NCEA Level 3 Calculus (Integration)
17. The Fundamental Theorem of Calculus

Goal for this week

To understand how (definite) integration and differentiation are related.

Finally, we have the punchline. We state the theorems first, and then give the (optional) proofs at the end.

Statements of theorems

"You can tell it's important because it has a name.
You can tell it's ***very*** *important because it has a* ***pompous*** *name."*

Theorem (First Fundamental Theorem of Calculus (FTC1))
Suppose f is a continuous function, and suppose F is any antiderivative of f (so $F' = f$). Then,

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b.$$

In other words, the definite integral of a function can be found by evaluating the indefinite integrals at the endpoints — geometrically, the total accumulated slope of a function over an interval is just the height gained by the function over the interval. This actually follows from a differently intuitive result:

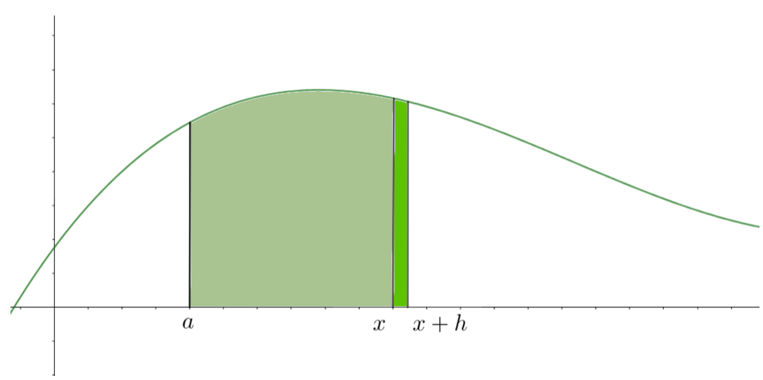
Theorem (Second Fundamental Theorem of Calculus (FTC2))
Suppose f is a continuous function. Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

For some intuition, we can consider the following graph of $y = f(x)$. The shaded area is the value of $A(x) = \int_a^x f(t)\,\mathrm{d}t$; then we use the fact that the area of the darker shaded area is approximated by $hf(x)$ (base times height) to see that

$$\frac{A(x+h) - A(x)}{h} \approx \frac{hf(x)}{h} = f(x).$$

If we take limits, as we do in the proof of this theorem (given below), then this approximation becomes exact: the rate of change of the area under a curve is simply the height of the curve.



Joke. A mathematics professor was lecturing to a class of students. As he wrote something on the board, he said to the class "Of course, this is immediately obvious." Upon seeing the blank stares of the students, he turned back to contemplate what he had just written. He began to pace back and forth, deep in thought. After about 10 minutes, just as the silence was beginning to become uncomfortable, he brightened, turned to the class and said, "Yes, it IS obvious."

These theorems are *fundamental* because they show us that the two main operations of calculus, integration and differentiation, are deeply related and are (in some sense) inverses of each other. To be absolutely clear, without this theorem we would have absolutely no justification in calling anti-derivatives 'integrals' — the second theorem tells us that if we make the upper bound of our definite integral vary then we obtain an anti-derivative for the expression underneath the integral sign, and the first theorem tells us that if we take the definite integral of the derivative of some function then we so happen to obtain anti-derivatives back out the other side! (In fact, notice that the indefinite integral of f is just $\int_{f^{-1}(C)}^{x} f(x)\,\mathrm{d}x$, where C is just our constant of integration!)

We also have the following theorem, which allows us to combine definite integrals together. In order to get some kind of geometric intuition, please draw a diagram or find some kind of intuitive explanation for each!

Theorem. *Suppose f and g are functions and λ is a real constant. Then, if the relevant integrals are defined, we have:*

1. $\displaystyle \lambda \int_a^b f(x)\,\mathrm{d}x = \int_a^b \lambda f(x)\,\mathrm{d}x.$

2. $\displaystyle \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x = \int_a^b f(x) + g(x)\,\mathrm{d}x.$

3. $\displaystyle \int_a^a f(x)\,\mathrm{d}x = 0.$

4. $\displaystyle \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x.$

Note that the areas below a curve are assigned *negative* area!

