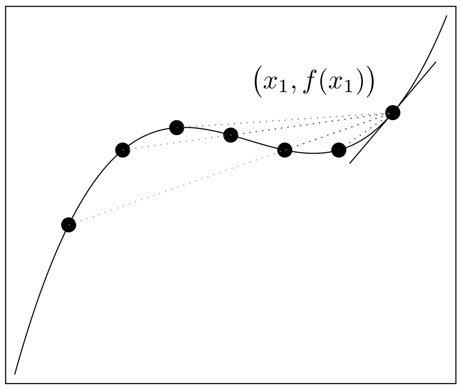
NCEA Level 3 Calculus (Differentiation)
2. Limits

Last week we looked at the derivative from the point of view of a ‘slope function’, and remarked that this definition is not really useful at all because it is dependent on some notion of ‘slope’ for functions which aren’t lines.

Consider, for example, the graph of a function f. A good definition of ‘slope’ for the graph at the point $(x_1, f(x_1))$ is the slope of the line at that point which ‘just touches’ the curve at that point — we can obtain this line by drawing another point (x_0, y_0) on the curve and joining it to $(x_1, f(x_1)$ by a line and then slowly moving x_0 towards x_1, redrawing the line at each point until x_0 and x_1 become the same. This process is illustrated below, where the dotted lines approach the solid tangent line as the point on the curve approaches the place we want to take the tangent.



The slope of the line at each point is simply

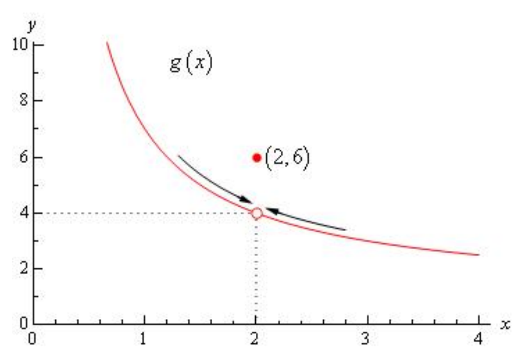
$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

and so the slope of the tangent line at x_0 will be obtained when we set Δx to zero. Unfortunately, we can’t do this easily because it would involve a division by zero and so we have to make use of the concept of a limit.

Suppose we consider some function f, and we notice that whenever its input approaches some value x_0 (from either direction) its output also approaches some value y_0, and that we can get its output as close as we like to y_0 by making the input appropriately close to x_0. Then we say that the **limit** of f as its input approaches x_0 is y_0, or (symbolically)

$$\lim_{x \to x_0} f(x) = y_0.$$

Consider the following function:



Although the *value* of the function at 2 is 6, the *limit* of the function at 2 is $\lim_{x \to 2} g(x) = 4$. When taking a limit we don’t care what the function does at the point — only what it looks like it *should* do based on the points around it. The limit of a function at a point is a property of the area around the point **and not a property of the point itself**.

You can also think of $\lim_{x\to x_0} f(x)$ as being the unique value that we could pick for $f(x_0)$ such that the function around that point has 'no gaps'. If there is no such unique value, there is no limit at the point x_0.

Limits happen to have a few simple properties.

Theorem. *If f and g are functions and the limits of f and g at x_0 exist, then:*

1. $\lambda \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} [\lambda f(x)]$ *(where λ is a constant);*
2. $\lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x) = \lim\limits_{x\to x_0} [f(x)+g(x)]$*;*
3. $\left(\lim\limits_{x\to x_0} f(x)\right)\left(\lim\limits_{x\to x_0} g(x)\right) = \lim\limits_{x\to x_0} [f(x)g(x)]$*;*
4. $\dfrac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$ *(if $g(x) \neq 0$ around the point we take the limit); and*
5. $f(\lim\limits_{x\to x_0} g(x)) = \lim\limits_{x\to x_0} [f(g(x))]$ *(if f is continuous: that is, at every point $\lim_{x\to x_0} f(x) = f(x_0)$).*

Examples. Using these limit laws, we can find some limits reasonably easily.

