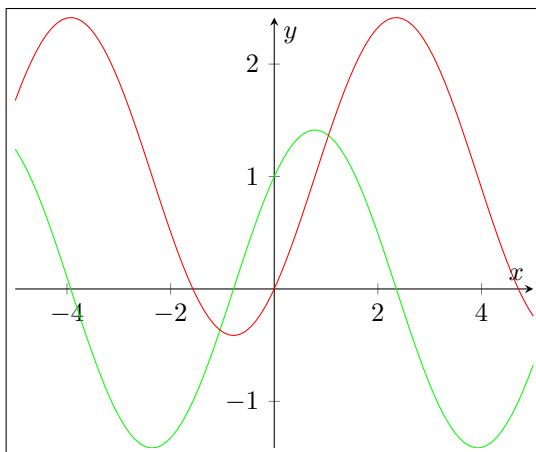


# NCEA Level 3 Calculus

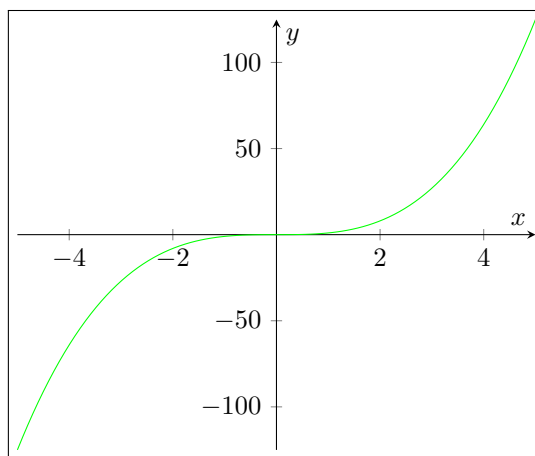
## Solutions to Homeworks

### 1. The Derivative

1. Green: derivative of function. Red: original function.

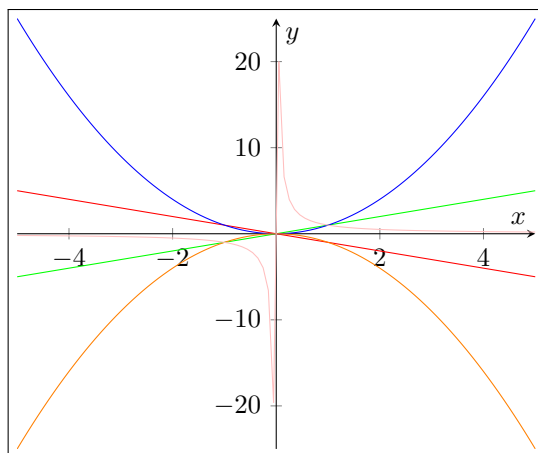


2. (a) At a min or max, the function is momentarily horizontal and so has slope zero; so  $m = 0$ .  
(b) Consider a graph like the following at  $(0, 0)$ :

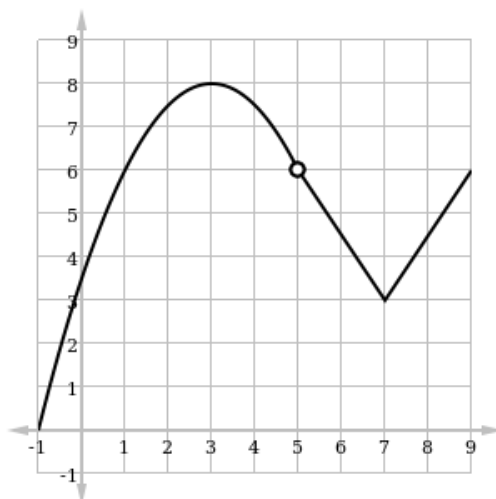


### 2. Limits

1. Increasing: the graph of the function is sloping up (green). Decreasing: the graph of the function is sloping down (red). Concave up: the graph of the function is increasing in slope (it is like a cup  $\cup$ ) (blue). Concave down: the graph of the function is decreasing in slope (it is like a cap  $\cap$ ) (orange). Continuous: the function has no holes (all of them except pink).



2. (a)  $\lim_{x \rightarrow -2} f(x) = 0$ ,  $\lim_{x \rightarrow 2} f(x) = -0.5$ .  
 (b) No, it approaches different values from the left and the right.  
 (c) Yes, because the function is continuous there.  
 (d)  $(-\infty, -3)$ ,  $(-3, -1)$ ,  $(-1, 2)$ ,  $(2, \infty)$ .  
 (e) -3, -1, 2.
3. For example,



### 3. Derivatives of Common Functions

1. (a)  $2x + 1/x$   
 (b)  $t^2 x^{t-1}$   
 (c)  $\cos x + \sin x$   
 (d)  $\frac{4}{5} x^{-1/5} = \frac{4}{5 \sqrt[5]{x}}$
2. The power  $x$  is not constant.
3. Consider  $x^{-n}$ . The first derivative is  $-n x^{-n-1}$ , the second is  $n(n+1) x^{-n-2}$ , and so the  $n$ th is  $(-1)^n n(n+1) \cdots (2n-1)(2n) x^{-2n} = (-1)^n \frac{(2n)!}{(n-1)!} x^{-2n}$ . [This can be proved via induction.]

4. (a) Note first that  $10^t = e^{t \ln 10}$ , so  $P = P_0 + e^{t \ln 10}$  and  $\frac{dP}{dt} = (\ln 10)e^{t \ln 10} = (\ln 10)10^t$ . At  $t = 100$ , we have  $\frac{dP}{dt} = 2.3 \times 10^{100}$ .
- (b) Real-world populations don't grow exponentially forever if there are finite resources (e.g. food).

#### 4. The Chain Rule

1.  $\frac{dy}{dx} = \frac{-\csc^2 x}{2\sqrt{\cot x}}$
2. (a) Simply apply the chain rule twice.  
 (b)  $y' = 5x^4(\cos x^5)(-\sin \sin x^5)(\cos \cos \sin x^5)(-\sin \sin \cos \sin x^5)(\cos \cos \sin \cos \sin x^5)$ .
3. (a)  $f'(\theta) = -2 \sin 2\theta$  and  $g'(\theta) = -4 \sin \theta \cos \theta = -2 \sin 2\theta$ , so  $f' = g'$  as they agree everywhere.  
 (b) Since  $f$  and  $g$  have the same derivative, they differ only by a constant. But  $f(0) = 1 = g(0)$ , so that constant is zero; hence  $f = g$ .

#### 5. The Product and Quotient Rules

1. (a)  $\cos x \ln x + \frac{\sin x}{x}$   
 (b)  $\sec kx + kx \sec kx \tan kx$   
 (c)  $\frac{-\pi(\sin \pi\theta + \cos \pi\theta) \sin \pi\theta - \pi(\cos \pi\theta - \sin \pi\theta) \cos \pi\theta}{(\sin \pi\theta + \cos \pi\theta)^2}$   
 (d)  $(\cos t)(3 \sin^2 t)(-\sin(\sin^3 t))(4 \cos^3 \sin^3 t)$ .

