Solutions

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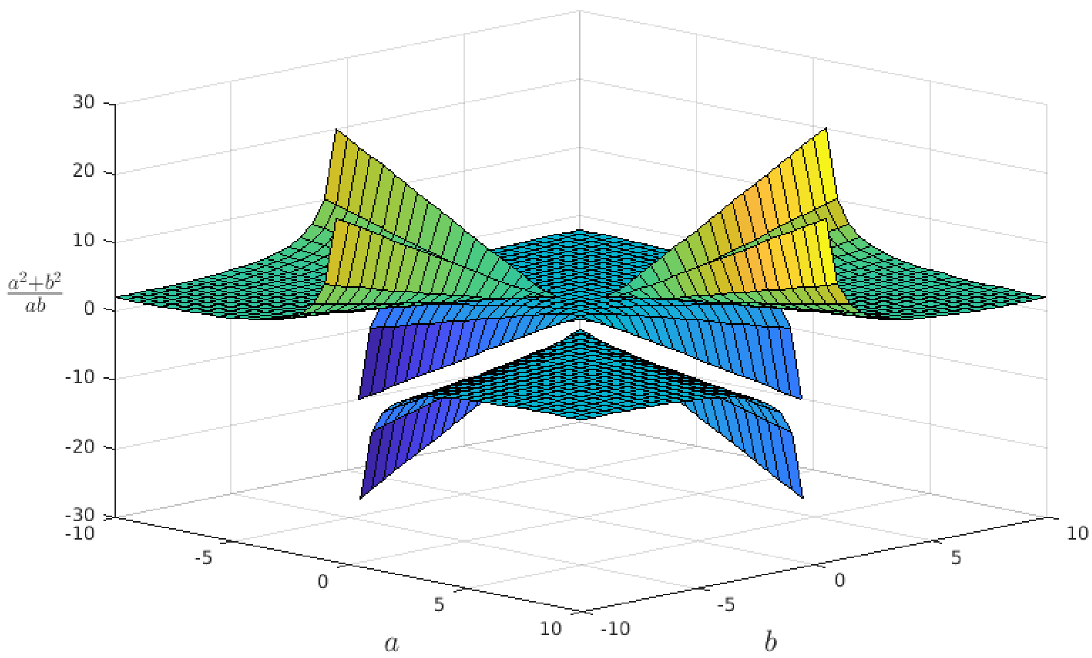

Contents

Section 1: Introduction

What does it really mean to *solve* an equation? These notes attempt to briefly present an outline to Level 3 and Scholarship methods for both finding solutions to equations and interpreting those solutions.

We will begin with simple linear equations, and will work our way up to arbitrary polynomials. Geometric and algebraic interpretations will be presented, along with a number of examples and exercises.

Philosophically, we have made the decision to leave a number of important results (chiefly de Moirve's theorem) as exercises. However, these results are required knowledge for the later problems — so the reader should at least glance at previous problems before attempting later ones.

Some exercises are much more difficult than others, and the difficulty does not always increase within a section (i.e. sometimes the first exercises can be quite hard). The number of stars by each problem is roughly indicative of a mixture of difficulty and time required. There is, however, no guarantee that the author's idea of a difficult question will match up with the reader's idea of a difficult question!

Note that there are fully worked answers for all problems, and often these solutions contain insights or additional information not in the main text. They are designed to be read in conjunction with the problems (but have a go yourself before reading the solutions). Additional information on the problems themselves, as well as a discussion of some notation, can be found in the introduction to the solutions.

The culmination of the text comes in sections 6 and 8; the former is a digression into more pure mathematics (developing a theory of the roots of unity), and the latter is a more applied section (developing the solution of the general cubic). Both should be easily within reach of the enthusiastic student. The section on geometry, 5, also includes a selection of material beyond that needed for level 3.

Section 2: Quadratic Equations

Even earlier than the Greeks, mathematics was being done in ancient Babylon. Several ancient clay tablets from the Old Babylonian period (between 1800 and 1600 BCE) contain mathematical problems and solutions. Such a problem, given on the tablet illustrated in section 2, is (translated of course):[2]

> I have added up seven times the side of my square and eleven times the area, getting 6;15.

The number 6;15 is in sexagesimal notation; it means $6 + \frac{15}{60} = 6.25$. We can therefore rewrite this problem as follows, where x is the length of the side of the square and x^2 is the area:

$$11x^2 + 7x = 6.25$$

Rearranging this we have $11x^2 + 7x - 6.25 = 0$, and so we see that our problem is simply to find the y-intercepts of a parabola, graphed as fig. 3. We find that the only positive solution to the equation is $x = 0.5$ — or 0;30 in Babylonian notation.

This problem is an example of a quadratic equation — an equation where the highest power of x is 2. All quadratic equations can be put into the form $ax^2 + bx + c = 0$, where a, b, and c are constants.

All of the equations which we will consider this year can be put into the form $f(x) = 0$ for some function f. The values x which satisfy the equation are called *solutions* of the equation, or *roots* or *zeroes* of the function.

The Quadratic Formula

In fact, there is a general formula to solve *any* quadratic equation. The following proof of this illustrates the idea of *completing the square*, which is a way to reduce the difficult problem of solving a general quadratic equation to a simpler problem: that of solving $y^2 = m$ for y given a value for m. A second proof is given as an exercise at the end of the section.

Theorem (Quadratic Formula)**.** *A quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0$) has at most two solutions, which are given by*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Proof. We want to transform our equation so that it looks like $(\cdots)^2 = d$; then we can just take the square root of both sides and solve. Let us naïvely guess that the bracket will look like $a(x + b/2a)^2$. If we expand this, we have

$$a\left(x + \frac{b}{2a}\right)^2 = ax^2 + bx + \frac{b^2}{4a}.$$

Comparing with our original equation, we are almost there: if we add on the constant term $c - \frac{b^2}{4a}$, all the unwanted material vanishes and we are left with the correct expression. With a little bit of manipulation, we obtain the required explicit formula for the solutions:

$$\begin{aligned} 0 = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\ \frac{b^2}{4a} + c = a\left(x + \frac{b}{2a}\right)^2 \\ \pm\sqrt{\frac{b^2}{4a^2} + \frac{c}{a}} - \frac{b}{2a} = x \\ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = x. \end{aligned}$$

□

The expression $b^2 - 4ac = \Delta_2$ is known as the *discriminant* of the quadratic, and it determines the nature of the solutions; if $\Delta_2 = 0$, there is one repeated root, while if $\Delta_2 > 0$ there are two real roots. We discuss the case where $\Delta_2 < 0$ later on.

[1] By Goran tek-en - Own work. Based on: Karte von Mesopotamien, Mesopotamia Syria. CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=30851043`

[2] Translation from [11].

Figure 1: Babylon in ancient Mesopotamia.[1]

Factors

To create a quadratic equation with given solutions, we write down a linear expression for each solution that evaluates to zero when the solution is substituted in, and then multiply them. For example, if we wanted a quadratic equation with the solutions $x = 2$ and $x = 3$, we take the two expressions $x - 2$ and $x - 3$, multiply the left hand sides together (obtaining $(x-2)(x-3) = x^2 - 5x + 6$), and set it to zero.

This works because if we substitute in one of our original solutions (in our example, 2 or 3) then one of the two parts of the left hand side will become zero and so the entire left hand side becomes zero.

In general, this idea works for higher-degree equations (like cubics, quartics, and so on). We can make a cubic with the solutions $x = -3$, $x = 9$, and $x = 13$ by writing $0 = (x+3)(x-9)(x-13)$.

We can even multiply together higher-degree polynomials to create a polynomial that has the roots of all of them — if (for example) we take the equation $(x^2 - 4)(x^2 - 9) = 0$, it will have the four solutions $x = \pm 2$ and $x = \pm 3$.

The polynomials which multiply together together to form a larger polynomial are known as *factors* of the larger polynomial, and the process of splitting a polynomial into factors is known as *factorising*.

Exercises

1. Find an example of a **quadratic equation** with the solutions: (a) $x = 7$ and $x = 4$, (b) $x = 7$ and $x = -4$, (c) $x = -7$ and $x = -4$, and (d) the single solution $x = 3$.

2. Use the discriminant Δ_2 of the following quadratics to find the number of distinct real roots each one has, without explicitly calulating those roots.

   (a) $3x^2 + 6x + 3 = 0$

   (b) $x^2 + 10x + 1 = 0$

   (c) $x^2 + 5x + 9 = 0$

   (d) $x^2 - \frac{14x}{3} + \frac{49}{9} = 0$

3. Use the *difference of two squares* identity $x^2 - b^2 = (x-b)(x+b)$ to factorise and hence solve the following equations for x:

