Solutions to L3 Calculus Differentiation Exam 3

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Question One

Part (a)

i. $f'(x) = -\frac{1}{2\sqrt{x}} e^{-\sqrt{x}}$ (or accept in form with indices) (1 mark)

ii. $g'(x) = \dfrac{\frac{2}{3}\ln x (x+1)^{-2/3} - \frac{2}{x}(x+1)^{1/3}}{\ln^2 x}$ (or simplified) (1 mark)

Part (b)

i. $v_x(t) = 2\cos 2t$, $v_y(t) = (1 + 2\cos 2t)\cos(t + \sin 2t)$ so

$$v(t) = \big(2\cos 2t, (1 + 2\cos 2t)\cos(t + \sin 2t)\big)$$

(3 marks)

ii.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1 + 2\cos 2t)\cos(t + \sin 2t)}{2\cos 2t}$$

This derivative represents the instantaneous change in the y-ordinate of the particle as its x-ordinate varies. (3 marks)

Question Two

Part (a)

First, we have $\frac{\mathrm{d}y}{\mathrm{d}x}5y^4 + 8xy + 4x^2\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2 + 6x^2y + 2x^3\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, so it follows that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{8xy + 3x^2 + 6x^2y}{5y^4 + 4x^2 + 2x^3}$$

and at the given point the tangent line has slope $m = \frac{37}{29}$. Hence the tangent line is described by

$$(y - 1) = \frac{37}{29}(x + 5) \Rightarrow y = \frac{37}{29}x + \frac{214}{19}.$$

(3 marks)

Part (b)

i. We have $\frac{\mathrm{d}V}{\mathrm{d}t} = -0.2$. Then $\frac{R}{r} = \frac{H}{h}$ so $r = \frac{Rh}{H}$ and $V(h) = \frac{\pi}{3}r^2h = \frac{R^2\pi}{3H^2}h^3$. Hence $\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{R^2\pi}{H^2}h^2$, $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{0.2H^2}{R^2h^2\pi}$, so at $h = 3$ the depth of water is decreasing at a rate of $0.0442\,\mathrm{m\,s^{-1}}$. (3 marks)

ii. Let the rate of pumping be k. We now have $\frac{\mathrm{d}V}{\mathrm{d}t} = k - 0.2$, so $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{H^2(k-0.2)}{R^2h^2\pi}$ and at $h = 2$, $0.1 = 0.4974(k - 0.2)$ and $k = 0.40\,\mathrm{m^3\,s^{-1}}$. (2 marks)

Question Three

Part (a)

Since $\varphi(x)$ is undefined for all $x < 0$, we cannot take the limit $\lim_{x \to 0} \varphi(x)$ as the function cannot tend to a single value from both sides. (2 marks)

Part (b)

Let the three sides of the triangle be x, $\frac{1}{2}(P - x)$, and $\frac{1}{2}(P - x)$. Then the height of the triangle is given by

$$h = \sqrt{\left(\frac{P - x}{2}\right)^2 - \left(\frac{x}{2}\right)^2} = \frac{1}{2}\sqrt{P^2 - 2Px}$$

and the area is

$$A = \frac{1}{2}xh = \frac{1}{4}\sqrt{P^2x^2 - 2Px^3}.$$

We wish to maximise this, so we take the derivative:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{P^2x - 3Px^2}{4\sqrt{P^2x^2 - 2Px^3}}.$$

Setting to zero, we have $P^2 - 3Px = 0$ and so $x = \frac{P}{3}$; so the triangle needs to be equilateral.

Part (c)

Profit P is income minus costs. Total income is Dc, total cost is $5 + \frac{5D}{2}$, so:

$$P = D\left(c - \frac{5}{2}\right) - 5 = 30\left(c - \frac{5}{2}\right)e^{-c/2} - 5$$

Taking the derivative, $\frac{\mathrm{d}P}{\mathrm{d}c} = 30e^{-c/2} - 15\left(c - \frac{5}{2}\right)e^{-c/2}$. Hence $30 = 15(c - \frac{5}{2})$ and $c = \$4.50$. (3 marks)