NCEA Level 3 Calculus (Differentiation)
10. Inverse Functions

We need to take a quick pitstop this week to deal with one more differentiation rule before we can start looking at a few more applications next week and then some interesting functions in higher dimensions later on.

Definition. A function is called **one-to-one** (or **injective**) if $f(x) = f(y)$ implies that $x = y$ (i.e. two different inputs can never give the same output/the function passes the horizontal-line test).

Let f be a function that is one-to-one. Then the **inverse** of f is the (unique) function f^{-1} such that

$$f(x) = y \iff f^{-1}(y) = x.$$

In other words, $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$.

Example. Here are some functions with their inverses:

Function	**Inverse**	**Notes**
e^x	$\ln x$	Note that $\ln x$ is defined only when $x > 0$ since $e^x > 0$ for all real x.
$\sin x$	$\sin^{-1} x$	Note that $\sin^{-1} x$ is only defined when $-\pi < x \leq \pi$ since otherwise $\sin x$ is not one-to-one.
$\cos x$	$\cos^{-1} x$	Note that $\cos^{-1} x$ is only defined when $-\pi < x \leq \pi$ since otherwise $\cos x$ is not one-to-one.
$\tan x$	$\tan^{-1} x$	Note that $\tan^{-1} x$ is defined for all x (why?), and so $\tan^{-1} x \neq \frac{\sin^{-1} x}{\cos^{-1} x}$.
x^2	$\sqrt{x}$	When x is positive.

The graph of the inverse of a function is the reflection of the graph of the original function around the line $x = y$ (essentially, we swap the x and y axes).

Theorem. *In general, if f is a function passing through (x, y), and f^{-1} is the inverse of f, then*

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(x)}.$$

Mnemonically, we can write this as

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}.$$

Proof. We have that $f(f^{-1}(y)) = y$. Taking the derivative of both sides, $f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$ and therefore $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$. □

This proof is not hard, but it is sometimes difficult to work out which x's and y's go where. We'll do a couple of examples now; the first one is one that we can already do, and so this gives us the advantage of knowing what the result should look like before we get there.

Example. Suppose f is defined by $y = f(x) = x^2$. Then $f^{-1}(y) = \sqrt{y}$. We evaluate it in three ways.

1. Power law: $f^{-1}(y) = y^{\frac{1}{2}}$ so $(f^{-1})(y) = \frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$.

2. Rigourous derivative of inverse: We have that $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$, so $(\sqrt{y})' = \frac{1}{2\sqrt{y}}$ (since $f'(x) = 2x$).

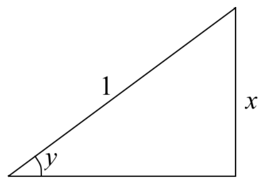
3. Mnemonic derivative of inverse: We wish to find $\frac{\mathrm{d}x}{\mathrm{d}y}$. Now $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ and so (by the mnemonic)

$$(f^{-1})'(y) = \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}.$$

Example. In order to illustrate the general process of finding the derivatives of inverse functions without symbol-pushing using the theorem above, let us now find the derivative of $y = \sin^{-1} x$.

$$\begin{aligned} y &= \sin^{-1} x \\ \sin y &= x \\ \frac{\mathrm{d}y}{\mathrm{d}x} \cos y &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos y} = \frac{1}{\cos \sin^{-1} x} = \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

The identity $\cos \sin^{-1} x = \sqrt{1 - x^2}$ comes from the following triangle:



By the same kind of calculation, we obtain the following table which gives the derivatives of inverses of the three primary trigonometric functions and their reciprocals.

Function	**Derivative**	**Function**	**Derivative**
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1 - x^2}}$	$\csc^{-1} x$	$-\dfrac{1}{x\sqrt{x^2 - 1}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1 - x^2}}$	$\sec^{-1} x$	$\dfrac{1}{x\sqrt{x^2 - 1}}$
$\tan^{-1} x$	$\dfrac{1}{1 + x^2}$	$\cot^{-1} x$	$-\dfrac{1}{x^2 + 1}$

Questions

1. Prove or disprove the following statements: $\boxed{\texttt{M}}$
   (a) The function $f : x \mapsto x^2 + x + 1$ is one-to-one (where x is real).
   (b) The function $g : x \mapsto 2^x$ is one-to-one (where x is a positive real).

2. Determine whether the following functions have inverses on the given interval: $\boxed{\texttt{M}}$
   (a) $x \mapsto x^3$ (on $\mathbb{R}$)
   (b) $y \mapsto y^4$ (on $\mathbb{R}$)
   (c) $y \mapsto y^4$ (for $y \geq 0$)
   (d) $y \mapsto y^4$ (for $y > 0$)
   (e) $\theta \mapsto \cos^{-1} \theta$ (on $\mathbb{R}$)
   (f) $\theta \mapsto \cos^{-1} \theta$ (for $-1 \leq \theta \leq 1$)

3. True or false: $\boxed{\texttt{A}}$
   (a) $\cos^{-1} x = \frac{1}{\cos x}$
   (b) If $x > 0$ then $(\ln x)^6 = 6 \ln x$
   (c) $\tan^{-1}(-1) = \frac{3\pi}{4}$ (think about which arm of $\tan x$ we're talking about)
   (d) The inverse of $f(x) = e^{3x}$ is $f^{-1}(x) = \frac{1}{3} \ln x$.

4. Find the derivative of $f(x) = \ln(e^x)$ in two different ways. $\boxed{\texttt{A}}$

5. Find y' if: A
   (a) $y = \sin^{-1} 2x$
   (b) $x = \sin^2 y$
   (c) $y = x + \tan^{-1} y$
   (d) $y = \ln \sin x - \frac{1}{2} \sin^2 x$
   (e) $y = 24 \arctan x + \arcsin \sqrt{x}$
   (f) $y = \sqrt{\sec^{-1} 2x}$

6. Find the local extrema, areas of concavity, and inflection points of the following functions; hence sketch their graphs. E
   (a) $y = e^x \sin x$ for $-\pi < x < \pi$
   (b) $y = x + \ln(x^2 + 1)$
   (c) $y = \sin^{-1}(1/x)$

7. Justify intuitively, without invoking the happy coincidence that our notation for derivatives looks like a fraction, the statement that $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$. E

8. If $f'(x) = \tan^{-1} x$, find $(f^{-1})'(x)$. E

9. Using the definition of ln as the inverse of $\exp : x \mapsto e^x$, show that $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$. *Note that the usual definition of* ln *goes in the reverse of this; we will go in that direction in a few weeks.* E

10. Prove the formulae for the derivatives of $\cos^{-1}$ and $\tan^{-1}$, using a similar method to that for $\sin^{-1} x$. S

11. Scholarship 2012: Consider the equation $x^n = \tan(ny)$, where n is a constant. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. S

12. Scholarship 2017: The functions sinh and cosh are defined as follows. S

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right),$$
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right).$$

The inverse function of sinh is denoted by $\sinh^{-1}$. By implicit differentiation, or otherwise, show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}.$$