

Level Three Conic Sections — Notes and solutions

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The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months—when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado! A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it... yet it finally surrounds the resistant substance.

(Grothendieck)

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Preface

This is a set of notes and outlines of solutions to the various exercises in the conic section notes.

Notes on Section 2: Basic results

Exercise 1: Polar form.

Show that, if X is on the directrix of a conic such that $OX \perp \ell$, then the polar equation of the conic with respect to this axis and origin O is

$$\frac{l}{r} = 1 + \varepsilon \cos \theta.$$

Conclude that:

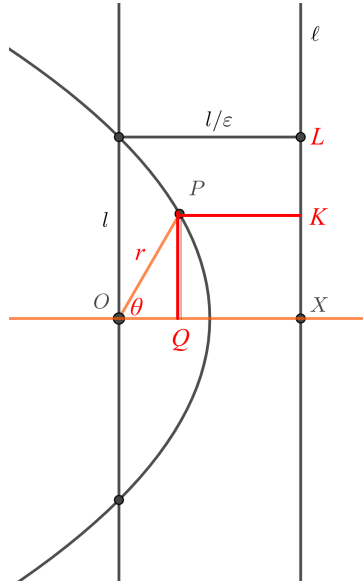
1. Every conic is symmetric with respect to OX .
2. The ellipse is a closed and bounded curve (i.e. it does not extend towards infinity).
3. The parabola is unbounded, but is connected.
4. The hyperbola consists of two separate branches, each extending to infinity, given by $-\alpha < \theta < \alpha$ and $\alpha < \theta < 2\pi - \alpha$ (where $\alpha = \operatorname{arcsec}(-\varepsilon)$).

Before answering this, I do need to concede that at this stage we have no good definition for a closed curve, or a connected curve, or for a branch/component of a curve. However, the curves we deal with are so simple that we can say that:

- A closed curve is one which is periodic in θ .
- A component/branch of a curve is a set of points on the curve such that (1) each point in the set corresponds to a value of θ taken from some open interval (a, b) ; and (2) each point corresponding to a value of θ from the interval is on the curve and has a finite radius.
- A connected curve is one which consists of a single component.

A fairly general and accessible version (restricted to curves in the real plane) can be found in books on multivariate calculus and/or the geometry of curves; see for example the first couple of chapters of do Carmo.¹ However, the ‘proper’ generalisation comes from topology: see for example Munkres (§23–24).

Let us consider the following diagram.



We have the following: $|OQ| = |OX| - |QX|$. But $|OQ| = r \cos \theta$, $|OX| = l/\varepsilon$, and $|QX| = |PK|$; thus $r \cos \theta = l/\varepsilon - |PK|$. But since P is on the conic, $\varepsilon = |OP|/|PK| = r/|PK|$; hence $r \cos \theta = l/\varepsilon - r/\varepsilon$ and thus $l/r = 1 - \varepsilon \cos \theta$.

¹*Differential geometry of curves and surfaces.*

For (1), we need to check that the graph is symmetric about OX ; this ray is our polar axis, and so we just need to check that (r, θ) is on \mathcal{C} iff $(r, -\theta)$ is on the curve. But $\cos \theta = \cos(-\theta)$, which is sufficient.

For (2), we need to do some work with inequalities. From the polar equation, $r = \frac{l}{1+\varepsilon \cos \theta}$; so we need to show that this fraction is bounded above (i.e. given any eccentricity ε and any length l then there exists some K — which depends on ε and l — such that $|r| < K$ for all possible θ). Let us therefore fix a given eccentricity ε . Since the conic is an ellipse, we have $0 \leq \varepsilon < 1$.

Since ε is real, there exists some positive κ satisfying $\varepsilon < \kappa < 1$.² Hence we have that $0 < \varepsilon < \kappa$, and thus $-\kappa < -\varepsilon < 0$; so $1 - \kappa < 1 - \varepsilon < 1$, and thus $1 + \varepsilon \cos \theta \geq 1 - \varepsilon > 1 - \kappa$.

From this, we have that $\frac{1}{1-\varepsilon \cos \theta} < \frac{1}{1-\kappa}$, and hence $r < \frac{l}{1-\kappa} < \infty$.

Further, $1 + \varepsilon \cos \theta \leq 1 + \varepsilon < 2$, so $\frac{1}{2} < \frac{1}{1+\varepsilon \cos \theta}$ and thus $\frac{l}{2} < r$. Combining our two r -involving inequalities, we obtain

$$\frac{l}{2} < r < \frac{l}{1-\kappa};$$

so r (and thus the curve) is bounded. The closedness of the curve is trivial, since r is periodic (with period dividing 2π).

