

# NCEA Level 3 Calculus (Differentiation)

## 8. Optimisation

Recall from Level 2 that a *local maximum* of a function  $f$  is some point  $(x, f(x))$  such that, for a sufficiently small interval  $I$  around  $x$ , for all  $y \in I$   $f(y) \leq f(x)$ ; a *local minimum* is defined in a similar way (see exercises).

**Example.** The function  $x \mapsto x^2$  has a local minimum at  $(0, 0)$ .

**Example.** The function  $x \mapsto 2x^3 + 15x^2 + 36x + 2$  has a local maximum at  $(-3, -25)$  and a local minimum at  $(-2, -26)$ .

**Example.** The function  $x \mapsto \sin x$  has a local maximum at  $(2n\pi + \frac{\pi}{2}, 1)$  for every  $n \in \mathbb{Z}$ , and a local minimum at  $(2n\pi - \frac{\pi}{2}, -1)$  for every  $n \in \mathbb{Z}$ .

Local extrema are also sometimes called *relative extrema*.

It is possible to prove that if  $f$  is defined around some value  $x$ , if  $f$  has a relative extremum at  $(x, f(x))$ , and if  $f'$  is defined at  $x$ , then  $f'(x) = 0$  (Theorem 1.10 in the notes). Because of this, we define a *critical point* of a function  $f$  to be some value  $x$  such that either  $f'(x) = 0$ , or  $f'(x)$  is undefined. In the first case, we also call the value a *stationary point*.

**All local extrema occur at critical points, but not all critical points occur at extrema.**

**Example.** The function  $x \mapsto 2x^3 + 15x^2 + 36x + 2$  above has critical points  $x = -2$  and  $x = -3$ . Both of these are local extrema.

**Example.** The function  $x \mapsto x^3$  above has a critical point at  $x = 0$ , but does not have a local extrema there.

**Example.** The function  $x \mapsto \frac{1}{x}$  does not have a critical point at  $x = 0$ , **because it is not defined there.**

We can use the first derivative to classify extrema as either maxima or minima.

### The First Derivative Test

1. Determine all critical points of  $f$ .
2. Determine the sign of  $f'(x)$  to the left and right of each critical point  $x_0$ :
  - If  $f'(x)$  changes from positive to negative as we move from left to right across  $x_0$ , then  $f(x)$  has a local maximum at  $x_0$ .
  - If  $f'(x)$  changes from negative to positive as we move from left to right across  $x_0$ , then  $f(x)$  has a local minimum at  $x_0$ .
  - If  $f'(x)$  does not change sign across  $x_0$ , then  $f(x)$  does not have a relative extremum at  $x_0$  (e.g.  $y = x^3$ ).

Recall last week's material; using the second derivative, we can come up with a second test:

### The Second Derivative Test

1. Compute  $f'(x)$  and  $f''(x)$ .
2. Find all the stationary points of  $f$  by finding all the points  $x_0$  such that  $f'(x_0) = 0$ .
3. Determine the sign of  $f''(x)$  for each stationary point  $x_0$ :
  - If  $f''(x_0) < 0$ , then  $f(x)$  has a relative maximum at  $x_0$ .
  - If  $f''(x_0) > 0$ , then  $f(x)$  has a relative minimum at  $x_0$ .
  - If  $f''(x_0) = 0$ , then  $f(x)$  could have a relative maximum, a relative minimum, or neither.

**Example.** Find and classify the critical points of  $y = x^3 - 3x^2 + 6$ .

*Solution.* We have  $\frac{dy}{dx} = 3x^2 - 6x$  and  $\frac{d^2y}{dx^2} = 6x - 6$ . Hence the critical points are  $x = 0$  and  $x = 2$ . At the former point,  $\frac{d^2y}{dx^2} < 0$ , and so the point is a maximum; at the latter point,  $\frac{d^2y}{dx^2} > 0$  and so the point is a minimum.