   (a) $x^2 - 9 = 0$

   (b) $x^2 - 7 = 0$

   (c) $x^2 - 15 = 1$

   (d) $x^2 - 2ab = a^2 + b^2$

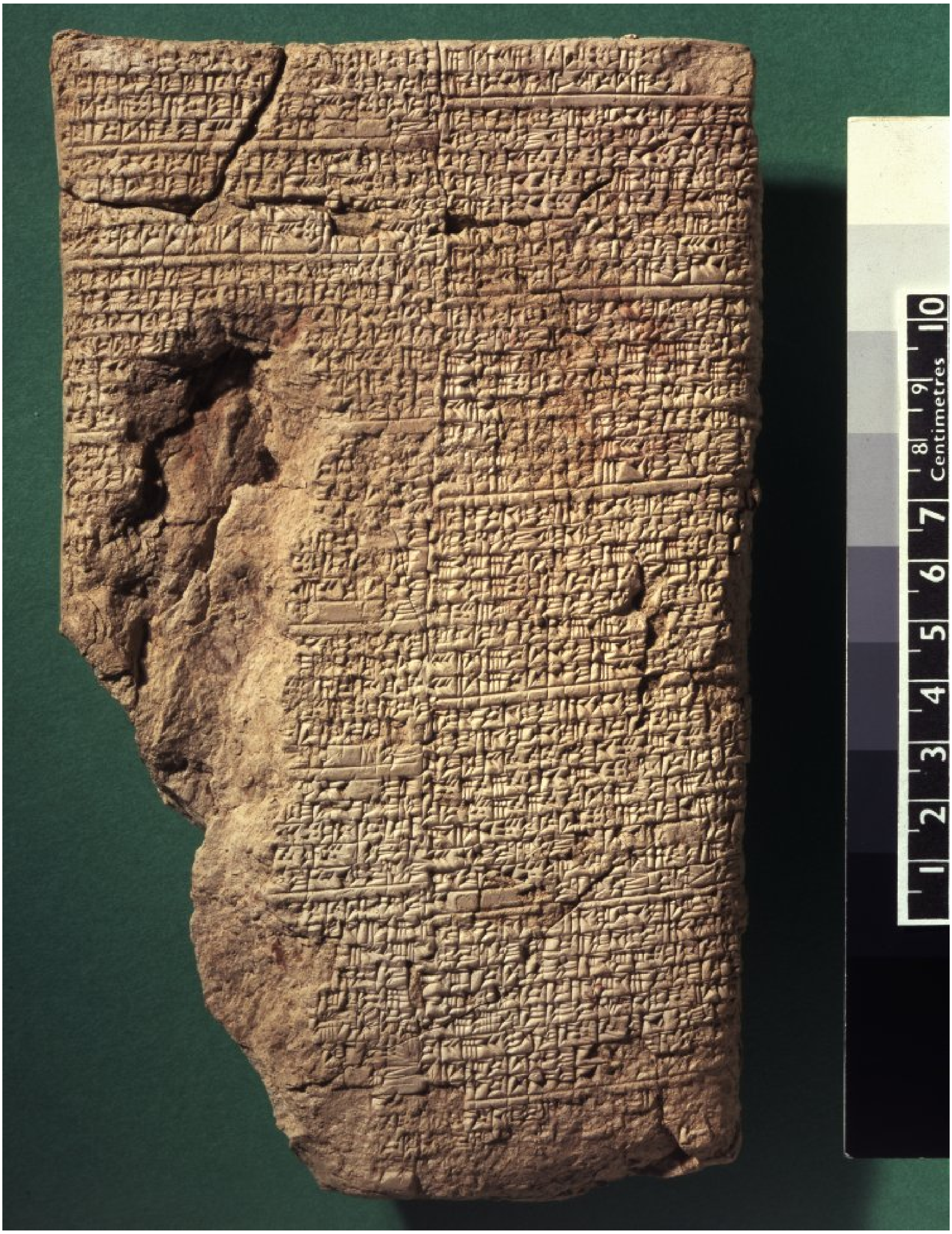

Figure 2: Tablet BM 13901; image from the British Museum.

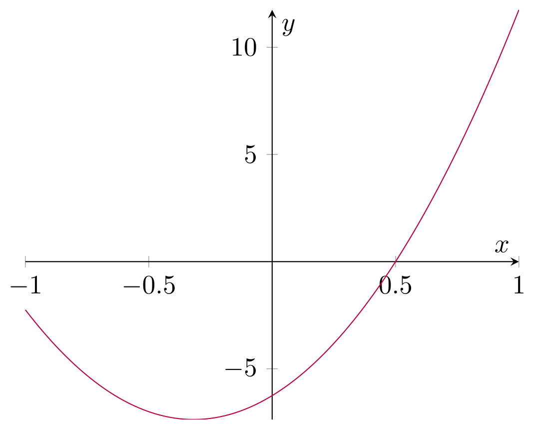

Figure 3: The graph of $y = 11x^2 + 7x - 6.25$.

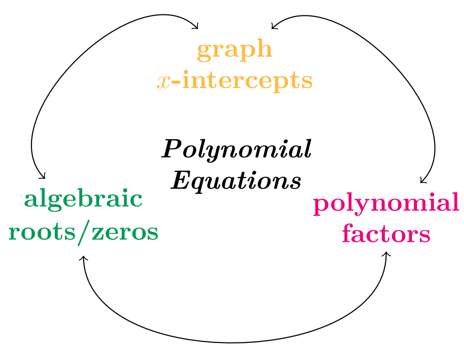

Figure 4: Relationships between concepts.

⋆4. Prove that $ax^2 + bx + c = Ax^2 + Bx + C$ implies that $a = A$, $b = B$, and $c = C$. This result allows us to *match coefficients*, an important tool which we can use to reason about the symmetries of polynomials.

5. Show that if α and β are the two solutions of $x^2 + bx + c = 0$, then we have $-b = \alpha + \beta$ and $c = \alpha\beta$.

6. Factorise $x^2 - 3x - 40$ by inspecting the coefficients and using the identity that a quadratic with the two solutions a and b is given by $(x - a)(x - b) = x^2 - (a + b)x + ab$ (note the change of sign in the factors).

7. For which values of k does the graph of the quadratic function $y = x^2 + (3k - 1)x + (2k + 10)$ not touch the x-axis?

8. Do the zeroes of a function uniquely identify that function? Why/why not?

⋆9. Solve the following equations in the real numbers: (a) $w^4 + 30w^2 + 29 = 0$, and (b) $3e^{2x} - 24e^x - 8 = 0$.

10. Write each of the following in the form $(x + p)^2 = q$ for some p and q, and hence find their solutions by completing the square.

   (a) $x^2 - 3x + 4 = 0$
   (b) $x^2 - 6x - 10 = 0$
   (c) $x^2 - 26x + 47 = 0$
   (d) $6x^2 - 12x + 13 = 0$
   (e) $-2x^2 + 3x + 5 = 0$

11. Suppose that $x^2 + bx + c = 0$ has two roots, α and β.

   (a) Show that $\alpha^2 + \beta^2 = b^2 - 2c$.
   (b) Show that $\Delta_2 \left[x^2 + bx + c\right] = (\alpha - \beta)^2$.

⋆12. Flesh out the following alternative proof of the quadratic formula from [6]. Let α and β be the two roots of the equation $x^2 + bx + c = 0$.

   (a) Then $x = \frac{1}{2}\left((\alpha + \beta) + (\alpha - \beta)\right) = \frac{1}{2}\left((\alpha + \beta) + \sqrt{(\alpha - \beta)^2}\right)$.
   (b) Note that $\sqrt{(\alpha - \beta)^2}$ has two values and show that taking the negative value still gives a root.
   (c) But $\alpha + \beta = -b$ and $(\alpha - \beta)^2 = b^2 - 4c$.
   (d) So $x = \frac{1}{2}\left(-b \pm \sqrt{b^2 - 4c}\right)$, which is the quadratic formula.

Section 3: Higher-degree Polynomials

Before we go any further, we must define the notion of a polynomial. A formal definition allows us to reason exactly about a mathematical object with more precision than our intuition.

Definition (Polynomial)**.** A *polynomial* is an expression that only has constant terms and terms involving powers of some variable. All polynomials can be written in the form $a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$, where x is the variable and $a_{n\ldots0}$ are the coefficients (and $a_n \neq 0$).

The value n is called the *degree* of the polynomial — i.e. the degree of a quadratic equation is 2. Given a polynomial p, we can write its degree as ∂p.

Factorising and Division

We can find solutions to general polynomial equations $p(x) = 0$ by factorising them and then finding the zeros of each factor. This is the reverse of the process outlined in the previous section for creating polynomials with an arbitrary number of solutions.

Solving a polynomial equation is equivalent to finding its factors.

For example, if we take the equation $x^2 + 3x + 2$ then we can see that two factors of this are $(x + 1)$ and $(x + 2)$ (because $x^2 + 3x + 2 = (x + 1)(x + 2)$). The two solutions are therefore $x = -1$ and $x = -2$.

Formally, we say that a polynomial $g(x)$ is a factor of a polynomial $p(x)$ if there is some polynomial $q(x)$ (the quotient) such that $p(x) = g(x) \cdot q(x)$. In this case, we say that $p(x)$ divided by $g(x)$ is $q(x)$. Even if $g(x)$ is not a factor of $p(x)$, it is possible to write that $p(x) = g(x) \cdot q(x) + r(x)$ for some remainder polynomial r if the degree of $g(x)$ is less than or equal to the degree of $p(x)$. This is similar to the division of integers: $11 = 2 \times 5 + 1$ where the factor is 2, the quotient is 5, and the remainder is 1.

Higher-degree polynomials are much harder to factorise and solve. However, once we find one factor we can divide it out to obtain a simpler polynomial. For example, take the cubic $x^3 + 2x^2 - x - 2 = 0$. We can see that one solution is $x = 1$ (it is usually a good idea to try simple solutions like 1 and 0 first), and therefore one factor of the equation is $(x - 1)$. If we divide out the cubic by this factor, we obtain that $x^3 + 2x^2 - x - 2 = (x - 1)(x^2 + 3x + 2)$, and we can solve the quadratic easily! The three solutions to this cubic are therefore $x = 1$, $x = -1$, and $x = -2$. Figure 5 shows both the cubic (in blue) and its factors (in purple) to illustrate this idea.

If we know that the solutions to a polynomial are integers, we need only try the factors of the constant term as this term is simply the product of the roots of the polynomial.

One algorithm to divide one polynomial by another is long division, such as the following calculation which provides us with an example of the division of a polynomial by another that is not a factor.

$$
\begin{array}{r|l}
 & \quad\;\; x^2 \qquad\; + 1 \\
\hline
x^2 - 1 & \;\; x^4 \qquad\; + x - 3 \\
 & - x^4 + x^2 \\
\cline{2-2}
 & \qquad\;\; x^2 + x - 3 \\
 & \qquad - x^2 \qquad + 1 \\
\cline{2-2}
 & \qquad\qquad\quad x - 2
\end{array}
$$

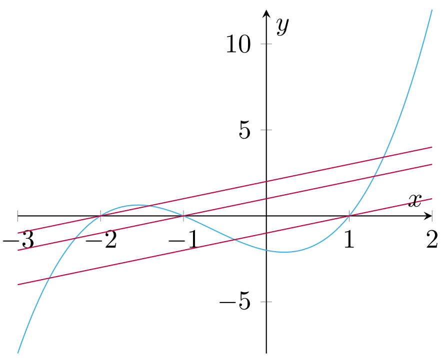

Figure 5: The solutions to an equation are exactly the solutions of its factors.

Here, we have divided $p(x) = x^4 + x - 3$ by $g(x) = x^2 - 1$ to obtain the quotient $q(x) = x^2 + 1$ and the remainder $r(x) = x - 2$ — i.e. we can write $x^4 + x - 3 = (x^2 - 1)(x^2 + 1) + x - 2$.

The Remainder Theorem

One important theorem that follows from the idea of polynomial division is the remainder theorem.

Theorem (Remainder Theorem)**.** *If we can write $p(x) = (x - a)q(x) + r$, then $p(a) = r$.*

Here we are dividing a polynomial $p(x)$ by some other linear equation $(x - a)$, to get a quotient $q(x)$ and a remainder r (which is just a number, not a polynomial, in this case). It follows that the remainder is just the value of $p(x)$ evaluated at a.

Proof. If we substitute a into the statement above, it simplifies and we obtain that $p(a) = (a-a)q(a)+r = 0 \cdot q(a) + r = r$. □

An important consequence of this theorem is that if a is a solution to the polynomial, then $r = p(a) = 0$ and so $(x - a)$ is a factor. As we expect, the solutions to an equation are the same as the solutions to the factors of the equation.

