NCEA Level 3 Calculus (Differentiation)
6. Tangent and Normal Lines

Goal for this week

To understand how calculus helps us approximate the graphs of functions using straight lines.

For the next few weeks, we will be studying the geometry and shape of functions using calculus. We begin in this section by taking a look at how closely functions can be approximated by straight lines around a point.

Tangent Lines

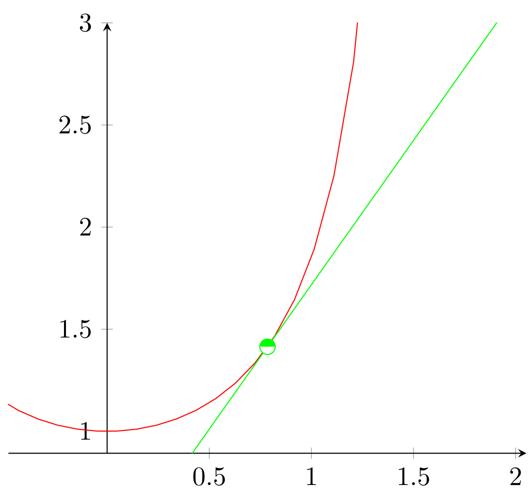
The **tangent line** of a function at a point is the unique line passing through that point such that the line has the same slope as the function at that point. Because we actually ended up defining slope based on the derivative, it follows that the equation of the tangent line to a curve at a point can be defined using the derivative.

More formally, if we have some function f which is differentiable at a point (x_0, y_0) then the tangent line to the function at that point has the equation

$$(y - y_0) = f'(x_0)(x - x_0).$$

Example. Consider the function $y = \sec x$. The derivative of this function is $y' = \tan x \sec x$; at the point $P = \left(\frac{\pi}{4}, \sqrt{2}\right)$, the slope is $\tan \frac{\pi}{4} \sec \frac{\pi}{4} = \sqrt{2}$.

The tangent line at P is therefore described by $y = \sqrt{2}(x - \frac{\pi}{4}) + \sqrt{2}$, or $y = \sqrt{2}x + \frac{4-\pi}{2\sqrt{2}}$.

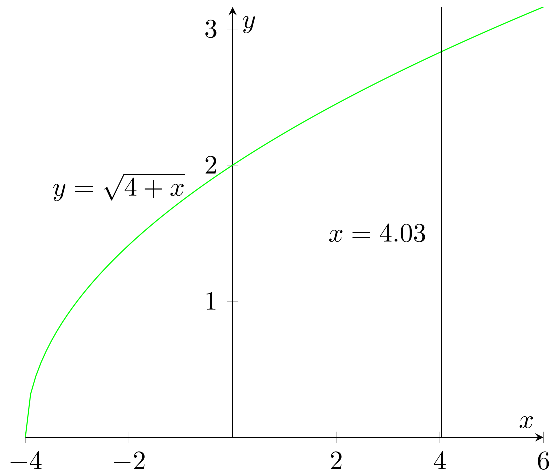


The tangent line of a function at a point is the best linear approximation to the function at that point. This means that if you know the value of a function and the derivative of that function at a point, then you can very easily approximate the other values of the function around that point: if $y_0 = f(x_0)$, and x is very close to x_0, then

$$f(x) \approx f'(x_0)(x - x_0) + y_0.$$

On the other hand, as the graph accompanying the example above shows, the tangent line is an awful approximation as we move further away from the point that we find the tangent line at. It is possible to obtain some measure of the error of a given approximation; this is explored, a little, in the exercises.

Example. Let us calculate $\sqrt{4.03}$ by hand(!). If we consider the function $f(x) = \sqrt{4+x}$, then $\sqrt{4.03} = f(0.03)$. Let's draw the situation out:



So we want the tangent line to f at the point $x = 0$. We have that $f'(x) = \frac{1}{2\sqrt{4+x}}$, and so $f'(0) = \frac{1}{4}$. The tangent line is the line through $(0, f(0)) = (0, 2)$ with gradient $\frac{1}{4}$, which has equation