Examples.

1. We calculate that $\int_0^1 \sqrt{x}\,\mathrm{d}x = \frac{2}{3}x^{3/2}\Big|_{x=0}^{1} = \frac{2}{3}1^{3/2} - \frac{2}{3}0^{3/2} = \frac{2}{3}$.

2. The definite integral $\int_0^\pi \cos x\,\mathrm{d}x$ is equal to 0; this is because the (negative) area under the $x-$axis exactly cancels the (positive) area above the $x-$axis. (Draw a picture.)

3. We calculate that $\int_0^2 2\,\mathrm{d}x = 2x\Big|_{x=0}^{2} = 2 \cdot 2 - 2 \cdot 0 = 4$. (Thus the integral gives us the correct value if we try to find the area of a square!)

Proofs

We will only prove the FTC for 'nice' functions. We actually prove the second FTC first as it is easier, and we will state the two theorems a little more carefully.

Theorem (FTC2)**.** *Suppose that f is a continuous function on the closed interval $[a, b]$.** *Then the function F defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

for all x in the closed interval $[a, b]$ is differentiable for all x such that $a < x < b$, and

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x).$$

Proof. Let us take the derivative in a straightforward manner.

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t}{h} = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,\mathrm{d}t}{h}.$$

Now, let $f(M)$ be the maximum value obtained by f on the closed interval $[x, x+h]$; let $f(m)$ be the minimum value. Interpreting the integral as an area, we have

$$hf(m) \le \int_x^{x+h} f(t)\,\mathrm{d}t \le hf(M) \implies f(m) \le \frac{1}{h} \int_x^{x+h} f(t)\,\mathrm{d}t \le f(M).$$

Now, as $h \to 0$ we must have $f(m) \to f(x)$ and $f(M) \to f(x)$ (because as we make the interval smaller, m and M move towards x). Hence

$$f(x) \le \frac{1}{h} \int_x^{x+h} f(t)\,\mathrm{d}t \le f(x)$$

and so $\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$. $\square$

Theorem (FTC1)**.** *Suppose f is continuous on the closed interval $[a, b]$, and suppose F is any antiderivative of f for all x such that $a < x < b$ (so $F'(x) = f(x)$ for all such x). Then,*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b.$$

Proof. Consider $\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$. In particular, $\int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of f and we can antidifferentiate both sides, obtaining

$$\int_a^x f(t)\,\mathrm{d}t = F(x) + C \tag{*}$$

(where C is some constant). Now substitute a for x in (*): we find that $0 = \int_a^a f(t)\,\mathrm{d}t = F(a) + C$, and in particular $-C = F(a)$. Substituting b for x in (*), we find that $\int_a^b f(t)\,\mathrm{d}t = F(b) + C = F(b) - F(a)$; and we are done. $\square$

*i.e. f is continuous at every x such that $a \le x \le b$.

Questions

1. Compute the following definite integrals. A

   (a) $\int_0^1 \mathrm{d}x$

   (b) $\int_{-1}^1 e^x \, \mathrm{d}x$

   (c) $\int_3^4 x^2 + 3x - 1 \, \mathrm{d}x$

   (d) $\int_0^1 x^n \, \mathrm{d}x$ for integer values of n.

2. Find the area underneath the given curves between the given bounds: A

   (a) $y = 6x^2 + 4x + 9$ between $x = 0$ and $x = 4$

   (b) $y = \sin x$ between $x = 0$ and $x = \pi$

   (c) $y = \sin x$ between $x = -\pi$ and $x = \pi$

   (d) $y = \cos x$ between $x = -\pi$ and $x = \pi$

   (e) $y = \frac{1}{x}$ between $x = 1$ and $x = 2$

3. Find all the problems in the following working. A

$$\int_1^{-1} \frac{\mathrm{d}x}{x} = \ln|-1| - \ln|1| = 0$$

4. Show that $\int \ln x \, \mathrm{d}x = x \ln x - x + C$. A

5. Let f be a function such that for all x, $f(-x) = -f(x)$. Such a function is called *odd*. Show that for all a, M

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = 0.$$

   What does this mean geometrically?

