NCEA Level 2 Mathematics
14. Anti-differentiation

The final mathematical topic that we will look at with regard to calculus is the inverse operation to differentiation: given a slope function (a derivative), we will find the original function.

Example. Suppose that we know that the derivative of some function is the function f' given by $f'(x) = 3x^2 + 2x + 1$. By reversing the power rule for derivatives, we can see that if we take f_0 to be $f_0(x) = x^3 + x^2 + x$ then we get the desired derivative (i.e. $f_0' = f'$). On the other hand, any function f_C such that $f_C(x) = x^3 + x^2 + x + C$ also has the same derivative.

From this example, we make two main observations:

Observation 1. If we know a differentiation rule, like $ax^r \mapsto arx^{r-1}$, then we can reverse it and say that if the derivative looks like ax^r then the original function looks like $\frac{a}{r+1}x^{r+1}$.

Observation 2. Given any derivative f', we have infinitely many functions that have f' as their derivative: if one is f, then $f + C$ for any constant C also has f' as its derivative, because

$$(f + C)' = f' + C' = f' + 0 = f'.$$

The set of antiderivatives of some function f' is denoted by

$$\int f'(x)\,\mathrm{d}x = f(x) + C.$$

The notation is unfortunate, but the best way to view it this year is as a pair of brackets: $\int$ and $\mathrm{d}x$. The mathematical operation of taking an antiderivative is called indefinite integration — if taking a function and finding its slope is splitting the function up into infinitely many pieces, then taking the antiderivative is packing all those infinitely many pieces back together into one integrated whole.

Example. Suppose that it is known that the function f

- has a derivative $f'(x) = 3x^5 + 4x^3 + 2x + 1$, and
- has a graph which passes through the point $(2, 1)$.

Then we know that all the possible candidates for f are given by

$$\int 3x^5 + 4x^3 + 2x + 1\,\mathrm{d}x = \frac{3}{6}x^6 + \frac{4}{4}x^4 + \frac{2}{2}x^2 + \frac{1}{1}x^1 + C = \frac{x^6}{2} + x^4 + x^2 + x + C.$$

We also know that $f(2) = 1$, so:

$$\begin{gathered} 1 = \frac{2^6}{2} + 2^4 + 2^2 + 2 + C \\ 1 = 2^5 + 2^4 + 2^2 + 2 + C = 32 + 16 + 4 + 2 + C = 54 + C \\ C = 1 - 54 = -53 \end{gathered}$$

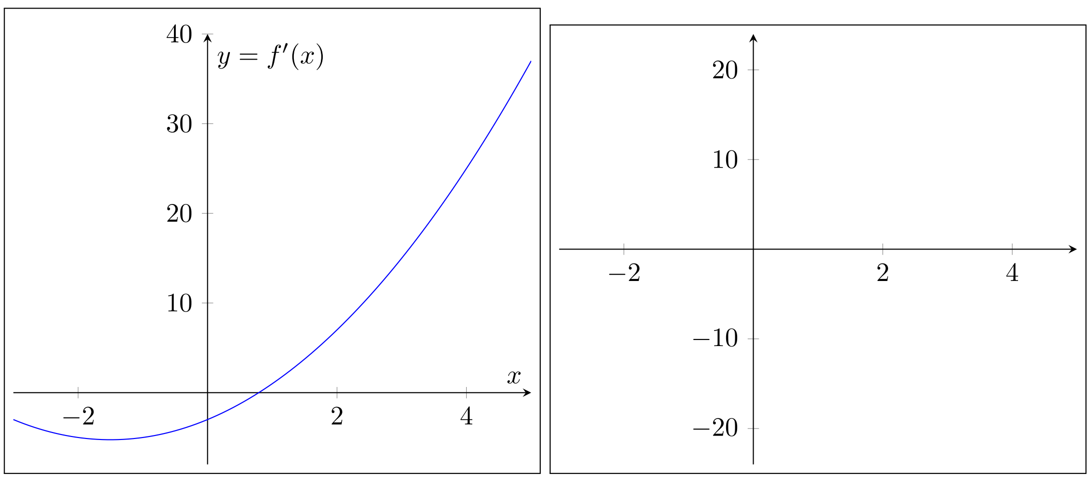
and hence $f(x) = \frac{x^6}{2} + x^4 + x^2 + x - 53$.

There is not much more to say about integration at this stage, because at Level 2 the geometric meaning of integration is no longer examinable. Suffice it to say, the operation of integration (as you will learn next year) is far deeper and more interesting than it first appears. The final problems in this week's problem set are an indication of this.

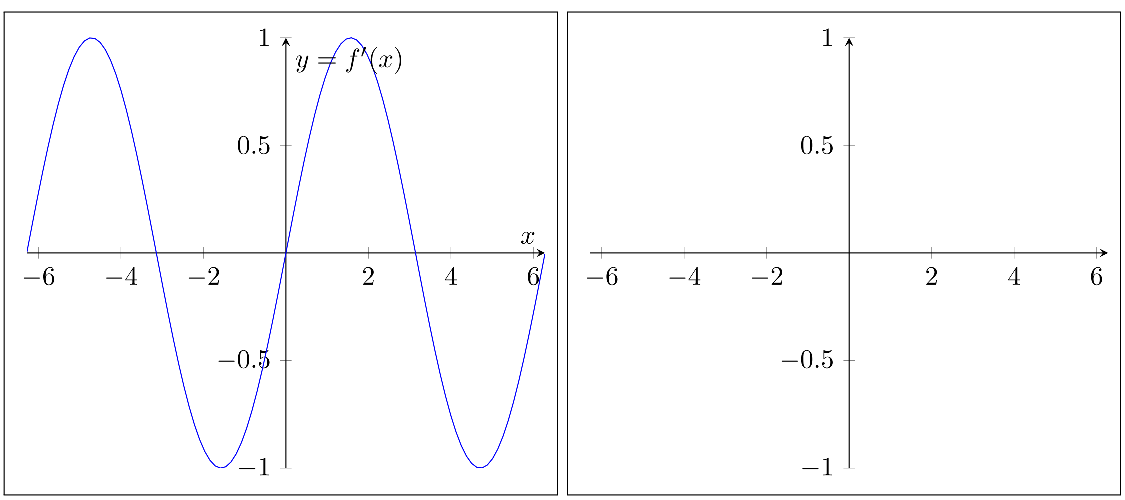
Questions

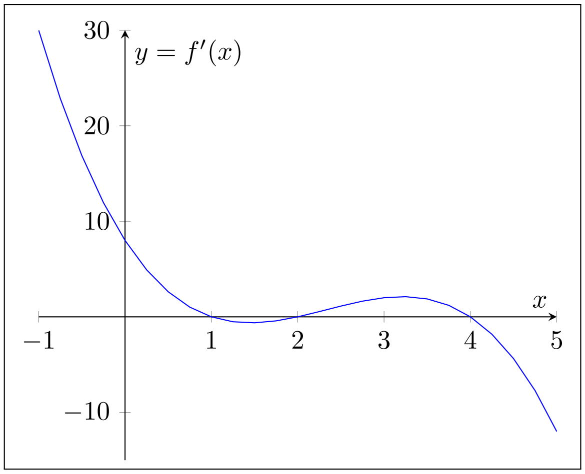
1. Find three functions that have derivatives equal to $x^2 - x$.
2. Find an antiderivative of $f(x) = 25x^4 + 12x^3 - x^{-2}$.
3. Evaluate $\int 12z^3 + 18z^{-4}\,\mathrm{d}z$.
4. Show that $x^3 + 3x + C$ is an antiderivative of $3x^2 + 3$.
5. Show that
$$\int 470x^4 + 2x + 1 + 6x^{-3}\,\mathrm{d}x = 94x^5 + x^2 + x - 3x^{-2} + C. \tag{1}$$
6. Evaluate the following indefinite integral:
$$\int -x^6 - \frac{1}{3\sqrt{x}} + \frac{x^{19}}{47} - \frac{2}{x^{-\frac{4}{5}}}\,\mathrm{d}x$$
7. Given the slope functions below, draw the antiderivative of each which passes through $(0, 0)$.

