Level Three Conic Sections

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WORK IN PROGRESS

O King, for traveling over the country, there are royal roads and roads for common citizens, but in geometry there is one road for all. (Menaechmus)

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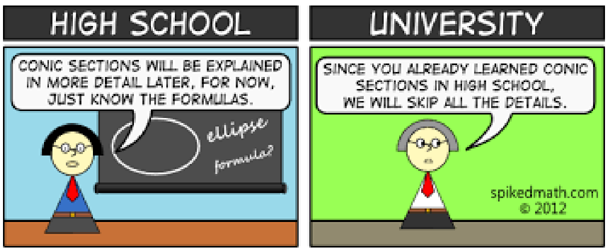
Preface

The conic sections are the simplest curves in the plane which are not simply straight lines. However, most secondary school treatments of the subject tend to present a set of vaguely connected case-by-case results, rather than any kind of coherent story. These notes are my attempt to avoid this.

Most of the content is presented as a series of exercises; however, even an enthusiastic Y13 student will require a significant amount of guidance. As always, though, it is important that the student struggles with the material on their own!

Prerequisites

These notes have perhaps the most 'formal' prerequisites out of all my Y13 notes. I will assume results from trigonometry, linear systems, calculus, and even algebra. However, this does not mean that the full power of these subjects are used. The most important prerequisites are actually Y11 and Y12 geometry because there are a number of results from there about triangles, circles, lines, and so forth that will be used here. My Y13 notes on trigonometry revise a number of these results, and so the reader is directed there initially.



Section 1: Introduction

The study of the conic sections (as with much of elementary plane and solid geometry) goes back to the ancient Greeks. It is thought that the first person to study the slices of a cone was Menaechmus, who lived on the Gallipoli Peninsula in around 350 BCE, in his solution of the Delian problem: the construction of a cube whose volume is twice that of a given cube. Anecdotally, it is to Menaechmus that the famous quote on the cover of these notes is attributed (usually in his disputed role as the tutor to Alexander the Great).

Euclid of Alexandria published his book *Elements*, one of the most influential mathematical texts of all time, in around 300 BCE (for perspective, this is around thirty years after the death of Alexander the Great). In it, he set down the set of basic results which can be proved true about circles and lines on a plane: while modern geometry has gone much further than Euclid (in considering geometry in more than two dimensions, and on surfaces much more complicated than the plane), the basic results which are taught in any introductory geometry course are usually treated in the Elements.

As well as studying circles and lines, Euclid wrote a treatise on the conic sections. Unfortunately, it no longer survives; however, Apollonius of Perga (a city in what is now Turkey) expanded Euclid's work and published his *Conics* the century after Euclid's death. Apollonius used Euclidean geometry to prove most of the classical theorems on the conic sections, including the many applications to physics that the subject holds.

Despite these ancient roots, it turns out that the subject is deep enough to still hold interest in the modern day. Indeed, we will use techniques from calculus (in other words, from the 16th and 17th century) and from complex analysis (the 18th and 19th century) in these notes; and the proper setting for conics turns out to be projective geometry, which dates from the 19th and 20th century — although we will only touch on this briefly here.

Conic sections have many beautiful applications; later on, we will discuss applications to optics and to celestial mechanics. For example, in the 17th century the German astronomer Johannes Kepler showed that all astronomical bodies follow, due to gravity, paths in the shape of conic sections.

