Level Three Calculus Worksheets

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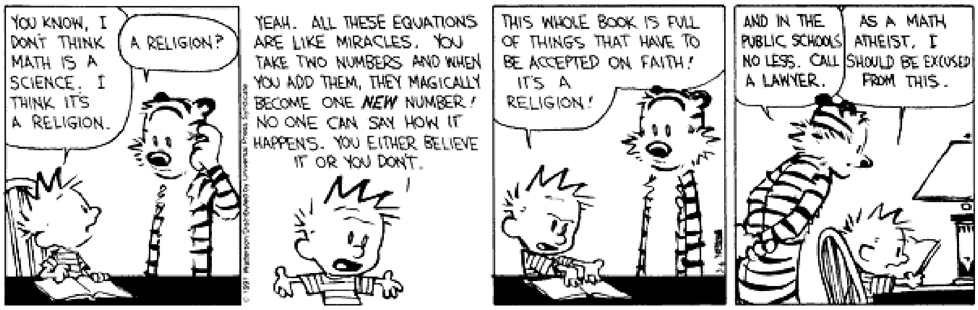
Contents

CHAPTER 1

Preamble

NCEA Level 3 Calculus
Introduction to the Notes



What is Calculus?

I will refrain from trying to advertise the subject to you, and will simply try to explain what calculus is and what kind of person uses it. Calculus is the broad study of:

- Continuous change;
- Slope, area, and volume; and
- Functions and relationships.

It has applications in physics, where calculus is the most natural language for Newtonian mechanics and classical electromagnetism; in chemistry and biology, where calculus can be used to model anything which changes over time (like rates of reaction, concentrations, and populations); in statistics (the study of probability distributions is just calculus); and in economics (I am assured). My own view, which I try to sprinkle throughout these notes, is mainly a mixture of geometric applications and physical intuition.

Within mathematics itself, calculus is the computational side of **real analysis**, the study of the properties of the real number system.

Mathematical Prerequisites

There are a number of things from Level 2 Mathematics that students should be comfortable with; generally, I assume in these notes a vague merit-level understanding of the core level 2 standards (by which I mean, the reader should be comfortable solving achieved level problems without guidance and have some idea how to approach more difficult problems):

- Level 2 Algebra: All material on quadratics (factorising, solving, discriminants), logs and exponents.
- Level 2 Calculus: Basic differentiation, geometric meaning of derivative (in particular, integration is *not* assumed)
- Level 2 Graphing: Recognising $x-/y-$shift of general functions, slope-intercept and point-slope form of linear equations, recognising period/frequency/amplitude/$x-/y-$shift from a trig function.
- Level 2 Trigonometry: Trig ratios, the Pythagorean theorem.
- Level 2 Simultaneous Equations: Solving linear and quadratic simultaneous equations.
- Level 2 Co-ordinate Geometry: Distances and linear equations.

Further into the notes, I also touch a bit on concepts covered in some of the other Level 3 standards, but not in so much detail that they need to be covered first:

- Level 3 Conics: Recognising forms from equations.

- Level 3 Algebra: Surds.
- Level 3 Trigonometry: Solving trig equations (including general solutions), reciprocal trig functions, use of trig identities (this latter mainly for E/S/OS-style integration problems).

In particular, no knowledge of complex algebra is assumed (or used) anywhere in the notes, and no knowledge from any L2 or L3 statistics standards is assumed.

I also do expect the reader to be able to draw graphs of various simple functions (e.g. parabolae, the three basic trig functions, logs and exponents, and so forth) 'by eye' — I realise that this is a little unrealistic, but at some point one must learn to become comfortable with the shapes and sizes of the objects we study. In order to help build this intuition, it is (very highly) recommended that the reader works through every example in detail, drawing pictures and so forth.

A Note on Problem Difficulty

One of the main goals for these notes is that they should be useful for students at all levels, from A to OS. Accordingly, the problems each week range from simple (most students should be able to just write down an answer without thinking too hard) to extremely non-trivial (it took **me** a while to work the problem, and I know this material quite well). If you can't do a problem, the best thing to do is to move on and come back to it — the problems don't always increase in difficulty. Of course, it is important to do a good number of problems **including some difficult ones**; you're not under exam conditions here and you're going to get an awful lot more out of a hard problem than an easy one!

I have marked the problems in the weekly worksheets (**not** the homework) with symbols relating vaguely to difficulty:

A M E S O

However, these are for my own reference and should not be taken to be accurate with respect to actual examinations.

Required content for Level 3

Some of the material goes beyond that required for NCEA Level 3; the following list gives some idea of the level of each sheet.

Differentiation

01. The Derivative
02. Limits
03. Derivatives of Common Functions
04. The Chain Rule
05. The Product and Quotient Rules
06. Tangent and Normal Lines
07. The Geometry of Functions
08. Optimisation
09. Implicit Differentiation
10. *Inverse Functions
11. Related Rates of Change

12. Parametric Functions
13. $^{*\perp}$Sequences and Series
14. †Differentiation Revision

Integration

15. Approximating Areas
16. Anti-differentiation
17. The Fundamental Theorem of Calculus
18. Substitution
19. Differential Equations
20. *Partial Fractions
21. *Integration by Parts
22. $^{*\perp}$Lengths, Volumes, and Areas
23. *Trigonometric Substitution
24. $^{\perp}$Kinematics
25. †Integration Revision
26. $^{*\perp\dagger}$☠More Interesting Problems

(*scholarship topic, ${}^{\perp}$interest topic, †revision, ☠reader discretion advised)
In particular, the author tends to follow the following rough guidelines:

Standard L3 student 1–9, 11, 12, 14, 15–19, 24, 25. No proofs; one week per sheet (so around two school terms).

Scholarship student All but 20, 23 and 26 (unless time available). Easy proofs. Leave some sheets as homework (e.g. cover the material of 3 – 5 in one week and leave a lot of problems for self study). Implicit differentiation and differential equations are revised in the conics notes, so some time can be saved there if needed. Sequences and series (13) and kinematics (24) are just revision from Y12, so can be left entirely as reading.

A Note on the Textbook

Many problems are taken from a couple of places:

- Stewart's Calculus (the current trendy textbook)
- Anton's Calculus (the old trendy textbook)
- Spivak's Calculus (the textbook you should use)
- University of Canterbury MATH199 lecture notes and problem sets
- Old NCEA/Scholarship exams

Most textbooks cover all the relevant material in the first few chapters (the first five or so in Stewart).

Homework

Every week has an associated homework sheet with a page of reading (five minutes or so) and a few questions (generally all pretty straightforward, but there is often a challenge question on there to keep you occupied).

I cannot emphasise enough how important it is to **do the homework**.

How to Read Mathematics

So we shall now explain how to read the book. The right way is to put it on your desk in the day, below your pillow at night, devoting yourself to the reading, and solving the exercises till you know it by heart. Unfortunately, I suspect the reader is looking for advice on how not to read, i.e. what to skip, and even better, how to read only some isolated highlights.

- Saharon Shelah, ‘Classification Theory and the Number of Non-Isomorphic Models’

A major part of Level 3 Mathematics is preparation for university-level study in pure mathematics or the hard or soft sciences. As such, this year we begin to expose you to some ‘real mathematics’ — not just the watered-down calculation and computation you’ve been doing since intermediate, but a real journey of discovery through one of humanity’s great fields of human experience. I guarantee that at primary school you enjoyed this kind of mathematics — for example, consider the non-obvious fact that $3 \times 5 = 5 \times 3$. Next year you will learn that there are some objects which do not have this property (commutativity).

There are a lot of definitions and theorems, but they allow us to capture our intuition of mathematical objects, and their behaviour in a more precise (and more correct) sense. It is important to read slowly and understand the statements as you go, as every piece strengthens the whole. I do not tend to repeat myself, so often you will find statements from earlier used later without comment.

I have also included proofs of a few of the theorems that we use, but rigorous proofs of many seemingly obvious ideas (like the fact that every function reaches a minimum and a maximum value on any closed interval) require some subtle properties of the real numbers and so it is more important to gain a geometric and intuitive feeling for many of the results rather than worrying too much.

Recommended reading for the interested: Paul Lockhart’s ‘A Mathematician’s Lament’.

NCEA Level 3 Calculus
Revision: Functions

Before we look at calculus proper, we need to revise a few things from our previous studies. Arguably, the most fundamental concept from L2 is that of a function, together with its graph.

Definition (Function)**.** A **function** is a relationship between two sets of things, called the *range* and the *domain*, such that everything in the range is related to exactly one thing in the domain. If f is a function which maps the value x to the value y, we write $f : x \mapsto y$, or $f(x) = y$. If f has range X and domain Y, we write $f : X \to Y$.

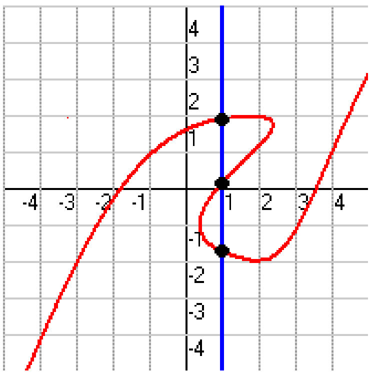
Two functions f and g are called equal *if and only if* the ranges of the two functions are the same *and* for every x in that range, $f(x) = g(x)$.

You can think of a function as a rule: it could be given by a formula, or by a list of inputs and outputs, or in any other way that one wants.

Definition (Graph)**.** If f is a function, then the set of all points (x, y) such that $y = f(x)$ is called the **graph** of the function.

Examples.

1. The map which takes a number x and spits out x^2 is a function — for every input, there is exactly one output. If we plot every point in the graph of this function, by plotting each input on the $x-$axis and the corresponding output on the $y-$axis, we obtain a parabola.
2. The curve graphed below is *not* the graph of a function, since for some inputs (like 1) the map has more than one output. We can check this by drawing vertical lines along the function, like that pictured: if a graph is a function, no vertical line can ever cross the curve more than once (this is the *vertical-line test*).

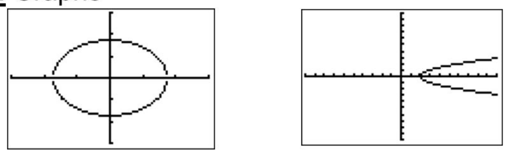


3. The map $f : x \mapsto \sin x$ is a function (and so are all the other triangle ratios). We could also define it by 'the function f such that $f(x) = \sin x$'. This function f can only produce numbers between 1 and -1; we say that its range is the interval from -1 to 1.
4. On the other hand, to ensure that we obtain functions the inverse trigonometric maps must be restricted to certain inputs: to take a particular example, since there are infinitely many x so that $\sin x = 1$, there are infinitely many possibilities for the value of $\arcsin x$. We will refrain from picking an explicit range for the inverse functions here and will generally just pick the most convenient at the time: it should be reasonably obvious from context.
5. The map $\iota : x \mapsto x$ is a function, called the *identity function*.
6. The map $\ln x$ is a function, but it is only defined when $x > 0$: we say that its *domain* is the positive real numbers.

7. The following are some more non-examples of functions.

Non – Examples of a Function

#1: Graphs



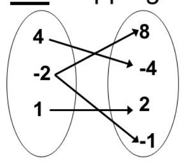
#2: Table

x	-1	2	1	0	-1
y	-5	3	2	-1	4

#3: Set

{(-1,2), (1,3), (-3, -1), (1, 4), (-4, -2), (2, 0)}

#4: Mapping



Revision Questions

1. Which of the following are functions?
   (a) $E(x) = 2^x$
   (b) $\phi : x \mapsto \frac{2}{x}$
   (c) The thing which maps every person to their youngest sibling.
   (d) The thing which sends every person to their youngest sibling that isn't themself.
   (e) $x \mapsto \lfloor x \rfloor$ (where $\lfloor x \rfloor$ is the largest integer less than or equal to x).
   (f) The relation that sends every person to their age.
2. I will define two functions, φ and ϑ, as follows:
   $$\varphi(x) = 2x - 7, \qquad \vartheta(\zeta) = \frac{1}{7}(14\zeta - 49).$$
   Explain why these functions are equal.
3. If $f(x) = x^2 + x$, find:
   (a) $f(1)$
   (b) $f(y)$
   (c) $f(x + h)$
4. Find the distance between $(-3, 4)$ and $(2, 1)$.
5. Three sides of a triangle are have lengths 8, 15, and 17.
   (a) Show that the triangle is right-angled.
   (b) Find the other two angles.
6. Factorise and solve $x^2 - 3x + 2 = 0$.
7. How many **lines** are there through the point $(2, 3)$ and the origin? Give the equations of all such lines.
8. Find the slope of the line $4x + 3y + 2 = 0$.

9. Find the solution to the following linear system:

$$\begin{aligned} 2x + y &= 7 \\ 3x - y &= 8 \end{aligned}$$

10. How many (real) solutions does $x^2 + 4x + 1$ have?
11. Draw $\sin(x)$, $\cos(x)$, $\tan(x)$, $\exp(x)$, and $\ln(x)$.
12. How many solutions does $\cos(3\pi x + 1) = 2$ have?
13. How many solutions does $\sin(3x) = \frac{1}{3}$ have?

CHAPTER 2

Differentiation

NCEA Level 3 Calculus (Differentiation)
1. The Derivative

Goal for this week

To remember why we care about derivatives, and remind ourselves of how slopes should behave.

Last year we met the derivative. Let us recall that the derivative of a function is, in some sense:

- A way to find the slope of another function.
- A function giving a rate of change.
- A way to find maxima and minima of a function.

The value of the derivative of a function at a point is the gradient/slope/rate of change of the function at that point. Over the next few sections, we will make this rather vague notion precise, allowing us to transform geometric questions about slope, curviness and speed into questions that can be solved via algebraic manipulation.

If f is a function, then the derivative is a function that takes some point x and gives us the slope of f at x. For this function, we write f' (read *eff prime*), so the slope of f at x is denoted by $f'(x)$. If we want to be explicit about the names of the quantities we are relating, then if $y = f(x)$ we write $f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ (read *dee y over dee x*); in this notation, the slope of the graph at $x = x_0$ is written as $\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{x=x_0}$ (but I tend to avoid this latter notation). In physics, especially in mechanics, the derivative of x with respect to time is often written like $\dot{x}$.

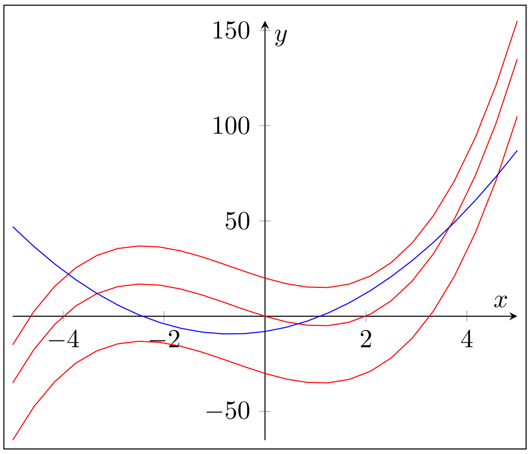
If a function does not have a well-defined derivative at some point, it is said to be non-differentiable at that point.

Technical Remark. Note that this isn't a proper definition for the derivative: all we've done is say that the derivative 'is the slope' of a function. However, we only have a definition of *slope* for linear functions. Our solution to this will be, next week, a definition of derivatives that doesn't mention slope explicitly but generalises the definition of slope. ♢

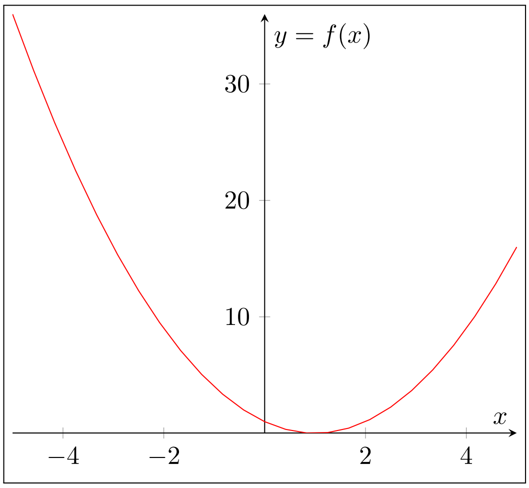
Example. Consider a function defined by $y = 2x + 1$. At every point, the slope of the graph of this function is 2; so the value of the derivative of this function is 2 at every point. (Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$.)

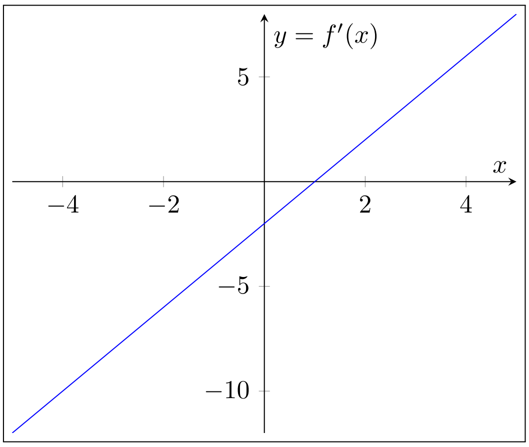
Example. The speed of a particle at any time is the rate of change of the displacement of the particle at that time; this can also be viewed as the slope of the displacement-time graph of the particle. Hence the speed of the particle is the derivative of the displacement. The derivative of speed is called the acceleration of a particle.

Example. In the diagram, the derivative of the red functions is shown as the blue function. Note that the derivative of the function depends only upon the shape, not the y-shift. Also, see that the derivative is positive as the function increases and is negative when the function is decreasing. The derivative is zero exactly where the function is 'flat'.



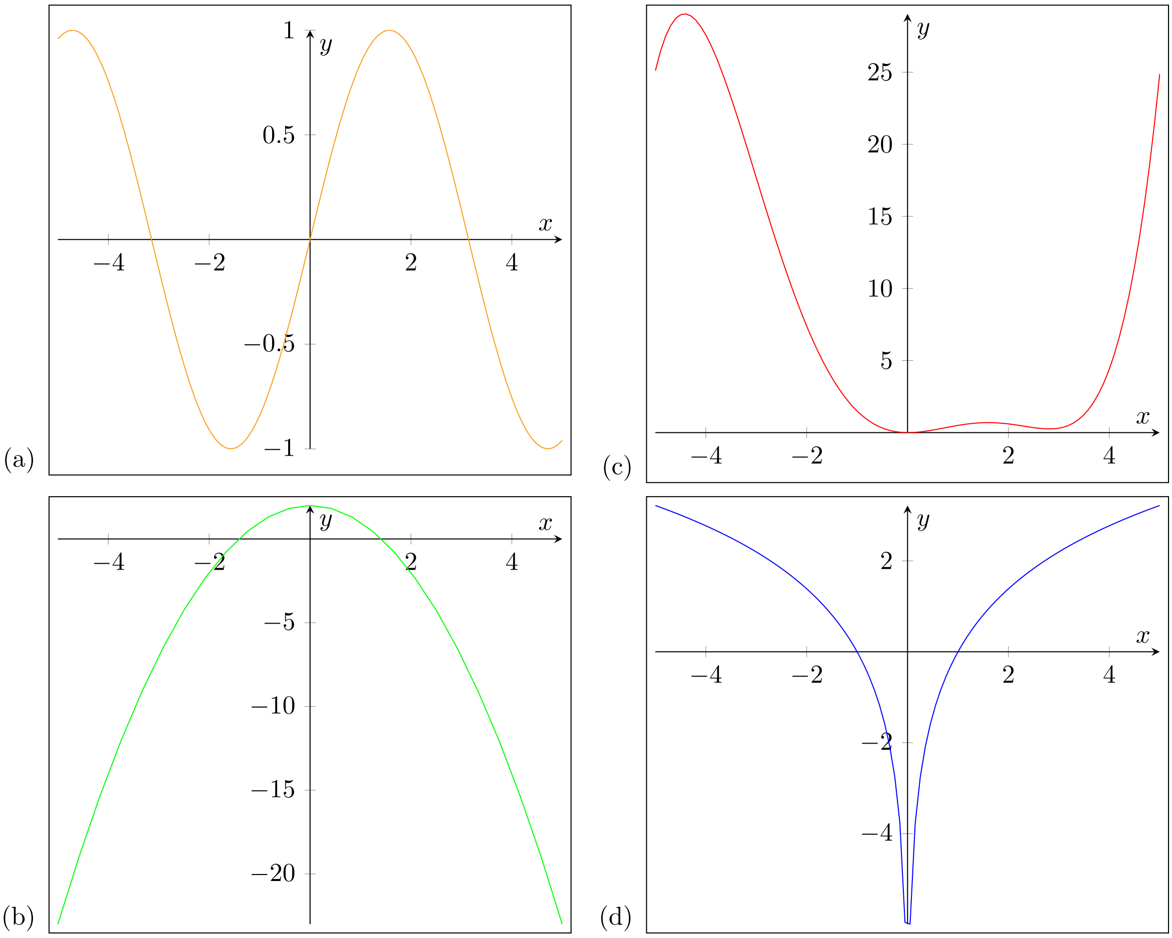
Example. The two diagrams below show the graphs of the function $f(x) = (x-1)^2$ and its derivative f'.





Questions

1. Why does the derivative not depend on the y-shift of a function?
2. Let f be a function. Describe the difference between f, f', $f(57)$, and $f'(57)$.
3. Draw the graph of the derivative of each graphed function.



4. Describe several ways in which a function f can fail to be differentiable at a point x, illustrating your examples with sketches.
5. Consider each of these functions in turn. Where is the derivative of each (i) negative, (ii) positive, (iii) zero, and (iv) undefined?
   (a) $x \mapsto x^2$
   (b) $x \mapsto \sin x$
   (c) $x \mapsto \tan x$
6. Describe the derivative of the function $x \mapsto \tan^{-1} x$.
7. Write down the slope of $y = mx + c$. Hence give $\frac{\mathrm{d}y}{\mathrm{d}x}$.
8. If f' is the derivative of some function f, describe the meaning of $(f')'$ in terms of rate of change (to save space, from now on we will write f'' for this second derivative).

9. If a function is periodic, with a period of T, what can you say about its derivative? M

10. Let f be a function. Suppose that it is known that $f'(3) = 9$, and $f(3) = 6$. A
    (a) What does the graph of $y = f(x)$ look like around $x = 3$?
    (b) Give the equation of the tangent line to $f(x)$ at $x = 3$.

11. The function floor maps any number x to the greatest integer less than or equal to x (so $\text{floor}(\pi) = 3$, for example). Draw a graph of $y = \text{floor}(x)$. Where is floor differentiable, and what is the derivative of floor at these points? M

12. When a hot water tap is turned on, the temperature T of the water depends on how long the water has been running. M
    (a) Sketch a possible graph of T as a function of the time t that the tap has been running.
    (b) Describe how the rate of change of T with respect to t varies as t increases.
    (c) Sketch a graph of the derivative of T with respect to t.

13. The rate of change of a population at a time t is directly proportional to the population $P(t)$ at that time, such that $\frac{\mathrm{d}P}{\mathrm{d}t} = P$. Draw a graph of the population over time if $P(0) \approx 1000$. M

14. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$. M
    (a) What is the meaning of the derivative $f'(5)$?
    (b) Suppose that there is an unlimited amount of space and nutrients. Which would you expect to be larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited does your answer change?

15. Consider an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is not a function: since both $(0, b)$ and $(0, -b)$ are members of the function, it fails the vertical line test. However, it would be nice to reason about its rate of change *as if it were* a function. Describe the slope of the ellipse as a particle traces the curve in an anticlockwise direction at a constant rate. M

NCEA Level 3 Calculus (Differentiation)
1. The Derivative (Homework)

Reading

Considering how many fools can calculate, it is surprising that it should be thought a difficult or a tedious task for any other fool to learn how to master the same tricks.

Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the text-books of advanced mathematics — and they are mostly clever fools — seldom take the time to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way.

Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fool the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can.

The preliminary terror, which chokes off most high school students from even attempting to learn how to calculate, can be abolished once and for all by simply stating what is the meaning — in common-sense terms — of the two principal symbols that are used in calculating.

These dreadful symbols are:

(1) d, which merely means "a little bit of".

Thus $\mathrm{d}x$ meas a little bit of x; or $\mathrm{d}u$ means a little bit of u. Ordinary mathematicians think it more polite to say "an element of", instead of "a little bit of". Just as you please. But you will find that these little bits (or elements) may be considered to be infinitesimally small.

(2) $\int$, which is merely a long S, and may be called (if you like) "the sum of".

Thus $\int \mathrm{d}x$ means the sum of all the little bits of x; or $\int \mathrm{d}t$ means the sum of all the little bits of t. Ordinary mathematicians call this symbol "the integral of". Now any fool can see that if x is considered to be made up of a lot of little bits, each of which is called $\mathrm{d}x$, if you add them all up together you get the sum of all the $\mathrm{d}x$'s (which is simply the same thing as the whole of x). The word "integral" simply means "the whole". If you think of the duration of time for one hour, you may (if you like) think of it cut up into 3600 little bits called seconds. The whole of the 3600 little bits added up together make one hour.

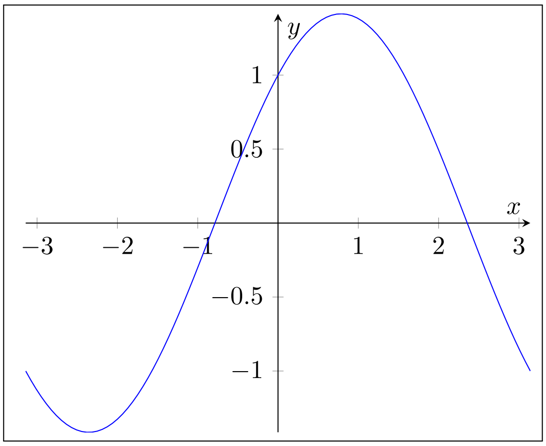
When you see an expression that begins with this terrifying symbol, you will henceforth know that it is put there merely to give you instructions that you are now to perform the operation (if you can) of totalling up all the little bits that are indicated by the symbols that follow.

That's all.

From *Calculus Made Easy*, by Silvanus P. Thompson and revised by Martin Gardner.

Questions

1. Given that the following graph is the derivative of a function passing through $(0, 0)$, draw the original function.



2. (a) At a maxima or minima, what value m does the derivative of a function take? Explain this geometrically.

   (b) Show, by example (i.e. draw a graph), that there exist functions such that they have a derivative equal to m at a point but not a minimum or maximum at that point.

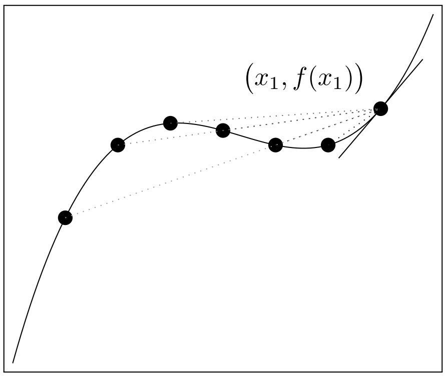
NCEA Level 3 Calculus (Differentiation)
2. Limits

Goal for this week

To use limits to give a proper definition for the derivative.

Last week we looked at the derivative from the point of view of a 'slope function', and remarked that this definition is not really useful at all because it is dependent on some notion of 'slope' for functions which aren't lines.

Consider, for example, the graph of a function f. A good definition of 'slope' for the graph at the point $(x_1, f(x_1))$ is the slope of the line at that point which 'just touches' the curve at that point — we can obtain this line by drawing another point (x_0, y_0) on the curve and joining it to $(x_1, f(x_1))$ by a line and then slowly moving x_0 towards x_1, redrawing the line at each point until x_0 and x_1 become the same. This process is illustrated below, where the dotted lines approach the solid tangent line as the point on the curve approaches the place we want to take the tangent.



The slope of the line at each point is simply

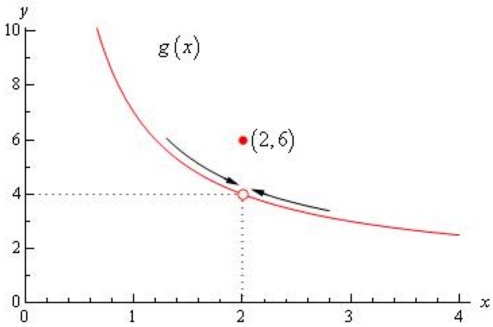
$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

and so the slope of the tangent line at x_0 will be obtained when we set Δx to zero. Unfortunately, we can't do this easily because it would involve a division by zero and so we have to make use of the concept of a limit.

Suppose we consider some function f, and we notice that whenever its input approaches some value x_0 (from either direction) its output also approaches some value y_0, and that we can get its output as close as we like to y_0 by making the input appropriately close to x_0. Then we say that the **limit** of f as its input approaches x_0 is y_0, or (symbolically)

$$\lim_{x \to x_0} f(x) = y_0.$$

Consider the following function:



Although the *value* of the function at 2 is 6, the *limit* of the function at 2 is $\lim_{x\to 2} g(x) = 4$. When taking a limit we don't care what the function does at the point — only what it looks like it *should* do based on the points around it. The limit of a function at a point is a property of the area around the point **and not a property of the point itself**.

You can also think of $\lim_{x\to x_0} f(x)$ as being the unique value that we could pick for $f(x_0)$ such that the function around that point has 'no gaps'. If there is no such unique value, there is no limit at the point x_0.

Limits happen to have a few simple properties.

Theorem. *If f and g are functions and the limits of f and g at x_0 exist, then:*

1. $\lambda \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} [\lambda f(x)]$ *(where λ is a constant);*
2. $\lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x) = \lim\limits_{x\to x_0} [f(x) + g(x)]$*;*
3. $\left(\lim\limits_{x\to x_0} f(x)\right)\left(\lim\limits_{x\to x_0} g(x)\right) = \lim\limits_{x\to x_0} [f(x)g(x)]$*;*
4. $\dfrac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$ *(if $g(x) \neq 0$ around the point we take the limit); and*
5. $f(\lim\limits_{x\to x_0} g(x)) = \lim\limits_{x\to x_0} [f(g(x))]$ *(if f is continuous: that is, at every point $\lim_{x\to x_0} f(x) = f(x_0)$).*

Examples. Using these limit laws, we can find some limits reasonably easily.

1. $\lim_{x\to 0} \frac{x}{x} = 1$ since as x gets closer and closer to 0, $\frac{x}{x} = 1$.
2. $\lim_{x\to 3} \frac{(x-2)(x-3)}{x-3} = 1$ since as x gets closer and closer to 3, the fraction gets arbitrarily close to 1.
3. $\lim_{x\to 0} \frac{1}{x}$ does not exist, since if we approach 0 from the left the function becomes arbitrarily negative and if we approach 0 from the right the function becomes arbitrarily positive — we do not approach the same value on both sides.
4. $\lim_{x\to\infty} \frac{1}{x} = 0$ since as x becomes arbitrarily large, $\frac{1}{x}$ becomes arbitrarily small.
5. $\lim_{x\to 0} \sqrt{x}$ does not exist, since $\sqrt{x}$ is undefined for $x < 0$.

With this new notation, we hereby define the **derivative** of f at x_0 to be the function f' described by

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

or equivalently

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Now that we have a proper definition, we can begin to calculate some derivatives.

Example. We will find the derivative of $f(x) = x^3$ at the point x using the definition.

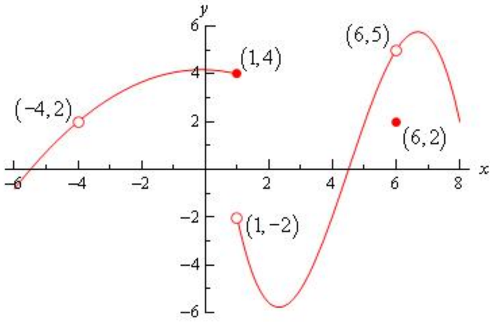
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\
&= 3x^2.
\end{aligned}$$

Questions

1. Guess the value of the following limit by evaluating the limuend for each x in $\{\pm 1, \pm 0.5, \pm 0.2, \pm 0.1, \pm 0.05, \pm 0.01\}$: A

$$\lim_{x \to 0} \frac{\sin x}{x + \tan x}$$

2. Consider the function f graphed below. A



   (a) For each of the following expressions, either give the value or explain why the expression is undefined.
      i. $f(-4)$
      ii. $\lim_{x \to -4} f(x)$
      iii. $f(1)$
      iv. $\lim_{x \to 1} f(x)$
   (b) Explain why the limit $\lim_{x \to 6} f(x)$ is not equal to $f(6)$.
   (c) At which points is f:
      i. Discontinuous?
      ii. Non-differentiable?

3. Evaluate the limit or explain why it does not exist: M

   (a) $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$
   (b) $\lim_{x \to 0} \frac{1}{x^3}$
   (c) $\lim_{x \to 9} \frac{1}{x^3}$
   (d) $\lim_{h \to 0} \frac{(2+h)^3-8}{h}$
   (e) $\lim_{x \to 4} \frac{x^2+5x+4}{x^2+3x-4}$
   (f) $\lim_{x \to \frac{\pi}{2}} \sin x$
   (g) $\lim_{x \to \infty} \sin x$
   (h) $\lim_{x \to \frac{\pi}{2}} \tan x$
   (i) $\lim_{x \to 0} \tan x$
   (j) $\lim_{x \to 0} \csc x$
   (k) $\lim_{x \to a} C$, where a and C are constants.
   (l) $\lim_{x \to -\infty} \tan^{-1} x$
   (m) $\lim_{y \to 0} \lim_{x \to 0} \frac{(x+y)(x-y)}{x^2-y^2}$
   (n) $\lim_{x \to \infty} 1/x$.
   (o) $\lim_{x \to \infty} \frac{2x}{x^2+1}$.
   (p) $\lim_{x \to \infty} \frac{x+2}{x-3}$.

4. Consider the function φ defined by M

$$\varphi(x) = 1 / \frac{1}{x - x}.$$

   Explain why neither $\varphi(\alpha)$ nor $\lim_{x \to \alpha} \varphi(x)$ exists for any real α.

5. Find the derivative of $x^2 + x$ from first principles. M

6. Find the derivative of $\sin x$ from first principles, given that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$. E

7. Show that $f(x) = |x-6|$ is not differentiable at $x = 6$. Find a formula for f'. E

8. Show that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ and $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ are equivalent definitions for the derivative at the point a of some function f. E

9. If $\lim_{x\to a}[f(x)+g(x)] = 2$ and $\lim_{x\to a}[f(x)-g(x)] = 1$, find $\lim_{x\to a} f(x)g(x)$. E

10. Last year, we defined the *exponent* a^r as follows: S

- If $r = 0$, then $a^r = 1$.
- If r is a natural number, then $a^r = a^{r-1} \cdot r$. (So $a^r = \underbrace{a \times \cdots \times a}_{r \text{ times}}$.)
- If r is a negative integer, then $a^r = \dfrac{1}{a^{-r}}$. (Note that $-r$ is positive.)
- If r is a rational number, so that $r = p/q$ in lowest form, then $a^r = a^{(p/q)} = \sqrt[q]{a^p}$ (where we take the positive root, if a choice needs to be made).

Give a reasonable definition for a^r where r is any real number. Use your definition to compute a reasonable approximation to 2^π (given that $\pi \approx 3.14159...$).

NCEA Level 3 Calculus (Differentiation)
2. Limits (Homework)

Reading

You may be wondering why we bother introducing the concept of limits: after all, we are simply replacing one handwavy picture-based definition (that of the derivative) with another! I will give the answer in two parts:

1. Limits are a more general and hence more useful concept; and
2. It is much easier to formally define a limit than a derivative.

Limits are more general

The obvious use of limit notation this year is to 'plug gaps' in functions; however, we can also (as you have seen) take limits of things towards infinity. This allows us to formalise things like infinite sums: we define the value of an infinite sum to be a special kind of limit.

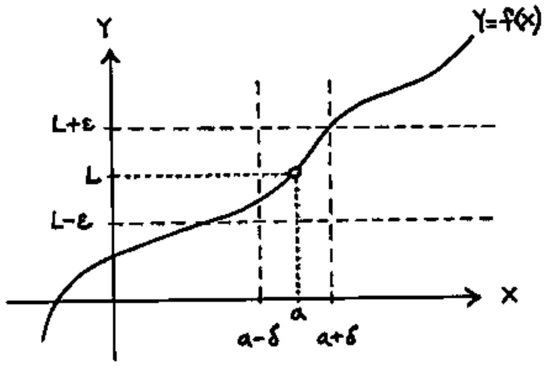
Limiting situations come up surprisingly often in physics and chemistry as well, if we want to look at the behaviour of a system in the long term: say the concentration of a particular compound in solution can be modelled by $C(t) = \frac{k}{t^2}$; then, if we wait a long time (i.e. let $t \to \infty$), we predict that the concentration becomes negligible.

It is easier to formally define a limit

Suppose that we have some function f such that

$$\lim_{x \to a} f(x) = L.$$

All we are saying here is (intuitively) that we can make the value of f *as close as we like* to being L, by taking x to be sufficiently close to a. I will not state the formal definition here (it is easy enough to find), except to state that it is a little stricter than this intuitive statement suggests (i.e. for all $x \neq a$ within the interval $(a - \delta, a + \delta)$ we must have $f(x)$ be within $(L - \varepsilon, L + \varepsilon)$).



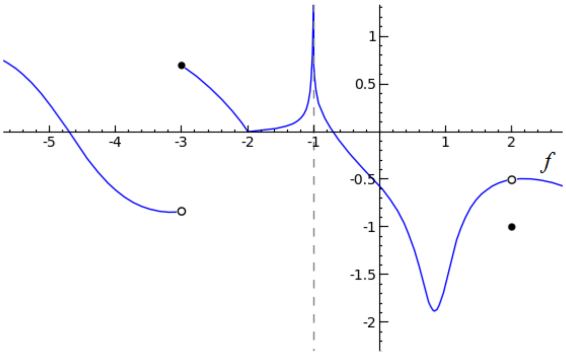
Questions

Derivatives and limits allow us to classify functions and their behaviour. Consider the following:

Properties of Functions

- A function is **increasing** if its derivative is positive.
- A function is **decreasing** if its derivative is negative.
- A function is **concave down** if its derivative is decreasing.
- A function is **concave up** if its derivative is increasing.
- A function f is **continuous** at a point a if $\lim_{x \to a} f(x) = f(a)$.

1. Describe all the function properties given above geometrically, and give an example of each.
2. Consider the function graphed below.



   (a) Find $\lim_{x \to -2} f(x)$ and $\lim_{x \to 2} f(x)$.
   (b) Does $\lim_{x \to -3} f(x)$ exist? Why/why not?
   (c) Does $\lim_{x \to 0} f(x)$ exist? Why/why not?
   (d) On what intervals is $f(x)$ continuous?
   (e) At what points is $f(x)$ not differentiable?

3. On an axis, sketch a graph of some function f that has the following features:
   - Is continuous for $0 < x < 5$ and $5 < x < 9$ and is discontinuous when $x = 5$
   - Is concave down ($f''(x) < 0$) for $0 < x < 5$
   - Has $f'(x) = 0$ at $(3, 8)$
   - Has $\lim_{x \to 5} f(x) = 6$.
   - Is not differentiable at $(7, 3)$.

NCEA Level 3 Calculus (Differentiation)
3. Derivatives of Common Functions

Goal for this week

To begin to actually calculate some simple derivatives.

Now that we understand the purpose and form of the derivative, we can begin to calculate the derivatives of some common functions. We list them here without proof, but it is easy to find the proofs yourself in a textbook or on the internet if you are interested.

Function	Derivative	Notes
x^n	nx^{n-1}	
e^x	e^x	Here, $e \approx 2.71828$ is Euler's number.
$\ln\|x\|$	$\frac{1}{x}$	Here, $\ln = \log_e$.
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\csc x$	$-\csc x \cot x$	
$\sec x$	$\sec x \tan x$	
$\cot x$	$-\csc^2 x$	

We also have several rules for finding the derivative of a more complicated function in terms of the derivatives of component functions.

Theorem. *Let f and g be functions, and let λ be a real constant. Then:*

1. $(\lambda)' = 0$*: The slope of a constant function is zero.*
2. $(\lambda f)' = \lambda f'$*: The slope of a scaled function is the scaled slope of the function.*
3. $(f+g)' = f' + g'$*: The slope of the sum of two functions is the sum of the individual slopes.*

These formulae can all be proved using the limit definition of the derivative, and it is a simple exercise to check them all. For example,

$$\begin{aligned}(f+g)'(x) = \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h} &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x).\end{aligned}$$

Note that the obvious product rule, $(fg)' = f'g'$ does **not** hold. We will discuss this further soon, although it should be noted that Leibniz initially believed this rule to be true! A counterexample is outlined in the exercises for you to work out.

Examples.

1. $\frac{\mathrm{d}}{\mathrm{d}x}[\sin x + \cos x] = \cos x - \sin x$.

2. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{3x^2+2x+1}{x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[3x+2+x^{-1}\right] = 3+0+(-1)x^{-2} = 3-\frac{1}{x^2}$.
3. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{\mathrm{d}}{\mathrm{d}x}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

Application. Many phenomena in physics can be modelled with sine waves; for example, if a particle on the end of a spring is moving with simple harmonic motion, then it has position $x = A\sin(\omega t+\phi)$; taking derivatives, we find that it has velocity $v = \frac{\mathrm{d}x}{\mathrm{d}t} = A\omega\cos(\omega t+\phi)$ and acceleration $a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -A\omega^2\sin(\omega t+\phi)$. In other words, it is always accelerating in the opposite direction to its movement!

Questions

1. Find the derivatives of $3x^3$, $2x^2$, and $6x^5$. Conclude that $(fg)' \neq f'g'$ in general. A

2. Find the derivatives of the following functions with respect to t: A
   (a) $y = 2t^3 + 3t^2$
   (b) $y = \sqrt{t}$
   (c) $y = (2t+1)(t-4)$
   (d) $g(t) = 4\sec t + 9\tan t$
   (e) $h(t) = \sqrt[5]{t} + 2\csc t - \ln t^3$
   (f) $\phi'(t) = \csc x + 12x^{1273} + 9$
   (g) $y = 2017t^{2016} + (t+2)^2$
   (h) $y = 940\sin t + \frac{1}{2}e^{t+2}$

3. Where is the function $x \mapsto x^3 - 2x^2 - x + 1$ increasing? A

4. Find the velocity v of a particle at time $t = 2\pi$ if its position function for $t > 0$ is $x = e^t - \sin t$. A

5. Find the slope of the tangent line to $y = x + \tan x$ at (π, π). A

6. It is **not** true that the derivative of $f(g(x))$ is $f'(g'(x))$. A
   (a) For a counterexample, consider $f(x) = x^2$ and $g(x) = x$; show that $f'(g'(x)) = 2$, but $\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = 2x$.
   (b) Compute the derivative of $\ln x^2$.

7. Suppose the derivative of a function is $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - x - 4$. What could the original function be? A

8. Find the 64th derivative of $\sin x$. M

9. Find the nth derivative of x^n. M

10. If $y = 2\sin 3x \cos 2x$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. (Hint: use an identity to rewrite this as a sum of functions.) M

11. For which values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent? M

12. Show that $y = 6x^3 + 5x - 3$ has no tangent line with a slope of 4. E

13. Find real values of α and β such that, if $y = \alpha \sin x + \beta \cos x$, then $y'' + y' - 2y = \sin x$. E

14. Consider a 12 m long ladder leaning against a wall such that the top of the ladder makes an angle θ with the wall. If this angle θ is varied, the distance D between the bottom of the ladder and the wall also changes. If $\theta = \pi/3$, what is the rate of change of D with respect to θ? E

15. Prove that the function φ given by $\varphi(x) = \frac{x^{101}}{101} + \frac{x^{51}}{51} + x + 1$ has no extreme values. E

16. The derivative is primarily a geometric concept, not an algebraic one. M
   (a) The area of a circle of radius r is $A = \pi r^2$. Find $\frac{\mathrm{d}A}{\mathrm{d}r}$. What do you notice?
   (b) Explain part (a) geometrically.
   (c) The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. Find an expression for the surface area.

17. We have the derivative of $\log_e$, but not for any other log base. Calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = \log_{10} x$. E

18. Prove the remaining two differentiation rules using the limit definition of the derivative. E

19. In this exercise, we will calculate the value of e based on solving the differential equation $f(x) = f'(x)$. Clearly one solution is $f(x) = 0$, but by drawing some pictures you should see that a reasonable guess for a more interesting solution would be an exponential function, $f(x) = a^x$. We just need to pick the right base. S

(a) Show that the derivative of a^x is given by

$$\lim_{h\to 0} a^x \left(\frac{a^h - 1}{h} \right).$$

(b) For a^x to be a solution to our differential equation, we need $\lim_{h\to 0} \frac{a^h-1}{h}$ to be 1. Show that

$$a = \lim_{h\to 0} (1+h)^{1/h}$$

works for this purpose.

(c) The expression in (b) is not the standard way to write this limit. Show that if $n = 1/h$, then

$$a = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

(d) By calculating the value of $\left(1 + \frac{1}{n}\right)^n$ for a suitably large value of n, obtain a numerical approximation for Euler's number, e.

NCEA Level 3 Calculus (Differentiation) 3. Derivatives of Common Functions (Homework)

Reading

Mathematics is fantastic. It is a subject where we do not have to take anyone's word or opinion. The truth is not determined by a higher authority who says 'because I say so', or because they saw it in a dream, the pixies at the bottom of their garden told them, or it came from some ancient mystical tradition. The truth is determined and justified with a mathematical proof.

A proof is an explanation of why a statement is true. More properly it is a convincing explanation of why the statement is true. By convincing I mean that it is convincing to a mathematician. (What that means is an important philosophical point which I am not going to get into; my interest is more in practical matters.)

Statements are usually proved by starting with some obvious statements, and proceeding by using small logical steps and applying definitions, axioms and previously established statements until the required statement results.

The mathematician's concept of proof is different to everyday usage. In everyday usage or in court for instance, proof is evidence that something is likely to be true. Mathematicians require more than this. We like to be 100% confident that a statement has been proved. We do not like to be 'almost certain'.

Having said that, how confident can we be that a theorem has been proved? Millions have seen a proof of Pythagoras' Theorem; we can be certain it is true. Proofs of newer results, however, may contain mistakes. I know from my own experience that some proofs given in books and research journals are in fact wrong.

From *How to Think Like a Mathematician*, by Kevin Houston.

For example, here are proofs of the three differentiation rules that we began to use this week.

Theorem. *Suppose f and g are functions which are differentiable at some point x, and suppose that λ is a real constant. Then:*

1. $(\lambda)' = 0$,
2. $(f+g)' = f'(x) + g'$, *and*
3. $(\lambda f)' = \lambda f'$.

