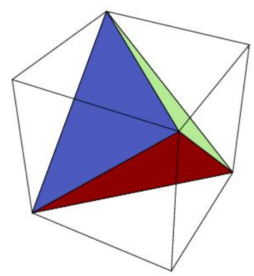
NCEA Level 3 Calculus (Integration)
26. More Interesting Problems

These problems do not just concern integration.

Questions

1. Find the equation of the line through the point $(3, 5)$ which cuts off the least area from the first quadrant. E
2. The area of a square is increasing at a constant rate of k m^2 per second. A tetrahedron (equilateral triangular pyramid), has a side length that is the same as that of the square at each instant. The initial volume of the tetrahedron was $1\,\mathrm{m}^3$. In terms of k, what is the volume of the tetrahedron three seconds after that? E
3. Consider the tetrahedron inscribed inside a cube, as in the figure. S

   The volume V of the cube at any instant t is increasing at a rate proportional to the value of V at that instant. The initial volume of the cube at $t = 0$ was 8 cubic units. What is the volume of the tetrahedron at time $t = 20$?
4. If $x \sin \pi x = \int_0^{x^2} f(t)\,\mathrm{d}t$, where f is continuous, find $f(4)$. [*Hint: you need not perform any integration.*] S
5. If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that S
   $$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.$$
   Hence, or otherwise, evaluate
   $$\lim_{x\to 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}.$$
6. (a) Consider the differential equation S
   $$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + 5\frac{\mathrm{d}\Phi}{\mathrm{d}t} + 6\Phi(t) = 0.$$
   Let f and g be the functions defined by $f(x) = e^{-2x}$ and $g(x) = e^{-3x}$.
      i. Show that all linear combinations of f and g are solutions to the differential equation.
      ii. Find the (unique) solution passing through $(0, 1)$ and $(1, 1)$.

   (b) More generally, consider the differential equation $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$. Let the zeroes of the quadratic polynomial $p(D) = aD^2 + bD + c$ be α and β. Show that all the linear combinations of $e^{\alpha x}$ and $e^{\beta x}$ are solutions to the differential equation. O

7. Compute the following definite integral. [*Hint: begin with a substitution.*] S

$$\int_0^{\pi/6} \sqrt{\tan\theta}\,\mathrm{d}\theta$$

8. (a) Consider the two functions $p(x) = 3x^5 - 5x^3 + 2x$ and $q(x) = 3x^5$. Show that their ratio approaches 1 as $x \to \infty$. E

   (b) Let $p(x)$ and $q(x) \neq 0$ be polynomials. Recall that the degree of a polynomial is the highest n such that x^n has a non-zero coefficient. Compute the limit S

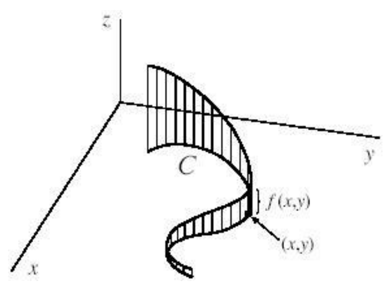
$$\lim_{x\to\infty} \frac{p(x)}{q(x)}$$

   if:

   i. the degree of $p(x)$ is less than that of $q(x)$.

   ii. the degree of $p(x)$ is greater than that of $q(x)$.

9. A definite integral calculates the between a curve and straight line, the x-axis. In a similar way, it is possible to use integration to calculate the area above a curved line and below a surface $z = f(x, y)$, like that in the figure. S



If the curve C is defined parametrically, that is $C(t) = (x(t), y(t))$, then the integral along the line can be calculated with the formula

$$\int_a^b f(x(t), y(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Compute the line integral of the function $f(x, y) = 2 + x^2y$ around the upper half of the unit circle.

10. The **sine integral** function is defined by S

$$\mathrm{Si}(x) = \begin{cases} \int_0^x \frac{\sin t}{t}\,\mathrm{d}t, & \text{for } x \neq 0; \\ 1, & \text{for } x = 0. \end{cases}$$

   (a) Recall that $\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a)$. Use this to show that $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f'(t)\,\mathrm{d}t = f'(x)$.

   (b) Find the x-coordinate of the first local maximum of Si to the right of the origin. Carefully prove that you have found a maximum.

   (c) Use the result in (a) to find an expression for the integral

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)\,\mathrm{d}t,$$

   where f is continuous and g and h are differentiable.

11. Minimise the function $f(x) = b \log_b N$ with respect to b, and show that the result is independent of the constant N.* E

12. We can calculate **improper integrals** (those where the bounds are infinite) as follows: S

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x.$$

Decide whether each of the integrals given below exists. If so, calculate its value; if not, explain why.