2.

$$\begin{aligned}
 F &= \frac{d}{dt} \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = a \frac{d}{dv} \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \\
 &= a \left( \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{m_0 v^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \right) \\
 &= a \left( \frac{m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} + \frac{m_0 v^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \right) \\
 &= m_0 a \left( \frac{c^2 \left(1 - \frac{v^2}{c^2}\right) + v^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \right) \\
 &= m_0 a \left( \frac{c^2 - v^2 + v^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \right) \\
 &= \frac{m_0 a}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}.
 \end{aligned}$$

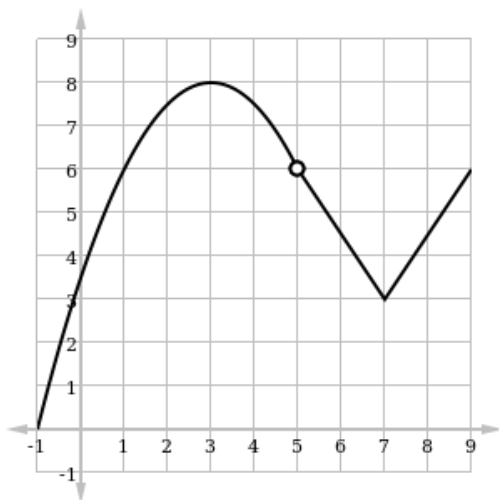
3. We wish to find  $\frac{d}{d\theta} \sin \text{rad}(\theta)$ , where  $\text{rad}(\theta) = \frac{\pi\theta}{180}$ ; so  $\frac{d(\text{rad})}{d\theta} = \frac{\pi}{180}$  and  $\frac{d}{d\theta} \sin \text{rad}(\theta) = \frac{\pi\theta}{180} \cos \text{rad}(\theta)$ . [The reason we have to do this is that the derivative of  $\sin$  uses the limit  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  which is false if  $x$  is in degrees.]

## 6. Tangent and Normal Lines

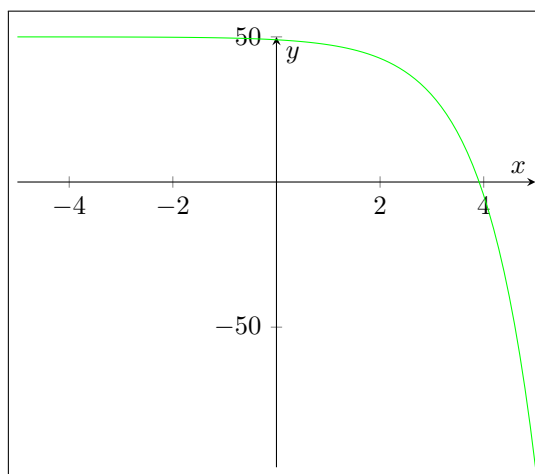
1. The normal to a curve  $f$  at a point  $(x_0, f(x_0))$  is the unique line passing through that point that is perpendicular to the tangent line of  $f$  at that point.
2.  $y' = \frac{-\sin(x+\pi)}{2\sqrt{\cos(x+\pi)}} + \cos x - 4 \sec x \tan^2 x e^{2 \tan^2 x}$ ; at  $x = \pi$ ,  $y' = -1$  and so the tangent line (best linear approximation) is  $y = -(x - \pi) = \pi - x$ .
3. Since the normal line has slope 3, the tangent line has slope  $-1/3$ . We can take any curve through  $(1, 0)$  with this slope, so we may as well take the tangent line itself:  $y = -\frac{1}{3}(x - 1) = \frac{1}{3} - \frac{1}{3}x$ .
4.  $\frac{dy}{dx} = \frac{1}{(1+3x)^{2/3}}$  and at  $x = 0$  the slope becomes 1. So the best linear approximation around the point  $(0, 1)$  is just  $\tilde{y} = x + 1$ . So at  $x = 0.01$ , we have  $\tilde{y} = 1.01$  as our approximate value of  $\sqrt[3]{1.03}$ . [The true value is around 1.0099, so we are not too far off.]

## 7. Higher Derivatives and the Geometry of a Function

- The second derivative tells us the concavity of a function: if the second derivative is positive, the function is curving up and if it is negative then the function is curving down.
- (a)  $f'(x) = 5x^4 - 5$ ,  $f''(x) = 20x^3$ .
  - (b)  $f'(x) = \frac{x^2-2x}{(x-1)^2}$ ,  $f''(x) = \frac{2x-2}{(x-1)^4}$ .
  - (c)  $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{4}x^{-3/4}$ ,  $f''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{16}x^{-5/4} = \frac{3}{16\sqrt[4]{x^5}} - \frac{1}{4\sqrt{x^3}}$ .
- (a) For example,



- (b) For example,



## 8. Optimisation

- At some point  $x$ , the distance between the two parabolae is  $\delta(x) = (x^2 + 1) - (x - x^2) = 2x^2 - x + 1$ . Taking the derivative, we find  $\delta'(x) = 4x - 1$  which has a single zero at  $x = 1/4$ ; by looking at the graph of the two parabolae, we see that this must be the location of the minimum distance  $\delta(1/4) = 7/8$  units.
- If  $y = 3x + 2 \cos x + 5$ , then  $\frac{dy}{dx} = 3 - 2 \sin x$ . Since  $1 \geq \sin x$ ,  $3 - 2 \sin x \geq 1$ . In particular, the function is everywhere increasing. Now, note that when  $x = -200\pi$ ,  $y = -600\pi + 7 < 0$ , and when  $x = 200\pi$ ,

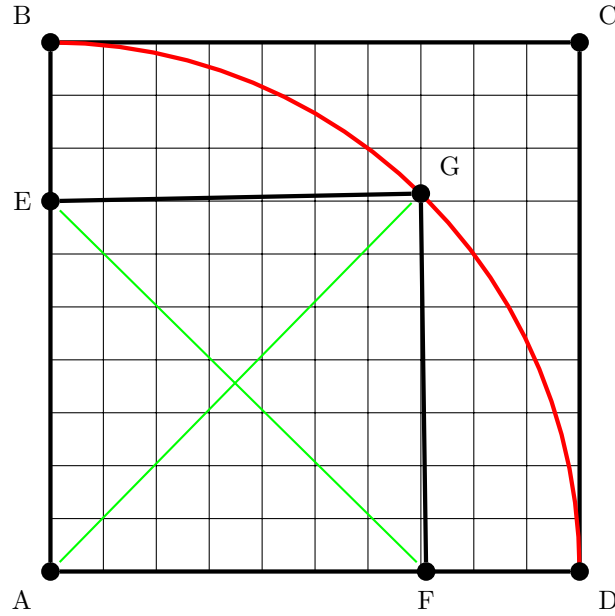
$y = 600\pi + 7 > 0$ . Since the function is continuous over this interval, it follows that at some point it passes through the  $y$ -axis and has at least one root; since it is increasing everywhere, it must have exactly one real root.

3. The area of such a rectangle will be  $A = 4xb\sqrt{1 - \frac{x^2}{a^2}}$ ; so

$$\frac{dA}{dx} = 4b\sqrt{1 - \frac{x^2}{a^2}} - \frac{4x^2b}{a^2\sqrt{1 - \frac{x^2}{a^2}}}.$$

Setting this to zero, we have  $a^2 = 2x^2$  and so  $2x = \sqrt{2}a$ . It follows that  $2y = b\sqrt{2}$ , and so the maximal area is  $2ab$ .

