

NCEA Level 3 Calculus (Integration)

17. The Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem)

Suppose f is a continuous function, and suppose F is any antiderivative of f (so $F' = f$). Then:

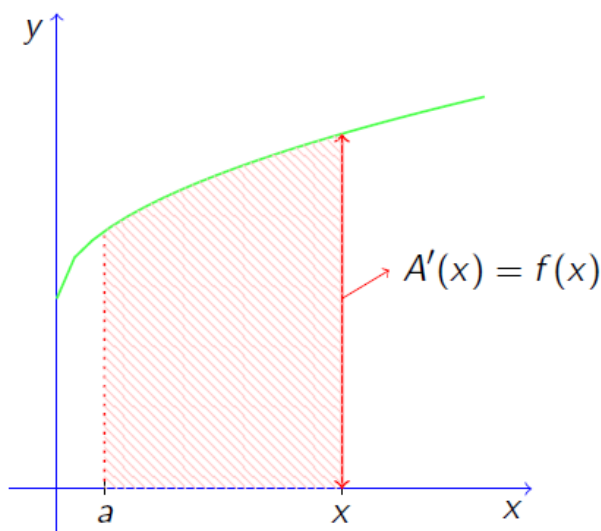
$$\int_a^b f(x) \, dx = F(b) - F(a) = F(x) \Big|_a^b$$

In other words, the definite integral of a function can be found by evaluating the indefinite integrals at the endpoints.

Theorem (Second Fundamental Theorem)

Suppose f is a continuous function. Then:

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$$



The rate of change of the area under a curve is simply the height of the curve.

We also have the following theorem:

Theorem. Suppose f, g are functions and λ is a real constant. Then:

1. $\lambda \int_a^b f(x) \, dx = \int_a^b \lambda f(x) \, dx.$
2. $\int_a^b f(x) \, dx + \int_a^b g(x) \, dx = \int_a^b f(x) + g(x) \, dx.$
3. $\int_a^a f(x) \, dx = 0.$
4. $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx.$

Note that the areas below a curve are assigned **NEGATIVE AREA**.

Example. Note that the definite integral $\int_0^\pi \cos(x) \, dx$ is equal to 0; this is because the (negative) area under the x -axis exactly cancels the (positive) area above the x -axis. (Draw a picture.)

Questions

1. A Compute the following definite integrals.

(a) $\int_0^1 dx$

(b) $\int_{-1}^1 e^x dx$

(c) $\int_3^4 x^2 + 3x - 1 dx$

(d) $\int_0^1 x^n dx$ for integer values of n .

2. A Find the area underneath the given curves between the given bounds:

(a) $y = 6x^2 + 4x + 9$ between $x = 0$ and $x = 4$

(b) $y = \sin x$ between $x = 0$ and $x = \pi$

(c) $y = \sin x$ between $x = -\pi$ and $x = \pi$

(d) $y = \cos x$ between $x = -\pi$ and $x = \pi$

(e) $y = \frac{1}{x}$ between $x = 1$ and $x = 2$

3. A Find all the problems in the following working.

$$\int_1^{-1} \frac{dx}{x} = \ln|-1| - \ln|1| = 0$$

4. A Show that $\int \ln x dx = x \ln x - x + C$.

5. M Let f be a function such that for all x , $f(-x) = -f(x)$. Such a function is called *odd*. Show that for all a ,

$$\int_{-a}^a f(x) dx = 0.$$

What does this mean geometrically?

6. M Let f be an odd function with period 2 such that $\int_0^1 f(x) dx = k$. Compute:

(a) $\int_{-1}^1 f(x) dx$

(b) $\int_0^{-1} f(x) dx$

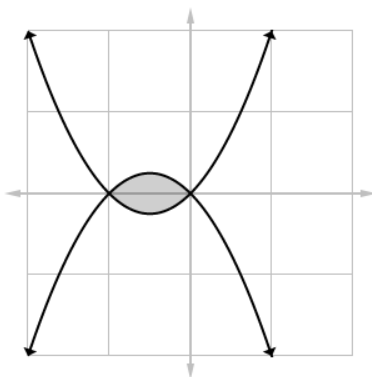
7. A Let f be a function such that for all x , $f(-x) = f(x)$. Such a function is called *even*. Show that for all a ,

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx.$$

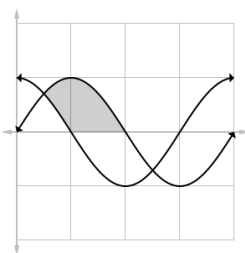
What does this mean geometrically?

8. M If $\int_{-2}^1 f(x) dx = 2$ and $\int_1^3 f(x) dx = -6$, what is the value of $\int_{-2}^3 f(x) dx$?

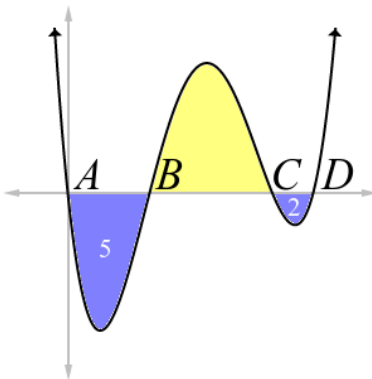
9. M Find the area between the curves $y = x^2 + x$ and $y = -x^2 - x$ shaded here.



10. M Find the area between the two curves $y = 1 + x^2$ and $y = 3 + x$.
11. M Find the area of the region bounded by $f(x) = 4$, $g(x) = \frac{e^x}{5}$, and $x = 0$.
12. M What is the area of the region between the graphs of $f(x) = 2x^2 + 5x$ and $g(x) = -x^2 - 6x + 4$ from $x = -4$ to $x = 0$?
13. E Find the area bounded by the curves $y = \sin x$ and $y = \cos x$ and the x -axis graphed here.



14. A Consider the function f graphed below; the total **unsigned** area between the curve and the x -axis is 10 square units. Find $\int_A^D f(x) dx$.



15. M (a) Sketch the graph of $y = |\sin x|$.
- (b) Compute $\int_0^{\pi/2} y dx$ using the FTC.
- (c) Hence, without doing any anti-differentiation, compute $\int_0^{2\pi} y dx$.

16. M Define $F(x)$ by

$$F(x) = \int_{\frac{\pi}{4}}^x \cos(2t) \, dt.$$

- (a) Use the Second Fundamental Theorem of Calculus to find $F'(x)$.
(b) Verify part (a) by integration and differentiation.
17. M Compute $\frac{d}{dx} \int_2^x t^t \, dt$.