Level Three Calculus

Second Edition

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https://github.com/aelzenaar/ncea-notes

Preface for the navigator

These notes are my second attempt at a coherent introduction to calculus at the level of NCEA Level 3 and NZ Scholarship.

I have made a few philosophical changes from the first edition:-

- I treat anti-differentiation at the same time as differentiation. (I do introduce the $\int$ notation and the term 'indefinite integral' here, though I would rather not.)
- The notes are split into "topic" chapters: *The basics* (the basic formal manipulations of derivatives and anti-derivatives), *Geometry of curves* (studying curves via differentiation), *Geometry of spaces* (definite integrals, the fundamental theorem of calculus, and arc lengths, surface area, and volumes), and *Motion and change* (differential equations).
- I have dropped many of the proofs; my justification for this change is threefold(!). Firstly, the students that 'need' the proofs will see them in a Stage I university course. Secondly, many of the proofs in elementary calculus often obscure a nugget of geometry behind the formal manipulation of limits, and so I would rather include intuitive geometric *justifications* for results in the space the proofs formerly were. Finally, most students at Level 3 are simply not ready for proofs: either they don't understand why proof is required, or their level of mathematical sophistication means that the proofs seem esoteric. I have included copious references to textbooks where proofs can be found.

I feel the need also to point out that these notes are incredibly geometric. *If you don't like teaching geometry, these are not the notes for you.*

Prerequisite material

Firstly, a hard fact: for a student to be successful in L3 calculus, they should have a good understanding of the material at L2 and earlier (I would generally expect that students with less than a merit in the level 2 algebra standard will struggle).

In these notes, I will use material from algebra and geometry at L2 or earlier liberally; I try to point it out when I use some of the more obsure results. I do not use any material from any of the level three standards, except trigonometry.

So, in general, the prerequisites and expectations for these notes are:-

- A good understanding of L2 coordinate geometry and algebra.
- A decent understanding of L3 trigonometry, *including the manipulation of identities.*

For some of the sections, knowledge of a little physics (L1 and/or L2) would be nice. I cover the material in the L2 calculus standard quickly so this is not formally a prerequisite, but a student who doesn't understand the material there well will struggle with these notes. Roughly speaking, the differentiation material there is more important.

I would strongly recommend revising the material on functions (section 4 of my own level 2 notes).

Recommended textbooks

I have used the following textbooks when writing these notes, in roughly increasing order of sophistication:

- *Calculus made easy*, by Sivanus P. Thompson and Martin Gardner. This book is perhaps at the correct level mathematically speaking for a Y12/13 student, but it is not very geometric. It is certainly worth looking at, though.
- *Calculus*, by James Stewart. This is one of the standard first-year computational calculus books. It has many examples and many exercises, but lacks soul.
- *Calculus*, by Michael Spivak. This is often called the 'One True Calculus Book',[1] but is more properly an introduction to real analysis. As such, it is too diffcult for all but the most motivated high school students.
- (For the sake of completeness,) *Advanced Calculus*, by L. H. Loomis and S. Sternberg. This is the author's favourite calculus book, but is eminently unsuitable for high school students of any motivation. It is incredibly geometric, wonderfully well written, and almost impenetrable. This book inspired these notes, philosophically speaking.

[1] For example, by the *Chicago undergraduate mathematics bibliography*: `https://www.ocf.berkeley.edu/~abhishek/chicmath.htm` (somewhat useful, if taken with a grain of salt).

Preface for the student

I don’t have much to say, really. I could spend time explaining why calculus is useful, but I won’t do that here because we’ll see a lot of examples of calculus ‘in the wild’ as we progress (just as a taster, we’ll look at some physics, some biology, some economics, and maybe even some statistics). I could equally well spend time trying to explain exactly what calculus *is* exactly, but this page is not large enough to contain such an exposition.

Instead, I will give some study advice.

Firstly, you must read the notes. You must sleep with them beneath your pillow. You must work through the examples yourself. You must do all the problems. You must ask questions.

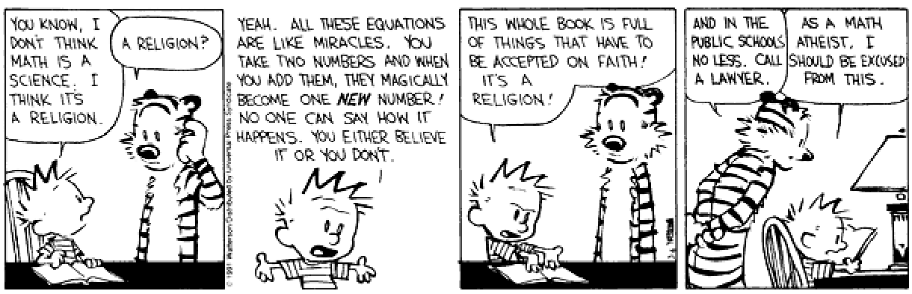
Because no student ever follows my first piece of advice, I will give you a second, easier option. For each topic, there are a few homework problems set. *At a minimum*, you should do all these problems. (But beware, if you *only* do these problems, you will be woefully underprepared for any situation you need calculus for.)

Secondly, draw pictures. I do my best to include lots of diagrams (some even in colour!), but one can never have too many pictures. (As a young girl called Alice once perceptively remarked, “What is the use of a book without pictures or conversations?”[2])

Thirdly, and I cannot stress this enough, *your exam grades do not matter.*[*citation needed*] It is perfectly possible to pass calculus exams without understanding the material, but if you do that (by, for example, trying to memorise everything in leu of understanding it), you are cheating yourself out of an education. If you understand the material, you will be prepared for every subject you may wish to take next year (and, as a bonus, you’ll pass the exam).

Let us begin.

[2]*Alice in Wonderland*, by Lewis Carroll.



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Chapter I

The basics

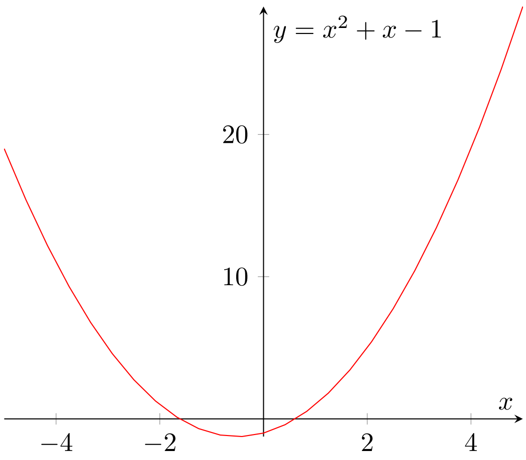


Figure 1: A parabola, $y = x^2 + x - 1$.

I.1 Tangent lines

Suppose we have two variables, y and x, such that any change in x produces a corresponding change in y. In the language we learned last year, we say that y is a function of x: that is, there is a function f such that $y = f(x)$.

Let us look at a few different functions.

Firstly, let's consider the function f defined by $f(x) = x^2 + x - 1$, graphed in figure 1. Let's look at how f behaves at a point x_0 by adding a small number h to the input and seeing how it changes the output.

$$f(x_0 + h) = (x_0 + h)^2 + (x_0 + h) - 1 = x_0^2 + 2x_0h + h^2 + x_0 + h - 1.$$

We're actually interested in the difference between our new output and our old output, because this gives us a measure of how quickly f changes when we move along the x-axis by h.

$$f(x_0 + h) - f(x_0) = (x_0^2 + 2x_0h + h^2 + x_0 + h - 1) - (x_0^2 + x_0 - 1) = 2x_0h + h + h^2$$

If h is small, then h^2 is miniscule and only makes up a very small part of $f(x_0 + h)$. We can therefore make the following approximation:

$$f(x_0 + h) - f(x_0) \approx (2x_0 + 1)h.$$

In other words, a small change in the input from x_0 to $x_0 + h$ produces a change in the output of the form $(2x_0 + 1)h$.

When we looked at straight lines in the past, they had a measure of *slope*: the ratio of 'rise' (change in output) to 'run' (change in input). For each x_0 that we feed into f here, we have a measure of the 'rise' of f over a very small distance, h. It makes sense, then, to define the slope of $y = f(x)$ at x_0 to be rise/run:

$$\frac{f(x_0 + h) - f(x_0)}{h} \approx \frac{(2x_0 + 1)h}{h} = 2x_0 + 1 \tag{I.1}$$

So when h is very small, the graph of f between x_0 and $x_0 + h$ looks like a straight line with slope $2x_0 + 1$.

Since the slope of our function defined in this way depends on the value of x_0, we have a new function which assigns to each point x_0 the slope of f around x_0. We denote this function by f', and call it the *derivative* of f. The slope of f at x_0 will be written $f'(x_0)$.

Another notation for the derivative is also common; because we are looking at changes in y divided by changes in x it sometimes makes sense to write the derivative as $\frac{\mathrm{d}y}{\mathrm{d}x}$ (where $\mathrm{d}x$ denotes, in some sense, a small change in x). The value of the derivative at x_0 is written as $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0}$. This notation is known as Leibniz notation, as it was first introduced in the 1600s by Gottfried Leibiz (a German

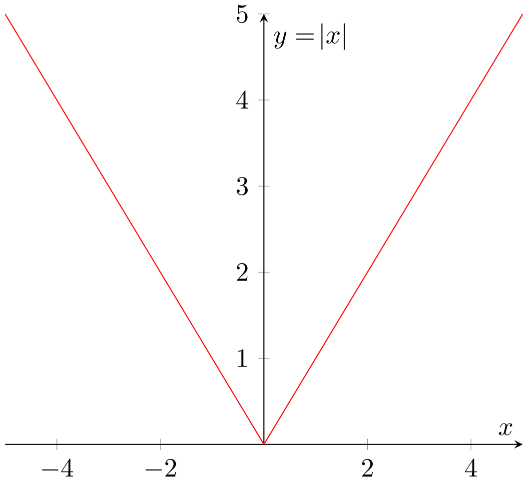


Figure 2: The absolute value function.

mathematician and philosopher who, it can be argued, was one of the first modern mathematicians to develop calculus in a sophisticated manner).

The derivative of the derivative of f is called the second derivative of f, and we write f'' for this new function. In general, the nth derivative of f is denoted by $f^{(n)}$ (it is the function produced by repeatedly differentiating f). In Leibniz notation, the nth derivative of $y = f(x)$ is $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$.

Definition. 1. If we can approximate the change in a function f around a point x_0 by writing $f(x_0 + h) - f(x_0) \approx mh$ for some constant m that depends on x_0 but not h, then f is said to be differentiable at x_0 with derivative $m = f'(x_0)$.

2. If $y = f(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$.

3. The line passing through $(x_0, f(x_0))$ with slope $f'(x_0)$ is called the *tangent line* to f at x_0.

We will look at a more complicated function now. Consider $y = \sin(x)$; we want to find its derivative, so we look at our output difference:

$$\sin(x+h) - \sin x = 2\cos\frac{(x+h)+x}{2}\sin\frac{(x+h)-x}{2} = 2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right).$$

For very small values of α, we have that $\sin\alpha \approx \alpha$ (as long as we measure α in radians).[1] In particular, $\sin(h/2) \approx h/2$. Further, if h is very small then $x + h/2 \approx x$. Making these two approximations, we find that

$$\sin(x+h) - \sin x \approx 2\cos(x)\cdot(h/2) = \cos(x)h.$$

Thus, for each x, we have

$$\frac{\sin(x+h) - \sin x}{h} \approx \cos x; \tag{I.2}$$

and as this approximation becomes arbitrarily precise as h gets closer to zero (because the closer h is to zero our approximations $h/2 \approx 0$ and $\sin(h/2) \approx h/2$ become better and better) we feel justified in saying that $\sin' = \cos$.

Finally, we will look at the absolute value function $x \mapsto |x|$ defined by

$$|x| = \begin{cases} x & \text{when } x \geq 0; \\ -x & \text{when } x < 0 \end{cases}.$$

This function is graphed in figure 2.

Let's try to calculate the slope of $y = |x|$.

- When we look around any positive x, the definition tells us that $|x+h| = x + h$. Thus $|x+h| - |x| = x + h - x = h$; hence $\frac{|x+h|-|x|}{h} = 1$, and so the derivative for any positive x is 1.

[1] See 2.13 in the trigonometry notes.

- When we look around any negative x, the definition tells us that $|x+h| = -x-h$. Thus $|x+h|-|x| = -x-h-(-x) = -h$; hence $\frac{|x+h|-|x|}{h} = -1$, and so the derivative for any negative x is -1.

But now we have a problem: at zero, if we take h to be a small positive number we find that $\frac{|x+h|-|x|}{h} = 1$; but if h is a small negative number, then $\frac{|x+h|-|x|}{h} = -1$. Since looking in different directions from the same point gives us different slopes, there is no good linear approximation to the curve $y = |x|$ at the point $(0,0)$. We will say that the curve is *non-differentiable* there.

Exercises and Problems

1. Let f be a function. Describe the difference between f, f', $f(57)$, and $f'(57)$.

2. The following data are temperatures T in a place measured hourly (t is the number of hours since noon, and T is measured in degrees celsius).

t	0	2	4	6	8	10	12
T	15	16	16	15	14	13	13.

   Since T is a function of t, we can consider the derivative T'. What does this derivative represent? What are the units of $T'(t)$? Give an approximate value for $T'(8)$.

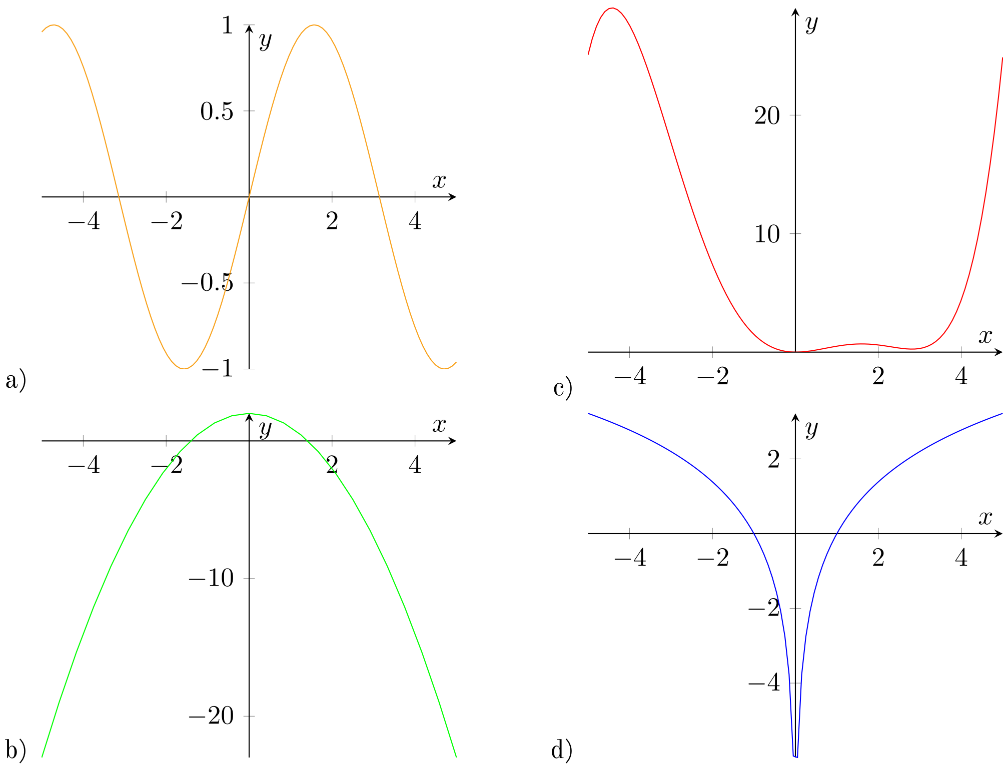
3. A particle is moving along a straight line, such that its displacement at time t is $s(t)$ (in metres, measured from some starting point).

   a) If t is measured in seconds, what are the units of $s'(t)$? What is the meaning of this quantity?

   b) The derivative of s' is denoted by s''. What are the units of $s''(t)$, and what is the meaning of this derivative?

4. Consider the curve $y = f(x)$ for some function x. What is the equation of the line joining the points $(x, f(x))$ and $(x+h, f(x+h))$? By letting h approach zero, give a justification for the definition of the tangent line to f at $(x, f(x))$. (Include a picture!)

5. The slope of a curve at a point tells us two things: whether it is sloping up or down, and the speed at which it is changing. Here are a few graphs of functions; sketch the graphs of their derivatives.



6. Describe several ways in which a function f can fail to be differentiable at a point x, illustrating your examples with sketches.

7. Consider each of these functions in turn. Where is the derivative of each (i) negative, (ii) positive, (iii) zero, and (iv) undefined?

   a) $x \mapsto x^2$
   b) $x \mapsto \sin x$
   c) $x \mapsto \tan x$

8. Justify: 'The derivative of f is the same as the derivative of $f + K$ for every constant K.'[2]

9. Above, we proved that if $y = x^2 + x - 1$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 1$. Write the equation of the tangent line to this curve at the point $(3, 11)$.

10. Let f be a function. Suppose that it is known that $f'(3) = 9$, and $f(3) = 6$.

    a) What does the graph of $y = f(x)$ look like around $x = 3$?
    b) Give the equation of the tangent line to $f(x)$ at $x = 3$.

11. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.

    a) What is the meaning of the derivative $f'(5)$?
    b) Suppose that there is an unlimited amount of space and nutrients. Which would you expect to be larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited does your answer change?

12. Prove that the only curves with constant slope are straight lines.

13. A ball dropped from a tower accelerates from rest at a constant rate, $-g$. If $h(t)$ is the height of the ball t seconds after it is dropped, what is its velocity t seconds after being dropped?

14. One model of population claims that the rate of change of a population P at a time is directly proportional to the size of the population at that time. In other words, $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(t)$ for some constant P. If $P(0) = 100$, sketch the population over time.

15. Consider an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is not a function: since both $(0, b)$ and $(0, -b)$ are members of the function, it fails the vertical line test. However, it would be nice to reason about its rate of change *as if it were* a function. Describe the slope of the ellipse as a particle traces the curve in an anticlockwise direction at a constant rate.

16. If $f(x) = x^3 + x$, calculate the derivative f' by writing $f(x + h) - f(x) \approx kh$ for some k.

17. We will calculate the derivative of $f(x) = \sqrt{x}$ at the point $(1, 1)$. To do this, consider the point $P = (1, 1)$ and the sequence of points $P_h = (1 + 1/h, \sqrt{1 + 1/h})$.

    a) Justify: we will obtain the tangent line to f at P by taking h larger and larger, so P_h gets closer and closer to P.
    b) Show that the slope of the line joining P and P_h is $\sqrt{h(h + 1)} - h$.
    c) Unfortunately, it is hard to see how this quantity behaves as h grows. Use the identity $a^2 - b^2 = (a + b)(a - b)$ to rewrite $\sqrt{h(h + 1)} - h$ as $\frac{h(h+1)-h^2}{\sqrt{h(h+1)}+h}$.
    d) Rewrite this fraction as $\frac{1}{\sqrt{1+1/h^2}+1}$ (so $\frac{f(1+1/h)-f(x)}{1/h} = \frac{1}{\sqrt{1+1/h^2}+1}$), and hence show that $f'(1) = 1/2$.

References

For a readable introduction to differentiation as the study of linear approximations analagous to what we see above, see the first few chapters of Thompson's *Calculus made easy*.

For many exercises on the behaviour of derivatives (as rates of change, and as slopes of tangent lines) see sections 2.1 and 2.2 of Stewart.

See also the section on calculus from my L2 notes.

[2] Here, $f + K$ denotes the function defined by $(f + K)(x) = f(x) + K$ for all x. Likewise, if f and g are functions we define $f + g$ to be the function satisfying $(f + g)(x) = f(x) + g(x)$ for all x.

Homework

Reading Considering how many fools can calculate, it is surprising that it should be thought a difficult or a tedious task for any other fool to learn how to master the same tricks.

Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the text-books of advanced mathematics — and they are mostly clever fools — seldom take the time to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way.

Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fool the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can.

The preliminary terror, which chokes off most high school students from even attempting to learn how to calculate, can be abolished once and for all by simply stating what is the meaning — in common-sense terms — of the two principal symbols that are used in calculating.

These dreadful symbols are:

(1) d, which merely means "a little bit of".

Thus $\mathrm{d}x$ meas a little bit of x; or $\mathrm{d}u$ means a little bit of u. Ordinary mathematicians think it more polite to say "an element of", instead of "a little bit of". Just as you please. But you will find that these little bits (or elements) may be considered to be infinitesimally small.

(2) $\int$, which is merely a long S, and may be called (if you like) "the sum of".

Thus $\int \mathrm{d}x$ means the sum of all the little bits of x; or $\int \mathrm{d}t$ means the sum of all the little bits of t. Ordinary mathematicians call this symbol "the integral of". Now any fool can see that if x is considered to be made up of a lot of little bits, each of which is called $\mathrm{d}x$, if you add them all up together you get the sum of all the $\mathrm{d}x$'s (which is simply the same thing as the whole of x). The word "integral" simply means "the whole". If you think of the duration of time for one hour, you may (if you like) think of it cut up into 3600 little bits called seconds. The whole of the 3600 little bits added up together make one hour.

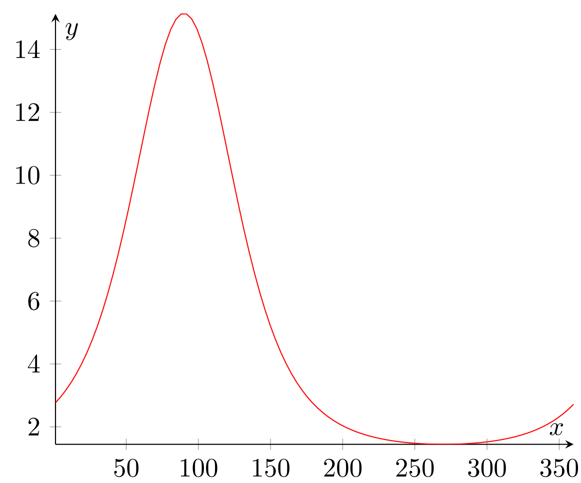
When you see an expression that begins with this terrifying symbol, you will henceforth know that it is put there merely to give you instructions that you are now to perform the operation (if you can) of totalling up all the little bits that are indicated by the symbols that follow.

That's all.

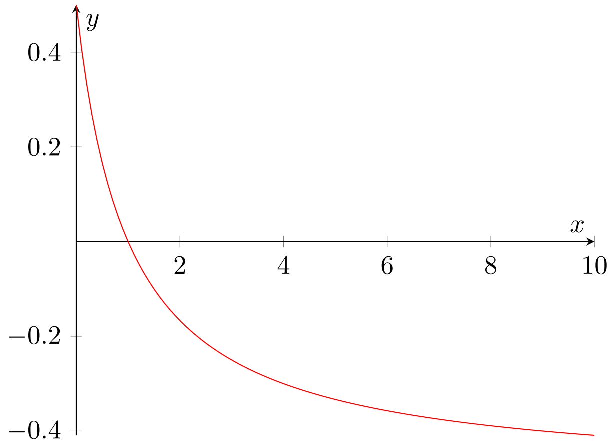
From *Calculus made easy*, by Silvanus P. Thompson and revised by Martin Gardner.

Problems

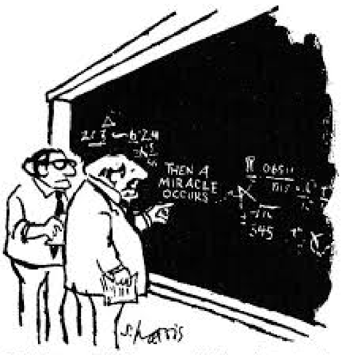
1. Draw the derivative of the following graphed function:



2. The following is the graph of the derivative of some function f. Sketch the graph of f, if $f(0) = 0$.



3. Show that if f and g are two functions, then $(f+g)' = f' + g'$.



"I think you should be more explicit here in step two."

Figure 3: Let's try to be less handwavy.

I.2 Limits at points

Last time, we defined the slope of a function f by considering the 'slope quotient'

$$\frac{f(x+h)-f(x)}{h}. \tag{I.3}$$

We said that a function was differentiable if we could approximate the top of this quotient by some expression of the form $f'(x)h$, and that the derivative of f at x was the number $f'(x)$.

This is a fairly intuitive definition, but there is a slight problem with it: we have no real way of knowing which approximations are 'valid'. For example, consider the function $f(x) = x^2$. The top of the difference quotient becomes $(x+h)^2 - x^2 = 2xh + h^2$. At this point, we wave our hands around and exclaim loudly that 'because h is small, h^2 is tiny and so we can just forget about it'. But waving our hands around is not a substitute for understanding what is going on!

This problem was rife in early discussions of calculus; the original way of dealing with it involved talking about 'infinitesimals' (this is how Leibniz thought about derivatives in the 1700s, for example) but it turns out that this makes more problems than it solves.

The modern solution involves what are called *limits*, which are a measure of how a function behaves around a point.

Definition. Suppose $f(x)$ is defined for all x around a point a (but not necessarily at a itself). Then we say that the limit of f as x approaches a equals L if we can make the values of $f(x)$ as close as we like to L by taking x to be close to (but not equal to) a. We write this symbolically as

$$\lim_{x\to a} f(x) = L,$$

or write that $f(x) \to L$ as $x \to a$.

The idea is that the limit of f at a is L if we can look at the graph of f, cover up the vertical line $x = a$, and use the behaviour of $y = f(x)$ in the neighbourhood of $x = a$ to guess what the graph looks like at that point. *The limit of f at a is dependent only on the points around a, not on the value (or lack thereof) of $f(a)$.*

Consider the graph of a function g given in figure 4. Although the *value* of the function at 2 is 6, the *limit* of the function at 2 is $\lim_{x\to 2} g(x) = 4$.

You can also think of $\lim_{x\to a} g(x)$ as being the unique value that we could pick for $g(a)$ such that the function around that point has 'no gaps'. If there is no such unique value, there is no limit at the point a.

A function a is called *continuous* at a point a if $\lim_{x\to a} f(x) = f(a)$; that is, if it takes the value at a which we would expect it to based on the points around a. The function g graphed in figure 4 is continuous at every point except $x = 2$.

Limits happen to have a few simple properties.

Theorem. *If f and g are functions and the limits of f and g at x_0 exist, then:*

1. $\lambda \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} [\lambda f(x)]$ *(where λ is a constant);*

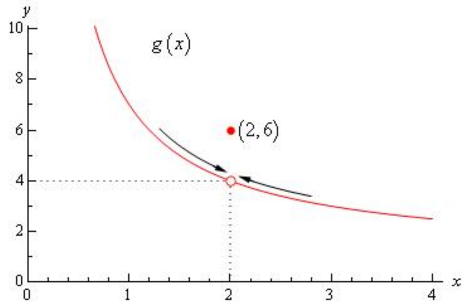


Figure 4: A function g where $\lim_{x\to 2} g(x) \neq g(x)$.

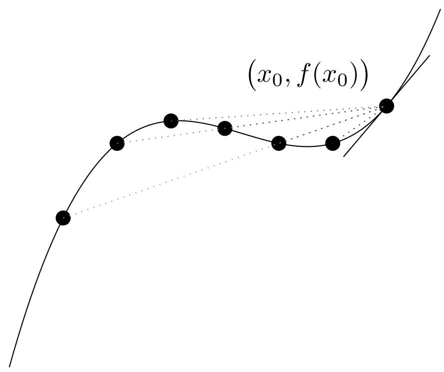


Figure 5: The tangent line as a limit of secant lines.

2. $\lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x) = \lim\limits_{x\to x_0} [f(x) + g(x)]$*;*

3. $\left(\lim\limits_{x\to x_0} f(x)\right)\left(\lim\limits_{x\to x_0} g(x)\right) = \lim\limits_{x\to x_0} [f(x)g(x)]$*;*

4. $\dfrac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$ *(if $g(x) \neq 0$ around the point we take the limit); and*

5. $f(\lim\limits_{x\to x_0} g(x)) = \lim\limits_{x\to x_0} [f(g(x))]$ *(if f is continuous).*

Examples. Using these limit laws, we can find some limits reasonably easily.

1. $\lim_{x\to 0} \frac{x}{x} = 1$ since as x gets closer and closer to 0, $\frac{x}{x} = 1$.

2. $\lim_{x\to 3} \frac{(x-2)(x-3)}{x-3} = 1$ since as x gets closer and closer to 3, the fraction gets arbitrarily close to 1.

3. $\lim_{x\to 0} \frac{1}{x}$ does not exist, since if we approach 0 from the left the function becomes arbitrarily negative and if we approach 0 from the right the function becomes arbitrarily positive — we do not approach the same value on both sides.

4. $\lim_{x\to\infty} \frac{1}{x} = 0$ since as x becomes arbitrarily large, $\frac{1}{x}$ becomes arbitrarily small.

5. $\lim_{x\to 0} \sqrt{x}$ does not exist, since $\sqrt{x}$ is undefined for $x < 0$.

We will now use limits to define derivatives in a way that is relatively easier to deal with. If f is a function, and we want to find the derivative at a point $(x_0, f(x_0))$, then we consider the slopes of the lines joining this point to other points $(x_0, f(x_0 + h))$. As $h \to 0$, these secant lines become closer and closer to being tangent lines at x_0; so if the limit of the slopes of the secant lines exist, it makes sense to define this as the derivative $f'(x_0)$. In other words,

$$f'(x_0) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \qquad \text{(I.4)}$$

See figure 5 for a visualisation of this.

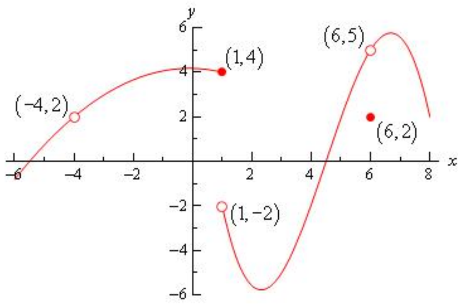
Just to bring ourselves back to our original formulation of derivatives as ways of computing linear approximations, we will say that $f(x+h)-f(x) \approx mh$ if $f(x+h)-f(x) = mh + \vartheta(h)$, where ϑ is some function such that $\vartheta(h)/h \to 0$ as $h \to 0$. (To see that this is the same thing as I.4, divide through by h and take the limit of both sides as $h \to 0$.)

Example. We will find the derivative of $f(x) = x^3$ at the point x using the definition.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\ &= 3x^2. \end{aligned}$$

Exercises and Problems

1. Consider the function f graphed below.



   a) For each of the following expressions, either give the value or explain why the expression is undefined.
      i. $f(-4)$
      ii. $\lim_{x\to -4} f(x)$
      iii. $f(1)$
      iv. $\lim_{x\to 1} f(x)$
   b) Explain why the limit $\lim_{x\to 6} f(x)$ is not equal to $f(6)$.
   c) At which points is f:
      i. Discontinuous?
      ii. Non-differentiable?

2. Evaluate the limit or explain why it does not exist:

   a) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$
   b) $\lim_{x\to 0} \frac{1}{x^3}$
   c) $\lim_{x\to 9} \frac{1}{x^3}$
   d) $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$
   e) $\lim_{x\to 4} \frac{x^2+5x+4}{x^2+3x-4}$
   f) $\lim_{x\to \frac{\pi}{2}} \sin x$
   g) $\lim_{x\to \infty} \sin x$
   h) $\lim_{x\to \frac{\pi}{2}} \tan x$
   i) $\lim_{x\to 0} \tan x$
   j) $\lim_{x\to 0} \csc x$
   k) $\lim_{x\to a} C$, where a and C are constants.
   l) $\lim_{x\to -\infty} \tan^{-1} x$
   m) $\lim_{y\to 0} \lim_{x\to 0} \frac{(x+y)(x-y)}{x^2-y^2}$
   n) $\lim_{x\to \infty} 1/x$.
   o) $\lim_{x\to \infty} \frac{2x}{x^2+1}$.
   p) $\lim_{x\to \infty} \frac{x+2}{x-3}$.

3. Show that $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ and $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ are equivalent definitions for the derivative at the point a of some function f.
4. If $\lim_{x\to a}[f(x)+g(x)]=2$ and $\lim_{x\to a}[f(x)-g(x)]=1$, find $\lim_{x\to a} f(x)g(x)$.
5. Explain why $\frac{x^2+x-6}{x-2}\neq x-3$, but $\lim_{x\to a}\frac{x^2+x-6}{x-2}=\lim_{x\to a}(x-3)$ for every a.
6. Prove that if f is differentiable at a then it is continuous at a.
7. Last year, we defined the *exponent* a^r as follows:
   - If $r=0$, then $a^r=1$.
   - If r is a natural number, then $a^r=a^{r-1}\cdot r$. (So $a^r=\underbrace{a\times\cdots\times a}_{r \text{ times}}$.)
   - If r is a negative integer, then $a^r=\dfrac{1}{a^{-r}}$. (Note that $-r$ is positive.)
   - If r is a rational number, so that $r=p/q$ in lowest form, then $a^r=a^{(p/q)}=\sqrt[q]{a^p}$ (where we take the positive root, if a choice needs to be made).

   Give a reasonable definition for a^r where r is any real number. Use your definition to compute a reasonable approximation to 2^π (given that $\pi\approx 3.14159...$).

References

For various exercises regarding limits, see sections 1.5 and 1.6 of Stewart. For a proper definition of limits (because the one given above is still handwavy: what do we mean by 'close'?) see Spivak (although this is beyond what the typical Y13 student needs).

For a very nice treatment of derivatives that defines $f'(x)$ to be the unique value for m satisfying $f(x+h)-f(x)=mh+\vartheta(h)$, see Loomis and Sternberg, sections 3.5 and 3.6. We will use the notation $f(x+h)\approx f'(x)h+f(x)$ in this kind of way repeatedly for the next few sections, but won't stop to make it too rigorous; in order to do so, replace it with the equivalent equality involving $\vartheta(h)$ and remember that $\vartheta(h)/h\to 0$ as $h\to 0$ by definition. (L & S calls functions like ϑ the 'little-oh' class of infinitesimals.)

Homework

Reading The notion of limits is incredibly fundamental to mathematics. In fact, the difference between the rational numbers and the real numbers is that, in the real numbers, every function which should have a limit at a point does have a limit at that point.

What do I mean by this?

