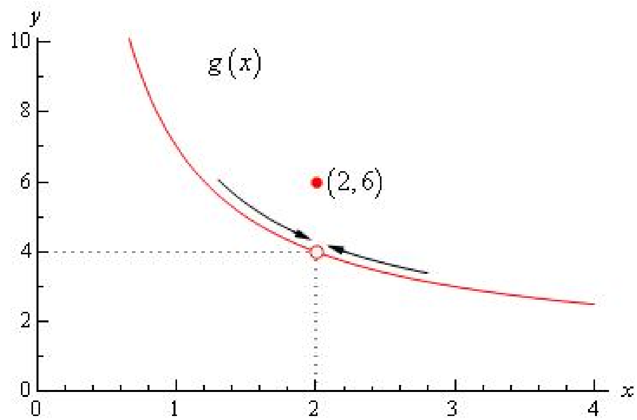
NCEA Level 3 Calculus (Differentiation)
2. Limits

How can we define the derivative in such a way that we can calculate with it? Recall that the *average gradient* (or average slope) of a function f over the interval $a \leq x \leq b$ is defined to be $m = \frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$. We wish to find the gradient at a single point; in order to do this, we will slowly move point b towards point a, letting Δx get closer and closer to zero. However, we cannot actually let $\Delta x = 0$ — and so we can only ever find an *approximation* to the instantaneous gradient without a bit of trickery!

Before moving on with the definition of the derivative, let's explore this idea of assigning values to a function by seeing what it looks like really close to the point we're interested in. We call this operation the *limit*.

Consider the following function:



Although the *value* of the function at 2 is 6, the *limit* of the function at 2 is $\lim_{x\to 2} g(x) = 4$ because when taking a limit we don't care what the function does at the point — only what it looks like it should do! Essentially, the limit of a function at a point is a property of the area around the point **and not a property of the point itself**.

You can also think of $\lim_{x\to x_0} f(x)$ as being the unique value that we could pick for $f(x_0)$ such that the function around that point has 'no gaps'.
Example. Here are a few algebraic examples of limit finding:

- $\lim_{x\to 0} \frac{x}{x} = 1$ since as x gets closer and closer to 0, $\frac{x}{x} = 1$.
- $\lim_{x\to 3} \frac{(x-2)(x-3)}{x-3} = 1$ since as x gets closer and closer to 3, the fraction gets arbitrarily close to 1.
- $\lim_{x\to 0} \frac{1}{x}$ does not exist, since if we approach 0 from the left the function becomes arbitrarily negative and if we approach 0 from the right the function becomes arbitrarily positive — we do not approach the same value on both sides.
- $\lim_{x\to \infty} \frac{1}{x} = 0$ since as x becomes arbitrarily large, $\frac{1}{x}$ becomes arbitrarily small.
- $\lim_{x\to 0} \sqrt{x}$ does not exist, since $\sqrt{x}$ is undefined for $x < 0$.

Now, let's revisit the derivative. We will define the derivative of a function f at a point a as the value of the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

which is equivalent to the definition above (see exercises).

Example. We will find the derivative of $f(x) = x^3$ at the point x using the definition.

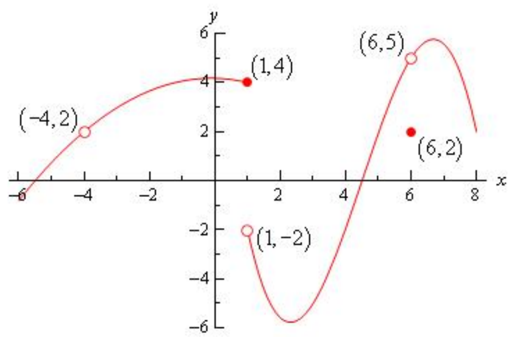
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\ &= 3x^2. \end{aligned}$$

Questions

1. A Guess the value of the following limit by evaluating the limuend for each x in $\{\pm 1, \pm 0.5, \pm 0.2, \pm 0.1, \pm 0.05, \pm 0.01\}$:

$$\lim_{x\to 0} \frac{\sin x}{x + \tan x}$$

2. A Consider the function f graphed below.



   (a) For each of the following expressions, either give the value or explain why the expression is undefined.
      i. $f(-4)$
      ii. $\lim_{x\to -4} f(x)$
      iii. $f(1)$
      iv. $\lim_{x\to 1} f(x)$

   (b) Explain why the limit $\lim_{x\to 6} f(x)$ is not equal to $f(6)$.

   (c) At which points is f:
      i. Discontinuous?
      ii. Non-differentiable?

