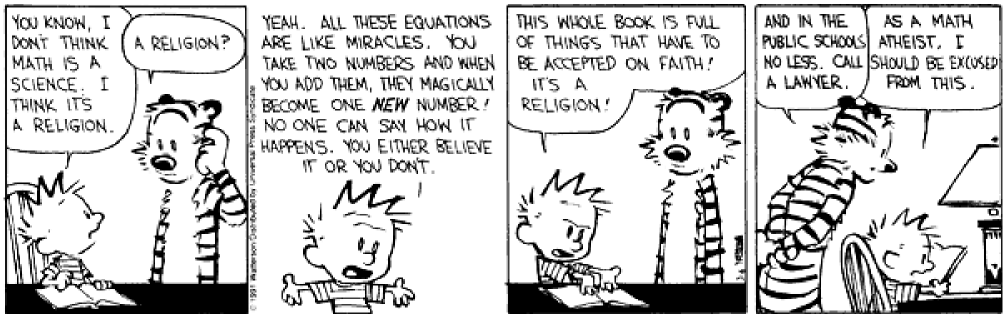
NCEA Level 3 Calculus
Introduction to the Notes



What is Calculus?

I will refrain from trying to advertise the subject to you, and will simply try to explain what calculus is and what kind of person uses it. Calculus is the broad study of:

- Continuous change;
- Slope, area, and volume; and
- Functions and relationships.

It has applications in physics, where calculus is the most natural language for Newtonian mechanics and classical electromagnetism; in chemistry and biology, where calculus can be used to model anything which changes over time (like rates of reaction, concentrations, and populations); in statistics (the study of probability distributions is just calculus); and in economics (I am assured). My own view, which I try to sprinkle throughout these notes, is mainly a mixture of geometric applications and physical intuition.

Within mathematics itself, calculus is the computational side of **real analysis**, the study of the properties of the real number system.

Mathematical Prerequisites

There are a number of things from Level 2 Mathematics that students should be comfortable with; generally, I assume in these notes a vague merit-level understanding of the core level 2 standards (by which I mean, the reader should be comfortable solving achieved level problems without guidance and have some idea how to approach more difficult problems):

- Level 2 Algebra: All material on quadratics (factorising, solving, discriminants), logs and exponents.
- Level 2 Calculus: Basic differentiation, geometric meaning of derivative (in particular, integration is *not* assumed)
- Level 2 Graphing: Recognising $x-/y-$shift of general functions, slope-intercept and point-slope form of linear equations, recognising period/frequency/amplitude/$x-/y-$shift from a trig function.
- Level 2 Trigonometry: Trig ratios, the Pythagorean theorem.
- Level 2 Simultaneous Equations: Solving linear and quadratic simultaneous equations.
- Level 2 Co-ordinate Geometry: Distances and linear equations.

Further into the notes, I also touch a bit on concepts covered in some of the other Level 3 standards, but not in so much detail that they need to be covered first:

- Level 3 Conics: Recognising forms from equations.

- Level 3 Algebra: Surds.
- Level 3 Trigonometry: Solving trig equations (including general solutions), reciprocal trig functions, use of trig identities (this latter mainly for E/S/OS-style integration problems).

In particular, no knowledge of complex algebra is assumed (or used) anywhere in the notes, and no knowledge from any L2 or L3 statistics standards is assumed.

I also do expect the reader to be able to draw graphs of various simple functions (e.g. parabolae, the three basic trig functions, logs and exponents, and so forth) 'by eye' — I realise that this is a little unrealistic, but at some point one must learn to become comfortable with the shapes and sizes of the objects we study. In order to help build this intuition, it is (very highly) recommended that the reader works through every example in detail, drawing pictures and so forth.

A Note on Problem Difficulty

One of the main goals for these notes is that they should be useful for students at all levels, from A to OS. Accordingly, the problems each week range from simple (most students should be able to just write down an answer without thinking too hard) to extremely non-trivial (it took **me** a while to work the problem, and I know this material quite well). If you can't do a problem, the best thing to do is to move on and come back to it — the problems don't always increase in difficulty. Of course, it is important to do a good number of problems **including some difficult ones**; you're not under exam conditions here and you're going to get an awful lot more out of a hard problem than an easy one!

