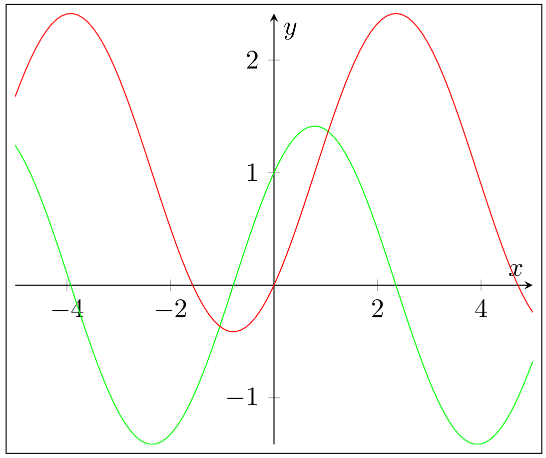
# NCEA Level 3 Calculus
# Solutions to Homeworks

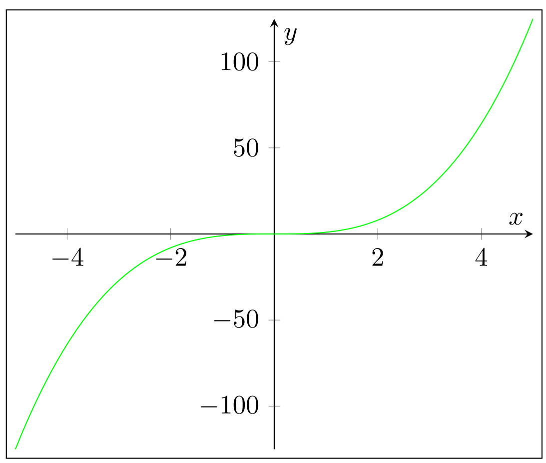
## 1. The Derivative

1. Green: derivative of function. Red: original function.



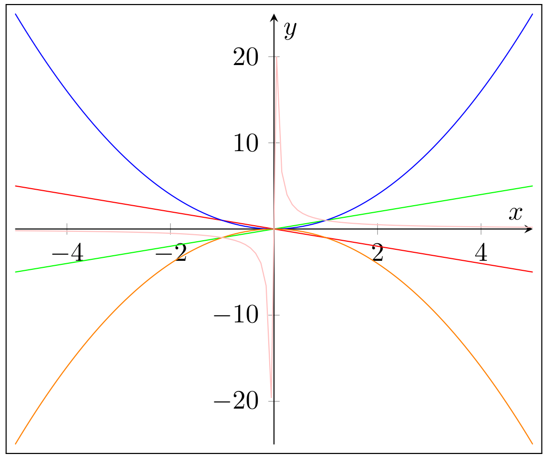

2. (a) At a min or max, the function is momentarily horizontal and so has slope zero; so $m = 0$.

   (b) Consider a graph like the following at $(0, 0)$:

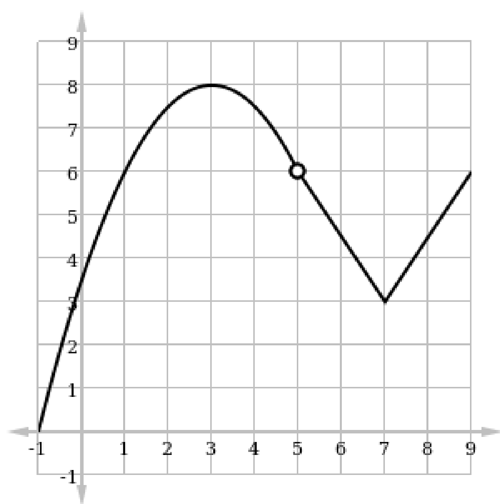



## 2. Limits

1. Increasing: the graph of the function is sloping up (green). Decreasing: the graph of the function is sloping down (red). Concave up: the graph of the function is increasing in slope (it is like a cup ∪) (blue). Concave down: the graph of the function is decreasing in slope (it is like a cap ∩) (orange). Continuous: the function has no holes (all of them except pink).




2. (a) $\lim_{x\to -2} f(x) = 0$, $\lim_{x\to 2} f(x) = -0.5$.
   (b) No, it approaches different values from the left and the right.
   (c) Yes, because the function is continuous there.
   (d) $(-\infty, -3)$, $(-3, -1)$, $(-1, 2)$, $(2, \infty)$.
   (e) -3, -1, 2.
3. For example,



## 3. Derivatives of Common Functions

1. (a) $2x + 1/x$
   (b) $t^2 x^{t-1}$
   (c) $\cos x + \sin x$
   (d) $\frac{4}{5}x^{-1/5} = \dfrac{4}{5\sqrt[5]{x}}$
2. The power $x$ is not constant.

