NCEA Level 3 Calculus (Integration)
20. Partial Fractions (Homework)

Reading

Perhaps I can best describe my journey of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but eventually you learn where each piece of furniture is. Finally after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of — and couldn't exist without — the many months of stumbling around in the dark that precede them.

From an interview by Nova of Andrew Wiles

Note: For the interested, a proof that one can always expand a rational function into partial fractions is outlined as exercise 11.1.13 in Artin (p. 441).

Questions

1. The *logistic equation* is used when modelling populations:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(1-P)$$

   (a) Find $P(t)$ explicitly.

   (b) Examine the behaviour of the population as $t \to \infty$. Graph the function.

   (c) Examine the behaviour of the population over time if you vary r (check $r = 0$, and $r < 0$ for example).

   (d) Do you think the logistic equation is a good model? Why/why not?

2. In the following, let $t = \tan\frac{x}{2}$ (where $|x| < \pi$).

   (a) Show that:

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \quad \text{and} \quad \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$$

   (b) Show that:

$$\cos x = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin x = \frac{2t}{1+t^2}$$

   (c) Show that:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}$$

   (d) Use the substitution t to evaluate:

      i. $\int (1-\cos x)^{-1}\,\mathrm{d}x$

      ii. $\int (3\sin x - 4\cos x)^{-1}\,\mathrm{d}x$