Well, consider $f(x)=\sqrt{x}$. Clearly we want this function to be continuous. If we apply f to the sequence

$$x_0=\frac{1^2}{1^2},x_1=\frac{14^2}{10^2},x_2=\frac{141^2}{100^2},x_3=\frac{1414^2}{1000^2},...,x_n=\frac{\lfloor 10^n\sqrt{2}\rfloor}{10^n},...$$

(where $\lfloor x\rfloor$ is the largest integer not larger than x) then each output is a rational number: $f(x_1)=14/10$, for example. Furthermore, these outputs get closer and closer together so $\lim_{n\to\infty} f(x_n)$ should exist. But the limit cannot exist in the rationals, because $\lim_{n\to\infty} f(x_n)=\sqrt{2}$, which is not rational!

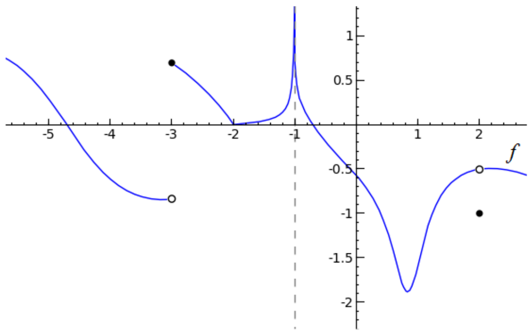
Formally, the way we get from the rational numbers (the numbers of the form a/b, where a and b are integers) to the real numbers is by taking the set of rationals and adding to it all the limits of Cauchy sequences — sequences where the adjacent terms get arbitrarily close together as we walk further along them.

Problems Derivatives and limits allow us to classify functions and their behaviour. Consider the following geometric properties:

- A function is *increasing* if its derivative is positive.
- A function is *decreasing* if its derivative is negative.
- A function is *concave down* if its derivative is decreasing.

- A function is *concave up* if its derivative is increasing.
- A function f is *continuous* at a point a if $\lim_{x\to a} f(x) = f(a)$.

1. Describe all the function properties given above geometrically, and give an example of each.
2. Consider the function graphed below.



   a) Find $\lim_{x\to -2} f(x)$ and $\lim_{x\to 2} f(x)$.
   b) Does $\lim_{x\to -3} f(x)$ exist? Why/why not?
   c) Does $\lim_{x\to 0} f(x)$ exist? Why/why not?
   d) On what intervals is $f(x)$ continuous?
   e) At what points is $f(x)$ not differentiable?

3. On an axis, sketch a graph of some function f that has the following features:
   - Is continuous for $0 < x < 5$ and $5 < x < 9$ and is discontinuous when $x = 5$
   - Is concave down ($f''(x) < 0$) for $0 < x < 5$
   - Has $f'(x) = 0$ at $(3, 8)$
   - Has $\lim_{x\to 5} f(x) = 6$.
   - Is not differentiable at $(7, 3)$.

I.3 Taking derivatives

Taking derivatives using the definitions quickly becomes unmanagable. Because of this, we want to produce a set of rules which will allow us to take derivatives of common functions.

Differentiation of polynomials

We begin with a few easy observations that will allow us to take derivatives of polynomials.

Theorem (Arithmetic of derivatives). *Let f and g be functions.*

1. *If $f(x) = \lambda$ for all x (where λ is a constant), then $f'(x) = 0$ for all x.*
2. *The derivative of $f + g$ is $f' + g'$ (i.e. for all x, $(f + g)'(x) = f'(x) + g'(x)$).*
3. *If λ is a constant, then $(\lambda f)' = \lambda f'$.*

Proof. 1. In this case, $\frac{f(x+h)-f(x)}{h} = \frac{\lambda-\lambda}{h} = 0$, so the difference quotient is always zero and so is the derivative.

2. If h is small, $(f+g)(x+h) - (f+g)(x) = f(x+h) + g(x+h) - f(x) - g(x) \approx f'(x)h + g'(x)h$ and so the derivative at x is $f'(x) + g'(x)$.
3. $(\lambda f)(x + h) - (\lambda f)(x) = \lambda(f(x + h) - f(x)) \approx \lambda(f'(x)h) = (\lambda f')(x)h$.

□

Now we consider $f(x) = x^n$ for integers n. Using the binomial theorem,

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \cdots - x^n}{h}$$

where every term hidden in the $\cdots$ includes an h^2 factor, so we obtain

$$f'(x) = \lim_{h\to 0} nx^{n-1} + h(\cdots) = nx^{n-1}. \tag{I.5}$$

In fact, although we proved this for integer n, it holds in general:

Theorem (Power law). *If $f(x) = x^\alpha$ then $f'(x) = \alpha x^{\alpha-1}$.*

Example. We can now differentiate every function of the form $f(x) = a_1x^{b_1} + \cdots + a_nx^{b_n}$: $f'(x) = b_1a_1x^{b_1-1} + \cdots + b_na_nx^{b_n-1}$. In particular, if $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2\sqrt{x}}$; if $g(x) = 2x^2 + 3$ then $g'(x) = 4x$; and if $h(x) = \frac{1}{x} + x^7$ then $h'(x) = -\frac{1}{x^2} + 7x^6$.

Trigonometric derivatives

We have already seen that $\sin' = \cos$; using similar reasoning, we can prove the following:

Theorem (Trigonometric derivatives).

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$

I will use the result $\sin' = \cos$ to prove that $\cos' = -\sin$, and we will prove the rest at a later time. Indeed, $\cos x = \sin(x + \pi/2)$; then $\frac{\mathrm{d}}{\mathrm{d}x} \cos x = \frac{\mathrm{d}}{\mathrm{d}x} \sin(x + \pi/2)$. But the graph of $\sin(x + \pi/2)$ is just the graph of $\sin x$, shifted to the left by $\pi/2$. Hence the slope of $\sin(x + \pi/2)$ is the same as the slope of $\sin x$, but shifted to the left by $\pi/2$; and the slope of $\sin x$ is $\cos x$. Hence:

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = \frac{\mathrm{d}}{\mathrm{d}x} \sin(x + \pi/2) = \cos(x + \pi/2) = -\sin x. \tag{I.6}$$

(This little trick I used here is explored in more detail in the L2 notes; we don't need it too often this year, because in a couple of sections we will learn a much more general way of dealing with this kind of situation.)

Application. Many phenomena in physics can be modelled with sine waves; for example, if a particle on the end of a spring is moving with simple harmonic motion, then it has position $x = A\sin(\omega t + \phi)$; taking derivatives, we find that it has velocity $v = \frac{\mathrm{d}x}{\mathrm{d}t} = A\omega\cos(\omega t + \phi)$ and acceleration $a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -A\omega^2\sin(\omega t + \phi)$. In other words, it is always accelerating in the opposite direction to its movement!

Exponential functions

The next function we want to consider here is $f(x) = a^x$, for constants a. We can compute that

$$f'(x) = \lim h \to 0 \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \left(\frac{a^h - 1}{h}\right).$$

So the exponential functions a^x have derivatives of the form Ka^x, where K is some constant. This begs the question, for which value of a (if any) does $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x$ (i.e. $K = 1$)? Well, we need to solve

$$\lim_{h\to 0} \left(\frac{a^h - 1}{h}\right) = 1$$

for a. We will begin by setting $u = 1/h$, so when $h \to 0$ we have $u \to \infty$. Thus

$$\lim_{u\to\infty} (a^{1/u} - 1)u = 1 = \lim_{u\to\infty} 1;$$

and applying the limit laws,

$$\begin{aligned}
\lim_{u\to\infty} (a^{1/u} - 1) &= \lim_{u\to\infty} \frac{1}{u}; \\
\lim_{u\to\infty} a^{1/u} &= \lim_{u\to\infty} \frac{1}{u} + 1; \\
a = \lim_{u\to\infty} (a^{1/u})^u &= \lim_{u\to\infty} \left(\frac{1}{u} + 1\right)^u.
\end{aligned}$$

It can be shown fairly easily that $\lim_{u\to\infty} \left(\frac{1}{u} + 1\right)^u$ does indeed exist (it has a value of 2.71828...), and we define its value to be e. Thus e is the base for the exponential function that is its own derivative: $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$. Often, we write $\exp(x) := e^x$.

Finally, note that if $K = \lim_{h\to 0} \left(\frac{a^h - 1}{h}\right) = \lim_{u\to\infty} u(a^{1/u} - 1)$ then

$$a = \lim_{u\to\infty} \left(\frac{K}{u} + 1\right)^u = \lim_{u\to\infty} \left(\left(\frac{K}{u} + 1\right)^{u/K}\right)^K = \lim_{(u/K)\to\infty} \left(\left(\frac{K}{u} + 1\right)^{u/K}\right)^K = e^K;$$

hence $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x K = a^x \log_e a$. (We normally write $\log_e = \ln$.)

Logarithmic derivatives

Finally, let us calculate $\frac{\mathrm{d}}{\mathrm{d}x} \ln x$. (This will allow us to find $\frac{\mathrm{d}}{\mathrm{d}x} \log_a x$ for all a, using the relationship $\log_a x = \frac{1}{\ln a} \ln x$.)

$$\ln'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h\to 0} \frac{1}{h} \ln\left(1 - \frac{h}{x}\right) = \lim_{h\to 0} \ln\left(1 - \frac{h}{x}\right)^{1/h}$$

Let $u = 1/h$; so as $h \to 0$, $u \to \infty$. Then, substituting, we obtain

$$\ln'(x) = \lim_{u\to\infty} \ln\left(1 - \frac{1}{ux}\right)^u.$$

Now, we use the fact that $\exp(\ln x) = x$:

$$e^{\ln'(x)} = \lim_{u\to\infty} \exp(\ln\left(1 - \frac{1}{ux}\right)^u) = \lim_{u\to\infty}\left(1 - \frac{1}{ux}\right)^u = \lim_{u\to\infty}\left(\left(1 - \frac{1}{ux}\right)^{ux}\right)^{(1/x)} = e^{1/x}$$

and thus $\ln'(x) = 1/x$.

Exercises and Problems

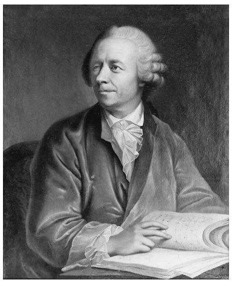
1. Find the derivatives of $3x^3$, $2x^2$, and $6x^5$. Conclude that $(fg)' \neq f'g'$ in general.
2. Find the derivatives of the following functions with respect to t:
   a) $y = 2t^3 + 3t^2$
   b) $y = \sqrt{t}$
   c) $y = (2t+1)(t-4)$
   d) $g(t) = 4\sec t + 9\tan t$
   e) $h(t) = \sqrt[5]{t} + 2\csc t - \ln t^3$
   f) $\phi'(t) = \csc x + 12x^{1273} + 9$
   g) $y = 2017t^{2016} + (t+2)^2$
   h) $y = 940\sin t + \frac{1}{2}e^{t+2}$
3. Where is the function $x \mapsto x^3 - 2x^2 - x + 1$ increasing?
4. Find the velocity v of a particle at time $t = 2\pi$ if its position function for $t > 0$ is $x = e^t - \sin t$.
5. Find the slope of the tangent line to $y = x + \tan x$ at (π, π).
6. Find a linear approximation $\tilde{f}$ to $f(x) = x^2 + x + 1$ at $(0, 1)$, and find some δ such that for all x satisfying $-\delta < x < \delta$, $-0.1 < \tilde{f}(x) - f(x) < 0.1$.
7. It is **not** true that the derivative of $f(g(x))$ is $f'(g'(x))$.
   a) For a counterexample, consider $f(x) = x^2$ and $g(x) = x$; show that $f'(g'(x)) = 2$, but $\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = 2x$.
   b) Compute the derivative of $\ln x^2$.
8. Suppose the derivative of a function is $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - x - 4$. What could the original function be?
9. Find the 64th derivative of $\sin x$.
10. Find the nth derivative of x^n.
11. If $y = 2\sin 3x \cos 2x$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. (Hint: use an identity to rewrite this as a sum of functions.)
12. For which values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?
13. Show that $y = 6x^3 + 5x - 3$ has no tangent line with a slope of 4.
14. Find real values of α and β such that, if $y = \alpha\sin x + \beta\cos x$, then $y'' + y' - 2y = \sin x$.
15. Consider a 12 m long ladder leaning against a wall such that the top of the ladder makes an angle θ with the wall. If this angle θ is varied, the distance D between the bottom of the ladder and the wall also changes. If $\theta = \pi/3$, what is the rate of change of D with respect to θ?
16. Prove that the function φ given by $\varphi(x) = \frac{x^{101}}{101} + \frac{x^{51}}{51} + x + 1$ never has a horizontal tangent line.
17. The derivative is primarily a geometric concept, not an algebraic one.
    a) The area of a circle of radius r is $A = \pi r^2$. Find $\frac{\mathrm{d}A}{\mathrm{d}r}$. What do you notice?
    b) Explain item (a) geometrically.
    c) The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. Find an expression for the surface area.

References

See sections 2.1 – 2.4 of Stewart. For a discussion of the exponential function and its relationship to compound interest and rates of growth, see Thompson chapter XIV.

Homework

Reading The number $e \approx 2.718$ is known as Euler's number, after the Swiss mathematician Leonhard Euler.



Leonhard Euler (1707–1783) was Switzerland's foremost scientist and one of the three greatest mathematicians of modern times (the other two being Gauss and Riemann).

He was perhaps the most prolific author of all time in any field. From 1727 to 1783 his writings poured out in a seemingly endless flood, constantly adding knowledge to every known branch of pure and applied mathematics, and also to many that were not known until he created them. He averaged about 800 printed pages a year throughout his long life, and yet he almost always had something worthwile to say and never seems long-winded. The publication of his complete works was started in 1911, and the end is not in sighr: it is now esitmated that 100 large volumes will be required for completion of the project. He suffered blindess during the last 17 years of his life, but with the aid of his powerful memory and fertile imagination, and with helpers to write his books and papers from dictation, he actually increased the already prodigious output of work.

Though he was not himself a teacher Euler has had a deeper influence on the teaching of mathematics than any other person. This came about chiefly through his three great treatises: *Introductio in Analysin Infinitorum* (1748); *Institutiones Calculi Differentialis* (1755); and *Institutiones Calculi Integralis* (1768–1794). There is considerable truth in the old saying that all elementary and advanced calculus textbooks since 1748 are essentially copies of Euler or copies of copies of Euler. These works summed up and codified the discoveries of his predecessors, and are full of Euler's own ideas. He extended and perfected plane and solid analytic geometry, introduced the analytic approach to trigonometry, and was responsible for the modern treatment of the functions ln and exp. It was though his work that the symbols e, π, and i became common currency for all mathematicians, as well as the functions sin and cos.

He was the first and greatest master of infinite series, products, and fractions; in 1736, he made the wonderful discovery that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6},$$

and also found the sums of the reciprocals of the fourth and sixth powers. (A closed form for the sum of the reciprocals of the cubes is still unknown.)

The foundations of classical mechanics had been laid down by Newton, but Euler was the principle architect. In his treatise of 1736 he was the first to explicitly introduce the concept of a point-like particle, and he was the also the first to study the acceleration of a particle moving along any curve and to use the notion of a vector in connection with velocity and acceleration. His continued successes in mathematical physics were so numerous, and his influence was so pervasive, that most of his discoveries are not credited to him at all and are taken for granted by physicists as part of the natural order of things.

Adapted from *Differential equations with applications and historical notes* (pp.136–146) by George F. Simmons (McGraw-Hill, 1991).

Problems

1. Differentiate with respect to x:

    a) $x^2 + \frac{1}{x}$

    b) tx^t

    c) $\sin x - \cos x$

    d) $\sqrt[5]{x^4}$

2. Explain why you cannot use the power rule to find the derivative of x^x.

3. Find the nth derivative of $\frac{1}{x^n}$.

4. Suppose a population grows exponentially with time, such that after t years the population is $P(t) = P_0 + 10^t$.

    a) Find the rate of change of the population at $t = 100$.

    b) Explain why this population model is unrealistic.

I.4 Anti-derivatives

We now want to begin to study the inverse of differentiation: the problem is, given a function f, to find some function F which has f as a derivative. Geometrically, we are given the rate of change of a function at every point, and we wish to recover the original function.

If $f = F'$, then F is said to be an *anti-derivative* of f.

First of all, we notice that if F is an anti-derivative of f then so is $F + C$ for any constant C; this is because $(F + C)' = F' + C' = F' + 0 = F' = f$. Thus when we take anti-derivatives there are infinitely many different solutions that all differ by a constant — we cannot recover the original function given a slope function without some more information. These functions are said to be the family of solutions that solve the differential equation $F' = f$; we will also write $F(x)+C = \int f(x)\,\mathrm{d}x$ in this case, and we say that F is the *indefinite integral* of f; in the expression $\int f(x)\,\mathrm{d}x$, $f(x)$ is called the *integrand*.

The reason for this terminology comes from the geometric meaning of the derivative — if F is an anti-derivative of f, then f is the slope function of F. This in turn means that each value of f tells us how quickly F is rising or falling at that point; roughly speaking, to get F back from f, we need to walk along f, adding up all these infinitesimal rises and falls of F — in other words, we need to take all the values of f, and 'integrate' (combine) them together to get back the form of F.