3. Consider $x^{-n}$. The first derivative is $-nx^{-n-1}$, the second is $n(n+1)x^{-n-2}$, and so the $n$th is $(-1)^n n(n+1)\cdots(2n-1)(2n)x^{-2n} = (-1)^n \frac{(2n)!}{(n-1)!}x^{-2n}$. [This can be proved via induction.]

4. (a) Note first that $10^t = e^{t\ln 10}$, so $P = P_0 + e^{t\ln 10}$ and $\frac{\mathrm{d}P}{\mathrm{d}t} = (\ln 10)e^{t\ln 10} = (\ln 10)10^t$. At $t = 100$, we have $\frac{\mathrm{d}P}{\mathrm{d}t} = 2.3 \times 10^{100}$.

   (b) Real-world populations don't grow exponentially forever if there are finite resources (e.g. food).

## 4. The Chain Rule

1. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\csc^2 x}{2\sqrt{\cot x}}$

2. (a) Simply apply the chain rule twice.

   (b) $y' = 5x^4(\cos x^5)(-\sin\sin x^5)(\cos\cos\sin x^5)(-\sin\sin\cos\sin x^5)(\cos\cos\sin\cos\sin x^5)$.

3. (a) $f'(\theta) = -2\sin 2\theta$ and $g'(\theta) = -4\sin\theta\cos\theta = -2\sin 2\theta$, so $f' = g'$ as they agree everywhere.

   (b) Since $f$ and $g$ have the same derivative, they differ only by a constant. But $f(0) = 1 = g(0)$, so that constant is zero; hence $f = g$.

## 5. The Product and Quotient Rules

1. (a) $\cos x \ln x + \dfrac{\sin x}{x}$

   (b) $\sec kx + kx \sec kx \tan kx$

   (c) $\dfrac{-\pi(\sin\pi\theta + \cos\pi\theta)\sin\pi\theta - \pi(\cos\pi\theta - \sin\pi\theta)\cos\pi\theta}{(\sin\pi\theta + \cos\pi\theta)^2}$

   (d) $(\cos t)(3\sin^2 t)(-\sin(\sin^3 t))(4\cos^3\sin^3 t)$.

2.

$$
\begin{aligned}
F = \frac{\mathrm{d}}{\mathrm{d}t}\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} &= a\frac{\mathrm{d}}{\mathrm{d}v}\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= a\left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m_0 v^2}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right) \\
&= a\left(\frac{m_0 c^2\left(1-\frac{v^2}{c^2}\right)}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} + \frac{m_0 v^2}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right) \\
&= m_0 a\left(\frac{c^2\left(1-\frac{v^2}{c^2}\right)+v^2}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right) \\
&= m_0 a\left(\frac{c^2-v^2+v^2}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right) \\
&= \frac{m_0 a}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}.
\end{aligned}
$$

3. We wish to find $\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\mathrm{rad}(\theta)$, where $\mathrm{rad}(\theta) = \frac{\pi\theta}{180}$; so $\frac{\mathrm{d}(\mathrm{rad})}{\mathrm{d}\theta} = \frac{\pi}{180}$ and $\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\mathrm{rad}(\theta) = \frac{\pi\theta}{180}\cos\mathrm{rad}(\theta)$. [The reason we have to do this is that the derivative of sin uses the limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$ which is false if $x$ is in degrees.]

## 6. Tangent and Normal Lines

1. The normal to a curve $f$ at a point $(x_0, f(x_0))$ is the unique line passing through that point that is perpendicular to the tangent line of $f$ at that point.
2. $y' = \frac{-\sin(x+\pi)}{2\sqrt{\cos(x+\pi)}} + \cos x - 4\sec x\tan^2 x e^{2\tan^2 x}$; at $x = \pi$, $y' = -1$ and so the tangent line (best linear approximation) is $y = -(x-\pi) = \pi - x$.
3. Since the normal line has slope 3, the tangent line has slope $-1/3$. We can take any curve through $(1,0)$ with this slope, so we may as well take the tangent line itself: $y = -\frac{1}{3}(x-1) = \frac{1}{3} - \frac{1}{3}x$.
4. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(1+3x)^{2/3}}$ and at $x = 0$ the slope becomes 1. So the best linear approximation around the point $(0,1)$ is just $\tilde{y} = x + 1$. So at $x = 0.01$, we have $\tilde{y} = 1.01$ as our approximate value of $\sqrt[3]{1.03}$. [The true value is around 1.0099, so we are not too far off.]

