NCEA Level 2 Mathematics
8. The Quadratic Formula

Solving Quadratics

Recall that a quadratic equation is one of the form $y = ax^2 + bx + c$ (where $a \neq 0$). A couple of weeks ago, we saw that we can always rearrange such an equation into vertex form; we do this by trying to rewrite it as a square plus a constant. This process is known as *completing the square*.

Our goal is to end up with something that looks like

$$y = \alpha(x + \beta)^2 + \gamma$$

where $(-\beta, \gamma)$ are the coordinates of the vertex of the parabola and α (as we have seen) is the 'scaling factor' that gives us the shape. If α is negative then the parabola opens downwards, and if α is positive then the parabola opens upwards.

If we expand the parabola equation, we obtain

$$\begin{aligned} y &= \alpha(x^2 + \beta x + \beta^2) + \gamma \\ &= \alpha x^2 + 2\alpha\beta x + \alpha\beta^2 + \gamma. \end{aligned}$$

By comparing coefficients, we see that:

$$\begin{aligned} a &= \alpha \\ b &= 2\alpha\beta = 2a\beta \\ c &= \alpha\beta^2 + \gamma = a\beta^2 + \gamma. \end{aligned}$$

Clearly, then, we have $\beta = b/2a$. Substituting this into the third equation, we have $c = a(b/2a)^2 + \gamma$ and so $\gamma = c - \frac{b^2}{4a}$.

Reasoning thusly, we see that

$$y = ax^2 + bx + c = \alpha(x + \beta)^2 + \gamma = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a},$$

which is what I asked you to prove in the homework when we looked at parabolae.

Example. Suppose we are given a rectangular plot of land and are told that the area of the land is $32\,\mathrm{km}^2$ and that one side of the land is $8\,\mathrm{km}$ longer than the other. In order to find the dimensions of the land, we have a quadratic equation which we can simplify:

$$x(x + 8) = 32 \implies x^2 + 8x - 32 = 0.$$

Completing the square, we have that

$$0 = (x + 4)^2 - 16 + (-32) = (x + 4)^2 - 48$$

and so $x = \sqrt{48} - 4 \approx 2.9\,\mathrm{km}$.

This example suggests that we can write down a formula for the value of x in any quadratic equation

$0 = ax^2 + bx + c$ by rewriting that equation in vertex form and solving for x:

$$
\begin{aligned}
0 = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
\frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2 \\
\pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = x + \frac{b}{2a} \\
-\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}} = x \\
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = x.
\end{aligned}
$$

We have therefore proved the following

Theorem (Quadratic formula)**.** *If $ax^2 + bx + c = 0$, then there are at most two distinct values for x. These values are given by*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

when they exist.

These values are called the *solutions*, the *roots*, or the *zeroes* of the equation.

Example. The width of a canal at ground level is 16 m. The sides of the canal can be modelled by a quadratic expression that would give a maximum depth of 16 m. However, the base of the canal is flat and has a width of 12 m. What is the depth of the canal?

Solution. Model the canal with $y = a(x + 8)(x - 8)$. This parabola passes through $(0, -16)$, so we have $-16 = a(+8)(-8)$ and hence $a = \frac{16}{64} = \frac{1}{4}$; so $y = \frac{1}{4}(x + 8)(x - 8)$.

The width of the parabola is 12 at the y–value corresponding to the x–values ± 6; at $x = \pm 6$, $y = -7$ and so the depth of the canal is 7 m.

