NCEA Level 2 Mathematics
10. Negative and Fractional Powers

Last week we defined the exponential function for powers which were whole numbers or zero, by defining $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$. We can make this definition more precise by making the following definition:*

Definition. If a is a number, then:

1. a^0 is defined to be 1.

2. a^n is defined to be $a \times a^{n-1}$, for integers $n > 0$.

One might easily ask if there is a way to extend this definition for non-whole-number powers; in fact, last week we implicitly used the fact that such an extension exists in solving some logarithmic equations (but relying on a calculator to 'know the definition' for us). Let us take inspiration from our recursive definition above, and try to 'pull ourselves up by our bootstraps' in steps: we will begin with negative powers.

So suppose we want to define what the value of a^{-n} is (where n is a positive integer). We can try to work out a plausible definition using the rules we want such a value to follow — for example, we want such a definition to obey the rule $a^b a^c = a^{b+c}$. In particular,

$$a^{-n} \times a^n = a^{(-n)+n} = a^0 = 1.$$

Hence a plausible definition for a^{-n} is $1/(a^n)$. This plausible definition also follows (to take another example) the rule $(a^b)^c = a^{bc}$, because $(a^{-n})^x = \left(\frac{1}{a^n}\right)^x = \frac{1}{a^{nx}} = a^{-nx}$ as we would expect.

So now we have a definition for all a^x, where x is an integer. The obvious next step is to look at rational powers; recall, a rational number is any number r that can be written in the form $r = \frac{p}{q}$, where p and q are both integers. As an aside, the following theorem is quite deep and perfectly accessible:-

Theorem. *There are real numbers which are not rational.*

Proof. In particular, we will show that any number x such that $x^2 = 2$ is irrational; for suppose that such an x can be written in the form $x = \frac{p}{q}$ where p and q are both positive integers. Then $2 = x^2 = \frac{p^2}{q^2}$, and hence $2q^2 = p^2$. But this implies that p^2 is even, and so p is itself even (because the squares of odd numbers are odd). Therefore, there is an integer n such that $p = 2n$. Substituting, we have $2q^2 = (2n)^2 = 4n^2$, and hence $q^2 = 2n^2$. But this means that q is even, and hence there is an integer m such that $q = 2m$; substituting, we have $(2m)^2 = 2n^2$, and hence $2m^2 = n^2$ and $2 = \frac{n^2}{m^2}$.

Notice, though, that $\frac{p^2}{q^2} = \frac{n^2}{m^2}$, but n and m were smaller than p and q respectively. Since we didn't say what p and q were to start with, this implies that for any pair of positive integers p and q such that $x = p/q$, there exist smaller positive integers n and m satisfying the same equation; and so we can repeat the whole process, finding two positive integers smaller than n and m, and so on *ad infinitum*.

But this is absurd: given any positive integer, there are only finitely many positive integers smaller than it! Thus our original assumption, that such integers p and q existed in the first place, must be false; so any number x such that $x^2 = 2$ cannot be rational. $\square$

Real numbers which are not rational are (rather unimaginatively) called *irrational*. Other numbers which are irrational include π, the square root of any prime number, and e.

Returning to our main theme, we want to define a^r, where $r = \frac{p}{q}$ is a rational number. Let us again work out a plausible definition using the rules we want such a number to follow; this time, we will use the 'power multiplication' rule:

$$\left(a^{p/q}\right)^q = a^{((p/q)\cdot q)} = a^p.$$

So we can define $a^{p/q}$ to be $\sqrt[q]{a^p}$. (If there's any confusion, we will more precisely define it to be the *positive* root; also, we require our rational number p/q to be written so that q is positive so that we don't have to worry about defining negative roots).

Our full definition so far looks like:

*By 'more precise' I mean 'we make it clearer what we mean by $\cdots$'.

Definition. If a is a number, then:

1. a^0 is defined to be 1.
2. a^n is defined to be $a \times a^{n-1}$, for integers $n > 1$.
3. a^{-n} is defined to be $\frac{1}{a^n}$, for integers $n > 1$.
4. $a^{p/q}$ is defined to be $\sqrt[q]{a^p}$, for rational numbers p/q such that $q > 0$.

Our final trick will be to define a^x for any real number x. Since we don't have the necessary machinery to do it properly this year, our definition will be vague. We use the fact that we want a^x to be continuous: that is, we want it to 'have no gaps' and 'not jump around unexpectedly'. Since x is real, we can always write it in decimal expansion: say

$$x = x_0 + 0.x_1x_2x_3 \dots x_n \dots = x_0 + \frac{x_1}{10} + \frac{x_2}{100} + \dots + \frac{x_n}{10^n} + \cdots$$

(where the notation $x_0 + 0.x_1x_2\dots$ means that x_0 is the 'integer part' of x and x_1, x_2 and so on are the digits of the decimal expansion). In particular, we have

$$a^x = a^{\left(x_0 + \frac{x_1}{10} + \frac{x_2}{100} + \dots + \frac{x_n}{10^n} + \cdots\right)} = a^{x_0} \times a^{x_1/10} \times \cdots \times a^{x_n/10^n} \times \cdots,$$

where we have already defined all the terms on the right — so we can define a^x to be 'the real number which we get closest to if we keep adding the terms on the right until infinity'. *This is obviously not precise, but just take my word for it that (a) it is possible to make the notion precise with a little more work, and (b) real powers are well-defined (that is, such a number always exists).*

Example.