This idea is made precise in the *fundamental theorem of calculus*, which we will state later on. This same theorem tells us that if f is a continuous function then there exists some anti-derivative F of f, and that this anti-derivative is unique up to a constant.

Differential equation		**Integral equation**
f is the slope function of F	$\Longleftrightarrow$	F is an anti-derivative of f
$f(x) = F'(x)$	$\Longleftrightarrow$	$F(x) + C = \displaystyle\int f(x)\,\mathrm{d}x$
$f(x) = \dfrac{\mathrm{d}y}{\mathrm{d}x}$	$\Longleftrightarrow$	$y + C = \displaystyle\int f(x)\,\mathrm{d}x$

There are a few simple rules that we can state right away. For example, we have the following power rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x} ax^n = nax^{n-1} \iff \int nax^{n-1}\,\mathrm{d}x = ax^n + C$$

so the anti-derivatives of ax^n are $\frac{a}{n+1}x^{n+1} + C$.

Looking at the inverse power law, we notice that there is an issue when we try to anti-differentiate $1/x$; the law tells us that $\int x^{-1}\,\mathrm{d}x = \frac{1}{1-1}x^0$, which is plainly nonsense. Luckily, last week we showed that $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = 1/x$, so $\int 1/x\,\mathrm{d}x = \ln x + C$.

Some more useful rules come by way of our differentiation arithmetic laws.

Theorem. *1.* $\int 0\,\mathrm{d}x = C$ *(the family of constant functions)*

2. $\int f(x) + g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$

3. $\int \lambda f(x)\,\mathrm{d}x = \lambda \int f(x)\,\mathrm{d}x$

Proof. 1. Firstly, $\frac{\mathrm{d}}{\mathrm{d}x}0 = 0$; so the anti-derivatives of 0 are $0 + C = C$.

2. Let F and G be anti-derivatives of f and g, so $F' = f$ and $G' = g$. Then $\int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x = F(x) + G(x) + C$ (*); but $\frac{\mathrm{d}}{\mathrm{d}x}[F(x) + G(x) + C] = f(x) + g(x)$ and so $F + G + C$ is an anti-derivative of $f + g$ and $\int f(x) + g(x)\,\mathrm{d}x = F(x) + G(x) + C$. Combining this with (*) we obtain the result.

3. Let F be an anti-derivative of f. Then $(\lambda F)' = \lambda(F') = \lambda f$, and thus λF is an anti-derivative of λf; so $\int \lambda f(x)\,\mathrm{d}x = \lambda F(x) = \lambda \int f(x)\,\mathrm{d}x$.

$\square$

Example. One anti-derivative of $y' = 3x^2 + 4$ is $x^3 + 4x$. Another is $x^3 + 4x + 1$. A third is $x^3 + 4x + 7$.

Unfortunately, there is no 'easy' way to anti-differentiate; we simply have to try to rearrange the function in some clever way until it looks like something that we know how to deal with.

Examples.

1. The most general antiderivative of $\sin x$ is $-\cos x + C$.
2. The most general antiderivative of $\tan x$ is $-\ln|\cos x| + C$.
3. $\int \frac{1}{x+3}\,\mathrm{d}x = \ln|x+3| + C$.
4. $\int \tan^2\theta\,\mathrm{d}\theta = \int \sec^2\theta - 1\,\mathrm{d}\theta = \tan\theta - \theta + C$.
5. $\int \frac{2x}{x^2+1}\,\mathrm{d}x = \ln\left|x^2+1\right| + C$.
6. $\int Ke^{Kx}\,\mathrm{d}x = e^{Kx} + C$ for all constants K.

Joke. Two mathematicians are in a bar. The first one says to the second that the average person knows very little about basic mathematics. The second one disagrees and claims that most people can cope with a reasonable amount of maths. The first mathematician goes off to the washroom, and in her absence the second calls over the waiter. She tells the waiter that in a few minutes, after her friend has returned, she will call him over and ask him a question. All he has to do is answer "one third x cubed." He repeats "one thir-dex cue?" She repeats "one third x cubed." He asks, "one thir dex cuebd?" "Yes, that's right," she says. So he agrees, and goes off mumbling to himself, "one thir dex cuebd...". The first mathematician returns and the second proposes asking the waiter for an anti-derivative to prove her point that most people do know something about basic maths; the first laughingly agrees. The second mathematician calls over the waiter and asks "what is the integral of x squared?" The waiter says "one third x cubed" and while walking away, turns back and says over his shoulder, "plus a constant!"

Exercises and Problems

1. For each expression, find the most general anti-derivative with respect to x.

   a) $2x$
   b) $x^2 + 3x + 1$
   c) $\frac{1}{2\sqrt{x}}$
   d) x^{-3}
   e) 10^x
   f) $\sec^2 x + \sqrt{x}$
   g) $x\sqrt{x}$
   h) $\sin x - \cos x$
   i) $\frac{2x^3+3x-\sqrt{x}}{\sqrt[3]{x}}$
   j) $\frac{1}{x^2} + e^x$
   k) $\sec^2(x+1)$

2. Verify the examples in the notes by differentiation.
3. Show that $\int 3x^2 + 4x + 5 + \frac{2}{x}\,\mathrm{d}x = x^3 + 2x^2 + 5x + \ln x^2 + C$.
4. If $\frac{\mathrm{d}y}{\mathrm{d}t} = 1.5\sqrt{t}$ and $y(4) = 10$, find $y(t)$ exactly.
5. Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$.
6. The velocity of a particle is given by $v(t) = 2t + 1$. Find its position at $t = 4$ if its position at $t = 0$ is $x = 0$.
7. The acceleration of a particle is given by $a(t) = 10\sin t + 3\cos t$. At $t = 0$, its position is $x = 0$; at $t = 2\pi$, its position is $x = 12$. Find its position at $t = \frac{\pi}{2}$.
8. Starting from rest, a car takes T seconds to reach its maximum speed, $v_{\max}$. A plausible model for the velocity of the car after t seconds is

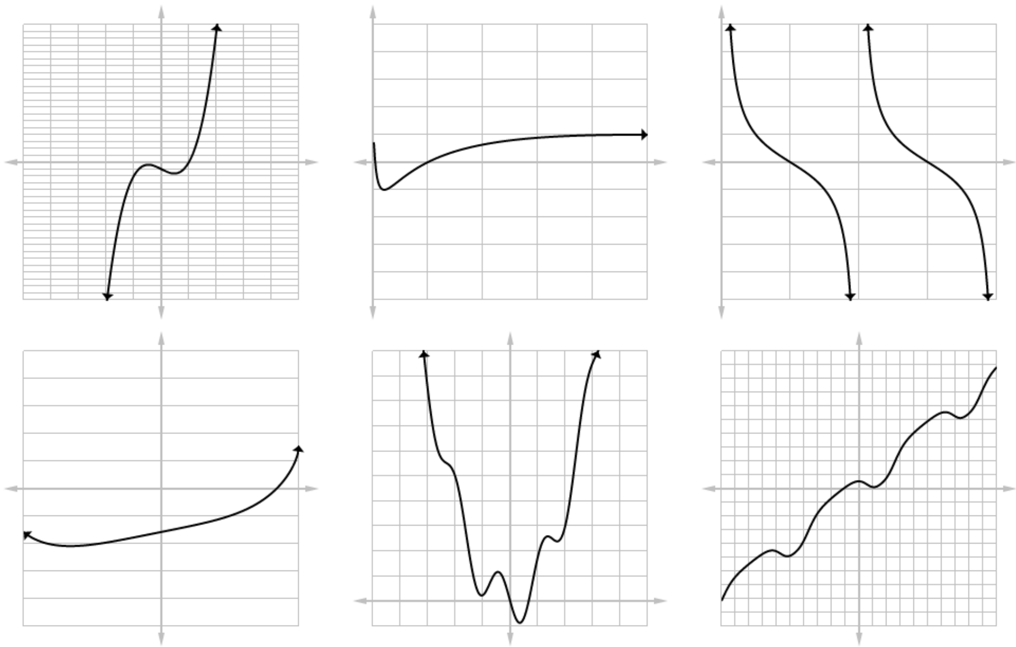
$$v(t) = \begin{cases} v_{\max}\left(\frac{2t}{T} - \frac{t^2}{T^2}\right) & t \le T, \\ v_{\max} & t \ge T. \end{cases}$$

   a) Write an expression for $a_{\max}$, the maximum acceleration attained by the car.

b) Show that the distance travelled by the car from the time it starts to the point it reaches its maximum speed is given by $s(t) = \frac{1}{3}a_{\max}T^2$.

9. Find all functions g such that $g'(x) = 4\sin x + \frac{2x^5-\sqrt{x}}{x}$.

10. For each function, sketch an anti-derivative passing through $(0,0)$:



11. Show that if F is an anti-derivative of f, G is an anti-derivative of g, and α and β are any constants, then $\alpha F + \beta G$ is an anti-derivative of $\alpha f + \beta g$.

12. Give an example of functions f and g such that if F and G are anti-derivatives of f and g respectively then FG is *not* an anti-derivative of fg.

References

Because of the way we're covering integration, most books will have problems which we can't do yet (for example, $\int 1/x \,\mathrm{d}x$). That said, many more anti-differentiation problems can be found in the following: Stewart, section 4.4; Thompson, chapter XVIII.

Homework

Reading In mathematics there are a lot of examples of operations which are easy to perform in one direction, but harder to reverse.

Easy	**Hard**
addition	subtraction
multiplication	division
expanding	factorising
exponents	logarithms
tying a knot	unravelling a knot
differentiation	anti-differentiation

One very important application of this idea is in cryptography. Most modern computer systems are dependent on something known as the RSA cipher, which essentially relies on the fact that it's much easier to multiply large primes together than it is to work out what primes divide a large integer.

Anti-differentiation in particular is very difficult, in the sense that there are rules that enable us to take the derivative of every combination of 'simple' functions (polynomials, exponentials, logarithms, trig functions, sums, products, functions of functions) — but there are some functions, made up of these building blocks, which do not have anti-derivatives of the same type.

For example, there is no anti-derivative of x^x which can be produced with simple functions. There is a function f such that $f'(x) = x^x$, we just can't write it down at all using any combination of these building blocks despite the function $x \mapsto x^x$ being made up of them.

Problems

1. Find the most general anti-derivative.

   a) $f(x) = x - 3$

   b) $f(x) = (x+1)(x+2)$

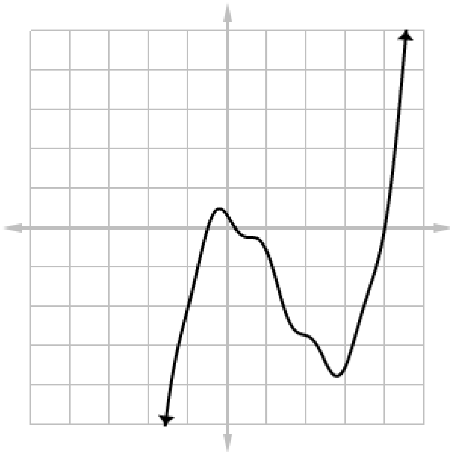
   c) $f(\theta) = 6\theta^2 - 7\sec^2\theta$

   d) $g(h) = \pi^2$

   e) $f(x) = x^{3.7} + \sqrt{x} + 7x^{\sqrt{7}-1}$

2. Given that the graph of φ passes through the point $(1, 6)$ and that the slope of its tangent line at $(x, \varphi(x))$ is $2x + 1$, find $\varphi(2)$.

3. This is the second derivative of g. Find g given that $g'(0) = 0$ and $g(0) = 1$.



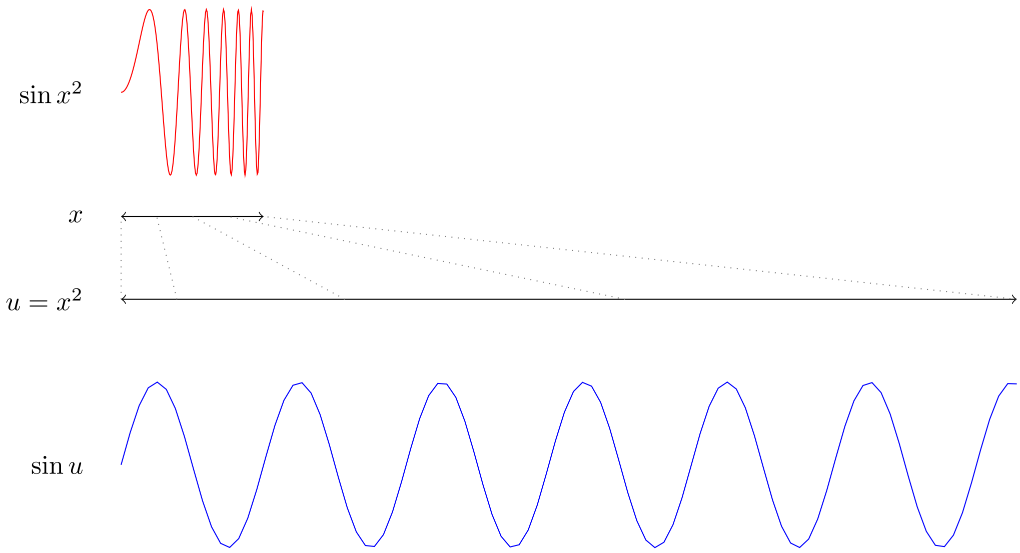


Figure 6: A graphic showing how the change of variables $u \leftrightarrow x^2$ stretches the graph of sin.

I.5 The chain rule

Consider the function $x \mapsto \sin(x^2)$. This function is made up of two functions, applied one after the other:

$$x \xmapsto{f} x^2 \xmapsto{g} \sin(x^2).$$

We often notate this function composition as $g \circ f$ (note that we evaluate from the right, so $(g \circ f)(x) = g(f(x))$).

Obviously the derivative of $\sin(x^2)$ is not just $\cos(2x)$, since the former has a horizontal tangent line at $x = \sqrt{\frac{\pi}{2}}$ but $\cos(\sqrt{2\pi}) \neq 0$. This shows us that, in general, the derivative of a function composition is not simply the composition of the derivatives.

In fact, it turns out that the derivative of $f \circ g$ is $g' \times (f' \circ g)$; in other words,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = g'(x) f'(g(x)).$$

This is known as the *chain rule*, since we are "chaining" together functions.

Let us convince ourselves that this rule is plausible. We can interpret the derivative $\frac{\mathrm{d}g}{\mathrm{d}x}$ as the rate of change of g with respect to x, and the derivative $\frac{\mathrm{d}f}{\mathrm{d}g}$ as the derivative of f with respect to small changes in the output of g; it is intuitive that if g changes twice as fast as x at some point, and f changes five times as fast as g, then f changes $2 \times 5 = 10$ times as fast as x.

The proof goes something like this: consider $f(g(x+h)) - f(g(x))$ for small h. We want to write this as mh, where m is only dependent on x. Now, since h is small, $g(x+h) \approx g(x) + g'(x)h$. Thus $f(g(x+h)) - f(g(x)) \approx f(g(x) + g'(x)h) - f(g(x))$. But if h is small enough, then $u = g'(x)h$ is small too; so, if $y = g(x)$, then

$$\begin{aligned} f(g(x+h)) - f(g(x)) &\approx f(g(x) + g'(x)h) - f(g(x)) \\ &= f(y+u) - f(y) \approx f'(y)u = f'(g(x))f'(x)h. \end{aligned}$$

Thus $(f \circ g)(x+h) - (f \circ g)(x) \approx f'(g(x))f'(x)h$, and so $(f \circ g)'(x) = f'(g(x))f'(x)$.

Examples.

1. The correct derivative of $\sin(x^2)$ is $2x\cos(x^2)$.

2. If $f(r) = \sqrt{r^2 - 3}$, then $f'(r) = 2r\frac{1}{2}\left(r^2 - 3\right)^{-1/2} = \dfrac{r}{\sqrt{r^2 - 3}}$.

3. If $g(x) = \sin((\sin^7 x^7 + 1)^7)$, then we compute:

$$g(x) = \sin\left(\left[\left(\sin x^7\right)^7 + 1\right]^7\right)$$
$$g'(x) = 7x^6 \cdot \cos x^7 \cdot 7\left(\sin x^7\right)^6 \cdot 7\left[\left(\sin x^7\right)^7 + 1\right] \cdot \cos\left(\left[\left(\sin x^7\right)^7 + 1\right]^7\right)$$

We can use the chain rule to relate rates of change together — for example, the area of a circle is given by $A = \pi r^2$ and so the rate of change of area with respect to radius $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$; but if r varies with respect to time then we can find the rate of change of the area with respect to time using the chain rule.

A useful mnemonic is (if x is a function of y which is itself a function of z)

$$\frac{\mathrm{d}x}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\mathrm{d}x}{\mathrm{d}z}. \qquad \text{(chain rule)}$$

We can also apply the inverse function rule for differentiation, which tells us that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}. \qquad \text{(inverse function rule)}$$

The inverse rule is easy to prove: if f is a function, we have that $f(f^{-1}(y)) = y$. Taking the derivative of both sides, $f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$ and therefore $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$.

These two operations allow us to rearrange equations as if $\frac{\mathrm{d}y}{\mathrm{d}x}$ were a fraction. There isn't much of a problem if you do think of it in this way, as long as you're careful.

