NCEA Level 3 Calculus (Differentiation)
11. Related Rates of Change

Moving on from optimisation and shape, we will now study the use of the derivative in modelling rates of change.

We can use the chain rule to relate rates of change together — for example, the area of a circle is given by $A = \pi r^2$ and so the rate of change of area with respect to radius $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$; but if r varies with respect to time then we can find the rate of change of the area with respect to time using the chain rule.

A useful mnemonic is (if x is a function of y which is itself a function of z)

$$\frac{\mathrm{d}x}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\mathrm{d}x}{\mathrm{d}z}. \qquad \text{(chain rule)}$$

We can also apply the inverse function rule for differentiation, which tells us that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}. \qquad \text{(inverse function rule)}$$

Although these two operations allow us to rearrange equations as if $\frac{\mathrm{d}y}{\mathrm{d}x}$ were a fraction, it is fairly important to remember that it is not a fraction and that whenever we treat it like one then it should be possible to justify that step more rigorously.

Example. A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1\,\mathrm{m\,s^{-1}}$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?

Solution. Let x be the distance of the bottom of the ladder from the wall, and let y be the height of the top of the ladder up the wall. We have $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ and $x = 3$; we also know that $y = \sqrt{25 - x^2}$, so:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{\sqrt{25 - x^2}} \cdot 1 \\ \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{x=3} &= -\frac{3}{\sqrt{25 - 9}} = -\frac{3}{4}.\end{aligned}$$

Hence the ladder is sliding down the wall at a rate of $-0.75\,\mathrm{m\,s^{-1}}$.

Example. The radius of a sphere is increasing at a rate of $\frac{\mathrm{d}r}{\mathrm{d}t} = -\ln(t - 1)$ metres per second. At what rate will the surface area of the sphere be growing at $t = 2$?

Solution. We have $\mathrm{SA} = 4\pi r^2$, so $\frac{\mathrm{dSA}}{\mathrm{d}r} = 8\pi r$ and

$$\frac{\mathrm{dSA}}{\mathrm{d}t} = \frac{\mathrm{dSA}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = -\ln(t-1) \times 8\pi r = 0.$$

The surface area of the sphere will be momentarily constant at $t = 2$.

Questions

For each of the following situations: (a) identify the quantities given in the problem; (b) identify the unknown; (c) draw a diagram; (d) write an equation relating the given quantities to an equation; (e) solve the problem.

1. Each side of a square is increasing at a rate of $6\,\mathrm{cm\,s^{-1}}$. At what rate is the area of the square increasing when the area of the square is $16\,\mathrm{cm}^2$? M
2. Gas is being forced into a spherical balloon at a rate of $400\,\mathrm{cm^3\,min^{-1}}$. How fast is the radius of the balloon increasing when the radius is 5 cm? M

3. If a snowball melts so that its surface area decreases at a rate of $1\,\mathrm{cm}^2\,\mathrm{min}^{-1}$. find the rate at which the diameter decreases when the diameter is $10\,\mathrm{cm}$. M

4. If $W = 4H^3S$ where $H = 3S^2 - 5$, find $\frac{\mathrm{d}W}{\mathrm{d}S}$ in terms of H and S. M

5. If $x^2 + y^2 + z^2 = 9$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$, and $\frac{\mathrm{d}y}{\mathrm{d}t} = 4$, find $\frac{\mathrm{d}z}{\mathrm{d}t}$ when $(x, y, z) = (2, 2, 1)$. M

6. The demand for an article varies inversely as the $\frac{5}{2}$ power of the selling price (i.e. if x is the number of articles sold and p is the price then $x = p^{-5/2}$). If the manufacture cost of one article is \$1, determine the selling price which will produce the greatest profit. E

7. A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle moves through the point $(1/3, 1)$, its x-ordinate increases at a rate of $\sqrt{10}\ \mathrm{cm\,s^{-1}}$. How fast is the distance from the particle to the origin changing at this instant? E

8. Gravel is dumped from a conveyor belt at a rate of $3\,\mathrm{m}^3\,\mathrm{min}^{-1}$, and forms a pile in the shape of a cone with equal height and base diameter. How fast is the height of the cone increasing when the pile is $3\,\mathrm{m}$ tall? M

9. The top of a ladder slides down a vertical wall at a rate of $0.15\,\mathrm{m\,s^{-1}}$. A the moment when the bottom of the ladder is $3\,\mathrm{m}$ from the bottom of the wall, it slides away from the wall at a rate of $0.2\,\mathrm{m\,s^{-1}}$. Find the length of the ladder. M

10. Two sides of a triangle have lengths $2\,\mathrm{m}$ and $3\,\mathrm{m}$. The angle between these sides is increasing at a rate of $4\,^\circ\,\mathrm{s}^{-1}$. How fast is the length of the third side changing when it is of length $4\,\mathrm{m}$? M

11. A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y-ordinate is decreasing at a rate of 3 units per second. How fast is the x-ordinate of the particle changing at that instant? E

12. The minute hand on a watch is $8\,\mathrm{mm}$ long and the second hand is $4\,\mathrm{mm}$ long. How fast is the distance between the tips of the hands changing at 1 o'clock? E

13. Scholarship 2015: A tank contains 200 litres of brine (solution of salt in water). Initially, the concentration is $0.5\,\mathrm{kg}$ of salt per litre. Brine containing $0.8\,\mathrm{kg}$ of salt per litre runs into the tank at a rate of 6 litres per minute. The mixture is kept thoroughly mixed and is running out at the same rate. S

    Find out how long it takes for the amount of salt in the tank to be $130\,\mathrm{kg}$.