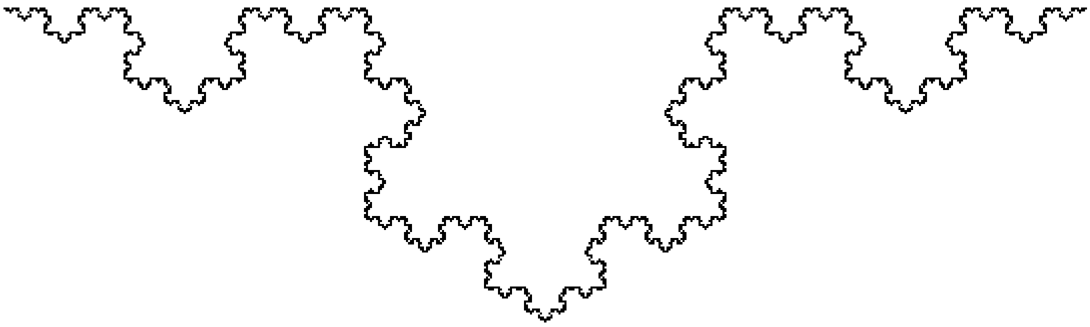
Level Two Mathematics



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Contents

CHAPTER 1

Preamble

NCEA Level 2 Mathematics
Preface

These notes present the NCEA Level 2 mathematics content from a mainly geometric standpoint, and so in some places are a little nonstandard. For example, the study of quadratic equations is approached by analysing quadratic graphs, and many of the exercises ask for intuitive and/or geometric explanations of many of the phenomena we study. This is because I see so many students entering L3 calculus with little to no ability to make use of the interplay between geometric and algebraic views of the same picture.

For example, I can cite the example of finding turning points of graphs: students remember very well that the way to solve this kind of exam problem is to "take the derivative and set it to zero", but most of them cannot explain why this works! This is foreign to me, because I don't remember not understanding this (and in fact I think the technique was taught to me geometrically first, which is really the only way to do it)! I find it hard to believe that teachers of L2 mathematics are so incompetent that they don't mention geometry at all when they introduce calculus, so maybe the problem is that when studying for exams the students just remember how to solve problems and neglect the "big picture": that memorising how to solve "types of problems" for an exam is both counterproductive and damaging because they don't remember the important ideas (about tangent lines, slopes, approximations, geometry) and then wonder why the "skills" they've learned (symbol pushing on an exam paper) are useless in the "real world"!

These notes also have another agenda: to introduce students in a calm way to mathematical proof. There is a definite increase in sophistication required for the later sections, but right from the first problem set students are asked to justify statements mathematically. I make no apology to those who want to use these notes but avoid forcing students to write proofs: it is simply how mathematics is done (and I don't think many of the exercises, if any, are out of the reach of the enthusiastic student).

I have tried to address many of these ideas in my student introduction as well.

Guide to the bibliography

The bibliography is a mixture of further reading and additional problemsets. I have not included many drill-type problems (like "solve for x given $x^2 + 3x - 20$") because they are easily found for those sections that need them: in particular, Spiegel is a good source of algebra drill problems and Foerster is a good source of trigonometry drill problems. For calculus and coordinate geometry, I have included Andree — although I know no good source that covers just the material in calculus needed for L2 and so it should be used with caution.

In terms of additional reading, most (all?) of the books are suitable for an enthusiastic Y12 student. I particularly recommend Lauwerier, Bóna, or any books on graph theory and the four-colour theorem for students interested in computer science and/or programming.

Many of the titles are popular mathematics books (e.g. the two by Bellos) that cover the material we see this year at a slightly lower level, and put it in context (although some of the topics should be taken with a grain of salt: Bellos includes chapters on such crank topics as the Golden Ratio).

List of sections with standards

Geometry

1. (2.1) Coordinate Geometry
2. (2.4) Arcs and Sectors of Circles
3. (2.4) Trigonometry

Algebra

4. (2.2) Functions
5. (2.2/2.6) Quadratic Modelling

6. (2.6/2.14) Simultaneous Equations
7. (2.6/2.14) Linear Inequations
8. (2.6) The Quadratic Formula
9. (2.2/2.6) Exponential and Logarithmic Functions
10. (2.2/2.6) Negative and Fractional Powers

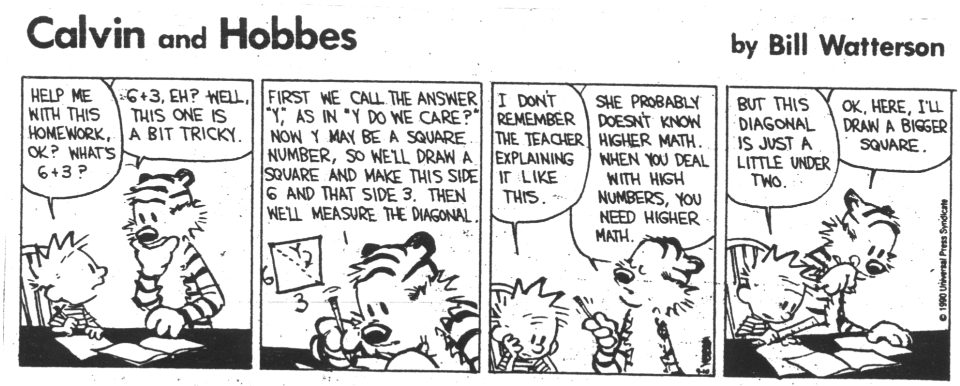
Calculus

11. (2.7) Slopes and Differentiation
12. (2.7) Tangent Lines and Approximation
13. (2.7) Turning Points Optimisation
14. (2.7) Anti-differentiation
15. (2.7) Kinematics and Rates of Change

Combinatorics

16. (Stats?) Counting and Combinatorics
17. (2.3) Number Sequences and Fractals
18. (2.5) Graphs and Networks

NCEA Level 2 Mathematics
Introduction to the Notes



What is mathematical proof?

A proof, that is, a mathematical argument, is a work of fiction, a poem. Its goal is to satisfy. A beautiful proof should explain, and it should explain clearly, deeply, and elegantly. A well-written, well-crafted argument should feel like a splash of cool water, and be a beacon of light — it should refresh the spirit and illuminate the mind. And it should be charming.

— From *A Mathematician's Lament*, by Paul Lockhart.

Mathematics is fantastic. It is a subject where we do not have to take anyone's word or opinion. The truth is not determined by a higher authority who says 'because I say so', or because they saw it in a dream, the pixies at the bottom of their garden told them, or it came from some ancient mystical tradition. The truth is determined and justified with a mathematical proof.

A proof is an explanation of why a statement is true. More properly it is a convincing explanation of why the statement is true. By convincing I mean that it is convincing to a mathematician. (What that means is an important philosophical point which I am not going to get into; my interest is more in practical matters.)

Statements are usually proved by starting with some obvious statements, and proceeding by using small logical steps and applying definitions, axioms and previously established statements until the required statement results.

The mathematician's concept of proof is different to everyday usage. In everyday usage or in court for instance, proof is evidence that something is likely to be true. Mathematicians require more than this. We like to be 100% confident that a statement has been proved. We do not like to be 'almost certain'.

Having said that, how confident can we be that a theorem has been proved? Millions have seen a proof of Pythagoras' Theorem; we can be certain it is true. Proofs of newer results, however, may contain mistakes. I know from my own experience that some proofs given in books and research journals are in fact wrong.

— From *How to Think Like a Mathematician*, by Kevin Houston.

Why am I expected to prove things?

As a student, you are expected to (try to) learn to think like a mathematician — and that means to justify and prove things. Some students (hopefully not you) think that this is somehow too hard: that being expected to think creatively is something that is best left out of the mathematics classroom. For that student, I give a number of reasons for the fact (and it is a fact) that creative mathematical thinking is both possible and necessary for the secondary school student.

1. If we don't justify our statements, or understand what we are doing, and simply plug numbers into a formula we are doing then we are not only not doing mathematics, but we are wasting our time! If

mathematics was just about plugging numbers into formulae, we wouldn't bother teaching it to students — because computers are far more reliable, and complain far less. You should be aiming to understand *why*, not simply to memorise *how*: and the idea of a proof is to *explain the why.*

2. As Paul Lockhart puts it in his classic book *A Mathematician's Lament*, we believe that secondary students are mature enough to have sophisticated opinions on literature and creative enough to be allowed to paint, write, and create music; so it isn't really a stretch to believe that secondary students are capable of doing their own mathematics.

3. A good argument is aesthetic (as some of you, who may be considering going to university to study law, literature, or another art, will well be aware); mathematicians have a reasonably well-developed idea of an aesthetic mathematical argument or mathematical theory, and it is accessible to you. Mathematics is one of the oldest forms of human expression, and some of the most elegant proofs that you will meet this year date back at least as far as the Greeks — and over the coming years, you will meet modern proofs in subjects that have a different aesthetic nature to (say) geometry.

4. A (correct) mathematical proof is absolute and forever, in that if a mathematical statement is proved then it cannot be overturned in thirty years like a theory of chemistry or biology might be upon the discovery of hitherto unknown evidence. Mathematical reasoning is not empirical (based simply on observation and the collection of evidence), it is based on logical inference and argument.

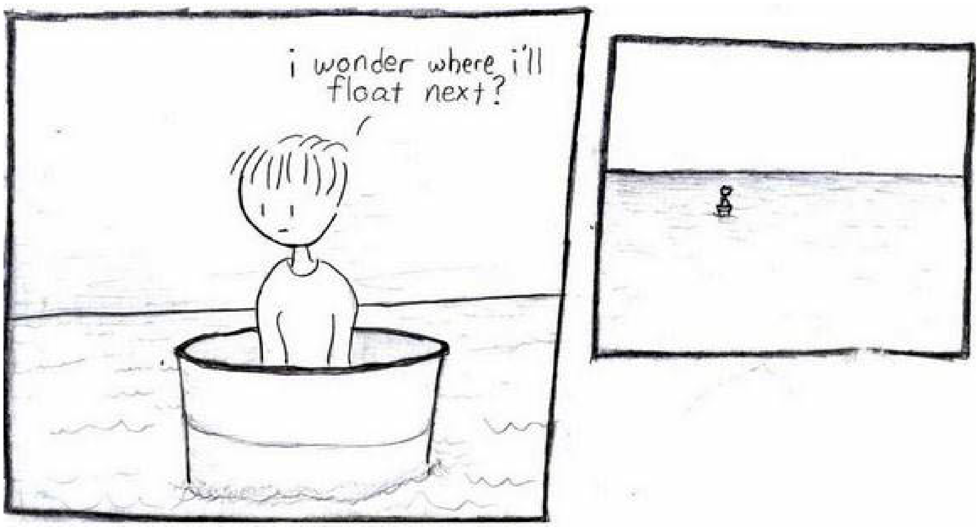
5. Mathematical reasoning is important in other artistic disciplines (history, linguistics, philosophy) and in the scientific disciplines (physics, chemistry, biology) — and is particulary intertwined with computer science.

Some advice

- Draw pictures, even if you are not strictly doing "geometry".
- Take the time to write clearly and slowly, because a piece of mathematics that is not clear and transparent to the reader is not mathematics at all.
- Always ask 'why', because nothing is every arbitrary: there will be a reason for everything, even if it is not explicitly spelled out in the notes. Ask why things are true, ask why particular problems are in the notes (because most of them have a particular goal, or a particular idea to illustrate that may not be immediately obvious).
- If you don't understand something (and by 'understand' I mean *really understand*, in that you can 'see' it in your mind and it has become almost obvious), ask questions and try to see things from another perspective.
- If you try a problem and you can't see a way through it immediately, come back later; usually, there is not an 'answer' that is to be calculated, but a process to be discovered or an argument to be developed — and it sometimes takes time for your mind to think unconsciously.
- Spend time on problems you think you can't do, because usually there is only a small flash of inspiration required; and anyway, the more difficult problems are far more interesting than the 'punch some buttons on the calculator' exercises, because they have a deeper meaning and are far more satisfying to complete. You might even surprise yourself.
- Always try to generalise: sure, it works for right-angled triangles: but what about *all* triangles?

The key is not calculations, it is comprehension. Calculations about triangles are not interesting, but the fact that such calculations are possible is not only interesting but almost counterintuitive; solving a quadratic equation is not interesting, but the idea of completing the square is interesting because it almost shouldn't work; and taking derivatives is not interesting, but the notion of describing geometry in a smooth and continuous way is, because it highlights the completeness of our number system. Look for the counterintuitive, not the boring routine; and try to see the beauty behind the arguments.

Good luck.



— https://www.xkcd.com/1/

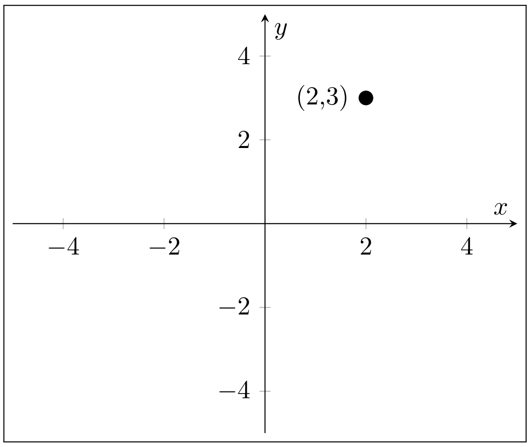
CHAPTER 2

Geometry

NCEA Level 2 Mathematics
1. Coordinate Geometry

The ancient Greeks were doing geometry at least as early as the 7th century BCE: Euclid's *Elements*, a mathematical book containing a collection of geometric propositions and proofs, is arguably the most influential mathematical work of all time and was regularly taught in schools until the 19th century. Of course, there have been some revolutions in geometry since the time of Euclid, and one of the most important was an idea due to René Descartes ("Renay Daycart") who first published the idea of using a coordinate system in 1637, in his book *La Géométrie*.

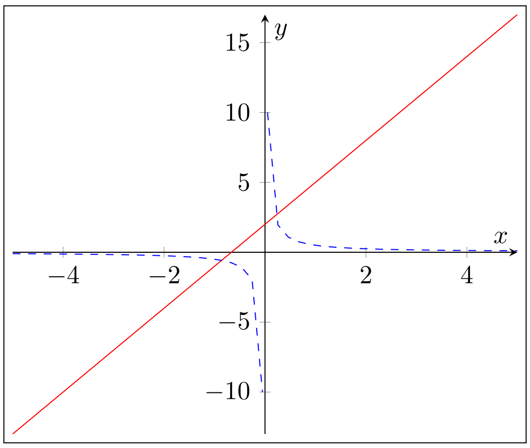
The idea is relatively simple: assign every point in the plane two numbers (coordinates) that describe its location with respect to the coordinate axes.



Traditionally, the two axes are the x- and y- axes; in three dimensions, we add a z-axis as well, at right angles to the first two.

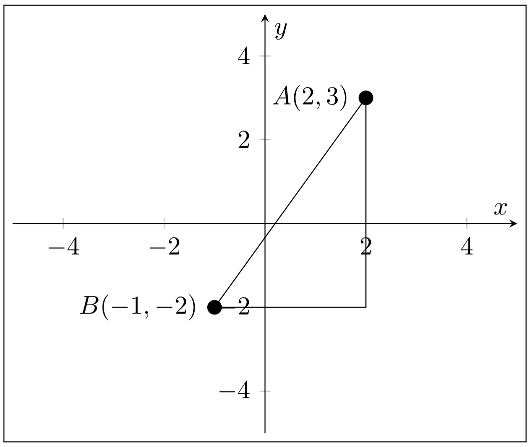
The reason this idea is so powerful is that now we can represent geometric objects algebraically, by writing down an equation so that the coordinates of every point of our object satisfies the equation.

For example, the equations $y = 3x + 2$ (red) and $y = \frac{1}{2x}$ (blue) are graphed here.



Points

One thing that we can do within a coordinate system is define the concept of "distance" between two points. We will do this by taking our two points and drawing a right-angled triangle, as follows:



Then the distance between the two points A and B is given by Pythagoras' theorem:-

$$d(A,B) = \sqrt{(3 - -2)^2 + (2 - -1)^2}.$$

More generally, the distance between two points $A(x_0, y_0)$ and $B(x_1, y_1)$ is given by

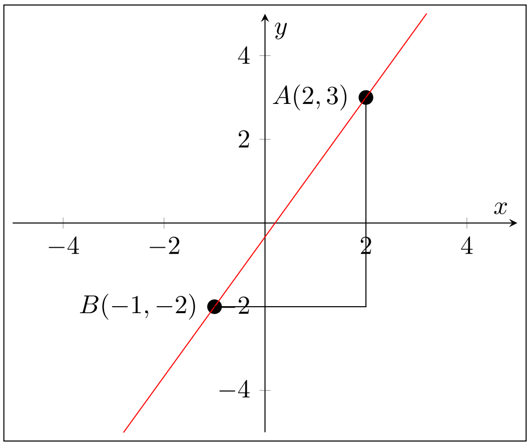
$$d(A,B) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

We can also find the midpoint of two points: the point precisely halfway between them. Clearly, if a point is halfway between two points on the plane then it is halfway between the two points on both coordinate axes; hence the point halfway between two points $A(x_0, y_0)$ and $B(x_1, y_1)$ is given by

$$m(A,B) = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right).$$

Lines

Turning to lines now, it is fairly obvious that every line (when we mention lines, we of course always mean *straight lines*) has a 'slope'. The slope (or 'gradient') of a line is a number which describes how 'steep' the line is. We define the 'slope' of a line to be the 'distance a point on the line travels upwards, as we move to the right by one unit length'. For example, consider the line in red here.



The slope of this line is $\frac{3--2}{2--1} = \frac{5}{3}$: every three steps to the right, we move five steps up — so if we move one step to the right, we move 5/3 of a step up.

We can use this idea to write down the equation of a line in general. Note that we have to take for granted one thing:

There is precisely one line through any two points.

We can't *prove* this within our conception of what it means to be a geometry, it is an axiom (something we assume without proof to be true). Note, however, that it is possible to define geometry in general in a different way so that this does become a theorem (something with a proof), but this requires other axioms beneath it that we cannot prove.

From our discussion above, we know that the slope of the line through the points (x_0, y_0) and (x_1, y_1) is given by

$$\frac{y_1 - y_0}{x_1 - x_0} \left(= \frac{\text{rise}}{\text{run}} \right).$$

We can now prove the following:

Theorem. *The unique line through the points (x_0, y_0) and (x_1, y_1) is given by the linear equation*

$$y - y_0 = \frac{y_1 - y_0}{x_0 - x_1}(x - x_0) = m(x - x_0),$$

where m is the slope of the line.

Proof. Let (x, y) be any point on the line. We need to find an expression for y in terms of x. However, note that the slope of the line between (x, y) and (x_0, y_0) is the same as the slope between (x_0, y_0) and (x_1, y_1) (because the line is straight). Therefore, we have

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_0 - x_1}$$

and rearranging this, we have

$$y - y_0 = \frac{y_1 - y_0}{x_0 - x_1}(x - x_0),$$

Q.E.D. □

For example, the line through $(2,1)$ and $(3,4)$ has slope $\frac{4-1}{3-2} = 3$, and equation

$$y - 1 = 3(x - 2).$$

On the other hand, we could swap around the two points and write the equation

$$y - 4 = 3(x - 3).$$

This is slightly annoying: both describe the same line, so why do we have two equations? Well, we can rearrange them:

$$\begin{aligned} y - 1 = 3(x-2) &\implies y = 3(x-2)+1 \implies y = 3x - 6 + 1 \implies y = 3x - 5; \\ y - 4 = 3(x-3) &\implies y = 3(x-3)+4 \implies y = 3x - 9 + 4 \implies y = 3x - 5. \end{aligned}$$

So the equations are the same if we carry out this process of putting them into the form

$$y = mx + c$$

where m is still the slope of the line, and c is the y-coordinate of the place where the line crosses the y-axis (since this point is the point on the line where $x = 0$ and so $y = m0 + c = c$).

Questions

1. Find the distance between the following pairs of points, drawing a diagram for each showing the geometric meaning of the thing you are calculating.
   (a) $(3,4)$ and $(6,8)$
   (b) $(3,0)$ and $(0,4)$
   (c) $(-1,1)$ and $(-4,-4)$
   (d) $(-4,-4)$ and $(-1,1)$
2. Justify the following statements mathematically, where A and B are points.
   (a) The distance from A to B is the same as the distance from B to A.
   (b) The distance from the midpoint of A and B ($m(A,B)$) to A is the same as the distance from $m(A,B)$ to B.
   (c) The equation of the line describing the y-axis is $x = 0$, and the equation of the line describing the x-axis is $y = 0$.
   (d) The slope of a horizontal line is zero.
   (e) A vertical line does not have a well-defined slope.
   (f) A line with a negative slope is sloping downwards.
3. Find the slope of the line through the following points.
   (a) $(5,7)$ and $(-2,1)$
   (b) $(-2,1)$ and $(5,7)$
   (c) $\left(\frac{2}{3},1\right)$ and $(4,2)$
   (d) $(2\pi,1)$ and $(0,\pi+1)$
4. Find the points where the line $y = -3x + 2$ crosses the x- and $y-$axes. These are called the x- and y-intercepts.
5. Find the point where the line $y = -x + 2$ crosses the line $y = x + 2$.

6. Justify intuitively the "triangle inequality", if A, B, and C are any three points:

$$d(A, B) \leq d(A, C) + d(C, B). \qquad \text{(Triangle inequality)}$$

7. Two lines are *perpendicular* if the angle between them is a right angle. Show that if the gradient of a line is m then the gradient of any perpendicular line is $-1/m$.

8. Two lines are *parallel* if they do not intersect. Show that two lines are parallel if and only if they have the same gradient.

9. In three dimensions, we can define the distance between two points (x_0, y_0, z_0) and (x_1, y_1, z_1) using the 3D version of Pythagoras' theorem:

$$d(A, B) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

(We will prove this version of Pythagoras' theorem in three dimensions in a few weeks.)

(a) Show that the distance between two points on the x, y-plane (the plane $z = 0$) reduces to the 2D version we gave above.

(b) Write down a plausible definition for the midpoint of A and B in three dimensions.

(c) Using your definition, find the midpoint of $A(2, -2, 5)$ and $B(6, 8, 9)$; is the distance between the midpoint and A half the distance between A and B?

10. Which of the following triangles ABC are isosceles or right-angled (or both)?

(a) $A = (1, 3) \quad B = (4, -2) \quad C = (1, -2)$

(b) $A = (1, -5) \quad B = (2, -1) \quad C = (9, -6)$

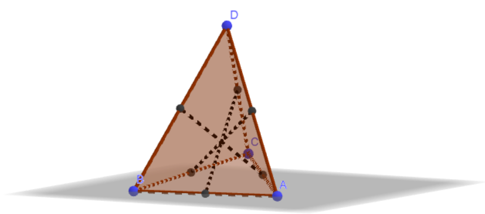
(c) $A = (4, 0) \quad B = (8, 2) \quad C = (8, -2)$

11. Let $ABCD$ be a quadrilateral. Show that the quadrilateral $WXYZ$, where

$$W = m(A, B)$$
$$X = m(B, C)$$
$$Y = m(C, D)$$
$$Z = m(D, A)$$

is a parallelogram.

12. Challenge problem. Consider a tetrahedron (see the figure below and `tetrahedron.ggb`). For each pair of opposite sides, draw a line segment connecting their midpoints.



(a) Show that all three such line segments intersect at a single point.

(b) In what ratio does the point of intersection divide each segment?

NCEA Level 2 Mathematics (Homework)
1. Coordinate Geometry

Reading

Go and watch...

https://www.youtube.com/watch?v=X1E7I7_r3Cw

What's it good for?

People use coordinate geometry for...

- Computer graphics: coordinate geometry is a compact, efficient, and simple way to store information about and manipulate complicated graphical data. Vector graphics programs, like Inkscape and Adobe Illustrator, store all their graphics as coordinate geometric objects as they take up less storage space, are easier to edit, and easier to scale and otherwise modify than storing objects pixel-by-pixel.

- Coordinate systems are used by physicists, geographers, and other people who need to describe the positions of objects in space relative to each other.

Questions

[This is a sample Ministry of Education L2 assessment task for this standard.]

Develop a general method for finding the co-ordinates of the reflection of any point in the mirror line joining the points $(0, 9/4)$ and $(9/2, 0)$ by completing the following steps:

1. Reflect the point $(4, 1)$ in the mirror line joining the points $(0, 9/4)$ and $(9/2, 0)$ and find the co-ordinates of the reflection;

2. If (a, b) is any point not on the mirror line, reflect the point (a, b) in the mirror line and find the co-ordinates of the reflection.

The quality of your discussion and reasoning will determine the overall grade. Show your calculations. Use appropriate mathematical statements. Clearly communicate your strategy and method at each stage of the solution.

NCEA Level 2 Mathematics
2. Arcs and Sectors of Circles

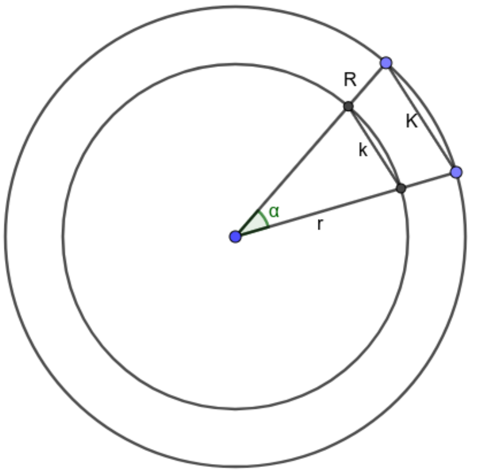
The Greeks studied two fundamental geometric objects: lines, and circles. Last week we looked at lines; this time, we will look at circles.

A circle is simply the set of all points that are at a distance r from a point (x_0, y_0): the point (x_0, y_0) is called the centre of the circle, and the distance r is the radius of the circle. If (x, y) is a point on the circle, then we have

$$d((x,y),(x_0,y_0)) = r \implies \sqrt{(x-x_0)^2 - (y-y_0)^2} = r$$

and the equation of the circle in cartesian coordinates is $(x-x_0)^2 + (y-y_0)^2 = r^2$.

Circumferii



Suppose the angle α is measured in degrees. Then the circumference of the inner circle is $c \approx \frac{180^\circ}{\alpha} k$, and the circumference of the outer circle is $C \approx \frac{180^\circ}{\alpha} K$. Now, the two triangles formed are similar since they have an identical angle and two sides with the same ratio of r/R; hence $k/K = r/R$. We can rewrite:

$$\frac{k}{K} = \frac{r}{R} \implies \frac{c\frac{\alpha}{180^\circ}}{C\frac{\alpha}{180^\circ}} \approx \frac{r}{R} \implies \frac{c}{C} \approx \frac{r}{R}.$$

As the size of the angle α becomes smaller and smaller, this approximation becomes exact: $\frac{c}{C} = \frac{r}{R}$, and so $\frac{c}{r} = \frac{C}{R}$. In other words, the ratio of the circumference of any circle to its radius is always the same. For historical reasons, we actually write this in terms of the diameter, and call the constant of proportionality π. We have therefore sketched a proof that

$$c = 2\pi r$$

for any circle with radius r and circumference c.

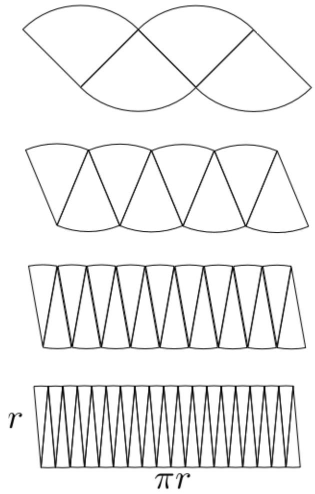
The number π is approximately equal to

$$3.14159265358979323846264\,33...$$

and in one of the exercises below you will calculate a first approximation to this value: it isn't just a number that is plucked out of thin air!

Areas

The other main result we have for circles is the area; by slicing the circle into smaller and smaller pieces, we can approximate the area of a circle with the area of a rectangle:



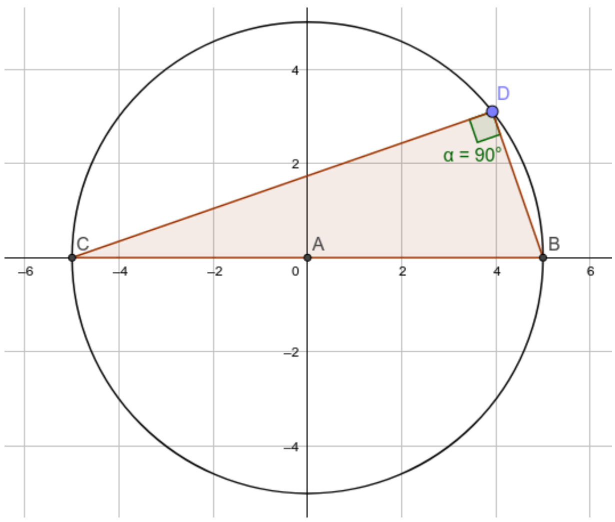
By means of this, we can intuitively justify that the area of a circle with radius r is

$$A = \pi r^2.$$

This idea of a limiting process will be made more clear next year (when you will be able to provide proper proofs of these facts), but hopefully these two results seem plausible.

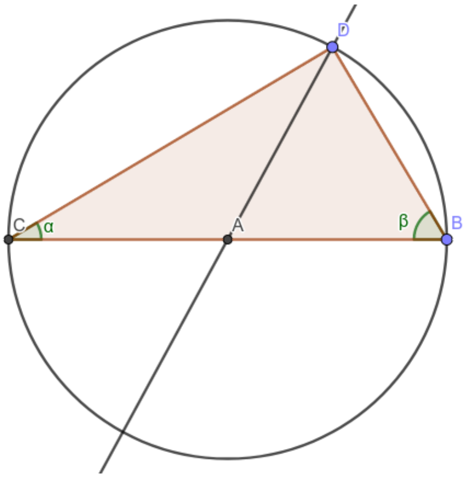
Thale's Theorem

As a taster for some of the other pretty theorems one can prove about circles, consider any circle; pick a diameter of the circle and any point on the circle itself; then the resulting triangle is always right-angled, as in the following diagram.



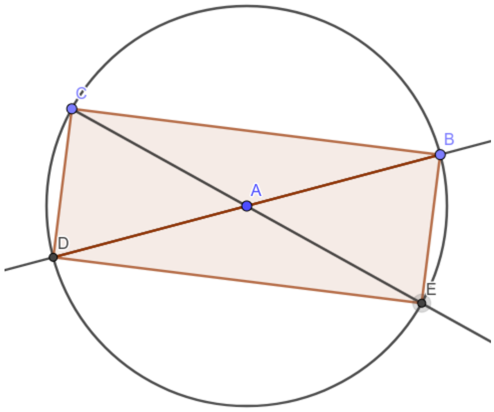
I actually know *two* proofs of this; we'll look at both!

Angle-y proof. The first proof is quite cute: we simply consider the next figure and do some angle pushing.



Clearly ACD and ABD are both isosceles, so $CDA = \alpha$ and $ADB = \beta$; hence $CDB = \alpha + \beta$, and so (using the fact that the internal angles of a triangle add to $180°$) we have $180° = \alpha + \beta + \alpha + \beta$; so $CDB = \alpha + \beta = 180°/2 = 90°$. $\square$

Rotate-y proof. The intuitive idea behind the second proof is that we take the triangle, 'rotate it around', and see that the resulting shape is a rectangle. In order to make this idea more precise, consider the following diagram.



Here, we took our initial triangle to be BCD, sitting on the diameter BD. Draw the line through C and A; clearly (since it passes through the centre) it is a diameter of the triangle. Hence the two diagonals of the quadrilateral $BCDE$ are the same length, and it is therefore a rectangle (so in particular DCB is a right angle). $\square$

Which proof do you prefer? Why?

Questions

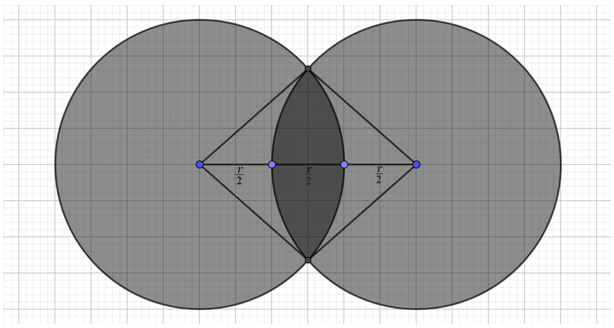
1. Suppose we take a circle of radius r, and cut out a slice with an angle α (like cutting a pizza). Draw a picture. What is the area of this slice, and what is the length of the arc of the circumference that is part of the slice? (Such a slice is called a *sector*.)