Section 2: Basic results

Definition. Let O be a fixed point and ℓ be some line not passing through O. A **conic** $\mathcal{C}$ is the locus of a point P such that, if K is the point on ℓ so that $\ell \perp PK$,

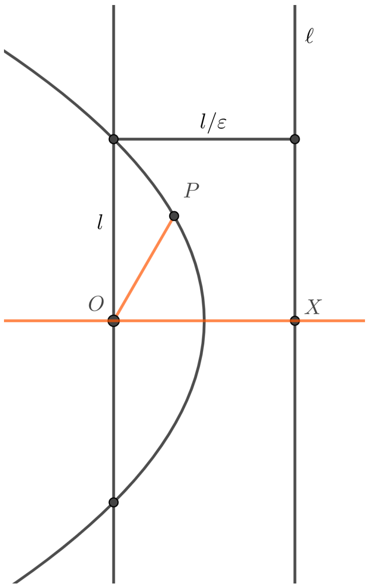
$$\varepsilon = \frac{|OP|}{|PK|}$$

where ε is some fixed constant.

The point O is called the **focus**, the line ℓ the **directrix**, and the constant ε the **eccentricity**. The **latus rectum** (Latin: straight side) is defined to be the chord of a conic through the focus and parallel to the directrix; let l be the length of the latus rectum.

The conic is variously called:

- an **ellipse** if $\varepsilon < 1$;
- a **parabola** if $\varepsilon = 1$; and
- an **hyperbola** if $\varepsilon > 1$.



Exercise 1: Polar form.

Show that, if X is on the directrix of a conic such that $OX \perp \ell$, then the polar equation of the conic with respect to this axis and origin O is

$$\frac{l}{r} = 1 + \varepsilon \cos\theta.$$

Conclude that:

1. Every conic is symmetric with respect to OX.
2. The ellipse is a closed and bounded curve (i.e. it does not extend towards infinity).
3. The parabola is unbounded, but is connected.
4. The hyperbola consists of two seperate branches, each extending to infinity, given by $-\alpha < \theta < \alpha$ and $\alpha < \theta < 2\pi - \alpha$ (where $\alpha = \operatorname{arcsec}(-\varepsilon)$).

Exercise 2: Rectangular form.

By squaring the polar form equation, show that the Cartesian equation for a conic, taking suitable axes and origin, is

$$x^2 + y^2 = (l - \varepsilon x)^2.$$

1. If $\varepsilon \neq 1$, and a suitable origin is chosen, show that the conic equation can be written in the form

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

for some real numbers a and b. The new location of the origin is called the **centre** of the conic. Conclude that:

   (a) Both the ellipse and hyperbola are symmetric across both Cartesian axes.

   (b) For an ellipse, the values $2a$ and $2b$ are the lengths of the chords through the origin along the x- and y-axes respectively.

   (c) The two branches of a hyperbola lie in opposite regions formed by the two lines (**asymptotes**)

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0.$$

   The value $2a$ is the length of the transverse axis of the hyperbola and the value $2b$ is the length of the conjugate axis: the two dimensions of the rectangle whose diagonals are the asymptotes and which is bounded by the hyperbola.

2. If $\varepsilon = 1$, show that the conic equation for the parabola can be written in the form

$$y^2 = 2l(\frac{1}{2}l - x).$$

By reflecting in a suitable vertical line, derive the standard form equation

$$y^2 = 2lx.$$

Given this latter equation, give the coordinates of the focus, and the Cartesian equation of the directrix.

Exercise 3: Parametric form.

Show that:

1. The ellipse $x^2/a^2 + y^2/b^2 = 1$ is parameterised by $(a \cos t, b \sin t)$ for $0 \leq t < 2\pi$.
2. The parabola $y^2 = 2lx$ is parameterised by $(2lt^2, 2lt)$.
3. The hyperbola $x^2/a^2 - y^2/b^2 = 1$ is parameterised by $(a \sec t, b \tan t)$ for $0 \leq t < 2\pi$.

Illustrate these parameterisations with a suitable computer program (e.g. Geogebra, or your favourite programming language).

Exercise 4: Focii of rectangular conics.

Given the equation

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

compute the coordinates of the focus, equation of the directrix, and eccentricity of the conic. Hence show that ellipses and hyperbolae have two focii and two directrices each. Where is the directrix of a circle?

