Assignment: Mathematical Writing Practice III

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1 Task

Suppose that $|z + w| = |z - w|$. Show that $\arg z - \arg w = \pm\frac{\pi}{2}$. You may, if necessary, use the result that $\arctan u - \arctan v = \arctan\left(\frac{u-v}{1+uv}\right)$.

Ensure that you write 'properly'. That means using complete sentences, justifying all logic, and aiming for clarity!

2 Hints

A list of things to think about:

- What can we assume? What are we trying to prove?
- How can we simplify the premises and the conclusion?
- Why are we given an extra piece of information? Could it be useful? Is it a hint about how we should attack the problem?
- What is special about x if $\arctan x = \frac{\pi}{2}$?

3 Example Answer

This problem is taken from *Solutions* (problem 19 from the Final Exercises).

We first let $z = a + bi$ and $w = c + di$. Then $|z + w| = +\sqrt{(a+c)^2 + (b+d)^2}$, and $|z - w| = +\sqrt{(a-c)^2 + (b-d)^2}$. Hence, we can write (by our assumption $|z+w| = |z-w|$) that $(a+c)^2 + (b+d)^2 = (a-c)^2 + (b-d)^2$, and therefore (by expanding and simplifying) that

$$\frac{ac}{bd} = -1. \tag{1}$$

We move now to simplify the thing which we are trying to prove: $\arg z - \arg w = \pm\frac{\pi}{2}$. An obvious first step is to substitute for w and z. We therefore obtain (by the definition of the argument of a complex number) that $\arctan\frac{b}{a} - \arctan\frac{d}{c} = \pm\frac{\pi}{2}$. Applying the result given about the difference of arctans, we can write our equation as

$$\pm\frac{\pi}{2} = \arctan\frac{\frac{b}{a} - \frac{d}{c}}{1 + \frac{bd}{ac}} \Rightarrow \tan\left(\pm\frac{\pi}{2}\right) = \frac{\frac{b}{a} - \frac{d}{c}}{1 + \frac{bd}{ac}}. \tag{2}$$

All that remains is to show that (1) implies (2). This can be done by noting that $\tan\left(\pm\frac{\pi}{2}\right)$ is undefined, and hence that $\frac{\frac{b}{a} - \frac{d}{c}}{1+\frac{bd}{ac}}$ being undefined is a sufficient condition for our conclusion (that $\arg z - \arg w = \pm\frac{\pi}{2}$) to hold.

We note that by (1), $1 + \frac{bd}{ac} = 0$ and hence the fraction is undefined. $\square$