NCEA Level 2 Mathematics (Integration I)

Reading

This year, you are expected to be able to integrate expressions involving powers of x, like $x^4 - \sqrt{x} + 9x^{-\frac{3}{4}}$, using the general rule

$$\int cx^n \,\mathrm{d}x = \frac{c}{n+1}x^{n+1} + C.$$

For example, the indefinite integral of the expression above is

$$\int x^4 - \sqrt{x} + 9x^{-\frac{3}{4}} = \frac{1}{5}x^5 - \frac{2}{3}x^{\frac{3}{2}} + 36\sqrt[4]{x} + C.$$

Last week, we looked at functions where the derivative was not defined, either at a single point or everywhere on the function. Similarly, there are functions which cannot be integrated — but in a slightly different sense. If you can write down a function in terms of what are called *elementary functions*, like x^4, $\sin x$, and so on, then it is always possible to find a derivative in terms of those functions. However, the same is not true for integration! For example, the derivative of the simple function $y = x^x$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = x^x + x^x \ln x$ (where $\ln x$ is a function we'll look at in a minute), but the integral $\int x^x \,\mathrm{d}x$ cannot be written down in a nice way (the integral exists, but we don't have a function to express it because we haven't got a function with the derivative x^x in our toolbox).

This year, we run into a little problem like this. Stop for a moment and try this problem: $\int x^{-1} \,\mathrm{d}x$.

See the problem? Our rule tells us that $\int x^{-1} \,\mathrm{d}x = \frac{1}{0}x^0 + C$, which doesn't make any sense. Unlike with derivatives, however, a failure to find the integral does not mean that it doesn't exist. On the other hand, we don't know any other functions which will differentiate to give x^{-1} so we can't integrate it!

So as mathematicians, we simply *define a new function which fits this criteria.* This is what sets mathematics apart from the sciences — if it turns out that we have a problem, we can often 'make up a solution' without reference to experiments or anything like that. We don't care whether this new function has any real applications to physics, or biology, or astronomy, or to anything else. We just saw a gap in the things we could do, and we defined it away!

Definition (Logarithmic Function)**.** The **natural logarithmic function** is the function defined by

$$\ln x = \int \frac{1}{x} \,\mathrm{d}x.$$

So this new function (which you may have seen before) solves our problem and allows us to integrate x^{-1}. It has some other useful properties too.

Recall that we can call $\sqrt{x}$ the *inverse* of x^2 because it undoes the effect of the squaring. Now, if we write $y = \ln x$, we can also find its inverse. Let's call it $x = \exp y$.

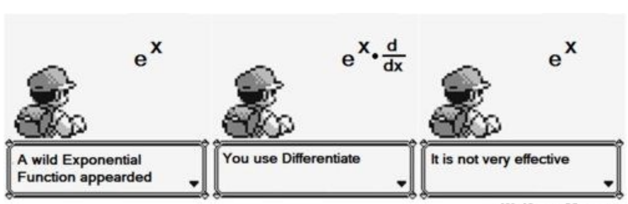
Definition (Exponential Function)**.** The **exponential function** is the inverse of $\ln x$ defined by

$$y = \exp x \iff x = \ln y.$$

What's the derivative of this new function? Well, let's work it out using the derivative of $\ln x$, which we already know. Suppose $y = \ln x$, so $x = \exp y$. Then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} \implies \frac{\mathrm{d}x}{\mathrm{d}y} = x = \exp y.$$

But notice that $\frac{\mathrm{d}x}{\mathrm{d}y}$ is simply the derivative of $x = \exp y$; so the exponential function turns out to be its own derivative!



Questions

1. Evaluate the following indefinite integral:

$$\int -x^6 - \frac{1}{3\sqrt{x}} + \frac{x^{19}}{47} - \frac{2}{x^{-\frac{4}{5}}}\,\mathrm{d}x$$

2. Find $f(x)$ if $f''(x) = -2 + 12x - 12x^2$, $f(0) = 4$, and $f'(0) = 12$.