Level Three Calculus

Second Edition

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https://github.com/aelzenaar/ncea-notes

Preface for the navigator

These notes are my second attempt at a coherent introduction to calculus at the level of NCEA Level 3 and NZ Scholarship.

I have made a few philosophical changes from the first edition:-

- I treat anti-differentiation at the same time as differentiation. (I do introduce the $\int$ notation and the term 'indefinite integral' here, though I would rather not.)
- The notes are split into "topic" chapters: *The basics* (the basic formal manipulations of derivatives and anti-derivatives), *Geometry of curves* (studying curves via differentiation), *Geometry of spaces* (definite integrals, the fundamental theorem of calculus, and arc lengths, surface area, and volumes), and *Motion and change* (differential equations).
- I have dropped many of the proofs; my justification for this change is threefold(!). Firstly, the students that 'need' the proofs will see them in a Stage I university course. Secondly, many of the proofs in elementary calculus often obscure a nugget of geometry behind the formal manipulation of limits, and so I would rather include intuitive geometric *justifications* for results in the space the proofs formerly were. Finally, most students at Level 3 are simply not ready for proofs: either they don't understand why proof is required, or their level of mathematical sophistication means that the proofs seem esoteric. I have included copious references to textbooks where proofs can be found.
- I have dropped one or two rather esoteric and old fashioned topics that I used to teach to scholarship students; most prominently, trig substitution. These have been replaced with a couple of new topics; one in particular is a much expanded explanation of Taylor polynomials (*not* Taylor series!). Note also that the main application of trig substitution is to integrate all rational functions: we expand our expression into its partial fraction form, we can integrate the terms with linear denominators easily, and then we need trig substitution to integrate the quadratic denominators. But after covering the L3 algebra topic (in particular, one needs to justify quickly something like $\operatorname{cis}(z) = \exp(iz)$) then it is possible to integrate these terms by factorising them into (complex) linear terms.

I feel the need also to point out that these notes are incredibly geometric. *If you don't like teaching geometry, these are not the notes for you.*

Prerequisite material

Firstly, a hard fact: for a student to be successful in L3 calculus, they should have a good understanding of the material at L2 and earlier (I would generally expect that students with less than a merit in the level 2 algebra standard will struggle).

In these notes, I will use material from algebra and geometry at L2 or earlier liberally; I try to point it out when I use some of the more obsure results. I do not use any material from any of the level three standards, except trigonometry.

So, in general, the prerequisites and expectations for these notes are:-

- A good understanding of L2 coordinate geometry and algebra.
- A decent understanding of L3 trigonometry, *including the manipulation of identities*.

For some of the sections, knowledge of a little physics (L1 and/or L2) would be nice. I cover the material in the L2 calculus standard quickly so this is not formally a prerequisite, but a student who doesn't understand the material there well will struggle with these notes. Roughly speaking, the differentiation material there is more important.

I would strongly recommend revising the material on functions (section 4 of my own level 2 notes).

Recommended textbooks

I have used the following textbooks when writing these notes, in roughly increasing order of sophistication:

- *Calculus made easy*, by Sivanus P. Thompson and Martin Gardner. This book is perhaps at the correct level mathematically speaking for a Y12/13 student, but it is not very geometric. It is certainly worth looking at, though.
- *Calculus*, by James Stewart. This is one of the standard first-year computational calculus books. It has many examples and many exercises, but lacks soul.
- *Calculus*, by Michael Spivak. This is often called the 'One True Calculus Book',[1] but is more properly an introduction to real analysis. As such, it is too diffcult for all but the most motivated high school students.
- (For the sake of completeness,) *Advanced Calculus*, by L. H. Loomis and S. Sternberg. This is the author's favourite calculus book, but is eminently unsuitable for high school students of any motivation.

[1] For example, by the *Chicago undergraduate mathematics bibliography*: `https://www.ocf.berkeley.edu/~abhishek/chicmath.htm` (somewhat useful, if taken with a grain of salt).

Preface for the student

I don't have much to say, really. I could spend time explaining why calculus is useful, but I won't do that here because we'll see a lot of examples of calculus 'in the wild' as we progress (just as a taster, we'll look at some physics, some biology, some economics, and maybe even some statistics). I could equally well spend time trying to explain exactly what calculus *is* exactly, but this page is not large enough to contain such an exposition.

Instead, I will give some study advice.

Firstly, you must read the notes. You must sleep with them beneath your pillow. You must work through the examples yourself. You must do all the problems. You must ask questions.

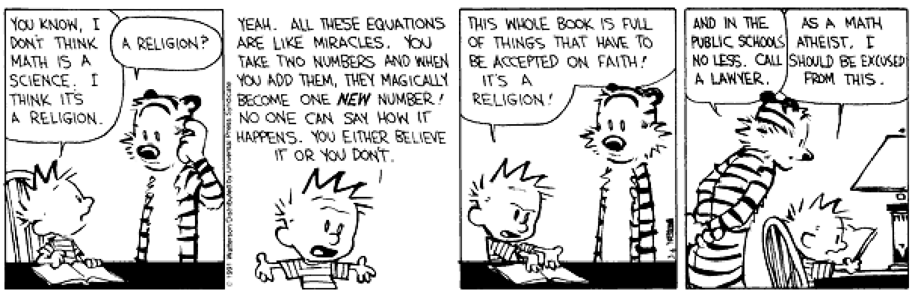
Because no student ever follows my first piece of advice, I will give you a second, easier option. For each topic, there are a few homework problems set. *At a minimum*, you should do all these problems. (But beware, if you *only* do these problems, you will be woefully underprepared for any situation you need calculus for.)

Secondly, draw pictures. I do my best to include lots of diagrams (some even in colour!), but one can never have too many pictures. (As a young girl called Alice once perceptively remarked, "What is the use of a book without pictures or conversations?"[2])

Thirdly, and I cannot stress this enough, *your exam grades do not matter.*[*citation needed*] It is perfectly possible to pass calculus exams without understanding the material, but if you do that (by, for example, trying to memorise everything in leu of understanding it), you are cheating yourself out of an education. If you understand the material, you will be prepared for every subject you may wish to take next year (and, as a bonus, you'll pass the exam).

Let us begin.

[2] *Alice in Wonderland*, by Lewis Carroll.



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Chapter I

The basics

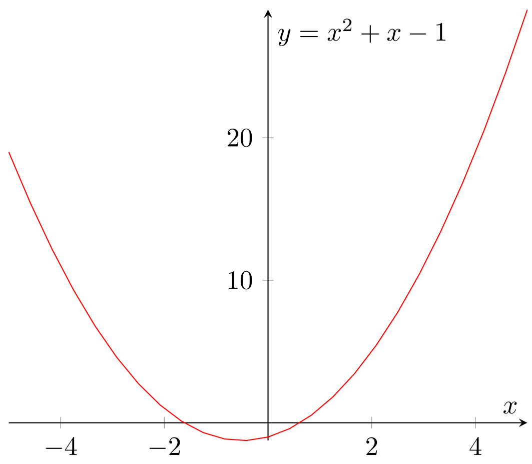


Figure 1: A parabola, $y = x^2 + x - 1$.

I.1 Tangent lines

Suppose we have two variables, y and x, such that any change in x produces a corresponding change in y. In the language we learned last year, we say that y is a function of x: that is, there is a function f such that $y = f(x)$.

Let us look at a few different functions.

Firstly, let's consider the function f defined by $f(x) = x^2 + x - 1$, graphed in figure 1. Let's look at how f behaves at a point x_0 by adding a small number h to the input and seeing how it changes the output.

$$f(x_0 + h) = (x_0 + h)^2 + (x_0 + h) - 1 = x_0^2 + 2x_0h + h^2 + x_0 + h - 1.$$

We're actually interested in the difference between our new output and our old output, because this gives us a measure of how quickly f changes when we move along the x-axis by h.

$$f(x_0 + h) - f(x_0) = (x_0^2 + 2x_0h + h^2 + x_0 + h - 1) - (x_0^2 + x_0 - 1) = 2x_0h + h + h^2$$

If h is small, then h^2 is miniscule and only makes up a very small part of $f(x_0 + h)$. We can therefore make the following approximation:

$$f(x_0 + h) - f(x_0) \approx (2x_0 + 1)h.$$

In other words, a small change in the input from x_0 to $x_0 + h$ produces a change in the output of the form $(2x_0 + 1)h$.

When we looked at straight lines in the past, they had a measure of *slope*: the ratio of 'rise' (change in output) to 'run' (change in input). For each x_0 that we feed into f here, we have a measure of the 'rise' of f over a very small distance, h. It makes sense, then, to define the slope of $y = f(x)$ at x_0 to be rise/run:

$$\frac{f(x_0 + h) - f(x_0)}{h} \approx \frac{(2x_0 + 1)h}{h} = 2x_0 + 1 \tag{I.1}$$

So when h is very small, the graph of f between x_0 and $x_0 + h$ looks like a straight line with slope $2x_0 + 1$.

Since the slope of our function defined in this way depends on the value of x_0, we have a new function which assigns to each point x_0 the slope of f around x_0. We denote this function by f', and call it the *derivative* of f. The slope of f at x_0 will be written $f'(x_0)$.

Another notation for the derivative is also common; because we are looking at changes in y divided by changes in x it sometimes makes sense to write the derivative as $\frac{\mathrm{d}y}{\mathrm{d}x}$ (where $\mathrm{d}x$ denotes, in some sense, a small change in x). The value of the derivative at x_0 is written as $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0}$. This notation is known as Leibniz notation, as it was first introduced in the 1600s by Gottfried Leibiz (a German

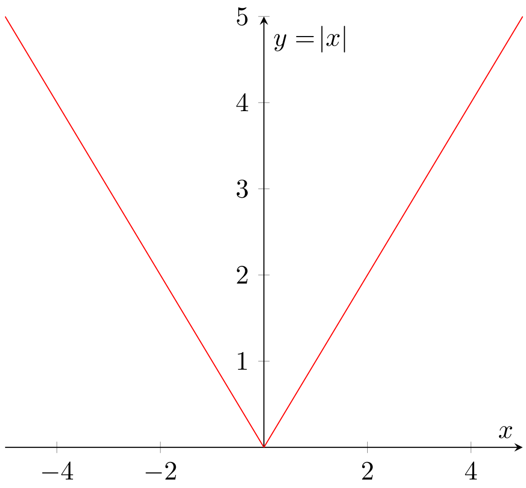


Figure 2: The absolute value function.

mathematician and philosopher who, it can be argued, was one of the first modern mathematicians to develop calculus in a sophisticated manner).

The derivative of the derivative of f is called the second derivative of f, and we write f'' for this new function. In general, the nth derivative of f is denoted by $f^{(n)}$ (it is the function produced by repeatedly differentiating f). In Leibniz notation, the nth derivative of $y = f(x)$ is $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$.

Definition. 1. If we can approximate the change in a function f around a point x_0 by writing $f(x_0 + h) - f(x_0) \approx mh$ for some constant m that depends on x_0 but not h, then f is said to be differentiable at x_0 with derivative $m = f'(x_0)$.

2. If $y = f(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$.

3. The line passing through $(x_0, f(x_0))$ with slope $f'(x_0)$ is called the *tangent line* to f at x_0.

We will look at a more complicated function now. Consider $y = \sin(x)$; we want to find its derivative, so we look at our output difference:

$$\sin(x+h) - \sin x = 2\cos\frac{(x+h)+x}{2}\sin\frac{(x+h)-x}{2} = 2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right).$$

For very small values of α, we have that $\sin\alpha \approx \alpha$ (as long as we measure α in radians).[1] In particular, $\sin(h/2) \approx h/2$. Further, if h is very small then $x + h/2 \approx x$. Making these two approximations, we find that

$$\sin(x+h) - \sin x \approx 2\cos(x)\cdot(h/2) = \cos(x)h.$$

Thus, for each x, we have

$$\frac{\sin(x+h) - \sin x}{h} \approx \cos x; \tag{I.2}$$

and as this approximation becomes arbitrarily precise as h gets closer to zero (because the closer h is to zero our approximations $h/2 \approx 0$ and $\sin(h/2) \approx h/2$ become better and better) we feel justified in saying that $\sin' = \cos$.

Finally, we will look at the absolute value function $x \mapsto |x|$ defined by

$$|x| = \begin{cases} x & \text{when } x \geq 0; \\ -x & \text{when } x < 0 \end{cases}.$$

This function is graphed in figure 2.

Let's try to calculate the slope of $y = |x|$.

- When we look around any positive x, the definition tells us that $|x+h| = x + h$. Thus $|x+h| - |x| = x + h - x = h$; hence $\frac{|x+h|-|x|}{h} = 1$, and so the derivative for any positive x is 1.

[1] See 2.13 in the trigonometry notes.

- When we look around any negative x, the definition tells us that $|x+h| = -x-h$. Thus $|x+h| - |x| = -x-h-(-x) = -h$; hence $\frac{|x+h|-|x|}{h} = -1$, and so the derivative for any negative x is -1.

But now we have a problem: at zero, if we take h to be a small positive number we find that $\frac{|x+h|-|x|}{h} = 1$; but if h is a small negative number, then $\frac{|x+h|-|x|}{h} = -1$. Since looking in different directions from the same point gives us different slopes, there is no good linear approximation to the curve $y = |x|$ at the point $(0,0)$. We will say that the curve is *non-differentiable* there.

Exercises and Problems

1. Let f be a function. Describe the difference between f, f', $f(57)$, and $f'(57)$.

2. The following data are temperatures T in a place measured hourly (t is the number of hours since noon, and T is measured in degrees celsius).

t	0	2	4	6	8	10	12
T	15	16	16	15	14	13	13.

   Since T is a function of t, we can consider the derivative T'. What does this derivative represent? What are the units of $T'(t)$? Give an approximate value for $T'(8)$.

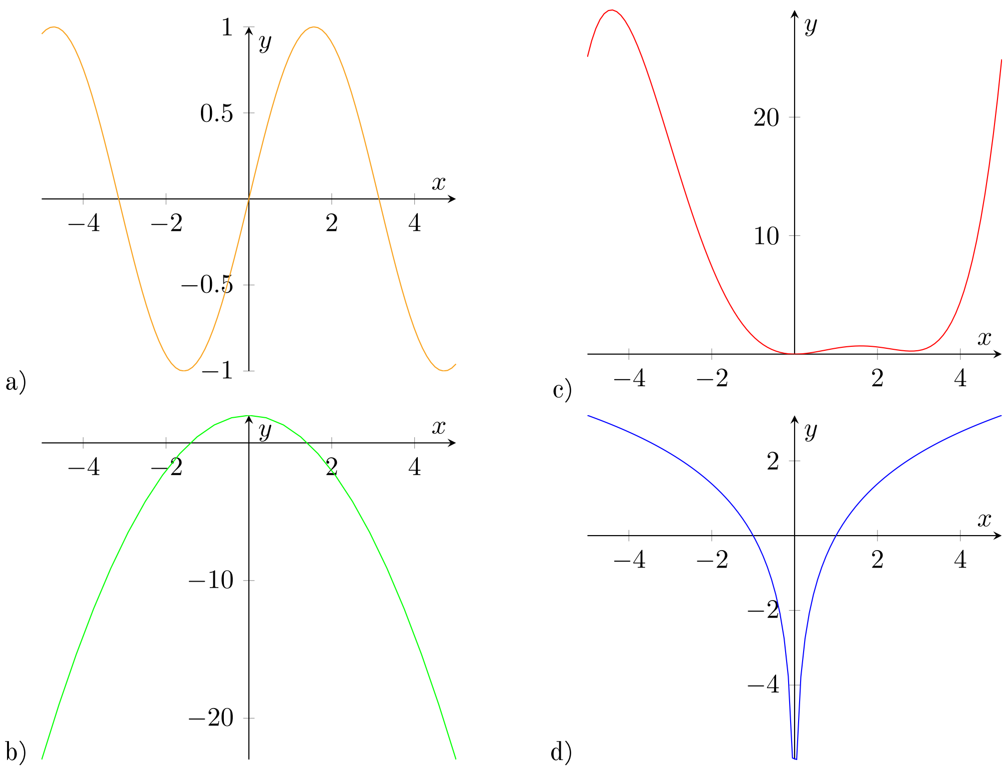
3. A particle is moving along a straight line, such that its displacement at time t is $s(t)$ (in metres, measured from some starting point).

   a) If t is measured in seconds, what are the units of $s'(t)$? What is the meaning of this quantity?

   b) The derivative of s' is denoted by s''. What are the units of $s''(t)$, and what is the meaning of this derivative?

4. Consider the curve $y = f(x)$ for some function x. What is the equation of the line joining the points $(x, f(x))$ and $(x+h, f(x+h))$? By letting h approach zero, give a justification for the definition of the tangent line to f at $(x, f(x))$. (Include a picture!)

5. The slope of a curve at a point tells us two things: whether it is sloping up or down, and the speed at which it is changing. Here are a few graphs of functions; sketch the graphs of their derivatives.



6. Describe several ways in which a function f can fail to be differentiable at a point x, illustrating your examples with sketches.
7. Consider each of these functions in turn. Where is the derivative of each (i) negative, (ii) positive, (iii) zero, and (iv) undefined?
   a) $x \mapsto x^2$
   b) $x \mapsto \sin x$
   c) $x \mapsto \tan x$
8. Justify: 'The derivative of f is the same as the derivative of $f + K$ for every constant K.'[2]
9. Above, we proved that if $y = x^2 + x - 1$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 1$. Write the equation of the tangent line to this curve at the point $(3, 11)$.
10. Let f be a function. Suppose that it is known that $f'(3) = 9$, and $f(3) = 6$.
    a) What does the graph of $y = f(x)$ look like around $x = 3$?
    b) Give the equation of the tangent line to $f(x)$ at $x = 3$.
11. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.
    a) What is the meaning of the derivative $f'(5)$?
    b) Suppose that there is an unlimited amount of space and nutrients. Which would you expect to be larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited does your answer change?
12. Prove that the only curves with constant slope are straight lines.
13. A ball dropped from a tower accelerates from rest at a constant rate, $-g$. If $h(t)$ is the height of the ball t seconds after it is dropped, what is its velocity t seconds after being dropped?
14. One model of population claims that the rate of change of a population P at a time is directly proportional to the size of the population at that time. In other words, $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(t)$ for some constant P. If $P(0) = 100$, sketch the population over time.
15. Consider an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is not a function: since both $(0, b)$ and $(0, -b)$ are members of the function, it fails the vertical line test. However, it would be nice to reason about its rate of change *as if it were* a function. Describe the slope of the ellipse as a particle traces the curve in an anticlockwise direction at a constant rate.
16. If $f(x) = x^3 + x$, calculate the derivative f' by writing $f(x + h) - f(x) \approx kh$ for some k.
17. We will calculate the derivative of $f(x) = \sqrt{x}$ at the point $(1, 1)$. To do this, consider the point $P = (1, 1)$ and the sequence of points $P_h = (1 + 1/h, \sqrt{1 + 1/h})$.
    a) Justify: we will obtain the tangent line to f at P by taking h larger and larger, so P_h gets closer and closer to P.
    b) Show that the slope of the line joining P and P_h is $\sqrt{h(h + 1)} - h$.
    c) Unfortunately, it is hard to see how this quantity behaves as h grows. Use the identity $a^2 - b^2 = (a + b)(a - b)$ to rewrite $\sqrt{h(h + 1)} - h$ as $\frac{h(h+1)-h^2}{\sqrt{h(h+1)}+h}$.
    d) Rewrite this fraction as $\frac{1}{\sqrt{1+1/h^2}+1}$ (so $\frac{f(1+1/h)-f(x)}{1/h} = \frac{1}{\sqrt{1+1/h^2}+1}$), and hence show that $f'(1) = 1/2$.

References

For a readable introduction to differentiation as the study of linear approximations analagous to what we see above, see the first few chapters of Thompson's *Calculus made easy*.

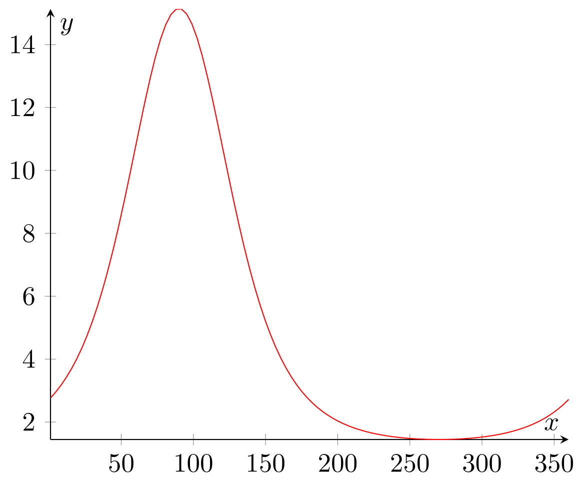
For many exercises on the behaviour of derivatives (as rates of change, and as slopes of tangent lines) see sections 2.1 and 2.2 of Stewart.

See also the section on calculus from my L2 notes.

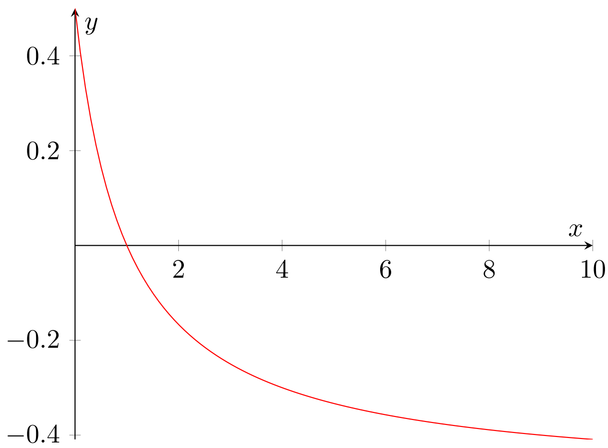
[2] Here, $f + K$ denotes the function defined by $(f + K)(x) = f(x) + K$ for all x. Likewise, if f and g are functions we define $f + g$ to be the function satisfying $(f + g)(x) = f(x) + g(x)$ for all x.