Forms of a Quadratic

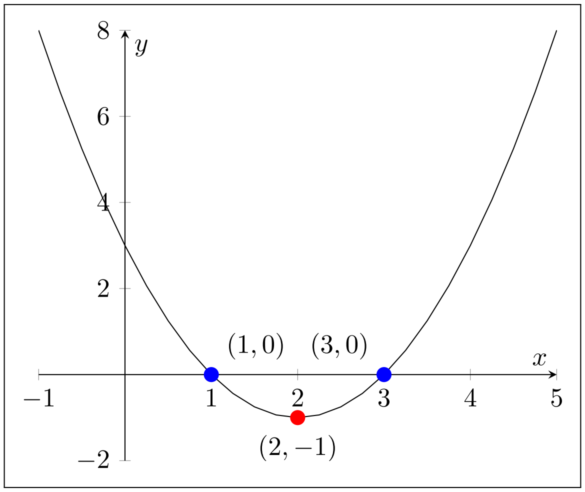
We have seen that there are three complementary ways of viewing the equation $y = ax^2 + bx + c$, each of which exhibits one particular characteristic of the function:

Form	**Exhibits**	**Example**
Expanded		$y = ax^2 + bx + c$
Factorised	roots/zeroes are α and β	$y = a(x-\alpha)(x-\beta)$
Completed square	vertex is at (x_0, y_0)	$y = a(x-x_0)^2 + y_0$

For example, consider the following equation:

$$y = (x-1)(x-3) = (x-2)^2 - 1 = x^2 - 4x + 3.$$

The function is graphed below, so that we can see graphically that each coloured form is an important geometric feature of the parabola described by the equation.



You need to be comfortable transforming between these three forms.

Note that some quadratics, like $x^2 + 1$, cannot be transformed into the factorised form: there are *no* real numbers α and β such that $x^2 + 1 = (x-\alpha)(x-\beta)$.

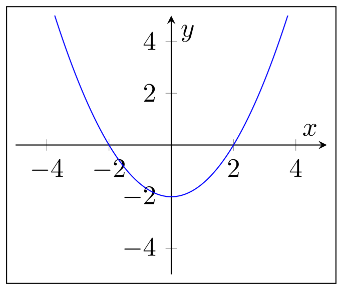
Classifying Roots

Let us look again at the vertex form of the general quadratic equation,

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

Solving the equation $0 = ax^2 + bx + c$ is equivalent to finding the x-intercepts of this parabola. The *number* of x-intercepts, and hence the number of solutions, must be at most two (because of the shape of the parabola), and can only be changed by shifting it up and down (changing the y-shift, $c - \frac{b^2}{4a}$).

Case I: two x-intercepts

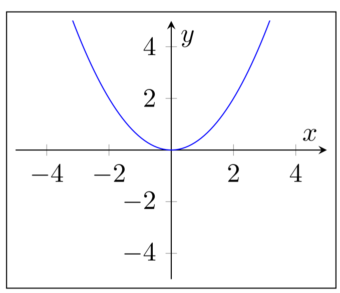


This happens in two situations:

- a is positive and $c - \frac{b^2}{4a}$ is less than zero. Hence $c < \frac{b^2}{4a}$, $4ac < b^2$, and $b^2 - 4ac > 0$.
- a is negative and $c - \frac{b^2}{4a}$ is greater than zero. Hence $c > \frac{b^2}{4a}$, $4ac < b^2$, and $b^2 - 4ac > 0$.

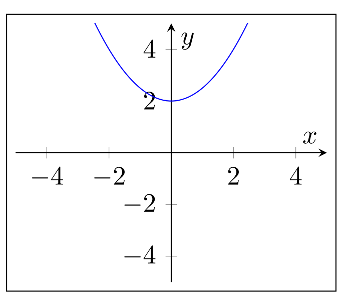
In either case, $b^2 - 4ac > 0$.

Case II: one x-intercept



This happens precisely when the vertex is sitting on the x-axis, so $c - \frac{b^2}{4a} = 0$ and $b^2 - 4ac = 0$.

Case III: no x-intercepts



This happens in two situations:

- a is positive and $c - \frac{b^2}{4a}$ is greater than zero. Hence $c > \frac{b^2}{4a}$, $4ac > b^2$, and $b^2 - 4ac < 0$.
- a is negative and $c - \frac{b^2}{4a}$ is less than zero. Hence $c < \frac{b^2}{4a}$, $4ac > b^2$, and $b^2 - 4ac < 0$.