Show that the following two properties can be used as alternative definitions for the ellipse and hyperbola:

AP1. Show that the ellipse with focii O_1 and O_2 is the locus of all points P such that $d(O_1, P) + d(P, O_2) = R$ for some constant R. Give R in terms of a and b.

AP2. Show that the hyperbola with focii O_1 and O_2 is the locus of all points P such that $d(O_1, P) - d(P, O_2) = R$ for some constant R. Give R in terms of a and b.

As an application of this definition, we will derive the 'reflection property' of the ellipse.

1. Suppose O_1 and O_2 are on the same side of a line ℓ. Show that the shortest path from O_1 to O_2 which touches the line ℓ is the broken line O_1XO_2, where X is on ℓ and the angles between O_1X and ℓ and between XO_2 and ℓ are equal. (Hint: you can do this with calculus, but that would be like killing a fly with a sledgehammer.)

2. Show that, if O_1 and O_2 are two focii of an ellipse and X is any point on the ellipse, then the broken line O_1XO_2 makes equal angles with the tangent line of the ellipse at X. (Hint: show that this broken line is the shortest path from O_1 to O_2 that touches the tangent line at X.)

3. Thus, given the law of reflection for waves, show that if a light source is placed at O_1 then every ray from the source will arrive at O_2 at precisely the same time.

4. Derive some result of this kind for parabolae by treating a parabola as an ellipse with one focus 'at infinity'. Suggest an application of this property related to, say, torches.

Section 3: Isometries

In the previous section, we saw geometrically that every ellipse or hyperbola is just a translated and rotated version of the graph of an equation with the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

(Herein, in this section we will write $p = 1/a^2$ and $q = 1/b^2$ for convenience.)

Here, we will show this relation using coordinates — the advantage of this approach is that it allows us to calculate precisely from the equation of a conic its position and angle with respect to the standard axes. The subtle point here is basically that the same curve will have different equations depending on its position in the coordinate system — in the previous section we chose our coordinate system in the 'nicest way possible' by geometric means, and now we are given a conic already sitting in some coordinate system which we want to understand.

In order to pursue this programme, we need to study the effect of rotations and translations on coordinate systems. We begin with rotations because it turns out that rotating before translating is easier for our purposes.

Let $\rho = \operatorname{cis}\theta$ be a complex number such that $|\rho| = 1$. If $z = x + yi$ is a point on the complex plane, then ρ acts by multiplication on z to rotate it around the origin by an angle θ; we can then calculate the resulting rectangular coordinates of ρz to see how a rotation affects our normal coordinate system.

$$\begin{aligned} z &= |z| \operatorname{cis}(\tan^{-1} x/y) \\ \rho z &= |z| \operatorname{cis}(\tan^{-1} x/y + \theta) \\ &= |z| \left[\cos(\tan^{-1} x/y + \theta) + i \sin(\tan^{-1} x/y + \theta) \right]. \end{aligned}$$

Using trig identities, we calculate

$$\begin{aligned} \cos(\tan^{-1} x/y + \theta) &= \cos(\tan^{-1} x/y) \cos\theta - \sin(\tan^{-1} x/y) \sin\theta \\ &= \frac{x}{\sqrt{x^2+y^2}} \cos\theta - \frac{y}{\sqrt{x^2+y^2}} \sin\theta \\ \sin(\tan^{-1} x/y + \theta) &= \sin(\tan^{-1} x/y) \cos\theta + \cos(\tan^{-1} x/y) \sin\theta \\ &= \frac{y}{\sqrt{x^2+y^2}} \cos\theta + \frac{x}{\sqrt{x^2+y^2}} \sin\theta \end{aligned}$$

and so

$$\begin{aligned} \rho z &= |z| \left[\cos(\tan^{-1} x/y + \theta) + i \sin(\tan^{-1} x/y + \theta) \right] \\ &= (x\cos\theta - y\sin\theta) + i(y\cos\theta + x\sin\theta). \end{aligned}$$

