NCEA Level 3 Calculus (Differentiation)
Prior Revision: Functions

Introduction to the Notes

Mathematical Prerequisites

There are a number of things from Level 2 Mathematics that students should be comfortable with; generally, I assume in these notes a vague merit-level understanding of the core level 2 standards (by which I mean, the reader should be comfortable solving achieved level problems without guidance and have some idea how to approach more difficult problems):

- Level 2 Algebra: All material on quadratics (factorising, solving, discriminants), logs and exponents.
- Level 2 Calculus: Basic differentiation, geometric meaning of derivative (in particular, integration is *not* assumed)
- Level 2 Graphing: Recognising $x-/y-$shift of general functions, slope-intercept and point-slope form of linear equations, recognising period/frequency/amplitude/$x-/y-$shift from a trig function.
- Level 2 Trigonometry: Trig ratios, the Pythagorean theorem.
- Level 2 Simultaneous Equations: Solving linear and quadratic simultaneous equations.
- Level 2 Co-ordinate Geometry: Distances and linear equations.

Further into the notes, I also touch a bit on concepts covered in some of the other Level 3 standards, but not in so much detail that they need to be covered first:

- Level 3 Conics: Recognising forms from equations.
- Level 3 Algebra: Surds.
- Level 3 Trigonometry: Solving trig equations (including general solutions), reciprocal trig functions, use of trig identities (this latter mainly for E/S/OS-style integration problems).

In particular, no knowledge of complex algebra is assumed (or used) anywhere in the notes.

No knowledge from any L2 or L3 statistics standards is assumed.

A Note on Problem Difficulty

One of the main goals for these notes is that they should be useful for students at all levels, from A to OS. Accordingly, the problems each week range from simple (most students should be able to just write down an answer without thinking too hard) to extremely non-trivial (it took **me** a while to work the problem, and I know this material quite well). If you can't do a problem, the best thing to do is to move on and come back to it — the problems don't always increase in detail. Of course, it is important to do a good number of problems **including some difficult ones**; you're not under exam conditions here and you're going to get an awful lot more out of a hard problem than an easy one!

I have marked the problems in the weekly worksheets (**not** the homework) with symbols relating vaguely to difficulty:

A M E S O

However, these are for my own reference and should not be taken to be accurate with respect to actual examinations.

A Note on the Textbook

Many problems are taken from a couple of places:

- Stewart's Calculus (the current trendy textbook)
- Anton's Calculus (the old trendy textbook)
- University of Canterbury MATH199 lecture notes and problem sets
- Old NCEA/Scholarship exams

Most textbooks cover all the relevant material in the first few chapters (the first five or so in Stewart).

Homework

Every week has an associated homework sheet with a page of reading (five minutes or so) and a few questions (generally all pretty straightforward, but there is often a challenge question on there to keep you occupied).

I cannot emphasise enough how important it is to **do the homework**.

How to Read Mathematics

So we shall now explain how to read the book. The right way is to put it on your desk in the day, below your pillow at night, devoting yourself to the reading, and solving the exercises till you know it by heart. Unfortunately, I suspect the reader is looking for advice on how not to read, i.e. what to skip, and even better, how to read only some isolated highlights.

- Saharon Shelah, 'Classification Theory and the Number of Non-Isomorphic Models'

A major part of Level 3 Mathematics is preparation for university-level study in pure mathematics or the hard or soft sciences. As such, this year we begin to expose you to some 'real mathematics' — not just the watered-down calculation and computation you've been doing since intermediate, but a real journey of discovery through one of humanity's great fields of human experience. I guarantee that at primary school you enjoyed this kind of mathematics — I hope, for example, that the fact that $3 \times 5 = 5 \times 3$ was once a non-obvious fact to you and that your teacher let you discover it for yourself. Next year you will learn that there are some objects which do not have this property (called commutativity).

There are a lot of definitions and theorems, but they allow us to capture our intuition of mathematical objects, their behaviour, and their beauty in a more precise (and more correct) sense. It is important to read slowly and understand the symbolism as you go, as every piece strengthens the whole (like a single hard-to-hear flute in a symphony).

Recommended reading for the interested: Paul Lockhart's 'A Mathematician's Lament'.

Welcome to Calculus

What is Calculus?

