NCEA Level 3 Calculus (Differentiation)
13. Sequences and Series

> **Goal for this week**
>
> To revise last year's material on sequences, especially how they behave 'at infinity'.

Sequences

So far, we have looked at functions that are continuous. This week, we take a short break from this and look at sequences and series. A sequence is, intuitively, a list of numbers written in a definite order:

$$(a_n) = (a_1, a_2, \ldots, a_i, \ldots).$$

Note that we index sequences starting at 1, unless stated otherwise. Theoretical computer scientists will often start at 0 instead.

Some examples of sequences include

$$(n) = (1, 2, 3, \ldots) \tag{1}$$

$$\left(\frac{n-1}{n}\right) = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right) \tag{2}$$

$$\left(\cos\frac{n\pi}{6}\right) = \left(1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots\right) \tag{3}$$

$$(1 - 0.2^n) = (0.8, 0.96, 0.992, 0.9984, \ldots) \tag{4}$$

$$\left(\frac{n^5}{n!}\right) = (0, 1, 16, 40.5, 42.6, 26.04, 10.8, \ldots) \tag{5}$$

$$(a_n) = (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots) \tag{6}$$

Clearly some of these sequences grow forever, while others settle down and become closer and closer to a particular value.

Definition. A sequence (a_n) has a limit L, written $\lim_{n\to\infty} a_n = L$, if we can make the terms a_i stay as close as we like to L by taking i sufficiently large. If (a_n) has a limit, then it is called **convergent**; otherwise, it is called **divergent**.

Examples (2) and (4) above are clearly convergent (they both converge to 1); example (5) is convergent to 0, even though it is initially increasing (because factorials grow faster than powers eventually). The others are all divergent:

(1) is clearly divergent because as $n \to \infty$, $a_n \to \infty$.

(3) is divergent because there is no number L such that a_i stays close to L as we increase i (it always jumps around).

(6) is divergent for the same reason as (3).

Some sequences can be written recursively.

$$a_1 = 1, a_{n+1} = 5a_n - 3 \quad (1, 2, 7, \ldots)$$

$$a_1 = 2, a_{n+1} = \frac{a_n}{1 + a_n} \quad \left(2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots\right)$$

Example. The **Fibonacci** sequence is defined by

$$(F_i) = \begin{cases} 1 & \text{if } i = 1 \text{ or } i = 2, \\ F_{i-1} + F_{i-2} & \text{otherwise.} \end{cases}$$

The first few elements of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21.

There is actually a closed-form formula for the Fibonacci sequence:

$$F_i = \frac{\left(1 + \sqrt{5}\right)^n - \left(1 - \sqrt{5}\right)^n}{2^n \sqrt{5}}.$$

More information about this sequence is given in the reading for the homework.

Series

From sequences, we move on to infinite sums (also known as series). Given some sequence (a_n), we have a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

Often it does not make sense to talk about the sum of infinitely many terms; obviously $1 + 2 + 3 + \cdots$ will increase forever. On the other hand, if we add the first n terms of

$$\left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots \right)$$

we obtain $1 - 1/2^n$, which converges to 1.

This suggests that we can consider the partial sums of a sequence (a_n)

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$s_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_n$$

and then say that the sum $\sum a_n$ exists if the sequence (s_n) converges.

An important class of series is the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

where $a \neq 0$ and r are constants. Clearly if $|r| \geq 1$ then the sum becomes $a + a + \cdots$ which converges; if $|r| < 1$, then we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$rs_n = \quad ar + ar^2 + ar^3 + \cdots + ar^n$$
$$s_n - rs_n = a - ar^n$$
$$s_n = \frac{a - ar^n}{1 - r}.$$

which converges to $a/(1 - r)$ as $n \to \infty$; so the sum exists.

Questions

1. Which of the following sequences are convergent (write the limit), and which are divergent (give a reason)? `M`

    (a) $((-1)^n)$

    (b) $(e^{1/n})$

    (c) $(n^2 e^{-n})$

    (d) $(3^n/(1+2^n))$

    (e) $a_1 = 6$, $a_{n+1} = a_n/n$

    (f) $(\cos 2\pi/n)$

    (g) $(\sin n!/n^n)$

    (h) $(\sin n^n/n!)$

2. Find the limit of the sequence $(\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots)$. `E`

3. Find the sum of the geometric series $1 + 1/2 + 1/4 + \cdots$. `A`

4. Find the sum of the geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$. `M`

5. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum. [Hint: calculate $\frac{1}{i} - \frac{1}{1+i}$.] `E`

6. Determine whether $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ is convergent or divergent. If it is convergent, find its sum. `E`

7. Consider the harmonic series `E`

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

    (a) Let s_n be the sequence of partial sums. Show that $s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{2}{2}$ [Hint: $1/3 > 1/4$.]