1. $\lim_{x\to 0} \frac{x}{x} = 1$ since as x gets closer and closer to 0, $\frac{x}{x} = 1$.
2. $\lim_{x\to 3} \frac{(x-2)(x-3)}{x-3} = 1$ since as x gets closer and closer to 3, the fraction gets arbitrarily close to 1.
3. $\lim_{x\to 0} \frac{1}{x}$ does not exist, since if we approach 0 from the left the function becomes arbitrarily negative and if we approach 0 from the right the function becomes arbitrarily positive — we do not approach the same value on both sides.
4. $\lim_{x\to\infty} \frac{1}{x} = 0$ since as x becomes arbitrarily large, $\frac{1}{x}$ becomes arbitrarily small.
5. $\lim_{x\to 0} \sqrt{x}$ does not exist, since $\sqrt{x}$ is undefined for $x < 0$.

With this new notation, we hereby define the **derivative** of f at x_0 to be the function f' described by

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0},$$

or equivalently

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

Now that we have a proper definition, we can begin to calculate some derivatives.

Example. We will find the derivative of $f(x) = x^3$ at the point x using the definition.

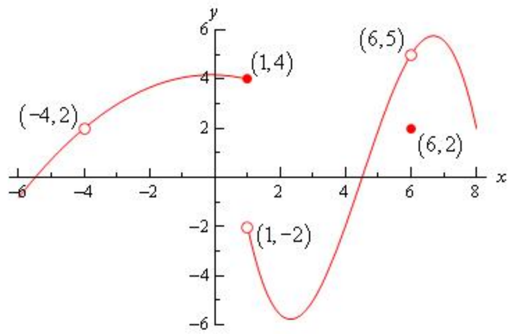
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\
&= 3x^2.
\end{aligned}$$

Questions

1. Guess the value of the following limit by evaluating the limuend for each x in $\{\pm 1, \pm 0.5, \pm 0.2, \pm 0.1, \pm 0.05, \pm 0.01\}$: $\boxed{\mathtt{A}}$

$$\lim_{x \to 0} \frac{\sin x}{x + \tan x}$$

2. Consider the function f graphed below. $\boxed{\mathtt{A}}$



   (a) For each of the following expressions, either give the value or explain why the expression is undefined.
      i. $f(-4)$
      ii. $\lim_{x \to -4} f(x)$
      iii. $f(1)$
      iv. $\lim_{x \to 1} f(x)$
   (b) Explain why the limit $\lim_{x \to 6} f(x)$ is not equal to $f(6)$.
   (c) At which points is f:
      i. Discontinuous?
      ii. Non-differentiable?

3. Evaluate the limit or explain why it does not exist: $\boxed{\mathtt{M}}$

   (a) $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$
   (b) $\lim_{x \to 0} \frac{1}{x^3}$
   (c) $\lim_{x \to 9} \frac{1}{x^3}$
   (d) $\lim_{h \to 0} \frac{(2+h)^3-8}{h}$
   (e) $\lim_{x \to 4} \frac{x^2+5x+4}{x^2+3x-4}$
   (f) $\lim_{x \to \frac{\pi}{2}} \sin x$
   (g) $\lim_{x \to \infty} \sin x$
   (h) $\lim_{x \to \frac{\pi}{2}} \tan x$
   (i) $\lim_{x \to 0} \tan x$
   (j) $\lim_{x \to 0} \csc x$
   (k) $\lim_{x \to a} C$, where a and C are constants.
   (l) $\lim_{x \to -\infty} \tan^{-1} x$
   (m) $\lim_{y \to 0} \lim_{x \to 0} \frac{(x+y)(x-y)}{x^2-y^2}$
   (n) $\lim_{x \to \infty} 1/x$.
   (o) $\lim_{x \to \infty} \frac{2x}{x^2+1}$.
   (p) $\lim_{x \to \infty} \frac{x+2}{x-3}$.

4. Consider the function φ defined by $\boxed{\mathtt{M}}$

$$\varphi(x) = 1/\frac{1}{x-x}.$$

   Explain why neither $\varphi(\alpha)$ nor $\lim_{x \to \alpha} \varphi(x)$ exists for any real α.

