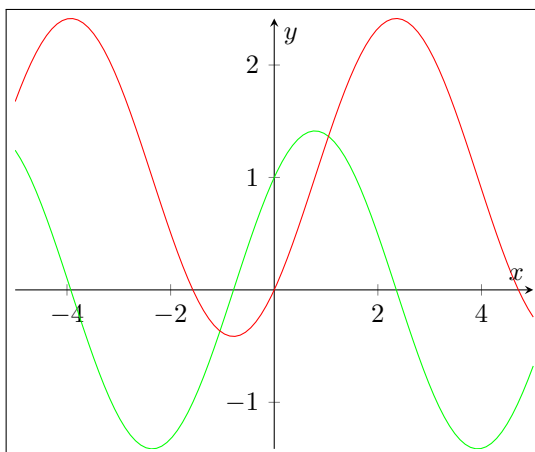


NCEA Level 3 Calculus

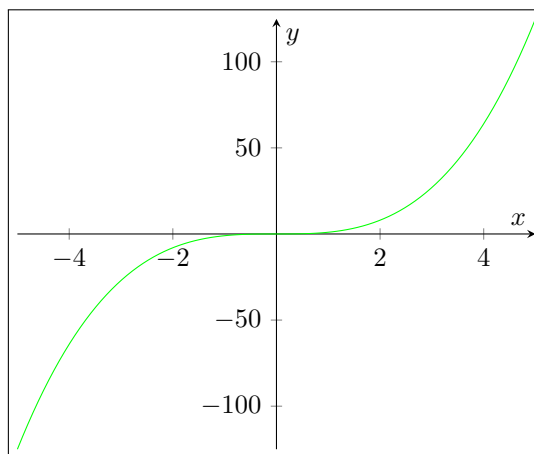
Solutions to Homeworks

1. The Derivative

1. Green: derivative of function. Red: original function.

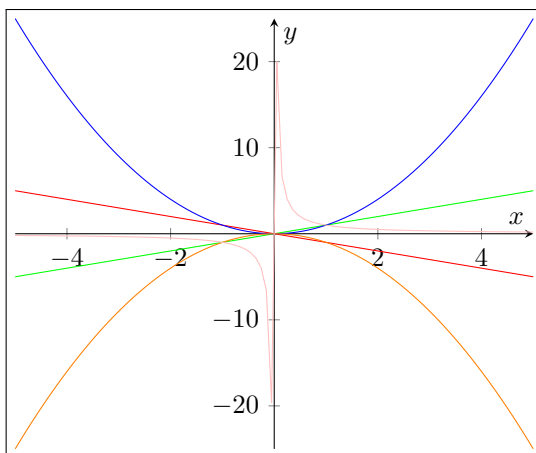


2. (a) At a min or max, the function is momentarily horizontal and so has slope zero; so $m = 0$.
(b) Consider a graph like the following at $(0, 0)$:

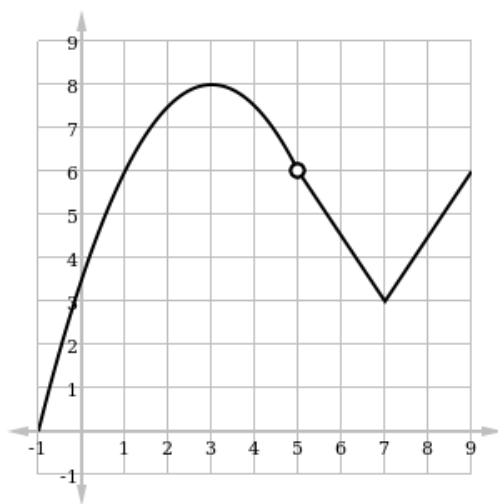


2. Limits

1. Increasing: the graph of the function is sloping up (green). Decreasing: the graph of the function is sloping down (red). Concave up: the graph of the function is increasing in slope (it is like a cup \cup) (blue). Concave down: the graph of the function is decreasing in slope (it is like a cap \cap) (orange). Continuous: the function has no holes (all of them except pink).



2. (a) $\lim_{x \rightarrow -2} f(x) = 0$, $\lim_{x \rightarrow 2} f(x) = -0.5$.
 (b) No, it approaches different values from the left and the right.
 (c) Yes, because the function is continuous there.
 (d) $(-\infty, -3)$, $(-3, -1)$, $(-1, 2)$, $(2, \infty)$.
 (e) -3, -1, 2.
3. For example,



3. Derivatives of Common Functions

1. (a) $2x + 1/x$
 (b) $t^2 x^{t-1}$
 (c) $\cos x + \sin x$
 (d) $\frac{4}{5} x^{-1/5} = \frac{4}{5\sqrt[5]{x}}$
2. The power x is not constant.

3. Consider x^{-n} . The first derivative is $-nx^{-n-1}$, the second is n^2x^{-n-2} , and so the n th is $(-1)^n n^n x^{-2n}$. [This can be proved via induction.]
4. (a) Note first that $10^t = e^{t \ln 10}$, so $P = P_0 + e^{t \ln 10}$ and $\frac{dP}{dt} = (\ln 10)e^{t \ln 10} = (\ln 10)10^t$. At $t = 100$, we have $\frac{dP}{dt} = 2.3 \times 10^{100}$.
- (b) Real-world populations don't grow exponentially forever if there are finite resources (e.g. food).

4. The Chain Rule

1. $\frac{dy}{dx} = \frac{-\csc^2 x}{2\sqrt{\cot x}}$
2. (a) Simply apply the chain rule twice.
 (b) $y' = 5x^4(\cos x^5)(-\sin \sin x^5)(\cos \cos \sin x^5)(-\sin \sin \cos \sin x^5)(\cos \cos \sin \cos \sin x^5)$.
3. (a) $f'(\theta) = -2 \sin 2\theta$ and $g'(\theta) = -4 \sin \theta \cos \theta = -2 \sin 2\theta$, so $f' = g'$ as they agree everywhere.
 (b) Since f and g have the same derivative, they differ only by a constant. But $f(0) = 1 = g(0)$, so that constant is zero; hence $f = g$.

5. The Product and Quotient Rules

1. (a) $\cos x \ln x + \frac{\sin x}{x}$
 (b) $\sec kx + kx \sec kx \tan kx$
 (c) $\frac{-\pi(\sin \pi\theta + \cos \pi\theta) \sin \pi\theta - \pi(\cos \pi\theta - \sin \pi\theta) \cos \pi\theta}{(\sin \pi\theta + \cos \pi\theta)^2}$
 (d) $(\cos t)(3 \sin^2 t)(-\sin(\sin^3 t))(4 \cos^3 \sin^3 t)$.

