NCEA Level 3 Trigonometry (exercise set)
7. Trigonometric Equations

Goal To understand the roots of the trigonometry functions.

1. Let a and b be real and define a function f by $f(\theta) = a\cos\theta + b\sin\theta$. We will study the behaviour of f.
   (a) Show that if α and β satisfy $f(\alpha) = f(\beta)$ then $\frac{b}{a} = \tan\frac{\alpha+\beta}{2}$.
   (b) Find all α such that $f(\alpha) = 0$.
   (c) Show that $f(\theta) \leq \sqrt{a^2+b^2}$ for all θ. Is equality ever attained?

2. Find all θ such that:
   (a) $64\sin^7\theta + \sin 7\theta = 0$.
   (b) $\cot^2\theta - 3\cot\theta + 2 = 0$. (Note: there is no exact value for $\arctan 1/2$, so leave your answer in this form.)
   (c) $\frac{\cos 3\theta}{\sec\theta} + \frac{\sin 3\theta}{\csc\theta} = \cos 2\theta$.

3. Find sine and cosine of $15°$ and $150°$ exactly.

We studied exact values of some trig functions in this section; we will now look at some exact formulae for π.

4. Prove *Machin's formula*, $\pi/4 = 4\arctan(1/5) - \arctan(1/239)$.

5. The formula we will consider here was discovered by John Wallis in 1655.
   (a) Recall that if a polynomial $p(x) = p_0 + p_1 x + \cdots + p_n x^n$ has roots $\alpha_1, ..., \alpha_n$ then we can write $p(x) = (x-\alpha_1)\cdots(x-\alpha_n)$.
   Convince yourself that the following infinite product (due to Euler) is plausible.
   $$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots$$
   [One can prove this rigorously: it is a special case of the so-called *Weierstraß factorisation theorem*.]
   (b) Let $x = \pi/2$, and use the formula in (a) to show that
   $$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots.$$
   (c) Calculate π by hand to a few decimal places.

Additional reading Hobson chapter VI.