Example (Inverse function rule). Let us find $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x$. If $y = \tan^{-1} x$, then $x = \tan y$; the inverse function rule tells us that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{\sec^2 y} = \cos^2 y$. Substituting for y, $\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos\tan^{-1} x)^2 = \frac{1}{x^2+1}$.[3]

Example. A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1\,\mathrm{m\,s^{-1}}$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?

Solution. Let x be the distance of the bottom of the ladder from the wall, and let y be the height of the top of the ladder up the wall. We have $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ and $x = 3$; we also know that $y = \sqrt{25 - x^2}$, so:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{\sqrt{25 - x^2}} \cdot 1$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{x=3} = -\frac{3}{\sqrt{25 - 9}} = -\frac{3}{4}.$$

Hence the ladder is sliding down the wall at a rate of $-0.75\,\mathrm{m\,s^{-1}}$.

Example. The radius of a sphere is increasing at a rate of $\frac{\mathrm{d}r}{\mathrm{d}t} = -\ln(t-1)$ metres per second. At what rate will the surface area of the sphere be growing at $t = 2$?

Solution. We have $\mathrm{SA} = 4\pi r^2$, so $\frac{\mathrm{dSA}}{\mathrm{d}r} = 8\pi r$ and

$$\frac{\mathrm{dSA}}{\mathrm{d}t} = \frac{\mathrm{dSA}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = -\ln(t-1) \times 8\pi r = 0.$$

The surface area of the sphere will be momentarily constant at $t = 2$.

[3] We proved that $\cos\tan^{-1} x = \frac{1}{\sqrt{x^2+1}}$ in the trigonometry notes.

Exercises and Problems

1. Identify the inner and outer functions, but don't try to differentiate.

   a) $\sqrt{\sin x}$
   b) $\sin\cos\tan x$
   c) $(2x+3)^{17}$
   d) $97(x+2)^2$
   e) $\ln\sin x$
   f) $\frac{1}{\sqrt{23x-x^2}}$

2. Differentiate with respect to t:

   a) $(2t+3)^{3000}$
   b) $\sin\ln t$
   c) $\sqrt{t^3+10t^2+3}$
   d) $\csc e^t$
   e) $\sin^3 t + 14\ln(3t)$
   f) $\sin\sin\sin t$
   g) $\cot(t+\sec t)$
   h) $\sin^2((t+\sin t)^2)$
   i) $\ln\sqrt{t+9}$
   j) $\sqrt{t}+\frac{1}{\sqrt[3]{t^4}}$
   k) $e^{\sec(t^2)}$
   l) $\sin\sqrt{t+\tan t}$

3. The derivative of a function is $2\cos 2x$. What could the original function be?

4. Differentiate $y=\sin^2 x+\cos^2 x$, and hence prove that $\sin^2 x+\cos^2 x=1$.

5. Suppose that the displacement of a particle on a vibrating spring is given by $x(t)=5+\frac{1}{8}\sin(5\pi t)$, where x is measured in centimetres and t in seconds.

   a) Find the velocity of the particle at time t.
   b) At which times is the particle momentarily stationary?

6. Find a linear approximation $\tilde{f}$ around 0 for $f(x)=\sqrt[4]{1+2x}$; then calculate δ such that for all x satisfying $-\delta<h<\delta$, $-0.1<\tilde{f}(x)-f(x)<0.1$.

7. The volume of a spherical balloon at a time t is given by $V(t)=\frac{4}{3}\pi r^2$, and its radius, changing over time, is given by $r(t)$. Find $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of $\frac{\mathrm{d}r}{\mathrm{d}t}$.

8. If $F(x)=f(3f(4f(x)))$, where $f(0)=0$ and $f'(0)=2$, find $F'(0)$.

9. Let α and s be functions. Compute $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\alpha(s(t))$.

10. Suppose $f(x)=g(x+g(a))$ for some differentiable function g and constant a. Find $f'(x)$.

11. The depth of water at the end of a jetty in a harbour varies with time due to the tides. The depth of the water is given by the formula

$$W=4.5-1.2\cos\frac{\pi t}{6}$$

   where W is the water depth in metres, and t is the time in hours after midnight.

   a) What is the rate of change of water depth 5 hours after midnight?
   b) When is the first time after $t=0$ that the tide changes direction?
   c) At that time, is the water changing from rising to falling or from falling to rising?

12. In physics, the rate of change of momentum of an object is proportional to the force needed to effect that change: if p is the momentum of the object as a function of time, $F=\frac{\mathrm{d}p}{\mathrm{d}t}$. The momentum of a particular object, oscillating back and forth along a line, is given by $p=mA\sin(\omega t+\phi)\,\mathrm{kg\,m\,s^{-1}}$ (where m, A, ω, and ϕ are various constants). What is the force acting on the object at $t=10$?

13. Find the 73rd derivative of $\sin 6x$.

14. Each side of a square is increasing at a rate of $6\,\mathrm{cm\,s^{-1}}$. At what rate is the area of the square increasing when the area of the square is $16\,\mathrm{cm}^2$?

15. Gas is being forced into a spherical balloon at a rate of $400\,\mathrm{cm^3\,min^{-1}}$. How fast is the radius of the balloon increasing when the radius is $5\,\mathrm{cm}$?

16. If a snowball melts so that its surface area decreases at a rate of $1\,\mathrm{cm^2\,min^{-1}}$. find the rate at which the diameter decreases when the diameter is $10\,\mathrm{cm}$.

17. If $x^2 + y^2 + z^2 = 9$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$, and $\frac{\mathrm{d}y}{\mathrm{d}t} = 4$, find $\frac{\mathrm{d}z}{\mathrm{d}t}$ when $(x, y, z) = (2, 2, 1)$.

18. A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle moves through the point $(1/3, 1)$, its x-ordinate increases at a rate of $\sqrt{10}\ \mathrm{cm\,s^{-1}}$. How fast is the distance from the particle to the origin changing at this instant?

19. Gravel is dumped from a conveyor belt at a rate of $3\,\mathrm{m^3\,min^{-1}}$, and forms a pile in the shape of a cone with equal height and base diameter. How fast is the height of the cone increasing when the pile is $3\,\mathrm{m}$ tall?

20. The top of a ladder slides down a vertical wall at a rate of $0.15\,\mathrm{m\,s^{-1}}$. A the moment when the bottom of the ladder is $3\,\mathrm{m}$ from the bottom of the wall, it slides away from the wall at a rate of $0.2\,\mathrm{m\,s^{-1}}$. Find the length of the ladder.

21. Two sides of a triangle have lengths $2\,\mathrm{m}$ and $3\,\mathrm{m}$. The angle between these sides is increasing at a rate of $4\,^\circ\,\mathrm{s^{-1}}$. How fast is the length of the third side changing when it is of length $4\,\mathrm{m}$?

22. A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y-ordinate is decreasing at a rate of 3 units per second. How fast is the x-ordinate of the particle changing at that instant?

23. The minute hand on a watch is $8\,\mathrm{mm}$ long and the hour hand is $4\,\mathrm{mm}$ long. How fast is the distance between the tips of the hands changing at 1 o'clock?

24. If $f(\theta) = \sin^{-1}(\theta)$, compute $\frac{\mathrm{d}}{\mathrm{d}\theta} f(\theta)$ and $\frac{\mathrm{d}}{\mathrm{d}x} f(x^4)$.

25. Recall that the *absolute value* of x, denoted $|x|$, is the value obtained by 'throwing away the sign' of x.

    a) Prove that
    $$\frac{\mathrm{d}}{\mathrm{d}x}|x| = \frac{x}{|x|}.$$
    [*Hint: Write* $|x| = \sqrt{x^2}$.]

    b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of both f and f'.

    c) If $\frac{\mathrm{d}x}{\mathrm{d}t} = \left|\alpha'(t)\right|$, find $\frac{\mathrm{d}^2 t}{\mathrm{d}x^2}$.

26. Note that on the formula sheet, the anti-derivative of $1/x$ is given as $\ln|x|$, not just $\ln x$.

    a) Compute $\frac{\mathrm{d}}{\mathrm{d}x} \ln|x|$ if $x < 0$, and hence justify formally why $\frac{\mathrm{d}}{\mathrm{d}x} \ln|x| = 1/x$.

    b) Draw $y = \ln|x|$ and $y = 1/x$ on the same pair of axes, and hence justify intuitively why $\frac{\mathrm{d}}{\mathrm{d}x} \ln|x| = 1/x$.

27. Soon, we will be studying the product rule for derivatives. It is possible, though not particularly usual, to prove it using simply the basic derivatives from the last section and the chain rule; in this exercise, you will do just that.

    Suppose that f and g are functions, and consider the function F defined by $F(x) = \big(f(x) + g(x)\big)^2$.

    a) Calculate $F'(x)$ using the chain rule.

    b) Calculate $F'(x)$ by multiplying out the square and differentiating the polynomial that results. (In particular, note that $\frac{\mathrm{d}}{\mathrm{d}x} 2(fg)(x) = 2(fg)'(x)$).

    c) Compare parts (a) and (b).

References

For an approach to the chain rule similar to the one taken here, see chapter IX of Thompson. See also sections 2.5 and 2.8 of Stewart.

Homework

Reading Historically, when Leibniz conceived of the notation, $\frac{\mathrm{d}y}{\mathrm{d}x}$ was supposed to be a quotient: it was the quotient of the "infinitesimal change in y produced by the change in x" divided by the "infinitesimal change in x".

However, the formulation of calculus with infinitesimals in the usual setting of the real numbers leads to a lot of problems. For one thing, infinitesimals can't exist in the usual setting of real numbers! This is because the real numbers satisfy an important property, called the Archimedean property: given any positive real number $\varepsilon > 0$, no matter how small, and given any positive real number $N > 0$, no matter how big, there exists a natural number n such that $n\varepsilon > N$. An "infinitesimal" ξ is supposed to be so small that no matter how many times you add it to itself, it never gets to 1, contradicting the Archimedean property.

Other problems: Leibniz defined the tangent to the graph of $y = f(x)$ at $x = a$ by saying "Take the point $(a, f(a))$; then add an infinitesimal amount to a, $a + \mathrm{d}x$, and take the point $(a + dx, f(a + \mathrm{d}x))$, and draw the line through those two points." But if they are two different points on the graph, then it's not a tangent, and if it's just one point, then you can't define the line because you just have one point. That's just two of the problems with infinitesimals.

So calculus was essentially rewritten from the ground up in the following 200 years to avoid these problems, and you are seeing the results of that rewriting. Because of that rewriting, the derivative is no longer a quotient, now it's a limit:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Because we cannot express this limit-of-a-quotient as a-quotient-of-the-limits (both numerator and denominator go to zero), then the derivative is not a quotient. However, Leibniz' notation is very suggestive and very useful; even though derivatives are not really quotients, in many ways they behave as if they were quotients. So we have the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}y}{\mathrm{d}x}$$

which looks very natural if you think of the derivatives as "fractions". You have the inverse function theorem, which tells you that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 1/\frac{\mathrm{d}y}{\mathrm{d}x}$$

which is again almost "obvious" if you think of the derivatives as fractions. So, because the notation is so nice and so suggestive, we keep the notation even though the notation no longer represents an actual quotient: it now represents a single limit. Even though we write $\frac{\mathrm{d}y}{\mathrm{d}x}$ as if it were a fraction, and many computations look like we are working with it like a fraction, it isn't really a fraction (it just plays one on television).

There is a way of getting around the logical difficulties with infinitesimals; this is called nonstandard analysis. It's pretty difficult to explain how one sets it up, but you can think of it as creating two classes of real numbers: the ones you are familiar with, that satisfy things like the Archimedean property, the least-upper-bound property, and so on, and then you add another, separate class of real numbers that includes infinitesimals and a bunch of other things. If you do that, then you can, if you are careful, define derivatives exactly like Leibniz, in terms of infinitesimals and actual quotients; if you do that, then all the rules of calculus that make use of $\frac{\mathrm{d}y}{\mathrm{d}x}$ as if it were a fraction are justified because, in that setting, it is a fraction. Still, one has to be careful because you have to keep infinitesimals and regular real numbers separate and not let them get confused, or you can run into some serious problems.

Arturo Magidin (`https://math.stackexchange.com/users/742/arturo-magidin`), Is $\frac{\mathrm{d}y}{\mathrm{d}x}$ not a ratio? (adapted), URL (version: 2017-09-15): `https://math.stackexchange.com/q/21209`

Problems

1. If $y = \sqrt{\cot x} - \sqrt{\cot a}$ (where a is constant), find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

2. a) Show that if $y = f(g(h(x)))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) \cdot g'(h(x)) \cdot f'(g(h(x)))$.
   b) Calculate the derivative of $y = \sin\cos\sin\cos\sin x^5$.

3. We will prove the double angle formula for cosine from the double angle formula for sine. Suppose $f(\theta) = \cos 2\theta$, and $g(\theta) = 1 - 2\sin^2\theta$.
   a) Show that $f' = g'$. (You may assume that $\sin 2\theta = 2\sin\theta\cos\theta$.)
   b) Verify that f and g agree at $\theta = 0$, and conclude that $f = g$.

4. If V is the volume of a cube with edge length x and the cube expands as time passes, find $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of $\frac{\mathrm{d}x}{\mathrm{d}t}$.

5. A water tank has the shape of an inverted circular cone with base radius $2\,\mathrm{m}$ and height $4\,\mathrm{m}$. If water is being pumped into the tank at a rate of $2\,\mathrm{m}^3\,\mathrm{min}^{-1}$, find the rate at which the water level is rising when the water is $3\,\mathrm{m}$ deep.

6. A boat is pulled into a dock by a rope attached to the bow of the boad and passing through a pulley on the dock that is $1\,\mathrm{m}$ higher than the bow of the boat. If the rope is pulled in at a rate of $1\,\mathrm{m\,s}^{-1}$, how fast is the boat approaching the dock when it is $8\,\mathrm{m}$ from the dock?

I.6 Substitution

Recall that the chain rule for differentiation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x).$$

In other words, $f(g(x))$ is an anti-derivative of $f'(g(x))g'(x)$ and so we can write

$$\int f'(g(x))g'(x)\,\mathrm{d}x = f(g(x)) + C.$$

To make this rule easier to apply in practice, we often perform what is known as a change of variables. We let $u = g(x)$, and then $\frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$. Substituting this in, we obtain

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

and then the rule is just the statement that we can 'cancel' the $\mathrm{d}x$'s, producing

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(u)\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int f'(u)\,\mathrm{d}u = f(u) + C = f(g(x)) + C.$$

This rule, which gives us a kind of chain rule for integration, is called *substitution*, or the *inverse chain rule*. It can be thought of as a change in coordinate system from an x-based system to one based on u, and we have to 'resize' our curve based on how much u stretches the coordinate system compared to x — and this 'stretch factor' is simply $\frac{\mathrm{d}u}{\mathrm{d}x}$.

Examples.

1. Suppose we wish to find $\int \sin x \cos x\,\mathrm{d}x$. Then let $u = \sin x$, so $\mathrm{d}u = \cos x\,\mathrm{d}x$ and

$$\int \sin x \cos x\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C.$$

   In this case, we also could have used a trigonometric identity.

2. Suppose we wish to find $\int xe^{x^2}\,\mathrm{d}x$. We can let $u = x^2$, and then $du = 2x\,\mathrm{d}x \Rightarrow \mathrm{d}x = \frac{du}{2x}$. Hence:

$$\int xe^{x^2}\,\mathrm{d}x = \int \frac{1}{2}e^u\,\mathrm{d}u = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

3. Suppose we wish to find $\int \frac{4}{x}(\ln x)^3\,\mathrm{d}x$. We let $u = \ln x$, and then $\mathrm{d}u = \frac{\mathrm{d}x}{x}$. Hence:

$$\int \frac{4}{x}(\ln x)^3\,\mathrm{d}x = 4\int u^3\,\mathrm{d}u = u^4 + C = (\ln u)^4 + C.$$

Exercises and Problems

1. Find the following indefinite integrals. (Remember, the indefinite integral of f, $\int f(x)\,\mathrm{d}x$, is the family of anti-derivatives of f.)

   a) $\int \sin 2x\,\mathrm{d}x$
   b) $\int \tan x\,\mathrm{d}x$
   c) $\int 3x\cos x\,\mathrm{d}x$
   d) $\int \frac{\cos x}{\sin x + 1}\,\mathrm{d}x$
   e) $\int (4x - 44)^{2019}\,\mathrm{d}x$
   f) $\int 4x\sqrt{x^2+3}\,\mathrm{d}x$
   g) $\int (3x-4)^2\,\mathrm{d}x$
   h) $\int \frac{x}{x^2+1}\,\mathrm{d}x$
   i) $\int \frac{2}{4x+3}\,\mathrm{d}x$
   j) $\int e^{2x+1}\,\mathrm{d}x$

k) $\int \sec 4x \tan 4x \, \mathrm{d}x$

l) $\int 2\cos x + \sin 2x \, \mathrm{d}x$

m) $\int -2x \csc^2(3x^2) \, \mathrm{d}x$

n) $\int \frac{3}{x^3} - \frac{4}{x+1} \, \mathrm{d}x$

o) $\int e^{x/2} + \frac{2}{x} \, \mathrm{d}x$

p) $\int x^2 \sec^2 x^3 + 9 \, \mathrm{d}x$

q) $\int -\csc(\tan x)\cot(\tan x)\sec^2 x \, \mathrm{d}x$

r) $\int \frac{\cos x - \sin x}{\cos x + \sin x} \, \mathrm{d}x$

s) $\int \frac{2017}{x \ln x} \, \mathrm{d}x$

t) $\int \tan x + \frac{1}{\tan x} \, \mathrm{d}x$

u) $\int (\cos x)(\sin\sin x)(\cos\cos\sin x) \, \mathrm{d}x$

2. By using the substitution $x = \sin\theta$, find

$$\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x.$$

3. Evaluate $\int \cos^5 x \, \mathrm{d}x$ using the substitution $t = \sin x$.

4. Find $\int \tan\theta \, \mathrm{d}\theta$ and $\int \cot\theta \, \mathrm{d}\theta$.

5. Complete the following working:

$$\begin{aligned} \int \sec x \, \mathrm{d}x &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, \mathrm{d}x \\ &= \int \frac{\cdots}{\sec x + \tan x} \, \mathrm{d}x \\ \text{Let } u = \ldots & \\ &= \int \frac{1}{\cdots} \, \mathrm{d}u \\ &= \ldots \end{aligned}$$

6. Find an anti-derivative of $\csc x$. (Hint: consider the previous problem.)

7. The velocity of a particle at time t is given by $v = \dfrac{\cos(\sqrt{2t+1})}{\sqrt{2t+1}}$. What is the position of the particle at time $t = 5$, given that $x(0.5) = 0$? (Recall that $v = \frac{\mathrm{d}x}{\mathrm{d}t}$.)