2.

$$\begin{aligned}
 F &= \frac{d}{dt} \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = a \frac{d}{dv} \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \\
 &= a \left(\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{m_0 v^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \right) \\
 &= a \left(\frac{m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} + \frac{m_0 v^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \right) \\
 &= m_0 a \left(\frac{c^2 \left(1 - \frac{v^2}{c^2}\right) + v^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \right) \\
 &= m_0 a \left(\frac{c^2 - v^2 + v^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \right) \\
 &= \frac{m_0 a}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}.
 \end{aligned}$$

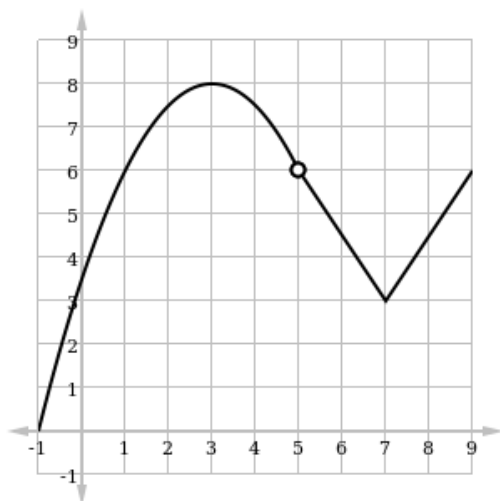
3. We wish to find $\frac{d}{d\theta} \sin \text{rad}(\theta)$, where $\text{rad}(\theta) = \frac{\pi\theta}{180}$; so $\frac{d(\text{rad})}{d\theta} = \frac{\pi}{180}$ and $\frac{d}{d\theta} \sin \text{rad}(\theta) = \frac{\pi\theta}{180} \cos \text{rad}(\theta)$. [The reason we have to do this is that the derivative of \sin uses the limit $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ which is false if x is in degrees.]

6. Tangent and Normal Lines

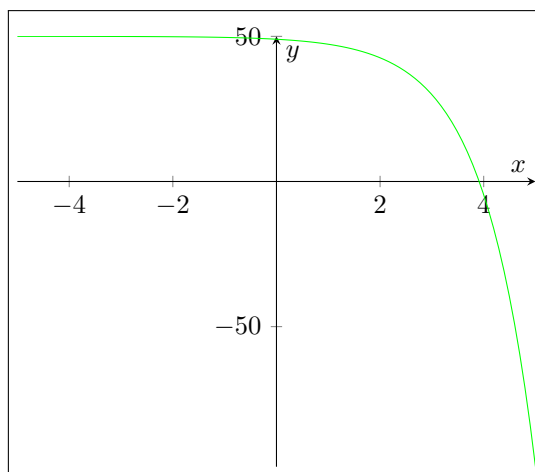
1. The normal to a curve f at a point $(x_0, f(x_0))$ is the unique line passing through that point that is perpendicular to the tangent line of f at that point.
2. $y' = \frac{-\sin(x+\pi)}{2\sqrt{\cos(x+\pi)}} + \cos x - 4 \sec x \tan^2 x e^{2 \tan^2 x}$; at $x = \pi$, $y' = -1$ and so the tangent line (best linear approximation) is $y = -(x - \pi) = \pi - x$.
3. Since the normal line has slope 3, the tangent line has slope $-1/3$. We can take any curve through $(1, 0)$ with this slope, so we may as well take the tangent line itself: $y = -\frac{1}{3}(x - 1) = \frac{1}{3} - \frac{1}{3}x$.
4. $\frac{dy}{dx} = \frac{1}{(1+3x)^{2/3}}$ and at $x = 0$ the slope becomes 1. So the best linear approximation around the point $(0, 1)$ is just $\tilde{y} = x + 1$. So at $x = 0.01$, we have $\tilde{y} = 1.01$ as our approximate value of $\sqrt[3]{1.03}$. [The true value is around 1.0099, so we are not too far off.]

7. Higher Derivatives and the Geometry of a Function

- The second derivative tells us the concavity of a function: if the second derivative is positive, the function is curving up and if it is negative then the function is curving down.
- $f'(x) = 5x^4 - 5$, $f''(x) = 20x^3$.
 - $f'(x) = \frac{x^2-2x}{(x-1)^2}$, $f''(x) = \frac{2x-2}{(x-1)^4}$.
 - $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{4}x^{-3/4}$, $f''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{16}x^{-5/4} = \frac{3}{16\sqrt[4]{x^5}} - \frac{1}{4\sqrt{x^3}}$.
- For example,



- For example,



8. Optimisation

- At some point x , the distance between the two parabolae is $\delta(x) = (x^2 + 1) - (x - x^2) = 2x^2 - x + 1$. Taking the derivative, we find $\delta'(x) = 4x - 1$ which has a single zero at $x = 1/4$; by looking at the graph of the two parabolae, we see that this must be the location of the minimum distance $\delta(1/4) = 7/8$ units.

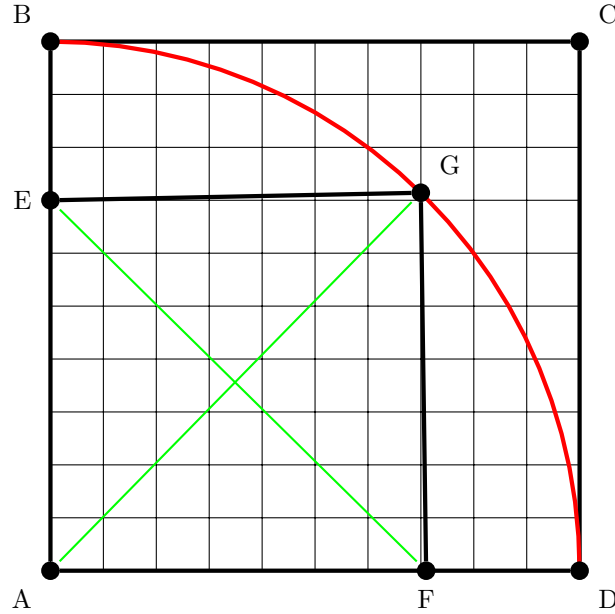
2. If $y = 3x + 2 \cos x + 5$, then $\frac{dy}{dx} = 3 - 2 \sin x$. Since $1 \geq \sin x$, $3 - 2 \sin x \geq 1$. In particular, the function is everywhere increasing. Now, note that when $x = -200\pi$, $y = -600\pi + 7 < 0$, and when $x = 200\pi$, $y = 600\pi + 7 > 0$. Since the function is continuous over this interval, it follows that at some point it passes through the y -axis and has at least one root; since it is increasing everywhere, it must have exactly one real root.