2. Notice that the formulae derived above all have ugly factors of 360°. Define one radian to be the angle such that the arc length of the sector defined by that angle is just r, the radius of the circle. Radians are, in many ways, a much more natural angle measurement unit.
   (a) Draw a picture to show this geometrically.
   (b) Show that one radian is precisely $\frac{\pi}{180}$ degrees.
   (c) How many radians are in a full circle?
   (d) Show that, in radians, the formulae derived in question 1 above simplify dramatically.

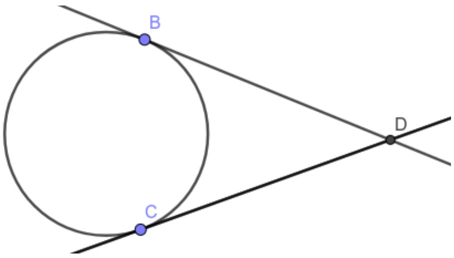
3. Suppose a circle has area 49π. What is the arc length of a sector of this circle with area 25π?

4. Show that if the angle subtended by a chord at the centre is $90°$ then $\ell = \sqrt{2}r$, where ℓ is the length of the chord.

5. Find the area of the shaded region in the diagram below.

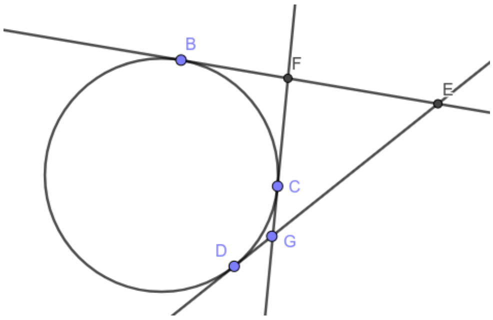


6. Suppose that two lines tanget to a circle at points B and C intersect at a point D, as shown. Show that the two segments BD and CD have equal lengths.



7. A simple definition of a circle is the set of all points $P = (x, y)$ such that $d(A, P) = r$ (where A is the centre of the circle and r is the radius of the circle). Use this definition to write an equation for the circle of radius r centred at (x_0, y_0).

8. In the following figure, all three lines are tangent to the circle. If the length of the segment BE is 5, what is the perimeter of the triangle FGE? [Hint: use the previous result above.]



9. (a) Find the area of the largest square that one can fit inside a circle of radius r.
   (b) Find the area of the smallest square that fits outside a circle of radius r.
   (c) Hence show that $2 < \pi < 4$.
   (d) How might you improve your estimate of π?

NCEA Level 2 Mathematics (Homework)
2. Arcs and Sectors of Circles

Reading

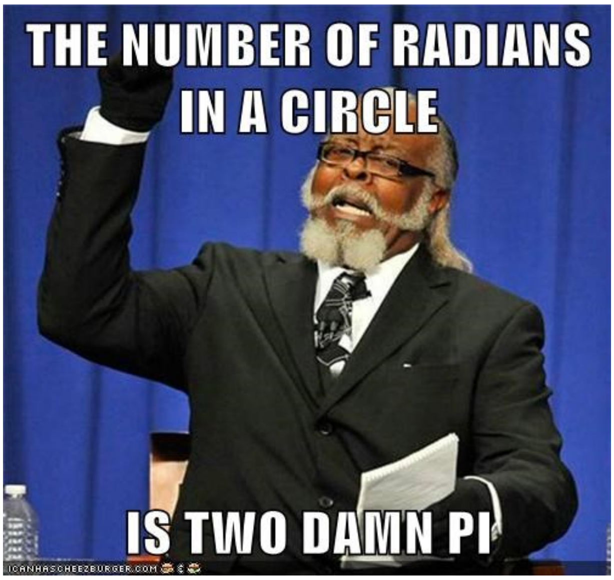
Go and watch...

https://www.youtube.com/watch?v=QncgmzH6yQU

What's it good for?

People use the geometry of circles for...

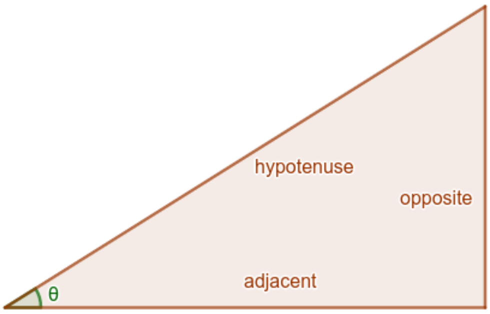
- Physics: physicists and engineers often want to model things that rotate, and proper definitions of rotational speed and acceleration require the use of the geometry of arc lengths and sector areas.
- Mathematics: the idea of a 'limiting process', where we take sums of things that we let become infinitesimally small, is a fundamental idea that underpins entire branches of mathematics and allows us to formally define the concepts of area and volume, and enables us to better understand things which are continuous.



Questions

1. A car tire has diameter 53 cm. A detector measures that a particular point on the tire tread rotates past 1061 times per second.
   (a) What speed is the car travelling at?
   (b) Is this setup practical and/or useful? Explain.
2. What is the radius of a circle such that the sector of area $\frac{\pi}{3}$ has arc length $\frac{\pi}{3}$?

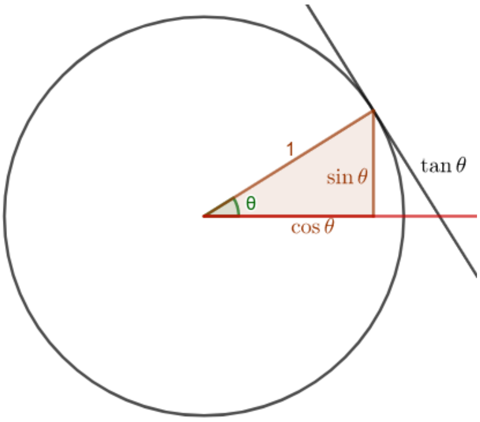
NCEA Level 2 Mathematics
3. Trigonometry



We are now going to look at triangles inside circles. Now, last year we learned that any triangles with two equal angles are similar; in particular, if we take ratios of sides, we obtain the same value. This means that if we have any right-angled triangle with angle θ like the one above, then the ratios $\frac{\text{opposite}}{\text{hypotenuse}}$, $\frac{\text{adjacent}}{\text{hypotenuse}}$, and $\frac{\text{opposite}}{\text{adjacent}}$ all depend only on the angle θ; we call them the sine, cosine, and tangent of the angle respectively:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}.$$

In particular, if we draw our triangle inside a unit circle then we can draw the following:



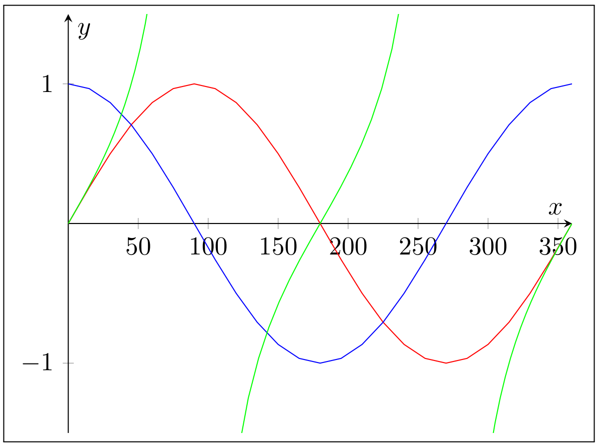
In fact, we can take this as our definition of sin and cos. To show that $\tan\theta$ is indeed the line segment marked there, first notice that since the large triangle is right-angled, the angle at the intersection of the horizontal line and the tangent line is $90° - \theta$; so the other non-right-angle in the smaller triangle is θ. Hence the hypotenuse of the triangle is $\frac{\text{adjacent}}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$, as proposed.

Note also that, from this diagram, we have

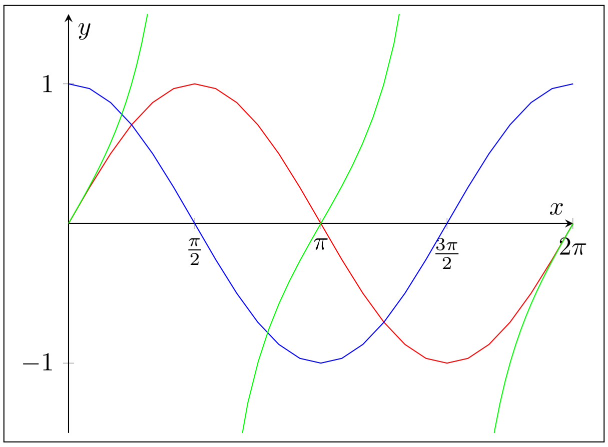
$$\sin^2\theta + \cos^2\theta = 1$$

for every angle θ.

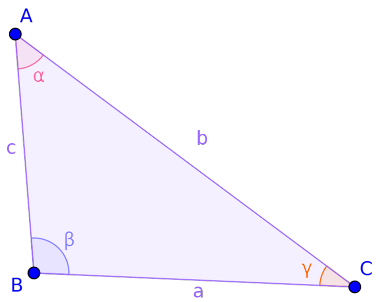
Since the sin of an angle is just the height of the point above the x-axis in the diagram above, we have that $-1 \leq \sin\theta \leq 1$; similarly, $-1 \leq \cos\theta \leq 1$. Note that when $\theta = 90°$, the tangent line becomes horizontal and so never intersects the x-axis: so $\tan 90°$ is undefined. We can even graph $\sin\theta$ (red), $\cos\theta$ (blue), and $\tan\theta$ (green):



If we graph them in radians, only the labels on the x-axis change:



Let us now begin to look at more general triangles:



Theorem (Sine rule)**.** *In any triangle, with the angles and sides labelled as above, we have*

$$\frac{\alpha}{\sin a} = \frac{\beta}{\sin b} = \frac{\gamma}{\sin c}.$$

Proof. Drop an altitude from B to AC, creating two new right-angled triangles. Then the length of this line can be calculated using both of the resulting right-angled triangles: so $c \sin \alpha = a \sin \gamma$ and $\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$. This proves the theorem. □

Theorem (Cosine rule)**.** *In any triangle, with the angles and sides labelled as above, we have*

$$a^2 = b^2 + c^2 - bc \cos \alpha.$$

Proof. Drop an altitude from B to AC, creating two new right-angled triangles. Then the length b can be split into two lengths, $c\cos\alpha$ and $b - c\cos\alpha$; the length of the altitude is $c\sin\alpha$. Now, apply the Pythagorean theorem to the triangle including the angle γ:

$$a^2 = (b - c\cos\alpha)^2 + c^2\sin^2\alpha = b^2 - 2bc\cos\alpha + c^2\cos^2\alpha + c^2\sin^2\alpha = b^2 + c^2 - 2bc\cos\alpha.$$

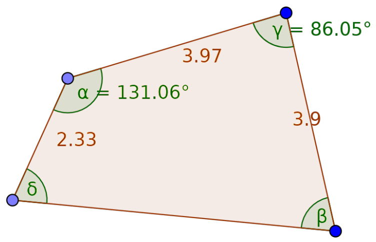
□

Questions

1. A beam of gamma rays is to be used to treat a tumor known to be 5.7 cm beneath the patient's skin. To avoid damaging a vital organ, the radiologist moves the source over by 8.3 cm.
   (a) At what angle to the patient's skin must the radiologist aim the gamma ray source to hit the tumor?
   (b) How far will the beam have to travel through the patient's body before reaching the tumor?

2. A surveyor measures the three sides of a triangular field and finds that they are 117, 165, and 257 metres long.
   (a) What is the measure of the largest angle of the field?
   (b) What is the area of the field?

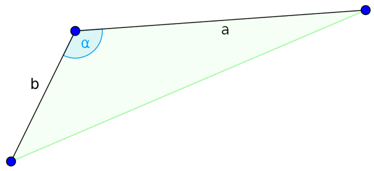
3. A field has the shape of a quadrilateral (four-sided shape) that is *not* a rectangle. Three sides measure 2.33, 3.97, and 3.9 kilometres, and two angles measure $\alpha = 131.06°$ and $\gamma = 86.05°$ (as in the figure).



   (a) By dividing the quadrilateral into two triangles, find its area.
   (b) Find the length of the fourth side.
   (c) Find the measures of the other two angles, β and δ.

4. Suppose that the country of Parah has just launched two satellites. The government of Noya sends aloft its most reliable astronaut, Ivan Advantage, to observe the satellites.
   (a) As Ivan approaches the two satellites, he finds that one of them is 8 km away from him and the other is at 11 km. He measures the angle between them to be 120°. How far apart are the satellites?
   (b) Three ships are assigned to rescue Ivan as his spacecraft plunges into the ocean. The ships are at the vertices of a triangle with sides of 5, 7, and 10 kilometres.
      i. Find the measure of the largest angle of this triangle.
      ii. Find the area of ocean in the triangular region bounded by the three ships.
   (c) To welcome their returning hero, the Noyans give Ivan a parade. The parade goes between the three cities of Triy, Kwe, and Stin. These cities are at the vertices of an equilateral triangle. The roads connecting them are straight, level, and direct, and the parade goes at a constant speed with no stops. From Triy to Kwe takes 80 min and from Kwe to Stin takes 80 min, but from Stin back to Triy along the third side of the triangle takes one hour and 20 minutes. How do you explain this discrepancy in times?

5. Find the area of the triangle below.



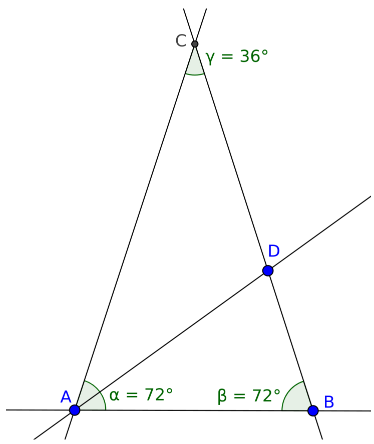
6. For each item below, decide whether or not such a triangle exists. If at least one does, how many exist?
   (a) Exactly one angle greater than $90°$.
   (b) Two angles greater than $\pi/2$.
   (c) Two sides of length 200,000.
   (d) Three sides of length 200,000.
   (e) Sides of length 90, 30, and 30.

7. Prove that, if a quadrilateral has equal diagonals, then it is a rectangle. (We used this fact last week!)

8. This question requires you to find exact values for trig functions *without* using a calculator. [Schol 1999]
   (a) Sketch an equilateral triangle of side 2 units. Hence, or otherwise, show that
      - $\cos\frac{\pi}{3} = \frac{1}{2}$,
      - $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, and
      - $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

   (b) Sketch a right angled isosceles triangle whose equal sides enclose the right angle and are of length 1. Hence, or otherwise, show that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

9. Consider the 75-75-36 triangle ABC given in the figure. The angle α has been bisected into two angles, and the resulting line meets the triangle at D.



   (a) Show that ABC and ABD are similar triangles.
   (b) Hence, or otherwise, show that $\frac{AB}{BD} = \frac{AB+BD}{AB}$.
   (c) Show that the ratio of the long side of the triangle to the short side of the triangle is $\frac{AB}{BD} = \frac{1+\sqrt{5}}{2} = \phi$.
   (d) Show that $\cos 72° = \frac{1}{2\phi}$.
   (e) Find $\sin 36°$ and $\sin 72°$.

NCEA Level 2 Mathematics (Homework)
3. Trigonometry

Reading

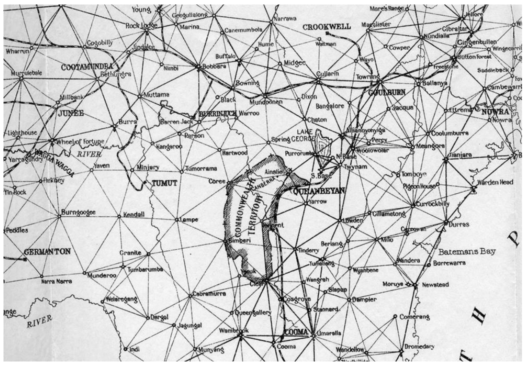
Go and watch...

`https://www.youtube.com/watch?v=o6KlpIWhbcw`

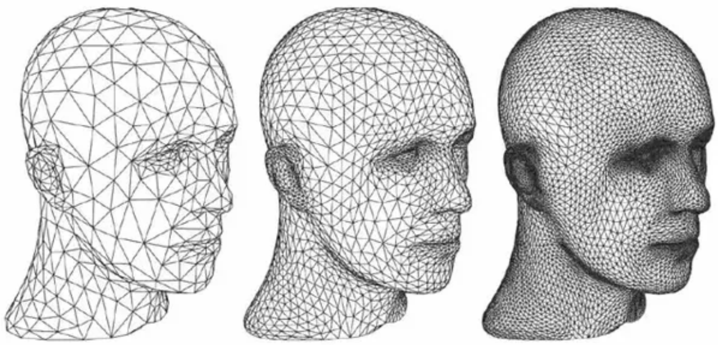
What's it good for?

People use the geometry of triangles for...

- Physics: physicists and engineers use trigonometry to calculate how forces and stresses are transmitted in systems that involve angles.
- Geography: triangulation is used to measure the areas and topography of large areas of land.



- Computer graphics: complicated 3D shapes are usually modelled with a large number of triangles or other polygons, to simplify and speed up shading and other algorithms.



Questions

[This is a sample Ministry of Education L2 assessment task for this standard.]

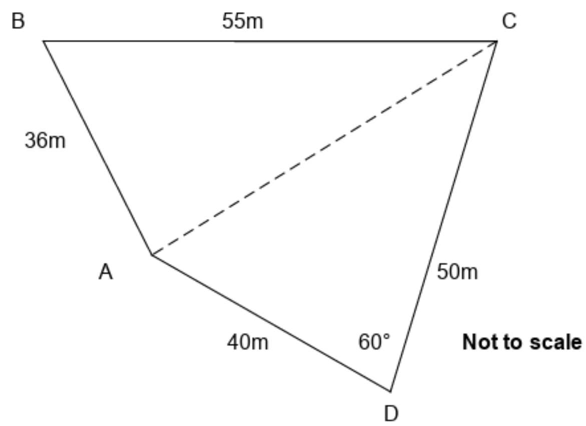
Your school is selling some unused land to raise money for a new gymnasium. The buyer will only purchase the land if the school can demonstrate that the land can be subdivided into four sections of at least $400\,\mathrm{m}^2$ each. The buyer will not purchase the land if all of the sections are triangular. This assessment activity requires you to determine the dimensions of appropriate subdivisions so that the sale can proceed.

Given the land diagram below,

1. Calculate the length of the pipeline running through the land.
2. Demonstrate that the land can be divided into four sections each of more than $400\,\mathrm{m}^2$ such that not all are triangular.
3. Show one possible way of dividing the land into four sections each of more than $400\,\mathrm{m}^2$ such that not all are triangular.

The quality of your reasoning, using a range of methods, and how well you link this context to your solutions will determine the overall grade. Clearly communicate your method using appropriate mathematical statements so that the new owner can easily verify the dimensions of the sections.

Resource 1: Land Diagram



The unused land bounded by ABCD needs to be split into four sections.
The pipeline running through the land is between points A and C.

CHAPTER 3

Algebra

NCEA Level 2 Mathematics
4. Functions

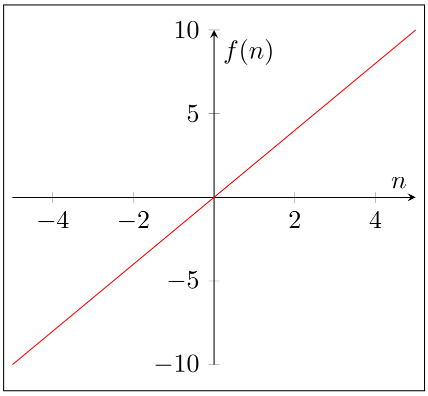
One of the most fundamental concepts in mathematics is that of a function. A function is a relationship between two sets of things, called the *range* and the *domain*, such that everything in the range is related to exactly one thing in the domain. You can think of a function as a rule: it could be given by a formula, or by a list of inputs and outputs, or in any other way that one likes.

If f is a function, and it relates x to y, then we write $f(x) = y$. In this notation, x is the *argument* or *input* and y is the *result* or *output*.

The above expression $f(x) = y$ is suggestive: we can graph functions by graphing every pair of numbers (x, y) which satisfies this equation.

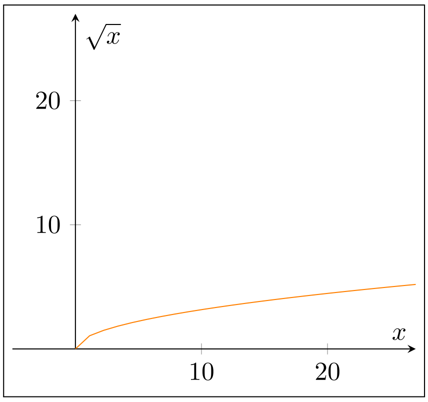
Example.

1. Suppose for every number n we associate its double, $2n$. This is a perfectly good function, which we can call f: then $f(n) = 2n$, and $f(2) = 2 \cdot 2 = 4$.

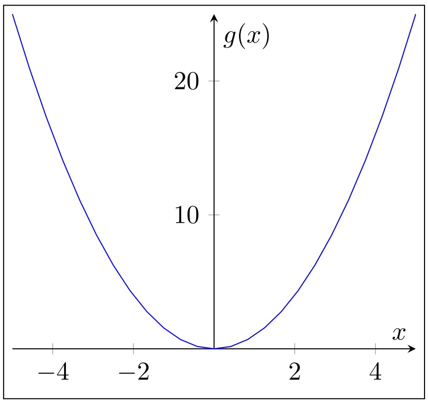


2. Suppose for every number x we associate the number which, when squared, gives x. This is *not* a well-defined function: for example, what number do we associate with 4: 2, or -2? What number do we associate with -1?

3. On the other hand, suppose for every positive number x we associate its positive square root. This time we do have a well-defined function. Note that its domain and range are both the positive numbers, and for every number in the domain there is precisely one number in the range.



4. Let $g(x) = x^2$. This is also a perfectly good function; every number has exactly one square. Note that $g(-2) = g(2) = 4$; this is allowed, but if a function f has the property that if x and y are different then $f(x) \neq f(y)$ then f is called one-to-one. The functions in (1) and (3) above are both one-to-one.



Notice that this is a square root on its side... can you explain why?

One-to-one functions are useful, because they have an *inverse.* That is, given any number in the *range* we can find the number in the *domain* that maps to it. If f is a one-to-one function, then its inverse is written as f^{-1}.

Example.

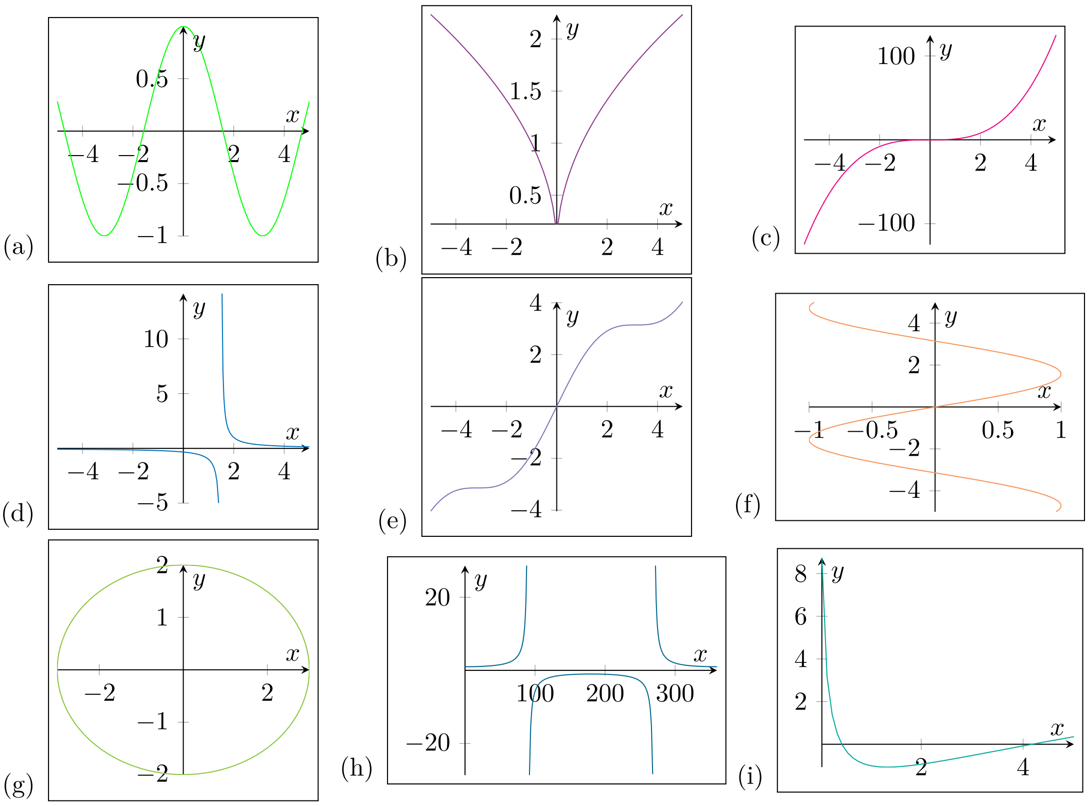
1. If $f(x) = 2x$, then $f^{-1}(x) = \frac{1}{2}x$.
2. If $g(x) = x^2$, then g does not have an inverse over all the numbers; but if we restrict the domain of g to the positive numbers, then it does have an inverse $g^{-1}(x) = +\sqrt{x}$.
3. If $h(x) = \sin x$ (and the range of x is restricted to $0 < x \leq 2\pi$) then h is a perfectly good function with range $-1 \leq h(x) \leq 1$; its inverse is $\sin^{-1}$, which takes a triangle ratio and returns the appropriate angle.

Questions

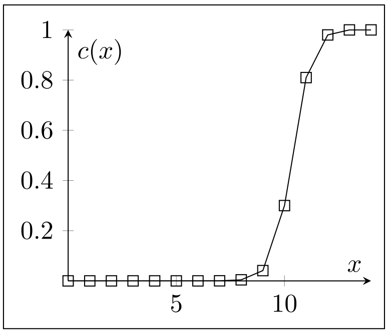
1. Justify the following statements with mathematical reasoning.
   (a) The graph of a function passes the vertical line test: any vertical line drawn on the graph crosses the function in at most one point.
   (b) The graph of a one-to-one function passes the horizontal line test: any horizontal line drawn on the graph crosses the function in at most one point.
   (c) The graph of the inverse of a function is the reflection of the graph of the function across the line $x = y$.
   (d) If we define f by
   $$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$
   then f is a function but it is not one-to-one.
   (e) If f and g are functions, and the range of g is the same as the domain of f, and we define a new function $(f \circ g)$ by $(f \circ g)(x) = f(g(x))$, then $(f \circ g)$ is a function with the same range as f and the same domain as g.

2. The following define functions only if they have a domain which is restricted. What is the largest possible domain for each, so that they are still functions? What is the range of each?
   (a) $f(x) = 1/x$
   (b) $g(\theta) = \tan\theta$
   (c) $h(x) = \left(\dfrac{x}{x-2}\right)^2$

3. Which of the following graphs are graphs of functions? Of those, which are one-to-one?



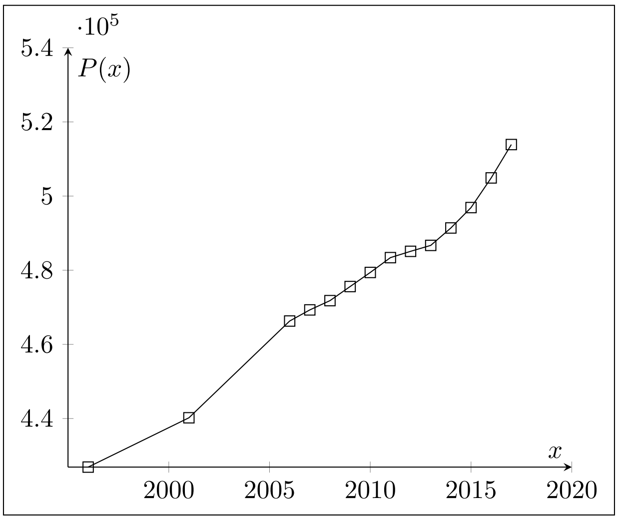
4. We will now look at shifting graphs around.
   (a) Graph f, if $f(x) = x^2$.
   (b) On the same paper, graph $f(x-1)$ and $f(x+1)$.
   (c) Graph $f(x) - 1$ and $f(x) + 1$.
   (d) Graph $2f(x)$ and $f(2x)$.
   (e) Make a conjecture about the relationship between the following graphs, if f is now any function and a, b, c, and d are any numbers:
      i. $y = f(x)$
      ii. $y = f(x+a)$
      iii. $y = f(x) + b$
      iv. $y = cf(x)$
      v. $y = f(dx)$
   (f) Explain intuitively why, when we add numbers inside the function argument, the graph shifts 'the wrong way'.
   (g) Graph $y = 3\sin(\frac{1}{2}x + 1) - 2$.
5. If I take $y = x^2$ and I want to shift and stretch its graph so that the vertex is sitting at $(-2, 3)$ and it goes through $(1, 1)$, can I do that? If so, what is the equation of the function that gives me the new graph?
6. The following graph is of the function c, where $c(x)$ is the proportion of a particular chemical species in solution at pH x (pH is, roughly speaking, a measure of acidity). *



   (a) Is this function invertible? That is, if I measure the concentration of this species at any point, can I identify precisely what the pH of the solution is?
   (b) Explain why knowing the proportion of a chemical species in water at a given pH might be useful; why is mathematical modelling useful in such a situation?

*From *Quantitative Chemical Analysis*, 7th edition, by Daniel Harris (page 233).

7. The following graph is of the function P, where $P(x)$ is the population of the Greater Wellington region in the year x.† Note that the y-axis does *not* start at zero, and is given in units of hundreds of thousands.



(a) Explain why extrapolating using this graph to find $P(2020)$ is probably justified.

(b) Give an example of a function where the values of the function above any point cannot be extrapolated from knowing the values of the function before the point. Do you think that extrapolation is normally possible for mathematical models of the natural world?

(c) Describe how the rate of population growth changes as we move forwards in time from 1996 to 2017.

(d) Given the real-world context of this graph, how would you expect the slope of the graph to change:

i. in the medium term (up to, say, 2040), and

ii. in the long term (say around the year 2100).

†Statistics are the Statistics NZ 'Subnational population estimates (RC, AU), by age and sex, at 30 June 1996, 2001, 2006-17 (2017 boundaries)' (retrieved 19 June 2018).

NCEA Level 2 Mathematics (Homework)
4. Functions

Reading

Go and watch...

`https://www.youtube.com/watch?v=PtLnwvH4kuE`

What's it good for?

People use functions and mathematical modelling for...

- Statistics, engineering, and the sciences: extrapolation from a set of data and prediction of results from future experiments and other situations is a key part of the scientific method.
- Theoretical physics: in particular, the mathematics behind quantum mechanics is heavily dependent on linear transformations, a particular class of function.
- Mathematics: as I mentioned in the notes, functions are a key idea in mathematics because they allow us to describe things like curves in space, rates of change, and relationships between objects. Graphs and diagrams of functions are often another way to view a concept and let us gain more clarity. Normally, we don't talk about functions in general, but restrict ourselves to subclasses of functions with nice properties (preserving distance, or more generally 'closeness'; preserving algebraic properties like addition or multiplication; etc.).

Questions

[This is from a sample Ministry of Education L2 assessment task for this standard.]

Place cones at the following co-ordinates, in metres, with the positive y axis pointing due north:

Cone	Location	Cone	Location	Cone	Location
A	$(-14, 1)$	D	$(7, 6)$	G	$(7, 0)$
B	$(-16, 6)$	E	$(7, 10)$	H	$(17, -3)$
C	$(3, 6)$	F	$(10, 10)$	I	$(7, 1)$

Give equations for each of the following curves:

1. Start from a point one metre to the north of cone A. Ride in a straight line to a point two metres to the north of cone B.
2. Starting from the end of line 1, weave around cones B, C, and D, such that the maximum distance south of cone C is the same as the maximum distance north of cones B and D, passing through the point $(5, 6)$ following a curve of the form $f(x) = A\sin(x - 5) + C$.

As revision from L1, expand and simplify the following:

1. $(x - 2)^2(x + 8)$
2. $(5x - 4)(x + 2)(x + 1)$

NCEA Level 2 Mathematics
5. Quadratic Modelling

A linear function is one of the form

$$f(x) = mx + c;$$

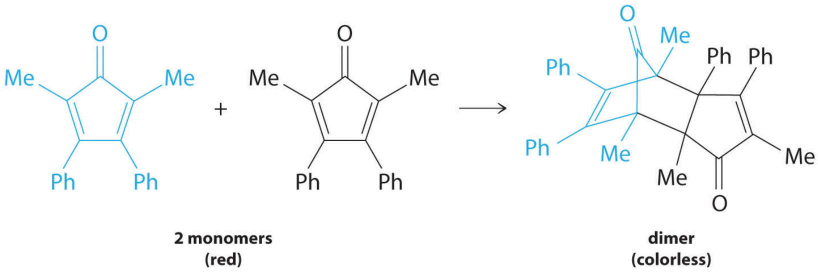
we have already seen that the graph of such a function is a straight line.

