Scholarship Calculus

There are five questions, worth a total of **??** marks.
Attempt ALL questions, showing all working.
Choose ONE option ONLY from 2(b) and 2(c).
Read questions carefully before attempting them.
Marks are available for partial answers.
The amount of time expected to be spent per question may not necessarily correlate "nicely" to the number of marks.
Diagrams may be used to support answers.
Candidates who do not provide diagrams for some questions may be disadvantaged.
Some marks are given for clarity and neatness of solutions or proofs.

Time Allowed:	Three Hours
Scholarship:	45 marks
Outstanding:	65 marks

Run LaTeX again to produce the table

Available Grades: *No Scholarship* *Scholarship* *Outstanding*

Question 1

(a) Consider the sequence of functions (5)

$$f_n(x) = nx(1-x^2)^n \qquad (0 \le x \le 1,\ n = 1, 2, 3, ...).$$

Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x \ne \int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x.$$

(b) Compute the following definite integral. (5)

$$\int_0^1 \sin^3 x \cos^4 x + \sin^4 x \cos^3 x\,\mathrm{d}x.$$

(c) Suppose that f is a function satisfying (5)

$$\begin{cases} \dfrac{f(x)}{2f'(x)} = 3(x^3 - 2x^2 - x + 2), \\ f(3) = 1. \end{cases}$$

Find $f(x)$ explicitly. You need only calculate any constants of integration to three decimal places.

Question 2

(a) Consider the cubic equation $p(x) = x^3 + px + q$ (where p and q are real). (10)

i. Let the three roots of $p(x)$ be α, β, and γ. Show that

$$(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 = -4p^3 - 27q^2.$$

ii. Find the nature of the roots of $p(x)$ in the cases where $-4p^3 - 27q^2$ is greater than, less than, or equal to zero.

Answer ONE of (b) and (c).

(b) Find a formula for $\binom{n-1}{k} - \binom{n-1}{k-1}$ in terms of $\binom{n}{k}$. (5)

(c) Minimise $F(x) = x^3 + y^4 + z^5$ with respect to the variables x, y, and z subject to: (5)

$$\begin{gathered} x + y + z \ge 1 \\ x - y + z \le 2 \\ z \ge 2 \\ 0.75x - 5.87y - 5.78z \ge -31.74. \end{gathered}$$

A graph of these planes is provided on the final page of this examination.

Question 3

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x)(\sec^{-1} x)$ in terms of x only. (5)

(b) Consider a circle of radius r that is rolling around the inside of a circle of radius R ($R > r$). Let $p = (x, y)$ be the point on the inner circle that is initially touching the outer circle. Show that (5)

$$x = (R-r)\cos\theta + \cos\left(\frac{R-r}{r}\theta\right),$$
$$y = (R-r)\sin\theta - \sin\left(\frac{R-r}{r}\theta\right).$$

A diagram is provided on the final page of this examination.

(c) The Steiner inellipse is the unique ellipse inscribed in a triangle and tangent to the midpoint of each side. Find the coordinates of the two focii of the Steiner inellipse inscribed in the triangle with vertices $(1, 7)$, $(7, 5)$, and $(3, 1)$. (5)

Question 4

(a) i. By differentiating $f(x)g(x)$, show that (5)

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x.$$

ii. Find an antiderivative of

$$\frac{\sqrt{4x^2-9}}{x^2}.$$

(b) Discuss the significance of the fundamental theorem of calculus. You should write approximately half a page. (5)

(c) An **homomorphism** is a function f such that $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for *all* real numbers a and b. Show that the only homomorphisms which are polynomials are $f(x) = 0$ and $f(x) = x$. (5)

Question 5

You may use the following theorem in answering this question. **Do not attempt to prove this theorem.**