3. The area of such a rectangle will be $A = 4xb\sqrt{1 - \frac{x^2}{a^2}}$; so

$$\frac{dA}{dx} = 4b\sqrt{1 - \frac{x^2}{a^2}} - \frac{4x^2b}{a^2\sqrt{1 - \frac{x^2}{a^2}}}.$$

Setting this to zero, we have $a^2 = 2x^2$ and so $2x = \sqrt{2}a$. It follows that $2y = b\sqrt{2}$, and so the maximal area is $2ab$.

4. Consider the following diagram.



It should be clear that $AG = 1$; call $\angle AEG = \theta$ and $\angle AFG = \phi$, and let $AE = EG = e$ and $AF = GF = f$. By the cosine rule, we have $1 = 2e^2(1 - \cos \theta)$ and $1 = 2f^2(1 - \cos \phi)$. Now, the area of the triangle $\triangle AEG$ is given by $\frac{1}{2}\sqrt{e^2 - \frac{1}{4}}$; the area of $\triangle AFG$ is given by $\frac{1}{2}\sqrt{f^2 - \frac{1}{4}}$. Since $AEFG$ is a (convex) quadrilateral with two right angles, $\theta + \phi = \pi$. Putting this all together, the area of the quadrilateral is $A = \frac{1}{2}\sqrt{e^2 - \frac{1}{4}} + \frac{1}{2}\sqrt{f^2 - \frac{1}{4}}$. We have that $e^2 = \frac{1}{2(1 - \cos \theta)}$ and $f^2 = \frac{1}{2(1 - \cos(\pi - \theta))} = \frac{1}{2(1 + \cos(\theta))}$, so the area in terms of θ is

$$A = \frac{1}{2}\sqrt{\frac{1}{2(1 - \cos \theta)} - \frac{1}{4}} + \frac{1}{2}\sqrt{\frac{1}{2(1 + \cos \theta)} - \frac{1}{4}}.$$

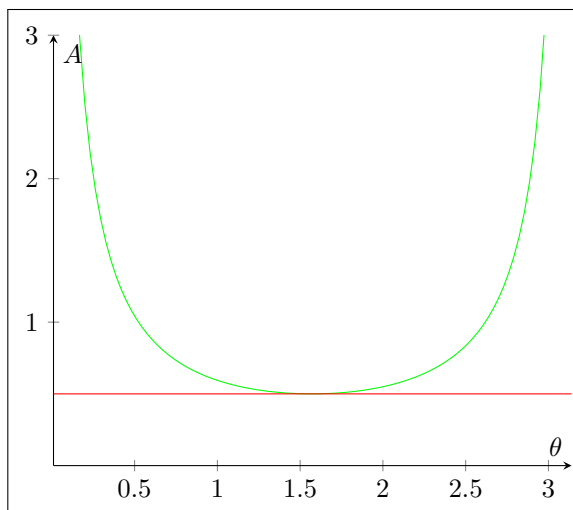
Taking the derivative, we obtain

$$\frac{dA}{d\theta} = \frac{1}{2} \frac{\sin \theta}{4(\cos \theta + 1)^2 \sqrt{\frac{1}{2(1 + \cos \theta)} - \frac{1}{4}}} - \frac{1}{2} \frac{\sin \theta}{4(1 - \cos \theta)^2 \sqrt{\frac{1}{2(1 - \cos \theta)} - \frac{1}{4}}};$$

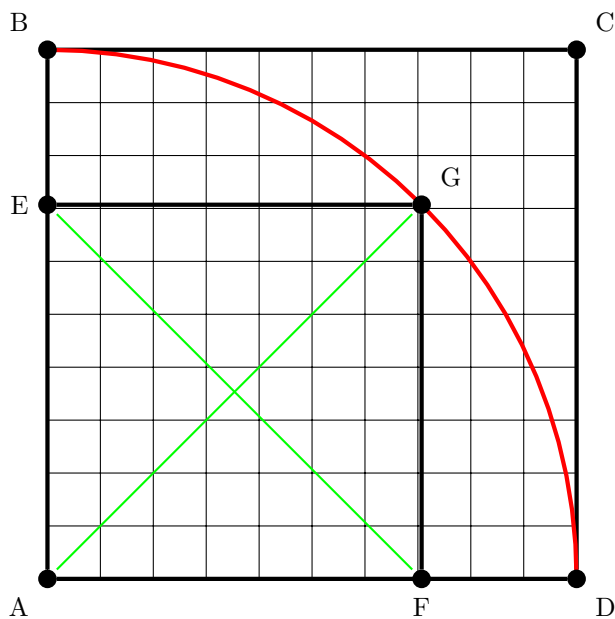
Now we set this to zero. We know that $0 < \theta < \pi$, so $\sin \theta \neq 0$ and hence

$$\begin{aligned}
 4(\cos \theta + 1)^2 \sqrt{\frac{1}{2(1 + \cos \theta)} - \frac{1}{4}} &= 4(1 - \cos \theta)^2 \sqrt{\frac{1}{2(1 - \cos \theta)} - \frac{1}{4}} \\
 \sqrt{\frac{1}{2} - \frac{\cos \theta + 1}{4}} &= \sqrt{\frac{1}{2} - \frac{1 - \cos \theta}{4}} \\
 \cos \theta &= -\cos \theta \\
 \cos \theta &= 0
 \end{aligned}$$

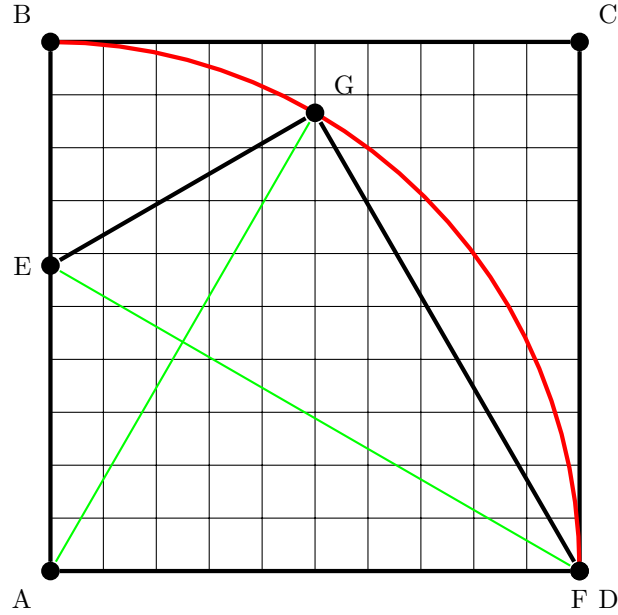
Hence $\theta = \phi = \pi/2$. Immediate calculation shows that $e = f = \frac{1}{\sqrt{2}}$; we thus have a square with side length $1/\sqrt{2}$, and area $\frac{1}{2}$. Is this a maximum or a minimum? We cheat by graphing the area versus θ :



so we obviously have the minimum area:



