NCEA Level 2 Mathematics
9. Exponential and Logarithmic Functions

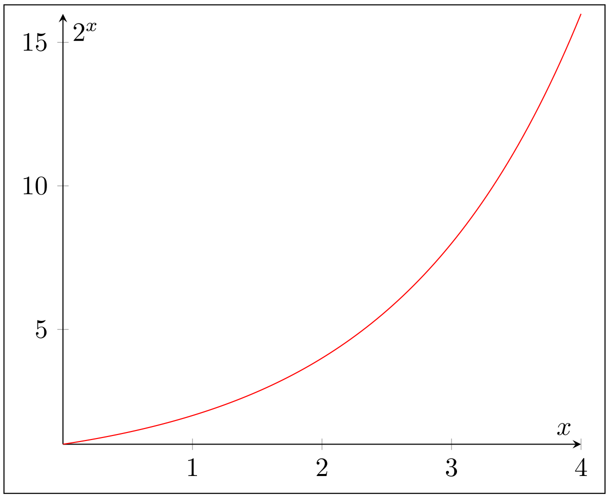
Exponentials

A particular species of bacteria reproduces by splitting in two every hour; if we start with one bacterium, after one hour we will have two; after two hourse, we will have four; after three hours, eight; and after n hours, we will have

$$2^n := \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}}$$

bacteria.

In general, equations of the form $y = a^x$ are called *exponential* equations.



Last year, we learned that exponents have the following properties:

1. $a^b \times a^c = \underbrace{a \times a \times \cdots \times a}_{b \text{ times}} \times \underbrace{a \times a \times \cdots \times a}_{c \text{ times}} = \underbrace{a \times a \times \cdots \times a}_{(b+c) \text{ times}} = a^{b+c}$.
2. $a^b \div a^c = \underbrace{a \times a \times \cdots \times a}_{b \text{ times}} \div \underbrace{a \div a \div \cdots \div a}_{c \text{ times}} = \underbrace{a \times a \times \cdots \times a}_{(b-c) \text{ times}} = a^{b-c}$.
3. $\left(a^b\right)^c = \underbrace{a^b \times a^b \times \cdots \times a^b}_{c \text{ times}} = \underbrace{a \times a \times \cdots \times a}_{bc \text{ times}} = a^{bc}$.
4. $a^1 = a$.
5. $a = a^1 = a^{0+1} = a^0 a^1 = a^0 a$, so $a^0 = 1$.

Note that we have some danger hiding in the background with these proofs: namely, if the powers are not whole numbers (or zero), they become meaningless! What does it mean to take 2 multiplied by itself π times? The solution, which we will look at briefly next week, is to define the function $x \mapsto a^x$ in a series of steps; we have already defined it when x is a natural number (or zero), and next week we will properly define it when x is an integer or rational number in general. Unfortunately, we won't have the necessary machinery to define it for any real number until next year.*

* For future reference, this is exercise 11 on the second L3 calculus worksheet.

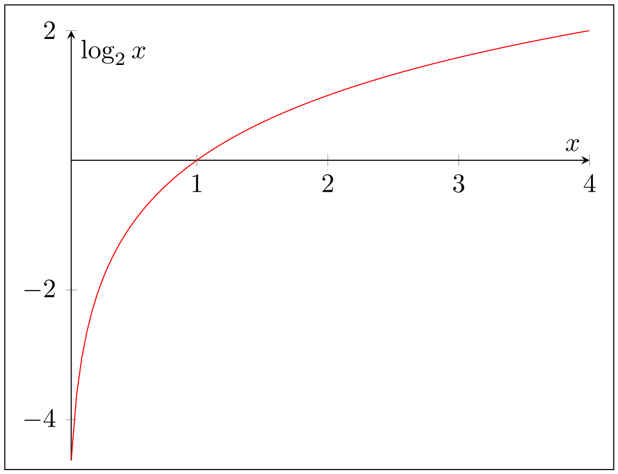
Logarithms

Suppose, on the other hand, we wish to know after how many hours we will have 1024 bacteria: we wish to find x such that $2^x = 1024$. This value is called the *logarithm* of 1024 with respect to 2, and we write $x = \log_2 1024$. In general, we have (as a definition),

$$y = a^x \iff x = \log_a y.$$

The quantity a is called the *base*, and is always positive.

Note that the function $x \mapsto \log_a x$ is the inverse of the function $x \mapsto a^x$.



If the base of a logarithm is 10, then we often don't write the base: so $\log 1000 = 3$, because $10^3 = 1000$.

