Solutions to L3 Calculus Integration Exam 1

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Question One

Part (a)

(i)

$$\int \frac{3t^2+2t}{\sqrt{t}}\,\mathrm{d}t = \int 3t^{3/2}+2t^{1/2}\,\mathrm{d}t = \frac{6}{5}t^{5/2}+\frac{4}{3}t^{3/2}+C.$$

(1 mark)

(ii)

$$\int 2\sin 2x\sin(\cos 2x)\,\mathrm{d}x = \sin\cos 2x + C.$$

(1 mark)

Part (b)

$$y = \frac{1}{\ln 2}\int \frac{1}{x+2} = \frac{\ln(x+2)}{\ln 2}+C$$

Since $3 = \frac{\ln 0+2}{\ln 2}+C = 1+C$, $C = 2$. Hence $y = \frac{\ln(x+2)}{\ln 2}+2$, and when $x=-1, y = \frac{\ln(-1+2)}{\ln 2}+2 = 2$. (3 marks)

Part (c)

$$\int\limits_0^{10} f(t)\,\mathrm{d}t = \int\limits_0^{y}\,\mathrm{d}t + \int\limits_x^{10} f(t)\,\mathrm{d}t - \int\limits_x^{y} f(t)\,\mathrm{d}t = 4+3-2 = 5$$

(2 marks)

Part (d)

We integrate along the x-axis from $x=-1$ to $x=0$ and then from $x=0$ to $x=1$. Over the first half, the radius of each semicircle is $r=x+1$, and over the second half the radius of each semicircle is $r=-x+1$. Hence our volume will be

$$\int\limits_{-1}^{0} \frac{1}{2}\pi(x+1)^2\,\mathrm{d}x + \int\limits_0^1 \frac{1}{2}\pi(-x+1)^2\,\mathrm{d}x = \frac{\pi}{6}(x+1)^3\Big|_{-1}^{0} - \frac{\pi}{6}(-x+1)^3\Big|_0^1 = \frac{\pi}{3}$$

(5 marks)

Question Two

Part (a)

$$\int\limits_{\pi/4}^{\pi/3} \csc^2\theta\,\mathrm{d}\theta = -\cot\theta\Big|_{\pi/4}^{\pi/3} = 1-\frac{1}{\sqrt{3}} \approx 0.4226$$

(2 marks)

Part (b)

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x - x^2 + \frac{2(1+\pi^2)}{\pi}x + \left(\frac{3\pi^2}{4}+2\right)\mathrm{d}x$$

$$= -\cos x - \frac{1}{3}x^3 + \frac{(1+\pi^2)}{\pi}x^2 + \left(\frac{3\pi^2}{4}+2\right)x\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$\approx 86.25748 - 22.01398 \approx 64.24350$$

(3 marks)

Part (c)

(i)

$$\begin{aligned}\int \frac{\mathrm{d}P}{1-\frac{P}{M}} &= \int \mathrm{d}t \\ -M\ln\left(1-\frac{P}{M}\right) &= t + C \\ \ln\left(1-\frac{P}{M}\right) &= -Mt + C \\ 1-\frac{P}{M} &= Ke^{-Mt} \\ P &= M\left(1-Ke^{-Mt}\right)\end{aligned}$$

At $t=0$, $P=100$. So $100 = M(1-Ke)$ and $K = \frac{M-100}{Me}$.
Hence the explicit formula for P is

$$P = M\left(1 - \frac{M-100}{M}e^{-Mt-1}\right) = M + \frac{(100-M)}{e^{Mt+1}}.$$

(4 marks)

(ii) At $t=0$, $\frac{\mathrm{d}P}{\mathrm{d}t} = 1 - \frac{P}{M} = 1$. Hence:

$$\begin{aligned}1 = 1 - \frac{M + \frac{(100-M)}{e^{Mt+1}}}{M} &= \frac{(100-M)}{Me} \\ M &= \frac{100}{e+1}\end{aligned}$$

and therefore

$$P = M + \frac{(100-M)}{e^{Mt+1}} = \frac{100}{e+1} + \frac{\left(100 - \frac{100}{e+1}\right)}{e^{\frac{100t}{e+1}+1}}.$$

At $t=100$, $P \approx 26.89$ — or around 27 animals. (3 marks)

Question Three

Part (a)

Applying Simpson's rule:

$$\frac{1}{3} \cdot \frac{6-0}{6} \cdot \big[3.2 + 1.1 + 4(2.7 + 1.7 + 1.0) + 2(1.9 + 1.3)\big] \approx 10.76.$$

(2 mark)

Part (b)

$$\int e^x (15 + e^x)^{2017} + 3 \,\mathrm{d}x = \frac{(15 + e^x)^{2018}}{2018} + 3x + C.$$

(2 marks)

Part (c)

$$\begin{aligned} \int \frac{\mathrm{d}y}{y+2} &= -\int n\pi \sin(xn\pi) \,\mathrm{d}x \\ \ln(y+2) &= \cos(xn\pi) + C \\ y &= Ke^{\cos(xn\pi)} - 2 \end{aligned}$$

Since $y = 0$ when $x = 0$, we have $2 = Ke$ and so $K = \frac{2}{e}$. Hence $y = 2e^{\cos(xn\pi)-1} - 2$, and when $x = \frac{1}{n}$, $y = 2e^{\cos(\pi)-1} - 2 = 2e^{-2} - 2 = \frac{2}{e^2} - 2 \approx -1.729$. (4 marks)

Part (d)

Let $x = \sin\theta$; then $\mathrm{d}x = \cos\theta \,\mathrm{d}\theta$, and:

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x &= \int \frac{\cos\theta}{\sqrt{1-(\sin\theta)^2}} \,\mathrm{d}\theta \\ &= \int \frac{\cos\theta}{\cos\theta} \,\mathrm{d}\theta \\ &= \int \mathrm{d}\theta \\ &= \theta = \sin^{-1} x. \end{aligned}$$

(4 marks)