Note that $\theta \geq \pi/2$, because otherwise $e > 1$. Suppose we take $\theta = 2\pi/3$; here is the graphed figure (with area 1.1547):



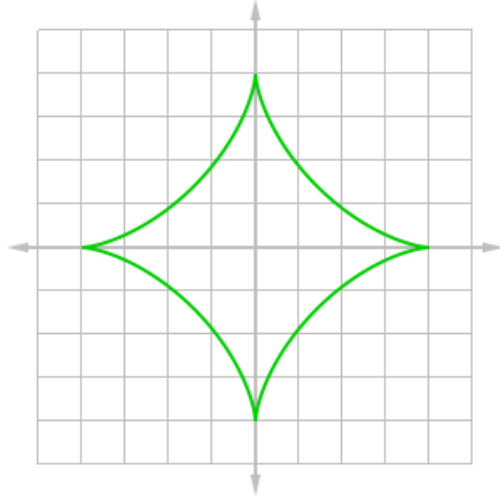
This is the maximum area, since if we increase θ any more it requires $f > 1$.

9. Related Rates

1. $V = [x(t)]^3$, so $\frac{dV}{dt} = 3\frac{dx}{dt}[x(t)]^2$.
2. The volume of a cone is $\frac{\pi}{3}r^2h$. Comparing similar triangles, if the water is at a height h then it forms a cone with radius $r = h/2$. Hence when the water is at a height h it has volume $V(t) = \frac{\pi[h(t)]^3}{24}$, and so $\frac{dV}{dt} = \frac{\pi[h(t)]^2}{8} \frac{dh}{dt}$. We know that $\frac{dV}{dt} = 2$, so solving for $\frac{dh}{dt}$ we have that $\frac{dh}{dt} = \frac{16}{\pi[h(t)]^2}$ and when $h = 3$ the height is rising at a rate of 0.57 m min^{-1} .
3. Let x be the hypotenuse of the formed triangle, and let y be the horizontal distance from the boat to the jetty so that $y = \sqrt{x^2 - 1}$. Then $\frac{dy}{dt} = \frac{x}{\sqrt{x^2 - 1}} \frac{dx}{dt} = \frac{\sqrt{y^2 + 1}}{y}$. So at $y = 8$, $\frac{dy}{dt} = \frac{\sqrt{65}}{8} \approx 1.0078 \text{ m s}^{-1}$.

10. Parametric Functions

1. (a) $\frac{dx}{dt} = 4t^3 - 6t^2 + 4t$, $\frac{dy}{dt} = 3t^2 - 1$, $\frac{dy}{dx} = \frac{3t^2 - 1}{4t^3 - 6t^2 + 4t}$, $\frac{d^2y}{dx^2} = -\frac{3t^4 - 6t^2 + 3t - 1}{(4t^3 - 6t^2 + 4t)t^2(2t^2 - 3t + 2)^2}$.
 (b) $\frac{dx}{dt} = -\sin t - 4\sin 2t$, $\frac{dy}{dt} = \cos t + 4\cos 2t$, $\frac{dy}{dx} = -\frac{\cos t + 4\cos 2t}{\sin t + 4\sin 2t}$, $\frac{d^2y}{dx^2} = \frac{12\cos(t) + 33}{(\sin t - 4\sin 2t)(8\cos(t) + 1)^2(\cos(t)^2 - 1)}$
2. We have $t^2 = (x - 1)^2$, so $y = e^{(x-1)^2}$ and $\frac{dy}{dx} = 2(x - 1)e^{(x-1)^2}$. At $x = 2$, $\frac{dy}{dx} = 2e$; so the best linear approximation is $y - e = 2e(x - 2)$, or $y = e(2x - 3)$.
3. (a) Should look something like this:



(b) $\frac{dx}{dt} = -12 \sin t \cos^2 t$, $\frac{dy}{dt} = 12 \cos t \sin^2 t$, so the slope at some t is simply

$$\frac{dy}{dx} = \frac{12 \cos t \sin^2 t}{-12 \sin t \cos^2 t} = -\frac{\sin t}{\cos t}.$$

(c) Cusps will be at precisely those points with turning points in the x or y direction (for $0 \leq t \leq 2\pi$). In other words, places where either $\sin t$ or $\cos t$ vanishes. These are at $t \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$; substituting these into the equation gives us the four points $(\pm 4, 0)$ and $(0, \pm 4)$.

11. Implicit Differentiation

1. (a) $y' = \frac{3x^2 + 6x}{2y}$.

(b) $(1 + y') \sin(x + y) = 2 - 2y' \implies y' = \frac{2 - \sin(x+y)}{2 + \sin(x+y)}$.

(c) $y' = \frac{20x^3 - 2x}{2y}$.

2. $2x + 2y + 2xy' - 2yy' + 1 = 0 \implies y' = \frac{-1 - 2x - 2y}{2x - 2y}$, so $y'(1, 2) = \frac{-1 - 2 - 4}{2 - 4} = 7/2$; hence the slope of the normal is $-2/7$, and the equation of the normal line is $y - 2 = -\frac{2}{7}(x - 1)$.

3. We have $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0$; suppose we have a tangent line passing through $(x_0, (\sqrt{c} - \sqrt{x_0})^2)$. Then the equation of this tangent is $y - (\sqrt{c} - \sqrt{x_0})^2 = -\frac{\sqrt{c} - \sqrt{x_0}}{\sqrt{x_0}}(x - x_0)$. When $y = 0$ we obtain the x -intercept; $0 = (\sqrt{c} - \sqrt{x_0})^2 - \frac{\sqrt{c} - \sqrt{x_0}}{\sqrt{x_0}}(x - x_0)$ and so $x = \sqrt{x_0 c}$. Similarly, when $x = 0$ we obtain $y = \sqrt{x_0 c} - x_0$. Their sum is therefore $2\sqrt{x_0 c} - x_0 = 2\sqrt{c}(\sqrt{c} - \sqrt{x_0}) - (\sqrt{c} - \sqrt{x_0})^2 = c$.

12. Kinematics

1. (a) $v = \frac{dh}{dt} = 122.5 - 9.8t$, so the initial velocity of the flare is 122.5 m s^{-1} .

(b) Zero.

(c) When $v = 0$, $t = 12$ and the height at this time is around 764 metres.