6. Let f be an odd function with period 2 such that $\int_0^1 f(x) \, \mathrm{d}x = k$. Compute: M

   (a) $\int_{-1}^1 f(x) \, \mathrm{d}x$

   (b) $\int_0^{-1} f(x) \, \mathrm{d}x$

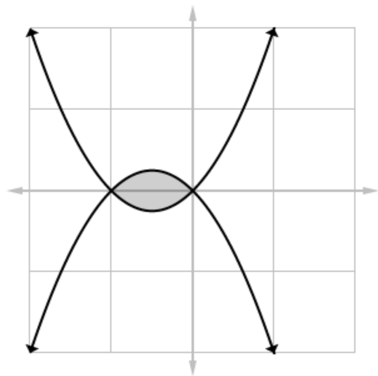
7. Let f be a function such that for all x, $f(-x) = f(x)$. Such a function is called *even*. Show that for all a, A

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = 2 \int_0^a f(x) \, \mathrm{d}x.$$

   What does this mean geometrically?

8. If $\int_{-2}^1 f(x) \, \mathrm{d}x = 2$ and $\int_1^3 f(x) \, \mathrm{d}x = -6$, what is the value of $\int_{-2}^3 f(x) \, \mathrm{d}x$? M

9. Find the area between the curves $y = x^2 + x$ and $y = -x^2 - x$ shaded here. M

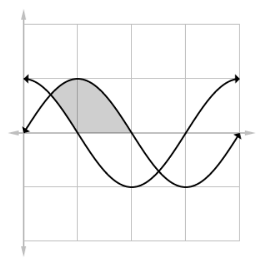


10. Find the area between the two curves $y = 1 + x^2$ and $y = 3 + x$. M

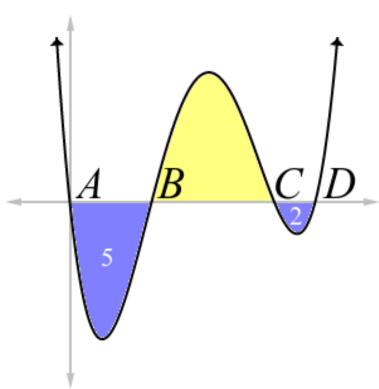
11. Find the area of the region bounded by $f(x) = 4$, $g(x) = \frac{e^x}{5}$, and $x = 0$. M

12. What is the area of the region between the graphs of $f(x) = 2x^2 + 5x$ and $g(x) = -x^2 - 6x + 4$ from $x = -4$ to $x = 0$? M

13. Find the area bounded by the curves $y = \sin x$ and $y = \cos x$ and the x−axis graphed here. E



14. Consider the function f graphed below; the total **unsigned** area between the curve and the x−axis is 10 square units. Find $\int_A^D f(x)\,\mathrm{d}x$. A



15. (a) Sketch the graph of $y = |\sin x|$. M

    (b) Compute $\int_0^{\pi/2} y\,\mathrm{d}x$ using the FTC.

    (c) Hence, without doing any anti-differentiation, compute $\int_0^{2\pi} y\,\mathrm{d}x$.

16. Define $F(x)$ by $\boxed{\texttt{M}}$

$$F(x) = \int_{\frac{\pi}{4}}^{x} \cos(2t)\,\mathrm{d}t.$$

(a) Use the second fundamental theorem of calculus to find $F'(x)$.

(b) Verify part (a) by integration and differentiation.

17. Compute $\frac{\mathrm{d}}{\mathrm{d}x}\int_2^x t^t\,\mathrm{d}t$. $\boxed{\texttt{M}}$

18. Scholarship 2014: Find exact expressions for the areas of the three labelled regions bounded by the two curves $y = 9\csc^2 x$ and $y = 16\sin^2 x$ between $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ shown below. $\boxed{\texttt{S}}$