Proof. We prove these using the properties of the limits.

$$\begin{aligned}
(\lambda)'(x) &= \lim_{h\to 0} \frac{\lambda - \lambda}{h} = 0.\\
(f+g)'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}\\
&= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}\\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\\
&= f'(x) + g'(x).\\
(\lambda f)'(x) &= \lim_{h\to 0} \frac{(\lambda f)(x+h) - (\lambda f)(x)}{h}\\
&= \lim_{h\to 0} \frac{\lambda(f(x+h) - f(x))}{h}\\
&= \lambda \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\\
&= \lambda f'(x).
\end{aligned}$$

□

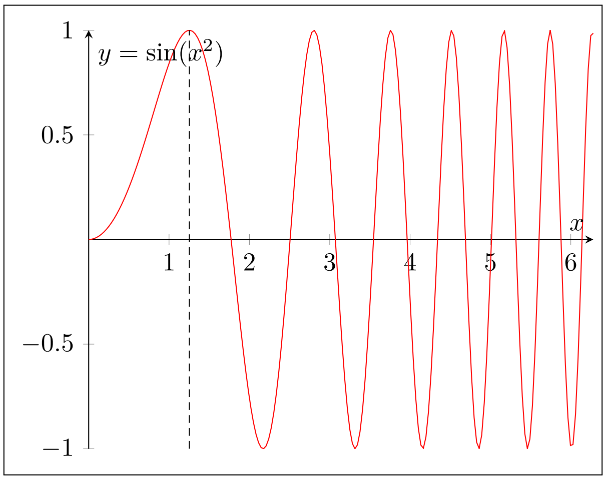
Questions

1. Differentiate with respect to x:
   (a) $x^2 + \ln x$
   (b) tx^t
   (c) $\sin x - \cos x$
   (d) $\sqrt[5]{x^4}$
2. Explain why you cannot use the power rule to find the derivative of x^x.
3. Find the nth derivative of $\frac{1}{x^n}$.
4. (More difficult!) Suppose a population grows exponentially with time, such that after t years the population $P = P_0 + 10^t$.
   (a) Given that the derivative of e^{cx} is ce^{cx} when c is constant, find the rate of change of the population at $t = 100$.
   (b) Explain why this population model is unrealistic.

NCEA Level 3 Calculus (Differentiation)
4. The Chain Rule

Goal for this week

To be able to take derivatives of functions of functions.



Consider the function $x \mapsto \sin(x^2)$. This function is made up of two functions, applied one after the other:

$$x \overset{f}{\mapsto} x^2 \overset{g}{\mapsto} \sin(x^2).$$

We often notate this *function composition* as $g \circ f$ (note that we evaluate from the right, so $(g \circ f)(x) = g(f(x))$).

Obviously the derivative of $\sin(x^2)$ is not just $\cos(2x)$, since the former has a stationary point at $x = \sqrt{\frac{\pi}{2}}$ but $\cos(\sqrt{2\pi}) \neq 0$. This shows us that, in general, the derivative of a function composition is not simply the composition of the derivatives.

In fact, it turns out that the derivative of $f \circ g$ is $g'(f' \circ g)$; in other words,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = g'(x) f'(g(x)).$$

This is known as the **chain rule**, since we are "chaining" together functions.

Before proving the chain rule, let us convince ourselves that it is plausible. We can interpret the derivative $\frac{\mathrm{d}g}{\mathrm{d}x}$ as the rate of change of g with respect to x, and the derivative $\frac{\mathrm{d}f}{\mathrm{d}g}$ as the derivative of f with respect to small changes in g; it is intuitive that if g changes twice as fast as x at some point, and f changes five times as fast as g, then f changes $2 \times 5 = 10$ times as fast as x.

The actual proof (given below) matches this intuition quite well.

Examples.

1. The correct derivative of $\sin(x^2)$ is $2x\cos(x^2)$.
2. If $f(r) = \sqrt{r^2 - 3}$, then $f'(r) = 2r\frac{1}{2}\left(r^2 - 3\right)^{-1/2} = \dfrac{r}{\sqrt{r^2 - 3}}$.
3. If $g(x) = \sin((\sin^7 x^7 + 1)^7)$, then we compute:

$$g(x) = \sin\left(\left[\left(\sin x^7\right)^7 + 1\right]^7\right)$$

$$g'(x) = 7x^6 \cdot \cos x^7 \cdot 7\left(\sin x^7\right)^6 \cdot 7\left[\left(\sin x^7\right)^7 + 1\right] \cdot \cos\left(\left[\left(\sin x^7\right)^7 + 1\right]^7\right)$$

This result can probably be simplified, however the point is to evaluate the derivative chain from inside to outside in a systematic fashion.

Proof of the chain rule (optional). The proof is a little fiddly, and comes in two parts. Recall that in the work on limits, we found that an alternative definition of the derivative of f at x was

$$f'(x) = \lim_{k \to x} \frac{f(x) - f(k)}{x - k}.$$

Now, suppose we wish to find the derivative of $f \circ g$ at x. In the first case, suppose that g is not constant around x (in other words, we can zoom in 'far enough' towards x so that for all k in the zoomed in area, $g(k) \neq g(x)$). Then:

$$\begin{aligned}
\lim_{k \to x} \frac{f(g(x)) - f(g(k))}{x - k} &= \lim_{k \to x} \frac{f(g(x)) - f(g(k))}{g(x) - g(k)} \cdot \frac{g(x) - g(k)}{x - k} \\
&= \lim_{k \to x} \frac{f(g(x)) - f(g(k))}{g(x) - g(k)} \cdot \lim_{k \to x} \frac{g(x) - g(k)}{x - k} \\
&= f'(g(x))g'(x),
\end{aligned}$$

(noting that as $k \to x$, $g(k) \to g(x)$). This calculation only works when g is not constant around x, because if g *is* constant around x then for all k sufficiently close to x, $g(x) - g(k) = 0$ and the limit does not exist.

To deal with this case, assume that g *is* constant around x. Then $g'(x) = 0$, and also for all h close enough to zero we have $g(x + h) = g(x)$. Then

$$(f \circ g)'(x) = \lim_{h \to 0} \frac{f(g(x + h)) - f(g(x))}{h} = \lim_{h \to 0} \frac{f(g(x)) - f(g(x))}{h} = 0 = 0 \times f'(g(x)) = g'(x)f'(g(x)).$$

□

Questions

1. Identify the inner and outer functions, but do not attempt to differentiate. A
   (a) $\sqrt{\sin x}$
   (b) $\sin\cos\tan x$
   (c) $(2x+3)^{17}$
   (d) $97(x+2)^2$
   (e) $\ln\sin x$
   (f) $\frac{1}{\sqrt{23x-x^2}}$

2. Differentiate with respect to t: A
   (a) $(2t+3)^{3000}$
   (b) $\sin\ln t$
   (c) $\sqrt{t^3+10t^2+3}$
   (d) $\csc e^t$
   (e) $\sin^3 t + 14\ln(3t)$
   (f) $\sin\sin\sin t$
   (g) $\cot(t+\sec t)$
   (h) $\sin^2((t+\sin t)^2)$
   (i) $\ln\sqrt{t+9}$
   (j) $\sqrt{t}+\frac{1}{\sqrt[3]{t^4}}$
   (k) $e^{\sec(t^2)}$
   (l) $\sin\sqrt{t+\tan t}$

3. The derivative of a function is $2\cos 2x$. What could the original function be? A

4. Differentiate $y=\sin^2 x+\cos^2 x$, and hence prove that $\sin^2 x+\cos^2 x=1$. M

5. Suppose that the displacement of a particle on a vibrating spring is given by $x(t)=5+\frac{1}{8}\sin(5\pi t)$, where x is measured in centimetres and t in seconds. A
   (a) Find the velocity of the particle at time t.
   (b) At which times is the particle momentarily stationary?

6. The volume of a spherical balloon at a time t is given by $V(t)=\frac{4}{3}\pi r^2$, and its radius, changing over time, is given by $r(t)$. Find $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of $\frac{\mathrm{d}r}{\mathrm{d}t}$.

7. If $F(x)=f(3f(4f(x)))$, where $f(0)=0$ and $f'(0)=2$, find $F'(0)$. M

8. Suppose $f(x)=g(x+g(a))$ for some differentiable function g and constant a. Find $f'(x)$. A

9. The depth of water at the end of a jetty in a harbour varies with time due to the tides. The depth of the water is given by the formula
$$W=4.5-1.2\cos\frac{\pi t}{6}$$
   where W is the water depth in metres, and t is the time in hours after midnight.
   (a) What is the rate of change of water depth 5 hours after midnight? A
   (b) When is the first time after $t=0$ that the tide changes direction? M
   (c) At that time, is the water changing from rising to falling or from falling to rising? E

10. In physics, the rate of change of momentum of an object is proportional to the force needed to effect that change: if p is the momentum of the object as a function of time, $F=\frac{\mathrm{d}p}{\mathrm{d}t}$. The momentum of a particular object, oscillating back and forth along a line, is given by $p=mA\sin(\omega t+\phi)\,\mathrm{kg\,m\,s^{-1}}$ (where m, A, ω, and ϕ are various constants). What is the force acting on the object at $t=10$? M

11. The force F (in newtons) acting at an angle θ with the horizontal that is needed to drag a mass of W kilograms along a horizontal surface at a constant velocity is given by

$$F = \frac{\mu W}{\cos\theta + \mu \sin\theta}$$

where μ is the coefficient of static friction (a constant).

   (a) If $W = 200\,\mathrm{kg}$ and $\mu = 0.2$, find $\frac{\mathrm{d}F}{\mathrm{d}\theta}$ when $\theta = \frac{\pi}{6}\,\mathrm{rad}$. A

   (b) Suppose now that θ is a function of time, so that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.5\,\mathrm{rad/s}$. Find $\frac{\mathrm{d}F}{\mathrm{d}t}$. M

12. Find the 73rd derivative of $\sin 6x$. E

13. Recall that the *absolute value* of x, denoted $|x|$, is the value obtained by 'throwing away the sign' of x. E

   (a) Prove that
$$\frac{\mathrm{d}}{\mathrm{d}x}|x| = \frac{x}{|x|}.$$
   [*Hint: Write* $|x| = \sqrt{x^2}$.]

   (b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of both f and f'.

14. In the next section, we will be studying the product rule for derivatives. It is possible, though not particularly usual, to prove it using simply the basic derivatives from the last section and the chain rule; in this exercise, you will do just that. S

   Suppose that f and g are functions, and consider the function F defined by $F(x) = \big(f(x) + g(x)\big)^2$.

   (a) Calculate $F'(x)$ using the chain rule.

   (b) Calculate $F'(x)$ by multiplying out the square and differentiating the polynomial that results. (In particular, note that $\frac{\mathrm{d}}{\mathrm{d}x}2(fg)(x) = 2(fg)'(x)$).

   (c) Compare parts (a) and (b).

NCEA Level 3 Calculus (Differentiation)
4. The Chain Rule (Homework)

Reading

Suppose we have the function $y = \sin(x^2)$ which we saw in the tutorial. Consider what happens if we let x change by a small amount, to $x + \mathrm{d}x$. Then y will change to $y + \mathrm{d}y$, and we have

$$y + \mathrm{d}y = \sin((x + \mathrm{d}x)^2) = \sin(x^2 + 2x\,\mathrm{d}x + \mathrm{d}x^2).$$

Let's let $\mathrm{d}x$ become really small; so $\mathrm{d}x^2$ becomes even smaller. In fact, we will take $\mathrm{d}x$ to be so small that $\mathrm{d}x^2 \to 0$. Then we can basically ignore it, and continue computing:

$$\sin(x^2 + 2x\,\mathrm{d}x + \mathrm{d}x^2) = \sin(x^2 + 2x\,\mathrm{d}x) = \sin(x^2)\cos(2x\,\mathrm{d}x) + \cos(x^2)\sin(2x\,\mathrm{d}x).$$

Now, when t is small, $\sin(t) \approx t$. Similarly, $\cos(t) \approx 1$. Since $\mathrm{d}x$ is small, we apply these approximations:

$$\sin(x^2)\cos(2x\,\mathrm{d}x) + \cos(x^2)\sin(2x\,\mathrm{d}x) = \sin(x^2) + 2x\,\mathrm{d}x\cos(x^2).$$

But recall that this is equal to $y + \mathrm{d}y$; so

$$\begin{aligned} y + \mathrm{d}y &= \sin(x^2) + 2x\,\mathrm{d}x\cos(x^2) \\ \mathrm{d}y &= 2x\,\mathrm{d}x\cos(x^2) \\ \tfrac{\mathrm{d}y}{\mathrm{d}x} &= 2x\cos(x^2). \end{aligned}$$

Note that this computation is most definitely **not rigorous** — we don't justify why we can ignore $\mathrm{d}x^2$ but not $\mathrm{d}x$, and we don't define what it even means to be "small enough to ignore"! However, it does at least suggest that, intuitively, the chain rule does what we expect it to do.

This calculation can be made rigorous if we define infinitesimals and rules for calculating with them, and this branch of mathematics is known as non-standard analysis. However, for the remainder of this year we will continue to base our discussions of calculus on limits and inequalities because they are easier to make rigorous, despite the initial barriers to intuition.

Questions

1. If $y = \sqrt{\cot x} - \sqrt{\cot a}$ (where a is constant), find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
2. (a) Show that if $y = f(g(h(x)))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) \cdot g'(h(x)) \cdot f'(g(h(x)))$.
   (b) Calculate the derivative of $y = \sin\cos\sin\cos\sin x^5$.
3. We will prove the double angle formula for cosine from the double angle formula for sine. Suppose $f(\theta) = \cos 2\theta$, and $g(\theta) = 1 - 2\sin^2\theta$.
   (a) Show that $f' = g'$. (You may assume that $\sin 2\theta = 2\sin\theta\cos\theta$.)
   (b) Verify that f and g agree at $\theta = 0$, and conclude that $f = g$.

NCEA Level 3 Calculus (Differentiation)
5. The Product and Quotient Rules

Goal for this week

To be able to take derivatives of things multiplied by other things, and to be able to decide which differentiation rules to use for a problem.

Two sections ago we saw that the derivative of a product is not simply the product of the derivatives; for example, take $(x)(x)$. If we differentiate each term and multiply, we obtain 1; however, the derivative of x^2 is (of course) $2x$. Ensure you understand why this is a counterexample to the naive rule $(fg)' = f'g'$ before continuing.

Suppose f and g are functions; then the *real* product rule is

$$(fg)' = gf' + fg'.$$

Proof. We simply apply the limit laws to the definition of the derivative and use a little trick (note that the expression highlighted in purple is exactly zero).

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x)}{h} \\
&= f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f(x)g'(x) + g(x)f'(x).
\end{aligned}$$

$\square$

Example. Consider $y = 2t\sin t$. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = 2\sin t + 2t\cos t$.

We can also write a rule for the derivative of a quotient of functions. You will be asked to prove it as an exercise, using the product rule.

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Example. Consider $f(s) = \sqrt{\frac{s^2+1}{s^2+4}}$. Then

$$f'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s^1+1}{s^2+4}\right] \cdot \frac{1}{2\sqrt{\frac{s^2+1}{s^2+4}}} = \frac{(s^2+4)(2s) - (s^2+1)(2s)}{(s^2+4)^2} \cdot \frac{\sqrt{s^2+4}}{2\sqrt{s^2+1}} = \frac{6s}{(s^2+4)^{\frac{3}{2}}(s^2+1)^{\frac{1}{2}}}.$$

With our rules (sum, chain, and product, together with our basic derivatives), we can now differentiate almost any combination of functions that we are currently aware of. The process of differentiation is entirely mechanical, and can be easily performed by a computer. As such, learning to differentiate more complicated combinations of functions is very similar to learning how to add, multiply, and perform long division, and is only a matter of practice.

Like basic arithmetic, differentiating functions is not interesting on its own; but being able to differentiate will allow us to talk more clearly about the geometry and behaviour of functions.

Questions

1. In each case, find $\frac{\mathrm{d}y}{\mathrm{d}t}$. A

   (a) $y = \left(3 + 2t^2\right)^4$

   (b) $y = \dfrac{t^3}{\ln t}$

   (c) $y = t\sqrt{t}$

   (d) $y = 2t \sin t - (t^2 - 2)\cos t$

   (e) $y = \dfrac{t}{\sqrt{a^2 - t^2}}$ (a constant)

   (f) $y = \dfrac{1}{8}t^8\left(1 - t^2\right)^{-4}$

   (g) $y = e^t \ln t$

   (h) $y = \log\left[1 + \dfrac{t^2 + 3t + 17}{t^{17}}\right]$

   (i) $y = \sin\left[e^{\tan t} \ln \tan t\right]$

   (j) $y = \dfrac{3t - 2}{\sqrt{2t + 1}}$

   (k) $y = \dfrac{\sec 2t}{1 + \tan 2t}$

   (l) $y = \dfrac{(t-1)(t-4)}{(t-2)(t-3)}$

   (m) $y = t\sin^2(\cos\sqrt{\sin \pi t})$

   (n) $y = \sqrt[5]{t \tan t}$

   (o) $y = \dfrac{(t+\lambda)^4}{t^4 + \lambda^4}$ (λ constant)

2. If $f(x) = e^{-x}$, find $f(0) + xf'(0)$. A

3. Show that $\frac{\mathrm{d}}{\mathrm{d}x} e^{\tan x} e^{-\cot x} = \left(\frac{\mathrm{d}}{\mathrm{d}x} e^{\tan x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} e^{-\cot x}\right)$. Reconcile this with our statement above that the naive product rule does not work in general. M

4. Suppose f and g are functions (g not the zero function). Write $\frac{f}{g} = fg^{-1}$ and prove the quotient rule from the product rule. M

5. Show that $y = xe^{-x}$ satisfies the differential equation $xy' = (1 - x)y$. M

6. If $y = \ln \frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}$, find y''. M

7. Find the equation of the tangent line to the graph of $y = \ln \cos \frac{x-1}{x}$ at the point $(1, 0)$. M

8. Show that $y = (1 + x + \ln x)^{-1}$ satisfies the differential equation $xy' = y(y \ln x - 1)$. M

9. Find the angle at which $y = x^2 \ln[(x-2)^2]$ cuts the x-axis at the point $(0, 0)$. E

10. When $x = 0$, is the curve $y = (x + 20)^2(2x^2 - 3)^6 - \ln \sin(x - \frac{\pi}{2})$ concave up or concave down? M

11. If $y = \frac{e^x}{\sin x}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 - \cot x)$. M

12. Show that if f, g, and h are functions then $(fgh)' = f'gh + fg'h + fgh'$. M

13. Suppose $f(x) = f(-x)$ for all x in the domain of f. Prove that $f'(x) = -f'(-x)$ for all x in the domain of $f'(x)$. E

14. Consider the function defined by $f(x) = x^x$. E

    (a) Rewrite f in the form $f(x) = e^{x \ln x}$, and hence find $f'(x)$.

    (b) Find $\frac{\mathrm{d}y}{\mathrm{d}t}$ if $y = (t^2 + 3)^{(t^2+3)}$.

15. A circle that closely fits points on a local section of a curve can be drawn for any continuous curve. The radius of curvature of the curve is defined as the radius of the approximating circle, which changes as we move around the curve. E

$$\text{radius of curvature} = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|}$$

Find the radius of curvature of the curve $y = e^{-x}\sin x$ at the point $(0, 0)$.

16. Find $f'(x+3)$ if $f(x+3) = (x+5)^7$. E

17. The number a is called a **double root** of some polynomial function f if $f(x) = (x-a)^2 g(x)$ for some polynomial g. Prove that a is a double root of f if and only if a is a root of both f and f'. E

18. Show that there is no function of the form E

$$f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0 + \frac{b_1}{x} + \cdots + \frac{b_m}{x^m}$$

such that $f'(x) = 1/x$.

19. Let's try to generalise the product rule for nth derivatives of products. Recall that the notation $f^{(n)}(x)$ means the nth derivative of f with respect to x. Assume in each case that the relevant derivatives exist. S
    (a) Find a formula for $(fg)'' = (fg)^{(2)}$ in terms of the derivatives of f and g.
    (b) What about $(fg)^{(3)}$?
    (c) If you can, prove Leibniz' formula for the nth derivative of a product using *induction*:

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$$

NCEA Level 3 Calculus (Differentiation)
5. The Product and Quotient Rules (Homework)

Reading

Even next year, when you will be expected to know the derivative rules, it is not necessary to remember the quotient rule because it is just a special case of the product rule. However, here is a little rhyme:

If it's the quotient rule you wish to know,
It's low-de-high less high-de-low.
Then draw the line and down below,
Denominator squared will go.[*]

Questions

1. Find the derivatives:
   (a) $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = \sin x \ln x$.
   (b) $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = x \sec kx$ (k constant).
   (c) $\frac{\mathrm{d}f}{\mathrm{d}\theta}$ if $f(\theta) = \frac{\cos \pi\theta}{\sin \pi\theta + \cos \pi\theta}$.
   (d) $\frac{\mathrm{d}y}{\mathrm{d}t}$ if $y = \cos^4(\sin^3 t)$.
2. The force F acting on a body with mass m and velocity v is the rate of change of momentum, $F = \frac{\mathrm{d}}{\mathrm{d}t}[mv]$. If m is constant, this becomes $F = ma$, where $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ is the acceleration of the body. However, due to relativistic effects, the mass of a particle varies with v as
$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$
where m_0 is the rest mass of the body and c is the speed of light. Show that
$$F = \frac{m_0 a}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}.$$
3. Recall that if θ is given in degrees, then $\frac{\pi\theta}{180}$ is the equivalent angle in radians. Find the derivative of $\sin\theta$ if θ is given in degrees.

[*]Quoted in *Mathematical Apocrypha* by Steven G. Krantz (p.36).

NCEA Level 3 Calculus (Differentiation)
6. Tangent and Normal Lines

Goal for this week

To understand how calculus helps us approximate the graphs of functions using straight lines.

For the next few weeks, we will be studying the geometry and shape of functions using calculus. We begin in this section by taking a look at how closely functions can be approximated by straight lines around a point.

Tangent Lines

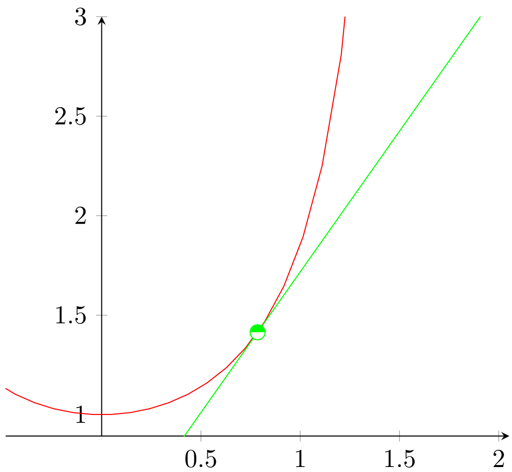
The **tangent line** of a function at a point is the unique line passing through that point such that the line has the same slope as the function at that point. Because we actually ended up defining slope based on the derivative, it follows that the equation of the tangent line to a curve at a point can be defined using the derivative.

More formally, if we have some function f which is differentiable at a point (x_0, y_0) then the tangent line to the function at that point has the equation

$$(y - y_0) = f'(x_0)(x - x_0).$$

Example. Consider the function $y = \sec x$. The derivative of this function is $y' = \tan x \sec x$; at the point $P = \left(\frac{\pi}{4}, \sqrt{2}\right)$, the slope is $\tan\frac{\pi}{4}\sec\frac{\pi}{4} = \sqrt{2}$.

The tangent line at P is therefore described by $y = \sqrt{2}(x - \frac{\pi}{4}) + \sqrt{2}$, or $y = \sqrt{2}x + \frac{4-\pi}{2\sqrt{2}}$.

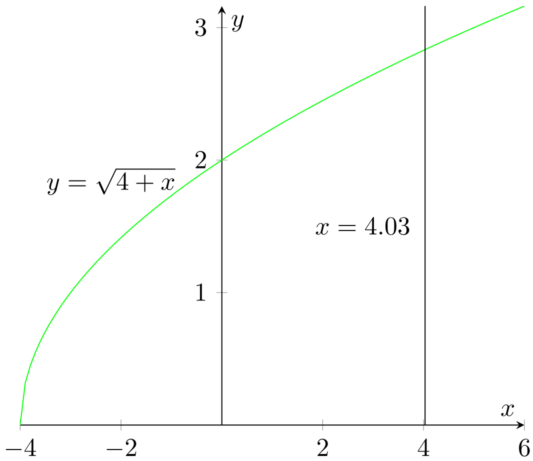


The tangent line of a function at a point is the best linear approximation to the function at that point. This means that if you know the value of a function and the derivative of that function at a point, then you can very easily approximate the other values of the function around that point: if $y_0 = f(x_0)$, and x is very close to x_0, then

$$f(x) \approx f'(x_0)(x - x_0) + y_0.$$

On the other hand, as the graph accompanying the example above shows, the tangent line is an awful approximation as we move further away from the point that we find the tangent line at. It is possible to obtain some measure of the error of a given approximation; this is explored, a little, in the exercises.

Example. Let us calculate $\sqrt{4.03}$ by hand(!). If we consider the function $f(x) = \sqrt{4+x}$, then $\sqrt{4.03} = f(0.03)$. Let's draw the situation out:



So we want the tangent line to f at the point $x = 0$. We have that $f'(x) = \frac{1}{2\sqrt{4+x}}$, and so $f'(0) = \frac{1}{4}$. The tangent line is the line through $(0, f(0)) = (0, 2)$ with gradient $\frac{1}{4}$, which has equation

$$y - 2 = \frac{1}{4}x.$$

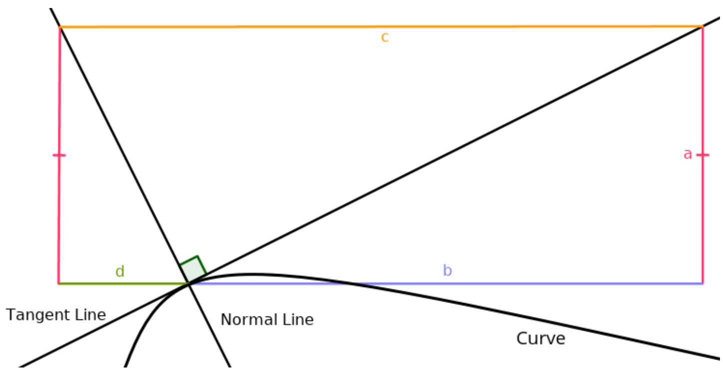
Hence $\sqrt{4.03} \approx \frac{1}{4} \times 0.03 + 2 = 2.0075$ — and, as promised, all of these calculations can be done without a calculator. (According to my calculator, $\sqrt{4.03} \approx 2.007486$ and so we are not far off at all.)

To do: add a discussion about the error of an approximation.

Normal Lines

If a function has a tangent line at a particular point, then the line perpendicular to the tangent line at that point is called the **normal line**.

Theorem. *If a tangent line has slope m, then the normal line to it has slope $m^{\perp} = -1/m$.*

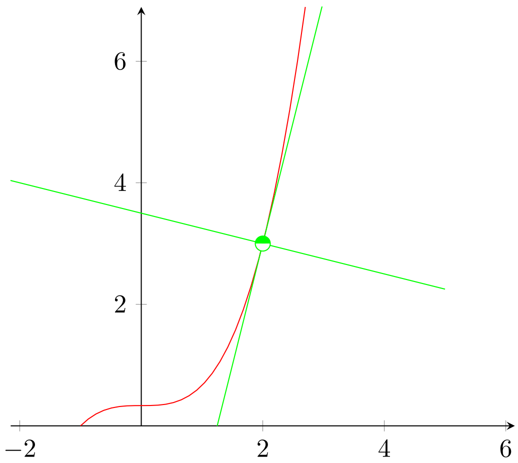


Proof. Consider the line shown in the diagram above, and let $m = \frac{a}{b}$ be the slope of the tangent line. So the length of the hypotenuse of the ab triangle is $\sqrt{a^2 + b^2}$. But $c = d + b$. Hence the length of the hypotenuse of the ad triangle can be found in two ways:

$$\sqrt{(d+b)^2 - \left(\sqrt{a^2+b^2}\right)^2} = \sqrt{a^2 + d^2}.$$

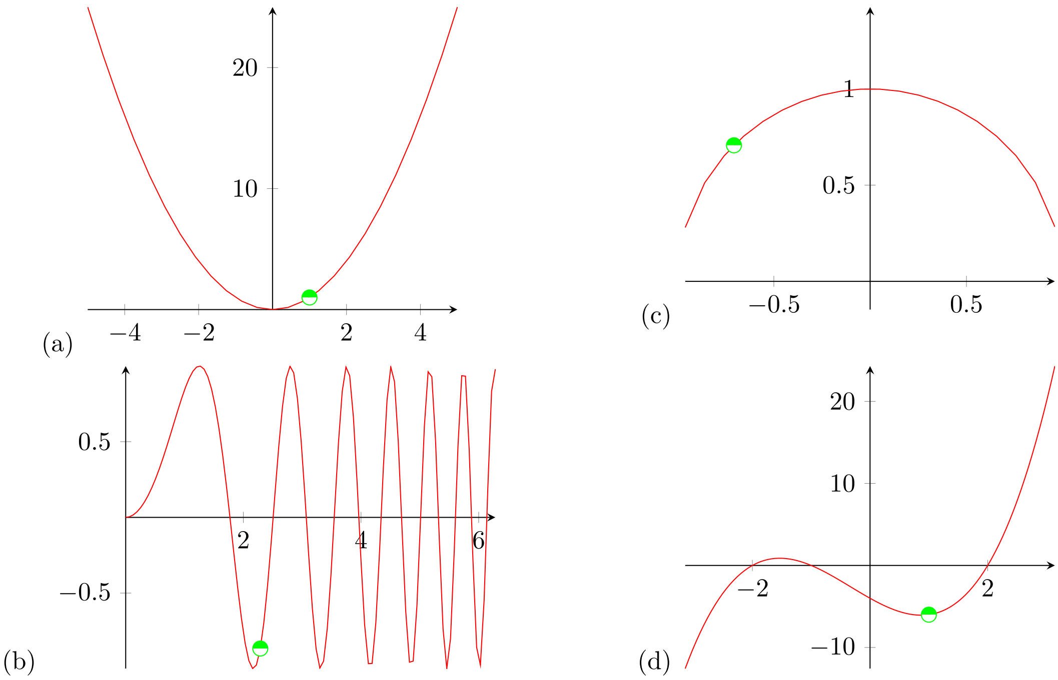
We can square both sides to obtain $d^2 + 2db + b^2 - a^2 - b^2 = a^2 + d^2$, and therefore $db = a^2$. Hence $d/a = a/b$; but $a/b = m$, and so the slope of the normal line is simply $-\frac{a}{d} = -\frac{1}{m}$. $\square$

Example. Consider the function f defined by $f(x) = \frac{x^3+1}{3}$. Then $f'(x) = x^2$, and the tangent line to the function at $(2, 3)$ is $(y - 3) = 4(x - 2)$, or $y = 4x - 5$. By the theorem above, the slope of the normal line to the function at that point is $-1/24$.



Questions

1. Draw the tangent and normal lines to each function at the indicated points. A



2. Find the tangent and normal lines to the following functions at the given points. M
   (a) $x \mapsto \sin x$ at $(0, 0)$
   (b) $x \mapsto \sin x$ at $(\pi, 0)$
   (c) $x \mapsto e^x$ at $(0, 1)$
   (d) $x \mapsto \sec x$ at $(0, 0)$
   (e) $x \mapsto x^2$ at $(1, 1)$
   (f) $x \mapsto \sqrt{x}$ at $(1, 1)$
   (g) $x \mapsto (x^4 - 3x^2 + 5)^3$ at $(0, 125)$
   (h) $x \mapsto \cos \tan x$ at $(\pi, 1)$
   (i) $x \mapsto \left(x + \frac{1}{x^2}\right)^{\sqrt{7}}$ at $(1, 2^{\sqrt{7}})$.
3. Find the equation of the tangent line to $y = x + \tan x$ at (π, π). M
4. Find an equation for the normal line to the curve $y = \frac{1}{\sqrt{x^2 - x}}$ at $(2, \frac{1}{\sqrt{2}})$. M
5. Consider the curve $y = \tan(2 \sin x)$.
   (a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \cos x \sec^2(2 \sin x)$. A
   (b) Find the equation of: M
      i. The tangent to the curve at $(\pi, 0)$
      ii. The normal to the curve at $(0, 0)$
6. Find the best linear approximation to $y = 3x^3 + 2x + 4$ around $x = 2$. M

7. Find the point(s) on the graph of the function $y = x^2$ such that the slope of the normal to the curve at that point is $m^{\perp} = -1$. M

8. The tangents to the curve $y = \frac{1}{4}(x-2)^2$ at points P and $Q = (6, 4)$ are perpendicular. What is the x-ordinate of P? E

9. Consider the surface described by $z = x^2 + 2\sin(y^2 + 1) + 2$. S
   (a) Find the derivative of z with respect to x, and find the tangent line in the x direction at $(0, 0)$.
   (b) Find the derivative of z with respect to y, and find the tangent line in the y direction at $(0, 0)$.
   (c) Hence describe the tangent *plane* of the surface at $(0, 0)$.

10. Consider the function $y = \sin x$.
    (a) Find the best linear approximation to this function around $(0, 0)$. M
    (b) Find the percentage error of this approximation to 1dp when $x = \pi$. M
    (c) At what point on the curve $(x \geq 0)$ does the percentage error of the approximation rise above 100%? E

11. (a) Find two points on the graph of $y = 1/x$ that share a common normal line. M
    (b) Show that there are no more such points. M
    (c) Show that there are no two points on the graph that share a common tangent line. M
    (d) Repeat parts (a)-(c) for the general hyperbola-like curve $y = x^{-n}$ (where n is a positive integer). E
    (e) What is the situation for the even more general case of the curve $y = x^r$, where r is any real number? S

12. Consider the quartic polynomial $p(x) = 2x^4 - 4x^3 - 23x^2 + 84x - 61$.
    (a) Find the best linear approximation to p around the point $(2, 15)$. M
    (b) Find the unique quadratic polynomial $q(x)$ such that $q(2) = p(2)$, $q'(2) = p'(2)$, and $q''(2) = p''(2)$. This is the best quadratic approximation to p at the point $(2, 15)$. S
    (c) Show that the best quadratic approximation to a function f at the point (x_0, y_0) is given by O

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2.$$

NCEA Level 3 Calculus (Differentiation)
6. Tangent and Normal Lines (Homework)

Reading

Go and watch...

`https://www.youtube.com/watch?v=6sNeE-mMYB8`

Questions

1. Write a definition for the normal line to a curve f at a point (x_0, y_0).
2. Find the best linear approximation to the curve $y = \sqrt{\cos(x+\pi)} + \sin x - e^{2(\tan x)^2}$ at the point $(\pi, 0)$.
3. Find a function such that the normal line to the function at $(1, 0)$ has the equation $y = 3x - 3$.
4. Find a linear approximation to the curve $y = \sqrt[3]{1+3x}$ around $x = 0$, and determine an approximate value for $\sqrt[3]{1.03}$.

NCEA Level 3 Calculus (Differentiation)
7. The Geometry of Functions

Goal for this week

To follow on from last week by using calculus to describe the geometry of the graph of a function.

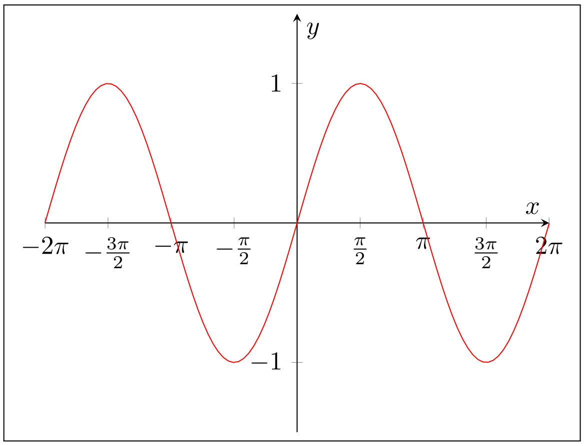
This week, we move on from approximating functions with straight lines to actually studying the geometry of the curves themselves. Back in the homework on limits, I gave you the following definitions:

Definition.

1. A function is **increasing** if its derivative is positive.
2. A function is **decreasing** if its derivative is negative.
3. A function is **concave down** if its derivative is decreasing.
4. A function is **concave up** if its derivative is increasing.
5. A function f is **continuous** at a point a if $\lim_{x\to a} f(x) = f(a)$.

Examples.

1. The function $x \mapsto x^2$ is concave up everywhere, increasing for $x > 0$, and decreasing when $x < 0$.
2. The function $x \mapsto \sin x$ is concave down when $(2n)\pi < x < (2n+1)\pi$, and concave up when $(2n+1)\pi < x < (2n+2)\pi$ (for all integers n).



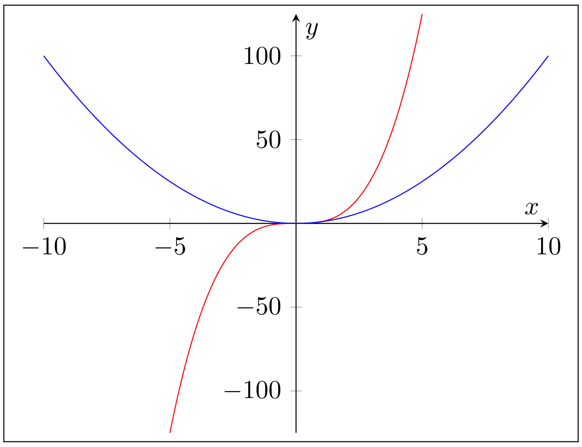
We can tell if a function is increasing (getting larger) or decreasing (getting smaller) around a point by looking at the derivative at that point. In order to look for concavity we must look at the derivative of the derivative (the second derivative). If a function is concave up (curving up), the second derivative is positive; if a function is concave down at a point, then the second derivative is negative. These facts are not ones which you should memorise, but ones which you should be able to reason about yourself based on the graph of a function.

A point where a function changes from concave up to concave down (or vice versa) is known as an **inflection point**.

Example. The function $x \mapsto x^3$ has an inflection point at $(0,0)$; to the left of this point, the function is concave down (the second derivative is negative) and to the right the function is concave up (the second derivative is positive).

In general, functions of the form $f(x) = x^n$ (for integer $n \geq 0$) have some fairly symmetric properties:

- If n is even, then $y = f(x)$ is even around the x-axis (i.e. $f(-x) = f(x)$), has a minimum at $(0,0)$, and tends to $+\infty$ in both directions. (See the function graphed in blue in the diagram below.)
- If n is odd, then $y = x^n$ is odd around the x-axis (i.e. $f(-x) = -f(x)$), has an inflection point at $(0,0)$, and tends to $-\infty$ towards the left and $+\infty$ towards the right. (See the function graphed in red in the diagram below.)

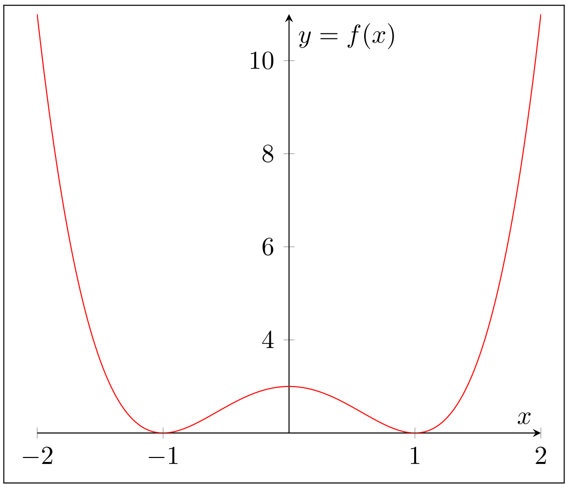


We call the value of the first derivative the *slope* or *gradient* of the function at a point; similarly, the second derivative measures how "curvy" a function is and we call its value the *concavity* of a function at a point.

We also want to study continuity: a function is continuous, intuitively, if its graph can be drawn without picking a pen up off the page. It is perhaps unfortunate that the concepts of continuity and differentiability are not the same. The details of this are worked in the exercises.

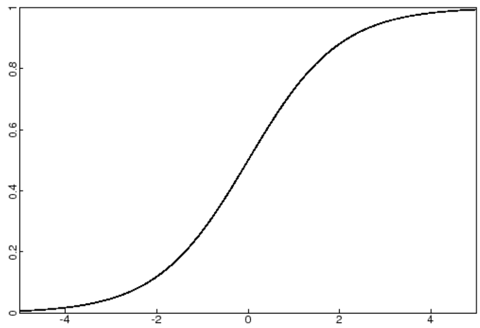
Example. Consider the function defined by $f(x) = x^4 - 2x^2 + 3$. Find the intervals on which f is increasing or decreasing, find the intervals of concavity, and find any inflection points.

Solution. We have $f'(x) = 4x^3 - 4x$. This function is zero at $x \in \{-1, 0, 1\}$, and so (since the function is a positive cubic) f will be decreasing when $x < -1$, increasing when $-1 < x < 0$, decreasing when $0 < x < 1$, and increasing when $1 < x$. We also have $f''(x) = 12x^2 - 4$ and so $f''(x) = 0$ when $x = \pm\frac{1}{\sqrt{3}}$. Hence the function is concave up when $x < -\frac{1}{\sqrt{3}}$, concave down when $|x| < \frac{1}{\sqrt{3}}$, and concave up when $x > \frac{1}{\sqrt{3}}$. The inflection points will be $x = \pm 1\sqrt{3}$.


$y = f(x)$
10
8
6
4
x
-2
-1
1
2

Questions

1. Find the second derivative of the following functions. A
   (a) $y = x^2 + x$
   (b) $f(x) = \sin x$
   (c) $g(x) = \cot(3x^2 + 5)$
   (d) $y = \frac{\sin mx}{x}$
   (e) $y = 4\sin^2 x$
   (f) $y = \tan^2(\sin\theta)$
   (g) $y = \tan\sqrt{1-x}$
2. Find the concavity of the function $y = \frac{x^2-1}{x^2+1}$ at $(0, -1)$. A
3. Find the intervals on which the following functions are increasing or decreasing, and find their intervals of concavity. M
   (a) $y = x^2 + 1$
   (b) $y = 2x^3 + 3x^2 - 36x$
   (c) $G(x) = x - 4\sqrt{x}$
4. The following function is known as the *logistic curve* and is used for population modelling. Find the intervals of concavity, and label any inflection points. A



5. The graph of $y = f(x)$ (where f is a continuous function) is concave up for all $x < 0$, concave down for $x > 0$, and decreasing everywhere. M
   (a) Sketch the graph of $y = f(x)$.
   (b) What can you say about $f'(x)$ and $f''(x)$ for $x < 0$ and $x > 0$?
   (c) What about $x = 0$?
6. Find a value of k such that the function F is continuous at $x = -3$, where E

$$F(x) = \begin{cases} \frac{x^2-9}{x+3} & \text{if } x \neq -3, \\ k & \text{if } x = -3. \end{cases}$$

7. Show whether or not the function g is continuous at the three points 2, 3, and 4, where E

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leq 2, \\ 2 - x & \text{if } 2 < x \leq 3, \\ x - 4 & \text{if } 3 < x \leq 4, \\ \pi & \text{if } x \geq 4. \end{cases}$$

8. Find all values of α such that Φ is continuous everywhere, where E

$$\Phi(x) = \begin{cases} x+1 & \text{if } x \leq \alpha, \\ x^2 & \text{if } x > \alpha. \end{cases}$$

9. Sketch a function satisfying the given criteria. M
   (a) i. Vertical asymptote at $x = 0$,
   ii. $f'(x) > 0$ if $x < -2$,
   iii. $f'(x) < 0$ if $x > -2$ $(x \neq 0)$,
   iv. $f''(x) < 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$.
   (b) i. $f'(0) = f'(2) = f'(4) = 0$,
   ii. $f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
   iii. $f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
   iv. $f''(x) > 0$ if $1 < x < 3$,
   v. $f''(x) < 0$ if $x < 1$ or $x > 3$.

10. A curve is defined by the function $f(x) = e^{-(x-k)^2}$. Find, in terms of k, the x-ordinates for which $f''(x) = 0$. E

11. We will prove that differentiability of f at a implies continuity of f at a; expand the following and use the limit laws to show that $\lim_{x \to a} f(x) - f(a) = 0$, carefully indicating where you use the existence of the derivative. S

$$\left[\lim_{a \to x} f(x) - f(a)\right]\left[\lim_{a \to x} \frac{x-a}{x-a}\right]$$

12. Give an example of a function which is continuous but not differentiable at some point. E

13. Scholarship 2010: Recall that the points of inflection of a curve are places where the second derivative changes sign. These are typically, **but not always**, points at which the second derivative is zero. S

    Consider the curve $y = \sqrt[3]{x} e^{-x^2}$.

    Write the second derivative in the form $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (ax^4 + bx^2 + x)e^{-x^2}x^{-5/3}$, and hence find the x-ordinates of the points of inflection of the curve.

14. Scholarship 2004: (You may wish to remind yourself how to perform long division of polynomials.) Consider the function S

$$y = \frac{x^2}{1+x^2},$$

    where $-1 \leq x \leq 1$. The gradient at the point $x = 1$ is $\frac{1}{2}$.

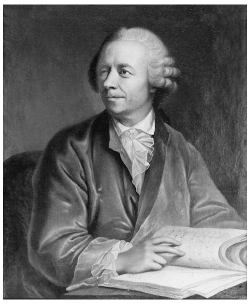
    Hence show that there is a point with $\frac{1}{4} \leq x \leq \frac{1}{2}$ where the gradient is also $\frac{1}{2}$.

15. Scholarship 2013: A function f is **even** if $f(-x) = f(x)$ for all x in its domain, and **odd** if $f(-x) = f(-x)$ for all x in its domain. S
    (a) Describe which polynomials are even, which are odd, and which are neither.
    (b) Suppose that g is any even differentiable function defined for all real numbers (not necessarily a polynomial). Use the limit definition of the derivative to prove that g' is odd.

NCEA Level 3 Calculus (Differentiation)
7. The Geometry of Functions (Homework)

Reading

The 'proper' name for the subject we are beginning to look at now is *differential geometry.* One of the pioneers of this subject was Leonhard Euler.



Leonhard Euler (1707–1783) was Switzerland's foremost scientist and one of the three greatest mathematicians of modern times (the other two being Gauss and Riemann).