4. Consider the following diagram.



It should be clear that  $AG = 1$ ; call  $\angle AEG = \theta$  and  $\angle AFG = \phi$ , and let  $AE = EG = e$  and  $AF = GF = f$ . By the cosine rule, we have  $1 = 2e^2(1 - \cos \theta)$  and  $1 = 2f^2(1 - \cos \phi)$ . Now, the area of the triangle  $\triangle AEG$  is given by  $\frac{1}{2}\sqrt{e^2 - \frac{1}{4}}$ ; the area of  $\triangle AFG$  is given by  $\frac{1}{2}\sqrt{f^2 - \frac{1}{4}}$ . Since  $AEFG$  is a (convex) quadrilateral with two right angles,  $\theta + \phi = \pi$ . Putting this all together, the area of the quadrilateral is  $A = \frac{1}{2}\sqrt{e^2 - \frac{1}{4}} + \frac{1}{2}\sqrt{f^2 - \frac{1}{4}}$ . We have that  $e^2 = \frac{1}{2(1 - \cos \theta)}$  and  $f^2 = \frac{1}{2(1 - \cos(\pi - \theta))} = \frac{1}{2(1 + \cos(\theta))}$ , so the area in terms of  $\theta$  is

$$A = \frac{1}{2}\sqrt{\frac{1}{2(1 - \cos \theta)} - \frac{1}{4}} + \frac{1}{2}\sqrt{\frac{1}{2(1 + \cos \theta)} - \frac{1}{4}}.$$

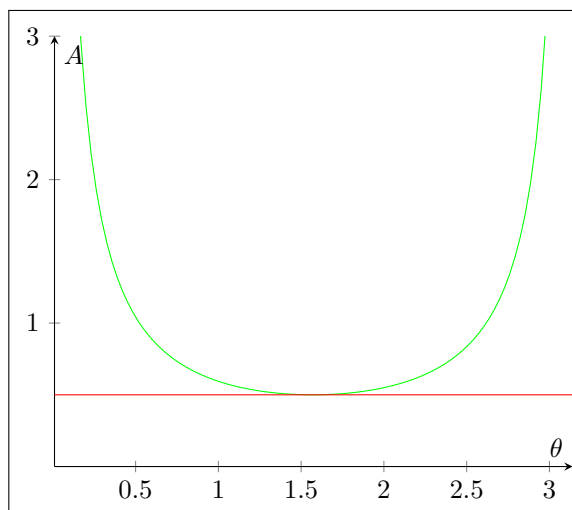
Taking the derivative, we obtain

$$\frac{dA}{d\theta} = \frac{1}{2} \frac{\sin \theta}{4(\cos \theta + 1)^2 \sqrt{\frac{1}{2(1 + \cos \theta)} - \frac{1}{4}}} - \frac{1}{2} \frac{\sin \theta}{4(1 - \cos \theta)^2 \sqrt{\frac{1}{2(1 - \cos \theta)} - \frac{1}{4}}};$$

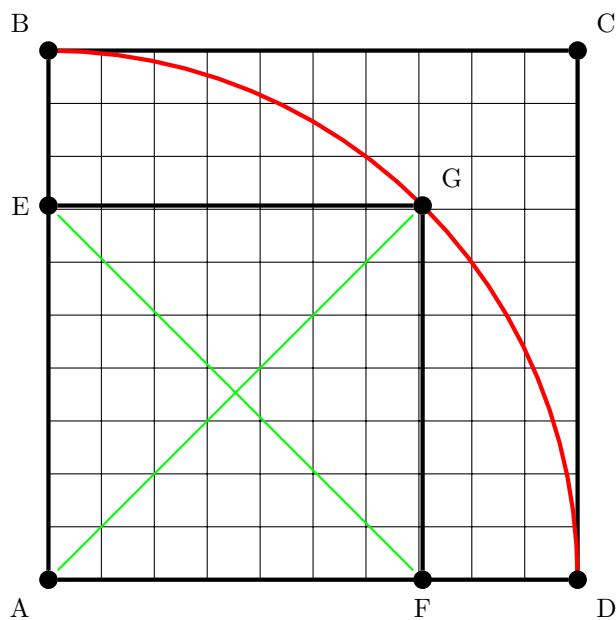
Now we set this to zero. We know that  $0 < \theta < \pi$ , so  $\sin \theta \neq 0$  and hence

$$\begin{aligned}
 4(\cos \theta + 1)^2 \sqrt{\frac{1}{2(1 + \cos \theta)} - \frac{1}{4}} &= 4(1 - \cos \theta)^2 \sqrt{\frac{1}{2(1 - \cos \theta)} - \frac{1}{4}} \\
 \sqrt{\frac{1}{2} - \frac{\cos \theta + 1}{4}} &= \sqrt{\frac{1}{2} - \frac{1 - \cos \theta}{4}} \\
 \cos \theta &= -\cos \theta \\
 \cos \theta &= 0
 \end{aligned}$$

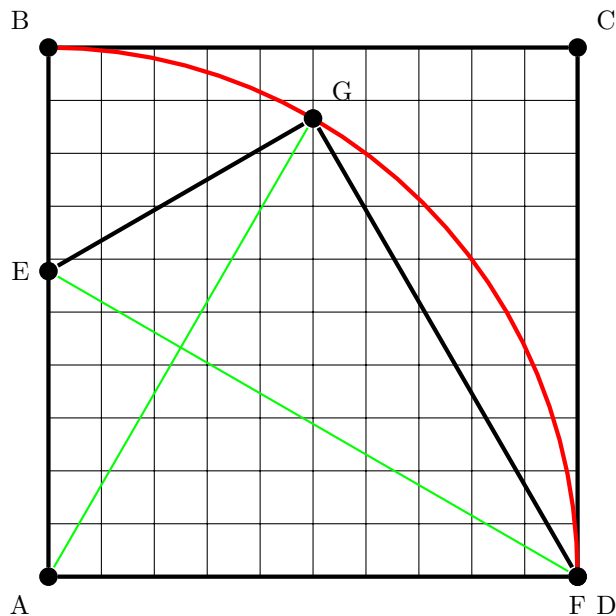
Hence  $\theta = \phi = \pi/2$ . Immediate calculation shows that  $e = f = \frac{1}{\sqrt{2}}$ ; we thus have a square with side length  $1/\sqrt{2}$ , and area  $\frac{1}{2}$ . Is this a maximum or a minimum? We cheat by graphing the area versus  $\theta$ :



so we obviously have the minimum area:



Note that  $\theta \geq \pi/2$ , because otherwise  $e > 1$ . Suppose we take  $\theta = 2\pi/3$ ; here is the graphed figure (with area 1.1547):



This is the maximum area, since if we increase  $\theta$  any more it requires  $f > 1$ .

