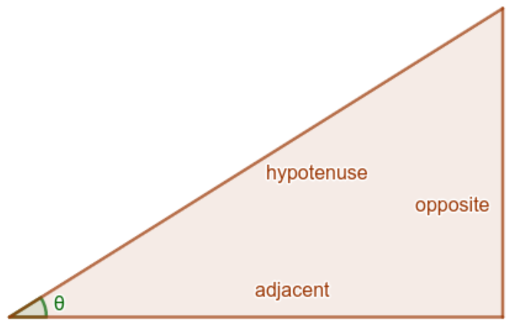
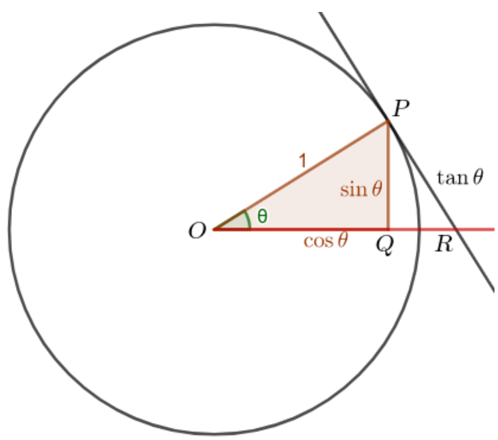
NCEA Level 2 Mathematics
3. Trigonometry



We are now going to look at triangles inside circles. Now, last year we learned that any triangles with two equal angles are similar; in particular, if we take ratios of sides, we obtain the same value. This means that if we have any right-angled triangle with angle θ like the one above, then the ratios $\frac{\text{opposite}}{\text{hypotenuse}}$, $\frac{\text{adjacent}}{\text{hypotenuse}}$, and $\frac{\text{opposite}}{\text{adjacent}}$ all depend only on the angle θ; we call them the sine, cosine, and tangent of the angle respectively:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}.$$

In particular, if we draw our triangle inside a unit circle then we can draw the following:



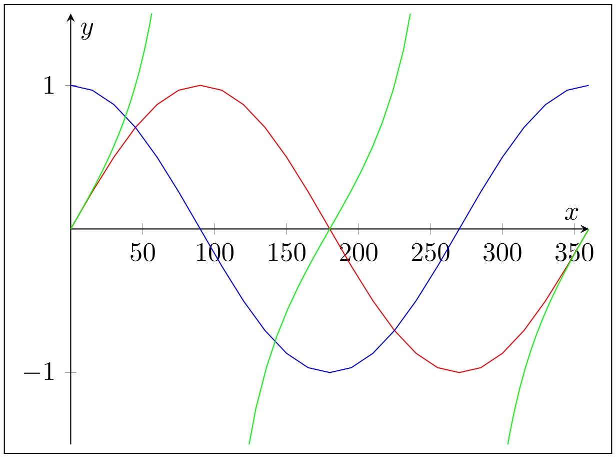
In fact, we can take this as our definition of sin and cos. To show that $\tan\theta$ is indeed the line segment marked, first notice that since the triangle OPR is right-angled, the angle at the intersection of the horizontal line and the tangent line is $90° - \theta$; so the other non-right-angle in the triangle PQR is θ. Hence the hypotenuse of PQR is $\frac{\text{adjacent}}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$, as proposed.

Note also that, from this diagram, we have

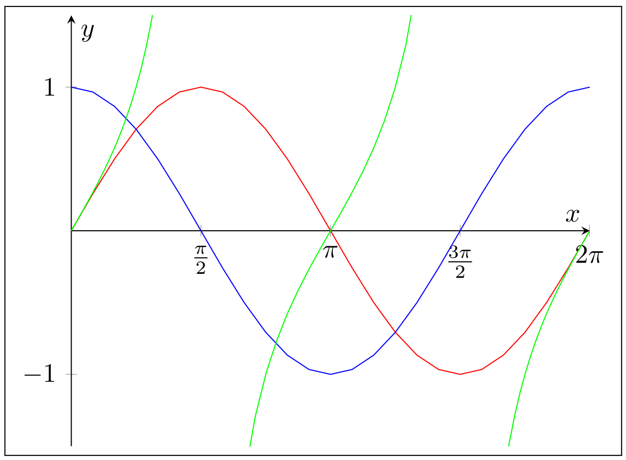
$$\sin^2\theta + \cos^2\theta = 1$$

for every angle θ.