The natural next step is to consider quadratic functions: those of the form

$$f(x) = ax^2 + bx + c.$$

Last year, we saw that the graphs of quadratic functions are parabolae; so we can use quadratic equations to model situations which are vaguely parabolic.

Example.



The following table gives the instantaneous rates of reaction for the dimerization reaction above.

Time (min)	**Concentration of reactant** (M)	**Instantaneous reaction rate** (M/min)
0	0.0054	
10	0.0044	8.0×10^{-5}
26	0.0034	5.0×10^{-5}
44	0.0027	3.1×10^{-5}
70	0.0020	1.8×10^{-5}
120	0.0014	8.0×10^{-6}

It is known that this reaction is second-order: that is, the reaction rate is modelled by a quadratic function of the reaction concentration. Let's call the reaction rate at a particular concentration $R(C)$; so

$$R(C) = XC^2 + YC + Z.$$

By using the values in the table above, we have that

$$\begin{aligned} 8.0 \times 10^{-5} &= X \cdot (0.0044)^2 + Y(0.0044) + Z \\ 5.0 \times 10^{-5} &= X \cdot (0.0034)^2 + Y(0.0034) + Z \\ 3.1 \times 10^{-5} &= X \cdot (0.0027)^2 + Y(0.0027) + Z. \end{aligned}$$

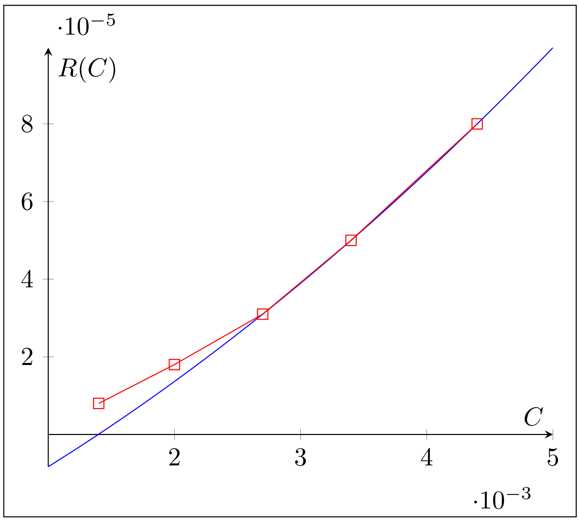
Using `matlab` to find X, Y, and Z we have

```
>> syms X Y Z;
>> eqn1 = 8.0e-5 == X * (0.0044)^2 + Y * (0.0044) + Z;
>> eqn2 = 5.0e-5 == X * (0.0034)^2 + Y * (0.0034) + Z;
>> eqn3 = 3.1e-5 == X * (0.0027)^2 + Y * (0.0027) + Z;
>> [A,B] = equationsToMatrix([eqn1,eqn2,eqn3],[X,Y,Z]);
>> linsolve(A,B);
```

```
ans =

                                          6148914691236521356/3658604241285734375
    364782697339246960843864593812277/215965875539156003306215539579289600
-4531426852830508602047900590172259/168723340264965627582980890296320000
```

So our model is
$$R(C) \approx 1.6807C^2 + 0.01689C - 2.686 \times 10^{-5}.$$
Graphing our model (blue) next to the original data (red), we see that the match is reasonable.



This is useful because it allows us to predict the reaction rate for concentrations that we have not measured; alternatively, we can predict the reactant concentration given the reaction rate (which can be easily measured).

Some situations require us to find a minimum or a maximum value. Suppose, for example, that we are asked the following.

Example. Find the dimensions of a rectangle with perimeter 1000 m such that the area is maximised.

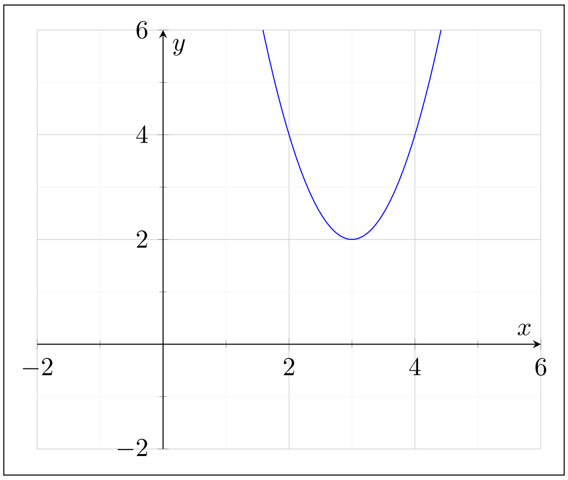
Solution. Later this year, we will learn a systematic way to solve optimisation problems like this. However, using calculus here would be like using a machine gun to kill a fly (although for some reason it is the favoured method of economics students)!

Let us call the two side lengths x and y; so we have that $1000 = 2x + 2y$ (so $500 = x + y$), and we want to minimise $A = xy$. Substituting, we have $A = x(500 - x) = 500x - x^2$. The maximum value will be the vertex of this parabola, so we need to put this formula into the form $A = -(x - b)^2 + c$; if we expand this, then $A = -x^2 + 2xb - b^2 + c$ and so we want $2b = 500$ and $c - b^2 = 0$. From these, we see that $b = 250$ and $c = (250)^2$. Hence $A = -(x - 250)^2 + (250)^2$; the vertex is at $x = 250$, and so $y = 250$: the area will be maximised when the rectangle is a square.

Questions

You might want to use a calculator or a computer to help you with some of the calculations: for example, solving systems of equations or quadratic equations.

1. Give the equation of the following parabola:



2. (a) Find the vertices of the parabolae $y = x^2 + 4x - 9$ and $y = -4x^2 + 2x - 1$.
   (b) More generally, show that the vertex of the parabola $y = x^2 + bx + c$ is
$$\left(-\frac{b}{2}, c - \frac{b^2}{4}\right).$$

3. (a) Find the equation of the parabola with vertex $(2, 3)$ that passes through $(9, 1)$.
   (b) More generally, show that the equation of the parabola with vertex $V(x_0, y_0)$ passing through $A(x_1, y_1)$ is
$$y = \frac{y_1}{(x_1 - x_0)^2 + y_0}(x - x_0) + y_0.$$

4. One application of mathematical modelling is in spectroscopy: it is possible to measure the absorbance of light by substances, which is proportional to the amount of the substance present.

Amount of protein (µg)	**Absorbance**
0	0.099
5.0	0.185
10.0	0.282
15.0	0.345
20.0	0.425
25.0	0.483

   (a) Use the values for 0, 5.0, and 25.0 micrograms to write down an equation for a parabolic model of this data.
   (b) How accurately does this model predict the absorbance for 15 micrograms?
   (c) A better model, taking into account all six data points, is found by computer to be
$$\text{amount of protein} = -0.000131429A^2 + 0.0188686A + 0.0982286.$$
   If the absorbance is measured experimentally to be 0.3, how much protein is present in the sample?

5. Show that the x-value of the vertex of a parabola is always halfway between the x-intercepts.

NCEA Level 2 Mathematics (Homework)
5. Quadratic Modelling

Reading

Go and watch...

https://www.youtube.com/watch?v=hoh4TmPzu1w

What's it good for?

People use parabolae for...

- Engineering, and the sciences: modelling situations where a quantity is roughly proportional to the square of another. In physics, a perfect projectile follows the path of a parabola.
- Mathematics: a parabola is a specific case of a conic section (the others are the hyperbola, the circle, and the ellipse). Conic sections are geometric figures that were systematically studied by the Greeks.

Questions

1. Show that if $y = ax^2 + bx + c$, then
$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$
(You might want to start with the complicated equation and simplify it, rather than going the other way.)
2. Explain why this shows that changing b and c only shifts the parabola around the plane rather than changing the size. In other words, it is possible for us to shift any parabola anywhere we want without affecting the side.
3. In fact, we have shown that every quadratic can be written in the form $y = a(x - \alpha)^2 + \beta$; since changing α and β only shifts the parabola around, the only parameter that can change the size is a. Therefore, if we have a parabola $y = ax^2 + bx + c$ then it has the same shape as $y = ax^2$. Show that if (x_1, y_1) is on the parabola $y = ax^2$, then $(\frac{b}{a}x_1, \frac{b}{a}y_1)$ is on the parabola $y = bx^2$ (where a and b are any nonzero numbers).
4. Conclude that any parabola can be mapped onto any other parabola by shifting it around the plane (changing the location of the vertex) and scaling it in both directions by an appropriate constant.

NCEA Level 2 Mathematics
6. Simultaneous Equations

Let's go back to last week, where we had the following example of a quadratic model.
Example. The following table gives the instantaneous rates of reaction for the a particular chemical reaction.

Time (min)	**Concentration of reactant** (M)	**Instantaneous reaction rate** (M/min)
0	0.0054	
10	0.0044	8.0×10^{-5}
26	0.0034	5.0×10^{-5}
44	0.0027	3.1×10^{-5}
70	0.0020	1.8×10^{-5}
120	0.0014	8.0×10^{-6}

It is known that the reaction rate is modelled by a quadratic function of the reaction concentration. Let's call the reaction rate at a particular concentration $R(C)$; so

$$R(C) = XC^2 + YC + Z.$$

By using the values in the table above, we have that

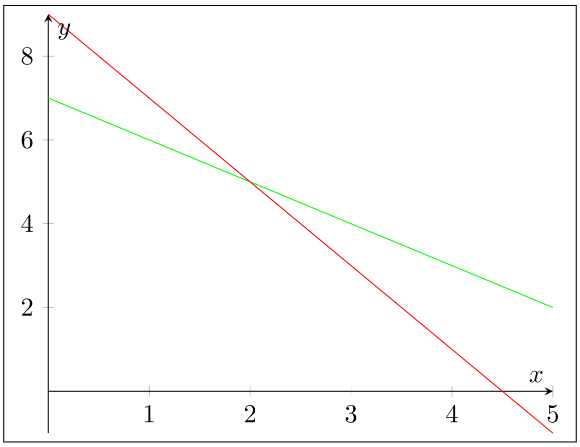
$$\begin{aligned}
8.0 \times 10^{-5} &= X \cdot (0.0044)^2 + Y(0.0044) + Z \\
5.0 \times 10^{-5} &= X \cdot (0.0034)^2 + Y(0.0034) + Z \\
3.1 \times 10^{-5} &= X \cdot (0.0027)^2 + Y(0.0027) + Z.
\end{aligned}$$

I'll leave off the end, because the point is we used a computer to solve this system of equations. This week, we'll learn a *systematic* method for solving such systems: not because it's easier than using a computer, but because it's interesting to see what's going on geometrically. (In my Y13 notes, there's a lot more detail — and proofs, which I'll skip this year.)

The main idea to get your head around this year is that a system of equations is *geometric*.
Example. Consider the system of equations

$$\begin{cases} x + y = 7 \\ 5x + 10y = 45. \end{cases}$$

If we plot all the values (x, y) which satisfy these equations, we obtain:



The key point here is that the point which satisfies both solutions is simply the geometric point of intersection.

In order to find this point algebraically, we can solve the first equation for y: $y = 7 - x$. We can then substitute this into the other: $5x + 10(7 - x) = 45$. Simplifying, we have

$$\begin{aligned} 5x + 70 - 10x &= 45 \\ 25 &= 5x \\ 5 &= x. \end{aligned}$$

Let's think about this a little more explicitly: since the same pair of values (x, y) makes both equations true, then if we can find y in terms of x using one equation then we can substitute it straight into the other one *because the symbol y represents the same thing in both.*

In another situation, we might have a line and a parabola.
Example (Extract from the sample L2 assessment for this topic.)**.** The student committee is planning the upcoming Performing Arts Showcase at your school this year. They are trying to determine how much they should make the price of adult tickets this year.

Determine the price they should set the adult tickets in 2012 if they want to make a profit of \$3210 from ticket sales, given that:

- The cost of a child ticket is planned to be \$5 and the cost of an adult ticket is not determined yet.
- The relationship between the expected number of child tickets and adult tickets to be sold can be modelled by $200x + y^2 = 80000$, where x represents the number of child tickets and y represents the number of adult tickets.
- The planning committee wants to make \$3210 from ticket sales.
- There needs to be only one possibility for the number of child and adult tickets to be sold.

Solution. The total profit that will be made is $5x + my$, where m is the cost of an adult ticket (yet to be determined); we therefore have three important pieces of information: $5x + my = 3210$, $200x + y^2 = 80000$, and the curves of these two equations intersect at only one point. Solving the first for x, we have $x = \frac{3210-my}{5}$; substituting, we have

$$\begin{aligned} 200\left(\frac{3210 - my}{5}\right) + y^2 &= 80000 \\ 128400 - 40my + y^2 &= 80000 \\ 48400 + 40my + y^2 &= 0 \end{aligned}$$

When we learned about parabolae, we learned that we could transform them into vertex-intercept form. Let's do that here, because our goal is to find a value m such that the vertex of $48400 + 40my + y^2 = x$ is sitting exacty on the y-axis — our parabola is flipped sideways, but the idea is the same. In fact, to make it clearer let's look at the 'graphy' form:*

$$(y + 20m)^2 + 48400 - 400m^2 = x.$$

This parabola is sitting on the y-axis when $48400 - 400m^2$ is zero (because this is the x-shift); hence $m = 11$, and the committee should sell the adult tickets for \$11 each.

*The idea of switching from looking at something in an 'equationy' sense and turning something into a parameter is actually quite a powerful idea that's applicable in many different situations.

Questions

1. Graph and solve the system of equations

$$\begin{cases} 1 = x + y, \\ 2 = 2x - y. \end{cases}$$

2. Let us explore how many solutions we can obtain for simple systems. If

$$\begin{cases} a = bx + cy, \\ p = qx + ry \end{cases}$$

is a system of simultaneous equations in x and y then depending on the constants we choose there are three possible situations:

- We can have no pairs (x, y) that satisfy both equations.
- We can have precisely one pair (x, y) that satisfies both equations.
- We can have infinitely many pairs (x, y) that satisfy both equations.

(We will prove this next year.)

(a) Draw a diagram showing each situation geometrically.

(b) Give an example of a system for each case, taking care to show that your systems have the desired number of solutions.

3. Graph (using a computer or a calculator) and solve the system of equations

$$\begin{cases} y = 2x + 3, \\ x^2 + 2xy - 1 = 0. \end{cases}$$

4. [Extract from the sample L2 assessment for this topic.] For the performing arts showcase example above, you must also provide the planning committee with the change in the adult tickets sales in 2011 compared to 2010. You are given that:

- **2010 Performing Arts Showcase**
  - The total number of tickets sold was 400.
  - The relationship between the number of child tickets and adult tickets sold can be modelled by $x^2 + y = 22750$, where x represents the number of child tickets and y represents the number of adult tickets.
- **2011 Performing Arts Showcase**
  - The cost of a child ticket was \$5 and the cost of an adult ticket was \$10.
  - The relationship between the number of child tickets and adult tickets sold can be modelled by $xy = 1000 + 100x$, where x represents the number of child tickets and y represents the number of adult tickets.
  - More than 300 tickets were sold.
  - The money generated from ticket sales was \$2050.

5. For which values of c does the system of equations

$$\begin{cases} y^2 = 2x^2 + xy \\ y = cx - 2 \end{cases}$$

have precisely two solutions?

6. Find the parabola passing through the three points $(0, 0)$, $(1, 1)$, and $(3, 0)$.

7. The graph of the equation $xy = a$ (where a is a number) is called a *hyperbola*.

   (a) Show that $xy = a$ and $xy = b$ never intersect if $a \neq b$.

   (b) Find the point of intersection between the curves

$$\begin{cases} x(y + 1) = 4 \\ y = 2x - 3. \end{cases}$$

8. The line $y = 2x - 3$ intersects the circle $x^2 - 6x + y^2 = 0$ precisely twice.

   (a) Find the coordinates of the centre of the circle.

   (b) Find the points of intersection.

   (c) How could the constant term 3 of the line be changed such that the line becomes tangent with the circle?

NCEA Level 2 Mathematics (Homework)
6. Systems of Equations

Reading

Go and watch...

https://www.youtube.com/watch?v=aOT_bG-vWyg

What's it good for?

People use systems of equations for...

- Engineering, and the sciences: if a set of unknown quantities are all interrelated (for example, concentrations in a chemical equilibrium).
- Mathematics: solving systems of linear equations is the motivating example for *linear algebra*, which forms the algebraic basis of geometry and is itself used in engineering and the sciences for modelling systems.
- Computer graphics: finding intersection points of curves is often a problem that needs to be solved in computer graphics systems (e.g. working out where a ray of light hits a surface).

Questions

1. Find all the solutions to the system of equations

$$\begin{cases} 4x^2 + 16x + y^2 + 15 = 4xy + 8y \\ y = 2x + 3. \end{cases}$$

2. Graph the above system. If we replace the linear equation with $y = cx + 3$, for which values of c does the system have no solutions? Is it possible to pick a value of c such that the system has exactly one solution?

NCEA Level 2 Mathematics
7. Linear Inequalities

An equation is a statement which says that two quantities are identical. If we don't want to be so precise, we can talk about inequalities: statements which tell us about the *relative size* of two quantities. More precisely, if a and b are two quantities then:

$a = b$	a is identical to b
$a \neq b$	a is not identical to b
$a \leq b$	either a is identical to b, or a is smaller than b
$a < b$	a is not identical to b and a is smaller than b

This allows us to impose an *ordering* structure onto the integers, as well as the algebraic structure that they already had. We will look at the interplay between the two in the exercises.

Questions

1. Justify the following statements with mathematical reasoning (where a, b, and c are quantities):
   (a) Precisely one of $a < b$, $a = b$, or $a > b$ is true.
   (b) If $a \leq b$ and $b \leq c$ then $a \leq c$.
   (c) If $a \leq b$ and $b \leq a$ then $a = b$.
2. Using number lines, explain why
   (a) $(-3) + (-4) = -7$;
   (b) $(-2) + 4 = 2$;
   (c) $(-10) + 4 = -6$.
3. Consider the following multiplication table.

a		b	ab
2	$\times$	5	10
2	$\times$	4	8
2	$\times$	3	6
2	$\times$	2	4
2	$\times$	1	2
2	$\times$	0	0
2	$\times$	-1	
2	$\times$	-2	
2	$\times$	-10	

   (a) What is the pattern in the final column?
   (b) Fill in the final three lines of the table by continuing the pattern.
   (c) Based on this table, is it more reasonable for the product of a **negative by a positive** to be positive or negative?
   (d) Using this definition, fill in the first five lines of the next table:

a		b	ab
-3	$\times$	4	
-3	$\times$	3	
-3	$\times$	2	
-3	$\times$	1	
-3	$\times$	0	
-3	$\times$	-1	
-3	$\times$	-2	
-3	$\times$	-10	

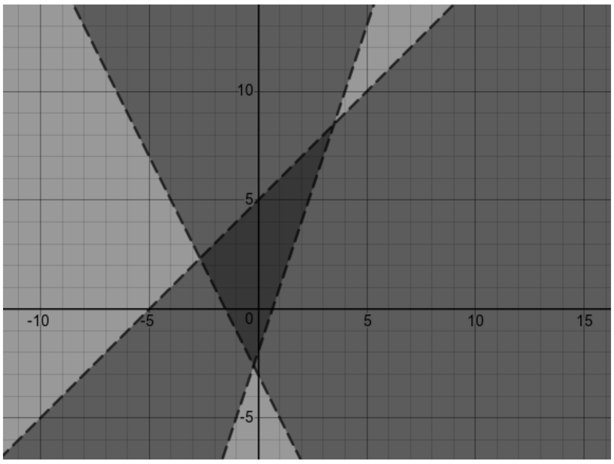
(e) Again using the pattern we see as we move down the final column, fill in the last three rows.

(f) Based on this table, is it more reasonable for the product of a **negative by a negative** to be positive or negative?

4. Justify the following statements with mathematical reasoning (where a, b, c, and d are quantities) — you may want to draw number lines, it makes things easier to visualise:
   (a) If $a \leq b$ and c is *positive* then $ac \leq bc$.
   (b) If $a \leq b$ and c is *negative* then $ac \geq bc$.
   (c) If $a \leq b$ and c is any quantity then $a + c \leq b + c$.
   (d) If $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.
   (e) If $a \leq b$ and $c \leq d$ then we cannot make any statement about the relative values of $a + d$ and $b + c$. [Hint: consider $1 \leq 2$ and $1 \leq 1$ as $a \leq b$ and $c \leq d$ respectively, then swap them around.]

5. We will now look at inequalities which involve variables.
   (a) For each of the following inequations, graph all the possible values of x and y that satisfy it.
      i. $4 + x < 3$
      ii. $3x + 2 \geq 2$
      iii. $x \geq y$
      iv. $x \leq y$
      v. $3x + 9y \leq 1$
      vi. $2x + y \geq 0$
   (b) Graph all possible values of x and y satisfying each of the following *sets* of inequalities. (The resulting region of the plane is called the *feasible region* of the system.)
      i. $x < y$, $x > y$, and $x < 2y$
      ii. $x \leq 2$, $x \geq -1$, $y \leq x$, $y \geq x - 3$

6. Consider the following graphed system of inequalities.



(a) Explicitly write down the three inequalities that have been graphed.

(b) What are the coordinates of the three intersection points?

7. (a) Show that no point simultaneously satisfies both of $y \geq 2x + 1$ and $y \leq 2x - 3$.

(b) Show that if $y \leq Ax + B$ is any linear inequality in x and y, then the feasible region of this inequality overlaps with at least one of the inequalities in (a).

8. The **arithmetic mean** of two numbers a and b is $\frac{a+b}{2}$; the **geometric mean** of a and b is $\sqrt{ab}$.

(a) Calculate the arithmetic and geometric means for several pairs of numbers. Make a conjecture about the relative order of the two means (is one always bigger than the other)?

(b) Show that

$$\left(\frac{a+b}{2}\right)^2 - ab = \left(\frac{a-b}{2}\right)^2.$$

(c) Suppose that a and b are positive numbers. Using part (b), or otherwise, show that the geometric mean of a and b is always less than or equal to their arithmetic mean. When are the two equal?

(d) Investigate the cases where a and b are both negative, or one is negative and one is positive. (Hint: one of these cases makes no sense.)

NCEA Level 2 Mathematics (Homework)
7. Linear Inequalities

Reading

Go and watch...

https://www.youtube.com/watch?v=ij-EK-MZv2Q

What's it good for?

People use systems of inequalities for...

- Economics: the setting of prices and production outputs in companies can often be modelled by a set of inequalities and an equation to maximise or minimise within the feasible region (this is called *linear programming*).
- Engineering: engineering problems like fluid flow through a network can also be modelled with linear programming.

Questions

1. Find the simplest set of inequalities that describes the same feasible region as the following system.

$$\begin{cases} 3x \leq 2 \\ x \leq -y \\ 3x \geq -1 - 3y \\ 8x \geq -2 \\ y \leq 5 \\ x \geq y + 2. \end{cases}$$

2. Consider the following system, the feasible region of which is a square.

$$\begin{cases} x \leq 1 \\ x \geq -1 \\ y \leq 1 \\ y \geq -1 \end{cases}$$

At what point(s) within the feasible region is $x + 2y$ largest?

NCEA Level 2 Mathematics
8. The Quadratic Formula

Solving Quadratics

Recall that a quadratic equation is one of the form $y = ax^2 + bx + c$ (where $a \neq 0$). A couple of weeks ago, we saw that we can always rearrange such an equation into vertex form; we do this by trying to rewrite it as a square plus a constant. This process is known as *completing the square*.

Our goal is to end up with something that looks like

$$y = \alpha(x + \beta)^2 + \gamma$$

where $(-\beta, \gamma)$ are the coordinates of the vertex of the parabola and α (as we have seen) is the 'scaling factor' that gives us the shape. If α is negative then the parabola opens downwards, and if α is positive then the parabola opens upwards.

If we expand the parabola equation, we obtain

$$\begin{aligned} y &= \alpha(x^2 + \beta x + \beta^2) + \gamma \\ &= \alpha x^2 + 2\alpha\beta x + \alpha\beta^2 + \gamma. \end{aligned}$$

By comparing coefficients, we see that:

$$\begin{aligned} a &= \alpha \\ b &= 2\alpha\beta = 2a\beta \\ c &= \alpha\beta^2 + \gamma = a\beta^2 + \gamma. \end{aligned}$$

Clearly, then, we have $\beta = b/2a$. Substituting this into the third equation, we have $c = a(b/2a)^2 + \gamma$ and so $\gamma = c - \frac{b^2}{4a}$.

Reasoning thusly, we see that

$$y = ax^2 + bx + c = \alpha(x + \beta)^2 + \gamma = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a},$$

which is what I asked you to prove in the homework when we looked at parabolae.

Example. Suppose we are given a rectangular plot of land and are told that the area of the land is $32\,\mathrm{km}^2$ and that one side of the land is $8\,\mathrm{km}$ longer than the other. In order to find the dimensions of the land, we have a quadratic equation which we can simplify:

$$x(x + 8) = 32 \implies x^2 + 8x - 32 = 0.$$

Completing the square, we have that

$$0 = (x + 4)^2 - 16 + (-32) = (x + 4)^2 - 48$$

and so $x = \sqrt{48} - 4 \approx 2.9\,\mathrm{km}$.

This example suggests that we can write down a formula for the value of x in any quadratic equation

$0 = ax^2 + bx + c$ by rewriting that equation in vertex form and solving for x:

$$\begin{aligned}
0 = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
\frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2 \\
\pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = x + \frac{b}{2a} \\
-\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}} = x \\
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = x.
\end{aligned}$$

We have therefore proved the following

Theorem (Quadratic formula)**.** *If $ax^2 + bx + c = 0$, then there are at most two distinct values for x. These values are given by*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

when they exist.

These values are called the *solutions*, the *roots*, or the *zeroes* of the equation.

Example. The width of a canal at ground level is 16 m. The sides of the canal can be modelled by a quadratic expression that would give a maximum depth of 16 m. However, the base of the canal is flat and has a width of 12 m. What is the depth of the canal?

Solution. Model the canal with $y = a(x + 8)(x - 8)$. This parabola passes through $(0, -16)$, so we have $-16 = a(+8)(-8)$ and hence $a = \frac{16}{64} = \frac{1}{4}$; so $y = \frac{1}{8}(x + 8)(x - 8)$. Suppose we shift the parabola by c so that the roots become ± 6; then c would be the distance below ground that the parabola has a width of 12 m.

$$\begin{aligned}
\frac{1}{4}(x + 8)(x - 8) + c = \frac{1}{4}(x + 6)(x - 6) \\
x^2 - 64 + 4c = x^2 - 36
\end{aligned}$$

Hence $4c = 64 - 36 = 28$ and $c = 7$ m.

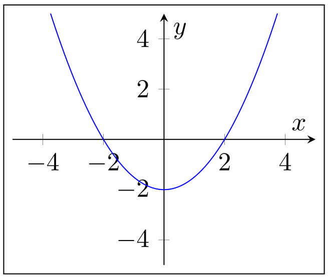
Classifying Roots

Let us look again at the vertex form of the general quadratic equation,

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

Solving the equation $0 = ax^2 + bx + c$ is equivalent to finding the x-intercepts of this parabola. The *number* of x-intercepts, and hence the number of solutions, must be at most two (because of the shape of the parabola), and can only be changed by shifting it up and down (changing the y-shift, $c - \frac{b^2}{4a}$).

Case I: two x-intercepts

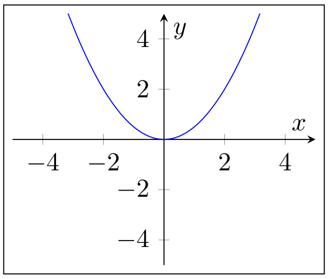


This happens in two situations:

- a is positive and $c - \frac{b^2}{4a}$ is less than zero. Hence $c < \frac{b^2}{4a}$, $4ac < b^2$, and $b^2 - 4ac > 0$.
- a is negative and $c - \frac{b^2}{4a}$ is greater than zero. Hence $c > \frac{b^2}{4a}$, $4ac < b^2$, and $b^2 - 4ac > 0$.

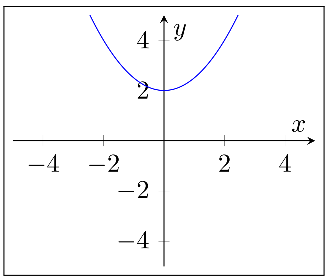
In either case, $b^2 - 4ac > 0$.

Case II: one x-intercept



This happens precisely when the vertex is sitting on the x-axis, so $c - \frac{b^2}{4a} = 0$ and $b^2 - 4ac = 0$.

Case III: no x-intercepts



This happens in two situations:

- a is positive and $c - \frac{b^2}{4a}$ is greater than zero. Hence $c > \frac{b^2}{4a}$, $4ac > b^2$, and $b^2 - 4ac < 0$.
- a is negative and $c - \frac{b^2}{4a}$ is less than zero. Hence $c < \frac{b^2}{4a}$, $4ac > b^2$, and $b^2 - 4ac < 0$.

In either case, $b^2 - 4ac < 0$.

Notice that the quantity $b^2 - 4ac$ tells us the nature of the roots in every case; it is known as the *discriminant* of the quadratic (and I denote it by Δ_2). We have therefore proved the following

Theorem. *Suppose $f(x) = ax^2 + bx + c$. Then:*

- *If $b^2 - 4ac < 0$, then $f(x) = 0$ has no solutions.*
- *If $b^2 - 4ac = 0$, then $f(x) = 0$ has precisely one solution.*
- *If $b^2 - 4ac > 0$, then $f(x) = 0$ has precisely two solutions.*

If we look at the quadratic equation again,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

notice that the discriminant appears underneath the square root sign and so it doesn't need to be memorised seperately.

Questions

Solving Quadratics

1. For each quadatic equation,
   - rewrite it into vertex form by completing the square if required;
   - graph the parabola it describes; and
   - calculate the x-intercepts of the parabola.

   (a) $y = x^2 + 1$

   (b) $y = x^2 + x$

   (c) $y = x^2 - 4x + 4$

   (d) $y = x^2 + 2x + 3$

   (e) $y = -x^2 + 4x - 2$

   (f) $y = 2x^2 + 2x + 2$

2. Show that if $x^2 - bx + c = 0$, then b is the sum of the solutions of the equation.

3. This question is revision from Level 1.

   (a) Justify, with mathematical reasoning, the following statement: the roots of the equation $(x - \alpha)(x - \beta) = 0$ are α and β.

   (b) Give a quadratic equation with roots -1 and 6.

4. Find all the y-intercepts of $-(x^2 + 2x - 3)(4x^2 - 6x + 2) = y$.

5. Factorise and solve $5x^2 - 9x - 2 = 0$.

6. Consider the quadratic equation $x^2 + bx + c = 0$.

   (a) Calculate b and c such that the quadratic equation has solutions -1 and 3.

   (b) Find the location of the vertex of the corresponding parabola, $y = x^2 + bx + c$.

7. Solve $\frac{x^2+5x+2}{x+2} = 3$.

8. Talia used timber to form the exterior sides of her rectangular garden. The length of the garden is x metres, and its area is $50\,\mathrm{m}^2$.

   (a) Show that the perimeter of the garden is given by $2x + \frac{100}{x}$.

   (b) If she uses 33 m of timber to build the sides, find the dimensions of the garden.

9. David and Sione are competing in a cycle race of 150 km. Sione cycles on average 4 km per hour faster than David, and finishes half an hour earlier than David. Find David's average speed. *You MUST use algebra to solve this problem. Note that average speed* $= \frac{distance}{time}$.

10. Simplify fully $\dfrac{2x^2 - 8}{x^2 - 2x - 8}$.

11. The equation $(x + 2) - 3\sqrt{x + 2} - 4 = 0$ has only one real solution. Find the value of x.

12. Check, by direct substitution, that both

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are solutions of $ax^2 + bx + c = 0$.

Classifying Roots

13. Without explicitly computing them, how many solutions does each quadratic equation have? Don't use the discriminant to decide for all four.
    (a) $0 = x^2 + 2$
    (b) $3 = 3x^2 + 3x$
    (c) $1 = -x^2 - 2x$
    (d) $0 = 2x^2 - 12x + 18$
14. Find k such that $x^2 + 3kx = 2$ has precisely one solution.
15. The equation $(2x - 3)(x + 4) = k$ has only one real solution; find the value of k.
16. Find all t such that the parabolae described by $tx^2 + x + 1$ and $-2x - tx + 1$ meet at precisely one point.
17. By considering the quadratic formula, give another proof that the discriminant 'encodes' the nature of the roots of the quadratic.
18. The quadratic equation $mx^2 - (m + 2)x + 2 = 0$ has two positive real roots. Find the possible value(s) of m, and the roots of the equation.
19. For what values of k does the parabola described by
$$y = x^2 + (3x - 1)x + (2k + 10)$$
never touch the x-axis?
20. Find the possible values of d if one or more real solutions exist for $x^2 + 5x - 1 = d(x^2 + 1)$. Interpret your answer geometrically.
21. Find expressions in terms of m and n for the roots of the equation
$$\frac{x - m}{x - n} = \frac{2(x + m)}{x + n}.$$
Give an inequality in terms of m and n, so that the equation has two distinct roots.