**Example.** Find two numbers whose difference is 100 and whose product is a minimum.

*Solution.* Let the two numbers be  $x$  and  $x + 100$ . We wish to minimise  $y = x(x + 100)$ ; clearly  $y' = 2x + 100$ , and so  $x = 50$  is a critical point. To the left of  $x = 50$ , the derivative is negative; to the right, the derivative is positive. Hence  $x = 50$  is indeed a minimum. The two required numbers are therefore 50 and 150.

**Example.** Find and classify the critical points of  $y = (x - 1)^2 + \ln x$ .

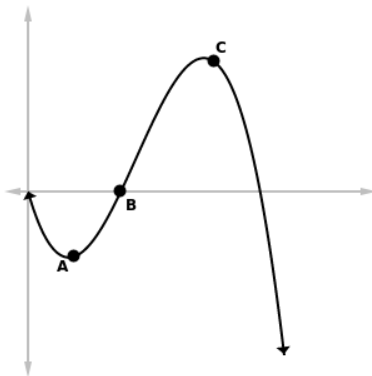
*Solution.* The derivative is  $y' = 2x - 2 + \frac{1}{x}$ . We therefore have one critical point at  $x = 0$  (where  $y'$  is undefined); this is an asymptote. Setting  $y' = 0$ , we have  $0 = 2x - 2 + \frac{1}{x} = 2x^2 - 2x + 1$  which has no real roots. Hence  $x = 0$  is the only critical point, and the curve has no local extrema.

**Example.** A rectangular plot of land is to be fenced using two varieties of fence. Two opposite sides will use fences selling for \$3 per metre, while the other two sides will use cheaper fence selling for \$2 per metre. Given that the total budget is \$1200, what is the greatest area of land which can be fenced?

*Solution.* Let  $x$  be the length of one of the expensive sides; then the length of one of the cheaper sides is  $\frac{1}{2}(1200 - 3x)$ , and the total area is  $A = \frac{1}{2}x(1200 - 3x) = \frac{1}{2}(1200x - 3x^2)$ . Hence  $\frac{dA}{dx} = 600 - 3x$ . We wish to find the maximum area, so set  $\frac{dA}{dx} = 0$ ; hence  $3x = 600$  and  $x = 200$ . Note that the second derivative is always negative, so this stationary point must be a maximum as required. The length of the other side will be  $\frac{1}{2}(1200 - 600) = 300$ , and so the maximum area is  $300 \times 200 = 60000$  square metres.

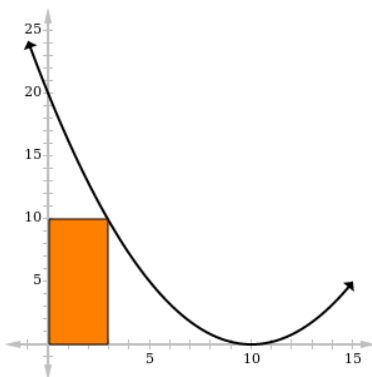
## Questions

1. A Write down a definition of a local minimum similar to that given for a maximum.
2. A Show that  $f(x) = x^4$  has  $f''(0) = 0$  but not a point of inflection at  $x = 0$  (in fact, it has a minimum at that point).
3. A Describe the advantages and disadvantages of the first and second derivative tests for local extrema.
4. M Describe the local extrema, concavity, and points of inflection of the function  $f(x) = x^4 - 4x^3$ .
5. A Consider the following graph:



Find the signs of  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at the three points  $A$ ,  $B$ , and  $C$ .

6. M Find all the local extrema of the following curves in the given intervals, and classify them as maxima, minima, or neither.
  - (a)  $f(x) = \sin x - \cos x$  on the interval  $0 < x < \pi$
  - (b)  $g(x) = x^3 - x^2 + x - 1$  on the interval  $-\infty < x < \infty$
7. M The sum of two positive numbers  $x$  and  $y$  is 16. Find the smallest possible value for  $S = x^2 + y^2$ .
8. M A box with an open top is to be constructed from a square piece of cardboard with a side length of 3 m by cutting out a square from each of the four corners and bending up the sides. Find the dimensions of the resultant box of maximum volume.
9. M Find the dimensions of a rectangle with area  $1000 \text{ m}^2$  such that the perimeter is minimised.
10. E A large orange rectangle is to be drawn with one corner sitting on the origin and the opposite corner lying on the curve  $y = 0.2(x - 10)^2$ . What is the maximum possible area of the rectangle?