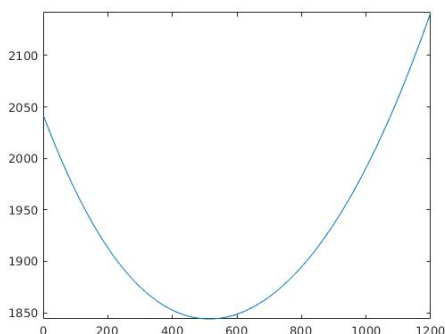
2. Let x be the distance from the point on the beach directly away from B. Then the total distance travelled is simply $D = \sqrt{600^2 + x^2} + \sqrt{800^2 + (1200 - x)^2}$; taking the derivative:

$$\frac{dD}{dx} = \frac{x}{\sqrt{600^2 + x^2}} - \frac{1200 - x}{\sqrt{800^2 + (1200 - x)^2}}$$

Setting to zero, we have

$$\begin{aligned} x\sqrt{800^2 + (1200 - x)^2} &= (1200 - x)\sqrt{600^2 + x^2} \\ 800^2 x^2 + (1200 - x)^2 x^2 &= (1200 - x)^2 (600^2 + x^2) \\ 0 &= 1200^2 600^2 - 600^2 2400x + (600^2 - 800^2)x^2 \\ x &\in \{-3600, 3600/7\}. \end{aligned}$$

Since $x \geq 0$, $x = 3600/7 \approx 514$. The total distance travelled is therefore around 1844 metres. By graphing D versus x , we see that this is indeed the required medium:

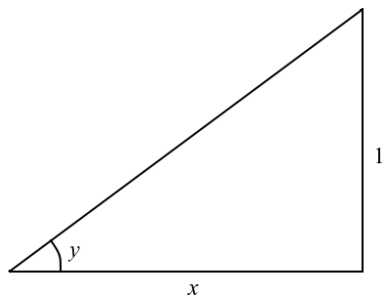


13. Inverse Functions

1. (a) $y' = \frac{2x}{1+x^4}$.
- (b) $f'(x) = \frac{1}{1+x^2}$.
- (c) $g'(x) = \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{1-x}} \frac{1}{1+(\sin^{-1} \sqrt{x})^2}$.
- 2.

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{2} \tan^{-1} x + \frac{1}{4} \ln \frac{(x+1)^2}{x^2+1} \right) &= \frac{1}{2+2x^2} + \frac{1}{4} \frac{x^2+1}{(x+1)^2} \frac{2-2x^2}{(x^2+1)^2} \\ &= \frac{1}{2+2x^2} + \frac{1}{2} \frac{1-x^2}{(x+1)^2(x^2+1)} \\ &= \frac{1}{2} \frac{(x+1)^2 + 1 - x^2}{(x+1)^2(1+x^2)} \\ &= \frac{1}{2} \frac{2x+2}{(x+1)^2(1+x^2)} \\ &= \frac{1}{(x+1)(x^2+1)}. \end{aligned}$$

3. Let $y = \cot^{-1} x$, so that $x = \cot y$ and $\frac{dx}{dy} = -\csc^2 y$; hence $\frac{dy}{dx} = -\sin^2 y$. Consider the following triangle:



So $\sin y = \frac{1}{\sqrt{1+x^2}}$, and $\frac{dy}{dx} = -\frac{1}{1+x^2}$.

14. Differentiation Revision

1. (a) $f'(x) = (2017 \times 3)x^{2016} - \frac{1}{19x^{20}} + \frac{1}{2017 \cdot 2017 \sqrt[2017]{(x+2)^{2016}}}$.

(b) $f'(h) = \pi r^2$.

(c) $f'(\theta) = -\frac{\mu mg(\mu \cos \theta - \sin \theta)}{(\mu \sin \theta + \cos \theta)^2}$.

(d) $f'(g) = \frac{(g^2 + \ln g) \cos g - (2g + 1/g) \sin g}{(g^2 + \ln g)^2}$.

(e) $3f(x) + 3xf'(x) + 2f(x)f'(x) = \frac{3+f(x)-xf'(x)}{[3+f(x)]^2}$ so $f'(x) = \frac{3+f(x)-3[3+f(x)]^2 f(x)}{3[3+f(x)]^2 x + 2[3+f(x)]^2 f(x) + x}$.

2. Let θ be the angle of the kite string, and let x be the horizontal distance to the kite along the ground (so the length of the string is $\sqrt{50^2 + x^2}$). Then $\sin \theta = 50/x$, so $\cos \theta \frac{d\theta}{dt} = -\frac{50}{x^2} \frac{dx}{dt}$. When the length of the string is 100, $x \approx 86.6$; so $\cos \theta = x/100 \approx 0.866$. Substituting $\frac{dx}{dt} = 2$, we have $\frac{d\theta}{dt} = -\frac{100}{86.6^2} \cdot \frac{1}{0.866} = -0.0154$.

3. The surface area of a cone is $\mathcal{S} = \pi r \sqrt{h^2 + r^2}$; we also have $27 = \frac{1}{3} \pi r^2 h$, so $r^2 = \frac{81}{\pi h}$ and

$$\mathcal{S} = \pi \sqrt{\frac{81}{\pi h} \left(h^2 + \frac{81}{\pi h} \right)} = \pi \sqrt{\frac{81h}{\pi} + \frac{81^2}{\pi^2 h^2}}$$

$$\frac{d\mathcal{S}}{dh} = \frac{\pi \left(\frac{81}{\pi} - 2 \frac{81^2}{\pi^2 h^3} \right)}{2 \sqrt{\frac{81h}{\pi} + \frac{81^2}{\pi^2 h^2}}}$$

In order to find a minimum, we set this derivative to zero and obtain $0 = \frac{81}{\pi} - 2 \frac{81^2}{\pi^2 h^3}$, so

$$h = \sqrt[3]{2 \frac{81}{\pi}} \approx 3.722 \text{ cm.}$$

From this, we find $r = \sqrt{81/\pi h} = 2.63 \text{ cm}$.

4. We begin by parameterising the hyperbola; completing the square, we can transform our equation into standard form:

$$\frac{(x-1)^2}{3} - \frac{y^2}{3} = 1$$

A parameterisation of this is $(1 + \sqrt{3} \sec t, \sqrt{3} \tan t)$. Now, given any point (x_0, y_0) we wish to minimise $\mathcal{D}(t) = \sqrt{(x_0 - 1 - \sqrt{3} \sec t)^2 + (y_0 - \sqrt{3} \tan t)^2}$ with respect to t .