Homework problems

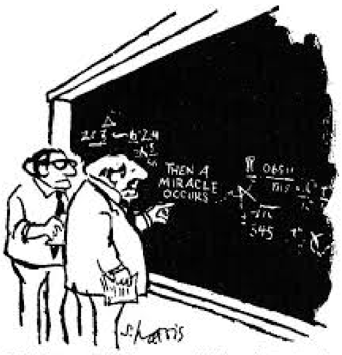
1. Draw the derivative of the following graphed function:



2. The following is the graph of the derivative of some function f. Sketch the graph of f, if $f(0) = 0$.



3. Show that if f and g are two functions, then $(f+g)' = f' + g'$.



"I think you should be more explicit here in step two."

Figure 3: Let's try to be less handwavy.

I.2 Limits at points

Last time, we defined the slope of a function f by considering the 'slope quotient'

$$\frac{f(x+h)-f(x)}{h}. \tag{I.3}$$

We said that a function was differentiable if we could approximate the top of this quotient by some expression of the form $f'(x)h$, and that the derivative of f at x was the number $f'(x)$.

This is a fairly intuitive definition, but there is a slight problem with it: we have no real way of knowing which approximations are 'valid'. For example, consider the function $f(x) = x^2$. The top of the difference quotient becomes $(x+h)^2 - x^2 = 2xh + h^2$. At this point, we wave our hands around and exclaim loudly that 'because h is small, h^2 is tiny and so we can just forget about it'. But waving our hands around is not a substitute for understanding what is going on!

This problem was rife in early discussions of calculus; the original way of dealing with it involved talking about 'infinitesimals' (this is how Leibniz thought about derivatives in the 1700s, for example) but it turns out that this makes more problems than it solves.

The modern solution involves what are called *limits*, which are a measure of how a function behaves around a point.

Definition. Suppose $f(x)$ is defined for all x around a point a (but not necessarily at a itself). Then we say that the limit of f as x approaches a equals L if we can make the values of $f(x)$ as close as we like to L by taking x to be close to (but not equal to) a. We write this symbolically as

$$\lim_{x\to a} f(x) = L,$$

or write that $f(x) \to L$ as $x \to a$.

The idea is that the limit of f at a is L if we can look at the graph of f, cover up the vertical line $x = a$, and use the behaviour of $y = f(x)$ in the neighbourhood of $x = a$ to guess what the graph looks like at that point. *The limit of f at a is dependent only on the points around a, not on the value (or lack thereof) of $f(a)$.*

Consider the graph of a function g given in figure 4. Although the *value* of the function at 2 is 6, the *limit* of the function at 2 is $\lim_{x\to 2} g(x) = 4$.

You can also think of $\lim_{x\to a} g(x)$ as being the unique value that we could pick for $g(a)$ such that the function around that point has 'no gaps'. If there is no such unique value, there is no limit at the point a.

A function a is called *continuous* at a point a if $\lim_{x\to a} f(x) = f(a)$; that is, if it takes the value at a which we would expect it to based on the points around a. The function g graphed in figure 4 is continuous at every point except $x = 2$.

Limits happen to have a few simple properties.

Theorem. *If f and g are functions and the limits of f and g at x_0 exist, then:*

1. $\lambda \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} [\lambda f(x)]$ *(where λ is a constant);*

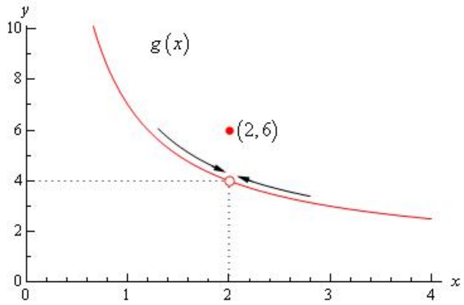


Figure 4: A function g where $\lim_{x\to 2} g(x) \neq g(x)$.

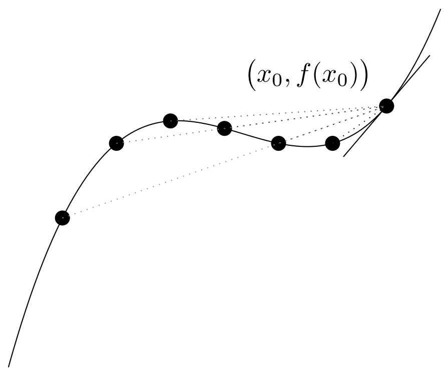


Figure 5: The tangent line as a limit of secant lines.

2. $\lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = \lim_{x\to x_0} [f(x) + g(x)]$*;*

3. $\left(\lim_{x\to x_0} f(x)\right)\left(\lim_{x\to x_0} g(x)\right) = \lim_{x\to x_0} [f(x)g(x)]$*;*

4. $\dfrac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \lim_{x\to x_0} \dfrac{f(x)}{g(x)}$ *(if $g(x) \neq 0$ around the point we take the limit); and*

5. $f(\lim_{x\to x_0} g(x)) = \lim_{x\to x_0} [f(g(x))]$ *(if f is continuous).*

Examples. Using these limit laws, we can find some limits reasonably easily.

1. $\lim_{x\to 0} \frac{x}{x} = 1$ since as x gets closer and closer to 0, $\frac{x}{x} = 1$.

2. $\lim_{x\to 3} \frac{(x-2)(x-3)}{x-3} = 1$ since as x gets closer and closer to 3, the fraction gets arbitrarily close to 1.

3. $\lim_{x\to 0} \frac{1}{x}$ does not exist, since if we approach 0 from the left the function becomes arbitrarily negative and if we approach 0 from the right the function becomes arbitrarily positive — we do not approach the same value on both sides.

4. $\lim_{x\to\infty} \frac{1}{x} = 0$ since as x becomes arbitrarily large, $\frac{1}{x}$ becomes arbitrarily small.

5. $\lim_{x\to 0} \sqrt{x}$ does not exist, since $\sqrt{x}$ is undefined for $x < 0$.

We will now use limits to define derivatives in a way that is relatively easier to deal with. If f is a function, and we want to find the derivative at a point $(x_0, f(x_0))$, then we consider the slopes of the lines joining this point to other points $(x_0, f(x_0 + h))$. As $h \to 0$, these secant lines become closer and closer to being tangent lines at x_0; so if the limit of the slopes of the secant lines exist, it makes sense to define this as the derivative $f'(x_0)$. In other words,

$$f'(x_0) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{I.4}$$

See figure 5 for a visualisation of this.

Just to bring ourselves back to our original formulation of derivatives as ways of computing linear approximations, we will say that $f(x+h) - f(x) \approx mh$ if $f(x+h) - f(x) = mh + \vartheta(h)$, where ϑ is some function such that $\vartheta(h)/h \to 0$ as $h \to 0$. (To see that this is the same thing as I.4, divide through by h and take the limit of both sides as $h \to 0$.)

Example. We will find the derivative of $f(x) = x^3$ at the point x using the definition.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\ &= 3x^2. \end{aligned}$$

Another view of limits (optional)

The notion of limits is incredibly fundamental to mathematics. In fact, the difference between the rational numbers and the real numbers is that, in the real numbers, every function which should have a limit at a point does have a limit at that point.

What do I mean by this?

Well, consider $f(x) = \sqrt{x}$. Clearly we want this function to be continuous. If we apply f to the sequence

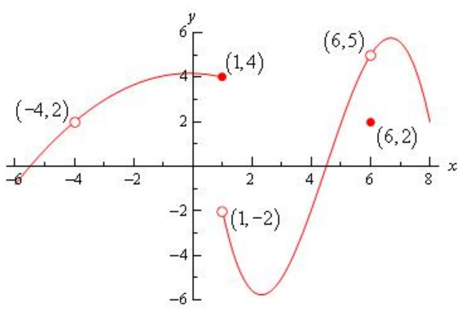
$$x_0 = \frac{1}{1}, x_1 = \frac{14}{10^1}, x_2 = \frac{141}{10^2}, x_3 = \frac{1414}{10^3}, \ldots, x_n = \frac{\lfloor 10^n\sqrt{2}\rfloor}{10^n}, \ldots$$

(where $\lfloor x \rfloor$ is the largest integer not larger than x) then each output is a rational number: $f(x_1) = 14/10$, for example. Furthermore, these outputs get closer and closer together so $\lim_{n\to\infty} f(x_n)$ should exist. But the limit cannot exist in the rationals, because $\lim_{n\to\infty} f(x_n) = \sqrt{2}$, which is not rational!

Formally, the way we get from the rational numbers (the numbers of the form a/b, where a and b are integers) to the real numbers is by taking the set of rationals and adding to it all the limits of Cauchy sequences — sequences where the adjacent terms get arbitrarily close together as we walk further along them.

Exercises and Problems

1. Consider the function f graphed below.



   a) For each of the following expressions, either give the value or explain why the expression is undefined.
      i. $f(-4)$
      ii. $\lim_{x\to -4} f(x)$
      iii. $f(1)$

iv. $\lim_{x\to 1} f(x)$

b) Explain why the limit $\lim_{x\to 6} f(x)$ is not equal to $f(6)$.

c) At which points is f:

i. Discontinuous?

ii. Non-differentiable?

2. Evaluate the limit or explain why it does not exist:

a) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

b) $\lim_{x\to 0} \frac{1}{x^3}$

c) $\lim_{x\to 9} \frac{1}{x^3}$

d) $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$

e) $\lim_{x\to 4} \frac{x^2+5x+4}{x^2+3x-4}$

f) $\lim_{x\to \frac{\pi}{2}} \sin x$

g) $\lim_{x\to \infty} \sin x$

h) $\lim_{x\to \frac{\pi}{2}} \tan x$

i) $\lim_{x\to 0} \tan x$

j) $\lim_{x\to 0} \csc x$

k) $\lim_{x\to a} C$, where a and C are constants.

l) $\lim_{x\to -\infty} \tan^{-1} x$

m) $\lim_{y\to 0} \lim_{x\to 0} \frac{(x+y)(x-y)}{x^2-y^2}$

n) $\lim_{x\to \infty} 1/x$.

o) $\lim_{x\to \infty} \frac{2x}{x^2+1}$.

p) $\lim_{x\to \infty} \frac{x+2}{x-3}$.

3. Show that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ and $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ are equivalent definitions for the derivative at the point a of some function f.

4. If $\lim_{x\to a}[f(x)+g(x)] = 2$ and $\lim_{x\to a}[f(x)-g(x)] = 1$, find $\lim_{x\to a} f(x)g(x)$.

5. Explain why $\frac{x^2+x-6}{x-2} \neq x-3$, but $\lim_{x\to a} \frac{x^2+x-6}{x-2} = \lim_{x\to a}(x-3)$ for every a.

6. Prove that if f is differentiable at a then it is continuous at a.

7. Last year, we defined the *exponent* a^r as follows:

- If $r = 0$, then $a^r = 1$.
- If r is a natural number, then $a^r = a^{r-1} \cdot r$. (So $a^r = \underbrace{a \times \cdots \times a}_{r \text{ times}}$.)
- If r is a negative integer, then $a^r = \dfrac{1}{a^{-r}}$. (Note that $-r$ is positive.)
- If r is a rational number, so that $r = p/q$ in lowest form, then $a^r = a^{(p/q)} = \sqrt[q]{a^p}$ (where we take the positive root, if a choice needs to be made).

Give a reasonable definition for a^r where r is any real number. Use your definition to compute a reasonable approximation to 2^π (given that $\pi \approx 3.14159...$).

References

For various exercises regarding limits, see sections 1.5 and 1.6 of Stewart. For a proper definition of limits (because the one given above is still handwavy: what do we mean by 'close'?) see Spivak (although this is beyond what the typical Y13 student needs).

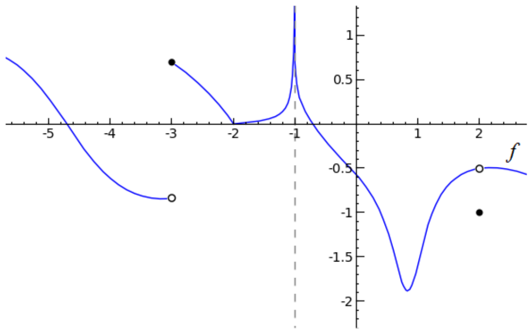
For a very nice treatment of derivatives that defines $f'(x)$ to be the unique value for m satisfying $f(x+h) - f(x) = mh + \vartheta(h)$, see Loomis and Sternberg, sections 3.5 and 3.6. We will use the notation $f(x+h) \approx f'(x)h + f(x)$ in this kind of way repeatedly for the next few sections, but won't stop to make it too rigorous; in order to do so, replace it with the equivalent equality involving $\vartheta(h)$ and remember that $\vartheta(h)/h \to 0$ as $h \to 0$ by definition. (L & S calls functions like ϑ the 'little-oh' class of infinitesimals.)

Homework problems

Derivatives and limits allow us to classify functions and their behaviour. Consider the following geometric properties:

- A function is *increasing* if its derivative is positive.
- A function is *decreasing* if its derivative is negative.
- A function is *concave down* if its derivative is decreasing.
- A function is *concave up* if its derivative is increasing.
- A function f is *continuous* at a point a if $\lim_{x\to a} f(x) = f(a)$.

1. Describe all the function properties given above geometrically, and give an example of each.
2. Consider the function graphed below.



   a) Find $\lim_{x\to -2} f(x)$ and $\lim_{x\to 2} f(x)$.
   b) Does $\lim_{x\to -3} f(x)$ exist? Why/why not?
   c) Does $\lim_{x\to 0} f(x)$ exist? Why/why not?
   d) On what intervals is $f(x)$ continuous?
   e) At what points is $f(x)$ not differentiable?

3. On an axis, sketch a graph of some function f that has the following features:
   - Is continuous for $0 < x < 5$ and $5 < x < 9$ and is discontinuous when $x = 5$
   - Is concave down ($f''(x) < 0$) for $0 < x < 5$
   - Has $f'(x) = 0$ at $(3, 8)$
   - Has $\lim_{x\to 5} f(x) = 6$.
   - Is not differentiable at $(7, 3)$.

I.3 Taking derivatives

Taking derivatives using the definitions quickly becomes unmanagable. Because of this, we want to produce a set of rules which will allow us to take derivatives of common functions.

Differentiation of polynomials

We begin with a few easy observations that will allow us to take derivatives of polynomials.

Theorem (Arithmetic of derivatives). *Let f and g be functions.*

1. *If $f(x) = \lambda$ for all x (where λ is a constant), then $f'(x) = 0$ for all x.*
2. *The derivative of $f + g$ is $f' + g'$ (i.e. for all x, $(f + g)'(x) = f'(x) + g'(x)$).*
3. *If λ is a constant, then $(\lambda f)' = \lambda f'$.*

Proof. 1. In this case, $\frac{f(x+h)-f(x)}{h} = \frac{\lambda-\lambda}{h} = 0$, so the difference quotient is always zero and so is the derivative.

2. If h is small, $(f+g)(x+h)-(f+g)(x) = f(x+h)+g(x+h)-f(x)-g(x) \approx f'(x)h+g'(x)h$ and so the derivative at x is $f'(x) + g'(x)$.
3. $(\lambda f)(x + h) - (\lambda f)(x) = \lambda(f(x + h) - f(x)) \approx \lambda(f'(x)h) = (\lambda f')(x)h$.

□

Now we consider $f(x) = x^n$ for integers n. Using the binomial theorem,

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \cdots - x^n}{h}$$

where every term hidden in the $\cdots$ includes an h^2 factor, so we obtain

$$f'(x) = \lim_{h\to 0} nx^{n-1} + h(\cdots) = nx^{n-1}. \tag{I.5}$$

In fact, although we proved this for integer n, it holds in general:

Theorem (Power law). *If $f(x) = x^\alpha$ then $f'(x) = \alpha x^{\alpha-1}$.*

Example. We can now differentiate every function of the form $f(x) = a_1x^{b_1} + \cdots + a_nx^{b_n}$: $f'(x) = b_1a_1x^{b_1-1} + \cdots + b_na_nx^{b_n-1}$. In particular, if $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2\sqrt{x}}$; if $g(x) = 2x^2 + 3$ then $g'(x) = 4x$; and if $h(x) = \frac{1}{x} + x^7$ then $h'(x) = -\frac{1}{x^2} + 7x^6$.

Trigonometric derivatives

We have already seen that $\sin' = \cos$; using similar reasoning, we can prove the following:

Theorem (Trigonometric derivatives).

Function	**Derivative**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$

I will use the result $\sin' = \cos$ to prove that $\cos' = -\sin$, and we will prove the rest at a later time. Indeed, $\cos x = \sin(x + \pi/2)$; then $\frac{\mathrm{d}}{\mathrm{d}x} \cos x = \frac{\mathrm{d}}{\mathrm{d}x} \sin(x + \pi/2)$. But the graph of $\sin(x + \pi/2)$ is just the graph of $\sin x$, shifted to the left by $\pi/2$. Hence the slope of $\sin(x + \pi/2)$ is the same as the slope of $\sin x$, but shifted to the left by $\pi/2$; and the slope of $\sin x$ is $\cos x$. Hence:

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = \frac{\mathrm{d}}{\mathrm{d}x} \sin(x + \pi/2) = \cos(x + \pi/2) = -\sin x. \tag{I.6}$$

(This little trick I used here is explored in more detail in the L2 notes; we don't need it too often this year, because in a couple of sections we will learn a much more general way of dealing with this kind of situation.)

Application. Many phenomena in physics can be modelled with sine waves; for example, if a particle on the end of a spring is moving with simple harmonic motion, then it has position $x = A\sin(\omega t + \phi)$; taking derivatives, we find that it has velocity $v = \frac{\mathrm{d}x}{\mathrm{d}t} = A\omega\cos(\omega t + \phi)$ and acceleration $a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -A\omega^2 \sin(\omega t + \phi)$. In other words, it is always accelerating in the opposite direction to its movement!

Exponential functions

The next function we want to consider here is $f(x) = a^x$, for constants a. We can compute that

$$f'(x) = \lim h \to 0 \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \left(\frac{a^h - 1}{h} \right).$$

So the exponential functions a^x have derivatives of the form Ka^x, where K is some constant. This begs the question, for which value of a (if any) does $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x$ (i.e. $K = 1$)? Well, we need to solve

$$\lim_{h\to 0} \left(\frac{a^h - 1}{h} \right) = 1$$

for a. We will begin by setting $u = 1/h$, so when $h \to 0$ we have $u \to \infty$. Thus

$$\lim_{u\to\infty} (a^{1/u} - 1)u = 1 = \lim_{u\to\infty} 1;$$

and applying the limit laws,

$$\begin{aligned} \lim_{u\to\infty} (a^{1/u} - 1) &= \lim_{u\to\infty} \frac{1}{u}; \\ \lim_{u\to\infty} a^{1/u} &= \lim_{u\to\infty} \frac{1}{u} + 1; \\ a = \lim_{u\to\infty} (a^{1/u})^u &= \lim_{u\to\infty} \left(\frac{1}{u} + 1 \right)^u. \end{aligned}$$

It can be shown fairly easily that $\lim_{u\to\infty} \left(\frac{1}{u} + 1\right)^u$ does indeed exist (it has a value of 2.71828...), and we define its value to be e. Thus e is the base for the exponential function that is its own derivative: $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$. Often, we write $\exp(x) := e^x$.

Finally, note that if $K = \lim_{h\to 0} \left(\frac{a^h - 1}{h} \right) = \lim_{u\to\infty} u(a^{1/u} - 1)$ then

$$a = \lim_{u\to\infty} \left(\frac{K}{u} + 1 \right)^u = \lim_{u\to\infty} \left(\left(\frac{K}{u} + 1 \right)^{u/K} \right)^K = \lim_{(u/K)\to\infty} \left(\left(\frac{K}{u} + 1 \right)^{u/K} \right)^K = e^K;$$

hence $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x K = a^x \log_e a$. (We normally write $\log_e = \ln$.)

Logarithmic derivatives

Finally, let us calculate $\frac{\mathrm{d}}{\mathrm{d}x} \ln x$. (This will allow us to find $\frac{\mathrm{d}}{\mathrm{d}x} \log_a x$ for all a, using the relationship $\log_a x = \frac{1}{\ln a} \ln x$.)

$$\ln'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h\to 0} \frac{1}{h} \ln \left(1 - \frac{h}{x} \right) = \lim_{h\to 0} \ln \left(1 - \frac{h}{x} \right)^{1/h}$$

Let $u = 1/h$; so as $h \to 0$, $u \to \infty$. Then, substituting, we obtain

$$\ln'(x) = \lim_{u\to\infty} \ln\left(1 - \frac{1}{ux}\right)^u.$$

Now, we use the fact that $\exp(\ln x) = x$:

$$e^{\ln'(x)} = \lim_{u\to\infty} \exp(\ln\left(1 - \frac{1}{ux}\right)^u) = \lim_{u\to\infty}\left(1 - \frac{1}{ux}\right)^u = \lim_{u\to\infty}\left(\left(1 - \frac{1}{ux}\right)^{ux}\right)^{(1/x)} = e^{1/x}$$

and thus $\ln'(x) = 1/x$.