## 7. Higher Derivatives and the Geometry of a Function

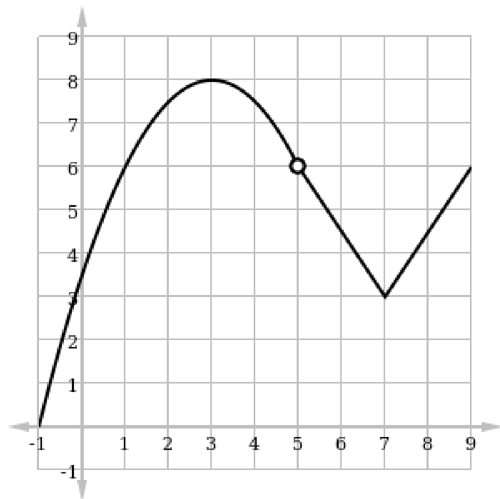
1. The second derivative tells us the concavity of a function: if the second derivative is positive, the function is curving up and if it is negative then the function is curving down.

2. (a) $f'(x) = 5x^4 - 5$, $f''(x) = 20x^3$.

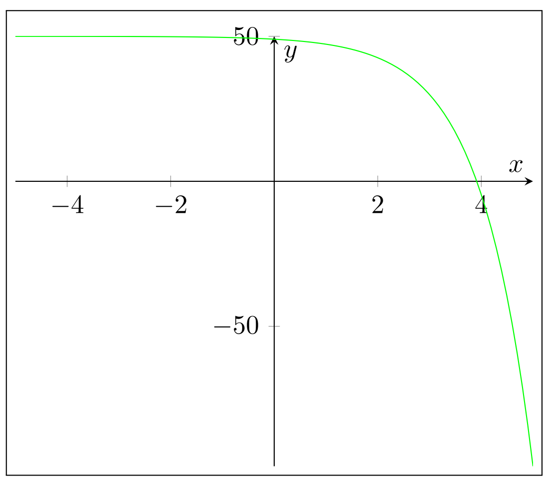
   (b) $f'(x) = \frac{x^2-2x}{(x-1)^2}$, $f''(x) = \frac{2x-2}{(x-1)^4}$.

   (c) $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{4}x^{-3/4}$, $f''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{16}x^{-5/4} = \frac{3}{16\sqrt[4]{x^5}} - \frac{1}{4\sqrt{x^3}}$.

3. (a) For example,




   (b) For example,




## 8. Optimisation

1. At some point $x$, the distance between the two parabolae is $\delta(x) = (x^2 + 1) - (x - x^2) = 2x^2 - x + 1$. Taking the derivative, we find $\delta'(x) = 4x - 1$ which has a single zero at $x = 1/4$; by looking at the graph of the two parabolae, we see that this must be the location of the miniumum distance $\delta(1/4) = 7/8$ units.

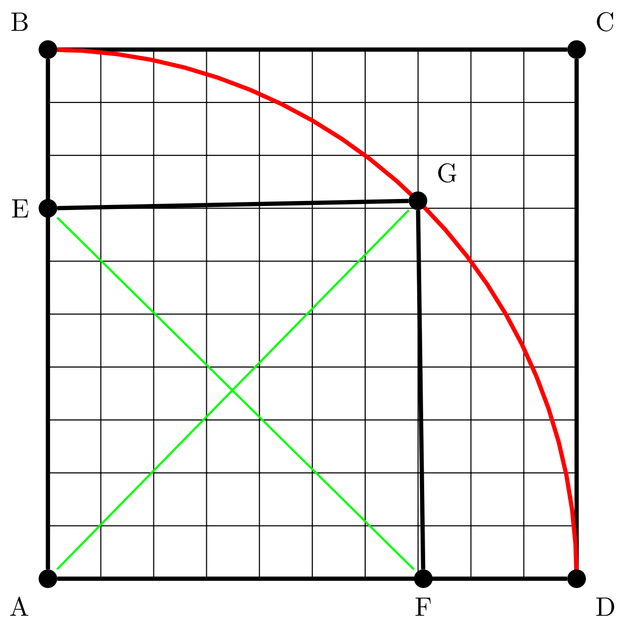
2. If $y = 3x + 2\cos x + 5$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 3 - 2\sin x$. Since $1 \geq \sin x$, $3 - 2\sin x \geq 1$. In particular, the function is everywhere increasing. Now, note that when $x = -200\pi$, $y = -600\pi + 7 < 0$, and when $x = 200\pi$, $y = 600\pi + 7 > 0$. Since the function is continuous over this interval, it follows that at some point it passes through the $y$-axis and has at least one root; since it is increasing everywhere, it must have exactly one real root.

