

Solutions to L3 Calculus Integration Exam 3

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Question One

Part (a)

- i. Note that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, so we must have

$$\int \frac{1}{\sqrt{2x+2}} dx = \int \frac{2}{2\sqrt{2x+2}} dx = \sqrt{2x+2} + C.$$

- ii. Let $u = \tan x$ so $du = \sec^2 x dx$ and our integral becomes

$$\int \sec(u) \tan(u) du = \sec u + C = \sec \tan x + C.$$

Part (b)

First, we find the left-hand integration limit: $1/x = 1/x^2$ so $1 = 1/x$ and $x = 1$.

$$\begin{aligned} \int_1^2 \frac{1}{x} - \frac{1}{x^2} dx &= \left[\ln x + \frac{1}{x} \right]_1^2 \\ &= (\ln 2 - \ln 1) + \left(\frac{1}{2} - 1 \right) \\ &= \ln \frac{1}{2} - \frac{1}{2}. \end{aligned}$$

Part (c)

Separation of variables.

$$\begin{aligned} \int \frac{\cos y}{\sin y} dy &= \int \frac{\sin x}{\cos x} dx \\ \ln \sin y &= -\ln \cos x + C \\ \sin y &= Ke^{\ln \sec x} = K \sec x \\ y &= \sin^{-1}(K \sec x). \end{aligned}$$

(where $K = e^C$.)

Question Two

Part (a)

x	$(x^2e^{-x})^2$
1.0	0.1353
1.5	0.2520
2.0	0.2931
2.5	0.2632
3.0	0.2008
3.5	0.1368
4.0	0.0859
4.5	0.0506
5.0	0.0284

Note that $n = 8$ (number of intervals is one less than numbers of points), $a = 1.0$, $b = 5.0$. Hence:

$$\begin{aligned}\int_1^5 (x^2e^{-x})^2 dx &\approx \frac{5-1}{8} [0.1353 + 0.0284 + 4(0.2520 + 0.2632 + 0.1368 + 0.0506) + \\ &\quad + 2(0.2931 + 0.2008 + 0.0859)] \\ &= 2.0669\end{aligned}$$

and the required value is 6.4932.

Part (b)

We have $5 = \left[\frac{1}{2}(\ln x)^2 \right]_1^k = \frac{1}{2}(\ln k)^2$, so $k = \exp(\sqrt{10})$.

Part (c)

Let $B(t)$ be the bank balance after t years. Then:

$$\frac{dB}{dt} = 0.04B,$$

so $\ln B = 0.04t + C \Rightarrow B = B_0e^{0.04t}$ ($B_0 = e^C$). We have $t = 4$ and $B_0 = 2500$, so $B(4) = 2500e^{0.16} = \$2933.78$.

Question Three

Part (a)

Computing:

$$\begin{aligned}\frac{\int_0^{12} 20 + 8 \sin\left(\frac{\pi t}{12}\right) dt}{12} &= \frac{1}{12} \left[20t - \frac{96}{\pi} \cos\left(\frac{\pi t}{12}\right) \right] \Big|_0^{12} \\ &= \frac{1}{12} \left[\left(240 + \frac{96}{\pi} \right) + \frac{96}{\pi} \right] \\ &= 25^\circ\text{C}.\end{aligned}$$

Part (b)

The area of the cross-section at height x is $A = (3 - 0.1x)(4 - 0.2x) = 0.02x^2 - x + 12$; hence the total volume is:

$$\int_0^{18} 0.02x^2 - x + 12 dx = \frac{0.02 \times 18^3}{3} - \frac{18^2}{2} + 12 \times 18 = 92.88 \text{ m}^3.$$

Part (c)

$$\begin{aligned}1 &= \int_m^{2m} x \cos(mx^2) dx \\ &= \left[\frac{1}{2m} \sin(mx^2) \right] \Big|_m^{2m} \\ &= \frac{1}{2m} (\sin 4m^3 - \sin m^3) \\ &= \frac{1}{2m} 2 \cos \frac{5m^3}{2} \sin \frac{3m^3}{2} \\ m &= \cos \frac{5m^3}{2} \sin \frac{3m^3}{2}.\end{aligned}$$

Since $\forall x$ we have $\cos x \leq 1$ and $\sin x \leq 1$, it follows that $\cos \frac{5m^3}{2} \sin \frac{3m^3}{2} \leq 1$ and so $m \leq 1$.