Exercises and Problems

1. Find the derivatives of $3x^3$, $2x^2$, and $6x^5$. Conclude that $(fg)' \neq f'g'$ in general.
2. Find the derivatives of the following functions with respect to t:
   a) $y = 2t^3 + 3t^2$
   b) $y = \sqrt{t}$
   c) $y = (2t+1)(t-4)$
   d) $g(t) = 4\sec t + 9\tan t$
   e) $h(t) = \sqrt[5]{t} + 2\csc t - \ln t^3$
   f) $\phi'(t) = \csc x + 12x^{1273} + 9$
   g) $y = 2017t^{2016} + (t+2)^2$
   h) $y = 940\sin t + \frac{1}{2}e^{t+2}$
3. Where is the function $x \mapsto x^3 - 2x^2 - x + 1$ increasing?
4. Find the velocity v of a particle at time $t = 2\pi$ if its position function for $t > 0$ is $x = e^t - \sin t$.
5. Find the slope of the tangent line to $y = x + \tan x$ at (π, π).
6. Find a linear approximation $\tilde{f}$ to $f(x) = x^2 + x + 1$ at $(0, 1)$, and find some δ such that for all x satisfying $-\delta < x < \delta$, $-0.1 < \tilde{f}(x) - f(x) < 0.1$.
7. It is **not** true that the derivative of $f(g(x))$ is $f'(g'(x))$.
   a) For a counterexample, consider $f(x) = x^2$ and $g(x) = x$; show that $f'(g'(x)) = 2$, but $\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = 2x$.
   b) Compute the derivative of $\ln x^2$.
8. Suppose the derivative of a function is $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - x - 4$. What could the original function be?
9. Find the 64th derivative of $\sin x$.
10. Find the nth derivative of x^n.
11. If $y = 2\sin 3x \cos 2x$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. (Hint: use an identity to rewrite this as a sum of functions.)
12. For which values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?
13. Show that $y = 6x^3 + 5x - 3$ has no tangent line with a slope of 4.
14. Find real values of α and β such that, if $y = \alpha\sin x + \beta\cos x$, then $y'' + y' - 2y = \sin x$.
15. Consider a 12 m long ladder leaning against a wall such that the top of the ladder makes an angle θ with the wall. If this angle θ is varied, the distance D between the bottom of the ladder and the wall also changes. If $\theta = \pi/3$, what is the rate of change of D with respect to θ?
16. Prove that the function φ given by $\varphi(x) = \frac{x^{101}}{101} + \frac{x^{51}}{51} + x + 1$ never has a horizontal tangent line.
17. The derivative is primarily a geometric concept, not an algebraic one.
   a) The area of a circle of radius r is $A = \pi r^2$. Find $\frac{\mathrm{d}A}{\mathrm{d}r}$. What do you notice?
   b) Explain item (a) geometrically.
   c) The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. Find an expression for the surface area.

References

See sections 2.1 – 2.4 of Stewart. For a discussion of the exponential function and its relationship to compound interest and rates of growth, see Thompson chapter XIV.

Homework problems

1. Differentiate with respect to x:
   a) $x^2 + \frac{1}{x}$
   b) tx^t
   c) $\sin x - \cos x$
   d) $\sqrt[5]{x^4}$
2. Explain why you cannot use the power rule to find the derivative of x^x.
3. Find the nth derivative of $\frac{1}{x^n}$.
4. Suppose a population grows exponentially with time, such that after t years the population is $P(t) = P_0 + 10^t$.
   a) Find the rate of change of the population at $t = 100$.
   b) Explain why this population model is unrealistic.

I.4 Anti-derivatives

We now want to begin to study the inverse of differentiation: the problem is, given a function f, to find some function F which has f as a derivative. Geometrically, we are given the rate of change of a function at every point, and we wish to recover the original function.

If $f = F'$, then F is said to be an *anti-derivative* of f.

First of all, we notice that if F is an anti-derivative of f then so is $F + C$ for any constant C; this is because $(F + C)' = F' + C' = F' + 0 = F' = f$. Thus when we take anti-derivatives there are infinitely many different solutions that all differ by a constant — we cannot recover the original function given a slope function without some more information. These functions are said to be the family of solutions that solve the differential equation $F' = f$; we will also write $F(x)+C = \int f(x)\,\mathrm{d}x$ in this case, and we say that F is the *indefinite integral* of f; in the expression $\int f(x)\,\mathrm{d}x$, $f(x)$ is called the *integrand*.

The reason for this terminology comes from the geometric meaning of the derivative — if F is an anti-derivative of f, then f is the slope function of F. This in turn means that each value of f tells us how quickly F is rising or falling at that point; roughly speaking, to get F back from f, we need to walk along f, adding up all these infinitesimal rises and falls of F — in other words, we need to take all the values of f, and 'integrate' (combine) them together to get back the form of F.

This idea is made precise in the *fundamental theorem of calculus*, which we will state later on. This same theorem tells us that if f is a continuous function then there exists some anti-derivative F of f, and that this anti-derivative is unique up to a constant.

Differential equation		**Integral equation**
f is the slope function of F	$\iff$	F is an anti-derivative of f
$f(x) = F'(x)$	$\iff$	$F(x) + C = \displaystyle\int f(x)\,\mathrm{d}x$
$f(x) = \dfrac{\mathrm{d}y}{\mathrm{d}x}$	$\iff$	$y + C = \displaystyle\int f(x)\,\mathrm{d}x$

There are a few simple rules that we can state right away. For example, we have the following power rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}ax^n = nax^{n-1} \iff \int nax^{n-1}\,\mathrm{d}x = ax^n + C$$

so the anti-derivatives of ax^n are $\frac{a}{n+1}x^{n+1} + C$.

Looking at the inverse power law, we notice that there is an issue when we try to anti-differentiate $1/x$; the law tells us that $\int x^{-1}\,\mathrm{d}x = \frac{1}{1-1}x^0$, which is plainly nonsense. Luckily, last week we showed that $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = 1/x$, so $\int 1/x\,\mathrm{d}x = \ln x + C$.

Some more useful rules come by way of our differentiation arithmetic laws.

Theorem.

1. *$\int 0\,\mathrm{d}x = C$ (the family of constant functions)*
2. *$\int f(x) + g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$*
3. *$\int \lambda f(x)\,\mathrm{d}x = \lambda \int f(x)\,\mathrm{d}x$*

Proof.

1. Firstly, $\frac{\mathrm{d}}{\mathrm{d}x}0 = 0$; so the anti-derivatives of 0 are $0 + C = C$.
2. Let F and G be anti-derivatives of f and g, so $F' = f$ and $G' = g$. Then $\int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x = F(x) + G(x) + C$ (*); but $\frac{\mathrm{d}}{\mathrm{d}x}[F(x) + G(x) + C] = f(x) + g(x)$ and so $F + G + C$ is an anti-derivative of $f + g$ and $\int f(x) + g(x)\,\mathrm{d}x = F(x) + G(x) + C$. Combining this with (*) we obtain the result.
3. Let F be an anti-derivative of f. Then $(\lambda F)' = \lambda(F') = \lambda f$, and thus λF is an anti-derivative of λf; so $\int \lambda f(x)\,\mathrm{d}x = \lambda F(x) = \lambda \int f(x)\,\mathrm{d}x$.

□

Example. One anti-derivative of $y' = 3x^2 + 4$ is $x^3 + 4x$. Another is $x^3 + 4x + 1$. A third is $x^3 + 4x + 7$.

Unfortunately, there is no 'easy' way to anti-differentiate; we simply have to try to rearrange the function in some clever way until it looks like something that we know how to deal with.

Examples.

1. The most general antiderivative of $\sin x$ is $-\cos x + C$.
2. The most general antiderivative of $\tan x$ is $-\ln|\cos x| + C$.
3. $\int \frac{1}{x+3}\,\mathrm{d}x = \ln|x+3| + C$.
4. $\int \tan^2\theta\,\mathrm{d}\theta = \int \sec^2\theta - 1\,\mathrm{d}\theta = \tan\theta - \theta + C$.
5. $\int \frac{2x}{x^2+1}\,\mathrm{d}x = \ln\left|x^2+1\right| + C$.
6. $\int Ke^{Kx}\,\mathrm{d}x = e^{Kx} + C$ for all constants K.

Inverse problems (optional)

In mathematics there are a lot of examples of operations which are easy to perform in one direction, but harder to reverse.

Easy	**Hard**
addition	subtraction
multiplication	division
expanding	factorising
exponents	logarithms
tying a knot	unravelling a knot
differentiation	anti-differentiation

One very important application of this idea is in cryptography. Most modern computer systems are dependent on something known as the RSA cipher, which essentially relies on the fact that it's much easier to multiply large primes together than it is to work out what primes divide a large integer.

Anti-differentiation in particular is very difficult, in the sense that there are rules that enable us to take the derivative of every combination of 'simple' functions (polynomials, exponentials, logarithms, trig functions, sums, products, functions of functions) — but there are some functions, made up of these building blocks, which do not have anti-derivatives of the same type.

For example, there is no anti-derivative of x^x which can be produced with simple functions. There is a function f such that $f'(x) = x^x$, we just can't write it down at all using any combination of these building blocks despite the function $x \mapsto x^x$ being made up of them.

Exercises and Problems

1. For each expression, find the most general anti-derivative with respect to x.

   a) $2x$
   b) $x^2 + 3x + 1$
   c) $\frac{1}{2\sqrt{x}}$
   d) x^{-3}
   e) 10^x
   f) $\sec^2 x + \sqrt{x}$
   g) $x\sqrt{x}$
   h) $\sin x - \cos x$
   i) $\frac{2x^3+3x-\sqrt{x}}{\sqrt[3]{x}}$
   j) $\frac{1}{x^2} + e^x$
   k) $\sec^2(x+1)$

2. Verify the examples in the notes by differentiation.
3. Show that $\int 3x^2 + 4x + 5 + \frac{2}{x}\,\mathrm{d}x = x^3 + 2x^2 + 5x + \ln x^2 + C$.
4. If $\frac{\mathrm{d}y}{\mathrm{d}t} = 1.5\sqrt{t}$ and $y(4) = 10$, find $y(t)$ exactly.
5. Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$.
6. The velocity of a particle is given by $v(t) = 2t + 1$. Find its position at $t = 4$ if its position at $t = 0$ is $x = 0$.

7. The acceleration of a particle is given by $a(t) = 10\sin t + 3\cos t$. At $t = 0$, its position is $x = 0$; at $t = 2\pi$, its position is $x = 12$. Find its position at $t = \frac{\pi}{2}$.

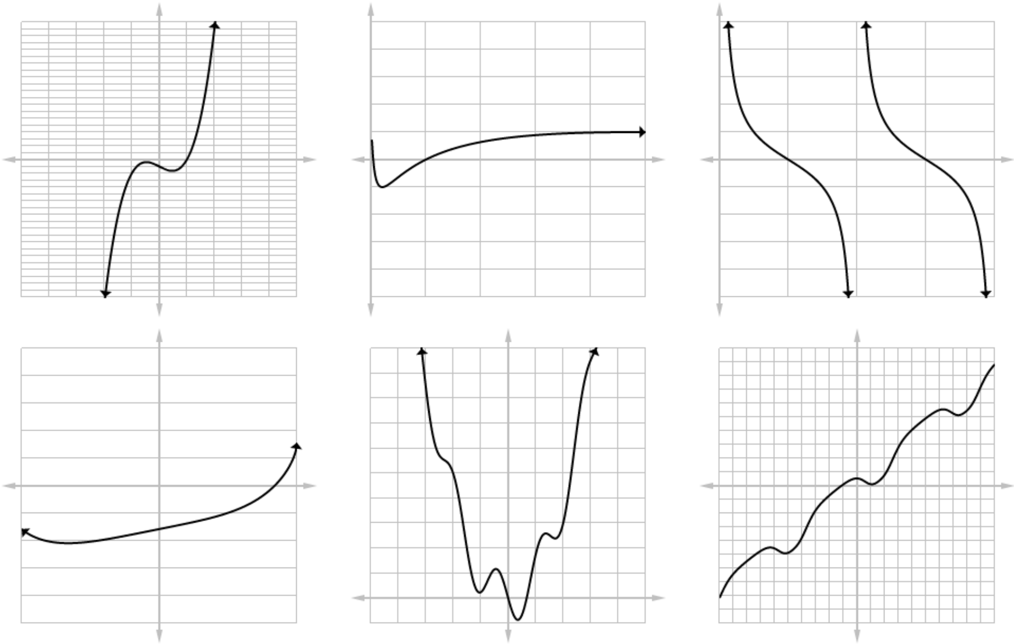
8. Starting from rest, a car takes T seconds to reach its maximum speed, $v_{\max}$. A plausible model for the velocity of the car after t seconds is

$$v(t) = \begin{cases} v_{\max}\left(\frac{2t}{T} - \frac{t^2}{T^2}\right) & t \leq T, \\ v_{\max} & t \geq T. \end{cases}$$

   a) Write an expression for $a_{\max}$, the maximum acceleration attained by the car.
   b) Show that the distance travelled by the car from the time it starts to the point it reaches its maximum speed is given by $s(t) = \frac{1}{3}a_{\max}T^2$.

9. Find all functions g such that $g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$.

10. For each function, sketch an anti-derivative passing through $(0, 0)$:



11. Show that if F is an anti-derivative of f, G is an anti-derivative of g, and α and β are any constants, then $\alpha F + \beta G$ is an anti-derivative of $\alpha f + \beta g$.

12. Give an example of functions f and g such that if F and G are anti-derivatives of f and g respectively then FG is *not* an anti-derivative of fg.

References

Because of the way we're covering integration, most books will have problems which we can't do yet (for example, $\int \tan x \,\mathrm{d}x$). That said, many more anti-differentiation problems can be found in the following: Stewart, section 4.4; Thompson, chapter XVIII.

Homework problems

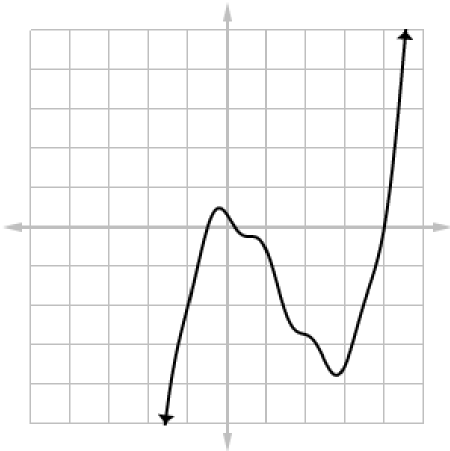
1. Find the most general anti-derivative.

   a) $f(x) = x - 3$
   b) $f(x) = (x+1)(x+2)$
   c) $f(\theta) = 6\theta^2 - 7\sec^2\theta$
   d) $g(h) = \pi^2$

e) $f(x) = x^{3.7} + \sqrt{x} + 7x^{\sqrt{7}-1}$

2. Given that the graph of φ passes through the point $(1, 6)$ and that the slope of its tangent line at $(x, \varphi(x))$ is $2x + 1$, find $\varphi(2)$.

3. This is the second derivative of g. Find g given that $g'(0) = 0$ and $g(0) = 1$.



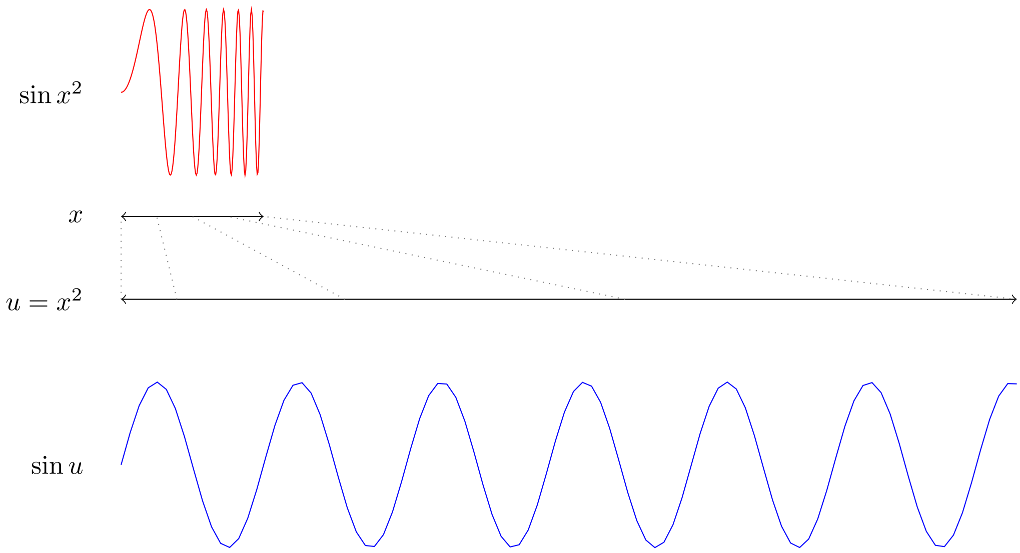


Figure 6: A graphic showing how the change of variables $u \leftrightarrow x^2$ stretches the graph of sin.

I.5 The chain rule

Consider the function $x \mapsto \sin(x^2)$. This function is made up of two functions, applied one after the other:

$$x \overset{f}{\mapsto} x^2 \overset{g}{\mapsto} \sin(x^2).$$

We often notate this function composition as $g \circ f$ (note that we evaluate from the right, so $(g \circ f)(x) = g(f(x))$).

Obviously the derivative of $\sin(x^2)$ is not just $\cos(2x)$, since the former has a horizontal tangent line at $x = \sqrt{\frac{\pi}{2}}$ but $\cos(\sqrt{2\pi}) \neq 0$. This shows us that, in general, the derivative of a function composition is not simply the composition of the derivatives.

In fact, it turns out that the derivative of $f \circ g$ is $g' \times (f' \circ g)$; in other words,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = g'(x) f'(g(x)).$$

This is known as the *chain rule*, since we are "chaining" together functions.

Let us convince ourselves that this rule is plausible. We can interpret the derivative $\frac{\mathrm{d}g}{\mathrm{d}x}$ as the rate of change of g with respect to x, and the derivative $\frac{\mathrm{d}f}{\mathrm{d}g}$ as the derivative of f with respect to small changes in the output of g; it is intuitive that if g changes twice as fast as x at some point, and f changes five times as fast as g, then f changes $2 \times 5 = 10$ times as fast as x.

The proof goes something like this: consider $f(g(x+h)) - f(g(x))$ for small h. We want to write this as mh, where m is only dependent on x. Now, since h is small, $g(x+h) \approx g(x) + g'(x)h$. Thus $f(g(x+h)) - f(g(x)) \approx f(g(x) + g'(x)h) - f(g(x))$. But if h is small enough, then $u = g'(x)h$ is small too; so, if $y = g(x)$, then

$$\begin{aligned} f(g(x+h)) - f(g(x)) &\approx f(g(x) + g'(x)h) - f(g(x)) \\ &= f(y+u) - f(y) \approx f'(y)u = f'(g(x))f'(x)h. \end{aligned}$$

Thus $(f \circ g)(x+h) - (f \circ g)(x) \approx f'(g(x))f'(x)h$, and so $(f \circ g)'(x) = f'(g(x))f'(x)$.

Examples.

1. The correct derivative of $\sin(x^2)$ is $2x\cos(x^2)$.

2. If $f(r) = \sqrt{r^2 - 3}$, then $f'(r) = 2r\frac{1}{2}\left(r^2 - 3\right)^{-1/2} = \dfrac{r}{\sqrt{r^2 - 3}}$.

3. If $g(x) = \sin((\sin^7 x^7 + 1)^7)$, then we compute:

$$\begin{aligned} g(x) &= \sin\left(\left[\left(\sin x^7\right)^7 + 1\right]^7\right) \\ g'(x) &= 7x^6 \cdot \cos x^7 \cdot 7\left(\sin x^7\right)^6 \cdot 7\left[\left(\sin x^7\right)^7 + 1\right] \cdot \cos\left(\left[\left(\sin x^7\right)^7 + 1\right]^7\right) \end{aligned}$$

We can use the chain rule to relate rates of change together — for example, the area of a circle is given by $A = \pi r^2$ and so the rate of change of area with respect to radius $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$; but if r varies with respect to time then we can find the rate of change of the area with respect to time using the chain rule.

A useful mnemonic is (if x is a function of y which is itself a function of z)

$$\frac{\mathrm{d}x}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\mathrm{d}x}{\mathrm{d}z}. \tag{chain rule}$$

We can also apply the inverse function rule for differentiation, which tells us that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}. \tag{inverse function rule}$$

The inverse rule is easy to prove: if f is a function, we have that $f(f^{-1}(y)) = y$. Taking the derivative of both sides, $f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$ and therefore $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$.

These two operations allow us to rearrange equations as if $\frac{\mathrm{d}y}{\mathrm{d}x}$ were a fraction. There isn't much of a problem if you do think of it in this way, as long as you're careful.

Example (Inverse function rule)**.** Let us find $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x$. If $y = \tan^{-1} x$, then $x = \tan y$; the inverse function rule tells us that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{\sec^2 y} = \cos^2 y$. Substituting for y, $\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos\tan^{-1} x)^2 = \frac{1}{x^2+1}$.[3]

Example. A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1\,\mathrm{m\,s^{-1}}$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?

Solution. Let x be the distance of the bottom of the ladder from the wall, and let y be the height of the top of the ladder up the wall. We have $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ and $x = 3$; we also know that $y = \sqrt{25 - x^2}$, so:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{\sqrt{25 - x^2}} \cdot 1 \\ \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{x=3} &= -\frac{3}{\sqrt{25 - 9}} = -\frac{3}{4}. \end{aligned}$$

Hence the ladder is sliding down the wall at a rate of $-0.75\,\mathrm{m\,s^{-1}}$.

Example. The radius of a sphere is increasing at a rate of $\frac{\mathrm{d}r}{\mathrm{d}t} = -\ln(t-1)$ metres per second. At what rate will the surface area of the sphere be growing at $t = 2$?

Solution. We have $\mathrm{SA} = 4\pi r^2$, so $\frac{\mathrm{dSA}}{\mathrm{d}r} = 8\pi r$ and

$$\frac{\mathrm{dSA}}{\mathrm{d}t} = \frac{\mathrm{dSA}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = -\ln(t-1) \times 8\pi r = 0.$$

The surface area of the sphere will be momentarily constant at $t = 2$.

[3] We proved that $\cos\tan^{-1} x = \frac{1}{\sqrt{x^2+1}}$ in the trigonometry notes.