11. M A window consisting of a rectangle topped with a semicircle is to have a fixed perimeter  $p$ . Find the radius of the semicircle in terms of  $p$  if the total area is to be maximised.
12. E A thin wire of length  $L$  is cut in two and the resulting lengths are bent to make a square and an equilateral triangle. Where should the wire be cut to make the total area of the shapes (a) a maximum and (b) a minimum?
13. E Find the point on the line  $y = 2x + 3$  closest to the origin.
14. E Find the point on the curve  $y = \sqrt{x}$  closest to  $(3, 0)$ .
15. M The rate in which photosynthesis takes place for a species of phytoplankton is modeled by the function

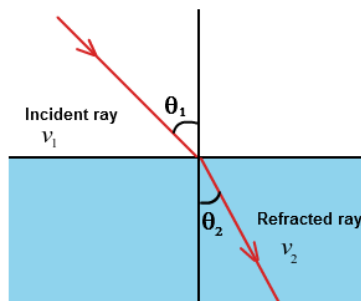
$$P = \frac{100I}{I^2 + I + 4}$$

For which light intensity  $I$  is  $P$  a maximum?

16. E By finding the  $x$ - and  $y$ - intercepts, the asymptotes, the critical points, the intervals of increase and decrease, the intervals of concavity, and any other important points, sketch the following functions (199):
  - (a)  $f(x) = \frac{x^2}{4-x^2}$
  - (b)  $f(x) = \frac{4x}{x^2+1}$  [Hint: consider what happens to  $f(x)$  as  $x \rightarrow \pm\infty$ .]
  - (c)  $f(x) = \frac{x^2-4x+5}{x-2} = x - 2 + \frac{1}{x-2}$  [Hint: consider what happens to  $f(x) - (x - 2)$  as  $x \rightarrow \pm\infty$ .]
17. E A cone with height  $h$  is inscribed in a larger cone of height  $H$  such that the vertex of the small cone is at the centre of the base of the larger cone. Show that the maximum volume of the smaller cone occurs when  $h = \frac{1}{3}H$ .
18. E Let  $v_1$  be the velocity of light in air and  $v_2$  be the velocity of light in water. A ray of light will travel from a point  $A$  in the air to a point  $B$  in the water by a path  $ACB$  that minimises the time taken. Show that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

where  $\theta_1$  is the angle of incidence and  $\theta_2$  is the angle of refraction.



19. E Show that the polynomial  $p(x) = 10x^3 + x^2 + x - 34$  has exactly one real zero.
20. E A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one third of the sheet on each side by an angle  $\theta$ . What angle should be chosen in order to obtain the maximum possible volume?
21. E A steel pipe is carried around a right-angled corner from a hallway 3 m wide into a hallway 2 m wide. What is the length of the longest pipe that can be carried horizontally around the corner? [Hint: this is actually a minimisation problem, despite the wording.]

22. E Find and classify the critical points of  $h(x) = x^4 + x^3 + cx^2$ .
23. S Show that  $\frac{x^2+1}{x} \geq 2$ ; hence (or otherwise) show that  $\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geq 8$ .
24. S Scholarship 2013: Prince Ruperts drops are made by dropping molten glass into cold water. A mathematical model for a drop as a volume of revolution uses  $y = \sqrt{\phi(e^{-x} - e^{-2x})}$  for  $x \geq 0$ , where  $\phi$  is the golden ratio  $\phi = \frac{1+\sqrt{5}}{2}$ .
- (a) Where is the modelled drop widest, and how wide is it there?
- (b) The drop changes shape at a point  $B$ , where the concavity of the function is zero. Use

$$\frac{d^2y}{dx^2} = \sqrt{\phi} \frac{e^{2x} - 6e^x + 4}{y^2 e^{4x}}$$

to find the exact  $x$ -ordinate of  $B$ .