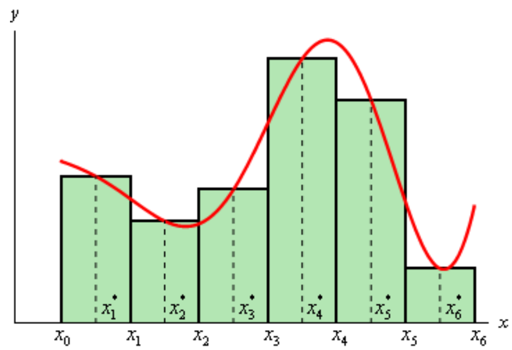
# NCEA Level 3 Calculus (Integration)
# 15. Approximating Areas

**Goal for this week**

To work out roughly how big the area underneath a graph is.

We now move from studying the geometry and shape of curves themselves to studying the geometry and shape of the areas bounded by curves. Just as for differentiation, we begin by finding finite approximations: in this case, of areas. We will mainly be interested in finding the areas between a curve and the $x$-axis.

## Approximating area using rectangles




We start with the simplest and most naive approximation: using a bunch of rectangles under the curve. Suppose we wish to find the area under the curve $y = f(x)$ between the two lines $x = a$ and $x = b$ by splitting it up into $n$ rectangles of width $\Delta x$. If we pick a value $x_i^*$ inside each rectangle, as shown in the diagram, the approximate area is

$$f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

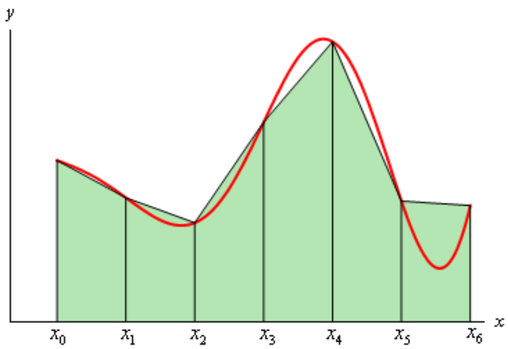
Usually, we pick each $x_i^*$ to be either the left-hand or right-hand edge of each rectangle. Note that we also have $\Delta x = \frac{b-a}{n}$ and so our left-hand approximation becomes

$$L_n = \frac{b-a}{n}\left[f(x_0) + \cdots + f(x_{n-1})\right]$$

and the right-hand approximation is

$$R_n = \frac{b-a}{n}\left[f(x_1) + \cdots + f(x_n)\right].$$

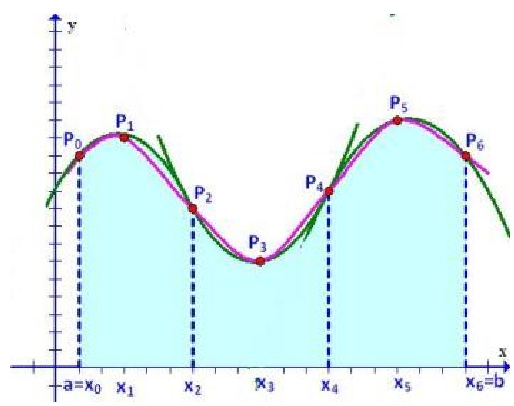
### Approximating area using trapezoids




A more useful approximation is found when we inscribe trapezoids into the curve. This can be done by taking the average of the left-hand and right-hand rectangle approximations (recall that the area of a trapezoid is the average of the area of two rectangles). When we do this, we obtain

$$T_n = \frac{b-a}{2n}\left(f(x_0) + f(x_n) + 2\left[f(x_2) + \cdots + f(x_{n-1})\right]\right).$$

### Approximating area using parabolae




An even better approximation (for smooth functions) is formed when we use parabolae to estimate the area. The resulting formula is called **Simpson's rule**. Note that $n$ must be even to use Simpson's rule.

$$S_n = \frac{b-a}{3n}\left(f(x_0) + f(x_n) + 4\left[f(x_1) + f(x_3) + \cdots + f(x_{n-1})\right] + 2\left[f(x_2) + \cdots + f(x_{n-2})\right]\right).$$

### The Definite Integral

It is obvious that all three methods of approximating area above will approach the 'real' area of the curve as $n \to \infty$. We call the true area under a curve $y = f(x)$ from $x = a$ to $x = b$ the **definite integral** of the curve from $a$ to $b$, and we notate it as*

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} T_n = \lim_{n\to\infty} S_n = \int_a^b f(x)\,\mathrm{d}x.$$

The large $\int$ is an elongated 'S', which stands for *sum* — we are summing all of the little areas from $a$ to $b$. Note the similarity to the notation above: we replace $\sum$ with $\int$, $x_i^*$ with $x$, and $\Delta x$ with $\mathrm{d}x$.

