

Solutions to L3 Calculus Integration Exam 1

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Question One

Part (a)

(i)

$$\int \frac{3t^2 + 2t}{\sqrt{t}} dt = \int 3t^{3/2} + 2t^{1/2} dt = \frac{6}{5}t^{5/2} + \frac{4}{3}t^{3/2} + C.$$

(1 mark)

(ii)

$$\int 2 \sin 2x \sin(\cos 2x) dx = \sin \cos 2x + C.$$

(1 mark)

Part (b)

$$y = \frac{1}{\ln 2} \int \frac{1}{x+2} = \frac{\ln(x+2)}{\ln 2} + C$$

Since $3 = \frac{\ln 0+2}{\ln 2} + C = 1 + C$, $C = 2$. Hence $y = \frac{\ln(x+2)}{\ln 2} + 2$, and when $x = -1$, $y = \frac{\ln(-1+2)}{\ln 2} + 2 = 2$. (3 marks)

Part (c)

$$\int_0^{10} f(t) dt = \int_0^y dt + \int_x^{10} f(t) dt - \int_x^y f(t) dt = 4 + 3 - 2 = 5$$

(2 marks)

Part (d)

We integrate along the x -axis from $x = -1$ to $x = 0$ and then from $x = 0$ to $x = 1$. Over the first half, the radius of each semicircle is $r = x + 1$, and over the second half the radius of each semicircle is $r = -x + 1$. Hence our volume will be

$$\int_{-1}^0 \frac{1}{2} \pi (x+1)^2 dx + \int_0^1 \frac{1}{2} \pi (-x+1)^2 dx = \frac{\pi}{6} (x+1)^3 \Big|_{-1}^0 - \frac{\pi}{6} (-x+1)^3 \Big|_0^1 = \frac{\pi}{3}$$

(5 marks)

Question Two

Part (a)

$$\int_{\pi/4}^{\pi/3} \csc^2 \theta d\theta = -\cot \theta \Big|_{\pi/4}^{\pi/3} = 1 - \frac{1}{\sqrt{3}} \approx 0.4226$$

(2 marks)

Part (b)

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x - x^2 + \frac{2(1+\pi^2)}{\pi}x + \left(\frac{3\pi^2}{4} + 2\right) dx \\ = -\cos x - \frac{1}{3}x^3 + \frac{(1+\pi^2)}{\pi}x^2 + \left(\frac{3\pi^2}{4} + 2\right)x \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\ \approx 86.25748 - 22.01398 \approx 64.24350 \end{aligned}$$

(3 marks)

Part (c)

(i)

$$\begin{aligned} \int \frac{dP}{1 - \frac{P}{M}} &= \int dt \\ -M \ln \left(1 - \frac{P}{M}\right) &= t + C \\ \ln \left(1 - \frac{P}{M}\right) &= -Mt + C \\ 1 - \frac{P}{M} &= Ke^{-Mt} \\ P &= M \left(1 - Ke^{-Mt}\right) \end{aligned}$$

At $t = 0$, $P = 100$. So $100 = M(1 - Ke)$ and $K = \frac{M-100}{Me}$.
Hence the explicit formula for P is

$$P = M \left(1 - \frac{M-100}{M}e^{-Mt-1}\right) = M + \frac{(100-M)}{e^{Mt+1}}.$$

(4 marks)

(ii) At $t = 0$, $\frac{dP}{dt} = 1 - \frac{P}{M} = 1$. Hence:

$$\begin{aligned} 1 &= 1 - \frac{M + \frac{(100-M)}{e^{Mt+1}}}{M} = \frac{(100-M)}{Me} \\ M &= \frac{100}{e+1} \end{aligned}$$

and therefore

$$P = M + \frac{(100-M)}{e^{Mt+1}} = \frac{100}{e+1} + \frac{\left(100 - \frac{100}{e+1}\right)}{e^{\frac{100t}{e+1}+1}}.$$

At $t = 100$, $P \approx 26.89$ — or around 27 animals. (3 marks)

Question Three

Part (a)

Applying Simpson's rule:

$$\frac{1}{3} \cdot \frac{6-0}{6} \cdot [3.2 + 1.1 + 4(2.7 + 1.7 + 1.0) + 2(1.9 + 1.3)] \approx 10.76.$$

(2 mark)

Part (b)

$$\int e^x(15 + e^x)^{2017} + 3 \, dx = \frac{(15 + e^x)^{2018}}{2018} + 3x + C.$$

(2 marks)

Part (c)

$$\begin{aligned} \int \frac{dy}{y+2} &= - \int n\pi \sin(xn\pi) \, dx \\ \ln(y+2) &= \cos(xn\pi) + C \\ y &= Ke^{\cos(xn\pi)} - 2 \end{aligned}$$

Since $y = 0$ when $x = 0$, we have $2 = Ke$ and so $K = \frac{2}{e}$. Hence $y = 2e^{\cos(xn\pi)-1} - 2$, and when $x = \frac{1}{n}$, $y = 2e^{\cos(\pi)-1} - 2 = 2e^{-2} - 2 = \frac{2}{e^2} - 2 \approx -1.729$. (4 marks)

Part (d)

Let $x = \sin \theta$; then $dx = \cos \theta \, d\theta$, and:

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}} \, dx &= \int \frac{\cos \theta}{\sqrt{1-(\sin \theta)^2}} \, d\theta \\ &= \int \frac{\cos \theta}{\cos \theta} \, d\theta \\ &= \int d\theta \\ &= \theta = \sin^{-1} x. \end{aligned}$$

(4 marks)