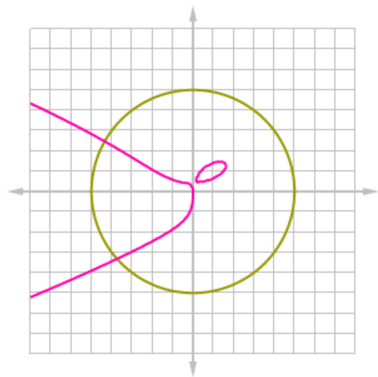
NCEA Level 3 Calculus (Differentiation)
9. Implicit Differentiation

This week, we will take a break from applications and look at some more interesting kinds of curves. Consider the following equations:

$$x^2 + y^2 = 25 \text{ and } x^3 + y^4 = 5xy - 2x$$

We can graph all the values (x, y) which satisfy these equations; here we have drawn the graph of the first equation (the circle) in green and the graph of the second (the weird disconnected one) in purple.



We can solve for the first y, as $y = \pm\sqrt{25 - x^2}$; however, the second is much harder to solve and so we cannot find its derivative using the techniques we have studied so far. These equations are examples of **implicit functions** of x. Note that neither is a 'real' function since they both fail the vertical-line test.

The key observation here is that *differentiation is an operation*, similar to addition. Just like we can add 3 to both sides of the true equation $2 + 4 = 6$ to obtain another true equation $2 + 3 + 4 = 3 + 6$, we can differentiate both sides of an equation to obtain another true equation. The only catch is that we must remember that y is a function of x and so we must employ the chain rule.

Example. If $x^2 + y^2 = 25$, by differentiating both sides with respect to x we obtain $2x + \frac{\mathrm{d}y}{\mathrm{d}x}2y = 0$ and therefore we have $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. Note that this depends on both x and y which makes sense: at $x = 0$, for example, we have two gradients (both of which are zero).

Example. If $x^3 + y^4 = 5xy - 2x$, then by differentiating both sides with respect to x we obtain $3x^2 + \frac{\mathrm{d}y}{\mathrm{d}x}4y^3 = 5y + 5x\frac{\mathrm{d}y}{\mathrm{d}x} - 2$ (being careful to use the product and chain rules in differentiating). Hence we have that the derivative is:

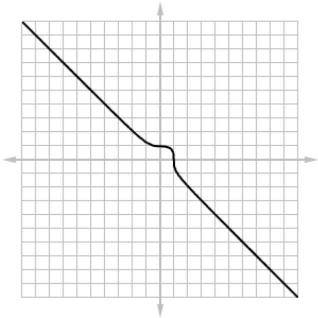
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y - 3x^2 - 2}{4y^3 - 5x}$$

Be careful to always specify which is the variable which you are differentiating with respect to.

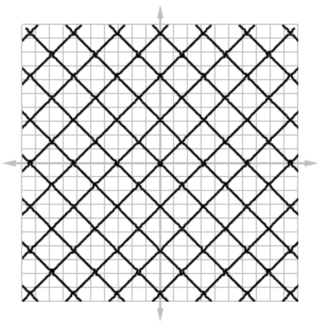
Questions

1. In each case, look at the cool pictures and find y': M

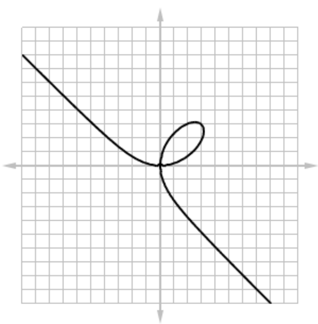
(a) $x^3 + y^3 = 1$



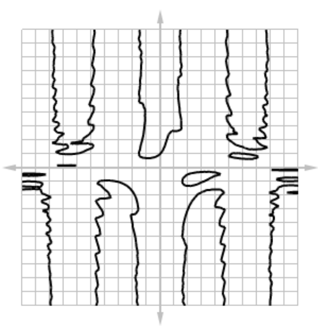
(b) $\sin^2 y + \cos^2 x = 1$



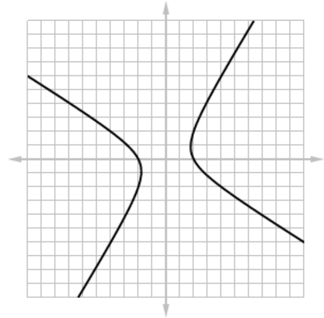
(c) $x^3 + y^3 = 6xy$ (the folium of Descartes)