*Technically, this is the **Riemann integral**; there are other kinds of integral.

The $\mathrm{d}x$ is *not a number*; it is merely a piece of notation which tells us which variable we are taking area with respect to. For example, $\int_a^b x^2 + y^3 \,\mathrm{d}x \neq \int_a^b x^2 + y^3 \,\mathrm{d}y$. At this stage in time, you can think of the notation as a pair of fancy brackets: the $\int_{z_0}^{z_1}$ corresponds to $\big($, and the $\mathrm{d}z$ corresponds to $\big)$. It makes no sense to have one without the other (or, for that matter, to have $\int$ without the bounds).

## Questions

1. Estimate the area under the graph of $f(x) = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an overestimate or an underestimate? A

2. Estimate the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ using four approximating rectangles and left endpoints. Sketch the graph and the rectangles. Is your estimate an overestimate or an underestimate? A

3. Using the trapezoidal rule with $n = 4$, estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$. Repeat with $n = 8$. Compare the two results. A

4. Use Simpson's rule with $n = 10$ to estimate the area under the graph of $y = e^{x^2}$ from $x = 0$ to $x = 1$. A

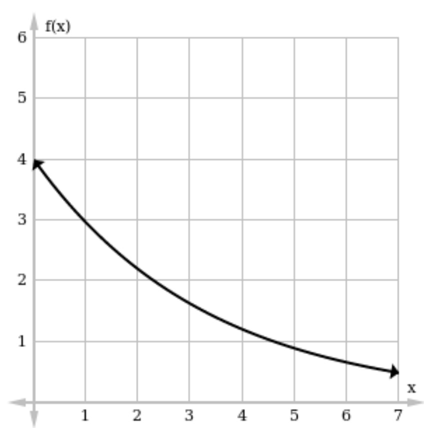
5. Use both the trapezoidal rule and Simpson's rule (both with $n = 10$) to compute the area under the graph of $y = \sqrt{z}e^{-z}$ from $z = 0$ to $z = 1$. A

6. Using four rectangles, find the approximate area under the the graph of $f(x)$ from $x = 1$ to $x = 5$ given the following table. A

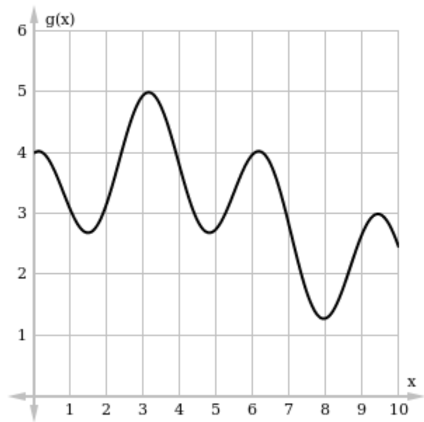
| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 1.0 | 2.4 | 3.5 | 4.0 |
| 1.5 | 2.9 | 4.0 | 4.1 |
| 2.0 | 3.3 | 4.5 | 3.9 |
| 2.5 | 3.6 | 5.0 | 3.5 |
| 3.0 | 3.8 | | |

7. Find the area of a circle of radius 4 using Simpson's rule with $n = 4$. What is the percentage error of the estimate? M

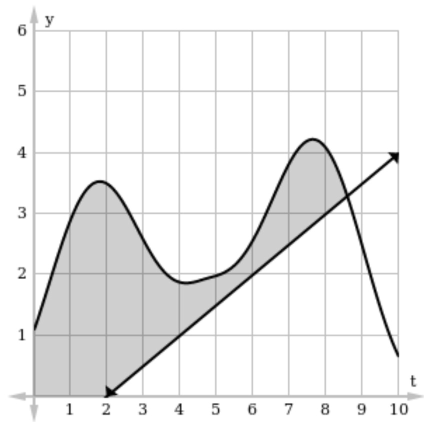
8. Approximate the area under $y = f(x)$ from $x = 0$ to $x = 6$ using Simpson's rule. A



9. Approximate the area under $y = f(x)$ from $x = 0$ to $x = 10$ using the trapezoidal rule. A




10. Approximate the shaded area using the trapezoidal rule. M




11. Show that $T_n = \frac{1}{2}(L_n + R_n)$. M

12. (Lifted straight from a Level 2 worksheet.) A
    (a) Draw the line $y = 2t + 1$ and use geometry to find the area under this line, above the $t$-axis, and between the vertical lines $t = 1$ and $t = 3$ (i.e. find $\int_1^3 2t + 1\,\mathrm{d}t$ using geometry).
    (b) If $x > 1$, let $A(x)$ be the area of the region that lies under the line $y = 2t+1$ between $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$ (i.e. find $A(x) = \int_1^x 2t + 1\,\mathrm{d}t$ using geometry).
    (c) Find $A'(x)$. What do you notice?