3. The area of such a rectangle will be $A = 4xb\sqrt{1 - \frac{x^2}{a^2}}$; so

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 4b\sqrt{1 - \frac{x^2}{a^2}} - \frac{4x^2 b}{a^2\sqrt{1 - \frac{x^2}{a^2}}}.$$

Setting this to zero, we have $a^2 = 2x^2$ and so $2x = \sqrt{2}a$. It follows that $2y = b\sqrt{2}$, and so the maximal area is $2ab$.

4. Consider the following diagram.




It should be clear that $AG = 1$; call $\angle AEG = \theta$ and $\angle AFG = \phi$, and let $AE = EG = e$ and $AF = GF = f$. By the cosine rule, we have $1 = 2e^2(1 - \cos\theta)$ and $1 = 2f^2(1 - \cos\phi)$. Now, the area of the triangle $\triangle AEG$ is given by $\frac{1}{2}\sqrt{e^2 - \frac{1}{4}}$; the area of $\triangle AFG$ is given by $\frac{1}{2}\sqrt{f^2 - \frac{1}{4}}$. Since $AEFG$ is a (convex) quadrilateral with two right angles, $\theta + \phi = \pi$. Putting this all together, the area of the quadrilateral is $A = \frac{1}{2}\sqrt{e^2 - \frac{1}{4}} + \frac{1}{2}\sqrt{f^2 - \frac{1}{4}}$. We have that $e^2 = \frac{1}{2(1-\cos\theta)}$ and $f^2 = \frac{1}{2(1-\cos(\pi-\theta))} = \frac{1}{2(1+\cos(\theta))}$, so the area in terms of $\theta$ is

$$A = \frac{1}{2}\sqrt{\frac{1}{2(1-\cos\theta)} - \frac{1}{4}} + \frac{1}{2}\sqrt{\frac{1}{2(1+\cos\theta)} - \frac{1}{4}}.$$

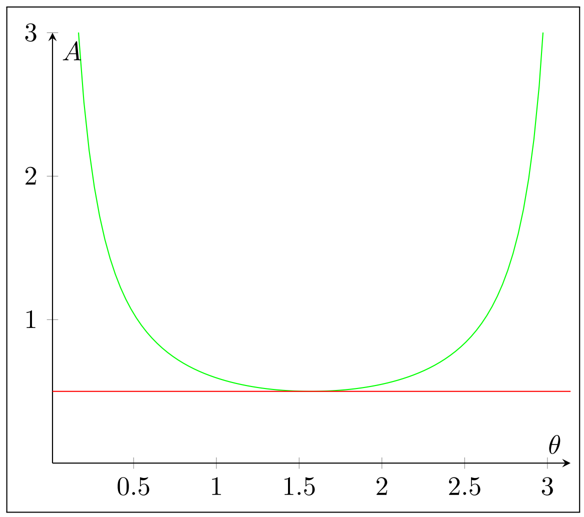
Taking the derivative, we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{1}{2}\frac{\sin\theta}{4(\cos\theta + 1)^2\sqrt{\frac{1}{2(1+\cos\theta)} - \frac{1}{4}}} - \frac{1}{2}\frac{\sin\theta}{4(1 - \cos\theta)^2\sqrt{\frac{1}{2(1-\cos\theta)} - \frac{1}{4}}};$$