If we can divide out a polynomial multiple times by a factor, we call the root(s) corresponding to that factor 'repeated.' The *multiplicity* of some root α of $p(x) = 0$ is the number of times that the factor $(x - \alpha)$ appears in $p(x)$: i.e. the highest number n such that the division of $p(x)$ by $(x - \alpha)^n$ gives a remainder of zero. The multiplicity, in some sense, quantifies the number of times that a root is repeated.

A second application of the remainder theorem is less obvious — we can use it to evaluate polynomials. For example, we can use the remainder theorem to find $f(4)$ if $f(x) = 4x^{17} - 4x^{16} + 3x^4 - 6x + 12$ by dividing $f(x)$ by $(x - 4)$ and taking the remainder.

Exercises

1. Find three different polynomials with variable x that have the two roots $x = 2$ and $x = 3$.
2. Show that $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ divided by $x^3 + 7$ gives a quotient of $x^3 + x^2 + x - 6$ and a remainder of $-6x^2 - 6x - 41$ by expanding and simplifying $(x^3+7)(x^3+x^2+x-6)+(-6x^2-6x-41)$.
3. Divide, finding the quotient and remainder polynomials: (a) $x^2 - 4$ by $x - 2$, (b) $x^2 - 4$ by $x - 3$, and (c) $t^7 - t^3 + 5$ by $t^3 + 7$.
4. If $x = 3$ is one zero of $x^3 - 3x^2 - 4x + 12$, find the other two.
5. Solve $x^3 - x^2 - 3x + 3 = 0$.
6. Find the roots of $x^4 - x^3 - 43x^2 + 85x - 42$.
7. How many distinct solutions does
$$(x^2 - 2x - 24)(x^2 + 5x) = (x^2 - 2x - 24)(4x + 12)$$
have?
8. Show that $t = 4$ is a zero of $t^4 - (6 + \sqrt{3})t^3 + 6\sqrt{3}t^2 + 32t - 32\sqrt{3}$.
9. Find the remainder after dividing $x^7 + 5x - 9$ by $(x - 6)$.
10. Find four polynomials $p_a(x)$, $p_b(x)$, $p_c(x)$, $p_d(x)$ with integer coefficients such that: (a) $p_a\left(\frac{1}{2}\right) = 0$; (b) $p_b\left(\frac{1}{2} + \frac{1}{2}\sqrt{3}\right) = 0$; (c) $p_c\left(2i - \sqrt{2}\right) = 0$; and (d) $p_d\left(\sqrt{i} + \frac{1}{\sqrt[3]{2}}\right) = 0$.

*11. If $x^2 + bx + c$ and $x^2 + dx + e$ have a common factor of $(x - p)$, show that $\frac{e-c}{b-d} = p$.

12. Let $p(x) = (x^2 - 25)^5$. One root of $p(x)$ is $x = 5$. What is the multiplicity of this root?
13. Is $(x + 3)$ a factor of $2x^3 + x^2 - 5x + 7$?
14. Use the remainder theorem to compute $f(3)$ for $f(x) = x^4 + x - 10$.

15. Show that if $\alpha \neq 0$ and β are roots of $x^n - x = 0$ (for $n > 1$), then α^{-1} and $\alpha\beta$ are also roots. Why does this not imply that $x^2 - x = 0$ and $x^3 - x = 0$ have at least four roots?

*16. Elliptic curves are a form of cubic; they are equations of the form $y^2 = x^3 + ax + b$.

 (a) Find the x-intercepts of $y^2 = x^3 - 2x$.

 (b) Find the z-intercepts of $y^2 = x^3 - \frac{4}{3}x - \frac{16}{27}$, given that $z = x - \frac{1}{3}$.

 (c) Consider an elliptic curve $\mathcal{E}$, and let P and Q be two rational points (i.e. points whose coordinates are rational) which are lying on the curve. Let $\mathcal{L}$ be the chord line uniquely determined by P and Q. Show that if $\mathcal{L}$ and $\mathcal{E}$ intersect at a third point R, then this third point is rational.

**17. The polynomial $x^3 + px - 1$ has three real non-zero roots, α, β, and γ.

 (a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ in terms of p, and hence show that p is negative.

 (b) Find the cubic polynomial with coefficients in terms of p with the roots α^2, β^2, and γ^2.

18. Take the general cubic, $at^3 + bt^2 + ct + d$. Show that the substitution $t = y - \frac{b}{3a}$ will give a cubic in y with no quadratic term (this is known as a Tschirnhaus substitution and is often the first step to create a general formula to solve the cubic).

19. Show that $\sqrt{2} + \sqrt{3} = \sqrt{5 + \sqrt{6}}$.

**20. Prove the following identity.[3]

$$\sqrt[3]{-18 + \sqrt{325}} + \sqrt[3]{-18 - \sqrt{325}} = -3$$

21. Let $w = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are rational. Find rational numbers p, q, r and s such that
$$w = p + q(\sqrt{2} + \sqrt{3}) + r(\sqrt{2} + \sqrt{3})^2 + s(\sqrt{2} + \sqrt{3})^3.$$

22. Show that there are no integers r and s such that $\sqrt{2} = \frac{r}{s}\sqrt{3}$.

[3] See chapter 1 of [10] for historical context.

Section 4: Complex Numbers

> [Mathematics consists of] true facts about imaginary objects.
>
> (Philip Davis and Reuben Hersh)

Now that we have somewhat developed the theory of polynomial equations, a sensible question to ask ourselves is the following.

When do polynomial equations have solutions?

The answer to this simple question is actually quite nuanced, and occupied mathematicians in Europe for centuries.

Consider, for example, the simple quadratic equation

$$x^2 - 2 = 0.$$

Notice that the coefficients of this equation are integers, but the solutions are not — in fact, the solutions ($\pm\sqrt{2}$) are not even rational! Notice also that the number of solutions is 2, the same as the degree of the polynomial.

Let us take a look at another example,

$$x^3 - x = 0.$$

Again, the degree of this polynomial is three — as is the number of solutions that we obtain from it.

These two examples, as well as our previous work, suggest that the number of solutions of a polynomial is the same as its degree. There is one problem with this, illustrated by another simple equation:

$$x^2 + 1 = 0.$$

If we try to solve this equation, we end up with a seemingly nonsense result: that $x = \pm\sqrt{-1}$. Since there is no real number with a negative square, it seems like we need to throw out our pretty result. However, remember that we have already seen an example of an equation where the solution is out of reach if we only look for solutions that are the 'same kind' as the coefficients.

Just as we extend the natural numbers to the integers, the integers to the rationals, and the rationals to the reals, we can extend our number system further so that the equation $x^2 + 1 = 0$ has a solution. We begin by defining i to be a square root of -1 (and setting $-i$ to be the other square root); then the set of all numbers of the form $a + bi$ (for any real numbers a and b) is called the set of *complex numbers*; this system of writing them is called *rectangular form*.

The complex numbers are generally considered to have been first used in Girolamo Cardano's *Ars Magna* in 1545, where he introduces them only to dismiss them as 'as subtle as they are useless'. Despite this, modern mathematics has fully accepted the existence of complex numbers for two main reasons: firstly they do not introduce any contradictions into basic arithmetic or any theory that we care about, and secondly they allow us to state the following historic theorem.

Theorem (Fundamental Theorem of Algebra)**.** *Let $p(x)$ be a polynomial with complex coefficients. Then, counting repeated roots, there are exactly ∂p complex roots of $p(x)$.*

The proof of the Fundamental Theorem unfortunately requires concepts and techniques far beyond the scope of this book. All proofs (that I am aware of) require the use of analysis (i.e. there is no pure-algebra proof).[4]

The first rigorous proof of the theorem was published by French mathematician Jean-Robert Argand in 1814; the name is somewhat incorrect in the modern era as the study of algebra is no longer purely devoted to the properties of the complex number field. An analogue of the theorem in a more modern context is Kronecker's Theorem, which states that a polynomial with coefficients in set of numbers in which we can add, subtract, multiply, and divide has at least one solution in a 'bigger' set of numbers that contains the original set.

Given a complex number $z = a + bi$, we call a the *real part* of z and b the *imaginary part* of z (writing $\mathfrak{Re}\,(z)$ and $\mathfrak{Im}\,(z)$ respectively). The term 'imaginary' is purely historical — the complex numbers exist in exactly the same way that a number like $\pi = 3.14159...$ or $e = 2.718...$ exists.

[4] Stewart gives a proof in [10], and Artin gives two(!) proofs in [1] §9.9.

We say that two complex numbers are equal **if and only if both the real and imaginary parts of those numbers are equal**, and we perform arithmetic on complex numbers in the same way that we perform arithmetic on the reals, remembering that $i^2 = -1$.

Complex numbers also have a geometric interpretation. If we set up a mapping $a + bi \mapsto (a, b)$ it is easy to see that we have an exact correspondence between complex numbers and points on a Cartesian plane. This kind of diagram is called an *Argand diagram*, named after French mathematician Jean-Robert Argand.

For convenience, we also often associate with each complex number the unique *vector* pointing from the origin to the location of the point on the Argand plane. Given a complex number $z = a+bi$, the length of its associated vector is called its *modulus*; this is written $|z|$, and (by the Pythagorean theorem) we have the simple result that $|z| = \sqrt{a^2 + b^2}$. The meaning of this is exactly the same as that of the absolute value of a real number (and in fact the modulus of a real number is exactly its absolute value) — despite this, note that while there is a natural ordering on the real numbers (for example, $-2 < 0 < \pi < \sqrt{17}$) there can be no such ordering on the complex numbers. It is completely nonsensical to talk about any complex number being 'bigger' than any other.

The five points marked on the Argand diagram in fig. 6 are i, 1, $3 + 2i$, $-1 - i$, and $2i - 1$. See if you can label each point on the diagram itself.

Addition and Subtraction

Due to the way we constructed the complex numbers, there are natural rules for arithmetic. Suppose we have two complex numbers, $u = a + bi$ and $v = c + di$. We have the following rules for addition and subtraction by collecting the real and imaginary terms together:

$$\begin{aligned} u + v &= (a + c) + (b + d)i, \text{ and} \\ u - v &= (a - c) + (b - d)i. \end{aligned}$$

Geometrically, we are simply taking the vectors associated with each point and adding them.

Before we discuss multiplication of complex numbers, we will discuss one more simple but important operation: that of complex conjugation. Our intent is to take the geometric property of reflection (across the real axis), and give it an algebraic meaning; in particular, the *complex conjugate* of $z = a + bi$ is $\overline{z} = a - bi$. An initial application of this concept is the simple result that if a polynomial has real coefficents, any complex roots must come in conjugate pairs (you are asked to prove this in the special case of quadratic polynomials as exercise 27, and in the general case as exercise 9.33).

