NCEA Level 3 Calculus
Revision: Functions

Before we look at calculus proper, we need to revise a few things from our previous studies. Arguably, the most fundamental concept from L2 is that of a function, together with its graph.

Definition (Function)**.** A **function** is a relationship between two sets of things, called the *range* and the *domain*, such that everything in the range is related to exactly one thing in the domain. If f is a function which maps the value x to the value y, we write $f : x \mapsto y$, or $f(x) = y$. If f has range X and domain Y, we write $f : X \to Y$.

Two functions f and g are called equal *if and only if* the ranges of the two functions are the same *and* for every x in that range, $f(x) = g(x)$.

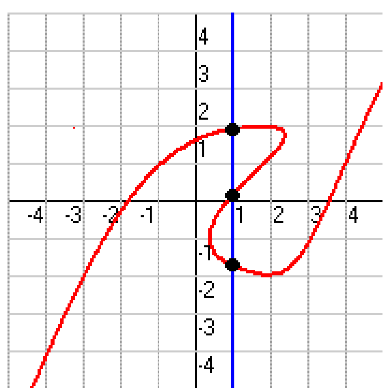
You can think of a function as a rule: it could be given by a formula, or by a list of inputs and outputs, or in any other way that one wants.

Definition (Graph)**.** If f is a function, then the set of all points (x, y) such that $y = f(x)$ is called the **graph** of the function.

Examples.

1. The map which takes a number x and spits out x^2 is a function — for every input, there is exactly one output. If we plot every point in the graph of this function, by plotting each input on the $x-$axis and the corresponding output on the $y-$axis, we obtain a parabola.

2. The curve graphed below is *not* the graph of a function, since for some inputs (like 1) the map has more than one output. We can check this by drawing vertical lines along the function, like that pictured: if a graph is a function, no vertical line can ever cross the curve more than once (this is the *vertical-line test*).



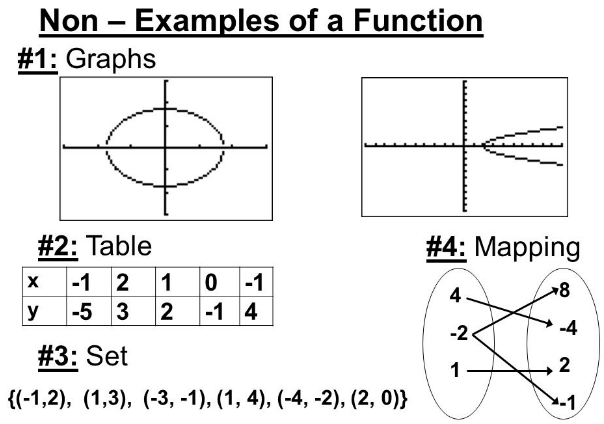
3. The map $f : x \mapsto \sin x$ is a function (and so are all the other triangle ratios). We could also define it by 'the function f such that $f(x) = \sin x$'. This function f can only produce numbers between 1 and -1; we say that its range is the interval from -1 to 1.

4. On the other hand, to ensure that we obtain functions the inverse trigonometric maps must be restricted to certain inputs: to take a particular example, since there are infinitely many x so that $\sin x = 1$, there are infinitely many possibilities for the value of $\arcsin x$. We will refrain from picking an explicit range for the inverse functions here and will generally just pick the most convenient at the time: it should be reasonably obvious from context.

5. The map $\iota : x \mapsto x$ is a function, called the *identity function*.

6. The map $\ln x$ is a function, but it is only defined when $x > 0$: we say that its *domain* is the positive real numbers.

7. The following are some more non-examples of functions.



Revision Questions

1. Which of the following are functions?
   (a) $E(x) = 2^x$
   (b) $\phi : x \mapsto \frac{2}{x}$
   (c) The thing which maps every person to their youngest sibling.
   (d) The thing which sends every person to their youngest sibling that isn't themself.
   (e) $x \mapsto \lfloor x \rfloor$ (where $\lfloor x \rfloor$ is the largest integer less than or equal to x).
   (f) The relation that sends every person to their age.

2. I will define two functions, φ and ϑ, as follows:

$$\varphi(x) = 2x - 7, \qquad \vartheta(\zeta) = \frac{1}{7}(14\zeta - 49).$$

   Explain why these functions are equal.

3. If $f(x) = x^2 + x$, find:
   (a) $f(1)$
   (b) $f(y)$
   (c) $f(x+h)$

4. Find the distance between $(-3, 4)$ and $(2, 1)$.

5. Three sides of a triangle are have lengths 8, 15, and 17.
   (a) Show that the triangle is right-angled.
   (b) Find the other two angles.

6. Factorise and solve $x^2 - 3x + 2 = 0$.

7. How many **lines** are there through the point $(2, 3)$ and the origin? Give the equations of all such lines.

8. Find the slope of the line $4x + 3y + 2 = 0$.

9. Find the solution to the following linear system:

$$\begin{aligned} 2x + y &= 7 \\ 3x - y &= 8 \end{aligned}$$

10. How many (real) solutions does $x^2 + 4x + 1$ have?
11. Draw $\sin(x)$, $\cos(x)$, $\tan(x)$, $\exp(x)$, and $\ln(x)$.
12. How many solutions does $\cos(3\pi x + 1) = 2$ have?
13. How many solutions does $\sin(3x) = \frac{1}{3}$ have?