# Solutions to L3 Calculus Integration Exam 2

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# Question One

## Part (a)

### (i)

$$\int_{\pi/4}^{\pi/2} 2\csc 2x \cot 2x \,\mathrm{d}x = -\csc 2x \Big|_{\pi/4}^{\pi/2} \qquad = \frac{-1}{\sin\pi} - \frac{-1}{\sin\frac{\pi}{2}} = 1.$$

(2 marks)

### (ii)

$$\begin{aligned}\int_1^4 t\left(\frac{1.5}{\sqrt{t}} + 12\right)\mathrm{d}t &= \int_1^4 1.5t^{0.5} + 12t \,\mathrm{d}t \\ &= t^{1.5} + 6t^2 \Big|_1^4 \\ &= t\sqrt{t} + 6t^2 \Big|_1^4 \\ &= (4\cdot 2 + 6\cdot 16) - (1+6) = 97.\end{aligned}$$

(2 marks)

## Part (b)

$$V = \int \frac{k}{t+1}\,\mathrm{d}t = k\ln(t+1) + C.$$

Now, when $t = 0$ we have $V = 0.5$. So $0.5 = k\ln 1 + C = C$, and $V = k\ln(t+1) + 0.5$. We also know that $2V_0 = V_3$ so $1 = k(\ln 4) + 0.5$ and $k = \frac{1}{2\ln 4} = \frac{1}{4\ln 2}$. Therefore $V = \frac{\ln(t+1)}{4\ln 2} + 0.5$; we want $t$ when $V = 2$ and L2 algebra shows that $t = 63$ minutes. (4 marks)

## Part (c)

$$\begin{aligned}\int_1^2 (-\frac{1}{2}x^2 + \frac{3}{2}x) - (\frac{1}{2}x^2 - \frac{3}{2}x + 2)\,\mathrm{d}x &= \int_1^2 -x^2 + 3x - 2\,\mathrm{d}x \\ &= -\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x\Big|_1^2 \quad = \frac{8}{3} + 6 - 4 + \frac{1}{3} - \frac{3}{2} + 2 \\ &= 5.5.\end{aligned}$$

The area between the curves is 5.5 square units. (3 marks)

# Question Two

## Part (a)

**(i)**

$$\int \frac{2x+1}{x^2+x}\,\mathrm{d}x = \ln(x^2+x)+C$$

(1 mark)

**(ii)**

$$\begin{aligned}
\int \cos^4\theta\,\mathrm{d}\theta &= \int (\cos^2\theta)^2\,\mathrm{d}\theta \\
&= \frac{1}{4}\int (\cos 2\theta + 1)^2\,\mathrm{d}\theta \\
&= \frac{1}{4}\int \cos^2 2\theta + 2\cos 2\theta + 1\,\mathrm{d}\theta \\
&= \frac{1}{8}\int \cos 4\theta + 4\cos 2\theta + 3\,\mathrm{d}\theta \\
&= \frac{1}{8}\left(\frac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right) + C \\
&= \frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3\theta}{8} + C.
\end{aligned}$$

(3 marks)

## Part (b)

$$\begin{aligned}
A &= \int_A^B f(x)-g(x)\,\mathrm{d}x + \int_B^C g(x)-f(x)\,\mathrm{d}x + \int_C^D f(x)-g(x)\,\mathrm{d}x \\
32 &= 2 + \int_B^C g(x)-f(x)\,\mathrm{d}x + 10 \\
20 &= \int_B^C g(x)-f(x)\,\mathrm{d}x \\
-20 &= \int_B^C f(x)-g(x)\,\mathrm{d}x
\end{aligned}$$

(3 marks)

## Part (c)

$$\int \frac{\mathrm{d}y}{y+1} = \int \sin 2\pi t \,\mathrm{d}t$$
$$\ln(y+1) = -\frac{1}{2\pi}\cos 2\pi t + C$$
$$y = Ke^{-\frac{\cos 2\pi t}{2\pi}} - 1.$$

At $t = 0$, $y = 1$. Hence $2 = Ke^{-\frac{1}{2\pi}}$ and $K = 2e^{\frac{1}{2\pi}}$. So:

$$\begin{aligned} y &= 2e^{\frac{1-\cos 2\pi t}{2\pi}} - 1 \\ &= 2e^{\frac{1-\cos\pi}{2\pi}} - 1 \\ &= 2e^{\frac{1}{\pi}} - 1 \\ &\approx 1.796... \end{aligned}$$

(4 marks)

# Question Three

## Part (a)

$$\begin{aligned} \pi \int_0^1 (e^{-x}+1)^2 \,\mathrm{d}x &= \pi \int_0^1 e^{-2x} + 2e^{-x} + 1 \,\mathrm{d}x \\ &= \pi - \frac{1}{2}e^{-2x} - 2e^{-x} + x \Big|_0^1 \\ &= \pi[(-\frac{1}{2}e^{-2} - 2e^{-1} + 1) - (-\frac{1}{2} - 2)] \\ &= \pi\left(\frac{7}{2} - \frac{1}{2e^2} - \frac{2}{e}\right) \\ &\approx 8.4715. \end{aligned}$$

(3 marks)

## Part (b)

Finding velocity $\int 12t + 12 \,\mathrm{d}t = 6t^2 + 12t + C$; at $t = 0$, $v = 0$ so $C = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 6t^2 + 12t$. Then $\int 6t^2 + 12t \,\mathrm{d}t = 2t^3 + 6t + C'$; at $t = 0$, $x = 3$ so $C' = 3$ and $x = 2t^3 + 6t^2 + 3$. Hence $x(10) = 2603$. (3 marks)

## Part (c)

$$\int \frac{\mathrm{d}T}{T - T_0} = \int -k \,\mathrm{d}t$$
$$\ln(T - T_0) = -kt + C$$
$$T = Ke^{-kt} + T_0$$

When $t = 0$, $230 = T = K$. Since $T_0 = 18$, $30 = 230e^{-kt} + 18$ and $t \approx \frac{2.95}{k}$. (4 marks)

## Part (d)

The integration bounds include two asymptotes, at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. (1 mark)