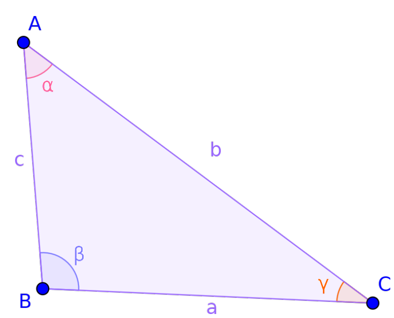
Since the sin of an angle is just the height of the point above the x-axis in the diagram above, we have that $-1 \leq \sin\theta \leq 1$; similarly, $-1 \leq \cos\theta \leq 1$. Note that when $\theta = 90°$, the tangent line becomes horizontal and so never intersects the x-axis: so $\tan 90°$ is undefined. We can even graph $\sin\theta$ (red), $\cos\theta$ (blue), and $\tan\theta$ (green):



If we graph them in radians, only the labels on the x-axis change:



Let us now begin to look at more general triangles:



Theorem (Sine rule)**.** *In any triangle, with the angles and sides labelled as above, we have*

$$\frac{\alpha}{\sin a} = \frac{\beta}{\sin b} = \frac{\gamma}{\sin c}.$$

Proof. Drop an altitude from B to AC, creating two new right-angled triangles. Then the length of this line can be calculated using both of the resulting right-angled triangles: so $c \sin \alpha = a \sin \gamma$ and $\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$. This proves the theorem. □

Theorem (Cosine rule)**.** *In any triangle, with the angles and sides labelled as above, we have*

$$a^2 = b^2 + c^2 - bc \cos \alpha.$$

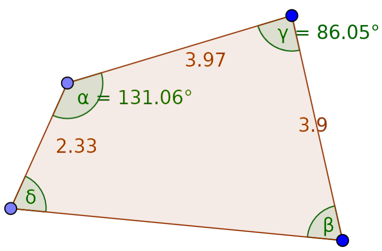
Proof. Drop an altitude from B to AC, creating two new right-angled triangles. Then the length b can be split into two lengths, $c\cos\alpha$ and $b - c\cos\alpha$; the length of the altitude is $c\sin\alpha$. Now, apply the Pythagorean theorem to the triangle including the angle γ:

$$a^2 = (b - c\cos\alpha)^2 + c^2\sin^2\alpha = b^2 - 2bc\cos\alpha + c^2\cos^2\alpha + c^2\sin^2\alpha = b^2 + c^2 - 2bc\cos\alpha.$$

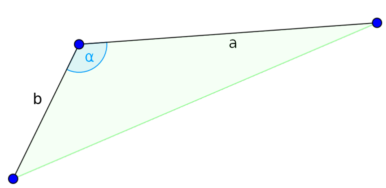
□

Questions

1. A beam of gamma rays is to be used to treat a tumor known to be 5.7 cm beneath the patient's skin. To avoid damaging a vital organ, the radiologist moves the source over by 8.3 cm.
   (a) At what angle to the patient's skin must the radiologist aim the gamma ray source to hit the tumor?
   (b) How far will the beam have to travel through the patient's body before reaching the tumor?
2. A surveyor measures the three sides of a triangular field and finds that they are 117, 165, and 257 metres long.
   (a) What is the measure of the largest angle of the field?
   (b) What is the area of the field?
3. A field has the shape of a quadrilateral (four-sided shape) that is *not* a rectangle. Three sides measure 2.33, 3.97, and 3.9 kilometres, and two angles measure $\alpha = 131.06°$ and $\gamma = 86.05°$ (as in the figure).



   (a) By dividing the quadrilateral into two triangles, find its area.
   (b) Find the length of the fourth side.
   (c) Find the measures of the other two angles, β and δ.
4. A surveyor is standing on top of a peak. She can see two prominent peaks ahead of her, and from previous measurements she knows that one of them is 8 km away from her and the other is 11 km away. She measures the angle between them to be 120°. How far apart are the two peaks (measured *along the ground*):
   (a) If that they have same height?
   (b) If the surveyor and the closer peak are at the same height, but the peak which is further away is 200 m higher?
5. Consider a triangle with sides of 5, 7, and 10 kilometres.
   (a) Find the measure of the largest angle of this triangle.
   (b) Find the area of the triangle.
6. Find the area of the triangle below.