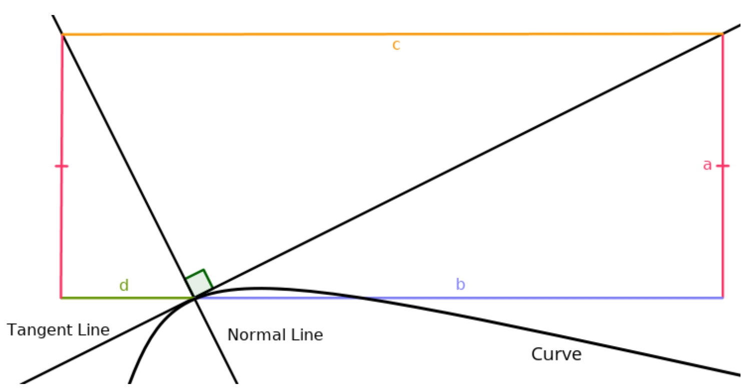
$$y - 2 = \frac{1}{4}x.$$

Hence $\sqrt{4.03} \approx \frac{1}{4} \times 0.03 + 2 = 2.0075$ — and, as promised, all of these calculations can be done without a calculator. (According to my calculator, $\sqrt{4.03} \approx 2.007486$ and so we are not far off at all.)

Normal Lines

If a function has a tangent line at a particular point, then the line perpendicular to the tangent line at that point is called the **normal line**.

Theorem. *If a tangent line has slope m, then the normal line to it has slope $m^{\perp} = -1/m$.*

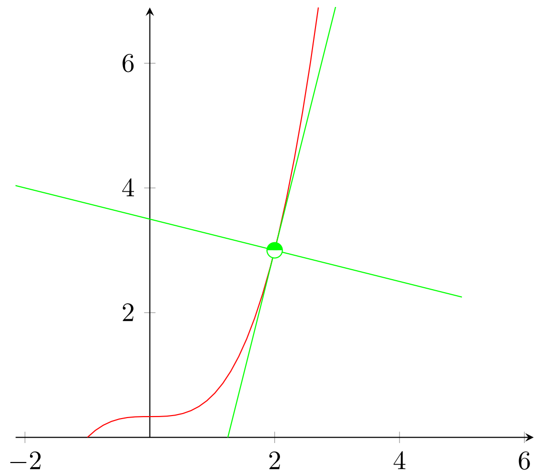


Proof. Consider the line shown in the diagram above, and let $m = \frac{a}{b}$ be the slope of the tangent line. So the length of the hypotenuse of the ab triangle is $\sqrt{a^2 + b^2}$. But $c = d + b$. Hence the length of the hypotenuse of the ad triangle can be found in two ways:

$$\sqrt{(d+b)^2 - \left(\sqrt{a^2+b^2}\right)^2} = \sqrt{a^2 + d^2}.$$

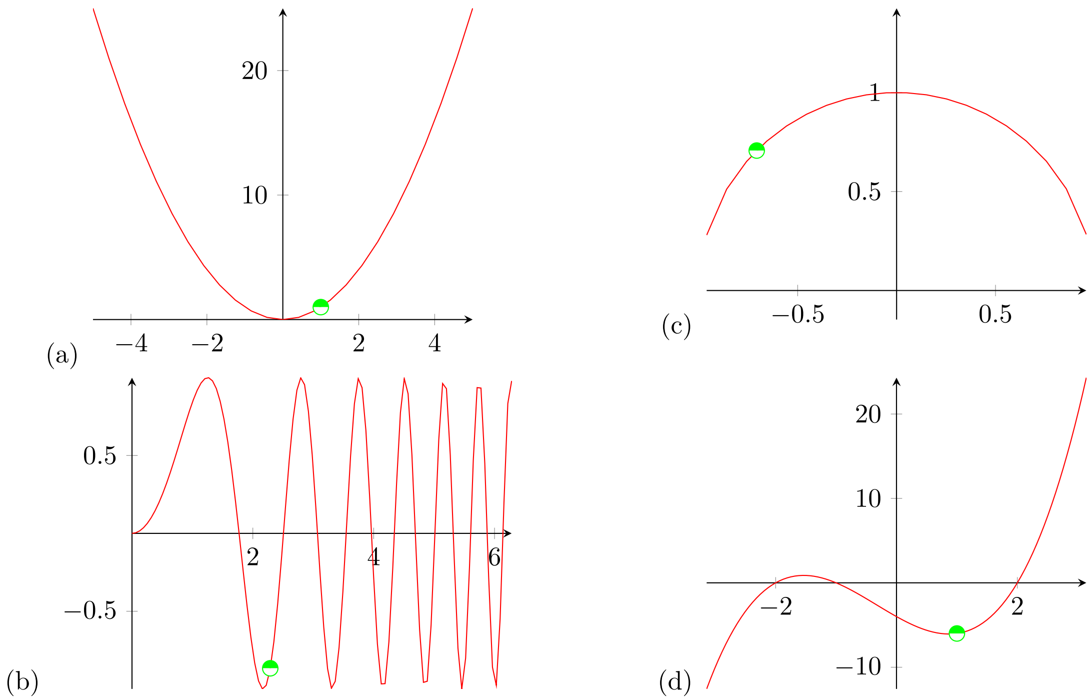
We can square both sides to obtain $d^2 + 2db + b^2 - a^2 - b^2 = a^2 + d^2$, and therefore $db = a^2$. Hence $d/a = a/b$; but $a/b = m$, and so the slope of the normal line is simply $-\frac{a}{d} = -\frac{1}{m}$. $\square$

