Scholarship Calculus

Question 1: Conics of Clarity

Consider the two conics:

$$x^2 - y^2 + 4y - 20 = 0$$
$$x^2 - 8x - 8y^2 = 0$$

(a) Classify and describe both conics.

(b) Find the point(s) of intersection of the two conics.

Question 2: Polynomials of Pleasure

Consider the equation $x^5 = 1$. One of the solutions is obviously $x = 1$; the other four solutions are complex.

(a) Draw all five solutions to this equation on an Argand diagram, labelling the angles between them (and label the non-real roots).

(b) Given that $x - 1$ is a factor of the equation, show that for any complex root α such that $\alpha^5 = 1$ then $\alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$.

(c) i. Show that the minimal polynomial with roots $\alpha^2 + \alpha^3$ and $\alpha + \alpha^4$ is equivalent to $p(x) = x^2 - x(\alpha + \alpha^2 + \alpha^3 + \alpha^4) + (\alpha + \alpha^4)(\alpha^2 + \alpha^3)$.

ii. Simplify $p(x)$ to obtain a polynomial with real coefficients.

Question 3: Multiplying Midgets

We can model a simple population (of animals, fungi, bacteria, and so on) using the *logistic equation*. The logistic equation is given by

$$\frac{dy}{dx} = ky(1 - \frac{y}{L})$$

where k and L are constants. L is called the carring capacity of the population.

(a) Describe the behaviour of a population as y tends to L (without solving the differential equation). Does this behaviour make sense in the context of a population? Why might L be called the "carrying capacity"?

(b) i. Find the general solution of the differential equation for y.

ii. Find the particular solution where $L = 5$, $k = 2$, and $y(0) = 8$.