7. For each item below, decide whether or not such a triangle exists. If at least one does, how many exist?
   (a) Exactly one angle greater than $90°$.
   (b) Two angles greater than $\pi/2$.
   (c) Two sides of length 200,000.
   (d) Three sides of length 200,000.
   (e) Sides of length 90, 30, and 30.

8. Prove that, if a quadrilateral has equal diagonals, then it is a rectangle. (We used this fact last week!)

9. Let A and B be points in three dimensional space. Show that

$$d(A,B) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

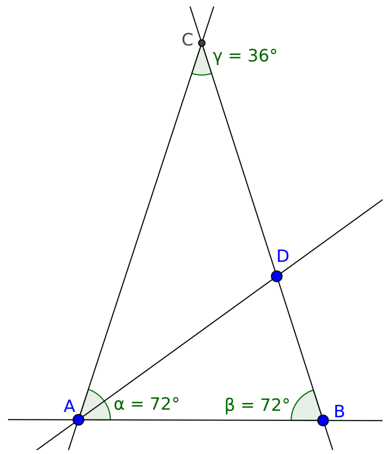
10. Let ABC be a triangle.*
    (a) Let $X = m(A,B)$, $Y = m(B,C)$, and $Z = (C,A)$ be the midpoints of the sides; then the lines CX, AY, and BZ are called the *medians* of the triangle. Show that the three medians always intersect at a single point N (the *centroid*).
    (b) Let k be the perpendicular bisector of AB, ℓ be the perpendicular bisector of BC, and m be the perpendicular bisector of CA. Show that k, ℓ, and m intersect at a single point O (the *circumcentre*). Show that O is the centre of the circle passing through A, B, and C.
    (c) Let λ be the line passing through A that bisects the angle of the triangle at A. Define μ and ν similarly as the angle bisectors at B and C. Show that λ, μ, and ν intersect at a single point P (the *incentre*). Show that P is the centre of the circle which is tangent to the three sides of the triangle.
    (d) Let ρ be the line through A perpendicular to BC; define σ and τ similarly to be lines through B and C. These lines are known as the *altitudes* of the triangle. Show that ρ, σ, and τ intersect at a single point (the *orthocentre*).
    (e) Show that for any triangle, the circumcentre, centroid, and orthocentre all lie on a single line (known as the *Euler line* of the triangle).
    (f) Show that for an isoceles triangle, the Euler line is the line of symmetry.
    (g) Show that the incentre lies on the Euler line exactly when the triangle is isoceles.
    (h) Show that for an equilateral triangle, the orthocentre, centroid, incentre, and circumcentre coincide. (We may simply call this point the centre.)

11. This question requires you to find exact values for trig functions *without* using a calculator. [Schol 1999]
    (a) Sketch an equilateral triangle of side 2 units. Hence, or otherwise, show that $\cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, and $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.
    (b) Sketch a right angled isosceles triangle whose equal sides enclose the right angle and are of length 1. Hence, or otherwise, show that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

*This question may be done fairly easily with coordinate geometry. However, infinitely more elegant solutions may be found by using the machinery of Euclidean geometry. See, for example, sections 1.4 to 1.6 of Coxeter's *Intro. to Geometry.*

12. Consider a regular tetrahedron (a four-sided shape such that each side is an equilateral triangle). For each vertex, draw the line joining it to the centre of the opposite side. By a problem in the section on coordinate geometry, the four lines must meet at a single point O. Show, by an exact calculation, that the angle at the point O between any line and any other line is around $109.5°$. (This is the bond angle between adjacent hydrogens in methane, CH_4.)

13. Consider the 75-75-36 triangle ABC given in the figure. The angle α has been bisected into two angles, and the resulting line meets the triangle at D.



(a) Show that ABC and ABD are similar triangles.

(b) Hence, or otherwise, show that $\frac{AB}{BD} = \frac{AB+BD}{AB}$.

(c) Show that the ratio of the long side of the triangle to the short side of the triangle is $\frac{AB}{BD} = \frac{1+\sqrt{5}}{2} = \phi$.

(d) Show that $\cos 72° = \frac{1}{2\phi}$.

(e) Find $\sin 36°$ and $\sin 72°$.