NCEA Level 3 Calculus (Differentiation)
7. Higher Derivatives and the Geometry of a Function

Back in the homework on limits, I gave you the following definitions:

Properties of Functions

- A function is **increasing** if its derivative is positive.
- A function is **decreasing** if its derivative is negative.
- A function is **concave down** if its derivative is decreasing.
- A function is **concave up** if its derivative is increasing.
- A function f is **continuous** at a point a if $\lim_{x \to a} f(x) = f(a)$.

Example. The function $x \mapsto x^2$ is concave up everywhere.

Example. The function $x \mapsto \sin x$ is concave down when $(2n)\pi < x < (2n+1)\pi$, and concave up when $(2n+1)\pi < x < (2n+2)\pi$ (for all integers n). Draw a diagram to see this.

We can tell if a function is increasing (getting larger) or decreasing (getting smaller) around a point by looking at the derivative at that point. However, in order to look for concavity we must look at the derivative of the derivative (the second derivative). If a function is concave up (curving up), the second derivative is positive; if a function is concave down at a point, then the second derivative is negative.

A point where a function changes from concave up to concave down (or vice versa) is known as an **inflection point**.

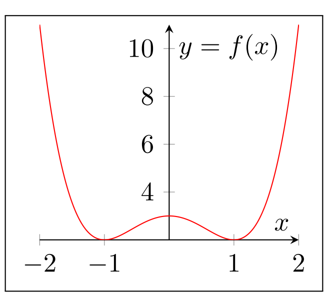
Example. The function $x \mapsto x^3$ has an inflection point at $(0, 0)$; to the left of this point, the function is concave down (the second derivative is negative) and to the right the function is concave up (the second derivative is positive).

We call the value of the first derivative the *slope* or *gradient* of the function at a point; similarly, the second derivative measures how "curvy" a function is and we call its value the *concavity* of a function at a point.

Unfortunately, there is no nice method for checking for continuity involving the derivatives, beyond the fact (proved in the exercises) that if a function is differentiable at a point then it is continuous there. At this level, however, it should be fairly obvious at which points a function is discontinuous.

Example. *Question.* Consider the function defined by $f(x) = x^4 - 2x^2 + 3$. (a) Find the intervals on which f is increasing or decreasing. (b) Find the intervals of concavity and the inflection points.

Solution. We have $f'(x) = 4x^3 - 4x$. This function is zero at $x \in \{-1, 0, 1\}$, and so (since the function is a positive cubic) f will be decreasing when $x < -1$, increasing when $-1 < x < 0$, decreasing when $0 < x < 1$, and increasing when $1 < x$. We also have $f''(x) = 12x^2 - 4$ and so $f''(x) = 0$ when $x = \pm\frac{1}{\sqrt{3}}$. Hence the function is concave up when $x < -\frac{1}{\sqrt{3}}$, concave down when $|x| < \frac{1}{\sqrt{3}}$, and concave up when $x > \frac{1}{\sqrt{3}}$. The inflection points will be $x = \pm 1\sqrt{3}$.



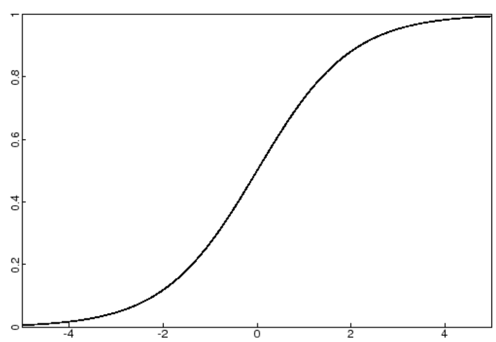
Questions

1. $\boxed{\texttt{A}}$ Find the second derivative of the following functions.
   (a) $y = x^2 + x$
   (b) $f(x) = \sin x$
   (c) $g(x) = \cot(3x^2 + 5)$
   (d) $y = \frac{\sin mx}{x}$
   (e) $y = 4\sin^2 x$
   (f) $y = \tan^2(\sin\theta)$
   (g) $y = \tan\sqrt{1-x}$

2. $\boxed{\texttt{A}}$ Find the concavity of the function $y = \frac{x^2-1}{x^2+1}$ at $(0, -1)$.

3. $\boxed{\texttt{M}}$ Find the intervals on which the following functions are increasing or decreasing, and find their intervals of concavity.
   (a) $y = x^2 + 1$
   (b) $y = 2x^3 + 3x^2 - 36x$
   (c) $G(x) = x - 4\sqrt{x}$

4. $\boxed{\texttt{A}}$ The following function is known as the *logistic curve* and is used for population modelling. Find the intervals of concavity, and label any inflection points.



5. $\boxed{\texttt{M}}$ Sketch a function satisfying the given criteria.
   (a) i. Vertical asymptote at $x = 0$,
   ii. $f'(x) > 0$ if $x < -2$,
   iii. $f'(x) < 0$ if $x > -2$ $(x \neq 0)$,
   iv. $f''(x) < 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$.
   (b) i. $f'(0) = f'(2) = f'(4) = 0$,
   ii. $f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
   iii. $f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
   iv. $f''(x) > 0$ if $1 < x < 3$,
   v. $f''(x) < 0$ if $x < 1$ or $x > 3$.

6. $\boxed{\texttt{E}}$ A curve is defined by the function $f(x) = e^{-(x-k)^2}$. Find, in terms of k, the x-ordinates for which $f''(x) = 0$.

7. $\boxed{\text{S}}$ We will prove that differentiability of f at a implies continuity of f at a; expand the following and use the limit laws to show that $\lim_{x\to a} f(x) - f(a) = 0$, carefully indicating where you use the existence of the derivative.

$$\left[\lim_{a\to x} f(x) - f(a)\right]\left[\lim_{a\to x} \frac{x-a}{x-a}\right]$$

8. $\boxed{\text{S}}$ Scholarship 2010: Recall that the points of inflection of a curve are places where the second derivative changes sign. These are typically, **but not always**, points at which the second derivative is zero.

   Consider the curve $y = \sqrt[3]{x}e^{-x^2}$.

   Write the second derivative in the form $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (ax^4 + bx^2 + x)e^{-x^2}x^{-5/3}$, and hence find the x-ordinates of the points of inflection of the curve.

9. $\boxed{\text{S}}$ Scholarship 2004: (You may wish to remind yourself how to perform long division of polynomials.) Consider the function

$$y = \frac{x^2}{1+x^2},$$

   where $-1 \le x \le 1$. The gradient at the point $x = 1$ is $\frac{1}{2}$.

   Hence show that there is a point with $\frac{1}{4} \le x \le \frac{1}{2}$ where the gradient is also $\frac{1}{2}$.