In other words, if the point (x, y) is rotated about the origin by an angle θ, then

$$(x, y) \overset{\rho}{\mapsto} (x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta). \tag{2}$$

Considering $ax^2 + bx + cxy + dy + ey^2$, then, we want to rotate (x, y) by some θ so that some terms vanish. Doing a long computation, we find that

$$\begin{aligned} ax^2 + bx + cxy + dx + ey^2 \mapsto\ & a(x\cos\theta - y\sin\theta)^2 + b(x\cos\theta - y\sin\theta) \\ & + c(x\cos\theta - y\sin\theta)(y\cos\theta + x\sin\theta) \\ & + d(y\cos\theta + x\sin\theta) + e(y\cos\theta + x\sin\theta)^2 \\ =\ & x^2(a\cos^2\theta + \frac{c}{2}\sin 2\theta + e\sin^2\theta) + x(b\cos\theta + d\sin\theta) \\ & + xy((e-a)\sin 2\theta + c\cos 2\theta) \\ & + y(-b\sin\theta + d\cos\theta) + y^2(a\sin^2\theta - \frac{c}{2}\sin 2\theta + e\cos^2\theta) \end{aligned}$$

and, looking at this, we see that an easy candidate we can try to get rid of is the xy term. In fact, if we want this term to be zero we need only solve

$$(e-a)\sin 2\theta + c\cos 2\theta = 0 \tag{3}$$

which is easy: $\frac{-c}{e-a} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$, and so in order to remove the xy term we need only rotate our coordinate system by

$$\theta = \frac{1}{2}\tan^{-1}\frac{c}{a-e}. \tag{4}$$

Making the coordinate system change $(x, y) \mapsto (x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta)$ therefore leaves us with something that looks like $ax^2 + bx + dy + ey^2 = 1$, where x and y are now coordinates in our rotated coordinate system and where the constants a to e are not necessarily the same as before.

We already know that if $y = f(x)$ is graphed, then we can shift the graph up by x_0 and to the right by y_0 by suitable transformations of the coordinates: $y - y_0 = f(x - x_0)$ has the shifted graph. Last year, we performed similar transformations on parabolae by completing the square, and so this is the technique which we will use now. We proceed as we did last year (but now completing the square in both x and y):

$$ax^2 + bx + dy + ey^2 = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + e\left(\frac{d}{2e} + y\right)^2 - \frac{d^2}{4e} = 1.$$

If we now let $(x, y) \mapsto \left(x - \frac{b}{2a}, y - \frac{d}{2e}\right)$, then we have got a coordinate transformation that removes all the linear terms and leaves us with something like

$$ax^2 + ey^2 = 1 + \frac{b^2}{4a} + \frac{d^2}{4e}$$

and upon division of both sides by $1 + \frac{b^2}{4a} + \frac{d^2}{4e}$ we end up, as promised, with something of the form $px^2 + qy^2 = 1$. Note that our coordinate system has completely changed: the new curve has the same shape as our original curve, but sits inside our new coordinate system much more naturally than it sat in our old coordinate system.

Exercise 5: Further thoughts on this argument.

You are asked to think critically about the above argument (so reread it and check the computations) and to extend it (so not only must you read it, but understand it).

1. Clearly $y = x^2$ cannot be written in the canonical form. Where does the proof above fall down for a parabola?
2. What happens if our ellipse is a circle?
3. We have now classified the quadratic equations in two variables corresponding to hyperbolae and ellipses ($px^2 \pm qy^2 = 1$) and parabolae ($y^2 = kx^2$). What other forms can the graph of such a quadratic equation take? (Hint: you should be able to find three other kinds, and show that every quadratic equation is of one of the six kinds.)

Exercise 6: Computations.

