NCEA Level 3 Calculus (Differentiation)
4. The Chain Rule (Homework)

Reading

Suppose we have the function $y = \sin(x^2)$ which we saw in the tutorial. Consider what happens if we let x change by a small amount, to $x + \mathrm{d}x$. Then y will change to $y + \mathrm{d}y$, and we have

$$y + \mathrm{d}y = \sin((x + \mathrm{d}x)^2) = \sin(x^2 + 2x\,\mathrm{d}x + \mathrm{d}x^2).$$

Let's let $\mathrm{d}x$ become really small; so $\mathrm{d}x^2$ becomes even smaller. In fact, we will take $\mathrm{d}x$ to be so small that $\mathrm{d}x^2 \to 0$. Then we can basically ignore it, and continue computing:

$$\sin(x^2 + 2x\,\mathrm{d}x + \mathrm{d}x^2) = \sin(x^2 + 2x\,\mathrm{d}x) = \sin(x^2)\cos(2x\,\mathrm{d}x) + \cos(x^2)\sin(2x\,\mathrm{d}x).$$

Now, when t is small, $\sin(t) \approx t$. Similarly, $\cos(t) \approx 1$. Since $\mathrm{d}x$ is small, we apply these approximations:

$$\sin(x^2)\cos(2x\,\mathrm{d}x) + \cos(x^2)\sin(2x\,\mathrm{d}x) = \sin(x^2) + 2x\,\mathrm{d}x\cos(x^2).$$

But recall that this is equal to $y + \mathrm{d}y$; so

$$\begin{aligned} y + \mathrm{d}y &= \sin(x^2) + 2x\,\mathrm{d}x\cos(x^2) \\ \mathrm{d}y &= 2x\,\mathrm{d}x\cos(x^2) \\ \tfrac{\mathrm{d}y}{\mathrm{d}x} &= 2x\cos(x^2). \end{aligned}$$

Note that this computation is most definitely **not rigorous** — we don't justify why we can ignore $\mathrm{d}x^2$ but not $\mathrm{d}x$, and we don't define what it even means to be "small enough to ignore"! However, it does at least suggest that, intuitively, the chain rule does what we expect it to do.

This calculation can be made rigorous if we define infinitesimals and rules for calculating with them, and this branch of mathematics is known as non-standard analysis. However, for the remainder of this year we will continue to base our discussions of calculus on limits and inequalities because they are easier to make rigorous, despite the initial barriers to intuition.

Questions

1. If $y = \sqrt{\cot x} - \sqrt{\cot a}$ (where a is constant), find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
2. (a) Show that if $y = f(g(h(x)))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) \cdot g'(h(x)) \cdot f'(g(h(x)))$.
   (b) Calculate the derivative of $y = \sin\cos\sin\cos\sin x^5$.
3. We will prove the double angle formula for cosine from the double angle formula for sine. Suppose $f(\theta) = \cos 2\theta$, and $g(\theta) = 1 - 2\sin^2\theta$.
   (a) Show that $f' = g'$. (You may assume that $\sin 2\theta = 2\sin\theta\cos\theta$.)
   (b) Verify that f and g agree at $\theta = 0$, and conclude that $f = g$.