He was perhaps the most prolific author of all time in any field. From 1727 to 1783 his writings poured out in a seemingly endless flood, constantly adding knowledge to every known branch of pure and applied mathematics, and also to many that were not known until he created them. He averaged about 800 printed pages a year throughout his long life, and yet he almost always had something worthwile to say and never seems long-winded. The publication of his complete works was started in 1911, and the end is not in sighr: it is now esitmated that 100 large volumes will be required for completion of the project. He suffered blindess during the last 17 years of his life, but with the aid of his powerful memory and fertile imagination, and with helpers to write his books and papers from dictation, he actually increased the already prodigious output of work.

Though he was not himself a teacher Euler has had a deeper influence on the teaching of mathematics than any other person. This came about chiefly through his three great treatises: *Introductio in Analysin Infinitorum* (1748); *Institutiones Calculi Differentialis* (1755); and *Institutiones Calculi Integralis* (1768–1794). There is considerable truth in the old saying that all elementary and advanced calculus textbooks since 1748 are essentially copies of Euler or copies of copies of Euler. These works summed up and codified the discoveries of his predecessors, and are full of Euler's own ideas. He extended and perfected plane and solid analytic geometry, introduced the analytic approach to trigonometry, and was responsible for the modern treatment of the functions ln and exp. It was though his work that the symbols e, π, and i became common currency for all mathematicians, as well as the functions sin and cos.

He was the first and greatest master of infinite series, products, and fractions; in 1736, he made the wonderful discovery that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6},$$

and also found the sums of the reciprocals of the fourth and sixth powers. (A closed form for the sum of the reciprocals of the cubes is still unknown.)

The foundations of classical mechanics had been laid down by Newton, but Euler was the principle architect. In his treatise of 1736 he was the first to explicitly introduce the concept of a point-like particle, and he was the also the first to study the acceleration of a particle moving along any curve and to use the notion of a vector in connection with velocity and acceleration. His continued successes in mathematical physics were so numerous, and his influence was so pervasive, that most of his discoveries are not credited to him at all and are taken for granted by physicists as part of the natural order of things.

Adapted from *Differential equations with applications and historical notes* (pp.136–146) by George F. Simmons (McGraw-Hill, 1991).

Questions

1. Explain, with sketches, the geometric meaning of the second derivative.
2. Find the second derivative of the following functions.
   (a) $f(x) = x^5 - 5x + 3$
   (b) $f(x) = \frac{x^2}{x-1}$
   (c) $f(x) = \sqrt{x} - \sqrt[4]{x}$
3. Sketch a function satisfying the given criteria.
   (a) (hint: your result should be an odd function)
      i. $f'(1) = f'(-1) = 0$,
      ii. $f'(x) < 0$ if $|x| < 1$,
      iii. $f'(x) > 0$ if $1 < |x| < 2$,
      iv. $f'(x) = -1$ if $|x| > 2$.
   (b) i. $f'(x) < 0$,
      ii. $f''(x) < 0$.

NCEA Level 3 Calculus
Differentiation Assignment

1. (a) (2 points) If $y = 3x^2 + 2x - \frac{1}{\sqrt{x+1}} + \frac{e^x}{\sin x}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   (b) An important mathematical skill is the ability to write down examples of objects satisfying certain properties.

      i. (2 points) Draw the graph of a function f passing through $(0, 1)$ such that $f'(x) < 0$ for all x, but $f''(x) > 0$ for all x.

      ii. (1 point) Give an explicit, simple example of such a function.

   (c) (3 points) Show that a solution to the differential equation
$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx(1-x)$$
   is given by
$$x(t) = \frac{1}{1 + \left(\frac{1}{x_0} - 1\right) e^{-rt}}.$$

2. Let f be the function defined by
$$f(x) = \sin(\tan(x + \pi/6) + \cos(4\pi)) \ln(1/x).$$

   (a) (2 points) Compute the derivative of $\tan(x + \pi/6) + \cos(4\pi)$ with respect to x.

   (b) (3 points) Write down explicitly $f'(x)$.

   (c) (3 points) Give the equation of the best linear approximation to $f(x)$ at $\left(\pi, -\ln(\pi)\sin\left(1 + \frac{1}{\sqrt{3}}\right)\right)$, giving constants to three decimal places.

3. (a) (2 points) Compute the derivative of $(x + 1)^3$ using the definition of the derivative.

   (b) (6 points) You may use the following lemma in answering this question. **Do not attempt to prove this lemma.**

   > **Squeeze Lemma**
   > If f, g, and h are all functions such that $f(x) \leq g(x) \leq h(x)$ for all x in the domain of definition of the three functions, and if $\lim_{x\to h} f(x) = \lim_{x\to h} h(x) = L$, then $\lim_{x\to h} g(x) = L$.

      i. Draw a sketch that displays the intuition behind the squeeze lemma.

      ii. Show that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$. [Hint: use the fact that $-1 \leq \sin x \leq 1$.]

NCEA Level 3 Calculus (Differentiation)
8. Optimisation

> **Goal for this week**
>
> To give a simple application of the geometry we looked at last week.

Recall from Level 2 that a **local maximum** of a function f is some point $(x, f(x))$ such that, for a sufficiently small interval around x, whenever y is in the interval then $f(y) \leq f(x)$. A **local minimum** is defined in a similar way. Local extrema are also sometimes called **relative extrema**.

Many optimisation problems in applied mathematics can be reduced to finding relative extrema.

Examples.

1. The function $x \mapsto x^2$ has a local minimum at $(0, 0)$.
2. The function $x \mapsto 2x^3 + 15x^2 + 36x + 2$ has a local maximum at $(-3, -25)$ and a local minimum at $(-2, -26)$.
3. The function $x \mapsto \sin x$ has a local maximum at $(2n\pi + \frac{\pi}{2}, 1)$ for every integer n, and a local minimum at $(2n\pi - \frac{\pi}{2}, 1)$ for every integer n.

For classification, we have the following theorem which links the location of relative extrema to the value of the derivative. Rather than memorising the proof, you should remember the geometric idea:- the derivative is changing from a positive value to a negative value (or vice versa), and so must pass through zero.

Theorem (Fermat's theorem)**.** *Let f be a function; suppose x_0 is a point in the interior of the domain of f, and that f has a relative extremum at $(x_0, f(x_0))$. Then $f'(x_0) = 0$.*

The proof of this is relatively straightforward; we just need the concept of left- and right-handed limits. Recall that, roughly speaking, a function has a limit at a point if it approaches the same value from both the left and the right. For left- and right-handed limits, we only require the function to approach a value from one side or the other.

Proof of Fermat's theorem (optional). Suppose f attains a relative maximum at x_0. Then for all h sufficiently close to zero, we have $f(x_0 + h) - f(x_0) \leq 0$. Hence, if $h < 0$, we have $\frac{f(x_0+h)-f(x_0)}{h} \geq 0$ (i.e. the derivative to the left is positive) and if $h > 0$, we have $\frac{f(x_0+h)-f(x_0)}{h} \leq 0$ (i.e. the derivative to the right is negative). Taking left- and right-hand limits around zero, we have the following chain of inequalities:

$$f'(x_0) = \lim_{h \to 0-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0 \geq \lim_{h \to 0+} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

Note that we use the fact that the derivative exists at x_0 and so the left- and right-hand limits both tend to the same value. Then the desired result, $f'(x) = 0$, follows directly.

For a relative minimum, the proof is essentially the same but with some inequalities swapped. □

Motivated by this theorem, we define a **critical point** of a function f to be some value x in the domain of f such that either $f'(x) = 0$, or $f'(x)$ is undefined. In the first case, we also call the value a **stationary point**. All local extrema occur at critical points, but not all critical points occur at extrema.

Examples.

1. The function $x \mapsto 2x^3 + 15x^2 + 36x + 2$ above has critical points $x = -2$ and $x = -3$. Both of these are local extrema.
2. The function $x \mapsto x^3$ above has a critical point at $x = 0$, but does not have a local extrema there.
3. The function $x \mapsto \frac{1}{x}$ *does not* have a critical point at $x = 0$, **because it is not defined there**.

Classifying Critical Points

We can use the first derivative to classify extrema as either maxima or minima.

1. Determine all critical points of f.
2. Determine the sign of $f'(x)$ to the left and right of each critical point x_0:
   - If $f'(x)$ changes from positive to negative as we move from left to right across x_0, then $f(x)$ has a local maximum at x_0.
   - If $f'(x)$ changes from negative to positive as we move from left to right across x_0, then $f(x)$ has a local minimum at x_0.
   - If $f'(x)$ does not change sign across x_0, then $f(x)$ does not have a relative extremum at x_0 (e.g. $y = x^3$).

On the other hand, using the second derivative, we can come up with a second test:

1. Compute $f'(x)$ and $f''(x)$.
2. Find all the stationary points of f by finding all the points x_0 such that $f'(x_0) = 0$.
3. Determine the sign of $f''(x)$ for each stationary point x_0:
   - If $f''(x_0) < 0$, then $f(x)$ has a relative maximum at x_0.
   - If $f''(x_0) > 0$, then $f(x)$ has a relative minimum at x_0.
   - If $f''(x_0) = 0$, then $f(x)$ could have a relative maximum, a relative minimum, or neither.

Example. Find and classify the critical points of $y = x^3 - 3x^2 + 6$.

Solution. We have $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 6$. Hence the critical points are $x = 0$ and $x = 2$. At the former point, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, and so the point is a maximum; at the latter point, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ and so the point is a minimum.

Example. Find two numbers whose difference is 100 and whose product is a minimum.

Solution. Let the two numbers be x and $x+100$. We wish to minimise $y = x(x+100)$; clearly $y' = 2x+100$, and so $x = -50$ is a critical point. To the left of $x = -50$, the derivative is negative; to the right, the derivative is positive. Hence $x = -50$ is indeed a minimum. The two required numbers are therefore -50 and 50.

Example. Find and classify the critical points of $y = (x-1)^2 + \ln x$.

Solution. The derivative is $y' = 2x - 2 + \frac{1}{x}$. We therefore have one critical point at $x = 0$ (where y' is undefined); this is an asymptote. Setting $y' = 0$, we have $0 = 2x - 2 + \frac{1}{x} = 2x^2 - 2x + 1$ which has no real roots. Hence $x = 0$ is the only critical point, and the curve has no local extrema.

Example. A rectangular plot of land is to be fenced using two varieties of fence. Two opposite sides will use fences selling for \$3 per metre, while the other two sides will use cheaper fence selling for \$2 per metre. Given that the total budget is \$1200, what is the greatest area of land which can be fenced?

Solution. Let x be the length of one of the expensive sides; then the length of one of the cheaper sides is $\frac{1}{2}(1200 - 3x)$, and the total area is $A = \frac{1}{2}x(1200 - 3x) = \frac{1}{2}(1200x - 3x^2)$. Hence $\frac{\mathrm{d}A}{\mathrm{d}x} = 600 - 3x$. We wish to find the maximum area, so set $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$; hence $3x = 600$ and $x = 200$. Note that the second derivative is always negative, so this stationary point must be a maximum as required. The length of the other side will be $\frac{1}{2}(1200 - 600) = 300$, and so the maximum area is $300 \times 200 = 60000$ square metres.

Questions

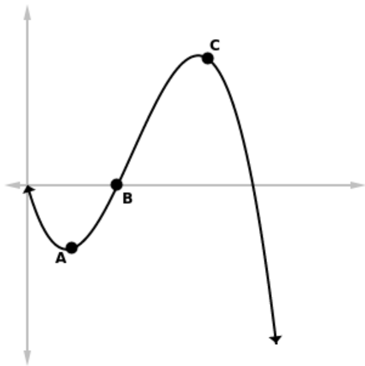
1. Write down a definition of a local minimum similar to that given for a maximum. A

2. Show that $f(x) = x^4$ has $f''(0) = 0$ but not a point of inflection at $x = 0$ (in fact, it has a minimum at that point). A

3. Describe the advantages and disadvantages of the first and second derivative tests for local extrema. A

4. Describe the local extrema, concavity, and points of inflection of the function $f(x) = x^4 - 4x^3$. M

5. Consider the following graph: A



Find the signs of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the three points A, B, and C.

6. Find all the local extrema of the following curves in the given intervals, and classify them as maxima, minima, or neither. M
   (a) $f(x) = \sin x - \cos x$ on the interval $0 < x < \pi$
   (b) $g(x) = x^3 - x^2 + x - 1$ on the interval $-\infty < x < \infty$

7. The sum of two positive numbers x and y is 16. Find the smallest possible value for $S = x^2 + y^2$. M

8. A box with an open top is to be constructed from a square piece of cardboard with a side length of 3 m by cutting out a square from each of the four corners and bending up the sides. Find the dimensions of the resultant box of maximum volume. M

9. Find the dimensions of a rectangle with area $1000\,\mathrm{m}^2$ such that the perimeter is minimised. M

10. A window consisting of a rectangle topped with a semicircle is to have a fixed perimeter p. Find the radius of the semicircle in terms of p if the total area is to be maximised. M

11. A thin wire of length L is cut in two and the resulting lengths are bent to make a square and an equilateral triangle. Where should the wire be cut to make the total area of the shapes (a) a maximum and (b) a minimum? E

12. Find the point on the line $y = 2x + 3$ closest to the origin. E

13. Find the point on the curve $y = \sqrt{x}$ closest to $(3, 0)$. E

14. By finding the x- and y- intercepts, the asymptotes, the critical points, the intervals of increase and decrease, the intervals of concavity, and any other important points, sketch the following functions (199): E
    (a) $f(x) = \frac{x^2}{4-x^2}$

(b) $f(x) = \frac{4x}{x^2+1}$ [*Hint: consider what happens to $f(x)$ as $x \to \pm\infty$.*]

(c) $f(x) = \frac{x^2-4x+5}{x-2} = x - 2 + \frac{1}{x-2}$ [*Hint: consider what happens to $f(x) - (x-2)$ as $x \to \pm\infty$.*]

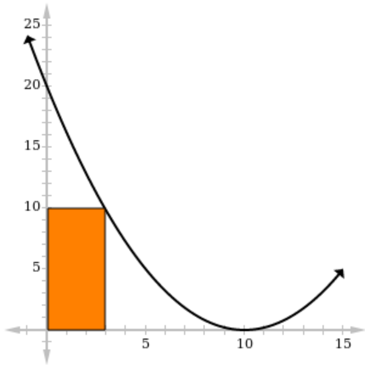
15. A cone with height h is inscribed in a larger cone of height H such that the vertex of the small cone is at the centre of the base of the larger cone. Show that the maximum volume of the smaller cone occurs when $h = \frac{1}{3}H$. E

16. Show that the polynomial $p(x) = 10x^3 + x^2 + x - 34$ has exactly one real zero. E

17. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one third of the sheet on each side by an angle θ. What angle should be chosen in order to obtain the maximum possible volume? E

18. A steel pipe is carried around a right-angled corner from a hallway 3 m wide into a hallway 2 m wide. What is the length of the longest pipe that can be carried horizontally around the corner? [*Hint: this is actually a minimisation problem, despite the wording.*] E

19. A large orange rectangle is to be drawn with one corner sitting on the origin and the opposite corner lying on the curve $y = 0.2(x-10)^2$. What is the maximum possible area of the rectangle? E



20. Show that $\frac{x^2+1}{x} \geq 2$; hence (or otherwise) show that $\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geq 8$. S

21. Scholarship 2013: Prince Ruperts drops are made by dropping molten glass into cold water. A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi(e^{-x} - e^{-2x})}$ for $x \geq 0$, where ϕ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. S

(a) Where is the modelled drop widest, and how wide is it there?

(b) The drop changes shape at a point B, where the concavity of the function is zero. Use

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{\phi}\frac{e^{2x} - 6e^x + 4}{y^2e^{4x}}$$

to find the exact x-ordinate of B.

22. Scholarship 2014: A family of functions is built from two functions f and g, with a new function h_p defined for each value of p, $0 \leq p \leq 1$: S

$$f(x) = 2 + \sin x$$
$$g(x) = 26 + \sin x$$
$$h_p(x) = [f(x)]^{1-p}[g(x)]^p.$$

Define a fourth function S, where $S(p)$ is the difference between the maximum and the minimum values of h_p. Find the exact value of p that maximises S.

Note that if a is constant, $\frac{\mathrm{d}}{\mathrm{d}x}a^x = (\ln a)a^x$.

NCEA Level 3 Calculus (Differentiation)
8. Optimisation (Homework)

Reading

Our students are not fencing in farm fields, cutting wires and folding them, or designing windows, so they are often uninspired by the optimization problems we give them. They seem like something that "someone, somewhere" might use, but the examples feel distant.

What are good examples of constrained optimization problems (perhaps not simple!) that today's students might actually encounter in their lives?

Bad Optimization Problems

I thought that Jack M made an interesting comment about this question:

> There aren't any. There may be situations where it's possible to apply optimization to solve a problem you've encountered, but in none of these cases is it honestly worth the effort of solving the problem analytically. I optimize path lengths every day when I walk across the grass on my way to classes, but I'm not going to get out a notebook and calculate an optimal route just to save myself twelve seconds of walking every morning. Mathematics beyond basic arithmetic is simply not useful in ordinary life. But I'm not sure if that's exactly what you mean. — JackM

To some extent, I agree with this comment. With few exceptions, mathematics beyond basic arithmetic is simply not useful in everyday life. Students know this, and you'll have trouble convincing them otherwise.

Because of this, I've always found "everyday"-style calculus problems a little artificial. Consider the following problem from Stewart's Calculus:

> A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

The proper response to this question is: who cares? Is there any reason to calculate this length precisely? Why would anyone ever use calculus to compute this? If you have an actual building and an actual ladder, you could just try it and see if the ladder fits. If you don't have a specific ladder in mind (e.g. you are buying a ladder), the thing to do would be to draw the situation on paper and then use a ruler to estimate the minimum length. Of course, it's neat that you can use calculus to solve this problem precisely, but this is more of a curiosity than a legitimate application.

Chris specifically mentions the farmer fence problem, the wire-cutting problem, and the Norman window problem as not relevant to the students' lives. I agree—none of these problems are relevant. But it's not because the students aren't farmers, or wire-cutters, or architects. Even in a class full of future farmers, the fence problem would still be bad, because farmers don't use calculus to plan their fences.

Good Optimization Problems

What calculus is useful for is science, economics, engineering, industrial operations, finance, and so forth. That is, it's useful for all the things that make our society run. Most students who take calculus at a university are planning to go into one of these fields, so calculus will be relevant in their lives — specifically in their future studies and in their professions.

Here's something that's closer to a real-life optimization problem:

> When a critically damped RLC circuit is connected to a voltage source, the current I in the circuit varies with time according to the equation
>
> $$I = \left(\frac{V}{L}\right) te^{-Rt/(2L)}$$
>
> where V is the applied voltage, L is the inductance, and R is the resistance (all of which are constant).
>
> Suppose an RLC circuit with a resistance of $30.0\,\Omega$ and an inductance of $0.400\,\mathrm{H}$ is attached to a $12.0\,\mathrm{V}$ voltage source. Find the maximum current that will occur in the circuit.

This is at least close to something that a physics or engineering student might actually come across in their future studies. It's real in a way that the farmer fence problem isn't, and even students who don't plan to study physics can sense that this is a legitimate application. (By the way, if you have good students, you might even ask them to come up with a formula for the maximum current, without giving them specific numbers for V, L and R. This has the advantage that it can't simply be solved using a graphing calculator.)

Of course, this isn't actually a constrained optimization problem — it's just an optimization problem. I'm not actually aware of any place in science that simple constrained optimization problems arise, although there are examples from economics (maximizing utility), finance (optimal portfolios), and industrial design (e.g. shape of a can type problems). When I cover constrained optimization in calculus, I usually stick to industrial-type problems (best cans, best shipping crates/boxes, best pipeline across a river, etc.), but that's probably just because I don't know enough about economics or finance to make up problems that involve them.

Finally, I should mention that I've never found the optimization portion of Calculus I particularly compelling. It's good to introduce the idea of optimization, but setting the derivative equal to zero isn't actually a very useful optimization technique by itself. It only really works for simple formulas—for anything complicated it just replaces one essentially numerical problem (finding the maximum of a function) with another (finding roots of a function). I agree that it should be covered, but it's far from the most important application of calculus.

From `https://matheducators.stackexchange.com/questions/1550`.

Questions

1. What is the minimum vertical distance between the parabolae $y = x^2 + 1$ and $y = x - x^2$?

2. Show that $3x + 2\cos x + 5 = 0$ has exactly one real root by showing that it is increasing everywhere and crosses the x-axis somewhere between two values of x.

3. Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

4. Let $ABCD$ be a square piece of paper with sides of length 1 m. A quarter-circle is drawn from B to D with centre A. The piece of paper is folded along EF, with E on AB and F on AD, so that A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF can have. (Hint: you may want to introduce a coordinate system.)

NCEA Level 3 Calculus (Differentiation)
9. Implicit Differentiation

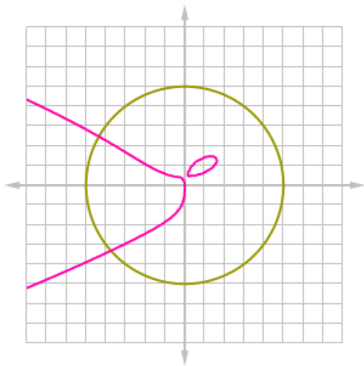
Goal for this week

To learn to differentiate functions which describe more complicated curves.

This week, we will take a break from applications and look at some more interesting kinds of curves. Consider the following equations:

$$x^2 + y^2 = 25 \text{ and } x^3 + y^4 = 5xy - 2x$$

We can graph all the values (x, y) which satisfy these equations; here we have drawn the graph of the first equation (the circle) in green and the graph of the second (the weird disconnected one) in purple.



We can solve for the first y, as $y = \pm\sqrt{25 - x^2}$; however, the second is much harder to solve and so we cannot find its derivative using the techniques we have studied so far. These equations are examples of **implicit functions** of x. Note that neither is a 'real' function since they both fail the vertical-line test.

The key observation here is that *differentiation is an operation*, similar to addition. Just like we can add 3 to both sides of the true equation $2 + 4 = 6$ to obtain another true equation $2 + 3 + 4 = 3 + 6$, we can differentiate both sides of an equation to obtain another true equation. The only catch is that we must remember that y is a function of x and so we must employ the chain rule.

Example. If $x^2 + y^2 = 25$, by differentiating both sides with respect to x we obtain $2x + \frac{\mathrm{d}y}{\mathrm{d}x} 2y = 0$ and therefore we have $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. Note that this depends on both x and y which makes sense: at $x = 0$, for example, we have two gradients (both of which are zero).

Example. If $x^3 + y^4 = 5xy - 2x$, then by differentiating both sides with respect to x we obtain $3x^2 + \frac{\mathrm{d}y}{\mathrm{d}x} 4y^3 = 5y + 5x\frac{\mathrm{d}y}{\mathrm{d}x} - 2$ (being careful to use the product and chain rules in differentiating). Hence we have that the derivative is:

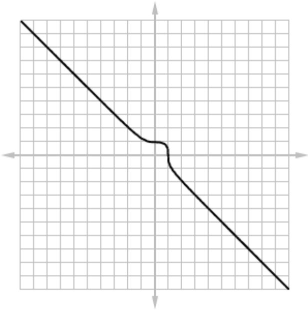
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y - 3x^2 - 2}{4y^3 - 5x}$$

It is very important to always specify the variable with respect to which you are differentiating.

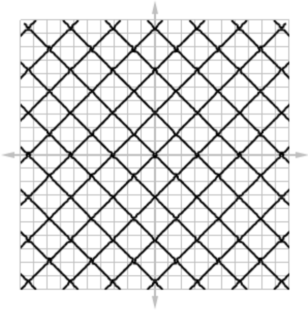
Questions

1. In each case, look at the cool pictures and find y': M

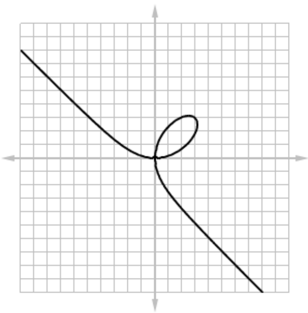
(a) $x^3 + y^3 = 1$



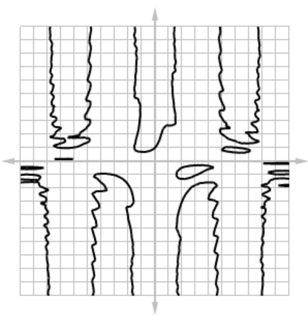
(b) $\sin^2 y + \cos^2 x = 1$



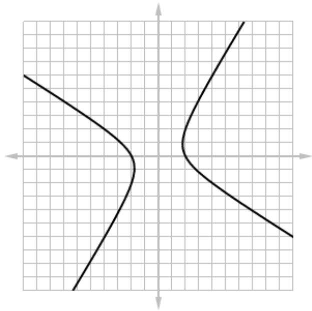
(c) $x^3 + y^3 = 6xy$ (the folium of Descartes)



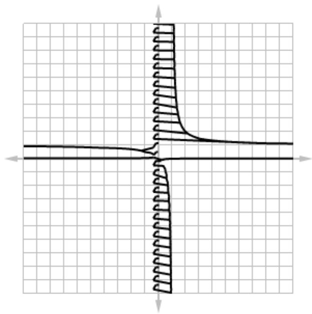
(d) $y \cos x = 1 + \sin(xy)$



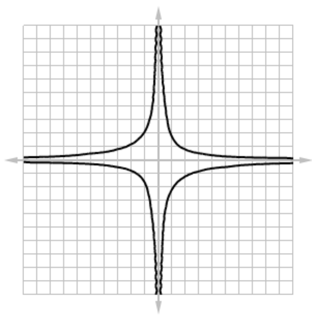
(e) $x^2 + xy - y^2 = 4$



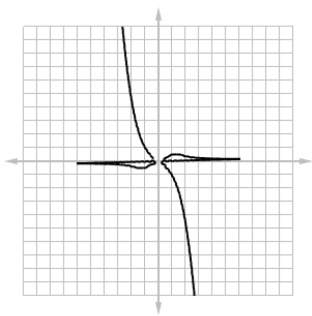
(f) $\frac{1}{x} + \frac{1}{y} = 1$



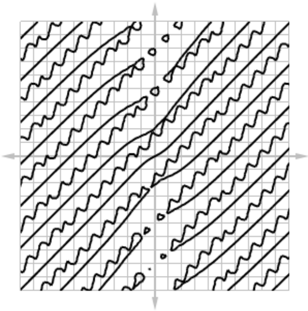
(g) $x^2y^2 + x \sin y = 4$



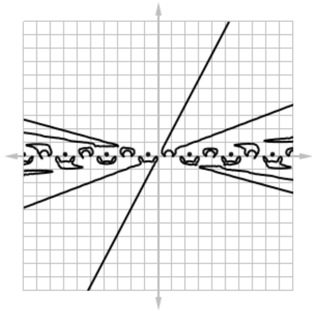
(h) $x^4y^2 - x^3y + 2xy^3 = 0$



(i) $\tan(x-y) = \frac{y}{1+x^2}$



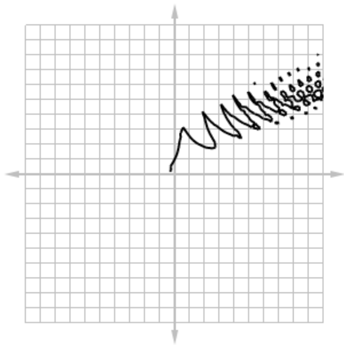
(j) $\sin\left(\frac{x}{y}\right) = \frac{1}{2}$



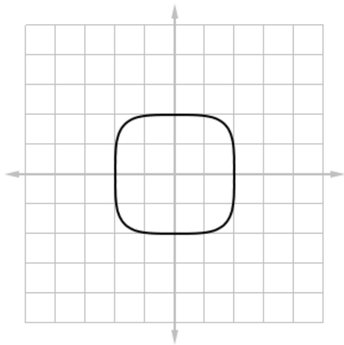
2. Consider the circle $x^2 + y^2 = 1$. Find the equation of the tangent to the curve at $(\sqrt{2}, \sqrt{2})$. M

3. The ellipse $x^2 + 3y^2 = 36$ has two tangent lines passing through the point $(12, 3)$. Find both. *This question is similar to one from the 2015 Scholarship paper.* S

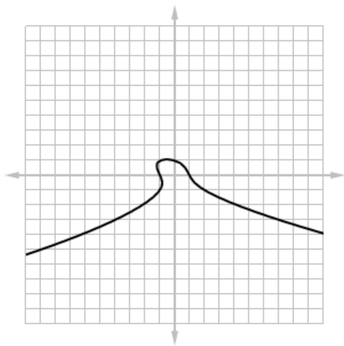
4. Find x' and y' if $\ln(y) = \sin(xy) + \frac{x}{y}$. M



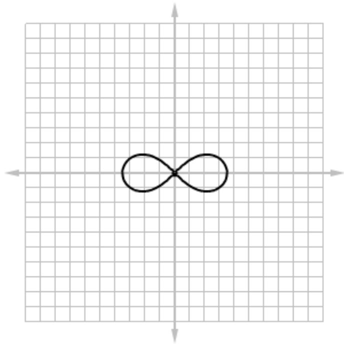
5. Find y'' if $x^4 + y^4 = 16$. M



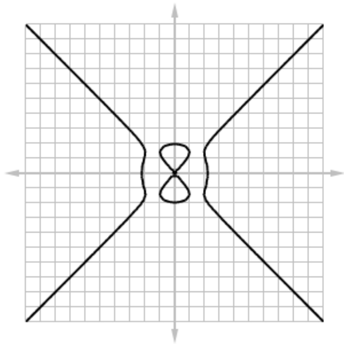
6. If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$. M



7. Find a tangent line to the curve $2(x^2+y^2)^2 = 25(x^2-y^2)$ at the point $(3,1)$. This curve is known as a lemniscate. M



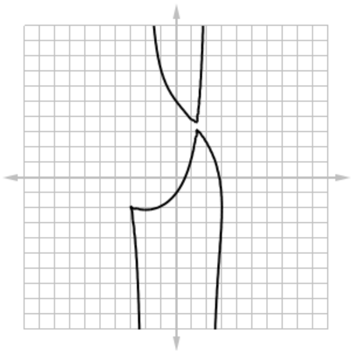
8. Find a tangent line to the curve $y^2(y^2-4) = x^2(x^2-5)$ at the point $(0,-2)$. This curve is known as a devil's curve. M



9. Consider the ellipse $x^2 - xy + y^2 = 3$.
   (a) Find the points where the ellipse crosses the x-axis. A
   (b) Show that the tangent lines of the curve at these points are parallel. M
   (c) Find the maximum and minimum points of the curve. E
10. Consider a circle C that is tangent to $3x+4y-12=0$ at $(0,3)$ and contains $(2,-1)$. Set up equations that would determine the centre (h,k) and radius r of C. E
11. The Bessel function of order 0, $y = J(x)$, satisfies the differential equation S
$$xy'' + y' + xy = 2$$
for all values of x. The value of the function at 0 is $J(0) = 1$.
   (a) Find $J'(0)$.
   (b) Use implicit differentiation to find $J''(0)$.
12. Scholarship 2018: Suppose a circle with centre O is drawn, and a point A is picked within the circle. Where should a point P be placed on the circumference of the circle such that the interior angle of the triangle OAP at P is maximised? S

13. Consider the following family of curves, known as Durer's shell curves (shown here for $a = 2$, $b = 3$): S

$$(x^2 + xy + ax - b^2)^2 = (b^2 - x^2)(x - y + a)^2.$$

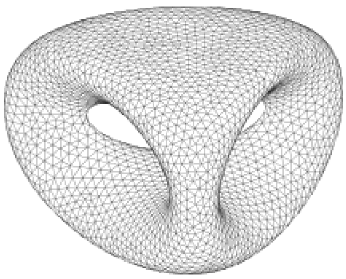


(a) For which value(s) of b does the curve become a straight line?

(b) Suppose that we restrict $a = \frac{b}{2}$. Find all non-differentiable points on the curve.

14. Moving into three dimensions, let us consider the surface described by O

$$2y(y^2 - 3x^2)(1 - z^2) + (x^2 + y^2)^2 - (9z^2 - 1)(1 - z^2) = 0.$$



(a) Verify that the point $\left(1, 1, \frac{1}{\sqrt{3}}\right)$ is on the surface.

(b) Find the values of $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$ at this point (holding y and x constant, respectively). What do these derivatives represent?

(c) Write down the equations of the tangent lines to the surface in the y and x directions.

(d) Find an equation for the unique plane containing both tangent lines. Describe what this plane represents geometrically.

NCEA Level 3 Calculus (Differentiation)
9. Implicit Differentiation (Homework)

Reading

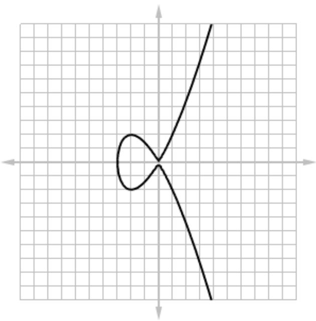
Underpinning all our work this week was the idea that if an implicit formula in x and y is 'nice enough' then there is a way to find a 'function' from x to y that we can differentiate. This rather vague notion is formalised by the rather important *implicit function theorem*, which states that if an equation $F(x, y) = 0$ has solution $(x, y) = (a, b)$ then, under certain conditions*, the equation implicitly defines in some region around x a function with a continuous derivative that takes the value b at $x = a$ — in other words, there is some function whose graph is the graph of $F(x, y) = 0$ for all of the points around the point (a, b) that we care about.

The proof of the theorem is long and we won't attempt it here; the important thing to take away is that this notion of taking a graph and then looking at the function which it describes in a small region is well-defined, and well-defined in such a way that we can do calculus on it.

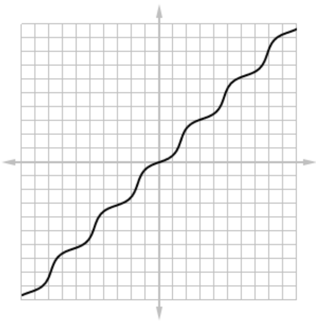
Questions

1. Find y' in each case:

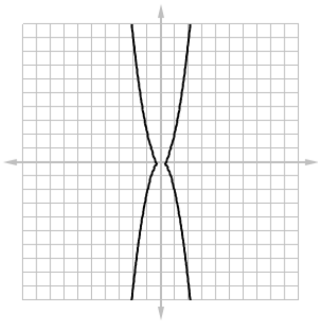
   (a) $y^2 = x^3 + 3x^2$ (Tschirnhausen cubic)

   (b) $\sin(x + y) = 2x - 2y$

   (c) $y^2 = 5x^4 - x^2$ (kampyle of Eudoxus)

2. Find the equation of the normal line to the curve $x^2 + 2xy - y^2 + x = 2$ at the point $(1, 2)$.
3. Show that the sum of the x and y intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is just c.

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* Essentially, that the graph of the function is not 'vertical' at (a, b).

NCEA Level 3 Calculus (Differentiation)
10. Inverse Functions

Goal for this week

To learn to find the derivative of the inverse of a function if we know the derivative of the function.

We need to take a quick pitstop this week to deal with one more differentiation rule before we can start looking at a few more applications next week and then some interesting functions in higher dimensions later on.

Definition. A function is called **one-to-one** (or **injective**) if $f(x) = f(y)$ implies that $x = y$ (i.e. two different inputs can never give the same output/the function passes the horizontal-line test).

Let f be a function that is one-to-one. Then the **inverse** of f is the (unique) function f^{-1} such that

$$f(x) = y \iff f^{-1}(y) = x.$$

In other words, $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$.

Example. Here are some functions with their inverses:

Function	Inverse	Notes
e^x	$\ln x$	Note that $\ln x$ is defined only when $x > 0$ since $e^x > 0$ for all real x.
$\sin x$	$\sin^{-1} x$	Note that $\sin^{-1} x$ is only defined when $-\pi < x \leq \pi$ since otherwise $\sin x$ is not one-to-one.
$\cos x$	$\cos^{-1} x$	Note that $\cos^{-1} x$ is only defined when $-\pi < x \leq \pi$ since otherwise $\cos x$ is not one-to-one.
$\tan x$	$\tan^{-1} x$	Note that $\tan^{-1} x$ is defined for all x (why?), and so $\tan^{-1} x \neq \frac{\sin^{-1} x}{\cos^{-1} x}$.
x^2	$\sqrt{x}$	When x is positive.

The graph of the inverse of a function is the reflection of the graph of the original function around the line $x = y$ (essentially, we swap the x and y axes).

Theorem. *In general, if f is a function passing through (x, y), and f^{-1} is the inverse of f, then*

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(x)}.$$

Mnemonically, we can write this as

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}.$$

Proof. We have that $f(f^{-1}(y)) = y$. Taking the derivative of both sides, $f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$ and therefore $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$. □

This proof is not hard, but it is sometimes difficult to work out which x's and y's go where. We'll do a couple of examples now; the first one is one that we can already do, and so this gives us the advantage of knowing what the result should look like before we get there.

Example. Suppose f is defined by $y = f(x) = x^2$. Then $f^{-1}(y) = \sqrt{y}$. We evaluate it in three ways.

1. Power law: $f^{-1}(y) = y^{\frac{1}{2}}$ so $(f^{-1})(y) = \frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$.

2. Rigourous derivative of inverse: We have that $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$, so $(\sqrt{y})' = \frac{1}{2\sqrt{y}}$ (since $f'(x) = 2x$).

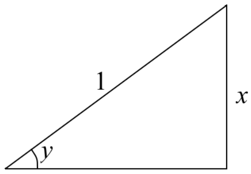
3. Mnemonic derivative of inverse: We wish to find $\frac{\mathrm{d}x}{\mathrm{d}y}$. Now $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ and so (by the mnemonic)

$$(f^{-1})'(y) = \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}.$$

Example. In order to illustrate the general process of finding the derivatives of inverse functions without symbol-pushing using the theorem above, let us now find the derivative of $y = \sin^{-1} x$.

$$\begin{aligned}
y &= \sin^{-1} x \\
\sin y &= x \\
\frac{\mathrm{d}y}{\mathrm{d}x} \cos y &= 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos y} = \frac{1}{\cos \sin^{-1} x} = \frac{1}{\sqrt{1 - x^2}}.
\end{aligned}$$

The identity $\cos \sin^{-1} x = \sqrt{1 - x^2}$ comes from the following triangle:



By the same kind of calculation, we obtain the following table which gives the derivatives of inverses of the three primary trigonometric functions and their reciprocals.

Function	**Derivative**	**Function**	**Derivative**
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\csc^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$	$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\cot^{-1} x$	$-\frac{1}{x^2+1}$

Questions

1. Prove or disprove the following statements: M
   (a) The function $f : x \mapsto x^2 + x + 1$ is one-to-one (where x is real).
   (b) The function $g : x \mapsto 2^x$ is one-to-one (where x is a positive real).

2. Determine whether the following functions have inverses on the given interval: M
   (a) $x \mapsto x^3$ (on $\mathbb{R}$)
   (b) $y \mapsto y^4$ (on $\mathbb{R}$)
   (c) $y \mapsto y^4$ (for $y \geq 0$)
   (d) $y \mapsto y^4$ (for $y > 0$)
   (e) $\theta \mapsto \cos^{-1} \theta$ (on $\mathbb{R}$)
   (f) $\theta \mapsto \cos^{-1} \theta$ (for $-1 \leq \theta \leq 1$)

3. True or false: A
   (a) $\cos^{-1} x = \frac{1}{\cos x}$
   (b) If $x > 0$ then $(\ln x)^6 = 6 \ln x$
   (c) $\tan^{-1}(-1) = \frac{3\pi}{4}$ (think about which arm of $\tan x$ we're talking about)
   (d) The inverse of $f(x) = e^{3x}$ is $f^{-1}(x) = \frac{1}{3} \ln x$.
4. Find the derivative of $f(x) = \ln(e^x)$ in two different ways. A
5. Find y' if: A
   (a) $y = \sin^{-1} 2x$
   (b) $x = \sin^2 y$
   (c) $y = x + \tan^{-1} y$
   (d) $y = \ln \sin x - \frac{1}{2} \sin^2 x$
   (e) $y = 24 \arctan x + \arcsin \sqrt{x}$
   (f) $y = \sqrt{\sec^{-1} 2x}$
6. Find the local extrema, areas of concavity, and inflection points of the following functions; hence sketch their graphs. E
   (a) $y = e^x \sin x$ for $-\pi < x < \pi$
   (b) $y = x + \ln(x^2 + 1)$
   (c) $y = \sin^{-1}(1/x)$
7. Justify intuitively, without invoking the happy coincidence that our notation for derivatives looks like a fraction, the statement that $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$. E
8. If $f'(x) = \tan^{-1} x$, find $(f^{-1})'(x)$. E
9. Using the definition of ln as the inverse of $\exp : x \mapsto e^x$, show that $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$. *Note that the usual definition of* ln *goes in the reverse of this; we will go in that direction in a few weeks.* E
10. Prove the formulae for the derivatives of $\cos^{-1}$ and $\tan^{-1}$, using a similar method to that for $\sin^{-1} x$. S
11. Scholarship 2012: Consider the equation $x^n = \tan(ny)$, where n is a constant. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. S
12. Scholarship 2017: The functions sinh and cosh are defined as follows. S

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right),$$
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right).$$

The inverse function of sinh is denoted by $\sinh^{-1}$. By implicit differentiation, or otherwise, show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}.$$

NCEA Level 3 Calculus (Differentiation)
10. Inverse Functions (Homework)

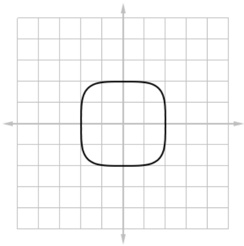
Reading

Go and watch...

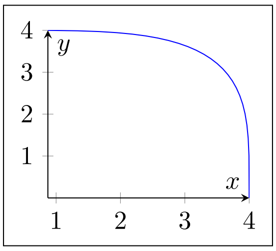
`https://www.youtube.com/watch?v=J-BdwKtsnYs`

There is an important theorem in analysis that is relatively easy to guess is true within the limited range of functions we have looked at: the inverse function theorem. The inverse function theorem essentially states that if a function is continuous at some point, and if its derivative exists and is continuous at that point, then the function has an inverse around that point.

For example, consider the curve $x^4 + y^4 = 16$ graphed here. Notice that the curve fails the horizontal line test everywhere (so does not have an inverse) and fails the vertical line test everywhere (so is not even a function).



This curve can be represented by the function $\theta \mapsto (\pm 4\sqrt{\cos\theta}, \pm 4\sqrt{\sin\theta})$ for $0 \le t < 2\pi$. Let's zoom in and look at the first quadrant.



In particular, if we restrict ourselves to this region then the curve is both a function *and* is 1-1 — so this section of the function has an inverse. This occurs because each y-value corresponds to a particular θ-value in this segment, and then we can (using our parameterisation) map that θ-value onto the corresponding x-value.

Questions

1. Find the derivatives with respect to x:
   (a) $y = \tan^{-1}(x^2)$
   (b) $\tan f(x) = x$
   (c) $g(x) = \arctan(\arcsin\sqrt{x})$
2. Show that
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\tan^{-1}x + \frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}\right) = \frac{1}{(1+x)(1+x^2)}$$
3. Prove that $\frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}x = -\frac{1}{x^2+1}$.

NCEA Level 3 Calculus (Differentiation)
11. Related Rates of Change

Goal for this week

To use the chain rule for problem-solving.

Our goal next week is to begin to look at more types of curves. In order to do this, there is a bit of machinery that we need to become confident with; as well as looking at the inverses of functions (which we did last week), we need to learn more about how compositions of functions cooperate with derivatives.

We can use the chain rule to relate rates of change together — for example, the area of a circle is given by $A = \pi r^2$ and so the rate of change of area with respect to radius $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$; but if r varies with respect to time then we can find the rate of change of the area with respect to time using the chain rule.

A useful mnemonic is (if x is a function of y which is itself a function of z)

$$\frac{\mathrm{d}x}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\mathrm{d}x}{\mathrm{d}z}. \qquad \text{(chain rule)}$$

We can also apply the inverse function rule for differentiation, which tells us that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}. \qquad \text{(inverse function rule)}$$

These two operations allow us to rearrange equations as if $\frac{\mathrm{d}y}{\mathrm{d}x}$ were a fraction. There isn't much of a problem if you do think of it in this way, as long as you're careful.

Example. A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1\,\mathrm{m\,s^{-1}}$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?

Solution. Let x be the distance of the bottom of the ladder from the wall, and let y be the height of the top of the ladder up the wall. We have $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ and $x = 3$; we also know that $y = \sqrt{25 - x^2}$, so:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{\sqrt{25 - x^2}} \cdot 1$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{x=3} = -\frac{3}{\sqrt{25 - 9}} = -\frac{3}{4}.$$

Hence the ladder is sliding down the wall at a rate of $-0.75\,\mathrm{m\,s^{-1}}$.

Example. The radius of a sphere is increasing at a rate of $\frac{\mathrm{d}r}{\mathrm{d}t} = -\ln(t-1)$ metres per second. At what rate will the surface area of the sphere be growing at $t = 2$?

Solution. We have $\mathrm{SA} = 4\pi r^2$, so $\frac{\mathrm{dSA}}{\mathrm{d}r} = 8\pi r$ and

$$\frac{\mathrm{dSA}}{\mathrm{d}t} = \frac{\mathrm{dSA}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = -\ln(t-1) \times 8\pi r = 0.$$

The surface area of the sphere will be momentarily constant at $t = 2$.