NCEA Level 2 Mathematics (Homework)
8. The Quadratic Formula

Reading

Go and watch...

https://www.youtube.com/watch?v=v-pyuaThp-c

What's it good for?

People use quadratic equations for...

- Engineering, economics, and the sciences: modelling situations (a perfect projectile follows a parabolic arc, a parabolic mirror reflects all light from the focus into parallel rays and is used in telescopes and radio dishes...)
- Mathematics: every polynomial is a product of quadratic and linear factors, and the fact that the quadratic formula even exists is actually quite surprising (there is no equivalent to 'completing the square' for cubics or anything more complicated).

Questions

1. Find the nature of the roots of the equation $x^2 + 3kx - 28$, if:
   (a) $k < 0$,
   (b) $k = 0$, and
   (c) $k > 0$.
2. Find the values of m for which one root of the equation $4x^2 = mx - 5$ is thrice the other root.
3. Suppose the quadratic equation $x^2 + bx + c = 0$ has the two roots α and β. Show that $bc = -\alpha^2\beta - \alpha\beta^2$.
4. The lengths of three sides of a right-angled triangle are $x - 2$, $2x$, and $x + 6$. If it is known that the length of the longest side is $x + 6$, compute x and give the length of the shortest side explicitly.

NCEA Level 2 Mathematics
9. Exponential and Logarithmic Functions

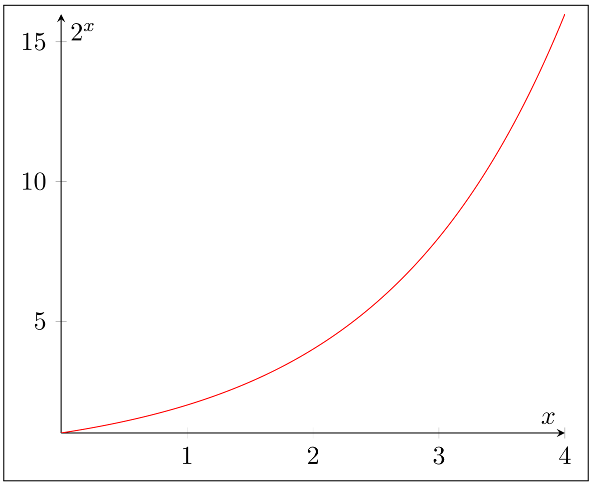
Exponentials

A particular species of bacteria reproduces by splitting in two every hour; if we start with one bacterium, after one hour we will have two; after two hourse, we will have four; after three hours, eight; and after n hours, we will have

$$2^n := \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}}$$

bacteria.

In general, equations of the form $y = a^x$ are called *exponential* equations.



Last year, we learned that exponents have the following properties:

1. $a^b \times a^c = \underbrace{a \times a \times \cdots \times a}_{b \text{ times}} \times \underbrace{a \times a \times \cdots \times a}_{c \text{ times}} = \underbrace{a \times a \times \cdots \times a}_{(b+c) \text{ times}} = a^{b+c}$.
2. $a^b \div a^c = \underbrace{a \times a \times \cdots \times a}_{b \text{ times}} \div \underbrace{a \div a \div \cdots \div a}_{c \text{ times}} = \underbrace{a \times a \times \cdots \times a}_{(b-c) \text{ times}} = a^{b-c}$.
3. $\left(a^b\right)^c = \underbrace{a^b \times a^b \times \cdots \times a^b}_{c \text{ times}} = \underbrace{a \times a \times \cdots \times a}_{bc \text{ times}} = a^{bc}$.
4. $a^1 = a$.
5. $a = a^1 = a^{0+1} = a^0 a^1 = a^0 a$, so $a^0 = 1$.

Note that we have some danger hiding in the background with these proofs: namely, if the powers are not whole numbers (or zero), they become meaningless! What does it mean to take 2 multiplied by itself π times? The solution, which we will look at briefly next week, is to define the function $x \mapsto a^x$ in a series of steps; we have already defined it when x is a natural number (or zero), and next week we will properly define it when x is an integer or rational number in general. Unfortunately, we won't have the necessary machinery to define it for any real number until next year.*

* For future reference, this is exercise 11 on the second L3 calculus worksheet.

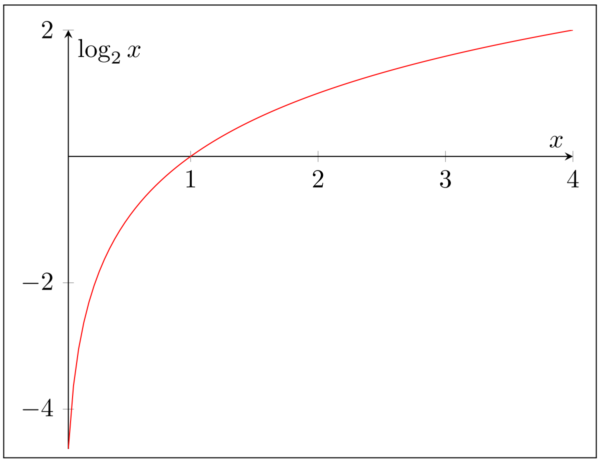
Logarithms

Suppose, on the other hand, we wish to know after how many hours we will have 1024 bacteria: we wish to find x such that $2^x = 1024$. This value is called the *logarithm* of 1024 with respect to 2, and we write $x = \log_2 1024$. In general, we have (as a definition),

$$y = a^x \iff x = \log_a y.$$

The quantity a is called the *base*, and is always positive.

Note that the function $x \mapsto \log_a x$ is the inverse of the function $x \mapsto a^x$.



If the base of a logarithm is 10, then we often don't write the base: so $\log 1000 = 3$, because $10^3 = 1000$.
The following logarithm laws can be derived from the exponent laws:

1. $\log_a x + \log_a y = \log_a xy$
2. $\log_a x - \log_a y = \log_a \frac{x}{y}$
3. $\log_a x^n = n \log_a x$
4. $\log_a 1 = 0$
5. $\log_a a = 1$
6. $\log_b x = \dfrac{\log_a x}{\log_a b}$ (change-of-base)

Example. Some elementary examples:

1. $\log_2 x = 10$ implies that $2^{10} = x$ and $x = 1024$.
2. $\log_x 49 = 2$ implies that $x^2 = 49$ and so $x = 7$.

Most applications of exponential and logarithmic equations outside of mathematics itself are to do with rates of change and rates of growth. This is because the rate of change of an exponential function is itself exponential, and so the exponential function will show up anywhere that a rate of change of a quantity is related directly to the amount of the quantity.

Example. A computer depreciates continuously in value from \$4699 to \$1500 over a period of 4.25 years. The value in dollars, y, of the computer t years after its value was \$4699 can be modelled by a function of the form

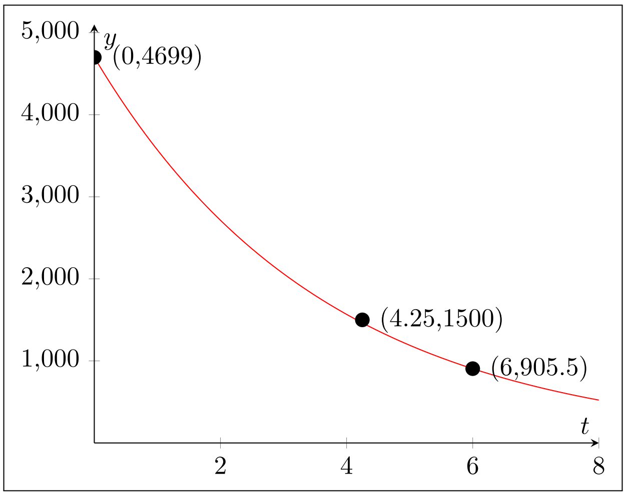
$$y = Ar^t,$$

where r is a constant. What is the value of the computer after six years?

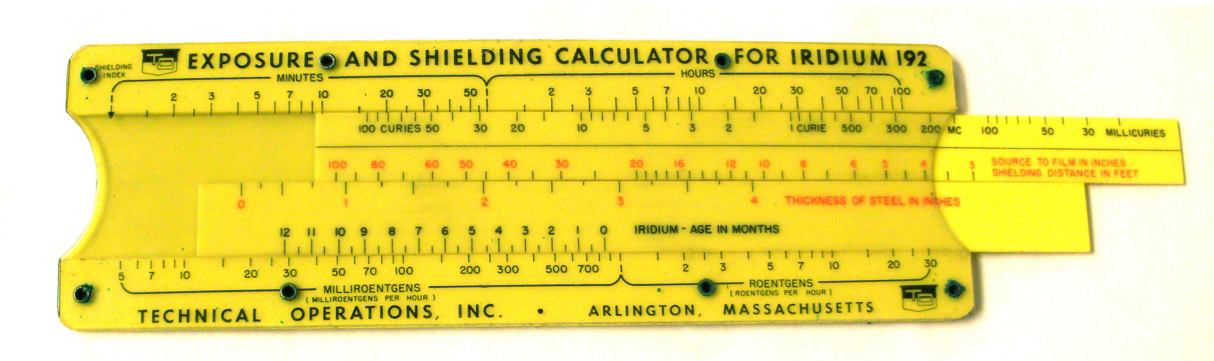
Solution. At $t = 0$, $y = 4699$ and so $4699 = Ar^0 = A$. On the other hand, we have $1500 = Ar^{4.25} = 4699 \cdot r^{4.25}$. Hence

$$\begin{aligned}\frac{1500}{4699} &= r^{4.25}\\ \log \frac{1500}{4699} &= \log r^{4.25}\\ \log \frac{1500}{4699} &= 4.25 \log r\\ r &= 10^{\left(\frac{1}{4.25}\log\frac{1500}{4699}\right)} \approx 0.76.\end{aligned}$$

Here, we used log base 10 because it happens to be on the calculator (we could have used any base); we then plugged the numbers into the calculator without worrying too much what powers that are fractions 'mean' (we'll discuss them next week). In the end, we found that a model for the value of the computer after t years is $y = 4699 \cdot 0.76^t$, and so after six years the computer is worth only around \$905.5.



Prior to the invention of the electronic calculator, mechanical devices called *slide rules* (see picture) were used by those who needed to make computations with large numbers. These devices consisted of two logarithmic scales next to each other, labelled with numbers; then the multiplication of a and b could be done by finding the lengths $\log a$ and $\log b$ on the slide rule, adding the two lengths together, and then using the rule $\log a + \log b = \log(ab)$ to read off the answer.



Questions

1. Write $2\log 3 - 3\log 2$ as the log of a single number.

2. Solve $8^{x+1} = 4^{2x-5}$ for x.

3. How many words with 4 letters of a 26 letter alphabet are possible?

4. Intuitively justify the following statements.
   (a) Multiplication $(n \times x)$ is repeated addition $(+n)$.
   (b) Exponentiation (n^x) is repeated multiplication $(\times n)$.
   (c) Division $(x \div n)$ counts 'how many' $(+n)$s.
   (d) Logarithms $(\log_n x)$ count 'how many' $(\times n)$s.

5. Simplify $\frac{4\log u^3}{\log u}$.

6. Solve for y, if $\log_2(y^{-6}) = (\log_2 y)^2 + 8$.

7. If the formula $P = A(0.75)^t$ models the amount P of a drug (in milligrams) in the bloodstream t hours after it is ingested, and the initial amount ingested is 500 mg, how long does it take for the amount of drug in the bloodstream to reduce by half?

8. Many population models are exponential.

Year	World population (billion)
1804	1
1927	2
1963	3
1974	4
1987	5
1999	6
2011	7

   (a) Assume the world population in the year t (CE) can be modelled with an exponential equation of the form
   $$P = P_0 r^{ct}.$$
   Find P_0, r, and c using the data from 1804, 1927, and 1963 (the three earliest years given above).
   (b) Write another model, using the three *latest* years above. Compare the two models.
   (c) Using the second model, calculate a projected terrestrial population in 2024 and in 2100.
   (d) How accurate do you think an exponential model will be in the long term?

9. Find all values of x satisfying $6(\log_8 x)^2 + 2\log_8 x - 4 = 0$.

10. Luka says that the equation $\log_x(4x + 12) = 2$ has only one solution. Is he correct?

11. Prove the logarithm laws, using the exponent laws. For example,
$$\log_a x + \log_a y = \log_a(a^{\log_a x + \log_a y}) = \log_a(a^{\log_a x} a^{\log_a y}) = \log_a xy = \log_a(a^{\log_a xy}) = \log_a xy.$$

12. When a bank loans money, the interest is *compounded*; that is, you earn interest on the interest you have already been charged.
    (a) Show that adding $x\%$ to a debt is equivalent to multiplying the debt by $(1 + \frac{x}{100})$.
    (b) Suppose interest is calculated after each year (that is to say, it is compounded annually). If the initial debt is \$100, and the annual compound interest is 20%, what is the amount owed by the end of the first three years?
    (c) Now, to simplify matters, we will say that our initial loan is \$1, and the interest rate is \$100 per annum. After one year, the value of the loan will double to \$2.
        i. If the bank calculates compound interest every six months instead of every year (so our interest is 50% per six months), show that we owe an extra \$0.25 after one year.
        ii. If the bank compounds interest every month, show that our total owed is now \$2.6130 after one year.
        iii. How much will we owe if the bank compounds interest every day?
        iv. In general, show that if we divide the annual percentage rate by n and compound it n times then the end-of-year balance of the loan is $(1 + \frac{1}{n})^n$.
    (d) From your working in part (c) above, we can conclude that as n increases (that is, if the bank compounds interest with shorter and shorter time intervals) then the total owed after one year climbs closer and closer to \$2.7182818.... This number, which is fundamental in mathematics, is known as Euler's constant, or e. Show that if a bank compounds interest continuously, with an interest rate of $x\%$ per annum and initial loan L, then after t years the total owed is

$$L \times e^{xt/100}.$$

NCEA Level 2 Mathematics (Homework)
9. Exponential and Logarithmic Functions

Reading

Go and watch...

https://www.youtube.com/watch?v=N-7tcTIrers

What's it good for?

People use exponential and logarithmic equations for...

- Chemistry, physics, engineering: whenever the rate of growth or rate of decline of a quantity is proportional to (or inversely proportional to) the amount of quantity present, the quantity is an exponential or logarithmic function of time. (This includes rates of chemical reaction, rates of capacitor charge/discharge, the position of a damped spring over time, and many other examples.)

Questions

1. Thirty minutes after a patient is administered his first dose of a medication, the amount of medication in his bloodstream reaches 224 mg. The amount of the medication in the bloodstream decreases continuously by 20% each hour. The amount of the medication M mg in the patient's bloodstream after it is administered can be modelled by the function

$$M = 224 \times 0.8^{t-0.5}$$

   where t is the time in hours since the drug was first administered.

   (a) Explain what 0.8 means in this function.

   (b) Give the initial amount of medication administered.

   (c) A second dose of the medication can be administered some time later, and again the amount of the medication in the patient's bloodstream from the second dose can be modelled by the same function as that for the first. The total amount of the drug in the blood stream must never exceed 300 mg. How long after administering the first dose can the second dose be administered?

2. Here are some revision questions on topics we have already covered.

   (a) Rearrange the following formula to make x the subject: $\frac{4x}{5} = \frac{y(x+3)}{2}$.

   (b) Show that the solutions of $x^2 + x - 56 = 0$ are four times those of $4x^2 + x - 14 = 0$.

   (c) Find the relationship between the solutions of the equations $dx^2 + ex + f = 0$ and $x^2 + ex + df = 0$ where d, e, and f are real numbers.

   (d) Consider the equation $(3x + 1)^2 = -7$.

      i. Explain why it has no real solutions; explain what this means graphically.

      ii. Compute the discriminant of the equation, and explain why this further supports your answer to (i).

NCEA Level 2 Mathematics
10. Negative and Fractional Powers

Last week we defined the exponential function for powers which were whole numbers or zero, by defining $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$. We can make this definition more precise by making the following definition:*

Definition. If a is a number, then:

1. a^0 is defined to be 1.
2. a^n is defined to be $a \times a^{n-1}$, for integers $n > 0$.

One might easily ask if there is a way to extend this definition for non-whole-number powers; in fact, last week we implicitly used the fact that such an extension exists in solving some logarithmic equations (but relying on a calculator to 'know the definition' for us). Let us take inspiration from our recursive definition above, and try to 'pull ourselves up by our bootstraps' in steps: we will begin with negative powers.

So suppose we want to define what the value of a^{-n} is (where n is a positive integer). We can try to work out a plausible definition using the rules we want such a value to follow — for example, we want such a definition to obey the rule $a^b a^c = a^{b+c}$. In particular,

$$a^{-n} \times a^n = a^{(-n)+n} = a^0 = 1.$$

Hence a plausible definition for a^{-n} is $1/(a^n)$. This plausible definition also follows (to take another example) the rule $(a^b)^c = a^{bc}$, because $(a^{-n})^x = \left(\frac{1}{a^n}\right)^x = \frac{1}{a^{nx}} = a^{-nx}$ as we would expect.

So now we have a definition for all a^x, where x is an integer. The obvious next step is to look at rational powers; recall, a rational number is any number r that can be written in the form $r = \frac{p}{q}$, where p and q are both integers. As an aside, the following theorem is quite deep and perfectly accessible:-

Theorem. *There are real numbers which are not rational.*

Proof. In particular, we will show that any number x such that $x^2 = 2$ is irrational; for suppose that such an x can be written in the form $x = \frac{p}{q}$ where p and q are both positive integers. Then $2 = x^2 = \frac{p^2}{q^2}$, and hence $2q^2 = p^2$. But this implies that p^2 is even, and so p is itself even (because the squares of odd numbers are odd). Therefore, there is an integer n such that $p = 2n$. Substituting, we have $2q^2 = (2n)^2 = 4n^2$, and hence $q^2 = 2n^2$. But this means that q is even, and hence there is an integer m such that $q = 2m$; substituting, we have $(2m)^2 = 2n^2$, and hence $2m^2 = n^2$ and $2 = \frac{n^2}{m^2}$.

Notice, though, that $\frac{p^2}{q^2} = \frac{n^2}{m^2}$, but n and m were smaller than p and q respectively. Since we didn't say what p and q were to start with, this implies that for any pair of positive integers p and q such that $x = p/q$, there exist smaller positive integers n and m satisfying the same equation; and so we can repeat the whole process, finding two positive integers smaller than n and m, and so on *ad infinitum.*

But this is absurd: given any positive integer, there are only finitely many positive integers smaller than it! Thus our original assumption, that such integers p and q existed in the first place, must be false; so any number x such that $x^2 = 2$ cannot be rational. □

Real numbers which are not rational are (rather unimaginatively) called *irrational.* Other numbers which are irrational include π, the square root of any prime number, and e.

Returning to our main theme, we want to define a^r, where $r = \frac{p}{q}$ is a rational number. Let us again work out a plausible definition using the rules we want such a number to follow; this time, we will use the 'power multiplication' rule:

$$\left(a^{p/q}\right)^q = a^{((p/q)\cdot q)} = a^p.$$

So we can define $a^{p/q}$ to be $\sqrt[q]{a^p}$. (If there's any confusion, we will more precisely define it to be the *positive* root; also, we require our rational number p/q to be written so that q is positive so that we don't have to worry about defining negative roots).

Our full definition so far looks like:

*By 'more precise' I mean 'we make it clearer what we mean by $\cdots$'.

Definition. If a is a number, then:

1. a^0 is defined to be 1.
2. a^n is defined to be $a \times a^{n-1}$, for integers $n > 1$.
3. a^{-n} is defined to be $\frac{1}{a^n}$, for integers $n > 1$.
4. $a^{p/q}$ is defined to be $\sqrt[q]{a^p}$, for rational numbers p/q such that $q > 0$.

Our final trick will be to define a^x for any real number x. Since we don't have the necessary machinery to do it properly this year, our definition will be vague. We use the fact that we want a^x to be continuous: that is, we want it to 'have no gaps' and 'not jump around unexpectedly'. Since x is real, we can always write it in decimal expansion: say

$$x = x_0 + 0.x_1x_2x_3 \dots x_n \dots = x_0 + \frac{x_1}{10} + \frac{x_2}{100} + \dots + \frac{x_n}{10^n} + \dots$$

(where the notation $x_0 + 0.x_1x_2\dots$ means that x_0 is the 'integer part' of x and x_1, x_2 and so on are the digits of the decimal expansion). In particular, we have

$$a^x = a^{\left(x_0 + \frac{x_1}{10} + \frac{x_2}{100} + \dots + \frac{x_n}{10^n} + \dots\right)} = a^{x_0} \times a^{x_1/10} \times \dots \times a^{x_n/10^n} \times \dots ,$$

where we have already defined all the terms on the right — so we can define a^x to be 'the real number which we get closest to if we keep adding the terms on the right until infinity'. *This is obviously not precise, but just take my word for it that (a) it is possible to make the notion precise with a little more work, and (b) real powers are well-defined (that is, such a number always exists).*

Example.

1. $2^{3/2} = \sqrt[2]{2^3} = \sqrt{8}$.
2. $4^{-1/2} = \frac{1}{4^{1/2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$.
3. $27^{5/3} = \left(27^{1/3}\right)^5 = \left(\sqrt[3]{27}\right)^5 = 3^5 = 243$.
4. $2^\pi \approx 2^3 \times 2^{1/10} \times 2^{4/100} \times 2^{1/1000} \times 2^{5/10000} \approx 8.8244$. (my calculator tells me that $2^\pi \approx 8.8249$, so this approximation isn't even that bad!)

Questions

1. Graph the equation $y = \vartheta^x$ for different values of ϑ:

   $$\vartheta = 10 \quad 2 \quad 1 \quad 1/2 \quad 1/10 \quad 0 \quad -1/10 \quad -1/2 \quad -1 \quad -2 \quad -10$$

   (a) What do you notice? Compare and contrast the different curves. Is there any point which all 11 curves pass through?

   (b) When ϑ is negative, the curve is an *exponential decay* curve; when ϑ is positive, the curve is an *exponential growth* curve. Conjecture some situations where an exponential decay or growth curve might be a good model for some situation.

2. Make a conjecture about the value of 0^0: should it be zero (because $0^n = 0$ for all n), or one (because $n^0 = 1$ for all n)? It might be helpful to graph $y = x^x$ for very small positive and negative values of x.

3. Justify the following statements with mathematical reasoning:

   (a) $\sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$ (where p and $q > 0$ are integers).

(b) If r and s are rational numbers, then $a^r \times a^s = a^{(r+s)}$ (recall we only proved this rule last week for integer powers).

4. Simplify the following, writing your answer with positive exponents:
   (a) $\frac{(4a^3)^2}{b^3} \times \frac{2b^2}{(2a)^2}$
   (b) $\frac{5x^2y}{2} \div \frac{10x}{y^2}$
   (c) $(2a^7 \times 50a^3)^{-1/2}$
   (d) $\frac{6m^5}{\sqrt{9m^{16}}}$
   (e) $\sqrt{\frac{\left(16a^{(2/3)}\right)^{(3/2)}}{a^{-1/2}}}$

5. Verify that the multiplication terms further to the right in the expression

$$a^{x_0} \times a^{x_1/10} \times a^{x_2/100} \times \cdots \times a^{x_n/10^n} \times \cdots$$

get closer and closer to 1. (Hint: each x_i, for $i > 0$, is a single digit and thus less than 10.) Hence justify why only taking a few of the first terms usually gives a good approximation to the 'real value' of $a^{x_0+0.x_1x_2\cdots}$.

6. A graph with Cartesian equation of the form $y = a(x - x_0)^{-1} + c$ is a *hyperbola*.
   (a) Suppose a hyperbola passes through the points $(-1, 0)$, $(0, -1)$, and $(3, 2)$. Find the constants a, x_0, and c and give the equation of the hyperbola.
   (b) Show that there is some value μ such that the hyperbola does not touch the line $x = \mu$. This line is called the *vertical asymptote* of the hyperbola.
   (c) Show that there is some value λ such that the hyperbola does not touch the line $y = \lambda$. This line is called the *horizontal asymptote* of the hyperbola.
   (d) Graph the hyperbola, using your graphing device of choice; describe the behaviour of the graph *around* the two asymptote lines.
   (e) Graph the equation $y = x^{-n}$ for different values of n; what do you notice?
   (f) Show that the hyperbola with vertical asymptote 'at infinity' is just a straight line $y = c$. (Hint: notice that in the hyperbola equation, $x = x_0$ is the vertical asymptote and 'substitute' $x_0 = \infty$ into the equation.) Is this what you expect intuitively?

7. Challenge question. Consider the equation $6^{2x} + m \cdot 6^x + n = 0$, where $n \leq 0$.
   (a) Prove that the equation has precisely two solutions for 6^x.
   (b) Show that only one of these solutions is valid for finding a solution for x if m is positive.

NCEA Level 2 Mathematics (Homework)
10. Negative and Fractional Powers

Reading

Editorial note: rather than boring you with more 'applications of exponents' (I listed most of the interesting ones last week), here's a little history.

In the May 1690 issue of the *Acta Eroditorum*, Jakob Bernoulli, the discoverer of e, revisited a question that had been puzzling mathematicians for a century. What is the correct geometry of the shape made by a piece of string when it is hanging between two points? This curve — called the 'catenary', from the Latin *catena*, chain — is produced when material is suspended by its own weight.

The curve whose identity Jakob Bernoulli so ardently sought turned out to have a hidden ingredient, e, the number he had uncovered in a different context [compound interest].

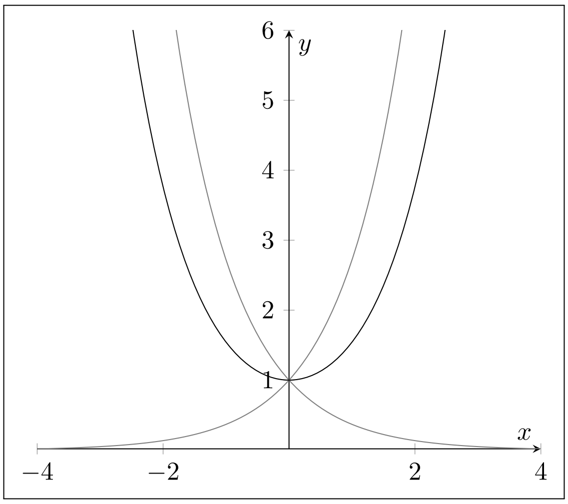
In modern notation, the equation for the catenary is

$$y = \frac{e^{ax} + e^{-ax}}{2a},$$

where a is a constant that changes the scale of the curve. The bigger a is, the further apart the two ends of the hanging string are.

Look closely at the equation. The term e^{ax} represents pure exponential growth, and the term e^{-ax} pure exponential decay. The equation adds them together, and then divides by two, which is a familiar arithmetical operation: adding two values and then halving the result is what we do when we want to find their average. In other words, the catenary is the average of the curves of exponential growth and decay, as illustrated below. Every point on the U falls exactly halfway between the two exponential curves.

Whenever we see a circle, we see π, the ratio of the circumference to the diameter. And whenever we see a hanging chain, a dangling spider's thread or the dip of an empty washing line, we see e.



Adapted from *Alex Through the Looking-Glass*, by Alex Bellos (pp.150-2).

Questions

1. Expand and simplify, writing with only positive exponents.
   (a) $\dfrac{(3 + x^{3/2})(3 - x^{3/2})}{x^{-4}}$
   (b) $\dfrac{x(y^3 + \sqrt{y}) + y^3}{y} - xy^{-(1/2)}$
2. Find all solutions to $8x^3 + 64 = 16\sqrt{8}x^{3/2}$.
3. Challenge question: we have seen that $\sqrt{2}$ is irrational; that is, $2^{1/2}$ is irrational and so it is possible for a^b to be irrational when both a and b are rational. Is it possible for a^b to be *rational* when both a and b are *irrational*?

CHAPTER 4

Calculus

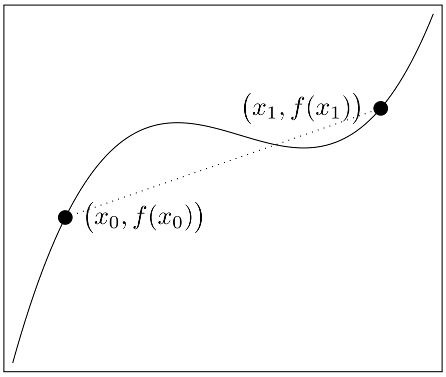
NCEA Level 2 Mathematics
11. Slopes and Differentiation

The main theme this year so far has been geometry and modelling the real world. This week, we move from modelling the stationary world of fields, fences, and compound interest to the world of continuity and motion.

We have already seen that if we model a changing quantity with a linear equation of the form $y = mx + c$, then it is possible to talk about the rate of change of that quantity by means of the slope of the graph of the equation.

Example. Suppose that the amount of sand in a pile (in kilograms, perhaps) is modelled by the equation mass $= 3(\text{time}) + 1$. Then the slope of the graph of mass versus time is 3, and so it makes sense to say that the rate of change of mass is three kilograms per unit of time.

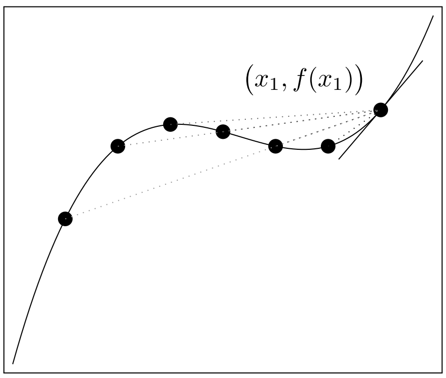
Our goal this week is to describe the slopes of functions which are not necessarily lines. Let us look at some arbitrary function f, and let us draw the graph $y = f(x)$. Then, recalling that we originally defined slope to be $m = \frac{\text{change in } y}{\text{change in } x}$, we can look at the slope between $y_0 = f(x_0)$ and $y_1 = f(x_1)$:



The average slope betwen $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is just the slope of the dotted line, which is simply

$$\frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

If we move the point (x_0, y_0) closer to (x_1, y_1) along the curve, then it makes sense that the slope of the dotted line will match the slope of $y = f(x)$ at the point (x_1, y_1) better and better (the dotted lines in the diagram below). If we imagine moving the point (x_0, y_0) until it coincides with the point (x_1, y_1), then the 'average' slope (indicated by the solid line below) will in face be the actual slope of the curve at x_1.



Example. Suppose we look at the curve $y = x^2$. The two points $(0, 0)$ and $(2, 4)$ are on this curve; the average slope of the curve between $x = 0$ and $x = 2$ is therefore $\frac{4-0}{2-0} = 2$.

Now, suppose we want to find the actual slope of the parabola at $x = 2$. We will do this in a slightly tricky way: finding the average slope between $x = 2$ and $x = 2 + h$, and then setting h to zero. The coordinates of the point on the parabola at $x = 2 + h$ will be $(2 + h, (2 + h)^2) = (2 + h, 4 + 4h + h^2)$ and so the average slope that we are looking for is

$$\frac{\text{rise}}{\text{run}} = \frac{(4 + 4h + h^2) - 4}{(2 + h) - 2} = \frac{4h + h^2}{h} = 4 + h;$$

hence the average slope between $x = 2$ and $x = 2 + h$ when $h = 0$ (i.e. when the two points are the same), and thus the actual slope of $y = x^2$ at $x = 2$, is 4.

More generally, suppose we are to find the slope of the parabola at (x_0, x_0^2). To do this, we will find the average slope between this point and the point $(x_0 + h, (x_0 + h)^2) = (x_0 + h, x_0^2 + 2x_0 h + h^2)$. The calculation runs as follows:

$$\frac{\text{rise}}{\text{run}} = \frac{(x_0^2 + 2x_0 h + h^2) - x_0^2}{(x_0 + h) - x_0} = \frac{2x_0 h + h^2}{h} = 2x_0.$$

Does this function give us the slope we expect? For $x < 0$, the parabola is downward-sloping (if we move to the right, we move down) and so we would expect the slope to be negative; and for $x > 0$, the parabola is upwards-sloping and so we would expect the slope to be positive. The function $x \mapsto 2x$ satisfies both these criteria, and so this is reassuring: our algebraic slope-finding seems to have given a reasonable answer.

Hence, given the function $f : x \mapsto x^2$, we can write down another function $f' : x \mapsto 2x$ that gives the slope of the graph $y = f(x)$ at any point we choose.

It is, in fact, possible with most functions f we have met to write down another function f' such that the second function (called the *derivative* of f) gives the slope of $y = f(x)$ at any point we choose.