For (3), we need to show the parabola $r = \frac{l}{1+\cos \theta}$ is unbounded (note $\varepsilon = 1$). But this is clearly true, for if $\theta \rightarrow \pi$ then $r \rightarrow \infty$. The parabola is connected, since $\theta = \pi$ is the only discontinuity, and the curve is periodic otherwise: to see this more clearly, let θ vary from $-\pi$ to π ; the result covers the entire parabola, and is only discontinuous at the endpoints.

Now for (4) we proceed similarly to (3); the curve is periodic with period 2π , and $r \rightarrow \infty$ precisely when $1 + \varepsilon \cos \theta \rightarrow 0$; this latter phenomenon occurs when $\cos \theta = -1/\varepsilon$, which is equivalent to $\sec \theta = -\varepsilon$. Fix $\alpha = \operatorname{arcsec}(-\varepsilon)$; then all of $-\alpha$, α , and $2\pi - \alpha$ satisfy $\sec(\theta) = -\varepsilon$, and these three points are the only points in the interval $[-\alpha, 2\pi - \alpha]$ which satisfy this. If θ ranges over this interval we cover the entire curve, and $r \rightarrow \infty$ only at the endpoints and at $\theta = \alpha$. Hence the curve is in two components (since it is continuous everywhere in the interval except these three points), and tends to infinity at the claimed angles.

Exercise 2: Rectangular form.

By squaring the polar form equation, show that the Cartesian equation for a conic, taking suitable axes and origin, is

$$x^2 + y^2 = (l - \varepsilon x)^2.$$

1. If $\varepsilon \neq 1$, and a suitable origin is chosen, show that the conic equation can be written in the form

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

for some real numbers a and b . The new location of the origin is called the **centre** of the conic. Conclude that:

- (a) Both the ellipse and hyperbola are symmetric across both Cartesian axes.
- (b) For an ellipse, the values $2a$ and $2b$ are the lengths of the chords through the origin along the x - and y -axes respectively. (The longer of these is called the **major axis**, and the shorter the **minor axis**.)
- (c) The two branches of a hyperbola lie in opposite regions formed by the two lines (**asymptotes**)

$$\left(\frac{x}{a} - \frac{y}{b}\right) \left(\frac{x}{a} + \frac{y}{b}\right) = 0.$$

The value $2a$ is the length of the transverse axis of the hyperbola and the value $2b$ is the length of the conjugate axis: the two dimensions of the rectangle whose diagonals are the asymptotes and which is bounded by the hyperbola.

- (d) Further, show that as $x \rightarrow \pm\infty$, the branches of the hyperbola tend arbitrarily close to the asymptotes. (Hint: take limits.)

²We need **completeness** here.

2. If $\varepsilon = 1$, show that the conic equation for the parabola can be written in the form

$$y^2 = 2l\left(\frac{1}{2}l - x\right).$$

By reflecting in a suitable vertical line, derive the standard form equation

$$y^2 = 2lx.$$

Given this latter equation, give the coordinates of the focus, and the Cartesian equation of the directrix.

We will pick our origin to be O , our x -axis to be the ray in the direction of X (i.e. the polar axis), and the y -axis to be the line through O perpendicular to OX .

To compute the rectangular form equation take $r = l - \varepsilon r \cos \theta$, so $r^2 = l^2 - 2l\varepsilon r \cos \theta + \varepsilon^2 r^2 \cos^2 \theta$ and then (substituting) $x^2 + y^2 = l^2 - 2l\varepsilon x + \varepsilon^2 x^2 = (l - \varepsilon x)^2$.

Assume $\varepsilon \neq 1$. Then we collect terms and complete the square:

$$\begin{aligned} x^2 + y^2 = (l - \varepsilon x)^2 &\implies (1 - \varepsilon^2)x^2 + 2l\varepsilon x + y^2 = l^2 \\ &\implies x^2 + \frac{2l\varepsilon}{1 - \varepsilon^2}x + \frac{y^2}{1 - \varepsilon^2} = \frac{l^2}{1 - \varepsilon^2} \\ &\implies \left(x + \frac{l\varepsilon}{1 - \varepsilon^2}\right)^2 - \frac{l^2\varepsilon^2}{(1 - \varepsilon^2)^2} + \frac{y^2}{1 - \varepsilon^2} = \frac{l^2}{1 - \varepsilon^2} \\ &\implies \left(x + \frac{l\varepsilon}{1 - \varepsilon^2}\right)^2 + \frac{y^2}{1 - \varepsilon^2} = \frac{l^2}{(1 - \varepsilon^2)^2}. \end{aligned}$$

Remark. I will take some liberties here with regard to choosing b . In particular, we 'want' b to be such that b^2 is negative when we have a hyperbola, so we should take $b = i\beta$ (where β is half the length of the conjugate axis); on the other hand, it is much easier to interpret b as a length if it is real.