Questions

1. Each side of a square is increasing at a rate of $6\,\mathrm{cm\,s^{-1}}$. At what rate is the area of the square increasing when the area of the square is $16\,\mathrm{cm^2}$? M
2. Gas is being forced into a spherical balloon at a rate of $400\,\mathrm{cm^3\,min^{-1}}$. How fast is the radius of the balloon increasing when the radius is 5 cm? M

3. If a snowball melts so that its surface area decreases at a rate of $1\,\mathrm{cm}^2\,\mathrm{min}^{-1}$. find the rate at which the diameter decreases when the diameter is $10\,\mathrm{cm}$. M

4. If $W = 4H^3S$ where $H = 3S^2 - 5$, find $\frac{\mathrm{d}W}{\mathrm{d}S}$ in terms of H and S. M

5. If $x^2 + y^2 + z^2 = 9$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$, and $\frac{\mathrm{d}y}{\mathrm{d}t} = 4$, find $\frac{\mathrm{d}z}{\mathrm{d}t}$ when $(x, y, z) = (2, 2, 1)$. M

6. The demand for an article varies inversely as the $\frac{5}{2}$ power of the selling price (i.e. if x is the number of articles sold and p is the price then $x = p^{-5/2}$). If the manufacture cost of one article is \$1, determine the selling price which will produce the greatest profit. E

7. A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle moves through the point $(1/3, 1)$, its x-ordinate increases at a rate of $\sqrt{10}\ \mathrm{cm\,s}^{-1}$. How fast is the distance from the particle to the origin changing at this instant? E

8. Gravel is dumped from a conveyor belt at a rate of $3\,\mathrm{m}^3\,\mathrm{min}^{-1}$, and forms a pile in the shape of a cone with equal height and base diameter. How fast is the height of the cone increasing when the pile is $3\,\mathrm{m}$ tall? M

9. The top of a ladder slides down a vertical wall at a rate of $0.15\,\mathrm{m\,s}^{-1}$. A the moment when the bottom of the ladder is $3\,\mathrm{m}$ from the bottom of the wall, it slides away from the wall at a rate of $0.2\,\mathrm{m\,s}^{-1}$. Find the length of the ladder. M

10. Two sides of a triangle have lengths $2\,\mathrm{m}$ and $3\,\mathrm{m}$. The angle between these sides is increasing at a rate of $4\,^\circ\,\mathrm{s}^{-1}$. How fast is the length of the third side changing when it is of length $4\,\mathrm{m}$? M

11. A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y-ordinate is decreasing at a rate of 3 units per second. How fast is the x-ordinate of the particle changing at that instant? E

12. The minute hand on a watch is $8\,\mathrm{mm}$ long and the second hand is $4\,\mathrm{mm}$ long. How fast is the distance between the tips of the hands changing at 1 o'clock? E

13. Scholarship 2015: A tank contains 200 litres of brine (solution of salt in water). Initially, the concentration is $0.5\,\mathrm{kg}$ of salt per litre. Brine containing $0.8\,\mathrm{kg}$ of salt per litre runs into the tank at a rate of 6 litres per minute. The mixture is kept thoroughly mixed and is running out at the same rate. S

    Find out how long it takes for the amount of salt in the tank to be $130\,\mathrm{kg}$.

NCEA Level 3 Calculus (Differentiation)
11. Related Rates of Change (Homework)

Reading

Problem-Solving Strategy

1. Read the problem carefully.
2. Identify the quantities given in the problem.
3. Identify the unknown quantity.
4. Draw a diagram and assign symbols to all quantities.
5. Write an equation relative the various quantities together (if necessary, using substitution to eliminate unwanted variables).
6. Use the chain rule to differentiate and solve for the unknown.

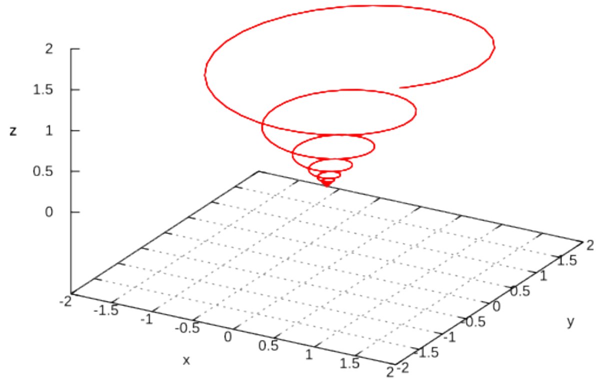
Questions

1. If V is the volume of a cube with edge length x and the cube expands as time passes, find $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of $\frac{\mathrm{d}x}{\mathrm{d}t}$.
2. A water tank has the shape of an inverted circular cone with base radius $2\,\mathrm{m}$ and height $4\,\mathrm{m}$. If water is being pumped into the tank at a rate of $2\,\mathrm{m}^3\,\mathrm{min}^{-1}$, find the rate at which the water level is rising when the water is $3\,\mathrm{m}$ deep.
3. A boat is pulled into a dock by a rope attached to the bow of the boad and passing through a pulley on the dock that is $1\,\mathrm{m}$ higher than the bow of the boat. If the rope is pulled in at a rate of $1\,\mathrm{m\,s}^{-1}$, how fast is the boat approaching the dock when it is $8\,\mathrm{m}$ from the dock?

NCEA Level 3 Calculus (Differentiation)
12. Parametric Functions

Goal for this week

To understand how calculus can be used to understand the geometry of a path through space.

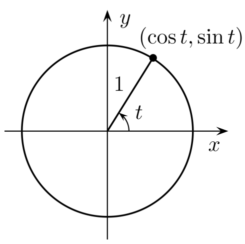


Some curves cannot be described simply with a function; for example, the above track of a particle is too complicated to analyse using any of the techniques which we have studied so far. One strategy which does work is to split the x, y, and z components apart and study them seperately. For example, we can **parameterise** the above curve as:

$$x(t) = e^{-t}\cos(10t)$$
$$y(t) = e^{-t}\sin(10t)$$
$$z(t) = e^{-t}.$$

With this example as our initial motivation, we move from discussing functions $\mathbb{R} \to \mathbb{R}$, and turn our attention to functions $\mathbb{R} \to \mathbb{R}^n$. This type of function is often called a **curve**; a set of component functions, like the one given above for the spiral, is called a **parameterisation**.

For a simpler example, consider the unit circle $x^2 + y^2 = 1$. By recalling the definitions of the trigonometric functions, we can parameterise the circle as $(x, y) = (\cos\theta, \sin\theta)$ for $0 \leq \theta < 2\pi$. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = -\cos\theta = -x$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \sin\theta = y$, so by the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{x}{y}$ — a much simpler calculation than taking the derivative of the square root required by working directly with the circle formula.



In order to justify this in general, we note that the output of a curve is a set of ordered pairs and is therefore (locally) a function from the first coordinate to the second coordinate. Suppose that we have some curve that is parametrised by $(x, y) = (f(t), g(t))$. Then $y = g(f^{-1}(x))$ and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{f'(f^{-1}(x))} \cdot g'(f^{-1}(x)) = \frac{1}{f'(t)} \cdot g'(t) = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}.$$

In order to find the second derivative, we replace y with $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}\frac{\mathrm{d}y}{\mathrm{d}x}}{\mathrm{d}x} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\mathrm{d}y}{\mathrm{d}x}\right) \cdot \frac{\mathrm{d}t}{\mathrm{d}x}\,.$$

Questions

1. In each case find $\frac{\mathrm{d}y}{\mathrm{d}x}$. M
   (a) $x = t\sin t$, $y = t^2 + t$
   (b) $x = 2\sec\theta$, $y = 3\tan\theta$
   (c) $x = \cos\theta$, $y = \cos 3\theta$
   (d) $x = e^{\sin\theta}$, $y = e^{\cos\theta}$

2. Find the equation of the chord joining the two points $t = 2$ and $t = 4$ on the curve $(x, y) = (2t - 3, t^3 + 6)$. A

3. Determine the point(s) of intersection of the curves γ and δ: M

$$\gamma : t \mapsto (t^2 - 2, t - 1)$$
$$\delta : t \mapsto (t, 2/t)$$

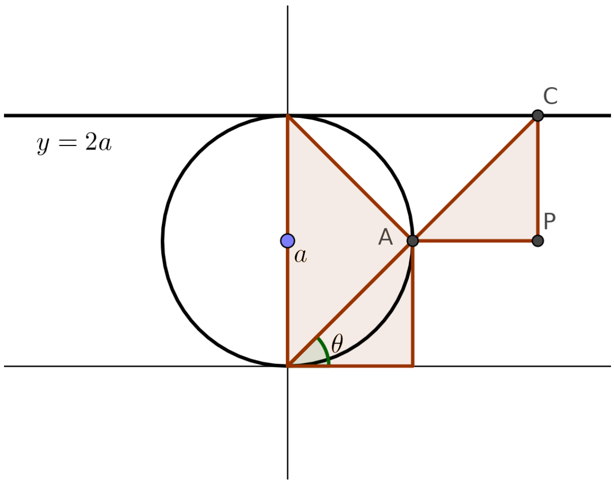
4. (a) If $y = 2t$ and $x = 4t^2$ define a curve, what is the gradient $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t? M
   (b) Show that this curve is a parabola. E

5. A curve has parametric equations $x = t^2 + 1$ and $y = t^3 + 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. M

6. Find the equation of the tangent to the curve $t \mapsto (2x^2 + 1, t^3 - 1)$ at $t = 2$. M

7. If $t \mapsto (x, y)$ is a parametric curve, find an expression for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ analagous to that found for the second derivative. E

8. A curve, called a *witch of Maria Agnesi*, consists of all possible positions of the point P in the diagram below. Show that the curve is given by $(x, y) = (2a\cot\theta, 2a\sin^2\theta)$ and find the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$. S



9. A particle moves through space over time; the position of the particle at time t is given by $(3\sin t, 2\cos t)$ $(0 \le t < 2\pi)$.
   (a) What is the component of the acceleration of the particle in the x direction at $t = \pi/4$? A
   (b) At what times is the particle stationary in the x direction? A
   (c) Is the particle ever momentarily totally stationary? M

10. Find the rightmost point on the curve $x = t - t^6$, $y = e^t$. E

11. For which values of t is the curve $x = \cos 2t$, $y = 3\cos t$ concave up? E

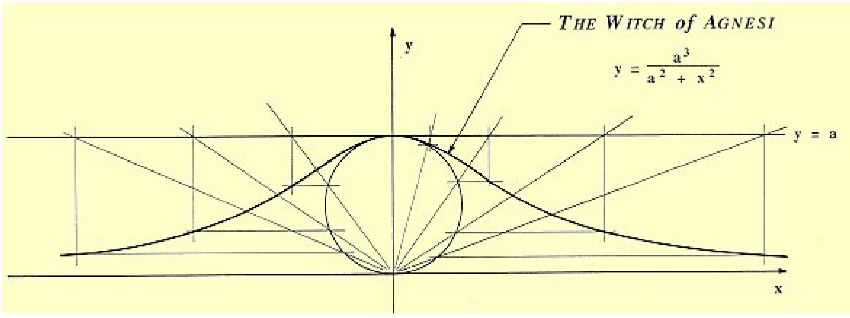
12. Show that the curve $\gamma : t \mapsto (\cos t, \sin t \cos t)$ has two tangents at (0,0) and find their equations. S

13. Scholarship 2000: The piriform is the curve defined by the equation $16y^2 = x^3(8 - x)$ where $x \geq 0$. S
    (a) Show that
    $$\begin{cases} x = 4(1 + \sin\theta) \\ y = 4(1 + \sin\theta)\cos\theta. \end{cases}$$
    are parametric equations for the piriform.
    (b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ, and show that $\theta = \frac{\pi}{6}$ is a stationary point of the curve.

14. We define a surface C parametrically in terms of two parameters, t and θ: O
    $$(x, y, z) = (t, t\cos\theta, t\sin\theta).$$
    (a) Show that the Cartesian equation for this surface is $x^2 = y^2 + z^2$. (This is a cone.)
    (b) Show that the intersection between C and the plane $z = 2$ is a hyperbola.
    (c) For what angle α does the intersection between C and the plane parametrically defined by
    $$(x, y, z) = (u\tan\alpha + 1, u, v)$$
    (for parameters u and v) become a parabola? (Hint: $x = y\tan\alpha + 1$, and z is arbitrary.)

NCEA Level 3 Calculus (Differentiation)
12. Parametric Functions (Homework)

Reading



Maria Gaetana Agnesi (1718 - 1799) was an Italian mathematician and philosopher, considered to be the first woman in the Western world to have achieved a reputation in mathematics.

Agnesi was the eldest child of a wealthy silk merchant who provided her with the best tutors available. She was an extremely precocious child who mastered Latin, Greek, Hebrew, and several modern languages at an early age, and her father liked to host gatherings where she could display her knowledge. *Propositiones philosophicae* ('Propositions of Philosophy'), a series of essays on natural philosophy and history based on her discussions before such gatherings, was published in 1738.

Agnesi's best-known work, *Instituzioni analitiche ad uso della gioventù italiana* (1748; 'Analytical Institutions for the Use of Italian Youth'), in two huge volumes, provided a remarkably comprehensive and systematic treatment of algebra and analysis, including such relatively new developments as integral and differential calculus. In this text is found a discussion of the Agnesi curve, a cubic curve known in Italian as versiera, which was confused with versicra ('witch') and translated into English as the 'Witch of Agnesi'. The French Academy of Sciences, in its review of the Instituzioni, stated that: 'We regard it as the most complete and best made treatise'. Pope Benedict XIV was similarly impressed and appointed Agnesi professor of mathematics at the University of Bologna in 1750.

However, Agnesi had turned increasingly to religion and never journeyed to Bologna. After the death of her father in 1752, she devoted herself almost exclusively to charitable work and religious studies. She established various hospices and died in one of the poorhouses that she had once directed.

From `https://www.britannica.com/biography/Maria-Gaetana-Agnesi`.

Questions

1. In each case find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.
   (a) $x = t^4 - 2t^3 + 2t^2,\ y = t^3 - t$
   (b) $x = \cos t + 2\cos 2t,\ y = \sin t + 2\sin 2t$

2. Find an equation for the tangent to the curve $x = 1 + \sqrt{t},\ y = e^{t^2}$ at the point $(2, e)$ by substituting one function into the other to eliminate the parameter.

3. (a) Use graphing software (e.g. `http://graphfree.com/`, change 'plot type' to 'parametric') to graph the curve $x = 4(\cos t)^3,\ y = 4(\sin t)^3$. This curve is called an *astroid* (as opposed to an asteroid).
   (b) Use calculus to find the slope of the tangent to the astroid in terms of t.
   (c) Use calculus to find the locations of the cusps (points).

NCEA Level 3 Calculus (Differentiation)
13. Sequences and Series

Goal for this week

To revise last year's material on sequences, especially how they behave 'at infinity'.

Sequences

So far, we have looked at functions that are continuous. This week, we take a short break from this and look at sequences and series. A sequence is, intuitively, a list of numbers written in a definite order:

$$(a_n) = (a_1, a_2, \dots, a_i, \dots).$$

Note that we index sequences starting at 1, unless stated otherwise. Theoretical computer scientists will often start at 0 instead.

Some examples of sequences include

$$(n) = (1, 2, 3, \dots) \tag{1}$$
$$\left(\frac{n-1}{n}\right) = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right) \tag{2}$$
$$\left(\cos\frac{n\pi}{6}\right) = \left(1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots\right) \tag{3}$$
$$(1 - 0.2^n) = (0.8, 0.96, 0.992, 0.9984, \dots) \tag{4}$$
$$\left(\frac{n^5}{n!}\right) = (0, 1, 16, 40.5, 42.6, 26.04, 10.8, \dots) \tag{5}$$
$$(a_n) = (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, \dots) \tag{6}$$

Clearly some of these sequences grow forever, while others settle down and become closer and closer to a particular value.

Definition. A sequence (a_n) has a limit L, written $\lim_{n\to\infty} a_n = L$, if we can make the terms a_i stay as close as we like to L by taking i sufficiently large. If (a_n) has a limit, then it is called **convergent**; otherwise, it is called **divergent**.

Examples (2) and (4) above are clearly convergent (they both converge to 1); example (5) is convergent to 0, even though it is initially increasing (because factorials grow faster than powers eventually). The others are all divergent:

(1) is clearly divergent because as $n \to \infty$, $a_n \to \infty$.

(3) is divergent because there is no number L such that a_i stays close to L as we increase i (it always jumps around).

(6) is divergent for the same reason as (3).

Some sequences can be written recursively.

$$a_1 = 1, a_{n+1} = 5a_n - 3 \quad (1, 2, 7, \dots)$$
$$a_1 = 2, a_{n+1} = \frac{a_n}{1 + a_n} \quad \left(2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \dots\right)$$

Example. The **Fibonacci** sequence is defined by

$$(F_i) = \begin{cases} 1 & \text{if } i = 1 \text{ or } i = 2, \\ F_{i-1} + F_{i-2} & \text{otherwise.} \end{cases}$$

The first few elements of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21.

There is actually a closed-form formula for the Fibonacci sequence:

$$F_i = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}.$$

More information about this sequence is given in the reading for the homework.

Series

From sequences, we move on to infinite sums (also known as series). Given some sequence (a_n), we have a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

Often it does not make sense to talk about the sum of infinitely many terms; obviously $1 + 2 + 3 + \cdots$ will increase forever. On the other hand, if we add the first n terms of

$$\left(\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots\right)$$

we obtain $1 - 1/2^n$, which converges to 1.

This suggests that we can consider the partial sums of a sequence (a_n)

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + \cdots + a_n = \sum_{i=1}^{n} a_n \end{aligned}$$

and then say that the sum $\sum a_n$ exists if the sequence (s_n) converges.

An important class of series is the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

where $a \neq 0$ and r are constants. Clearly if $|r| \geq 1$ then the sum becomes $a + a + \cdots$ which converges; if $|r| < 1$, then we have

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= \qquad ar + ar^2 + ar^3 + \cdots + ar^n \\ s_n - rs_n &= a - ar^n \\ s_n &= \frac{a - ar^n}{1 - r}. \end{aligned}$$

which converges to $a/(1-r)$ as $n \to \infty$; so the sum exists.

Questions

1. Which of the following sequences are convergent (write the limit), and which are divergent (give a reason)? M
   (a) $((-1)^n)$
   (b) $(e^{1/n})$
   (c) $(n^2 e^{-n})$
   (d) $(3^n/(1+2^n))$
   (e) $a_1 = 6$, $a_{n+1} = a_n/n$
   (f) $(\cos 2\pi/n)$
   (g) $(\sin n!/n^n)$
   (h) $(\sin n^n/n!)$

2. Find the limit of the sequence $(\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots)$. E

3. Find the sum of the geometric series $1 + 1/2 + 1/4 + \cdots$. A

4. Find the sum of the geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$. M

5. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum. [Hint: calculate $\frac{1}{i} - \frac{1}{1+i}$.] E

6. Determine whether $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ is convergent or divergent. If it is convergent, find its sum. E

7. Consider the harmonic series E
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots.$$
   (a) Let s_n be the sequence of partial sums. Show that $s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{2}{2}$ [Hint: $1/3 > 1/4$.]
   (b) Show that $s_8 > 1 + \frac{3}{2}$.
   (c) Prove that $s_{2^n} \geq 1 + \frac{n}{2}$ in general. Conclude that the harmonic series diverges.

8. Find c if $\sum_{n=1}^{\infty} (1+c)^{-n} = 2$. E

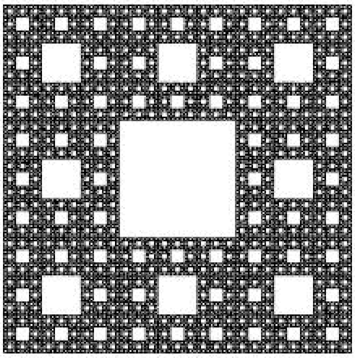
9. A power series is a series of the form S
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots.$$
   You can think of a power series as a polynomial with infinitely many terms. It can be shown that the normal sum rule for differentiation holds, so that $\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} c_n x^n$.
   (a) Find the derivative of the Bessel function, $J_0(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.
   (b) If $|x| < 1$, show that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.
   (c) Find a power series representation for $f(x) = 1/(1-x)^2$. [Hint: differentiate the equation in (b).]
   (d) Find a power series representation for $g(x) = \ln(1+x)$. [Hint: differentiate g, use (c), and antidifferentiate.]
   (e) When is the power series representation of g valid?

10. The Cantor set is constructed as follows. Start with the closed interval $[0, 1]$ and remove the open interval $(\frac{1}{3}, \frac{2}{3})$. This leaves the two closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$; remove the open middle third of each of these. Four intervals remain; remove the open middle third of each. Continue this process indefinitely, at each step removing the open middle third of each interval remaining from the previous step. The Cantor set is the set of numbers remaining from the interval $[0, 1]$ after this process has been carried out. S

(a) Show that the total length of all the removed intervals is 1.

(b) Despite this, there are still infinitely many numbers remaining. Exhibit some members of the Cantor set.

(c) The Sierpinski carpet is a two-dimensional version of the Cantor set; remove the centre one-ninth of a square of side 1, then the centres of the eight remaining squares, and so on. Show that the sum of the areas of the removed squares approaches 1. Hence the Sierpinski carpet has area zero.



11. Scholarship 2017: The sequence (a_n) is defined as follows:- S

$$\begin{cases} (a_1) = 2, \\ (a_2) = 7, \\ (a_{n+1}) = \frac{1}{2}(a_n + a_{n+1}) \quad \text{for } n \geq 2. \end{cases}$$

(a) Find an exact formula for the nth term of the sequence.

(b) What is the limit of a_n as $n \to +\infty$?

12. Consider the following limit (you may assume that it exists): S

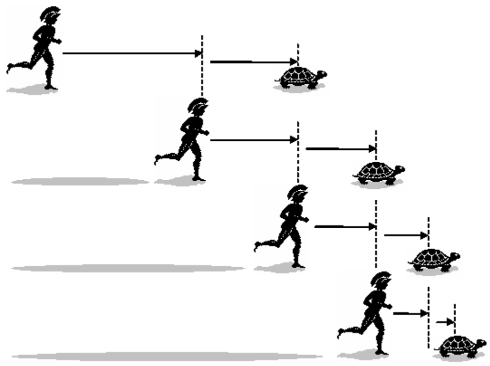
$$L = 0.999... = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{9}{10^i}$$

(a) Write $L = 0.9 + 0.09 + \cdots$, and make a conjecture about the value of L.

(b) Prove or disprove your conjecture from (a).

13. Zeno was a Greek philosopher active in the 5th century BCE. He presented a list of 'paradoxes', or apparent contradictions, including the following (adapted from Wikipedia): S

- Suppose Achilles is in a foot race with a tortoise. Achilles runs much faster than the tortoise, but the latter has a head start. By the time Achilles reaches the location that the tortoise started, the tortoise will have moved a small amount further on; similarly, by the time Achilles reaches the new location of the tortoise, it will have moved an even smaller distance further on; and by this reasoning it follows that Achilles can never overtake the tortoise.



- Suppose Homer wishes to walk to the end of a path. Before he can get there, he must get halfway there. Before he can get halfway there, he must get a quarter of the way there. Before traveling a quarter, he must travel one-eighth; before an eighth, one-sixteenth; and so on. So Homer cannot walk to the end of the path.
- For motion to occur, an object must change the position which it occupies. Consider an example of an arrow in flight. In any one (duration-less) instant of time, the arrow is neither moving to where it is, nor to where it is not. It cannot move to where it is not, because no time elapses for it to move there; it cannot move to where it is, because it is already there. In other words, at every instant of time there is no motion occurring. If everything is motionless at every instant, and time is entirely composed of instants, then motion is impossible.

Use your knowledge of limits to explain these apparent contradictions.

NCEA Level 3 Calculus (Differentiation)
13. Sequences and Series (Homework)

Reading

Leonardo of Pisa, Italian born, grew up in North Aftica, where his father Guilielmo was working as a diplomat on behalf of merchants trading at Bugia (modern Algeria). He accompanied his father on his numerous travels, encountered the Arabic system for writing numbers and understood its importance. In his *Liber Abbaci* of 1202 he writes: 'When my father, who had been appointed by his country as public notary in the customs at Bugia acting for the Pisan merchants going there, was in charge, he summoned me to him while I was still a child, and having an eye to usefulness and future convenience, desired me to stay there and receive instruction in the school of accounting. There, when I had been introduced to the art of the Indian's nine symbols through remarkable teaching, knowledge of the art very soon pleased me above all else.'

The book introduced the Hindu-Arabic notation to Europe, and formed a comprehensive arithmetic text, containing a wealth of material related to trade and currency conversion. Although it took several centuries for Hindu-Arabic notation to replace the traditional abacus, the advantages of a purely written system of calculation soon became apparent.

Leonardo is often known by his nickname 'Fibonacci', which means 'son of Bonaccio', but this name is not recorded befre the 18th century and was probably invented then by Guillaume Libri.

The third section of the *Liber Abbaci* contains a problem that seems to have originated with Leonardo: 'A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if in every month, each pair begets a new pair, which from the second month onwards becomes productive?'

This rather quirky problem leads to a curious, and famous, system of numbrs: 1, 2, 3, 5, 8, 13, 21, 34, 55, and so on. Each number is the sum of the preceeding two numbers. This is known as the Fibonacci sequence, and it turns up repeatedly in mathematics and in the natural world. In particular, many flowers have a Fibonacci number of petals. This is not a coincidence, but a consequence of the growth pattern of the plant and the geometry of the 'primordia' — tiny clumps of cells at the tip of the growing shoot that give rise to important structures, including petals.

Although Fibonacci's growth rule for rabbit populations is unrealistic, more general rules of a similar kind (called Leslie models) are used today for certain problems in population dynamics, the study of how animal populations change in size as the animals breed and die.

From *Taming the Infinite*, by Ian Stewart.

Questions

1. Determine whether the following sequences converge. If so, give the limit.
   (a) $(a_n) = \left(\frac{2+n^3}{1+2n^3}\right)$
   (b) $(b_n) = \left(9^{n+1}/10^n\right)$
2. An alternating series is a series whose terms are alternately positive and negative. For the following alternating series, write out the first few terms of the series and the first few terms of the series of partial sums. Does the series seem to converge? You need not find the limit.
   (a) $\sum_{n=1}^{\infty}(-1)^n \frac{2}{2n+1}$.
   (b) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{2n}{n+4}$.

NCEA Level 3 Calculus (Differentiation)
14. Differentiation Revision

Halfway there! Let's do a bit of revision.

Questions

1. True or False: A
   (a) If a function f is continuous around a point, then it is differentiable at that point.
   (b) If a function f is differentiable around a point, then it is continuous at that point.
   (c) If f and g are differentiable, then $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)] = f'(x)+g'(x)$.
   (d) If f and g are differentiable, then $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g'(x)$.
   (e) $\frac{\mathrm{d}}{\mathrm{d}x}(x+3)^2 = 2(x+3)$.
   (f) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2+3)^2 = 2(x^2+3)$.
   (g) $\frac{\mathrm{d}}{\mathrm{d}x}\tan^2 x = \frac{\mathrm{d}}{\mathrm{d}x}\sec^2 x$.
   (h) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$.
   (i) If $A = \pi r^2$, then $\mathrm{d}A\pi = 0$.

2. Find $\frac{\mathrm{d}y}{\mathrm{d}t}$ in each case. A
   (a) $y = t^2 + 3t$
   (b) $y = \frac{4-t}{3+t}$
   (c) $y = (t^4 - 3t^2 + 5)^2$
   (d) $y = (t+1)^{2017}$
   (e) $y = \frac{3t+4t^2}{\sqrt{t}}$
   (f) $y = \sin 2t$
   (g) $y = \sin^2 t$
   (h) $y = \sin t^2$
   (i) $y = \cos\tan t$
   (j) $y = (27t+3)^{2017}(t^2 - \sqrt{t})^{2020}$
   (k) $y = \left(t + \frac{1}{t^2}\right)^{\sqrt{7}}$

3. Find the equation of the tangent line to the curve $\sqrt{1+4\sin x}$ at the point $(0,1)$. M

4. Find y'' in each case: A
   (a) $y = 3x^3 + 2x + \sqrt{2x} + \frac{1}{x^2}$
   (b) $y = e^{2x}$
   (c) $y = \sqrt{4t+1}$
   (d) $y = 4\sin^2 x$

5. Find the nth derivative of e^{2x} (where n is a natural (counting) number). M

6. The height of a projectile after t seconds can be modelled by $h = 3t(t-10)$. At what time is the height of the projectile at a maximum? Use the second-derivative test to prove that you have found a maximum. E

7. Find $f'(x)$ if: M
   (a) $f(x) = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + 3$
   (b) $f(x) = 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + 7$

8. In each case, find y' in terms of x and y: M
   (a) $x^2 + y^2 = 4$
   (b) $x^2 + 4xy + y^2 = 13$
   (c) $xy^4 + x^2y = x + 3y$
   (d) $x^2 \cos y + \sin 2y = xy$

9. By differentiating the double-angle formula for cosine, M

$$\cos 2x = \cos^2 x - \sin^2 x,$$

   obtain the double-angle formula for the sine function.

10. Find f' in terms of g' if $f(x) = x^2 g(x)$. M

11. The volume of a cube is increasing at a rate of $10\,\mathrm{cm}^3\,\mathrm{min}^{-1}$. How fast is the surface area increasing when the length of an edge is $30\,\mathrm{cm}$? M

12. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$. M
    (a) Find the rate of change of volume with respect to height if the radius is constant.
    (b) Find the rate of change of volume with respect to radius if the height is constant.

13. A particle moves along a horizontal line such that its coordinate at time t is $x = \sqrt{b^2 + c^2t^2}$ $(t \geq 0)$, where b and c are positive constants. Find its velocity and acceleration functions. M

14. From first principles, show that: M
    (a) $\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x$
    (b) $\frac{\mathrm{d}}{\mathrm{d}x}[2x^3 + 2x] = 6x^2 + 2$

15. Find the derivative of $f(x) = \frac{4}{\sqrt{1-x}}$ from first principles. E

16. Each limit represents the derivative of a function f at a point a. Identify each function and point. E
    (a) $\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h}$
    (b) $\lim_{t\to 1} \frac{t^4+t-2}{t-1}$

17. Find the best linear approximation to $f(x) = \sqrt{25 - x^2}$ near $x = 3$. M

18. A balloon is rising at a constant speed of $2\,\mathrm{m\,s^{-1}}$. A girl is cycling along a straight road at a speed of $5\,\mathrm{m\,s^{-1}}$. When she passes under the balloon, it is $15\,\mathrm{m}$ above her. How fast is the distance between the person and the balloon increasing 3 seconds later? M

19. On a straight shoreline there is a tree and exactly opposite it, $100\,\mathrm{m}$ away in the sea, stands a lighthouse. A strong and thin spotlight on its top revolves at the rate of one revolution per 4 seconds, its light creating a running light spot on the shore. You stand on the shore $100\,\mathrm{m}$ from the tree. How fast does this spot move when it goes past you? E

20. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t for the following parametrically-defined curves. M
    (a) $t \mapsto (1 + e^{2t}, e^t)$

(b) $x = \tan t$, $y = \sec 2t$

(c) $x = \frac{t^2-10}{t^2+1}$, $y = tx$

21. Find the lowest point on the curve $\gamma : t \mapsto (t^3 - 3t, t^2 + t + 1)$. Prove you have found a minimum. E

22. Find the acceleration of a particle at time t if its displacement from the origin at time t is $-t^6 + 5t^4 + \sin t$. M

23. A piece of wire 10 m long is cut into two pieces; one is bent into a square and the other into a circle. Where should the wire be cut to ensure the total area of the enclosed shapes is (a) a minimum and (b) a maximum? E

24. A cone is made by cutting a sector out of a circle of paper of radius R and gluing together the edges of the cut. Find the maximum possible volume of the cone. M

25. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r. E

26. At which points does the 'bouncing wagon' curve $2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$ have horizontal tangents? E

27. Salt forms a cone as it falls from a conveyor belt. The slant of the cone forms an angle of 30° with the horizontal. The belt delivers salt at a rate of $2\,\mathrm{m}^3$ per minute. When the radius of the cone is ten metres, what is the rate of increase of the slant height (measured along the surface of the cone)? E

28. Suppose we take a circle of radius r, and inscribe within it a triangle such that each corner of the triangle is located on the circle. What is the maximum possible area of the triangle? S

29. Take derivatives of the following functions with respect to x: E
    (a) $f(x) = 2^x$
    (b) $g(x) = \log_{\log x} x$

30. Scholarship 2017: For $y = x^{(x^x)}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 2$. S

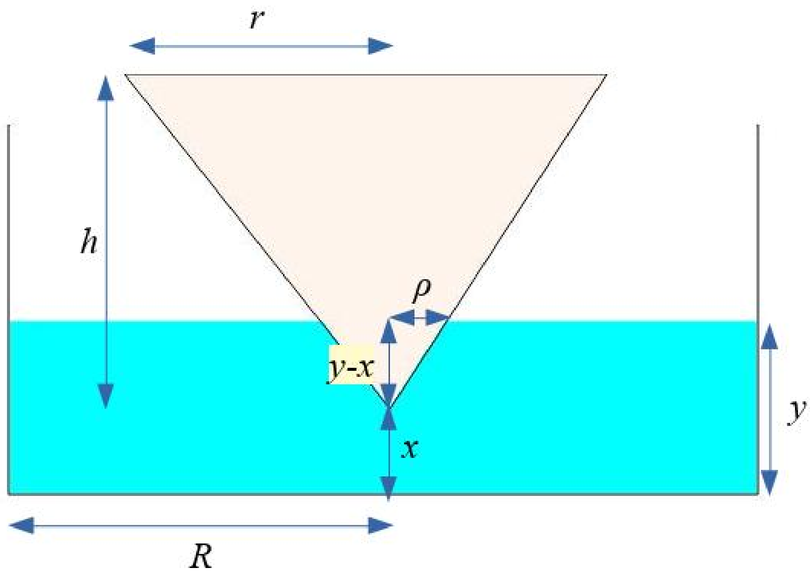
31. Scholarship 2015 (adapted): A car is driving along a road shaped like a parabola at night. The parabola has a vertex at the origin, and the car starts at a point 100 m west and 100 m north of the origin. S
    (a) Write an equation modelling the road as a parabola.
    (b) Find the general equation for the tangent line to the parabola at some point (x_0, y_0), and substitute into it the parabola equation to obtain an equation only in x, x_0, and y_0.
    (c) Suppose there is a statue of the Roman emperor Augustus located 100 m east and 50 m north of the origin. Write the equation for the tangent line of the parabola passing through the statue (so that it only depends on a value x on the parabola).
    (d) Hence find the single point (x, y) on the road where the headlights of the car illuminate the statue.

32. (Difficult) Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line. O

33. (Difficult) A cone of radius r centimetres and height h centimetres is lowered point first at a rate of $1\,\mathrm{cm}^2$ into a cylinder of radius R centimetres that is partially filled with water. How fast is the water level rising at the instant that the cone is fully submerged? S

Worked solution for problem:

(Difficult) A cone of radius r centimetres and height h centimetres is lowered point first at a rate of $1\,\mathrm{cm}^2$ into a cylinder of radius R centimetres that is partially filled with water. How fast is the water level rising at the instant that the cone is fully submerged?



Let V be the volume of water in the tank, which we know to be constant. We are given that $\frac{\mathrm{d}x}{\mathrm{d}t} = -1$. Using trigonometry, we find $\rho = \frac{r}{h}(y-x)$ so we have

$$\begin{aligned} V &= x\pi R^2 + (y-x)\pi R^2 - \frac{1}{3}\pi\rho^2(y-x) \\ &= \pi R^2 y - \frac{1}{3}\pi\frac{r^2}{h^2}(y-x)^3 \end{aligned}$$

Hence $\frac{\mathrm{d}V}{\mathrm{d}x} = \pi R^2 - \frac{\pi r^2}{h^2}(y-x)^2(1-\frac{\mathrm{d}x}{\mathrm{d}y})$; but V is constant, so any derivative of V is zero and

$$0 = \pi R^2 - \frac{\pi r^2}{h^2}(y-x)^2(1-\frac{\mathrm{d}x}{\mathrm{d}y}).$$

When the cone is exactly submerged, $y - x = h$. So $0 = R^2 - r^2(1-\frac{\mathrm{d}x}{\mathrm{d}y})$ and $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{r^2-R^2}{r^2}$; hence

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x} = -1 \times \frac{r^2}{r^2-R^2} = \frac{r^2}{R^2-r^2}.$$

NCEA Level 3 Calculus (Differentiation)
14. Differentiation Revision (Homework)

Reading

Go and watch...

`https://www.youtube.com/watch?v=fIaupXkpB00`

Questions

1. Find f' in each case:
   (a) $f(x) = 3x^{2017} + \frac{1}{x^{19}} + \sqrt[2017]{x+2}$
   (b) $f(h) = \pi r^2 h$
   (c) $f(\theta) = \frac{\mu m g}{\mu \sin\theta + \cos\theta}$
   (d) $f(g) = \frac{\sin g}{g^2 + \ln g}$
   (e) $3x \cdot f(x) + [f(x)]^2 = \frac{x}{3+f(x)}$
2. A kite 50 m above the ground moves horizontally at a speed of $2\,\mathrm{m\,s^{-1}}$. At what rate is the angle between the string and the horizontal decreasing when 100 m of string has been let out?
3. A cone-shaped paper drinking cup is to be made to hold $27\,\mathrm{cm}^3$ of water (or other liquid beverage). Find the height and radius of the cup that will use the smallest amount of paper.
4. Find the closest point on the hyperbola $x^2 - 2x - y^2 = 2$ to the following points. You may find it useful to complete the square and parameterise the hyperbola (it's easier than dealing with solving the hyperbola equation for x or y).
   (a) $(2, 1)$.
   (b) $(3, 1)$.

CHAPTER 3

Integration

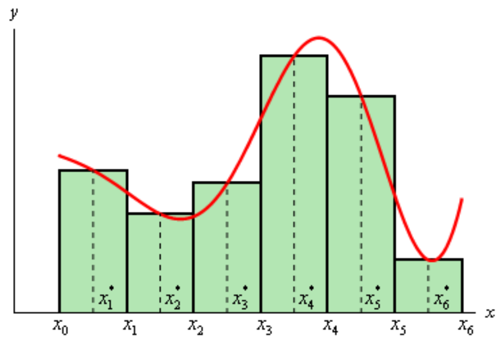
NCEA Level 3 Calculus (Integration)
15. Approximating Areas

Goal for this week

To work out roughly how big the area underneath a graph is.

We now move from studying the geometry and shape of curves themselves to studying the geometry and shape of the areas bounded by curves. Just as for differentiation, we begin by finding finite approximations: in this case, of areas. We will mainly be interested in finding the areas between a curve and the x-axis.

Approximating area using rectangles



We start with the simplest and most naive approximation: using a bunch of rectangles under the curve. Suppose we wish to find the area under the curve $y = f(x)$ between the two lines $x = a$ and $x = b$ by splitting it up into n rectangles of width Δx. If we pick a value x_i^* inside each rectangle, as shown in the diagram, the approximate area is

$$f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

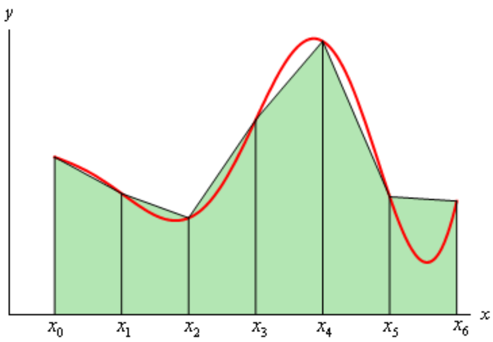
Usually, we pick each x_i^* to be either the left-hand or right-hand edge of each rectangle. Note that we also have $\Delta x = \frac{b-a}{n}$ and so our left-hand approximation becomes

$$L_n = \frac{b-a}{n}\left[f(x_0) + \cdots + f(x_{n-1})\right]$$

and the right-hand approximation is

$$R_n = \frac{b-a}{n}\left[f(x_1) + \cdots + f(x_n)\right].$$

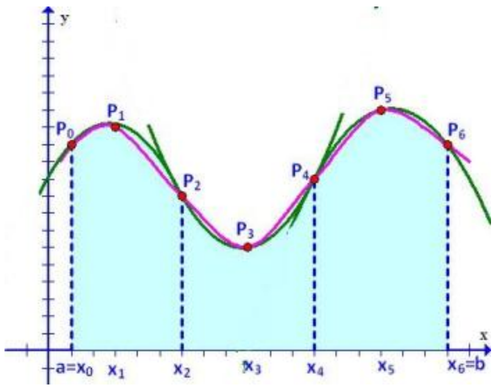
Approximating area using trapezoids



A more useful approximation is found when we inscribe trapezoids into the curve. This can be done by taking the average of the left-hand and right-hand rectangle approximations (recall that the area of a trapezoid is the average of the area of two rectangles). When we do this, we obtain

$$T_n = \frac{b-a}{2n}\Big(f(x_0) + f(x_n) + 2\left[f(x_2) + \cdots + f(x_{n-1})\right]\Big).$$

Approximating area using parabolae



An even better approximation (for smooth functions) is formed when we use parabolae to estimate the area. The resulting formula is called **Simpson's rule**. Note that n must be even to use Simpson's rule.

$$S_n = \frac{b-a}{3n}\Big(f(x_0) + f(x_n) + 4\left[f(x_1) + f(x_3) + \cdots + f(x_{n-1})\right] + 2\left[f(x_2) + \cdots + f(x_{n-2})\right]\Big).$$

The Definite Integral

It is obvious that all three methods of approximating area above will approach the 'real' area of the curve as $n \to \infty$. We call the true area under a curve $y = f(x)$ from $x = a$ to $x = b$ the **definite integral** of the curve from a to b, and we notate it as*

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} T_n = \lim_{n\to\infty} S_n = \int_a^b f(x)\,\mathrm{d}x.$$

The large $\int$ is an elongated 'S', which stands for *sum* — we are summing all of the little areas from a to b. Note the similarity to the notation above: we replace $\sum$ with $\int$, x_i^* with x, and Δx with $\mathrm{d}x$.

*Technically, this is the **Riemann integral**; there are other kinds of integral.

The $\mathrm{d}x$ is *not a number*; it is merely a piece of notation which tells us which variable we are taking area with respect to. For example, $\int_a^b x^2 + y^3 \,\mathrm{d}x \neq \int_a^b x^2 + y^3 \,\mathrm{d}y$. At this stage in time, you can think of the notation as a pair of fancy brackets: the $\int_{z_0}^{z_1}$ corresponds to $\big($, and the $\mathrm{d}z$ corresponds to $\big)$. It makes no sense to have one without the other (or, for that matter, to have $\int$ without the bounds).

Questions

1. Estimate the area under the graph of $f(x) = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an overestimate or an underestimate? A

2. Estimate the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ using four approximating rectangles and left endpoints. Sketch the graph and the rectangles. Is your estimate an overestimate or an underestimate? A

3. Using the trapezoidal rule with $n = 4$, estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$. Repeat with $n = 8$. Compare the two results. A

4. Use Simpson's rule with $n = 10$ to estimate the area under the graph of $y = e^{x^2}$ from $x = 0$ to $x = 1$. A

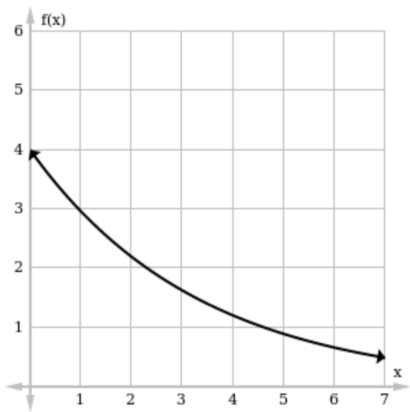
5. Use both the trapezoidal rule and Simpson's rule (both with $n = 10$) to compute the area under the graph of $y = \sqrt{z}e^{-z}$ from $z = 0$ to $z = 1$. A

6. Using four rectangles, find the approximate area under the the graph of $f(x)$ from $x = 1$ to $x = 5$ given the following table. A

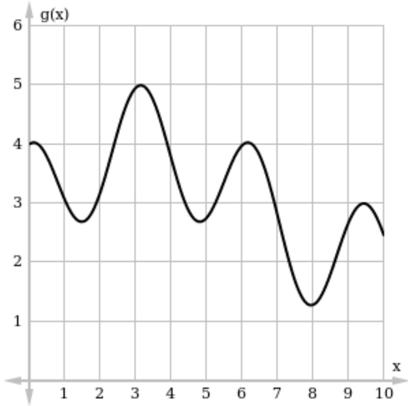
x	$f(x)$	x	$f(x)$
1.0	2.4	3.5	4.0
1.5	2.9	4.0	4.1
2.0	3.3	4.5	3.9
2.5	3.6	5.0	3.5
3.0	3.8		

7. Find the area of a circle of radius 4 using Simpson's rule with $n = 4$. What is the percentage error of the estimate? M

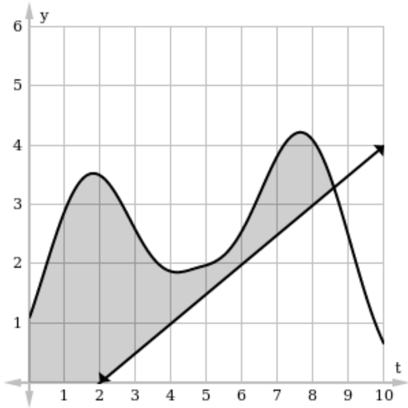
8. Approximate the area under $y = f(x)$ from $x = 0$ to $x = 6$ using Simpson's rule. A



9. Approximate the area under $y = f(x)$ from $x = 0$ to $x = 10$ using the trapezoidal rule. A



10. Approximate the shaded area using the trapezoidal rule. M



11. Show that $T_n = \frac{1}{2}(L_n + R_n)$. M

12. (Lifted straight from a Level 2 worksheet.) A
    (a) Draw the line $y = 2t + 1$ and use geometry to find the area under this line, above the t-axis, and between the vertical lines $t = 1$ and $t = 3$ (i.e. find $\int_1^3 2t + 1\,\mathrm{d}t$ using geometry).
    (b) If $x > 1$, let $A(x)$ be the area of the region that lies under the line $y = 2t+1$ between $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$ (i.e. find $A(x) = \int_1^x 2t + 1\,\mathrm{d}t$ using geometry).
    (c) Find $A'(x)$. What do you notice?

NCEA Level 3 Calculus (Integration)
15. Approximating Areas (Homework)

Reading

Engineering is like dancing; you don't learn it in a darkened lecture hall watching slides: you learn it by getting out on the dance floor and having your toes stepped on.
- Professor Jack Alford (1920–2006). The same can be said for doing definite integrals.

To really appreciate this book, one dedicated to the arcane art of calculating definite integrals, it is necessary (although perhaps it is not sufficient) that you be the sort of person who finds the following question fascinating, one right up there in a fierce battle with a hot cup of coffee and a sugar donut for first place on the list of sinful pleasures: without actually calculating x, show that if

$$x + \frac{1}{x} = 1$$

then it follows that

$$x^7 + \frac{1}{x^7} = 1.$$

Okay, I know what many (but, I hope, not *you*) are thinking at being confronted with a question like this: of what Earthly significance could such a problem possibly have? Well, none as far as I know, but its fascination (or not) for you provides (I think) excellent psychological insight into whether or not you should spend time and/or good money on this book. If the problem leaves someone confused, puzzled, or indifferent (maybe all three) then my advice to them would be to put this book down and to look instead for a good mystery novel, the latest Lincoln biography (there seems to be a new one every year — what could possibly be left unsaid?), or perhaps a vegetarian cookbook.