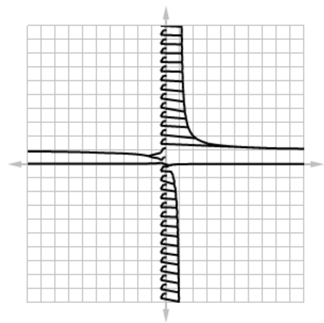
(d) $y \cos x = 1 + \sin(xy)$



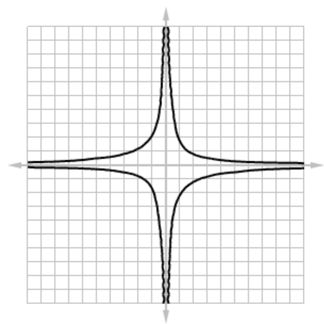
(e) $x^2 + xy - y^2 = 4$



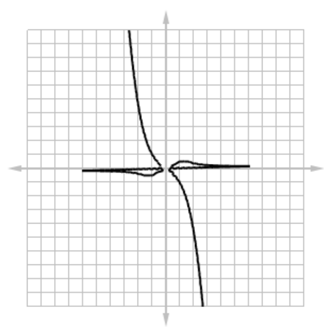
(f) $\frac{1}{x} + \frac{1}{y} = 1$



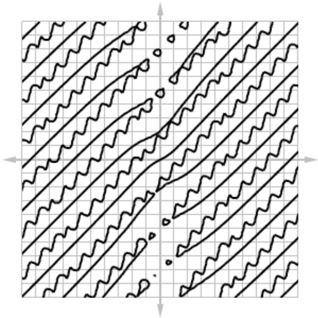
(g) $x^2y^2 + x \sin y = 4$



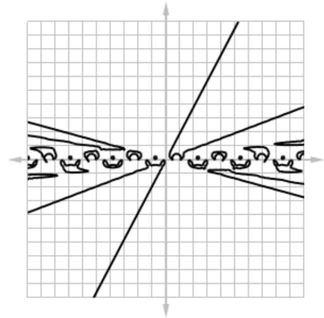
(h) $x^4y^2 - x^3y + 2xy^3 = 0$



(i) $\tan(x-y) = \frac{y}{1+x^2}$



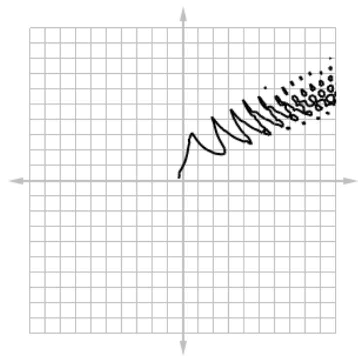
(j) $\sin\left(\frac{x}{y}\right) = \frac{1}{2}$



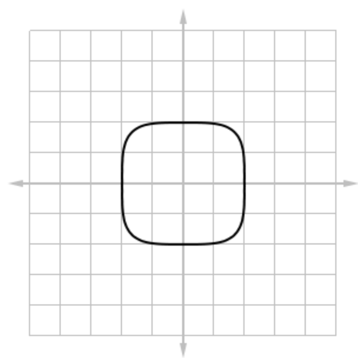
2. Consider the circle $x^2 + y^2 = 1$. Find the equation of the tangent to the curve at $(\sqrt{2}, \sqrt{2})$. M

3. The ellipse $x^2 + 3y^2 = 36$ has two tangent lines passing through the point $(12, 3)$. Find both. *This question is similar to one from the 2015 Scholarship paper.* S