## 9. Implicit Differentiation

- (a)  $y' = \frac{3x^2+6x}{2y}$ .
  - (b)  $(1+y') \sin(x+y) = 2 - 2y' \implies y' = \frac{2-\sin(x+y)}{2+\sin(x+y)}$ .
  - (c)  $y' = \frac{20x^3-2x}{2y}$ .
- $2x + 2y + 2xy' - 2yy' + 1 = 0 \implies y' = \frac{-1-2x-2y}{2x-2y}$ , so  $y'(1, 2) = \frac{-1-2-4}{2-4} = 7/2$ ; hence the slope of the normal is  $-2/7$ , and the equation of the normal line is  $y - 2 = -\frac{2}{7}(x - 1)$ .
- We have  $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0$ ; suppose we have a tangent line passing through  $(x_0, (\sqrt{c} - \sqrt{x_0})^2)$ . Then the equation of this tangent is  $y - (\sqrt{c} - \sqrt{x_0})^2 = -\frac{\sqrt{c}-\sqrt{x_0}}{\sqrt{x_0}}(x - x_0)$ . When  $y = 0$  we obtain the  $x$ -intercept;  $0 = (\sqrt{c} - \sqrt{x_0})^2 - \frac{\sqrt{c}-\sqrt{x_0}}{\sqrt{x_0}}(x - x_0)$  and so  $x = \sqrt{x_0}c$ . Similarly, when  $x = 0$  we obtain  $y = \sqrt{x_0}c - x_0$ . Their sum is therefore  $2\sqrt{x_0}c - x_0 = 2\sqrt{c}(\sqrt{c} - \sqrt{y_0}) - (\sqrt{c} - \sqrt{y_0})^2 = c$ .

## 10. Inverse Functions

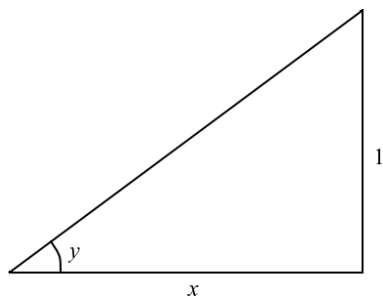
- (a)  $y' = \frac{2x}{1+x^4}$ .
  - (b)  $f'(x) = \frac{1}{1+x^2}$ .
  - (c)  $g'(x) = \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{1-x}} \frac{1}{1+(\sin^{-1} \sqrt{x})^2}$ .



2.

$$\begin{aligned}
 \frac{d}{dx} \left( \frac{1}{2} \tan^{-1} x + \frac{1}{4} \ln \frac{(x+1)^2}{x^2+1} \right) &= \frac{1}{2+2x^2} + \frac{1}{4} \frac{x^2+1}{(x+1)^2} \frac{2-2x^2}{(x^2+1)^2} \\
 &= \frac{1}{2+2x^2} + \frac{1}{2} \frac{1-x^2}{(x+1)^2(x^2+1)} \\
 &= \frac{1}{2} \frac{(x+1)^2+1-x^2}{(x+1)^2(1+x^2)} \\
 &= \frac{1}{2} \frac{2x+2}{(x+1)^2(1+x^2)} \\
 &= \frac{1}{(x+1)(x^2+1)}.
 \end{aligned}$$

3. Let  $y = \cot^{-1} x$ , so that  $x = \cot y$  and  $\frac{dx}{dy} = -\csc^2 y$ ; hence  $\frac{dy}{dx} = -\sin^2 y$ . Consider the following triangle:



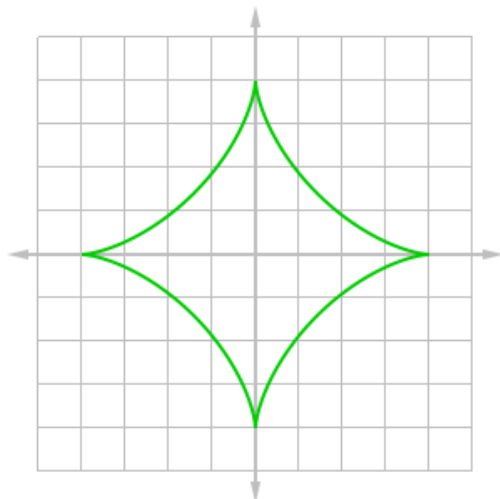
So  $\sin y = \frac{1}{\sqrt{1+x^2}}$ , and  $\frac{dy}{dx} = -\frac{1}{1+x^2}$ .

## 11. Related Rates of Change

1.  $V = [x(t)]^3$ , so  $\frac{dV}{dt} = 3\frac{dx}{dt}[x(t)]^2$ .
2. The volume of a cone is  $\frac{\pi}{3}r^2h$ . Comparing similar triangles, if the water is at a height  $h$  then it forms a cone with radius  $r = h/2$ . Hence when the water is at a height  $h$  it has volume  $V(t) = \frac{\pi[h(t)]^3}{24}$ , and so  $\frac{dV}{dt} = \frac{\pi[h(t)]^2}{8} \frac{dh}{dt}$ . We know that  $\frac{dV}{dt} = 2$ , so solving for  $\frac{dh}{dt}$  we have that  $\frac{dh}{dt} = \frac{16}{\pi[h(t)]^2}$  and when  $h = 3$  the height is rising at a rate of  $0.57 \text{ m min}^{-1}$ .
3. Let  $x$  be the hypotenuse of the formed triangle, and let  $y$  be the horizontal distance from the boat to the jetty so that  $y = \sqrt{x^2 - 1}$ . Then  $\frac{dy}{dt} = \frac{x}{\sqrt{x^2 - 1}} \frac{dx}{dt} = \frac{\sqrt{y^2 + 1}}{y}$ . So at  $y = 8$ ,  $\frac{dy}{dt} = \frac{\sqrt{65}}{8} \approx 1.0078 \text{ m s}^{-1}$ .

## 12. Parametric Functions

1. (a)  $\frac{dx}{dt} = 4t^3 - 6t^2 + 4t$ ,  $\frac{dy}{dt} = 3t^2 - 1$ ,  $\frac{dy}{dx} = \frac{3t^2 - 1}{4t^3 - 6t^2 + 4t}$ ,  $\frac{d^2y}{dx^2} = -\frac{3t^4 - 6t^2 + 3t - 1}{(4t^3 - 6t^2 + 4t)^2(2t^2 - 3t + 2)}$ .  
 (b)  $\frac{dx}{dt} = -\sin t - 4 \sin 2t$ ,  $\frac{dy}{dt} = \cos t + 4 \cos 2t$ ,  $\frac{dy}{dx} = -\frac{\cos t + 4 \cos 2t}{\sin t + 4 \sin 2t}$ ,  $\frac{d^2y}{dx^2} = \frac{12 \cos(t) + 33}{(\sin t - 4 \sin 2t)(8 \cos(t) + 1)^2(\cos(t)^2 - 1)}$
2. We have  $t^2 = (x - 1)^2$ , so  $y = e^{(x-1)^2}$  and  $\frac{dy}{dx} = 2(x - 1)e^{(x-1)^2}$ . At  $x = 2$ ,  $\frac{dy}{dx} = 2e$ ; so the best linear approximation is  $y - e = 2e(x - 2)$ , or  $y = e(2x - 3)$ .
3. (a) Should look something like this:



- (b)  $\frac{dx}{dt} = -12 \sin t \cos^2 t$ ,  $\frac{dy}{dt} = 12 \cos t \sin^2 t$ , so the slope at some  $t$  is simply

$$\frac{dy}{dx} = \frac{12 \cos t \sin^2 t}{-12 \sin t \cos^2 t} = -\frac{\sin t}{\cos t}.$$

- (c) Cusps will be at precisely those points with turning points in the  $x$  or  $y$  direction (for  $0 \leq t \leq 2\pi$ ). In other words, places where either  $\sin t$  or  $\cos t$  vanishes. These are at  $t \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ ; substituting these into the equation gives us the four points  $(\pm 4, 0)$  and  $(0, \pm 4)$ .