5. Find the derivative of $x^2 + x$ from first principles. $\boxed{\mathtt{M}}$

6. Find the derivative of $\sin x$ from first principles, given that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$. $\boxed{\mathtt{E}}$

7. Show that $f(x) = |x-6|$ is not differentiable at $x = 6$. Find a formula for f'. E

8. Show that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ and $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ are equivalent definitions for the derivative at the point a of some function f. E

9. If $\lim_{x\to a}[f(x)+g(x)] = 2$ and $\lim_{x\to a}[f(x)-g(x)] = 1$, find $\lim_{x\to a} f(x)g(x)$. E

10. Consider the following limit (you may assume that it exists): S

$$L = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{9}{10^i}$$

   (a) Write $L = 0.9 + 0.09 + \cdots$, and make a conjecture about the value of L.

   (b) Prove or disprove your conjecture from (a). [*Hint: you may wish to consider the following working.*]

$$\left(1 - \frac{1}{10}\right) L = 9\left(1 - \frac{1}{10}\right)\left(\frac{1}{10} + \frac{1}{100} + \cdots\right)$$

   (c) In general, consider the sum $S_n = \sum_{i=1}^{n} ar^i$ for some real constants a and r. Prove that

$$\lim_{n\to\infty} S_n = \frac{a(1-r^n)}{1-r}.$$

   [*Hint: use the same trick as in (b).*] Is there any restriction on r for this limit to exist?

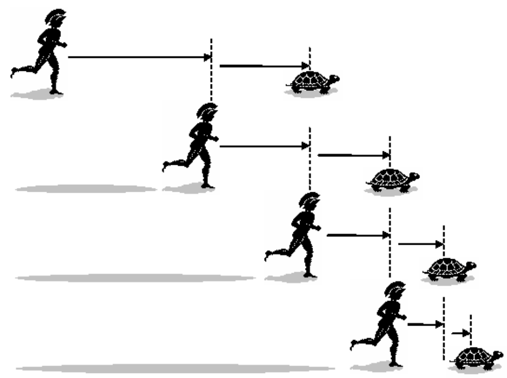
11. Last year, we defined the *exponent* a^r as follows: S

   - If $r = 0$, then $a^r = 1$.
   - If r is a natural number, then $a^r = a^{r-1} \cdot r$. (So $a^r = \underbrace{a \times \cdots \times a}_{r \text{ times}}$.)
   - If r is a negative integer, then $a^r = \dfrac{1}{a^{-r}}$. (Note that $-r$ is positive.)
   - If r is a rational number, so that $r = p/q$ in lowest form, then $a^r = a^{(p/q)} = \sqrt[q]{a^p}$ (where we take the positive root, if a choice needs to be made).

   Give a reasonable definition for a^r where r is any real number. Use your definition to compute a reasonable approximation to 2^π (given that $\pi \approx 3.14159...$).

12. Zeno was a Greek philosopher active in the 5th century BCE. He presented a list of ‘paradoxes’, or apparent contradictions, including the following (adapted from Wikipedia): S

- Suppose Achilles is in a foot race with a tortoise. Achilles runs much faster than the tortoise, but the latter has a head start. By the time Achilles reaches the location that the tortoise started, the tortoise will have moved a small amount further on; similarly, by the time Achilles reaches the new location of the tortoise, it will have moved an even smaller distance further on; and by this reasoning it follows that Achilles can never overtake the tortoise.



- Suppose Homer wishes to walk to the end of a path. Before he can get there, he must get halfway there. Before he can get halfway there, he must get a quarter of the way there. Before traveling a quarter, he must travel one-eighth; before an eighth, one-sixteenth; and so on. So Homer cannot walk to the end of the path.
- For motion to occur, an object must change the position which it occupies. Consider an example of an arrow in flight. In any one (duration-less) instant of time, the arrow is neither moving to where it is, nor to where it is not. It cannot move to where it is not, because no time elapses for it to move there; it cannot move to where it is, because it is already there. In other words, at every instant of time there is no motion occurring. If everything is motionless at every instant, and time is entirely composed of instants, then motion is impossible.

Use your knowledge of limits to explain these apparent contradictions.