Example. Let's try $f(x) = x^3$ now; we'll go a bit quicker this time.

$$\frac{\text{rise}}{\text{run}} = \frac{(x + h)^3 - x^3}{(x + h) - h} = \frac{x^3 + 3x^2 h + 3xh^2 + h^3 - x^3}{h} = 3x^2 + 3xh + h^2$$

and so (setting h to zero) it seems that the derivative of f is $f' : x \mapsto 3x^2$.

Does this make sense? Well, $y = x^3$ is always sloping upwards, and $3x^2$ is never negative, so yes — this does make sense.

With the basic examples out of the way, I will state (without proof, but the proof is similar to the ideas above) the following theorem.

Theorem (Power rule)**.** *If f is a function defined by $f(x) = ax^r$, where r is any real number, then the derivative of f is the function defined by $f'(x) = rax^{r-1}$.*

This matches our example above: for x^2, we obtained $2x = 2x^{2-1}$ and for x^3 we obtained $3x^2 = 3x^{3-1}$. Note though that this theorem holds when r is any real number. Hence:

Example.

1. The derivative of $3x^{2018}$ is $6054x^{2017}$.
2. The derivative of $9x^\pi$ is $9\pi x^{\pi-1}$.
3. The derivative of 97 is 0.
4. The derivative of $\sqrt{x}$ is $\frac{1}{2}x^{-1/2}$.
5. The derivative of $1/x$ is $-\frac{1}{x^2}$.

We also have the following theorem which allows us to combine derivatives:

Theorem. *If f and g are functions, and λ is any number, then*

1. $(\lambda)' = 0$ — *the derivative of a flat line/constant is zero.*
2. $(f+g)' = f' + g'$ — *the derivative of the sum of two functions is the sum of the derivatives.*
3. $(\lambda f)' = \lambda f'$ — *the derivative of a number times a function is the number times the derivative.*

Hence,

Example.

1. The derivative of $2x^2 + 3x$ is $4x + 3$.
2. The derivative of $19x^{\sqrt{2}} - 3$ is $19\sqrt{2}x^{\sqrt{2}-1}$.

Finally, we can obviously take derivatives with respect to variables that are not x; obviously the two functions $x \mapsto f(x)$ and $y \mapsto f(y)$ are the same function, and the derivative of $3y^2$ with respect to y is $6y$.

If we want to make the variable of differentiation clear, we can use the alternative Leibniz notation: if $y = f(x)$, then the derivative is

$$\text{either} \qquad f' \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x}.$$

If we take the derivative of the derivative, the result is called the second derivative and is notated by

$$\text{either} \qquad f'' \qquad \text{or} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

In general, the nth derivative is notated by

$$\text{either} \qquad f^{(n)} \qquad \text{or} \qquad \frac{\mathrm{d}^ny}{\mathrm{d}x^n}.$$

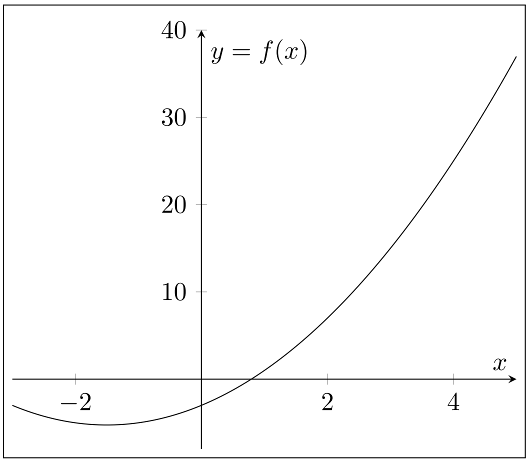
I am aware that the notation $f^{(n)}$ is awful as it looks like an exponent, and so never will never use it myself.

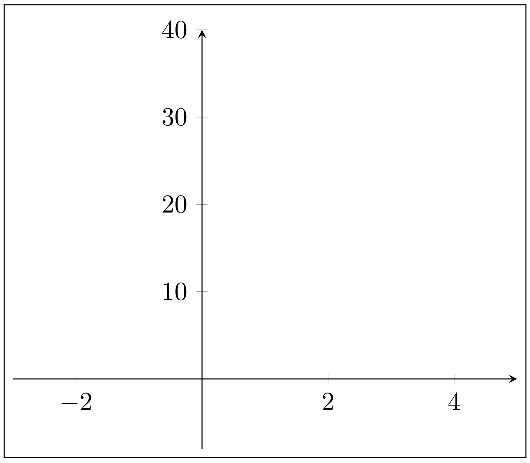
Questions

1. Since the sum rule for derivatives holds (i.e. $(f+g)' = f' + g'$), one might be tempted to guess that the product rule $(fg)' = f'g'$ also holds. Unfortunately, this is not the case.
   (a) Differentiate $f(x) = 4x^3$, $g(x) = 2x^2$, and $h(x) = 8x^5$.
   (b) Notice that $fg = h$, but $f'g' \neq h'$.
   (c) There is in fact a product rule for derivatives, but this year you won't need it. Find the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = (3x+2)(x^2+1)$.
2. Differentiate the function, with respect to x or the stated variable.
   (a) $a(x) = 186.5$
   (b) $b(x) = \sqrt{30}$
   (c) $c(x) = 5x - 1$
   (d) $z = -4x^{10}$
   (e) $e(x) = x^3 - 4x + 6$
   (f) $f(t) = \frac{1}{2}t^6 - 3t^4 + t$ with respect to t
   (g) $g(x) = x^2(1-2x)$
   (h) $h(x) = (x-2)(2x+3)$
   (i) $y = x^{-2/5}$
   (j) $B(y) = cy^{-6}$ with respect to y
3. The following part-questions require you to find higher derivatives of functions.
   (a) Find $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ if $x = 3t^2 + 4$.
   (b) Suppose $f(x) = 9x^2 + \frac{1}{x^3} + \sqrt{x}$. Find $f''(x)$.
   (c) Find $\frac{\mathrm{d}^3y}{\mathrm{d}t^3}$ if $y = 3x^{-1}$.
   (d) Find $f''(x)$ if $f'(x) = 9x^2 + 3x^{-2}$.
   (e) Find $a(t) = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}$ if $s(t) = 3t - 4t^{-1}$.
4. If we 'zoom out' from the graph of any polynomial, and ignore the 'wiggly bits' in the middle, the result always 'looks like' either the graph of $y = \pm x^2$, or the graph of $y = \pm x^3$.
   (a) Check this by zooming out from $y = 3x^6 + 2x^3 - 17x^2 + 3$ and $y = -2x^9 + 30x^2 - x$ on a graphing calculator.
   (b) Intuitively justify why the derivative of an odd polynomial (one which has an odd highest power of x) is an even polynomial, and the derivative of an even polynomial is an odd polynomial.
5. Where is the graph of $y = \frac{x^2}{2} - \frac{x^3}{3}$ increasing?

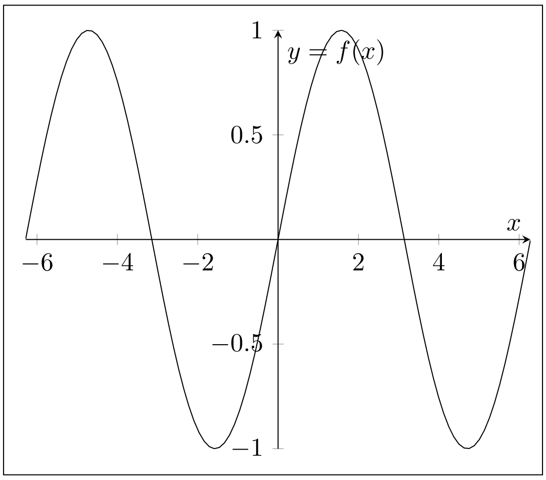
6. Draw the graph of the slope function (derivative) for each of the following graphed functions.

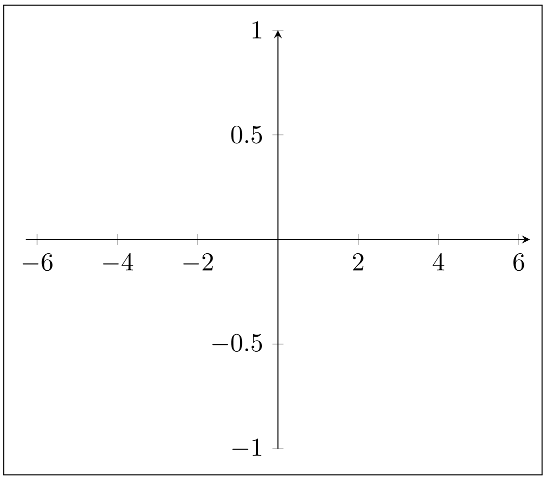
(a)





(b)





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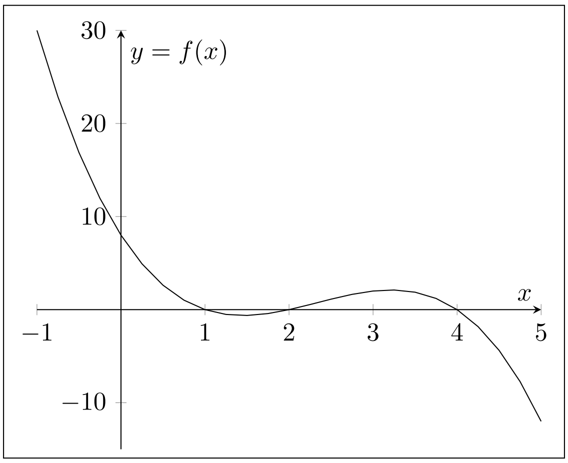
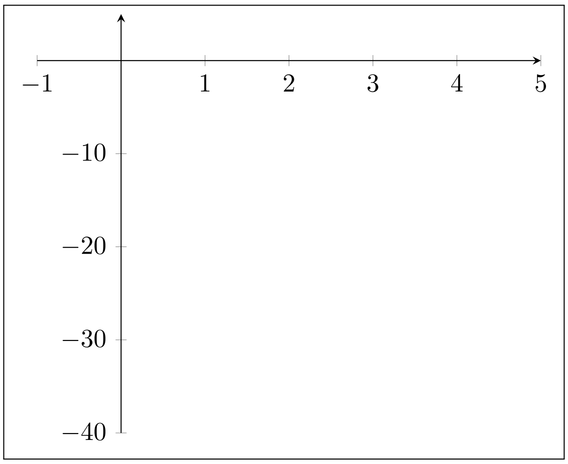
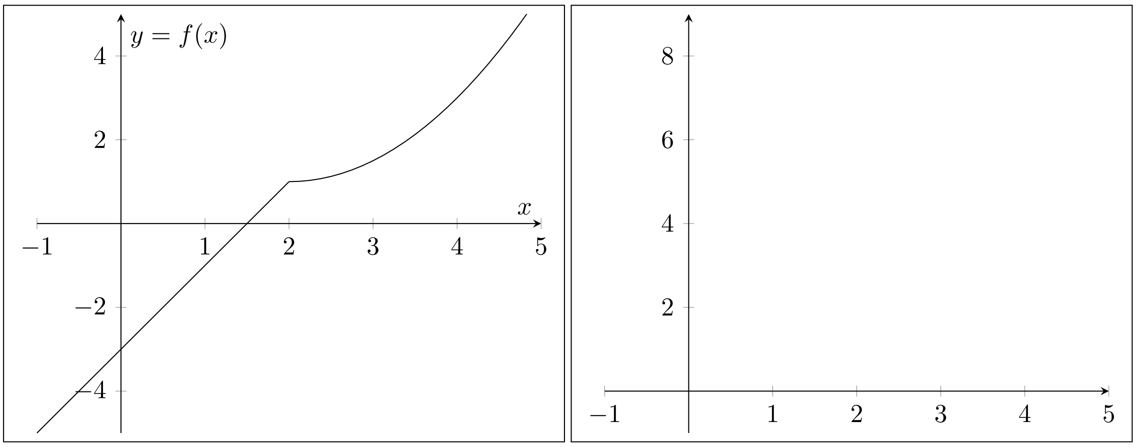
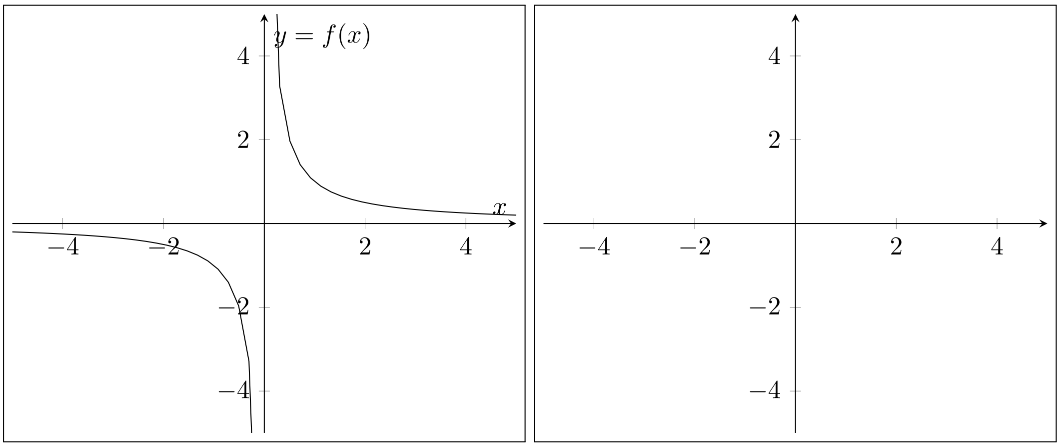
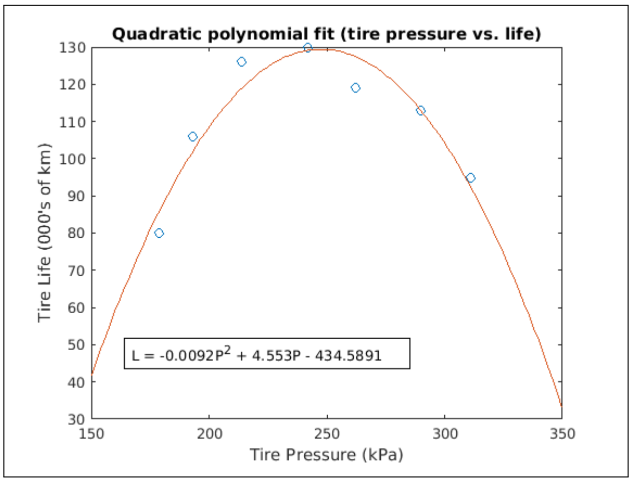




(d) 

(e) 

7. Car tires need to be inflated properly because overinflation or underinflation can cause premature treadwear. The graph shows tire life L (in thousands of kilometres) for a certain type of tire at various pressures P (in kPa), as well as a quadratic function that models the tire life.



Use the model to estimate $\frac{\mathrm{d}L}{\mathrm{d}P}$ when $P = 200$ and when $P = 300$. What is the meaning of the derivative? What is the significance of the sign of the derivatives?

8. A function f is given by $f(x) = 2 - 4x + 4x^2 + ax^3$. The gradient of the graph at the point where $x = 1$ is 3. Find the value of a.
9. Let f be a function of x defined by $f(x) = 3x^2 + 6x + 6$. Show that f is a solution of the *differential equation* $f(x) - f'(x) = 3x^2$.
10. Show that the function f of y defined by $f(y) = \frac{3}{y^2} + 2y$ is not differentiable at $y = 0$.
11. It is natural to ask if there is any function such that the function is its own derivative.
    (a) An obvious candidate is the function K defined by $K(x) = 0$. Show that $\frac{\mathrm{d}K}{\mathrm{d}x} = K(x)$.
    (b) For a more interesting example, we will look at the exponential functions $y = a^x$. Draw the gradient function of $y = 2^x$; explain why looking at the exponential functions is probably a productive way to answer our question.
    (c) Suppose we try to use the same technique as we used with the parabola. Show that the average slope between (x, a^x) and $(x + h, a^{x+h})$ is given by
    $$a^x \left(\frac{a^h - 1}{h} \right).$$
    (d) For a^x to be its own slope, we want $\frac{a^h-1}{h}$ to get closer and closer to 1 as h gets closer and closer to zero. Unfortunately we can't just substitute $h = 0$ straight in. Why?
    (e) Let $a = (1 + h)^{1/h}$ (where we take h very small), and show that $\frac{a^h-1}{h} = 1$. This suggests that a works as a base for our exponential-which-is-its-own-derivative if we let h get closer to zero here.
    (f) Again, we can't let $h = 0$ in a — so let $n = 1/h$, and show that $a = \left(1 + \frac{1}{n}\right)^n$.
    (g) If $n \to 0$, then $h \to \infty$. Therefore, our base we want is just $\left(1 + \frac{1}{n}\right)^n$ if we let n get infinitely large: the notation we use is
    $$\lim_{n} \left(1 + \frac{1}{n} \right)^n,$$
    which you can read as the *limit* with respect to n. Calculate this base for some large value of n. Is the number you see familiar?

NCEA Level 2 Mathematics (Homework)
11. Slopes and Differentiation

Reading

Go and watch...

https://www.youtube.com/watch?v=axZTv5YJssA

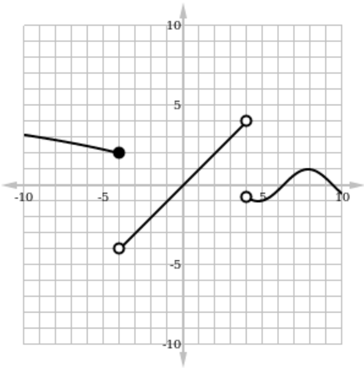
What's it good for?

People use calculus for...

- Engineering and physics: calculus is the natural language of any system which moves or changes over time (the movement of a vehicle or a robot arm, or the current in an electrical system).
- Chemistry and biology: rates of reaction and fluid pressure, population growth, and molecular kinetics are all examples of systems that are best understood using either calculus or further applications of calculus.
- Mathematics: the subjects which grow out of calculus (analysis and topology, for example) are fundamental geometric theories of space, distance, and transformation.

Questions

1. A function f is given by $f(x) = 2x^3 - 10x + 5$. Find the gradient of the graph of f at the point where $x = 2$.
2. Find the coordinates of the point on the curve $y = \frac{4}{x^2}$ where the gradient is 1.
3. Answer the following questions about this graph. Open circles denote locations where the function *is not* defined, while filled circles denote locations at the end of a segment where the function *is* defined.



   (a) What is the slope of the graph at $x = 8$?
   (b) Does the function have a derivative everywhere (i.e. can you guess the slope of the graph at every point)? If not, where does it fail to be differentiable?
   (c) At $x = -5$, is the derivative positive or negative?
   (d) What is the slope of the curve around $x = 0$?

NCEA Level 2 Mathematics
12. Tangent Lines and Approximations

One application of calculus is to find approximations to curves. Our goal is to write down a linear equation that approximates any given curve (around a given point). This is useful if (for example) we are given a complicated function like

$$f(x) = [\sin(x^{100})]^{[\cos(\sin x^2)]}.$$

This function is so weird that the graphing software I use cannot even graph it properly. The value of this function at $x = 0$ is very easy to calculate:

$$f(0) = [\sin(0^{100})]^{[\cos(\sin 0^2)]} = 0^{\cos(0)} = 0$$

However, as soon as we try to calculate other values it becomes difficult.

Suppose we want to know the value of a function near a point that it's easy to find the value of the function at — for example, $f(0.001)$. Our goal is to to draw the line through the easy point, with the same slope as the function at that point, and then work out what our desired x-value maps to using this easy function.

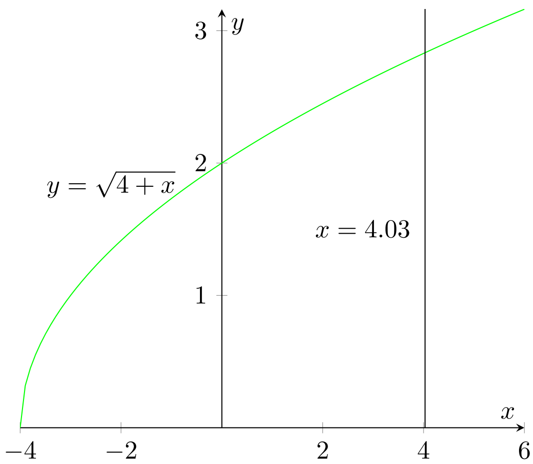
Definition. Let $y = f(x)$ be a curve, and let f be differentiable at some x-value x_0. Then the *tangent line* to the curve at x_0 is simply the line through the point $(x_0, f(x_0))$ that has the same slope as the curve at that point.

From our work on coordinate geometry, we know that the equation of this line is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Let's take an example.

Example. Let us calculate $\sqrt{4.03}$ by hand(!). If we consider the function $f(x) = \sqrt{4+x}$, then $\sqrt{4.03} = f(0.03)$. Let's draw the situation out:



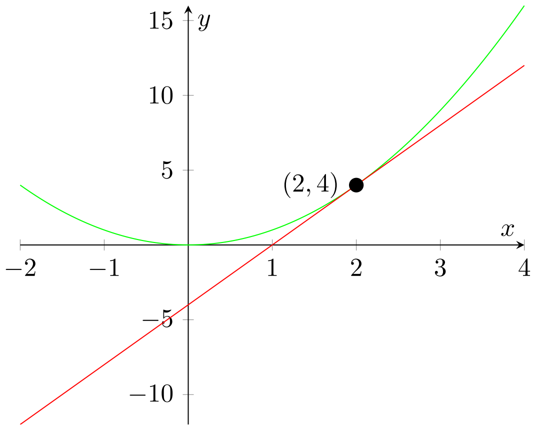
So we want the tangent line to f at the point $x = 0$. We have that $f'(x) = \frac{1}{2\sqrt{4+x}}$ (why? we can easily take the derivative of $\sqrt{x}$, it's just $\frac{1}{2\sqrt{x}}$ — and the graph of f is just the graph of $\sqrt{x}$ but shifted four units to the left, so we just shift the slope function itself four units to the left to match up), and so $f'(0) = \frac{1}{4}$. The tangent line is the line through $(0, f(0)) = (0, 2)$ with gradient $\frac{1}{4}$, which has equation

$$y - 2 = \frac{1}{4}x.$$

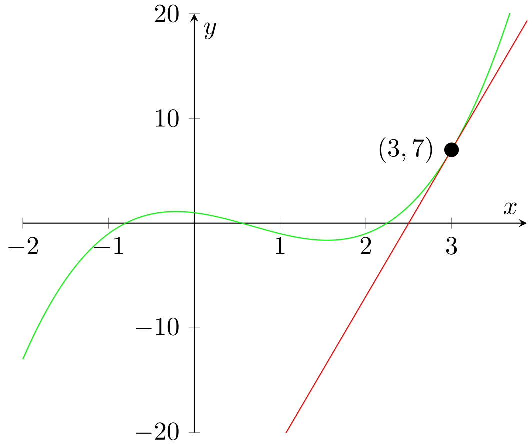
Hence $\sqrt{4.03} \approx \frac{1}{4} \times 0.03 + 2 = 2.0075$ — and, as promised, all of these calculations can be done without a calculator. (According to my calculator, $\sqrt{4.03} \approx 2.007486$ and so we are not far off at all.)

Now, with all the motivation out of the way, we will just look at some more simple examples.

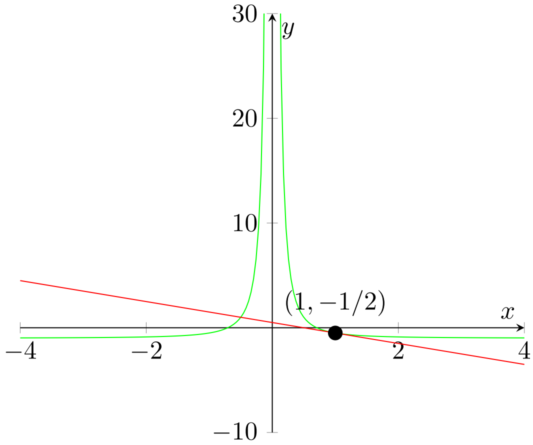
Example. Consider $y = x^2$. At $(2, 4)$, the tangent line has slope 4 and hence equation $y - 4 = 4(x - 2)$:



Example. Consider $y = x^3 - 2x^2 - x + 1$. At $(3, 7)$, the tangent line has slope 14 and hence equation $y - 7 = 14(x - 3)$:

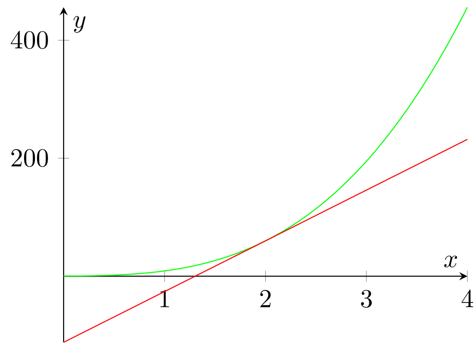


Example. Consider $y = \frac{1}{2x^2} - 1$. At $(1, -1/2)$, the tangent line has slope -1 and hence equation $y + 1/2 = -(x - 1)$:



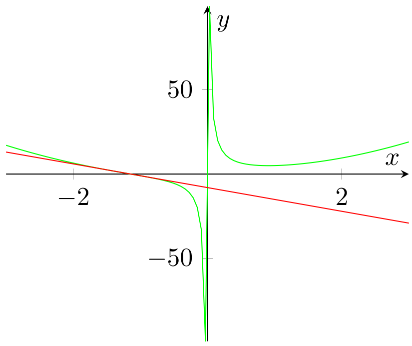
Questions

1. Why is there no tangent line to $y = x^2$ at the point $(0, -1)$?
2. Consider the function $f(x) = 7x^3 + 2x$.



   (a) What is the slope of the graph of $y = f(x)$ around $x = 2$?
   (b) Give the equation of the tangent line to the graph at $x = 2$.

3. Consider the function $g(x) = 2x^2 + \frac{3}{x}$.



   (a) What is the slope of the graph of $y = g(x)$ around $x = -1$?
   (b) Give the equation of the tangent line to the graph at $x = -1$.
   (c) The normal line to a graph at a point is the line going through that point that lies at right angles to the graph (and hence to the tangent line to the graph).
      i. Consider the line with slope m going through (x_0, y_0); it has equation $(y - y_0) = m(x - x_0)$. What is the slope of the line at right angles to it going through the same point?
      ii. Give the equation of the normal line to the graph of $y = g(x)$ at $x = -1$.

4. (a) Find the slope function of $y = x^3 + 8x^2 + 22x - 21$ in two different ways:
      i. By rewriting the function as $y = (x-3)^3 + (x-3)^2 + (x-3)$ and then differentiating $z^3 + z^2 + z$;
      ii. By simply taking the derivative of the original function in its expanded form.
   (b) Hence explain how the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is related geometrically to the derivative $\frac{\mathrm{d}}{\mathrm{d}z}[z^3 + z^2 + z]$.
   (c) Show that there are no points where the graph of y versus x has a horizontal tangent line.

5. To expand slightly on the previous question, consider now the graph $y = (x^2 + 1)^2 + (x^2 + 1)$.
   (a) By expanding the brackets, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
   (b) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 2(x^2 + 1) + 1$. (Where did this right-hand function come from?)

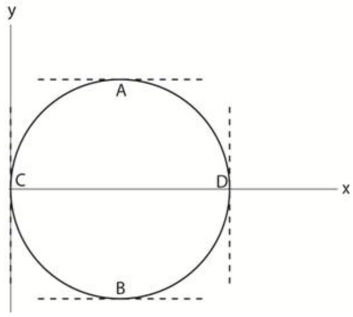
(c) Can you explain why our argument about 'shifting functions' does not work here? Hint: if we transform x to $x-3$, there is no shrinking or stretching going on — but this is not always the case.

6. A function f is differentiable at a point x if the value $f'(x)$ is well-defined.
   (a) Give some examples of functions which are *not* differentiable at some point.
   (b) Can you define differentiability in a different way, using tangent lines?
   (c) Is it ever possible for a function to have a horizontal normal line at any point? Explain how your answer is related to the idea of differentiability.
7. Consider the hyperbola $y = 1/x$.
   (a) Explain why the hyperbola has no tangent line at $x = 0$.
   (b) Show that the tangent lines to the hyperbola at $(-1,-1)$ and $(1,1)$ are parallel.
   (c) More generally, show that the tangent lines to the hyperbola at $(-x,-1/x)$ and $(x,1/x)$ are always parallel.
   (d) Are there any points on the hyperbola which share a common normal line (not simply a normal line with the same slope, but the same line full stop)? What about tangent lines? Hint: yes, and no.
8. Finally, here are some functions and points to find tangent lines at. If there is no tangent line at the point given, carefully explain why. Draw some graphs out as well.
   (a) $y = 3x^2 + 3x + 1$ at $(0,1)$.
   (b) $y = \sqrt{1-x^2}$ at $(1,0)$.
   (c) $y = 1/x^2$ at $(1,1)$.
   (d) $y = 1/x^2$ at $(2,1/4)$.
   (e) $y = \sqrt[4]{x^3}$ at $(2, \sqrt[4]{8})$.
   (f) $y = \sqrt[3]{x^2+2x+1}$ at $(0,1)$.(Hint: complete the square.)

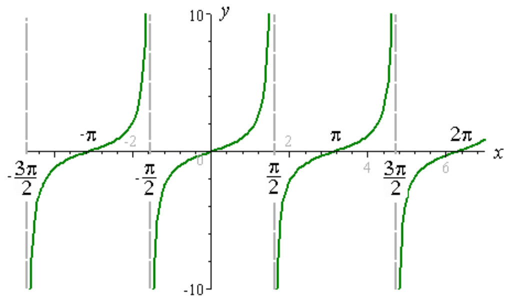
NCEA Level 2 Mathematics (Homework)
12. Tangent Lines and Approximations

Reading

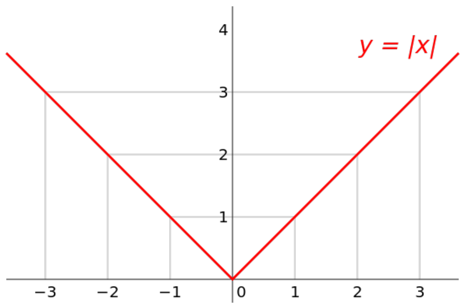
So far, we have only looked at 'nice' functions. However, it is possible to find functions which are not so nice, in the sense that the derivative does not exist at some point. What does this mean? It means that at some point, the graph of the function doesn't have a well-defined slope. For example, the function could become vertical (what is the slope of a vertical line?), or it could jump from one place to another without passing any of the points in between. Note that a function can be differentiable everywhere except one point, for example the absolute value function! Here are some examples of some non-differentiable functions.



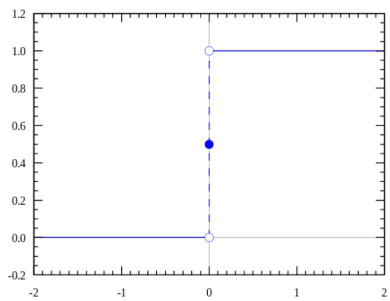
The circle is not differentiable at C or D because the tangent lines are vertical.



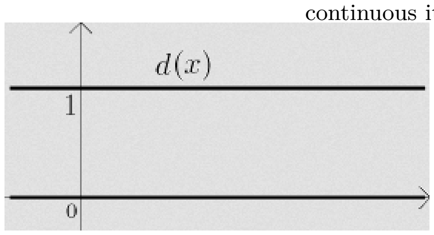
The tan function is not differentiable when the angle is an odd multiple of $\frac{\pi}{2}$ because the function is undefined there.



The absolute value function is not differentiable at $x = 0$ because it has no tangent line (try to draw one in!).



The Heaviside step function is the derivative of the absolute value (can you see this geometrically?) with $H(0) = \frac{1}{2}$, plugging the hole. However, because it is not continuous it is not differentiable at $x = 0$.



The Dirichlet function $d(x)$ takes the value 1 when x is irrational and 0 when x is rational, and so is continuous nowhere (there is a rational number between any two irrationals and vice versa, so the function jumps between 0 and 1 infinitely often). As you might expect, it is differentiable nowhere.

Questions

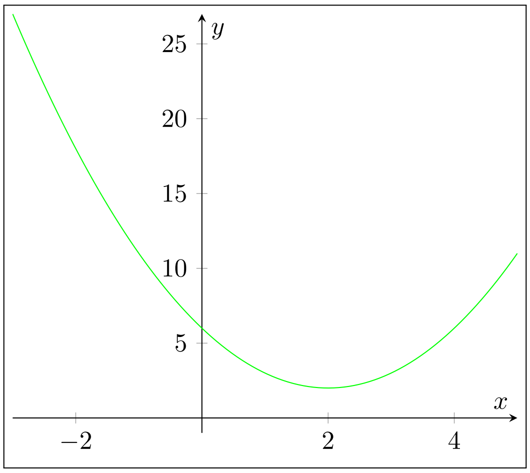
1. A function f is given by
$$f(x) = 2 - 4x + 5x^2 + ax^3. \tag{1}$$
The gradient of the graph at the point where $x = 1$ is 3. Find the value of a.