### 13. Sequences and Series

- Converges to  $1/2$ .
  - Diverges:  $9^{n+1}/10^n = 9^{n+1}/(9+1)^n = 9^{n+1}/(9^n + \dots) \rightarrow \infty$ .
- The series is  $2/3 - 2/5 + 2/7 - 2/9 + \dots$ . It has partial sums  $2/3, 4/15, 58/105, \dots$ . Converges to  $\pi/2$ .
  - The series is  $-2/5 + 4/6 - 6/7 + 8/8 - 10/9 + \dots$ . It has partial sums  $4/15, -62/125, \dots$ . Diverges (the terms added and subtracted keep growing, so partial sums become very positive and very negative alternately).

### 14. Differentiation Revision

- $f'(x) = (2017 \times 3)x^{2016} - \frac{1}{19x^{20}} + \frac{1}{2017 \cdot 2017 \sqrt{(x+2)^{2016}}}$ .
  - $f'(h) = \pi r^2$ .
  - $f'(\theta) = -\frac{\mu mg(\mu \cos \theta - \sin \theta)}{(\mu \sin \theta + \cos \theta)^2}$ .
  - $f'(g) = \frac{(g^2 + \ln g) \cos g - (2g + 1/g) \sin g}{(g^2 + \ln g)^2}$ .
  - $3f(x) + 3xf'(x) + 2f(x)f'(x) = \frac{3+f(x)-xf'(x)}{[3+f(x)]^2}$  so  $f'(x) = \frac{3+f(x)-3[3+f(x)]^2 f(x)}{3[3+f(x)]^2 x + 2[3+f(x)]^2 f(x) + x}$ .
- Let  $\theta$  be the angle of the kite string, and let  $x$  be the horizontal distance to the kite along the ground (so the length of the string is  $\sqrt{50^2 + x^2}$ ). Then  $\sin \theta = 50/x$ , so  $\cos \theta \frac{d\theta}{dt} = -\frac{50}{x^2} \frac{dx}{dt}$ . When the length of the string is 100,  $x \approx 86.6$ ; so  $\cos \theta = x/100 \approx 0.866$ . Substituting  $\frac{dx}{dt} = 2$ , we have  $\frac{d\theta}{dt} = -\frac{100}{86.6^2} \cdot \frac{1}{0.866} = -0.0154$ .

3. The surface area of a cone is  $\mathcal{S} = \pi r \sqrt{h^2 + r^2}$ ; we also have  $27 = \frac{1}{3} \pi r^2 h$ , so  $r^2 = \frac{81}{\pi h}$  and

$$\begin{aligned}\mathcal{S} &= \pi \sqrt{\frac{81}{\pi h} \left( h^2 + \frac{81}{\pi h} \right)} = \pi \sqrt{\frac{81h}{\pi} + \frac{81^2}{\pi^2 h^2}} \\ \frac{d\mathcal{S}}{dh} &= \frac{\pi \left( \frac{81}{\pi} - 2 \frac{81^2}{\pi^2 h^3} \right)}{2 \sqrt{\frac{81h}{\pi} + \frac{81^2}{\pi^2 h^2}}}\end{aligned}$$

In order to find a minimum, we set this derivative to zero and obtain  $0 = \frac{81}{\pi} - 2 \frac{81^2}{\pi^2 h^3}$ , so

$$h = \sqrt[3]{2 \frac{81}{\pi}} \approx 3.722 \text{ cm.}$$

From this, we find  $r = \sqrt{81/\pi h} = 2.63 \text{ cm.}$

4. We begin by parameterising the hyperbola; completing the square, we can transform our equation into standard form:

$$\frac{(x-1)^2}{3} - \frac{y^2}{3} = 1$$

A parameterisation of this is  $(1 + \sqrt{3} \sec t, \sqrt{3} \tan t)$ . Now, given any point  $(x_0, y_0)$  we wish to minimise  $\mathcal{D}(t) = \sqrt{(x_0 - 1 - \sqrt{3} \sec t)^2 + (y_0 - \sqrt{3} \tan t)^2}$  with respect to  $t$ .

- (a) Firstly, consider  $(x_0, y_0) = (2, 1)$ . Then  $\mathcal{D}(t) = \sqrt{(1 - \sqrt{3} \sec t)^2 + (1 - \sqrt{3} \tan t)^2}$ . Taking the derivative, we find that:

$$\frac{d\mathcal{D}}{dt} = \frac{(\sqrt{3} \sec t - 1)(\sqrt{3} \sec t \tan t) + (\sqrt{3} \tan t - 1)(\sqrt{3} \sec^2 t)}{\sqrt{(1 - \sqrt{3} \sec t)^2 + (1 - \sqrt{3} \tan t)^2}}$$

Using MATLAB to compute the solution of  $\frac{d\mathcal{D}}{dt} = 0$ ,

```
vpasolve((sqrt(3) * sec(t) - 1)*(sqrt(3) * sec(t) * tan(t))
== (1 - sqrt(3)*tan(t))*(sqrt(3)*(sec(t))^2),t)
```

we find  $t \approx 0.3759$ ; so  $(x, y) = (2.8621, 0.6835)$ .

- (b) Note that  $(3, 1)$  is already on the hyperbola. ☺

## 15. Approximating Areas

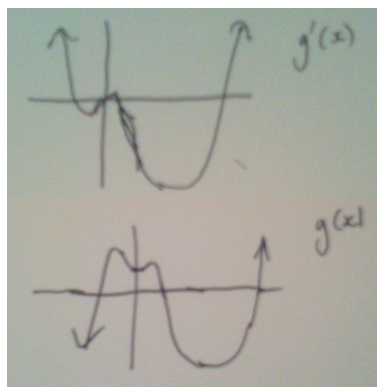
1. I use Simpson's rule with  $n = 8$ :

$$\int_0^{1.6} g(x) dx \approx \frac{0.2}{3} (12.1 + 13.2 + 4(11.6 + 11.1 + 12.2 + 13.0) + 2(11.3 + 11.7 + 12.6)) = 19.21.$$

2. Measure the height of the shaded area at each point (using  $n = 10$  is probably easiest), collapsing the empty area down (e.g. the height of the function at  $x = -1$  is just  $3 + 1 = 4$ ). Then use some numerical integration method.