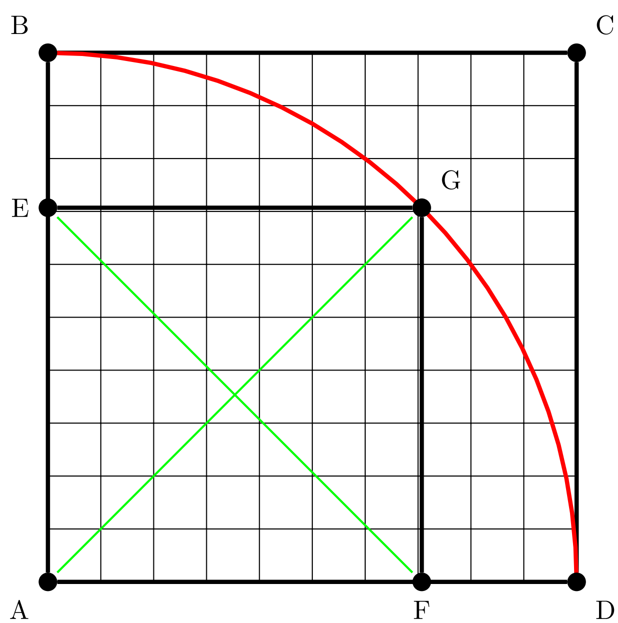
Now we set this to zero. We know that $0 < \theta < \pi$, so $\sin\theta \neq 0$ and hence

$$4(\cos\theta+1)^2\sqrt{\frac{1}{2(1+\cos\theta)}-\frac{1}{4}} = 4(1-\cos\theta)^2\sqrt{\frac{1}{2(1-\cos\theta)}-\frac{1}{4}}$$
$$\sqrt{\frac{1}{2}-\frac{\cos\theta+1}{4}} = \sqrt{\frac{1}{2}-\frac{1-\cos\theta}{4}}$$
$$\cos\theta = -\cos\theta$$
$$\cos\theta = 0$$

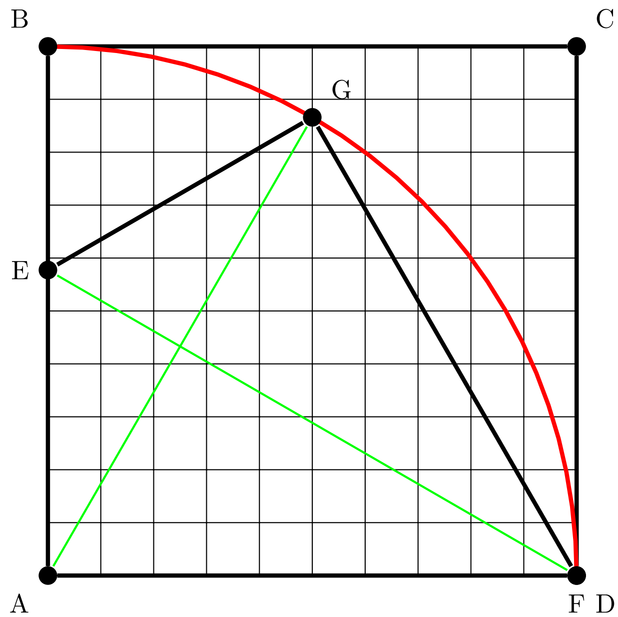
Hence $\theta = \phi = \pi/2$. Immediate calculation shows that $e = f = \frac{1}{\sqrt{2}}$; we thus have a square with side length $1/\sqrt{2}$, and area $\frac{1}{2}$. Is this a maximum or a minimum? We cheat by graphing the area versus $\theta$:




so we obviously have the minimum area:




Note that $\theta \geq \pi/2$, because otherwise $e > 1$. Suppose we take $\theta = 2\pi/3$; here is the graphed figure (with area 1.1547):




This is the maximum area, since if we increase $\theta$ any more it requires $f > 1$.

