NCEA Level 3 Calculus (Integration)
26. More Interesting Problems

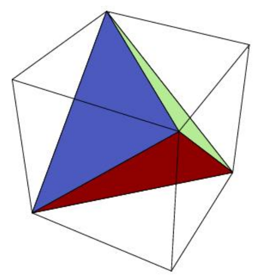
These problems do not just concern integration.

Questions

1. Find the equation of the line through the point $(3, 5)$ which cuts off the least area from the first quadrant. E

2. The area of a square is increasing at a constant rate of k m^2 per second. A tetrahedron (equilateral triangular pyramid), has a side length that is the same as that of the square at each instant. The initial volume of the tetrahedron was $1\,\mathrm{m}^3$. In terms of k, what is the volume of the tetrahedron three seconds after that? E

3. Consider the tetrahedron inscribed inside a cube, as in the figure. S



   The volume V of the cube at any instant t is increasing at a rate proportional to the value of V at that instant. The initial volume of the cube at $t = 0$ was 8 cubic units. What is the volume of the tetrahedron at time $t = 20$?

4. If $x \sin \pi x = \int_0^{x^2} f(t)\,\mathrm{d}t$, where f is continuous, find $f(4)$. [*Hint: you need not perform any integration.*] S

5. If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that S

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.$$

   Hence, or otherwise, evaluate

$$\lim_{x \to 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}.$$

6. (a) Consider the differential equation S

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + 5\frac{\mathrm{d}\Phi}{\mathrm{d}t} + 6\Phi(t) = 0.$$

   Let f and g be the functions defined by $f(x) = e^{-2x}$ and $g(x) = e^{-3x}$.
   i. Show that all linear combinations of f and g are solutions to the differential equation.
   ii. Find the (unique) solution passing through $(0, 1)$ and $(1, 1)$.

   (b) More generally, consider the differential equation $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$. Let the zeroes of the quadratic polynomial $p(D) = aD^2 + bD + c$ be α and β. Show that all the linear combinations of $e^{\alpha x}$ and $e^{\beta x}$ are solutions to the differential equation. O

7. Compute the following definite integral. [*Hint: begin with a substitution.*] S

$$\int_0^{\pi/6} \sqrt{\tan\theta}\,\mathrm{d}\theta$$

8. (a) Consider the two functions $p(x) = 3x^5 - 5x^3 + 2x$ and $q(x) = 3x^5$. Show that their ratio approaches 1 as $x \to \infty$. E

   (b) Let $p(x)$ and $q(x) \neq 0$ be polynomials. Recall that the degree of a polynomial is the highest n such that x^n has a non-zero coefficient. Compute the limit S

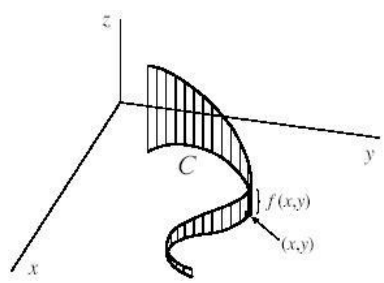
$$\lim_{x\to\infty} \frac{p(x)}{q(x)}$$

   if:

   i. the degree of $p(x)$ is less than that of $q(x)$.

   ii. the degree of $p(x)$ is greater than that of $q(x)$.

9. A definite integral calculates the between a curve and straight line, the x-axis. In a similar way, it is possible to use integration to calculate the area above a curved line and below a surface $z = f(x, y)$, like that in the figure. S



If the curve C is defined parametrically, that is $C(t) = (x(t), y(t))$, then the integral along the line can be calculated with the formula

$$\int_a^b f(x(t), y(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Compute the line integral of the function $f(x, y) = 2 + x^2y$ around the upper half of the unit circle.

10. The **sine integral** function is defined by S

$$\mathrm{Si}(x) = \begin{cases} \int_0^x \frac{\sin t}{t}\,\mathrm{d}t, & \text{for } x \neq 0; \\ 1, & \text{for } x = 0. \end{cases}$$

   (a) Recall that $\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a)$. Use this to show that $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f'(t)\,\mathrm{d}t = f'(x)$.

   (b) Find the x-coordinate of the first local maximum of Si to the right of the origin. Carefully prove that you have found a maximum.