Exercises and Problems

1. Identify the inner and outer functions, but don't try to differentiate.

   a) $\sqrt{\sin x}$
   b) $\sin\cos\tan x$
   c) $(2x+3)^{17}$
   d) $97(x+2)^2$
   e) $\ln\sin x$
   f) $\frac{1}{\sqrt{23x-x^2}}$

2. Differentiate with respect to t:

   a) $(2t+3)^{3000}$
   b) $\sin\ln t$
   c) $\sqrt{t^3+10t^2+3}$
   d) $\csc e^t$
   e) $\sin^3 t + 14\ln(3t)$
   f) $\sin\sin\sin t$
   g) $\cot(t+\sec t)$
   h) $\sin^2((t+\sin t)^2)$
   i) $\ln\sqrt{t+9}$
   j) $\sqrt{t}+\frac{1}{\sqrt[3]{t^4}}$
   k) $e^{\sec(t^2)}$
   l) $\sin\sqrt{t+\tan t}$

3. The derivative of a function is $2\cos 2x$. What could the original function be?

4. Differentiate $y=\sin^2 x+\cos^2 x$, and hence prove that $\sin^2 x+\cos^2 x=1$.

5. Suppose that the displacement of a particle on a vibrating spring is given by $x(t)=5+\frac{1}{8}\sin(5\pi t)$, where x is measured in centimetres and t in seconds.

   a) Find the velocity of the particle at time t.
   b) At which times is the particle momentarily stationary?

6. Find a linear approximation $\tilde{f}$ around 0 for $f(x)=\sqrt[4]{1+2x}$; then calculate δ such that for all x satisfying $-\delta<h<\delta$, $-0.1<\tilde{f}(x)-f(x)<0.1$.

7. The volume of a spherical balloon at a time t is given by $V(t)=\frac{4}{3}\pi r^2$, and its radius, changing over time, is given by $r(t)$. Find $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of $\frac{\mathrm{d}r}{\mathrm{d}t}$.

8. If $F(x)=f(3f(4f(x)))$, where $f(0)=0$ and $f'(0)=2$, find $F'(0)$.

9. Find the local extrema, areas of concavity, and inflection points of the following functions; hence sketch their graphs.

   a) $y=e^x\sin x$ for $-\pi<x<\pi$
   b) $y=x+\ln(x^2+1)$
   c) $y=\sin^{-1}(1/x)$

10. The depth of water at the end of a jetty in a harbour varies with time due to the tides. The depth of the water is given by the formula

$$W=4.5-1.2\cos\frac{\pi t}{6}$$

where W is the water depth in metres, and t is the time in hours after midnight.

   a) What is the rate of change of water depth 5 hours after midnight?
   b) When is the first time after $t=0$ that the tide changes direction?
   c) At that time, is the water changing from rising to falling or from falling to rising?

11. In physics, the rate of change of momentum of an object is proportional to the force needed to effect that change: if p is the momentum of the object as a function of time, $F = \frac{\mathrm{d}p}{\mathrm{d}t}$. The momentum of a particular object, oscillating back and forth along a line, is given by $p = mA\sin(\omega t + \phi)\,\mathrm{kg\,m\,s^{-1}}$ (where m, A, ω, and ϕ are various constants). What is the force acting on the object at $t = 10$?

12. Find the 73rd derivative of $\sin 6x$.

13. Each side of a square is increasing at a rate of $6\,\mathrm{cm\,s^{-1}}$. At what rate is the area of the square increasing when the area of the square is $16\,\mathrm{cm}^2$?

14. Gas is being forced into a spherical balloon at a rate of $400\,\mathrm{cm^3\,min^{-1}}$. How fast is the radius of the balloon increasing when the radius is $5\,\mathrm{cm}$?

15. If a snowball melts so that its surface area decreases at a rate of $1\,\mathrm{cm^2\,min^{-1}}$. find the rate at which the diameter decreases when the diameter is $10\,\mathrm{cm}$.

16. If $x^2 + y^2 + z^2 = 9$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$, and $\frac{\mathrm{d}y}{\mathrm{d}t} = 4$, find $\frac{\mathrm{d}z}{\mathrm{d}t}$ when $(x, y, z) = (2, 2, 1)$.

17. A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle moves through the point $(1/3, 1)$, its x-ordinate increases at a rate of $\sqrt{10}\ \mathrm{cm\,s^{-1}}$. How fast is the distance from the particle to the origin changing at this instant?

18. Gravel is dumped from a conveyor belt at a rate of $3\,\mathrm{m^3\,min^{-1}}$, and forms a pile in the shape of a cone with equal height and base diameter. How fast is the height of the cone increasing when the pile is $3\,\mathrm{m}$ tall?

19. The top of a ladder slides down a vertical wall at a rate of $0.15\,\mathrm{m\,s^{-1}}$. A the moment when the bottom of the ladder is $3\,\mathrm{m}$ from the bottom of the wall, it slides away from the wall at a rate of $0.2\,\mathrm{m\,s^{-1}}$. Find the length of the ladder.

20. Two sides of a triangle have lengths $2\,\mathrm{m}$ and $3\,\mathrm{m}$. The angle between these sides is increasing at a rate of $4\ {}^\circ\,\mathrm{s^{-1}}$. How fast is the length of the third side changing when it is of length $4\,\mathrm{m}$?

21. A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y-ordinate is decreasing at a rate of 3 units per second. How fast is the x-ordinate of the particle changing at that instant?

22. The minute hand on a watch is $8\,\mathrm{mm}$ long and the hour hand is $4\,\mathrm{mm}$ long. How fast is the distance between the tips of the hands changing at 1 o'clock?

23. If $f(\theta) = \sin^{-1}(\theta)$, compute $\frac{\mathrm{d}}{\mathrm{d}\theta} f(\theta)$ and $\frac{\mathrm{d}}{\mathrm{d}x} f(x^4)$.

24. Recall that the *absolute value* of x, denoted $|x|$, is the value obtained by 'throwing away the sign' of x.

    a) Prove that
    $$\frac{\mathrm{d}}{\mathrm{d}x}|x| = \frac{x}{|x|}.$$
    [*Hint: Write* $|x| = \sqrt{x^2}$.]

    b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of both f and f'.

    c) If $\frac{\mathrm{d}x}{\mathrm{d}t} = |\alpha'(t)|$, find $\frac{\mathrm{d}^2 t}{\mathrm{d}x^2}$.

25. Note that on the formula sheet, the anti-derivative of $1/x$ is given as $\ln|x|$, not just $\ln x$.

    a) Compute $\frac{\mathrm{d}}{\mathrm{d}x}\ln|x|$ if $x < 0$, and hence justify formally why $\frac{\mathrm{d}}{\mathrm{d}x}\ln|x| = 1/x$.

    b) Draw $y = \ln|x|$ and $y = 1/x$ on the same pair of axes, and hence justify intuitively why $\frac{\mathrm{d}}{\mathrm{d}x}\ln|x| = 1/x$.

26. Soon, we will be studying the product rule for derivatives. It is possible, though not particularly usual, to prove it using simply the basic derivatives from the last section and the chain rule; in this exercise, you will do just that.

    Suppose that f and g are functions, and consider the function F defined by $F(x) = \big(f(x) + g(x)\big)^2$.

a) Calculate $F'(x)$ using the chain rule.

b) Calculate $F'(x)$ by multiplying out the square and differentiating the polynomial that results. (In particular, note that $\frac{\mathrm{d}}{\mathrm{d}x}2(fg)(x) = 2(fg)'(x)$).

c) Compare parts (a) and (b).

References

For an approach to the chain rule similar to the one taken here, see chapter IX of Thompson. See also sections 2.5 and 2.8 of Stewart.

Homework problems

1. If $y = \sqrt{\cot x} - \sqrt{\cot a}$ (where a is constant), find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

2. a) Show that if $y = f(g(h(x)))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) \cdot g'(h(x)) \cdot f'(g(h(x)))$.

   b) Calculate the derivative of $y = \sin\cos\sin\cos\sin x^5$.

3. We will prove the double angle formula for cosine from the double angle formula for sine. Suppose $f(\theta) = \cos 2\theta$, and $g(\theta) = 1 - 2\sin^2\theta$.

   a) Show that $f' = g'$. (You may assume that $\sin 2\theta = 2\sin\theta\cos\theta$.)

   b) Verify that f and g agree at $\theta = 0$, and conclude that $f = g$.

4. If V is the volume of a cube with edge length x and the cube expands as time passes, find $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of $\frac{\mathrm{d}x}{\mathrm{d}t}$.

5. A water tank has the shape of an inverted circular cone with base radius $2\,\mathrm{m}$ and height $4\,\mathrm{m}$. If water is being pumped into the tank at a rate of $2\,\mathrm{m}^3\,\mathrm{min}^{-1}$, find the rate at which the water level is rising when the water is $3\,\mathrm{m}$ deep.

6. A boat is pulled into a dock by a rope attached to the bow of the boad and passing through a pulley on the dock that is $1\,\mathrm{m}$ higher than the bow of the boat. If the rope is pulled in at a rate of $1\,\mathrm{m\,s}^{-1}$, how fast is the boat approaching the dock when it is $8\,\mathrm{m}$ from the dock?

I.6 Substitution

Recall that the chain rule for differentiation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x).$$

In other words, $f(g(x))$ is an anti-derivative of $f'(g(x))g'(x)$ and so we can write

$$\int f'(g(x))g'(x)\,\mathrm{d}x = f(g(x)) + C.$$

To make this rule easier to apply in practice, we often perform what is known as a change of variables. We let $u = g(x)$, and then $\frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$. Substituting this in, we obtain

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

and then the rule is just the statement that we can 'cancel' the $\mathrm{d}x$'s, producing

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(u)\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int f'(u)\,\mathrm{d}u = f(u) + C = f(g(x)) + C.$$

This rule, which gives us a kind of chain rule for integration, is called *substitution*, or the *inverse chain rule*. It can be thought of as a change in coordinate system from an x-based system to one based on u, and we have to 'resize' our curve based on how much u stretches the coordinate system compared to x — and this 'stretch factor' is simply $\frac{\mathrm{d}u}{\mathrm{d}x}$.

Examples.

1. Suppose we wish to find $\int \sin x \cos x\,\mathrm{d}x$. Then let $u = \sin x$, so $\mathrm{d}u = \cos x\,\mathrm{d}x$ and

$$\int \sin x \cos x\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C.$$

   In this case, we also could have used a trigonometric identity.

2. Suppose we wish to find $\int xe^{x^2}\,\mathrm{d}x$. We can let $u = x^2$, and then $du = 2x\,\mathrm{d}x \Rightarrow \mathrm{d}x = \frac{du}{2x}$. Hence:

$$\int xe^{x^2}\,\mathrm{d}x = \int \frac{1}{2}e^u\,\mathrm{d}u = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

3. Suppose we wish to find $\int \frac{4}{x}(\ln x)^3\,\mathrm{d}x$. We let $u = \ln x$, and then $\mathrm{d}u = \frac{\mathrm{d}x}{x}$. Hence:

$$\int \frac{4}{x}(\ln x)^3\,\mathrm{d}x = 4\int u^3\,\mathrm{d}u = u^4 + C = (\ln u)^4 + C.$$

Exercises and Problems

1. Find the following indefinite integrals. (Remember, the indefinite integral of f, $\int f(x)\,\mathrm{d}x$, is the family of anti-derivatives of f.)

   a) $\int \sin 2x\,\mathrm{d}x$

   b) $\int \tan x\,\mathrm{d}x$

   c) $\int 3x\cos x\,\mathrm{d}x$

   d) $\int \frac{\cos x}{\sin x + 1}\,\mathrm{d}x$

   e) $\int (4x - 44)^{2019}\,\mathrm{d}x$

   f) $\int 4x\sqrt{x^2+3}\,\mathrm{d}x$

   g) $\int (3x-4)^2\,\mathrm{d}x$

   h) $\int \frac{x}{x^2+1}\,\mathrm{d}x$

   i) $\int \frac{2}{4x+3}\,\mathrm{d}x$

   j) $\int e^{2x+1}\,\mathrm{d}x$

k) $\displaystyle\int \sec 4x \tan 4x \,\mathrm{d}x$

l) $\displaystyle\int 2\cos x + \sin 2x \,\mathrm{d}x$

m) $\displaystyle\int -2x\csc^2(3x^2)\,\mathrm{d}x$

n) $\displaystyle\int \frac{3}{x^3} - \frac{4}{x+1}\,\mathrm{d}x$

o) $\displaystyle\int e^{x/2} + \frac{2}{x}\,\mathrm{d}x$

p) $\displaystyle\int x^2\sec^2 x^3 + 9\,\mathrm{d}x$

q) $\displaystyle\int -\csc(\tan x)\cot(\tan x)\sec^2 x\,\mathrm{d}x$

r) $\displaystyle\int \frac{\cos x - \sin x}{\cos x + \sin x}\,\mathrm{d}x$

s) $\displaystyle\int \frac{2017}{x\ln x}\,\mathrm{d}x$

t) $\displaystyle\int \tan x + \frac{1}{\tan x}\,\mathrm{d}x$

u) $\displaystyle\int (\cos x)(\sin\sin x)(\cos\cos\sin x)\,\mathrm{d}x$

2. By using the substitution $x = \sin\theta$, find

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

3. Evaluate $\int \cos^5 x\,\mathrm{d}x$ using the substitution $t = \sin x$.

4. Find $\int \tan\theta\,\mathrm{d}\theta$ and $\int \cot\theta\,\mathrm{d}\theta$.

5. Complete the following working:

$$\begin{aligned}
\int \sec x\,\mathrm{d}x &= \int \sec x\frac{\sec x + \tan x}{\sec x + \tan x}\,\mathrm{d}x \\
&= \int \frac{\dots}{\sec x + \tan x}\,\mathrm{d}x \\
\text{Let } u &= \dots \\
&= \int \frac{1}{\dots}\,\mathrm{d}u \\
&= \dots
\end{aligned}$$

6. Find an anti-derivative of $\csc x$. (Hint: consider the previous problem.)

7. The velocity of a particle at time t is given by $v = \dfrac{\cos(\sqrt{2t+1})}{\sqrt{2t+1}}$. What is the position of the particle at time $t = 5$, given that $x(0.5) = 0$? (Recall that $v = \frac{\mathrm{d}x}{\mathrm{d}t}$.)

8. Consider the following indefinite integral:

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

   a) Show, using the inverse function rule for differentiation, that the anti-derivatives of $\frac{1}{\sqrt{1-x^2}}$ are $\sin^{-1} x + C$.

   b) Compute the indefinite integral a different way, using the substitution $x = \sin\theta$.

   c) Find the anti-derivatives of

$$f(x) = \frac{-1}{2\sqrt{x-x^2}}.$$

   (Hint: try to substitute $u = \sqrt{1-x}$.)

9. Compute the following:

   a) $\displaystyle\int \frac{x^2(5x^2+4x-3)}{x^5+x^4-x^3+1}\,\mathrm{d}x$‘

   b) $\displaystyle\int \frac{x^2+1}{x(x^2+3)}\,\mathrm{d}x$

The previous problem involved finding anti-derivatives of *rational functions*: those of the form $\frac{P(x)}{Q(x)}$ for polynomials P and Q. In general, it is possible to find anti-derivatives of all such functions by writing them as sums of fractions with linear or quadratic denominators; this is known as *expansion via partial fractions*.

10. Some more interesting problems:

    a) Rewrite in the form $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$ and integrate:
    $$\int \frac{4x}{x^3 - x^2 - x + 1} \, \mathrm{d}x.$$

    b) Use the obvious substitution and divide through:
    $$\int \frac{\sqrt{x+1}}{x} \, \mathrm{d}x.$$

11. Recall that $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{x^2+1}$.

    a) Rewrite the given rational function as follows:
    $$\frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} = \frac{A}{3x-1} + \frac{Bx + C}{x^2 + 1}$$

    b) Hence (or otherwise) compute:
    $$\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} \, \mathrm{d}x.$$

12. Use appropriate substitutions to evaluate:

    a) $\displaystyle\int \frac{\cos\theta}{\sin^2\theta + 4\sin\theta - 5} \, \mathrm{d}\theta$

    b) $\displaystyle\int \frac{e^{3x}}{e^{2x} + 4} \, \mathrm{d}t$

    c) $\displaystyle\int \frac{5 + 2\ln x}{x(1 + \ln x)^2}$

13. In the following, let $t = \tan\frac{x}{2}$ (where $|x| < \pi$). We will apply the techniques from the last few problems in the notes, calculating some anti-derivatives of rational functions by expanding them as sums of fractions.

    a) Show that:
    $$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \quad \text{and} \quad \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$$

    b) Show that:
    $$\cos x = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin x = \frac{2t}{1+t^2}$$

    c) Show that:
    $$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}$$

    d) Use the substitution t to evaluate:

       i. $\int (1 - \cos x)^{-1} \, \mathrm{d}x$

       ii. $\int (3\sin x - 4\cos x)^{-1} \, \mathrm{d}x$

References

For exercises and notes on substitution, see Thompson chapter XX (the section on substitution is two or three pages in). For partial fractions, see chapter XIII.

For the interested, a proof that one can always expand a rational function into partial fractions is outlined as exercise 11.1.13 in Artin (p. 441).

Homework problems

1. Calculate the following indefinite integrals:

   a) $\int -\csc 3x \cot 3x \, \mathrm{d}x$

   b) $\int -x \sec^2 3x^2 \, \mathrm{d}x$

   c) $\int \frac{\sqrt{x}+3x-2}{x} \, \mathrm{d}x$

   d) $\int \sin^3 x \cos^2 x \, \mathrm{d}x$ (Hint: use $\sin^2 x = 1 - \cos^2 x$ to rewrite the integrand.)

2. Recall that $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}$. Find $\int \frac{x}{1+x^4} \, \mathrm{d}x$.

3. Let y be a function of x, and let x in turn be a function of t. If $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 0$, and if $x(t) = 7t + e^t$, find an explicit expression for $y(t)$.

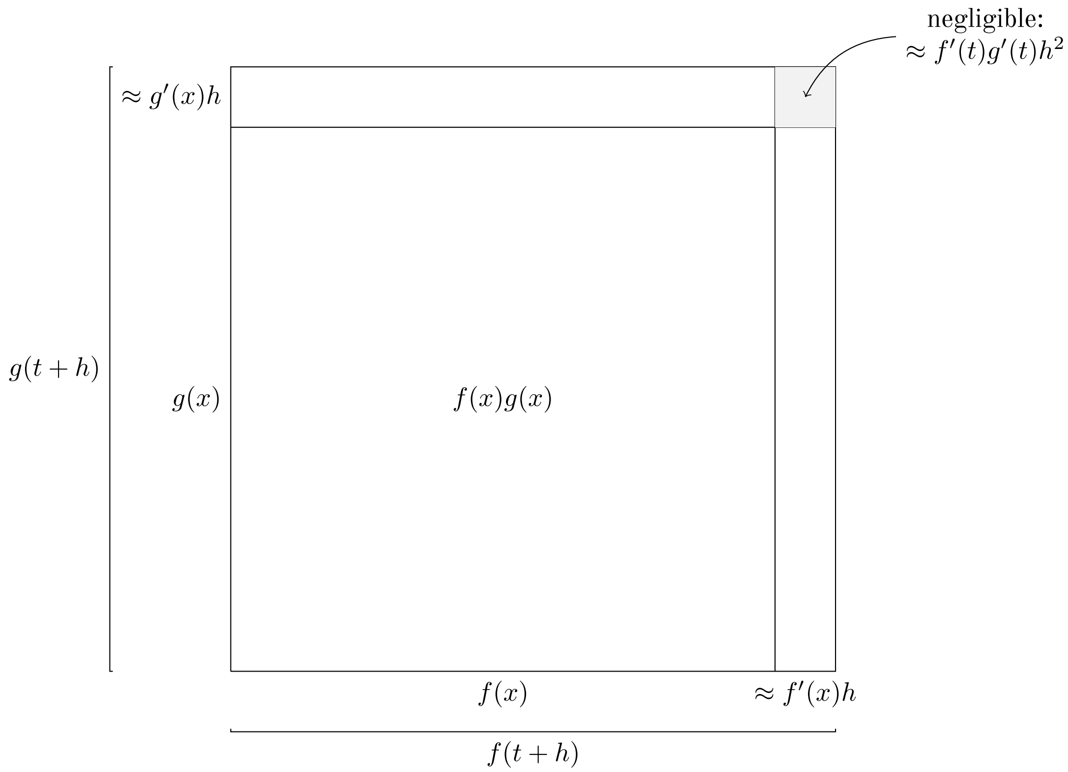


Figure 7: The approximate errors in the product rule estimation.

I.7 The product law

So far, we can differentiate functions which are made up of sums and compositions of polynomials, trig functions, and ln and exp. However, the following function will leave us lost and confused if we try to compute its derivative directly:

$$f(x) = (\sin x)(\cos x) \qquad f'(x) = ?$$

In this particular case, we can use the identity $\sin 2x = 2\sin x\cos x$ to rewrite $f(x) = \frac{1}{2}\sin 2x$ and then apply the chain rule to find that $f'(x) = \cos 2x$. However, in general we don't have nice things like trig identities; thus, we need a rule to differentiate products of functions.

First of all, we notice that $(fg)' \neq (f')(g')$.[4] Indeed, for the function f defined above, $(\sin x)'(\cos x)' = -\sin x\cos x$; this is zero at $x = 1$, but we have already seen that the derivative of f is $\cos 2x$, which is equal to 1 when $x = 1$.