In either case, $b^2 - 4ac < 0$.

Notice that the quantity $b^2 - 4ac$ tells us the nature of the roots in every case; it is known as the *discriminant* of the quadratic (and I denote it by Δ_2). We have therefore proved the following

Theorem. *Suppose $f(x) = ax^2 + bx + c$. Then:*

- *If $b^2 - 4ac < 0$, then $f(x) = 0$ has no solutions.*
- *If $b^2 - 4ac = 0$, then $f(x) = 0$ has precisely one solution.*
- *If $b^2 - 4ac > 0$, then $f(x) = 0$ has precisely two solutions.*

If we look at the quadratic equation again,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

notice that the discriminant appears underneath the square root sign and so it doesn't need to be memorised seperately.

Questions

Solving Quadratics

1. For each quadatic equation,
   - rewrite it into vertex form by completing the square if required;
   - graph the parabola it describes; and
   - calculate the x-intercept(s) of the parabola, if it has any.

   (a) $y = x^2 + 1$
   (b) $y = x^2 + x$
   (c) $y = x^2 - 4x + 4$
   (d) $y = x^2 + 2x + 3$
   (e) $y = -x^2 + 4x - 2$
   (f) $y = 2x^2 + 2x + 2$
2. Show that if $x^2 - bx + c = 0$, then b is the sum of the solutions of the equation.
3. This question is revision from Level 1.
   (a) Justify, with mathematical reasoning, the following statement: the roots of the equation $(x - \alpha)(x - \beta) = 0$ are α and β.
   (b) Give a quadratic equation with roots -1 and 6.
4. Find all the y-intercepts of $-(x^2 + 2x - 3)(4x^2 - 6x + 2) = y$.
5. Factorise and solve $5x^2 - 9x - 2 = 0$.
6. Consider the quadratic equation $x^2 + bx + c = 0$.
   (a) Calculate b and c such that the quadratic equation has solutions -1 and 3.

(b) Find the location of the vertex of the corresponding parabola, $y = x^2 + bx + c$.

7. Solve $\frac{x^2+5x+2}{x+2} = 3$.

8. Talia used timber to form the exterior sides of her rectangular garden. The length of the garden is x metres, and its area is $50\,\text{m}^2$.
   (a) Show that the perimeter of the garden is given by $2x + \frac{100}{x}$.
   (b) If she uses 33 m of timber to build the sides, find the dimensions of the garden.

9. David and Sione are competing in a cycle race of 150 km. Sione cycles on average 4 km per hour faster than David, and finishes half an hour earlier than David. Find David's average speed. *You MUST use algebra to solve this problem. Note that average speed* $= \frac{\textit{distance}}{\textit{time}}$.

10. Simplify fully $\dfrac{2x^2 - 8}{x^2 - 2x - 8}$.

11. The equation $(x + 2) - 3\sqrt{x + 2} - 4 = 0$ has only one real solution. Find the value of x.

12. Check, by direct substitution, that both

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are solutions of $ax^2 + bx + c = 0$.

13. (a) Suppose that it is known that one solution of $x^2 + bx + c$ is four times the other (i.e. the two solutions are α and 4α). Show that $c = 4b^2/25$.
   (b) Conversely, show that one solution of $x^2 + bx + \frac{4b^2}{25}$ is four times the other, no matter the value of b.
   (c) Show that one solution of $3x^2 + 15x + 12$ is four times the other.

14. (Challenge problem.) Let AB be the diameter of a circle centred at O. Draw the circles with diameters AO and OB; draw a third circle centred at T, tangent to all three existing circles. If the radius of the circle at T is 8, what is the length $d(A, B)$?