   (c) Use the result in (a) to find an expression for the integral

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)\,\mathrm{d}t,$$

   where f is continuous and g and h are differentiable.

11. Minimise the function $f(x) = b\log_b N$ with respect to b, and show that the result is independent of the constant N.* E

12. We can calculate **improper integrals** (those where the bounds are infinite) as follows: S

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x.$$

Decide whether each of the integrals given below exists. If so, calculate its value; if not, explain why.

(a) $\displaystyle\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ (b) $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ (c) $\displaystyle\int_1^\infty \sin x\,\mathrm{d}x$

13. (a) Show that $F(x) = \tan^{-1} x$ is an anti-derivative of $f(x) = \frac{1}{1+x^2}$ in the following ways: S
    i. Differentiate $F(x)$ and simplify to give $f(x)$.
    ii. Use the substitution $x = \tan\theta$ to integrate $f(x)$ and simplify to give $F(x)$.

    (b) Recall that $22/7$ is often given as a rough approximation to π. Consider the integral O

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x,$$

and hence show that $22/7 > \pi$.†

14. Consider the operator $\mathcal{L}$ defined by S

$$\mathcal{L}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\left[f\left(e^x\right)\right].$$

(a) Show that $\mathcal{L}x^n = n$ and that $\mathcal{L}[u(x)]^n = n\mathcal{L}u(x)$.
(b) Find an expression for $\mathcal{L}[u(x)v(x)]$ and $\mathcal{L}[u(x)/v(x)]$.
(c) Find an expression for $\mathcal{L}[u(x)+v(x)]$.
(d) For which y is $\mathcal{L}y = y$?

15. Compute the following indefinite integrals: S

(a) $\displaystyle\int \frac{\sin\frac{1}{x}}{x^2}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{\ln x\cos x - \frac{\sin x}{x}}{(\ln x)^2}\,\mathrm{d}x$

16. (Harder!) Suppose ι is a function defined as O

$$\iota(x) = \int_0^x t^t \sin(t)\,\mathrm{d}t.$$

(a) Write down the domain and range of ι (caution: be careful for negative fractional x).

* Dudley, *Mathematical Cranks*, p.52.
† Nahin, *Inside Interesting Integrals*, pp.23-4.

(b) Suppose I is an antiderivative of $t^t \sin(t)$. Show that

$$\iota^{-1}(x) = I^{-1}(x + I(0))$$

is the only possibility for the inverse of ι, if the inverse exists.

(c) Find the derivative of ι with respect to x. Hence, show that ι changes from decreasing to increasing at an odd number of points within the interval $(16, 20)$. Conclude that ι^{-1} is not a function, and hence ι has no well-defined inverse.

17. A while ago (when we talked about the product and quotient rules), I claimed that the radius of the circle best approximating a continuous curve around a point (x, y) is given by 0

$$\text{radius of curvature} = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|}.$$

Let us attempt to prove this.

(a) Let f be a continuous function at x such that the second derivative of f at x exists. By recalling our work on approximations, explain why knowing up to the second derivative of f should be enough to find the 'best circular approximation' of f at $(x, f(x))$.

(b) Consider the circle of radius r centred at (x_0, y_0). Suppose that this circle passes through the point (x_1, y_1); suppose further that the first derivative of the y-ordinate of the circle with respect to the x-ordinate is m, and that the second derivative is c. Write down expressions for r, x_0, and y_0 in terms of x_1, y_1, m, and c.

(c) Use part (b) to write down the radius of the unique circle passing through $(x, f(x))$ with matching first and second derivatives to f.

18. We prove that π is irrational.[‡] Suppose that $\pi = \frac{a}{b}$ where a and b are positive integers. Let n be a positive integer, and define 0

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$
$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x).$$

(a) Show that the value of f and every derivative of f at $x = 0$ and $x = a/b$ is integral. Conclude that $F(\pi) + F(0)$ is an integer.

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}[F'(x)\sin x - F(x)\cos x] = f(x)\sin x$, and hence that

$$\int_0^\pi f(x)\sin x = F(\pi) + F(0). \tag{*}$$

(c) Prove that $0 < f(x)\sin x < \frac{\pi^n a^n}{n!}$, and conclude that (*) is positive but arbitrarily small for sufficiently large n.

(d) Derive a contradiction from (a) and (c).

[‡] Ivan Niven, *A simple proof that π is irrational.*