8. Consider the following indefinite integral:

$$\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x.$$

   a) Show, using the inverse function rule for differentiation, that the anti-derivatives of $\frac{1}{\sqrt{1-x^2}}$ are $\sin^{-1} x + C$.
   b) Compute the indefinite integral a different way, using the substitution $x = \sin\theta$.
   c) Find the anti-derivatives of

$$f(x) = \frac{-1}{2\sqrt{x - x^2}}.$$

   (Hint: try to substitute $u = \sqrt{1-x}$.)

9. Compute the following:

   a) $\int \frac{x^2(5x^2+4x-3)}{x^5+x^4-x^3+1} \, \mathrm{d}x$‘

   b) $\int \frac{x^2+1}{x(x^2+3)} \, \mathrm{d}x$

The previous problem involved finding anti-derivatives of *rational functions*: those of the form $\frac{P(x)}{Q(x)}$ for polynomials P and Q. In general, it is possible to find anti-derivatives of all such functions by writing them as sums of fractions with linear or quadratic denominators; this is known as *expansion via partial fractions*.

10. Some more interesting problems:

    a) Rewrite in the form $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1}$ and integrate:
    $$\int \frac{4x}{x^3-x^2-x+1}\,\mathrm{d}x.$$
    b) Use the obvious substitution and divide through:
    $$\int \frac{\sqrt{x+1}}{x}\,\mathrm{d}x.$$

11. Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{x^2+1}$.

    a) Rewrite the given rational function as follows:
    $$\frac{x^2+x-2}{3x^3-x^2+3x-1} = \frac{A}{3x-1}+\frac{Bx+C}{x^2+1}$$
    b) Hence (or otherwise) compute:
    $$\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,\mathrm{d}x.$$

12. Use appropriate substitutions to evaluate:

    a) $\displaystyle\int \frac{\cos\theta}{\sin^2\theta+4\sin\theta-5}\,\mathrm{d}\theta$
    b) $\displaystyle\int \frac{e^{3x}}{e^{2x}+4}\,\mathrm{d}t$
    c) $\displaystyle\int \frac{5+2\ln x}{x(1+\ln x)^2}$

13. In the following, let $t=\tan\frac{x}{2}$ (where $|x|<\pi$). We will apply the techniques from the last few problems in the notes, calculating some anti-derivatives of rational functions by expanding them as sums of fractions.

    a) Show that:
    $$\cos\left(\frac{x}{2}\right)=\frac{1}{\sqrt{1+t^2}} \quad\text{and}\quad \sin\left(\frac{x}{2}\right)=\frac{t}{\sqrt{1+t^2}}$$
    b) Show that:
    $$\cos x = \frac{1-t^2}{1+t^2} \quad\text{and}\quad \sin x = \frac{2t}{1+t^2}$$
    c) Show that:
    $$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{2}{1+t^2}$$
    d) Use the substitution t to evaluate:
        i. $\int(1-\cos x)^{-1}\,\mathrm{d}x$
        ii. $\int(3\sin x-4\cos x)^{-1}\,\mathrm{d}x$

References

For exercises and notes on substitution, see Thompson chapter XX (the section on substitution is two or three pages in). For partial fractions, see chapter XIII.

For the interested, a proof that one can always expand a rational function into partial fractions is outlined as exercise 11.1.13 in Artin (p. 441).

Figure 7: Sofia Kovalevskaya (public domain)

Homework

Reading Sofia Kovalevskaya (1850–1891) was the daughter of an artillery general and a member of the Russian nobility. It so happened that her nursery walls had been papered with pages from lecture notes on analysis. At the age of 11 she took a close look at her wallpaper and taught herself calculus. She became attracted to mathematics, preferring it to all other areas of study. Her father tried to stop her, but she carried on regardless, reading an algebra book when her parents were sleeping.

In order to travel and obtain an education, she was obliged to marry, but the marriage was never a success. In 1869 she studied mathematics in Heidelberg, but because female students were not permitted, she had to pursuade the university to let her attend lectures on an unofficial basis. She showed impressive mathematical talent, and in 1871 she went to Berlin, where she studied under the great analyst Karl Weierstrass. Again, she was not permitted to be an official student, but Weierstrass gave her private lessons.

She carried out original research, and by 1874 Weierstrass said that her work was suitable for a doctorate. She had written three papers, on differential equations, elliptic functions, and the rings of Saturn. In the same year Göttingen University awarded her a doctoral degree. The differential equations paper was published in 1875.

In 1878 she had a daughter, but returned to mathematics in 1880, working on the refraction of light. In 1883 her husband, from whom she had separated, committed suicide and she spend more and more time working on her mathematics to assuage her feelings of guilt. She obtained a university position in Stockholm, giving lectures in 1884. In 1889 she became the third female professor ever at a European university, after Maria Agnesi (who never took up her post) and the physicist Laura Bassi. Here she did research on the motion of a rigid body, entered it for a prize offered by the Academy of Sciences in 1886, and won. The jury found the work so brilliant that they increased the prize money. Subsequent work on the same topic was awarded a prize by the Swedish Academy of Sciences, and led to her being elected to the Imperial Academy of Sciences.

From *Taming the Infinite*, by Ian Stewart (Quercus, 2008).

Problems

1. Calculate the following indefinite integrals:

   a) $\int -\csc 3x \cot 3x \, \mathrm{d}x$

   b) $\int -x \sec^2 3x^2 \, \mathrm{d}x$

   c) $\int \frac{\sqrt{x}+3x-2}{x} \, \mathrm{d}x$

   d) $\int \sin^3 x \cos^2 x \, \mathrm{d}x$ (Hint: use $\sin^2 x = 1 - \cos^2 x$ to rewrite the integrand.)

2. Recall that $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}$. Find $\int \frac{x}{1+x^4} \, \mathrm{d}x$.

3. Let y be a function of x, and let x in turn be a function of t. If $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 0$, and if $x(t) = 7t + e^t$, find an explicit expression for $y(t)$.

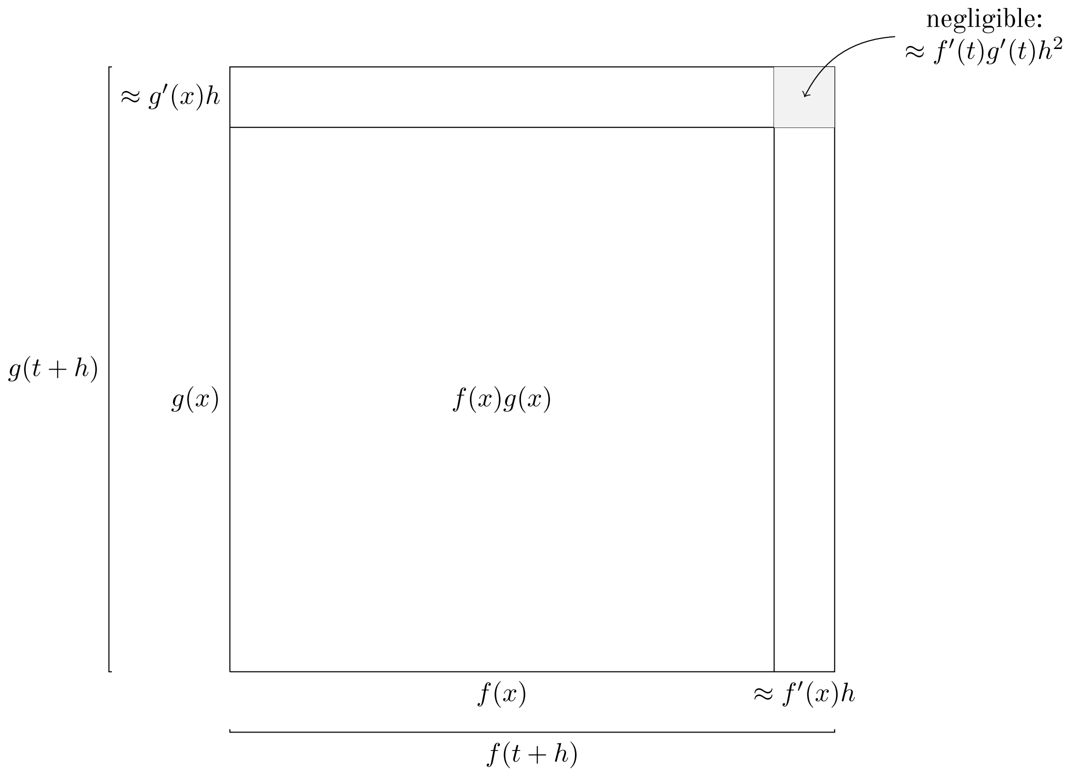


Figure 8: The approximate errors in the product rule estimation.

I.7 The product law

So far, we can differentiate functions which are made up of sums and compositions of polynomials, trig functions, and ln and exp. However, the following function will leave us lost and confused if we try to compute its derivative directly:

$$f(x) = (\sin x)(\cos x) \qquad f'(x) = ?$$

In this particular case, we can use the identity $\sin 2x = 2\sin x \cos x$ to rewrite $f(x) = \frac{1}{2}\sin 2x$ and then apply the chain rule to find that $f'(x) = \cos 2x$. However, in general we don't have nice things like trig identities; thus, we need a rule to differentiate products of functions.

First of all, we notice that $(fg)' \neq (f')(g')$.[4] Indeed, for the function f defined above, $(\sin x)'(\cos x)' = -\sin x \cos x$; this is zero at $x = 1$, but we have already seen that the derivative of f is $\cos 2x$, which is equal to 1 when $x = 1$.

We will try to derive one by estimation; consider the following difference:

$$(fg)(x+h) - fg(x) = f(x+h)g(x+h) - f(x)g(x)$$

We may assume that f and g are differentiable at x, and so we can approximate them with their derivatives,

$$\begin{aligned} f(x+h)g(x+h) - f(x)g(x) &\approx (f'(x)h + f(x))(g'(x)h + g(x)) - f(x)g(x) \\ &= f'(x)g'(x)h^2 + f'(x)g(x)h + f(x)g'(x)h. \end{aligned}$$

Applying the reasoning we developed a few sections ago, we note that as $h \to 0$, the approximation becomes exact.[5] Taking the limit of our difference quotient, we find that

$$\begin{aligned} \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} &= \lim_{h\to 0} \frac{f'(x)g'(x)h^2 + f'(x)g(x)h + f(x)g'(x)h}{h} \\ &= f'(x)g(x) + f(x)g'(x); \end{aligned}$$

[4] To save ink, I will write (fg) for the function defined by $(fg)(x) = f(x)g(x)$.

[5] Technical note. Recall that we defined $f(x) \approx g(x)$ if $f(x) = g(x) + \vartheta(h)$ where $\vartheta(h)/h \to 0$ as $h \to 0$. One can therefore make the reasoning here rigorous by carrying through the ϑ's that give us the approximations for $f(x+h)$ and $g(x+h)$, checking that we end up with an estimation term that is also satisfies the ϑ condition.

and we have justified the following

Theorem (Product law). *If f and g are differentiable at x, then*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Example. Consider $y = 2t\sin t$. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = 2\sin t + 2t\cos t$.

With our rules (sum, chain, and product, together with our basic derivatives), we can now differentiate almost any combination of functions that we are currently aware of. The process of differentiation is entirely mechanical, and can be easily performed by a computer. As such, learning to differentiate more complicated combinations of functions is very similar to learning how to add, multiply, and perform long division, and is only a matter of practice.

Example. Let $f(x) = \sin x^2 + e^{10x^2+3x+e^x} + \frac{2x+3}{\ln x}$. We can split this into three derivative-taking problems by applying the sum rule; so

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\sin x^2 + \frac{\mathrm{d}}{\mathrm{d}x}e^{10x^2+3x+e^x} + \frac{\mathrm{d}}{\mathrm{d}x}\frac{2x+3}{\ln x}$$

Taking these in turn, $\sin x^2 \mapsto 2x\cos x^2$ (applying the chain rule, since we have a function composition) and $e^{10x^2+3x+e^x} \mapsto (20x+3+e^x)e^{10x^2+3x+e^x}$ (applying the chain rule again). Finally, note that $\frac{2x+3}{\ln x} = (2x+3)[(\ln x)^{-1}]$ and so we need to apply the product rule:

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x}(2x+3)[(\ln x)^{-1}] &= \left(\tfrac{\mathrm{d}}{\mathrm{d}x}(2x+3)\right)\left((\ln x)^{-1}\right) + (2x+3)\left(\tfrac{\mathrm{d}}{\mathrm{d}x}(\ln x)^{-1}\right) \\
&= 2(\ln x)^{-1} + (2x+3)\left(-1(\ln x)^{-2}\cdot\tfrac{\mathrm{d}}{\mathrm{d}x}\ln x\right) \\
&= 2(\ln x)^{-1} + (2x+3)\left(-1(\ln x)^{-2}\cdot\frac{1}{x}\right) \\
&= \frac{2}{\ln x} - \frac{2x+3}{x(\ln x)^2} \\
&= \frac{2x\ln x - 2x - 3}{x(\ln x)^2}.
\end{aligned}$$

Thus the derivative of f is

$$f'(x) = 2x\cos x^2 + (20x+3+e^x)e^{10x^2+3x+e^x} + \frac{2x\ln x - 2x - 3}{x(\ln x)^2}.$$

Exercises and Problems

1. In each case, find $\frac{\mathrm{d}y}{\mathrm{d}t}$.

   a) $y = \left(3+2t^2\right)^4$
   b) $y = \dfrac{t^3}{\ln t}$
   c) $y = t\sqrt{t}$
   d) $y = 2t\sin t - (t^2-2)\cos t$
   e) $y = \dfrac{t}{\sqrt{a^2-t^2}}$ (a constant)
   f) $y = \dfrac{1}{8}t^8\left(1-t^2\right)^{-4}$
   g) $y = e^t\ln t$
   h) $y = \log\left[1 + \dfrac{t^2+3t+17}{t^{17}}\right]$
   i) $y = \sin\left[e^{\tan t}\ln\tan t\right]$
   j) $y = \dfrac{3t-2}{\sqrt{2t+1}}$
   k) $y = \dfrac{\sec 2t}{1+\tan 2t}$
   l) $y = \dfrac{(t-1)(t-4)}{(t-2)(t-3)}$
   m) $y = t\sin^2(\cos\sqrt{\sin\pi t})$
   n) $y = \sqrt[5]{t\tan t}$
   o) $y = \dfrac{(t+\lambda)^4}{t^4+\lambda^4}$ (λ constant)

2. If $f(x) = e^{-x}$, find $f(0) + xf'(0)$.

3. Show that $\frac{\mathrm{d}}{\mathrm{d}x} e^{\tan x} e^{-\cot x} = \left(\frac{\mathrm{d}}{\mathrm{d}x} e^{\tan x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} e^{-\cot x}\right)$. Reconcile this with our statement above that the naive product rule does not work in general.

4. The altitude h of a triangle is increasing at a constant rate of $1\,\mathrm{cm}\,\mathrm{min}^{-1}$ while the area A increases at a constant rate of $2\,\mathrm{cm}^2\,\mathrm{min}^{-1}$. At what rate is the length b of the base of the triangle increasing when $h = 10\,\mathrm{cm}$ and $A = 100\,\mathrm{cm}^2$?

5. Show that if f and g are differentiable at x, such that $g(x) \neq 0$, we have
$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$
This is often called the *quotient law*.

6. Show that $y = xe^{-x}$ satisfies the differential equation $xy' = (1-x)y$.

7. If $y = \ln \frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}$, find y''.

8. Find the equation of the tangent line to the graph of $y = \ln\cos\frac{x-1}{x}$ at the point $(1, 0)$.

9. Show that $y = (1 + x + \ln x)^{-1}$ satisfies the differential equation $xy' = y(y\ln x - 1)$.

10. Find the angle at which $y = x^2 \ln[(x-2)^2]$ cuts the x-axis at the point $(0, 0)$.

11. When $x = 0$, is the curve $y = (x+20)^2(2x^2-3)^6 - \ln\sin(x - \frac{\pi}{2})$ concave up or concave down?

12. If $y = \frac{e^x}{\sin x}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 - \cot x)$.

13. Show that if f, g, and h are functions then $(fgh)' = f'gh + fg'h + fgh'$.

14. Suppose $f(x) = f(-x)$ for all x in the domain of f. Prove that $f'(x) = -f'(-x)$ for all x in the domain of $f'(x)$.