But, if your pen is already out and scrawled pages of calculations are beginning to pile-up on your desk, then by gosh you *are* just the sort of person for whom I wrote this book. More specifically, I've written with three distinct types of readers in mind: (1) physics/engineering/math students in their undergraduate years; (2) professors looking for interesting lecture material; and (3) nonacademic professionals looking for a 'good technical read.'

There are two possible concerns associated with calculating definite integrals that we should address with no delay. First, do *real* mathematicians actually do that sort of thing? Isn't mere *computation* the dirty business (best done out of sight, in the shadows of back-alleys so as not to irreparably damage the young minds of impressionable youths) of grease-covered engineers with leaky pens in their shirts, or of geeky physicists in rumbled pants and chalk dust on their noses? Isn't it in the deep, clear ocean of analytical proofs and theorems where we find *real* mathematicians, swimming like powerful, sleek seals? As an engineer, myself, I find that attitude just a bit elitist, and so I am pleased to point to the pleasure in computation that many of the great mathematicians enjoyed, from Newton to the present day.

Let me give you two examples of that. First, the reputation of the greatest English mathematician of the first half of the twentieth century, G.H. Hardy (1877–1947), partially rests on his phenomenal skill at doing definite integrals. (Hardy appears in numerous places in this book.) And second, the hero of this book (Riemann) is best known today for (besides his integral) his formulation of the greatest unsolved problem in mathematics, about which I'll tell you lots more at the end of the book. But after his death, when his private notes on that very problem were studied, it was found that imbedded in all the deep theoretical stuff was a calculation of $\sqrt{2}$. To 38 decimal places!

The other concern I occasionally hear strikes me as just plain crazy; the complaint that there is no end to definite integrals. (This should, instead, be a cause for joy.) You can fiddle with integrands, and with upper and lower limits, in an uncountable infinity of ways, goes the grumbling, so what's the point of calculating definite integrals since you can't possibly do them all? I hope writing this concern out in words is sufficient to make clear its ludicrous nature. We can never do all possible definite integrals, so why bother doing any? Well, what's next—you can't possibly add together all possible pairs of the real numbers, so why bother learning to add? Like I said — that's nuts!

What makes doing the specific integrals in this book of value aren't the specific answers we'll obtain, but rather the tricks (excuse me, the methods) we'll use in obtaining those answers; methods you may be able to use in evaluating the integrals you will encounter in the future in your own work. Many of the integrals I'll show you do have important uses in mathematical physics and engineering, but others are included just because they look, at first sight, to be so damn tough that it's a real kick to see how they simply crumble away when attacked with *the right trick*.

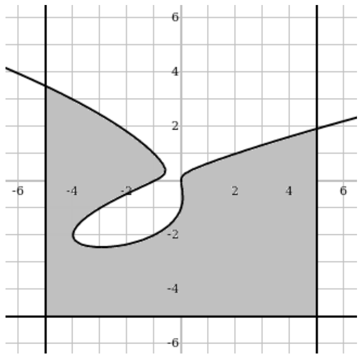
From *Inside Interesting Integrals*, by Paul J. Nahin.

Questions

1. Using the following table of values, estimate $\int_0^{1.6} g(x)\,\mathrm{d}x$.

x	$g(x)$	x	$g(x)$
0.0	12.1	1.0	12.2
0.2	11.6	1.2	12.6
0.4	11.3	1.4	13.0
0.6	11.1	1.6	13.2
0.8	11.7		

2. Estimate the shaded area.



3. Approximate the value of π with a numeric integration. Use your ingenuity. (For convenience of checking your result, $\pi = 3.141592....$)

NCEA Level 3 Calculus (Integration)
16. Anti-differentiation

Goal for this week

To practice undoing differentiation.

For some time, we have been hinting that finding area was in some way the inverse to finding slope. In last week's questions, the curtain was pulled back a little further when we calculated an area function for a variable endpoint, took the derivative, and got the original function back. Next week, all will be revealed; but first, we need to do a little bit more work. This week, we're going to look at **anti-differentiation**: finding the original function given its derivative.

Example. One antiderivative of $y' = 3x^2 + 4$ is $x^3 + 4x$. Another is $x^3 + 4x + 1$. A third is $x^3 + 4x + 7$. Obviously every function of the form $y = x^3 + 4x + C$ for some constant C will differentiate to the given y', so we must remember always to make this clear.

We also call antiderivatives **indefinite integrals**, and in this notation the above example is

$$\int 3x^2 + 4 \,\mathrm{d}x = x^3 + 4x + C.$$

Unfortunately, there is no 'easy' way to anti-differentiate; we simply have to try to rearrange the function in some clever way until it looks like something that we know how to deal with.

Joke. Two mathematicians are in a bar. The first one says to the second that the average person knows very little about basic mathematics. The second one disagrees and claims that most people can cope with a reasonable amount of maths. The first mathematician goes off to the washroom, and in his absence the second calls over the waitress. He tells her that in a few minutes, after his friend has returned, he will call her over and ask her a question. All she has to do is answer "one third x cubed." She repeats "one thir–dex cue?" He repeats "one third x cubed." She asks, "one thir dex cuebd?" "Yes, that's right," he says. So she agrees, and goes off mumbling to herself, "one thir dex cuebd...". The first guy returns and the second proposes a bet to prove his point, that most people do know something about basic math. He says he will ask the blonde waitress an integral, and the first laughingly agrees. The second man calls over the waitress and asks "what is the integral of x squared?" The waitress says "one third x cubed" and while walking away, turns back and says over her shoulder, "plus a constant!"

Examples.

1. The most general antiderivative of $\sin x$ is $-\cos x + C$.
2. The most general antiderivative of $\tan x$ is $-\ln|\cos x| + C$.
3. $\int \frac{1}{x+3} \,\mathrm{d}x = \ln|x + 3| + C$.
4. $\int \tan^2 \theta \,\mathrm{d}\theta = \int \sec^2 \theta - 1 \,\mathrm{d}\theta = \tan \theta - \theta + C$.
5. $\int \frac{2x}{x^2+1} \,\mathrm{d}x = \ln\left|x^2 + 1\right| + C$.
6. $\int Ke^{Kx} \,\mathrm{d}x = e^{Kx} + C$ for all constants K.

Questions

1. For each expression, find the most general antiderivative with respect to x. A

(a) $2x$

(b) x^{-3}

(c) 0

(d) $\sec^2 x + \sqrt{x}$

(e) $x\sqrt{x}$

(f) $\sin x - \cos x$

(g) $\frac{2x^3+3x-\sqrt{x}}{\sqrt[3]{x}}$

(h) $\frac{1}{x^2} + e^x$.

2. Verify the examples in the notes by differentiation. A

3. Show that $\int 3x^2 + 4x + 5 + 2e^{2x}\,\mathrm{d}x = x^3 + 2x^2 + 5x + e^{2x} + C$. A

4. Find y' when $y = 3 + \sin(2x+4)$ and hence find $\int 2\cos(2x+4)\,\mathrm{d}x$. A

5. If $\frac{\mathrm{d}y}{\mathrm{d}t} = 1.5\sqrt{t}$ and $y(4) = 10$, find $y(t)$ exactly. A

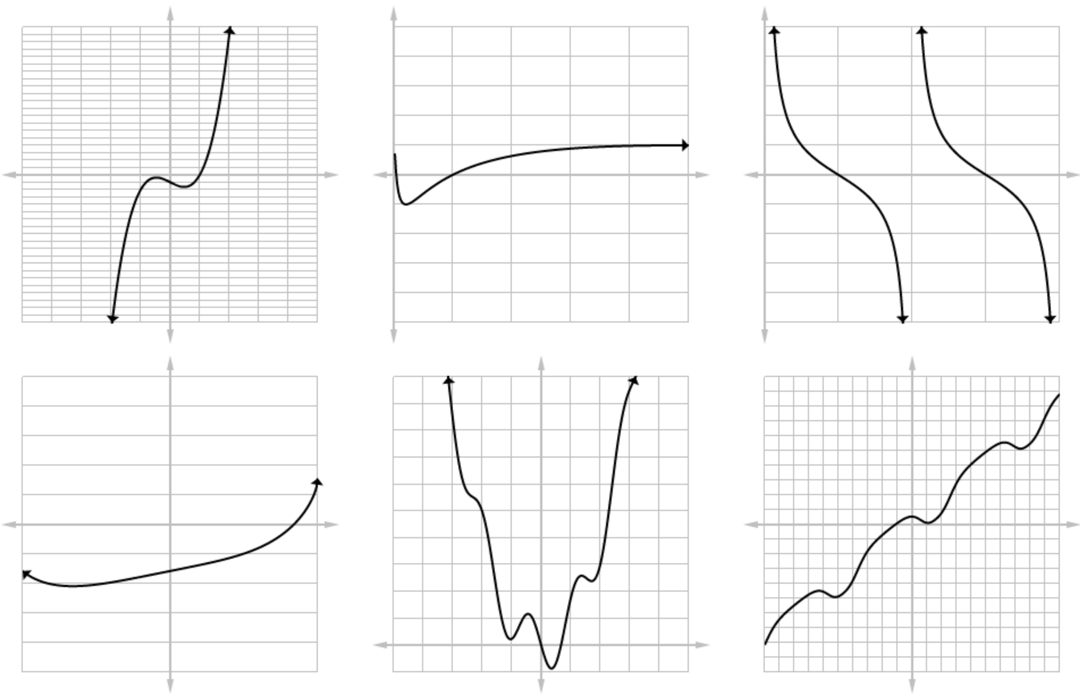
6. Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$. M

7. The velocity of a particle is given by $v(t) = 2t + 1$. Find its position at $t = 4$ if its position at $t = 0$ is $x = 0$. A

8. The acceleration of a particle is given by $a(t) = 10\sin t + 3\cos t$. At $t = 0$, its position is $x = 0$; at $t = 2\pi$, its position is $x = 12$. Find its position at $t = \frac{\pi}{2}$. M

9. Find all functions g such that $g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$. A

10. For each function, sketch an antiderivative passing through $(0,0)$: M



11. Show that if F is an anti-derivative of f, G is an anti-derivative of g, and α and β are any constants, then $\alpha F + \beta G$ is an anti-derivative of $\alpha f + \beta g$. E

12. Give an example of functions f and g such that if F and G are anti-derivatives of f and g respectively then FG is *not* an anti-derivative of fg. E

13. Note that on the formula sheet, the anti-derivative of $1/x$ is given as $\ln|x|$, not just $\ln x$. E
    (a) Compute $\frac{\mathrm{d}}{\mathrm{d}x} \ln|x|$ if $x < 0$, and hence justify formally why $\frac{\mathrm{d}}{\mathrm{d}x} \ln|x| = 1/x$.
    (b) Draw $y = \ln|x|$ and $y = 1/x$ on the same pair of axes, and hence justify intuitively why $\frac{\mathrm{d}}{\mathrm{d}x} \ln|x| = 1/x$.

NCEA Level 3 Calculus (Integration)
16. Anti-differentiation (Homework)

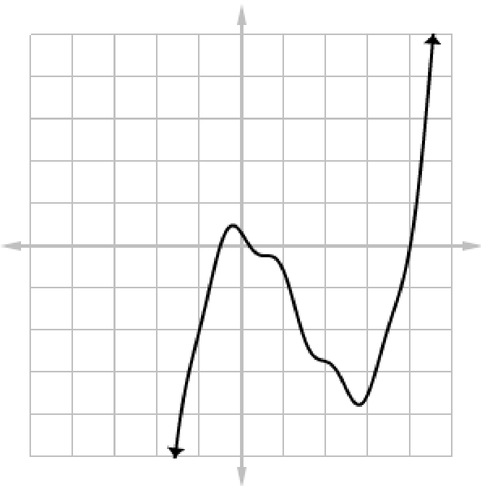
Reading

Go and watch...

https://www.youtube.com/watch?v=7dcDuVyzb8Y

Questions

1. Find the most general antiderivative.
   (a) $f(x) = x - 3$
   (b) $f(x) = (x + 1)(x + 2)$
   (c) $f(\theta) = 6\theta^2 - 7\sec^2\theta$
   (d) $g(h) = \pi^2$
   (e) $f(x) = x^{3.7} + \sqrt{x} + 7x^{\sqrt{7}-1}$
2. Given that the graph of φ passes through the point $(1, 6)$ and that the slope of its tangent line at $(x, \varphi(x))$ is $2x + 1$, find $\varphi(2)$.
3. This is the second derivative of g. Find g given that $g'(0) = 0$ and $g(0) = 1$.



NCEA Level 3 Calculus (Integration)
17. The Fundamental Theorem of Calculus

Goal for this week

To understand how (definite) integration and differentiation are related.

Finally, we have the punchline. We state the theorems first, and then give the (optional) proofs at the end.

Statements of theorems

"You can tell it's important because it has a name.
*You can tell it's **very** important because it has a **pompous** name."*

Theorem (First Fundamental Theorem of Calculus (FTC1))
Suppose f is a continuous function, and suppose F is any antiderivative of f (so $F' = f$). Then,

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b.$$

In other words, the definite integral of a function can be found by evaluating the indefinite integrals at the endpoints — geometrically, the total accumulated slope of a function over an interval is just the height gained by the function over the interval. This actually follows from a differently intuitive result:

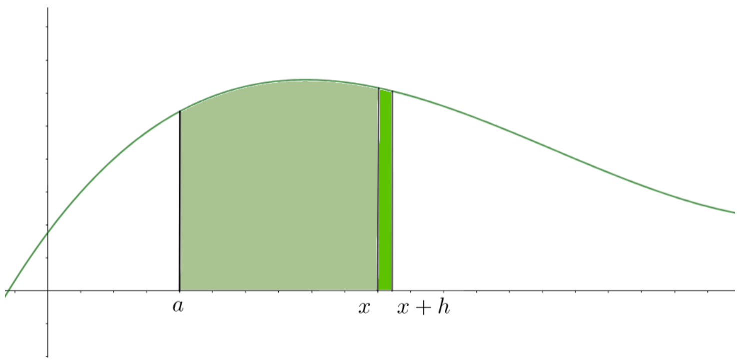
Theorem (Second Fundamental Theorem of Calculus (FTC2))
Suppose f is a continuous function. Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

For some intuition, we can consider the following graph of $y = f(x)$. The shaded area is the value of $A(x) = \int_a^x f(t)\,\mathrm{d}t$; then we use the fact that the area of the darker shaded area is approximated by $hf(x)$ (base times height) to see that

$$\frac{A(x+h) - A(x)}{h} \approx \frac{hf(x)}{h} = f(x).$$

If we take limits, as we do in the proof of this theorem (given below), then this approximation becomes exact: the rate of change of the area under a curve is simply the height of the curve.



Joke. A mathematics professor was lecturing to a class of students. As he wrote something on the board, he said to the class "Of course, this is immediately obvious." Upon seeing the blank stares of the students, he turned back to contemplate what he had just written. He began to pace back and forth, deep in thought. After about 10 minutes, just as the silence was beginning to become uncomfortable, he brightened, turned to the class and said, "Yes, it IS obvious."

These theorems are *fundamental* because they show us that the two main operations of calculus, integration and differentiation, are deeply related and are (in some sense) inverses of each other. To be absolutely clear, without this theorem we would have absolutely no justification in calling anti-derivatives 'integrals' — the second theorem tells us that if we make the upper bound of our definite integral vary then we obtain an anti-derivative for the expression underneath the integral sign, and the first theorem tells us that if we take the definite integral of the derivative of some function then we so happen to obtain anti-derivatives back out the other side! (In fact, notice that the indefinite integral of f is just $\int_{f^{-1}(C)}^{x} f(x)\,\mathrm{d}x$, where C is just our constant of integration!)

We also have the following theorem, which allows us to combine definite integrals together. In order to get some kind of geometric intuition, please draw a diagram or find some kind of intuitive explanation for each!

Theorem. *Suppose f and g are functions and λ is a real constant. Then, if the relevant integrals are defined, we have:*

1. $\lambda \int\limits_a^b f(x)\,\mathrm{d}x = \int\limits_a^b \lambda f(x)\,\mathrm{d}x.$
2. $\int\limits_a^b f(x)\,\mathrm{d}x + \int\limits_a^b g(x)\,\mathrm{d}x = \int\limits_a^b f(x) + g(x)\,\mathrm{d}x.$
3. $\int\limits_a^a f(x)\,\mathrm{d}x = 0.$
4. $\int\limits_a^b f(x)\,\mathrm{d}x + \int\limits_b^c f(x)\,\mathrm{d}x = \int\limits_a^c f(x)\,\mathrm{d}x.$

Note that the areas below a curve are assigned *negative* area!

Examples.

1. We calculate that $\int_0^1 \sqrt{x}\,\mathrm{d}x = \frac{2}{3}x^{3/2}\Big|_{x=0}^{1} = \frac{2}{3}1^{3/2} - \frac{2}{3}0^{3/2} = \frac{2}{3}.$
2. The definite integral $\int_0^\pi \cos x\,\mathrm{d}x$ is equal to 0; this is because the (negative) area under the $x-$axis exactly cancels the (positive) area above the $x-$axis. (Draw a picture.)
3. We calculate that $\int_0^2 2\,\mathrm{d}x = 2x\Big|_{x=0}^{2} = 2\cdot 2 - 2\cdot 0 = 4$. (Thus the integral gives us the correct value if we try to find the area of a square!)

Proofs

We will only prove the FTC for 'nice' functions. We actually prove the second FTC first as it is easier, and we will state the two theorems a little more carefully.

Theorem (FTC2)**.** *Suppose that f is a continuous function on the closed interval $[a, b]$.[*] Then the function F defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

for all x in the closed interval $[a, b]$ is differentiable for all x such that $a < x < b$, and

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

Proof. Let us take the derivative in a straightforward manner.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t}{h} = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,\mathrm{d}t}{h}.$$

Now, let $f(M)$ be the maximum value obtained by f on the closed interval $[x, x+h]$; let $f(m)$ be the minimum value. Interpreting the integral as an area, we have

$$hf(m) \le \int_x^{x+h} f(t)\,\mathrm{d}t \le hf(M) \implies f(m) \le \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t \le f(M).$$

Now, as $h \to 0$ we must have $f(m) \to f(x)$ and $f(M) \to f(x)$ (because as we make the interval smaller, m and M move towards x). Hence

$$f(x) \le \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t \le f(x)$$

and so $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$. □

Theorem (FTC1)**.** *Suppose f is continuous on the closed interval $[a, b]$, and suppose F is any antiderivative of f for all x such that $a < x < b$ (so $F'(x) = f(x)$ for all such x). Then,*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b.$$

Proof. Consider $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$. In particular, $\int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of f and we can antidifferentiate both sides, obtaining

$$\int_a^x f(t)\,\mathrm{d}t = F(x) + C \tag{*}$$

(where C is some constant). Now substitute a for x in (*): we find that $0 = \int_a^a f(t)\,\mathrm{d}t = F(a) + C$, and in particular $-C = F(a)$. Substituting b for x in (*), we find that $\int_a^b f(t)\,\mathrm{d}t = F(b) + C = F(b) - F(a)$; and we are done. □

[*] i.e. f is continuous at every x such that $a \le x \le b$.

Questions

1. Compute the following definite integrals. A
   (a) $\int_0^1 \mathrm{d}x$
   (b) $\int_{-1}^1 e^x \,\mathrm{d}x$
   (c) $\int_3^4 x^2 + 3x - 1 \,\mathrm{d}x$
   (d) $\int_0^1 x^n \,\mathrm{d}x$ for integer values of n.

2. Find the area underneath the given curves between the given bounds: A
   (a) $y = 6x^2 + 4x + 9$ between $x = 0$ and $x = 4$
   (b) $y = \sin x$ between $x = 0$ and $x = \pi$
   (c) $y = \sin x$ between $x = -\pi$ and $x = \pi$
   (d) $y = \cos x$ between $x = -\pi$ and $x = \pi$
   (e) $y = \frac{1}{x}$ between $x = 1$ and $x = 2$

3. Find all the problems in the following working. A

$$\int_1^{-1} \frac{\mathrm{d}x}{x} = \ln|-1| - \ln|1| = 0$$

4. Show that $\int \ln x \,\mathrm{d}x = x \ln x - x + C$. A

5. Let f be a function such that for all x, $f(-x) = -f(x)$. Such a function is called *odd*. Show that for all a, M

$$\int_{-a}^{a} f(x) \,\mathrm{d}x = 0.$$

   What does this mean geometrically?

6. Let f be an odd function with period 2 such that $\int_0^1 f(x) \,\mathrm{d}x = k$. Compute: M
   (a) $\int_{-1}^1 f(x) \,\mathrm{d}x$
   (b) $\int_0^{-1} f(x) \,\mathrm{d}x$

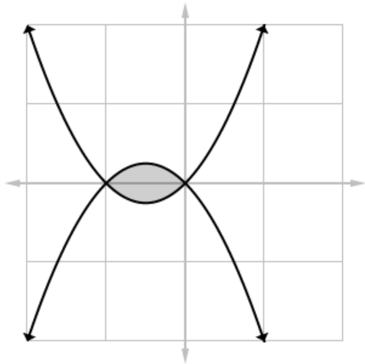
7. Let f be a function such that for all x, $f(-x) = f(x)$. Such a function is called *even*. Show that for all a, A

$$\int_{-a}^{a} f(x) \,\mathrm{d}x = 2 \int_0^a f(x) \,\mathrm{d}x.$$

   What does this mean geometrically?

8. If $\int_{-2}^1 f(x) \,\mathrm{d}x = 2$ and $\int_1^3 f(x) \,\mathrm{d}x = -6$, what is the value of $\int_{-2}^3 f(x) \,\mathrm{d}x$? M

9. Find the area between the curves $y = x^2 + x$ and $y = -x^2 - x$ shaded here. M

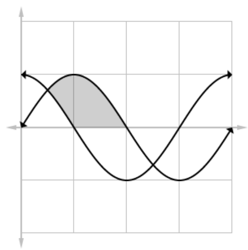


10. Find the area between the two curves $y = 1 + x^2$ and $y = 3 + x$. M

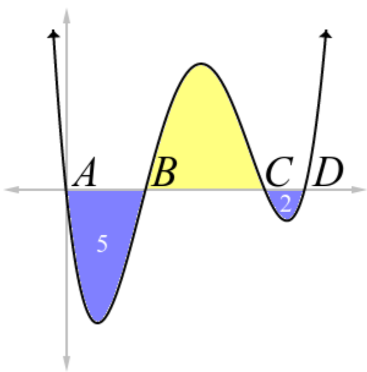
11. Find the area of the region bounded by $f(x) = 4$, $g(x) = \frac{e^x}{5}$, and $x = 0$. M

12. What is the area of the region between the graphs of $f(x) = 2x^2 + 5x$ and $g(x) = -x^2 - 6x + 4$ from $x = -4$ to $x = 0$? M

13. Find the area bounded by the curves $y = \sin x$ and $y = \cos x$ and the $x-$axis graphed here. E



14. Consider the function f graphed below; the total **unsigned** area between the curve and the $x-$axis is 10 square units. Find $\int_A^D f(x)\, dx$. A



15. (a) Sketch the graph of $y = |\sin x|$. M

    (b) Compute $\int_0^{\pi/2} y\, dx$ using the FTC.

    (c) Hence, without doing any anti-differentiation, compute $\int_0^{2\pi} y\, dx$.

16. Define $F(x)$ by M

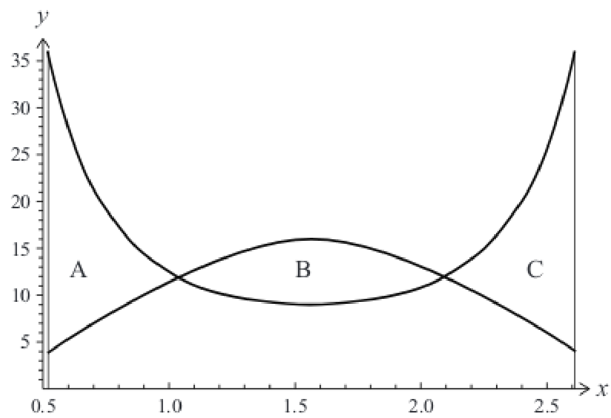
$$F(x) = \int_{\frac{\pi}{4}}^{x} \cos(2t)\,\mathrm{d}t.$$

(a) Use the second fundamental theorem of calculus to find $F'(x)$.

(b) Verify part (a) by integration and differentiation.

17. Compute $\frac{\mathrm{d}}{\mathrm{d}x}\int_2^x t^t\,\mathrm{d}t$. M

18. Scholarship 2014: Find exact expressions for the areas of the three labelled regions bounded by the two curves $y = 9\csc^2 x$ and $y = 16\sin^2 x$ between $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ shown below. S



NCEA Level 3 Calculus (Integration)
17. The Fundamental Theorem of Calculus (Homework)

Reading

Go and watch...

https://www.youtube.com/watch?v=FnJqaIESC2s

Questions

1. Evaluate
$$\int_0^{\frac{\pi}{4}} \sec^2\theta \,\mathrm{d}\theta$$
2. If $\int_1^3 f(x)\,\mathrm{d}x = 7$ and $\int_2^3 f(x)\,\mathrm{d}x = -3$, what is the value of $\int_1^2 f(x)\,\mathrm{d}x$?
3. Find the area enclosed between the graphs of $3y = x^2$ and $y = 2x$.

NCEA Level 3 Calculus (Integration)
18. Substitution

Goal for this week

To practice integrating functions by undoing the chain rule.

Recall that the **chain rule** for differentiation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x).$$

Since integration is (in some sense) the inverse of differentiation, we can write (by applying the fundamental theorem of calculus)

$$\int f'(g(x))g'(x)\,\mathrm{d}x = f(g(x)) + C.$$

For a mnemonic, we can let $u = g(x)$. Then $\mathrm{d}u = g'(x)\,\mathrm{d}x$* and so, by the rule we just wrote down, we have

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(u)\,\mathrm{d}u = f(u) + C = f(g(x)) + C.$$

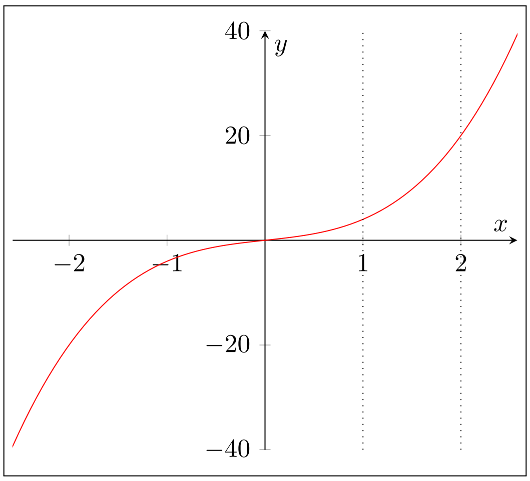
In Leibniz notation, we have

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{\mathrm{d}f}{\mathrm{d}g}\,\mathrm{d}g = \int f'(g)\,\mathrm{d}g = f(g) + C = f(g(x)) + C,$$

and so one can intuitively think about this (here we substitute g out) as the cancellation of differentials underneath an integral sign.

This rule, which gives us a kind of chain rule for integration, is called **substitution**, or the **inverse chain rule**. It can be thought of as a change in coordinate system from an x-based system to one based on u, and we have to 'resize' our area based on how much u stretches the coordinate system compared to x — and this 'stretch factor' is simply $\frac{\mathrm{d}u}{\mathrm{d}x}$.

Example. For example, consider $\int_1^2 2x(x^2+1)\,\mathrm{d}x$; we are finding the area shown here between the dotted lines.

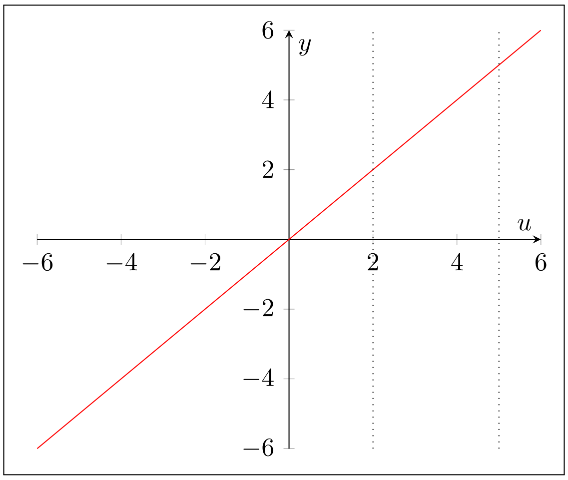


*again, this is just a mnemonic: it *is* possible to make $\mathrm{d}x$ meaningful (it is what is known as a *differential form*), but all we are really doing is applying the chain rule.

Let us make the substitution $u = x^2 + 1$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and our integral becomes

$$\int_1^2 2x(x^2+1)\,\mathrm{d}x = \int_{u^{-1}(2)}^{u^{-1}(5)} \frac{\mathrm{d}u}{\mathrm{d}x} u(x)\,\mathrm{d}x = \int_2^5 u\,\mathrm{d}u.$$

We can graph our region of integration again.



This new coordinate system, which is $2x$ times as large as the older one, is much simpler to integrate inside!

Examples.

1. Suppose we wish to find $\int \sin x \cos x\,\mathrm{d}x$. Then let $u = \sin x$, so $\mathrm{d}u = \cos x\,\mathrm{d}x$ and

$$\int \sin x \cos x\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C.$$

2. In this case, we also could have used a trigonometric identity. Suppose we wish to find $\int xe^{x^2}\,\mathrm{d}x$. We can let $u = x^2$, and then $du = 2x\,\mathrm{d}x \Rightarrow \mathrm{d}x = \frac{du}{2x}$. Hence:

$$\int xe^{x^2}\,\mathrm{d}x = \int \frac{1}{2}e^u\,\mathrm{d}u = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

3. Suppose we wish to find $\int \frac{4}{x}(\ln x)^3\,\mathrm{d}x$. We let $u = \ln x$, and then $\mathrm{d}u = \frac{\mathrm{d}x}{x}$. Hence:

$$\int \frac{4}{x}(\ln x)^3\,\mathrm{d}x = 4\int u^3\,\mathrm{d}u = u^4 + C = (\ln u)^4 + C.$$

Questions

1. Find the following indefinite integrals. A

   (a) $\int \sin 2x\,\mathrm{d}x$
   (b) $\int (4x-44)^{2019}\,\mathrm{d}x$
   (c) $\int 4x\sqrt{x^2+3}\,\mathrm{d}x$
   (d) $\int (3x-4)^2\,\mathrm{d}x$
   (e) $\int \frac{x}{x^2+1}\,\mathrm{d}x$
   (f) $\int \frac{2}{4x+3}\,\mathrm{d}x$
   (g) $\int e^{2x+1}\,\mathrm{d}x$
   (h) $\int \sec 4x \tan 4x\,\mathrm{d}x$
   (i) $\int 2\cos x + \sin 2x\,\mathrm{d}x$
   (j) $\int -2x\csc^2(3x^2)\,\mathrm{d}x$

(k) $\int \frac{3}{x^3} - \frac{4}{x+1}\,\mathrm{d}x$

(l) $\int e^{x/2} + \frac{2}{x}\,\mathrm{d}x$

(m) $\int x^2 \sec^2 x^3 + 9\,\mathrm{d}x$

(n) $\int -\csc(\tan x)\cot(\tan x)\sec^2 x\,\mathrm{d}x$

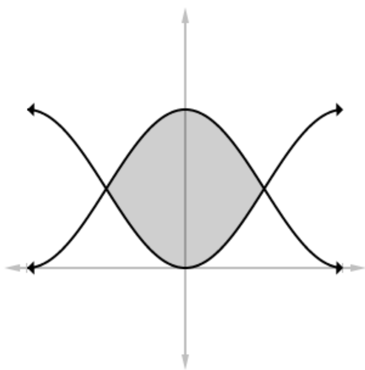
(o) $\int \frac{\cos x - \sin x}{\cos x + \sin x}\,\mathrm{d}x$

(p) $\int \frac{2017}{x \ln x}\,\mathrm{d}x$

2. By using the substitution $x = \sin\theta$, find M

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

3. Compute the following definite integrals: M
   (a) $\int_0^1 xe^{-x^2}\,\mathrm{d}x$
   (b) $\int_{-\pi/3}^{\pi/3} x^4 \sin x\,\mathrm{d}x$ (hint: no substitution is required)
   (c) $\int_0^1 \cos(\pi t/2)\,\mathrm{d}t$
   (d) $\int_0^1 (3t-1)^{50}\,\mathrm{d}t$
   (e) $\int_0^1 \sqrt[3]{1+7x}\,\mathrm{d}x$
   (f) $\int_0^1 \frac{\mathrm{d}x}{1+\sqrt{x}}\,\mathrm{d}x$
   (g) $\int_{-1}^2 x(x-1)^3\,\mathrm{d}x$
   (h) $\int_0^3 x\sqrt{1+x^2}\,\mathrm{d}x$
4. Find the area enclosed by the curve $y = 4\sin 3x \cos x$ and the $x-$axis from $x = 0$ to $x = \frac{\pi}{3}$. M
5. Find k such that $\int_0^k e^{2x}\,\mathrm{d}x = 40$. E
6. Calculate the area enclosed by the curve $y = \frac{3x-2}{x+4}$ and the lines $y = 0$, $x = 1$, and $x = 5$. E
7. Find the area between the curves $y = \sin^2 kx$ and $y = \cos^2 kx$ shaded below. E



8. Find $\int \tan\theta\,\mathrm{d}\theta$ and $\int \cot\theta\,\mathrm{d}\theta$. M

9. Complete the following working: A

$$\begin{aligned}
\int \sec x \, \mathrm{d}x &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, \mathrm{d}x \\
&= \int \frac{\dots}{\sec x + \tan x} \, \mathrm{d}x \\
\text{Let } u = \dots & \\
&= \int \frac{1}{\dots} \, \mathrm{d}u \\
&= \dots
\end{aligned}$$

10. Show that M

$$\int x^4 \sin x \, \mathrm{d}x = 4x(x^2 - 6x)\sin x - (x^4 - 12x^2 + 24)\cos x + C.$$

11. If $y = x\sqrt{\sin x^3 + \cos x^3}$, find $\pi \int_0^1 y^2 \, \mathrm{d}x$. M

12. The velocity of a particle at time t is given by $v = \frac{\cos(\sqrt{2t+1})}{\sqrt{2t+1}}$. What is the position of the particle at time $t = 5$, given that $x(0.5) = 0$? (Recall that $v = \frac{\mathrm{d}x}{\mathrm{d}t}$.) M

13. Evaluate $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, \mathrm{d}x$ [*Hint: use the substitution* $x = \frac{\pi}{2} - u$ *and add the result to the original integral.*] S

14. Scholarship 1999: S
    (a) Evaluate $\int \cos^5 x \, \mathrm{d}x$ using the substitution $t = \sin x$.
    (b) i. If $f(x) = \cos^5 x$, what are $f(0)$, $f'(0)$, and $f''(0)$?
        ii. Hence evaluate a, b, and c in the approximation $\cos^5 x \approx a + bx + cx^2$.
        iii. Use this to give an approximation for $\int \cos^5 x \, \mathrm{d}x$.
    (c) Evaluate $\int_0^{0.6} \cos^5 x \, \mathrm{d}x$ to three significant figures, using:
        i. The exact integration in (a).
        ii. The expression in (b)(iii).
        iii. Simpson's rule.

NCEA Level 3 Calculus (Integration)
18. Substitution (Homework)

Reading

Historically, when Leibniz conceived of the notation, $\frac{\mathrm{d}y}{\mathrm{d}x}$ was supposed to be a quotient: it was the quotient of the "infinitesimal change in y produced by the change in x" divided by the "infinitesimal change in x".

However, the formulation of calculus with infinitesimals in the usual setting of the real numbers leads to a lot of problems. For one thing, infinitesimals can't exist in the usual setting of real numbers! This is because the real numbers satisfy an important property, called the Archimedean property: given any positive real number $\varepsilon > 0$, no matter how small, and given any positive real number $N > 0$, no matter how big, there exists a natural number n such that $n\varepsilon > N$. An "infinitesimal" ξ is supposed to be so small that no matter how many times you add it to itself, it never gets to 1, contradicting the Archimedean property.

Other problems: Leibniz defined the tangent to the graph of $y = f(x)$ at $x = a$ by saying "Take the point $(a, f(a))$; then add an infinitesimal amount to a, $a + \mathrm{d}x$, and take the point $(a + dx, f(a + \mathrm{d}x))$, and draw the line through those two points." But if they are two different points on the graph, then it's not a tangent, and if it's just one point, then you can't define the line because you just have one point. That's just two of the problems with infinitesimals.

So calculus was essentially rewritten from the ground up in the following 200 years to avoid these problems, and you are seeing the results of that rewriting. Because of that rewriting, the derivative is no longer a quotient, now it's a limit:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Because we cannot express this limit-of-a-quotient as a-quotient-of-the-limits (both numerator and denominator go to zero), then the derivative is not a quotient. However, Leibniz' notation is very suggestive and very useful; even though derivatives are not really quotients, in many ways they behave as if they were quotients. So we have the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}y}{\mathrm{d}x}$$

which looks very natural if you think of the derivatives as "fractions". You have the inverse function theorem, which tells you that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 1/\frac{\mathrm{d}y}{\mathrm{d}x}$$

which is again almost "obvious" if you think of the derivatives as fractions. So, because the notation is so nice and so suggestive, we keep the notation even though the notation no longer represents an actual quotient: it now represents a single limit. Even though we write $\frac{\mathrm{d}y}{\mathrm{d}x}$ as if it were a fraction, and many computations look like we are working with it like a fraction, it isn't really a fraction (it just plays one on television).

There is a way of getting around the logical difficulties with infinitesimals; this is called nonstandard analysis. It's pretty difficult to explain how one sets it up, but you can think of it as creating two classes of real numbers: the ones you are familiar with, that satisfy things like the Archimedean property, the least-upper-bound property, and so on, and then you add another, separate class of real numbers that includes infinitesimals and a bunch of other things. If you do that, then you can, if you are careful, define derivatives exactly like Leibniz, in terms of infinitesimals and actual quotients; if you do that, then all the rules of calculus that make use of $\frac{\mathrm{d}y}{\mathrm{d}x}$ as if it were a fraction are justified because, in that setting, it is a fraction. Still, one has to be careful because you have to keep infinitesimals and regular real numbers separate and not let them get confused, or you can run into some serious problems.

Arturo Magidin (`https://math.stackexchange.com/users/742/arturo-magidin`), Is $\frac{\mathrm{d}y}{\mathrm{d}x}$ not a ratio? (adapted), URL (version: 2017-09-15): `https://math.stackexchange.com/q/21209`

Questions

1. Calculate the following indefinite integrals:
   (a) $\int -\csc 3x \cot 3x \, \mathrm{d}x$
   (b) $\int -x \sec^2 3x^2 \, \mathrm{d}x$
   (c) $\int \frac{\sqrt{x}+3x-2}{x} \, \mathrm{d}x$
   (d) $\int \sin^3 x \cos^2 x \, \mathrm{d}x$ (hint: use $\sin^2 x = 1 - \cos^2 x$ to rewrite the integrand)
2. Find the area enclosed between the curve $y = 2 \sin 5x \cos 3x$ and the $x-$axis from $x = 0$ to $x = \frac{\pi}{6}$.
3. Recall that $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}$. Find $\int \frac{x}{1+x^4} \, \mathrm{d}x$.

NCEA Level 3 Calculus (Integration)
19. Differential Equations

Goal for this week

To learn to solve simple equations involving derivatives.

Many physical problems can be expressed by writing different rates of change in terms of each other. For example, for a spring pulled a distance x away from its equilibrium point we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

for some constant k; and for a falling stone at distance r from the centre of the Earth, we have

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{K}{r^2}$$

where K is some constant. These kinds of equations are known as **differential equations**.

Suppose $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$. It would be nice if we could find y in terms of x only. This is in fact possible, using substitution:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= f(x)g(y) \\ \Rightarrow \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x} &= f(x) \\ \Rightarrow \int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int f(x)\,\mathrm{d}x. \end{aligned}$$

Now, let $G(y)$ be an antiderivative of $\frac{1}{g(y)}$ (with respect to y). By the chain rule, then,

$$\frac{\mathrm{d}}{\mathrm{d}x}G(y) = \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}$$

and so

$$\int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = G(y) = \int \frac{1}{g(y)}\,\mathrm{d}y.$$

Hence we have

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

This way of solving differential equations is called **separation of variables**.

Example. Suppose we know that $y\frac{\mathrm{d}y}{\mathrm{d}x} = e^x$. Then we can separate the variables:

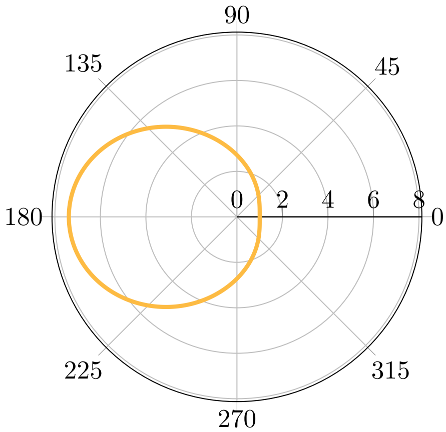
$$\begin{aligned} \int y\,\mathrm{d}y &= \int e^x\,\mathrm{d}x \\ \Rightarrow \frac{1}{2}y^2 &= e^x + C \\ \Rightarrow y^2 &= 2e^x + C. \end{aligned}$$

If we know that the curve passes through $(0, 0)$, then $0 = 2e^0 + C$ and $C = -2$, so $y^2 = 2e^x - 2$.

To check our answer, let us now use implicit differentiation to differentiate this curve. We have $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2e^x$ so and $y\frac{\mathrm{d}y}{\mathrm{d}x} = e^x$ as expected: our solution is correct.

If we have a differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} + A(x)y + B(x) = 0$ for any functions A and B, then for every point (x_0, y_0) in the plane, there exists a unique solution f whose graph $y = f(x)$ passes through (x_0, y_0).

Example. Suppose that $r(\theta)$ is implicitly defined by $\frac{\mathrm{d}r}{\mathrm{d}\theta} = r\sin\theta$ with the condition $r(\pi) = e$. Separating variables, we have $\int \frac{\mathrm{d}r}{r} = \int \sin\theta\,\mathrm{d}\theta$; so $\ln|r| = -\cos\theta + C$ and therefore $r = Ke^{-\cos\theta}$ for some constant K. But $e = Ke^{-\cos\pi} = Ke^0 = K$; so $r(\theta) = ee^{-\cos\theta} = e^{1-\cos\theta}$. Graphing this:



Questions

1. Find y in terms of x in each case, if each curve passes through $(1, 1)$: M

   (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = yx$
   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} + x = yx$
   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} + y = yx$
   (d) $\sqrt{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$
   (e) $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+2)^2$
   (f) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2+1}{2y}e^x$
   (g) $\frac{\mathrm{d}y}{\mathrm{d}x} = x\cos^2 y$
   (h) $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \tan y$
   (i) $2y\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + 2x + 1$
   (j) $\sin y \frac{\mathrm{d}y}{\mathrm{d}x} = 3x$
   (k) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x(e^{x^2}+2)}{6y^2}$

2. (a) Show that one antiderivative of $f(x) = x\sin x\,\mathrm{d}x$ is $F(x) = \sin x - x\cos x$. E

   (b) Find $y(\pi)$ if $y(0) = \pi$ and
$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \theta y \sin\theta.$$

3. Newton's law of cooling states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings; in other words, if the temperature of the body at time t is T and the ambient temperature is T_∞ then $\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_\infty)$ (where k is some constant.) E

   (a) A loaf of bread is taken from the oven at a temperature of 400 °C and is set down on a bench in an area with an ambient temperature of 20 °C. It is found that it takes ten minutes for the loaf to cool to half its initial temperature. How many minutes will it take for the loaf to cool to 30 °C?

   (b) A detective is called to the scene of a crime where a dead body has just been found. She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and is found to be 24 °C. The detective checks the programmable thermostat and finds that the room has been kept at a constant 20 °C for the past 3 days.

   After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 22 °C. This last temperature reading was taken exactly one hour after the first one. Assuming that the victim's body temperature was normal (37.5 °C) prior to death, at what time did the victim die?

4. We will apply calculus concepts to chemical rates of reaction. E
   (a) A *first-order reaction* is one whose rate depends linearly on the concentration of one reactant A; in other words, $-\frac{\mathrm{d}[A]}{\mathrm{d}t} = k[A]$.
   One example of a first-order reaction is the decomposition of hydrogen peroxide:
   $$2\,\mathrm{H_2O_2(aq)} \longrightarrow 2\,\mathrm{H_2O} + \mathrm{O_2(g)}$$
   What percentage of hydrogen peroxide will have decomposed after 600 seconds, if the reaction constant is $k = 6.40 \times 10^{-5}\,\mathrm{s^{-1}}$?
   (b) If a reaction depends linearly on the concentrations of two reactants, A and B (as in $\mathrm{A} + \mathrm{B} \longrightarrow \mathrm{C}$) then the rate of reaction is given by
   $$-\frac{\mathrm{d}[A]}{\mathrm{d}t} = -\frac{\mathrm{d}[B]}{\mathrm{d}t} = \frac{\mathrm{d}[C]}{\mathrm{d}t} = k[A][B].$$
   If we consider the reaction $\mathrm{NO_2} + \mathrm{CO} \longrightarrow \mathrm{CO_2} + \mathrm{NO}$, the rate is experimentally found to be *second-order* in the reactant $\mathrm{NO_2}$ and independent of the concentration of carbon monoxide. It follows that the rate of reaction is
   $$\frac{\mathrm{d}[\mathrm{NO_2}]}{\mathrm{d}t} = -k[\mathrm{NO_2}]^2$$
   where k is some constant.
   Initially, the concentration of $\mathrm{NO_2}$ is $2.0\,\mathrm{mol\,L^{-1}}$; after ten minutes, the concentration has decreased to $1.0\,\mathrm{mol\,L^{-1}}$. How long will it take for the concentration to become $0.5\,\mathrm{mol\,L^{-1}}$?
5. It is known that the motion of a particle is described by the differential equation M
   $$v = \frac{4\sin(2t)}{x}.$$
   Initially, the particle is two metres away from the origin in the positive $x-$direction. Find the particle's position after ten seconds.
6. Suppose that $y'(x) = e^{x+2y}$, and $y(0) = 0$. Find $y(x)$ explicitly. M
7. Assume that the rate of reproduction of some population P is proportional to the number of pairs of individuals; so M
   $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2.$$
   Show that the size of the population becomes infinitely large in a finite time.
8. Recall from physics that **simple harmonic motion** occurs when the force F on an object is proportional to (and opposite in direction to) its displacement k from some zero point ($F = -kx$). Recall also that Newton's second law tells us that the acceleration felt by an object of mass m is given by $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{F}{m}$. We wish to find a formula for x, the displacement of the object, at time t. We have: M
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{kx}{m}$$
   Show that $x = A\cos(\sqrt{k/m}\cdot t)$ is a solution of the given differential equation.
9. Read the following result, which you don't need to prove. E

   Theorem. *Suppose P, Q, and R are constants, and suppose (x_0, y_0) and (x_0, y_0') are points. Then there exists a unique function f satisfying the following conditions:*
   $$\begin{cases} f''(x) + Pf'(x) + Qf(x) + R = 0 \\ f(x_0) = y_0 \\ f'(x_0) = y_0'. \end{cases}$$

(a) Suppose that $y = s(x)$ and $y = c(x)$ satisfy $y'' + y = 0$, and that:
- $s(0) = 0$, $s'(0) = 1$
- $c(0) = 1$, $c'(1) = 0$.

