NCEA Level 3 Calculus (Differentiation)
1. The Derivative

What is the derivative?

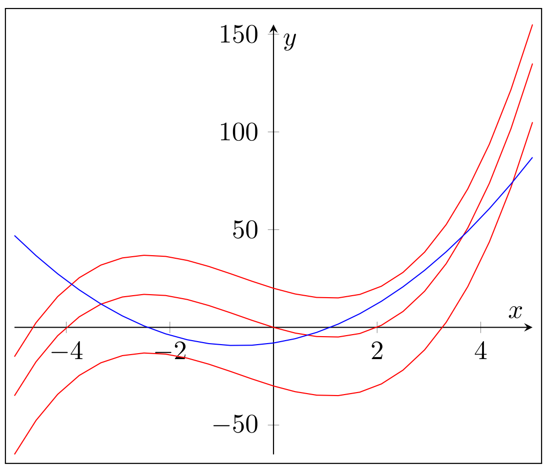
- A way to find the slope of another function.
- A function giving a rate of change.
- A way to find maxima and minima of a function.

Essentially, the value of the derivative of a function at a point is the gradient/slope/rate of change of that function at that point.

Example. Consider a function defined by $y = 2x + 1$. At every point, the slope of the graph of this function is 2; so the value of the derivative of this function is 2 at every point.

Example. The speed of a particle at any time is the rate of change of the displacement of the particle at that time; this can also be viewed as the slope of the displacement-time graph of the particle. Hence the speed of the particle is the derivative of the displacement.

Example. In the diagram, the derivative of the red functions is shown as the blue function. Note that the derivative of the function depends only upon the shape, not the y-shift. Also, see that the derivative is positive as the function increases and is negative when the function is decreasing. The derivative is zero exactly where the function is 'flat'.

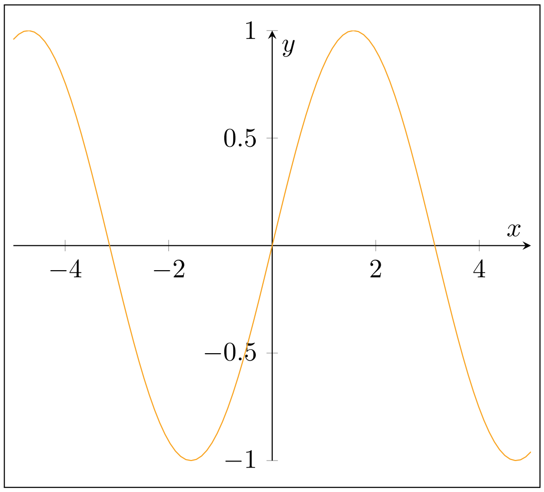


Questions

1. $\boxed{\texttt{A}}$ Why does the derivative not depend on the y-shift of a function?

2. $\boxed{\texttt{A}}$ Draw the derivative of each graphed function.

(a)



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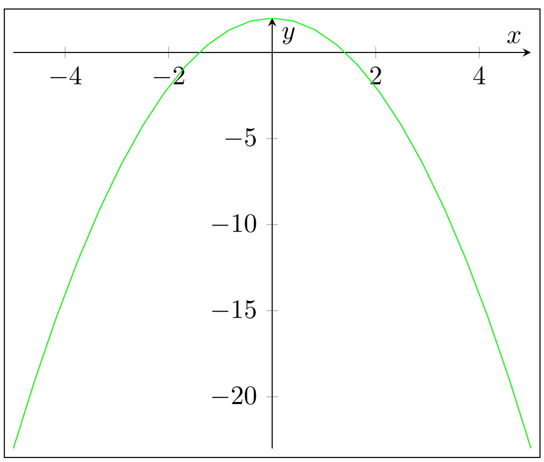

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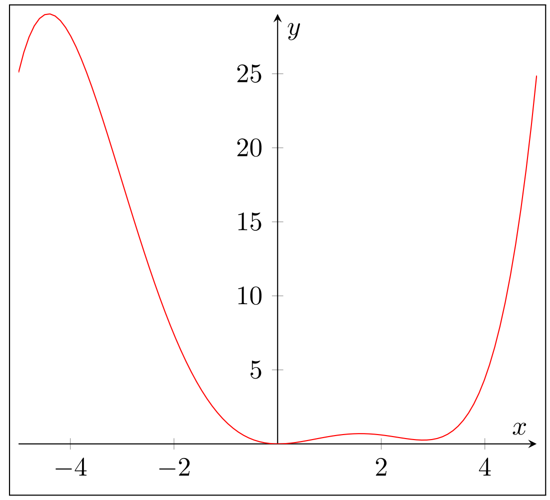

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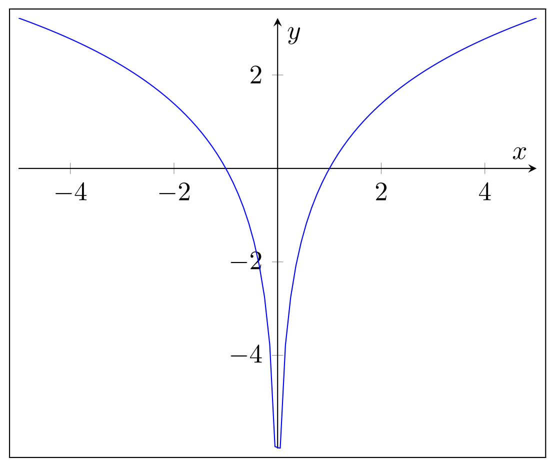

3. $\boxed{\texttt{A}}$ Consider each of these functions in turn. Where is the derivative of each (i) negative, (ii) positive, (iii) zero, and (iv) undefined?
   (a) $x \mapsto x^2$
   (b) $x \mapsto \sin x$
   (c) $x \mapsto \tan x$

4. $\boxed{\texttt{A}}$ Describe the derivative of the function $x \mapsto \tan^{-1} x$.

5. $\boxed{\texttt{M}}$ Consider the function $f : t \mapsto \text{floor}(t)$. Where is f differentiable? (Define $\text{floor}(t)$ to be the greatest integer less than or equal to t.)

6. $\boxed{\texttt{M}}$ When a hot water tap is turned on, the temperature T of the water depends on how long the water has been running.
   (a) Sketch a possible graph of T as a function of the time t that the tap has been running.

(b) Describe how the rate of change of T w.r.t. t varies as t increases.

(c) Sketch a graph of the derivative of T w.r.t. t.

7. $\boxed{\texttt{M}}$ The rate of change of a population at a time t is directly proportional to the population $P(t)$ at that time, such that $\frac{\mathrm{d}P}{\mathrm{d}t} = P$. Draw a graph of the population over time if $P(0) \approx 1000$.

8. $\boxed{\texttt{M}}$ Consider a continuous function $f : \mathbb{R} \to \mathbb{R}$ which has n roots (all of multiplicity one) in some interval I. How many roots must its derivative have in I?

9. $\boxed{\texttt{M}}$ If a function is periodic, with a period of T, what can you say about its derivative?

10. $\boxed{\texttt{M}}$ Consider an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is not a function: since both $(0, b)$ and $(0, -b)$ are members of the function, it fails the vertical line test. However, it would be nice to reason about its rate of change *as if it were* a function. Describe the slope of the ellipse as a particle traces the curve in an anticlockwise direction at a constant rate.

11. $\boxed{\texttt{M}}$ The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.

    (a) What is the meaning of the derivative $f'(5)$? What are its units?

    (b) Suppose that there is an unlimited amount of space and nutrients. Which would you expect to be larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited does your answer change?