2. Give the equation of the tangent line to the curve
$$y = 3x^3 - \sqrt{x} + \frac{1}{x^3}$$
at the point $(1, 3)$.

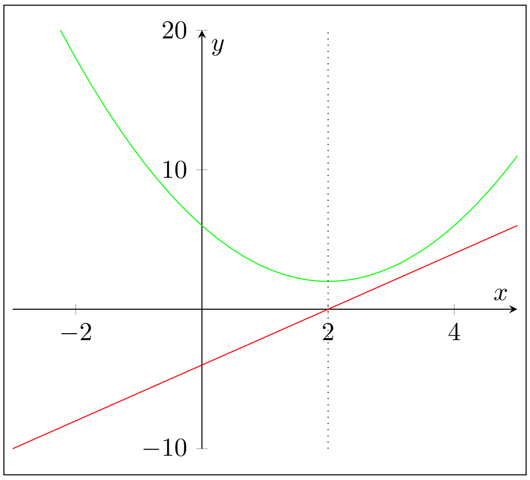
NCEA Level 2 Mathematics
13. Turning Points and Optimisation

Many problems reduce down to finding points where the graph of a function changes direction from increasing to decreasing, or vice-versa. These points are called turning points, or extreme points, of the graph. We can further classify turning points into local minima and local maxima.

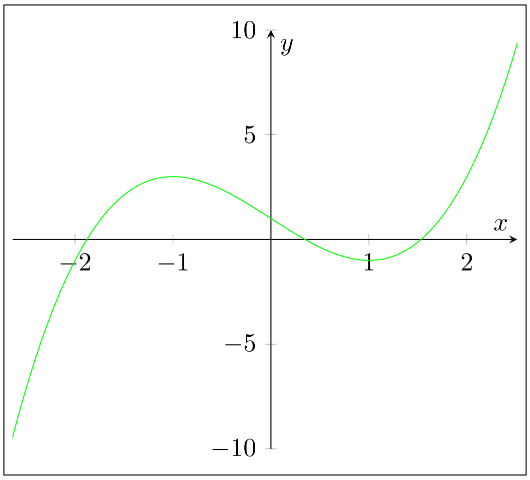
Example. Consider $y = (x-2)^2 + 2$, graphed here.



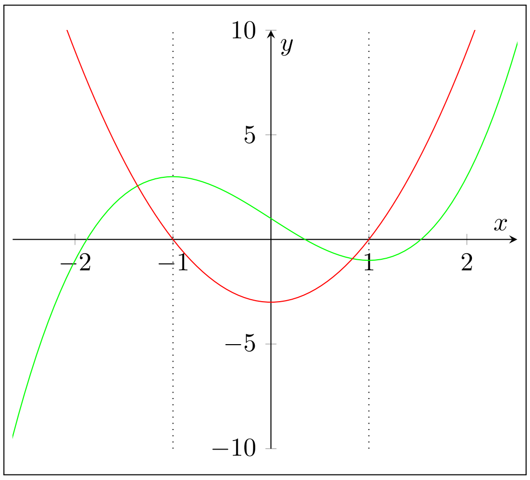
This function has a minimum at $(2, 2)$, because the function changes from decreasing to increasing there. We can rephrase this by saying that the derivative (in red) changes from negative to positive there, and in particular the derivative is exactly zero there.



Example. Consider $y = x^3 - 3x + 1$, graphed here.



This function has a minimum at $(1, -1)$, and a maximum at $(-1, 3)$. Again, if we plot the derivative $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 3$ in the same plot we see that the derivative is zero at the turning points.



Based on these two examples, we can state the following theorem (but again, we won't prove it here):

Theorem. *If the graph of $y = f(x)$ has a turning point (maximum or minimum) at $x = x_0$, then $f'(x_0) = 0$ or f is not differentiable at x_0.*[*]

We can look at this from a tangent line perspective as well: at a maximum or a minimum, the graph is 'flat' and so the tangent line is horizontal there — so the slope of the tangent line and of the function is zero there.

On the other hand, note that $f'(x_0) = 0$ does not necessarily imply that f has a turning point at x_0. For example, consider $y = x^3$: this function has no turning points, but $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$ is zero at $(0, 0)$. Places where a function has a zero (or nonexistent) derivative are called critical points.

If the derivative at a critical point exists, but the point is not a maximum or a minimum, then the point is known as an inflection point. If we look at an entire interval, there may be multiple maxima or minima; the largest maximum is known as the global maximum of the function on the interval, and the smallest minimum is known as the global minimum of the function on that interval. Maxima and minima which are not global maxima or minima are known as local maxima or minima; a function might have no global maxima (take

[*]For an example of this second case, consider the graph of $y = |x|$.

$y = x^2$ for example), but on any given closed interval (i.e. an interval which includes its endpoints) it always attains a local maximum or minimum.

Example. Let us find the critical points of the graph of $y = 4x^3 - 6x^2 + 2x - 1$ and classify them. The critical points are precisely those places where the slope of the graph is zero — i.e. the roots of the derivative,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 - 12x + 2.$$

Setting the derivative to zero, we can use the quadratic formula to find the two roots are

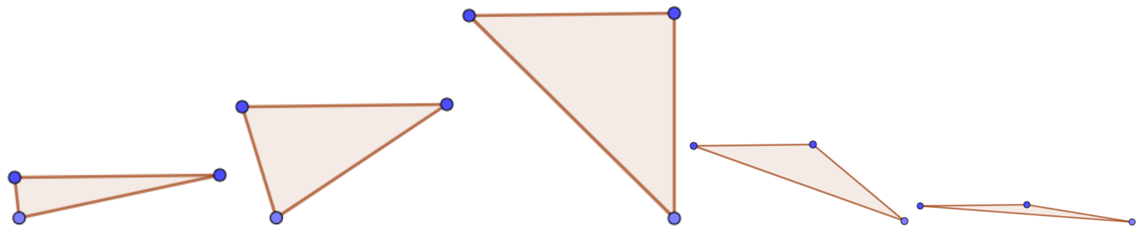
$$x = \frac{12 \pm \sqrt{144 - 4 \cdot 12 \cdot 2}}{2 \cdot 12} = \frac{12 \pm \sqrt{48}}{24} = \frac{1}{2} \pm \frac{\sqrt{12}}{6}.$$

Since the cubic has a positive leading coefficient, we know that it will go from $-\infty$ in the bottom left to $+\infty$ in the top right, and hence the two turning points will be a maximum and a minimum respectively.

Alternatively, to see this one could note that the derivative is positive to the left of the left-hand turning point, negative between the two, and positive again to the right of the right-hand turning point: so the graph changes from increasing, to decreasing, and back to increasing, and so we must have a maximum and a minimum in that order.

Let us look at one final geometric example.

Example. Suppose we have an isosceles triangle ABC, where $|AB| = |AC| = x$ and $|BC| = y$. Clearly if we spread the edges apart, the area increases and then decreases:



The area of the isosceles triangle is given by

$$A = \frac{1}{2} \times y \times \sqrt{x^2 - \frac{y^2}{4}}.$$

Now, we can't actually take the derivative of this function this year — but area is always positive, so the minimum of the area squared will be at the same place as the minimum of the area. Hence we want to minimise

$$A^2 = \frac{1}{4}y^2 \left(x^2 - \frac{y^2}{4} \right) = \frac{1}{4}x^2y^2 - \frac{1}{16}y^4$$

with respect to y (since x is fixed).

We have that $\frac{\mathrm{d}A^2}{\mathrm{d}y} = (1/2)x^2y - (1/4)y^3$; setting this to zero, we have that $x^2y = (1/2)y^3$. Since we know that $y \neq 0$ (that corresponds to a minimum, not a maximum) we can divide through, and so $x^2 = (1/2)y^2$ — so the minimum occurs when $y = x\sqrt{2}$.

Of course, this corresponds to the situation where the angle between the two equal sides is a right angle (for then $\sqrt{x^2 + x^2} = x\sqrt{2}$, as we just found is the case) — this is what we would expect from the symmetry of the triangle and from playing around with different pictures of an isoceles triangle. If we draw the triangle inside a circle, it is the halfway point between the two zero-area possibilities.

Questions

1. Consider the function graphed below.

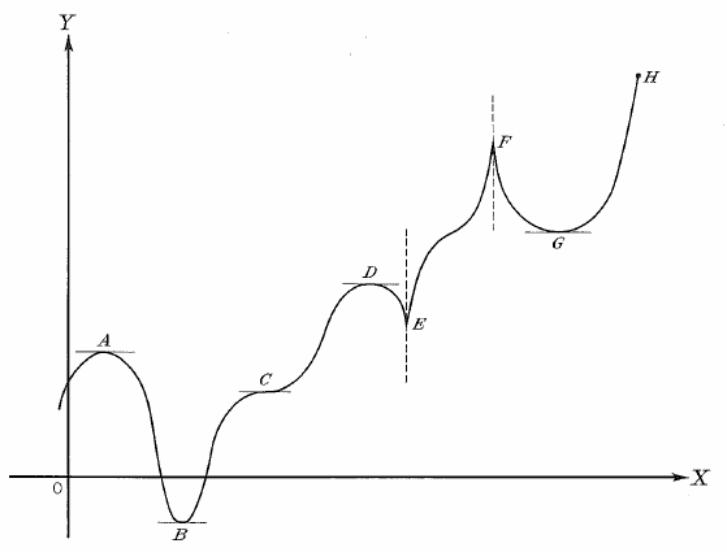


Figure 4-5

For each point A to H:

- Explain why the point is a critical point of the function by either giving the slope of the function at that point, or explaining why the derivative does not exist at that point.
- Label the point as an inflection point, a maximum, or a minimum.
- If the point is a maximum or a minimum, label it as either local or global.

2. Sketch graphs with the following properties:
   (a) A maximum at $x = 3$ and a minimum at $x = 6$.
   (b) No maximums or minimums, but critical points at $x = 2$ and $x = 4$.
   (c) Maximums at $x = 2$ and $x = 4$ but no minimums.

3. Find the maximum and minimum points of the function g defined by $g(x) = 2x^3 + x^2 + 2x$.

4. The function F, where $F(x) = x^{\frac{4}{5}}(x-4)^2$, has critical points at $x = 0$, $x = \frac{8}{7}$, and $x = 4$. Classify each one as a maximum, a minimum, or neither.

5. Show that the turning points of $y = x^4 - x^2$ are, alternately, minimum, maximum, and minimum.

6. Find the extreme values (if any) of the following functions of x:
   (a) $y = x^5$
   (b) $y = \frac{1}{x}$
   (c) $y = x^2 - 1$
   (d) $y = 2x^3 - 21x^2 + 72x + 18$
   (e) $f(x) = x^{10} - 4$
   (f) $y = \frac{1}{\sqrt{x}} + x^2$
   (g) $y = x^3 - x - 1$
   (h) $y = x^3 - x^2 + x - 1$
   (i) $f(x) = \frac{1}{x} + x - x^2$
   (j) $y = 16$
   (k) $y = \frac{x^{-2}+x^2}{2x}$
   (l) $x = \frac{y-2}{x+3}$

7. Find the maximum value of the derivative of $2x^2 - x^3$.

8. Prove that the function φ given by $\varphi(x) = \frac{x^{101}}{101} + \frac{x^{51}}{51} + x + 1$ has no extreme values.

9. Show that:
   (a) $y = x^4 + 3x^3 - x^2 - x + 20$ does not pass through the x-axis.
   (b) $x^3 - 5x + 100 = 0$ has only one real solution.
   (c) $x^3 - x^2 - x + 1$ has exactly two roots.

10. A farmer needs to create a rectangular field with a fence. He has 500 m of fencing available, and a building is on one side of the field (so that side does not need fencing). Determine the dimensions of the field to enclose the greatest area.

11. Suppose that a rectangle has perimeter p. Show that if the rectangle is to have the greatest possible area then it must be a square.

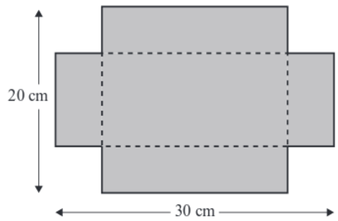
12. Find the maximum value of $y = -x^2 + 6x - 5$, both
    (a) *without* calculus; and
    (b) *with* calculus.

13. Find k so that $y = x^3 + kx^2 + x + 1$ has (a) two, (b) one, and (c) zero turning points.

14. The graph of $f(x) = x^3 + ax^2 + bx + 2$ has turning points when $x = -1$ and $x = 3$. Find the values of a and b.

15. Prove that the graph of $y = x^3(3 - x)$ has a local maximum when $x = \frac{9}{4}$. Carefully justify that the turning point *is* a local maximum.

16. Find the maximum volume of an open box (i.e. a box with base and sides, but no lid) that can be made from a rectangular piece of cardboard measuring 20 cm by 30 cm by removing the corner squares and folding along the dotted lines. Carefully justify that this is the maximum volume.



17. In an area surrounding a farming airstrip there is a height restriction for fireworks of 50 m. The height h metres above the ground reached by a firework t seconds after it is fired can be modelled by the function

$$h = 20t - 5t^2. \qquad (1)$$

Will the firework break the 50 m limit?

18. By noting that the derivative changes from positive to negative at a maximum, and from negative to positive at a minimum, suggest a criteria for classifying turning points using the second derivative.

NCEA Level 2 Mathematics (Homework)
13. Tangent Lines and Approximations

Reading

Go and watch...

`https://www.youtube.com/watch?v=F5RyVWI4Onk`

Polya's four-step approach to problem solving

1. *Preparation:* Understand the problem
   - Learn the necessary underlying mathematical concepts
   - Consider the terminology and notation used in the problem:
     (a) What sort of a problem is it?
     (b) What is being asked?
     (c) What do the terms mean?
     (d) Is there enough information or is more information needed?
     (e) What is known or unknown?
   - Rephrase the problem in your own words.
   - Write down specific examples of the conditions given in the problem.
2. *Thinking Time:* Devise a plan
   - You must start somewhere so try something.
   - How are you going to attack the problem?
   - Possible strategies: (i.e. reach into your bag of tricks.)
     (a) Draw pictures
     (b) Use a variable and choose helpful names for variables or unknowns.
     (c) Be systematic.
     (d) Solve a simpler version of the problem.
     (e) Guess and check. Trial and error. Guess and test. (Guessing is OK as long as you can back it up later.)
     (f) Look for a pattern or patterns.
     (g) Make a list.
   - Once you understand what the problem is, if you are stumped or stuck, set the problem aside for a while. Your subconscious mind may keep working on it.
   - Moving on to think about other things may help you stay relaxed, flexible, and creative rather than becoming tense, frustrated, and forced in your efforts to solve the problem.
3. *Insight:* Carry out the plan
   - Once you have an idea for a new approach, jot it down immediately. When you have time, try it out and see if it leads to a solution.
   - If the plan does not seem to be working, then start over and try another approach. Often the first approach does not work. Do not worry, just because an approach does not work, it does not mean you did it wrong. You actually accomplished something, knowing a way does not work is part of the process of elimination.

- Once you have thought about a problem or returned to it enough times, you will often have a flash of insight: a new idea to try or a new perspective on how to approach solving the problem.
- The key is to ***keep trying until something works***.

4. *Verification:* Look back

- Once you have a potential solution, check to see if it works.
  (a) Dd you answer the question?
  (b) Is your result reasonable?
  (c) Double check to make sure that all of the conditions related to the problem are satisfied.
  (d) Double check any computations involved in finding your solution.
- If you find that your solution does not work, there may only be a simple mistake. Try to fix or modify your current attempt before scrapping it. Remember what you tried — it is likely that at least part of it will end up being useful.
- Is there another way of doing the problem which may be simpler? (You need to become flexible in your thinking. There usually is not one right way.)
- Can the problem or method be generalized so as to be useful for future problems?

Questions

1. A cylindrical tube (with open ends) is to be made from a sheet of paper with area $25\,\mathrm{cm}^2$. What should the dimensions of the tube be in order to maximise the volume of the tube? Justify that you have found a maximum.
2. A function f is given by $f(x) = 2x^3 - 3ax^2 + 6bx - 2$. The function has two turning points, at $x = 2$ and at $x = 3$. What are the values of a and b?
3. Suppose a wire of length ℓ is cut into two pieces, one of length x and one of length $\ell - x$. One piece is used to form the circumference of a circle, and the other is used to form the perimeter of a square. How long should the length x be in order to ensure that the total area of the circle and the square is minimised? Carefully justify that you have found a minimum.

NCEA Level 2 Mathematics
14. Anti-differentiation

The final mathematical topic that we will look at with regard to calculus is the inverse operation to differentiation: given a slope function (a derivative), we will find the original function.

Example. Suppose that we know that the derivative of some function is the function f' given by $f'(x) = 3x^2 + 2x+1$. By reversing the power rule for derivatives, we can see that if we take f_0 to be $f_0(x) = x^3+x^2+x$ then we get the desired derivative (i.e. $f_0' = f'$). On the other hand, any function f_C such that $f_C(x) = x^3+x^2+x+C$ also has the same derivative.

From this example, we make two main observations:

Observation 1 (Inverse power rule)**.** *If we know a differentiation rule, like $ax^r \mapsto arx^{r-1}$, then we can reverse it and say that if the derivative looks like ax^r then the original function looks like $\frac{a}{r+1}x^{r+1}$.*

Observation 2. *Given any derivative f', we have infinitely many functions that have f' as their derivative: if one is f, then $f + C$ for any constant C also has f' as its derivative, because*

$$(f + C)' = f' + C' = f' + 0 = f'.$$

The set of antiderivatives of some function f' is denoted by

$$\int f'(x)\,\mathrm{d}x = f(x) + C.$$

The notation is unfortunate, but the best way to view it this year is as a pair of brackets: $\int$ and $\mathrm{d}x$. The mathematical operation of taking an antiderivative is called indefinite integration — if taking a function and finding its slope is splitting the function up into infinitely many pieces, then taking the antiderivative is packing all those infinitely many pieces back together into one integrated whole.

Example. Suppose that it is known that the function f

- has a derivative $f'(x) = 3x^5 + 4x^3 + 2x + 1$, and
- has a graph which passes through the point $(2, 1)$.

Then we know that all the possible candidates for f are given by

$$\int 3x^5 + 4x^3 + 2x + 1\,\mathrm{d}x = \frac{3}{6}x^6 + \frac{4}{4}x^4 + \frac{2}{2}x^2 + \frac{1}{1}x^1 + C = \frac{x^6}{2} + x^4 + x^2 + x + C.$$

We also know that $f(2) = 1$, so:

$$\begin{gathered} 1 = \frac{2^6}{2} + 2^4 + 2^2 + 2 + C \\ 1 = 2^5 + 2^4 + 2^2 + 2 + C = 32 + 16 + 4 + 2 + C = 54 + C \\ C = 1 - 54 = -53 \end{gathered}$$

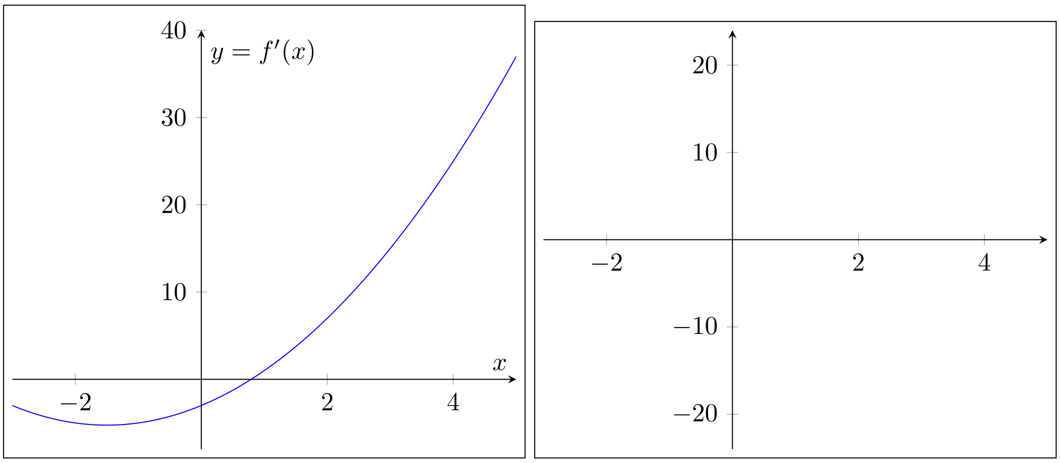
and hence $f(x) = \frac{x^6}{2} + x^4 + x^2 + x - 53$.

There is not much more to say about integration at this stage, because at Level 2 the geometric meaning of integration is no longer examinable. Suffice it to say, the operation of integration (as you will learn next year) is far deeper and more interesting than it first appears. The final problems in this week's problem set are an indication of this.

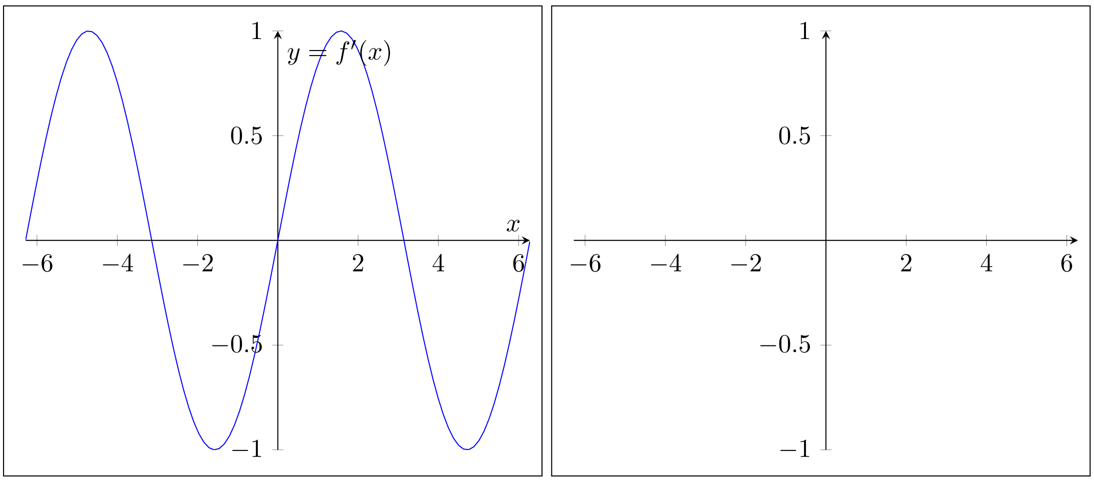
Questions

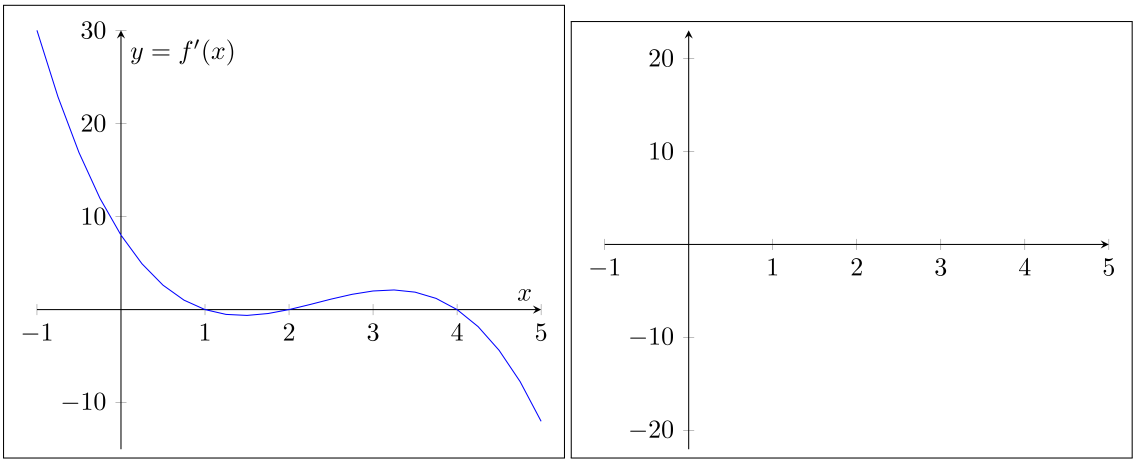
1. Find three functions that have derivatives equal to $x^2 - x$.
2. Find an antiderivative of $f(x) = 25x^4 + 12x^3 - x^{-2}$.
3. Evaluate $\int 12z^3 + 18z^{-4}\,\mathrm{d}z$.
4. Show that $x^3 + 3x + C$ is an antiderivative of $3x^2 + 3$.
5. Show that
$$\int 470x^4 + 2x + 1 + 6x^{-3}\,\mathrm{d}x = 94x^5 + x^2 + x - 3x^{-2} + C. \tag{1}$$
6. Evaluate the following indefinite integral:
$$\int -x^6 - \frac{1}{3\sqrt{x}} + \frac{x^{19}}{47} - \frac{2}{x^{-\frac{4}{5}}}\,\mathrm{d}x$$
7. Given the slope functions below, draw the antiderivative of each which passes through $(0, 0)$.

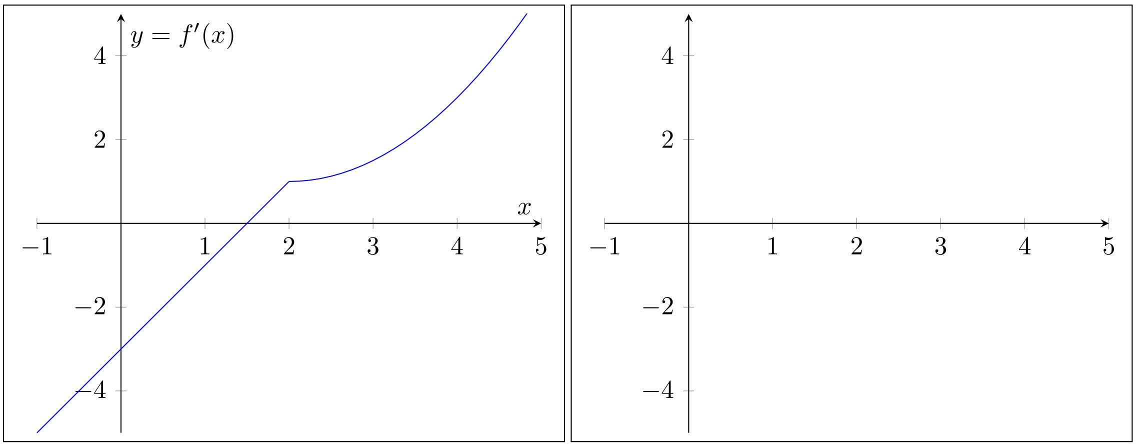
(a)



(b)



(c) 

(d) 

8. (a) By drawing graphs, show that it is plausible that $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$. (You may assume for the remainder of this question that this derivative is correct.)
   (b) Find all possible functions ψ such that

$$\psi'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

   (c) Suppose that we know that $\psi(0) = -8$. Find ψ.

9. Find g if $g'(x) = x\sqrt{x}$ and $g(1) = 2$.

10. Suppose that θ is a function of x such that

$$\theta'(x) = 8x^3 + 3x^2 + ax,$$

where a is a constant. Given that $\theta(0) = 9$ and $\theta(-1) = 14$, find θ and a.

11. Find $f(x)$ if $f''(x) = -2 + 12x - 12x^2$, $f(0) = 4$, and $f'(0) = 12$.

12. Suppose that f is a function of x given by $f(x) = x^3 - Ax^2 + 3x - B$, where A and B are real constants, which passes through $(0, -4)$ and has a critical point at $x = 1$. Find $f(x)$ exactly.

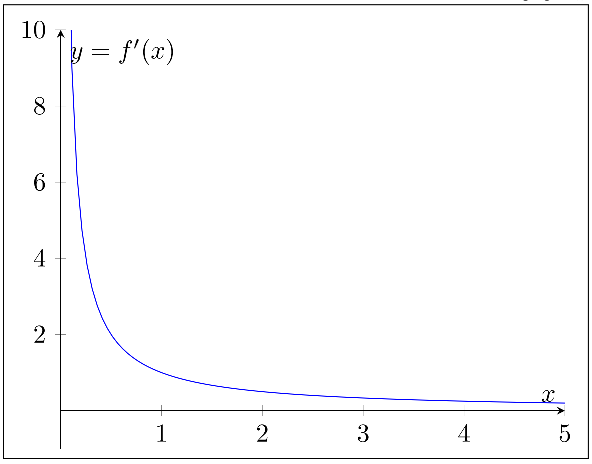
13. Suppose that y is a function of x given by

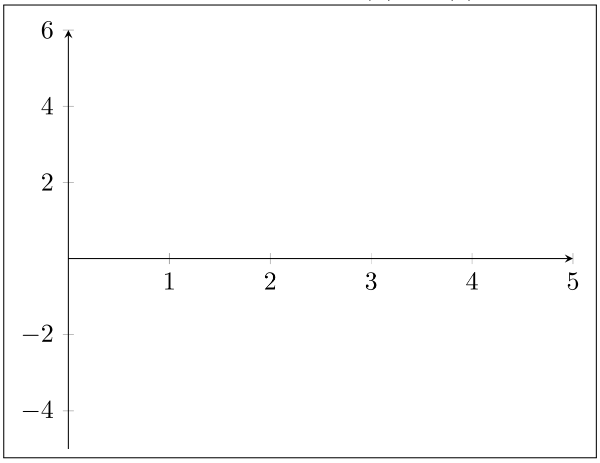
$$y = x^3 + Bx^2 + Cx + 1,$$

and that the graph of y has a minimum at $(3, 0)$.

Find B and C.

14. Every function which is made up of bits of the form ax^r added up can be differentiated using the power rule. However, not every function of that form has an anti-derivative that can be calculated using the inverse power rule.
    (a) Show that no such function f differentiates to give $f'(x) = \frac{1}{x}$. (Hint: try to integrate using the reverse power rule).
    (b) It is a consequence of a theorem of analysis that a function f exists so that $f'(x) = \frac{1}{x}$, even though it is not of the form above. Given the following graph of f', draw the graph of $y = f(x)$ if $f(1) = 0$.





    (c) The function which you graphed in part (b) is given the special name ln; so you have drawn the graph of $y = \ln(x)$. Let us consider the inverse function of ln, which we will call nl for the time being (because it's like ln backwards). Given the rule that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$, we can find the derivative of nl from the derivative of ln.
        i. Write $x = \ln y$, and give $\frac{\mathrm{d}x}{\mathrm{d}y}$.
        ii. Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y$.
        iii. Conclude that, since $x = \ln y \iff y = \mathrm{nl}\, x$, it must be the case that $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{nl}\, x = \mathrm{nl}\, x$.
    (d) So nl is its own derivative — a situation we already looked at! Clearly nl is not the zero function; we saw that there was only one other function which was its own derivative. What was it? (Hint: check the end of sheet 11.)
    (e) If ln is the inverse of the type of function that you found nl to be, what kind of function is it?
    (f) The letters ln stand for *logarithme naturel* (French). Why do you think this is?

15. (Advertisement for Level 3!)
    (a) Draw the line $y = 2t + 1$ and use geometry to find the area under this line, above the t-axis, and between the vertical lines $t = 1$ and $t = 3$.
    (b) If $x > 1$, let $A(x)$ be the area of the region that lies under the line $y = 2t + 1$ between $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$.
    (c) Find $A'(x)$. What do you notice?

NCEA Level 2 Mathematics (Homework)
14. Anti-differentiation

Reading

Go and watch...

`https://www.youtube.com/watch?v=j4hW7AwETZA`

What's it good for?

As I kind of hinted in the notes and in the final problem in the problemset, the geometric meaning of integration is to do with area. In fact, integration is the general way to find volumes, lengths, areas, and in general any kind of extent in space. People use integration for...

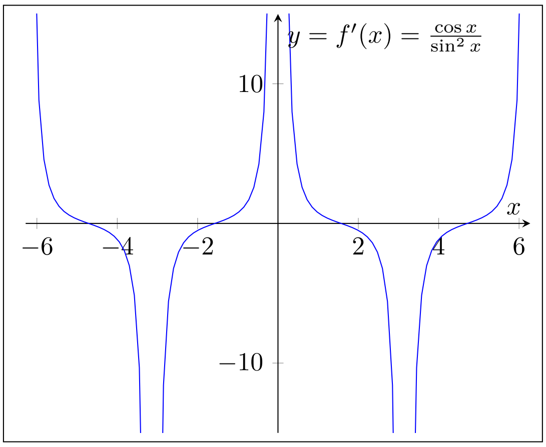
- The sciences and engineering: integration, both in its guise as "undoing slope-finding" and in its guise as an area-finding device, is heavily used in all the sciences (either explicitly, as in physics, or implicitly, as in chemistry and biology). Differential equations, equations involving functions and their derivatives, are also often found in these subjects as rates of change play an important role in engineering and in science; integrals are used to solve differential equations. In physics especially, multi-dimensional integrals and derivatives play an important role in the theories of the universe.