First, read the following quote:

> The attitude adopted in this book is that while we expect to get numbers out of the machine, we also expect to take action based on them, and, therefore we need to understand thoroughly what numbers may, or may not, mean. To cite the author's favorite motto,
> "The purpose of computing is insight, not numbers," although some people claim,
> "The purpose of computing numbers is not yet in sight."
> There is an innate risk in computing because "to compute is to sample, and one then enters the domain of statistics with all its uncertainties."
>
> — Richard W. Hamming, *Introduction to applied numerical analysis*, McGraw-Hill 1971, p.31.
> (Quoted in `https://mathoverflow.net/a/7164`.)

1. Classify the following conics:
   (a) $\frac{10}{4}x^2 + 3xy + \frac{10}{4}y^2 = 1$
   (b) $4x^2 + 24xy + 11y^2 = 5$
2. What happens to the equation for the hyperbola $x^2 - y^2 = a^2$ upon rotation by $\pi/4$?

3. Give an equation for the parabola with vertex $(0, 2)$, and with axis $y = x + 2$ (opening into the positive x-half plane).

4. Find the two focii of an ellipse passing through $(0, 0)$, $(0, 1)$, and $(1, 2)$.

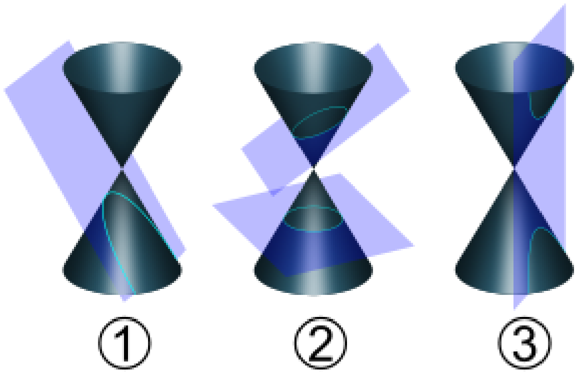
Remark. Felix Klein (a German mathematician who studied, among other things, geometric transformations) defined a geometry as a set of points together with a set of transformations of those points, such that two figures (two collections of points) are said to be 'equivalent' if there exists some invertible transformation mapping the figures onto each other.

For example, in normal Euclidean geometry (our current playground), we say two figures are 'congruent' if they can be placed on top of each other to exactly cover each other: in other words (and more precisely), two figures are congruent if one can be mapped to the other via a series of translations, rotations, and reflections — precisely the case which we have studied in this section. (Such transformations are called *isometries.*) Since all these maps preserve lengths and angles, these things are well-defined in Euclidean geometry.

A less familiar geometry is affine geometry, in which we also allow dilations (scaling) — so all circles are equivalent in affine geometry. This does mean that lengths become meaningless: any line segment can be mapped onto any other line segment, no matter its length. Angles also become meaningless (why?). However, enough properties are left to provide a rich venue for geometry: for example, parallel lines are still parallel after an affine transformation, and so any result which only uses parallelism in its proof makes sense here.

Other geometries include Möbius geometry, where figures are equivalent if they can be mapped onto each other by an isometry or a circle inversion (a reflection through a circle), and projective geometry, which is a further loosening of the conditions on an affine geometry such that even parallel lines make no sense and the only results left are those which just involve incidence and intersection. Perhaps surprisingly, projective geometry is in some sense the most rich geometry of all.

Section 4: Cones



We now move from the planar definitions of the conics to the 'cone-slicing' definitions. Essentially, our goal is to show that the conics are just slices of a cone (hence the name), as in the diagram above.[1]

Exercise 7: Defining a cone.

We will define a cone to be the locus of all lines in three dimensional space which pass through the origin and through a point on some fixed curve in the plane $z = 1$. Note: to show a given object is a cone, one must guess a good defining curve (usually by intersecting it with the given plane) and then do two things: (N1) show that every line through the origin and a point on the guessed curve lies on the object, and (N2) show that every point on the object lies on such a line.