Multiplication

There is a rule for multiplying complex numbers that is similar to the rules for addition; we simply collect like terms and remember that $i^2 = -1$ in order to obtain the result

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

This way of writing the result is a little esoteric and has no nice geometric meaning. In order to remedy this problem, we rewrite our complex numbers in *polar form.* Instead of writing the number in

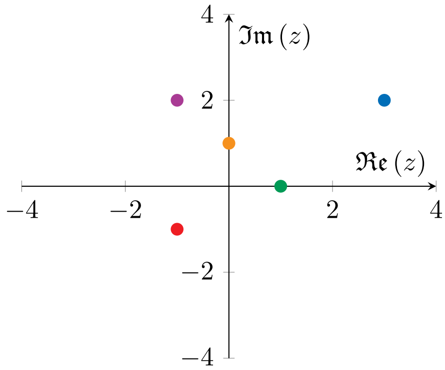

Figure 6: Five points on an Argand diagram.

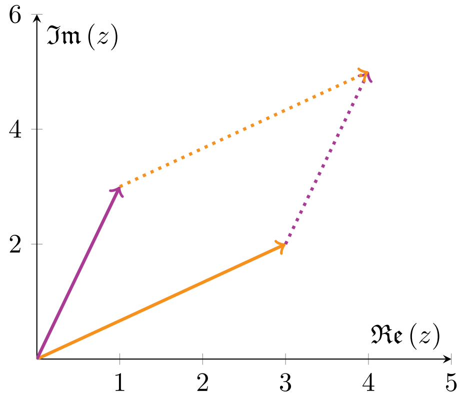

Figure 7: The addition of complex numbers.

the form $z = a + bi$, where a and b are the distances from the two axes, we write $z = r \operatorname{cis} \theta$, where r is the distance of the number from the origin (simply the modulus $|z|$ again) and θ is the angle that number's vector makes with the x-axis (known as the *argument* of z, and written as $\arg(z)$). The function $\operatorname{cis} \theta$ is defined to be $\cos \theta + i \sin \theta$.

It is simple to see that $\theta = \tan^{-1} \frac{b}{a}$; we have already got a formula for r.

This new notation simplifies multiplication significantly, because you will show as an exercise that

$$(r \operatorname{cis} \theta)(t \operatorname{cis} \varphi) = (rt) \operatorname{cis}(\theta + \varphi).$$

For example, if we were to multiply $3 \operatorname{cis} \pi$ by $4 \operatorname{cis} \pi$, we would obtain $12 \operatorname{cis} 2\pi = 12$. This is as expected, as $p \operatorname{cis} \pi = -p$, and $-3 \times -4 = 12$.

Note that any complex number has an infinite number of different representations in polar form, simply by adding 2π (an entire rotation around the origin) to its argument. This fact comes into play later on when we try to solve equations involving powers of complex numbers.

Exponentiation

Finally, we ask ourselves what a definition of complex exponentiation should be. There are two important behaviours of the real exponential function which we want to preserve: the sum-to-product rule ($\exp(x + y) = \exp(x)\exp(y)$) and the differentiation rule ($\exp'(x) = \exp(x)$).

If we define the complex exponential in such a way that these rules hold, then we will have $\exp(x+iy) = \exp(x)\exp(iy)$; in particular, we need only make a sensible definition for $\exp(iy)$. Further, we have already seen a function that satisfies the sum rule: $\operatorname{cis}(x + y) = \operatorname{cis}(x)\operatorname{cis}(y)$. Thus, we make a guess that a decent definition for our exponential function will be $\exp(iy) = \operatorname{cis}(y)$.

We now check that this definition satisfies the differentiation rule: we have not formally defined how differentiation should work over the complex numbers, but cis is only a function of real variables and so

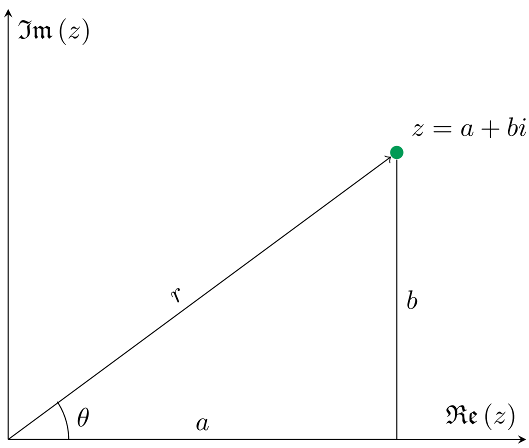

Figure 8: The polar form of a complex number.

we can differentiate it using our normal definitions:

$$\frac{\mathrm{d}}{\mathrm{d}y}[\cos y + i \sin y] = -\sin y + i \cos y = i(i \sin y + \cos y) = i \operatorname{cis} y.$$

Thus, if we make this definition, then $\exp(iy) = i \exp(iy)$ which is the 'usual' rule for exponent differentiation.

We therefore make the definition formally, known as *Euler's formula*, that

$$\operatorname{cis} \theta = \exp(i\theta)$$

and thus

$$\exp(x + iy) = e^x \operatorname{cis} y;$$

we will also allow ourselves to use the standard notation, $\exp(z) = e^z$.

Philosophically, we make the note that complex exponentiation is very elegant, in the sense that it is simply a function which transforms numbers from rectangular form into polar form by 'wrapping them around the origin'!

From Euler's formula we obtain the following equation, which relates five fundamental mathematical constants in one expression and which some have called the most beautiful equation in mathematics:

$$e^{i\pi} + 1 = 0. \qquad \text{(Euler's Identity)}$$

Exercises

1. Evaluate the following expressions, and plot the answers on an Argand diagram:
   (a) $(3 + 2i) + (6 - 2i)$
   (b) $24 - (6 + 2i)$
   (c) $2(2 + i) + 6i - 7$
2. If we add two real numbers, can we obtain an imaginary number? If we add two imaginary numbers, can we obtain a real number?
3. Let $v = 3 - 7i$ and $w = -4 + 6i$.
   (a) Find the real numbers p and q such that $pv + qw = 6.5 - 11i$.
   (b) Show that any complex number z can be written as $z = pv + qw$ for some real p and q.
4. Solve the quadratic equation $x^2 + 4 = 0$.
5. Prove that $z + \overline{z} = 2 \cdot \mathfrak{Re}\,(z)$ and $z - \overline{z} = 2i \cdot \mathfrak{Im}\,(z)$.
6. Verify the following properties of conjugation.
   (a) $\overline{\overline{z}} = z$

Figure 9: From `http://brownsharpie.courtneygibbons.org/?p=848`.

(b) $\overline{w} + \overline{z} = \overline{w + z}$

(c) $\overline{w}\overline{z} = \overline{wz}$

7. Find i^{957}.

8. Show that $|a + bi| \geq |a|$ and $|a + bi| \geq |b|$.

9. Find $(3 + 2i)(6 + 8i)$ in rectangular form.

10. (a) Convert $1 + i$ into polar form.

    (b) Find $(1 + i)(\sqrt{2} \operatorname{cis} \frac{3\pi}{4})$ in both polar form and rectangular form.

11. Compute $(6 \operatorname{cis} \frac{23\pi}{24})(9 \operatorname{cis} \frac{14\pi}{17})$, leaving your answer in polar form.

12. (a) Prove that $(r \operatorname{cis} \theta)(t \operatorname{cis} \varphi) = (rt) \operatorname{cis}(\theta + \varphi)$.

    (b) Describe the geometric meaning of the multiplication of complex numbers.

13. Let $u = 2 \operatorname{cis} \frac{\pi}{2}$ and $v = 3 \operatorname{cis} \frac{3\pi}{2}$. Plot u, v, and uv on an Argand diagram.

14. Prove **de Moivre's Theorem**: $(r \operatorname{cis} \theta)^n = (r^n) \operatorname{cis}(n\theta)$.

15. Show that if $u = r \operatorname{cis} \theta$ and $v = t \operatorname{cis} \varphi$ then $\frac{u}{v} = \frac{r}{t} \operatorname{cis}(\theta - \varphi)$.

16. Using de Moirve's Theorem, prove that for complex numbers w and m and integers n and m, (a) $w^n w^m = w^{n+m}$, and (b) $(w^n)^m = w^{nm}$.

17. Convert $w = 1 + \sqrt{3}i$ into polar form, and calculate w^3.

18. Show that for any complex number z, the product $z\overline{z}$ is both real and non-negative. Hence show that $(x - z)(x - \overline{z})$ has only real coeffients.

19. Let x be a real number. Show that, for all integers n, $\operatorname{cis} \frac{2x\pi}{n} = \operatorname{cis} \frac{2(x+n)\pi}{n}$.

20. For which complex numbers is z^2 real? What about z^3?

21. Transform $\frac{a+bi}{c+di}$ so that the only imaginary part is in the numerator.

22. Find $(a + bi)^{-1}$ in rectangular form.

23. Write the complex number $\left(\frac{4i^7-i}{1+2i}\right)^2$ in the form $a + bi$, where a and b are real numbers.

24. (a) Prove that a number z is real if and only if $\overline{z} = z$.

    (b) Hence, or otherwise, show that $z\overline{w} + w\overline{z}$ is always real.

    (c) Show that $z\overline{w} + w\overline{z} \leq 2|w||z|$.

25. Show that if $z = a + ib$ then $\sqrt{z\overline{z}} = |z|$.

⋆26. If $\zeta = \sqrt{\frac{1}{2}(a + \sqrt{a^2 + b^2})} + i\sqrt{\frac{1}{2}(-a + \sqrt{a^2 + b^2})}$ is a complex number, find ζ^2 in the form $p + iq$ (where a, b, p, and q are real).

⋆27. If $z = x + iy$, and $az^2 + bz + c = 0$, show that $a\overline{z}^2 + b\overline{z} + c = 0$ if a, b, and c are real. (This exercise is generalised in 9.33.)

⋆28. Use Euler's identity to find $\ln(-1)$, and hence $\ln(-x)$ for real x.

⋆29. Prove that for every positive integer n, $(-1 + \sqrt{3}i)^{3n} + (-1 - \sqrt{3}i)^{3n} = 2^{3n+1}$.

30. Show that $y_1(x) = e^{ix} + e^{-ix}$ and $y_2(x) = 2\cos x$ are both solutions of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0$$

with initial conditions $y(0) = 2$ and $y'(0) = 0$. (Also see 32 below.)

31. Find $\sqrt{i}$ in rectangular form.

32. (a) Show that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and that $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.
    (b) If $x + x^{-1} = 2\cos\theta$, find $x^n + x^{-n}$ in terms of n and θ.

33. (a) Show that $(2 \pm i)^3 = 2 \pm 11i$.
    (b) Simplify fully $\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$.
    (c) Show that (b) is a root of the cubic equation $t^3 - 15t - 4 = 0$, and hence find all three solutions.

