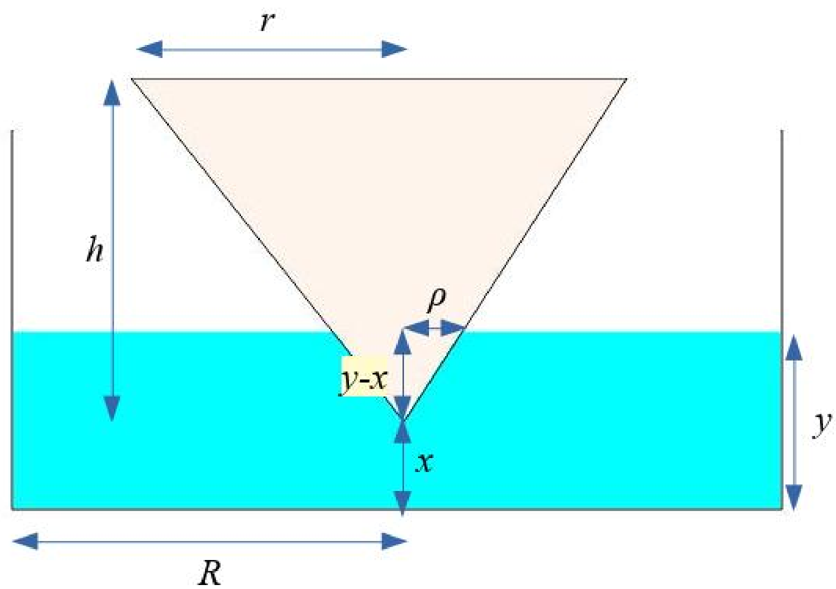
Worked solution for problem:

(Difficult) A cone of radius r centimetres and height h centimetres is lowered point first at a rate of $1\,\mathrm{cm}^2$ into a cylinder of radius R centimetres that is partially filled with water. How fast is the water level rising at the instant that the cone is fully submerged?



Let V be the volume of water in the tank, which we know to be constant. We are given that $\frac{\mathrm{d}x}{\mathrm{d}t} = -1$. Using trigonometry, we find $\rho = \frac{r}{h}(y-x)$ so we have

$$\begin{aligned} V &= x\pi R^2 + (y-x)\pi R^2 - \frac{1}{3}\pi\rho^2(y-x) \\ &= \pi R^2 y - \frac{1}{3}\pi\frac{r^2}{h^2}(y-x)^3 \end{aligned}$$

Hence $\frac{\mathrm{d}V}{\mathrm{d}x} = \pi R^2 - \frac{\pi r^2}{h^2}(y-x)^2(1-\frac{\mathrm{d}x}{\mathrm{d}y})$; but V is constant, so any derivative of V is zero and

$$0 = \pi R^2 - \frac{\pi r^2}{h^2}(y-x)^2(1-\frac{\mathrm{d}x}{\mathrm{d}y}).$$

When the cone is exactly submerged, $y - x = h$. So $0 = R^2 - r^2(1-\frac{\mathrm{d}x}{\mathrm{d}y})$ and $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{r^2-R^2}{r^2}$; hence

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x} = -1 \times \frac{r^2}{r^2-R^2} = \frac{r^2}{R^2-r^2}.$$