NCEA Level 2 Mathematics
7. Linear Inequalities

An equation is a statement which says that two quantities are identical. If we don't want to be so precise, we can talk about inequalities: statements which tell us about the *relative size* of two quantities. More precisely, if a and b are two quantities then:

$a = b$	a is identical to b
$a \neq b$	a is not identical to b
$a \leq b$	either a is identical to b, or a is smaller than b
$a < b$	a is not identical to b and a is smaller than b

This allows us to impose an *ordering* structure onto the integers, as well as the algebraic structure that they already had. We will look at the interplay between the two in the exercises.

Questions

1. Justify the following statements with mathematical reasoning (where a, b, and c are quantities):
   (a) Precisely one of $a < b$, $a = b$, or $a > b$ is true.
   (b) If $a \leq b$ and $b \leq c$ then $a \leq c$.
   (c) If $a \leq b$ and $b \leq a$ then $a = b$.
2. Using number lines, explain why
   (a) $(-3) + (-4) = -7$;
   (b) $(-2) + 4 = 2$;
   (c) $(-10) + 4 = -6$.
3. Consider the following multiplication table.

a		b	ab
2	$\times$	5	10
2	$\times$	4	8
2	$\times$	3	6
2	$\times$	2	4
2	$\times$	1	2
2	$\times$	0	0
2	$\times$	-1	
2	$\times$	-2	
2	$\times$	-10	

   (a) What is the pattern in the final column?
   (b) Fill in the final three lines of the table by continuing the pattern.
   (c) Based on this table, is it more reasonable for the product of a **negative by a positive** to be positive or negative?
   (d) Using this definition, fill in the first five lines of the next table:

a		b	ab
-3	$\times$	4	
-3	$\times$	3	
-3	$\times$	2	
-3	$\times$	1	
-3	$\times$	0	
-3	$\times$	-1	
-3	$\times$	-2	
-3	$\times$	-10	

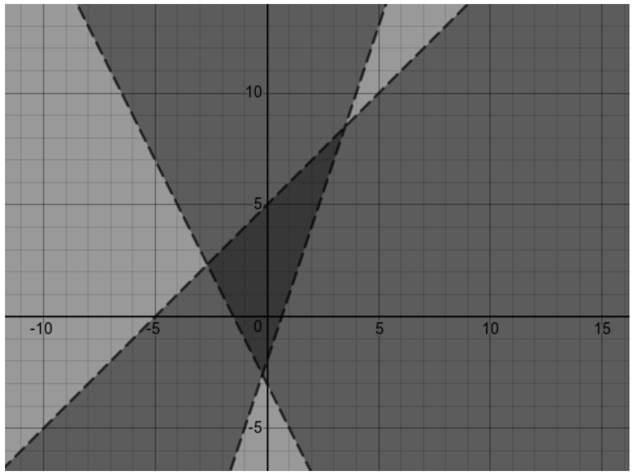
(e) Again using the pattern we see as we move down the final column, fill in the last three rows.

(f) Based on this table, is it more reasonable for the product of a **negative by a negative** to be positive or negative?

4. Justify the following statements with mathematical reasoning (where a, b, c, and d are quantities) — you may want to draw number lines, it makes things easier to visualise:
   (a) If $a \leq b$ and c is *positive* then $ac \leq bc$.
   (b) If $a \leq b$ and c is *negative* then $ac \geq bc$.
   (c) If $a \leq b$ and c is any quantity then $a + c \leq b + c$.
   (d) If $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.
   (e) If $a \leq b$ and $c \leq d$ then we cannot make any statement about the relative values of $a + d$ and $b + c$. [Hint: consider $1 \leq 2$ and $1 \leq 1$ as $a \leq b$ and $c \leq d$ respectively, then swap them around.]

5. We will now look at inequalities which involve variables.
   (a) For each of the following inequations, graph all the possible values of x and y that satisfy it.
      i. $4 + x < 3$
      ii. $3x + 2 \geq 2$
      iii. $x \geq y$
      iv. $x \leq y$
      v. $3x + 9y \leq 1$
      vi. $2x + y \geq 0$
   (b) Graph all possible values of x and y satisfying each of the following *sets* of inequalities. (The resulting region of the plane is called the *feasible region* of the system.)
      i. $x < y$, $x > y$, and $x < 2y$
      ii. $x \leq 2$, $x \geq -1$, $y \leq x$, $y \geq x - 3$

6. Consider the following graphed system of inequalities.



   (a) Explicitly write down the three inequalities that have been graphed.

   (b) What are the coordinates of the three intersection points?

7. (a) Show that no point simultaneously satisfies both of $y \geq 2x + 1$ and $y \leq 2x - 3$.

   (b) Show that if $y \leq Ax + B$ is any linear inequality in x and y, then the feasible region of this inequality overlaps with at least one of the inequalities in (a).

8. The **arithmetic mean** of two numbers a and b is $\frac{a+b}{2}$; the **geometric mean** of a and b is $\sqrt{ab}$.

   (a) Calculate the arithmetic and geometric means for several pairs of numbers. Make a conjecture about the relative order of the two means (is one always bigger than the other)?

   (b) Show that

$$\left(\frac{a+b}{2}\right)^2 - ab = \left(\frac{a-b}{2}\right)^2.$$

   (c) Suppose that a and b are positive numbers. Using part (b), or otherwise, show that the geometric mean of a and b is always less than or equal to their arithmetic mean. When are the two equal?

   (d) Investigate the cases where a and b are both negative, or one is negative and one is positive. (Hint: one of these cases makes no sense.)