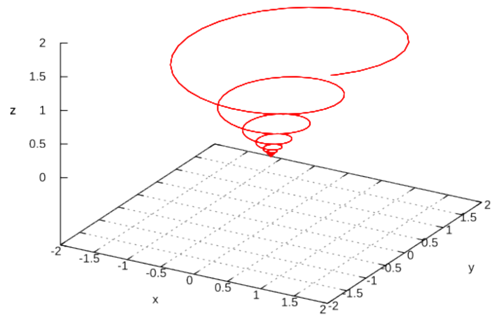
NCEA Level 3 Calculus (Differentiation)
10. Parametric Functions



Some curves cannot be described simply with a function; for example, the above track of a particle is too complicated to analyse using any of the techniques which we have studied so far. One strategy which does work is to split the x, y, and z components apart and study them seperately. For example, we can *parameterise* the above curve as:

$$\begin{aligned} x(t) &= e^{-t}\cos(10t) \\ y(t) &= e^{-t}\sin(10t) \\ z(t) &= e^{-t}. \end{aligned}$$

For a simper example, consider the unit circle $x^2 + y^2 = 1$. By recalling the definitions of the trigonometric functions, we can parameterise the circle as $(x, y) = (\cos\theta, \sin\theta)$ for $0 \leq \theta < 2\pi$. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = -\cos\theta = -x$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \sin\theta = y$, so by the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{x}{y}$ — a much simpler calculation than taking the derivative of the square root required by working directly with the circle formula.

In general, we have

$$d\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$$

In order to find the second derivative, we replace y with $\frac{\mathrm{d}y}{\mathrm{d}x}$:

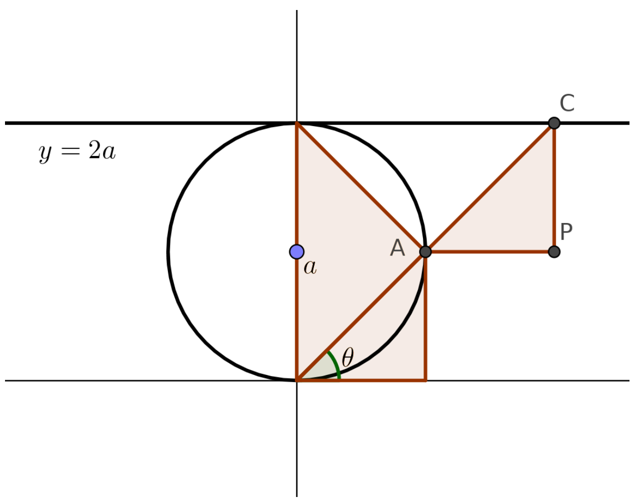
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}\frac{\mathrm{d}y}{\mathrm{d}x}}{\mathrm{d}x} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}\right) \cdot \frac{\mathrm{d}t}{\mathrm{d}x}\,.$$

Questions

1. $\boxed{\texttt{M}}$ In each case find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
   (a) $x = t\sin t$, $y = t^2 + t$
   (b) $x = 2\sec\theta$, $y = 3\tan\theta$
   (c) $x = \cos\theta$, $y = \cos 3\theta$
   (d) $x = e^{\sin\theta}$, $y = e^{\cos\theta}$
2. $\boxed{\texttt{A}}$ Find the equation of the chord joining the two points $t = 2$ and $t = 4$ on the curve $(x, y) = (2t - 3, t^3 + 6)$.
3. $\boxed{\texttt{M}}$ Determine the point(s) of intersection of the curves γ and δ:

$$\gamma : t \mapsto (t^2 - 2, t - 1)$$
$$\delta : t \mapsto (t, 2/t)$$

4. (a) $\boxed{\texttt{M}}$ If $y = 2t$ and $x = 4t^2$ define a curve, what is the gradient $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t?
   (b) $\boxed{\texttt{E}}$ Show that this curve is a parabola.
5. $\boxed{\texttt{M}}$ A curve has parametric equations $x = t^2 + 1$ and $y = t^3 + 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.
6. $\boxed{\texttt{M}}$ Find the equation of the tangent to the curve $t \mapsto (2x^2 + 1, t^3 - 1)$ at $t = 2$.
7. $\boxed{\texttt{E}}$ If $t \mapsto (x, y)$ is a parametric curve, find an expression for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ analagous to that found for the second derivative.
8. $\boxed{\texttt{S}}$ A curve, called a *witch of Maria Agnesi*, consists of all possible positions of the point P in the diagram below. Show that the curve is given by $(x, y) = (2a\cot\theta, 2a\sin^2\theta)$ and find the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$.



9. A particle moves through space over time; the position of the particle at time t is given by $(3\sin t, 2\cos t)$ $(0 \leq t < 2\pi)$.
   (a) $\boxed{\texttt{A}}$ What is the component of the acceleration of the particle in the x direction at $t = \pi/4$?
   (b) $\boxed{\texttt{A}}$ At what times is the particle stationary in the x direction?
   (c) $\boxed{\texttt{M}}$ Is the particle ever momentarily totally stationary?

10. $\boxed{\text{E}}$ Find the rightmost point on the curve $x = t - t^6$, $y = e^t$.

11. $\boxed{\text{E}}$ For which values of t is the curve $x = \cos 2t$, $y = 3\cos t$ concave up?

12. $\boxed{\text{S}}$ Show that the curve $\gamma : t \mapsto (\cos t, \sin t \cos t)$ has two tangents at (0,0) and find their equations.

13. $\boxed{\text{S}}$ Scholarship 2000: The piriform is the curve defined by the equation $16y^2 = x^3(8 - x)$ where $x \geq 0$.

    (a) Show that

    $$\begin{cases} x = 4(1 + \sin\theta) \\ y = 4(1 + \sin\theta)\cos\theta. \end{cases}$$

    are parametric equations for the piriform.

    (b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ, and show that $\theta = \frac{\pi}{6}$ is a stationary point of the curve.