NCEA Level 3 Trigonometry (exercise set)
2. Definitions

Goal To use some of the fundamental properties of sin and cos.

1. We can use proposition 2.7 to prove various identities relating sin and cos.
   (a) Prove that for all θ, $\cos\left(\frac{\pi}{2}+\theta\right)+\cos\left(\frac{\pi}{2}-\theta\right)+\sin(\pi+\theta)+\sin\theta=0$.
   (b) Prove that, if α, β, and γ are the interior angles of a triangle, then $\sin(\gamma/2)=\cos[(\alpha+\beta)/2]$.

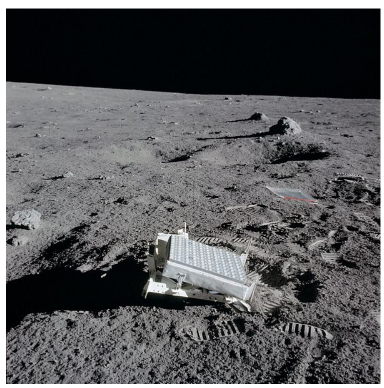
2. Some basic practical applications of trigonometry occur when surveying land.

   From the top of a tower, 36 m high, the corners A, B, C of a triangular plane in the horizontal plane through the bottom of the tower are observed to have respective bearings (measured anticlockwise from due north)

$$72°18' \qquad 104°37' \qquad 157°13'$$

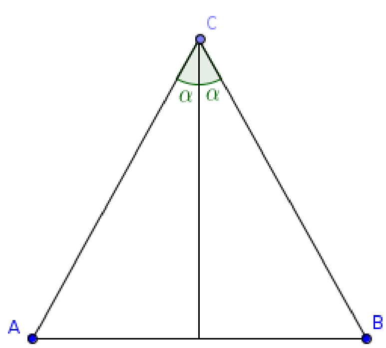
   (where the notation $24°4'$ means "24 degrees, plus four-sixtieths of a degree"); the angles of depression of the corners (angles between the vertical and the line of sight) are respectively $34°$, $21°$, and $43°$. What is the area of the field?

3. In the late 1960's and early 1970's, the Apollo crewed moon missions left mirrors on the surface of the moon. This means that the distance from the earth to the moon can be measured by firing a beam of light at the mirror, and measuring the time it takes to return.



   The speed of light is around $3.00 \times 10^8\,\mathrm{m\,s^{-1}}$.
   (a) Such an experiment finds that the time taken from emission of light to the detection of the reflected pulse is 2.5 s. How far away is the moon, approximately?
   (b) A 10-cent coin (diameter 20 mm) is found to cover the moon exactly when held at a distance 2.2 m away from the eye. What is the approximate diameter of the moon?

4. Draw an isosceles triangle, with equal sides 1 and included angle 2α; draw the perpendicular line from the included angle to the base.



Let D be the intersection point between BC and the perpendicular line through A. Show that:

- The base length, AB, is $2\sin\alpha$;
- The perpendicular segment length, AD, is $\sin 2\alpha$;
- The interior angle at B is $\pi/2 - \alpha$.

Hence, by considering the triangle ABD, show that $\sin 2\alpha = 2\sin\alpha\cos\alpha$.

5. We will now generalise question 4. An *isometry* is a mapping ι in the plane (i.e. a function taking points to points) which preserves lengths; in other words, ι is an isometry if for each pair of points A and B, $|AB| = \big|\iota(A)\iota(B)\big|$. (Recall that if f is a function/map (they mean the same thing) then $f(x)$ denotes the output of f when fed the input x.)

   (a) Let $X = (x_1, x_2)$ be any point except $(0,0)$. Define the mapping T_X to be the function that sends each point $P = (p_1, p_2)$ to $T_X(P) = (p_1 + x_1, p_2 + x_2)$.

   - Show that T_X is an isometry (called the *translation* through X).
   - Show that there is **no** point F such that $T_X(F) = F$.
   - Draw the action of T_X on points in the plane. ('Draw a picture showing what T_X does.')

   (b) Fix some angle $\theta \neq 0$. Show that the mapping R_θ, defined by

   $$R_\theta\big((x,y)\big) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

   is an isometry. Does there exist any point F such that $R_\theta(F) = F$? How many? Draw the action of F on points in the plane.

   (c) Let θ and ϕ be two angles. Show that

   $$R_\phi\Big(R_\theta\big((1,0)\big)\Big) = (\cos\theta\cos\phi - \sin\theta\sin\phi, \cos\theta\sin\phi + \sin\theta\cos\phi).$$

   By considering the geometric meaning of R_θ, conclude that $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ and $\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$. [1]

Additional reading Hobson, chapter III. For a nice treatment of isometries, see Coxeter chapter 3.

[1] Note: a proper proof of this last identity is given as theorem 4.1 in the notes. The issue with this heuristic argument here is that we have not got a formal notion of 'rotation' that allows us to conclude that if we do a rotation by one angle, followed by a rotation by another, then we do indeed get a rotation through the sum of the angles. If we take the above to be the *definition* of a rotation, then when we prove this identity using the properties of sin and cos then we will have *shown* that the composition of two rotations is the rotation whose angle is the sum of the first two. Combined with the fact that the map is an isometry that fixes a single point, we will have proved that the map R_θ does indeed satisfy all three properties that it 'should' satisfy in order to be called a rotation. If you don't get what the problem is here, it's not the end of the world, it is a little subtle.