NCEA Level 2 Mathematics
4. Functions

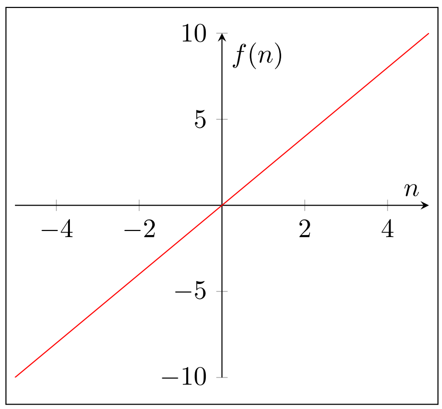
One of the most fundamental concepts in mathematics is that of a function. A function is a relationship between two sets of things, called the *range* and the *domain*, such that everything in the range is related to exactly one thing in the domain. You can think of a function as a rule: it could be given by a formula, or by a list of inputs and outputs, or in any other way that one likes.

If f is a function, and it relates x to y, then we write $f(x) = y$. In this notation, x is the *argument* or *input* and y is the *result* or *output*.

The above expression $f(x) = y$ is suggestive: we can graph functions by graphing every pair of numbers (x, y) which satisfies this equation.

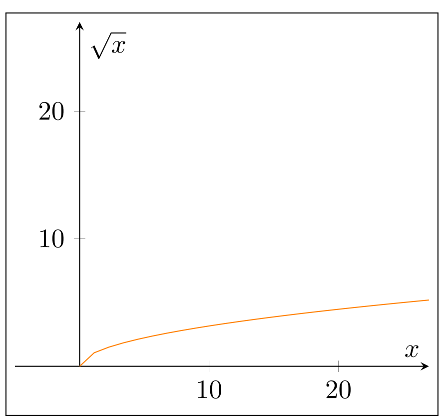
Example.

1. Suppose for every number n we associate its double, $2n$. This is a perfectly good function, which we can call f: then $f(n) = 2n$, and $f(2) = 2 \cdot 2 = 4$.

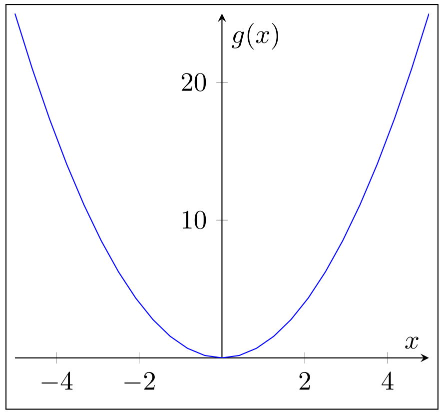


2. Suppose for every number x we associate the number which, when squared, gives x. This is *not* a well-defined function: for example, what number do we associate with 4: 2, or -2? What number do we associate with -1?

3. On the other hand, suppose for every positive number x we associate its positive square root. This time we do have a well-defined function. Note that its domain and range are both the positive numbers, and for every number in the domain there is precisely one number in the range.



4. Let $g(x) = x^2$. This is also a perfectly good function; every number has exactly one square. Note that $g(-2) = g(2) = 4$; this is allowed, but if a function f has the property that if x and y are different then $f(x) \neq f(y)$ then f is called one-to-one. The functions in (1) and (3) above are both one-to-one.



Notice that this is a square root on its side... can you explain why?

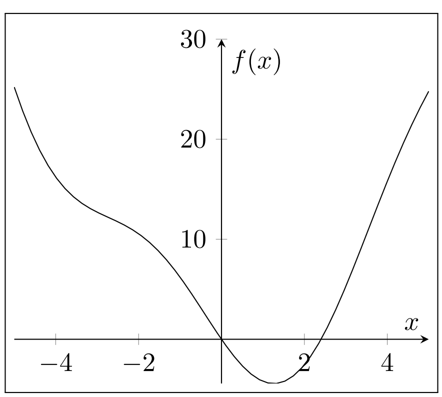
One-to-one functions are useful, because they have an *inverse*. That is, given any number in the *range* we can find the number in the *domain* that maps to it. If f is a one-to-one function, then its inverse is written as f^{-1}.

Example.