34. You do not need the fundamental theorem of algebra for this exercise.
    (a) Prove that all cubic equations with real coefficients must have exactly three roots in the complex numbers.
    (b) Let $p(x)$ be a polynomial of degree n such that there exists some complex number ζ such that $p(\zeta) = 0$. Show that $p(x) = 0$ has exactly n solutions (counting repeated roots).

Section 5: Geometry

> Algebra is the offer made by the devil to the mathematician... All you need to do, is give me your soul: give up geometry. (Michael Atiyah)

We have already seen that we can view the complex numbers as points in the plane, $\mathbb{R}^2$. This means that we can carry out geometric operations algebraically using complex numbers. For example, the length of the line joining two points z_1 and z_2 is simply $|z_1 - z_2|$.

We define the *locus* of an equation to be the set of all points satisfying that equation; for example, the locus of $x^2 + y^2 = 1$ is the set of all points making up the unit circle. We can use our correspondence between points on a plane and complex numbers to reason about the locus of a complex equation; for example, the locus of the equation $|z| = 1$ is the set of all points at a unit distance from the origin: it is another way of talking about the unit circle. (If you write $z = x + iy$, then $|z| = x^2 + y^2$ and the comparison becomes even more obvious.)

More interestingly, consider the set of all points z such that for some real number t and for a pair of fixed points z_1 and z_2, $z = tz_1 + (1 - t)z_2$. The locus of this set is the line joining z_1 and z_2. To prove this, write $z = x + iy$, $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$; then the equation tells us that

$$x + yi = tx_1 + ity_1 + x_2 + iy_2 - tx_2 - ity_2$$
$$x = t(x_1 - x_2) + x_2, \qquad y = t(y_1 - y_2) + y_2$$

From the second line, we see that $t = \frac{x - x_2}{x_1 - x_2}$ and thus

$$y = \frac{x - x_2}{x_1 - x_2}(y_1 - y_2) + y_2.$$

Writing $m = \frac{y_1 - y_2}{x_1 - x_2}$, we see that we have the equation of a line passing through (x_2, y_2) with slope m.

Now, we will consider two non-zero complex numbers w and z; we will call the two points *orthogonal* if the lines passing from w and z to the origin meet at right angles. The claim is that the two points are orthogonal precisely when $\mathfrak{Re}\,(w)\,\mathfrak{Re}\,(z) + \mathfrak{Im}\,(w)\,\mathfrak{Im}\,(z) = 0$. Suppose that the two points are orthogonal; then, by Pythagoras' theorem, we have that $|w - z|^2 = |w|^2 + |z|^2$. Expanding this, we obtain

$$(w - z)(\overline{w} - \overline{z}) = w\overline{w} + z\overline{z}$$
$$z\overline{w} + w\overline{z} = 0.$$

But $z\overline{w} + w\overline{z} = z\overline{w} + \overline{z\overline{w}}$, and hence (applying exercise 4.5) we have $2\mathfrak{Re}\,(z\overline{w}) = 0$. But $\mathfrak{Re}\,(z\overline{w}) = \mathfrak{Re}\,(z)\,\mathfrak{Re}\,(w) + \mathfrak{Im}\,(z)\,\mathfrak{Im}\,(w)$, and so the condition claimed holds.

On the other hand, suppose $\mathfrak{Re}\,(z)\,\mathfrak{Re}\,(w) + \mathfrak{Im}\,(z)\,\mathfrak{Im}\,(w) = 0$. We form the triangle with sides $|w|$, $|z|$, and $|w - z|$ with an angle θ at the origin. By the cosine rule, $|w - z|^2 = |w|^2 + |z|^2 - 2|w||z|\cos\theta$. Using the same argument as above, we have $z\overline{w} + w\overline{z} = -2|w||z|\cos\theta$; but the left hand side is zero, and thus (since $|w|$ and $|z|$ are non-zero) we have $\cos\theta = 0$ and $\theta = \pm\pi/2$ as claimed.

We call the expression $\mathfrak{Re}\,(w)\,\mathfrak{Re}\,(z) + \mathfrak{Im}\,(w)\,\mathfrak{Im}\,(z)$ the *dot product* of w and z; we write it (w, z). One important property of the dot product is that $(u + v, w) = (u, w) + (v, w)$.

The dot product is intimately connected with the geometry of the complex plane. As well as our result about orthogonality above, we have two more properties:

1. The modulus of z is $|z| = \sqrt{(z, z)}$.
2. If θ is the angle at the origin between w and z (i.e. $\theta = \arg w - \arg z$), then $\cos\theta = \frac{(w,z)}{|w\|z|}$.

Suppose we have two nonzero complex numbers, w_0 and z_0. We want to find the closest point z on the line generated by z_0 (i.e. the line passing through z_0 and the origin) to the point w_0. This point is called the *projection* of w_0 onto z_0, and we denote it by $z = \mathrm{proj}_{z_0}(w_0)$.

The calculation of the projection is based on the dot product, and proceeds as follows. Consider $h = w_0 - tz_0$, where t is some real number. Then $(h, z_0) = (w_0, z_0) - (tz_0, z_0) = (w_0, z_0) - t|z_0|^2$; so h is perpendicular to tz_0 precisely when $t = \frac{(w_0,z_0)}{|z_0|^2} = \frac{(w_0,z_0)}{(z_0,z_0)}$. It follows that

$$\mathrm{proj}_{z_0}(w_0) = z_0\frac{(w_0, z_0)}{(z_0, z_0)}.$$

This result tells us that the geometric meaning of the dot product (a, b) is just the length of the 'shadow' cast onto a by b, multiplied by the squared length of a. But the dot product is symmetric, so this is also the length of the 'shadow' cast onto b by a, multiplied by the squared length of b. If both a and b have length 1, then this length is just the cosine of the angle between them. (See fig. 10.)

As a bonus, if we are given any fixed complex number z_0 then we can write every vector w_0 in the plane in the form $w_0 = tz_0 + z_0^\perp$ where z_0 and $z_0^\perp$ are perpendicular: just take $t = \frac{(w_0,z_0)}{(z_0,z_0)}$ and $z_0^\perp = w_0 - tz_0$.

Another important geometric property of the complex numbers is the triangle inequality; if a and b are complex numbers, then $|a + b| \leq |a| + |b|$. Geometrically, this just states that if two sides of a triangle have lengths $|a|$ and $|b|$ then the third side has length at most $|a| + |b|$ (fig. 11).

Exercises

1. Show that, if a and b are fixed complex numbers, then $|z - a| = |z - b|$ describes the line which cuts the midpoint of the segment between a and b at a right angle.

2. Let $v = 1 + i$ and $z = x + iy$ for any real numbers x and y.

   (a) Show that the equation $|z - v| = |vz|$ represents a circle, and find its centre and radius.

   (b) Find the point of intersection of the circle with the straight line $|z - v| = |z + v|$.

3. Find the locus of all z such that for some fixed a, $z - a$ is perpendicular to $z + a$.

4. Show that if u, v, and w are complex numbers, and if λ and μ are real numbers, then:

   (a) $(u, v) = (v, u)$

   (b) $(\lambda u + \mu v, w) = \lambda(u, w) + \mu(v, w)$

⋆5. Show that if b is a complex number and a and c are real numbers, then the locus of all complex numbers z satisfying $a(z, z) + (b, z) + c = 0$ is a circle. (Hint: if z was real, you would complete the square.)

6. Let X be a set of complex numbers. If $\rho \neq 0$ and t are complex numbers, define $\rho X + t$ to be the set of all complex numbers of the form $\rho x + t$ for some x in the set X. Show that (a) if X is a line, then $\rho X + t$ is a line; (b) if X is a circle, then $\rho X + t$ is a circle.

7. Let p be a positive real number, and let Γ be the locus of points z satisfying $|z - p| = c\mathfrak{Re}\,(z)$. Show that Γ is:

   (a) an ellipse if $0 < c < 1$;

   (b) a parabola if $c = 1$;

   (c) a hyperbola if $c > 1$.

8. A set of complex numbers is called *convex* if for every pair of complex numbers in the set the line segment joining them is also in the set.

   (a) Show that the set of all z where $|z| < 1$ is convex.

   (b) Show that the set of all z where $0 < |z| < 1$ is not convex.

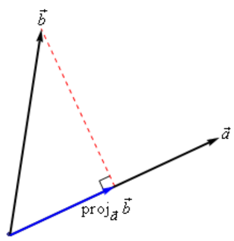

Figure 10: The projection of b onto a. From `http://tutorial.math.lamar.edu/Classes/CalcII/DotProduct.aspx`.

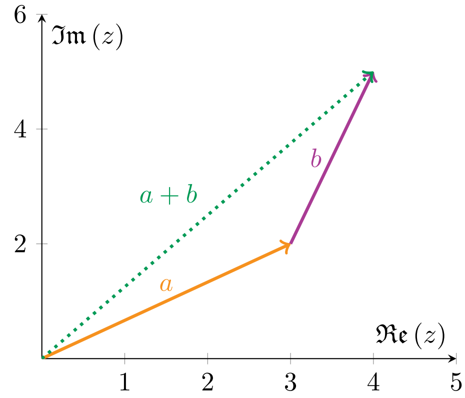

Figure 11: The triangle inequality.

9. A set of complex numbers is called *star shaped* if there is some point c in the set such that for every other point p in the set, the segment joining p to c lies in the set. Show that every convex set is star shaped.

**10. If S is a set of finitely many complex numbers $z_1, ..., z_n$, the *convex hull* of S is the smallest convex set containing every point of S. Show that a point w is in the convex hull of S precisely when there exist n non-negative real numbers, $\lambda_1, ..., \lambda_n$, such that $\lambda_1 + \cdots + \lambda_n = 1$, and

$$w = \lambda_1 z_1 + \cdots \lambda_n z_n.$$

The points w are called *convex combinations* of the z_i.

*11. The following inequality, which holds for all real numbers a_1, a_2, ..., a_n, b_1, ..., b_n, is known as the Cauchy-Schwarz inequality.

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \qquad \text{(Cauchy-Schwarz)}$$

The inequality is a very useful result in analysis; there are several different elementary proofs of it.

(a) Show that if $a > 0$, then $ax^2 + bx + c \geq 0$ for all x if and only if $b^2 - 4ac \leq 0$.

(b) Prove the Cauchy-Schwarz inequality by considering the expression $(a_1x+b_1)^2+\cdots+(a_nx+b_n)^2$, collecting terms, and applying (a).

(c) As a special case of the inequality, show that if w and z are complex numbers then $(w, z)^2 \leq |w|^2|z|^2$.

(d) Hence show the *triangle inequality*: $|a + b| \leq |a| + |b|$ for complex numbers a and b.

12. Multiplication by i rotates a point by $\frac{\pi}{2}$ around the origin.

(a) A generalisation of this allows us to rotate a point z around an arbitrary point a by that angle: $z' = a + i(z - a)$. Justify this formula.