1. Give various non-examples of cones, especially surfaces satisfying critera N1 but not N2 and those satisfying N2 but not N1.
2. Show that $x^2 + y^2 = cz^2$ is a cone whose defining curve is a circle; show that the line through the origin and the centre of this circle cuts the $z = 1$ plane at right angles. (This cone is called a **right circular cone**.)
3. Show that $ax^2 + by^2 + cz^2 = 0$ is a cone for every value of a, b, and c. What is its defining curve?

Exercise 8: Algebraic cone-slicing.

Show that all six kinds of conic sections which you found in exercise 5 are indeed obtained by slicing cones with planes.

Hints:

- Fix a cone, say $x^2 + y^2 = cz^2$, and then slice it with various planes.
- The plane through (x_0, y_0, z_0) which is orthogonal to the line between $(0, 0, 0)$ and (a, b, c) is given by
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$
If you use this fact, you need not prove it.

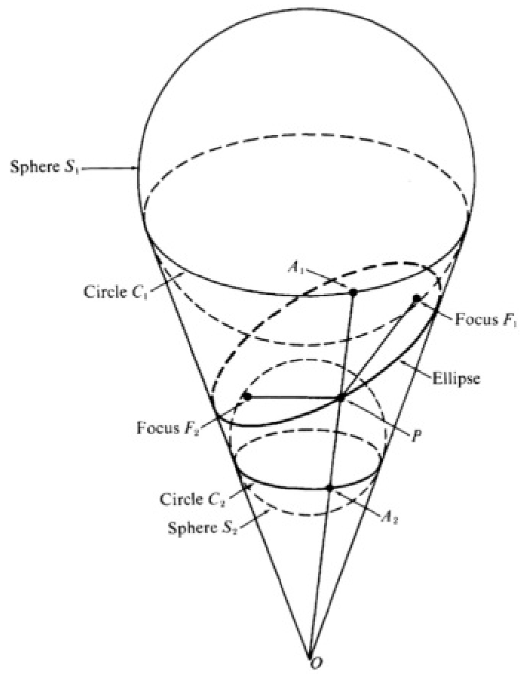
Exercise 9: Geometric cone-slicing: the icecream proof.

In the previous exercise, we had to use coordinates to prove that every slice of a cone is a conic. We will now discover a geometric proof that an ellipse is just a slice of a right circular cone, in such a way that we will understand why such slices obey the focii properties of the ellipse.[2]

Let us take our plane and slice our cone with it. Pick any point P in the intersection of the two surfaces (i.e. on the curve we wish to show is an ellipse). In addition, we will place two spheres S_1 and S_2 into our cone such that they are tangent to both the cone and to the plane. Let F_1 and F_2 be the two points of contact between the plane and the spheres, and let C_1 and C_2 be the circles of contact between the cone and the spheres.

[1] By Pbroks13 - Own work, CC BY 3.0, `https://commons.wikimedia.org/w/index.php?curid=5919064`

[2] Tom M. Apostol, *Linear Algebra: A first course* (pp.78–9). John Wiley & Sons (1997).



1. Show that if X is a point outside some sphere, and if M and N are points on the sphere such that XM and XN are tangent lines to the sphere, then $|XM| = |XN|$. (Hint: consider the cross-section of the sphere obtained by slicing it with the plane of the triangle XMN.)

2. Suppose the vertex of our cone is O. Show that if A_1 and A_2 are the points of intersection between the line OP and the two circles C_1 and C_2 then $|PF_1| = |PA_1|$ and $|PF_2| = |PA_2|$.

3. Show that $|PA_1| + |PA_2|$ is independent of the point P.

4. Conclude that $|PF_1| + |PF_2|$ is constant for every value of P, and therefore that the locus of P is an ellipse with focii F_1 and F_2.

To do: same result but for hyperbolae.

Section 5: Orthogonal families of conics

In this section, we will consider families of conics with the same focii, and we will find all the curves which are orthogonal to each family.

