NCEA Level 3 Calculus (Differentiation)
8. Optimisation

Goal for this week

To give a simple application of the geometry we looked at last week.

Recall from Level 2 that a **local maximum** of a function f is some point $(x, f(x))$ such that, for a sufficiently small interval around x, whenever y is in the interval then $f(y) \leq f(x)$. A **local minimum** is defined in a similar way. Local extrema are also sometimes called **relative extrema**.

Many optimisation problems in applied mathematics can be reduced to finding relative extrema.

Examples.

1. The function $x \mapsto x^2$ has a local minimum at $(0, 0)$.
2. The function $x \mapsto 2x^3 + 15x^2 + 36x + 2$ has a local maximum at $(-3, -25)$ and a local minimum at $(-2, -26)$.
3. The function $x \mapsto \sin x$ has a local maximum at $(2n\pi + \frac{\pi}{2}, 1)$ for every integer n, and a local minimum at $(2n\pi - \frac{\pi}{2}, 1)$ for every integer n.

For classification, we have the following theorem which links the location of relative extrema to the value of the derivative. Rather than memorising the proof, you should remember the geometric idea:- the derivative is changing from a positive value to a negative value (or vice versa), and so must pass through zero.

Theorem (Fermat's theorem)**.** *Let f be a function; suppose x_0 is a point in the interior of the domain of f, and that f has a relative extremum at $(x_0, f(x_0))$. Then $f'(x_0) = 0$.*

The proof of this is relatively straightforward; we just need the concept of left- and right-handed limits. Recall that, roughly speaking, a function has a limit at a point if it approaches the same value from both the left and the right. For left- and right-handed limits, we only require the function to approach a value from one side or the other.

Proof of Fermat's theorem (optional). Suppose f attains a relative maximum at x_0. Then for all h sufficiently close to zero, we have $f(x_0 + h) - f(x_0) \leq 0$. Hence, if $h < 0$, we have $\frac{f(x_0+h)-f(x_0)}{h} \geq 0$ (i.e. the derivative to the left is positive) and if $h > 0$, we have $\frac{f(x_0+h)-f(x_0)}{h} \leq 0$ (i.e. the derivative to the right is negative). Taking left- and right-hand limits around zero, we have the following chain of inequalities:

$$f'(x_0) = \lim_{h \to 0-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0 \geq \lim_{h \to 0+} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

Note that we use the fact that the derivative exists at x_0 and so the left- and right-hand limits both tend to the same value. Then the desired result, $f'(x) = 0$, follows directly.

For a relative minimum, the proof is essentially the same but with some inequalities swapped. □

Motivated by this theorem, we define a **critical point** of a function f to be some value x in the domain of f such that either $f'(x) = 0$, or $f'(x)$ is undefined. In the first case, we also call the value a **stationary point**. All local extrema occur at critical points, but not all critical points occur at extrema.

Examples.

1. The function $x \mapsto 2x^3 + 15x^2 + 36x + 2$ above has critical points $x = -2$ and $x = -3$. Both of these are local extrema.
2. The function $x \mapsto x^3$ above has a critical point at $x = 0$, but does not have a local extrema there.
3. The function $x \mapsto \frac{1}{x}$ *does not* have a critical point at $x = 0$, **because it is not defined there**.

Classifying Critical Points

We can use the first derivative to classify extrema as either maxima or minima.

1. Determine all critical points of f.
2. Determine the sign of $f'(x)$ to the left and right of each critical point x_0:
   - If $f'(x)$ changes from positive to negative as we move from left to right across x_0, then $f(x)$ has a local maximum at x_0.
   - If $f'(x)$ changes from negative to positive as we move from left to right across x_0, then $f(x)$ has a local minimum at x_0.
   - If $f'(x)$ does not change sign across x_0, then $f(x)$ does not have a relative extremum at x_0 (e.g. $y = x^3$).