    (b) Show that $s_8 > 1 + \frac{3}{2}$.

    (c) Prove that $s_{2^n} \geq 1 + \frac{n}{2}$ in general. Conclude that the harmonic series diverges.

8. Find c if $\sum_{n=1}^{\infty} (1+c)^{-n} = 2$. `E`

9. A power series is a series of the form `S`

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots .$$

    You can think of a power series as a polynomial with infinitely many terms. It can be shown that the normal sum rule for differentiation holds, so that $\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} c_n x^n$.

    (a) Find the derivative of the Bessel function, $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$.

    (b) If $|x| < 1$, show that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

    (c) Find a power series representation for $f(x) = 1/(1-x)^2$. [Hint: differentiate the equation in (b).]

    (d) Find a power series representation for $g(x) = \ln(1+x)$. [Hint: differentiate g, use (c), and antidifferentiate.]

    (e) When is the power series representation of g valid?

10. The Cantor set is constructed as follows. Start with the closed interval $[0,1]$ and remove the open interval $(\frac{1}{3}, \frac{2}{3})$. This leaves the two closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$; remove the open middle third of each of these. Four intervals remain; remove the open middle third of each. Continue this process indefinitely, at each step removing the open middle third of each interval remaining from the previous step. The Cantor set is the set of numbers remaining from the interval $[0,1]$ after this process has been carried out.

(a) Show that the total length of all the removed intervals is 1.

(b) Despite this, there are still infinitely many numbers remaining. Exhibit some members of the Cantor set.

(c) The Sierpinski carpet is a two-dimensional version of the Cantor set; remove the centre one-ninth of a square of side 1, then the centres of the eight remaining squares, and so on. Show that the sum of the areas of the removed squares approaches 1. Hence the Sierpinski carpet has area zero.
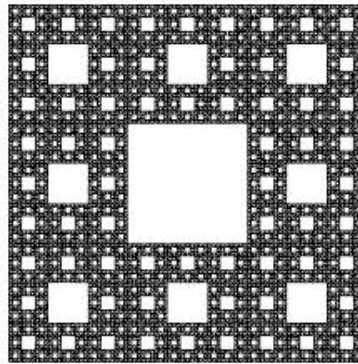
11. Scholarship 2017: The sequence (a_n) is defined as follows:-

$$\begin{cases} (a_1) = 2, \\ (a_2) = 7, \\ (a_{n+1}) = \frac{1}{2}(a_n + a_{n+1}) \quad \text{for } n \geq 2. \end{cases}$$

(a) Find an exact formula for the nth term of the sequence.

(b) What is the limit of a_n as $n \to +\infty$?

12. Consider the following limit (you may assume that it exists):

$$L = 0.999... = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{9}{10^i}$$

   (a) Write $L = 0.9 + 0.09 + \cdots$, and make a conjecture about the value of L.
   (b) Prove or disprove your conjecture from (a).

13. Zeno was a Greek philosopher active in the 5th century BCE. He presented a list of 'paradoxes', or
    apparent contradictions, including the following (adapted from Wikipedia):

   • Suppose Achilles is in a foot race with a tortoise. Achilles runs much faster than the tortoise, but
     the latter has a head start. By the time Achilles reaches the location that the tortoise started, the
     tortoise will have moved a small amount further on; similarly, by the time Achilles reaches the new
     location of the tortoise, it will have moved an even smaller distance further on; and by this reasoning
     it follows that Achilles can never overtake the tortoise.
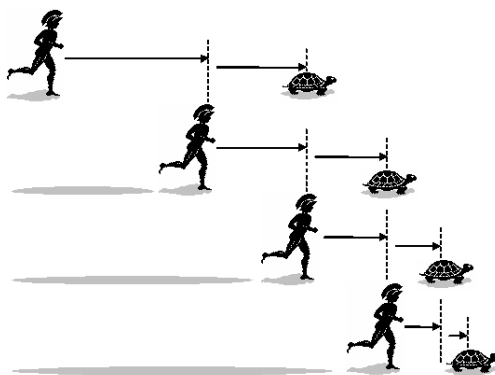
   • Suppose Homer wishes to walk to the end of a path. Before he can get there, he must get halfway
     there. Before he can get halfway there, he must get a quarter of the way there. Before traveling a
     quarter, he must travel one-eighth; before an eighth, one-sixteenth; and so on. So Homer cannot
     walk to the end of the path.

   • For motion to occur, an object must change the position which it occupies. Consider an example of
     an arrow in flight. In any one (duration-less) instant of time, the arrow is neither moving to where
     it is, nor to where it is not. It cannot move to where it is not, because no time elapses for it to move
     there; it cannot move to where it is, because it is already there. In other words, at every instant of
     time there is no motion occurring. If everything is motionless at every instant, and time is entirely
     composed of instants, then motion is impossible.

   Use your knowledge of limits to explain these apparent contradictions.