Definition. Two curves are said to be **orthogonal** if, at every point where they intersect, they have tangent lines which meet at right angles. Equivalently, two curves are orthogonal if at each point of intersection the normal line of one is the tangent line of the other.

Let us first revise a result which you should now have proved twice (both in Y12 coordinate geometry and in Y13 calculus).

Exercise 10: Orthogonality.

Show that, if $y = f(x)$ is a simple curve (i.e. a curve which does not cross itself) that is differentiable at $P = (x_0, y_0)$, then there is a unique line N passing through P that is orthogonal to the curve at P; further, the slope of N is $-\frac{1}{f'(x_0)}$.

If the reader is feeling clever, she should prove it in two different ways. (Hint: you can prove it via drawing triangles — this is the proof given in the Y13 calculus notes — or by rotating your line.)

Remark. In the exercise above, we implicitly assumed in the statement that our curve was the graph of a function; however, this need not be the case. We only need the curve to be a function locally (i.e. for there to exist a small ball about (x_0, y_0) within which our curve is the graph of a function). You should check that your proof only uses local properties of the graph of f!

Exercise 11: Homogenous differential equations.

Suppose y is a function of x. Recall that a differential equation is said to be **separable** if it can be written in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)Q(y)$, where P and Q are functions of x and y alone.

1. Show, using the chain rule (i.e. without treating $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a fraction), that $\int \frac{1}{Q(y)}\,\mathrm{d}y = \int \frac{\mathrm{d}}{x}$.
2. If a curve with derivative $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin x}{\cos y}$ passes through $(0, 0)$, which y values does it take when $x = \pi$?
3. Find an explicit formula for a function f defined for all positive numbers such that when $y = f(x)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{1-x}$ and $f(1) = 0$.

The differential equations we wish to consider later in this section are, unfortunately, *not* separable. However, they are nice enough that we can apply a change of variables to *make* them separable.

We say that a function of two variables $P(x, y)$ is **homogenous of degree** n if $P(tx, ty) = t^n P(x, y)$ for all x and y.

4. Suppose that $P(x, y)$ and $Q(x, y)$ are both homogenous of degree n. Show that $f(x, y) = \frac{P(x,y)}{Q(x,y)}$ is homogenous of degree 0.
5. Let us consider the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, where f is as in the previous question. Let $t = 1/x$; show that, using the substitution $z = y/x$ and homogeneity, we obtain the separable equation

$$z + x\frac{\mathrm{d}z}{\mathrm{d}x} = f(1, z) = \frac{P(1, z)}{Q(1, z)}.$$

   Find the solution of this equation.

6. Using this technique, solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2-2y^2}{xy}$. (Be sure to check that the technique is applicable: i.e. that the right hand side is homogenous of degree zero.)

In a break from form, I will do a nice easy example to start with. Suppose we consider the family of curves consisting of all the circles whose centre is the origin; it should be intuitively clear that the only curves orthogonal to all of these circles at once are the lines through the origin.

As a first step, we know that every such circle has equation $x^2 + y^2 = R^2$ for some constant R. Given any particular circle, its derivative is given by $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$; since the parameter R is absent from this equation, it describes every such circle. (We have shown that every such circle has this derivative, but you should check yourself, by integrating, that every equation of this form is a circle centred at the origin.)

We want to find all the curves such that, at every point (x, y) on our curve, the tangent line of our curve is the normal line of the circle centred at the origin and passing through that point. To do this, we simply need to solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}$$

and this is not only separable but easy: $\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x}$ and thus $\ln|x| = \ln|y| + C$, or $x = Ky$; and this is just the family of all lines through the origin, as we guessed initially.

For convenience, if $\mathcal{F}$ is a family of curves then we will call the family of curves orthogonal to $\mathcal{F}$ the **orthogonal complement** of $\mathcal{F}$.

Exercise 12: Orthogonal families of curves.