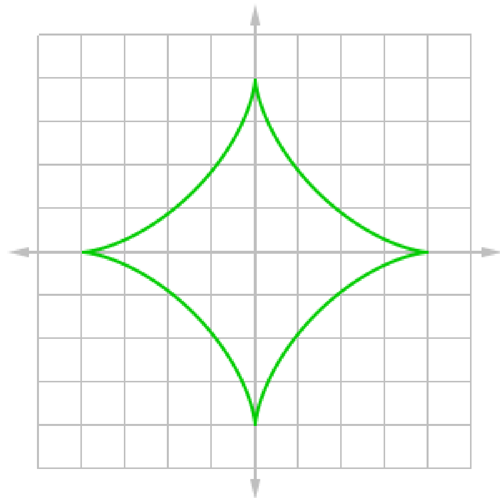
## 9. Related Rates

1. $V = [x(t)]^3$, so $\frac{\mathrm{d}V}{\mathrm{d}t} = 3\frac{\mathrm{d}x}{\mathrm{d}t}[x(t)]^2$.
2. The volume of a cone is $\frac{\pi}{3}r^2h$. Comparing similar triangles, if the water is at a height $h$ then it forms a cone with radius $r = h/2$. Hence when the water is at a height $h$ it has volume $V(t) = \frac{\pi[h(t)]^3}{24}$, and so $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi[h(t)]^2}{8}\frac{\mathrm{d}h}{\mathrm{d}t}$. We know that $\frac{\mathrm{d}V}{\mathrm{d}t} = 2$, so solving for $\frac{\mathrm{d}h}{\mathrm{d}t}$ we have that $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{16}{\pi[h(t)]^2}$ and when $h = 3$ the height is rising at a rate of $0.57\,\mathrm{m\,min^{-1}}$.
3. Let $x$ be the hypotenuse of the formed triangle, and let $y$ be the horizontal distance from the boat to the jetty so that $y = \sqrt{x^2-1}$. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{x}{\sqrt{x^2-1}}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sqrt{y^2+1}}{y}$. So at $y = 8$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\sqrt{65}}{8} \approx 1.0078\,\mathrm{m\,s^{-1}}$.

## 10. Parametric Functions

1. (a) $\frac{\mathrm{d}x}{\mathrm{d}t} = 4t^3 - 6t^2 + 4t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3t^2-1}{4t^3-6t^2+4t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{3t^4-6t^2+3t-1}{(4t^3-6t^2+4t)t^2(2t^2-3t+2)^2}$.

   (b) $\frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t - 4\sin 2t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \cos t + 4\cos 2t$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cos t+4\cos 2t}{\sin t+4\sin 2t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{12\cos(t)+33}{(\sin t-4\sin 2t)\left(8\cos(t)+1\right)^2\left(\cos(t)^2-1\right)}$
2. We have $t^2 = (x-1)^2$, so $y = e^{(x-1)^2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2(x-1)e^{(x-1)^2}$. At $x = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2e$; so the best linear approximation is $y - e = 2e(x-2)$, or $y = e(2x-3)$.
3. (a) Should look something like this:



(b) $\frac{\mathrm{d}x}{\mathrm{d}t} = -12\sin t\cos^2 t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 12\cos t\sin^2 t$, so the slope at some $t$ is simply

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{12\cos t\sin^2 t}{-12\sin t\cos^2 t} = -\frac{\sin t}{\cos t}.$$

(c) Cusps will be at precisely those points with turning points in the $x$ or $y$ direction (for $0 \le t \le 2\pi$). In other words, places where either $\sin t$ or $\cos t$ vanishes. These are at $t \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$; substituting these into the equation gives us the four points $(\pm 4, 0)$ and $(0, \pm 4)$.

## 11. Implicit Differentiation

1. (a) $y' = \frac{3x^2+6x}{2y}$.

   (b) $(1+y')\sin(x+y) = 2 - 2y' \implies y' = \frac{2-\sin(x+y)}{2+\sin(x+y)}$.

   (c) $y' = \frac{20x^3-2x}{2y}$.

2. $2x + 2y + 2xy' - 2yy' + 1 = 0 \implies y' = \frac{-1-2x-2y}{2x-2y}$, so $y'(1,2) = \frac{-1-2-4}{2-4} = 7/2$; hence the slope of the normal is $-2/7$, and the equation of the normal line is $y - 2 = -\frac{2}{7}(x-1)$.

3. We have $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0$; suppose we have a tangent line passing through $(x_0, (\sqrt{c} - \sqrt{x_0})^2)$. Then the equation of this tangent is $y - (\sqrt{c} - \sqrt{x_0})^2 = -\frac{\sqrt{c}-\sqrt{x_0}}{\sqrt{x_0}}(x - x_0)$. When $y = 0$ we obtain the $x$-intercept; $0 = (\sqrt{c} - \sqrt{x_0})^2 - \frac{\sqrt{c}-\sqrt{x_0}}{\sqrt{x_0}}(x - x_0)$ and so $x = \sqrt{x_0 c}$. Similarly, when $x = 0$ we obtain $y = \sqrt{x_0 c} - x_0$. Their sum is therefore $2\sqrt{x_0 c} - x_0 = 2\sqrt{c}(\sqrt{c} - \sqrt{y_0}) - (\sqrt{c} - \sqrt{y_0})^2 = c$.