We will try to derive one by estimation; consider the following difference:

$$(fg)(x+h) - fg(x) = f(x+h)g(x+h) - f(x)g(x)$$

We may assume that f and g are differentiable at x, and so we can approximate them with their derivatives,

$$\begin{aligned} f(x+h)g(x+h) - f(x)g(x) &\approx (f'(x)h + f(x))(g'(x)h + g(x)) - f(x)g(x) \\ &= f'(x)g'(x)h^2 + f'(x)g(x)h + f(x)g'(x)h. \end{aligned}$$

Applying the reasoning we developed a few sections ago, we note that as $h \to 0$, the approximation becomes exact.[5] Taking the limit of our difference quotient, we find that

$$\begin{aligned} \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} &= \lim_{h\to 0} \frac{f'(x)g'(x)h^2 + f'(x)g(x)h + f(x)g'(x)h}{h} \\ &= f'(x)g(x) + f(x)g'(x); \end{aligned}$$

[4] To save ink, I will write (fg) for the function defined by $(fg)(x) = f(x)g(x)$.

[5] Technical note. Recall that we defined $f(x) \approx g(x)$ if $f(x) = g(x) + \vartheta(h)$ where $\vartheta(h)/h \to 0$ as $h \to 0$. One can therefore make the reasoning here rigorous by carrying through the ϑ's that give us the approximations for $f(x+h)$ and $g(x+h)$, checking that we end up with an estimation term that is also satisfies the ϑ condition.

and we have justified the following

Theorem (Product law). *If f and g are differentiable at x, then*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Example. Consider $y = 2t\sin t$. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = 2\sin t + 2t\cos t$.

With our rules (sum, chain, and product, together with our basic derivatives), we can now differentiate almost any combination of functions that we are currently aware of. The process of differentiation is entirely mechanical, and can be easily performed by a computer. As such, learning to differentiate more complicated combinations of functions is very similar to learning how to add, multiply, and perform long division, and is only a matter of practice.

Example. Let $f(x) = \sin x^2 + e^{10x^2+3x+e^x} + \frac{2x+3}{\ln x}$. We can split this into three derivative-taking problems by applying the sum rule; so

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\sin x^2 + \frac{\mathrm{d}}{\mathrm{d}x}e^{10x^2+3x+e^x} + \frac{\mathrm{d}}{\mathrm{d}x}\frac{2x+3}{\ln x}$$

Taking these in turn, $\sin x^2 \mapsto 2x\cos x^2$ (applying the chain rule, since we have a function composition) and $e^{10x^2+3x+e^x} \mapsto (20x+3+e^x)e^{10x^2+3x+e^x}$ (applying the chain rule again). Finally, note that $\frac{2x+3}{\ln x} = (2x+3)[(\ln x)^{-1}]$ and so we need to apply the product rule:

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x}(2x+3)[(\ln x)^{-1}] &= \left(\tfrac{\mathrm{d}}{\mathrm{d}x}(2x+3)\right)\left((\ln x)^{-1}\right) + (2x+3)\left(\tfrac{\mathrm{d}}{\mathrm{d}x}(\ln x)^{-1}\right) \\
&= 2(\ln x)^{-1} + (2x+3)\left(-1(\ln x)^{-2}\cdot \tfrac{\mathrm{d}}{\mathrm{d}x}\ln x\right) \\
&= 2(\ln x)^{-1} + (2x+3)\left(-1(\ln x)^{-2}\cdot \frac{1}{x}\right) \\
&= \frac{2}{\ln x} - \frac{2x+3}{x(\ln x)^2} \\
&= \frac{2x\ln x - 2x - 3}{x(\ln x)^2}.
\end{aligned}$$

Thus the derivative of f is

$$f'(x) = 2x\cos x^2 + (20x+3+e^x)e^{10x^2+3x+e^x} + \frac{2x\ln x - 2x - 3}{x(\ln x)^2}.$$

Exercises and Problems

1. In each case, find $\frac{\mathrm{d}y}{\mathrm{d}t}$.

   a) $y = \left(3+2t^2\right)^4$
   b) $y = \dfrac{t^3}{\ln t}$
   c) $y = t\sqrt{t}$
   d) $y = 2t\sin t - (t^2-2)\cos t$
   e) $y = \dfrac{t}{\sqrt{a^2-t^2}}$ (a constant)
   f) $y = \dfrac{1}{8}t^8\left(1-t^2\right)^{-4}$
   g) $y = e^t\ln t$
   h) $y = \log\left[1 + \dfrac{t^2+3t+17}{t^{17}}\right]$
   i) $y = \sin\left[e^{\tan t}\ln\tan t\right]$
   j) $y = \dfrac{3t-2}{\sqrt{2t+1}}$
   k) $y = \dfrac{\sec 2t}{1+\tan 2t}$
   l) $y = \dfrac{(t-1)(t-4)}{(t-2)(t-3)}$
   m) $y = t\sin^2(\cos\sqrt{\sin\pi t})$
   n) $y = \sqrt[5]{t\tan t}$
   o) $y = \dfrac{(t+\lambda)^4}{t^4+\lambda^4}$ (λ constant)

2. If $f(x) = e^{-x}$, find $f(0) + xf'(0)$.

3. Show that $\frac{\mathrm{d}}{\mathrm{d}x} e^{\tan x} e^{-\cot x} = \left(\frac{\mathrm{d}}{\mathrm{d}x} e^{\tan x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} e^{-\cot x}\right)$. Reconcile this with our statement above that the naive product rule does not work in general.

4. The altitude h of a triangle is increasing at a constant rate of $1\,\mathrm{cm}\,\mathrm{min}^{-1}$ while the area A increases at a constant rate of $2\,\mathrm{cm}^2\,\mathrm{min}^{-1}$. At what rate is the length b of the base of the triangle increasing when $h = 10\,\mathrm{cm}$ and $A = 100\,\mathrm{cm}^2$?

5. Show that if f and g are differentiable at x, such that $g(x) \neq 0$, we have

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

This is often called the *quotient law.*

Quoted in *Mathematical Apocrypha* by Steven G. Krantz (p.36):

If it's the quotient rule you wish to know,
It's low-de-high less high-de-low.
Then draw the line and down below,
Denominator squared will go.

6. Show that $y = xe^{-x}$ satisfies the differential equation $xy' = (1-x)y$.

7. If $y = \ln \frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}$, find y''.

8. Find the equation of the tangent line to the graph of $y = \ln\cos\frac{x-1}{x}$ at the point $(1, 0)$.

9. Show that $y = (1 + x + \ln x)^{-1}$ satisfies the differential equation $xy' = y(y\ln x - 1)$.

10. Find the angle at which $y = x^2 \ln[(x-2)^2]$ cuts the x-axis at the point $(0, 0)$.

11. When $x = 0$, is the curve $y = (x+20)^2(2x^2-3)^6 - \ln\sin(x - \frac{\pi}{2})$ concave up or concave down?

12. If $y = \frac{e^x}{\sin x}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 - \cot x)$.

13. Show that if f, g, and h are functions then $(fgh)' = f'gh + fg'h + fgh'$.

14. Suppose $f(x) = f(-x)$ for all x in the domain of f. Prove that $f'(x) = -f'(-x)$ for all x in the domain of $f'(x)$.

15. Consider the function defined by $f(x) = x^x$.

    a) Rewrite f in the form $f(x) = e^{x\ln x}$, and hence find $f'(x)$.

    b) Find $\frac{\mathrm{d}y}{\mathrm{d}t}$ if $y = (t^2+3)^{(t^2+3)}$.

16. Prove the product rule a different way by writing $f(x)g(x) = e^{\ln(f(x)g(x))}$.

17. Find $f'(x+3)$ if $f(x+3) = (x+5)^7$.

18. The number a is called a **double root** of some polynomial function f if $f(x) = (x-a)^2 g(x)$ for some polynomial g. Prove that a is a double root of f if and only if a is a root of both f and f'.

References

Chapter VI of Thompson; section 2.3 of Stewart.

Homework problems

1. Find the derivatives:

    a) $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = \sin x \ln x$.

    b) $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = x \sec kx$ (k constant).

    c) $\frac{\mathrm{d}f}{\mathrm{d}\theta}$ if $f(\theta) = \frac{\cos \pi\theta}{\sin \pi\theta + \cos \pi\theta}$.

    d) $\frac{\mathrm{d}y}{\mathrm{d}t}$ if $y = \cos^4(\sin^3 t)$.

2. Find an expression for $(fg)''(x)$ in terms of $f'(x)$ and $g'(x)$.

3. Suppose a liquid is oozing from a corner across a rectangular ridged surface that makes it easier to flow in one direction than the other; call the corner $(0, 0)$, and suppose that the ridges are in the y–direction: so we would expect the flow to be faster towards increasing y compared to increasing x. As the total volume V of liquid oozed increases, the flow rate increases due to the pressure. Suppose that the rate of ooze is constant, at $\frac{\mathrm{d}V}{\mathrm{d}t} = 3\,\mathrm{m}^3\,\mathrm{h}^{-1}$. A measurement shows that the flow rates of the liquid at its edges in the two directions are $\frac{\mathrm{d}x}{\mathrm{d}t} = V^{-1/2} e^{k(V^{1/2})}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = e^{kV}$, for some small constant $k \approx 0.24$. (Both rates are in metres per hour.)

    a) Assuming that the liquid covers the surface uniformly, and that the area covered is roughly rectangular, what is the area covered after three hours (the initial volume being zero), and what is the rate of change of the area covered at that time? [Useless hint: you will need to take some anti-derivatives at some point, and you should need to use the product and chain rules for derivatives eventually as well.]

    b) (Even more funner question.) Suppose the room measures ten metres by ten metres; when the liquid reaches the wall in the y–direction, suppose that the full 'force' of the liquid is now pushing in the x–direction and the flow rate in that direction is the sum of the original flow rates: so $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = V^{-1/2} e^{k(V^{1/2})} + e^{kV}$.
    Calculate how long the liquid will take to reach the wall in the y–direction, and then (taking into account the changed flow rates) work out how long the liquid takes to fill the entire floor area of the room. [Even more useless hint: the final answer should be large.]

I.8 Anti-differentiation by parts

We have already seen that, by reversing the chain rule, we can anti-differentiate some function compositions. Similarly, we can reverse the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$
$$f(x)g(x) = \int f'(x)g(x)\,\mathrm{d}x + \int f(x)g'(x)\,\mathrm{d}x.$$

This result is normally written in the form

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x \tag{I.7}$$

and is known as *integration by parts*. We often write it in Leibniz notation, where it looks like $\int \frac{\mathrm{d}u}{\mathrm{d}x} v\,\mathrm{d}x = uv - \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x$.

Examples. 1. Consider $\int x\sin x\,\mathrm{d}x$, which does not yield to any obvious change of variable. Let $u = x$, and let $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$. So $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$, and $v = -\cos x$. Hence:

$$\int x\sin x\,\mathrm{d}x = -x\cos x + \int \cos x\,\mathrm{d}x = -x\cos x + \sin x + C,$$

where C is an arbitrary constant. Check that $(-x\cos x + \sin x)' = x\sin x$.

2. Now we will anti-differentiate $x^2 \sin 2x$. We work as follows:

$$\begin{aligned}\int x^2\sin 2x\,\mathrm{d}x &= -\frac{x^2\cos 2x}{2} + \int x\cos 2x\,\mathrm{d}x\\ &= -\frac{x^2\cos 2x + x\sin 2x}{2} - \int \frac{1}{2}\sin 2x\\ &= -\frac{x^2\cos 2x + x\sin 2x}{2} + \frac{1}{4}\cos 2x + C.\end{aligned}$$

The aim is to end up with an easier integral than the one that was started with.

Exercises and Problems

1. Compute the following indefinite integrals.

   a) $\int xe^x\,\mathrm{d}x$
   b) $\int x^2e^{2x}\,\mathrm{d}x$
   c) $\int \ln x\,\mathrm{d}x$
   d) $\int t^5\ln t\,\mathrm{d}t$
   e) $\int t^3e^{-t^2}\,\mathrm{d}t$
   f) $\int \sin\ln y\,\mathrm{d}y$
   g) $\int x\tan^2 x\,\mathrm{d}x$
   h) $\int \frac{te^t+te^{-t}}{2}\,\mathrm{d}t$
   i) $\int \cos\sqrt{t}\,\mathrm{d}t$
   j) $\int \theta^3\cos(\theta^2)\,\mathrm{d}\theta$
   k) $\int (x^2+1)e^{-x}\,\mathrm{d}x$

2. Consider $\int f'(x)g(x)\,\mathrm{d}x$; show that, for integration by parts, we can take any anti-derivative of f' for f.

3. Prove that
$$\int \cos^n x\,\mathrm{d}x = \frac{1}{n}\sin x\cos^{(n-1)} x + \frac{n-1}{n}\int \cos^{(n-2)} x\,\mathrm{d}x$$

4. Evaluate $\int (\ln x)^2\,\mathrm{d}x$.

5. A particle moving in one dimension has a velocity function $v(t) = t^2e^{-t}$ (where t is in seconds). What is its displacement from its starting position after three minutes?

6. Scholarship 2016: A curve passing through the point $(1,1)$ has the property that at each point (x,y) on the curve, the gradient of the curve is $x - 2y$; that is, $\frac{\mathrm{d}y}{\mathrm{d}x} = x - 2y$.

a) Show that $\frac{\mathrm{d}}{\mathrm{d}x} e^{2x} y = x e^{2x}$.

b) Hence, or otherwise, find the equation of the curve.

7. Find $I(x) = \int e^x \cos x \,\mathrm{d}x$.

8. Evaluate $\int \sin 4x \cos 5x \,\mathrm{d}x$ in two different ways.

9. Find an anti-derivative of $(\sin^{-1} x)^2$.

10. Recall that $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}$. Find $\int \tan^{-1} x \,\mathrm{d}x$.

11. We integrate $\int 1/x \,\mathrm{d}x$ by parts:

$$\int \frac{1}{x} \,\mathrm{d}x = \frac{1}{x} \cdot x - \int -\frac{1}{x^2} \cdot x \,\mathrm{d}x = 1 + \int \frac{1}{x} \,\mathrm{d}x$$

Cancelling the indefinite integral from both sides, we have $0 = 1$. Explain.

References

Thompson, chapter XX; Stewart, section 7.1.

Homework problems

1. Compute the following indefinite integrals.

   a) $\int x \cos 5x \,\mathrm{d}x$

   b) $\int \cos x \ln \sin x \,\mathrm{d}x$

   c) $\int \cos \sqrt{x} \,\mathrm{d}x$

2. a) Prove that $\int (\ln x)^n \,\mathrm{d}x = x(\ln x)^n - \int (\ln x)^{(n-1)} \,\mathrm{d}x$.

   b) Find $\int (\ln x)^3 \,\mathrm{d}x$.

Chapter II

The geometry of curves

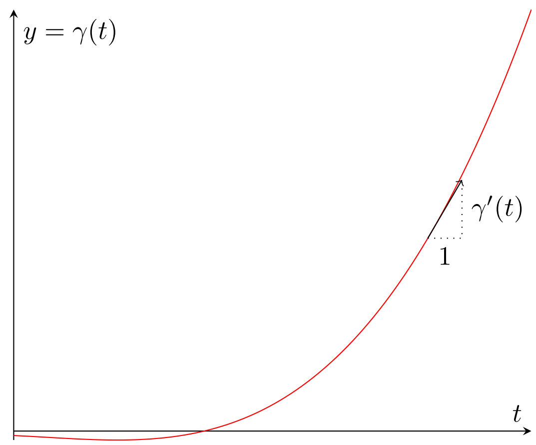


Figure 8: Slope as a measure of direction.

II.9 The geometry of graphs of functions

In this chapter we want to study the geometry of curves: direction, speed, bending, and twisting. This field of study is properly called *differential geometry*.

In this section, we will discuss the geometry of the graphs of functions, and then over the next few sections we will define and then talk about curves with more interesting and complex properties.

Slope and concavity

Suppose γ is a function. If we walk along the graph of γ, then at each point $(t, \gamma(t))$ we have a measure of the direction that we are pointing: we are pointing at an angle $\arctan \gamma'(t)$ to the $x-$axis.

We can essentially differentiate (if you pardon the pun) three different kinds of behaviour:

Definition. Let γ be a function. Then:

1. If $\gamma'(a) > 0$, γ is said to be *increasing* at a.
2. If $\gamma'(a) = 0$, γ is said to be *stationary* at a.
3. If $\gamma'(a) < 0$, γ is said to be *decreasing* at a.

However, this does not yet give us any information about the curvature of the graph of γ: the amount of 'bending' taking place. Based on our studies so far, it would make some sense to define curvature to be the rate of change of slope. Unfortunately, it turns out that this definition is 'incomplete' in some technical sense (we will discuss this briefly when we talk about arc length). However, the second derivative γ'' is useful to us on its own merits; instead of curvature, we will call it *concavity*.

Definition. Let γ be a function. Then:

1. If $\gamma''(a) > 0$, γ is said to be *concave up* (or *convex*) at a.
2. If $\gamma''(a) = 0$, a is said to be an *inflection point* of γ.
3. If $\gamma''(a) < 0$, γ is said to be *concave down* (or simply *concave*) at a.

If γ is concave up for all points a , then we call the function as a whole concave up (and likewise for concave down functions).

Examples.

1. The function $x \mapsto x^2$ is concave up everywhere, increasing for $x > 0$, and decreasing when $x < 0$.
2. The function $x \mapsto \sin x$ is concave down when $(2n)\pi < x < (2n+1)\pi$, and concave up when $(2n+1)\pi < x < (2n+2)\pi$ (for all integers n). See figure 9

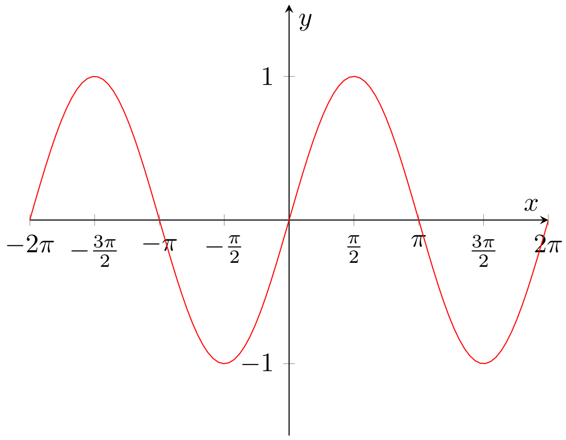


Figure 9: The sine function.

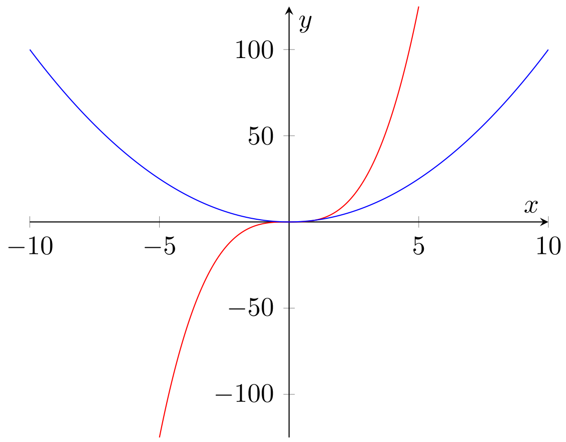


Figure 10: Graphs of x^n for odd n (red) and even n (blue).

3. The function $x \mapsto x^3$ has an inflection point at $(0, 0)$; to the left of this point, the function is concave down (the second derivative is negative) and to the right the function is concave up (the second derivative is positive).

4. In general, functions of the form $f(x) = x^n$ (for integer $n \geq 0$) have some fairly symmetric properties:

   - If n is even, then $y = f(x)$ is even around the x-axis (i.e. $f(-x) = f(x)$), has a minimum at $(0, 0)$, and tends to $+\infty$ in both directions. (See the function graphed in blue in figure 10.)

   - If n is odd, then $y = x^n$ is odd around the x-axis (i.e. $f(-x) = -f(x)$), has an inflection point at $(0, 0)$, and tends to $-\infty$ towards the left and $+\infty$ towards the right. (See the function graphed in red in the figure.)

Example. Consider the function defined by $f(x) = x^4 - 2x^2 + 3$ (figure 11). Find the intervals on which f is increasing or decreasing, find the intervals of concavity, and find any inflection points.

Solution. We have $f'(x) = 4x^3 - 4x$. This function is zero at $x \in \{-1, 0, 1\}$, and so (since the function is a positive cubic) f will be decreasing when $x < -1$, increasing when $-1 < x < 0$, decreasing when $0 < x < 1$, and increasing when $1 < x$. We also have $f''(x) = 12x^2 - 4$ and so $f''(x) = 0$ when $x = \pm\frac{1}{\sqrt{3}}$. Hence the function is concave up when $x < -\frac{1}{\sqrt{3}}$, concave down when $|x| < \frac{1}{\sqrt{3}}$, and concave up when $x > \frac{1}{\sqrt{3}}$. The inflection points will be $x = \pm 1\sqrt{3}$.

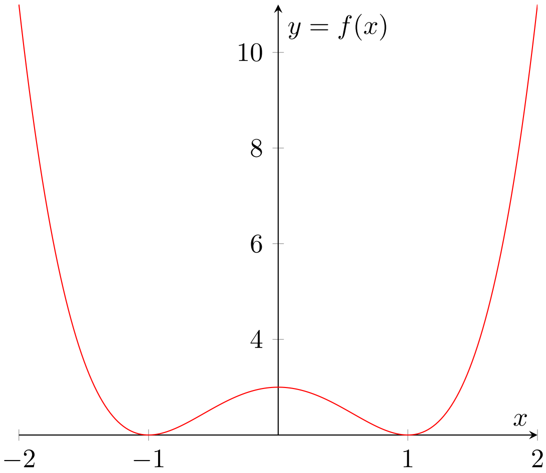


Figure 11: The graph of $f(x) = x^4 - 2x^2 + 3$.

Continuity

We have already mentioned continuity, when we discussed limits. We would like to call a function continuous, intuitively, if its graph can be drawn without picking a pen up off the page. This is insufficient: we cannot prove continuity of any function via this definition! The precise definition of continuity was given by Bernard Bolzano in the early 1800s, and is one of the greatest historical breakthroughs in mathematics. His definition, for us, is best stated in terms of limits:

Definition. A function f is said to be *continuous* at a if $\lim_{x \to a} f(x) = f(a)$. If f is continous for every a in its domain, the function as a whole is said to be continuous.