Example. Consider the function f defined by $f(x) = \frac{x^3+1}{3}$. Then $f'(x) = x^2$, and the tangent line to the function at $(2, 3)$ is $(y - 3) = 4(x - 2)$, or $y = 4x - 5$. By the theorem above, the slope of the normal line to the function at that point is $-1/24$.



Questions

1. Draw the tangent and normal lines to each function at the indicated points. A



2. Find the tangent and normal lines to the following functions at the given points. M
   (a) $x \mapsto \sin x$ at $(0, 0)$
   (b) $x \mapsto \sin x$ at $(\pi, 0)$
   (c) $x \mapsto e^x$ at $(0, 1)$
   (d) $x \mapsto \sec x$ at $(0, 0)$
   (e) $x \mapsto x^2$ at $(1, 1)$
   (f) $x \mapsto \sqrt{x}$ at $(1, 1)$
   (g) $x \mapsto (x^4 - 3x^2 + 5)^3$ at $(0, 125)$
   (h) $x \mapsto \cos \tan x$ at $(\pi, 1)$
   (i) $x \mapsto \left(x + \frac{1}{x^2}\right)^{\sqrt{7}}$ at $(1, 2^{\sqrt{7}})$.

3. Find the equation of the tangent line to $y = x + \tan x$ at (π, π). M

4. Find an equation for the normal line to the curve $y = \frac{1}{\sqrt{x^2 - x}}$ at $(2, \frac{1}{\sqrt{2}})$. M

5. Consider the curve $y = \tan(2 \sin x)$.
   (a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \cos x \sec^2(2 \sin x)$. A
   (b) Find the equation of: M
      i. The tangent to the curve at $(\pi, 0)$
      ii. The normal to the curve at $(0, 0)$

6. Find the best linear approximation to $y = 3x^3 + 2x + 4$ around $x = 2$. M

7. Find the point(s) on the graph of the function $y = x^2$ such that the slope of the normal to the curve at that point is $m^{\perp} = -1$. M

8. The tangents to the curve $y = \frac{1}{4}(x-2)^2$ at points P and $Q = (6, 4)$ are perpendicular. What is the x-ordinate of P? E

9. Consider the function $y = \sin x$.
   (a) Find the best linear approximation to this function around $(0, 0)$. M
   (b) Find the percentage error of this approximation to 1dp when $x = \pi$. M
   (c) At what point on the curve ($x \geq 0$) does the percentage error of the approximation rise above 100%? E

10. (a) Find two points on the graph of $y = 1/x$ that share a common normal line. M
   (b) Show that there are no more such points. M
   (c) Show that there are no two points on the graph that share a common tangent line. M
   (d) Repeat parts (a)-(c) for the general hyperbola-like curve $y = x^{-n}$ (where n is a positive integer). E
   (e) What is the situation for the even more general case of the curve $y = x^r$, where r is any real number? S

11. Consider the quartic polynomial $p(x) = 2x^4 - 4x^3 - 23x^2 + 84x - 61$.
   (a) Find the best linear approximation to p around the point $(2, 15)$. M
   (b) Find the unique quadratic polynomial $q(x)$ such that $q(2) = p(2)$, $q'(2) = p'(2)$, and $q''(2) = p''(2)$. This is the best quadratic approximation to p at the point $(2, 15)$. S
   (c) Show that the best quadratic approximation to a function f at the point (x_0, y_0) is given by O

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2.$$