On the other hand, using the second derivative, we can come up with a second test:

1. Compute $f'(x)$ and $f''(x)$.
2. Find all the stationary points of f by finding all the points x_0 such that $f'(x_0) = 0$.
3. Determine the sign of $f''(x)$ for each stationary point x_0:
   - If $f''(x_0) < 0$, then $f(x)$ has a relative maximum at x_0.
   - If $f''(x_0) > 0$, then $f(x)$ has a relative minimum at x_0.
   - If $f''(x_0) = 0$, then $f(x)$ could have a relative maximum, a relative minimum, or neither.

Example. Find and classify the critical points of $y = x^3 - 3x^2 + 6$.

Solution. We have $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 6$. Hence the critical points are $x = 0$ and $x = 2$. At the former point, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, and so the point is a maximum; at the latter point, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ and so the point is a minimum.

Example. Find two numbers whose difference is 100 and whose product is a minimum.

Solution. Let the two numbers be x and $x+100$. We wish to minimise $y = x(x+100)$; clearly $y' = 2x+100$, and so $x = -50$ is a critical point. To the left of $x = -50$, the derivative is negative; to the right, the derivative is positive. Hence $x = -50$ is indeed a minimum. The two required numbers are therefore -50 and 50.

Example. Find and classify the critical points of $y = (x-1)^2 + \ln x$.

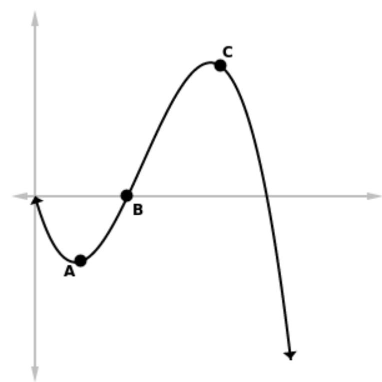
Solution. The derivative is $y' = 2x - 2 + \frac{1}{x}$. We therefore have one critical point at $x = 0$ (where y' is undefined); this is an asymptote. Setting $y' = 0$, we have $0 = 2x - 2 + \frac{1}{x} = 2x^2 - 2x + 1$ which has no real roots. Hence $x = 0$ is the only critical point, and the curve has no local extrema.

Example. A rectangular plot of land is to be fenced using two varieties of fence. Two opposite sides will use fences selling for \$3 per metre, while the other two sides will use cheaper fence selling for \$2 per metre. Given that the total budget is \$1200, what is the greatest area of land which can be fenced?

Solution. Let x be the length of one of the expensive sides; then the length of one of the cheaper sides is $\frac{1}{2}(1200 - 3x)$, and the total area is $A = \frac{1}{2}x(1200 - 3x) = \frac{1}{2}(1200x - 3x^2)$. Hence $\frac{\mathrm{d}A}{\mathrm{d}x} = 600 - 3x$. We wish to find the maximum area, so set $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$; hence $3x = 600$ and $x = 200$. Note that the second derivative is always negative, so this stationary point must be a maximum as required. The length of the other side will be $\frac{1}{2}(1200 - 600) = 300$, and so the maximum area is $300 \times 200 = 60000$ square metres.

Questions

1. Write down a definition of a local minimum similar to that given for a maximum. A
2. Show that $f(x) = x^4$ has $f''(0) = 0$ but not a point of inflection at $x = 0$ (in fact, it has a minimum at that point). A
3. Describe the advantages and disadvantages of the first and second derivative tests for local extrema. A
4. Describe the local extrema, concavity, and points of inflection of the function $f(x) = x^4 - 4x^3$. M
5. Consider the following graph: A