In these notes, I will reverse this notation here and take b to be the *length* which is half the y -chord (for both the hyperbola and the ellipse), and $\beta = ib$ to be the purely imaginary number which has the same modulus. This notation is not standard. ■

Set $a = \frac{l}{1 - \varepsilon^2}$, and $\beta = \sqrt{1 - \varepsilon^2}a$ (where β will be imaginary, in the case of a hyperbola). Then set $b = |\beta|$.

Let us make a change of coordinates, sending $(x, y) \mapsto (x', y') = (x - \varepsilon a)$; then our equation becomes

$$(N1) \quad \frac{x'^2}{a^2} \pm \frac{y'^2}{b^2} = 1$$

where the sign \pm is the same as the sign of $1 - \varepsilon^2$; more precisely, our equation becomes

$$(N2) \quad \frac{x'^2}{a^2} + \frac{y'^2}{\beta^2} = 1$$

but the former is more usual.

Note also that $l = \frac{\beta^2}{a}$, in this coordinate system.

If (x, y) satisfies the equation N2, by the properties of squaring all four sign permutations of $(\pm x, \pm y)$ satisfy it as well; so the curve is indeed symmetric across both axes.

Now, suppose we have an ellipse; so the sign in the equation is $+$. The x -intercepts of the curve are $x = \pm a$, so the chord along the x -axis has length $2a$. By the same reasoning, the y -intercepts are $y = \pm b$, so the y -axis chord has length $2b$.

Now let us consider the case of a hyperbola. We have shifted our origin, and so we need to compute a new polar form, depending on a and b . Since $r^2 = x^2 + y^2$ and $\tan \theta = \frac{y}{x}$, we have $r^2 = x^2(1 + \tan^2 \theta) = x^2 \sec^2 \theta$. Substituting:

$$1 = \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{r^2}{a^2 \sec^2 \theta} - \frac{r^2 \tan^2 \theta}{b^2 \sec^2 \theta} = r^2 \left(\frac{b^2 - a^2 \tan^2 \theta}{a^2 b^2 \sec^2 \theta} \right)$$

and hence

$$(N3) \quad r = \frac{ab|\sec \theta|}{\sqrt{b^2 - a^2 \tan^2 \theta}}$$

is our new polar form (let us call it **central polar form**; in these notes, we will call the polar form from exercise 1 the **focus polar form**).

Remark. Let us translate this into the use of imaginary axes, so we can use the same formula for the ellipse as well. We have the following:

$$1 = \frac{x^2}{a^2} + \frac{y^2}{\beta^2} = \frac{r^2}{a^2 \sec^2 \theta} + \frac{r^2 \tan^2 \theta}{\beta^2 \sec^2 \theta} = r^2 \left(\frac{\beta^2 + a^2 \tan^2 \theta}{a^2 \beta^2 \sec^2 \theta} \right)$$

and hence

$$(N4) \quad r = \left| \frac{ab \sec \theta}{\sqrt{\beta^2 + a^2 \tan^2 \theta}} \right| = \frac{ab |\sec \theta|}{\sqrt{|\beta^2 + a^2 \tan^2 \theta|}}.$$

We can now “nicely” write down an equation for ε in terms of a and β : since $a = \frac{l}{1-\varepsilon^2}$ and $l = \frac{\beta^2}{a}$, we have that $\varepsilon^2 = \frac{a^2 - \beta^2}{a^2}$; this becomes $\varepsilon^2 = \frac{a^2 + b^2}{a^2}$ for an ellipse, and $\varepsilon^2 = \frac{a^2 - b^2}{a^2}$ for a hyperbola. (This use of imaginary axes for the hyperbola thus wraps up the two different formulae for the eccentricity into a single formula, which is nice.) ■

Since we have only performed a translation, our curve is still in two disconnected branches. The equation for r is periodic with period 2π , and is undefined precisely when $b^2 - a^2 \tan^2 \theta \leq 0$. The ‘boundaries’ of this area are thus the bounding lines of the hyperbola, and are the lines described by $b^2 - a^2 \tan^2 \theta = 0$, or $\tan \theta = \pm \frac{b}{a}$; in other words, $y = \pm \frac{b}{a}x$.