It is perhaps unfortunate that the concepts of continuity and differentiability are not the same. The details of this are studied in exercise 11.

Arc length and curvature (optional)

The second derivative measures the curvature of a graph based on our position as we walk along the x–axis. A little thought suggests that it might be more natural to measure the curvature based on our position as we walk along the graph itself: if we take our graph and we slant it, for example, this doesn't change the visual 'bendiness' of the graph but it does change the relative position of the graph above the x–axis and thus the values of $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ change.

Let us fix some point on our curve; as we walk along our curve, we measure the distance we travel from this point. After we stick our graph into a coordinate system, this length (the *arc length*) becomes a function of our position.

We will begin with a circle, which we might guess behaves very nicely. I will define the curvature κ of the circle to be the rate of change of the angle θ that our tangent line makes with the x–axis as we walk along the curve, increasing s: in other words, $\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}$.

Consider figure 12. We are approximating a tangent line at B with the line AB, and then varying θ by a small amount, k. As our secant line AB approaches a tangent line, the angle at B between AB and the radius becomes a right angle. Similarly, the angle at C with the radius is a right angle for small k. Thus the angle at B in the triangle OBC is approximately $\pi/2 - k$, and the angle at C is $\pi/2$; thus the angle at O is k.

We therefore have that $kr \approx s(\theta + k) - s(\theta)$, and so $\frac{\mathrm{d}s}{\mathrm{d}\theta} = r$; therefore $\kappa = 1/r$.

If f is some arbitrary function, and we consider the graph $y = f(x)$, we can calculate $\frac{\mathrm{d}\theta}{\mathrm{d}s}$ at each point (we will do this calculation in a second). At each point, we can associate a circle with the same curvature; this circle is called the *osculating circle*, the radius r of the circle is the *radius of curvature* of our graph at that point, and have will say that the *curvature* of the graph at the point x is $\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s} = 1/r$.

So now we will play the same game as above, but now with an arbitrary curve. Consider the function in figure 13; we will again denote the angle at x with the horizontal by θ. As we increase

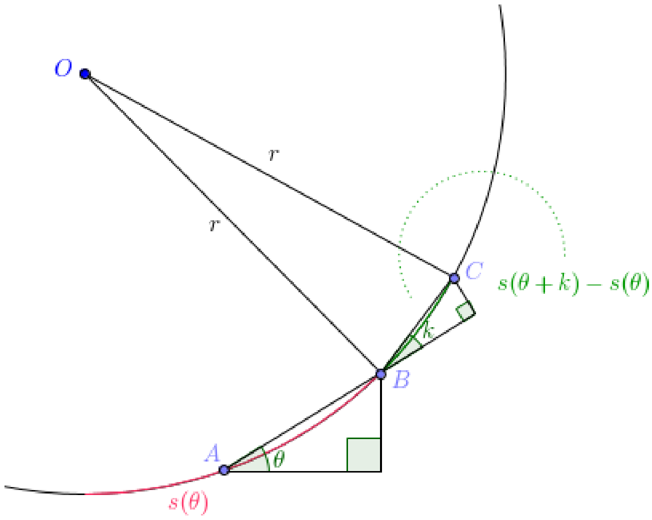


Figure 12: Calculating the curvature of a circle.

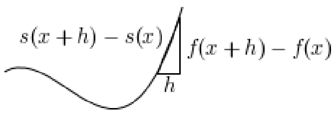


Figure 13: Calculating the curvature of a function.

x by h, we increase s by $s(x+h)-s(x)$. By definition of the derivative, $\frac{\mathrm{d}s}{\mathrm{d}x}h \approx s(x+h)-s(x) = \sqrt{h^2+(f(x+h)-f(x))^2}$; dividing through,

$$\frac{\mathrm{d}s}{\mathrm{d}x} \approx \sqrt{1+\left(\frac{f(x+h)-f(x)}{h}\right)^2};$$

and taking the limit $h \to 0$, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+\tan^2\theta} = \sec\theta. \tag{II.1}$$

Furthermore, since $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta$ we have

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\tan\theta = \frac{\mathrm{d}\theta}{\mathrm{d}x}\sec^2\theta$$

and hence

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\sec^2\theta}{\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)}. \tag{II.2}$$

Finally, by definition we have that the radius of curvature is (combining equations II.1 and II.2)

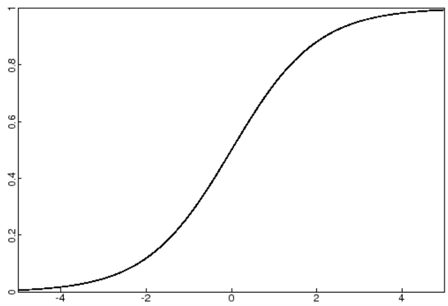
$$r = \frac{\mathrm{d}s}{\mathrm{d}\theta} = \frac{\mathrm{d}s}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\sec^3\theta}{\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)};$$

and the curvature is therefore (again by definition)

$$\kappa(x) = \frac{1}{r} = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\sec^3\theta} = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left(\sqrt{1+\tan^2\theta}\right)^3} = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}}.$$

Exercises and Problems

1. The following function is known as the *logistic curve* and is used for population modelling. Find the intervals of concavity, and label any inflection points.



2. Find the second derivative of the following functions.

   a) $y = x^2 + x$
   b) $f(x) = \sin x$
   c) $g(x) = \cot(3x^2 + 5)$
   d) $y = \frac{\sin mx}{x}$
   e) $y = 4\sin^2 x$
   f) $y = \tan^2(\sin\theta)$
   g) $y = \tan\sqrt{1-x}$

3. Find the concavity of the function $y = \frac{x^2-1}{x^2+1}$ at $(0, -1)$.

4. Find the intervals on which the following functions are increasing or decreasing, and find their intervals of concavity.

   a) $y = x^2 + 1$
   b) $y = 2x^3 + 3x^2 - 36x$
   c) $G(x) = x - 4\sqrt{x}$

5. The graph of $y = f(x)$ (where f is a continuous function) is concave up for all $x < 0$, concave down for $x > 0$, and decreasing everywhere.

   a) Sketch the graph of $y = f(x)$.
   b) What can you say about $f'(x)$ and $f''(x)$ for $x < 0$ and $x > 0$?
   c) What about $x = 0$?

6. Find a value of k such that the function F is continuous at $x = -3$, where

$$F(x) = \begin{cases} \frac{x^2-9}{x+3} & \text{if } x \neq -3, \\ k & \text{if } x = -3. \end{cases}$$

7. Show whether or not the function g is continuous at the three points $(2, g(2))$, $(3, g(3))$, and $(4, g(4))$, where

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leq 2, \\ 2 - x & \text{if } 2 < x \leq 3, \\ x - 4 & \text{if } 3 < x \leq 4, \\ \pi & \text{if } x \geq 4. \end{cases}$$

8. Find all values of α such that Φ is continuous everywhere, where

$$\Phi(x) = \begin{cases} x + 1 & \text{if } x \leq \alpha, \\ x^2 & \text{if } x > \alpha. \end{cases}$$

9. Sketch a function satisfying the given criteria.

   a)
   - Vertical asymptote at $x = 0$,
   - $f'(x) > 0$ if $x < -2$,
   - $f'(x) < 0$ if $x > -2$ ($x \neq 0$),
   - $f''(x) < 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$.

   b)
   - $f'(0) = f'(2) = f'(4) = 0$,
   - $f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
   - $f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
   - $f''(x) > 0$ if $1 < x < 3$,
   - $f''(x) < 0$ if $x < 1$ or $x > 3$.

10. A curve is defined by the function $f(x) = e^{-(x-k)^2}$. Find, in terms of k, the x-ordinates for which $f''(x) = 0$.

11. It turns out that if a function f is differentiable at a then f is always continuous at a, but the converse is not true: there exist continuous functions that are not differentiable. (In fact, there exist functions that are continuous everywhere but differentiable nowhere.)

    a) We will prove that differentiability of f at a implies continuity of f at a; expand the following and use the limit laws to show that $\lim_{x\to a} f(x) - f(a) = 0$, carefully indicating where you use the existence of the derivative.

    $$\left[\lim_{a\to x} f(x) - f(a)\right]\left[\lim_{a\to x} \frac{x-a}{x-a}\right]$$

    b) Give an example of a function which is continuous but not differentiable at some point.

12. We will do some studies of convexity that may be familiar to students who have looked at the exercises on convexity in the algebra notes. We assume that all functions are continuous and differentiable everywhere for simplicity.

    a) Show that if f is a convex function, and if $P = (p, f(p))$ and $Q = (q, f(q))$ are any two distinct points on the graph of f, then for every point $X = (x_1, x_2)$ on the line segment $\overline{PQ}$, $x_2 \geq f(x_1)$ and equality is only obtained at the endpoints.

    b) Show that if f is a convex function, and if $P = (p, f(P))$ is a point on the graph of f, then for every point (x_1, x_2) on the tangent line to f at P, $x_2 \leq f(x_1)$ and equality is only obtained at P.

    c) Prove similar statements to (a) and (b) in the case that f is a concave function. [Hint: there is not much work involved, as long as one ponders the function $-f$.]

13. Scholarship 2010: Recall that the points of inflection of a curve are places where the second derivative changes sign. These are typically, **but not always**, points at which the second derivative is zero.

    Consider the curve $y = \sqrt[3]{x}e^{-x^2}$.

    Write the second derivative in the form $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (ax^4 + bx^2 + x)e^{-x^2}x^{-5/3}$, and hence find the x-ordinates of the points of inflection of the curve.

14. Scholarship 2004: (You may wish to remind yourself how to perform long division of polynomials.) Consider the function

    $$y = \frac{x^2}{1+x^2},$$

    where $-1 \leq x \leq 1$. The gradient at the point $x = 1$ is $\frac{1}{2}$.

    Hence show that there is a point with $\frac{1}{4} \leq x \leq \frac{1}{2}$ where the gradient is also $\frac{1}{2}$.

15. Scholarship 2013: A function f is **even** if $f(-x) = f(x)$ for all x in its domain, and **odd** if $f(-x) = f(-x)$ for all x in its domain.

    a) Describe which polynomials are even, which are odd, and which are neither.

b) Suppose that g is any even differentiable function defined for all real numbers (not necessarily a polynomial). Use the limit definition of the derivative to prove that g' is odd.

16. Recall that we can define the derivative of f by $\mathsf{D}f(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x}$. We will generalise this, by writing $\mathsf{SD}f(x) = \lim_{h \to 0} \frac{f(x+h)-f(x-h)}{2h}$. This is called the *symmetric derivative* of f.

a) In fact, we defined the derivative of f at x to be the unique f' such that $f(x+h) \approx f(x) + hf'(x)$ for small h.[1]
Show that, for small h, if $f'(x)$ exists then $f(x+h) - f(x-h) \approx 2hf'(h)$. Hence conclude that $\mathsf{SD}f(x) = \mathsf{D}f(x)$ whenever the latter exists.

b) The converse is not true: show that if we define $f(x) = |x|$, then $\mathsf{SD}f(0)$ exists but $\mathsf{D}f(0)$ does not.

c) Define the *second* symmetric derivative of f by

$$\mathsf{SD}^2 f(x) = \lim_{h \to 0} \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h} = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Show that whenever $f''(x) = \mathsf{D}^2 f(x)$ exists then $\mathsf{SD}^2 f(x)$ exists and has the same value; show that the converse does not hold (i.e. the existence of the second symmetric derivative does not imply the existence of the usual second derivative) by considering a suitable function, such as

$$\operatorname{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0. \end{cases}$$

17. One may recall from one of the L1 externals that we can recover a quadratic equation given a table of its values. Suppose we know that the following table gives points on the graph of $f(x) = ax^2 + bx + c$.

x	$f(x)$
0	-5
1	2
2	15

Define the *discrete first and second derivatives* of f by $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^2 f(x) = \Delta f(x+1) - \Delta f(x)$. According to the god-given material in L1, we know that if f is a quadratic, then $a = \frac{1}{2}\Delta^2 f(x)$ (for any choice of x); in this example, we can fill in the table as follows:-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
1	-5	7	6
2	2	13	
3	15		

Hence $a = 3$. We can then write (since we know f is a quadratic) $bx + c = f(x) - 3x^2$, which tells us that $b \cdot 1 + c = -8$ and $b \cdot 2 + c = -10$; hence $b = (-10 - {}^{-}8)/1 = -2$ and $c = -6$.

a) Justify the above steps. (Possible approach: $hf''(x) \approx f'(x+h) - f'(x)$; set $h = 1$, and work out what fudge factor $\vartheta(h)$ we have.)

b) Develop a theory of discrete first and second derivatives. (Possible routes of study could include: finding a geometric meaning of the discrete derivatives; defining discrete nth derivatives; studying the relationship between the discrete derivatives and the usual derivatives. You may also want to generalise my definition: instead of $f(x+1) - f(x)$, perhaps one might like to look at $[f(x+k) - f(x)]/k$ (sans limit).)

18. These problems relate to the optional section on curvature and arc length.

a) If $y = f(x)$, formula II.1 gives us the derivative of arc length with respect to distance along the x-axis. What is the arc length along the graph of the function $f(x) = \frac{1}{3}x^3 - x$ between the vertical lines $x = 0$ and $x = 5$?

[1] Then we defined the notation $\varphi(x, h) \approx \psi(x, h)$ to mean that $\varphi(x) = \psi(x) + \vartheta(h)$ for some function ϑ satisfying $\vartheta(h)/h \to 0$ as $h \to 0$.

b) Repeat (a) for the function $g(x) = \ln|\sec x|$.

c) What is the curvature of a straight line?

d) Calculate the curvature $\kappa(x)$ of the function $f(x) = x^2$ at the points $x = 0$, $x = 2$, and $x = 4$. Draw the osculating circles at each of these points. What happens to $\kappa(x)$ as $x \to \infty$?

References

A good introduction to the geometry of curves, and differential geometry in general, is *Differential Geometry of Curves and Surfaces* by Manfredo P. do Carmo.

For a discussion of the history of continuity, see J F Harper (2016): Defining continuity of real functions of real variables, BSHM Bulletin: Journal of the British Society for the History of Mathematics, DOI:10.1080/17498430.2015.1116053 (`http://homepages.ecs.vuw.ac.nz/~harper/harper16.pdf`).

Discrete derivatives (see exercise 17) are useful when taking derivatives numerically (say we have a table of numbers defining a function f, but we don't have a nice formula for it). The kinds of things one may want to search for in a library catalogue are "difference calculus" or "discrete calculus".

Homework problems

1. Explain, with sketches, the geometric meaning of the second derivative.

2. Find the second derivative of the following functions.

   a) $f(x) = x^5 - 5x + 3$

   b) $f(x) = \frac{x^2}{x-1}$

   c) $f(x) = \sqrt{x} - \sqrt[4]{x}$

3. Sketch a function satisfying the given criteria.

   a) (hint: your result should be an odd function)
      - $f'(1) = f'(-1) = 0$,
      - $f'(x) < 0$ if $|x| < 1$,
      - $f'(x) > 0$ if $1 < |x| < 2$,
      - $f'(x) = -1$ if $|x| > 2$.

   b)
      - $f'(x) < 0$,
      - $f''(x) < 0$.

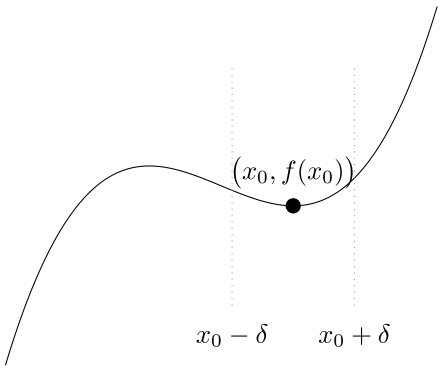


Figure 14: A local minimum of f at x_0 is a point lower than every other point in some interval around it.

II.10 Optimisation problems

Suppose we need to choose a rectangle with a fixed perimeter, but a maximal area. By considering the symmetry of the problem, it is almost clear that if a solution exists then it *should* be a square.

Indeed, suppose that our rectangle has side lengths x and y, and perimeter $\mathcal{P}$. Then $\mathcal{P} = 2(x+y)$, and so the area of our rectangle (the quantity we wish to maximise) is

$$\mathcal{A} = xy = \frac{x(\mathcal{P} - 2x)}{2} = \frac{1}{2}\mathcal{P}x - x^2.$$

Noting that the graph of the function $\mathcal{A}(x)$ is a parabola, opening downwards, it is clear that our desired maximum is the vertex; completing the square, we find that

$$\mathcal{A} = \frac{1}{16}\mathcal{P}^2 - \left(x - \frac{1}{4}\mathcal{P}\right)^2$$

and thus the vertex has x−value $\mathcal{P}/4$; immediately, $x = y$ and we see that our guess, of a square optimal shape, was correct.

We were able to solve this problem because it ended up being equivalent to finding the maximum of a quadratic, which is easy to solve in this way with only a little effort. However, consider the following problem, which is essentially the same but now in three dimensions:

Problem. Suppose a rectangular prism has two square faces of side length x, and four rectangular faces with side lengths x and y; so the volume of the prism is x^2y. Let the perimeter $\mathcal{P} = 8x + 4y$ of the prism be fixed. Find x and y satisfying this constraint such that the volume is maximised.

Considering $V = x^2y$, we substitute in our constraints:

$$V = x^2\left(\frac{\mathcal{P} - 8x}{4}\right) = \frac{\mathcal{P}}{2}x^2 - 2x^3$$

and now we need to find the maximum value of a cubic — which is not something we can do easily geometrically.

These types of problems are our motivation for the study in this section.

Definition. If f is a function, then f is said to have a *local maximum* at x_0 if there exists some (small) number δ such that, for all x between $x_0 - \delta$ and $x_0 + \delta$, $f(x) \leq f(x_0)$.

Similarly, f is said to have a *local minimum* at x_0 if there exists some number δ such that, for all x between $x_0 - \delta$ and $x_0 + \delta$, $f(x) \geq f(x_0)$.

Local maxima and local minima are, collectively, called *local extrema*. If we can take δ as large as we like in either definition (e.g. if $f(x) \geq f(x_0)$ for all possible x anywhere in the real numbers), we replace ‘local’ with *global*.

For an illustration, see figure 14.

Many optimisation problems in applied mathematics can be reduced to finding relative extrema.

Examples. As in our first example above, many functions have local extrema that can be found with a little intelligent thought.

1. The function $x \mapsto x^2$ has a local minimum at $(0,0)$.
2. The function $x \mapsto 2x^3 + 15x^2 + 36x + 2$ has a local maximum at $(-3, -25)$ and a local minimum at $(-2, -26)$.
3. The function $x \mapsto \sin x$ has a local maximum at $(2n\pi + \frac{\pi}{2}, 1)$ for every integer n, and a local minimum at $(2n\pi - \frac{\pi}{2}, 1)$ for every integer n.

The main theorem is the following, which is a simple extension of the ideas from the last section on the geometry of the graphs of functions. The basic geometric idea is that at a local extrema, the graph is changing from increasing to decreasing (or vice versa) and thus the derivative is changing from a positive value to a negative value (or vice versa), and so must pass through zero.

Theorem (Fermat). *Let f be a function; suppose x_0 is a point in the interior of the domain of f (i.e. $f(x)$ is defined for all x close to x_0 on both sides), that f is differentiable at x_0, and that f has a relative extremum at x_0. Then $f'(x_0) = 0$.*

Proof. Suppose x_0 is a local maximum of f. Then for all $x < x_0$ such that x is 'sufficiently close'[2] to x_0, $f(x) \le f(x_0)$. Thus

$$\frac{f(x_0) - f(x)}{x_0 - x} \ge 0,$$

since it is the quotient of two non-negative numbers.

Similarly, for all $x > x_0$ sufficiently close to x_0, $f(x) \le f(x_0)$ and so

$$\frac{f(x_0) - f(x)}{x_0 - x} \le 0.$$

Since $\lim_{x\to x_0} \frac{f(x_0)-f(x)}{x_0-x}$ exists, the quantity $\frac{f(x_0)-f(x)}{x_0-x}$ must tend to the same value whether we approach from the left or from the right; and the only possible such value is zero. $\square$

Motivated by this theorem, we define a *critical point* of a function f to be some value x in the domain of f such that either $f'(x) = 0$, or $f'(x)$ is undefined. In the first case, we also call the value a *stationary point*. All local extrema occur at critical points, but not all critical points occur at extrema.

Examples.

1. The function $x \mapsto 2x^3 + 15x^2 + 36x + 2$ above has critical points $x = -2$ and $x = -3$. Both of these are local extrema.
2. The function $x \mapsto x^3$ above has a critical point at $x = 0$, but does not have a local extrema there.
3. The function $x \mapsto \frac{1}{x}$ **does not** have a critical point at $x = 0$, **because it is not defined there**.

Utilising this technique, we can write down a recipe like the following to find extrema of a function f mechanically:

1. Compute f'.
2. Find the points where f is defined but f' is not
3. Find all the points where f' is zero.
4. Find all the points x where f is defined at x and all points on one side of x, but not the other.
5. Then all the local extrema of f will be included in the lists of points in (2)–(3), and so we check each manually.

There are two problems with this recipe. Firstly, the list of possible extrema may include points which are not actually extrema and so we need a way to decide quickly whether a point is an extrema or a false alarm. This is something we will address in a moment, so a more pressing concern is the second: finding zeroes of a function, in this case f', is just as hard as finding extrema — so we are only replacing one difficult problem with another.

[2] i.e. x lies within the δ-interval in the definition of a local maximum

The geometry of critical points

We can use the first derivative to classify extrema as either maxima or minima.

