

## NCEA Level 3 Calculus (Integration)

### 23. Trigonometric Substitution

Consider the integral

$$\int_0^1 x^3 \sqrt{1-x^2} \, dx.$$

There is no obvious easy substitution to simplify this integral, and integration by parts could work but will require a lot of work with no guaranteed payoff. However, recall that  $\sqrt{1-\sin^2 \theta} = \cos \theta$ ; this identity suggests that we could perhaps substitute  $x = \sin \theta$  in order to obtain  $dx = \cos \theta \, d\theta$  and so

$$\begin{aligned} \int_0^1 x^3 \sqrt{1-x^2} \, dx &= \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^2 \theta \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \sin \theta (1 - \cos^2 \theta) \cos \theta \, d\theta \\ &= \int_0^{\frac{\pi}{2}} (\cos^2 \theta - \cos^4 \theta) \sin \theta \, d\theta \end{aligned}$$

Now, letting  $u = \cos \theta$  we obtain

$$\begin{aligned} \int_0^{\frac{\pi}{2}} (\cos^2 \theta - \cos^4 \theta) \sin \theta \, d\theta &= - \int_1^0 u^2 - u^4 \, du \\ &= \frac{1}{3} u^3 - \frac{1}{5} u^5 \Big|_{u=0}^1 \\ &= \frac{1}{3} - \frac{1}{5} = \frac{2}{15}. \end{aligned}$$

Here is a table of trig substitutions:

Integrand	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$

**Example.** Consider  $I = \int \frac{dx}{\sqrt{9+x^2}}$ . Let  $x = 3 \tan \theta$  so  $dx = 3 \sec^2 \theta \, d\theta$  and:

$$\begin{aligned} I &= \int \frac{3 \sec^2 \theta}{3 \sqrt{1 + \tan^2 \theta}} \, d\theta = \int \sec \theta \, d\theta \\ &= \ln(\sec \theta + \tan \theta) + C = \ln(\sec \tan^{-1}(x/3) + x/3) + C \\ &= \ln \left( \sqrt{\left(\frac{x}{3}\right)^2 + 1} + \frac{x}{3} \right) + C. \end{aligned}$$

## Questions

1. Find the following integrals:

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(a)  $\int \frac{x^2-9}{x^3} dx$

(b)  $\int \frac{dx}{\sqrt{x^2+a^2}}$

(c)  $\int_0^3 x^2(9-x^2) dx$

(d)  $\int_0^1 x\sqrt{1-x^4} dx$

(e)  $\int_{\sqrt{2}}^2 \frac{dx}{t^3\sqrt{t^2-1}}$

(f)  $\int \frac{\sqrt{25x^2-4}}{x} dx$

2. Use the integral  $2 \int_r^{-r} \sqrt{r^2-x^2} dx$  to find the area of a circle of radius  $r$ .

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3. Scholarship 2005: Find, in terms of  $r$ , the area between the ellipse  $x^2 + 16(y-r)^2 = r^2$  and the circle  $x^2 + y^2 = r^2$ . You may use the substitution  $x = r \sin u$  to find the integral  $\int \sqrt{r^2-x^2} dx$ .

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4. By integrating, verify that

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$$\int_0^x \sqrt{a^2-t^2} dt = \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2}.$$

5. A charged rod of length  $L$  produces a electric field at the point  $(a, b)$  given by

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$$E(a, b) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\epsilon_0(x^2+b^2)^{3/2}} dx.$$

Evaluate this integral to find an explicit expression for  $E(a, b)$ .

6. One of these integrations should be done by partial fractions and one by trig substitution. Do them both.

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$$\int \frac{dx}{(4x^2+9)^2} \quad \int \frac{x^3}{x^2+x-6} dx$$

7. Check this working (the substitution  $x = 3 \sin \theta$  is used). Find any mistakes.

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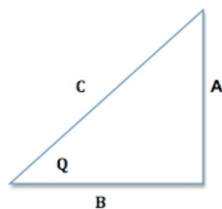
$$\begin{aligned} I &= \int \frac{\sqrt{9-x^2}}{x^2} dx = \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta \\ &= \int \frac{1-\sin^2 \theta}{\sin^2 \theta} d\theta = \int \csc^2 \theta - 1 d\theta \\ &= -\cot \theta - \theta = -\cot\left(\sin^{-1}(x/3)\right) - \sin(x/3) \\ &= \frac{\sqrt{9-x^2}}{x} - \sin(x/3). \end{aligned}$$

8. A water storage tank has the shape of a cylinder with diameter 10 m. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 m, what percentage of the total capacity is being used?
9. When writing this worksheet I went on the internet and found this. Find the mistake(s), and do the integral.

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$$\int \frac{7}{x^2 \sqrt{36 - 25x^2}} dx$$



For trigonometric substitution to solve the above integral, fill in the blanks below using the picture of the triangle given.

side A =  $5x$

side B =  $6$

side C =  $\sqrt{36 - 25x^2}$

$\frac{5x}{6} = \tan(Q)$

$\frac{5}{6} dx = \frac{1}{\cos(Q)^2} dQ$

$\frac{\sqrt{36 - 25x^2}}{6} = \frac{1}{\cos(Q)}$

Submit Answer

Incorrect. Tries 1/8 Previous Tries