- (a) Firstly, consider $(x_0, y_0) = (2, 1)$. Then $\mathcal{D}(t) = \sqrt{(1 - \sqrt{3} \sec t)^2 + (1 - \sqrt{3} \tan t)^2}$. Taking the derivative, we find that:

$$\frac{d\mathcal{D}}{dt} = \frac{(\sqrt{3} \sec t - 1)(\sqrt{3} \sec t \tan t) + (\sqrt{3} \tan t - 1)(\sqrt{3} \sec^2 t)}{\sqrt{(1 - \sqrt{3} \sec t)^2 + (1 - \sqrt{3} \tan t)^2}}$$

Using MATLAB to compute the solution of $\frac{d\mathcal{D}}{dt} = 0$,

$$\text{vpasolve}((\text{sqrt}(3) * \sec(t) - 1) * (\text{sqrt}(3) * \sec(t) * \tan(t)) \\ == (1 - \text{sqrt}(3) * \tan(t)) * (\text{sqrt}(3) * (\sec(t))^2), t)$$

we find $t \approx 0.3759$; so $(x, y) = (2.8621, 0.6835)$.

- (b) Note that $(3, 1)$ is already on the hyperbola. ☺

15. Approximating Areas

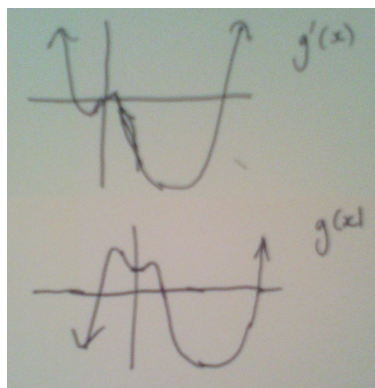
1. I use Simpson's rule with $n = 8$:

$$\int_0^{1.6} g(x) dx \approx \frac{0.2}{3} (12.1 + 13.2 + 4(11.6 + 11.1 + 12.2 + 13.0) + 2(11.3 + 11.7 + 12.6)) = 19.21.$$

2. Measure the height of the shaded area at each point (using $n = 10$ is probably easiest), collapsing the empty area down (e.g. the height of the function at $x = -1$ is just $3 + 1 = 4$). Then use some numerical integration method.
3. Like 2. but simpler.

16. Anti-differentiation

1. (a) $F(x) = \frac{1}{2}x^2 - 3x + C$
 - (b) $f(x) = x^2 + 3x + 2$, so $F(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C$
 - (c) $F(\theta) = 2\theta^3 - 7 \tan \theta + C$
 - (d) $G(h) = \pi^2 h$
 - (e) $F(x) = \frac{x^{4.7}}{4.7} + \frac{2}{3}\sqrt{x^3} + \sqrt{7}x^{\sqrt{7}}$
2. $\varphi(x) = x^2 + x + C$; but $\varphi(1) = 6$, so $1 + 1 + C = 6$ and $C = 4$. Hence $\varphi(x) = x^2 + x + 4$, and $\varphi(2) = 10$.
3. See following image.



17. The Fundamental Theorem of Calculus

1. $\int_0^{\pi/4} \sec^2 \theta \, d\theta = [\tan \theta]_0^{\pi/4} = 1.$
2. $\int_1^2 f(x) \, dx = \int_1^3 f(x) \, dx - \int_2^3 f(x) \, dx = 10.$
3. First we find the intersection points; we have $6x = x^2$, so $x \in \{0, 6\}$. Hence we compute

$$\int_0^6 2x - \frac{x^2}{3} \, dx = \left[x^2 - \frac{x^3}{9} \right]_0^6 = 36 - 6^3/9 = 12.$$

18. Substitution

1. (a) $\frac{\csc 3x}{3} + C.$
(b) $-\frac{\tan 3x^2}{6} + C.$
(c) $2\sqrt{x} + 3x - 2\ln x + C.$
(d) $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$
2. Use trig identity: $2 \sin 5x \cos 3x = \sin 8x + \sin 2x$. Then

$$\int_0^{\pi/6} \sin 8x + \sin 2x \, dx = \left[-\frac{\cos 8x}{8} - \frac{\cos 2x}{2} \right]_0^{\pi/6} = 0.4375.$$

3. $\frac{1}{2} \tan^{-1} x^2 + C.$ (Substitute $u = x^2$.)

19. Differential Equations

1. (a) $\int e^y \, dy = \int e^t \, dt$, so $e^y = e^t + C$ and $y = \ln(e^t + C).$
(b) $\int \frac{dy}{y^2} = \int t \, dt$, so $-\frac{1}{y} = \frac{1}{2}t^2 + \frac{C}{2}$ and $y = -\frac{2}{t^2 + C}.$
(c) $\int \sec^2 y \, dy = \int dt$, so $\tan y = t + C$ and $y = \tan^{-1}(t + C).$
(d) $\int \sin y \, dy = \int -t \cos t \, dt$, so $-\cos y = \cos t - t \sin t - C$ (by the hint) and $y = \cos^{-1}(t \sin t - \cos t + C).$
2. Using Newton's law of cooling, $\frac{dT}{dt} = k(T - T_\infty)$ (where T_∞ is the ambient temperature). Solving this differential equation, we find $\int \frac{1}{T - T_\infty} \, dT = \int k \, dt$ and so $T = T_0 e^{kt} + T_\infty$. We have $T_\infty = 30^\circ$, and $T_0 = 100^\circ$; also, at $t = 3$ we have $T = 70$ so $70 = 100e^{3k} + 30$; hence $k = \frac{\ln 0.4}{3} = -0.31$ and by direct substitution $T = 100e^{-0.31t} + 30$. Let $T = 31$; then $t = 14.86$ and so the temperature will drop to 31° after around fifteen minutes.
3. (a) We have $\frac{dV}{dt} = \text{rate in} - \text{rate out} = 3 - kV$. Hence $\int \frac{1}{3 - kV} \, dV = \int dt$, so $-\frac{\ln(3 - kV)}{k} = t + C$ and $V = \frac{3 - Ke^{-kt}}{k}$. At $t = 0$, $V = 100$; so $100k = (3 - K)$. We also have $kV = 3$ where $V = 120$, so $k = 1/40 = 0.025$. Hence $2.5 = 3 - K$ and $K = 0.5$. It immediately follows that

$$V = \frac{3 - 0.5e^{-0.025t}}{0.025}$$

and at $t = 10$, $V = 104$ litres.

- (b) The rate of water flow out is $kV = 3 - 0.5e^{-0.025t}$, which is always less than 3 (the rate in). In fact, as $t \rightarrow \infty$, the volume tends to 120 L and the rate in tends to equal the rate out.

