NCEA Level 3 Trigonometry (exercise set)
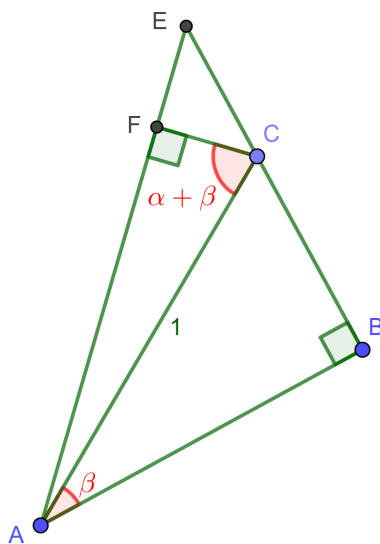# 4. Identity Fishing

Goal   To practice proving additional identities relating the trigonometric functions.

1. Prove the following identites true for all angles x.

    (a) $\sec 2x = \dfrac{\sec^2 x}{2 - \sec^2 x}$

    (b) $\cos x + \sin x = \dfrac{\cos 2x}{\cos x - \sin x}$

2. Write an expression for $\cos 4x$ in terms of only $\cos x$ (and powers and multiples of $\cos x$).

3. Fix two numbers $0 < \alpha < \pi/2$ and $0 < \beta < \pi/2$. Consider the following diagram.

    We have two right triangles, where ABC has an angle β and hypotenuse $|AC| = 1$. From C we measure an angle $\alpha + \beta$, and extend this line to meet a line from C at a right angle (so both interior angles at F are right). Hence $|AB| = \cos\beta$, $|BC| = \sin\beta$, and $|FC| = \cos(\alpha + \beta)$.

    (a) Show that the interior angle at E is α.

    (b) Show that $|EC| = \frac{\cos(\alpha+\beta)}{\sin\alpha}$.

    (c) Use (b) and the discussion below the diagram to show that

    $$\frac{|AB|}{|BE|} = \frac{\cos\beta}{\sin\beta + \frac{\cos(\alpha+\beta)}{\sin\alpha}}.$$

    (d) Use (a) to show that $\frac{|AB|}{|BE|} = \frac{\sin\alpha}{\cos\alpha}$.

    (e) Prove that $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

4. Find all the points x such that $\sin^2 x > \cos^2 x$.

5. If a, b, c, and d are the internal angles of a quadrilateral, show that

$$\cos(a+b+c) + \cos(b+c+d) + \cos(c+d+a) + \cos(d+a+b) = -4\cos\frac{a+b}{2}\cos\frac{a+c}{2}\cos\frac{a+d}{2}.$$

Additional reading   Hobson, chapter IV, V.