Calculus is the study of:

- Continuous change.
- Slope, area, and volume.
- Functions and relationships.

It has applications in:

- Physics and chemistry.
- Probability theory.
- Population theory.
- Economics (I am assured).

In pure mathematics, calculus can be seen as the computational side of a pretty subject called **real analysis**.

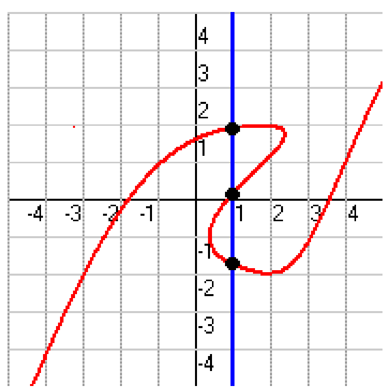
Revision of Functions

The most fundamental concept in calculus (and in mathematics as a whole) is that of a *function*.

Definition (Function)**.** A function is a something which takes a set of numbers (for example, the real numbers $\mathbb{R}$) and assigns to each one exactly one number (which could be the same or different).

Example. The map which takes a number x and spits out x^2 is a function — for every input, there is exactly one output. If we *graph* this function, we plot its input on the $x-$axis and its output on the $y-$axis and obtain a parabola.

Example. The curve graphed below is *not* a function, since for some inputs (like 1) it has more than one output. We can check this by drawing vertical lines along the function, like that pictured: if a graph is a function, no vertical line can ever cross the curve more than once (this is the *vertical-line test*).

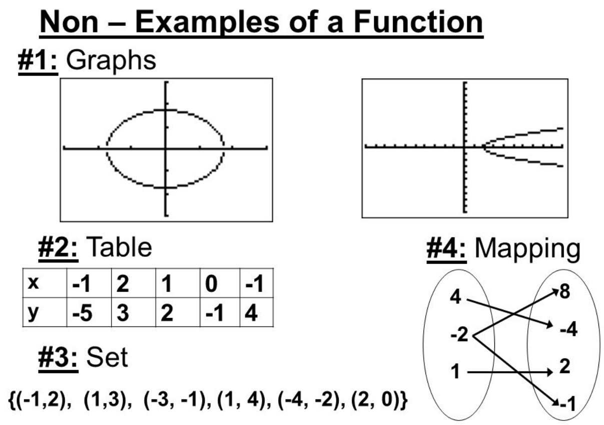


Example. The map $f : x \mapsto \sin x$ is a function. We could also define it by 'the function f such that $f(x) = \sin x$'. This function f can only produce numbers between 1 and -1; we say that its *range* is the interval from -1 to 1.

Example. The map $\iota : x \mapsto x$ is a function, called the *identity function*.

Example. The map $\ln x$ is a function, but it is only defined when $x > 0$: we say that its *domain* is the positive real numbers.

Example. Some more non-examples:



Questions

1. Which of the following are functions?
   (a) $E(x) = 2^x$
   (b) $\phi : x \mapsto \frac{2}{x}$
   (c) The thing which maps every person to their youngest sibling.
   (d) The thing which sends every person to their youngest sibling that isn't themself.
   (e) $x \mapsto \lfloor x \rfloor$ (the floor map).
   (f) The relation that sends every person to their age.
2. Find the distance between $(-3, 4)$ and $(2, 1)$.
3. Three sides of a triangle are have lengths 8, 15, and 17.
   (a) Show that the triangle is right-angled.
   (b) Find the other two angles.
4. Factorise and solve $x^2 - 3x + 2 = 0$.
5. How many **lines** are there through the point $(2, 3)$ and the origin? Give the equations of all such lines.
6. Find the slope of the line $4x + 3y + 2 = 0$.
7. Find the solution to the following linear system:

$$\begin{aligned} 2x + y &= 7 \\ 3x - y &= 8 \end{aligned}$$

8. How many (real) solutions does $x^2 + 4x + 1$ have?
9. Draw $\sin(x)$, $\cos(x)$, $\tan(x)$, $\exp(x)$, and $\ln(x)$.
10. How many solutions does $\cos(3\pi x + 1) = 2$ have?
11. How many solutions does $\sin(3x) = \frac{1}{3}$ have?