NCEA Level 3 Calculus (Differentiation)
14. Differentiation Revision

Halfway there! Let's do a bit of revision.

Questions

1. True or False: A
   (a) If a function f is continuous around a point, then it is differentiable at that point.
   (b) If a function f is differentiable around a point, then it is continuous at that point.
   (c) If f and g are differentiable, then $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$.
   (d) If f and g are differentiable, then $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g'(x)$.
   (e) $\frac{\mathrm{d}}{\mathrm{d}x}(x+3)^2 = 2(x+3)$.
   (f) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2+3)^2 = 2(x^2+3)$.
   (g) $\frac{\mathrm{d}}{\mathrm{d}x}\tan^2 x = \frac{\mathrm{d}}{\mathrm{d}x}\sec^2 x$.
   (h) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$.
   (i) If $A = \pi r^2$, then $\mathrm{d}A\pi = 0$.

2. Find $\frac{\mathrm{d}y}{\mathrm{d}t}$ in each case. A
   (a) $y = t^2 + 3t$
   (b) $y = \frac{4-t}{3+t}$
   (c) $y = (t^4 - 3t^2 + 5)^2$
   (d) $y = (t+1)^{2017}$
   (e) $y = \frac{3t+4t^2}{\sqrt{t}}$
   (f) $y = \sin 2t$
   (g) $y = \sin^2 t$
   (h) $y = \sin t^2$
   (i) $y = \cos\tan t$
   (j) $y = (27t+3)^{2017}(t^2 - \sqrt{t})^{2020}$
   (k) $y = \left(t + \frac{1}{t^2}\right)^{\sqrt{7}}$

3. Find the equation of the tangent line to the curve $\sqrt{1 + 4\sin x}$ at the point $(0, 1)$. M

4. Find y'' in each case: A
   (a) $y = 3x^3 + 2x + \sqrt{2x} + \frac{1}{x^2}$
   (b) $y = e^{2x}$
   (c) $y = \sqrt{4t+1}$
   (d) $y = 4\sin^2 x$

5. Find the nth derivative of e^{2x} (where n is a natural (counting) number). M

6. The height of a projectile after t seconds can be modelled by $h = 3t(t-10)$. At what time is the height of the projectile at a maximum? Use the second-derivative test to prove that you have found a maximum. E

7. Find $f'(x)$ if: $\boxed{\texttt{M}}$
   (a) $f(x) = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + 3$
   (b) $f(x) = 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + 7$

8. In each case, find y' in terms of x and y: $\boxed{\texttt{M}}$
   (a) $x^2 + y^2 = 4$
   (b) $x^2 + 4xy + y^2 = 13$
   (c) $xy^4 + x^2y = x + 3y$
   (d) $x^2\cos y + \sin 2y = xy$

9. By differentiating the double-angle formula for cosine, $\boxed{\texttt{M}}$
$$\cos 2x = \cos^2 x - \sin^2 x,$$
obtain the double-angle formula for the sine function.

10. Find f' in terms of g' if $f(x) = x^2 g(x)$. $\boxed{\texttt{M}}$

11. The volume of a cube is increasing at a rate of $10\,\mathrm{cm}^3\,\mathrm{min}^{-1}$. How fast is the surface area increasing when the length of an edge is $30\,\mathrm{cm}$? $\boxed{\texttt{M}}$

12. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$. $\boxed{\texttt{M}}$
   (a) Find the rate of change of volume with respect to height if the radius is constant.
   (b) Find the rate of change of volume with respect to radius if the height is constant.