## 16. Anti-differentiation

- (a)  $F(x) = \frac{1}{2}x^2 - 3x + C$   
(b)  $f(x) = x^2 + 3x + 2$ , so  $F(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C$   
(c)  $F(\theta) = 2\theta^3 - 7\tan\theta + C$   
(d)  $G(h) = \pi^2 h$   
(e)  $F(x) = \frac{x^{4.7}}{4.7} + \frac{2}{3}\sqrt{x^3} + \sqrt{7}x\sqrt{7}$
- $\varphi(x) = x^2 + x + C$ ; but  $\varphi(1) = 6$ , so  $1 + 1 + C = 6$  and  $C = 4$ . Hence  $\varphi(x) = x^2 + x + 4$ , and  $\varphi(2) = 10$ .
- See following image.



## 17. The Fundamental Theorem of Calculus

- $\int_0^{\pi/4} \sec^2 \theta \, d\theta = [\tan \theta] \Big|_0^{\pi/4} = 1.$
- $\int_1^2 f(x) \, dx = \int_1^3 f(x) \, dx - \int_2^3 f(x) \, dx = 10.$
- First we find the intersection points; we have  $6x = x^2$ , so  $x \in \{0, 6\}$ . Hence we compute

$$\int_0^6 2x - \frac{x^2}{3} \, dx = \left[ x^2 - \frac{x^3}{9} \right] \Big|_0^6 = 36 - 6^3/9 = 12.$$

## 18. Substitution

- (a)  $\frac{\csc 3x}{3} + C.$   
(b)  $-\frac{\tan 3x^2}{6} + C.$   
(c)  $2\sqrt{x} + 3x - 2\ln x + C.$   
(d)  $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$
- Use trig identity:  $2 \sin 5x \cos 3x = \sin 8x + \sin 2x$ . Then

$$\int_0^{\pi/6} \sin 8x + \sin 2x \, dx = \left[ -\frac{\cos 8x}{8} - \frac{\cos 2x}{2} \right] \Big|_0^{\pi/6} = 0.4375.$$

- $\frac{1}{2} \tan^{-1} x^2 + C.$  (Substitute  $u = x^2$ .)

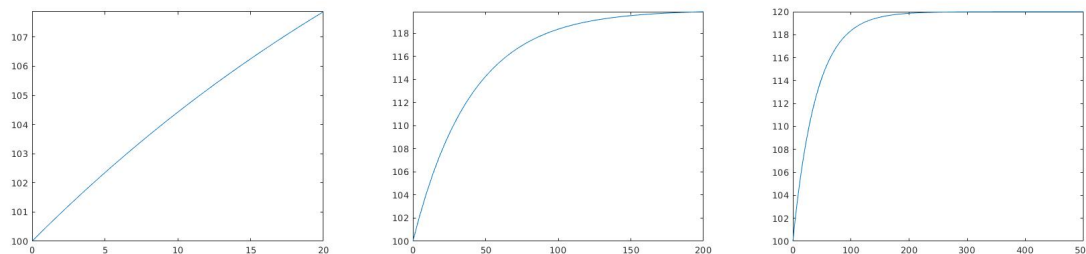
## 19. Differential Equations

- $\int e^y dy = \int e^t dt$ , so  $e^y = e^t + C$  and  $y = \ln(e^t + C)$ .
  - $\int \frac{dy}{y^2} = \int t dt$ , so  $-\frac{1}{y} = \frac{1}{2}t^2 + \frac{C}{2}$  and  $y = -\frac{2}{t^2 + C}$ .
  - $\int \sec^2 y dy = \int dt$ , so  $\tan y = t + C$  and  $y = \tan^{-1}(t + C)$ .
  - $\int \sin y dy = \int -t \cos t dt$ , so  $-\cos y = \cos t - t \sin t - C$  (by the hint) and  $y = \cos^{-1}(t \sin t - \cos t + C)$ .
- Using Newton's law of cooling,  $\frac{dT}{dt} = k(T - T_\infty)$  (where  $T_\infty$  is the ambient temperature). Solving this differential equation, we find  $\int \frac{1}{T - T_\infty} dT = \int k dt$  and so  $T = T_0 e^{kt} + T_\infty$ . We have  $T_\infty = 30^\circ$ , and  $T_0 = 100^\circ$ ; also, at  $t = 3$  we have  $T = 70$  so  $70 = 100e^{3k} + 30$ ; hence  $k = \frac{\ln 0.4}{3} = -0.31$  and by direct substitution  $T = 100e^{-0.31t} + 30$ . Let  $T = 31$ ; then  $t = 14.86$  and so the temperature will drop to  $31^\circ$  after around fifteen minutes.
- We have  $\frac{dV}{dt} = \text{rate in} - \text{rate out} = 3 - kV$ . Hence  $\int \frac{1}{3 - kV} dV = \int dt$ , so  $-\frac{\ln(3 - kV)}{k} = t + C$  and  $V = \frac{3 - Ke^{-kt}}{k}$ . At  $t = 0$ ,  $V = 100$ ; so  $100k = (3 - K)$ . We also have  $kV = 3$  where  $V = 120$ , so  $k = 1/40 = 0.025$ . Hence  $2.5 = 3 - K$  and  $K = 0.5$ . It immediately follows that

$$V = \frac{3 - 0.5e^{-0.025t}}{0.025}$$

and at  $t = 10$ ,  $V = 104$  litres.

- The rate of water flow out is  $kV = 3 - 0.5e^{-0.025t}$ , which is always less than 3 (the rate in). In fact, as  $t \rightarrow \infty$ , the volume tends to 120 L and the rate in tends to equal the rate out.

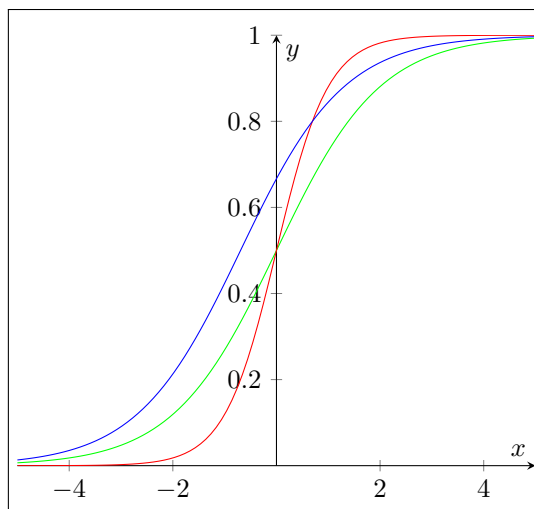


## 20. Partial Fractions

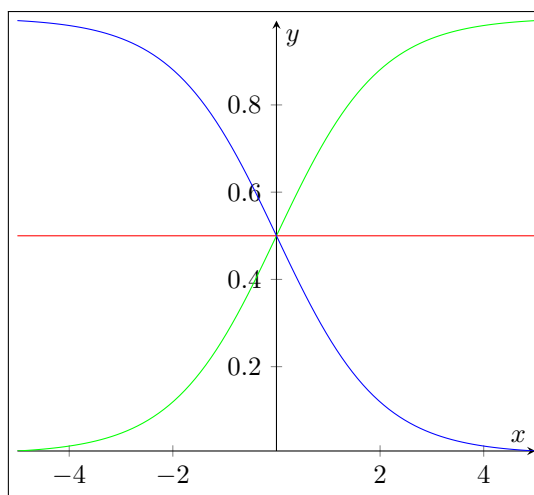
- 

$$\begin{aligned} \int r dt &= \int \frac{dP}{P(1-P)} = \\ &= \int \frac{1}{P} + \frac{1}{1-P} dP \\ rt + C &= \ln \frac{P}{1-P} \\ Ke^{rt} &= \frac{P}{1-P} \\ \frac{Ke^{rt}}{1 + Ke^{rt}} &= P. \end{aligned}$$

- It should be clear that as  $t \rightarrow \infty$ ,  $P \rightarrow 1$ . (If we look at  $\frac{dP}{dt} = \frac{rP}{P_\infty}(P_\infty - P)$ ,  $P \rightarrow P_\infty$ .)  
Green:  $r = K = 1$ ; red:  $r = 2$ ,  $K = 1$ ; blue:  $r = 1$ ,  $K = 2$ .