15. Consider the function defined by $f(x) = x^x$.
    a) Rewrite f in the form $f(x) = e^{x\ln x}$, and hence find $f'(x)$.
    b) Find $\frac{\mathrm{d}y}{\mathrm{d}t}$ if $y = (t^2+3)^{(t^2+3)}$.

16. Prove the product rule a different way by writing $f(x)g(x) = e^{\ln(f(x)g(x))}$.

17. Find $f'(x+3)$ if $f(x+3) = (x+5)^7$.

18. The number a is called a **double root** of some polynomial function f if $f(x) = (x-a)^2 g(x)$ for some polynomial g. Prove that a is a double root of f if and only if a is a root of both f and f'.

References

Chapter VI of Thompson; section 2.3 of Stewart.

Homework

Reading

If it's the quotient rule you wish to know,
It's low-de-high less high-de-low.
Then draw the line and down below,
Denominator squared will go.

Quoted in *Mathematical Apocrypha* by Steven G. Krantz (p.36).

Problems

1. Find the derivatives:

   a) $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = \sin x \ln x$.

   b) $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = x \sec kx$ (k constant).

   c) $\frac{\mathrm{d}f}{\mathrm{d}\theta}$ if $f(\theta) = \frac{\cos \pi\theta}{\sin \pi\theta + \cos \pi\theta}$.

   d) $\frac{\mathrm{d}y}{\mathrm{d}t}$ if $y = \cos^4(\sin^3 t)$.

2. Find an expression for $(fg)''(x)$ in terms of $f'(x)$ and $g'(x)$.

3. Suppose a liquid is oozing from a corner across a rectangular ridged surface that makes it easier to flow in one direction than the other; call the corner $(0, 0)$, and suppose that the ridges are in the y–direction: so we would expect the flow to be faster towards increasing y compared to increasing x. As the total volume V of liquid oozed increases, the flow rate increases due to the pressure. Suppose that the rate of ooze is constant, at $\frac{\mathrm{d}V}{\mathrm{d}t} = 3\,\mathrm{m}^3\,\mathrm{h}^{-1}$. A measurement shows that the flow rates of the liquid at its edges in the two directions are $\frac{\mathrm{d}x}{\mathrm{d}t} = V^{-1/2}e^{k(V^{1/2})}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = e^{kV}$, for some small constant $k \approx 0.24$. (Both rates are in metres per hour.)

   a) Assuming that the liquid covers the surface uniformly, and that the area covered is roughly rectangular, what is the area covered after three hours (the initial volume being zero), and what is the rate of change of the area covered at that time? [Useless hint: you will need to take some anti-derivatives at some point, and you should need to use the product and chain rules for derivatives eventually as well.]

   b) (Even more funner question.) Suppose the room measures ten metres by ten metres; when the liquid reaches the wall in the y–direction, suppose that the full 'force' of the liquid is now pushing in the x–direction and the flow rate in that direction is the sum of the original flow rates: so $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = V^{-1/2}e^{k(V^{1/2})} + e^{kV}$.
   Calculate how long the liquid will take to reach the wall in the y–direction, and then (taking into account the changed flow rates) work out how long the liquid takes to fill the entire floor area of the room. [Even more useless hint: the final answer should be large.]

I.8 Anti-differentiation by parts

We have already seen that, by reversing the chain rule, we can anti-differentiate some function compositions. Similarly, we can reverse the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$
$$f(x)g(x) = \int f'(x)g(x)\,\mathrm{d}x + \int f(x)g'(x)\,\mathrm{d}x.$$

This result is normally written in the form

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x \tag{I.7}$$

and is known as *integration by parts*. We often write it in Leibniz notation, where it looks like $\int \frac{\mathrm{d}u}{\mathrm{d}x} v\,\mathrm{d}x = uv - \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x$.

Examples. 1. Consider $\int x\sin x\,\mathrm{d}x$, which does not yield to any obvious change of variable. Let $u = x$, and let $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$. So $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$, and $v = -\cos x$. Hence:

$$\int x\sin x\,\mathrm{d}x = -x\cos x + \int \cos x\,\mathrm{d}x = -x\cos x + \sin x + C,$$

where C is an arbitrary constant. Check that $(-x\cos x + \sin x)' = x\sin x$.

2. Now we will anti-differentiate $x^2\sin 2x$. We work as follows:

$$\begin{aligned}\int x^2\sin 2x\,\mathrm{d}x &= -\frac{x^2\cos 2x}{2} + \int x\cos 2x\,\mathrm{d}x\\ &= -\frac{x^2\cos 2x + x\sin 2x}{2} - \int \frac{1}{2}\sin 2x\\ &= -\frac{x^2\cos 2x + x\sin 2x}{2} + \frac{1}{4}\cos 2x + C.\end{aligned}$$

The aim is to end up with an easier integral than the one that was started with.

Exercises and Problems

1. Compute the following indefinite integrals.

   a) $\int xe^x\,\mathrm{d}x$
   b) $\int x^2e^{2x}\,\mathrm{d}x$
   c) $\int \ln x\,\mathrm{d}x$
   d) $\int t^5\ln t\,\mathrm{d}t$
   e) $\int t^3e^{-t^2}\,\mathrm{d}t$
   f) $\int \sin\ln y\,\mathrm{d}y$
   g) $\int x\tan^2 x\,\mathrm{d}x$
   h) $\int \frac{te^t+te^{-t}}{2}\,\mathrm{d}t$
   i) $\int \cos\sqrt{t}\,\mathrm{d}t$
   j) $\int \theta^3\cos(\theta^2)\,\mathrm{d}\theta$
   k) $\int (x^2+1)e^{-x}\,\mathrm{d}x$

2. Consider $\int f'(x)g(x)\,\mathrm{d}x$; show that, for integration by parts, we can take any anti-derivative of f' for f.

3. Prove that
$$\int \cos^n x\,\mathrm{d}x = \frac{1}{n}\sin x\cos^{(n-1)} x + \frac{n-1}{n}\int \cos^{(n-2)} x\,\mathrm{d}x$$

4. Evaluate $\int (\ln x)^2\,\mathrm{d}x$.

5. A particle moving in one dimension has a velocity function $v(t) = t^2e^{-t}$ (where t is in seconds). What is its displacement from its starting position after three minutes?

6. Scholarship 2016: A curve passing through the point $(1,1)$ has the property that at each point (x,y) on the curve, the gradient of the curve is $x - 2y$; that is, $\frac{\mathrm{d}y}{\mathrm{d}x} = x - 2y$.

   a) Show that $\frac{\mathrm{d}}{\mathrm{d}x} e^{2x} y = x e^{2x}$.
   b) Hence, or otherwise, find the equation of the curve.

7. Find $I(x) = \int e^x \cos x \,\mathrm{d}x$.

8. Evaluate $\int \sin 4x \cos 5x \,\mathrm{d}x$ in two different ways.

9. Find an anti-derivative of $(\sin^{-1} x)^2$.

10. Recall that $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}$. Find $\int \tan^{-1} x \,\mathrm{d}x$.

11. We integrate $\int 1/x \,\mathrm{d}x$ by parts:

$$\int \frac{1}{x} \,\mathrm{d}x = \frac{1}{x} \cdot x - \int -\frac{1}{x^2} \cdot x \,\mathrm{d}x = 1 + \int \frac{1}{x} \,\mathrm{d}x$$

    Cancelling the indefinite integral from both sides, we have $0 = 1$. Explain.

References

Thompson, chapter XX; Stewart, section 7.1.

Homework

Reading `https://www.youtube.com/watch?v=-reFBJ4R9iA`

Problems

1. Compute the following indefinite integrals.
   a) $\int x \cos 5x \,\mathrm{d}x$
   b) $\int \cos x \ln \sin x \,\mathrm{d}x$
   c) $\int \cos \sqrt{x} \,\mathrm{d}x$
2. a) Prove that $\int (\ln x)^n \,\mathrm{d}x = x(\ln x)^n - \int (\ln x)^{(n-1)} \,\mathrm{d}x$.
   b) Find $\int (\ln x)^3 \,\mathrm{d}x$.

Chapter II

The geometry of curves

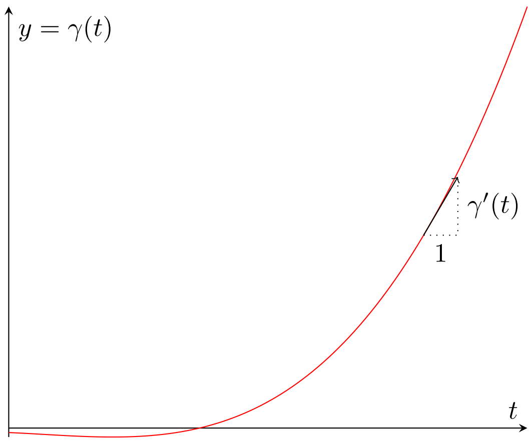


Figure 9: Slope as a measure of direction.

II.9 The geometry of graphs of functions

In this chapter we want to study the geometry of curves: direction, speed, bending, and twisting. This field of study is properly called *differential geometry*.

In this section, we will discuss the geometry of the graphs of functions, and then over the next few sections we will define and then talk about curves with more interesting and complex properties.

Slope and concavity

Suppose γ is a function. If we walk along the graph of γ, then at each point $(t, \gamma(t))$ we have a measure of the direction that we are pointing: we are pointing at an angle $\arctan \gamma'(t)$ to the x–axis.

We can essentially differentiate (if you pardon the pun) three different kinds of behaviour:

Definition. Let γ be a function. Then:

1. If $\gamma'(a) > 0$, γ is said to be *increasing* at a.
2. If $\gamma'(a) = 0$, γ is said to be *stationary* at a.
3. If $\gamma'(a) < 0$, γ is said to be *decreasing* at a.

However, this does not yet give us any information about the curvature of the graph of γ: the amount of 'bending' taking place. Based on our studies so far, it would make some sense to define curvature to be the rate of change of slope. Unfortunately, it turns out that this definition is 'incomplete' in some technical sense (we will discuss this briefly when we talk about arc length). However, the second derivative γ'' is useful to us on its own merits; instead of curvature, we will call it *concavity*.

Definition. Let γ be a function. Then:

1. If $\gamma''(a) > 0$, γ is said to be *concave up* (or *convex*) at a.
2. If $\gamma''(a) = 0$, a is said to be an *inflection point* of γ.
3. If $\gamma''(a) < 0$, γ is said to be *concave down* (or simply *concave*) at a.

If γ is concave up for all points a , then we call the function as a whole concave up (and likewise for concave down functions).

Examples. 1. The function $x \mapsto x^2$ is concave up everywhere, increasing for $x > 0$, and decreasing when $x < 0$.

2. The function $x \mapsto \sin x$ is concave down when $(2n)\pi < x < (2n + 1)\pi$, and concave up when $(2n + 1)\pi < x < (2n + 2)\pi$ (for all integers n). See figure 10

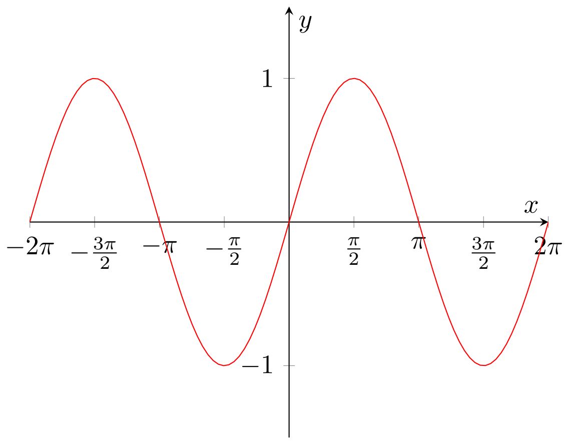


Figure 10: The sine function.

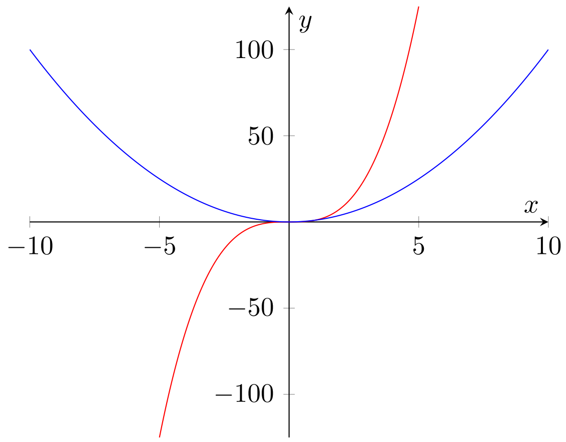


Figure 11: Graphs of x^n for odd n (red) and even n (blue).

3. The function $x \mapsto x^3$ has an inflection point at $(0, 0)$; to the left of this point, the function is concave down (the second derivative is negative) and to the right the function is concave up (the second derivative is positive).

4. In general, functions of the form $f(x) = x^n$ (for integer $n \geq 0$) have some fairly symmetric properties:

   - If n is even, then $y = f(x)$ is even around the x-axis (i.e. $f(-x) = f(x)$), has a minimum at $(0, 0)$, and tends to $+\infty$ in both directions. (See the function graphed in blue in figure 11.)

   - If n is odd, then $y = x^n$ is odd around the x-axis (i.e. $f(-x) = -f(x)$), has an inflection point at $(0, 0)$, and tends to $-\infty$ towards the left and $+\infty$ towards the right. (See the function graphed in red in the figure.)

Example. Consider the function defined by $f(x) = x^4 - 2x^2 + 3$ (figure 12). Find the intervals on which f is increasing or decreasing, find the intervals of concavity, and find any inflection points.

Solution. We have $f'(x) = 4x^3 - 4x$. This function is zero at $x \in \{-1, 0, 1\}$, and so (since the function is a positive cubic) f will be decreasing when $x < -1$, increasing when $-1 < x < 0$, decreasing when $0 < x < 1$, and increasing when $1 < x$. We also have $f''(x) = 12x^2 - 4$ and so $f''(x) = 0$ when $x = \pm\frac{1}{\sqrt{3}}$. Hence the function is concave up when $x < -\frac{1}{\sqrt{3}}$, concave down when $|x| < \frac{1}{\sqrt{3}}$, and concave up when $x > \frac{1}{\sqrt{3}}$. The inflection points will be $x = \pm 1\sqrt{3}$.

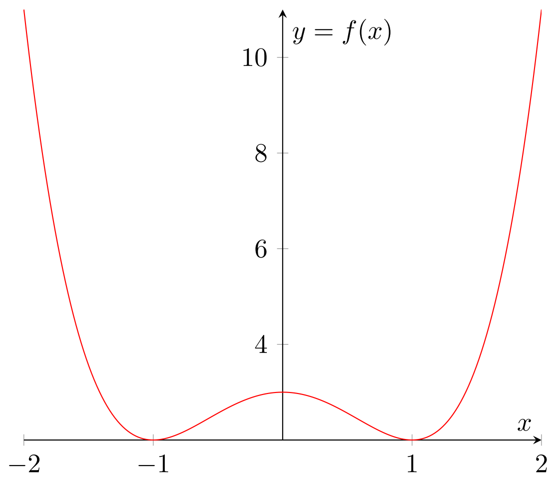


Figure 12: The graph of $f(x) = x^4 - 2x^2 + 3$.

Continuity

We have already mentioned continuity, when we discussed limits. We would like to call a function continuous, intuitively, if its graph can be drawn without picking a pen up off the page. This is insufficient: we cannot prove continuity of any function via this definition! The precise definition of continuity was given by Bernard Bolzano in the early 1800s, and is one of the greatest historical breakthroughs in mathematics. His definition, for us, is best stated in terms of limits:

Definition. A function f is said to be *continuous* at a if $\lim_{x \to a} f(x) = f(a)$. If f is continous for every a in its domain, the function as a whole is said to be continuous.

It is perhaps unfortunate that the concepts of continuity and differentiability are not the same. The details of this are studied in exercise 11.

Arc length and curvature (optional)

The second derivative measures the curvature of a graph based on our position as we walk along the x–axis. A little thought suggests that it might be more natural to measure the curvature based on our position as we walk along the graph itself: if we take our graph and we slant it, for example, this doesn't change the visual 'bendiness' of the graph but it does change the relative position of the graph above the x–axis and thus the values of $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ change.

Let us fix some point on our curve; as we walk along our curve, we measure the distance we travel from this point. After we stick our graph into a coordinate system, this length (the *arc length*) becomes a function of our position.

We will begin with a circle, which we might guess behaves very nicely. I will define the curvature κ of the circle to be the rate of change of the angle θ that our tangent line makes with the x–axis as we walk along the curve, increasing s: in other words, $\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}$.

Consider figure 13. We are approximating a tangent line at B with the line AB, and then varying θ by a small amount, k. As our secant line AB approaches a tangent line, the angle at B between AB and the radius becomes a right angle. Similarly, the angle at C with the radius is a right angle for small k. Thus the angle at B in the triangle OBC is approximately $\pi/2 - k$, and the angle at C is $\pi/2$; thus the angle at O is k.