Show, using only this information, that $s' = c$ and $c' = -s$.

(b) Show, again using only the information in (a) and the theorem, that:
i. $[s(x)]^2 + [c(x)]^2 = 1$ (we have actually already done this)
ii. $s(x + a) = s(x)c(a) + c(x)s(a)$
iii. $2s(x)c(a) = s(x + a) + s(x - a)$

(c) Explicitly write down functions s and c satisfying the above.

10. Scholarship 2000: The piriform is the curve defined by the equation $16y^2 = x^3(8 - x)$ where $x \geq 0$. By solving the differential equation S
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2(6 - x)}{8y}$$
($y = 0$ when $x = 0$), show the piriform is the solution.

11. Scholarship 2015: Determine all differentiable equations of the form $y = f(x)$ which have the properties: S
$$f'(x) = (f(x))^3 \text{ and } f(0) = 2$$

12. If you are more interested in nice geometry than applications, fill in the details here. Consider the family of curves consisting of all circles tangent to the y-axis at $(0, 0)$: that is, all circles with equations of the form O
$$(x - r)^2 + y^2 = r^2.$$
(a) Show that $2(x - r) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x-r}{y}$.

(b) By substituting the original equation, eliminate the parameter r.

(c) Thus we have found a differential equation satisfied by all of the circles in the family. We now want to find the set of curves such that each of our new curves is orthogonal to all of the curves in the original family: that is, the set of curves such that each intersects every circle at right angles. We thus need to solve
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy}{x^2 - y^2}.$$
Justify this.

(d) Use the substitution $z = y/x$, and rewrite the differential equation in terms of z and x only. The result should be separable; solve it as usual, and graph the resulting family of functions to check that they are indeed the ones we are looking for.

NCEA Level 3 Calculus (Integration)
19. Differential Equations (Homework)

Reading

So far, we have looked at equations where the unknown is either a number or a point in n-dimensional space (that is, a sequence of n numbers). In order to generate these equations, we took various combinations of the basic arithmetical operations and applied them to our unknowns.

Here, for comparison, are two well-known differential equations, the first "ordinary" and the second "partial":

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k^2x = 0,$$
$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right).$$

The first is the equation for simple harmonic motion, which has the general solution $x(t) = A\sin kt + B\cos kt$; the second is the heat equation which describes the way that the distribution of heat in a physical medium changes with time.

For many reasons, differential equations represent a jump in sophistication. One is that the unknowns are *functions*, which are much more complicated objects than numbers or n-dimensional points. (For example, the first equation above asks what function x of t has the property that if you differentiate it twice then you get $-k^2$ times the original function.) A second is that the basic operations one performs on functions include differentiation and integration, which are considerably less "basic" than addition and multiplication. A third is that differential equations which can be solved in "closed form," that is, by means of a formula for the unknown function f, are the exception rather than the rule, even when the equations are natural and important.

From 'The Princeton Companion to Mathematics', I.4 §1.5

Questions

1. Solve the following equations for $y(t)$:
   (a) $e^{y-t}\frac{\mathrm{d}y}{\mathrm{d}t} = 1$
   (b) $\frac{\mathrm{d}y}{\mathrm{d}t} = ty^2$
   (c) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\sec^2 y}$
   (d) $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{t}{\sec t \sin y}$ (*Hint: first show that* $\frac{\mathrm{d}}{\mathrm{d}x}[\cos x - x\sin x] = -x\cos x$)
2. A copper ball with temperature 100 °C is dropped into a basin of water with constant temperature 30 °C. After 3 minutes the temperature of the ball has decreased to 70 °C. When will it reach a temperature of 31 °C?
3. Consider a tank of water. The rate of flow of water into the tank is a constant $3\,\mathrm{L\,s^{-1}}$; the flow out is directly proportional to the volume of water in the tank. Initially, the volume of water in the tank is 100 L; if the volume were to increase to $120\,\mathrm{L\,s^{-1}}$, the rate of water flowing out would exactly balance the rate of water flowing in.
   (a) Form a differential equation and find the volume of water after ten minutes.
   (b) Does the outward rate of flow ever become greater than the incoming rate of flow?

NCEA Level 3 Calculus
Integration Assignment

1. Compute the indefinite integrals.
   (a) (2 points) $\displaystyle\int x \cdot \cos(x^2) \cdot \sin(\sin x^2)\,\mathrm{d}x$
   (b) (2 points) $\displaystyle\int \pi t \csc^2(2t^2)\,\mathrm{d}t$
   (c) (2 points) $\displaystyle\int \frac{\sqrt{j} + 3j^5 + 3j^6 + 3j^7 + 2}{2j^7}\,\mathrm{d}j$
   (d) (2 points) $\displaystyle\int \frac{\ln t^2}{t}\,\mathrm{d}t$

Solution:

(a) Let $u = \sin x^2$. Then $\mathrm{d}u = 2x \cos x^2\,\mathrm{d}x$ and our integral becomes $\int \frac{1}{2} \sin u\,\mathrm{d}u = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos\sin x^2 + C$. This could also be done by two substitutions, $u = x^2$ and then $v = \sin u$.

(b) Let $u = 2t^2$. Then $\mathrm{d}u = 4t\,\mathrm{d}t$ and our integral becomes $\int \frac{\pi}{4}\csc^2(u)\,\mathrm{d}u = -\frac{\pi}{4}\cot u + C = -\frac{\pi}{4}\cot(2x^2) + C$.

(c) We simplify to find that our integral becomes $\frac{1}{2}\int j^{-6.5} + 3j^{-2} + 3j^{-1} + 3 + 2j^{-7}\,\mathrm{d}j = \frac{1}{2}\left(-\frac{j^{-5.5}}{5.5} - 3j^{-1} + 3\ln|j| + 3j - \frac{1j^{-6}}{3}\right) + C = \frac{1}{2}\left(-\frac{1}{5.5\sqrt{j^{11}}} - \frac{3}{j} + 3\ln|j| + 3j - \frac{1}{3j^6}\right) + C$.

(d) First note that $\ln t^2 = 2\ln t$. Then let $u = \ln t$ so $\mathrm{d}u = \frac{1}{t}\,\mathrm{d}t$ and the integral becomes $\int 2u\,\mathrm{d}u = u^2 + C = (\ln t)^2 + C$.

2. We will prove the identity $1 + \tan^2 x = \sec^2$.
   (a) (1 point) Calculate $\frac{\mathrm{d}}{\mathrm{d}x}\sec^2 x$.
   (b) (4 points) Using the substitution $u = \tan x$, integrate your answer to part (a). Conclude that $\sec^2 x = \tan^2 x + C$ for some constant C.
   (c) (2 points) Find the value of C and conclude the identity above.

Solution:

(a) Using the chain rule the required derivative is $2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x$.

(b) We find $\int 2\sec^2 x \tan x\,\mathrm{d}x$. Let $u = \tan x$. Then $\mathrm{d}u = \sec^2 x$, and our integral becomes $\int 2u\,\mathrm{d}u = u^2 + C = \tan^2 x + C$. But from (a) we have $\int 2\sec^2 x \tan x = \sec^2 x + C'$. Hence $\tan^2 x + C = \sec^2 x + C'$ and the two differ only by a constant.

(c) The identity must hold for all x, and so we set $x = 0$. Then $\sec^2 0 = \tan^2 0 + C$ and $C = 1$. Hence we have $\sec^2 x = \tan^2 x + 1$ as expected.

3. (2 points) Find the area bounded by the curve $y = 3x^2 + x - 2$ and the $x-$axis.

Solution: The curve can be factored as $y = (3x - 2)(x + 1)$ and so the $x-$intercepts are $x = -1$ and $x = \frac{2}{3}$. We must therefore find $\int_{-1}^{2/3} 3x^2 + x - 2 \,\mathrm{d}x = x^3 + 0.5x^2 - 2x \Big|_{-1}^{2/3} = -\frac{125}{54} \approx -2.315$.

4. (2 points) If $\int_{-1}^{2} 3f(x)\,\mathrm{d}x = 9$ and $\int_{-1}^{3} f(x)\,\mathrm{d}x = 1$, find $\int_{2}^{3} f(x)\,\mathrm{d}x$.

Solution: $\int_{2}^{3} f(x)\,\mathrm{d}x = \int_{-1}^{3} f(x)\,\mathrm{d}x - \frac{1}{3}\int_{-1}^{2} 3f(x)\,\mathrm{d}x = 1 - 3 = -2$.

5. (a) (3 points) Compute $\int_0^R 2\pi r\,\mathrm{d}r$. Interpret your answer (you may wish to draw a diagram).
   (b) (2 points) Find the volume of a sphere of radius R by integration; the surface area of a sphere of radius r is given by $SA = 4\pi r^2$.

Solution:

(a) $\int_0^R 2\pi r\,\mathrm{d}r = \pi r^2 \Big|_0^R = \pi R^2$, which is the area of a circle of radius R. This makes sense as we are summing up all the circumferi of circles radiating out from the centre of our larger circle: we expect to get the full area.

(b) Same reasoning: $\int_0^R 4\pi r^2\,\mathrm{d}r = 4/3\pi r^3 \Big|_0^R = 4/3\pi R^3$.

6. (5 points) Find $y(\sqrt{\pi/2})$ if $y(0) = 0$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\sin(x^2)\cot y.$$

Solution: Separating variables, we have $\int \tan y\,\mathrm{d}y = \int x\sin x^2\,\mathrm{d}x$. The RHS is simply $-\frac{1}{2}\cos x^2 + C$ (substitute x^2 out), and we can rewrite the LHS as $\tan y = \frac{\sin y}{\cos y}$ so (using the substitution $\cos y$); hence overall we have got $-\ln|\cos y| = -\frac{1}{2}\cos x^2 + C$. But $y(0) = 0$ so $-\ln 1 = -\frac{1}{2} + C$ and $\ln 1 = 0$ so $C = 0.5$ and $|\cos y| = e^{\frac{1}{2}\cos x^2 - \frac{1}{2}}$. At $x = \sqrt{\pi/2}$, the RHS becomes $e^{-0.5}$ and so $\cos y = \pm e^{-0.5}$; therefore we have two solutions for y: $y = \cos^{-1}(\pm e^{-0.5})$.

NCEA Level 3 Calculus (Integration)
20. Partial Fractions

Goal for this week

To learn to integrate ratios of polynomials.

Definition (Rational Function)**.** A **rational function** is a function f which can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

for suitable polynomials p and q (where $q \neq 0$).

We can already integrate some rational functions; in particular, those of the form $f(x) = p'(x)/p(x)$:

$$\int \frac{p'(x)}{p(x)}\,\mathrm{d}x = \ln\left|p(x)\right|.$$

This week we will learn a technique that, in theory, allows us to integrate *all* rational functions. To understand the idea, note that we can easily integrate all functions of the form

$$f(x) = \frac{A}{(ax+b)^n}$$

for real constants A, a, b, and n.

Our task is simply to 'deconstruct' arbitrary fractions into this form.

Example.

$$\frac{2}{x-4} + \frac{3}{x+1} = \frac{5x-10}{x^2-3x-4}$$

so

$$\int \frac{5x-10}{x^2-3x-4}\,\mathrm{d}x = \int \frac{2}{x-4} + \frac{3}{x+1}\,\mathrm{d}x = 2\ln|x-4| + 2\ln|x-4|.$$

In effect, the technique of partial fractions is the reverse of this: we *decompose* the more complex rational function into two or more functions which are easier to integrate.

Let $f(x) = \dfrac{p(x)}{q(x)}$. Then we have four cases

1. $q(x)$ is the product of distinct linear factors.
2. $q(x)$ is the product of linear factors, some of which are repeated.
3. $q(x)$ is the product of distinct factors, some of which are irreducible quadratics.
4. $q(x)$ contains a repeated irreducible quadratic factor.

Note: the degree of p must be less than the degree of q, so you may need to use long division before applying the technique of partial fractions.

We consider only the first two cases here. See Stewart §7.4 for the others.

Type I: Distinct linear factors

Suppose that $q(x) = (\alpha_1 x + \beta_1) + \cdots + (\alpha_n x + \beta_n)$. Then the partial fraction decomposition is of the form

$$\sum_{i=1}^{n} \frac{A_i}{\alpha_i x + \beta_i}.$$

Example.

$$\frac{11x-2}{6x^2+x-1} = \frac{11x-2}{(2x+1)(3x-1)} = \frac{A}{2x+1} + \frac{B}{3x-1}.$$

So $11x - 2 = A(3x-1) + B(2x+1)$. Let $x = 1/3$, so $B = \frac{11/3-2}{2/3+1} = 1$; then let $x = -1/2$, so $A = \frac{11/2+2}{3/2+1} = 3$.

Hence

$$\int \frac{11x-2}{6x^2+x-1}\,\mathrm{d}x = \int \frac{3}{2x+1} + \frac{1}{3x-1}\,\mathrm{d}x = \frac{3}{2}\ln|2x+1| + \frac{1}{2}\ln|3x-1| + C.$$

Type II: Repeated linear factors

Suppose some factor $(\alpha_i x + \beta_i)^r$ appears in the factorisation of $Q(x)$. Then the partial fraction decomposition will include

$$\sum_{j=1}^{r} \frac{A_{i_j}}{(\alpha_i x + \beta_i)^j}.$$

Example. Consider $\int \frac{2x+4}{x^3-2x^2}\,\mathrm{d}x$. We wish to find a partial fraction expansion:

$$\begin{aligned}\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{2x+4}{x^3-2x^2} &\iff 2x+4 = Ax(x-2) + B(x-2) + Cx^2 \\ &\iff 2x+4 = (A+C)x^2 + (B-2A)x - 2B\end{aligned}$$

Matching coefficients, we find $B = -2$, $A = -2$, and $C = 2$. Then:

$$\begin{aligned}\int \frac{2x+4}{x^3-2x^2}\,\mathrm{d}x &= \int \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}\,\mathrm{d}x \\ &= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C \\ &= 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C.\end{aligned}$$

Questions

1. Find $\displaystyle\int \frac{A}{(ax+b)^n}\,\mathrm{d}x$ if A, a, b, and n are real constants. S

2. Evaluate, using partial fractions: S

   (a) $\displaystyle\int \frac{3x-1}{(x-3)(x+4)}$

   (b) $\displaystyle\int \frac{1}{x^2-3x-4}$

   (c) $\displaystyle\int \frac{1}{x^2-6x-7}$

   (d) $\displaystyle\int \frac{11x+17}{2x^2+7x-4}$

   (e) $\displaystyle\int \frac{5x-10}{x^2-3x-4}$

   (f) $\displaystyle\int \frac{x+7}{x^2-x-6}$

   (g) $\displaystyle\int \frac{1}{x^2+5x+6}$

   (h) $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}$

3. Some more interesting problems: S

   (a) Rewrite in the form $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1}$ and integrate:
   $$\int \frac{4x}{x^3-x^2-x+1}\,\mathrm{d}x.$$

   (b) Use the obvious substitution and divide through:
   $$\int \frac{\sqrt{x+1}}{x}\,\mathrm{d}x.$$

4. Use appropriate substitutions to evaluate: S

   (a) $\displaystyle\int \frac{\cos\theta}{\sin^2\theta+4\sin\theta-5}\,\mathrm{d}\theta$

   (b) $\displaystyle\int \frac{e^{3x}}{e^{2x}+4}\,\mathrm{d}t$

   (c) $\displaystyle\int \frac{5+2\ln x}{x(1+\ln x)^2}$

5. Don't use a sledgehammer to kill a fly, and compute the following: E
   (a) $\displaystyle\int \frac{x^2(5x^2+4x-3)}{x^5+x^4-x^3+1}\,\mathrm{d}x$‘
   (b) $\displaystyle\int \frac{x^2+1}{x(x^2+3)}\,\mathrm{d}x$
6. We have already computed $\int \sec x\,\mathrm{d}x$ via a bit of a trick, but we could also use partial fractions. O
   (a) Show that $\sec x = \frac{\cos x}{1-\sin^2 x}$.
   (b) Hence, or otherwise, compute $\int \sec x\,\mathrm{d}x$.
7. Solve the following differential equation for $y(x)$: S
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-a^4}{x^2-a^2}$$
8. Scholarship 2008: S
   (a) $\dfrac{A}{x} + \dfrac{B}{P-x} = \dfrac{1}{x(P-x)}$ where x is a variable and P is a constant. Find A and B in terms of P.
   (b) When a rumour about a teacher is started at a school of size P students, it spreads at a rate (in students per day) that is proportional to the product of the number of students who know the rumour, N, and those who do not. Find an expression for the number of students N who know the rumour after t days.
   (c) For a particular rumour about a teacher, 0.5% of students know the rumour initially. The principal will need to act to stop the rumour once more than half the school's students know it. When $\dfrac{1}{5}$ of the students know the rumour, the number who know the rumour is increasing at a rate of $0.08P$ students per day. How long will it be before the principal must act?
9. Scholarship 2015: The rate of spread of a rumour at a particular school is proportional to both the number of students who know a rumour, S, and the number of students who do not. If N is the total number of students in the school, then $\frac{\mathrm{d}S}{\mathrm{d}t} = kS(N-S)$. Initially, two students knew the rumour. Show that the number of students who know the rumour at time t is $S(t) = \dfrac{N}{1+\frac{1}{2}e^{-kNt}(N-2)}$. O
10. Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{x^2+1}$. O
    (a) Find a partial expansion of the given rational function as follows:
$$\frac{x^2+x-2}{3x^3-x^2+3x-1} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$
    (b) Hence (or otherwise) compute:
$$\int_{\pi/4}^{\pi/3} \frac{x^2+x-2}{3x^3-x^2+3x-1}\,\mathrm{d}x.$$
11. (Revenge of the limits.) Compute the following series. [*Hint: this sheet is on partial fractions.*] S
$$\lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n(n+1)}\right)$$

NCEA Level 3 Calculus (Integration)
20. Partial Fractions (Homework)

Reading

Perhaps I can best describe my journey of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but eventually you learn where each piece of furniture is. Finally after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of — and couldn't exist without — the many months of stumbling around in the dark that precede them.

From an interview by Nova of Andrew Wiles

Note: For the interested, a proof that one can always expand a rational function into partial fractions is outlined as exercise 11.1.13 in Artin (p. 441).

Questions

1. The *logistic equation* is used when modelling populations:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(1-P)$$

   (a) Find $P(t)$ explicitly.

   (b) Examine the behaviour of the population as $t \to \infty$. Graph the function.

   (c) Examine the behaviour of the population over time if you vary r (check $r = 0$, and $r < 0$ for example).

   (d) Do you think the logistic equation is a good model? Why/why not?

2. In the following, let $t = \tan \frac{x}{2}$ (where $|x| < \pi$).

   (a) Show that:

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \quad \text{and} \quad \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$$

   (b) Show that:

$$\cos x = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin x = \frac{2t}{1+t^2}$$

   (c) Show that:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}$$

   (d) Use the substitution t to evaluate:

      i. $\int (1-\cos x)^{-1}\,\mathrm{d}x$

      ii. $\int (3\sin x - 4\cos x)^{-1}\,\mathrm{d}x$

NCEA Level 3 Calculus (Integration)
21. Integration by Parts

Goal for this week

To practice integrating functions by undoing the product rule.

The substitution rule is the inverse of the chain rule; similarly, there is an inverse of the product rule.

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \\ \iff &\int f'(x)g(x) + f(x)g'(x)\,\mathrm{d}x = f(x)g(x) \\ \iff &\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \end{aligned}$$

Mnemonically,

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Example. Consider $\int x \sin x\,\mathrm{d}x$, which does not yield to any obvious change of variable. Let $u = x$, and let $\mathrm{d}v = \sin x\,\mathrm{d}x$. So $\mathrm{d}u = \mathrm{d}x$, and $v = -\cos x$. Hence:

$$\int x\sin x\,\mathrm{d}x = -x\cos x + \int \cos x\,\mathrm{d}x = -x\cos x + \sin x + C,$$

where C is an arbitrary constant. Check that $(-x\cos x + \sin x)' = x \sin x$.

The aim is to end up with an easier integral than the one that was started with. A good choice for u is usually (in descending order of usefulness):

1. Logarithms
2. Powers of x
3. Exponentials
4. Trig functions

Questions

1. Compute the following indefinite integrals. S
   (a) $\int xe^x\,\mathrm{d}x$
   (b) $\int x^2e^{2x}\,\mathrm{d}x$
   (c) $\int \ln x\,\mathrm{d}x$
   (d) $\int p^5 \ln p\,\mathrm{d}p$
   (e) $\int t^3e^{-t^2}\,\mathrm{d}t$
   (f) $\int \sin \ln y\,\mathrm{d}y$
   (g) $\int x\tan^2 x\,\mathrm{d}x$
2. Prove that S

$$\int \cos^n x\,\mathrm{d}x = \frac{1}{n}\sin x \cos^{(n-1)} x + \frac{n-1}{n}\int \cos^{(n-2)} x\,\mathrm{d}x$$

3. If $I_n = \int_0^n x^n e^x \, \mathrm{d}x$, write down an explicit general formula for I_n. S

4. Evaluate $\int (\ln x)^2 \, \mathrm{d}x$. S

5. Compute $\int_0^\lambda te^{-\lambda t} \, \mathrm{d}t$. S

6. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, and $f'(4) = 3$. Evaluate $\int_1^4 xf''(x) \, \mathrm{d}x$. S

7. A particle moving in one dimension has a velocity function $v(t) = t^2 e^{-t}$ (where t is in seconds). What is its displacement from its starting position after three minutes? S

8. Find the area bounded by $y = x^2 \ln x$ and $y = 4 \ln x$ S

9. Scholarship 2012:
   (a) Find $\frac{\mathrm{d}}{\mathrm{d}x}[x \cos x]$ and use this result to find $\int x \sin x \, \mathrm{d}x$. E
   (b) Hence find the value of $\int_0^{n\pi} x \sin x \, \mathrm{d}x$ for integer values of n. S

10. Scholarship 2016: O
    (a) A function $f(x)$, where x is a real number, is defined implicitly by the formula

$$f(x) = x - \int_0^{\pi/2} f(x) \sin(x) \, \mathrm{d}x.$$

    Find the explicit expression for $f(x)$ in simplest form.
    (b) A curve passing through the point $(1, 1)$ has the property that at each point (x, y) on the curve, the gradient of the curve is $x - 2y$; that is, $\frac{\mathrm{d}y}{\mathrm{d}x} = x - 2y$.
       i. Show that $\frac{\mathrm{d}}{\mathrm{d}x} e^{2x} y = xe^{2x}$.
       ii. Hence, or otherwise, find the equation of the curve.

11. It is well known that S

$$\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x = \sqrt{\pi}.$$

    Using this result, show that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} \, \mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

12. Find $I = \int e^x \cos x \, \mathrm{d}x$. O

13. Recall that $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}$. Find $\int \tan^{-1} x \, \mathrm{d}x$. S

14. We integrate $\int 1/x \, \mathrm{d}x$ by parts: S

$$\int \frac{1}{x} \, \mathrm{d}x = \frac{1}{x} \cdot x - \int -\frac{1}{x^2} \cdot x \, \mathrm{d}x = 1 + \int \frac{1}{x} \, \mathrm{d}x$$

    Cancelling the indefinite integral from both sides, we have $0 = 1$. Explain.

NCEA Level 3 Calculus (Integration)
21. Integration by Parts (Homework)

Reading

Go and watch...

https://www.youtube.com/watch?v=-reFBJ4R9iA

Questions

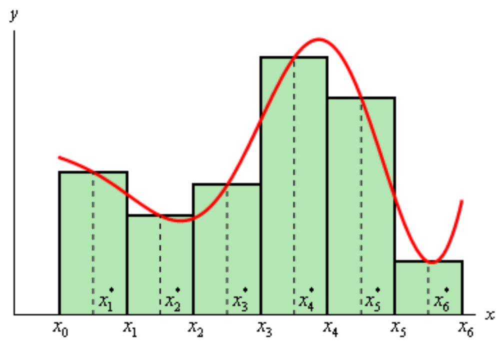
1. Compute the following indefinite integrals.
   (a) $\int x \cos 5x \, \mathrm{d}x$
   (b) $\int \cos x \ln \sin x \, \mathrm{d}x$
   (c) $\int \cos \sqrt{x} \, \mathrm{d}x$
2. Evaluate:
   (a) $$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2) \, \mathrm{d}\theta$$
   (b) $$\int_0^1 (x^2 + 1)e^{-x} \, \mathrm{d}x$$
3. (a) Prove that $\int (\ln x)^n \, \mathrm{d}x = x(\ln x)^n - \int (\ln x)^{(n-1)} \, \mathrm{d}x$.
   (b) Find $\int (\ln x)^3 \, \mathrm{d}x$.

NCEA Level 3 Calculus (Integration)
22. Lengths, Volumes, and Areas

Goal for this week

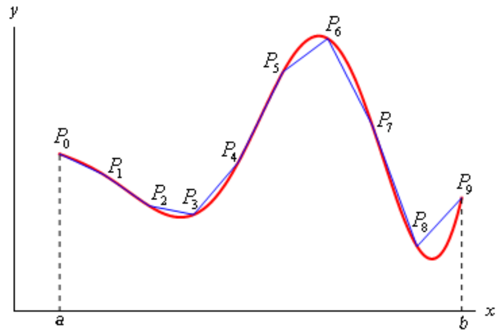
To use an integral as a fancy sum.

Recall that the definite integral is simply a way of calculating the area bounded by a curve. We defined it to be an infinite sum of areas under the curve that individually tended to zero.



Today we will investigate finding volumes, curve lengths, and areas of surfaces by integration.

Curve Lengths



Suppose we have a function f and we wish to find the length *measured along the curve* between two points a and b. Split the interval into pieces of length Δx; then the arc length over each subdivision is approximated by $\sqrt{\Delta x^2 + (f'(x_i^*)\Delta x)^2}$ where x_i is a point inside the subdivisioon. We therefore have the sum over the whole interval (a, b) (where the total number of subdivisions is n):

$$\sum_{i=0}^{n} \sqrt{\Delta x^2 + (f'(x_i^*)\Delta x)^2} = \sum_{i=0}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x.$$

But if we let $n \to \infty$, this is exactly the following integral:

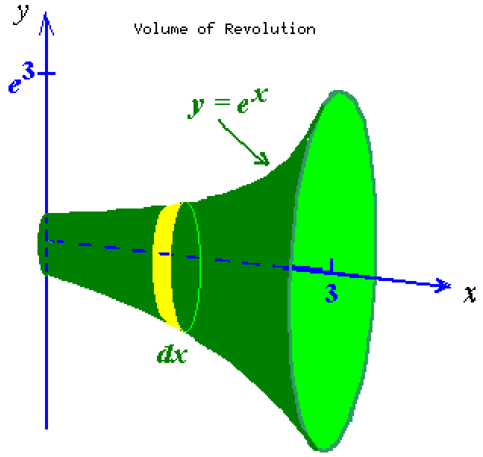
$$R = \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x.$$

Example. We wish to find the length along the curve $y = \ln \cos x$ along the interval $(0, \pi/3)$. We have the following integral (noting that $y' = -\frac{\sin x}{\cos x} = -\tan x$):

$$R = \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, \mathrm{d}x = \int_0^{\pi/3} \sec x\, \mathrm{d}x = \int_0^{\pi/3} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, \mathrm{d}x = \int_1^{2+\sqrt{3}} \frac{1}{u}\, \mathrm{d}u = \ln(2 + \sqrt{3}) \approx 1.3170.$$

The integral of $\sec$ *is tricky; it can also be done by partial fractions (multiply top and bottom of* $1/\cos x$ *by* $\cos x$ *and substitute* $u = \sin x$*).*

Volumes of Revolution



We can carry out the same procedure to find a volume of revolution; the volume of each small disc can be approximated with $\pi[f(x_i^*)]^2 \Delta x$, and so we have

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=0}^{n} \pi[f(x_i^*)]^2 \Delta x \\ &= \int_a^b \pi[f(x)]^2\, \mathrm{d}x. \end{aligned}$$

Surface Areas of Revolution

We can model the surface area of revolution of a curve is similarly found by modelling the volume as a disc of radius $f(x_i^*)$ and length $\sqrt{1 + (f'(x_i^*))^2}\Delta x$:

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=0}^{n} 2\pi f(x_i^*)\sqrt{1 + (f'(x_i^*))^2}\Delta x \\ &= \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, \mathrm{d}x. \end{aligned}$$

Questions

1. Determine the length of: M

   (a) $x = \frac{2}{3}(y-1)^{3/2}$ between $1 \leq y \leq 4$.

   (b) $y = \ln \sec x$ between $0 \leq x \leq \frac{\pi}{4}$.

2. The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface. M

3. (a) Suppose f is a function of x, and that you know that the graph $y = f(x)$ is a straight line. Furthermore, assume that $f(0) = 0$ and $f(h) = r$ where h and r are constants. Find a formula for f, and draw its graph. E

   (b) Find a formula for the volume enclosed by rotating the graph $y = f(x)$ around the x-axis between the origin ($x = 0$) and $x = h$. Sketch a diagram showing the volume.

4. The cartesian equation for a circle of radius r is $y^2 = r^2 - x^2$. Compute the volume of revolution of the circle from $x = -r$ to $x = r$, and hence write down the formula for the volume of a sphere of radius r. E

5. Find the volume of the solid obtained by rotating the region bounded by the given curves around the x-axis. Sketch the region and the solid. M

   (a)
   $$\begin{cases} y = 2 - \frac{1}{2}x \\ y = 0 \\ x = 1 \\ x = 2 \end{cases}$$

   (b)
   $$\begin{cases} y = x - x^2 \\ y = 0 \end{cases}$$

   (c)
   $$\begin{cases} y = \sqrt{25 - x^2} \\ y = 0 \\ x = 2 \\ x = 4 \end{cases}$$

6. Find the volume of rotation of the region bounded by $y = \sin x$ and $y = \cos x$ around the line $y = -1$, where $0 \leq x \leq \pi/4$. E

7. The ellipse M
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where } a > b$$
   is rotated around the x-axis to form an ellipsoid. Find the surface area of the ellipsoid.

8. Use Simpson's rule with $n = 8$ to estimate the volume of the solid resulting when the region enclosed by the curves M
   $$\begin{cases} y = \sin^8 x \\ y = 2x/\pi \\ x = 0 \\ x = \pi/2 \end{cases}$$
   is revolved around the x-axis.

9. A vase is created by rotating the curve S

$$x = \frac{1}{200}y^3 - \frac{1}{10}y^2 + \frac{3}{2}y + \frac{5}{3}$$

around the y-axis for $0 \leq y \leq 20$ (y is in centimetres).

(a) Find a function $V(\alpha)$ for the volume of water in the vase if it is filled up to $y = \alpha$.

(b) Water flows into the vase at a rate of $10\,\mathrm{cm}^3\,\mathrm{min}^{-1}$; water flows out at a rate directly proportional to the square root of the volume in the vase at time t.

i. The initial volume of water in the vase at time $t = 0$ is $3\,\mathrm{cm}^3$. Find the initial height of the water.

ii. After three minutes, the volume of water in the vase is $3.6\,\mathrm{cm}^3$. Will the vase ever fill completely? If so, how long does it take?

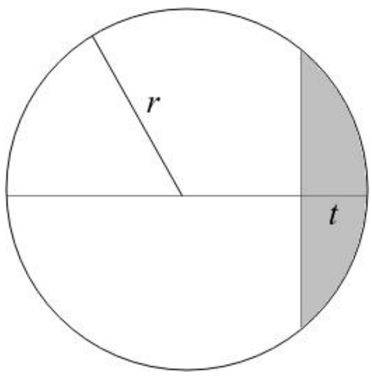
10. Use integration to find the volume of a cylinder by taking slices along the cylinder's axis. E

11. Use integration to find the volume of a cone by taking slices along the cone's axis. E

12. Compute the integral S

$$\int_{r-t}^{t} \sqrt{r^2 - x^2}\,\mathrm{d}x$$

and hence write down an expression for the slice of width t of a circle of radius r (figure).



13. The base of a solid S is a circle of radius r. Cross-sections perpendicular to the base are squares. What is the volume of S? S

14. A cathedral dome is designed with three semicircular supports of radius r such that each horizontal cross-section is a regular hexagon. Show that the volume of the dome is $r^3\sqrt{3}$. S

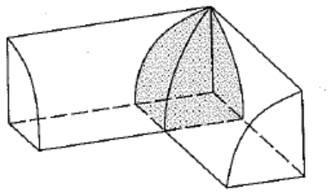
15. The integral E

$$V = \int_0^3 2\pi x^5\,\mathrm{d}x$$

represents the volume of a solid. Describe the solid.

16. Consider a cylindrical hole of length h drilled through the centre of a sphere. Find the volume $V(h)$ of the remaining solid. *Hint: you should find that V is independent of the size of the sphere.* O

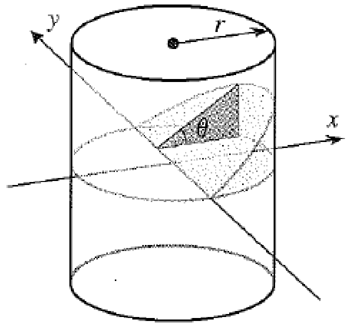
17. Two identical right circular cylinders of radius r have axes that intersect at right angles. Find the volume of the intersection region (known as the *Steinmetz solid*). *Hint: an interesting portion of the intersection is shown in the figure.*[*] O



18. Use an integral to estimate the value of the sum S

$$\sum_{n=0}^{1000000} \sqrt{n}.$$

19. A wedge is cut from a right circular cylinder of radius r by two planes, one perpendicular to the axis of the cylinder and one at an angle θ with the first (as in the figure[†]). Find the volume of the wedge by slicing perpendicular to the y-axis. O



20. Scholarship 2000: The piriform is the curve defined by the equation $16y^2 = x^3(8-x)$ where $x \geq 0$. Find the volume of revolution obtained by rotating the piriform around the x-axis. S

21. Scholarship 2012: Stewie Griffin is a character from the television programme *Family Guy*. His head can be considered as a volume of revolution, turning a curve on an axis passing through his ears. Different volumes are obtained, depending on the shape of the rotated curve. O

    Assuming the head has width $2w$ and height $2h$, find the **ratio** of the volume obtained using a parabolic curve to the volume obtained using a semi-elliptical curve.

22. Scholarship 2013: Prince Rupert's drops are made by dropping molten glass into cold water. A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi(e^{-x} - e^{-2x})}$ for $x \geq 0$, where ϕ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. S

    (a) Show that the volume of the drop between $x = 0$ and $x = \ln(p)$ is $V = \frac{\pi\phi}{2}\left(\frac{p-1}{p}\right)^2$.

    (b) Hence or otherwise, explain why the volume of the drop is never more than some upper limit V_L, no matter how long its tail.

23. Scholarship 2015: Find the area of the surface of revolution obtained when the graph of $f(x) = x^3 + \frac{1}{12x}$, from $x = 1$ to $x = 3$, is revolved $360°$ around the x-axis. S

[*] Figure from Anton, Early Trans. 10th Ed., p 431.
[†] Ibid.

NCEA Level 3 Calculus (Integration)
22. Lengths, Volumes, and Areas (Homework)

Reading

Absolute continuity of motion is not comprehensible to the human mind. Laws of motion of any kind only become comprehensible to man when he examines arbitrarily selected elements of that motion; but at the same time, a large proportion of human error comes from the arbitrary division of continuous motion into discontinuous elements.

There is a well-known so-called sophism of the ancients consisting in this, that Achilles could never catch up with a tortoise he was following in spite of the fact that he travelled ten times as fast as the tortoise. By the time Achilles has covered the distance that separated him from the tortoise, the tortoise has covered one-tenth of tat distance ahead of him: when Achiles has covered that tenth, the tortoise has covered another one-hundredth, and so on for ever. This problem seemed to the ancients insoluble. The absurd answer (that Achilles could never overtake the tortoise) resulted from this: that motion was arbitrarily divided into discontinuous elements, whereas the motion both of Achilles and of the tortoise was continuous.

By adopting smaller and smaller elements of motion we only approach a solution of the problem, but never reach it. Only when we have admitted the conception of the infinitely small, and the resulting geometrical progression with a common ratio of one-tenth, and have found the sum of this progression to infinity, do we reach a solution to this problem. A modern branch of mathematics, having achieved the art of dealing with the infinitely small, can now yield solutions in other more complex problems of motion, which used to appear insoluble.

This modern branch of mathematics, unknown to the ancients, when dealing with problems of motion, admits the conception of the infinitely small, and so conforms to the chief condition of motion (absolute continuity) and thereby corrects the inevitable error which the human mind cannot avoid when dealing with separate elements of motion instead of examining continuous motion.

In seeking the laws of historical movement just the same thing happens.

The movement of humanity, arising as it does from innumerable arbitrary human wills, is continuous.

To understand the laws of this continuous human movement is the aim of history. But to arrive at these laws, resulting from the sum of all those human wills, man's mind postulates arbitrary and disconnected units. The first method of history is to take an arbitrarily selected series of continuous events and examine it apart from others, though there is and can be no *beginning* to any event, for one event always flows uninterruptedly from another. The second method is to consider the actions of some one man — a king or a commander — as equivalent to the sum of many individual wills; whereas the sum of individual wills is never expressed by the activity of a single historical personage.

Historical science in its endeavour to draw nearer to truth continually takes smaller and smaller units for examination. But however small the unit it takes, we begin to feel that to take any unit disconnected from others, or to assume a *beginning* to any phenomenon, or to say that the will of many men is expressed by the actions of any one historic personage, is in itself false.

It needs no critical exertion to reduce utterly to dust any deductions drawn from history. It is merely necessary to select some larger or smaller unit as the subject of observation — as criticism has every right to do, seeing that whatever unit history observes must always be arbitrarily selected.

Only by taking an infinitesimally small unit for observation (the differential of history, that is, the individual tendencies of men) and attaining to the art of integrating them (that is, finding the sum of those infinitesimals) can we hope to arrive at the laws of history.

From book 3, part 3, chapter 1 of *War and Peace*, by Leo Tolstoy.
Extra reading:
`http://teachers.dadeschools.net/akoski/downloads/Reading_Journals/Essays/tolstoy_integration.pdf`
(*Tolstoy's Integration Metaphor*, S. Ahearn, 2004.)

Questions

1. Find the volume of revolution around the x-axis of the hyperbola $y = \frac{1}{x}$ from $x = 1$ to $x = 2$.
2. Find the area of the surface created by rotating the curve $y = \sin x$ around the x-axis ($0 \le x \le \pi$). What is the radius of the circle with the same area?
3. Write down a formula for the volume of a square-based pyramid with base side length L and height H.
4. Scholarship 2017: The length S of a curve expressed in polar coordinates is given by

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta.$$

   Find the length of the entire curve $r = a(1 - \cos\theta)$ in terms of the constant a.

   [*Note: you may wish to graph the curve first.*]

NCEA Level 3 Calculus (Integration)
23. Trigonometric Substitution

Goal for this week

To practice using a special kind of substitution which is sometimes useful.

Consider the integral

$$\int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}x.$$

There is no obvious easy substitution to simplify this integral, and integration by parts could work but will require a lot of work with no guranteed payoff. However, recall that $\sqrt{1-\sin^2\theta} = \cos\theta$; this identity suggests that we could perhaps substitute $x = \sin\theta$ in order to obtain $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$ and so

$$\begin{aligned}\int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\,\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \sin\theta(1-\cos^2\theta)\cos\theta\,\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} (\cos^2\theta - \cos^4\theta)\sin\theta\,\mathrm{d}\theta\end{aligned}$$

Now, letting $u = \cos\theta$ we obtain

$$\begin{aligned}\int_0^{\frac{\pi}{2}} (\cos^2\theta - \cos^4\theta)\sin\theta\,\mathrm{d}\theta &= -\int_1^0 u^2 - u^4\,\mathrm{d}u \\ &= \frac{1}{3}u^3 - \frac{1}{5}u^5\Big|_{u=0}^{1} \\ &= \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.\end{aligned}$$

Here is a table of trig substitutions:

Integrand	**Substitution**	**Identity**
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Integrals requiring trig substitution to solve turn up quite often in physics, especially electromagnetism.

Example. Consider $I = \int \frac{\mathrm{d}x}{\sqrt{9+x^2}}$. Let $x = 3\tan\theta$ so $\mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta$ and:

$$\begin{aligned}
I &= \int \frac{3\sec^2\theta}{3\sqrt{1+\tan^2\theta}}\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta \\
&= \ln(\sec\theta + \tan\theta) + C = \ln(\sec\tan^{-1}(x/3) + x/3) + C \\
&= \ln\left(\sqrt{\left(\frac{x}{3}\right)^2 + 1} + \frac{x}{3}\right) + C.
\end{aligned}$$

Questions

1. Find the following integrals: S

   (a) $\int \frac{x^2-9}{x^3}\,\mathrm{d}x$

   (b) $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}}$

   (c) $\int_0^3 x^2(9-x^2)\,\mathrm{d}x$

   (d) $\int_0^1 x\sqrt{1-x^4}\,\mathrm{d}x$

   (e) $\int_{\sqrt{2}}^2 \frac{\mathrm{d}x}{t^3\sqrt{t^2-1}}$

   (f) $\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x$

2. Use the integral $2\int_r^{-r}\sqrt{r^2-x^2}\,\mathrm{d}x$ to find the area of a circle of radius r. S

3. Scholarship 2005: Find, in terms of r, the area between the ellipse $x^2+16(y-r)^2=r^2$ and the circle $x^2+y^2=r^2$. You may use the substitution $x=r\sin u$ to find the integral $\int\sqrt{r^2-x^2}\,\mathrm{d}x$. S

4. By integrating, verify that S

$$\int_0^x \sqrt{a^2-t^2}\,\mathrm{d}t = \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)+\frac{1}{2}x\sqrt{a^2-x^2}.$$

5. A charged rod of length L produces a electric field at the point (a,b) given by S

$$E(a,b)=\int_{-a}^{L-a}\frac{\lambda b}{4\pi\varepsilon_0(x^2+b^2)^{3/2}}\,\mathrm{d}x.$$

   Evaluate this integral to find an explicit expression for $E(a,b)$.

6. One of these integrations should be done by partial fractions and one by trig substitution. Do them both. O

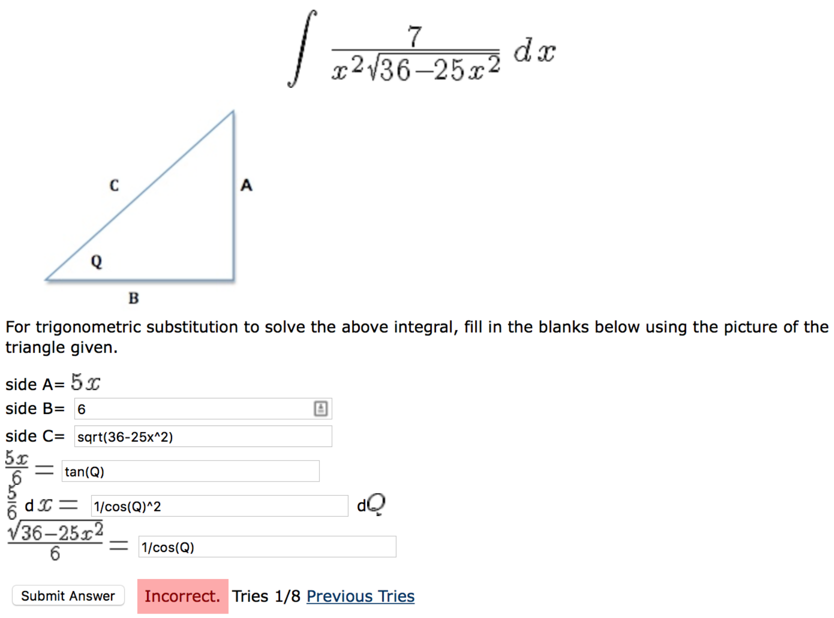
$$\int\frac{\mathrm{d}x}{(4x^2+9)^2}\qquad\qquad\int\frac{x^3}{x^2+x-6}\,\mathrm{d}x$$

7. Check this working (the substitution $x=3\sin\theta$ is used). Find any mistakes. S

$$\begin{aligned}
I &= \int\frac{\sqrt{9-x^2}}{x^2}\,\mathrm{d}x = \int\frac{\cos^2\theta}{\sin^2\theta}\,\mathrm{d}\theta\\
&= \int\frac{1-\sin^2\theta}{\sin^2\theta}\,\mathrm{d}\theta = \int\csc^2\theta-1\,\mathrm{d}\theta\\
&= -\cot\theta-\theta = -\cot\left(\sin^{-1}(x/3)\right)-\sin(x/3)\\
&= \frac{\sqrt{9-x^2}}{x}-\sin(x/3).
\end{aligned}$$

8. A water storage tank has the shape of a cylinder with diameter 10 m. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 m, what percentage of the total capacity is being used? S

9. When writing this worksheet I went on the internet and found this. Find the mistake(s), and do the integral. S



NCEA Level 3 Calculus (Integration)
23. Trigonometric Substitution (Homework)

Reading

Whereas Newton had concentrated on finding the derivatives of given functions and on the inverse process, the recognition that limits of sums like those in definite integrals can be obtained by reversing differentiation is due primarily to Gottfried Wilhelm Leibniz (1646–1716). Leibniz' career contrasts sharply with Newton's. Newton had undertaken the study of mathematics and physics early in life and had pursued these two fields almost exclusively, although he did make minor contributions to chemistry and theology. His career as a professor gave him the opportunity to concentrate. Leibniz started by studying law at the University of Leipzig, the city in which he grew up. He secured a doctor's degree in law at the University of Altdorf in 1666. His first position was that of ambassador for the Elector of Mainz, and until 1672 his interest in mathematics was secondary. In 1672, during a trip to Paris, he met Huygens (the astronomer and mathematician), who acquanted Leibniz with current scientific problems and activities. Leibniz's interests were deeply stirred, and thereafter he devoted much time to mathematics.