(b) Consider a treasure map with the following instructions:

> From the statue of Richard Seddon, go to the kauri tree (counting your steps), and then turn exactly 90°left and walk the same number of steps to the point g'. Returning to the statue, walk to the beech tree (again counting your steps). Turn right by 90°, and walk the same number of steps to point g''. The treasure is buried exactly at the midpoint of the line joining g' and g''.

Given that the kauri tree is at $(0, 0)$, the beech tree is at $(10, 0)$, and the statue is somewhere on the line $y = 2009$, find the location of the treasure.

*13. A line in $\mathbb{C}^2$ (the plane with complex coordinates) is defined to be the locus of a linear equation $ax + by + c = 0$ where a, b, and c are complex constants. Prove that, given two distinct points (x_0, y_0) and (x_1, y_1) in $\mathbb{C}^2$, there is a **unique** line through those two points. *Hint: it is certainly **not** true that there is a unique linear equation whose graph includes both points.*

*14. Investigate the locus of $\left\{z = r\,\mathrm{cis}\,\theta : r = \cos\left(\frac{n}{d}\theta\right)\right\}$ for different values of n and d.

Section 6: Roots of Unity

Let us take the equation $z^3 = 1$. We know that this equation has exactly three complex roots, and of these we already know that the only real root is $z = 1$. How can we find the two non-real roots?

Noting that $1 = 1 \operatorname{cis} 2k\pi$ for all natural numbers k, we can apply de Moivre's Theorem to show that $1^{\left(\frac{1}{3}\right)} = \operatorname{cis} \frac{2k\pi}{3}$.

We can then set n to 0, 1, and 2 to obtain our three roots of the original equation: $z = 1$, $z = \operatorname{cis} \frac{2\pi}{3}$, and $z = \operatorname{cis} \frac{4\pi}{3}$ respectively. Note that if we set k to any higher number (3, for example) we obtain one of the roots we already have (e.g. $\operatorname{cis} \frac{6\pi}{3} = 1$), since we have gone 'around the circle'.

If we look at the roots geometrically (on an Argand diagram like that in fig. 12), we see that the nth roots of 1 will be arranged in a circle of radius 1 centred on the origin, and the angles between them will be exactly $\frac{2\pi}{n}$. If n is even, both 1 and -1 will both be real roots, but if n is odd then 1 will be the only real root. In general, the sum of all n nth roots of unity is zero (see exercise 6).

Any integer power of an nth root of unity is also an nth root of unity. However, not all nth roots will 'generate' all the other ones in this way; for example, $\operatorname{cis} \frac{2\pi}{3}$ is a sixth root of unity but will miss out every second sixth root if we raise it to integer powers. Roots which *do* generate all the others are called *primitive roots of unity.*

We formally define a primitive nth root of unity to be a complex number ω such that $\omega^n = 1$ but $\omega^k \neq 1$ for all $k < n$. In other words, if we list all the roots of unity in order for $n = 1,\ 2,\ 3$ and so on, then a root is only primitive for that value of n for which it first appears.

For example, $\mu = \operatorname{cis} \frac{2\pi}{3}$ is a primitive third root of unity (since the smallest nonzero n such that $\mu^n = 1$ is $n = 3$).

We now prove that, as we claimed, the primitive nth roots of unity 'generate' all the other nth roots of unity:

Theorem. *Given the polynomial $z^n = 1$, with primitive root ω, all solutions are given by ω^k for $0 \leq k \leq (n-1)$. In other words, the integer powers of a primitive nth root of unity must be* all *the nth roots of unity.*

Proof. We first prove that all the powers ω^k for k defined above are distinct. Suppose that there are two values for k, say $k = a$ and $k = b$, such that $\omega^a = \omega^b$ and $a \neq b$. Then we have that $\omega^{a-b} = 1$ and therefore $a - b = 0$ (the other possibility here would be that $a - b$ were equal to some non-zero multiple of n, none of which are possible values of k) so $a = b$.

Since there are n possible values for k, there are n distinct powers of ω which are all roots of the polynomial. However, the polynomial is of degree n and so has exactly n roots — thus, the powers of ω are exactly the roots of the polynomial. $\square$

Roots of unity allow us to find all the nth roots of any number easily. Suppose $a^n = x$; then the nth roots of x will be $a = \omega^k \sqrt[n]{x}$ $(0 \leq k < n)$ where ω is a primitive nth root of unity and $\sqrt[n]{x}$ is any nth root of x.

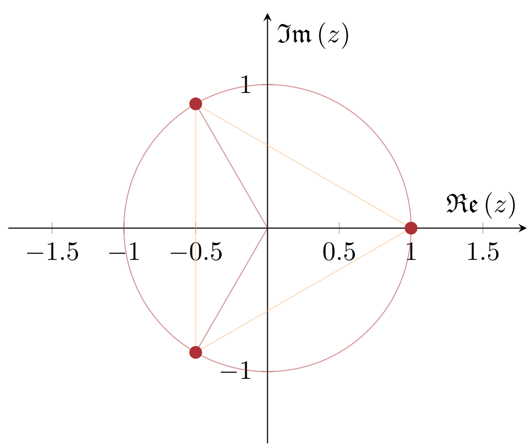

Figure 12: The cube roots of 1.

Exercises

For the exercises marked †, it may be useful to use the result that for all integers a and b there exist integers m and n such that $am + bn = \gcd(a, b)$ (where the gcd of two numbers is the largest integer that divides into both of them). Two integers a and b are coprime if they share no divisors (i.e. if $\gcd(a, b) = 1$).

1. Let $p(z) = z^5 - 1$.
   (a) Find exactly each of the roots of $p(z)$.
   (b) Let α be the root of p with the smallest non-zero positive argument. Show explicitly that the roots can be written as 1, α, α^2, α^3, and α^4.
2. Find all solutions of $z^3 + n = 0$, where n is a positive real number, in exact form in terms of n.
3. Solve for z if $(z - 3)^7 = 1$.
4. Find the fifth roots of $4 + 4i$ in polar form, and draw them on an Argand diagram. Hence find integers p and q such that $(p + qi)^5 = (4 + 4i)$.
5. Write down all of the primitive sixth roots of unity. What about the primitive fifth roots of unity?

*6. (a) Let α be a complex root of $x^3 = 1$. Show by computation that $\alpha^2 + \alpha + 1 = 0$.
   (b) In general, prove that the sum of all n nth roots of unity is zero (for $n > 1$).

*7. Find the product of all n nth roots of unity.

*8. Solve $(z + 1)^3 = 8$ for z and show that the sum of the solutions is -3.

*9. Given that $a = b + kn$ for some integer k, show that $z^a = z^b$ where z is a primitive nth root of unity.

10. Prove that the product of an ath root of unity by a bth root of unity is an abth root of unity.

†*11. Prove the following: Let a and b be coprime integers. Then all the abth roots of unity can be obtained as products of ath roots of unity and bth roots of unity.

*12. The theorem stated in this section requires ω to be a **primitive** nth root of unity in order for all the nth roots of unity to be powers of ω. Why do we need this restriction?

13. Prove the following: Let a and b be coprime integers. Then $x^a - 1 = 0$ and $x^b - 1 = 0$ have only the trivial root $x = 1$ in common.
14. (a) Prove the converse of the theorem in the text: i.e. show that if ζ generates all the kth roots of unity then it is a primitive kth root of unity.
    (b) Let ζ be the root of $p(x) = x^k - 1$ with smallest positive argument. Show that ζ is a primitive kth root of unity.

†15. Let α be the kth root of unity with smallest positive argument. Show that the primitive kth roots of unity are precisely α^n where $0 < n < k$ and $\gcd(n, k) = 1$.

16. The fifth-degree polynomial $p(x)$, where $p(k) = 0$, has as its roots the vertices of a regular pentagon centred around $(\frac{1}{2}k, 0)$. Give $p(x)$ such that all coefficients are real.

**17. Show that all solutions to $(z + 1)^n = z^n$ lie on the line $\mathfrak{Re}\,(z) = -\frac{1}{2}$.

18. Find all the third roots of 2.
19. A *group* is a set G together with some operation $\cdot$ satisfying the following:
    (a) For all a, b in G, $a \cdot b$ is in G.
    (b) For all a, b, c in G, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
    (c) There is some element e in G such that for all a in G, $a \cdot e = a$.
    (d) For every element a in G there is some b in G such that $a \cdot b = e$.

    Show that the set of all nth roots of unity form a group under multiplication.
20. Suppose $\sigma(n)$ is the function that sends n to the sum of its divisors. For example, the divisors of 4 are 1, 2, and 4; so $\sigma(4) = 1 + 2 + 4 = 7$. Prove that if p is prime then $\sigma(p^n) = \frac{p^{n+1}-1}{p-1}$.

Section 7: The Double-Triangle Problem

As an application of our work on primitive roots, we solve the *double-triangle problem* which appeared in the 2009 New Zealand Scholarship examination.

(b) Six points are shown in the Argand diagram in Figure 2. They are the roots of $p(x)$, a degree 6 polynomial with real coefficients.

The points lie on two concentric circles centred at the origin, and are the vertices of equilateral triangles, as shown in the figure.

The positive real root of $p(x)$ is k.

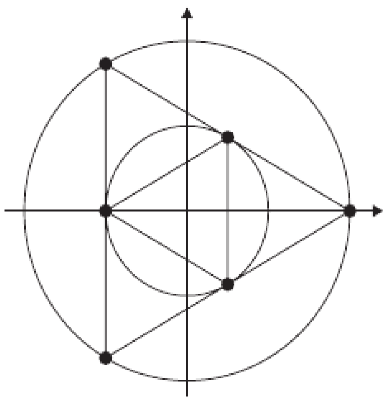

Figure 2: Argand diagram showing the roots of $p(x)$.

(i) List the **exact** roots of $p(x) = 0$. Hence or otherwise write $p(x)$.
It need not be expanded, but should not contain complex terms.

The question states that the positive real solution is $z_1 = k = k \operatorname{cis} 0$. We can therefore see that the other two solutions on the outer circle will be given by $z_2 = k \operatorname{cis} \frac{2\pi}{3}$ and $z_3 = k \operatorname{cis} \frac{-2\pi}{3}$.

The negative real solution will have the same real part as the two complex solutions on the outer circle; we can calculate this by taking $z_4 = k \cos \frac{2\pi}{3} = -\frac{k}{2}$. We can find the other two solutions on the inside circle by rotating our negative real solution by $\frac{2\pi}{3}$, obtaining $z_5 = \frac{k}{2} \operatorname{cis}(\pi - \frac{2\pi}{3}) = \frac{k}{2} \operatorname{cis} \frac{\pi}{3}$ and $z_6 = \frac{k}{2} \operatorname{cis} \frac{-\pi}{3}$.