## 12. Sequences and Series

1. (a) Converges to $1/2$.

   (b) Diverges: $9^{n+1}/10^n = 9^{n+1}/(9+1)^n = 9^{n+1}/(9^n + \cdots) \to \infty$.

2. (a) The series is $2/3 - 2/5 + 2/7 - 2/9 + \cdots$. It has partial sums $2/3, 4/15, 58/105, \ldots$. Converges to $\pi/2$.

(b) The series is $-2/5+4/6-6/7+8/8-10/9+\cdots$. It has partial sums $4/15, -62/125, \ldots$. Diverges (the terms added and subtracted keep growing, so partial sums become very positive and very negative alternately).

## 13. Inverse Functions

1. (a) $y' = \frac{2x}{1+x^4}$.

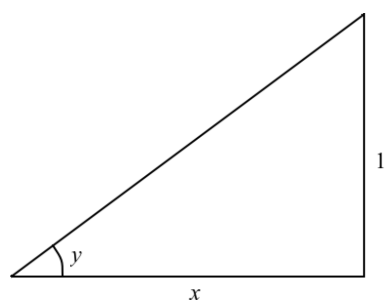
   (b) $f'(x) = \frac{1}{1+x^2}$.

   (c) $g'(x) = \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{1-x}} \frac{1}{1+(\sin^{-1}\sqrt{x})^2}$.

2.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\tan^{-1}x + \frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}\right) &= \frac{1}{2+2x^2} + \frac{1}{4}\frac{x^2+1}{(x+1)^2}\frac{2-2x^2}{(x^2+1)^2} \\
&= \frac{1}{2+2x^2} + \frac{1}{2}\frac{1-x^2}{(x+1)^2(x^2+1)} \\
&= \frac{1}{2}\frac{(x+1)^2+1-x^2}{(x+1)^2(1+x^2)} \\
&= \frac{1}{2}\frac{2x+2}{(x+1)^2(1+x^2)} \\
&= \frac{1}{(x+1)(x^2+1)}.
\end{aligned}
$$

3. Let $y = \cot^{-1}x$, so that $x = \cot y$ and $\frac{\mathrm{d}x}{\mathrm{d}y} = -\csc^2 y$; hence $\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin^2 y$. Consider the following triangle:




So $\sin y = \frac{1}{\sqrt{1+x^2}}$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{1+x^2}$.

## 14. Differentiation Revision

1. (a) $f'(x) = (2017\times 3)x^{2016} - \frac{1}{19x^{20}} + \frac{1}{2017\sqrt[2017]{(x+2)^{2016}}}$.

   (b) $f'(h) = \pi r^2$.

   (c) $f'(\theta) = -\frac{\mu mg(\mu\cos\theta-\sin\theta)}{(\mu\sin\theta+\cos\theta)^2}$.

   (d) $f'(g) = \frac{(g^2+\ln g)\cos g-(2g+1/g)\sin g}{(g^2+\ln g)^2}$.

   (e) $3f(x)+3xf'(x)+2f(x)f'(x) = \frac{3+f(x)-xf'(x)}{[3+f(x)]^2}$ so $f'(x) = \frac{3+f(x)-3[3+f(x)]^2 f(x)}{3[3+f(x)]^2x+2[3+f(x)]^2f(x)+x}$.

2. Let $\theta$ between the angle of the kite string, and let $x$ be the horizontal distance to the kite along the ground (so the length of the string is $\sqrt{50^2+x^2}$). Then $\sin\theta = 50/x$, so $\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{50}{x^2}\frac{\mathrm{d}x}{\mathrm{d}t}$. When the length of the string is 100, $x \approx 86.6$; so $\cos\theta = x/100 \approx 0.866$. Substituting $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$, we have $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{100}{86.6^2}\cdot\frac{1}{0.866} = -0.0154$.

3. The surface area of a cone is $\mathcal{S} = \pi r\sqrt{h^2+r^2}$; we also have $27 = \frac{1}{3}\pi r^2 h$, so $r^2 = \frac{81}{\pi h}$ and

$$\mathcal{S} = \pi\sqrt{\frac{81}{\pi h}\left(h^2+\frac{81}{\pi h}\right)} = \pi\sqrt{\frac{81h}{\pi}+\frac{81^2}{\pi^2h^2}}$$

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}h} = \frac{\pi\left(\frac{81}{\pi}-2\frac{81^2}{\pi^2h^3}\right)}{2\sqrt{\frac{81h}{\pi}+\frac{81^2}{\pi^2h^2}}}$$

In order to find a minimum, we set this derivative to zero and obtain $0 = \frac{81}{\pi} - 2\frac{81^2}{\pi^2h^3}$, so

$$h = \sqrt[3]{2\frac{81}{\pi}} \approx 3.722\,\mathrm{cm}.$$

From this, we find $r = \sqrt{81/\pi h} = 2.63\,\mathrm{cm}$.