1. $2^{3/2} = \sqrt[2]{2^3} = \sqrt{8}$.
2. $4^{-1/2} = \frac{1}{4^{1/2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$.
3. $27^{5/3} = \left(27^{1/3}\right)^5 = \left(\sqrt[3]{27}\right)^5 = 3^5 = 243$.
4. $2^\pi \approx 2^3 \times 2^{1/10} \times 2^{4/100} \times 2^{1/1000} \times 2^{5/10000} \approx 8.8244$. (my calculator tells me that $2^\pi \approx 8.8249$, so this approximation isn't even that bad!)

Questions

1. Graph the equation $y = \vartheta^x$ for different values of ϑ:

   $$\vartheta = 10 \qquad 2 \qquad 1 \qquad 1/2 \qquad 1/10 \qquad 0 \qquad -1/10 \qquad -1/2 \qquad -1 \qquad -2 \qquad -10$$

   (a) What do you notice? Compare and contrast the different curves. Is there any point which all 11 curves pass through?

   (b) When ϑ is negative, the curve is an *exponential decay* curve; when ϑ is positive, the curve is an *exponential growth* curve. Conjecture some situations where an exponential decay or growth curve might be a good model for some situation.

2. Make a conjecture about the value of 0^0: should it be zero (because $0^n = 0$ for all n), or one (because $n^0 = 1$ for all n)? It might be helpful to graph $y = x^x$ for very small positive and negative values of x.

3. Justify the following statements with mathematical reasoning:

   (a) $\sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$ (where p and $q > 0$ are integers).

(b) If r and s are rational numbers, then $a^r \times a^s = a^{(r+s)}$ (recall we only proved this rule last week for integer powers).

4. Evaluate $\sqrt{27^{-2/3}} + 5^{2/3} \cdot 5^{1/3}$.

5. A student was asked to evaluate $x + 2y + \sqrt{(x-2y)^2}$ for $(x, y) = (2, 4)$. They wrote

$$x + 2y + \sqrt{(x-2y)^2} = x + 2y + x - 2y = 2x$$

and thus obtained the value $2x = 2 \cdot 2 = 4$ for their answer. Were they correct?

6. Simplify the following, writing your answer with positive exponents:
   (a) $\frac{(4a^3)^2}{b^3} \times \frac{2b^2}{(2a)^2}$
   (b) $\frac{5x^2y}{2} \div \frac{10x}{y^2}$
   (c) $(2a^7 \times 50a^3)^{-1/2}$
   (d) $\frac{6m^5}{\sqrt{9m^{16}}}$
   (e) $\sqrt{\frac{\left(16a^{(2/3)}\right)^{(3/2)}}{a^{-1/2}}}$

7. Evaluate $\log_{1/4} 16$, $\log_8 4$, and $\log \sqrt[4]{10}$.

8. Verify that the multiplication terms further to the right in the expression

$$a^{x_0} \times a^{x_1/10} \times a^{x_2/100} \times \cdots \times a^{x_n/10^n} \times \cdots$$

get closer and closer to 1. (Hint: each x_i, for $i > 0$, is a single digit and thus less than 10.) Hence justify why only taking a few of the first terms usually gives a good approximation to the 'real value' of $a^{x_0 + 0.x_1x_2\cdots}$.

9. A graph with Cartesian equation of the form $y = a(x - x_0)^{-1} + c$ is a *hyperbola*.
   (a) Suppose a hyperbola passes through the points $(-1, 0)$, $(0, -1)$, and $(3, 2)$. Find the constants a, x_0, and c and give the equation of the hyperbola.
   (b) Show that there is some value μ such that the hyperbola does not touch the line $x = \mu$. This line is called the *vertical asymptote* of the hyperbola.
   (c) Show that there is some value λ such that the hyperbola does not touch the line $y = \lambda$. This line is called the *horizontal asymptote* of the hyperbola.
   (d) Graph the hyperbola, using your graphing device of choice; describe the behaviour of the graph *around* the two asymptote lines.
   (e) Graph the equation $y = x^{-n}$ for different values of n; what do you notice?
   (f) Show that the hyperbola with vertical asymptote 'at infinity' is just a straight line $y = c$. (Hint: notice that in the hyperbola equation, $x = x_0$ is the vertical asymptote and 'substitute' $x_0 = \infty$ into the equation.) Is this what you expect intuitively?

10. Challenge question. Consider the equation $6^{2x} + m \cdot 6^x + n = 0$, where $n \leq 0$.
    (a) Prove that the equation has precisely two solutions for 6^x.
    (b) Show that only one of these solutions is valid for finding a solution for x if m is positive.