## NCEA Level 3 Calculus Integration Assignment

1. Compute the indefinite integrals.

(a) (2 points) 
$$\int x \cdot \cos(x^2) \cdot \sin(\sin x^2) dx$$

(b) (2 points) 
$$\int \pi t \csc^2(2t^2) dt$$

(c) (2 points) 
$$\int \frac{\sqrt{j} + 3j^5 + 3j^6 + 3j^7 + 2}{2j^7} \, dj$$

(d) (2 points) 
$$\int \frac{\ln t^2}{t} dt$$

## Solution:

(a) Let  $u = \sin x^2$ . Then  $du = 2x \cos x^2 dx$  and our integral becomes  $\int \frac{1}{2} \sin u du = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos \sin x^2 + C$ . This could also be done by two substitutions,  $u = x^2$  and then  $v = \sin u$ .

(b) Let  $u=2t^2$ . Then  $du=4t\,dt$  and our integral becomes  $\int \frac{\pi}{4}\csc^2(u)\,du=-\frac{\pi}{4}\cot u+C=-\frac{\pi}{4}\cot(2x^2)+C$ .

(c) We simplify to find that our integral becomes  $\frac{1}{2} \int j^{-6.5} + 3j^{-2} + 3j^{-1} + 3 + 2j^{-7} dj = \frac{1}{2} \left( -\frac{j^{-5.5}}{5.5} - 3j^{-1} + 3\ln|j| + 3j - \frac{1j^{-6}}{3} \right) + C = \frac{1}{2} \left( -\frac{1}{5.5} \sqrt{j^{11}} - \frac{3}{j} + 3\ln|j| + 3j - \frac{1}{3j^6} \right) + C.$ 

(d) First note that  $\ln t^2 = 2 \ln t$ . Then let  $u = \ln t$  so  $du = \frac{1}{t} dt$  and the integral becomes  $\int 2u du = u^2 + C = (\ln t)^2 + C$ .

2. We will prove the identity  $1 + \tan^2 x = \sec^2 x$ 

(a) (1 point) Calculate  $\frac{d}{dx} \sec^2 x$ .

(b) (4 points) Using the substitution  $u = \tan x$ , integrate your answer to part (a). Conclude that  $\sec^2 x = \tan^2 x + C$  for some constant C.

(c) (2 points) Find the value of C and conclude the identity above.

## **Solution:**

(a) Using the chain rule the required derivative is  $2 \sec x \cdot \sec x \tan x = 2 \sec^2 x \tan x$ .

(b) We find  $\int 2 \sec^2 x \tan x \, dx$ . Let  $u = \tan x$ . Then  $du = \sec^2 x$ , and our integral becomes  $\int 2u \, du = u^2 + C = \tan^2 x + C$ . But from (a) we have  $\int 2 \sec^2 x \tan x = \sec^2 x + C'$ . Hence  $\tan^2 x + C = \sec^2 x + C'$  and the two differ only by a constant.

- (c) The identity must hold for all x, and so we set x = 0. Then  $\sec^2 0 = \tan^2 0 + C$  and C = 1. Hence we have  $\sec^2 x = \tan^2 x + 1$  as expected.
- 3. (2 points) Find the area bounded by the curve  $y = 3x^2 + x 2$  and the x-axis.

**Solution:** The curve can be factored as y = (3x - 2)(x + 1) and so the x-intercepts are x = -1 and  $x = \frac{2}{3}$ . We must therefore find  $\int_{-1}^{2/3} 3x^2 + x - 2 \, dx = x^3 + 0.5x^2 - 2x \Big|_{-1}^{2/3} = -\frac{125}{54} \approx -2.315$ .

4. (2 points) If  $\int_{-1}^{2} 3f(x) dx = 9$  and  $\int_{-1}^{3} f(x) dx = 1$ , find  $\int_{2}^{3} f(x) dx$ .

**Solution:**  $\int_{2}^{3} f(x) dx = \int_{-1}^{3} f(x) dx - \frac{1}{3} \int_{-1}^{2} 3f(x) dx = 1 - 3 = -2.$ 

- 5. (a) (3 points) Compute  $\int_0^R 2\pi r \, dr$ . Interpret your answer (you may wish to draw a diagram).
  - (b) (2 points) Find the volume of a sphere of radius R by integration; the surface area of a sphere of radius r is given by  $SA = 4\pi r^2$ .

## Solution:

- (a)  $\int_0^R 2\pi r \, dr = \pi r^2 \Big|_0^R = \pi R^2$ , which is the area of a circle of radius R. This makes sense as we are summing up all the circumferi of circles radiating out from the centre of our larger circle: we expect to get the full area.
- **(b)** Same reasoning:  $\int_0^R 4\pi r^2 dr = 4/3\pi r^3 \Big|_0^R = 4/3\pi R^3$ .
- 6. (5 points) Find  $y(\sqrt{\pi/2})$  if y(0) = 0 and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\sin(x^2)\cot y.$$

**Solution:** Separating variables, we have  $\int \tan y \, \mathrm{d}y = \int x \sin x^2 \, \mathrm{d}x$ . The RHS is simply  $-\frac{1}{2} \cos x^2 + C$  (substitute  $x^2$  out), and we can rewrite the LHS as  $\tan y = \frac{\sin y}{\cos y}$  so (using the substitution  $\cos y$ ); hence overall we have got  $-\ln|\cos y| = -\frac{1}{2} \cos x^2 + C$ . But y(0) = 0 so  $-\ln 1 = -\frac{1}{2} + C$  and  $\ln 1 = 0$  so C = 0.5 and  $|\cos y| = e^{\frac{1}{2} \cos x^2 - \frac{1}{2}}$ . At  $x = \sqrt{\pi/2}$ , the RHS becomes  $e^{-0.5}$  and so  $\cos y = \pm e^{-0.5}$ ; therefore we have two solutions for y:  $y = \cos^{-1}(\pm e^{-0.5})$ .