NCEA Level 3 Calculus (Integration)
16. Anti-differentiation

A few weeks ago when we looked at kinematics, I hinted that finding area was in some way the inverse to finding slope. In last week's questions, the curtain was pulled back a little further when we calculated an area function for a variable endpoint, took the derivative, and got the original function back. Next week, all will be revealed; but first, we need to do a little bit more work. This week, we're going to look at *anti-differentiation*: finding the original function given its derivative.

Example. One antiderivative of $y' = 3x^2 + 4$ is $x^3 + 4x$. Another is $x^3 + 4x + 1$. A third is $x^3 + 4x + 7$. Obviously every function of the form $y = x^3 + 4x + C$ for some constant C will differentiate to the given y', so we must remember always to make this clear.

We also call antiderivatives **indefinite integrals**, and in this notation the above example is

$$\int 3x^2 + 4 \,\mathrm{d}x = x^3 + 4x + C.$$

Example. The most general antiderivative of $\sin x$ is $-\cos x + C$.
Example. The most general antiderivative of $\tan x$ is $-\ln|\cos x| + C$.
Example. $\int \frac{1}{x+3} \,\mathrm{d}x = \ln|x+3| + C$.
Example. $\int \tan^2\theta \,\mathrm{d}\theta = \int \sec^2\theta - 1 \,\mathrm{d}\theta = \tan\theta - \theta + C$.
Example. $\int \frac{2x}{x^2+1} \,\mathrm{d}x = \ln\left|x^2+1\right| + C$.
Example. $\int Ke^{Kx} \,\mathrm{d}x = e^{Kx} + C$ for all constants K.

Questions

1. A In each case, find the most general antiderivative.
   (a) $f(x) = 2x$
   (b) $f(x) = x^{-3}$
   (c) $f(x) = 0$
   (d) $f(x) = \sec^2(x) + \sqrt{x}$
   (e) $f(x) = x\sqrt{x}$
   (f) $f(t) = \sin t - \cos t$
2. A Verify the above examples by differentiation.
3. A Show that $\int 3x^2 + 4x + 5 + 2e^{2x} \,\mathrm{d}x = x^3 + 2x^2 + 5x + e^{2x} + C$.
4. A Find y' when $y = 3 + \sin(2x+4)$ and hence find $\int 2\cos(2x+4)\,\mathrm{d}x$.
5. A If $\frac{\mathrm{d}y}{\mathrm{d}t} = 1.5\sqrt{t}$ and $y(4) = 10$, find $y(t)$ exactly.
6. M Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$.
7. A The velocity of a particle is given by $v(t) = 2t + 1$. Find its position at $t = 4$ if its position at $t = 0$ is $x = 0$.
8. M The acceleration of a particle is given by $a(t) = 10\sin t + 3\cos t$. At $t = 0$, its position is $x = 0$; at $t = 2\pi$, its position is $x = 12$. Find its position at $t = \frac{\pi}{2}$.
9. A Find all functions g such that $g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$.

10. $\boxed{\texttt{M}}$ For each function, sketch an antiderivative passing through $(0,0)$: