NCEA Level 3 Calculus (Integration)
18. Substitution

Recall that the **chain rule** for differentiation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x).$$

Since integration is (in some sense) the inverse of differentiation, we can write (by applying the fundamental theorem of calculus)

$$\int f'(g(x))g'(x)\,\mathrm{d}x = f(g(x)) + C.$$

For a mnemonic, we can let $u = g(x)$. Then $\mathrm{d}u = g'(x)\,\mathrm{d}x^*$ and so, by the rule we just wrote down, we have

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(u)\,\mathrm{d}u = f(u) + C = f(g(x)) + C.$$

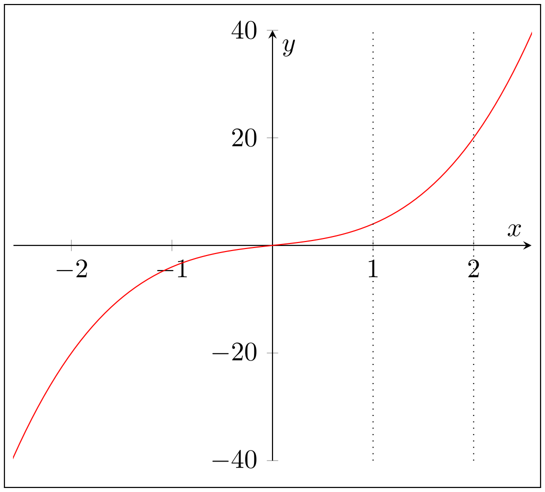
In Leibniz notation, we have

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{\mathrm{d}f}{\mathrm{d}g}\,\mathrm{d}g = \int f'(g)\,\mathrm{d}g = f(g) + C = f(g(x)) + C,$$

and so one can intuitively think about this (here we substitute g out) as the cancellation of differentials underneath an integral sign.

This rule, which gives us a kind of chain rule for integration, is called **substitution**, or the **inverse chain rule**. It can be thought of as a change in coordinate system from an x-based system to one based on u, and we have to 'resize' our area based on how much u stretches the coordinate system compared to x — and this 'stretch factor' is simply $\frac{\mathrm{d}u}{\mathrm{d}x}$.

Example. For example, consider $\int_1^2 2x(x^2+1)\,\mathrm{d}x$; we are finding the area shown here between the dotted lines.

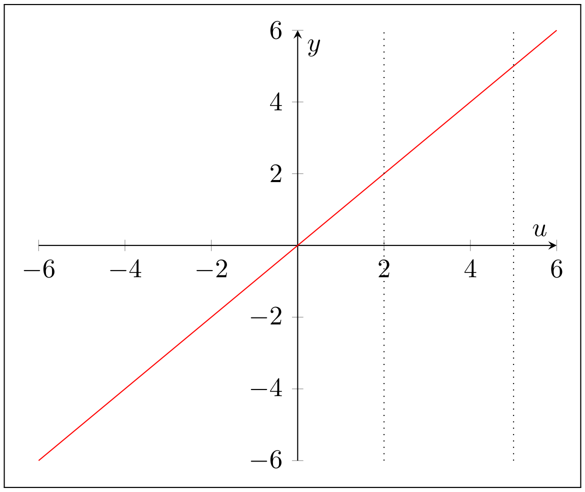


Let us make the substitution $u = x^2 + 1$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and our integral becomes

$$\int_1^2 2x(x^2+1)\,\mathrm{d}x = \int_{u^{-1}(2)}^{u^{-1}(5)} \frac{\mathrm{d}u}{\mathrm{d}x}u(x)\,\mathrm{d}x = \int_2^5 u\,\mathrm{d}u.$$

We can graph our region of integration again.

*again, this is just a mnemonic: it *is* possible to make $\mathrm{d}x$ meaningful (it is what is known as a *differential form*), but all we are really doing is applying the chain rule.



This new coordinate system, which is $2x$ times as large as the older one, is much simpler to integrate inside!

Examples.

1. Suppose we wish to find $\int \sin x \cos x \,\mathrm{d}x$. Then let $u = \sin x$, so $\mathrm{d}u = \cos x \,\mathrm{d}x$ and

$$\int \sin x \cos x \,\mathrm{d}x = \int u \,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C.$$

2. In this case, we also could have used a trigonometric identity. Suppose we wish to find $\int xe^{x^2} \,\mathrm{d}x$. We can let $u = x^2$, and then $du = 2x \,\mathrm{d}x \Rightarrow \mathrm{d}x = \frac{du}{2x}$. Hence:

$$\int xe^{x^2} \,\mathrm{d}x = \int \frac{1}{2}e^u \,\mathrm{d}u = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

3. Suppose we wish to find $\int \frac{4}{x}(\ln x)^3 \,\mathrm{d}x$. We let $u = \ln x$, and then $\mathrm{d}u = \frac{\mathrm{d}x}{x}$. Hence:

$$\int \frac{4}{x}(\ln x)^3 \,\mathrm{d}x = 4\int u^3 \,\mathrm{d}u = u^4 + C = (\ln u)^4 + C.$$

