Here we prove the FTC for 'nice' functions. We actually prove the second FTC first as it is easier.

Theorem (Second Fundamental Theorem)**.** *Suppose that f is a continuous function on the closed interval $[a, b]$.*[*] *Then:*

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

Proof. Let us take the derivative in a straightforward manner.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t}{h} = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,\mathrm{d}t}{h}.$$

Now, let $f(M)$ be the maximum value obtained by f on the closed interval $[x, x+h]$; let $f(m)$ be the minimum value. Interpreting the integral as an area, we have

$$hf(m) \le \int_x^{x+h} f(t)\,\mathrm{d}t \le hf(M) \implies f(m) \le \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t \le f(M).$$

Now, as $h \to 0$ we must have $f(m) \to f(x)$ and $f(M) \to f(x)$ (because as we make the interval smaller, m and M move towards x). Hence

$$f(x) \le \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t \le f(x)$$

and so $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$. □

Theorem (First Fundamental Theorem)**.** *Suppose f is continuous on the closed interval $[a, b]$, and suppose F is any antiderivative of f (so $F' = f$). Then:*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b$$

Proof. Consider $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$. In particular, $\int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of f and we can antidifferentiate both sides, obtaining

$$\text{(*)}\qquad \int_a^x f(t)\,\mathrm{d}t = F(x) + C$$

(where C is some constant). Now substitute a for x in (*): we find that $0 = \int_a^a f(t)\,\mathrm{d}t = F(a) + C$, and in particular $-C = F(a)$. Substituting b for x in (*), we find that $\int_a^b f(t)\,\mathrm{d}t = F(b) + C = F(b) - F(a)$; and we are done. □

Exercise. Explain why we cannot calculate $\int_0^1 1/x^2\,\mathrm{d}x$ via the version of the fundamental theorem proved here. Then calculate $\lim_{\alpha\to 0}\int_\alpha^1 1/x^2\,\mathrm{d}x$; why is this allowed?

[*] i.e. f is continuous at every x such that $a \le x \le b$.