(a) $\displaystyle\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ (b) $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ (c) $\displaystyle\int_1^\infty \sin x\,\mathrm{d}x$

13. (a) Show that $F(x) = \tan^{-1} x$ is an anti-derivative of $f(x) = \frac{1}{1+x^2}$ in the following ways: S
    i. Differentiate $F(x)$ and simplify to give $f(x)$.
    ii. Use the substitution $x = \tan\theta$ to integrate $f(x)$ and simplify to give $F(x)$.

    (b) Recall that $22/7$ is often given as a rough approximation to π. Consider the integral O

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x,$$

and hence show that $22/7 > \pi$.†

14. Consider the operator $\mathcal{L}$ defined by S

$$\mathcal{L}f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln\left[f\left(e^x\right)\right].$$

(a) Show that $\mathcal{L}x^n = n$ and that $\mathcal{L}[u(x)]^n = n\mathcal{L}u(x)$.

(b) Find an expression for $\mathcal{L}[u(x)v(x)]$ and $\mathcal{L}[u(x)/v(x)]$.

(c) Find an expression for $\mathcal{L}[u(x) + v(x)]$.

(d) For which y is $\mathcal{L}y = y$?

15. Compute the following indefinite integrals: S

(a) $\displaystyle\int \frac{\sin\frac{1}{x}}{x^2}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{\ln x \cos x - \frac{\sin x}{x}}{(\ln x)^2}\,\mathrm{d}x$

16. (Harder!) Suppose ι is a function defined as O

$$\iota(x) = \int_0^x t^t \sin(t)\,\mathrm{d}t.$$

(a) Write down the domain and range of ι (caution: be careful for negative fractional x).

* Dudley, *Mathematical Cranks*, p.52.
† Nahin, *Inside Interesting Integrals*, pp.23-4.

(b) Suppose I is an antiderivative of $t^t \sin(t)$. Show that

$$\iota^{-1}(x) = I^{-1}(x + I(0))$$

is the only possibility for the inverse of ι, if the inverse exists.

(c) Find the derivative of ι with respect to x. Hence, show that ι changes from decreasing to increasing at an odd number of points within the interval $(16, 20)$. Conclude that ι^{-1} is not a function, and hence ι has no well-defined inverse.

17. A while ago (when we talked about the product and quotient rules), I claimed that the radius of the circle best approximating a continuous curve around a point (x, y) is given by 0

$$\text{radius of curvature} = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|}.$$

Let us attempt to prove this.

(a) Let f be a continuous function at x such that the second derivative of f at x exists. By recalling our work on approximations, explain why knowing up to the second derivative of f should be enough to find the 'best circular approximation' of f at $(x, f(x))$.

(b) Consider the circle of radius r centred at (x_0, y_0). Suppose that this circle passes through the point (x_1, y_1); suppose further that the first derivative of the y-ordinate of the circle with respect to the x-ordinate is m, and that the second derivative is c. Write down expressions for r, x_0, and y_0 in terms of x_1, y_1, m, and c.

(c) Use part (b) to write down the radius of the unique circle passing through $(x, f(x))$ with matching first and second derivatives to f.

18. We prove that π is irrational.[‡] Suppose that $\pi = \frac{a}{b}$ where a and b are positive integers. Let n be a positive integer, and define 0

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$
$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x).$$

(a) Show that the value of f and every derivative of f at $x = 0$ and $x = a/b$ is integral. Conclude that $F(\pi) + F(0)$ is an integer.

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}[F'(x)\sin x - F(x)\cos x] = f(x)\sin x$, and hence that

$$\int_0^\pi f(x)\sin x = F(\pi) + F(0). \qquad (*)$$

(c) Prove that $0 < f(x)\sin x < \frac{\pi^n a^n}{n!}$, and conclude that (*) is positive but arbitrarily small for sufficiently large n.

(d) Derive a contradiction from (a) and (c).

[‡] Ivan Niven, *A simple proof that π is irrational.*

Written Questions

It is possible for scholarship exams to include written questions; the following are some examples of the subjects and style of question.

1. Discuss the significance of the fundamental theorem of calculus.
2. Many natural phenomena are modelled well by the differential equation $y = \frac{\mathrm{d}y}{\mathrm{d}x}$. Discuss the reasons for this.
3. Compare and contrast the geometric and algebraic viewpoints of the complex numbers.
4. One of the main ideas of calculus is the approximation of complex phenomena with simpler models. Discuss, with examples.