1. Determine all critical points of f.
2. Determine the sign of $f'(x)$ to the left and right of each critical point x_0:
   - If $f'(x)$ changes from positive to negative as we move from left to right across x_0, then $f(x)$ has a local maximum at x_0.
   - If $f'(x)$ changes from negative to positive as we move from left to right across x_0, then $f(x)$ has a local minimum at x_0.
   - If $f'(x)$ does not change sign across x_0, then $f(x)$ does not have a relative extremum at x_0 (e.g. $y = x^3$).

On the other hand, using the second derivative, we can come up with a different test:

1. Compute $f'(x)$ and $f''(x)$.
2. Find all the stationary points of f by finding all the points x_0 such that $f'(x_0) = 0$.
3. Determine the sign of $f''(x)$ for each stationary point x_0:
   - If $f''(x_0) < 0$ (i.e. f is concave down), then $f(x)$ has a relative maximum at x_0.
   - If $f''(x_0) > 0$ (i.e. f is concave up), then $f(x)$ has a relative minimum at x_0.
   - If $f''(x_0) = 0$ (i.e. f has an inflection point at x_0), then $f(x)$ could have a relative maximum, a relative minimum, or neither.

Example. Find and classify the critical points of $y = x^3 - 3x^2 + 6$.

Solution. We have $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 6$. Hence the critical points are $x = 0$ and $x = 2$. At the former point, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, and so the point is a maximum; at the latter point, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ and so the point is a minimum.

Example. Find two numbers whose difference is 100 and whose product is a minimum.

Solution. Let the two numbers be x and $x + 100$. We wish to minimise $y = x(x + 100)$; clearly $y' = 2x + 100$, and so $x = -50$ is a critical point. To the left of $x = -50$, the derivative is negative; to the right, the derivative is positive. Hence $x = -50$ is indeed a minimum. The two required numbers are therefore -50 and 50.

Example. Find and classify the critical points of $y = (x - 1)^2 + \ln x$.

Solution. The derivative is $y' = 2x - 2 + \frac{1}{x}$. We therefore have one critical point at $x = 0$ (where y' is undefined); this is an asymptote. Setting $y' = 0$, we have $0 = 2x - 2 + \frac{1}{x} = 2x^2 - 2x + 1$ which has no real roots. Hence $x = 0$ is the only critical point, and the curve has no local extrema.

Example. A rectangular plot of land is to be fenced using two varieties of fence. Two opposite sides will use fences selling for \$3 per metre, while the other two sides will use cheaper fence selling for \$2 per metre. Given that the total budget is \$1200, what is the greatest area of land which can be fenced?

Solution. Let x be the length of one of the expensive sides; then the length of one of the cheaper sides is $\frac{1}{2}(1200-3x)$, and the total area is $A = \frac{1}{2}x(1200-3x) = \frac{1}{2}(1200x-3x^2)$. Hence $\frac{\mathrm{d}A}{\mathrm{d}x} = 600-3x$. We wish to find the maximum area, so set $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$; hence $3x = 600$ and $x = 200$. Note that the second derivative is always negative, so this stationary point must be a maximum as required. The length of the other side will be $\frac{1}{2}(1200 - 600) = 300$, and so the maximum area is $300 \times 200 = 60000$ square metres.

Example (Electric circuits)**.** A battery of constant voltage $\mathcal{E}$ and internal resistance $\mathfrak{r}$ is connected to an external resistance R_{load}. For what external resistance will the power P dissipated by the external resistance be maximal? (Possible application: we want to build a lamp to be connected to a particular voltage; what resistance should the filament be for maximum brightness?)

Solution. We have that $P = I^2 R$, where I is the current through the circuit. Then $I = \frac{\mathcal{E}}{\mathfrak{r}+R}$, so $P = \mathcal{E}^2 \frac{R}{(\mathfrak{r}+R)^2}$. In order to maximise this we will compute $\frac{\mathrm{d}P}{\mathrm{d}R}$:

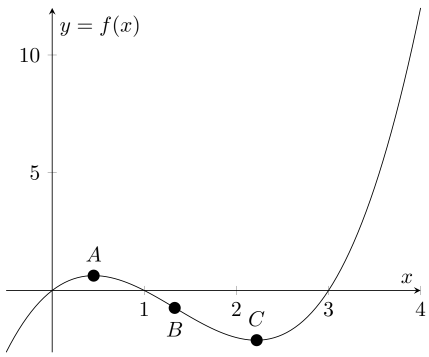
$$\frac{\mathrm{d}P}{\mathrm{d}R} = \mathcal{E}^2 \frac{\mathfrak{r} - R}{(\mathfrak{r} + R)^3}, \qquad \frac{\mathrm{d}^2 P}{\mathrm{d}R^2} = \mathcal{E}^2 \frac{3R - 3\mathfrak{r} - \mathfrak{r}R - R^2}{(\mathfrak{r} + R)^4}.$$

Clearly $\frac{\mathrm{d}P}{\mathrm{d}R} = 0$ precisely when $R = \mathfrak{r}$; and in this case, $\frac{\mathrm{d}^2 P}{\mathrm{d}R^2} = \mathcal{E}^2(-2\mathfrak{r}^2)((2\mathfrak{r})^4) < 0$, so this point is indeed a maximum for P.

Thus the load should be given resistance equal to the internal resistance of the battery.

Exercises and Problems

1. Describe the advantages and disadvantages of the first and second derivative tests for local extrema.
2. Describe the local extrema, concavity, and points of inflection of the function $f(x) = x^4 - 4x^3$.
3. Consider the following graph:



   Find the signs of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at the three points A, B, and C.
4. Find all the local extrema of the following curves in the given intervals, and classify them as maxima, minima, or neither.
   a) $f(x) = \sin x - \cos x$ on the interval $0 < x < \pi$
   b) $g(x) = x^3 - x^2 + x - 1$ on the interval $-\infty < x < \infty$
5. The sum of two positive numbers x and y is 16. Find the smallest possible value for $S = x^2 + y^2$.
6. A box with an open top is to be constructed from a square piece of cardboard with a side length of 3 m by cutting out a square from each of the four corners and bending up the sides. Find the dimensions of the resultant box of maximum volume.
7. Find the dimensions of a rectangle with area $1000\,\mathrm{m}^2$ such that the perimeter is minimised.
8. A window consisting of a rectangle topped with a semicircle is to have a fixed perimeter p. Find the radius of the semicircle in terms of p if the total area is to be maximised.
9. A thin wire of length L is cut in two and the resulting lengths are bent to make a square and an equilateral triangle. Where should the wire be cut to make the total area of the shapes (a) a maximum and (b) a minimum?
10. Find the point on the line $y = 2x + 3$ closest to the origin.
11. Find the point on the curve $y = \sqrt{x}$ closest to $(3, 0)$.
12. By finding the $x-$ and $y-$ intercepts, the asymptotes, the critical points, the intervals of increase and decrease, the intervals of concavity, and any other important points, sketch the following functions (199):
    a) $f(x) = \frac{x^2}{4-x^2}$

b) $f(x) = \frac{4x}{x^2+1}$ [*Hint: consider what happens to $f(x)$ as $x \to \pm\infty$.*]

c) $f(x) = \frac{x^2-4x+5}{x-2} = x - 2 + \frac{1}{x-2}$ [*Hint: consider what happens to $f(x) - (x-2)$ as $x \to \pm\infty$.*]

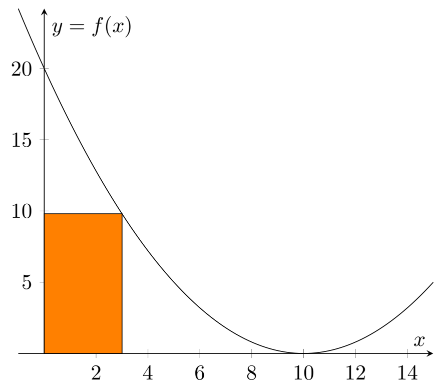
13. A cone with height h is inscribed in a larger cone of height H such that the vertex of the small cone is at the centre of the base of the larger cone. Show that the maximum volume of the smaller cone occurs when $h = \frac{1}{3}H$.

14. Show that the polynomial $p(x) = 10x^3 + x^2 + x - 34$ has exactly one real zero.

15. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one third of the sheet on each side by an angle θ. What angle should be chosen in order to obtain the maximum possible volume?

16. A steel pipe is carried around a right-angled corner from a hallway 3 m wide into a hallway 2 m wide. What is the length of the longest pipe that can be carried horizontally around the corner? [*Hint: this is actually a minimisation problem, despite the wording.*]

17. A large orange rectangle is to be drawn with one corner sitting on the origin and the opposite corner lying on the curve $y = 0.2(x-10)^2$. What is the maximum possible area of the rectangle?



18. Show that $\frac{x^2+1}{x} \geq 2$; hence (or otherwise) show that $\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geq 8$.

19. Scholarship 2013: Prince Ruperts drops are made by dropping molten glass into cold water. A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi(e^{-x} - e^{-2x})}$ for $x \geq 0$, where ϕ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

a) Where is the modelled drop widest, and how wide is it there?

b) The drop changes shape at a point B, where the concavity of the function is zero. Use

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sqrt{\phi}\frac{e^{2x} - 6e^x + 4}{y^2 e^{4x}}$$

to find the exact x-ordinate of B.

20. Scholarship 2014: A family of functions is built from two functions f and g, with a new function h_p defined for each value of p, $0 \leq p \leq 1$:

$$f(x) = 2 + \sin x$$
$$g(x) = 26 + \sin x$$
$$h_p(x) = [f(x)]^{1-p}[g(x)]^p.$$

Define a fourth function S, where $S(p)$ is the difference between the maximum and the minimum values of h_p. Find the exact value of p that maximises S.

Note that if a is constant, $\frac{\mathrm{d}}{\mathrm{d}x}a^x = (\ln a)a^x$.

21. Note that the second-derivative test may be inconclusive: if $f'(x_0) = 0$, and $f''(x_0) = 0$, then we may have an inflection point (like in the case of $x \mapsto x^3$), or a local extrema (like $x \mapsto x^4$). Looking at the second example here, one might conjecture that if we have a local extrema then $f^{(n)}(x_0) > 0$ for some large enough n (in the case of x^4, the fourth derivative is positive).

    Define f by
$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

    a) Show that f is differentiable at zero.
    b) Show that, for all $n > 0$, $f^{(n)}(0) = 0$. (Implicit here is the existence of the nth derivative.)
    c) Show that f has a global minimum at zero.
    d) Justify that there can thus never be a rule, based simply on checking nth derivatives, for proving that any function has a local minimum or maximum at a point. Hence our conjecture is false.

22. Find x, y, and z positive real numbers such that the volume $V = xyz$ of the rectangular prism with these sidelengths is maximised, given that the perimeter of the prism is $4x + 4y + 4z = 12$ and the surface area is $2xy + 2yz + 2zx = 6$.

References

For a significant number of optimisation exercises of the mechanical sort, see Stewart sections 4.1 and 4.7.

For more interesting problems, see Spivak chapter 11.

One avenue for further reading is the subject of the calculus of variations. See, for example, L&S section 3.15, as well as (for example) *An introduction to the Calculus of Variations*, by Hans Sagan (Dover, 1992).

Homework problems

1. What is the minimum vertical distance between the parabolae $y = x^2 + 1$ and $y = x - x^2$?

2. Show that $3x + 2\cos x + 5 = 0$ has exactly one real root by showing that it is increasing everywhere and crosses the x-axis somewhere between two values of x.

3. Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

4. Let $ABCD$ be a square piece of paper with sides of length 1 m. A quarter-circle is drawn from B to D with centre A. The piece of paper is folded along EF, with E on AB and F on AD, so that A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF can have. [*Hint: you may want to introduce a coordinate system.*]

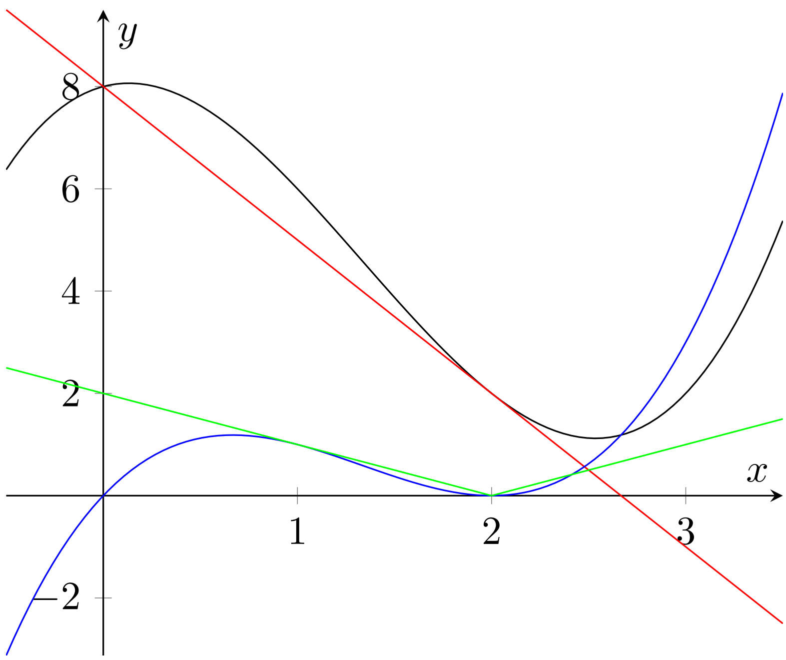


Figure 15: The graph of a function $f(x)$ (black), its tangent line $\tilde{f}(x)$ at 2 (red), $f(x) - \tilde{f}(x)$ (blue), and $y = |x - 2|$ (green).

II.11 Approximations

Our original definition of the derivative was motivated in part by finding the 'best' linear approximation to a curve at a given point. Indeed, if f is a function then, for all h such that $f(x_0 + h)$ is defined, we have

$$f(x_0 + h) - f(x_0) = hf'(x_0) + \vartheta(h)$$

where $\lim_{h\to 0} \vartheta(h)/h = 0$; in particular, for any x where f is defined (moreover, we need f to be defined everywhere *between* x and x_0) we have

$$f(x) - f(x_0) = (x - x_0)f'(x_0) + \vartheta(x - x_0)$$

where we have simply replaced h with $x - x_0$. We already defined the tangent line to f at x_0 to be the line given by $y - f(x_0) = (x - x_0)f'(x_0)$; so the above equation tells us that if (x, y) is on the tangent line then

$$f(x) - y = [(x - x_0)f'(x_0) + \vartheta(x - x_0) + f(x_0)] - [(x - x_0)f'(x_0) + f(x_0)] = \vartheta(x - x_0)$$

and so we have that

$$\frac{f(x) - y}{x - x_0} = \frac{\vartheta(x - x_0)}{x - x_0} \xrightarrow{x - x_0 \to 0} 0:$$

in other words, as we move our point x of interest closer and closer to x_0, the 'error' in the tangent line approximation vanishes at a faster rate than our movement. Figure 15 illustrates these relationships; notice that at points x close to the point of tangency (in this case $x_0 = 2$), $\tilde{f}(x) - f(x)$ is approaching zero faster than the distance $|x - x_0|$.

Given that the tangent line is, in this sense, a good linear approximation, it is natural to ask the following question:

If f is differentiable at x_0, what is the best polynomial approximation to f around the point x_0?

As a reminder, a polynomial is a function p of the form $p(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$ where $p_n \neq 0$. The various p_k are called the *coefficients*, and n is the *degree* of p.

This question is an important one, because if we can answer it then we can always replace differentiable functions (which are difficult to calculate — what is sin(32.341)?) with polynomials (which only require a finite number of multiplications and additions to calculate) with only a small loss of information.

One important use for this is in physics; we will see later on (when we look at differential equations) that in order to model something like a swinging pendulum, we need to justify why $\sin x \approx x$ for small x. In engineering applications too, it is often useful to replace complicated sums and products of things like trig functions with quadratics and cubics that have the same 'shape' around a given point of interest but are much easier to calculate with.

Our plan of attack will be, given a function f that is differentiable (at least) n times at some point x_0, to find a polynomial

$$p(x) = p_n(x - x_0)^n + p_{n-1}(x - x_0)^{n-1} + \cdots + p_1(x - x_0) + p_0$$

satisfying the following conditions:

$$\begin{aligned} p(x_0) &= f(x_0) \\ p'(x_0) &= f'(x_0) \\ &\vdots \\ p^{(n)}(x_0) &= f^{(n)}(x_0). \end{aligned}$$

(For convenience, the zeroth derivative of f is f itself.)

Our argument above shows that the linear polynomial approximation at x_0 is $p(x) = f'(x_0)(x - x_0) + f(x_0)$. We will need the following lemma:

Lemma. *Let $p(x) = p_n(x - x_0)^n + p_{n-1}(x - x_0)^{n-1} + \cdots + p_1(x - x_0) + p_0$ be a polynomial. Then $p_k = \frac{p^{(k)}(x_0)}{k!}$ for all $0 \leq k \leq n$.*

Proof. We have the following:

$$\begin{aligned} p(x) &= p_n(x - x_0)^n + \cdots + p_3(x - x_0)^3 + p_2(x - x_0)^2 + p_1(x - x_0) + p_0 &\implies p(x_0) = p_0 \\ p'(x) &= np_n(x - x_0)^{n-1} + \cdots + 3p_3(x - x_0)^2 + 2p_2(x - x_0) + p_1 &\implies p'(x_0) = p_1 \\ p''(x) &= n(n-1)p_n(x - x_0)^{n-2} + \cdots + (3 \cdot 2)p_2(x - x_0) + 2p_2 &\implies p''(x_0) = 2p_2 \\ p^{(3)}(x) &= n(n-1)(n-2)p_n(x - x_0)^{n-3} + \cdots + (3 \cdot 2)p_2 &\implies p^{(3)}(x_0) = (2 \cdot 3)p_3 \end{aligned}$$

and in general, completing the pattern,

$$p^{(k)}(x_0) = (k!)p_k.$$

□

Thus, if we want to match f with a polynomial of degree n such that $f^{(k)}(x_0) = p^{(k)}(x_0)$ for every $0 \leq k \leq n$ we simply choose $p(x) = p_n(x - x_0)^n + p_{n-1}(x - x_0)^{n-1} + \cdots + p_1(x - x_0) + p_0$ such that

$$p_k = \frac{f^{(k)}(x_0)}{k!}.$$

This polynomial is called the nth *Taylor polynomial* of f at x_0; we might even write $\mathsf{T}_{n,x_0} f$ for this polynomial.

Example. 0, 1, 0, -1, 0, 1, Consider $f(x) = \sin x$. Then $f^{(0)}(0) = 0$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = -1$, and in general

$$f^{(n)}(0) = \begin{cases} 0 & n \text{ even} \\ (-1)^k & n = 2k + 1 \text{ odd} \end{cases}$$

In particular, $\mathsf{T}_{2k+1,0} f = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + \frac{(-1)^k}{(2k+1)!}x^{2k+1}$; the Taylor polynomials for $k = 0$ (red), $k = 1$ (green), and $k = 2$ (blue) are graphed in figure 16.

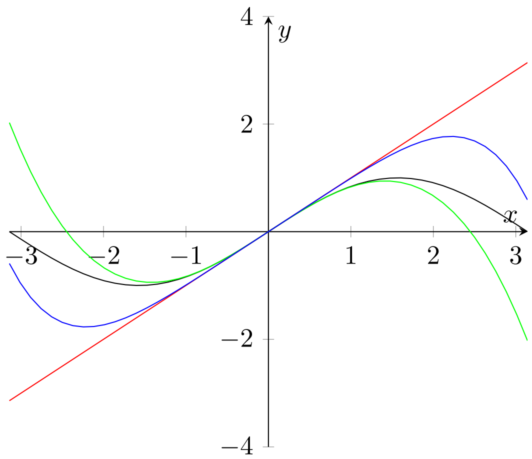


Figure 16: The first few Taylor polynomials of sine.

We can even do the same error estimation that we performed above, although it is a little tedious; we find that

$$\lim_{x \to x_0} \frac{f(x) - \mathsf{T}_{n,x_0} f(x)}{(x - x_0)^n} = 0 \tag{II.3}$$

(so $\mathsf{T}_{n,x_0} \to f(x)$ faster than $(x - x_0)^n \to 0$).

Note that this only tells us that the Taylor polynomials $\mathsf{T}_{n,x_0} f$ approximate f *very, very, very* close to the point x_0. It is not the case, in general, that the Taylor polynomials are a good approximation *around* the point x_0. For example, in one of the problems from the previous section we saw that

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is such that $f^{(n)}(0) = 0$ for all n, and thus $\mathsf{T}_{n,0} f(x) = 0$ for all n; so even for large n, the Taylor polynomials are a terrible approximation!

Exercises and Problems

1. Find the best quadratic approximation to $f(x) = 1/(1 + x^2)$ at zero.

2. Find the first, second, third, and nth Taylor polynomials of the following functions at zero.

   a) $x \mapsto e^x$
   b) $x \mapsto \cos x$
   c) $x \mapsto \frac{1}{1+x}$

3. a) According to problem set 6 of the trigonometry notes (remember those!), $\arctan a + \arctan b = \arctan\left[(a + b)/(1 - ab)\right]$ as long as the sum on the left is between $\pm\pi/2$. Show that $\pi/4 = 5 \arctan(1/5) - \arctan(1/239)$.
   b) Show that $\pi = 3.14159\ldots$.

4. a) Let $p(x)$ be a polynomial of degree n, and let x_0 be any point. Show that for all x, $p(x) = \mathsf{T}_{n,x_0} p(x)$.
   b) Write the function $p(x) = 22 - 49x + 35x^2 - 10x^3 + x^4$ as a polynomial in $(x - 3)$.