I have marked the problems in the weekly worksheets (**not** the homework) with symbols relating vaguely to difficulty:

A M E S O

However, these are for my own reference and should not be taken to be accurate with respect to actual examinations.

Required content for Level 3

Some of the material goes beyond that required for NCEA Level 3; the following list gives some idea of the level of each sheet.

Differentiation

01. The Derivative
02. Limits
03. Derivatives of Common Functions
04. The Chain Rule
05. The Product and Quotient Rules
06. Tangent and Normal Lines
07. The Geometry of Functions
08. Optimisation
09. Implicit Differentiation
10. *Inverse Functions
11. Related Rates of Change

12. Parametric Functions
13. $^{*\perp}$Sequences and Series
14. †Differentiation Revision

Integration

15. Approximating Areas
16. Anti-differentiation
17. The Fundamental Theorem of Calculus
18. Substitution
19. Differential Equations
20. *Partial Fractions
21. *Integration by Parts
22. $^{*\perp}$Lengths, Volumes, and Areas
23. *Trigonometric Substitution
24. $^{\perp}$Kinematics
25. †Integration Revision
26. $^{*\perp\dagger}$☠More Interesting Problems

(*scholarship topic, $^{\perp}$interest topic, †revision, ☠reader discretion advised)
In particular, the author tends to follow the following rough guidelines:

Standard L3 student 1–9, 11, 12, 14, 15–19, 24, 25. No proofs; one week per sheet (so around two school terms).

Scholarship student All but 20, 23 and 26 (unless time available). Easy proofs. Leave some sheets as homework (e.g. cover the material of 3 – 5 in one week and leave a lot of problems for self study). Implicit differentiation and differential equations are revised in the conics notes, so some time can be saved there if needed. Sequences and series (13) and kinematics (24) are just revision from Y12, so can be left entirely as reading.

A Note on the Textbook

Many problems are taken from a couple of places:

- Stewart's Calculus (the current trendy textbook)
- Anton's Calculus (the old trendy textbook)
- Spivak's Calculus (the textbook you should use)
- University of Canterbury MATH199 lecture notes and problem sets
- Old NCEA/Scholarship exams

Most textbooks cover all the relevant material in the first few chapters (the first five or so in Stewart).

Homework

Every week has an associated homework sheet with a page of reading (five minutes or so) and a few questions (generally all pretty straightforward, but there is often a challenge question on there to keep you occupied).

I cannot emphasise enough how important it is to **do the homework**.

How to Read Mathematics

So we shall now explain how to read the book. The right way is to put it on your desk in the day, below your pillow at night, devoting yourself to the reading, and solving the exercises till you know it by heart. Unfortunately, I suspect the reader is looking for advice on how not to read, i.e. what to skip, and even better, how to read only some isolated highlights.

- Saharon Shelah, 'Classification Theory and the Number of Non-Isomorphic Models'

A major part of Level 3 Mathematics is preparation for university-level study in pure mathematics or the hard or soft sciences. As such, this year we begin to expose you to some 'real mathematics' — not just the watered-down calculation and computation you've been doing since intermediate, but a real journey of discovery through one of humanity's great fields of human experience. I guarantee that at primary school you enjoyed this kind of mathematics — for example, consider the non-obvious fact that $3 \times 5 = 5 \times 3$. Next year you will learn that there are some objects which do not have this property (commutativity).

There are a lot of definitions and theorems, but they allow us to capture our intuition of mathematical objects, and their behaviour in a more precise (and more correct) sense. It is important to read slowly and understand the statements as you go, as every piece strengthens the whole. I do not tend to repeat myself, so often you will find statements from earlier used later without comment.

I have also included proofs of a few of the theorems that we use, but rigorous proofs of many seemingly obvious ideas (like the fact that every function reaches a minimum and a maximum value on any closed interval) require some subtle properties of the real numbers and so it is more important to gain a geometric and intuitive feeling for many of the results rather than worrying too much.

Recommended reading for the interested: Paul Lockhart's 'A Mathematician's Lament'.