> **Mean Value Theorem**
> Let a and b be real numbers such that $a < b$, and let f be a function satisfying two hypotheses:
>
> 1. f is continuous at all x such that $a < x < b$.
> 2. f is differentiable at all x such that $a \leq x \leq b$.
>
> Then there exists some number c such that
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(a) Verify that the function f defined by (5)

$$f(x) = \frac{x}{x+2}$$

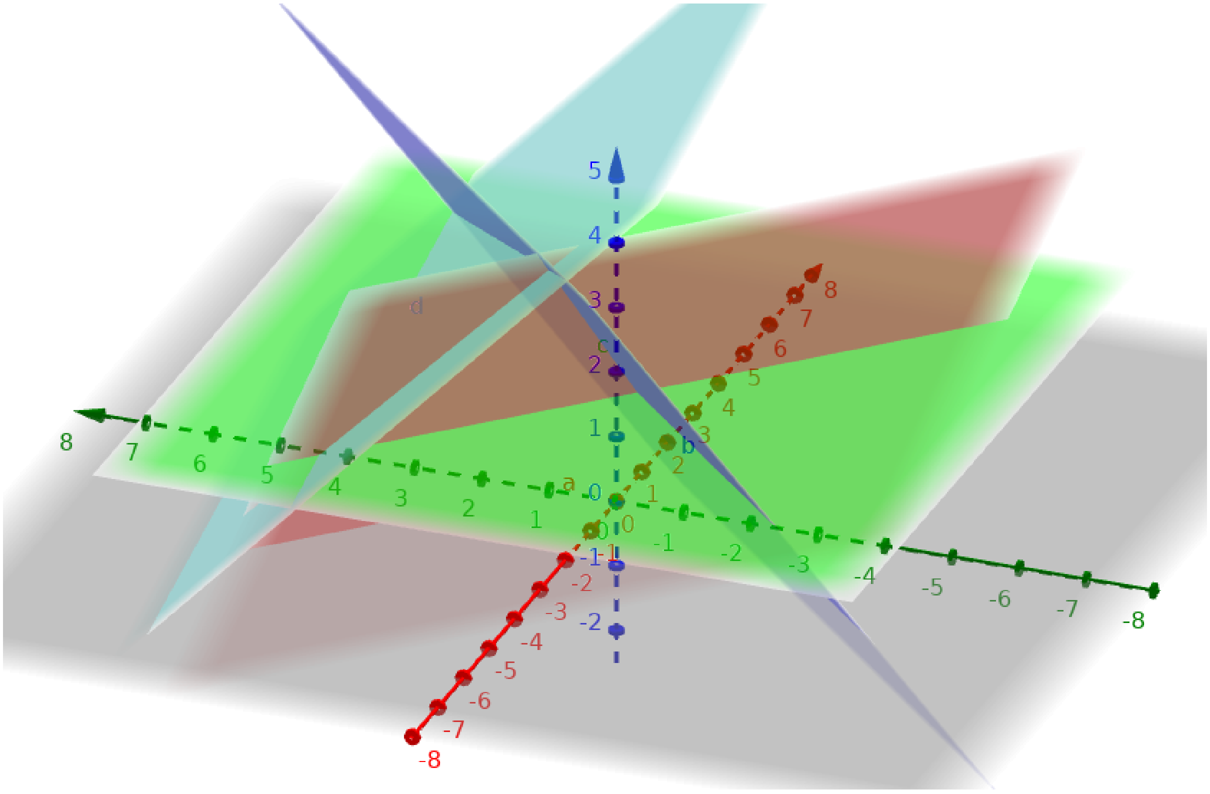
satisfies the hypotheses of the mean value theorem on the interval $1 < x < 4$. Find all points c that satisfy the conclusion of the mean value theorem.

(b) A point a is called a **fixed point** of a function f if $f(a) = a$. Suppose that $f'(x) \neq 1$ for all real numbers x; show that f has at most one fixed point. (5)

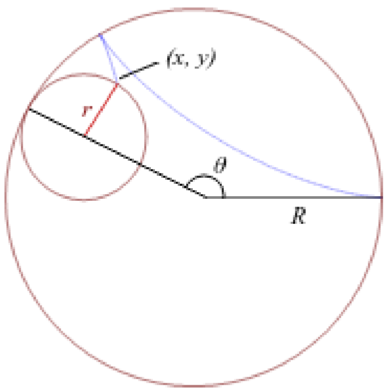
(c) Find a function f such that $f'(-1) = \frac{1}{2}$, $f'(0) = 0$, and $f''(x) > 0$ for all x, or prove that such a function cannot exist. (5)

Figures provided overleaf.

Useful Figures



Question 2(c)



Question 3(b)