The following logarithm laws can be derived from the exponent laws:

1. $\log_a x + \log_a y = \log_a xy$
2. $\log_a x - \log_a y = \log_a \frac{x}{y}$
3. $\log_a x^n = n \log_a x$
4. $\log_a 1 = 0$
5. $\log_a a = 1$
6. $\log_b x = \dfrac{\log_a x}{\log_a b}$ (change-of-base)

Example. Some elementary examples:

1. $\log_2 x = 10$ implies that $2^{10} = x$ and $x = 1024$.
2. $\log_x 49 = 2$ implies that $x^2 = 49$ and so $x = 7$.

Most applications of exponential and logarithmic equations outside of mathematics itself are to do with rates of change and rates of growth. This is because the rate of change of an exponential function is itself exponential, and so the exponential function will show up anywhere that a rate of change of a quantity is related directly to the amount of the quantity.

Example. A computer depreciates continuously in value from \$4699 to \$1500 over a period of 4.25 years. The value in dollars, y, of the computer t years after its value was \$4699 can be modelled by a function of the form

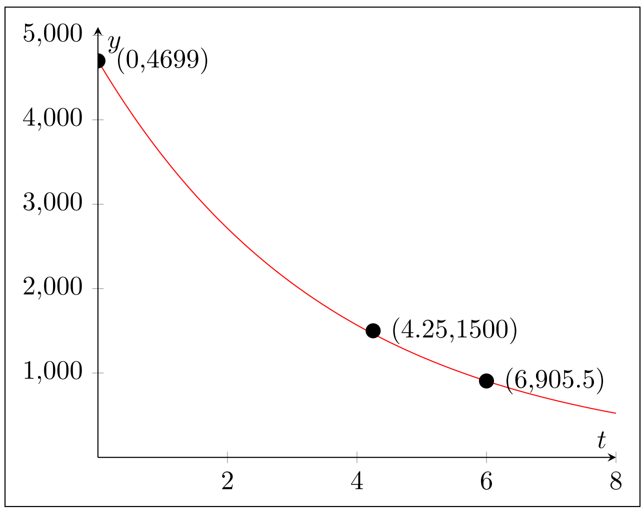
$$y = Ar^t,$$

where r is a constant. What is the value of the computer after six years?

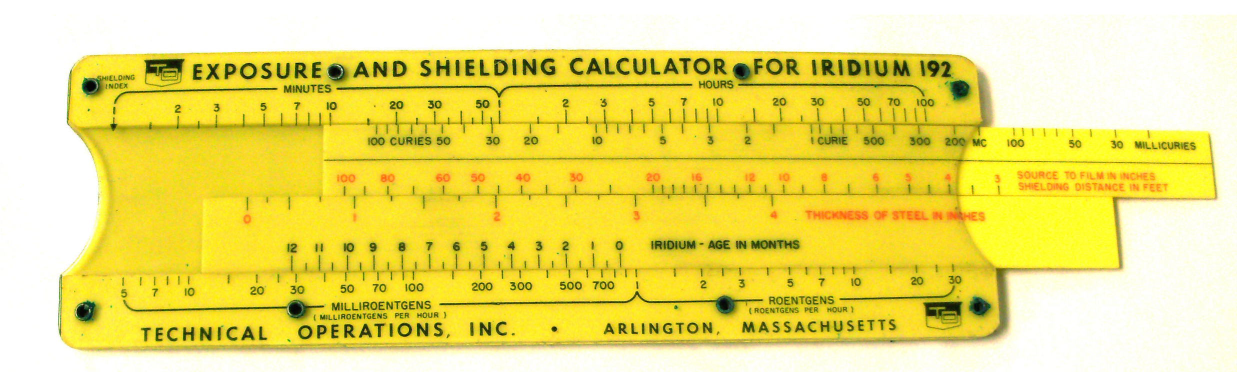
Solution. At $t = 0$, $y = 4699$ and so $4699 = Ar^0 = A$. On the other hand, we have $1500 = Ar^{4.25} = 4699 \cdot r^{4.25}$. Hence

$$\begin{aligned}
\frac{1500}{4699} &= r^{4.25} \\
\log \frac{1500}{4699} &= \log r^{4.25} \\
\log \frac{1500}{4699} &= 4.25 \log r \\
r = 10^{\left(\frac{1}{4.25} \log \frac{1500}{4699}\right)} &\approx 0.76.
\end{aligned}$$

Here, we used log base 10 because it happens to be on the calculator (we could have used any base); we then plugged the numbers into the calculator without worrying too much what powers that are fractions 'mean' (we'll discuss them next week). In the end, we found that a model for the value of the computer after t years is $y = 4699 \cdot 0.76^t$, and so after six years the computer is worth only around \$905.5.



