NCEA Level 2 Mathematics
12. Tangent Lines and Approximations

One application of calculus is to find approximations to curves. Our goal is to write down a linear equation that approximates any given curve (around a given point). This is useful if (for example) we are given a complicated function like

$$f(x) = [\sin(x^{100})]^{[\cos(\sin x^2)]}.$$

This function is so weird that the graphing software I use cannot even graph it properly. The value of this function at $x = 0$ is very easy to calculate:

$$f(0) = [\sin(0^{100})]^{[\cos(\sin 0^2)]} = 0^{\cos(0)} = 0$$

However, as soon as we try to calculate other values it becomes difficult.

Suppose we want to know the value of a function near a point that it's easy to find the value of the function at — for example, $f(0.001)$. Our goal is to to draw the line through the easy point, with the same slope as the function at that point, and then work out what our desired x-value maps to using this easy function.

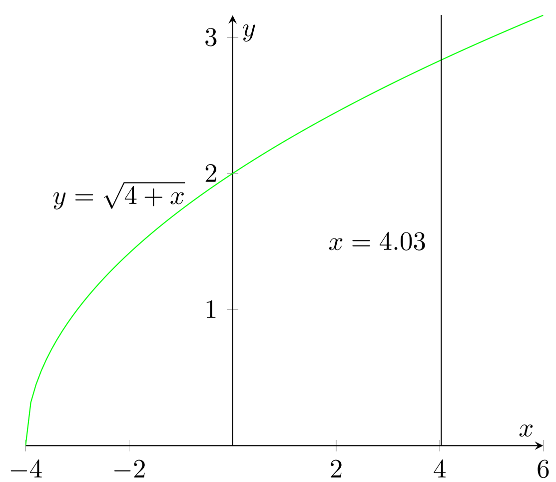
Definition. Let $y = f(x)$ be a curve, and let f be differentiable at some x-value x_0. Then the *tangent line* to the curve at x_0 is simply the line through the point $(x_0, f(x_0))$ that has the same slope as the curve at that point.

From our work on coordinate geometry, we know that the equation of this line is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Let's take an example.

Example. Let us calculate $\sqrt{4.03}$ by hand(!). If we consider the function $f(x) = \sqrt{4+x}$, then $\sqrt{4.03} = f(0.03)$. Let's draw the situation out:



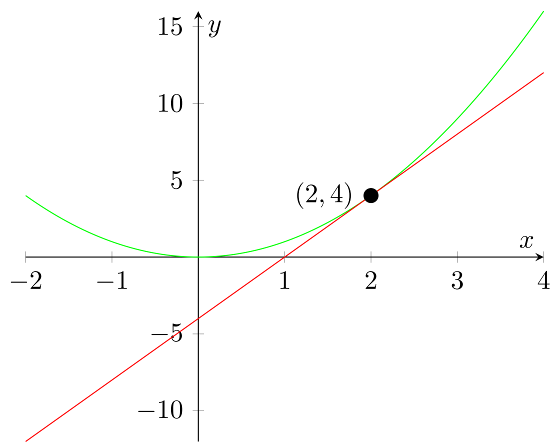
So we want the tangent line to f at the point $x = 0$. We have that $f'(x) = \frac{1}{2\sqrt{4+x}}$ (why? we can easily take the derivative of $\sqrt{x}$, it's just $\frac{1}{2\sqrt{x}}$ — and the graph of f is just the graph of $\sqrt{x}$ but shifted four units to the left, so we just shift the slope function itself four units to the left to match up), and so $f'(0) = \frac{1}{4}$. The tangent line is the line through $(0, f(0)) = (0, 2)$ with gradient $\frac{1}{4}$, which has equation

$$y - 2 = \frac{1}{4}x.$$

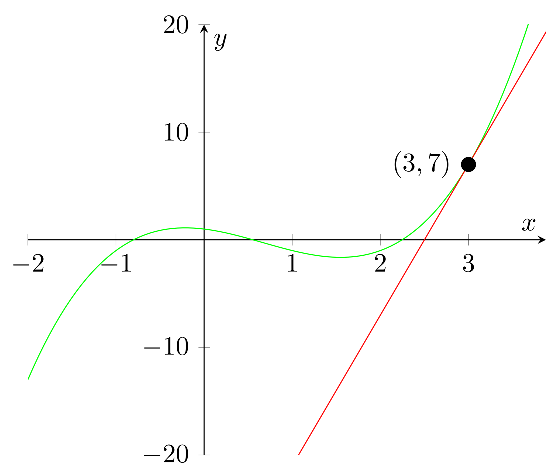
Hence $\sqrt{4.03} \approx \frac{1}{4} \times 0.03 + 2 = 2.0075$ — and, as promised, all of these calculations can be done without a calculator. (According to my calculator, $\sqrt{4.03} \approx 2.007486$ and so we are not far off at all.)

Now, with all the motivation out of the way, we will just look at some more simple examples.