Questions

1. Find the following indefinite integrals. A

(a) $\int \sin 2x \,\mathrm{d}x$

(b) $\int (4x - 44)^{2019} \,\mathrm{d}x$

(c) $\int 4x\sqrt{x^2 + 3} \,\mathrm{d}x$

(d) $\int (3x - 4)^2 \,\mathrm{d}x$

(e) $\int \frac{x}{x^2+1} \,\mathrm{d}x$

(f) $\int \frac{2}{4x+3} \,\mathrm{d}x$

(g) $\int e^{2x+1} \,\mathrm{d}x$

(h) $\int \sec 4x \tan 4x \,\mathrm{d}x$

(i) $\int 2\cos x + \sin 2x \,\mathrm{d}x$

(j) $\int -2x \csc^2(3x^2) \,\mathrm{d}x$

(k) $\int \frac{3}{x^3} - \frac{4}{x+1} \,\mathrm{d}x$

(l) $\int e^{x/2} + \frac{2}{x} \,\mathrm{d}x$

(m) $\int x^2 \sec^2 x^3 + 9 \,\mathrm{d}x$

(n) $\int -\csc(\tan x)\cot(\tan x)\sec^2 x \,\mathrm{d}x$

(o) $\int \frac{\cos x - \sin x}{\cos x + \sin x} \,\mathrm{d}x$

(p) $\int \frac{2017}{x \ln x} \,\mathrm{d}x$

2. By using the substitution $x = \sin\theta$, find M

$$\int \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x.$$

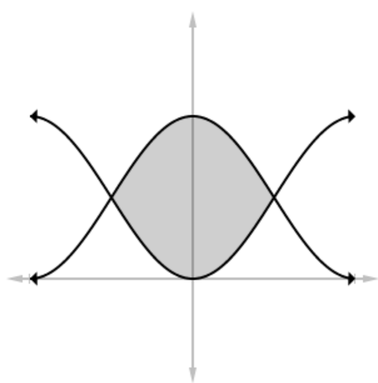
3. Compute the following definite integrals: M
   (a) $\int_0^1 xe^{-x^2}\,\mathrm{d}x$
   (b) $\int_{-\pi/3}^{\pi/3} x^4 \sin x\,\mathrm{d}x$ (hint: no substitution is required)
   (c) $\int_0^1 \cos(\pi t/2)\,\mathrm{d}t$
   (d) $\int_0^1 (3t-1)^{50}\,\mathrm{d}t$
   (e) $\int_0^1 \sqrt[3]{1+7x}\,\mathrm{d}x$
   (f) $\int_0^1 \frac{\mathrm{d}x}{1+\sqrt{x}}\,\mathrm{d}x$
   (g) $\int_{-1}^2 x(x-1)^3\,\mathrm{d}x$
   (h) $\int_0^3 x\sqrt{1+x^2}\,\mathrm{d}x$

4. Find the area enclosed by the curve $y = 4\sin 3x \cos x$ and the $x-$axis from $x=0$ to $x=\frac{\pi}{3}$. M

5. Find k such that $\int_0^k e^{2x}\,\mathrm{d}x = 40$. E

6. Calculate the area enclosed by the curve $y=\frac{3x-2}{x+4}$ and the lines $y=0$, $x=1$, and $x=5$. E

7. Find the area between the curves $y=\sin^2 kx$ and $y=\cos^2 kx$ shaded below. E



8. Find $\int \tan\theta\,\mathrm{d}\theta$ and $\int \cot\theta\,\mathrm{d}\theta$. M

9. Complete the following working: A

$$\begin{aligned}\int \sec x\,\mathrm{d}x &= \int \sec x \frac{\sec x+\tan x}{\sec x + \tan x}\,\mathrm{d}x\\ &= \int \frac{\dots}{\sec x+\tan x}\,\mathrm{d}x\\ \text{Let } u = \dots & \\ &= \int \frac{1}{\dots}\,\mathrm{d}u\\ &= \dots\end{aligned}$$

10. Show that M

$$\int x^4 \sin x\,\mathrm{d}x = 4x(x^2-6x)\sin x - (x^4-12x^2+24)\cos x + C.$$

11. If $y = x\sqrt{\sin x^3 + \cos x^3}$, find $\pi \int_0^1 y^2 \, \mathrm{d}x$. M

12. The velocity of a particle at time t is given by $v = \frac{\cos(\sqrt{2t+1})}{\sqrt{2t+1}}$. What is the position of the particle at time $t = 5$, given that $x(0.5) = 0$? (Recall that $v = \frac{\mathrm{d}x}{\mathrm{d}t}$.) M

13. Evaluate $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, \mathrm{d}x$ [*Hint: use the substitution $x = \frac{\pi}{2} - u$ and add the result to the original integral.*] S

14. Scholarship 1999: S
    (a) Evaluate $\int \cos^5 x \, \mathrm{d}x$ using the substitution $t = \sin x$.
    (b) i. If $f(x) = \cos^5 x$, what are $f(0)$, $f'(0)$, and $f''(0)$?
        ii. Hence evaluate a, b, and c in the approximation $\cos^5 x \approx a + bx + cx^2$.
        iii. Use this to give an approximation for $\int \cos^5 x \, \mathrm{d}x$.
    (c) Evaluate $\int_0^{0.6} \cos^5 x \, \mathrm{d}x$ to three significant figures, using:
        i. The exact integration in (a).
        ii. The expression in (b)(iii).
        iii. Simpson's rule.