- (c)  $r$  lets us vary how fast the population gets to the maximum. Green:  $r = K = 1$ ; red:  $r = 0$ ; blue:  $r = -1$ .



- (d) Write it yourself.
2. (a) Draw a triangle with angle  $x/2$ , hypotenuse  $\sqrt{1+t^2}$ , adjacent edge 1, and opposite edge  $t$ .  
 (b)

$$\sin x = 2 \sin(x/2) \cos(x/2) = \frac{2t}{1+t^2}$$

$$\cos x = (\cos(x/2))^2 - (\sin(x/2))^2 = \left( \frac{1}{\sqrt{1+t^2}} \right)^2 - \left( \frac{t}{\sqrt{1+t^2}} \right)^2 = \frac{1-t^2}{1+t^2}.$$

- (c) We have  $x = \tan^{-1} 2t$ , so the result follows immediately.
- (d) i. Let  $t = \tan(x/2)$ . Then, substituting, we have

$$\int \frac{1}{1 - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{1}{t^2} dt = -\frac{1}{t} + C = -\frac{1}{\tan \frac{x}{2}} + C.$$

ii. Similarly,

$$\begin{aligned}\int \frac{1}{3\frac{2t}{1+t^2} - 4\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt &= \int \frac{1}{3t-2+2t^2} dt = \int \frac{1}{(2t-1)(t+2)} dt \\ &= \frac{1}{5} \ln \frac{1-2t}{t+2} + C = \frac{1}{5} \ln \frac{1-2\tan \frac{x}{2}}{\tan \frac{x}{2}+2} + C.\end{aligned}$$

## 21. Integration by Parts

1. (a)

$$\int x \cos 5x \, dx = \frac{1}{5} x \sin 5x - \int \frac{1}{5} \sin 5x \, dx = \frac{1}{5} (x \sin 5x + \cos 5x) + C.$$

(b)

$$\begin{aligned}\int \cos x \ln \sin x \, dx &= \sin x \ln \sin x - \int \sin x \frac{\cos x}{\sin x} \, dx \\ &= \sin x \ln \sin x - \int \cos x \, dx = \sin x (\ln \sin x - 1) + C.\end{aligned}$$

(c) Let  $u = \sqrt{x}$ , so  $dx = 2u \, du$  and our integral becomes

$$\int 2u \cos u \, du = 2u \sin u - \int 2 \sin u \, du = 2u \sin u + 2 \cos u + C = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C.$$

2. (a) Let  $u = \theta^2$ , so our integral becomes  $\frac{1}{2} \int_{\pi/2}^{\pi} u \cos u \, du$ . From 1(c) above, we know that  $\int u \cos u \, du = u \sin u + \cos u + C$ . Hence the required result is

$$\frac{1}{2} \int_{\pi/2}^{\pi} u \cos u \, du = \frac{1}{2} [u \sin u + \cos u] \Big|_{u=\pi/2}^{\pi} = -\frac{1}{2} - \frac{\pi}{4}.$$

(b) We use integration by parts twice.

$$\begin{aligned}\int (x^2 + 1)e^{-x} \, dx &= -e^{-x}(x^2 + 1) + \int 2xe^{-x} \, dx \\ &= -e^{-x}(x^2 + 1) - 2xe^{-x} + \int 2e^{-x} \, dx \\ &= -e^{-x}(x^2 + 1) - 2xe^{-x} - 2e^{-x} + C.\end{aligned}$$

Hence the result we are looking for is  $3 - 6e^{-1}$ .

3. (a) Apply integration by parts to  $\int 1 \cdot (\ln x)^n \, dx$  by integrating 1 and differentiating  $(\ln x)^n$ .

(b) Applying (a), we find

$$\begin{aligned}\int (\ln x)^3 \, dx &= x(\ln x)^3 - \int (\ln x)^2 \, dx \\ &= x(\ln x)^3 - (x(\ln x)^2 - \int \ln x \, dx) \\ &= x(\ln x)^3 - (x(\ln x)^2 - (x \ln x - x)) \\ &= x(\ln x)^3 - x(\ln x)^2 + x \ln x - x.\end{aligned}$$

## 22. Lengths, Volumes, and Areas

1. We simply calculate the relevant integral:

$$\pi \int_1^2 x^{-2} dx = \pi [(-2^{-1}) - (-1^{-1})] = \frac{\pi}{2}.$$

2. Calculating the surface area:

$$\begin{aligned} 2\pi \int_0^\pi \sin x \sqrt{1 - \cos^2 x} dx &= 2\pi \int_0^\pi \sin^2 x dx \\ &= \pi [x - \sin 2x] \Big|_0^\pi \\ &= \pi^2 \end{aligned}$$

So the radius of the equivalent circle is  $\sqrt{\pi}$ .

3. Summing along the axis from base to point, each slice has an area  $\left(\frac{L}{H}x\right)^2 = \frac{L^2}{H^2}x^2$ ; hence the total volume is

$$V = \int_0^H \frac{L^2}{H^2} x^2 dx = \frac{1}{3} L^2 H.$$

4. We have  $r = a(1 - \cos \theta)$  so  $\frac{dr}{d\theta} = a \sin \theta$ . Hence:

$$\begin{aligned} S &= \int_0^{2\pi} \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta \\ &= \int_0^{2\pi} \sqrt{a^2 - 2a^2 \cos \theta + a^2(\sin^2 \theta + \cos^2 \theta)} d\theta \\ &= a\sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos \theta} d\theta. \end{aligned}$$

We turn our attention, then, to the integral  $\int \sqrt{1 - \cos \theta} d\theta$ . Let  $u = 1 - \cos \theta$ ; then  $du = \sin \theta d\theta$ ; but  $\sin \cos^{-1}(1 - u) = \sqrt{2u - u^2}$  (this can be verified by drawing a suitable triangle). Hence  $du = \sqrt{2u - u^2} d\theta$ , and

$$\begin{aligned} \int \sqrt{1 - \cos \theta} d\theta &= \int \frac{\sqrt{u}}{\sqrt{2u - u^2}} du \\ &= \int \frac{1}{\sqrt{2 - u}} \\ &= -2\sqrt{2 - u} + C \\ &= -2\sqrt{1 + \cos \theta} + C. \end{aligned}$$

