NCEA Level 3 Calculus (Integration)
16. Anti-differentiation

Goal for this week

To practice undoing differentiation.

For some time, we have been hinting that finding area was in some way the inverse to finding slope. In last week's questions, the curtain was pulled back a little further when we calculated an area function for a variable endpoint, took the derivative, and got the original function back. Next week, all will be revealed; but first, we need to do a little bit more work. This week, we're going to look at **anti-differentiation**: finding the original function given its derivative.

Example. One antiderivative of $y' = 3x^2 + 4$ is $x^3 + 4x$. Another is $x^3 + 4x + 1$. A third is $x^3 + 4x + 7$. Obviously every function of the form $y = x^3 + 4x + C$ for some constant C will differentiate to the given y', so we must remember always to make this clear.

We also call antiderivatives **indefinite integrals**, and in this notation the above example is

$$\int 3x^2 + 4 \, \mathrm{d}x = x^3 + 4x + C.$$

Unfortunately, there is no 'easy' way to anti-differentiate; we simply have to try to rearrange the function in some clever way until it looks like something that we know how to deal with.

Joke. Two mathematicians are in a bar. The first one says to the second that the average person knows very little about basic mathematics. The second one disagrees and claims that most people can cope with a reasonable amount of maths. The first mathematician goes off to the washroom, and in his absence the second calls over the waitress. He tells her that in a few minutes, after his friend has returned, he will call her over and ask her a question. All she has to do is answer "one third x cubed." She repeats "one thir–dex cue?" He repeats "one third x cubed." She asks, "one thir dex cuebd?" "Yes, that's right," he says. So she agrees, and goes off mumbling to herself, "one thir dex cuebd...". The first guy returns and the second proposes a bet to prove his point, that most people do know something about basic math. He says he will ask the blonde waitress an integral, and the first laughingly agrees. The second man calls over the waitress and asks "what is the integral of x squared?" The waitress says "one third x cubed" and while walking away, turns back and says over her shoulder, "plus a constant!"

Examples.

1. The most general antiderivative of $\sin x$ is $-\cos x + C$.
2. The most general antiderivative of $\tan x$ is $-\ln|\cos x| + C$.
3. $\int \frac{1}{x+3} \, \mathrm{d}x = \ln|x+3| + C$.
4. $\int \tan^2 \theta \, \mathrm{d}\theta = \int \sec^2 \theta - 1 \, \mathrm{d}\theta = \tan \theta - \theta + C$.
5. $\int \frac{2x}{x^2+1} \, \mathrm{d}x = \ln\left|x^2 + 1\right| + C$.
6. $\int Ke^{Kx} \, \mathrm{d}x = e^{Kx} + C$ for all constants K.

Questions

1. For each expression, find the most general antiderivative with respect to x. A

(a) $2x$

(b) x^{-3}

(c) 0

(d) $\sec^2 x + \sqrt{x}$

(e) $x\sqrt{x}$

(f) $\sin x - \cos x$

(g) $\frac{2x^3+3x-\sqrt{x}}{\sqrt[3]{x}}$

(h) $\frac{1}{x^2} + e^x$.

2. Verify the examples in the notes by differentiation. A

3. Show that $\int 3x^2 + 4x + 5 + 2e^{2x}\,\mathrm{d}x = x^3 + 2x^2 + 5x + e^{2x} + C$. A

4. Find y' when $y = 3 + \sin(2x+4)$ and hence find $\int 2\cos(2x+4)\,\mathrm{d}x$. A

5. If $\frac{\mathrm{d}y}{\mathrm{d}t} = 1.5\sqrt{t}$ and $y(4) = 10$, find $y(t)$ exactly. A

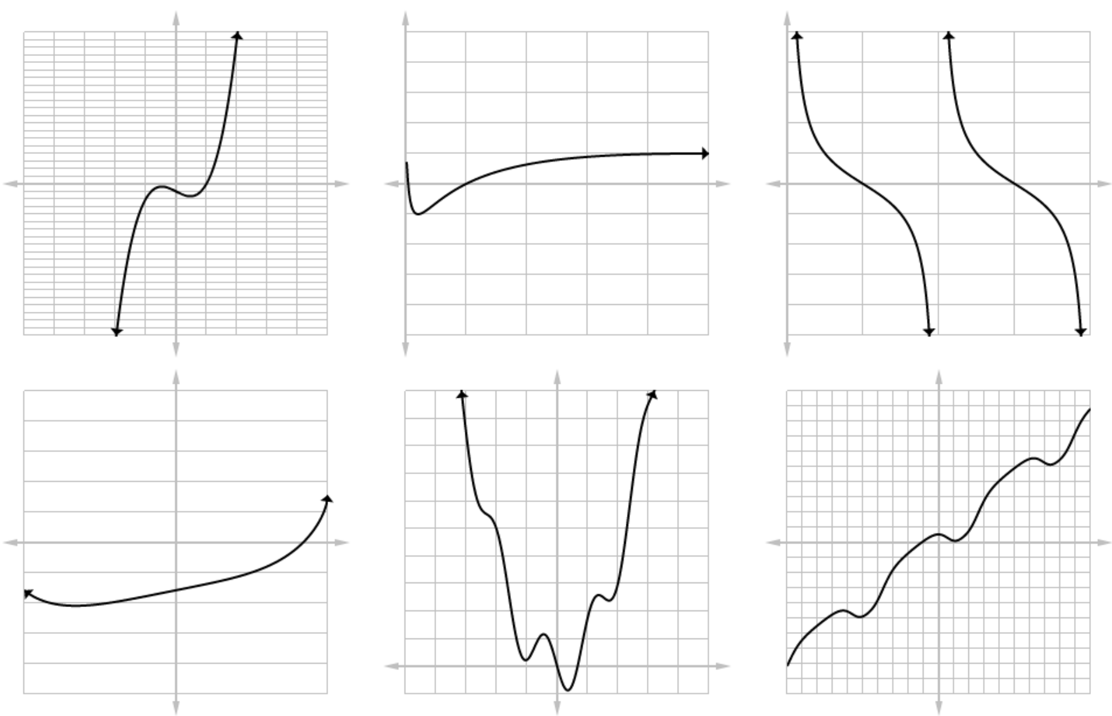
6. Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$. M

7. The velocity of a particle is given by $v(t) = 2t + 1$. Find its position at $t = 4$ if its position at $t = 0$ is $x = 0$. A

8. The acceleration of a particle is given by $a(t) = 10\sin t + 3\cos t$. At $t = 0$, its position is $x = 0$; at $t = 2\pi$, its position is $x = 12$. Find its position at $t = \frac{\pi}{2}$. M

9. Find all functions g such that $g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$. A

10. For each function, sketch an antiderivative passing through $(0, 0)$: M



11. Show that if F is an anti-derivative of f, G is an anti-derivative of g, and α and β are any constants, then $\alpha F + \beta G$ is an anti-derivative of $\alpha f + \beta g$. E

12. Give an example of functions f and g such that if F and G are anti-derivatives of f and g respectively then FG is *not* an anti-derivative of fg. E

13. Note that on the formula sheet, the anti-derivative of $1/x$ is given as $\ln|x|$, not just $\ln x$. E

(a) Compute $\frac{\mathrm{d}}{\mathrm{d}x} \ln|x|$ if $x < 0$, and hence justify formally why $\frac{\mathrm{d}}{\mathrm{d}x} \ln|x| = 1/x$.

(b) Draw $y = \ln|x|$ and $y = 1/x$ on the same pair of axes, and hence justify intuitively why $\frac{\mathrm{d}}{\mathrm{d}x} \ln|x| = 1/x$.