We (or rather you, the reader) will now show that when we take orthogonal complements of various families of conics, then we obtain other families of conics. As well as the results explicitly stated here, you should consider how the families are related to each other. Show that:

1. The family of NE-SW diagonal hyperbolae, $xy = c$, are orthogonal to the family of NW-SE diagonal hyperbolae, $xy = -c$.
2. The vertical parabolae, $y = ax^2$, are orthogonal to the family of ellipses $x^2 + \frac{y^2}{b} = 1$. (See `parabolae-and-ellipse.ggb`.)
3. The family of all circles tangent to the y-axis at the origin (i.e. $x^2 + y^2 = 2rx$) is orthogonal to the family of all circles tangent to the x-axis at the origin (i.e. $x^2 + y^2 = 2ry$).

Hence, complete the following: orthogonal complements of X are Y, taking X to be hyperbolae, parabolae, circles, and ellipses. (We have found the orthogonal complements of families of curves in particular places, but consider the hint after exercise 13.)

Finally, we have one rather pretty example.

Exercise 13: Circles of Apollonius.

Fix two points A and B. Let us consider the locus of all points P such that $\frac{|AP|}{|BP|} = \rho$, for positive numbers ρ; we will examine one particular perpendicular family.

1. Show that if we fix A and B and vary ρ then we have a family of circles whose centres lie on the line AB.
2. Show that the orthogonal complement of this family is the set of circles passing through A and B with centres on the perpendicular bisector of the segment AB.
3. Use Geogebra (or similar software) to draw a nice picture.

Hint: without loss of generality, you can suppose $A = (-1, 0)$ and $B = (0, 1)$. (Then the result follows for any choice of A and B, simply because dilations, translations, and rotations all preserve circles and orthogonality of lines.)

There are three ways to do this exercise that the author is aware of: perhaps the easiest for us now is to apply calculus like we have been doing throughout this section. For a geometric view on this exercise, see Coxeter §6.6; for a view from complex calculus, see Fisher §1.2.

Exercise 14: Bipolar coordinates (difficult).

(Prerequisite: you do need to understand exercise 13 quite well. You should also read the epilogue on hyperbolic trigonometric functions in the trigonometry notes.)

From your pictures in the previous exercise, you should notice that every point in the plane is at the intersection of two circles of apollonius. Let us fix $A = (-1, 0)$ and $B = (1, 0)$, as in the hint for the previous exercise; we will use these circles to define a new coordinate system for the plane.

Suppose $P = (x, y)$ lies at the intersection of two circles; then we define σ to be the angle APB, and $\tau = \ln \frac{|PA|}{|PB|}$. We then define the **bipolar coordinates** of P to be (σ, τ).

1. Show that
$$\begin{cases} x = \frac{\sinh \tau}{\cosh \tau - \cos \sigma} \\ y = \frac{\sin \sigma}{\cosh \tau - \cos \sigma}. \end{cases}$$
2. Identifying the plane $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$, show that this is equivalent to $x + iy = ai \cot\left(\frac{\sigma + i\tau}{2}\right)$. (Recall that we define $\cot z = \frac{e^{iz} - e^{-iz}}{ie^{iu} + ie^{-iu}}$.)

3. Identify the locus of all $P = (\sigma, \tau)$ such that (a) σ is held constant, or (b) τ is held constant.
4. Translate the standard equations of the conics into bipolar coordinates.

Section 6: Applications to physics

To do: gravity laws.

Section 7: Further reading

See references in text, as well as:

- H.S.M. Coxeter, *Introduction to geometry.* John Wiley & Sons (1961).
- Stephen D. Fisher, *Complex Variables.* Dover (1999).
- Keith Kendig, *Conics.* Mathematical Association of America (2005).
- Morris Kline, *Mathematics for the nonmathematician.* Dover (1985).
- George Salmon, *A treatise on conic sections.* Longmans, Green (1900).
- George F. Simmons, *Differential equations with applications and historical notes.* McGraw-Hill (1991).