NCEA Level 3 Calculus (Integration)
23. Trigonometric Substitution

Goal for this week

To practice using a special kind of substitution which is sometimes useful.

Consider the integral

$$\int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}x.$$

There is no obvious easy substitution to simplify this integral, and integration by parts could work but will require a lot of work with no guranteed payoff. However, recall that $\sqrt{1-\sin^2\theta} = \cos\theta$; this identity suggests that we could perhaps substitute $x = \sin\theta$ in order to obtain $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$ and so

$$\begin{aligned}
\int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \sin\theta(1-\cos^2\theta)\cos\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} (\cos^2\theta-\cos^4\theta)\sin\theta\,\mathrm{d}\theta
\end{aligned}$$

Now, letting $u = \cos\theta$ we obtain

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} (\cos^2\theta-\cos^4\theta)\sin\theta\,\mathrm{d}\theta &= -\int_1^0 u^2 - u^4\,\mathrm{d}u \\
&= \frac{1}{3}u^3 - \frac{1}{5}u^5\Big|_{u=0}^{1} \\
&= \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.
\end{aligned}$$

Here is a table of trig substitutions:

Integrand	**Substitution**	**Identity**
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Integrals requiring trig substitution to solve turn up quite often in physics, especially electromagnetism.

Example. Consider $I = \int \frac{\mathrm{d}x}{\sqrt{9+x^2}}$. Let $x = 3\tan\theta$ so $\mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta$ and:

$$\begin{aligned} I &= \int \frac{3\sec^2\theta}{3\sqrt{1+\tan^2\theta}}\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta \\ &= \ln(\sec\theta + \tan\theta) + C = \ln(\sec\tan^{-1}(x/3) + x/3) + C \\ &= \ln\left(\sqrt{\left(\frac{x}{3}\right)^2 + 1} + \frac{x}{3}\right) + C. \end{aligned}$$

Questions

1. Find the following integrals: $\boxed{\text{S}}$

   (a) $\int \frac{x^2-9}{x^3}\,\mathrm{d}x$

   (b) $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}}$

   (c) $\int_0^3 x^2(9-x^2)\,\mathrm{d}x$

   (d) $\int_0^1 x\sqrt{1-x^4}\,\mathrm{d}x$

   (e) $\int_{\sqrt{2}}^2 \frac{\mathrm{d}x}{t^3\sqrt{t^2-1}}$

   (f) $\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x$

2. Use the integral $2\int_r^{-r} \sqrt{r^2-x^2}\,\mathrm{d}x$ to find the area of a circle of radius r. $\boxed{\text{S}}$

3. Scholarship 2005: Find, in terms of r, the area between the ellipse $x^2+16(y-r)^2=r^2$ and the circle $x^2+y^2=r^2$. You may use the substitution $x=r\sin u$ to find the integral $\int \sqrt{r^2-x^2}\,\mathrm{d}x$. $\boxed{\text{S}}$

4. By integrating, verify that $\boxed{\text{S}}$

$$\int_0^x \sqrt{a^2-t^2}\,\mathrm{d}t = \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)+\frac{1}{2}x\sqrt{a^2-x^2}.$$

5. A charged rod of length L produces a electric field at the point (a,b) given by $\boxed{\text{S}}$

$$E(a,b)=\int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0(x^2+b^2)^{3/2}}\,\mathrm{d}x.$$

   Evaluate this integral to find an explicit expression for $E(a,b)$.

6. One of these integrations should be done by partial fractions and one by trig substitution. Do them both. $\boxed{\text{O}}$

$$\int \frac{\mathrm{d}x}{(4x^2+9)^2} \qquad \int \frac{x^3}{x^2+x-6}\,\mathrm{d}x$$

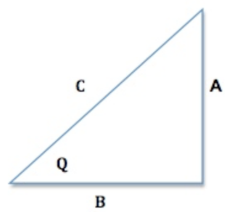
7. Check this working (the substitution $x=3\sin\theta$ is used). Find any mistakes. $\boxed{\text{S}}$

$$\begin{aligned}
I &= \int \frac{\sqrt{9-x^2}}{x^2}\,\mathrm{d}x = \int \frac{\cos^2\theta}{\sin^2\theta}\,\mathrm{d}\theta \\
&= \int \frac{1-\sin^2\theta}{\sin^2\theta}\,\mathrm{d}\theta = \int \csc^2\theta - 1\,\mathrm{d}\theta \\
&= -\cot\theta - \theta = -\cot\left(\sin^{-1}(x/3)\right) - \sin(x/3) \\
&= \frac{\sqrt{9-x^2}}{x} - \sin(x/3).
\end{aligned}$$

8. A water storage tank has the shape of a cylinder with diameter 10 m. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 m, what percentage of the total capacity is being used? S

9. When writing this worksheet I went on the internet and found this. Find the mistake(s), and do the integral. S

$$\int \frac{7}{x^2\sqrt{36-25x^2}}\,dx$$



For trigonometric substitution to solve the above integral, fill in the blanks below using the picture of the triangle given.

side A= $5x$
side B= 6
side C= sqrt(36-25x^2)
$\frac{5x}{6} =$ tan(Q)
$\frac{5}{6}$ d$x =$ 1/cos(Q)^2 dQ
$\frac{\sqrt{36-25x^2}}{6} =$ 1/cos(Q)

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