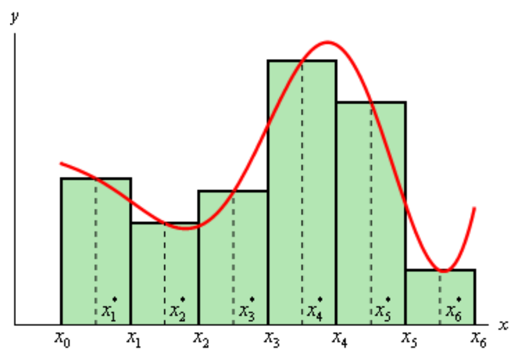
NCEA Level 3 Calculus (Integration)
22. Lengths, Volumes, and Areas

Goal for this week

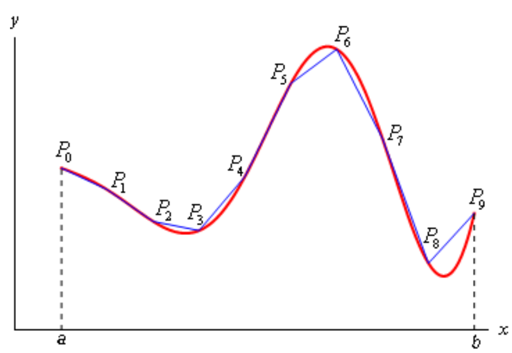
To use an integral as a fancy sum.

Recall that the definite integral is simply a way of calculating the area bounded by a curve. We defined it to be an infinite sum of areas under the curve that individually tended to zero.



Today we will investigate finding volumes, curve lengths, and areas of surfaces by integration.

Curve Lengths



Suppose we have a function f and we wish to find the length *measured along the curve* between two points a and b. Split the interval into pieces of length Δx; then the arc length over each subdivision is approximated by $\sqrt{\Delta x^2 + (f'(x_i^*)\Delta x)^2}$ where x_i is a point inside the subdivisioon. We therefore have the sum over the whole interval (a, b) (where the total number of subdivisions is n):

$$\sum_{i=0}^{n} \sqrt{\Delta x^2 + (f'(x_i^*)\Delta x)^2} = \sum_{i=0}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x.$$

But if we let $n \to \infty$, this is exactly the following integral:

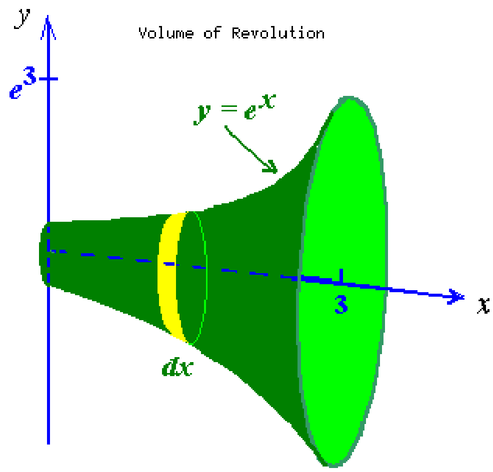
$$R = \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x.$$

Example. We wish to find the length along the curve $y = \ln \cos x$ along the interval $(0, \pi/3)$. We have the following integral (noting that $y' = -\frac{\sin x}{\cos x} = -\tan x$):

$$R = \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, \mathrm{d}x = \int_0^{\pi/3} \sec x \, \mathrm{d}x = \int_0^{\pi/3} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, \mathrm{d}x = \int_1^{2+\sqrt{3}} \frac{1}{u}\, \mathrm{d}u = \ln(2 + \sqrt{3}) \approx 1.3170.$$

The integral of $\sec$ *is tricky; it can also be done by partial fractions (multiply top and bottom of* $1/\cos x$ *by* $\cos x$ *and substitute* $u = \sin x$*).*

Volumes of Revolution



We can carry out the same procedure to find a volume of revolution; the volume of each small disc can be approximated with $\pi[f(x_i^*)]^2 \Delta x$, and so we have

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=0}^{n} \pi[f(x_i^*)]^2 \Delta x \\ &= \int_a^b \pi[f(x)]^2 \, \mathrm{d}x. \end{aligned}$$

Surface Areas of Revolution

We can model the surface area of revolution of a curve is similarly found by modelling the volume as a disc of radius $f(x_i^*)$ and length $\sqrt{1 + (f'(x_i^*))^2}\Delta x$:

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=0}^{n} 2\pi f(x_i^*) \sqrt{1 + (f'(x_i^*))^2} \Delta x \\ &= \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, \mathrm{d}x. \end{aligned}$$

Questions

1. Determine the length of: M
   (a) $x = \frac{2}{3}(y-1)^{3/2}$ between $1 \le y \le 4$.
   (b) $y = \ln \sec x$ between $0 \le x \le \frac{\pi}{4}$.

2. The arc of the parabola $y = x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis. Find the area of the resulting surface. M

3. (a) Suppose f is a function of x, and that you know that the graph $y = f(x)$ is a straight line. Furthermore, assume that $f(0) = 0$ and $f(h) = r$ where h and r are constants. Find a formula for f, and draw its graph. E
   (b) Find a formula for the volume enclosed by rotating the graph $y = f(x)$ around the x-axis between the origin ($x = 0$) and $x = h$. Sketch a diagram showing the volume.

