NCEA Level 3 Calculus
Integration Assignment

1. Compute the indefinite integrals.

   (a) (2 points) $\displaystyle\int x \cdot \cos(x^2) \cdot \sin(\sin x^2)\,\mathrm{d}x$

   (b) (2 points) $\displaystyle\int \pi t \csc^2(2t^2)\,\mathrm{d}t$

   (c) (2 points) $\displaystyle\int \frac{\sqrt{j} + 3j^5 + 3j^6 + 3j^7 + 2}{2j^7}\,\mathrm{d}j$

   (d) (2 points) $\displaystyle\int \frac{\ln t^2}{t}\,\mathrm{d}t$

Solution:

(a) Let $u = \sin x^2$. Then $\mathrm{d}u = 2x\cos x^2\,\mathrm{d}x$ and our integral becomes $\int \frac{1}{2}\sin u\,\mathrm{d}u = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos\sin x^2 + C$. This could also be done by two substitutions, $u = x^2$ and then $v = \sin u$.

(b) Let $u = 2t^2$. Then $\mathrm{d}u = 4t\,\mathrm{d}t$ and our integral becomes $\int \frac{\pi}{4}\csc^2(u)\,\mathrm{d}u = -\frac{\pi}{4}\cot u + C = -\frac{\pi}{4}\cot(2x^2) + C$.

(c) We simplify to find that our integral becomes $\frac{1}{2}\int j^{-6.5} + 3j^{-2} + 3j^{-1} + 3 + 2j^{-7}\,\mathrm{d}j = \frac{1}{2}\left(-\frac{j^{-5.5}}{5.5} - 3j^{-1} + 3\ln|j| + 3j - \frac{1j^{-6}}{3}\right) + C = \frac{1}{2}\left(-\frac{1}{5.5\sqrt{j^{11}}} - \frac{3}{j} + 3\ln|j| + 3j - \frac{1}{3j^6}\right) + C$.

(d) First note that $\ln t^2 = 2\ln t$. Then let $u = \ln t$ so $\mathrm{d}u = \frac{1}{t}\,\mathrm{d}t$ and the integral becomes $\int 2u\,\mathrm{d}u = u^2 + C = (\ln t)^2 + C$.

2. We will prove the identity $1 + \tan^2 x = \sec^2$.

   (a) (1 point) Calculate $\frac{\mathrm{d}}{\mathrm{d}x}\sec^2 x$.

   (b) (4 points) Using the substitution $u = \tan x$, integrate your answer to part (a). Conclude that $\sec^2 x = \tan^2 x + C$ for some constant C.

   (c) (2 points) Find the value of C and conclude the identity above.

Solution:

(a) Using the chain rule the required derivative is $2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x$.

(b) We find $\int 2\sec^2 x \tan x\,\mathrm{d}x$. Let $u = \tan x$. Then $\mathrm{d}u = \sec^2 x$, and our integral becomes $\int 2u\,\mathrm{d}u = u^2 + C = \tan^2 x + C$. But from (a) we have $\int 2\sec^2 x \tan x = \sec^2 x + C'$. Hence $\tan^2 x + C = \sec^2 x + C'$ and the two differ only by a constant.

(c) The identity must hold for all x, and so we set $x = 0$. Then $\sec^2 0 = \tan^2 0 + C$ and $C = 1$. Hence we have $\sec^2 x = \tan^2 x + 1$ as expected.

3. (2 points) Find the area bounded by the curve $y = 3x^2 + x - 2$ and the $x-$axis.

Solution: The curve can be factored as $y = (3x-2)(x+1)$ and so the $x-$intercepts are $x = -1$ and $x = \frac{2}{3}$. We must therefore find $\int_{-1}^{2/3} 3x^2 + x - 2 \, \mathrm{d}x = x^3 + 0.5x^2 - 2x \Big|_{-1}^{2/3} = -\frac{125}{54} \approx -2.315$.

4. (2 points) If $\int_{-1}^{2} 3f(x) \, \mathrm{d}x = 9$ and $\int_{-1}^{3} f(x) \, \mathrm{d}x = 1$, find $\int_{2}^{3} f(x) \, \mathrm{d}x$.

Solution: $\int_2^3 f(x) \, \mathrm{d}x = \int_{-1}^{3} f(x) \, \mathrm{d}x - \frac{1}{3} \int_{-1}^{2} 3f(x) \, \mathrm{d}x = 1 - 3 = -2$.

5. (a) (3 points) Compute $\int_0^R 2\pi r \, \mathrm{d}r$. Interpret your answer (you may wish to draw a diagram).

   (b) (2 points) Find the volume of a sphere of radius R by integration; the surface area of a sphere of radius r is given by $SA = 4\pi r^2$.

Solution:

(a) $\int_0^R 2\pi r \, \mathrm{d}r = \pi r^2 \Big|_0^R = \pi R^2$, which is the area of a circle of radius R. This makes sense as we are summing up all the circumferi of circles radiating out from the centre of our larger circle: we expect to get the full area.

(b) Same reasoning: $\int_0^R 4\pi r^2 \, \mathrm{d}r = 4/3\pi r^3 \Big|_0^R = 4/3\pi R^3$.

6. (5 points) Find $y(\sqrt{\pi/2})$ if $y(0) = 0$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x \sin(x^2) \cot y.$$

Solution: Separating variables, we have $\int \tan y \, \mathrm{d}y = \int x \sin x^2 \, \mathrm{d}x$. The RHS is simply $-\frac{1}{2} \cos x^2 + C$ (substitute x^2 out), and we can rewrite the LHS as $\tan y = \frac{\sin y}{\cos y}$ so (using the substitution $\cos y$); hence overall we have got $-\ln|\cos y| = -\frac{1}{2} \cos x^2 + C$. But $y(0) = 0$ so $-\ln 1 = -\frac{1}{2} + C$ and $\ln 1 = 0$ so $C = 0.5$ and $|\cos y| = e^{\frac{1}{2} \cos x^2 - \frac{1}{2}}$. At $x = \sqrt{\pi/2}$, the RHS becomes $e^{-0.5}$ and so $\cos y = \pm e^{-0.5}$; therefore we have two solutions for y: $y = \cos^{-1}(\pm e^{-0.5})$.