# Solutions to L3 Calculus Differentiation Exam 2

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# Question One

## Part (a)

$$\begin{aligned}\frac{\mathrm{d}D}{\mathrm{d}t} &= -\omega A\cos(kx-\omega t+\phi_0)\\ \frac{\mathrm{d}D}{\mathrm{d}x} &= kA\cos(kx-\omega t+\phi_0)\end{aligned}$$

(2 marks)

## Part (b)

$$\begin{aligned}\tan y &= e^x\\ \frac{\mathrm{d}y}{\mathrm{d}x}\sec^2 y &= e^x\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= e^x\cos^2 y\end{aligned}$$

So at $(0, \frac{\pi}{4})$ we have $\frac{\mathrm{d}y}{\mathrm{d}x} = e^0\cos^2\frac{\pi}{4} = \frac{1}{2}$. (3 marks)

## Part (c)

We have that the angle of the slope of the cone with the base is $\tan^{-1}\frac{H}{R}$ and so the height of the cylinder at radius $r$ is $h = (R-r)\frac{H}{R} = H(1-\frac{r}{R})$. The volume of the cylinder is $V = \pi r^2 h = \pi r^2 H(1-\frac{r}{R}) = H\pi r^2 - \frac{H\pi}{R}r^3$. Taking the derivative, $\frac{\mathrm{d}V}{\mathrm{d}r} = 2H\pi r - \frac{3H\pi}{R}r^2$; setting to zero, $2H\pi r = \frac{3H\pi}{R}r^2$ and so (since $r \neq 0$) $r = \frac{2R}{3}$.

We also have $\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = 2H\pi - \frac{6H\pi}{R}r$, so at our found radius $\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = 2H\pi - 4H\pi < 0$ and so the function is concave down at that point — we have indeed found a maximum.

(5 marks)

# Question Two

## Part (a)

### (i)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^{-x}}{x^2} - \frac{e^{-x}}{x} - \frac{\sin x}{x^2} + \frac{\cos x}{x}.$$

(2 mark)

### (ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{2}x^{3/2} + \frac{15}{2}x^{1/2} - \frac{1}{2}x^{-3/2}.$$

(1 mark)

## Part (b)

**(i)**

$$\frac{\mathrm{d}}{\mathrm{d}x}x^3 = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$$

(1 mark)

**(ii)** Noticing that we are just finding the value of the derivative of $x^3$ at $x = 2$, we evaluate $3x^2$ at 2 and obtain 12 as the value of the limit. Alternatively, one could expand the brackets and evaluate the limit algebraically. (2 marks)

## Part (c)

$$f'(x) = \frac{x^4}{4} - 2x^2 + 16$$
$$f''(x) = x^3 - 4x = x(x^2-4) = (x-2)x(x+2)$$

Since $f''(x)$ is a positive conic, it is negative on the intervals $x < -2$ and $0 < x < 2$ and positive on the intervals $-2 < x < 0$ and $x > 2$. Hence the function $f$ is concave up on the latter two intervals. (4 marks)

# Question Three

## Part (a)

$$g'(t) = 6t \cdot \frac{\pi}{2\sqrt{3t^2+4}} \cdot \cos(\pi\sqrt{3t^2+4}).$$

We therefore have $g'(2) = 12 \cdot \frac{\pi}{2\sqrt{16}} \cdot \cos(\pi\sqrt{16}) = 12 \cdot \frac{\pi}{8} \cdot 1 = \frac{3\pi}{2}$. (2 marks)

## Part (b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 9x^2 - 4x + 2$$

Therefore at the point $(0, -1)$ the gradient is $m = 2$ and the normal line will have a slope of $-\frac{1}{2}$. Hence the equation of the normal is $(y+1) = -\frac{1}{2}x$, or $y = -\frac{1}{2}x - 1$. (3 marks)

## Part (c)

We have $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}t} \cdot (-\frac{1}{\sqrt{S}}) = -\frac{3t+4}{\sqrt{S}}$. This is zero exactly when $3t+4 = 0$, which is when $t = -\frac{4}{3} < 0$. (2 marks)

## Part (d)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 36x + 90$$

Suppose $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$. Then $6x^2 - 36x + 86 = 0$. But $36^2 - 4 \times 6 \times 86 < 0$, so the gradient of the curve is never 4. (3 marks)