What is amazing about Leibniz is the vast quantity of first-rate contributions he made to other fields. Although his profession was law, his work in mathematics and philosophy ranks among the best the world has produced. He also did major work in mechanics, nautical science, optics, hydrostatics, logic, philology, geology, and theology, and was a pioneer in historical research. No subject pursued by intellectuals of his age was neglected; only Leibniz himself went unrecognised and neglected by his contemporaries.

Adapted from *Mathematics for the nonmathematician* (pp.406–7) by Morris Kline (Dover, 1985).

Questions

Compute the following integrals. Some may not require trig substitution.

1. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+4}}$
2. $\displaystyle\int \frac{x^6}{\sqrt{1-x^{14}}}\,\mathrm{d}x$
3. $\displaystyle\int \sqrt{9-x^2}\,\mathrm{d}x$
4. $\displaystyle\int x\sqrt{25x^2-4}\,\mathrm{d}x$
5. $\displaystyle\int \sqrt{4-9z^2}\,\mathrm{d}z$
6. $\displaystyle\int_0^{1/6} \frac{x^5}{(36x^2+1)^{3/2}}\,\mathrm{d}x$
7. $\displaystyle\int \frac{\ln x}{x^5}\,\mathrm{d}x$
8. $\displaystyle\int_1^2 \frac{5t-2}{2t^2-t-1}\,\mathrm{d}t$

NCEA Level 3 Calculus (Integration)
24. Kinematics

Goal for this week

To apply calculus to physics.

Calculus was independently developed by Sir Isaac Newton to describe mechanical motion in physics. This use is known as **kinematics** (from the Greek *kinein*, 'to move'). Suppose a particle moves from position x_0 to position x_1 over a time Δt. We call the ratio

$$\frac{x_1 - x_0}{\Delta t}$$

the **average velocity** of the particle; if we let $x_1 \to x_2$ (or let $\Delta t \to 0$), we obtain the derivative $\frac{\mathrm{d}x}{\mathrm{d}t} = v$, the **instantaneous velocity** of the particle at the point x (usually just abbreviated to velocity).

Similarly, the rate of change of velocity is called *acceleration*. We have $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$. (Out of interest, the third derivative of displacement is known as *jerk*, and the fourth is *jounce*.)

Now, suppose we know the velocity of a particle at each instant over a given time interval. Suppose we split the interval up into small intervals, each of length Δt. Then the total distance travelled is approximated by $\sum v\Delta t$, where the sum is taken for each small interval. If we make the intervals smaller, then clearly our approximation becomes better; and to obtain the true answer, we need only take an integral.

Displacement, s		$\int_{t_0}^{t_1} v \,\mathrm{d}t$
Velocity, v	$\frac{\mathrm{d}s}{\mathrm{d}t}$	$\int_{t_0}^{t_1} a \,\mathrm{d}t$
Acceleration, a	$\frac{\mathrm{d}v}{\mathrm{d}t}$	

We can prove the following **kinematic equations** if acceleration is kept constant over a time period Δt. These equations should be familar to all of those that took level 2 physics, and they are derived by finding areas underneath a velocity-time graph: in short, via calculus.

$$\begin{aligned} v_f &= v_i + a\Delta t \\ s &= v_i\Delta t + \frac{1}{2}a\Delta t^2 \\ v_f^2 &= v_i^2 + 2as \\ s &= \frac{v_f + v_i}{2}\Delta t \end{aligned}$$

Questions

All distances are given in m, and all times in s, unless otherwise stated.

1. A particle moves from $x = 2\,\mathrm{m}$ to $x = 3\,\mathrm{m}$ over $3\,\mathrm{s}$. What is its average velocity over that time? A
2. A particle moves from $(3, 4)$ to $(12, -3)$ over a time $10\,\mathrm{s}$. If the displacements are measured in metres, what is the magnitude of its average velocity during that period? A
3. An object A has a positive acceleration a, and a second object B has a negative acceleration $-a$. Both are moving in the same direction. Which of the following is *not* true? A
   (a) Object B is slowing down compared to object A.
   (b) Object B has a lower velocity than object A.

(c) At some point, object B will reach a velocity of zero and then start moving in the opposite direction.

(d) If object B is behind object A, the two will never cross paths.

4. Suppose a particle has a constant velocity of $34\,\mathrm{m\,s^{-1}}$. How long does it take for the particle to travel $150\,\mathrm{m}$? A

5. Derive the kinematic equations, by considering the integrals of a velocity function $v(t)$ with constant derivative a. M

6. The velocity v of an object t seconds after it moves from the origin is given by M

$$v(t) = 3t^2 - 6t - 24.$$

(a) Write down the formula for the acceleration of the particle after t seconds.

(b) Work out the initial velocity and acceleration.

(c) When is the object at rest momentarily?

(d) When did the object return to the origin?

(e) What was the total distance travelled by the object before it returned to the origin?

7. A well-wrapped food parcel is dropped from an aeroplane flying at a height of $500\,\mathrm{m}$ above the ground. The constant acceleration due to gravity is $-9.81\,\mathrm{m\,s^{-2}}$. Air resistance is negligible. M

(a) How long does it take for the food parcel to hit the ground?

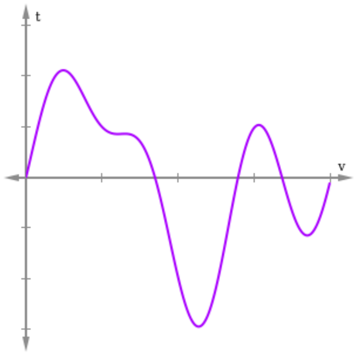
(b) How fast is the food parcel moving when it hits the ground?

8. A racing car travelling at $210\,\mathrm{km\,h^{-1}}$ skids for a distance of $150\,\mathrm{m}$ after its brakes are applied. The brakes provide a constant deceleration. M

(a) What is the deceleration in $\mathrm{m\,s^{-2}}$?

(b) How long does it take for the car to stop?

9. The following is a graph of the instantaneous velocity of an object moving in one dimension over time. M



(a) Draw the acceleration of the object over time.

(b) Draw the position of the object over time, if it was originally located at $x = 0$.

10. The velocity of an Olympic sprinter is modelled by M

$$v_x = a(1 - e^{-bt}),$$

where $a = 11.81\,\mathrm{m\,s^{-1}}$ and $b = 0.6887\,\mathrm{s^{-1}}$. Find an expression for the distance travelled after time t.

11. The displacement of an object moving in a straight line on either side of a fixed origin is given by E

$$s(t) = 2t^3 - 12t^2 + 18t + 3.$$

(a) Find the minimum velocity of the object. Carefully prove that you have found a minimum.

(b) What is the distance between the origin and the object when its velocity is at a minimum?

12. The acceleration of a rocket propelled washing machine is given by $\frac{\mathrm{d}v}{\mathrm{d}t} = 9t^3 - t^4 + t^{-3/2}$, where $0 \leq t \leq 10$. Find the distance which it has travelled after 10 seconds if its initial velocity (at $t = 0$) was $90\,\mathrm{m\,s^{-1}}$. M

13. The acceleration of an object is given by $a(t) = 0.2t + 0.3\sqrt{t}$ for $0 \leq t \leq 10$, where a is the acceleration of the object in $\mathrm{m\,s^{-2}}$ and t is the time in seconds from the instant that movement began. The object was moving with a velocity of $5\,\mathrm{m\,s^{-1}}$ when $t = 4$. How far was the object from its starting point after nine seconds? M

14. A ball is thrown straight up from the edge of the roof of a building, with initial velocity v_0. A second ball is dropped from the roof 1.00 s later. Both feel a constant acceleration due to gravity, $g = -9.81\,\mathrm{m\,s^{-2}}$. E

(a) Suppose the height of the building is 20.00 m. What must be the initial speed v_0 if both balls are to hit the ground at the same time?

(b) Consider a second building of unknown height h; if the first ball is thrown upwards with initial velocity v_1, and both balls hit the ground at the same time, give an expression for h in terms of v_1.

NCEA Level 3 Calculus (Integration)
24. Kinematics (Homework)

Reading

Kinematics may seem like an odd topic to end with, especially as it is more of a physics topic than a mathematics topic. The reason for its inclusion in these notes is by means of revision from level 2; and the reason it is in level 2, is so that you can apply calculus to L3 physics. If you are not studying physics, you may wonder why you should bother learning this particular application of calculus; the answer is that, historically, calculus began as an attempt to formalise mechanics and so a physical intuition can often be useful when solving problems that are not at first glance physical.

Beyond this, there is only one fundamental concept in this topic that you must remember: the derivative is just a rate of change. Velocity is rate of change of position, and acceleration is rate of change of velocity. If you slow down faster, your acceleration is more negative.

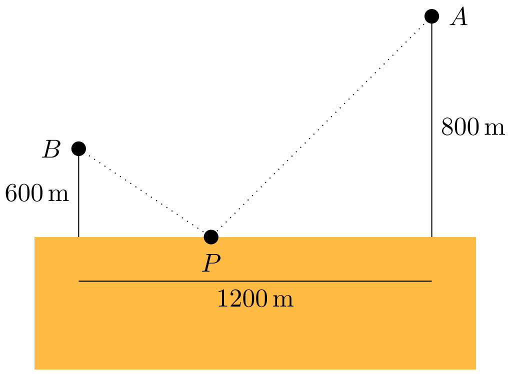
In terms of integration, if you have a certain velocity at a given point then that implies that over a given period of time, you travel a certain distance. It follows that if you add up all these instantaneous velocities, multiplying each by the infinitesimal time that you are travelling for each one, then you obtain the total distance you travel; that is, you see that $x = \int \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$.

Questions

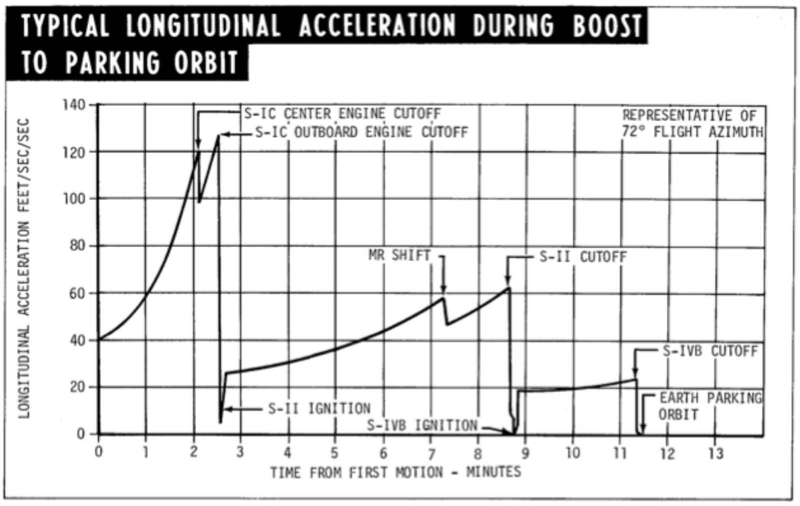
All distances are given in m, and all times in s, unless otherwise stated.

1. A distress flare is fired vertically into the air from a boat at sea. The height in metres of the flare t seconds after firing is given by
$$h = 122.5t - 4.9t^2.$$
   (a) What is the initial velocity of the flare?
   (b) At the peak of its flight, what is the vertical velocity of the flare?
   (c) What is the maximum height reached by the flare?

2. Part of the course for an ocean swim runs from bouy A to bouy B. Swimmers must come ashore on the at some point P along a long straight beach on the way. Bouy A is 800 m away from the beach, and bouy B is 600 m away from the beach. What is the least distance that a swimmer must swim? (Hint: minimise $PA + PB$.)



3. The following graph shows the acceleration of a rocket from launch until it reaches orbit. Given that the initial velocity of the rocket was $0\,\mathrm{ft\,s^{-1}}$, find the final velocity of the rocket.



NCEA Level 3 Calculus (Integration)
25. Integration Revision

Questions

1. True or false: A
   (a) $\int_a^b \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = y(b) - y(a)$
   (b) $\int_0^1 f(x)\,\mathrm{d}x + \int_0^1 g(x)\,\mathrm{d}x = \int_0^1 f(x) + g(x)\,\mathrm{d}x$
   (c) $\int_0^2 f(x)\,\mathrm{d}x + \int_0^1 f(x)\,\mathrm{d}x = \int_1^2 f(x)\,\mathrm{d}x$
   (d) $\int \sin(x)\,\mathrm{d}x \int \cos(x)\,\mathrm{d}x = \int \sin(x)\cos(x)\,\mathrm{d}x$
   (e) $\int \frac{1}{u}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x$
   (f) A definite integral always represents the area under a curve.
   (g) If $u = 2x$ then $\int \sqrt{2x}\,\mathrm{d}x = \int \sqrt{u}\,\mathrm{d}u$.
   (h) The indefinite integral of $\ln x$ is just $\ln x$.

2. Compute the following indefinite integrals: A
   (a) $\int \sin x\,\mathrm{d}x$
   (b) $\int \sec 3x \tan 3x\,\mathrm{d}x$
   (c) $\int \frac{2x^4 - x^2}{x^3}\,\mathrm{d}x$
   (d) $\int \sin x \cos x\,\mathrm{d}x$
   (e) $\int \sin^2 x\,\mathrm{d}x$
   (f) $\int \frac{1}{2u}\,\mathrm{d}u$
   (g) $\int \frac{2x + \sec^2 x}{2\sqrt{x^2 + \tan x}}\,\mathrm{d}x$
   (h) $\int \frac{t^4 - 1}{t - 1}\,\mathrm{d}t$
   (i) $\int \frac{t^{2017} + \sqrt{t^{2017}} + \sqrt[3]{t^{2017}}}{\sqrt[2017]{t^2}}\,\mathrm{d}t$
   (j) $\int \sec\theta\,\mathrm{d}\theta$ (hint: multiply through by $\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$)
   (k) $\int \sec x \tan x + \sec^2 x \tan x\,\mathrm{d}x$

3. Compute the definite integrals: A
   (a) $\int_0^1 x^2\,\mathrm{d}x$
   (b) $\int_0^{\pi/2} \tan(x/2)\,\mathrm{d}x$
   (c) $\int_0^4 x^3 + x^2 + \sqrt{x}\,\mathrm{d}x$
   (d) $\int_1^e \frac{1}{t}\,\mathrm{d}t$

4. Suppose $f'(x) + \frac{f(x)}{x} = 0$, and $f(1) = 1$. Find $f(x)$ explicitly. M

5. If $\int_A^B f(x)\,\mathrm{d}x = 3$ and $\int_A^C f(x)\,\mathrm{d}x = 4$, find $\int_B^C f(x)\,\mathrm{d}x$. M

6. A function ϕ, whose graph passes through the origin, is such that the slope of ϕ at any given point x is exactly $2\phi(x)$. Find ϕ exactly. M

7. Find A such that $\int_0^A \sin x\,\mathrm{d}x$ is maximised. E

8. Find B such that $\int_B^{B^2} x^2 - 4x - 4\,\mathrm{d}x$ is maximised. E

(Note on the above two questions: the problem of finding a function $q(t)$ such that the integral of some expression involving $q(t)$ and its derivative is minimised over a given interval is a fundamental problem in physics, surprisingly!)

9. Find the area of the region bounded by $y = 1 + x^2$, $y = -1 - x^2$, $x = 1$, and $x = -1$. M

10. Use integration to find the area enclosed between the curve $y = e^{2x} - \frac{1}{e^{3x}}$ and the lines $y = 0$, $x = 0$, and $x = 1.2$. M

11. A function f is *even* if $f(-x) = f(x)$ for all x in its domain. M
    (a) What geometric property does the graph of an even function have?
    (b) Suppose f is an even function with $\int_0^7 f(x)\,\mathrm{d}x = 20$. Find:
        i. $\int_{-7}^7 f(x)\,\mathrm{d}x$
        ii. $\int_0^7 3f(x) + 2\,\mathrm{d}x$

12. A function y is implicitly defined in terms of x in each case; find $y(x)$ explicitly. M
    (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2$, $y(1) = 1$.
    (b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{3y}$, $y(\pi/6) = 1$.

13. Use integration to find the area enclosed between $y = 1 - 0.2x^4$ and $y = 0.4x^4$. M

14. Mr Leibniz has a continer of oil and places it in the garage. Unfortunately, he puts the container on top of a sharp nail and it begins to leak. The rate of decrease of the volume of oil in the container is given by the differential equation $\frac{\mathrm{d}V}{\mathrm{d}t} = -kVt$, where V is the remaining volume of oil remaining after t hours have passed. The volume of oil in the container when it was placed in the garage was 3000 mL; after twenty hours have passed, the volume remaining is 2400 mL. How much (if any) oil will remain in the container after 96 hours have passed? E

15. An object has acceleration (in one dimension) $a(t) = 0.2t + 0.3\sqrt{t}$ for $0 \le t \le 10$. At $t = 4$, the velocity of the object is $5\,\mathrm{m\,s^{-1}}$. How far has the object travelled after nine seconds? M

16. The formula for integration by parts is $\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$. S
    (a) Find $\int \ln x\,\mathrm{d}x$ explicitly.
    (b) Hence, or otherwise, find $\int_1^2 4x \ln x\,\mathrm{d}x$.

17. (a) i. Show that $\frac{3}{x+1} - \frac{4}{x-2} = \frac{-x-10}{x^2-x-2}$. E
        ii. Hence, or otherwise, compute $\int_0^1 \frac{-x-10}{x^2-x-2}\,\mathrm{d}x$. E
    (b) Find β such that $\int_\beta^{2\beta} \frac{1}{x^2+5x+6}\,\mathrm{d}x = 1$. S

18. Use the trapezium rule to approximate the value of $\int_1^4 x^x\,\mathrm{d}x$. This function cannot be integrated in terms of elementary functions. A

19. The indefinite integral $\int e^{-x^2}\,\mathrm{d}x$ cannot be integrated in terms of elementary functions, but it can be shown that the definite integral $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$ has the value $\sqrt{\pi}$. Most of the curve lies within the bounds $-2 \le x \le 2$. Use Simpson's rule to approximate the value of $\int_{-2}^2 e^{-x^2}$, and compare this with the expected value. M

20. Consider the function implicitly defined by $y(x) = x + \int_0^{\sqrt{2}} y(x)\,\mathrm{d}x$. If the graph of the function includes the point $(0, 1)$, find $y(x)$ explicitly. M

21. An object is at 4° in a refrigerator. It is removed and placed on a shelf with an ambient temperature of 20°. After two minutes, the object has warmed to 5°. How long will it take for the object to reach 10°? [*Use Newton's law of cooling.*] E

22. A property owner assumes that the rate of increase of the value of his property at any time is proportional to the value, $\$V$, of the property at that time. E
    (a) Write the differential equation that expresses this statement.
    (b) The property was valued at \$365 000 in May 2012, and at \$382 000 in November 2013. Solve the differential equation from (a) to find the price that the owner paid for the property in May 2007 when he purchased the property, given his assumption is accurate.

23. Find x such that $\int_1^x \frac{\ln u}{u}\,\mathrm{d}u = 1$. M

24. The formula for surface area of the volume of revolution of $y = f(x)$ is $2\pi \int_a^b f(x)\sqrt{f'(x)+1}\,\mathrm{d}x$. Find the area of the surface obtained by rotating about the x-axis the part of the curve $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y$ that lies between $y = 1$ and $y = e$. E

25. The base of a solid is the region bounded by the parabolae $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid, if cross-sections perpendicular to the x-axis are squares with one side lying along the base. E

26. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. S

27. Scholarship 2016: Consider $I_n = \int_0^{\pi/2} \frac{\sin 2nx}{\sin x}\,\mathrm{d}x$, where $n \geq 0$. Show that $I_n - I_{n-1} = \frac{2(-1)^{n-1}}{2n-1}$. O

28. Scholarship 2010: A flower pattern is constructed by using a sinusoidal function $r(\theta)$ to define the distance from the origin to the curve at a radial angle θ. The x and y coordinates of a point on the curve are given by the following equations, where $0 \leq b \leq a$ and n is a positive integer (the number of petals). S

$$r(\theta) = a + b\sin(n\theta) \qquad x(\theta) = r(\theta)\cos(\theta) \qquad y(\theta) = r(\theta)\sin(\theta)$$

    (a) The area inside such a function is given by

$$A = \frac{1}{2}\int_0^{2\pi} (r(\theta))^2\,\mathrm{d}\theta$$

    when $r(\theta) \geq 0$.
    Show that the area of a flower pattern is $\pi\left(a^2 + \frac{1}{2}b^2\right)$.
    (b) A lemon squeezer with base radius a_0 and height H is made to the following specifications.
    At a height h (where $0 \leq h \leq H$) the cross-section is a flower pattern with

$$a(h) = \frac{H-h}{H}a_0 \text{ and } b(h) = \frac{h}{H}a(h).$$

    Use integration with respect to h to show that the volume of a lemon squeezer is **exactly** 5% **greater than** the volume of a cone with the same base and height.

29. Evaluate the following: O

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^x \left(\int_0^{\sin t} \sqrt{1+u^4}\,\mathrm{d}u\right)\mathrm{d}t$$

[*Hint: no integration is required. Use the FTC and the (differentiation) chain rule.*]

NCEA Level 3 Calculus (Integration)
25. Integration Revision (Homework)

Reading

Go and watch...

https://www.youtube.com/watch?v=iGI_LLb3rgg

Questions

1. Compute the following integrals.
   (a) $\int_1^2 \sin x \, \mathrm{d}x$
   (b) $\int \frac{u^2+1}{u^3+3u} \, \mathrm{d}u$
   (c) $\int_0^{\pi/6} \tan x \, \mathrm{d}x$
2. Suppose $y'(x) = \frac{3x^2+4x-4}{2y(x)-4}$ and $y(1) = 3$. Compute the possible values of y when x is 2.
3. Let $\omega(a, x) = \int_0^x \frac{a^3}{t^2+a^2} \, \mathrm{d}t$.
   (a) Find $\omega(a, x)$ explicitly in terms of a and x. You may wish to use the substitution $t = a \tan \theta$.
   (b) Compute $\omega(2, 2)$ exactly.
   (c) Find x such that $\omega(\sqrt{3}, x) = \pi$.
4. Scholarship 2016: Compute the following integral, giving your answer in exact form.

$$\int_{-\pi/2}^{\pi/2} (\sin^5 x + \cos^5 x) \, \mathrm{d}x$$

NCEA Level 3 Calculus (Integration)
26. More Interesting Problems

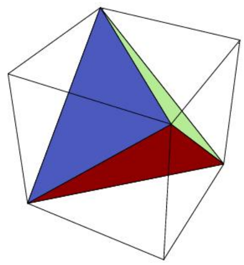
These problems do not just concern integration.

Questions

1. Find the equation of the line through the point $(3, 5)$ which cuts off the least area from the first quadrant. E

2. The area of a square is increasing at a constant rate of k m^2 per second. A tetrahedron (equilateral triangular pyramid), has a side length that is the same as that of the square at each instant. The initial volume of the tetrahedron was $1\,\mathrm{m}^3$. In terms of k, what is the volume of the tetrahedron three seconds after that? E

3. Consider the tetrahedron inscribed inside a cube, as in the figure. S



   The volume V of the cube at any instant t is increasing at a rate proportional to the value of V at that instant. The initial volume of the cube at $t = 0$ was 8 cubic units. What is the volume of the tetrahedron at time $t = 20$?

4. If $x \sin \pi x = \int_0^{x^2} f(t)\,\mathrm{d}t$, where f is continuous, find $f(4)$. [*Hint: you need not perform any integration.*] S

5. If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that S

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.$$

   Hence, or otherwise, evaluate

$$\lim_{x\to 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}.$$

6. (a) Consider the differential equation S

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + 5\frac{\mathrm{d}\Phi}{\mathrm{d}t} + 6\Phi(t) = 0.$$

   Let f and g be the functions defined by $f(x) = e^{-2x}$ and $g(x) = e^{-3x}$.
   i. Show that all linear combinations of f and g are solutions to the differential equation.
   ii. Find the (unique) solution passing through $(0, 1)$ and $(1, 1)$.

   (b) More generally, consider the differential equation $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$. Let the zeroes of the quadratic polynomial $p(D) = aD^2 + bD + c$ be α and β. Show that all the linear combinations of $e^{\alpha x}$ and $e^{\beta x}$ are solutions to the differential equation. O

7. Compute the following definite integral. [*Hint: begin with a substitution.*] S

$$\int_0^{\pi/6} \sqrt{\tan\theta}\,\mathrm{d}\theta$$

8. (a) Consider the two functions $p(x) = 3x^5 - 5x^3 + 2x$ and $q(x) = 3x^5$. Show that their ratio approaches 1 as $x \to \infty$. E

   (b) Let $p(x)$ and $q(x) \neq 0$ be polynomials. Recall that the degree of a polynomial is the highest n such that x^n has a non-zero coefficient. Compute the limit S

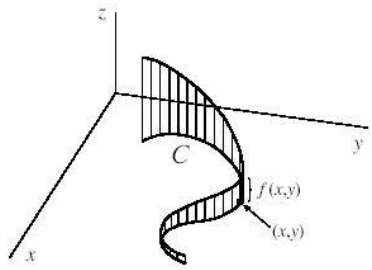
$$\lim_{x\to\infty} \frac{p(x)}{q(x)}$$

   if:

   i. the degree of $p(x)$ is less than that of $q(x)$.

   ii. the degree of $p(x)$ is greater than that of $q(x)$.

9. A definite integral calculates the between a curve and straight line, the x-axis. In a similar way, it is possible to use integration to calculate the area above a curved line and below a surface $z = f(x, y)$, like that in the figure. S



If the curve C is defined parametrically, that is $C(t) = (x(t), y(t))$, then the integral along the line can be calculated with the formula

$$\int_a^b f(x(t), y(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Compute the line integral of the function $f(x, y) = 2 + x^2 y$ around the upper half of the unit circle.

10. The **sine integral** function is defined by S

$$\mathrm{Si}(x) = \begin{cases} \int_0^x \frac{\sin t}{t}\,\mathrm{d}t, & \text{for } x \neq 0; \\ 1, & \text{for } x = 0. \end{cases}$$

   (a) Recall that $\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a)$. Use this to show that $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f'(t)\,\mathrm{d}t = f'(x)$.

   (b) Find the x-coordinate of the first local maximum of Si to the right of the origin. Carefully prove that you have found a maximum.

   (c) Use the result in (a) to find an expression for the integral

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)\,\mathrm{d}t,$$

   where f is continuous and g and h are differentiable.

11. Minimise the function $f(x) = b \log_b N$ with respect to b, and show that the result is independent of the constant N.* E

12. We can calculate **improper integrals** (those where the bounds are infinite) as follows: S

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x.$$

Decide whether each of the integrals given below exists. If so, calculate its value; if not, explain why.

(a) $\displaystyle\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ (b) $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ (c) $\displaystyle\int_1^\infty \sin x\,\mathrm{d}x$

13. (a) Show that $F(x) = \tan^{-1} x$ is an anti-derivative of $f(x) = \frac{1}{1+x^2}$ in the following ways: S
   i. Differentiate $F(x)$ and simplify to give $f(x)$.
   ii. Use the substitution $x = \tan\theta$ to integrate $f(x)$ and simplify to give $F(x)$.

   (b) Recall that $22/7$ is often given as a rough approximation to π. Consider the integral O

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x,$$

   and hence show that $22/7 > \pi$.†

14. Consider the operator $\mathcal{L}$ defined by S

$$\mathcal{L}f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln\left[f\left(e^x\right)\right].$$

   (a) Show that $\mathcal{L}x^n = n$ and that $\mathcal{L}[u(x)]^n = n\mathcal{L}u(x)$.
   (b) Find an expression for $\mathcal{L}[u(x)v(x)]$ and $\mathcal{L}[u(x)/v(x)]$.
   (c) Find an expression for $\mathcal{L}[u(x) + v(x)]$.
   (d) For which y is $\mathcal{L}y = y$?

15. Compute the following indefinite integrals: S

   (a) $\displaystyle\int \frac{\sin\frac{1}{x}}{x^2}\,\mathrm{d}x$

   (b) $\displaystyle\int \frac{\ln x \cos x - \frac{\sin x}{x}}{(\ln x)^2}\,\mathrm{d}x$

16. (Harder!) Suppose ι is a function defined as O

$$\iota(x) = \int_0^x t^t \sin(t)\,\mathrm{d}t.$$

   (a) Write down the domain and range of ι (caution: be careful for negative fractional x).

* Dudley, *Mathematical Cranks*, p.52.
† Nahin, *Inside Interesting Integrals*, pp.23-4.

(b) Suppose I is an antiderivative of $t^t \sin(t)$. Show that

$$\iota^{-1}(x) = I^{-1}(x + I(0))$$

is the only possibility for the inverse of ι, if the inverse exists.

(c) Find the derivative of ι with respect to x. Hence, show that ι changes from decreasing to increasing at an odd number of points within the interval $(16, 20)$. Conclude that ι^{-1} is not a function, and hence ι has no well-defined inverse.

17. A while ago (when we talked about the product and quotient rules), I claimed that the radius of the circle best approximating a continuous curve around a point (x, y) is given by 0

$$\text{radius of curvature} = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|}.$$

Let us attempt to prove this.

(a) Let f be a continuous function at x such that the second derivative of f at x exists. By recalling our work on approximations, explain why knowing up to the second derivative of f should be enough to find the 'best circular approximation' of f at $(x, f(x))$.

(b) Consider the circle of radius r centred at (x_0, y_0). Suppose that this circle passes through the point (x_1, y_1); suppose further that the first derivative of the y-ordinate of the circle with respect to the x-ordinate is m, and that the second derivative is c. Write down expressions for r, x_0, and y_0 in terms of x_1, y_1, m, and c.

(c) Use part (b) to write down the radius of the unique circle passing through $(x, f(x))$ with matching first and second derivatives to f.

18. We prove that π is irrational.[‡] Suppose that $\pi = \frac{a}{b}$ where a and b are positive integers. Let n be a positive integer, and define 0

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$
$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x).$$

(a) Show that the value of f and every derivative of f at $x = 0$ and $x = a/b$ is integral. Conclude that $F(\pi) + F(0)$ is an integer.

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}[F'(x)\sin x - F(x)\cos x] = f(x)\sin x$, and hence that

$$\int_0^\pi f(x)\sin x = F(\pi) + F(0). \tag{*}$$

(c) Prove that $0 < f(x)\sin x < \frac{\pi^n a^n}{n!}$, and conclude that (*) is positive but arbitrarily small for sufficiently large n.

(d) Derive a contradiction from (a) and (c).

[‡] Ivan Niven, *A simple proof that π is irrational.*

CHAPTER 4

Additional Material

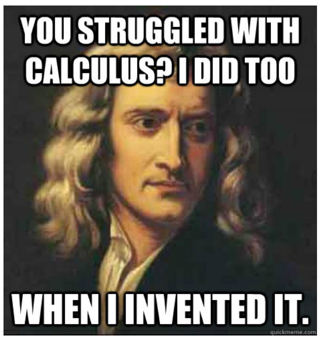
NCEA Level 3 Calculus
Exam Advice

Now is the time when I pass on my exam-taking wisdom in bullet-point list form.

- Don't cram the night before the exam. Get a good night's sleep and have a proper meal for breakfast.
- Prepare your exam bag the night before. Know the earthquake procedures and have an emergency pack.
- Stay calm and be confident.
- If you forget how numbers work, try counting your fingers and work upwards from there. (Quality advice from a former Scholarship candidate.)
- Draw a diagram if one isn't given.
- Don't study by reading examples, study by doing problems.
- Read through all the questions before attempting any of them.
- Have a problem-solving strategy.
- Begin by working out the part of the standard the question is asking about.
- Show all your work.
- Never leave a problem blank.
- If you're stuck, move on and do another question.
- Write sentences, not symbols. Use correct spelling, grammar. and punctuation??!
- **If the question gives you a hint, there's probably a reason for that!**

Scholarship exams especially are a competition between you and the examiner. They want to impress you with the difficulty of the questions that they throw at you, **but you can meet that challenge.** There is nothing that they can chuck your way that cannot be done with level three material (because they're not allowed to), and so the way to get through a scholarship question is usually to work methodically.

Remember that mathematics is difficult, and that it took hundreds of years for humanity to discover the concepts and techniques which you will apply in the three hours of an NZQA external examination.



Study Skills for Mathematics

(From the University of Cambridge)

[Examinations are] designed to test your knowledge of the courses you have attended rather than your ability to jump through mathematical hoops. Nevertheless, strategy matters. Extreme marks (either high or low) are available in mathematics examinations, which means that playing the cards you hold to best advantage is of vital importance.

Here are some thoughts; you've heard them all before, but that does not make them any less worth saying. The examinations may besome way off, but you will find that good examination technique can be acquired over the course of the year by making suitable preparations and developing good habits. (For example, the first two points assume that your year's work is in good order.)

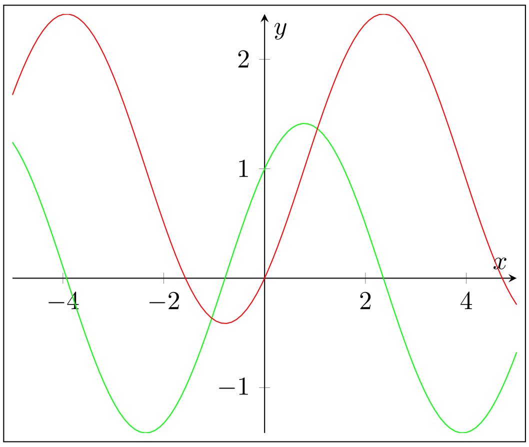
- For revision, work through examples while reading the relevant section of your notes (just reading is not enough).
- For last minute preparation, look through your supervision work to remind yourself how to do questions.
- In the examination, above all, stay cool — if it is hard for you, it is probably hard for everyone.
- Don't rush into a question — read the whole paper carefully and start with the question you feel most confident about.
- Analyse exactly what you are being asked to do; try to under- stand the hints (explicit and implicit); remember to distinguish between terms such as explain/prove/define/etc.
- Remember that different parts of a question are often linked (it is usually obvious from the notation and choice of variables).*
- Set out your answer legibly and logically (don't scribble down the first thought that comes into your head) — this not only helps you to avoid silly mistakes but also signals to the examiner that you know what you are doing (which can be effective even if you haven't the foggiest idea what you are doing).
- If you get stuck, state in words what you are trying to do and move on (at E-level, you don't get credit for merely stating intentions, but university examiners are generally grateful for any sign of intelligent life).

* This is less relevant for modern NCEA examinations (even the last 2-3 Scholarship exams).

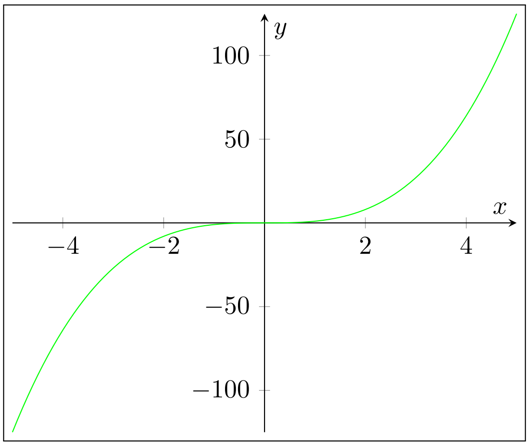
NCEA Level 3 Calculus
Solutions to Homeworks

1. The Derivative

1. Green: derivative of function. Red: original function.

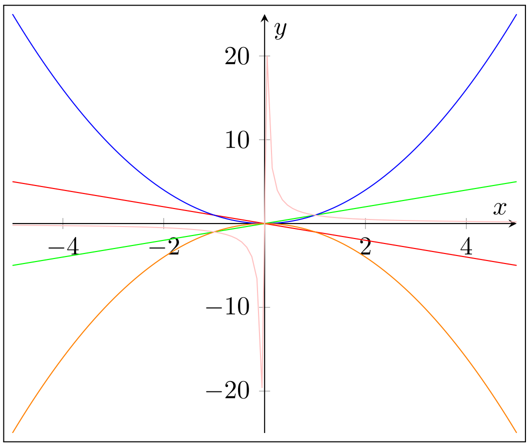


2. (a) At a min or max, the function is momentarily horizontal and so has slope zero; so $m = 0$.
   (b) Consider a graph like the following at $(0, 0)$:

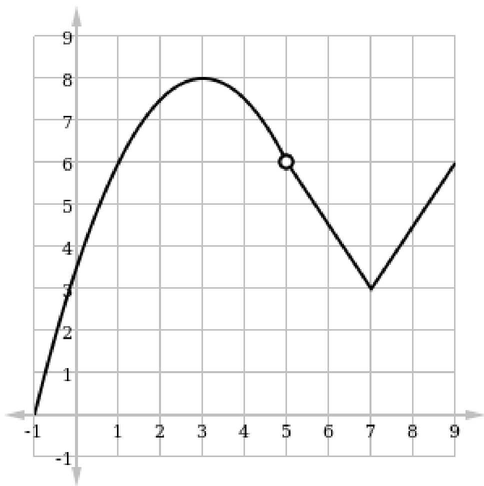


2. Limits

1. Increasing: the graph of the function is sloping up (green). Decreasing: the graph of the function is sloping down (red). Concave up: the graph of the function is increasing in slope (it is like a cup ∪) (blue). Concave down: the graph of the function is decreasing in slope (it is like a cap ∩) (orange). Continuous: the function has no holes (all of them except pink).



2. (a) $\lim_{x\to-2} f(x) = 0$, $\lim_{x\to 2} f(x) = -0.5$.
   (b) No, it approaches different values from the left and the right.
   (c) Yes, because the function is continuous there.
   (d) $(-\infty, -3)$, $(-3, -1)$, $(-1, 2)$, $(2, \infty)$.
   (e) -3, -1, 2.
3. For example,



3. Derivatives of Common Functions

1. (a) $2x + 1/x$
   (b) $t^2 x^{t-1}$
   (c) $\cos x + \sin x$
   (d) $\frac{4}{5}x^{-1/5} = \dfrac{4}{5\sqrt[5]{x}}$
2. The power x is not constant.
3. Consider x^{-n}. The first derivative is $-nx^{-n-1}$, the second is $n(n+1)x^{-n-2}$, and so the nth is $(-1)^n n(n+1)\cdots(2n-1)(2n)x^{-2n} = (-1)^n \frac{(2n)!}{(n-1)!}x^{-2n}$. [This can be proved via induction.]

4. (a) Note first that $10^t = e^{t\ln 10}$, so $P = P_0 + e^{t\ln 10}$ and $\frac{\mathrm{d}P}{\mathrm{d}t} = (\ln 10)e^{t\ln 10} = (\ln 10)10^t$. At $t = 100$, we have $\frac{\mathrm{d}P}{\mathrm{d}t} = 2.3 \times 10^{100}$.
   (b) Real-world populations don't grow exponentially forever if there are finite resources (e.g. food).

4. The Chain Rule

1. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\csc^2 x}{2\sqrt{\cot x}}$
2. (a) Simply apply the chain rule twice.
   (b) $y' = 5x^4(\cos x^5)(-\sin\sin x^5)(\cos\cos\sin x^5)(-\sin\sin\cos\sin x^5)(\cos\cos\sin\cos\sin x^5)$.
3. (a) $f'(\theta) = -2\sin 2\theta$ and $g'(\theta) = -4\sin\theta\cos\theta = -2\sin 2\theta$, so $f' = g'$ as they agree everywhere.
   (b) Since f and g have the same derivative, they differ only by a constant. But $f(0) = 1 = g(0)$, so that constant is zero; hence $f = g$.

5. The Product and Quotient Rules

1. (a) $\cos x \ln x + \dfrac{\sin x}{x}$
   (b) $\sec kx + kx \sec kx \tan kx$
   (c) $\dfrac{-\pi(\sin\pi\theta + \cos\pi\theta)\sin\pi\theta - \pi(\cos\pi\theta - \sin\pi\theta)\cos\pi\theta}{(\sin\pi\theta + \cos\pi\theta)^2}$
   (d) $(\cos t)(3\sin^2 t)(-\sin(\sin^3 t))(4\cos^3\sin^3 t)$.

2.

$$
\begin{aligned}
F = \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{m_0 v}{\sqrt{1-\dfrac{v^2}{c^2}}} &= a\,\frac{\mathrm{d}}{\mathrm{d}v}\,\frac{m_0 v}{\sqrt{1-\dfrac{v^2}{c^2}}} \\
&= a\left(\frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} + \frac{m_0 v^2}{c^2\left(1-\dfrac{v^2}{c^2}\right)^{3/2}}\right) \\
&= a\left(\frac{m_0 c^2\left(1-\dfrac{v^2}{c^2}\right)}{c^2\left(1-\dfrac{v^2}{c^2}\right)^{3/2}} + \frac{m_0 v^2}{c^2\left(1-\dfrac{v^2}{c^2}\right)^{3/2}}\right) \\
&= m_0 a\left(\frac{c^2\left(1-\dfrac{v^2}{c^2}\right)+v^2}{c^2\left(1-\dfrac{v^2}{c^2}\right)^{3/2}}\right) \\
&= m_0 a\left(\frac{c^2-v^2+v^2}{c^2\left(1-\dfrac{v^2}{c^2}\right)^{3/2}}\right) \\
&= \frac{m_0 a}{\left(1-\dfrac{v^2}{c^2}\right)^{3/2}}.
\end{aligned}
$$

3. We wish to find $\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\mathrm{rad}(\theta)$, where $\mathrm{rad}(\theta)=\frac{\pi\theta}{180}$; so $\frac{\mathrm{d(rad)}}{\mathrm{d}\theta}=\frac{\pi}{180}$ and $\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\mathrm{rad}(\theta)=\frac{\pi\theta}{180}\cos\mathrm{rad}(\theta)$. [The reason we have to do this is that the derivative of sin uses the limit $\lim_{x\to 0}\frac{\sin x}{x}=1$ which is false if x is in degrees.]

6. Tangent and Normal Lines

1. The normal to a curve f at a point $(x_0, f(x_0))$ is the unique line passing through that point that is perpendicular to the tangent line of f at that point.

2. $y' = \frac{-\sin(x+\pi)}{2\sqrt{\cos(x+\pi)}} + \cos x - 4\sec x\tan^2 x e^{2\tan^2 x}$; at $x=\pi$, $y'=-1$ and so the tangent line (best linear approximation) is $y=-(x-\pi)=\pi-x$.

3. Since the normal line has slope 3, the tangent line has slope $-1/3$. We can take any curve through $(1,0)$ with this slope, so we may as well take the tangent line itself: $y=-\frac{1}{3}(x-1)=\frac{1}{3}-\frac{1}{3}x$.

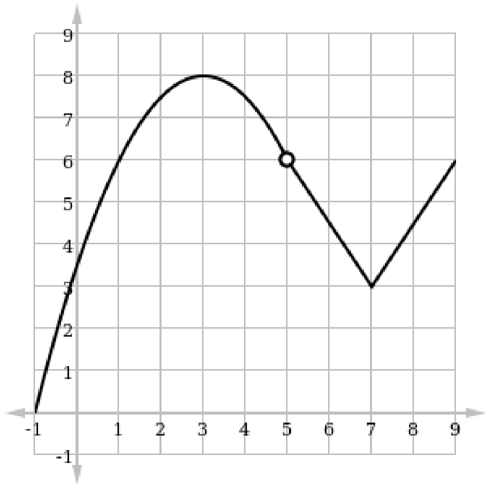
4. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{(1+3x)^{2/3}}$ and at $x=0$ the slope becomes 1. So the best linear approximation around the point $(0,1)$ is just $\tilde{y}=x+1$. So at $x=0.01$, we have $\tilde{y}=1.01$ as our approximate value of $\sqrt[3]{1.03}$. [The true value is around 1.0099, so we are not too far off.]

7. Higher Derivatives and the Geometry of a Function

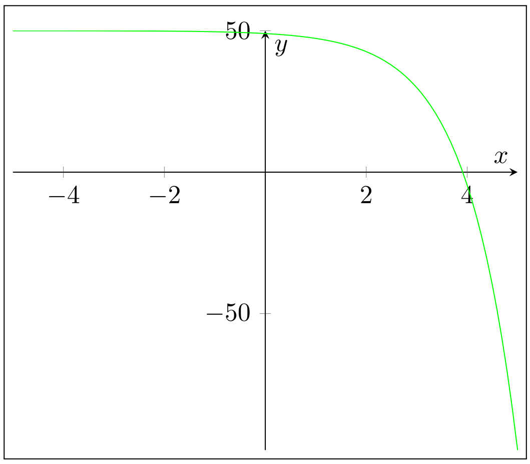
1. The second derivative tells us the concavity of a function: if the second derivative is positive, the function is curving up and if it is negative then the function is curving down.

2. (a) $f'(x) = 5x^4 - 5$, $f''(x) = 20x^3$.
   (b) $f'(x) = \frac{x^2-2x}{(x-1)^2}$, $f''(x) = \frac{2x-2}{(x-1)^4}$.
   (c) $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{4}x^{-3/4}$, $f''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{16}x^{-5/4} = \frac{3}{16\sqrt[4]{x^5}} - \frac{1}{4\sqrt{x^3}}$.

3. (a) For example,



   (b) For example,



8. Optimisation

1. At some point x, the distance between the two parabolae is $\delta(x) = (x^2+1) - (x - x^2) = 2x^2 - x + 1$. Taking the derivative, we find $\delta'(x) = 4x - 1$ which has a single zero at $x = 1/4$; by looking at the graph of the two parabolae, we see that this must be the location of the miniumum distance $\delta(1/4) = 7/8$ units.