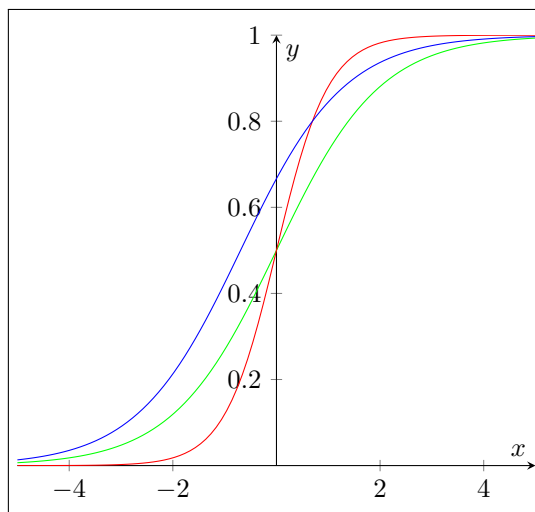
20. Partial Fractions

1. (a)

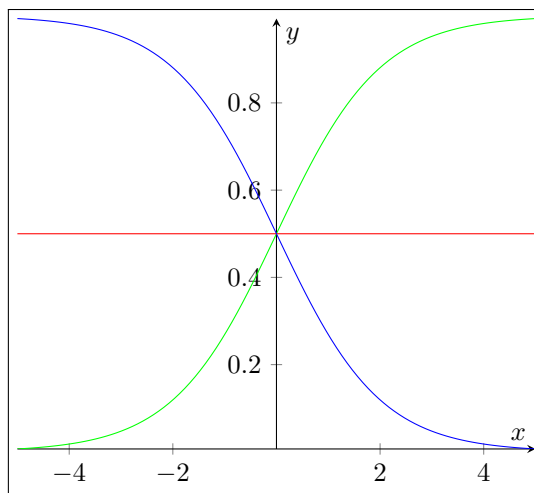
$$\begin{aligned}\int r \, dt &= \int \frac{dP}{P(1-P)} = \\ &= \int \frac{1}{P} + \frac{1}{1-P} \, dP \\ rt + C &= \ln \frac{P}{1-P} \\ Ke^{rt} &= \frac{P}{1-P} \\ \frac{Ke^{rt}}{1 + Ke^{rt}} &= P.\end{aligned}$$

- (b) It should be clear that as $t \rightarrow \infty$, $P \rightarrow 1$. (If we look at $\frac{dP}{dt} = \frac{rP}{P_\infty}(P_\infty - P)$, $P \rightarrow P_\infty$.)

Green: $r = K = 1$; red: $r = 2$, $K = 1$; blue: $r = 1$, $K = 2$.



- (c) r lets us vary how fast the population gets to the maximum. Green: $r = K = 1$; red: $r = 0$; blue: $r = -1$.



- (d) Write it yourself.
2. (a) Draw a triangle with angle $x/2$, hypotenuse $\sqrt{1+t^2}$, adjacent edge 1, and opposite edge t .
- (b)

$$\sin x = 2 \sin(x/2) \cos(x/2) = \frac{2t}{1+t^2}$$

$$\cos x = (\cos(x/2))^2 - (\sin(x/2))^2 = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 = \frac{1-t^2}{1+t^2}.$$

(c) We have $x = \tan^{-1} 2t$, so the result follows immediately.

(d) i. Let $t = \tan(x/2)$. Then, substituting, we have

$$\int \frac{1}{1 - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{1}{t^2} dt = -\frac{1}{t} + C = -\frac{1}{\tan \frac{x}{2}} + C.$$

ii. Similarly,

$$\begin{aligned} \int \frac{1}{3\frac{2t}{1+t^2} - 4\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt &= \int \frac{1}{3t - 2 + 2t^2} dt = \int \frac{1}{(2t-1)(t+2)} dt \\ &= \frac{1}{5} \ln \frac{1-2t}{t+2} + C = \frac{1}{5} \ln \frac{1-2\tan \frac{x}{2}}{\tan \frac{x}{2} + 2} + C. \end{aligned}$$

21. Integration by Parts

1. (a)

$$\int x \cos 5x \, dx = \frac{1}{5} x \sin 5x - \int \frac{1}{5} \sin 5x \, dx = \frac{1}{5} (x \sin 5x + \cos 5x) + C.$$

- (b)

$$\begin{aligned} \int \cos x \ln \sin x \, dx &= \sin x \ln \sin x - \int \sin x \frac{\cos x}{\sin x} \, dx \\ &= \sin x \ln \sin x - \int \cos x \, dx = \sin x (\ln \sin x - 1) + C. \end{aligned}$$

(c) Let $u = \sqrt{x}$, so $dx = 2u du$ and our integral becomes

$$\int 2u \cos u du = 2u \sin u - \int 2 \sin u du = 2u \sin u + 2 \cos u + C = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C.$$

2. (a) Let $u = \theta^2$, so our integral becomes $\frac{1}{2} \int_{\pi/2}^{\pi} u \cos u du$. From 1(c) above, we know that $\int u \cos u du = u \sin u + \cos u + C$. Hence the required result is

$$\frac{1}{2} \int_{\pi/2}^{\pi} u \cos u du = \frac{1}{2} [u \sin u + \cos u] \Big|_{u=\pi/2}^{\pi} = -\frac{1}{2} - \frac{\pi}{4}.$$

(b) We use integration by parts twice.

$$\begin{aligned} \int (x^2 + 1)e^{-x} dx &= -e^{-x}(x^2 + 1) + \int 2xe^{-x} dx \\ &= -e^{-x}(x^2 + 1) - 2xe^{-x} + \int 2e^{-x} dx \\ &= -e^{-x}(x^2 + 1) - 2xe^{-x} - 2e^{-x} + C. \end{aligned}$$

Hence the result we are looking for is $3 - 6e^{-1}$.

3. (a) Apply integration by parts to $\int 1 \cdot (\ln x)^n dx$ by integrating 1 and differentiating $(\ln x)^n$.
 (b) Applying (a), we find

$$\begin{aligned} \int (\ln x)^3 dx &= x(\ln x)^3 - \int (\ln x)^2 dx \\ &= x(\ln x)^3 - (x(\ln x)^2 - \int \ln x dx) \\ &= x(\ln x)^3 - (x(\ln x)^2 - (x \ln x - x)) \\ &= x(\ln x)^3 - x(\ln x)^2 + x \ln x - x. \end{aligned}$$

22. Lengths, Volumes, and Areas

1. We simply calculate the relevant integral:

$$\pi \int_1^2 x^{-2} dx = \pi [(-2^{-1}) - (-1^{-1})] = \frac{\pi}{2}.$$

2. Calculating the surface area:

$$\begin{aligned} 2\pi \int_0^{\pi} \sin x \sqrt{1 - \cos^2 x} dx &= 2\pi \int_0^{\pi} \sin^2 x dx \\ &= \pi [x - \sin 2x] \Big|_0^{\pi} \\ &= \pi^2 \end{aligned}$$

So the radius of the equivalent circle is $\sqrt{\pi}$.