Classifying Roots

15. Without explicitly computing them, how many solutions does each quadratic equation have? Don't use the discriminant to decide for all four.
   (a) $0 = x^2 + 2$
   (b) $3 = 3x^2 + 3x$
   (c) $1 = -x^2 - 2x$
   (d) $0 = 2x^2 - 12x + 18$

16. Find k such that $x^2 + 3kx - 2 = 0$ has precisely one solution.

17. The equation $(2x - 3)(x + 4) = k$ has only one real solution; find the value of k.

18. Find all t such that the parabolae described by $y = tx^2 + x + 1$ and $y = -2x^2 - tx + 1$ meet at precisely one point.

19. By considering the quadratic formula, give another proof that the discriminant 'encodes' the nature of the roots of the quadratic.

20. The quadratic equation $mx^2-(m+2)x+2=0$ has two positive real roots. Find the possible value(s) of m, and the roots of the equation.

21. For what values of k does the parabola described by

$$y=x^2+(3x-1)x+(2k+10)$$

never touch the x-axis?

22. Find the possible values of d if one or more real solutions exist for $x^2+5x-1=d(x^2+1)$. Interpret your answer geometrically.

23. How many real roots does x^3-4x^2+7x-4 have?

24. Find expressions in terms of m and n for the roots of the equation

$$\frac{x-m}{x-n}=\frac{2(x+m)}{x+n}.$$

Give an inequality, in terms of m and n, so that the equation has two distinct roots.

25. (a) Two positive numbers have sum 25 and product 136. What are the two numbers?
    (b) For which numbers S and P is it possible to find at least one pair of real numbers with sum S and product P?

26. Let α and β be the roots of x^2+bx+c.
    (a) Show that $\alpha^2+\beta^2=(-b)^2-2c$.
    (b) Conclude that $\Delta_2[x^2+bx+c]=(\alpha-\beta)^2$.

27. Suppose α and β are the two solutions of $x^2+ax+b=0$. Write $(\alpha+\beta)^3$ in terms of a and b.

28. Let ρ be positive. Suppose P and Q are points such that $|PQ|\leq 2\rho$; show that the two circles of radius ρ with centres P and Q intersect; show that they intersect in exactly one place if and only if $|PQ|=2\rho$.

29. (Baby algebraic geometry!) The equation $(x-2)^2+y^2=4$ describes a circle $\mathcal{C}$.
    (a) Show that $y=mx$ always intersects $\mathcal{C}$ in precisely two points, no matter the value of m.
    (b) Find all t such that $y=t(x-5)$ intersects the circle $\mathcal{C}$ precisely once.
    (c) We will now study the intersections of $\mathcal{C}$ with the family of curves $\mathcal{V}_b$ defined by the equation $y^2=x^2+bx$ (where b is a real number).
        i. Show that $\mathcal{C}$ and $\mathcal{V}_b$ always have an intersection at $(0,0)$, no matter what value b is given.
        ii. Show that if (x_0,y_0) is an intersection point of the two curves, then $(x_0,-y_0)$ is as well.
        iii. Show that there is a real number B_1 such that:
            - When $b<B_1$, $(0,0)$ is the only intersection point;
            - When $b=B_1$, the two curves have exactly two intersection points; and
            - When $B_1<b$, the two curves have exactly three intersection points.
        iv. Show that the equation of $\mathcal{V}_b$ can be written in the form

$$\frac{\left(x-\frac{b}{2}\right)^2}{(b/2)^2}-\frac{y^2}{(b/2)^2}=1.$$

    (d) Show that if $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (with $b\leq a$) describes a curve $\mathcal{E}$ then the two points

$$O_1=(+\sqrt{a^2-b^2},0) \qquad O_2=(-\sqrt{a^2-b^2},0)$$

have the property that for all points P,Q on the curve $\mathcal{E}$, $d(P,O_1)+d(P,O_2)=d(Q,O_1)+d(Q,O_2)$. (Compare with the final problem in section 5.)