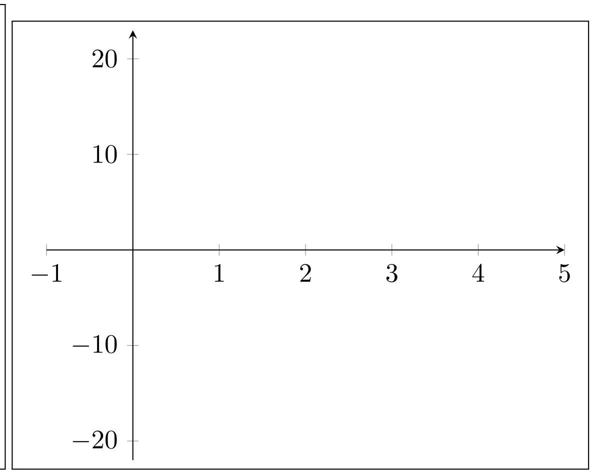
(a)

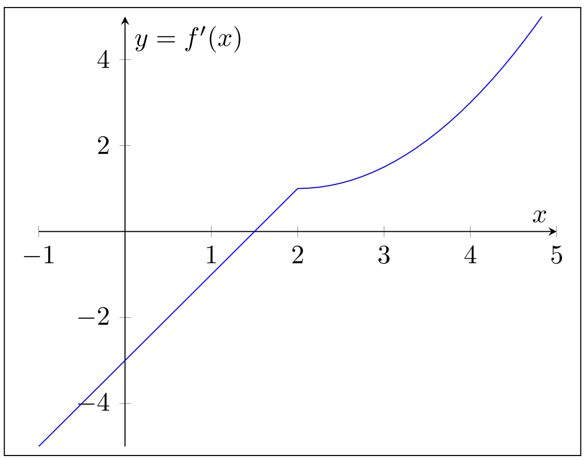


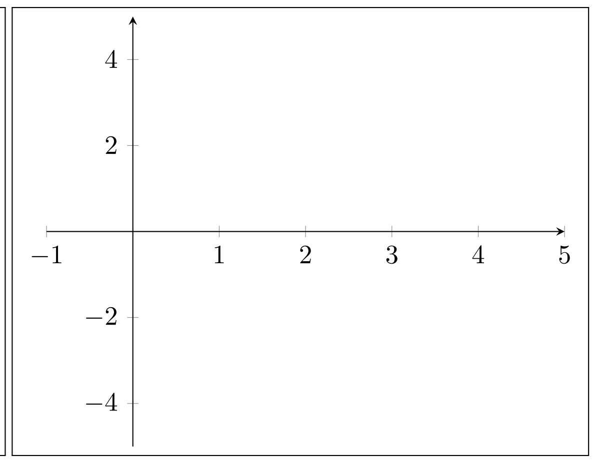
(b)



(c)




(d)




8. (a) By drawing graphs, show that it is plausible that $\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$. (You may assume for the remainder of this question that this derivative is correct.)

   (b) Find all possible functions ψ such that
$$\psi'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

   (c) Suppose that we know that $\psi(0) = -8$. Find ψ.

9. Find g if $g'(x) = x\sqrt{x}$ and $g(1) = 2$.

10. Suppose that θ is a function of x such that
$$\theta'(x) = 8x^3 + 3x^2 + ax,$$
where a is a constant. Given that $\theta(0) = 9$ and $\theta(-1) = 14$, find θ and a.