3. $\boxed{\texttt{M}}$ Evaluate the limit or explain why it does not exist:

(a) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

(b) $\lim_{x\to 0} \frac{1}{x^3}$

(c) $\lim_{x\to 9} \frac{1}{x^3}$

(d) $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$

(e) $\lim_{x\to 4} \frac{x^2+5x+4}{x^2+3x-4}$

(f) $\lim_{x\to \frac{\pi}{2}} \sin x$

(g) $\lim_{x\to \infty} \sin x$

(h) $\lim_{x\to \frac{\pi}{2}} \tan x$

(i) $\lim_{x\to 0} \tan x$

(j) $\lim_{x\to 0} \csc x$

(k) $\lim_{x\to a} C$, where a and C are constants.

(l) $\lim_{x\to -\infty} \tan^{-1} x$

(m) $\lim_{y\to 0} \lim_{x\to 0} \frac{(x+y)(x-y)}{x^2-y^2}$

(n) $\lim_{x\to \infty} 1/x$.

(o) $\lim_{x\to \infty} \frac{2x}{x^2+1}$.

(p) $\lim_{x\to \infty} \frac{x+2}{x-3}$.

4. $\boxed{\texttt{M}}$ Consider the function φ defined by

$$\varphi(x) = 1/\frac{1}{x-x}.$$

Explain why neither $\varphi(\alpha)$ nor $\lim_{x\to\alpha} \varphi(x)$ exists for any real α.

5. $\boxed{\texttt{M}}$ Find the derivative of $x^2 + x$ from first principles.

6. $\boxed{\texttt{E}}$ Find the derivative of $\sin x$ from first principles, given that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

7. $\boxed{\texttt{E}}$ Show that $f(x) = |x-6|$ is not differentiable at $x = 6$. Find a formula for f'.

8. $\boxed{\texttt{E}}$ Show that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ and $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ are equivalent definitions for the derivative at the point a of some function f.

9. $\boxed{\texttt{E}}$ If $\lim_{x\to a}[f(x)+g(x)] = 2$ and $\lim_{x\to a}[f(x)-g(x)] = 1$, find $\lim_{x\to a} f(x)g(x)$.

10. $\boxed{\texttt{S}}$ Consider the following limit (you may assume that it exists):

$$L = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{9}{10^i}$$

(a) Write $L = 0.9 + 0.09 + \cdots$, and make a conjecture about the value of L.

(b) Prove or disprove your conjecture from (a). [*Hint: you may wish to consider the following working.*]

$$\left(1-\frac{1}{10}\right) L = 9\left(1-\frac{1}{10}\right)\left(\frac{1}{10}+\frac{1}{100}+\cdots\right)$$

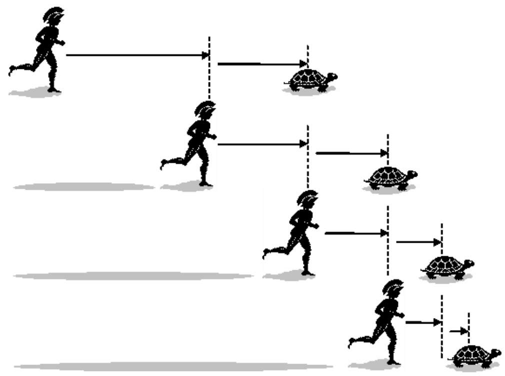
(c) In general, consider the sum $S_n = \sum_{i=1}^{n} ar^i$ for some real constants a and r. Prove that

$$\lim_{n\to\infty} S_n = \frac{a(1-r^n)}{1-r}.$$

[*Hint: use the same trick as in (b).*] Is there any restriction on r for this limit to exist?

11. $\boxed{S}$ Zeno was a Greek philosopher active in the 5th century BCE. He presented a list of 'paradoxes', or apparent contradictions, including the following (adapted from Wikipedia):

- Suppose Achilles is in a foot race with a tortoise. Achilles runs much faster than the tortoise, but the latter has a head start. By the time Achilles reaches the location that the tortoise started, the tortoise will have moved a small amount further on; similarly, by the time Achilles reaches the new location of the tortoise, it will have moved an even smaller distance further on; and by this reasoning it follows that Achilles can never overtake the tortoise.



- Suppose Homer wishes to walk to the end of a path. Before he can get there, he must get halfway there. Before he can get halfway there, he must get a quarter of the way there. Before traveling a quarter, he must travel one-eighth; before an eighth, one-sixteenth; and so on. So Homer cannot walk to the end of the path.

- For motion to occur, an object must change the position which it occupies. Consider an example of an arrow in flight. In any one (duration-less) instant of time, the arrow is neither moving to where it is, nor to where it is not. It cannot move to where it is not, because no time elapses for it to move there; it cannot move to where it is, because it is already there. In other words, at every instant of time there is no motion occurring. If everything is motionless at every instant, and time is entirely composed of instants, then motion is impossible.

Use your knowledge of limits to explain these apparent contradictions.