3. Summing along the axis from base to point, each slice has an area $\left(\frac{L}{H}x\right)^2 = \frac{L^2}{H^2}x^2$; hence the total volume is

$$V = \int_0^H \frac{L^2}{H^2}x^2 dx = \frac{1}{3}L^2H.$$

4. We have $r = a(1 - \cos \theta)$ so $\frac{dr}{d\theta} = a \sin \theta$. Hence:

$$\begin{aligned} S &= \int_0^{2\pi} \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta \\ &= \int_0^{2\pi} \sqrt{a^2 - 2a^2 \cos \theta + a^2(\sin^2 \theta + \cos^2 \theta)} d\theta \\ &= a\sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos \theta} d\theta. \end{aligned}$$

We turn our attention, then, to the integral $\int \sqrt{1 - \cos \theta} d\theta$. Let $u = 1 - \cos \theta$; then $du = \sin \theta d\theta$; but $\sin \cos^{-1}(1 - u) = \sqrt{2u - u^2}$ (this can be verified by drawing a suitable triangle). Hence $du = \sqrt{2u - u^2} d\theta$, and

$$\begin{aligned} \int \sqrt{1 - \cos \theta} d\theta &= \int \frac{\sqrt{u}}{\sqrt{2u - u^2}} du \\ &= \int \frac{1}{\sqrt{2 - u}} \\ &= -2\sqrt{2 - u} + C \\ &= -2\sqrt{1 + \cos \theta} + C. \end{aligned}$$

Therefore (and changing our integral to double the integral from 0 to π to avoid the problem of having a closed loop),

$$\begin{aligned} S &= 2a\sqrt{2} \int_0^{\pi} \sqrt{1 - \cos \theta} d\theta \\ &= 2a\sqrt{2} \left[-2\sqrt{1 + \cos \theta} \right]_0^{\pi} \\ &= 2a\sqrt{2} \left[(-2\sqrt{1 + \cos \pi}) - (-2\sqrt{1 + \cos 0}) \right] \\ &= 2a\sqrt{2} \left[(-2\sqrt{0}) - (-2\sqrt{2}) \right] \\ &= 2a\sqrt{2} \times 2\sqrt{2} = 8a. \end{aligned}$$

23. Trigonometric Substitution

These ones are tedious and can be checked by the computer, so I have not written full answers for all of them.

1. Let $x = 2 \tan \theta$, so $dx = 2 \sec^2 \theta$:

$$\int \frac{2 \sec^2 \theta}{\sqrt{x^2 + 4}} dx = \int \frac{\sec^2 \theta}{\sqrt{1 + \tan^2 \theta}} dx = \int \sec \theta dx = \ln \left(\sqrt{\left(\frac{x}{2}\right)^2 + 1} + \frac{x}{2} \right) + C.$$

2. First, let $u = x^7$ so $du = 7x^6 dx$ and our integral becomes

$$\frac{1}{7} \int \frac{du}{\sqrt{1-u^2}} = \frac{1}{14} \ln \frac{1+u}{1-u} + C = \frac{1}{14} \ln \frac{1+x^7}{1-x^7} + C.$$

3. Use $x = \frac{2}{5} \sec \theta$.
 4. Use $x = \frac{2}{3} \sin \theta$.
 5. Use $x = \frac{1}{6} \tan \theta$ and simplify.
 6. Use integration by parts; the resulting integral $-\frac{\ln x}{4x^4} + \int \frac{dx}{4x^5}$ is much simpler.
 7. Use partial fractions.

24. Formal Definitions of exp and ln

$$\int \frac{1}{\text{cabin}} d(\text{cabin}) = \log \text{cabin} + \text{sea}.$$

25. Integration Revision

1. (a) $\int_1^2 \sin x dx = [-\cos x]_1^2 = \cos 1 - \cos 2$.
 (b) $\int \frac{u^2+1}{u^3+3u} = \frac{1}{3} \ln(u^3+3u) + C$.
 (c) $\int_0^{\pi/6} \tan x dx = [\ln \sec x]_0^{\pi/6} \approx 0.1438$.
 2. We have $\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$, so $\int 2y-4 dy = \int 3x^2+4x-4 dx$. Hence $y^2-4y = x^3+2x^2-4x+C$; we also have $C = -2$, so $y^2-4y = x^3+2x-4x-2$. We are trying to find y if $x = 2$; so $y^2-4y = 8+4-4-2 = 6$. Solving $y^2-4y-6 = 0$, we find $y = \frac{4 \pm \sqrt{38}}{2}$.
 3. (a) Let $t = a \tan \theta$. Then:

$$\begin{aligned} \int \frac{a^3}{t^2+a^2} dt &= \int \frac{a^4 \sec^2 \theta}{a^2 \tan^2 \theta + a^2} d\theta \\ &= \int \frac{a^2 \sec^2 \theta}{\sec^2 \theta} d\theta \\ &= a^2 \theta = a^2 \tan^{-1} \left(\frac{t}{a} \right); \end{aligned}$$

$$\text{hence } \omega(a, x) = a^2 \tan^{-1} \left(\frac{x}{a} \right).$$

- (b) It follows that $\omega(2, 2) = 4 \tan^{-1} 1 = \pi$.

- (c) We wish to find x such that $\pi = 3 \tan^{-1} \left(\frac{x}{\sqrt{3}} \right)$; in other words, $x = \sqrt{3} \tan \left(\frac{\pi}{3} \right) = 3$.

4. Note first that $\int_{-\pi/2}^{\pi/2} \sin^5 x \, dx = 0$ since \sin^5 is odd. Then, we argue as follows:

$$\begin{aligned}\int \cos^5 x \, dx &= \int \cos x (1 - \sin^2 x)^2 \, dx \\&= \int (1 - t^2)^2 \, dt \quad (t = \sin x) \\&= \int 1 - 2t^2 + t^4 \, dt \\&= t - \frac{2}{3}t^3 + \frac{t^5}{5} + C \\&= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.\end{aligned}$$

Hence

$$\int_{-\pi/2}^{\pi/2} \sin^5 x \, dx = \left(\sin \frac{\pi}{2} - \frac{2}{3} \sin^3 \frac{\pi}{2} + \frac{1}{5} \sin^5 \frac{\pi}{2} \right) - \left(\sin \frac{-\pi}{2} - \frac{2}{3} \sin^3 \frac{-\pi}{2} + \frac{1}{5} \sin^5 \frac{-\pi}{2} \right).$$

But $\sin(\pi/2) = 1$; so we have $(1 - \frac{2}{3} + \frac{1}{5}) - (-1 + \frac{2}{3} - \frac{1}{5}) = 2(1 - \frac{2}{3} + \frac{1}{5}) = \frac{16}{15}$.