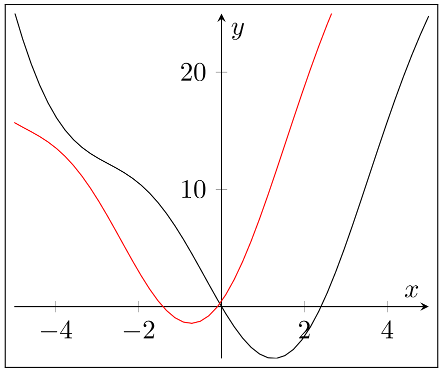
1. If $f(x) = 2x$, then $f^{-1}(x) = \frac{1}{2}x$.
2. If $g(x) = x^2$, then g does not have an inverse over all the numbers; but if we restrict the domain of g to the positive numbers, then it does have an inverse $g^{-1}(x) = +\sqrt{x}$.
3. If $h(x) = \sin x$ (and the range of x is restricted to $0 < x \leq 2\pi$) then h is a perfectly good function with range $-1 \leq h(x) \leq 1$; its inverse is $\sin^{-1}$, which takes a triangle ratio and returns the appropriate angle.

Transforming graphs

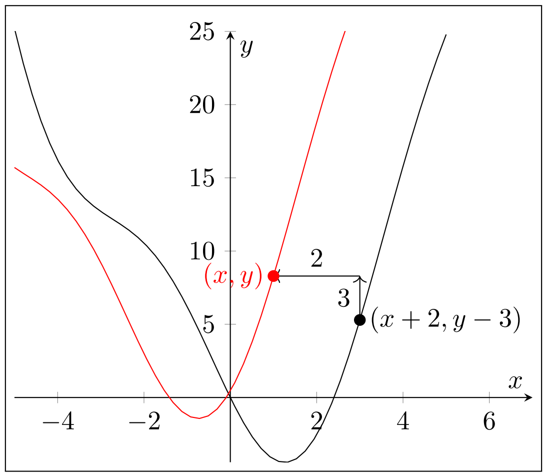
We will now look at shifting graphs around. Consider the graph drawn in black in the following; it depicts the function f.



Suppose we want to find the function whose graph is the same as the graph of f, but shifted to the left by 2 units, and up by 3 units, as drawn here in red.



Suppose a point (x, y) lies on the new graph. Then the point $(x + 2, y - 3)$ lies on the old graph:



Since $(x + 2, y - 3)$ lies on the graph of f, it must satisfy the equation

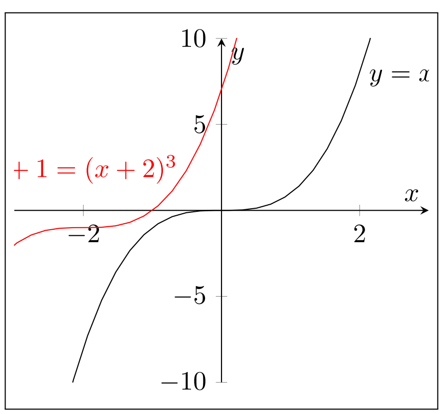
$$y\text{-ordinate} = f(x\text{-ordinate}).$$

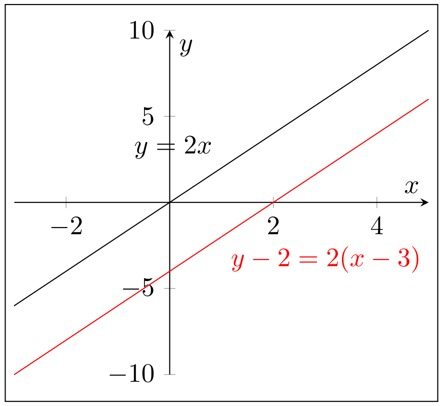
In particular, $y - 3 = f(x + 2)$.

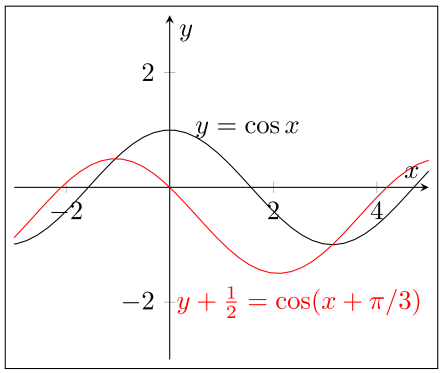
Thus, the equation corresponding to the new graph is $y - 3 = f(x + 2)$. This illustrates the following general principle:

If the graph of $y = f(x)$ is shifted to the right by a distance c and up by a distance d, then the resulting graph is the graph of $y - d = f(x - c)$.

Here are some examples:







Questions

1. Draw the graphs of the following equations. Within each lettered part, compare and contrast the graphs you obtain and explain why the things you observe occur.

   Use a mixture of techniques from: (a) writing down a table of values; (b) using a calculator or computer; (c) recognising the shape from earlier studies (e.g. last year you studied parabolae).