Prior to the invention of the electronic calculator, mechanical devices called *slide rules* (see picture) were used by those who needed to make computations with large numbers. These devices consisted of two logarithmic scales next to each other, labelled with numbers; then the multiplication of a and b could be done by finding the lengths $\log a$ and $\log b$ on the slide rule, adding the two lengths together, and then using the rule $\log a + \log b = \log(ab)$ to read off the answer.



Joke. The water receded and the Ark came to rest upon the land. Noah opened the doors and commanded the animals, "Go forth and multiply." The animals slowly departed the Ark except for two snakes that remained in the back. Again Noah proclaimed ,"Go forth and multiply" yet the two snakes did not move. Noah walked to the back of the Ark and asked, "Why have you not followed my command?". The snakes answered, "Noah, we cannot, because we are adders."

Noah then went out upon the land and felled several large trees; from these trees he made a four legged platform. He then went inside the Ark and carried the snakes outside and upon placing them on the platform, his words became true.

As everyone knows, adders can multiply using log tables.†

Questions

1. Intuitively justify the following statements.
   (a) Multiplication ($n \times x$) is repeated addition ($+n$).
   (b) Exponentiation (n^x) is repeated multiplication ($\times n$).
   (c) Division ($x \div n$) counts 'how many' ($+n$)s.
   (d) Logarithms ($\log_n x$) count 'how many' ($\times n$)s.

2. This question is a list of mechanical exercises. It is important to be fluent with the mechanical use of exponentials and logarithms; however, in order to get anything more than a low achieved it is not enough to just focus on this kind of problem. In particular, if you plan to continue with any kind of mathematical subject next year (calculus, statistics, physics, chemistry) then you *must* be doing a significant number of other problems.
   (a) Evaluate $\log_2 32$ and $\log_3 1/9$.
   (b) Write $2\log 3 - 3\log 2$ as the log of a single number.
   (c) Solve $8^{x+1} = 4^{2x-5}$ for x.
   (d) Write $\log_2 \sqrt[3]{\frac{\sqrt[3]{15}\,5^4}{3^3\,\sqrt[3]{9}}}$ as a single number.
   (e) Simplify $\frac{4\log u^3}{\log u}$.
   (f) Solve $2\log x = \log 16$ for x.
   (g) Solve $\log_{x-1}(4x-4) = 2$ for x.
   (h) Express $\log \dfrac{U^3V^2}{W^5}$ as an algebraic sum of logarithms.
   (i) Solve $4^{2x-1} = 5^{x+2}$ for x.
   (j) Solve for x if $210 = (10^x)^3$.
   (k) Find x if $a^x = 5^{x-1}$ (where a is some constant).
   (l) Solve $\log x = 2\log mx$ for x in terms of m.
   (m) If a sequence of numbers is given by $a_n = 2^n + 3$, show that the difference between the $(n-1)$th and nth terms of the sequence is $a_n - 3$.