5. Suppose f and g both have n derivatives at x_0. Let λ be a real number. Calculate:

   a) $\mathsf{T}_{n,x_0}(\lambda f)$
   b) $\mathsf{T}_{n,x_0}(f + g)$
   c) $\mathsf{T}_{n,x_0}(fg)$
   d) $\mathsf{T}_{n-1,x_0}(f')$

6. Prove formula II.3 above.

References

An introduction to the approximation of sufficiently differentiable functions by polynomials can be found in Spivak, chapter 19. With a little extra work one can work out what degree of polynomial is needed at a point x_0 to reduce the error to less than a specified amount; to do this one needs integration (which we have not yet seen), and the details are in Spivak.

For some discussion on example applications of higher-order (i.e. non-linear) Taylor approximations in physics and engineering, see `https://matheducators.stackexchange.com/q/2152`.

Homework problems

1. Find the best quartic (degree 4 polynomial) approximation to $f(x) = x^5 + x^3 + x$ about (a) 0 and (b) 1.

2. Square roots are difficult. Compute an approximation to $\sqrt{4.003}$ by hand. (Hint: expand $x \mapsto \sqrt{x}$ as a Taylor series about 4.)

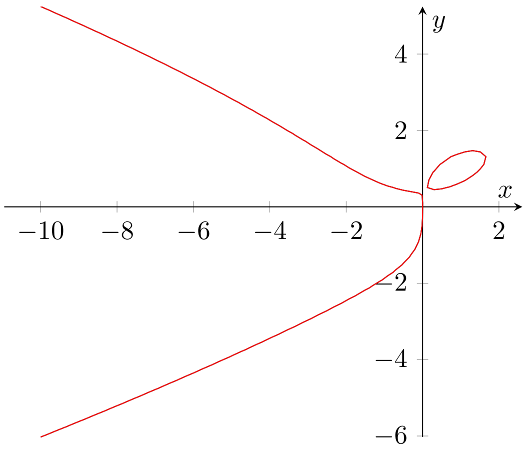


Figure 17: Two curves, which are not graphs of functions.

II.12 Geometry of implicit curves

We have played around now with the graphs of functions; these are a very important class of curve, but many useful curves are *not* graphs of functions. For example, the two curves displayed in figure 17 are not graphs of functions as they do not satisfy the 'vertical line test'.

The two curves were generated by plotting the set of all points (x, y) satisfying the following equations:

$$x^2 + y^2 - 25 = 0; \qquad \text{(black)}$$
$$x^3 + y^4 - 5xy + 2x = 0. \qquad \text{(red)}$$

This suggests that we should expand our definition of 'graph' from merely functions to arbitrary equations in two variables:-

Definition. Let f be a function of two variables; in other words, a function which takes two arguments, which we will usually call x and y (the value of f at this pair of arguments is written $f(x, y)$). Then the *implicit curve generated by* f is the set of all points (x, y) such that $f(x, y) = 0$.

For example, our two curves above are the implicit curves generated by the two functions

$$f(x, y) = x^2 + y^2 - 25 \text{ and} \qquad \text{(black)}$$
$$g(x, y) = x^3 + y^4 - 5xy + 2x. \qquad \text{(red)}$$

We would like to be able to talk about the geometry of curves like this using similar language to the language we have already developed; we would like to be able to say that the black circle has negative slope at $(4, 3)$ and positive slope at $(4, -3)$ for example.

Luckily if our curve of interest is 'sufficiently nice' at a given point (x_0, y_0) (follow the footnote for the technical meaning of this, though basically every simple thing you write down will be nice enough for us — we need the curve to be non-vertical, not cross itself, and not be too wiggly)[3] then a theorem called the *implicit function theorem* tells us that there is a function whose graph coincides with the curve around the point. In other words, if we have an implicit curve generated by f, and f is nice at (x_0, y_0), then there exists some function $\hat{f}$ defined around x_0 such that whenever x is very close to x_0, $(x, \hat{f}(x))$ is on the curve. Since $\hat{f}$ is differentiable in the usual way, and has a graph with the same shape as the curve at (x_0, y_0), it is reasonable to define the slope of the curve at (x_0, y_0) (which we will call $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_0,y_0)}$) to be the slope of the graph of $\hat{f}$ at x_0. In other words,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_0,y_0)} := \hat{f}'(x_0). \qquad \text{(II.4)}$$

[3] The function f needs to satisfy the hypotheses of the implicit function theorem; we need f to be continuously differentiable and we need $\frac{\partial f}{\partial y} \neq 0$ at every point of interest, i.e. the curve cannot be vertical there.

Stop and reread the above paragraph. Draw some pictures and convince yourself that the idea makes intuitive sense (you should be able to see that, geometrically, it makes sense). We will not prove the implicit function theorem, but you need to convince yourself that it is plausible.

We therefore have reduced our problem to finding this function $\hat{f}$. I will illustrate our general technique through a couple of examples.

Example. Consider the circle, $x^2 + y^2 = 25$; let us differentiate it at $A = (4, 3)$ and $B = (4, -3)$. The implicit function theorem tells us that there exist two functions, f and g, such that the graph of f is precisely some part of the circle around A and such that the graph of G is precisely some part of the circle around B.

Let us take f first. We have that $x^2 + f(x)^2 = 25$; we can differentiate both sides with respect to x, obtaining that $2x + 2f(x)f'(x) = 0$ (we need to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)^2]$) and hence $f'(x) = \frac{-2x}{2f(x)}$. But we are interested in $x = 4$; and by definition of f, we have $f(4) = 3$ (since f has the same graph as the curve around $(4, 3)$); so $f'(4) = \frac{-2\cdot 4}{2\cdot 3} = -\frac{4}{3}$. This is negative, as we expected.

For the case of point B, the same calculation tells us that $g'(x) = \frac{-2x}{2g(x)}$ and so $g'(4) = \frac{-2\cdot 4}{2\cdot -3} = \frac{4}{3}$.

Often we don't bother to write down the implicit function explicitly; we would write something like

$$x^2 + y^2 = 25 \implies 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

This is perfectly fine, as long as you remember (1) that y is actually treated as a function of x, and (2) that y is an *different* function of x depending on where you are on the curve.

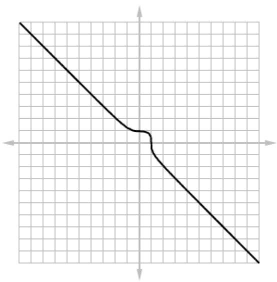
Example. If $x^3 + y^4 = 5xy - 2x$, then by differentiating both sides with respect to x we obtain $3x^2 + \frac{\mathrm{d}y}{\mathrm{d}x}4y^3 = 5y + 5x\frac{\mathrm{d}y}{\mathrm{d}x} - 2$ (being careful to use the product and chain rules in differentiating). Hence we have that the derivative is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y - 3x^2 - 2}{4y^3 - 5x}$$

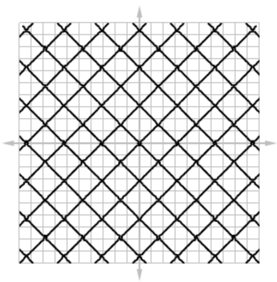
Exercises and Problems

1. In each case, look at the cool pictures and find an explicit formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

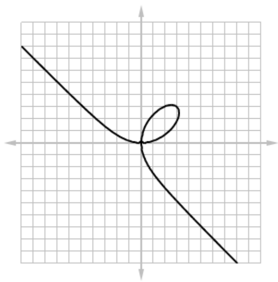
a) $x^3 + y^3 = 1$



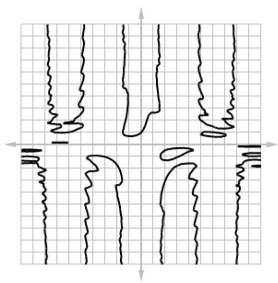
b) $\sin^2 y + \cos^2 x = 1$



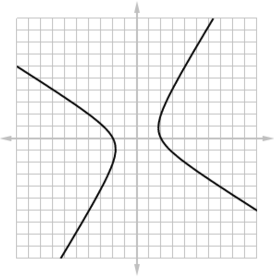
c) $x^3 + y^3 = 6xy$ (the folium of Descartes)



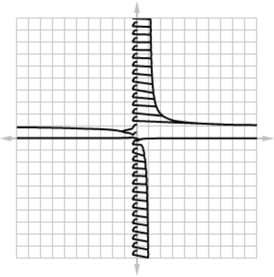
d) $y \cos x = 1 + \sin(xy)$



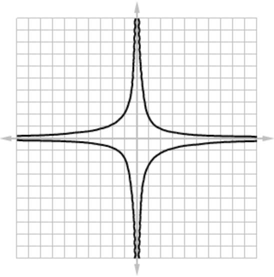
e) $x^2 + xy - y^2 = 4$



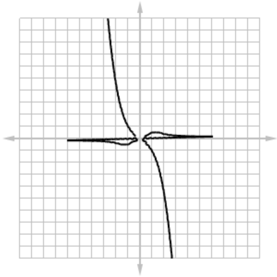
f) $\frac{1}{x} + \frac{1}{y} = 1$



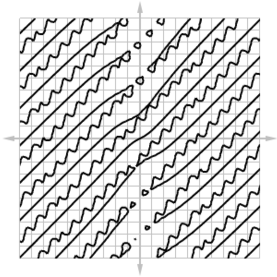
g) $x^2y^2 + x \sin y = 4$



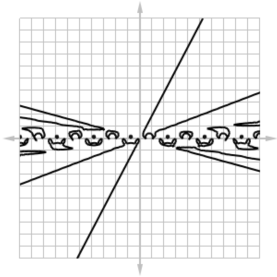
h) $x^4y^2 - x^3y + 2xy^3 = 0$



i) $\tan(x - y) = \frac{y}{1+x^2}$



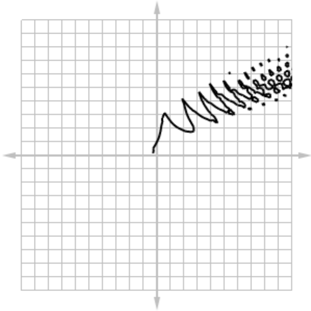
j) $\sin\left(\frac{x}{y}\right) = \frac{1}{2}$



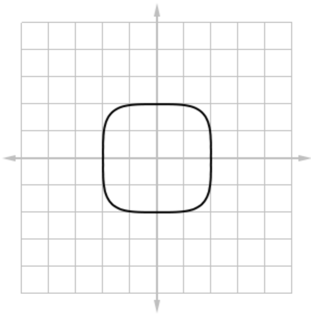
2. Consider the circle $x^2 + y^2 = 1$. Find the equation of the tangent to the curve at $(\sqrt{2}, \sqrt{2})$.

3. The ellipse $x^2 + 3y^2 = 36$ has two tangent lines passing through the point $(12, 3)$. Find both. *This question is similar to one from the 2015 Scholarship paper.*

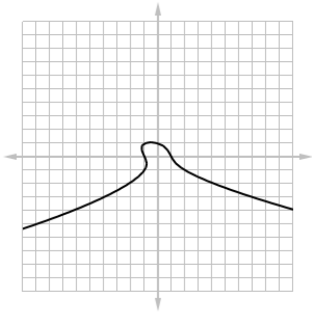
4. Find x' and y' if $\ln(y) = \sin(xy) + \frac{x}{y}$.



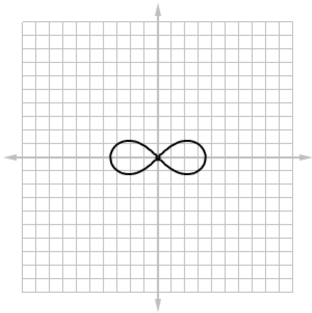
5. Find y'' if $x^4 + y^4 = 16$.



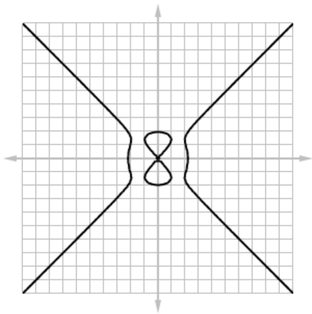
6. If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.



7. Find a tangent line to the curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point $(3, 1)$. This curve is known as a lemniscate.



8. Find a tangent line to the curve $y^2(y^2 - 4) = x^2(x^2 - 5)$ at the point $(0, -2)$. This curve is known as a devil's curve.



9. Consider the ellipse $x^2 - xy + y^2 = 3$.

   a) Find the points where the ellipse crosses the x-axis.

   b) Show that the tangent lines of the curve at these points are parallel.

   c) Find the maximum and minimum points of the curve.

10. Consider a circle C that is tangent to $3x + 4y - 12 = 0$ at $(0, 3)$ and contains $(2, -1)$. Set up equations that would determine the centre (h, k) and radius r of C.

11. The Bessel function of order 0, $y = J(x)$, satisfies the equation

$$xy'' + y' + xy = 2$$

for all values of x. The value of the function at 0 is $J(0) = 1$.

   a) Find $J'(0)$.

   b) Use implicit differentiation to find $J''(0)$.

12. Scholarship 2018: Suppose a circle with centre O is drawn, and a point A is picked within the circle. Where should a point P be placed on the circumference of the circle such that the interior angle of the triangle OAP at P is maximised?

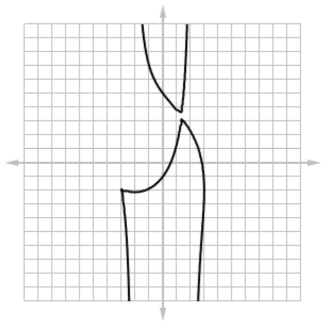
13. Scholarship 2012: Consider the equation $x^n = \tan(ny)$, where n is a constant. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x.

14. Scholarship 2017: The functions sinh and cosh are defined as follows.

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right),$$
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right).$$

The inverse function of sinh is denoted by $\sinh^{-1}$. By implicit differentiation, or otherwise, show that
$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}.$$

15. Consider the following family of curves, known as Durer's shell curves (shown here for $a = 2$, $b = 3$):
$$(x^2 + xy + ax - b^2)^2 = (b^2 - x^2)(x - y + a)^2.$$



a) For which value(s) of b does the curve become a straight line?

b) Suppose that we restrict $a = \frac{b}{2}$. Find all non-differentiable points on the curve.

References

For many more examples and exercises, see Stewart, § 2.6. Wikipedia also has a long list of interesting curves at `https://en.wikipedia.org/wiki/List_of_curves`.

For a proof of the implicit function theorem, see Loomis and Sternberg, § 3.11.

This subject properly belongs to the field of differential geometry; see the references for § 9.

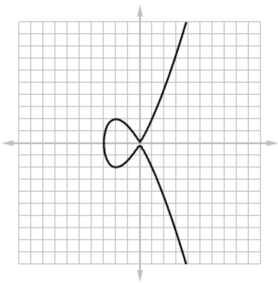
Homework problems

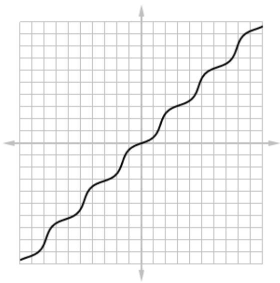
1. Find y' in each case:

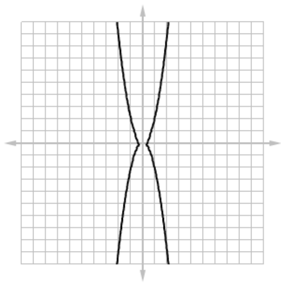
a) $y^2 = x^3 + 3x^2$ (Tschirnhausen cubic)

b) $\sin(x + y) = 2x - 2y$

c) $y^2 = 5x^4 - x^2$ (kampyle of Eudoxus)







2. Find the equation of the normal line to the curve $x^2 + 2xy - y^2 + x = 2$ at the point $(1, 2)$.

3. Show that the sum of the x and y intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is just c.

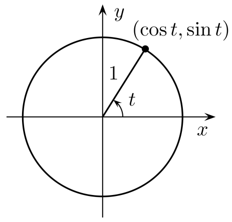


Figure 18: Parameterisation of the circle.

II.13 Geometry of parameterised curves

In the last section, we tried to describe the geometry of curves that can be described by implicit functions. There is another direction we can generalise which will give us some similarly nice results.

A very simple example of the technique we want to study can be generted by recalling the definition of the functions sin and cos. For each angle t take the line $\ell(t)$ through $(0, 0)$ which makes an anticlockwise angle t with the positive x–axis, and call the intersection point between $\ell(t)$ and the unit circle $P = (x, y)$; then set $\sin t = y$ and $\cos t = x$.

This definition sets up an exact correspondence between angles t ($-\pi \le t < \pi$) and points (x, y) on the unit circle — for each angle t we have precisely one point $(\cos t, \sin t)$ on the circle, and for each point (x, y) on the circle there is precisely one number $t = \arcsin t = \arccos t$.

In particular, we can say that the circle is the set of all points $(x, y) = (\cos t, \sin t)$; and this is just a function (well, a pair of functions) of one variable — increasing t walks around the circle anticlockwise. We can differentiate to find that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = \cos t$; thus, by the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\cos t}{-\sin t} = -\frac{x}{y}. \tag{II.5}$$

More generally, we want to study curves whose coordinates depend on a single parameter. If f and g are differentiable functions, the curve γ given by

$$\gamma(t) = (x, y) = (f(t), g(t)) \tag{II.6}$$

is called a *parameterised curve*.

Recall that the *implicit function theorem* states (roughly) that every curve can be sliced up into pieces which form the graphs of functions. For each piece, $y = g(f^{-1}(x))$ and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{f'(f^{-1}(x))} \cdot g'(f^{-1}(x)) = \frac{1}{f'(t)} \cdot g'(t) = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}.$$

In order to find the second derivative, we replace y with $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}\frac{\mathrm{d}y}{\mathrm{d}x}}{\mathrm{d}x} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}\right) \cdot \frac{\mathrm{d}t}{\mathrm{d}x}.$$

Exercises and Problems

1. In each case find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
   a) $x = t\sin t$, $y = t^2 + t$
   b) $x = 2\sec\theta$, $y = 3\tan\theta$
   c) $x = \cos\theta$, $y = \cos 3\theta$
   d) $x = e^{\sin\theta}$, $y = e^{\cos\theta}$
2. Find the equation of the chord joining the two points $t = 2$ and $t = 4$ on the curve $(x, y) = (2t - 3, t^3 + 6)$.
3. Determine the point(s) of intersection of the curves γ and δ defined by

$$\gamma : t \mapsto (t^2 - 2, t - 1),$$
$$\delta : t \mapsto (t, 2/t).$$

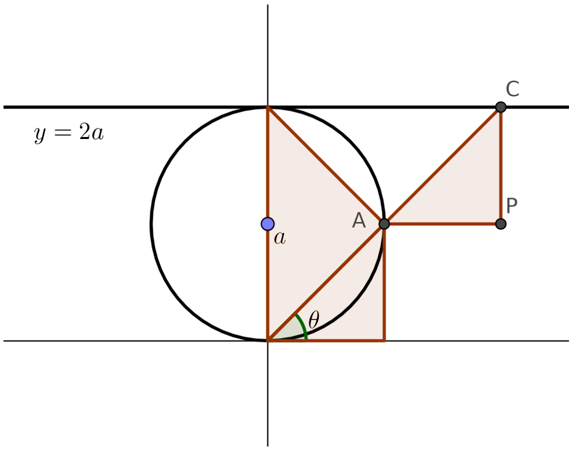
4. a) If $y = 2t$ and $x = 4t^2$ define a curve, what is the gradient $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t?
   b) Show that this curve is a parabola.

5. A curve has parametric equations $x = t^2 + 1$ and $y = t^3 + 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

6. Find the equation of the tangent to the curve $t \mapsto (2x^2 + 1, t^3 - 1)$ at $t = 2$.

7. If $t \mapsto (x, y)$ is a parametric curve, find an expression for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ analagous to that found for the second derivative.

8. A curve, called a *witch of Maria Agnesi*, consists of all possible positions of the point P in the diagram below. Graph the curve, show that the curve is given parametrically by $(x, y) = (2a \cot\theta, 2a \sin^2\theta)$, and find the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$.



9. A particle moves through space over time; the position of the particle at time t is given by $(3 \sin t, 2 \cos t)$ $(0 \leq t < 2\pi)$.
   a) What is the component of the acceleration of the particle in the x direction at $t = \pi/4$?
   b) At what times is the particle stationary in the x direction?
   c) Is the particle ever momentarily totally stationary?

10. Find the rightmost point on the curve $x = t - t^6$, $y = e^t$.

11. For which values of t is the curve $x = \cos 2t$, $y = 3\cos t$ concave up?

12. Show that the curve $\gamma : t \mapsto (\cos t, \sin t \cos t)$ has two tangents at (0,0) and find their equations.

13. a) Give a second parameterisation of the unit circle $x^2 + y^2 = 1$ by considering the set of all lines through $(-1, 0)$.
    b) Give a parameterisation for the cubic $x^3 - y^2 = 0$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ both using implicit differentiation and parametric differentiation, and check that the two results agree.

14. Scholarship 2000: The piriform is the curve defined by the equation $16y^2 = x^3(8 - x)$ where $x \geq 0$.
    a) Show that
    $$\begin{cases} x = 4(1 + \sin\theta) \\ y = 4(1 + \sin\theta)\cos\theta. \end{cases}$$
    are parametric equations for the piriform.
    b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ, and show that $\theta = \frac{\pi}{6}$ is a stationary point of the curve.

15. We define a surface C parametrically in terms of two parameters, t and θ:
$$(x, y, z) = (t, t\cos\theta, t\sin\theta).$$

a) Show that the Cartesian equation for this surface is $x^2 = y^2 + z^2$. (This is a cone.)

b) Show that the intersection between C and the plane $z = 2$ is a hyperbola.

c) For what angle α does the intersection between C and the plane parametrically defined by

$$(x, y, z) = (u \tan \alpha + 1, u, v)$$

(for parameters u and v) become a parabola? (Hint: $x = y \tan \alpha + 1$, and z is arbitrary.)

References

Homework problems

1. F