11. Find $f(x)$ if $f''(x) = -2 + 12x - 12x^2$, $f(0) = 4$, and $f'(0) = 12$.

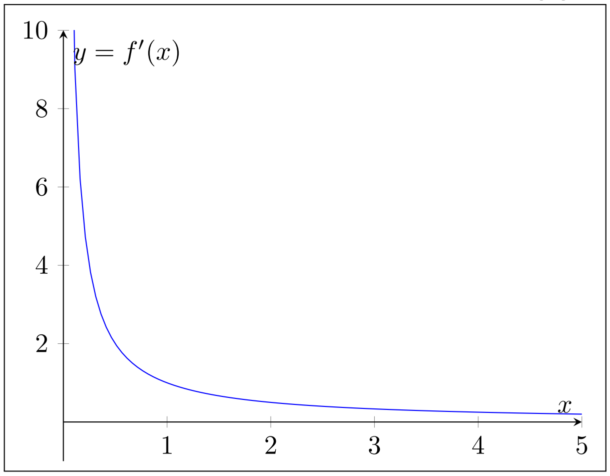
12. Suppose that f is a function of x given by $f(x) = x^3 - Ax^2 + 3x - B$, where A and B are real constants, which passes through $(0, -4)$ and has a critical point at $x = 1$. Find $f(x)$ exactly.

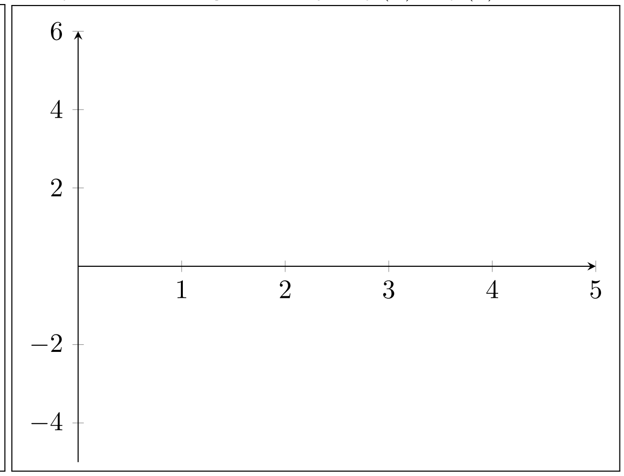
13. Suppose that y is a function of x given by
$$y = x^3 + Bx^2 + Cx + 1,$$
and that the graph of y has a minimum at $(3, 0)$.
Find B and C.

14. Every function which is made up of bits of the form ax^r added up can be differentiated using the power rule. However, the same is not true for differentiation.

   (a) Show that no such function f differentiates to give $f'(x) = \frac{1}{x}$. (Hint: try to integrate using the reverse power rule).

   (b) It is a consequence of a theorem of analysis that a function f exists so that $f'(x) = \frac{1}{x}$, even though it is not of the form above. Given the following graph of f', draw the graph of $y = f(x)$ if $f(1) = 0$.





   (c) The function which you graphed in part (b) is given the special name ln; so you have drawn the graph of $y = \ln(x)$. Let us consider the inverse function of ln, which we will call nl for the time being (because it's like ln backwards). Given the rule that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$, we can find the derivative of nl from the derivative of ln.

      i. Write $x = \ln y$, and give $\frac{\mathrm{d}x}{\mathrm{d}y}$.

      ii. Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x} = x$.

      iii. Conclude that, since $x = \ln y \iff y = \mathrm{nl}\, x$, it must be the case that $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{nl}\, x = \mathrm{nl}\, x$.

   (d) So nl is its own derivative — a situation we already looked at! Clearly nl is not the zero function; we saw that there was only one other function which was its own derivative. What was it? (Hint: check the end of sheet 11.)

   (e) If ln is the inverse of the type of function that you found nl to be, what kind of function is it?

   (f) The letters ln stand for *logarithme naturel* (French). Why do you think this is?

15. (Advertisement for Level 3!)

   (a) Draw the line $y = 2t + 1$ and use geometry to find the area under this line, above the t-axis, and between the vertical lines $t = 1$ and $t = 3$.

   (b) If $x > 1$, let $A(x)$ be the area of the region that lies under the line $y = 2t + 1$ between $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$.

   (c) Find $A'(x)$. What do you notice?