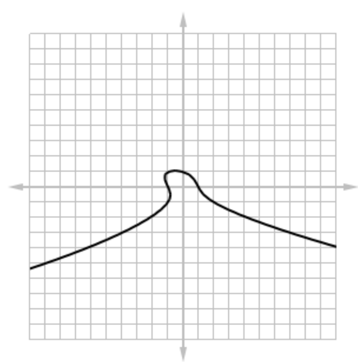
4. Find x' and y' if $\ln(y) = \sin(xy) + \frac{x}{y}$. M



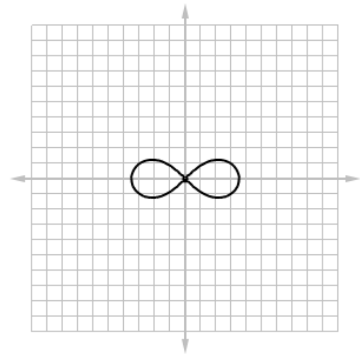
5. Find y'' if $x^4 + y^4 = 16$. M



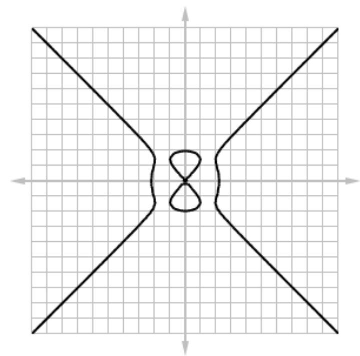
6. If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$. M



7. Find a tangent line to the curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point $(3, 1)$. This curve is known as a lemniscate. M



8. Find a tangent line to the curve $y^2(y^2 - 4) = x^2(x^2 - 5)$ at the point $(0, -2)$. This curve is known as a devil's curve. M



9. Consider the ellipse $x^2 - xy + y^2 = 3$.
   (a) Find the points where the ellipse crosses the x-axis. A
   (b) Show that the tangent lines of the curve at these points are parallel. M
   (c) Find the maximum and minimum points of the curve. E

10. Consider a circle C that is tangent to $3x + 4y - 12 = 0$ at $(0, 3)$ and contains $(2, -1)$. Set up equations that would determine the centre (h, k) and radius r of C. E

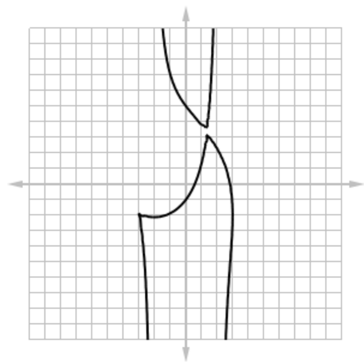
11. The Bessel function of order 0, $y = J(x)$, satisfies the differential equation S

$$xy'' + y' + xy = 2$$

for all values of x. The value of the function at 0 is $J(0) = 1$.
   (a) Find $J'(0)$.
   (b) Use implicit differentiation to find $J''(0)$.

12. Consider the following family of curves, known as Durer's shell curves (shown here for $a = 2$, $b = 3$): S

$$(x^2 + xy + ax - b^2)^2 = (b^2 - x^2)(x - y + a)^2.$$

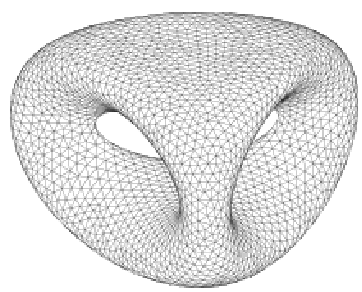


(a) For which value(s) of b does the curve become a straight line?

(b) Suppose that we restrict $a = \frac{b}{2}$. Find all non-differentiable points on the curve.

13. Moving into three dimensions, let us consider the surface described by O

$$2y(y^2 - 3x^2)(1 - z^2) + (x^2 + y^2)^2 - (9z^2 - 1)(1 - z^2) = 0.$$



(a) Verify that the point $\left(1, 1, \frac{1}{\sqrt{3}}\right)$ is on the surface.

(b) Find the values of $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$ at this point (holding y and x constant, respectively). What do these derivatives represent?

(c) Write down the equations of the tangent lines to the surface in the y and x directions.

(d) Find an equation for the unique plane containing both tangent lines. Describe what this plane represents geometrically.