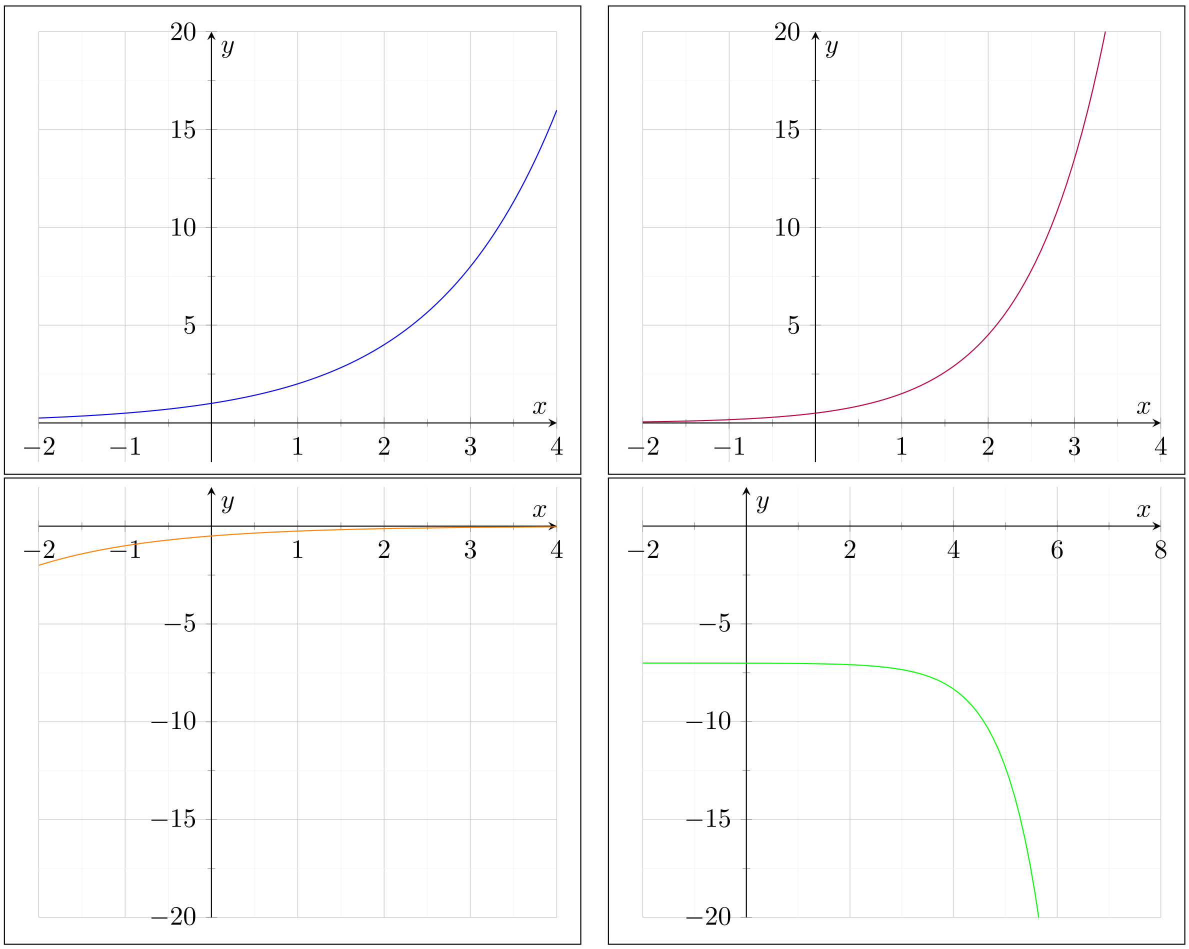
3. (a) Draw the graph of $y = 2^x$ and the graph of $y = 3^x$ on the same piece of graph paper and compare them. What about $y = 4^x$? Can you explain any patterns you see in terms of the exponent laws above?
   (b) Draw the graph of $y = \log_2(x)$ and the graph of $y = \log_3(x)$ on the same piece of graph paper and compare them.
   (c) Draw the graph of $y = \log_2(x)/\log_2(3)$ and the graph of $y = \log_3(x)$ on the same piece of graph paper and compare them. [Look at the final logarithm law above.]

† Attribution: `https://mathoverflow.net/a/1946`, although variations of this joke are widespread and numerous.

(d) Draw the graph of $y = \log_3(x)$, the graph of $y = \log_4(x)$, and the graph of $y = \log_5(x)$ on the same piece of graph paper. Are there any intersection points? Can you explain any that occur in terms of the logarithm laws?

(e) Draw the graph of $y = \log_5(x+2)$, the graph of $y = \log_5(x)$, and the graph of $y = \log_5(x-2)$ on the same piece of graph paper. Are there any intersection points? Can you explain any that occur in terms of the logarithm laws?

4. Give the equations of the following exponential graphs:



5. How many words with 4 letters of a 26 letter alphabet are possible?

6. Solve for y, if $\log_2(y^{-6}) = (\log_2 y)^2 + 8$.

7. Find all values of x satisfying $6(\log_8 x)^2 + 2\log_8 x - 4 = 0$.

8. Luka says that the equation $\log_x(4x + 12) = 2$ has only one solution. Is he correct?

9. If the formula $P = A(0.75)^t$ models the amount P of a drug (in milligrams) in the bloodstream t hours after it is ingested, and the initial amount ingested is 500 mg, how long does it take for the amount of drug in the bloodstream to reduce by half?

10. The graph of $y = a + b\log x + c(\log x)^2$ passes through $(1, 0)$, $(10, 7)$, and $(100, 13)$. What is the value of y when x is $1/10$?