Root	**Polar form**	**Rectangular form**
z_1	$k \operatorname{cis} 0$	k
z_2	$k \operatorname{cis} \frac{2\pi}{3}$	$-\frac{k}{2} + i\frac{k\sqrt{3}}{2}$
z_3	$k \operatorname{cis} \frac{-2\pi}{3}$	$-\frac{k}{2} - i\frac{k\sqrt{3}}{2}$
z_4	$\frac{k}{2} \operatorname{cis} \pi$	$-\frac{k}{2}$
z_5	$\frac{k}{2} \operatorname{cis} \frac{\pi}{3}$	$\frac{k}{4} + i\frac{k\sqrt{3}}{4}$
z_6	$\frac{k}{2} \operatorname{cis} \frac{-\pi}{3}$	$\frac{k}{4} - i\frac{k\sqrt{3}}{4}$

We now generate our polynomial using the techniques described in the section on quadratic equations:

$$\begin{aligned}
f(x) &= (x - z_1)(x - z_4) = (x - k)\left(x + \frac{k}{2}\right) \\
g(x) &= (x - z_2)(x - z_3) = \left(x - \left(-\frac{k}{2} + i\frac{k\sqrt{3}}{2}\right)\right)\left(x - \left(-\frac{k}{2} - i\frac{k\sqrt{3}}{2}\right)\right) \\
&= x^2 - \frac{k}{2}x + \frac{k^2}{4} \\
h(x) &= (x - z_5)(x - z_6) = \left(x - \left(\frac{k}{4} + i\frac{k\sqrt{3}}{4}\right)\right)\left(x - \left(\frac{k}{4} - i\frac{k\sqrt{3}}{4}\right)\right) \\
&= x^2 + kx + k^2 \\
\therefore\ p(x) &= f(x)g(x)h(x) = (x - k)\left(x + \frac{k}{2}\right)\left(x^2 - \frac{k}{2}x + \frac{k^2}{4}\right)\left(x^2 + kx + k^2\right)
\end{aligned}$$

Note that because all of the complex roots come in conjugate pairs the imaginary parts cancel leaving us with a sextic polynomial with real coefficients.

Exercises

Work through the problem, writing out all working clearly. Attempt the problem without looking at the solution. Describe the main ideas used in each step of the solution, and make sure you understand why each step is taken and why the solution is correct.

Are there any other possible ways of completing the problem? Is this the only possible solution?

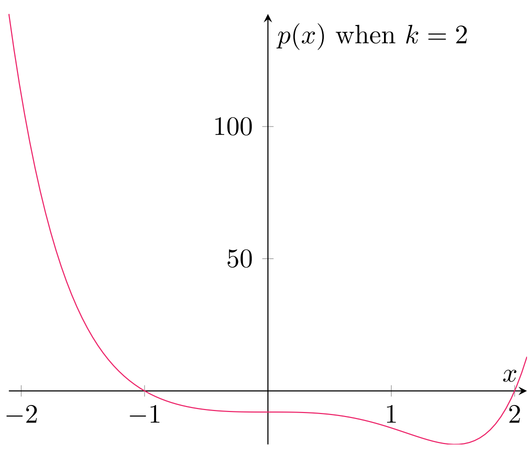

Figure 13: A polynomial with the desired roots.

Section 8: Solving the Cubic

This section presents a general solution for the cubic equation similar to the solution of the quadratic equation given in exercise 2.12. Be sure to complete that exercise before reading this section. The general outline of this proof is given in §14 of [6].

This particular method of solution was presented first by the French mathematician Alexandre-Théophile Vandermonde in 1771. However, it is believed that the first person to solve the general cubic equation was the Italian Scipio de Ferro who passed on at least part of his method to his student Antonio Fior. A solution was independently discovered at around the same time (in 1535) by Niccolo Fontana (also known as Tartaglia, the Stammerer), who was conviced to pass them on to another Italian, Girolamo Cardano. Cardano later (in 1545) published the solution in his book, *Ars Magna*, which also included a solution to the general quartic by Ludovico Ferrari.

Suppose we have some polynomial $t^3 - \sigma_1 t^2 + \sigma_2 t - \sigma_3$ (note the signs on the coefficients) with the three complex roots (not necessarily distinct) x, y, and z. So $\sigma_1 = x+y+z$, $\sigma_2 = xy + xz + yz$, and $\sigma_3 = xyz$, where σ_n is known as the nth elementary symmetric polynomial in x, y, and z.

We note the following three identities:

$$\begin{aligned} x^2z + xy^2 + yz^2 + x^2y + xz^2 + y^2z &= \sigma_1\sigma_2 - 3\sigma_3 \\ x^3 + y^3 + z^3 &= \sigma_1^3 - 3(\sigma_1\sigma_2 - 3\sigma_3) - 6\sigma_3 \\ x^2 + y^2 + z^2 &= \sigma_1^2 - 2\sigma_2 \end{aligned}$$

Finally, let α be a primitive cube root of 1.

Computations for the Solution

Note first that

$$\begin{aligned} x &= \frac{1}{3}\Big((x+y+z) + (x+\alpha y+\alpha^2 z) + (x+\alpha^2 y+\alpha z)\Big) \\ &= \frac{1}{3}\Big((x+y+z) + \sqrt[3]{(x+\alpha y+\alpha^2 z)^3} + \sqrt[3]{(x+\alpha^2 y+\alpha z)^3}\Big) \\ &= \frac{1}{3}\Big(\sigma_1 + \sqrt[3]{(x+\alpha y+\alpha^2 z)^3} + \sqrt[3]{(x+\alpha^2 y+\alpha z)^3}\Big) \end{aligned}$$

(remembering that the three cube roots of 1 add to 0).

Now, we must find expressions for $u = (x+\alpha y+\alpha^2 z)^3$ and $v = (x+\alpha^2 y+\alpha z)^3$ in terms of the coefficients of the polynomial, σ_n. Suppose that we can find uv and $u+v$ in terms of the elementary symmetric polynomials. Then we can find u and v using the quadratic formula.

So what are uv and $u+v$?

Expanding u and v individually, we find

$$\begin{aligned} u &= 3(x^2z + xy^2 + yz^2)\alpha^2 + 3(x^2y + xz^2 + y^2z)\alpha + x^3 + y^3 + z^3 + 6xyz, \text{ and} \\ v &= 3(x^2z + xy^2 + yz^2)\alpha + 3(x^2y + xz^2 + y^2z)\alpha^2 + x^3 + y^3 + z^3 + 6xyz. \end{aligned}$$

Figure 14: Niccolo Fontana (Tartaglia) (left) and Évariste Galois (right)

Adding these together, we have

$$\begin{aligned}
u+v = \quad & 3\alpha(x^2z+xy^2+yz^2+x^2y+xz^2+y^2z) \\
& +3\alpha^2(x^2z+xy^2+yz^2+x^2y+xz^2+y^2z) \\
& +12xyz+2(x^3+y^3+z^3) \\
= \quad & 3(\alpha+\alpha^2)(\sigma_1\sigma_2-3\sigma_3)+12\sigma_3+2(\sigma_1^3-3(\sigma_1\sigma_2-3\sigma_3)-6\sigma_3) \\
= \quad & -3(\sigma_1\sigma_2-3\sigma_3)+12\sigma_3+2(\sigma_1^3-3(\sigma_1\sigma_2-3\sigma_3)-6\sigma_3) \\
= \quad & 2\sigma_1^3-9\sigma_1\sigma_2+27\sigma_3.
\end{aligned}$$

To find uv, we proceed as follows:

$$\begin{aligned}
(x+\alpha y+\alpha^2z)(x+\alpha^2y+\alpha z) &= (xy+xz+yz)\alpha^2+(xy+xz+yz)\alpha+x^2+y^2+z^2 \\
&= (\alpha+\alpha^2)(\sigma_2)+\sigma_1^2-2\sigma_2 \\
&= \sigma_1^2-3\sigma_2 \\
&\Downarrow \\
uv &= (\sigma_1^2-3\sigma_2)^3.
\end{aligned}$$

Solution

So, to find the solutions of a cubic equation $t^3-\sigma_1t^2+\sigma_2t-\sigma_3=0$:

1. Calculate:

$$\begin{aligned}
u+v &= 2\sigma_1^3-9\sigma_1\sigma_2+27\sigma_3 \\
uv &= (\sigma_1^2-3\sigma_2)^3
\end{aligned}$$

2. Then calculate:

$$u,v=\frac{(u+v)\pm\sqrt{(u+v)^2-4uv}}{2}$$

3. Hence, we have nine possible solutions (one for each choice of cube root), of which three will work in the original equation (trial and error must be used at this point):

$$x=\frac{1}{3}\left(\sigma_1+\sqrt[3]{u}+\sqrt[3]{v}\right)$$

   Remember that the three cube roots of a number will be $\sqrt[3]{u}$, $\alpha\sqrt[3]{u}$, and $\alpha^2\sqrt[3]{u}$ where α is a complex cube root of unity.

A variation of this method also solves quartic equations. However, no general solution to the quintic equation (or any higher degree equation) exists in terms of radicals (terms under a $\sqrt{}$). The lack of a general solution for any polynomial with $\delta > 5$ was originally proved by Ruffini and Abel in the early 1800s, and a general study of the symmetries of the roots of polynomials (the beginnings of Galois theory) was first published (after being rejected twice) by the French Academy of Sciences in 1843 after their late author Évariste Galois was killed in a duel over a girl in 1832 (at the age of 20). For more historical details, see the introductory chapter of [10].

Clearing Up Some Technical Points

How did we know that uv and $u+v$ could be expressed in terms of the elementary symmetric polynomials (the coefficients)? Well, it so happens that uv and $u+v$ are symmetric in x, y, and z (this is not hard to check), and we have a theorem due to Sir Isaac Newton which states that:

Theorem (Fundamental Theorem on Symmetric Polynomials)**.** *Suppose that r_1, r_2, ..., r_n are the roots of some polynomial. Then every symmetric polynomial in r_1, r_2, ..., r_n can be expressed (uniquely) as a polynomial in the elementary symmetric functions σ_1, σ_2, ..., σ_n.*

Note that the elementary symmetric functions of order n are simply the coefficients of the polynomial $(x-a_1)(x-a_2)\cdots(x-a_n)$. For example, the second-order elementary symmetric functions are just $\sigma_1 = a_1 + a_2$ and $\sigma_2 = a_1 a_2$; the third-order elementary symmetric functions are just $\sigma_1 = a_1 + a_2 + a_3$, $\sigma_2 = a_1 a_2 + a_1 a_3 + a_2 a_3$, and $\sigma_3 = a_1 a_2 a_3$.