4. The cartesian equation for a circle of radius r is $y^2 = r^2 - x^2$. Compute the volume of revolution of the circle from $x = -r$ to $x = r$, and hence write down the formula for the volume of a sphere of radius r. E

5. Find the volume of the solid obtained by rotating the region bounded by the given curves around the x-axis. Sketch the region and the solid. M
   (a)
$$\begin{cases} y = 2 - \frac{1}{2}x \\ y = 0 \\ x = 1 \\ x = 2 \end{cases}$$
   (b)
$$\begin{cases} y = x - x^2 \\ y = 0 \end{cases}$$
   (c)
$$\begin{cases} y = \sqrt{25 - x^2} \\ y = 0 \\ x = 2 \\ x = 4 \end{cases}$$

6. Find the volume of rotation of the region bounded by $y = \sin x$ and $y = \cos x$ around the line $y = -1$, where $0 \le x \le \pi/4$. E

7. The ellipse M
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where } a > b$$
   is rotated around the x-axis to form an ellipsoid. Find the surface area of the ellipsoid.

8. Use Simpson's rule with $n = 8$ to estimate the volume of the solid resulting when the region enclosed by the curves M
$$\begin{cases} y = \sin^8 x \\ y = 2x/\pi \\ x = 0 \\ x = \pi/2 \end{cases}$$
   is revolved around the x-axis.

9. A vase is created by rotating the curve S

$$x = \frac{1}{200}y^3 - \frac{1}{10}y^2 + \frac{3}{2}y + \frac{5}{3}$$

around the y-axis for $0 \le y \le 20$ (y is in centimetres).

(a) Find a function $V(\alpha)$ for the volume of water in the vase if it is filled up to $y = \alpha$.

(b) Water flows into the vase at a rate of $10\,\mathrm{cm}^3\,\mathrm{min}^{-1}$; water flows out at a rate directly proportional to the square root of the volume in the vase at time t.

i. The initial volume of water in the vase at time $t = 0$ is $3\,\mathrm{cm}^3$. Find the initial height of the water.

ii. After three minutes, the volume of water in the vase is $3.6\,\mathrm{cm}^3$. Will the vase ever fill completely? If so, how long does it take?

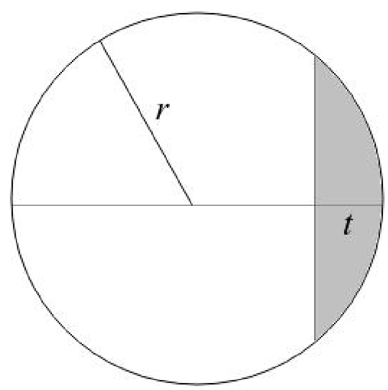
10. Use integration to find the volume of a cylinder by taking slices along the cylinder's axis. E

11. Use integration to find the volume of a cone by taking slices along the cone's axis. E

12. Compute the integral S

$$\int_{r-t}^{t} \sqrt{r^2 - x^2}\,\mathrm{d}x$$

and hence write down an expression for the slice of width t of a circle of radius r (figure).



13. The base of a solid S is a circle of radius r. Cross-sections perpendicular to the base are squares. What is the volume of S? S

14. A cathedral dome is designed with three semicircular supports of radius r such that each horizontal cross-section is a regular hexagon. Show that the volume of the dome is $r^3\sqrt{3}$. S

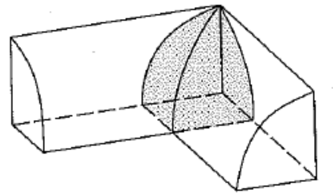
15. The integral E

$$V = \int_{0}^{3} 2\pi x^5\,\mathrm{d}x$$

represents the volume of a solid. Describe the solid.

16. Consider a cylindrical hole of length h drilled through the centre of a sphere. Find the volume $V(h)$ of the remaining solid. *Hint: you should find that V is independent of the size of the sphere.* O

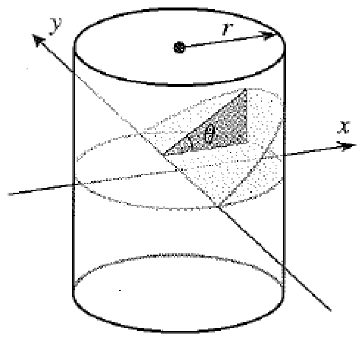
17. Two identical right circular cylinders of radius r have axes that intersect at right angles. Find the volume of the intersection region (known as the *Steinmetz solid*). *Hint: an interesting portion of the intersection is shown in the figure.** O



18. Use an integral to estimate the value of the sum S

$$\sum_{n=0}^{1000000} \sqrt{n}.$$

19. A wedge is cut from a right circular cylinder of radius r by two planes, one perpendicular to the axis of the cylinder and one at an angle θ with the first (as in the figure†). Find the volume of the wedge by slicing perpendicular to the y-axis. O



20. Scholarship 2000: The piriform is the curve defined by the equation $16y^2 = x^3(8-x)$ where $x \geq 0$. Find the volume of revolution obtained by rotating the piriform around the x-axis. S

21. Scholarship 2012: Stewie Griffin is a character from the television programme *Family Guy*. His head can be considered as a volume of revolution, turning a curve on an axis passing through his ears. Different volumes are obtained, depending on the shape of the rotated curve. O

    Assuming the head has width $2w$ and height $2h$, find the **ratio** of the volume obtained using a parabolic curve to the volume obtained using a semi-elliptical curve.

22. Scholarship 2013: Prince Rupert's drops are made by dropping molten glass into cold water. A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi(e^{-x} - e^{-2x})}$ for $x \geq 0$, where ϕ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. S

    (a) Show that the volume of the drop between $x = 0$ and $x = \ln(p)$ is $V = \frac{\pi\phi}{2}\left(\frac{p-1}{p}\right)^2$.

    (b) Hence or otherwise, explain why the volume of the drop is never more than some upper limit V_L, no matter how long its tail.

23. Scholarship 2015: Find the area of the surface of revolution obtained when the graph of $f(x) = x^3 + \frac{1}{12x}$, from $x = 1$ to $x = 3$, is revolved 360° around the x-axis. S

* Figure from Anton, Early Trans. 10th Ed., p 431.
† Ibid.