NCEA Level 3 Calculus (Integration)
21. Integration by Parts

The substitution rule is the inverse of the chain rule; similarly, there is an inverse of the product rule.

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x) \\ \iff & \int f'(x)g(x) + f(x)g'(x)\,\mathrm{d}x = f(x)g(x) \\ \iff & \int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \end{aligned}$$

Mnemonically,

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Example. Consider $\int x \sin x\,\mathrm{d}x$, which does not yield to any obvious change of variable. Let $u = x$, and let $\mathrm{d}v = \sin x\,\mathrm{d}x$. So $\mathrm{d}u = \mathrm{d}x$, and $v = -\cos x$. Hence:

$$\int x\sin x\,\mathrm{d}x = -x\cos x + \int \cos x\,\mathrm{d}x = -x\cos x + \sin x + C,$$

where C is an arbitrary constant. Check that $(-x\cos x + \sin x)' = x\sin x$.

The aim is to end up with an easier integral than the one that was started with. A good choice for u is usually (in descending order of usefulness):

1. Logarithms
2. Powers of x
3. Exponentials
4. Trig functions

Questions

1. Compute the following indefinite integrals. S
   (a) $\int xe^x\,\mathrm{d}x$
   (b) $\int x^2e^{2x}\,\mathrm{d}x$
   (c) $\int \ln x\,\mathrm{d}x$
   (d) $\int p^5 \ln p\,\mathrm{d}p$
   (e) $\int t^3e^{-t^2}\,\mathrm{d}t$
   (f) $\int \sin\ln y\,\mathrm{d}y$
   (g) $\int x\tan^2 x\,\mathrm{d}x$
2. Prove that S
$$\int \cos^n x\,\mathrm{d}x = \frac{1}{n}\sin x\cos^{(n-1)} x + \frac{n-1}{n}\int \cos^{(n-2)} x\,\mathrm{d}x$$
3. If $I_n = \int_0^n x^n e^x\,\mathrm{d}x$, write down an explicit general formula for I_n. S
4. Evaluate $\int (\ln x)^2\,\mathrm{d}x$. S

5. Compute $\int_0^\lambda te^{-\lambda t}\,\mathrm{d}t$. S

6. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, and $f'(4) = 3$. Evaluate $\int_1^4 xf''(x)\,\mathrm{d}x$. S

7. A particle moving in one dimension has a velocity function $v(t) = t^2e^{-t}$ (where t is in seconds). What is its displacement from its starting position after three minutes? S

8. Find the area bounded by $y = x^2 \ln x$ and $y = 4 \ln x$ S

9. Scholarship 2012:
   (a) Find $\frac{\mathrm{d}}{\mathrm{d}x}[x \cos x]$ and use this result to find $\int x \sin x\,\mathrm{d}x$. E
   (b) Hence find the value of $\int_0^{n\pi} x \sin x\,\mathrm{d}x$ for integer values of n. S

10. Scholarship 2016: O
    (a) A function $f(x)$, where x is a real number, is defined implicitly by the formula

$$f(x) = x - \int_0^{\pi/2} f(x)\sin(x)\,\mathrm{d}x.$$

    Find the explicit expression for $f(x)$ in simplest form.

    (b) A curve passing through the point $(1, 1)$ has the property that at each point (x, y) on the curve, the gradient of the curve is $x - 2y$; that is, $\frac{\mathrm{d}y}{\mathrm{d}x} = x - 2y$.
       i. Show that $\frac{\mathrm{d}}{\mathrm{d}x}e^{2x}y = xe^{2x}$.
       ii. Hence, or otherwise, find the equation of the curve.

11. It is well known that S

$$\int\limits_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

    Using this result, show that

$$\int\limits_{-\infty}^{\infty} x^2e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

12. Find $I = \int e^x \cos x\,\mathrm{d}x$. O

13. Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2}$. Find $\int \tan^{-1} x\,\mathrm{d}x$. S

14. We integrate $\int 1/x\,\mathrm{d}x$ by parts: S

$$\int \frac{1}{x}\,\mathrm{d}x = \frac{1}{x} \cdot x - \int -\frac{1}{x^2} \cdot x\,\mathrm{d}x = 1 + \int \frac{1}{x}\,\mathrm{d}x$$

    Cancelling the indefinite integral from both sides, we have $0 = 1$. Explain.