Basically, the theorem implies that if we have a symmetric expression in the roots of some polynomial, then we can write that expression in terms of the coefficients of the polynomial.

One example is the discriminant of a polynomial. We have already met the discriminant of a quadratic, $\Delta_2 = b^2 - 4ac$; we can more generally define the discriminant of an nth degree polynomial to be

$$\Delta_n[(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)] = \prod_{i<j}(\alpha_i - \alpha_j)^2$$

which is obviously symmetric in each α_i.

Exercises

**1. Check the author's algebra.

2. Solve $t^3 + t^2 - 89t + 231 = 0$.

3. Solve $t^3 + 21t^2 - 32t + 3510 = 0$.

4. Solve $2t^3 + 4it^2 + 58t - 84i = 0$.

5. In 1225, Leonardo of Pisa (Fibonacci) was asked by Holy Roman Emperor Frederick II to solve the cubic equation $x^3 + 2x^2 + 10x = 20$. His solution was

$$x = 1 + \frac{22}{60} + \frac{7}{60^2} + \frac{42}{60^3} + \frac{33}{60^4} + \frac{4}{60^5} + \frac{40}{60^6}.$$

   (a) Show that the equation has exactly one real root.

   (b) Use the method outlined in this section to find numerical approximations to the three roots of the polynomial.

6. Solve $t^3 - 15t - 4 = 0$ using the methods outlined in this section. See exercise 33 from the section on complex numbers.

7. Verify that uv and $u + v$ are symmetric in x, y, and z.

8. Read the historical introduction of Ian Stewart's *Galois Theory* [10].

9. The discriminant of the general quartic equation $q(x) = A(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ is given by the formula

$$\Delta_4[q(x)] = (\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2.$$

   Suppose that for a particular quartic $Q(x)$ with real coefficients, $\Delta_4[Q(x)] > 0$. What can you say about the number of real roots?

Section 9: Final Exercises

The shortest path between two truths in the real domain passes through the complex domain.
(Jacques Hadamard)

These exercises broadly cover the content of the book, and their difficulty varies! However, in general it is a good idea to look at the exercises in each section first as they often include some content themselves.

1. Is $(x-15)$ a factor of $(x^3-19x-30)$? Is (x^2+5x+6) a factor?
2. Factor completely $9x^4-13x^2+4$.
3. Solve $x^3+9x^2=60-8x$.
4. Find k such that $(x-4)$ is a factor of $x^3+7x^2-14x+k$.
5. Find a value of $k \neq 0$ such that $kx^2-6x+1=0$ will have just one root.
6. Find all sixth roots of i.
7. Find k such that $8-x+2\sqrt{2x+k}=0$ has exactly one real root.
8. Solve $(\alpha^2+2\alpha-4)(\alpha^7+1)=0$.
9. Solve $x^4+x^2+1=0$ for x.

*10. Solve $\beta^2+\beta+1=0$ for x if $\beta=x^2+x+1$.

*11. Let α, β, and γ be the three roots of $ax^3+bx^2+cx+d=0$. Prove that: (a) $\alpha+\beta+\gamma=\frac{-b}{a}$, (b) $\alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a}$, and (c) $\alpha\beta\gamma=\frac{-d}{a}$. Hence show that $\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2=\frac{bd}{a^2}$.

12. Solve $(z+1)^3=8(z-1)^3$ for z. Give exact answers in the form $a+ib$.
13. Graph the equation $|z|=3$ in the complex plane.
14. If $z=1+i$ and $w=\frac{1}{z}+i$, find the argument of w.

*15. If $\frac{z+2i}{z-2i}$ is purely imaginary, describe the possible values of z.

16. If $|z-1+2i|=|z+1|$ and $z=x+yi$, find an expression for y in terms of x (i.e. find the locus of z).
17. Sketch the region satisfied by $\mathfrak{Re}\,(z-i\bar{z})>2$.
18. If $x=2$ and $x=6$ are solutions of $p(x)=Ax^2+Bx+C$ and $p(0)=-4$, find A, B, and C.
19. If $w=2-3i$ is a zero of $3w^3-14w^2+Aw-26$ (where A is real), find A and the remaining two roots.

*20. Use de Moivre's Theorem to show that

(a) $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta=\cos^2\theta-\sin^2\theta$; and

(b) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ and $\cos 3\theta = 4\cos^3\theta-3\cos\theta$.

21. Use Euler's formula to prove that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$, and that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$.

**22. Show that $\arctan a+\arctan b=\arctan\frac{a+b}{1-ab}$.

*23. Suppose that $|z+w|=|z-w|$. Show that $\arg z-\arg w=\pm\frac{\pi}{2}$.

*24. If $3z^3+(2-3ai)z^2+(6+2bi)z+4$ has exactly one real root, what value must the quotient b/a take if both a and b are real? Find the real root.

*25. Find all possible values for θ if $\operatorname{cis}^2\theta+\operatorname{cis}\theta+1=0$.

26. Graph the locus of $\arg w=|w|$. What about $\arg w+|w|=1$?

27. Given a quadratic equation $x^2 + px + q$ and a root α, show that the other root β is given by $\beta = -p - \alpha$ and find a similar expression for finding two roots of a cubic given the third.

28. Suppose p is a quadratic (i.e. $p(x) = ax^2 + bx + c$ for some a, b, and c). Suppose further that $p(0) = 9$, and $p(3) = 0$. How many distinct roots does p have?

*29. Use the identity $x^2 + y^2 = (x - iy)(x + iy)$ to prove that if m and n are integers that can be written as the sum of two squares, then their product mn can also be written as a sum of two squares.

30. Solve the following system of equations:

$$\begin{aligned} x^2 + 4xy + y^2 &= 2 \\ x^2 - 2xy + y^2 &= -4 \end{aligned}$$

31. Suppose ω is a primitive cube root of unity. Show that

$$y_2 = \omega \sqrt[3]{\frac{1}{2}(-1+\sqrt{5})} + \omega^2 \sqrt[3]{\frac{1}{2}(-1-\sqrt{5})}$$

and

$$y_3 = \omega^2 \sqrt[3]{\frac{1}{2}(-1+\sqrt{5})} + \omega \sqrt[3]{\frac{1}{2}(-1-\sqrt{5})}$$

are complex conjugates.

32. Find all solutions to $x^{n-1} + x^{n-2} + \cdots + 1 = 0$, where n is a natural number.

33. (a) Let $p(x) = \sum_{r=0}^{n} p_r x^r$ be a polynomial with real coefficients. If $p(z) = 0$, then $p(\overline{z}) = 0$. (This is a generalisation of 4.27 to arbitrary degree polynomials.)
    (b) Let $p(x) = \sum_{r=0}^{n} p_r x^{2r}$ be a polynomial with real coefficients and only even powers of x. If $p(a + bi) = 0$, then $p(\pm a \pm bi) = 0$ for all possible combinations of $\pm$.

*34. Under which conditions will the equation $x^2 + a(1+i)x + b(1+i) = 0$ have one or more real solutions if both a and b are real?

35. Find the complex number z which satisfies $\arg(z - 1 - i) = -\frac{\pi}{6}$ and $\arg(z - 1 + i) = \frac{\pi}{6}$.

**36. Find all integer values of a and b such that $\frac{a^2+b^2}{ab}$ is an integer.

**37. Let w and z be complex numbers, and let $u = w + z$ and $v = w^2 + z^2$. Prove that w and z are real **if and only if** u and v are real and $u^2 \leq 2v$.

*38. Suppose $x + \frac{1}{x} = 1$.
    (a) Show, without calculating x, that we must necessarily have

$$x^7 + \frac{1}{x^7} = 1$$

    (b) Calculate the possible values of x and verify this fact.

**39. Calculate i^i.

40. Let $\mathbb{R}[\epsilon]$ be the real numbers together with some new element $\epsilon \neq 0$ such that $\epsilon^2 = 0$.
    (a) Does there exist β in $\mathbb{R}[\epsilon]$ such that $\beta\epsilon = 1$ (i.e. does ϵ have a multiplicative inverse)?
    (b) When does $(a + b\epsilon)^{-1}$ exist?
    (c) Solve $x^2 - 1 = 2\epsilon$ in $\mathbb{R}[\epsilon]$.

*41. Suppose $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{i=0}^{m} b_i x^i$ are polynomials. Show that

$$p(x)q(x) = \sum_{i=0}^{m+n} c_i x^i,$$

where each c_i is a constant. Give an expression for c_i in terms of the coefficients of $p(x)$ and $q(x)$.

**42. In this exercise, we will prove the division theorem that we stated without proof in section 3: if f and $g \neq 0$ are polynomials, then there exist unique polynomials q and r such that $\partial r < \partial g$ and

$$f(x) = g(x)q(x) + r(x).$$

(a) Consider the set S of all polynomials of the form $f(x) - g(x)q(x)$ for some polynomial $q(x)$. Show that S is non-empty (i.e. exhibit some polynomial in S).

(b) Explain why there must be some polynomial in S with minimal degree (that is, you cannot exhibit an infinite sequence of polynomials $p_1, p_2, \ldots$ in S such that $\partial p_1 > \partial p_2 > \cdots > \partial p_i > \cdots$).

(c) Pick such a polynomial of minimal degree; call it r. Fix also the associated q (i.e. now we have $f(x) = g(x)q(x) + r(x)$ for our fixed r and q). Show that if $\partial r \geq \partial g$, then it is possible to construct another polynomial in S with lower degree than r. Conclude that $\partial r < \partial g$.

(d) Show that q and r are uniquely determined by f and g.

43. Take the polynomial $x^2 = 4$ in the integers modulo 6 (i.e. the integers 0, 1, 2, 3, 4, 5 such that $5 + 1 = 0$). Solve for all possible values of x.

44. **Bonus exercise I:** In general, what numbers will work in the place of 3 and 9 in the XKCD comic at the bottom of the bibliography?

45. **Bonus exercise II:** Notice that $\frac{3}{16} - \frac{3}{19} = \frac{3}{16} \cdot \frac{3}{19}$. For which values of a, b, and d is the identity $\frac{a}{b} - \frac{a}{d} = \frac{a}{b} \cdot \frac{a}{d}$ true?

Section 10: Bibliography and Further Reading

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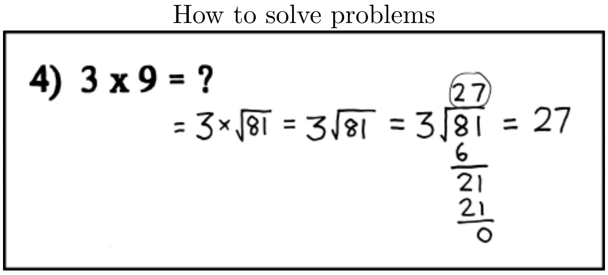
With thanks to Heydin Leeet for highlighting a typo in an exercise.

How to solve problems



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