Therefore (and changing our integral to double the integral from 0 to  $\pi$  to avoid the problem of having



a closed loop),

$$\begin{aligned}
 S &= 2a\sqrt{2} \int_0^{\pi} \sqrt{1 - \cos \theta} \, d\theta \\
 &= 2a\sqrt{2} \left[ -2\sqrt{1 + \cos \theta} \right]_0^{\pi} \\
 &= 2a\sqrt{2} \left[ (-2\sqrt{1 + \cos \pi}) - (-2\sqrt{1 + \cos 0}) \right] \\
 &= 2a\sqrt{2} \left[ (-2\sqrt{0}) - (-2\sqrt{2}) \right] \\
 &= 2a\sqrt{2} \times 2\sqrt{2} = 8a.
 \end{aligned}$$

## 23. Trigonometric Substitution

*These ones are tedious and can be checked by the computer, so I have not written full answers for all of them.*

1. Let  $x = 2 \tan \theta$ , so  $dx = 2 \sec^2 \theta$ :

$$\int \frac{2 \sec^2 \theta}{\sqrt{x^2 + 4}} dx = \int \frac{\sec^2 \theta}{\sqrt{1 + \tan^2 x}} dx = \int \sec \theta dx = \ln \left( \sqrt{\left(\frac{x}{2}\right)^2 + 1} + \frac{x}{2} \right) + C.$$

2. First, let  $u = x^7$  so  $du = 7x^6 dx$  and our integral becomes

$$\frac{1}{7} \int \frac{du}{\sqrt{1 - u^2}} = \frac{1}{14} \ln \frac{1 + u}{1 - u} + C = \frac{1}{14} \ln \frac{1 + x^7}{1 - x^7} + C.$$

3. Use  $x = \frac{2}{5} \sec \theta$ .
4. Use  $x = \frac{2}{3} \sin \theta$ .
5. Use  $x = \frac{1}{6} \tan \theta$  and simplify.
6. Use integration by parts; the resulting integral  $-\frac{\ln x}{4x^4} + \int \frac{dx}{4x^5}$  is much simpler.
7. Use partial fractions.

## 24. Kinematics

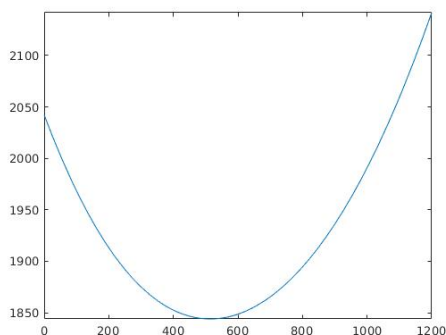
1. (a)  $v = \frac{dh}{dt} = 122.5 - 9.8t$ , so the initial velocity of the flare is  $122.5 \text{ m s}^{-1}$ .  
 (b) Zero.  
 (c) When  $v = 0$ ,  $t = 12$  and the height at this time is around 764 metres.
2. Let  $x$  be the distance from the point on the beach directly away from B. Then the total distance travelled is simply  $D = \sqrt{600^2 + x^2} + \sqrt{800^2 + (1200 - x)^2}$ ; taking the derivative:

$$\frac{dD}{dx} = \frac{x}{\sqrt{600^2 + x^2}} - \frac{1200 - x}{\sqrt{800^2 + (1200 - x)^2}}$$

Setting to zero, we have

$$\begin{aligned}
 x\sqrt{800^2 + (1200 - x)^2} &= (1200 - x)\sqrt{600^2 + x^2} \\
 800^2 x^2 + (1200 - x)^2 x^2 &= (1200 - x)^2 (600^2 + x^2) \\
 0 &= 1200^2 600^2 - 600^2 2400x + (600^2 - 800^2)x^2 \\
 x &\in \{-3600, 3600/7\}.
 \end{aligned}$$

Since  $x \geq 0$ ,  $x = 3600/7 \approx 514$ . The total distance travelled is therefore around 1844 metres. By graphing  $D$  versus  $x$ , we see that this is indeed the required medium:



3. Like question 2 from sheet 15:- we just pick an approximation for the integral (in this case, jagged edges means parallelograms won't be too bad).

## 25. Integration Revision

1. (a)  $\int_1^2 \sin x \, dx = [-\cos x]_1^2 = \cos 1 - \cos 2.$

(b)  $\int \frac{u^2+1}{u^3+3u} = \frac{1}{3} \ln(u^3 + 3u) + C.$

(c)  $\int_0^{\pi/6} \tan x \, dx = [\ln \sec x]_0^{\pi/6} \approx 0.1438.$

2. We have  $\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$ , so  $\int 2y-4 \, dy = \int 3x^2+4x-4 \, dx$ . Hence  $y^2-4y = x^3+2x^2-4x+C$ ; we also have  $C = -2$ , so  $y^2-4y = x^3+2x^2-4x-2$ . We are trying to find  $y$  if  $x = 2$ ; so  $y^2-4y = 8+4-4-2 = 6$ . Solving  $y^2-4y-6 = 0$ , we find  $y = \frac{4 \pm \sqrt{38}}{2}$ .

3. (a) Let  $t = a \tan \theta$ . Then:

$$\begin{aligned} \int \frac{a^3}{t^2 + a^2} \, dt &= \int \frac{a^4 \sec^2 \theta}{a^2 \tan^2 \theta + a^2} \, d\theta \\ &= \int \frac{a^2 \sec^2 \theta}{\sec^2 \theta} \, d\theta \\ &= a^2 \theta = a^2 \tan^{-1} \left( \frac{t}{a} \right); \end{aligned}$$

hence  $\omega(a, x) = a^2 \tan^{-1} \left( \frac{x}{a} \right).$

- (b) It follows that  $\omega(2, 2) = 4 \tan^{-1} 1 = \pi$ .

- (c) We wish to find  $x$  such that  $\pi = 3 \tan^{-1} \left( \frac{x}{\sqrt{3}} \right)$ ; in other words,  $x = \sqrt{3} \tan \left( \frac{\pi}{3} \right) = 3$ .

4. Note first that  $\int_{-\pi/2}^{\pi/2} \sin^5 x \, dx = 0$  since  $\sin^5$  is odd. Then, we argue as follows:

$$\begin{aligned}
 \int \cos^5 x \, dx &= \int \cos x (1 - \sin^2 x)^2 \, dx \\
 &= \int (1 - t^2)^2 \, dt \quad (t = \sin x) \\
 &= \int 1 - 2t^2 + t^4 \, dt \\
 &= t - \frac{2}{3}t^3 + \frac{t^5}{5} + C \\
 &= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.
 \end{aligned}$$

Hence

$$\int_{-\pi/2}^{\pi/2} \sin^5 x \, dx = \left( \sin \frac{\pi}{2} - \frac{2}{3} \sin^3 \frac{\pi}{2} + \frac{1}{5} \sin^5 \frac{\pi}{2} \right) - \left( \sin \frac{-\pi}{2} - \frac{2}{3} \sin^3 \frac{-\pi}{2} + \frac{1}{5} \sin^5 \frac{-\pi}{2} \right).$$

But  $\sin(\pi/2) = 1$ ; so we have  $(1 - \frac{2}{3} + \frac{1}{5}) - (-1 + \frac{2}{3} - \frac{1}{5}) = 2(1 - \frac{2}{3} + \frac{1}{5}) = \frac{16}{15}$ .