NCEA Level 3 Calculus (Differentiation)
1. The Derivative

Goal for this week

To remember why we care about derivatives, and remind ourselves of how slopes should behave.

Last year we met the derivative. Let us recall that the derivative of a function is, in some sense:

- A way to find the slope of another function.
- A function giving a rate of change.
- A way to find maxima and minima of a function.

The value of the derivative of a function at a point is the gradient/slope/rate of change of the function at that point. Over the next few sections, we will make this rather vague notion precise, allowing us to transform geometric questions about slope, curviness and speed into questions that can be solved via algebraic manipulation.

If f is a function, then the derivative is a function that takes some point x and gives us the slope of f at x. For this function, we write f' (read *eff prime*), so the slope of f at x is denoted by $f'(x)$. If we want to be explicit about the names of the quantities we are relating, then if $y = f(x)$ we write $f'(x) = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ (read *dee y over dee x*); in this notation, the slope of the graph at $x = x_0$ is written as $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0}$ (but I tend to avoid this latter notation). In physics, especially in mechanics, the derivative of x with respect to time is often written like $\dot{x}$.

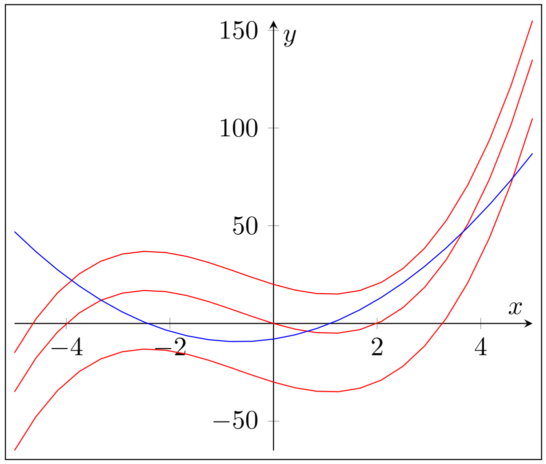
If a function does not have a well-defined derivative at some point, it is said to be non-differentiable at that point.

Technical Remark. Note that this isn't a proper definition for the derivative: all we've done is say that the derivative 'is the slope' of a function. However, we only have a definition of *slope* for linear functions. Our solution to this will be, next week, a definition of derivatives that doesn't mention slope explicitly but generalises the definition of slope. ♢

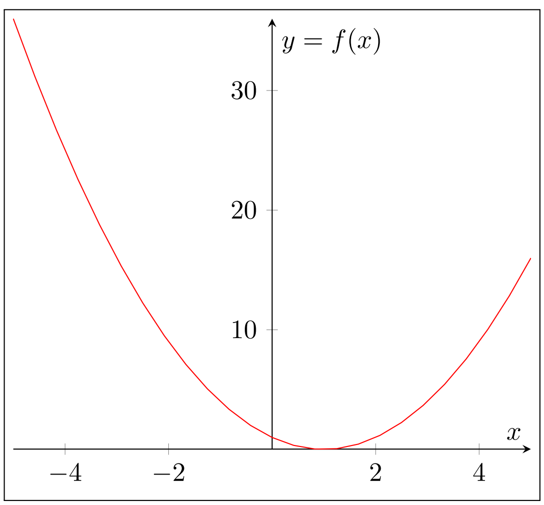
Example. Consider a function defined by $y = 2x + 1$. At every point, the slope of the graph of this function is 2; so the value of the derivative of this function is 2 at every point. (Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$.)

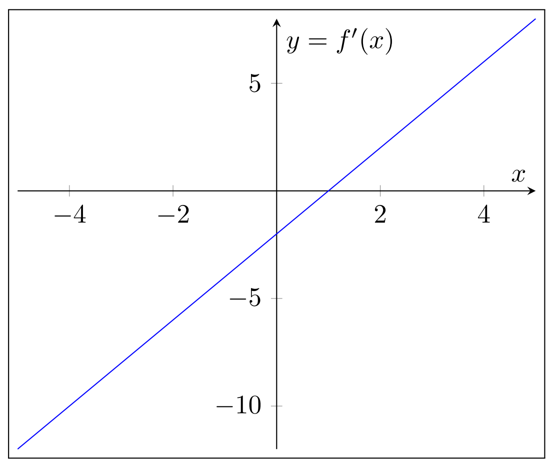
Example. The speed of a particle at any time is the rate of change of the displacement of the particle at that time; this can also be viewed as the slope of the displacement-time graph of the particle. Hence the speed of the particle is the derivative of the displacement. The derivative of speed is called the acceleration of a particle.

Example. In the diagram, the derivative of the red functions is shown as the blue function. Note that the derivative of the function depends only upon the shape, not the y-shift. Also, see that the derivative is positive as the function increases and is negative when the function is decreasing. The derivative is zero exactly where the function is 'flat'.



