NCEA Level 3 Calculus (Differentiation)
13. Inverse Functions

A function is one-to-one (or injective) if $f(x) = f(y)$ implies that $x = y$ (i.e. two different inputs can never give the same output/the function passes the horizontal-line test).
Definition. Let f be a function that is one-to-one. Then the **inverse** of f is the (unique) function f^{-1} such that
$$f(x) = y \iff f^{-1}(y) = x.$$
In other words, $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$.
Example. Here are some functions with their inverses:

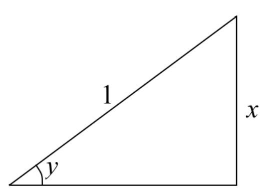
Function	Inverse	Notes
e^x	$\ln x$	Note that $\ln x$ is defined only when $x > 0$ since $e^x > 0$ for all real x.
$\sin x$	$\sin^{-1} x$	Note that $\sin^{-1} x$ is only defined when $-\pi < x \le \pi$ since otherwise $\sin x$ is not one-to-one.
$\cos x$	$\cos^{-1} x$	Note that $\cos^{-1} x$ is only defined when $-\pi < x \le \pi$ since otherwise $\cos x$ is not one-to-one.
$\tan x$	$\tan^{-1} x$	Note that $\tan^{-1} x$ is defined for all x (why?), and so $\tan^{-1} x \neq \frac{\sin^{-1} x}{\cos^{-1} x}$.
x^2	$\sqrt{x}$	When x is positive.

The graph of the inverse of a function is the reflection of the graph of the original function around the line $x = y$ (essentially, we swap the x and y axes).

In order to illustrate the general process of finding the derivatives of inverse functions, let us now find the derivative of $y = \sin^{-1} x$.

$$\begin{aligned} y &= \sin^{-1} x \\ \sin y &= x \\ \frac{\mathrm{d}y}{\mathrm{d}x} \cos y &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos y} = \frac{1}{\cos \sin^{-1} x} = \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

The identity $\cos \sin^{-1} x = \sqrt{1 - x^2}$ comes from the following triangle:



It is possible to write down an expression relating the derivative of f^{-1} to the derivative of f. However, in the immortal words of Michael Artin, "even mathematicians occasionally feel that isn't worthwhile making a precise formulation — that it is easier to consider each case as it comes along. This is one of those occasions."* (The general theorem is given in the exercises below for you to prove.)

* Artin, *Algebra* (1st ed), p.457.

By the same kind of calculation, we obtain the following table which gives the derivatives of inverses of the three primary trigonometric functions and their reciprocals.

Function	**Derivative**	**Function**	**Derivative**
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\csc^{-1} x$	$-\dfrac{1}{x\sqrt{x^2-1}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$	$\sec^{-1} x$	$\dfrac{1}{x\sqrt{x^2-1}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\cot^{-1} x$	$-\dfrac{1}{x^2+1}$

Questions

1. M Prove or disprove the following statements:
   (a) The function $f : x \mapsto x^2 + x + 1$ is one-to-one (where x is real).
   (b) The function $g : x \mapsto 2^x$ is one-to-one (where x is a positive real).
2. M Determine whether the following functions have inverses on the given interval:
   (a) $x \mapsto x^3$ (on $\mathbb{R}$)
   (b) $y \mapsto y^4$ (on $\mathbb{R}$)
   (c) $y \mapsto y^4$ (for $y \geq 0$)
   (d) $y \mapsto y^4$ (for $y > 0$)
   (e) $\theta \mapsto \cos^{-1}\theta$ (on $\mathbb{R}$)
   (f) $\theta \mapsto \cos^{-1}\theta$ (for $-1 \leq \theta \leq 1$)
3. A True or false:
   (a) $\cos^{-1} x = \frac{1}{\cos x}$
   (b) If $x > 0$ then $(\ln x)^6 = 6 \ln x$
   (c) $\tan^{-1}(-1) = \frac{3\pi}{4}$ (think about which arm of $\tan x$ we're talking about)
   (d) The inverse of $f(x) = e^{3x}$ is $f^{-1}(x) = \frac{1}{3}\ln x$.
4. A Find the derivative of $f(x) = \ln(e^x)$ in two different ways.
5. A Find y' if:
   (a) $y = \sin^{-1} 2x$
   (b) $x = \sin^2 y$
   (c) $y = x + \tan^{-1} y$
   (d) $y = \ln \sin x - \frac{1}{2}\sin^2 x$
   (e) $y = 24 \arctan x + \arcsin \sqrt{x}$
   (f) $y = \sqrt{\sec^{-1} 2x}$

6. $\boxed{\texttt{E}}$ Find the local extrema, areas of concavity, and inflection points of the following functions; hence sketch their graphs.

   (a) $y = e^x \sin x$ for $-\pi < x < \pi$

   (b) $y = x + \ln(x^2 + 1)$

   (c) $y = \sin^{-1}(1/x)$

7. $\boxed{\texttt{E}}$ Justify intuitively, without invoking the happy coincidence that our notation for derivatives looks like a fraction, the statement that $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$.

8. $\boxed{\texttt{E}}$ If $f'(x) = \tan^{-1} x$, find $(f^{-1})'(x)$.

9. $\boxed{\texttt{E}}$ Suppose f is a function, and f^{-1} is the inverse of f. Prove that $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$.

10. $\boxed{\texttt{S}}$ Prove the formulae for the derivatives of $\cos^{-1}$ and $\tan^{-1}$, using a similar method to that for $\sin^{-1} x$.

11. $\boxed{\texttt{S}}$ Scholarship 2012: Consider the equation $x^n = \tan(ny)$, where n is a constant. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x.