11. (a) A function f is defined by the following rule:

$$f(x) = \begin{cases} 20 & 0 \leq x \leq 5 \\ ax^2 - 10ax + (20 + 25a) & x > 5. \end{cases}$$

   i. What is the domain of f?
   ii. If $f(9) = 52$, what is the value of a?
   iii. What is the range of f?

(b) A function g is defined by the following rule:

$$g(x) = b + c \log_3 x.$$

   i. Choose b and c so that both the following are true:
      - The graphs of f and g meet at the point $(9, 52)$, and
      - $g(81) = 100$.
   ii. Find all the points (x, y) that lie on the graphs of both f and g.

12. Many population models are exponential.

Year	**World population (billion)**
1804	1
1927	2
1963	3
1974	4
1987	5
1999	6
2011	7

(a) Assume the world population in the year t (CE) can be modelled with an exponential equation of the form

$$P = P_0 r^{ct}.$$

Find P_0, r, and c using the data from 1804, 1927, and 1963 (the three earliest years given above).

(b) Write another model, using the three *latest* years above. Compare the two models.

(c) Using the second model, calculate a projected terrestrial population in 2024 and in 2100.

(d) How accurate do you think an exponential model will be in the long term?

13. Radioactive substances slowly decompose into lead. The rate at which the decomposition occurs is usually measured in terms of the *half-life*: the time taken for an amount of the substance to decay by a factor of $\frac{1}{2}$. It can be shown (using calculus) that if A_0 is the amount of material initially present, then the material present after t years can be modelled by

$$A(t) = A_0(2.72)^{-kt}$$

where k is a constant.

(a) Show that $k = \frac{\log_{2.72} 2}{\tau}$, where τ is the half-life in years.

(b) Radium has a half-life of 1590 yr. How long will it take for six grams of radium to decay to one gram?

(c) A new radioactive substance decays from 1 gram to 0.98 grammes in one year. What is the half-life of the substance?

(d) One of the most useful applications of this model is in dating ancient materials. Because of the action of cosmic rays in the atmosphere, there is always a certain percentage of a particular radioactive form of carbon in the atmosphere (namely, carbon-14). Since living matter is always interchanging material with the environment, the living matter also contains the same percentage of carbon-14. Upon death, the exchange with the environment ceases and so no new carbon-14 enters cells; the remaining carbon-14 continues to decay, and the amount of remaining carbon-14 can be detected using instrumentation. The model can then be used to estimate the rough date of death.

The half-life of carbon-14 is 5570 yr.

In a sample taken from the bone structure of a recently discovered mummy, it is found that only 1/10 of the original amount of carbon-14 remains. How old is the mummy?

14. Prove the logarithm laws, using the exponent laws. For example,

$$\log_a x + \log_a y = \log_a(a^{\log_a x + \log_a y}) = \log_a(a^{\log_a x} a^{\log_a y}) = \log_a xy = \log_a(a^{\log_a xy}) = \log_a xy.$$

15. When a bank loans money, the interest is *compounded*; that is, you earn interest on the interest you have already been charged.

(a) Show that adding $x\%$ to a debt is equivalent to multiplying the debt by $(1 + \frac{x}{100})$.

(b) Suppose interest is calculated after each year (that is to say, it is compounded annually). If the initial debt is \$100, and the annual compound interest is 20%, what is the amount owed by the end of the first three years?

(c) Now, to simplify matters, we will say that our initial loan is \$1, and the interest rate is \$100 per annum. After one year, the value of the loan will double to \$2.

i. If the bank calculates compound interest every six months instead of every year (so our interest is 50% per six months), show that we owe an extra \$0.25 after one year.

ii. If the bank compounds interest every month, show that our total owed is now \$2.6130 after one year.

iii. How much will we owe if the bank compounds interest every day?

iv. In general, show that if we divide the annual percentage rate by n and compound it n times then the end-of-year balance of the loan is $(1 + \frac{1}{n})^n$.

(d) From your working in part (c) above, we can conclude that as n increases (that is, if the bank compounds interest with shorter and shorter time intervals) then the total owed after one year climbs closer and closer to \$2.7182818.... This number, which is fundamental in mathematics, is known as Euler's constant, or e. Show that if a bank compounds interest continuously, with an interest rate of $x\%$ per annum and initial loan L, then after t years the total owed is

$$L \times e^{xt/100}.$$

Leonhard Euler (1707–1783) was a Swiss mathematician who made a vast number of contributions to mathematics: the Wikipedia page *List of things named after Leonhard Euler* contains over one hundred entries. We will meet the number e in particular a few more times this year, and we will also see a number of different problems associated with Euler.