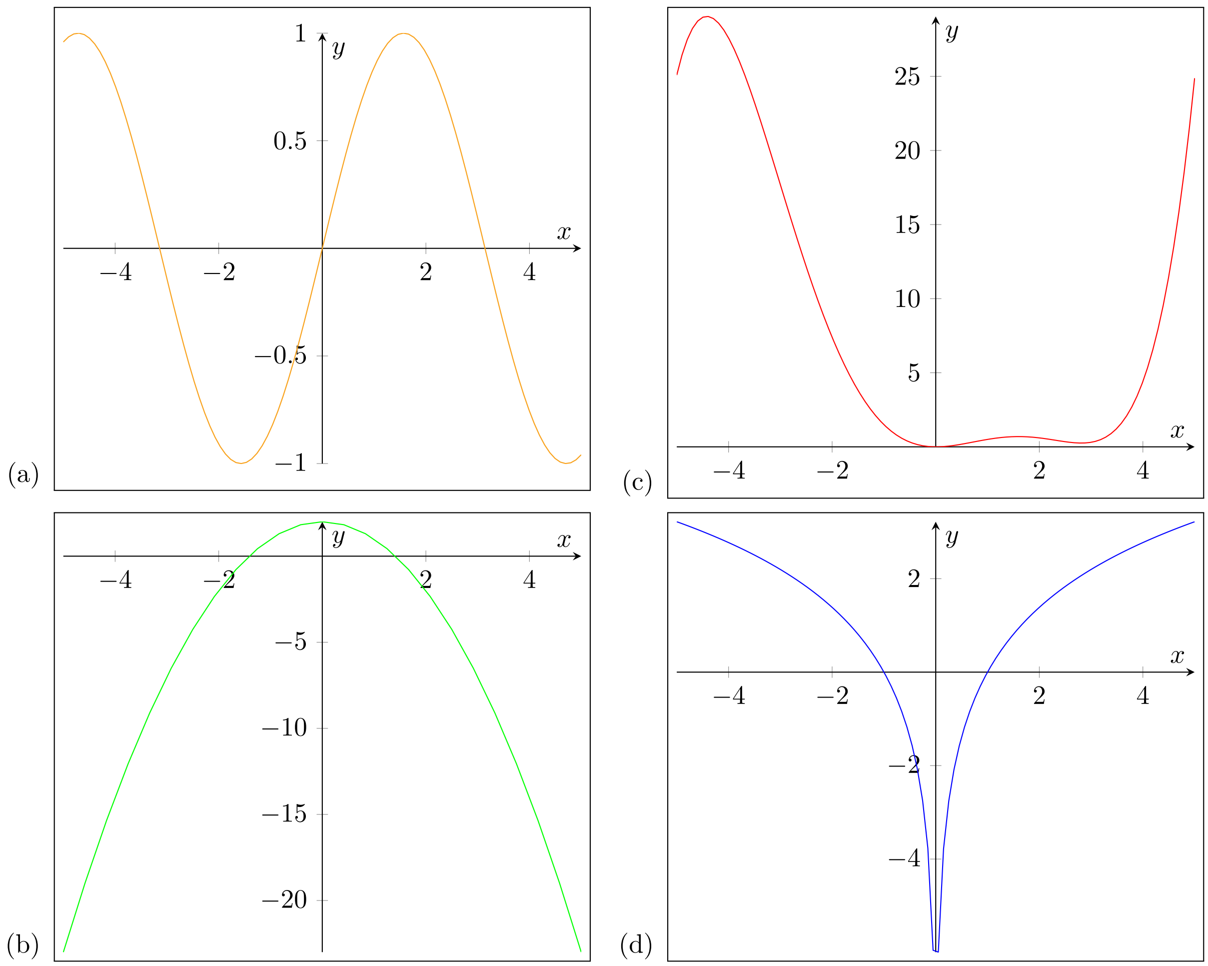
Example. The two diagrams below show the graphs of the function $f(x) = (x-1)^2$ and its derivative f'.





Questions

1. Why does the derivative not depend on the y-shift of a function?
2. Let f be a function. Describe the difference between f, f', $f(57)$, and $f'(57)$.
3. Draw the graph of the derivative of each graphed function.



4. Describe several ways in which a function f can fail to be differentiable at a point x, illustrating your examples with sketches.
5. Consider each of these functions in turn. Where is the derivative of each (i) negative, (ii) positive, (iii) zero, and (iv) undefined?
   (a) $x \mapsto x^2$
   (b) $x \mapsto \sin x$
   (c) $x \mapsto \tan x$
6. Describe the derivative of the function $x \mapsto \tan^{-1} x$.
7. Write down the slope of $y = mx + c$. Hence give $\frac{\mathrm{d}y}{\mathrm{d}x}$.
8. If f' is the derivative of some function f, describe the meaning of $(f')'$ in terms of rate of change (to save space, from now on we will write f'' for this second derivative).

9. If a function is periodic, with a period of T, what can you say about its derivative? M

10. Let f be a function. Suppose that it is known that $f'(3) = 9$, and $f(3) = 6$. A
    (a) What does the graph of $y = f(x)$ look like around $x = 3$?
    (b) Give the equation of the tangent line to $f(x)$ at $x = 3$.

11. The function floor maps any number x to the greatest integer less than or equal to x (so $\text{floor}(\pi) = 3$, for example). Draw a graph of $y = \text{floor}(x)$. Where is floor differentiable, and what is the derivative of floor at these points? M

12. When a hot water tap is turned on, the temperature T of the water depends on how long the water has been running. M
    (a) Sketch a possible graph of T as a function of the time t that the tap has been running.
    (b) Describe how the rate of change of T with respect to t varies as t increases.
    (c) Sketch a graph of the derivative of T with respect to t.

13. The rate of change of a population at a time t is directly proportional to the population $P(t)$ at that time, such that $\frac{\mathrm{d}P}{\mathrm{d}t} = P$. Draw a graph of the population over time if $P(0) \approx 1000$. M

14. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$. M
    (a) What is the meaning of the derivative $f'(5)$?
    (b) Suppose that there is an unlimited amount of space and nutrients. Which would you expect to be larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited does your answer change?

15. Consider an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is not a function: since both $(0, b)$ and $(0, -b)$ are members of the function, it fails the vertical line test. However, it would be nice to reason about its rate of change *as if it were* a function. Describe the slope of the ellipse as a particle traces the curve in an anticlockwise direction at a constant rate. M