We can now give the dimensions for the axes of the hyperbola. The x -intercepts of the hyperbola, as for the ellipse, are $x = \pm a$. Thus the four corners of the rectangle are (a, b) , $(-a, b)$, $(a, -b)$, and $(-a, -b)$. Thus the two side lengths of the rectangle, as claimed, are $2a$ (in the x -direction) and $2b$ (in the y -direction). [Note, in this situation we take b to be real, not imaginary.]

Finally, as $x \rightarrow \infty$, we need to show that the curve becomes arbitrarily close to the asymptotes. Indeed, we can easily show that $\frac{dy}{dx} = \frac{xb^2}{ya^2}$. Consider the portion of the hyperbola where $x > 0$ and $y > 0$. Here,

$$y = b\sqrt{\frac{x^2}{a^2} - 1}$$

and so the equation of the tangent to the curve at (x_0, y_0) is

$$y = \frac{x_0 b^2}{y_0 a^2}(x - x_0) + y_0 = \frac{x_0 b}{a^2 \sqrt{\frac{x_0^2}{a^2} - 1}}x - \frac{x_0^2 b}{a^2 \sqrt{\frac{x_0^2}{a^2} - 1}} + b\sqrt{\frac{x_0^2}{a^2} - 1}.$$

The slope of the tangent line, as $x_0 \rightarrow \infty$, is

$$\frac{x_0 b}{a^2 \sqrt{\frac{x_0^2}{a^2} - 1}} = \frac{b}{a^2 \sqrt{\frac{1}{a^2} - \frac{1}{x_0^2}}} \rightarrow \frac{b}{a^2 \sqrt{\frac{1}{a^2} - 0}} = \frac{b}{a}.$$

Further, the y -intercept becomes

$$-\frac{x_0^2 b}{a^2 \sqrt{\frac{x_0^2}{a^2} - 1}} + b\sqrt{\frac{x_0^2}{a^2} - 1} = \frac{ba^2(\frac{x_0^2}{a^2} - 1) - x_0^2 b}{a^2 \sqrt{\frac{x_0^2}{a^2} - 1}} = \frac{-ba^2}{a^2 \sqrt{\frac{x_0^2}{a^2} - 1}} \rightarrow 0$$

when $x_0 \rightarrow \infty$.

Thus the tangent line of the curve approaches $y = \frac{b}{a}x$ as $x \rightarrow \infty$, and hence the curve itself approaches this line. We can perform similar calculations for the various cases where $x < 0$ or $y < 0$ (or both).

For the parabola, since $\varepsilon = 1$, we obtain $x^2 + y^2 = (l - x)^2 = l^2 - 2lx + x^2$, and so $y^2 = 2l(\frac{1}{2}l - x)$. If we reflect in the line $x = l/4$, then we map

$$(x, y) \rightarrow (-(x - l/4) + l/4, y) = (l/2 - x, y)$$

and our equation becomes

$$y^2 = 2l\left(\frac{l}{2} - \frac{l}{2} + x\right) = 2lx.$$

We obtained the equation $y^2 = 2l(\frac{1}{2}l - x)$ using the focus polar form, and so in that form the focus was $(0, 0)$. In doing our reflection, we mapped $(0, 0) \rightarrow (l/2, 0)$; hence this is the location of the focus. In other words, if $y^2 = 4ax$, then $2l = 4a$ and $a = l/2$; so $(a, 0)$ is the focus. The directrix of the original form was located at $x = l/\varepsilon = l$; after the transformation, it is located at $x = -l/2$, or (if we write the equation as $y^2 = 4ax$) it is located at $x = -a$. (These results suggest that $y^2 = 4ax$ is the most natural form for the equation of the parabola.)

Notes on Section 3: Isometries

Notes on Section 4: Cones

Notes on Section 5: Orthogonal families of conics

Notes on Section 6: Applications to physics

Notes on Section 7: The intersection theorem

Appendix A: Further reading

See references in text, as well as:

- H.S.M. Coxeter, *Introduction to geometry*. John Wiley & Sons (1961).
- Stephen D. Fisher, *Complex variables*. Dover (1999).
- Robin Hartshorne, *Geometry: Euclid and beyond*. Springer (2000).
- Keith Kendig, *Conics*. Mathematical Association of America (2005).
- Morris Kline, *Mathematics for the nonmathematician*. Dover (1985).
- John M. Lee, *Axiomatic geometry*. American Mathematical Society (2017).
- George Salmon, *A treatise on conic sections*. Longmans, Green (1900).
- George F. Simmons, *Differential equations with applications and historical notes*. McGraw-Hill (1991).
- Marta Sved, *Journey into geometries*. Mathematical Association of America (1991).