixed perimeter p . Find the radius of the semicircle in terms of p if the total area is to be maximised.
13. E A thin wire of length L is cut in two and the resulting lengths are bent to make a square and an equilateral triangle. Where should the wire be cut to make the total area of the shapes (a) a maximum and (b) a minimum?
14. E Find the point on the line $y = 2x + 3$ closest to the origin.
15. E Find the point on the curve $y = \sqrt{x}$ closest to $(3, 0)$.
16. M The rate in which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}.$$

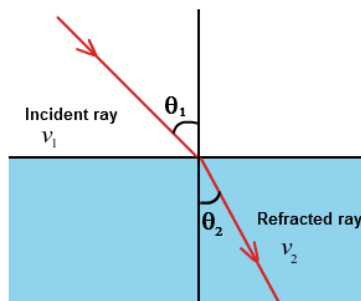
For which light intensity I is P a maximum?

17. E By finding the x - and y - intercepts, the asymptotes, the critical points, the intervals of increase and decrease, the intervals of concavity, and any other important points, sketch the following functions (199):
- (a) $f(x) = \frac{x^2}{4-x^2}$
- (b) $f(x) = \frac{4x}{x^2+1}$ [Hint: consider what happens to $f(x)$ as $x \rightarrow \pm\infty$.]
- (c) $f(x) = \frac{x^2-4x+5}{x-2} = x - 2 + \frac{1}{x-2}$ [Hint: consider what happens to $f(x) - (x - 2)$ as $x \rightarrow \pm\infty$.]

18. **E** A cone with height h is inscribed in a larger cone of height H such that the vertex of the small cone is at the centre of the base of the larger cone. Show that the maximum volume of the smaller cone occurs when $h = \frac{1}{3}H$.
19. **E** Let v_1 be the velocity of light in air and v_2 be the velocity of light in water. A ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimises the time taken. Show that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

where θ_1 is the angle of incidence and θ_2 is the angle of refraction.



20. **E** Show that the polynomial $p(x) = 10x^3 + x^2 + x - 34$ has exactly one real zero.
21. **E** A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one third of the sheet on each side by an angle θ . What angle should be chosen in order to obtain the maximum possible volume?
22. **E** A steel pipe is carried around a right-angled corner from a hallway 3 m wide into a hallway 2 m wide. What is the length of the longest pipe that can be carried horizontally around the corner? [*Hint: this is actually a minimisation problem, despite the wording.*]
23. **E** Find and classify the critical points of $h(x) = x^4 + x^3 + cx^2$.
24. **S** Show that $\frac{x^2+1}{x} \geq 2$; hence (or otherwise) show that $\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geq 8$.
25. **S** Scholarship 2013: Prince Ruperts drops are made by dropping molten glass into cold water. A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi(e^{-x} - e^{-2x})}$ for $x \geq 0$, where ϕ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.
- (a) Where is the modelled drop widest, and how wide is it there?
- (b) The drop changes shape at a point B , where the concavity of the function is zero. Use

$$\frac{d^2y}{dx^2} = \sqrt{\phi} \frac{e^{2x} - 6e^x + 4}{y^2 e^{4x}}$$

to find the exact x -ordinate of B .