2. If $y = 3x + 2\cos x + 5$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 3 - 2\sin x$. Since $1 \geq \sin x$, $3 - 2\sin x \geq 1$. In particular, the function is everywhere increasing. Now, note that when $x = -200\pi$, $y = -600\pi + 7 < 0$, and when $x = 200\pi$,

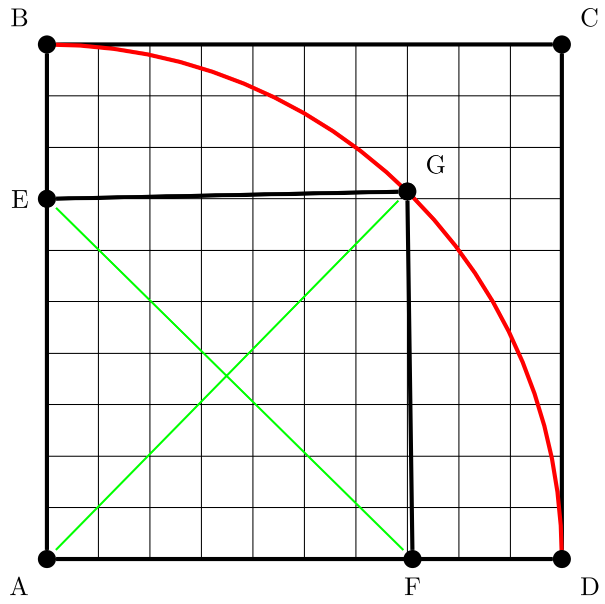
$y = 600\pi + 7 > 0$. Since the function is continuous over this interval, it follows that at some point it passes through the y-axis and has at least one root; since it is increasing everywhere, it must have exactly one real root.

3. The area of such a rectangle will be $A = 4xb\sqrt{1-\frac{x^2}{a^2}}$; so

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 4b\sqrt{1-\frac{x^2}{a^2}} - \frac{4x^2 b}{a^2\sqrt{1-\frac{x^2}{a^2}}}.$$

Setting this to zero, we have $a^2 = 2x^2$ and so $2x = \sqrt{2}a$. It follows that $2y = b\sqrt{2}$, and so the maximal area is $2ab$.

4. Consider the following diagram.



It should be clear that $AG = 1$; call $\angle AEG = \theta$ and $\angle AFG = \phi$, and let $AE = EG = e$ and $AF = GF = f$. By the cosine rule, we have $1 = 2e^2(1-\cos\theta)$ and $1 = 2f^2(1-\cos\phi)$. Now, the area of the triangle $\triangle AEG$ is given by $\frac{1}{2}\sqrt{e^2-\frac{1}{4}}$; the area of $\triangle AFG$ is given by $\frac{1}{2}\sqrt{f^2-\frac{1}{4}}$. Since $AEFG$ is a (convex) quadrilateral with two right angles, $\theta+\phi=\pi$. Putting this all together, the area of the quadrilateral is $A = \frac{1}{2}\sqrt{e^2-\frac{1}{4}} + \frac{1}{2}\sqrt{f^2-\frac{1}{4}}$. We have that $e^2 = \frac{1}{2(1-\cos\theta)}$ and $f^2 = \frac{1}{2(1-\cos(\pi-\theta))} = \frac{1}{2(1+\cos(\theta))}$, so the area in terms of θ is

$$A = \frac{1}{2}\sqrt{\frac{1}{2(1-\cos\theta)} - \frac{1}{4}} + \frac{1}{2}\sqrt{\frac{1}{2(1+\cos\theta)} - \frac{1}{4}}.$$

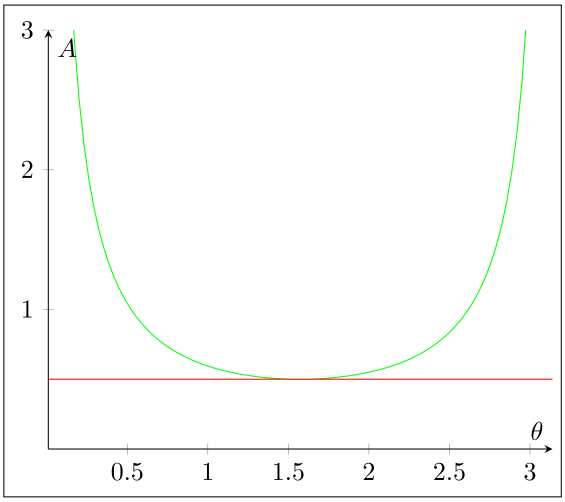
Taking the derivative, we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{1}{2}\frac{\sin\theta}{4(\cos\theta+1)^2\sqrt{\frac{1}{2(1+\cos\theta)}-\frac{1}{4}}} - \frac{1}{2}\frac{\sin\theta}{4(1-\cos\theta)^2\sqrt{\frac{1}{2(1-\cos\theta)}-\frac{1}{4}}};$$

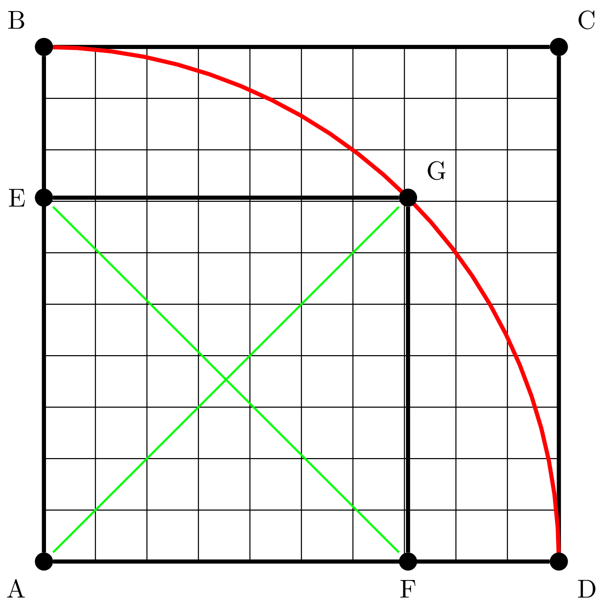
Now we set this to zero. We know that $0 < \theta < \pi$, so $\sin\theta \neq 0$ and hence

$$4(\cos\theta + 1)^2\sqrt{\frac{1}{2(1+\cos\theta)} - \frac{1}{4}} = 4(1-\cos\theta)^2\sqrt{\frac{1}{2(1-\cos\theta)} - \frac{1}{4}}$$
$$\sqrt{\frac{1}{2} - \frac{\cos\theta + 1}{4}} = \sqrt{\frac{1}{2} - \frac{1-\cos\theta}{4}}$$
$$\cos\theta = -\cos\theta$$
$$\cos\theta = 0$$

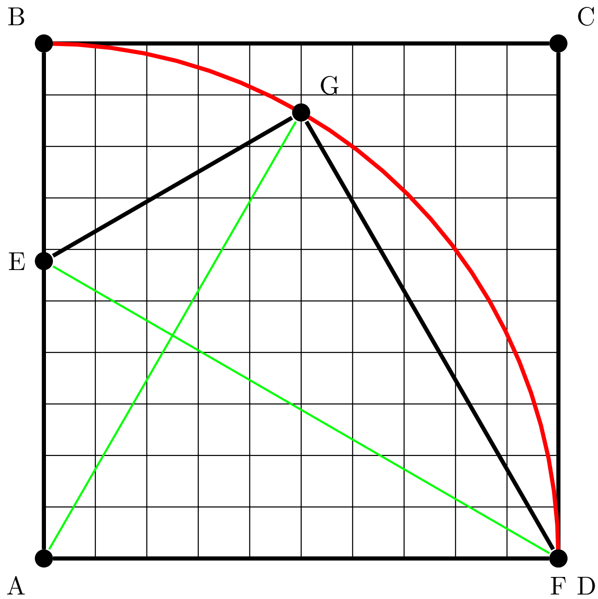
Hence $\theta = \phi = \pi/2$. Immediate calculation shows that $e = f = \frac{1}{\sqrt{2}}$; we thus have a square with side length $1/\sqrt{2}$, and area $\frac{1}{2}$. Is this a maximum or a minimum? We cheat by graphing the area versus θ:



so we obviously have the minimum area:



Note that $\theta \geq \pi/2$, because otherwise $e > 1$. Suppose we take $\theta = 2\pi/3$; here is the graphed figure (with area 1.1547):



This is the maximum area, since if we increase θ any more it requires $f > 1$.

9. Implicit Differentiation

1. (a) $y' = \frac{3x^2+6x}{2y}$.

   (b) $(1+y')\sin(x+y) = 2 - 2y' \implies y' = \frac{2-\sin(x+y)}{2+\sin(x+y)}$.

   (c) $y' = \frac{20x^3-2x}{2y}$.

2. $2x + 2y + 2xy' - 2yy' + 1 = 0 \implies y' = \frac{-1-2x-2y}{2x-2y}$, so $y'(1,2) = \frac{-1-2-4}{2-4} = 7/2$; hence the slope of the normal is $-2/7$, and the equation of the normal line is $y - 2 = -\frac{2}{7}(x-1)$.

3. We have $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0$; suppose we have a tangent line passing through $(x_0, (\sqrt{c} - \sqrt{x_0})^2)$. Then the equation of this tangent is $y - (\sqrt{c} - \sqrt{x_0})^2 = -\frac{\sqrt{c}-\sqrt{x_0}}{\sqrt{x_0}}(x - x_0)$. When $y = 0$ we obtain the x-intercept; $0 = (\sqrt{c} - \sqrt{x_0})^2 - \frac{\sqrt{c}-\sqrt{x_0}}{\sqrt{x_0}}(x - x_0)$ and so $x = \sqrt{x_0 c}$. Similarly, when $x = 0$ we obtain $y = \sqrt{x_0 c} - x_0$. Their sum is therefore $2\sqrt{x_0 c} - x_0 = 2\sqrt{c}(\sqrt{c} - \sqrt{y_0}) - (\sqrt{c} - \sqrt{y_0})^2 = c$.

10. Inverse Functions

1. (a) $y' = \frac{2x}{1+x^4}$.

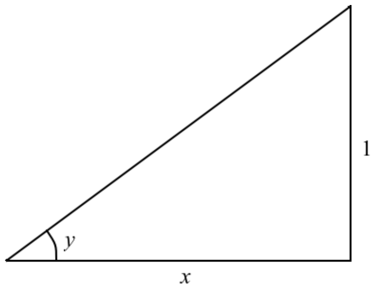
   (b) $f'(x) = \frac{1}{1+x^2}$.

   (c) $g'(x) = \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{1-x}} \frac{1}{1+(\sin^{-1}\sqrt{x})^2}$.

2.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\tan^{-1}x+\frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}\right) &= \frac{1}{2+2x^2}+\frac{1}{4}\frac{x^2+1}{(x+1)^2}\frac{2-2x^2}{(x^2+1)^2}\\ &= \frac{1}{2+2x^2}+\frac{1}{2}\frac{1-x^2}{(x+1)^2(x^2+1)}\\ &= \frac{1}{2}\frac{(x+1)^2+1-x^2}{(x+1)^2(1+x^2)}\\ &= \frac{1}{2}\frac{2x+2}{(x+1)^2(1+x^2)}\\ &= \frac{1}{(x+1)(x^2+1)}.\end{aligned}$$

3. Let $y=\cot^{-1}x$, so that $x=\cot y$ and $\frac{\mathrm{d}x}{\mathrm{d}y}=-\csc^2 y$; hence $\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin^2 y$. Consider the following triangle:



So $\sin y=\frac{1}{\sqrt{1+x^2}}$, and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{1+x^2}$.

11. Related Rates of Change

1. $V=[x(t)]^3$, so $\frac{\mathrm{d}V}{\mathrm{d}t}=3\frac{\mathrm{d}x}{\mathrm{d}t}[x(t)]^2$.

2. The volume of a cone is $\frac{\pi}{3}r^2h$. Comparing similar triangles, if the water is at a height h then it forms a cone with radius $r=h/2$. Hence when the water is at a height h it has volume $V(t)=\frac{\pi[h(t)]^3}{24}$, and so $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\pi[h(t)]^2}{8}\frac{\mathrm{d}h}{\mathrm{d}t}$. We know that $\frac{\mathrm{d}V}{\mathrm{d}t}=2$, so solving for $\frac{\mathrm{d}h}{\mathrm{d}t}$ we have that $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{16}{\pi[h(t)]^2}$ and when $h=3$ the height is rising at a rate of $0.57\,\mathrm{m\,min}^{-1}$.

3. Let x be the hypotenuse of the formed triangle, and let y be the horizontal distance from the boat to the jetty so that $y=\sqrt{x^2-1}$. Then $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{x}{\sqrt{x^2-1}}\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\sqrt{y^2+1}}{y}$. So at $y=8$, $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\sqrt{65}}{8}\approx 1.0078\,\mathrm{m\,s}^{-1}$.

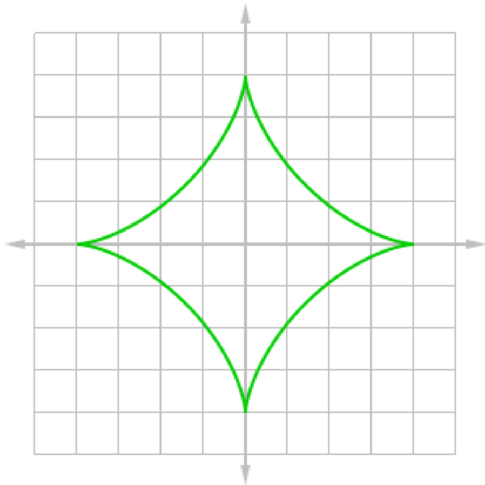
12. Parametric Functions

1. (a) $\frac{\mathrm{d}x}{\mathrm{d}t}=4t^3-6t^2+4t$, $\frac{\mathrm{d}y}{\mathrm{d}t}=3t^2-1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3t^2-1}{4t^3-6t^2+4t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{3t^4-6t^2+3t-1}{(4t^3-6t^2+4t)t^2(2t^2-3t+2)^2}$.

   (b) $\frac{\mathrm{d}x}{\mathrm{d}t}=-\sin t-4\sin 2t$, $\frac{\mathrm{d}y}{\mathrm{d}t}=\cos t+4\cos 2t$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\cos t+4\cos 2t}{\sin t+4\sin 2t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{12\cos(t)+33}{(\sin t-4\sin 2t)\left(8\cos(t)+1\right)^2\left(\cos(t)^2-1\right)}$

2. We have $t^2=(x-1)^2$, so $y=e^{(x-1)^2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=2(x-1)e^{(x-1)^2}$. At $x=2$, $\frac{\mathrm{d}y}{\mathrm{d}x}=2e$; so the best linear approximation is $y-e=2e(x-2)$, or $y=e(2x-3)$.

3. (a) Should look something like this:



(b) $\frac{\mathrm{d}x}{\mathrm{d}t} = -12\sin t\cos^2 t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 12\cos t\sin^2 t$, so the slope at some t is simply

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{12\cos t\sin^2 t}{-12\sin t\cos^2 t} = -\frac{\sin t}{\cos t}.$$

(c) Cusps will be at precisely those points with turning points in the x or y direction (for $0 \le t \le 2\pi$). In other words, places where either $\sin t$ or $\cos t$ vanishes. These are at $t \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$; substituting these into the equation gives us the four points $(\pm 4, 0)$ and $(0, \pm 4)$.

13. Sequences and Series

1. (a) Converges to $1/2$.
   (b) Diverges: $9^{n+1}/10^n = 9^{n+1}/(9+1)^n = 9^{n+1}/(9^n + \cdots) \to \infty$.
2. (a) The series is $2/3 - 2/5 + 2/7 - 2/9 + \cdots$. It has partial sums $2/3, 4/15, 58/105, \ldots$. Converges to $\pi/2$.
   (b) The series is $-2/5 + 4/6 - 6/7 + 8/8 - 10/9 + \cdots$. It has partial sums $4/15, -62/125, \ldots$. Diverges (the terms added and subtracted keep growing, so partial sums become very positive and very negative alternately).

14. Differentiation Revision

1. (a) $f'(x) = (2017 \times 3)x^{2016} - \frac{1}{19x^{20}} + \frac{1}{2017\sqrt[2017]{(x+2)^{2016}}}$.
   (b) $f'(h) = \pi r^2$.
   (c) $f'(\theta) = -\frac{\mu m g(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}$.
   (d) $f'(g) = \frac{(g^2 + \ln g)\cos g - (2g + 1/g)\sin g}{(g^2+\ln g)^2}$.
   (e) $3f(x) + 3xf'(x) + 2f(x)f'(x) = \frac{3+f(x)-xf'(x)}{[3+f(x)]^2}$ so $f'(x) = \frac{3+f(x)-3[3+f(x)]^2 f(x)}{3[3+f(x)]^2 x + 2[3+f(x)]^2 f(x) + x}$.
2. Let θ between the angle of the kite string, and let x be the horizontal distance to the kite along the ground (so the length of the string is $\sqrt{50^2 + x^2}$). Then $\sin\theta = 50/x$, so $\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{50}{x^2}\frac{\mathrm{d}x}{\mathrm{d}t}$. When the length of the string is 100, $x \approx 86.6$; so $\cos\theta = x/100 \approx 0.866$. Substituting $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$, we have $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{100}{86.6^2}\cdot\frac{1}{0.866} = -0.0154$.

3. The surface area of a cone is $\mathcal{S} = \pi r\sqrt{h^2 + r^2}$; we also have $27 = \frac{1}{3}\pi r^2 h$, so $r^2 = \frac{81}{\pi h}$ and

$$\mathcal{S} = \pi\sqrt{\frac{81}{\pi h}\left(h^2 + \frac{81}{\pi h}\right)} = \pi\sqrt{\frac{81h}{\pi} + \frac{81^2}{\pi^2 h^2}}$$

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}h} = \frac{\pi\left(\frac{81}{\pi} - 2\frac{81^2}{\pi^2 h^3}\right)}{2\sqrt{\frac{81h}{\pi} + \frac{81^2}{\pi^2 h^2}}}$$

In order to find a minimum, we set this derivative to zero and obtain $0 = \frac{81}{\pi} - 2\frac{81^2}{\pi^2 h^3}$, so

$$h = \sqrt[3]{2\frac{81}{\pi}} \approx 3.722\,\mathrm{cm}.$$

From this, we find $r = \sqrt{81/\pi h} = 2.63\,\mathrm{cm}$.

4. We begin by parameterising the hyperbola; completing the square, we can transform our equation into standard form:

$$\frac{(x-1)^2}{3} - \frac{y^2}{3} = 1$$

A parameterisation of this is $(1 + \sqrt{3}\sec t, \sqrt{3}\tan t)$. Now, given any point (x_0, y_0) we wish to minimise $\mathcal{D}(t) = \sqrt{(x_0 - 1 - \sqrt{3}\sec t)^2 + (y_0 - \sqrt{3}\tan t)^2}$ with respect to t.

(a) Firstly, consider $(x_0, y_0) = (2, 1)$. Then $\mathcal{D}(t) = \sqrt{(1 - \sqrt{3}\sec t)^2 + (1 - \sqrt{3}\tan t)^2}$. Taking the derivative, we find that:

$$\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t} = \frac{(\sqrt{3}\sec t - 1)(\sqrt{3}\sec t\tan t) + (\sqrt{3}\tan t - 1)(\sqrt{3}\sec^2 t)}{\sqrt{(1 - \sqrt{3}\sec t)^2 + (1 - \sqrt{3}\tan t)^2}}$$

Using MATLAB to compute the solution of $\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t} = 0$,

```
vpasolve((sqrt(3) * sec(t) - 1)*(sqrt(3) * sec(t) * tan(t))
     == (1 - sqrt(3)*tan(t))*(sqrt(3)*(sec(t))^2),t)
```

we find $t \approx 0.3759$; so $(x, y) = (2.8621, 0.6835)$.

(b) Note that $(3, 1)$ is already on the hyperbola. 😇

15. Approximating Areas

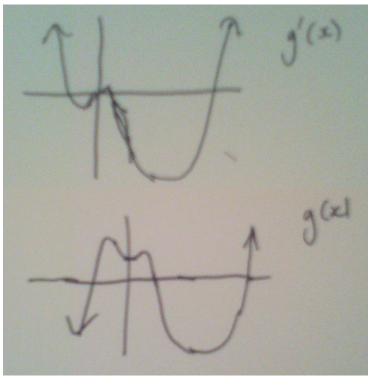
1. I use Simpson's rule with $n = 8$:

$$\int_0^{1.6} g(x)\,\mathrm{d}x \approx \frac{0.2}{3}(12.1 + 13.2 + 4(11.6 + 11.1 + 12.2 + 13.0) + 2(11.3 + 11.7 + 12.6)) = 19.21.$$

2. Measure the height of the shaded area at each point (using $n = 10$ is probably easiest), collapsing the empty area down (e.g. the height of the function at $x = -1$ is just $3 + 1 = 4$). Then use some numerical integration method.

16. Anti-differentiation

1. (a) $F(x) = \frac{1}{2}x^2 - 3x + C$
   (b) $f(x) = x^2 + 3x + 2$, so $F(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C$
   (c) $F(\theta) = 2\theta^3 - 7\tan\theta + C$
   (d) $G(h) = \pi^2 h$
   (e) $F(x) = \frac{x^{4.7}}{4.7} + \frac{2}{3}\sqrt{x^3} + \sqrt{7}x^{\sqrt{7}}$
2. $\varphi(x) = x^2 + x + C$; but $\varphi(1) = 6$, so $1 + 1 + C = 6$ and $C = 4$. Hence $\varphi(x) = x^2 + x + 4$, and $\varphi(2) = 10$.
3. See following image.



17. The Fundamental Theorem of Calculus

1. $\int_0^{\pi/4} \sec^2\theta \,\mathrm{d}\theta = [\tan\theta]\Big|_0^{\pi/4} = 1.$
2. $\int_1^2 f(x)\,\mathrm{d}x = \int_1^3 f(x)\,\mathrm{d}x - \int_2^3 f(x)\,\mathrm{d}x = 10.$
3. First we find the intersection points; we have $6x = x^2$, so $x \in \{0, 6\}$. Hence we compute

$$\int_0^6 2x - \frac{x^2}{3}\,\mathrm{d}x = [x^2 - \frac{x^3}{9}]\Big|_0^6 = 36 - 6^3/9 = 12.$$

18. Substitution

1. (a) $\frac{\csc 3x}{3} + C.$
   (b) $-\frac{\tan 3x^2}{6} + C.$
   (c) $2\sqrt{x} + 3x - 2\ln x + C.$
   (d) $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$
2. Use trig identity: $2\sin 5x\cos 3x = \sin 8x + \sin 2x$. Then

$$\int_0^{\pi/6} \sin 8x + \sin 2x\,\mathrm{d}x = \left[-\frac{\cos 8x}{8} - \frac{\cos 2x}{2}\right]\Bigg|_0^{\pi/6} = 0.4375.$$

3. $\frac{1}{2}\tan^{-1} x^2 + C$. (Substitute $u = x^2$.)

19. Differential Equations

1. (a) $\int e^y \, \mathrm{d}y = \int e^t \, \mathrm{d}t$, so $e^y = e^t + C$ and $y = \ln(e^t + C)$.

   (b) $\int \frac{\mathrm{d}y}{y^2} = \int t \, \mathrm{d}t$, so $-\frac{1}{y} = \frac{1}{2}t^2 + \frac{C}{2}$ and $y = -\frac{2}{t^2+C}$.

   (c) $\int \sec^2 y \, \mathrm{d}y = \int \mathrm{d}t$, so $\tan y = t + C$ and $y = \tan^{-1}(t + C)$.

   (d) $\int \sin y \, \mathrm{d}y = \int -t \cos t \, \mathrm{d}t$, so $-\cos y = \cos t - t \sin t - C$ (by the hint) and $y = \cos^{-1}(t \sin t - \cos t + C)$.

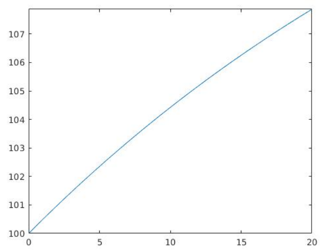
2. Using Newton's law of cooling, $\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_\infty)$ (where T_∞ is the ambient temperature). Solving this differential equation, we find $\int \frac{1}{T - T_\infty} \, \mathrm{d}T = \int k \, \mathrm{d}t$ and so $T = T_0 e^{kt} + T_\infty$. We have $T_\infty = 30°$, and $T_0 = 100°$; also, at $t = 3$ we have $T = 70$ so $70 = 100e^{3k} + 30$; hence $k = \frac{\ln 0.4}{3} = -0.31$ and by direct substitution $T = 100e^{-0.31t} + 30$. Let $T = 31$; then $t = 14.86$ and so the temperature will drop to $31°$ after around fifteen minutes.

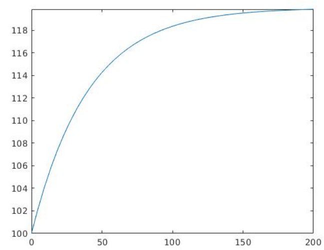
3. (a) We have $\frac{\mathrm{d}V}{\mathrm{d}t} = \text{rate in} - \text{rate out} = 3 - kV$. Hence $\int \frac{1}{3-kV} \, \mathrm{d}V = \int \mathrm{d}t$, so $-\frac{\ln(3-kV)}{k} = t + C$ and $V = \frac{3 - Ke^{-kt}}{k}$. At $t = 0$, $V = 100$; so $100k = (3 - K)$. We also have $kV = 3$ where $V = 120$, so $k = 1/40 = 0.025$. Hnce $2.5 = 3 - K$ and $K = 0.5$. It immediately follows that

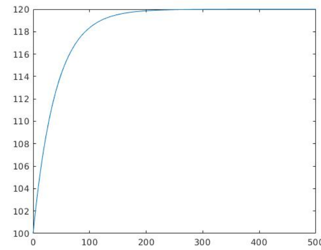
$$V = \frac{3 - 0.5e^{-0.025t}}{0.025}$$

   and at $t = 10$, $V = 104$ litres.

   (b) The rate of water flow out is $kV = 3 - 0.5e^{-0.025t}$, which is always less than 3 (the rate in). In fact, as $t \to \infty$, the volume tends to 120 L and the rate in tends to equal the rate out.





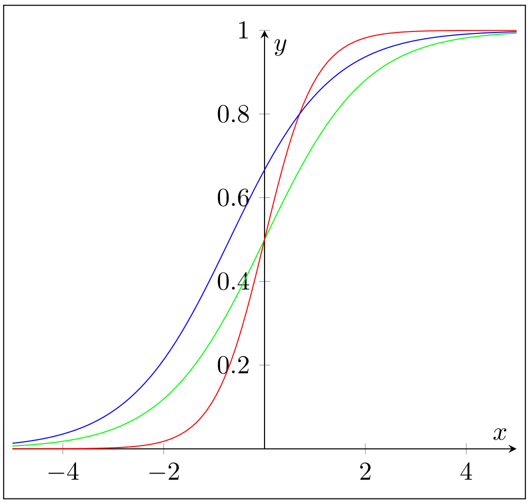


20. Partial Fractions

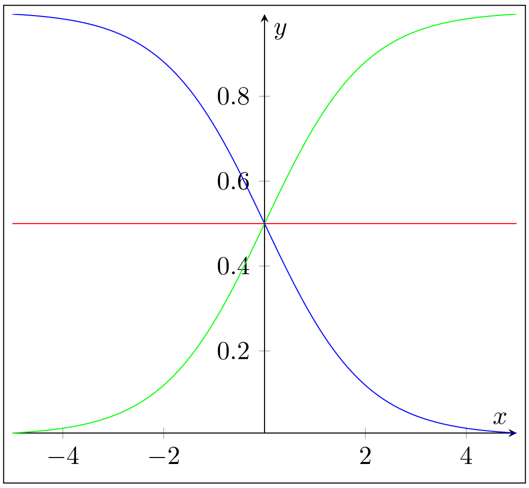
1. (a)

$$\begin{aligned} \int r \, \mathrm{d}t &= \int \frac{\mathrm{d}P}{P(1-P)} = \\ &= \int \frac{1}{P} + \frac{1}{1-P} \, \mathrm{d}P \\ rt + C &= \ln \frac{P}{1-P} \\ Ke^{rt} &= \frac{P}{1-P} \\ \frac{Ke^{rt}}{1 + Ke^{rt}} &= P. \end{aligned}$$

   (b) It should be clear that as $t \to \infty$, $P \to 1$. (If we look at $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{rP}{P_\infty}(P_\infty - P)$, $P \to P_\infty$.) Green: $r = K = 1$; red: $r = 2$, $K = 1$; blue: $r = 1$, $K = 2$.



(c) r lets us vary how fast the population gets to the maximum. Green: $r = K = 1$; red: $r = 0$; blue: $r = -1$.



(d) Write it yourself.

2. (a) Draw a triangle with angle $x/2$, hypotenuse $\sqrt{1+t^2}$, adjacent edge 1, and opposite edge t.

(b)

$$\sin x = 2\sin(x/2)\cos(x/2) = \frac{2t}{1+t^2}$$
$$\cos x = (\cos(x/2))^2 - (\sin(x/2))^2 = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 = \frac{1-t^2}{1+t^2}.$$

(c) We have $x = \tan^{-1} 2t$, so the result follows immediately.

(d) i. Let $t = \tan(x/2)$. Then, substituting, we have

$$\int \frac{1}{1 - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t = \int \frac{1}{t^2}\,\mathrm{d}t = -\frac{1}{t} + C = -\frac{1}{\tan\frac{x}{2}} + C.$$

ii. Similarly,

$$\int \frac{1}{3\frac{2t}{1+t^2} - 4\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t = \int \frac{1}{3t-2+2t^2}\,\mathrm{d}t = \int \frac{1}{(2t-1)(t+2)}\,\mathrm{d}t$$
$$= \frac{1}{5}\ln\frac{1-2t}{t+2} + C = \frac{1}{5}\ln\frac{1-2\tan\frac{x}{2}}{\tan\frac{x}{2}+2} + C.$$

21. Integration by Parts

1. (a)

$$\int x\cos 5x\,\mathrm{d}x = \frac{1}{5}x\sin 5x - \int \frac{1}{5}\sin 5x\,\mathrm{d}x = \frac{1}{5}(x\sin 5x + \cos 5x) + C.$$

(b)

$$\int \cos x\ln\sin x\,\mathrm{d}x = \sin x\ln\sin x - \int \sin x\frac{\cos x}{\sin x}\,\mathrm{d}x$$
$$= \sin x\ln\sin x - \int \cos x\,\mathrm{d}x = \sin x(\ln\sin x - 1) + C.$$

(c) Let $u = \sqrt{x}$, so $\mathrm{d}x = 2u\,\mathrm{d}u$ and our integral becomes

$$\int 2u\cos u\,\mathrm{d}u = 2u\sin u - \int 2\sin u\,\mathrm{d}u = 2u\sin u + 2\cos u + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$$

2. (a) Let $u = \theta^2$, so our integral becomes $\frac{1}{2}\int_{\pi/2}^{\pi} u\cos u\,\mathrm{d}u$. From 1(c) above, we know that $\int u\cos u\,\mathrm{d}u = u\sin u + \cos u + C$. Hence the required result is

$$\frac{1}{2}\int_{\pi/2}^{\pi} u\cos u\,\mathrm{d}u = \frac{1}{2}\left[u\sin u + \cos u\right]\Big|_{u=\pi/2}^{\pi} = -\frac{1}{2} - \frac{\pi}{4}.$$

(b) We use integration by parts twice.

$$\int (x^2+1)e^{-x}\,\mathrm{d}x = -e^{-x}(x^2+1) + \int 2xe^{-x}\,\mathrm{d}x$$
$$= -e^{-x}(x^2+1) - 2xe^{-x} + \int 2e^{-x}\,\mathrm{d}x$$
$$= -e^{-x}(x^2+1) - 2xe^{-x} - 2e^{-x} + C.$$

Hence the result we are looking for is $3 - 6e^{-1}$.

3. (a) Apply integration by parts to $\int 1\cdot(\ln x)^n\,\mathrm{d}x$ by integrating 1 and differentiating $(\ln x)^n$.

(b) Applying (a), we find

$$\int (\ln x)^3\,\mathrm{d}x = x(\ln x)^3 - \int (\ln x)^2\,\mathrm{d}x$$
$$= x(\ln x)^3 - (x(\ln x)^2 - \int \ln x\,\mathrm{d}x)$$
$$= x(\ln x)^3 - (x(\ln x)^2 - (x\ln x - x))$$
$$= x(\ln x)^3 - x(\ln x)^2 + x\ln x - x.$$

22. Lengths, Volumes, and Areas

1. We simply calculate the relevant integral:

$$\pi \int_1^2 x^{-2}\,\mathrm{d}x = \pi[(-2^{-1}) - (-1^{-1})] = \frac{\pi}{2}.$$

2. Calculating the surface area:

$$\begin{aligned} 2\pi \int_0^\pi \sin x\sqrt{1-\cos^2 x}\,\mathrm{d}x &= 2\pi \int_0^\pi \sin^2 x\,\mathrm{d}x \\ &= \pi\left[x - \sin 2x\right]\Big|_0^\pi \\ &= \pi^2 \end{aligned}$$

So the radius of the equivalent circle is $\sqrt{\pi}$.

3. Summing along the axis from base to point, each slice has an area $\left(\frac{L}{H}x\right)^2 = \frac{L^2}{H^2}x^2$; hence the total volume is

$$V = \int_0^H \frac{L^2}{H^2}x^2\,\mathrm{d}x = \frac{1}{3}L^2 H.$$

4. We have $r = a(1-\cos\theta)$ so $\frac{\mathrm{d}r}{\mathrm{d}\theta} = a\sin\theta$. Hence:

$$\begin{aligned} S &= \int_0^{2\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \sqrt{a^2 - 2a^2\cos\theta + a^2(\sin^2\theta + \cos^2\theta)}\,\mathrm{d}\theta \\ &= a\sqrt{2}\int_0^{2\pi} \sqrt{1-\cos\theta}\,\mathrm{d}\theta. \end{aligned}$$

We turn our attention, then, to the integral $\int \sqrt{1-\cos\theta}\,\mathrm{d}\theta$. Let $u = 1-\cos\theta$; then $\mathrm{d}u = \sin\theta\,\mathrm{d}\theta$; but $\sin\cos^{-1}(1-u) = \sqrt{2u-u^2}$ (this can be verified by drawing a suitable triangle). Hence $\mathrm{d}u = \sqrt{2u-u^2}\,\mathrm{d}\theta$, and

$$\begin{aligned} \int \sqrt{1-\cos\theta}\,\mathrm{d}\theta &= \int \frac{\sqrt{u}}{\sqrt{2u-u^2}}\,\mathrm{d}u \\ &= \int \frac{1}{\sqrt{2-u}} \\ &= -2\sqrt{2-u} + C \\ &= -2\sqrt{1+\cos\theta} + C. \end{aligned}$$

Therefore (and changing our integral to double the integral from 0 to π to avoid the problem of having a closed loop),

$$
\begin{aligned}
S &= 2a\sqrt{2}\int_0^{\pi}\sqrt{1-\cos\theta}\,\mathrm{d}\theta \\
&= 2a\sqrt{2}\left[-2\sqrt{1+\cos\theta}\right]\Big|_0^{\pi} \\
&= 2a\sqrt{2}\left[(-2\sqrt{1+\cos\pi}) - (-2\sqrt{1+\cos 0})\right] \\
&= 2a\sqrt{2}\left[(-2\sqrt{0}) - (-2\sqrt{2})\right] \\
&= 2a\sqrt{2}\times 2\sqrt{2} = 8a.
\end{aligned}
$$

23. Trigonometric Substitution

These ones are tedious and can be checked by the computer, so I have not written full answers for all of them.

1. Let $x = 2\tan\theta$, so $\mathrm{d}x = 2\sec^2\theta$:

$$\int \frac{2\sec^2\theta}{\sqrt{x^2+4}}\,\mathrm{d}x = \int \frac{\sec^2\theta}{\sqrt{1+\tan^2 x}}\,\mathrm{d}x = \int \sec\theta\,\mathrm{d}x = \ln\left(\sqrt{\left(\frac{x}{2}\right)^2+1}+\frac{x}{2}\right)+C.$$

2. First, let $u = x^7$ so $\mathrm{d}u = 7x^6\,\mathrm{d}x$ and our integral becomes

$$\frac{1}{7}\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \frac{1}{14}\ln\frac{1+u}{1-u}+C = \frac{1}{14}\ln\frac{1+x^7}{1-x^7}+C.$$

3. Use $x = \frac{2}{5}\sec\theta$.
4. Use $x = \frac{2}{3}\sin\theta$.
5. Use $x = \frac{1}{6}\tan\theta$ and simplify.
6. Use integration by parts; the resulting integral $-\frac{\ln x}{4x^4} + \int \frac{\mathrm{d}x}{4x^5}$ is much simpler.
7. Use partial fractions.

24. Kinematics

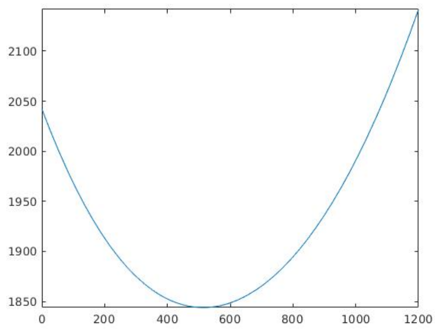
1. (a) $v = \frac{\mathrm{d}h}{\mathrm{d}t} = 122.5 - 9.8t$, so the initial velocity of the flare is $122.5\,\mathrm{m\,s^{-1}}$.
   (b) Zero.
   (c) When $v = 0$, $t = 12$ and the height at this time is around 764 metres.
2. Let x be the distance from the point on the beach directly away from B. Then the total distance travelled is simply $D = \sqrt{600^2+x^2} + \sqrt{800^2+(1200-x)^2}$; taking the derivative:

$$\frac{\mathrm{d}D}{\mathrm{d}x} = \frac{x}{\sqrt{600^2+x^2}} - \frac{1200-x}{\sqrt{800^2+(1200-x)^2}}$$

   Setting to zero, we have

$$
\begin{aligned}
x\sqrt{800^2+(1200-x)^2} &= (1200-x)\sqrt{600^2+x^2} \\
800^2x^2+(1200-x)^2x^2 &= (1200-x)^2(600^2+x^2) \\
0 &= 1200^2600^2 - 600^2 2400x + (600^2-800^2)x^2 \\
x &\in \{-3600, 3600/7\}.
\end{aligned}
$$

Since $x \geq 0$, $x = 3600/7 \approx 514$. The total distance travelled is therefore around 1844 metres. By graphing D versus x, we see that this is indeed the required medium:



3. Like question 2 from sheet 15:- we just pick an approximation for the integral (in this case, jagged edges means parallelograms won't be too bad).

25. Integration Revision

1. (a) $\int_1^2 \sin x \, \mathrm{d}x = [-\cos x]\Big|_1^2 = \cos 1 - \cos 2$.
   (b) $\int \frac{u^2+1}{u^3+3u} = \frac{1}{3}\ln(u^3+3u) + C$.
   (c) $\int_0^{\pi/6} \tan x \, \mathrm{d}x = [\ln \sec x]\Big|_0^{\pi/6} \approx 0.1438$.
2. We have $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2+4x-4}{2y-4}$, so $\int 2y - 4 \, \mathrm{d}y = \int 3x^2 + 4x - 4 \, \mathrm{d}x$. Hence $y^2 - 4y = x^3 + 2x^2 - 4x + C$; we also have $C = -2$, so $y^2 - 4y = x^3 + 2x - 4x - 2$. We are trying to find y if $x = 2$; so $y^2 - 4y = 8 + 4 - 4 - 2 = 6$. Solving $y^2 - 4y - 6 = 0$, we find $y = \frac{4 \pm \sqrt{38}}{2}$.
3. (a) Let $t = a \tan\theta$. Then:

$$\begin{aligned} \int \frac{a^3}{t^2 + a^2} \, \mathrm{d}t &= \int \frac{a^4 \sec^2\theta}{a^2 \tan^2 + a^2} \, \mathrm{d}\theta \\ &= \int \frac{a^2 \sec^2\theta}{\sec^2\theta} \, \mathrm{d}\theta \\ &= a^2\theta = a^2 \tan^{-1}\left(\frac{t}{a}\right); \end{aligned}$$

   hence $\omega(a, x) = a^2 \tan^{-1}\left(\frac{x}{a}\right)$.
   (b) It follows that $\omega(2, 2) = 4\tan^{-1} 1 = \pi$.
   (c) We wish to find x such that $\pi = 3\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$; in other words, $x = \sqrt{3}\tan\left(\frac{\pi}{3}\right) = 3$.

4. Note first that $\int_{-\pi/2}^{\pi/2} \sin^5 x \, \mathrm{d}x = 0$ since $\sin^5$ is odd. Then, we argue as follows:

$$\begin{aligned}
\int \cos^5 x \, \mathrm{d}x &= \int \cos x (1 - \sin^2 x)^2 \, \mathrm{d}x \\
&= \int (1 - t^2)^2 \, \mathrm{d}t \qquad (t = \sin x) \\
&= \int 1 - 2t^2 + t^4 \, \mathrm{d}t \\
&= t - \frac{2}{3} t^3 + \frac{t^5}{5} + C \\
&= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C.
\end{aligned}$$

Hence

$$\int\limits_{-\pi/2}^{\pi/2} \sin^5 x \, \mathrm{d}x = (\sin \frac{\pi}{2} - \frac{2}{3} \sin^3 \frac{\pi}{2} + \frac{1}{5} \sin^5 \frac{\pi}{2}) - (\sin \frac{-\pi}{2} - \frac{2}{3} \sin^3 \frac{-\pi}{2} + \frac{1}{5} \sin^5 \frac{-\pi}{2}).$$

But $\sin(\pi/2) = 1$; so we have $(1 - \frac{2}{3} + \frac{1}{5}) - (-1 + \frac{2}{3} - \frac{1}{5}) = 2(1 - \frac{2}{3} + \frac{1}{5}) = \frac{16}{15}$.

S

S–CALCF

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Scholarship 2015
Calculus

2.00 p.m. Tuesday 17 November 2015

FORMULAE AND TABLES BOOKLET

Refer to this booklet to answer the questions for Scholarship Calculus 93202Q.

Check that this booklet has pages 2–4 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

CALCULUS – USEFUL FORMULAE

ALGEBRA

Quadratics

If $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Logarithms

$y = \log_b x \Leftrightarrow x = b^y$

$\log_b(xy) = \log_b x + \log_b y$

$\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

$\log_b(x^n) = n\log_b x$

$\log_b x = \dfrac{\log_a x}{\log_a b}$

Complex numbers

$z = x + \mathrm{i}y$
$\quad = r\,\mathrm{cis}\,\theta$
$\quad = r(\cos\theta + \mathrm{i}\sin\theta)$

$\bar{z} = x - \mathrm{i}y$
$\quad = r\,\mathrm{cis}\,(-\theta)$
$\quad = r(\cos\theta - \mathrm{i}\sin\theta)$

$r = |z| = \sqrt{z\bar{z}} = \sqrt{(x^2 + y^2)}$

$\theta = \arg z$

where $\cos\theta = \dfrac{x}{r}$

and $\sin\theta = \dfrac{y}{r}$

De Moivre's Theorem

If n is any integer, then

$(r\,\mathrm{cis}\,\theta)^n = r^n\,\mathrm{cis}\,(n\theta)$

Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!r!}$$

Some values of $\binom{n}{r}$ are given in the table below.

n \ r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
11	1	11	55	165	330	462	462	330	165	55	11
12	1	12	66	220	495	792	924	792	495	220	66

COORDINATE GEOMETRY

Straight Line

Equation $y - y_1 = m(x - x_1)$

Circle

$(x-a)^2 + (y-b)^2 = r^2$

has a centre (a, b) and radius r

Parabola

$y^2 = 4ax$ or $(at^2, 2at)$

Focus $(a, 0)$ Directrix $x = -a$

Ellipse

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or $(a\cos\theta, b\sin\theta)$

Foci $(c, 0)$ $(-c, 0)$ where $b^2 = a^2 - c^2$

Eccentricity: $e = \dfrac{c}{a}$

Hyperbola

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $(a\sec\theta, b\tan\theta)$

asymptotes $y = \pm\dfrac{b}{a}x$

Foci $(c, 0)$ $(-c, 0)$ where $b^2 = c^2 - a^2$

Eccentricity: $e = \dfrac{c}{a}$

CALCULUS

Differentiation

$y = f(x)$	$\dfrac{dy}{dx} = f'(x)$
$\ln x$	$\dfrac{1}{x}$
e^{ax}	ae^{ax}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$

Integration

$f(x)$	$\int f(x)dx$
x^n	$\dfrac{x^{n+1}}{n+1} + c$ $(n \neq -1)$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + c$
$\dfrac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert + c$

First principles

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Parametric Function

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

Product Rule

$(f.g)' = f.g' + g.f'$ or if $y = uv$ then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Quotient Rule

$\left(\dfrac{f}{g}\right)' = \dfrac{g.f' - f.g'}{g^2}$ or if $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Composite Function or Chain Rule

$$\big(f(g)\big)' = f'(g).g'$$

or if $y = f(u)$ and $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

NUMERICAL METHODS

Trapezium Rule

$$\int_a^b f(x)\ dx \approx \frac{1}{2}h\left[y_0 + y_n + 2(y_1 + y_2 + ... + y_{n-1})\right]$$

where $h = \dfrac{b-a}{n}$ and $y_r = f(x_r)$

Simpson's Rule

$$\int_a^b f(x)\ dx \approx \frac{1}{3}h\left[y_0 + y_n + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})\right]$$

where $h = \dfrac{b-a}{n}$, $y_r = f(x_r)$ and n is even.

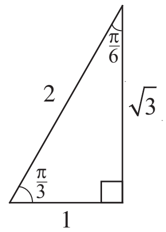
TRIGONOMETRY

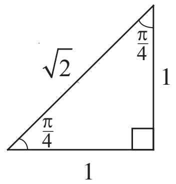
$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$





Sine Rule

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine Rule

$c^2 = a^2 + b^2 - 2ab\cos C$

Identities

$\cos^2\theta + \sin^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

General Solutions

If $\sin\theta = \sin\alpha$ then $\theta = n\pi + (-1)^n\alpha$

If $\cos\theta = \cos\alpha$ then $\theta = 2n\pi \pm \alpha$

If $\tan\theta = \tan\alpha$ then $\theta = n\pi + \alpha$

where n is any integer

Compound Angles

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

Double Angles

$\sin 2A = 2\sin A\cos A$

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A\end{aligned}$$

Products

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Sums

$\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

$\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

$\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

$\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

MEASUREMENT

Triangle

Area $= \dfrac{1}{2}ab\sin C$

Trapezium

Area $= \dfrac{1}{2}(a+b)h$

Sector

Area $= \dfrac{1}{2}r^2\theta$

Arc length $= r\theta$

Cylinder

Volume $= \pi r^2 h$

Curved surface area $= 2\pi rh$

Cone

Volume $= \dfrac{1}{3}\pi r^2 h$

Curved surface area $= \pi rl$ where $l =$ slant height

Sphere

Volume $= \dfrac{4}{3}\pi r^3$

Surface area $= 4\pi r^2$

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