   (a) i. $y = x^2$
   ii. $y = x^2 + 1$
   iii. $y = (x+1)^2$
   iv. $y - 1 = (x+1)^2$

   (b) i. $y = (x-3)^3$
   ii. $y = (x-3)^3 + 2$
   iii. $y = x^3$

   (c) i. $y = 2^x$
   ii. $y = 2^{x+2}$
   iii. $y = 2^{x-2}$
   iv. $y = 5 \times 2^x$

   (d) i. $y = \sin x$
   ii. $y = 3 \sin x$
   iii. $y = \cos x$
   iv. $y = \sin(x - \pi/2)$
   v. $y = 1 + \sin x$

2. Justify the following statements with mathematical reasoning.

(a) The graph of a function passes the vertical line test: any vertical line drawn on the graph crosses the function in at most one point.

(b) The graph of a one-to-one function passes the horizontal line test: any horizontal line drawn on the graph crosses the function in at most one point.

(c) The graph of the inverse of a function is the reflection of the graph of the function across the line $x = y$.

(d) If we define f by

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

then f is a function but it is not one-to-one.

(e) If f and g are functions, and the range of g is the same as the domain of f, and we define a new function $(f \circ g)$ by $(f \circ g)(x) = f(g(x))$, then $(f \circ g)$ is a function with the same range as f and the same domain as g.

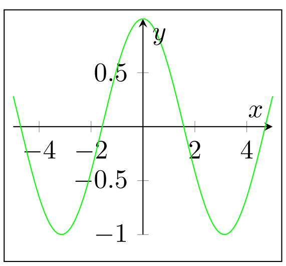
3. The following define functions only if they have a domain which is restricted. What is the largest possible domain for each, so that they are still functions? What is the range of each?

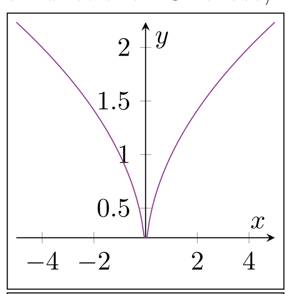
(a) $f(x) = 1/x$

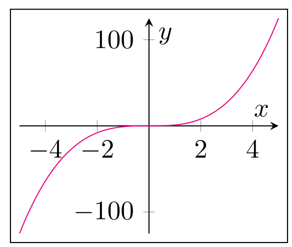
(b) $g(\theta) = \tan\theta$

(c) $h(x) = \left(\dfrac{x}{x-2}\right)^2$

4. Which of the following graphs are graphs of functions? Of those, which are one-to-one?

(a)


(b)


(c)


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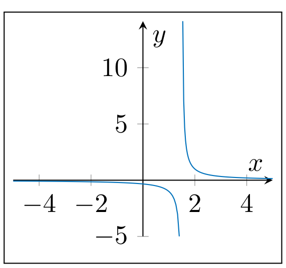

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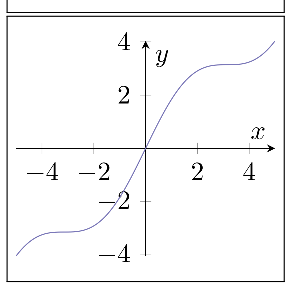
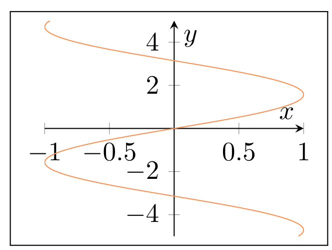
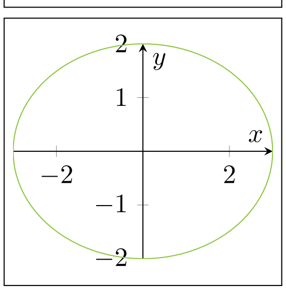
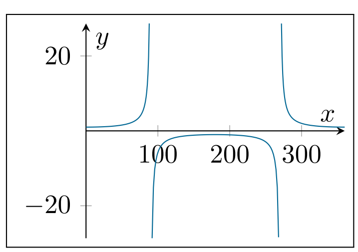
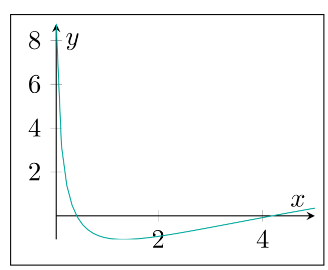


(f)


(g)


(h)


(i)


5. Graph the following functions.

(a) $y = 2^{x+1} - 3$

(b) $y = (x-3)^2 + 2$

(c) $y = \cos(x - \pi/2) + 1$

(d) $y = (x-1)(x-2)(x-3)$

(e) $y = 3\sin(\frac{1}{2}x + 1) - 2$

6. Is it possible to find K such that the function

$$f(x) = \begin{cases} x^3 - Kx + K & x \le 0 \\ \sin x & x \ge 0 \end{cases}$$

is well-defined at 0?