We therefore have that $kr \approx s(\theta + k) - s(\theta)$, and so $\frac{\mathrm{d}s}{\mathrm{d}\theta} = r$; therefore $\kappa = 1/r$.

If f is some arbitrary function, and we consider the graph $y = f(x)$, we can calculate $\frac{\mathrm{d}\theta}{\mathrm{d}s}$ at each point (we will do this calculation in a second). At each point, we can associate a circle with the same curvature; this circle is called the *osculating circle*, the radius r of the circle is the *radius of curvature* of our graph at that point, and have will say that the *curvature* of the graph at the point x is $\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s} = 1/r$.

So now we will play the same game as above, but now with an arbitrary curve. Consider the function in figure 14; we will again denote the angle at x with the horizontal by θ. As we increase

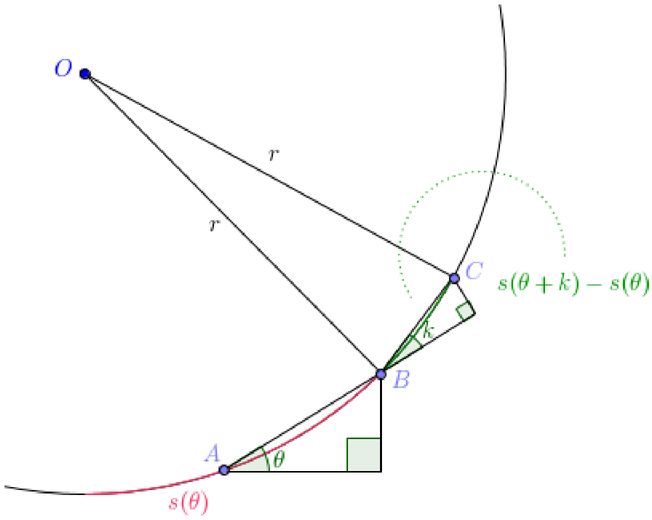


Figure 13: Calculating the curvature of a circle.

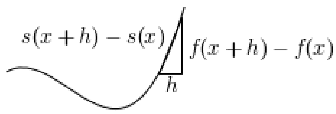


Figure 14: Calculating the curvature of a function.

x by h, we increase s by $s(x+h)-s(x)$. By definition of the derivative, $\frac{\mathrm{d}s}{\mathrm{d}x}h \approx s(x+h)-s(x) = \sqrt{h^2+(f(x+h)-f(x))^2}$; dividing through,

$$\frac{\mathrm{d}s}{\mathrm{d}x} \approx \sqrt{1+\left(\frac{f(x+h)-f(x)}{h}\right)^2};$$

and taking the limit $h \to 0$, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+\tan^2\theta} = \sec\theta. \tag{II.1}$$

Furthermore, since $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta$ we have

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\tan\theta = \frac{\mathrm{d}\theta}{\mathrm{d}x}\sec^2\theta$$

and hence

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\sec^2\theta}{\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)}. \tag{II.2}$$

Finally, by definition we have that the radius of curvature is (combining equations II.1 and II.2)

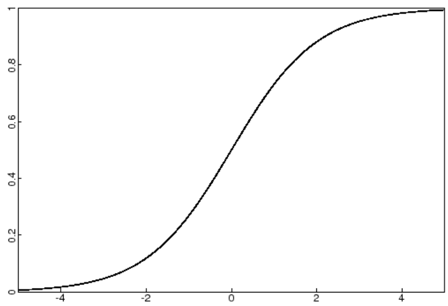
$$r = \frac{\mathrm{d}s}{\mathrm{d}\theta} = \frac{\mathrm{d}s}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\sec^3\theta}{\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)};$$

and the curvature is therefore (again by definition)

$$\kappa(x) = \frac{1}{r} = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\sec^3\theta} = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left(\sqrt{1+\tan^2\theta}\right)^3} = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}}.$$

Exercises and Problems

1. The following function is known as the *logistic curve* and is used for population modelling. Find the intervals of concavity, and label any inflection points.



2. Find the second derivative of the following functions.

   a) $y = x^2 + x$
   b) $f(x) = \sin x$
   c) $g(x) = \cot(3x^2 + 5)$
   d) $y = \frac{\sin mx}{x}$
   e) $y = 4\sin^2 x$
   f) $y = \tan^2(\sin\theta)$
   g) $y = \tan\sqrt{1-x}$

3. Find the concavity of the function $y = \frac{x^2-1}{x^2+1}$ at $(0, -1)$.

4. Find the intervals on which the following functions are increasing or decreasing, and find their intervals of concavity.

   a) $y = x^2 + 1$
   b) $y = 2x^3 + 3x^2 - 36x$
   c) $G(x) = x - 4\sqrt{x}$

5. The graph of $y = f(x)$ (where f is a continuous function) is concave up for all $x < 0$, concave down for $x > 0$, and decreasing everywhere.

   a) Sketch the graph of $y = f(x)$.
   b) What can you say about $f'(x)$ and $f''(x)$ for $x < 0$ and $x > 0$?
   c) What about $x = 0$?

6. Find a value of k such that the function F is continuous at $x = -3$, where

$$F(x) = \begin{cases} \frac{x^2-9}{x+3} & \text{if } x \neq -3, \\ k & \text{if } x = -3. \end{cases}$$

7. Show whether or not the function g is continuous at the three points $(2, g(2))$, $(3, g(3))$, and $(4, g(4))$, where

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leq 2, \\ 2 - x & \text{if } 2 < x \leq 3, \\ x - 4 & \text{if } 3 < x \leq 4, \\ \pi & \text{if } x \geq 4. \end{cases}$$

8. Find all values of α such that Φ is continuous everywhere, where

$$\Phi(x) = \begin{cases} x + 1 & \text{if } x \leq \alpha, \\ x^2 & \text{if } x > \alpha. \end{cases}$$

9. Sketch a function satisfying the given criteria.

   a)
      - Vertical asymptote at $x = 0$,
      - $f'(x) > 0$ if $x < -2$,
      - $f'(x) < 0$ if $x > -2$ $(x \neq 0)$,
      - $f''(x) < 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$.

   b)
      - $f'(0) = f'(2) = f'(4) = 0$,
      - $f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
      - $f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
      - $f''(x) > 0$ if $1 < x < 3$,
      - $f''(x) < 0$ if $x < 1$ or $x > 3$.

10. A curve is defined by the function $f(x) = e^{-(x-k)^2}$. Find, in terms of k, the x-ordinates for which $f''(x) = 0$.

11. It turns out that if a function f is differentiable at a then f is always continuous at a, but the converse is not true: there exist continuous functions that are not differentiable. (In fact, there exist functions that are continuous everywhere but differentiable nowhere.)

    a) We will prove that differentiability of f at a implies continuity of f at a; expand the following and use the limit laws to show that $\lim_{x\to a} f(x) - f(a) = 0$, carefully indicating where you use the existence of the derivative.

    $$\left[\lim_{a\to x} f(x) - f(a)\right]\left[\lim_{a\to x} \frac{x-a}{x-a}\right]$$

    b) Give an example of a function which is continuous but not differentiable at some point.

12. We will do some studies of convexity that may be familiar to students who have looked at the exercises on convexity in the algebra notes. We assume that all functions are continuous and differentiable everywhere for simplicity.

    a) Show that if f is a convex function, and if $P = (p, f(p))$ and $Q = (q, f(q))$ are any two distinct points on the graph of f, then for every point $X = (x_1, x_2)$ on the line segment $\overline{PQ}$, $x_2 \geq f(x_1)$ and equality is only obtained at the endpoints.

    b) Show that if f is a convex function, and if $P = (p, f(P))$ is a point on the graph of f, then for every point (x_1, x_2) on the tangent line to f at P, $x_2 \leq f(x_1)$ and equality is only obtained at P.

    c) Prove similar statements to (a) and (b) in the case that f is a concave function. [Hint: there is not much work involved, as long as one ponders the function $-f$.]

13. Scholarship 2010: Recall that the points of inflection of a curve are places where the second derivative changes sign. These are typically, **but not always**, points at which the second derivative is zero.

    Consider the curve $y = \sqrt[3]{x}e^{-x^2}$.

    Write the second derivative in the form $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (ax^4 + bx^2 + x)e^{-x^2}x^{-5/3}$, and hence find the x-ordinates of the points of inflection of the curve.

14. Scholarship 2004: (You may wish to remind yourself how to perform long division of polynomials.) Consider the function

    $$y = \frac{x^2}{1+x^2},$$

    where $-1 \leq x \leq 1$. The gradient at the point $x = 1$ is $\frac{1}{2}$.

    Hence show that there is a point with $\frac{1}{4} \leq x \leq \frac{1}{2}$ where the gradient is also $\frac{1}{2}$.

15. Scholarship 2013: A function f is **even** if $f(-x) = f(x)$ for all x in its domain, and **odd** if $f(-x) = f(-x)$ for all x in its domain.

    a) Describe which polynomials are even, which are odd, and which are neither.

b) Suppose that g is any even differentiable function defined for all real numbers (not necessarily a polynomial). Use the limit definition of the derivative to prove that g' is odd.

16. Recall that we can define the derivative of f by $\mathsf{D}f(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}$. We will generalise this, by writing $\mathsf{SD}f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$. This is called the *symmetric derivative* of f.

a) In fact, we defined the derivative of f at x to be the unique f' such that $f(x+h) \approx f(x) + hf'(x)$ for small h.[1]
Show that, for small h, if $f'(x)$ exists then $f(x+h) - f(x-h) \approx 2hf'(h)$. Hence conclude that $\mathsf{SD}f(x) = \mathsf{D}f(x)$ whenever the latter exists.

b) The converse is not true: show that if we define $f(x) = |x|$, then $\mathsf{SD}f(0)$ exists but $\mathsf{D}f(0)$ does not.

c) Define the *second* symmetric derivative of f by

$$\mathsf{SD}^2 f(x) = \lim_{h\to 0} \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h} = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Show that whenever $f''(x) = \mathsf{D}^2 f(x)$ exists then $\mathsf{SD}^2 f(x)$ exists and has the same value; show that the converse does not hold (i.e. the existence of the second symmetric derivative does not imply the existence of the usual second derivative) by considering a suitable function, such as

$$\operatorname{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0. \end{cases}$$

17. One may recall from one of the L1 externals that we can recover a quadratic equation given a table of its values. Suppose we know that the following table gives points on the graph of $f(x) = ax^2 + bx + c$.

x	$f(x)$
0	-5
1	2
2	15

Define the *discrete first and second derivatives* of f by $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^2 f(x) = \Delta f(x+1) - \Delta f(x)$. According to the god-given material in L1, we know that if f is a quadratic, then $a = \frac{1}{2}\Delta^2 f(x)$ (for any choice of x); in this example, we can fill in the table as follows:-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
1	-5	7	6
2	2	13	
3	15		

Hence $a = 3$. We can then write (since we know f is a quadratic) $bx + c = f(x) - 3x^2$, which tells us that $b \cdot 1 + c = -8$ and $b \cdot 2 + c = -10$; hence $b = (-10 - {}^{-}8)/1 = -2$ and $c = -6$.

a) Justify the above steps. (Possible approach: $hf''(x) \approx f'(x+h) - f'(x)$; set $h = 1$, and work out what fudge factor $\vartheta(h)$ we have.)

b) Develop a theory of discrete first and second derivatives. (Possible routes of study could include: finding a geometric meaning of the discrete derivatives; defining discrete nth derivatives; studying the relationship between the discrete derivatives and the usual derivatives. You may also want to generalise my definition: instead of $f(x+1) - f(x)$, perhaps one might like to look at $[f(x+k) - f(x)]/k$ (sans limit).)

18. These problems relate to the optional section on curvature and arc length.

a) If $y = f(x)$, formula II.1 gives us the derivative of arc length with respect to distance along the x-axis. What is the arc length along the graph of the function $f(x) = \frac{1}{3}x^3 - x$ between the vertical lines $x = 0$ and $x = 5$?

[1] Then we defined the notation $\varphi(x,h) \approx \psi(x,h)$ to mean that $\varphi(x) = \psi(x) + \vartheta(h)$ for some function ϑ satisfying $\vartheta(h)/h \to 0$ as $h \to 0$.

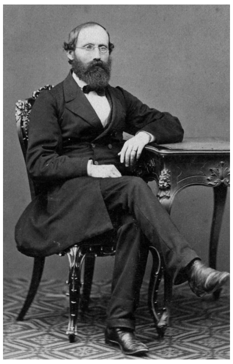


Figure 15: Bernhard Riemann (public domain)

b) Repeat (a) for the function $g(x) = \ln|\sec x|$.

c) What is the curvature of a straight line?

d) Calculate the curvature $\kappa(x)$ of the function $f(x) = x^2$ at the points $x = 0$, $x = 2$, and $x = 4$. Draw the osculating circles at each of these points. What happens to $\kappa(x)$ as $x \to \infty$?

19. Define f by

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

a) Show that f is differentiable at zero.

b) Show that, for all $n > 0$, $f^{(n)}(0) = 0$. (Implicit here is the existence of the nth derivative.)

c) Show that f has a global minimum at zero.

d) Justify that there can thus never be a rule, based simply on checking nth derivatives, for proving that any function has a local minimum or maximum at a point.

References

A good introduction to the geometry of curves, and differential geometry in general, is *Differential Geometry of Curves and Surfaces* by Manfredo P. do Carmo.

For a discussion of the history of continuity, see J F Harper (2016): Defining continuity of real functions of real variables, BSHM Bulletin: Journal of the British Society for the History of Mathematics, DOI:10.1080/17498430.2015.1116053 (`http://homepages.ecs.vuw.ac.nz/~harper/harper16.pdf`).

Discrete derivatives (see exercise 17) are useful when taking derivatives numerically (say we have a table of numbers defining a function f, but we don't have a nice formula for it). The kinds of things one may want to search for in a library catalogue are "difference calculus" or "discrete calculus".

Homework

Reading Bernhard Riemann (born September 17, 1826, Breselenz, Hanover [Germany] — died July 20, 1866, Selasca, Italy) was a German mathematician whose profound and novel approaches to the study of geometry laid the mathematical foundation for Albert Einstein's theory of relativity. He also made important contributions to the theory of functions, complex analysis, and number theory.

Riemann was born into a poor Lutheran pastor's family, and all his life he was a shy and introverted person. He was fortunate to have a schoolteacher who recognized his rare mathematical ability and lent him advanced books to read, including Legendre's *Number Theory* (1830). Riemann read the book in a week and then claimed to know it by heart. He went on to study mathematics at the University of Göttingen in 1846–47 and 1849–51 and at the University of Berlin (now the Humboldt University of Berlin) in 1847–49. He then gradually worked his way up the academic profession, through a succession of poorly paid jobs, until he became a full professor in 1859 and gained, for the first time in his life, a measure of financial security. However, in 1862, shortly after his marriage to

Elise Koch, Riemann fell seriously ill with tuberculosis. Repeated trips to Italy failed to stem the progress of the disease, and he died in Italy in 1866.

Riemann's visits to Italy were important for the growth of modern mathematics there; Enrico Betti in particular took up the study of Riemannian ideas. Ill health prevented Riemann from publishing all his work, and some of his best was published only posthumously — e.g., the first edition of Riemann's *Gesammelte mathematische Werke* (1876; "Collected Mathematical Works"), edited by Richard Dedekind and Heinrich Weber.

In 1854 Riemann presented his ideas on geometry for the official postdoctoral qualification at Göttingen; the elderly Gauss was an examiner and was greatly impressed. Riemann argued that the fundamental ingredients for geometry are a space of points (called today a manifold) and a way of measuring distances along curves in the space. He argued that the space need not be ordinary Euclidean space and that it could have any dimension (he even contemplated spaces of infinite dimension). Nor is it necessary that the surface be drawn in its entirety in three-dimensional space. A few years later this inspired the Italian mathematician Eugenio Beltrami to produce just such a description of non-Euclidean geometry, the first physically plausible alternative to Euclidean geometry. Riemann's ideas went further and turned out to provide the mathematical foundation for the four-dimensional geometry of space-time in Einstein's theory of general relativity. It seems that Riemann was led to these ideas partly by his dislike of the concept of action at a distance in contemporary physics and by his wish to endow space with the ability to transmit forces such as electromagnetism and gravitation.

Adapted from *Bernhard Riemann* (by Jeremy John Gray) in the Encyclopaedia Britannica, `https://www.britannica.com/biography/Bernhard-Riemann`.

Problems

1. Explain, with sketches, the geometric meaning of the second derivative.
2. Find the second derivative of the following functions.
   a) $f(x) = x^5 - 5x + 3$
   b) $f(x) = \frac{x^2}{x-1}$
   c) $f(x) = \sqrt{x} - \sqrt[4]{x}$
3. Sketch a function satisfying the given criteria.
   a) (hint: your result should be an odd function)
      - $f'(1) = f'(-1) = 0$,
      - $f'(x) < 0$ if $|x| < 1$,
      - $f'(x) > 0$ if $1 < |x| < 2$,
      - $f'(x) = -1$ if $|x| > 2$.
   b)
      - $f'(x) < 0$,
      - $f''(x) < 0$.