- Mathematics: the study of integrals can be done on two levels — as "recreational mathematics", where people try to solve hard integrals for fun (like solving a sudoku puzzle or a crossword), or as a deep subject known as measure theory — the theory of functions which measure things, and the different varieties of integrals that apply those functions onto space.

Questions

1. The following graph shows $f'(x)$, the derivative of a function f. Use the graph of the derivative to recreate the graph of the original function.

   Hint: Where must the original function be decreasing or increasing? Where will it have maximums and minimums? How fast does it change?



2. It is known that Φ is a function such that:
   - Φ has a stationary point at $x = 0$.
   - $\frac{d^2\Phi}{dx^2} = 60x^4 - 180x^2 + 48$.
   - $\Phi(2) = 0$.

   (a) Describe the nature of the stationary point at $x = 0$.

   (b) Find the locations of the other stationary points.

   (c) Find an exact expression for $\Phi(x)$.

NCEA Level 2 Mathematics
15. Kinematics and Rates of Change

The original development of calculus was as a tool for physicists to describe motion. The application itself is entirely natural, and there is no new mathematics involved, so this week should be nice and easy.

Let $x(t)$ be a function describing the position of some object at some time t. Then the velocity of the object, measured by the rate of change of position, is just $v(t) = x'(t)$. The acceleration of the object, which is just the rate of change of velocity, is $a(t) = v'(t)$. This is summed up in the following table:

Displacement, x		$\int v(t)\,\mathrm{d}t$
Velocity, v	$\frac{\mathrm{d}x}{\mathrm{d}t}$	$\int a(t)\,\mathrm{d}t$
Acceleration, a	$\frac{\mathrm{d}v}{\mathrm{d}t}$	

There is only one fundamental concept in this topic that you must remember: the derivative is just a rate of change. Velocity is rate of change of position, and acceleration is rate of change of velocity. If you slow down faster, your acceleration is more negative.

Do not try to memorise the above table, understand what it is saying physically.

Note that if you are taking physics, physicists sometimes have an annoying habit of writing $\dot{x}$ instead of $\frac{\mathrm{d}x}{\mathrm{d}t}$; so dotted variables may indicate time derivatives. While this notation is horrible and ugly, it is often forced upon us. I will not use it myself, and you should not expect to ever see it in a mathematics paper again; but be aware that this notation does exist.

A note about L3 physics

This topic appears in L2 mathematics for two main reasons: firstly, as an easy historical application of calculus; and secondly, so that you are well-prepared for level three physics. While calculus is not actually required for physics until university, a solid understanding of this year's calculus topics will enable you to make connections next year that may be otherwise obscure. If you are planning to do scholarship physics next year (and I urge you to consider it if you enjoy physics), then knowledge of calculus is a definite advantage as it can simplify some of the problems!

Questions

All distances are given in m, and all times in s, unless otherwise stated.

1. If a ball is thrown into the air with a velocity of $10\,\mathrm{m\,s^{-1}}$, its height y in metres after t seconds is given by
$$y = 10t - 4.9t^2. \tag{1}$$
   Find the vertical velocity of the ball when $t = 2$.

2. The velocity v $\mathrm{m\,s^{-1}}$ of an object t seconds after it passes a fixed point can be modelled by the function
$$v(t) = 4t^3 - t^2 + 2t.$$
   Find the equation for the acceleration of the object.

3. A tank is being filled with water. The height of the water, $h$cm, in the tank at any time t minutes after it began filling is given by $h = t^2 + 2t$. Find the rate that the height of the water is changing at three minutes after the tank begins to fill.

4. A balloon has an initial volume $5\,\mathrm{cm}^3$, and is being inflated at a rate given by $\frac{\mathrm{d}V}{\mathrm{d}t} = 4t$, where V is the volume of the balloon in cubic centimetres and t is the time in seconds since the balloon began to inflate. Give the volume of the balloon after ten seconds.

5. A projectile follows a path through space modelled by $y = 4x - x^2$. At what distance along the ground is it at its maximum height, and what is that height?

6. The distance, s, of a moving point from a given point P at a time t is given by
$$s(t) = \frac{1}{3}t^3 - 2t^2 - 12t.$$
Fully describe the acceleration of the object beginning from time $t = 0$.

7. A car begins to roll slowly down a sloping driveway, with velocity given by $v = 0.4t\,\mathrm{m\,s^{-1}}$ (where t is the time from the beginning of the car's roll). It is five seconds until someone notices the movement; in this time, how far does the car travel?

8. A child moves a Buzzy Bee™ toy forwards and backwards along a straight line. At time t, where $0 \leq t \leq 10$, the toy's position is modelled by $x = 3t - 1.3t^2 + 0.1t^3$.
   (a) At which time(s) is the toy stationary?
   (b) What is the acceleration of the toy at $t = 3$?
   (c) What displacement is the toy from the origin when the velocity of the toy is most negative?

9. The area of a circle varies if the radius r of the circle is changed. In this way, the area A of the circle is a function of r given by $A(r) = \pi r^2$.
   (a) What is the rate of change of the area of the circle with respect to the radius?
   (b) A piece of computer graphics software is drawing a circle so that the radius changes with time at a rate $\frac{\mathrm{d}r}{\mathrm{d}t} = 3t$. The initial radius of the circle is $2\,\mathrm{cm}$.
      i. Give the radius of the circle after three seconds.
      ii. After how long will the area of the circle reach $10\,\mathrm{cm}^2$?

10. A particle is moving through space along an axis. Its displacement from the origin at any time $t > 0$ is given by $s(t) = t^5 - 38t^4 + 560t^3 - 3982t^2 + 13599t - 17820$.
    (a) Find an expression for the velocity of the particle at time t, $v(t)$.
    (b) At what time is the particle moving with the most speed towards the origin, and how fast will it be moving at that time?
    (c) What is the acceleration of the particle at that time?
    (d) How many times does the particle change direction after $t = 0$?

11. An object is moving on a straight line; the point P lies on this line. Initially, the object is at point P and has a velocity of $4\,\mathrm{m\,s^{-1}}$. The acceleration of the object at a time t seconds after it leaves P is given by the function $a(t) = 2 - 6t$. How far from P is the object after three seconds?

12. Look at the following table of trigonometric derivatives.

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$

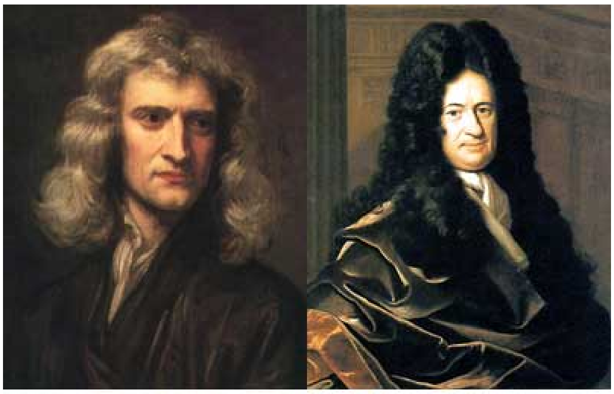
A spring oscillates such that the position of its end after a length of time t is given by $x = 2\sin(t)$. What is the approximate acceleration of its end at $t = 5$?

NCEA Level 2 Mathematics (Homework)
15. Kinematics

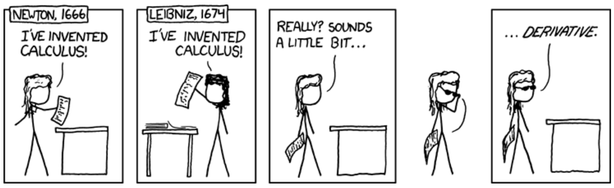
Reading

Go and watch...

`https://www.youtube.com/watch?v=pI62ANEGK6Q`



Sir Isaac Newton (left) and Gottfried Wilhelm von Leibniz (right)



Maybe it's time to tell the story of Isaac Newton and Gottfried Wilhelm Leibniz. If you've ever studied calculus, you know it was created independently by Newton and Leibniz. Few of us appreciate the full fury of the priority dispute behind that.

Newton was born in 1642, Leibniz four years later. Calculus is a means for calculating the way quantities vary with each other, rather than just the quantities themselves. The bare bones of that idea had been hatching (since the Greeks) before either Newton or Leibniz was born. But they each wrote a full system of calculus.

In 1665 Newton created his somewhat clumsy method of fluxions. He feared criticism and sat on his work 'til 1704. Then he published it as an appendix to his book on Optiks (the behaviour of light). The odd thing is, he began his fight with Leibniz long before he published anything. Leibniz wrote his calculus around 1673, and he used the notation we still use today – derivatives expressed as dy/dx, and so on. He too sat on his work for a long time. He published it in 1684 (still twenty years ahead of Newton!).

A surprised Newton took the offensive. But both men had cronies egging them on. Johann Bernoulli, who used Leibniz's calculus to maximize functions, goaded Leibniz into fighting Newton. Newton was surrounded by toadies whom Leibniz called the enfants perdus, the lost children. Newton choreographed the attack, and they carried the battle. They accused Leibniz of plagiarism, a charge that falls apart when you trace the details. In the end, Newton's campaign was effective and damaging. He emerged with the credit. But when people like Leonard Euler and the Bernoullis erected the field of applied analysis, they used Leibniz's calculus.

Leibniz worked in an astonishing variety of fields. He was first to state the conservation of energy. He worked for a reunification of Catholics and Protestants. That may well've been fed by his optimistic metaphysics. It was he who claimed we live in the best of all possible worlds. Voltaire, born when both Newton and Leibniz were on in years, wouldn't stand still for that. While his mistress, Emily de Breteuil, translated Newton's Principia into French, Voltaire wrote Candide. And Candide's friend Dr. Pangloss made vicious sport of Leibniz's optimism.

Leibniz died poor and dishonored, while Newton was given a state funeral. Yet history validates Leibniz. For as time passes, so does the potency of Newton's assault. And Leibniz gradually finds his place as one of the great thinkers of all time.

Adapted from `http://www.uh.edu/engines/epi1375.htm`.

Questions

All distances are given in m, and all times in s, unless otherwise stated.

1. A particle moves in space along a single axis, with velocity function $v(t) = t^2 + t - 12$ (where t is measured from some arbitrary starting point).
   (a) What is the acceleration of the particle at $t = 10\,\mathrm{s}$?
   (b) The particle is closest to the origin at $t = 3\,\mathrm{s}$.
      i. By considering $v(t)$, show that $t = 3$ is indeed a turning point for the graph of the position function $x(t)$ of the particle.
      ii. If the minimum distance between the particle and the origin is $300\,\mathrm{m}$, calculate the distance from the particle to the origin at $t = 10\,\mathrm{s}$.
2. A cubic equation is a polynomial of degree three — that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. Recall that a critical point is a point x where $f'(x) = 0$, or $f'(x)$ is undefined.
   i. Sketch examples of a cubic function with zero, one, and two critical points.
   ii. Prove that a cubic function can have a maximum of two critical points.

CHAPTER 5

Counting

NCEA Level 2 Mathematics
16. Counting and Combinatorics

We move from calculus, the study of the continuous, to combinatorics, the study of the discrete. We begin with a simple question: how many ways are there of picking a committee of three people, given a group of six people? Our first attempt at this is relatively naive: there are six choices for the first person in the committee, five for the second, and four for the third — so $6 \times 5 \times 4 = 120$ altogether. The problem with this is that we have overcounted: suppose our six people are labelled from A to F; then we have (for example) counted ABC and BAC separately (because in one we picked A first, and in the other we picked B first) despite the simple fact that they are the same committee!

In order to solve this problem, we need to count cleverer. Because the solution in this case works in general, we'll work it out in general to save time. So suppose we have a group of n people, and we want to pick a committee of r of them. Then, if we count different orderings separately, we obtain

$$n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

different committees (where the notation $n!$, read n bang or n factorial, denotes $n(n-1)\cdots 3 \cdot 2 \cdot 1$). This is the problem we solved above (and is called the number of permutations of r things out of n in total, or ${}^nP_r$); we just need to divide through by the number of times we counted each group. There are $r!$ different orderings for a set of r people, and so we counted each committee of r people $r!$ times in total — once for each ordering. Hence the total number of committees possible, up to ordering, is

$$\frac{n!}{(n-r)!r!} = \binom{n}{r} = {}^nC_r$$

which is read 'n choose r'. We have therefore proved:

Theorem. *The number of ways to choose r objects from a set of n, if we don't care about order, is given by $\binom{n}{r}$. If we care about order, the number of choices for the smaller set is given by ${}^nP_r$.*

There is an interesting pattern that we can make with these 'choice constants', known as *Pascal's triangle* after French mathematician Blaise Pascal.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \cdots & & & & &
\end{array}$$

The row number (starting from zero) is n, and the column number (from the left and starting from zero) is r. Let's look at some of the patterns here.

Observation 1. *The first and last column of each row is 1. In our counting notation, this is the observation that $\binom{n}{0} = \binom{n}{n} = 1$ for all sets of size n.*

Proof. We need to check that each observation is true for the whole table, not just the portion I gave above. This proof is left to you! □

Observation 2. *The second and $r-1$th column of each row is just n (except the first). In our counting notation, this is the observation that $\binom{n}{1} = \binom{n}{n-1} = n$.*

Proof. We need to check that each observation is true for the whole table, not just the portion I gave above. This proof is left to you! This one can be done either by using the factorial formula, or by counting. □

Observation 3. *The triangle is symmetric about the middle axis. In our counting notation, this is the observation that* $\binom{n}{r} = \binom{n}{n-r}$.

Proof 1: less satisfying.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$$

□

Proof 2: counting. Suppose we pick a subset of size r out of our set of size n. This also uniquely identifies a subset of size $n-r$: the elements we didn't take! Since there is a one-to-one correspondence between picking sets of size r and sets of size $n-r$, the number of choices for both sizes of subset must be the same. □

Observation 4. *Each number is given by the sum of the two numbers diagonally above from it. In our counting notation, this is the observation that* $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$.

Proof. Pick some object x in our larger set of size n. Then we are left with some set of size $n-1$ left over. If we want to pick a subset of size r from the set of size n, then we have two cases: either the subset we pick contains x, or it does not. If it contains x, we need to pick another $r-1$ things from the remaining $n-1$ in order to fill our set up; if it does not, then we need to pick up a full r things from the remaining $n-1$. In the first case, there are $\binom{n-1}{r-1}$ choices; in the second case, there are $\binom{n-1}{r}$ choices. □

Observation 5. *The sum of all the numbers in a row is* 2^n*:*

$$\begin{aligned}
2^0 &= 1 \\
2^1 &= 1+1 \\
2^2 &= 1+2+1 \\
2^3 &= 1+3+3+1 \\
2^4 &= 1+4+6+4+1 \\
2^5 &= 1+5+10+10+5+1
\end{aligned}$$

Proof. The sum of all the numbers in a row is the total number of ways we can choose subsets of size zero, plus subsets of size one, all the way up to subsets of size n: in short, the sum is just the total number of subsets of the larger set. But there are 2^n different subsets of the larger set (for each element, we either keep it or not: n choices, with two possible outcomes for each). □

I will only give one application of Pascal's triangle and the counting numbers here, but rest assured there are many others.

Example. Recognise this pattern?

1. $(x+y)^1 = \mathbf{1}x + \mathbf{1}y$
2. $(x+y)^2 = \mathbf{1}x^2 + \mathbf{2}xy + \mathbf{1}y^2$
3. $(x+y)^3 = \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3$
4. $(x+y)^4 = \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4$

Theorem (Binomial theorem)**.**

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}x^0y^n$$

Proof. Recall how expanding binomials work: we write

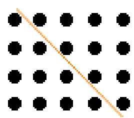
$$(x+y)^n = (x+y)(x+y)\cdots(x+y)$$

and then expand using the distributive law. Each term in the result consists of either x or y from the first bracket, either x or y from the second, and so on until we pick either x or y from the final bracket; so each term contains exactly n x's and y's, and there is one term for each possible combination of picking things from brackets. In fact, there will be $\binom{n}{r}$ terms with r x's — because that is the number of different ways to pick r x's out of the n brackets. The remaining $n-r$ things we picked for each of those terms must be y's; and we when we add up the terms where we pick 0 x's, 1 x, 2 x's, and so on, we obtain the desired result. $\square$

Because of this proof, the counting coefficients $\binom{n}{r}$ are more properly called *binomial coefficients*.

Questions

1. Given that $n!$ is the number of orderings of n objects, explain why $0! = 1$ is a reasonable definition.
2. I have a bucket of 10 numbered balls, and a bucket of 10 green balls that all look the same.
   (a) Suppose I pick a set of five balls from each. How many different ways are there of doing this if I count two sets the same when I can't tell them apart?
   (b) How many different ways are there of picking a set of ten balls, but this time I pick randomly from both buckets?
3. How many words of exactly ten characters are possible with an alphabet of 26 characters if:
   (a) Repetitions are allowed.
   (b) Repetitions are not allowed at all.
   (c) Repetitions are allowed, but adjacent characters must be different.
4. Repeat question 3, but now for words of ten characters or less.
5. Verify the binomial theorem for $(2+3)^4$, by both doing the exponent and by expanding out. (This allows us to split a large exponential operation into several smaller ones.)
6. (a) Add up $1+2+\cdots+99+100$.
   (b) Add up $1+2+\cdots+(n-1)+n$. How many different proofs of the resulting formula can you think of? Hint:



7. Justify observations 1 and 2 by counting cleverly.
8. Give another proof for observation 5 by expanding $(1+1)^n$ using the binomial theorem.
9. Prove that, if we alternately add $+$'s and $-$'s between the members of each row of Pascal's triangle, then the resulting sum is zero. For example, $1-4+6-4+1=0$. (Hint: $(1-1)^n = 0$.) Rephrase this in terms of binomial coefficients.
10. A company once had a "matching picture" contest in which the object was to match the pictures of four celebrities with their baby pictures. Contestents were able to enter as many times as they wished, and the first prize was \$10,000.
    (a) How many different entries would you need to send in to be sure of having all four pictures matched correctly?

(b) Does it seem reasonable to think that you would win \$10,000 if you sent in an entry with all the pictures correctly matched?

11. Let us calculate the number of ways there are to pick a team of seven out of a pool of twenty people, such that one of the members of the team is captain.
   (a) We can do this two ways; calculate both counts, and check they agree.
      i. Pick one captain out of twenty, and then six remaining players out of the 19 left.
      ii. Pick seven players out of twenty, and then one captain out of the seven.
   (b) Prove the following generalisation in two different ways:
$$\binom{n}{b}\binom{n-b}{c} = \binom{n}{b+c}\binom{b+c}{b}$$
      i. As in (a), double-count the number of ways to pick a team of $b+c$ players from n, where b players are 'special'.
      ii. Algebraically, with the factorial definition of binomial coefficients.
   (c) Notice that the actual quantity in part (b)i is the number ways of picking *three* subsets, of size x, y, and z, from a set of n, and the subsets *partition* the set: use up all the elements without overcounting, so $x+y+z=n$.
      i. Show that the number of ways to partition a set of size n using m smaller subsets, with sizes $r_1, r_2, ..., r_m$, is given by the *multinomial coefficient*
$$\left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_m \end{array} \right\} = \frac{n!}{r_1! r_2! \cdots r_m!}.$$
      Hint: order n, then chop off the first r_1 elements, then the next r_2 elements, and so on, taking account of ordering.
      ii. Show that the binomial coefficients are special cases of the multinomial coefficients, but that
$$\binom{n}{r} \neq \left\{ \begin{array}{c} n \\ r \end{array} \right\}$$
      in general (hence why I use different brackets).

12. We have defined the number e to be $\lim_n \left(1+\frac{1}{n}\right)^n$.
   (a) Show that the ith term in the expansion of $\left(1+\frac{1}{n}\right)^n$ for finite n is given by
$$\frac{n!}{(n-i!)i!n^i}.$$
   (b) Suppose we hold i fixed, and send n off to infinity. Show that
$$\frac{n!}{(n-i)!} = n(n-1)(n-2)\cdots(n-i+1)$$
   for all i, and conclude that
$$\frac{n!}{(n-i)!n^i}$$
   tends towards 1. (Hint: as n tends to infinity, but x remains finite, $n-x \approx n$.)
   (c) Hence, by expanding $\left(1+\frac{1}{n}\right)^n$ using the binomial theorem, show that
$$e = \lim_n \left(1+\frac{1}{n}\right)^n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{i!} + \cdots,$$
   and check that the two values roughly agree for $n = 10$.

NCEA Level 2 Mathematics (Homework)
16. Counting and Combinatorics

Reading

Go and watch...

`https://www.youtube.com/watch?v=aSsCUOmT-Bk`

What's it good for?

People use combinatorics for...

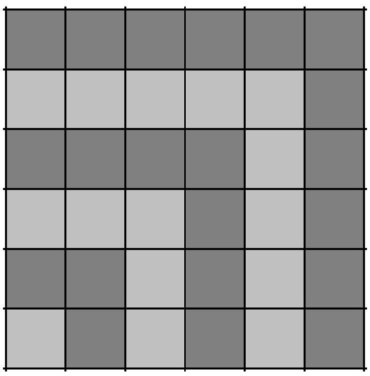
- Computer science: combinatorics (both the counting we did this week, and the work we'll do over the next few weeks) is a foundation of computer science as it allows the efficiency of various algorithms to be measured.
- Statistics and probability: if you want to measure probabilities, you need to be able to count how many ways each possibility can occur!
- Mathematics: combinatorics is one of the fastest-growing areas of modern mathematics, as many modern problems are phrased in terms of the discrete (that is, individual pieces and finite sets) rather than the continuous (infinite sets, as studied in calculus).

Questions

1. How many different ways are there to pick seven numbered tennis balls from a bucket of ten, and then order them?
2. Check that

$$\begin{gathered} 1 = 1 \\ 1 + 3 = 4 \\ 1 + 3 + 5 = 9 \\ 1 + 3 + 5 + 7 = 16 \\ 1 + 3 + 5 + 7 + 9 = 25 \\ \cdots \end{gathered}$$

Generalise (i.e. state and prove some result for all sums of odd numbers). Hint:



3. The first few powers of 11 are rows of Pascal's triangle (11, 121, 1331, ...). Is this a coincidence? (Hint: no.)
4. Extension question. If you have n pieces of spaghetti (each arbitrarily long), and you cross them such that every piece crosses every other piece exactly once, how many crossing points are there?

NCEA Level 2 Mathematics
17. Number Sequences and Fractals

A number sequence is an arrangement of numbers in which each successive number follows the last according to a uniform rule. More precisely, a number sequence is a correspondence between the counting numbers and a set of numbers: $(a_n) = a_1, a_2, a_3, \ldots$. Our goal in this section is twofold:

- First, we will try to get through all of the boring computations first;
- Second, we will refresh ourselves in the oasis of geometry after a walk in the desert of analysis.

Arithmetic Sequences

Arithmetic sequences are the simplest kind of interesting sequence.

Definition. An arithmetic sequence is a number sequence in which each successive term may be found by adding the same number; formally, a sequence is arithmetic if $a_{n+1} = a_n + k$ for every $n > 1$ and for some constant k.

Example. In the following sequence, $a_1 = 2$ and $a_{n+1} = a_n + 3$.

$$2, 5, 8, 14, 17, \cdots$$

For arithmetic sequences, if we know the initial value and the constant difference then we can find each number in the sequence easily.

Theorem. *If $a_1, a_2, \cdots$ is an arithmetic sequence with constant difference k, then $a_n = a_1 + (n-1)k$. (This is called the* general term *of the sequence.)*

Proof. $a_n = a_{(n-1)} + k = a_{(n-2)} + 2k = \cdots = a_{(n-(n-1))} + (n-1)k$. □

Suppose we want to find the *sum* of the first n values of some sequence $a_1, a_2, \ldots$. For a simple example, we turn to a problem from last week.

Example. We will find the sum of the first n counting numbers. Behold:

$$\begin{array}{ccccccccccccccccc} 1 & + & 2 & + & 3 & + & 4 & + & \cdots & + & (n-3) & + & (n-2) & + & (n-1) & + & n \\ n & + & (n-1) & + & (n-2) & + & (n-3) & + & \cdots & + & 4 & + & 3 & + & 2 & + & 1 \end{array}$$

Adding the two rows together and dividing by two, we obtain the result that

$$1 + \cdots + n = \frac{n(n+1)}{2}.$$

(This is called the nth partial sum of the series $1 + 2 + 3 + \cdots + n$.)

We now turn our attention to finding the nth partial sum of the series $a_1 + a_2 + a_3 + \cdots + a_n$. By the theorem above, we can rewrite this as

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_n &= [a_1 + (1-1)k] + [a_1 + (2-1)k] + [a_1 + (3-1)k] + \cdots + [a_1 + (n-1)k] \\ &= [a_1 + \cdots + a_1] + [0k + 1k + 2k + \cdots + (n-1)k] \\ &= na_1 + k[0 + 1 + 2 + \cdots + (n-1)] \\ &= na_1 + k\frac{n(n-1)}{2}. \end{aligned}$$

Hence, we have proved the following

Theorem. *The nth partial sum of the series $a_1 + a_2 + a_3 + \cdots + a_n$ is given by*

$$na_1 + k\frac{n(n-1)}{2}$$

(but you should memorise the idea of the proof, not the formula.)

Geometric Sequences

The next simplest kind of sequence after arithmetic sequences (where you add a constant term) is a geometric sequence (where you multiply by a constant term).

Definition. A geometric sequence is a number sequence in which each successive term may be found by multiplying by the same number; formally, a sequence is geometric if $a_{n+1} = ka_n$ for every $n > 1$ and for some constant k.

Example.

1. $a_1 = 1$, $a_n = 2a_{n-1}$: $1, 2, 4, 6, 8, ...$ (the binary sequence).
2. $a_1 = 100$, $a_n = \frac{1}{10}a_{n-1}$: $100, 10, 1, 0.1, 0.01, ...$
3. $a_1 = 1$, $a_n = -1a_{n-1}$: $1, -1, 1, -1, ...$

Theorem. *If $a_1, a_2, \cdots$ is an geometric sequence with constant ratio k, then:*

1. *the general term of the sequence is $a_n = a_1 k^{n-1}$.*
2. *the nth partial sum of the sequence is $s_n = a_1 \frac{1-k^n}{1-k}$.*

Proof.

1. Exercise.
2. This proof uses a little trick:

$$\begin{aligned} s_n &= a_1k^0 + a_1k^1 + a_1k^2 + \cdots + a_1k^{n-1} = a_1^n(1 + k + k^2 + \cdots + k^{n-1}) \\ (1-k)s_n &= a_1(1-k)(1 + k + k^2 + \cdots + k^{n-1}) \\ &= a_1[(1 + k + k^2 + \cdots + k^{n-1}) - k(1 + k + k^2 + \cdots + k^{n-1})] \\ &= a_1[(1 + k + k^2 + \cdots + k^{n-1}) - (k + k^2 + k^3 + \cdots + k^n)] \\ &= a_1[1 - k^n] \\ s_n &= a_1\frac{1-k^n}{1-k}. \end{aligned}$$

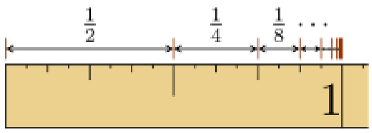
□

Geometric sequences are slightly more interesting than arithmetic sequences; if we add up all the terms of an arithmetic sequence, the resulting partial sums always grow towards $\pm\infty$. On the other hand, it is possible for the sum of all the terms of a geometric sequence to tend to some finite value. One case in which this happens is the following example.

Example. Consider the geometric sequence given by $a_n = \frac{1}{2}^n$:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

The sum $1 + 1/2 + 1/4 + \cdots$ converges to 1, as shown graphically here.



On the other hand, it is very easy for series to diverge (not tend towards any finite value): just take $k > 1$. More interestingly, we can consider the following sequence:

Example. Consider the geometric sequence given by $a_n = (-1)^n$:

$$1, -1, +1, -1, +1, \ldots$$

The sum $1 - 1 + 1 - 1 + \cdots$ does not converge as it jumps between $+1$ and -1 infinitely often.

Let us try to work out under which conditions the geometric series *does* converge. We want to look at the behaviour of the quantity $a_1 \frac{1-k^n}{1-k}$ as n grows. We make the following observations:

1. If $k = 1$, then the quantity is undefined: but then, the sequence looks like $a_1 + a_1 + \cdots$, which grows arbitrarily large and so the sum diverges.
2. If $k > 1$, then the sum also grows arbitrarily large and the series diverges.
3. If $k = -1$, then the sum diverges but now oscillates around zero instead of growing to infinity.
4. If $k < -1$, then the sum grows arbitrarily large and oscillates between being positive and negative, so diverges.
5. If $-1 < k < 1$, then k^n gets smaller as n increases — so tends to zero, and the sum of the series as $n \to \infty$ is given by $\lim a_n = a_1 \frac{1}{1-k}$.

In summary, we have seen that:

- The general term of an arithmetic sequence is $a_n = a_1 + (n-1)k$.
- The nth partial sum of an arithmetic series is $s_n = na_1 + k\frac{n(n-1)}{2}$.
- The general term of a geometric sequence is $a_n = a_1 k^{n-1}$.
- The nth partial sum of a geometric series is $s_n = a_1 \frac{1-k^n}{1-k}$.
- If $-1 < k < 1$ is the ratio of a geometric series, then the series converges to $\lim a_n = a_1 \frac{1}{1-k}$

These five facts are the important things to remember.

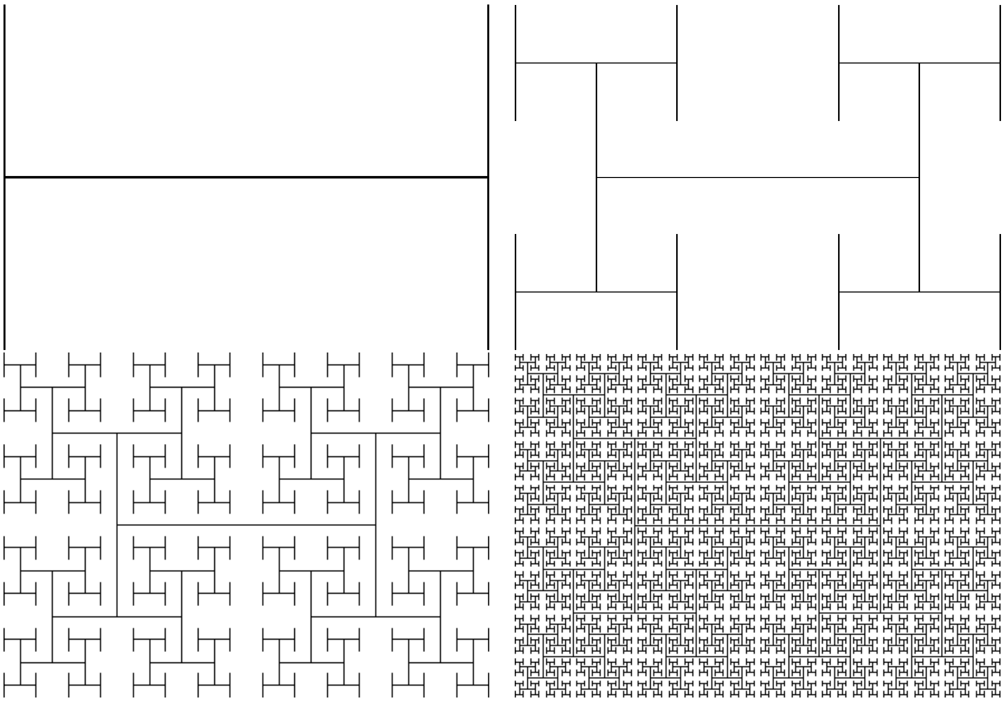
Fractals

We now move from sequences of numbers to sequences of geometric objects, with an very brief overview of fractals. Informally, a fractal is a geometric figure with the property of self-similarity: if you zoom in, it looks 'the same'. Many examples of fractal geometry can be found in nature; the traditional example is a coastline:

As we zoom in, the coastline reveals new 'wiggles' that we couldn't see at a larger scale — but it always looks wiggly in the same way, no matter how far you zoom in.

Construction 1 (H-fractal)**.** Begin at step one with a line segment of length 1. Then at each step n, add a line segment of length $\left(\frac{1}{\sqrt{2}}\right)^{n-1}$ to the endpoints of each line segment created at step $n-1$. (The lengths are purely aesthetic.)

The curve produced is the H-fractal; a Python script to draw it (`hcurve.py`) is given in the appendices. Below we have the curve after the second, fourth, eighth, and twelfth steps.



How many segments are present at the nth step? At the first step we have one, at the second three, at the third seven, at the fourth fifteen, and so on: the number seems to be $2^n - 1$ segments. How do we prove this? Well, suppose we have k endpoints at the $n-1$th step. Then at the nth step, we add k lines and hence $2k$ endpoints — so the number of endpoints doubles each time. Initially we have two endpoints, so the number of endpoints added at each step follows a geometric sequence with initial term 2 and ratio 2: at step n, we add 2^n endpoints. The sum of all these endpoints is simply $2\frac{1-2^n}{1-2} = 2(2^n - 1)$ (using our formula for the partial sum of a geometric series); and every line has two endpoints, so we must divide by two.