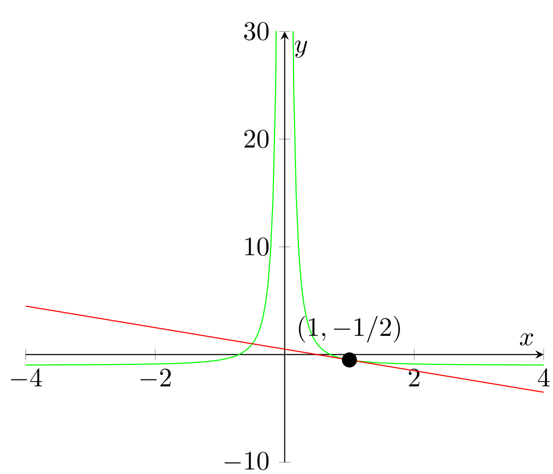
Example. Consider $y = x^2$. At $(2, 4)$, the tangent line has slope 4 and hence equation $y - 4 = 4(x - 2)$:



Example. Consider $y = x^3 - 2x^2 - x + 1$. At $(3, 7)$, the tangent line has slope 14 and hence equation $y - 7 = 14(x - 3)$:

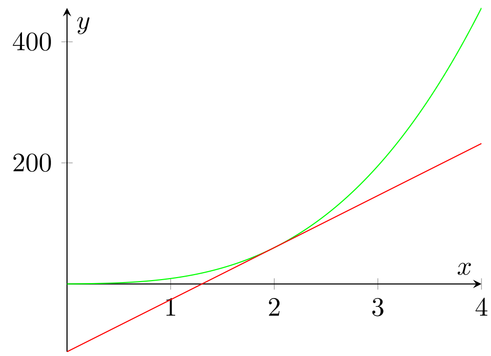


Example. Consider $y = \frac{1}{2x^2} - 1$. At $(1, -1/2)$, the tangent line has slope -1 and hence equation $y + 1/2 = -(x - 1)$:

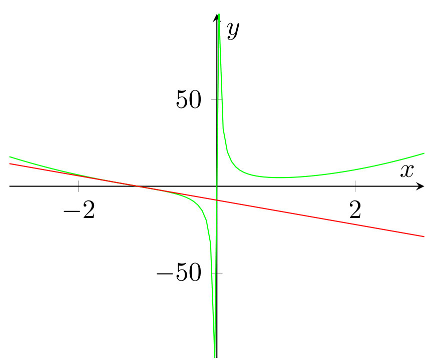


Questions

1. Why is there no tangent line to $y = x^2$ at the point $(0, -1)$?
2. Consider the function $f(x) = 7x^3 + 2x$.



   (a) What is the slope of the graph of $y = f(x)$ around $x = 2$?
   (b) Give the equation of the tangent line to the graph at $x = 2$.
3. Consider the function $g(x) = 2x^2 + \frac{3}{x}$.



   (a) What is the slope of the graph of $y = g(x)$ around $x = -1$?
   (b) Give the equation of the tangent line to the graph at $x = -1$.
   (c) The normal line to a graph at a point is the line going through that point that lies at right angles to the graph (and hence to the tangent line to the graph).
      i. Consider the line with slope m going through (x_0, y_0); it has equation $(y - y_0) = m(x - x_0)$. What is the slope of the line at right angles to it going through the same point?
      ii. Give the equation of the normal line to the graph of $y = g(x)$ at $x = -1$.
4. (a) Find the slope function of $y = x^3 + 8x^2 + 22x - 21$ in two different ways:
      i. By rewriting the function as $y = (x-3)^3 + (x-3)^2 + (x-3)$ and then differentiating $z^3 + z^2 + z$;
      ii. By simply taking the derivative of the original function in its expanded form.
   (b) Hence explain how the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is related geometrically to the derivative $\frac{\mathrm{d}}{\mathrm{d}z}[z^3 + z^2 + z]$.
   (c) Show that there are no points where the graph of y versus x has a horizontal tangent line.
5. To expand slightly on the previous question, consider now the graph $y = (x^2 + 1)^2 + (x^2 + 1)$.
   (a) By expanding the brackets, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
   (b) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 2(x^2 + 1) + 1$. (Where did this right-hand function come from?)

(c) Can you explain why our argument about 'shifting functions' does not work here? Hint: if we transform x to $x-3$, there is no shrinking or stretching going on — but this is not always the case.

6. A function f is differentiable at a point x if the value $f'(x)$ is well-defined.
   (a) Give some examples of functions which are *not* differentiable at some point.
   (b) Can you define differentiability in a different way, using tangent lines?
   (c) Is it ever possible for a function to have a horizontal normal line at any point? Explain how your answer is related to the idea of differentiability.

7. Consider the hyperbola $y = 1/x$.
   (a) Explain why the hyperbola has no tangent line at $x = 0$.
   (b) Show that the tangent lines to the hyperbola at $(-1, -1)$ and $(1, 1)$ are parallel.
   (c) More generally, show that the tangent lines to the hyperbola at $(-x, -1/x)$ and $(x, 1/x)$ are always parallel.
   (d) Are there any points on the hyperbola which share a common normal line (not simply a normal line with the same slope, but the same line full stop)? What about tangent lines? Hint: yes, and no.

8. Finally, here are some functions and points to find tangent lines at. If there is no tangent line at the point given, carefully explain why. Draw some graphs out as well.
   (a) $y = 3x^2 + 3x + 1$ at $(0, 1)$.
   (b) $y = \sqrt{1 - x^2}$ at $(1, 0)$.
   (c) $y = 1/x^2$ at $(1, 1)$.
   (d) $y = 1/x^2$ at $(2, 1/4)$.
   (e) $y = \sqrt[4]{x^3}$ at $(2, \sqrt[4]{8})$.
   (f) $y = \sqrt[3]{x^2 + 2x + 1}$ at $(0, 1)$.(Hint: complete the square.)