7. Is it possible to find K such that the function

$$f(x) = \begin{cases} x^3 - Kx + 3 & x \le 0 \\ K\cos x & x \ge 0 \end{cases}$$

is well-defined at 0?

8. What is the equation of the straight line through $(-2,-1)$ that meets $y = (x-3)(x-2)$ at its y–intercept?

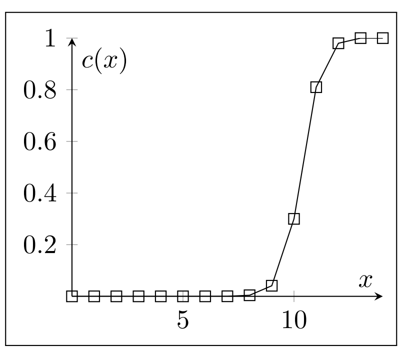
9. If I take $y = x^2$ and I want to shift and stretch (equally in each direction) its graph so that the vertex is sitting at $(-2, 3)$ and it goes through $(1, 1)$, can I do that? If so, what is the equation of the function that gives me the new graph?

10. If I take $y = 2^x$ and I want to shift and stretch (equally in each direction) its graph so that it passes through both $(-2, 3)$ and $(1, 1)$, can I do that? If so, what is the equation of the function that gives me the new graph?

11. Find A such that the graph of $f(x) = A \cdot 3^{x-1} + 2$ passes through the given point, or explain why no such A can be found. (a) $(0, 4)$; (b) $(0, 2)$; (c) $(0, -2)$; (d) $(0, -4)$.

12. Let $y = 3x^2 + bx + 9$. Find b such that the graph of the equation intersects $y = 2x^2$ at exactly one point.

13. The following graph is of the function c, where $c(x)$ is the proportion of a particular chemical species in solution at pH x (pH is, roughly speaking, a measure of acidity). *

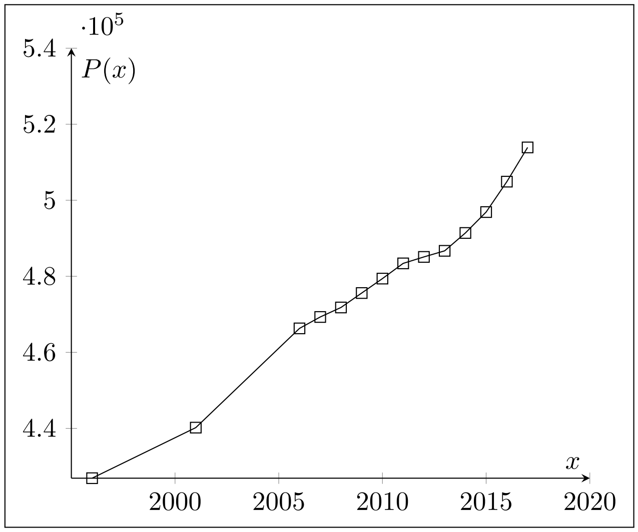


(a) Is this function invertible? That is, if I measure the concentration of this species at any point, can I identify precisely what the pH of the solution is?

(b) Explain why knowing the proportion of a chemical species in water at a given pH might be useful; why is mathematical modelling useful in such a situation?

*From *Quantitative Chemical Analysis*, 7th edition, by Daniel Harris (page 233).

14. The following graph is of the function P, where $P(x)$ is the population of the Greater Wellington region in the year x.[†] Note that the y-axis does *not* start at zero, and is given in units of hundreds of thousands.



(a) Explain why extrapolating using this graph to find $P(2020)$ is probably justified.

(b) Give an example of a function where the values of the function above any point cannot be extrapolated from knowing the values of the function before the point. Do you think that extrapolation is normally possible for mathematical models of the natural world?

(c) Describe how the rate of population growth changes as we move forwards in time from 1996 to 2017.

(d) Given the real-world context of this graph, how would you expect the slope of the graph to change:

i. in the medium term (up to, say, 2040), and

ii. in the long term (say around the year 2100).

[†]Statistics are the Statistics NZ 'Subnational population estimates (RC, AU), by age and sex, at 30 June 1996, 2001, 2006-17 (2017 boundaries)' (retrieved 19 June 2018).