13. A particle moves along a horizontal line such that its coordinate at time t is $x = \sqrt{b^2 + c^2t^2}$ $(t \geq 0)$, where b and c are positive constants. Find its velocity and acceleration functions. $\boxed{\texttt{M}}$

14. From first principles, show that: $\boxed{\texttt{M}}$
   (a) $\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x}[2x^3 + 2x] = 6x^2 + 2$

15. Find the derivative of $f(x) = \frac{4}{\sqrt{1-x}}$ from first principles. $\boxed{\texttt{E}}$

16. Each limit represents the derivative of a function f at a point a. Identify each function and point. $\boxed{\texttt{E}}$
   (a) $\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h}$
   (b) $\lim_{t\to 1} \frac{t^4+t-2}{t-1}$

17. Find the best linear approximation to $f(x) = \sqrt{25 - x^2}$ near $x = 3$. $\boxed{\texttt{M}}$

18. A balloon is rising at a constant speed of $2\,\mathrm{m\,s}^{-1}$. A girl is cycling along a straight road at a speed of $5\,\mathrm{m\,s}^{-1}$. When she passes under the balloon, it is $15\,\mathrm{m}$ above her. How fast is the distance between the person and the balloon increasing 3 seconds later? $\boxed{\texttt{M}}$

19. On a straight shoreline there is a tree and exactly opposite it, $100\,\mathrm{m}$ away in the sea, stands a lighthouse. A strong and thin spotlight on its top revolves at the rate of one revolution per 4 seconds, its light creating a running light spot on the shore. You stand on the shore $100\,\mathrm{m}$ from the tree. How fast does this spot move when it goes past you? $\boxed{\texttt{E}}$

20. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t for the following parametrically-defined curves. $\boxed{\texttt{M}}$
   (a) $t \mapsto (1 + e^{2t}, e^t)$

(b) $x = \tan t$, $y = \sec 2t$

(c) $x = \frac{t^2-10}{t^2+1}$, $y = tx$

21. Find the lowest point on the curve $\gamma : t \mapsto (t^3 - 3t, t^2 + t + 1)$. Prove you have found a minimum. E

22. Find the acceleration of a particle at time t if its displacement from the origin at time t is $-t^6 + 5t^4 + \sin t$. M

23. A piece of wire 10 m long is cut into two pieces; one is bent into a square and the other into a circle. Where should the wire be cut to ensure the total area of the enclosed shapes is (a) a minimum and (b) a maximum? E

24. A cone is made by cutting a sector out of a circle of paper of radius R and gluing together the edges of the cut. Find the maximum possible volume of the cone. M

25. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r. E

26. At which points does the 'bouncing wagon' curve $2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$ have horizontal tangents? E

27. Salt forms a cone as it falls from a conveyor belt. The slant of the cone forms an angle of 30° with the horizontal. The belt delivers salt at a rate of $2\,\mathrm{m}^3$ per minute. When the radius of the cone is ten metres, what is the rate of increase of the slant height (measured along the surface of the cone)? E

28. Suppose we take a circle of radius r, and inscribe within it a triangle such that each corner of the triangle is located on the circle. What is the maximum possible area of the triangle? S

29. Take derivatives of the following functions with respect to x: E

    (a) $f(x) = 2^x$

    (b) $g(x) = \log_{\log x} x$

30. Scholarship 2017: For $y = x^{(x^x)}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 2$. S

31. Scholarship 2015 (adapted): A car is driving along a road shaped like a parabola at night. The parabola has a vertex at the origin, and the car starts at a point 100 m west and 100 m north of the origin. S

    (a) Write an equation modelling the road as a parabola.

    (b) Find the general equation for the tangent line to the parabola at some point (x_0, y_0), and substitute into it the parabola equation to obtain an equation only in x, x_0, and y_0.

    (c) Suppose there is a statue of the Roman emperor Augustus located 100 m east and 50 m north of the origin. Write the equation for the tangent line of the parabola passing through the statue (so that it only depends on a value x on the parabola).

    (d) Hence find the single point (x, y) on the road where the headlights of the car illuminate the statue.

32. (Difficult) Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line. O

33. (Difficult) A cone of radius r centimetres and height h centimetres is lowered point first at a rate of $1\,\mathrm{cm}^2$ into a cylinder of radius R centimetres that is partially filled with water. How fast is the water level rising at the instant that the cone is fully submerged? S