   Find the signs of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the three points A, B, and C.
6. Find all the local extrema of the following curves in the given intervals, and classify them as maxima, minima, or neither. M
   (a) $f(x) = \sin x - \cos x$ on the interval $0 < x < \pi$
   (b) $g(x) = x^3 - x^2 + x - 1$ on the interval $-\infty < x < \infty$
7. The sum of two positive numbers x and y is 16. Find the smallest possible value for $S = x^2 + y^2$. M
8. A box with an open top is to be constructed from a square piece of cardboard with a side length of $3\,\mathrm{m}$ by cutting out a square from each of the four corners and bending up the sides. Find the dimensions of the resultant box of maximum volume. M
9. Find the dimensions of a rectangle with area $1000\,\mathrm{m}^2$ such that the perimeter is minimised. M
10. A window consisting of a rectangle topped with a semicircle is to have a fixed perimeter p. Find the radius of the semicircle in terms of p if the total area is to be maximised. M
11. A thin wire of length L is cut in two and the resulting lengths are bent to make a square and an equilateral triangle. Where should the wire be cut to make the total area of the shapes (a) a maximum and (b) a minimum? E
12. Find the point on the line $y = 2x + 3$ closest to the origin. E
13. Find the point on the curve $y = \sqrt{x}$ closest to $(3, 0)$. E
14. By finding the x- and y- intercepts, the asymptotes, the critical points, the intervals of increase and decrease, the intervals of concavity, and any other important points, sketch the following functions (199): E
    (a) $f(x) = \frac{x^2}{4-x^2}$

(b) $f(x) = \frac{4x}{x^2+1}$ [*Hint: consider what happens to $f(x)$ as $x \to \pm\infty$.*]

(c) $f(x) = \frac{x^2-4x+5}{x-2} = x - 2 + \frac{1}{x-2}$ [*Hint: consider what happens to $f(x) - (x-2)$ as $x \to \pm\infty$.*]

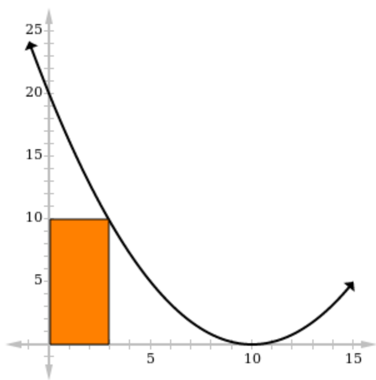
15. A cone with height h is inscribed in a larger cone of height H such that the vertex of the small cone is at the centre of the base of the larger cone. Show that the maximum volume of the smaller cone occurs when $h = \frac{1}{3}H$. E

16. Show that the polynomial $p(x) = 10x^3 + x^2 + x - 34$ has exactly one real zero. E

17. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one third of the sheet on each side by an angle θ. What angle should be chosen in order to obtain the maximum possible volume? E

18. A steel pipe is carried around a right-angled corner from a hallway 3 m wide into a hallway 2 m wide. What is the length of the longest pipe that can be carried horizontally around the corner? [*Hint: this is actually a minimisation problem, despite the wording.*] E

19. A large orange rectangle is to be drawn with one corner sitting on the origin and the opposite corner lying on the curve $y = 0.2(x-10)^2$. What is the maximum possible area of the rectangle? E



20. Show that $\frac{x^2+1}{x} \geq 2$; hence (or otherwise) show that $\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geq 8$. S

21. Scholarship 2013: Prince Ruperts drops are made by dropping molten glass into cold water. A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi(e^{-x} - e^{-2x})}$ for $x \geq 0$, where ϕ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. S

(a) Where is the modelled drop widest, and how wide is it there?

(b) The drop changes shape at a point B, where the concavity of the function is zero. Use

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sqrt{\phi}\frac{e^{2x} - 6e^x + 4}{y^2 e^{4x}}$$

to find the exact x-ordinate of B.

22. Scholarship 2014: A family of functions is built from two functions f and g, with a new function h_p defined for each value of p, $0 \leq p \leq 1$: S

$$f(x) = 2 + \sin x$$
$$g(x) = 26 + \sin x$$
$$h_p(x) = [f(x)]^{1-p}[g(x)]^p.$$

Define a fourth function S, where $S(p)$ is the difference between the maximum and the minimum values of h_p. Find the exact value of p that maximises S.

Note that if a is constant, $\frac{\mathrm{d}}{\mathrm{d}x}a^x = (\ln a)a^x$.