An interesting property of this curve is that, as well as having 'infinite length', it is space-filling — when we complete the construction at infinity, the curve will cover every point in the rectangle that it is bounded by.[*]

Other methods of producing fractals involve removing portions of a figure rather than adding portions.

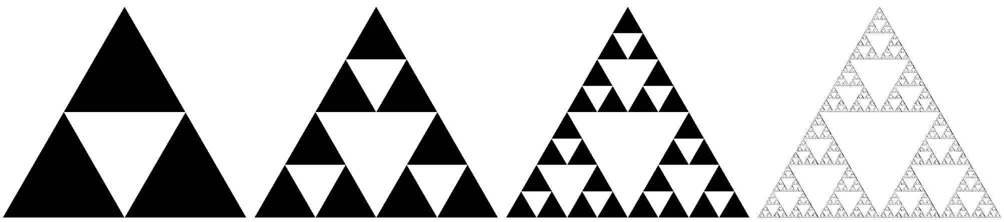
Construction 2 (Cantor set)**.** At step 1, begin with a unit segment; then at step n, remove the middle third of each of the segments remaining from the previous step.

This construction, pictured below, was first discovered by Henry John Stephen Smith in 1874 and entered mainstream mathematical knowledge in 1883 due to Georg Cantor (the father of set theory). The Cantor set itself, produced by continuing the construction to infinity, still has infinitely many points but has (in a precise sense) zero length. A Python script to draw it (`cantor.py`) is given in the appendices.

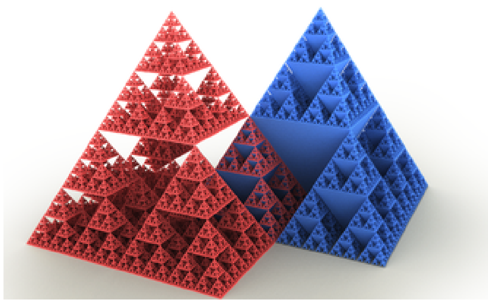
Our initial length is 1; at each step we remove precisely 1/3 of the remaining length, or equivalently we keep 2/3 of the remaining length; so the length of the set at step n is given by the geometric sequence with general term $\left(\frac{2}{3}\right)^{n-1}$. In particular, since 2/3 is less than 1, if we take $n \to \infty$ the sequence goes to zero.

Construction 3 (Sierpinski triangle)**.** At step 1, we start with a filled equilateral triangle. At the nth step, split each filled triangle into four equilateral triangles and remove the central triangle.

If this construction is carried on forever, the result is the Sierpinski triangle. This fractal was first described by a Polish mathematician, Waclaw Sierpinski, in 1916, and is a generalisation of the Cantor construction to two dimensions. Similarly to the Cantor set, the measure of the Sierpinski triangle (in this case the area) is zero but it still contains infinitely many points! A Python script to draw it (`sierp.py`) is given in the appendices; below, we have the triangle after two, three, four, and nine iterations.



One possible generalisation to three dimensions is the Sierpinski tetrahedron, produced by removing pyramids from a pyramid; two are pictured below.[†]



[*]More precisely, we can pick some number N such that after the Nth step, the curve comes within any distance δ of any point within the rectangle that we want. Obviously if δ is small then we need N to be (very) large, but the point is that it's theoretically possible!

[†]CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=647064`

Questions

Sequences and Series

1. The following are snippets of sequences that are either arithmetic or geometric. Give the general term of each. (In each case, $a_1 = 1$.)
   (a) $\ldots, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \cdots$
   (b) $\ldots, 5, 7, 9, \ldots$
   (c) $\ldots, -10, 100, -1000, \ldots$
   (d) $\ldots, 0.02116, 0.0097336, 0.004477456, \ldots$
2. Prove that the general term of a geometric sequence (a_n) with constant ratio k is $a_1 k^{n-1}$.
3. Compare the formulae that give the general term of an arithmetic sequence and a geometric sequence.
4. The Fibonacci sequence is the sequence defined by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-2} + f_{n-1}$. The first few numbers in the sequence are $1, 1, 2, 3, 5, 8, 13, 21, \cdots$.
   (a) The Fibonacci sequence was studied by Leonardo of Pisa (Fibonacci) in the middle ages, in his book *Liber Abaci* ("Book of Calculation"). In this book appeared the following problem:

   A pair of rabbits one month old are too young to produce more rabbits, but suppose that in their second month and every month thereafter they produce a new pair. If each new pair of rabbits does the same, and none of the rabbits die, how many pairs of rabbits will there be at the beginning of each month?

   Show that the solution is given by the Fibonacci sequence.
   (b) Prove that the Fibonacci sequence is neither an arithmetic nor a geometric sequence.
   (c) It turns out that, despite not being a geometric sequence, the ratios of adjacent Fibonacci numbers tend to a constant value. Taking this to be true without proof, we will calculate what this ratio is.
      i. Justify why, if n is very large, $\frac{f_{n+1}}{f_n} \approx \frac{f_n}{f_{n-1}}$.
      ii. Show that, if $x = \frac{f_n+1}{f_n}$, then $x \approx 1 + \frac{1}{x}$.
      iii. Show that $x = \frac{1}{2} + \frac{\sqrt{5}}{2}$. (This value is usually called φ, the golden ratio.)
      iv. Experimentally verify that this is the approximate ratio between adjacent values for the first few values of the Fibonacci series.
      v. Explain why we have *not* proved that this is the eventual ratio between adjacent values of the Fibonacci sequence.
5. According to legend, the game of chess was invented by an ancient Indian minister for his ruler; the ruler was impressed, and asked the minister what reward he wanted, and the minister requested that the ruler take a chessboard and give him one grain of wheat on the first square, two grains on the second, four on the third, eight on the fourth, and so on. The ruler laughed it off as a meager prize for such a brilliant invention.
   (a) How much wheat did the minister ask for?
   (b) The weight of a grain of wheat is around 65 mg. Compare the weight of the wheat on the first half of the chessboard to the weight on the second half.

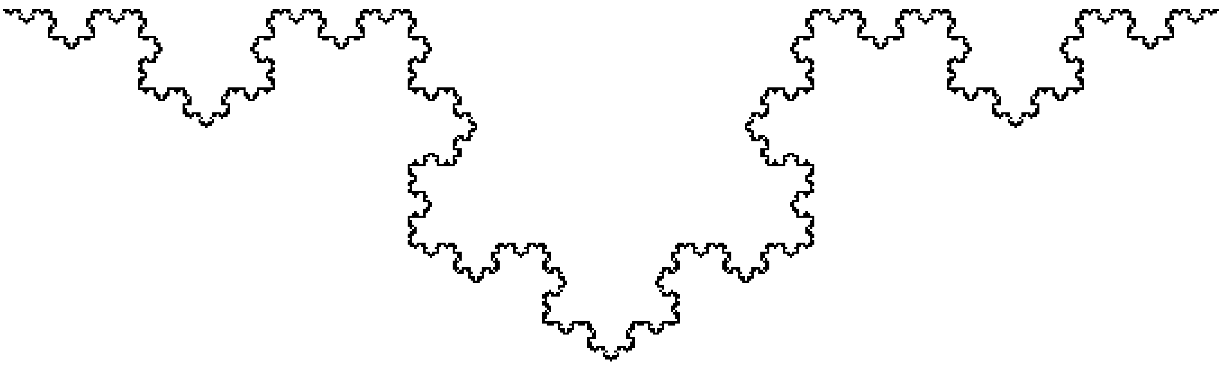
Fractals

6. What is the total length of the H-fractal after n steps?

7. Consider the figure given of the Cantor set, where each iteration is added on below the previous terms in order to form a fractal figure. Supposing that we continue this on for n steps (so the figure above is of step 6 of this process), and the height of each term is constant at 1 unit.
   (a) What is the total area of the fractal after n steps?
   (b) What happens to the area as n tends towards infinity? Does the area converge to some finite number?
8. Show that the area of the Sierpinski triangle is zero, by calculating the total area of triangles that are removed.
9. The Sierpinski triangle is very amenable to generalisation.
   (a) Give a construction for a fractal produced in a similar way to the Sierpinski triangle, but instead of subdividing a triangle into triangles subdivide a square into squares. (The resulting fractal is known as the Sierpinski carpet.) What is the area of this figure after n steps?
   (b) Generalise part (a) to three dimensions, by removing cubes from a cube. (The resulting cube is known as the Menger sponge.) Find the volume after n steps.
10. The Koch curve is produced by continuing the following construction to infinity:

    Construction 4. At step 1, we have a unit segment. At the nth step, we replace the middle third of each segment from the previous step with the upright sides of an equilateral triangle, so the segment is replaced by four segments with total length 4/3 of the segment length.

    (a) Below is the seventh iteration of the Koch curve (see `koch.py` in the appendices). Draw the first few iterations.



    (b) Show that the length of the curve as $n \to \infty$ becomes infinite.
    (c) This curve is an example of a **continuous** curve that has **no tangent line anywhere**. Justify both bolded claims.
    (d) Show that, if we start with an equilateral triangle and then add our equilateral triangles always on the outside, then the area enclosed by the final figure is precisely 8/5 of the original triangle area. The resulting figure, the Koch snowflake, is therefore a finite area bounded by a curve of infinite length.

H-fractal: `hcurve.py`

```
import tkinter as tk
from math import sqrt

def hcurve(x, y, length, scale, horizontal, iterations, canvas):
  if(iterations == 0):
    return

  if horizontal == True:
    x0 = x - length/2
    x1 = x + length/2
    y0 = y1 = y
  else:
    y0 = y - length/2
    y1 = y + length/2
    x0 = x1 = x

  canvas.create_line(x0, y0, x1, y1)

  hcurve(x0, y0, length*scale, scale, not(horizontal), iterations - 1, canvas)
  hcurve(x1, y1, length*scale, scale, not(horizontal), iterations - 1, canvas)

w = tk.Canvas(width=500,height=500)
hcurve(250, 250, 220, 1/sqrt(2), True, 12, w)
w.grid()
w.mainloop()
```

Cantor set: `cantor.py`

```
import tkinter as tk
from math import sqrt

def comb(x, y, length, height, iterations, canvas):
  print(length)
  if(iterations == 0):
    return

  x0 = x - length/2
  x1 = x + length/2 + 1
  y0 = y - height/2
  y1 = y + height/2 + 1

  w.create_rectangle(x0, y0, x1, y1, fill='black')

  comb(x - length/3, y + height + 10, length/3, height, iterations - 1, canvas)
  comb(x + length/3, y + height + 10, length/3, height, iterations - 1, canvas)


w = tk.Canvas(width=1400,height=500)
comb(700, 20, 1300, 20, 6, w)
w.grid()
w.mainloop()
```

Sierpinski triangle: sierp.py

```
import tkinter as tk
from math import sqrt

def sierp(x, y, length, iterations, canvas):
  if(iterations == 0):
    return

  xleft = x - length/2
  xright = x + length/2
  yleft = yright = y + length*sqrt(3)/2

  canvas.create_polygon(x, y, xleft, yleft, xright, yright, fill = 'black')
  canvas.create_polygon((x+xleft)/2, (y+yleft)/2,
                        x, yleft, (x+xright)/2, (y+yright)/2, fill = 'white')

  sierp((x+xleft)/2, (y+yleft)/2, length/2, iterations - 1, canvas)
  sierp((x+xright)/2, (y+yright)/2, length/2, iterations - 1, canvas)
  sierp(x,y, length/2, iterations - 1, canvas)

w = tk.Canvas(width=600,height=600)
sierp(300, 20, 550, 5, w)
w.grid()
w.mainloop()
```

Koch curve: koch.py

```
import tkinter as tk
from math import sqrt, atan2, cos, sin

def koch(coords, where, iterations):
  x0 = coords[where][0]
  y0 = coords[where][1]
  x1 = coords[where + 1][0]
  y1 = coords[where + 1][1]
  length = sqrt((x1 - x0)**2 + (y1 - y0)**2)
  r = length*1/((sqrt(3)))
  theta = atan2(y1 - y0, x1 - x0)
  phi = atan2(1, sqrt(3))
  xnew1 = x0 + length/3 * cos(theta)
  ynew1 = y0 + length/3 * sin(theta)
  xnew2 = x0 + r * cos(theta + phi)
  ynew2 = y0 + r * sin(theta + phi)
  xnew3 = x0 + 2*length/3 * cos(theta)
  ynew3 = y0 + 2*length/3 * sin(theta)

  coords.insert(where + 1, [xnew1, ynew1])
  coords.insert(where + 2, [xnew2, ynew2])
  coords.insert(where + 3, [xnew3, ynew3])

  if(iterations > 1):
    coords = koch(coords, where, iterations - 1)
    coords = koch(coords, where + 4**(iterations-1), iterations - 1)
    coords = koch(coords, where + 2*4**(iterations-1), iterations - 1)
    coords = koch(coords, where + 3*4**(iterations-1), iterations - 1)
  return coords

w = tk.Canvas(width=600,height=600)
coords = koch([[30, 300],[600 - 30, 300]], 0, 6)
for i in range(0, len(coords) - 1):
  w.create_line(coords[i][0], coords[i][1], coords[i + 1][0], coords[i + 1][1])
w.grid()
w.mainloop()
```

NCEA Level 2 Mathematics (Homework)
17. Number Sequences and Fractals

Reading

Go and watch...

Series of three videos:
https://www.youtube.com/watch?v=ahXIMUkSXX0
https://www.youtube.com/watch?v=lOIP_Z_-0Hs
https://www.youtube.com/watch?v=14-NdQwKz9w

What's it good for?

People use sequences, series, and fractals for...

- Science: the study of fractals and chaotic patterns are increasingly important in modern science. According to Wikipedia,[a] phenomena known to have fractal features include:
  - River networks
  - Fault lines
  - Mountain ranges
  - Craters
  - Lightning bolts
  - Coastlines
  - Mountain goat horns
  - Trees
  - Algae
  - Geometrical optics
  - Animal coloration patterns
  - Romanesco broccoli
  - Pineapple
  - Heart rates
  - Heart sounds
  - Earthquakes
  - Snowflakes
  - Psychological subjective perception
  - Crystals
  - Blood vessels and pulmonary vessels
  - Ocean waves
  - DNA
  - Soil pores
  - Rings of Saturn
  - Proteins
  - Surfaces in turbulent flows

[b]

- Mathematics: The behaviour of finite sequences and series is connected with combinatorics (like we saw last week and will see next week), while the behaviour of infinite sequences and series is connected with calculus.

[a] https://en.wikipedia.org/wiki/Fractal#Natural_phenomena_with_fractal_features
[b] By Jon Sullivan, http://pdphoto.org/PictureDetail.php?mat=pdef&pg=8232.

Questions

[This is a sample Ministry of Education L2 assessment task for this standard.]

This assessment activity requires you to create a fractal and use sequences and series to investigate features of the shape. Features of fractals include such things as length, area, number of items, volume.

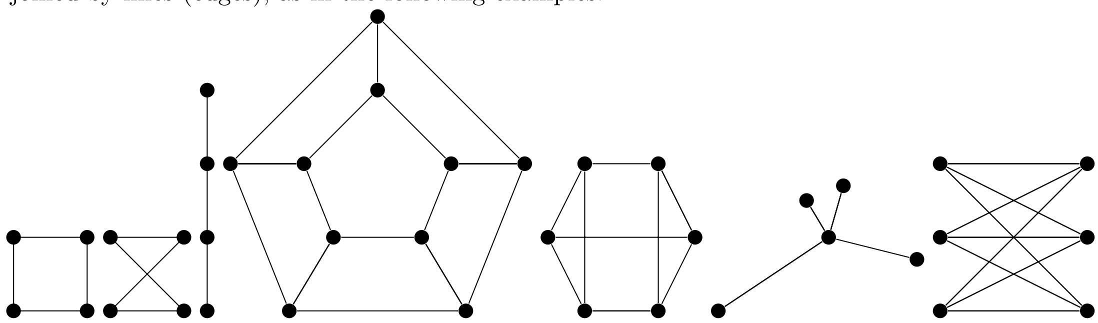
Create your own fractal. Include:

- Details of how the fractal is created, i.e. the initial unit segment or shape, and how your fractals are formed, including diagrams.
- The values generated for at least three stages (after the initial stage) of the fractal for at least two of the features of the fractal.
- The totals for at least two features of the fractal for any given stage.
- Describe what will happen to the values and totals for each feature as the number of iterations increases.
- For your chosen features, will there be a point where the next iteration makes no significant difference to the feature? Describe the conditions under which this might happen.

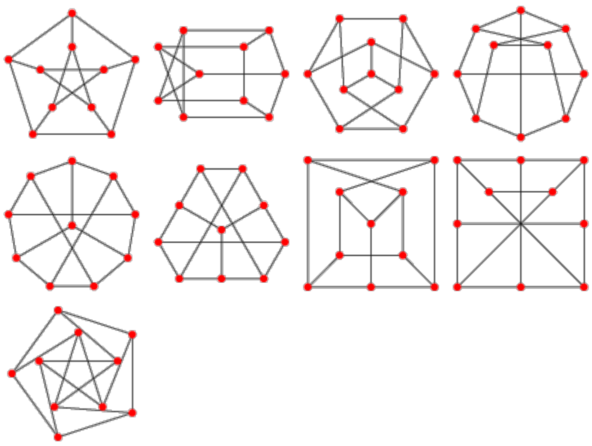
The quality of your reasoning and how well you link this context to generalisations of arithmetic and geometric sequences will determine the overall grade. Include calculations, diagrams or formulae, as appropriate. Clearly communicate your method using correct mathematical statements where appropriate.

NCEA Level 2 Mathematics
18. Graphs and Networks

The final topic that we will look at is the theory of graphs. A graph is a set of points (called vertices) that are joined by lines (edges), as in the following examples:



As we see at the left, the same graph can be drawn in different ways. The following nine drawings are all of the same graph, the Petersen graph.



In this topic, we will only consider graphs that are *finite*.

Definition.

- Two vertices are *adjacent* if there is an edge joining them.
- The *order* of a vertex is the number of edges incident to it.
- If we label the vertices, then each edge can be identified by its endpoints: if an edge joins a and b, we call the edge ab.
- A *path* on a graph between two vertices a and b is an ordered set of edges $av_1, v_1v_2, \dots, v_nb$ such that no edge is repeated.
- A *cycle* is a path on the graph between a and itself.
- If every two vertices are connected by some path, then the graph is called *connected*.
- If the graph has no cycles, it is called a *tree*.

Traversibility

Questions about paths and traversibility have been asked about graphs for hundreds of years. In 1736, Leonhard Euler solved the following problem:

The city of Königsberg in Prussia was set on a river, and included two large islands connected by bridges as in the following diagram.

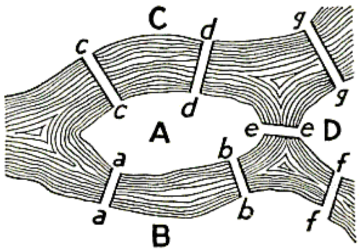
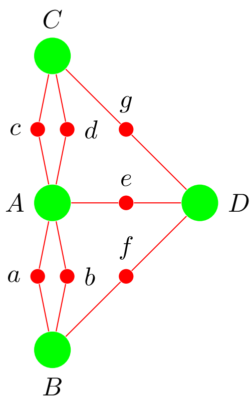


Figure 98. *Geographic Map: The Königsberg Bridges.*

Is it possible to walk a path in the city such that every bridge is crossed precisely once?

The answer, as we will see, is no. We begin by drawing a graph of the situation in order to eliminate all of the extraneous details beyond the connectedness of the city.



A path that traverses every edge of a graph exactly once is called an Eulerian path (and an Eulerian path that is also a cycle is called an Eulerian cycle).

Theorem (Euler)**.** *A connected graph has an Eulerian path if and only if it has either zero or two vertices with odd order.*

We will prove the 'only if' half here: that a graph has an Eulerian path only if it has zero or two vertices with odd order. The 'only if' part was proved by Carl Hierholzer in 1873.

Proof of necessity. Suppose a graph has an Eulerian path. Consider some vertex v that the path passes through. Then:

1. If the path does not have an endpoint on v, then v has even order because every edge at the vertex is an edge of the path, and each time the path enters the vertex it leaves (so all the edges at v can be paired up).

2. If the path has precisely one endpoint on v, it has odd order, because all the edges but the endpoint edge can be paired up as in (1).

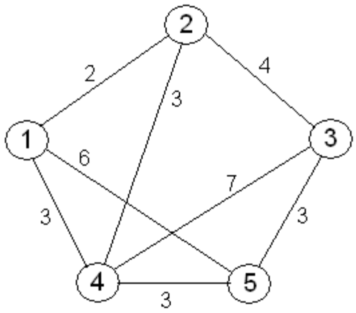
3. If the path has two endpoints at v, then v has even order, because we can pair the two endpoints together and then pair the other edges as in (1).

Every vertex has odd order, except if a vertex falls under case (2). But this can happen at most twice (a path has only two endpoints), and if both endpoints are at different vertices then case (2) applies to both. Hence a graph with an Eulerian path has either zero or two vertices of odd order. □

Hence the Königsberg challenge cannot be solved: there are more than two vertices of odd order.

Weighted graphs

Graphs can be used to model situations in subjects including computer science, scheduling, linguistics, and biology. For example, suppose some company has five distribution centres and wants to find an economical shipping pattern. Let's draw a graph with five vertices and label the edges with the cost of sending a truck (in hundreds of dollars) between the two joined centres:



(If we label edges like this, the graph becomes a *weighted graph*. The numbers are referred to as costs, weights, or distances.)

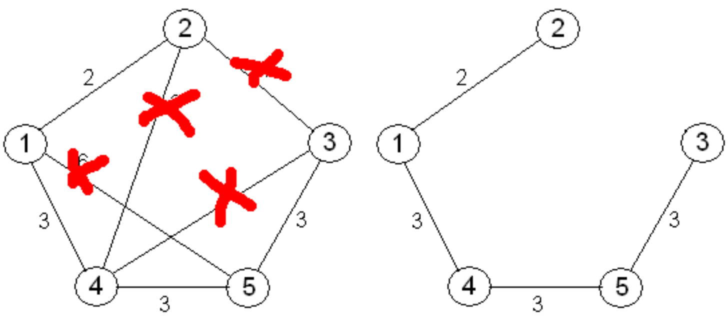
Suppose the company wants to send packages between every pair of distribution centrs, but does not want to pay to run all of the routes graphed. What are the edges that we can safely delete? In practice, we want to find a minimum spanning tree for this graph: a subgraph, without cycles, so that the total cost of the subgraph is a minimum. (Finding this graph would allow this company to minimise expendiature while not losing any connections.) One algorithm to find the minimum spanning tree is the reverse-delete algorithm:

Algorithm (Reverse-delete)**.**

1. List all of the edges in descending weight-order.
2. Examine each edge in order, starting with the most expensive. If deleting this edge disconnects the graph, do not remove the edge; otherwise, remove the edge.

The remaining graph is a minimum spanning tree.

For example, in this case we obtain the following minimum spanning tree:



On the other hand, suppose the company does not run all its routes constantly and instead wants to find the cheapest route for a particular package from one centre to another, utilising any of the eight edges above. This can be done with Dijkstra's algorithm:

Algorithm (Dijkstra)**.** Suppose we want to find the shortest path from some vertex a to some vertex b. In fact, this algorithm will give us the shortest path from a to *any* other vertex!

Label vertex a with zero and every other vertex as ∞; then set a as the current vertex, and:

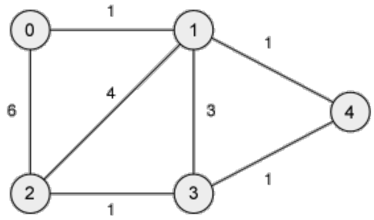
1. Consider every unvisited vertex v directly adjacent to the current vertex, and update the label of v to the smallest value out of:
   - Either the current value on v, or
   - The sum of the current value on the current vertex and the distance from the current vertex to v.
2. If the vertex v has been relabeled by step (1), then mark with an arrow the edge joining the current vertex with v and unmark any other edges on v that have been marked in previous steps.
3. Label the current vertex as visited.
4. If every vertex is visited, then we halt; otherwise, set the current vertex to the vertex with minimal label and repeat from step (1).

Each vertex v is now labeled with the minimum distance from a to v, and the shortest path from a to v is marked by the arrows.

Questions

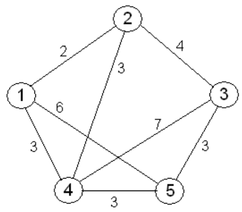
1. Justify, with mathematical reasoning, the following statements.
   (a) A graph that is not connected has no Eulerian paths.
   (b) If a graph has no vertices of odd order, all Eulerian paths are circuits; if there are two vertices with odd order, all Eulerian paths begin at one and end at the other.
   (c) The sum of all the orders of all the vertices must be even.
2. Consider the Königsberg bridge graph.
   (a) Show that it is possible to exhibit a Eulerian path on the graph resulting from adding a single extra bridge.
   (b) Show that, no matter where the bridge is placed, there will still be no Eulerian circuit.
3. Suppose there are three houses on a flat plane (or plain), and each needs to be connected to water, gas, and electricity.
   (a) Show that this is impossible without two connections crossing, without using a third dimension or running a connection through a house.
   (b) Show that, on a torus (a doughnut), it *is* possible.
4. Legend has it that an ancient Indian lord had five sons, and he allowed them to split his land up after his death between them, on the proviso that the land of each son must be in one piece, and must share a boundary with the land of all four other sons. Why is this funny?

5. Consider the following weighted graph G.

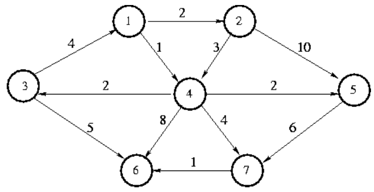


(a) Give an example of a graph with more than one minimal spanning tree.
(b) Show that G has precisely one minimal spanning tree.
(c) Find a vertex v of G such that the minimal spanning tree of the graph also gives the shortest paths from v to every other vertex.

6. Find the cheapest shipping path between distribution centres 1 and 3 in the example of a weighted graph from above (reproduced here).

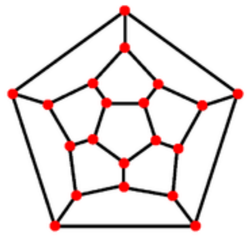


7. Suppose that, in addition to weights on edges of a graph, we assign a direction: perhaps we are modelling the direction and velocity of some fluid flow. Consider the following directed graph:



(a) Call a directed graph connected if we can give a directed path from every vertex a to every other vertex b (i.e. a path with arrows from a to b). Explain why this directed graph is not connected.
(b) Find the shortest path from vertex 4 to every other vertex.

8. A **Hamiltonian path** is a path on a graph such that every vertex appears on the path exactly once. If the two endpoints of the path are adjacent, then the path is called a **Hamiltonian cycle**. Find a Hamiltonian cycle on the dodecahedron graph:



The existence of Hamiltonian paths on graphs is much more difficult than the existence of Eulerian paths. One recent result (from 2005) is that a graph with n vertices has a Hamiltonian path if, for every pair of non-adjacent vertices, the sum of their degrees and the distance between them is greater than n.

NCEA Level 2 Mathematics (Homework)
18. Graphs and Networks

Reading

Go and watch...

https://www.youtube.com/watch?v=CruQylWSfoU

What's it good for?

People use graphs and networks for...

- Computer science: modelling computer networks, modelling connections between objects (Google's search algorithm is based on weighted graphs).
- Engineering: electrical circuits can be modelled with weighted graphs.
- Linguistics: syntax trees and semantic networks are used when modelling language and the changes of languages over time.
- Physics and chemistry: graphs can be used to model crystal structures or the structures of molecules (especially in organic chemistry).
- Mathematics: graph theory has tight links with topology and knot theory. The four-colour theorem (the vertices of any graph that can be drawn with no edge crossings can be coloured with at most four colours so no adjacent vertices have the same colour) was only proved in the latter half of the 20th century, and was the first major proof to involve considerable work by computers.

Questions

[This is a sample internal assessment task for this standard, based on those provided by the Ministry of Education.]

This assessment activity requires you to apply graphs and networks to a real-world situation. Read the entire activity before beginning work.

The Lower North Island Logistics Company (LNILC) wants to streamline its business by redesigning its logistics network. In Resource A, a list of town/city pairs in the current network and travel times by truck between them are given.

1. Draw a weighted graph modelling the information in Resource A.

2. Find the shortest route (in terms of travel time) between Carterton and Marton.

3. The cost of providing a truck between two of the city pairs is directly proportional to the travel time.

   (a) Determine any links that will (i) be in *every* minimal spanning tree, and (ii) will be in *no* minimal spanning tree.

   (b) Determine the smallest possible network that covers all the destinations, and costs the least to maintain.

4. LNILC is considering signing a deal with KiwiRail that provides rail services between the following city pairs:

   - Wellington to Palmerston North: 135 minutes
   - Palmerston North to Whanganui: 50 minutes
   - Whanganui to New Plymouth: 50 minutes
   - Palmerston North to Hastings: 110 minutes
   - Wellington to Carterton: 80 minutes
   - Carterton to Masterton: 18 minutes
   - Masterton to Hastings: 140 minutes
   - Masterton to Palmerston North: 65 minutes

   How do your answers above change, incorporating these new links?

5. Produce a final report, incorporating your findings above, that could recommend:

   - If any truck links should not be maintained;
   - Which, if any, rail links should be included in the future network;
   - Any other changes that could be made to improve the cost-effectiveness of the entire network, and minimise the time taken for packages to travel between particular major towns and cities. (For example, you may recommend that a particular link be kept despite not appearing in your minimum spanning tree if it significantly reduces the distance between two key destinations.) Resource B may be useful for this.

The quality of your reasoning and how well you link this context to graph and network methods will determine the overall grade. Include calculations, diagrams or formulae, as appropriate. Clearly communicate your method using correct mathematical statements where appropriate.

Resource A: town-city pairs with times

All links are bi-directional and all travel times are in minutes.

City 1	City 2	Travel time by road
Wellington	Porirua	27
Wellington	Lower Hutt	22
Lower Hutt	Wainuiomata	12
Lower Hutt	Upper Hutt	22
Porirua	Upper Hutt	28
Porirua	Kapiti	30
Kapiti	Upper Hutt	68
Upper Hutt	Martinborough	49
Upper Hutt	Carterton	51
Carterton	Masterton	15
Martinborough	Masterton	43
Masterton	Dannevirke	77
Masterton	Palmerston North	79
Kapiti	Otaki	20
Otaki	Levin	18
Levin	Palmerston North	40
Levin	Whanganui	74
Palmerston North	Whanganui	56
Palmerston North	Fielding	16
Fielding	Marton	25
Marton	Whanganui	29
Palmerston North	Dannevirke	54
Dannevirke	Napier	89
Dannevirke	Hastings	74
Napier	Hastings	24
Whanganui	Hawera	68
Hawera	Stratford	26
Stratford	New Plymouth	33
Hawera	New Plymouth	90

Resource B: major towns and cities in the Lower North Island

The following urban areas served by LNILC have populations over 10,000:

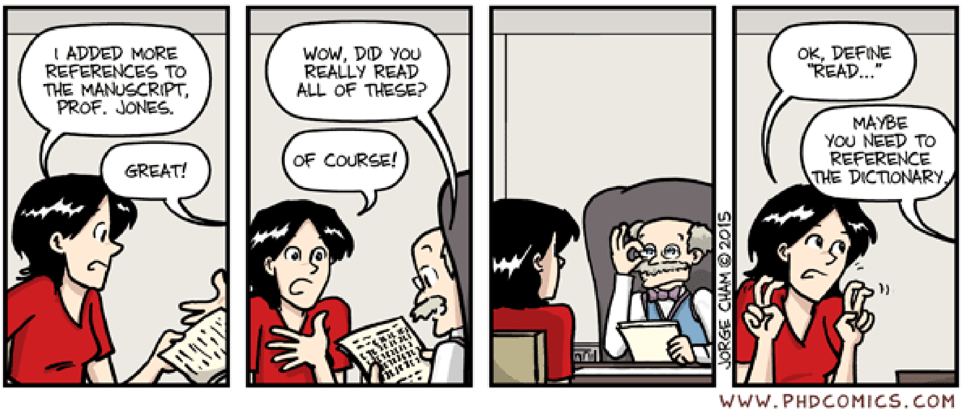
- Wellington (207,900)*
- Napier-Hastings (133,000)*
- Lower Hutt (104,700)
- Palmerston North (85,300)
- New Plymouth (57,500)*
- Porirua (55,900)
- Upper Hutt (43,200)
- Kapiti (42,300)
- Whanganui (40,300)*
- Masterton (21,800)
- Levin (20,900)
- Fielding (16,550)

(*Urban area served by a port.)

CHAPTER 6

Addendum

NCEA Level 2 Mathematics Bibliography



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