Scholarship Calculus

Question 1: Complex Numbers

(a) i. Find all eighth roots of 256.

ii. Find the polar form of $\alpha = 2 - 3i$.

iii. If $\beta = 3 + 7i$, and $\gamma = 9 + 11i$, find the unique fourth-degree polynomial with real coefficients that has both β and γ as roots.

iv. Solve for x if $x^4 + i = 0$.

(b) If $\zeta = \sqrt{\frac{1}{2}(a + \sqrt{a^2 + b^2})} + i\sqrt{\frac{1}{2}(-a + \sqrt{a^2 + b^2})}$ is a complex number (with $i = \sqrt{-1}$ and $b, c \in \mathbb{R}$), find ζ^2 in the form $p + iq$.

Question 2: Systems of Equations

(a) Solve the following systems of equations:

i. $\begin{cases} x + y = 8 \\ x - y = 0 \end{cases}$

ii. $\begin{cases} 2x - 5y = 6 \\ 5x - 2y = 12 \end{cases}$

iii. $\begin{cases} x - 2y + z = 7 \\ x - y + z = 4 \\ 2x + y - 3z = -4 \end{cases}$

iv. $\begin{cases} 3x + 3y = 2 \\ \dfrac{x^2 + y^2}{xy} = -2 \end{cases}$

(b) A teacher sets 99 homework problems for her Calculus class each week, of three different types. The number of questions of each type given in week n are represented by x_n, y_n, and z_n respectively.
Suppose that this techer uses the following system of linear equations to vary the number of questions of each type given each week:

$$\begin{aligned} x_{n+1} &= 0.8x_n + 0.7y_n + 0.6z_n \\ y_{n+1} &= 0.1x_n + 0.2y_n + 0.4z_n \\ z_{n+1} &= 0.1x_n + 0.1y_n \end{aligned}$$

Her class notice that the number of questions of each type stabilises after several weeks - in the long run, they notice that $x_{n+1} = x_n$, $y_{n+1} = y_n$, and $z_{n+1} = z_n$.
How many questions of each type will the teacher give each week once the numbers stabilise?

(c) Find c such that $x - y + 2 = 0$, $3x - 3y + 7 = 0$, and $3x + 2y + x = 0$ shall meet at a single point.

Question 3: Derivatives

(a) A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km away from P?

(b) If f and g are differential functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$$

Note: you may wish to start from the right hand side and expand the derivatives using first principles.

(c) Brain weight B as a function of body weight W in fish has been modelled by the function $B = 0.007W^{\frac{2}{3}}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimetres) is $W = 0.12L^{2.53}$. If, over 10 Myr the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast has this species' brain growing when the average length was 18 cm?

Question 4: Conic Sections

(a) Determine equations of three different lines, all of which pass through the point $(2, -6)$.

(b) Determine the locus of a point which is always as far from the x axis as it is from the point $(1, 3)$.

(c) The earth moves in an elliptical path about the sun with the sun at one focus of the ellipse. If the distance of the earth from the sun varies from 4.6×10^7 km to 6.98×10^7 km, find the equation of the ellipse.

(d) A satellite, P, is travelling anticlockwise around an elliptical orbit with centre O. The elliptical orbit may be represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the satellite P is represented by the point $(a\cos\theta, b\sin\theta)$.

i. Draw a diagram showing P, θ, a, and b, where O $= (0, 0)$.

ii. The satellite P shines a microwave beam in the directions perpendicular to its direction of motion. Show that, when $\theta = \frac{\pi}{4}$, the beam cuts the vertical plane (the y-axis) at a vertical distance $\frac{\sqrt{2}a^2}{2b}$ below the level of the satellite.

Question 5: Functions

Definition: A function f is called *even* if for all x in the domain of f, $f(-x) = f(x)$ - i.e. the function is symmetric about the y-axis. A function f is called *odd* if for all x in the domain of f, $f(-x) = -f(x)$.

(a) Decide if each of the following function is even, odd, or neither:

i. $f(x) = x^3 + x$

ii. $g(x) = 1 - x^2$

iii. $h(x) = 2x - x^2$

(b) **Definition:** Given some function f which sends x to y, we can define a function $f^{-1}(x)$ which sends y to x. This function is the *inverse* of f.

Prove that the inverse of an odd function is itself odd.