4. We begin by parameterising the hyperbola; completing the square, we can transform our equation into standard form:

$$\frac{(x-1)^2}{3} - \frac{y^2}{3} = 1$$

A parameterisation of this is $(1+\sqrt{3}\sec t, \sqrt{3}\tan t)$. Now, given any point $(x_0, y_0)$ we wish to minimise $\mathcal{D}(t) = \sqrt{(x_0-1-\sqrt{3}\sec t)^2+(y_0-\sqrt{3}\tan t)^2}$ with respect to $t$.

   (a) Firstly, consider $(x_0, y_0) = (2, 1)$. Then $\mathcal{D}(t) = \sqrt{(1-\sqrt{3}\sec t)^2+(1-\sqrt{3}\tan t)^2}$. Taking the derivative, we find that:

$$\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t} = \frac{(\sqrt{3}\sec t-1)(\sqrt{3}\sec t\tan t)+(\sqrt{3}\tan t-1)(\sqrt{3}\sec^2 t)}{\sqrt{(1-\sqrt{3}\sec t)^2+(1-\sqrt{3}\tan t)^2}}$$

   Using MATLAB to compute the solution of $\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t} = 0$,

```
vpasolve((sqrt(3) * sec(t) - 1)*(sqrt(3) * sec(t) * tan(t))
        == (1 - sqrt(3)*tan(t))*(sqrt(3)*(sec(t))^2),t)
```

   we find $t \approx 0.3759$; so $(x, y) = (2.8621, 0.6835)$.

   (b) Note that $(3, 1)$ is already on the hyperbola. 😇

## 15. Approximating Areas

1. I use Simpson's rule with $n = 8$:

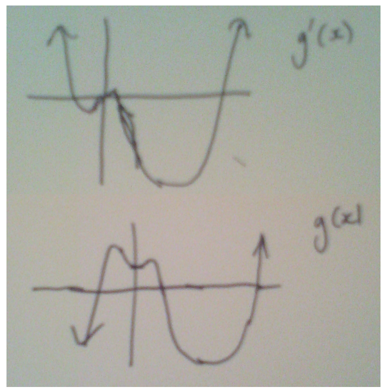
$$\int_0^{1.6} g(x)\,\mathrm{d}x \approx \frac{0.2}{3}(12.1+13.2+4(11.6+11.1+12.2+13.0)+2(11.3+11.7+12.6)) = 19.21.$$

2. Measure the height of the shaded area at each point (using $n = 10$ is probably easiest), collapsing the empty area down (e.g. the height of the function at $x = -1$ is just $3+1 = 4$). Then use some numerical integration method.

3. Like 2. but simpler.

## 16. Anti-differentiation

1. (a) $F(x) = \frac{1}{2}x^2 - 3x + C$
   (b) $f(x) = x^2 + 3x + 2$, so $F(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C$
   (c) $F(\theta) = 2\theta^3 - 7\tan\theta + C$
   (d) $G(h) = \pi^2 h$
   (e) $F(x) = \frac{x^{4.7}}{4.7} + \frac{2}{3}\sqrt{x^3} + \sqrt{7}x^{\sqrt{7}}$
2. $\varphi(x) = x^2 + x + C$; but $\varphi(1) = 6$, so $1 + 1 + C = 6$ and $C = 4$. Hence $\varphi(x) = x^2 + x + 4$, and $\varphi(2) = 10$.
3. See following image.




## 17. The Fundamental Theorem of Calculus

1. $\int_0^{\pi/4} \sec^2\theta \, \mathrm{d}\theta = [\tan\theta]\Big|_0^{\pi/4} = 1.$
2. $\int_1^2 f(x)\,\mathrm{d}x = \int_1^3 f(x)\,\mathrm{d}x - \int_2^3 f(x)\,\mathrm{d}x = 10.$
3. First we find the intersection points; we have $6x = x^2$, so $x \in \{0, 6\}$. Hence we compute

$$\int_0^6 2x - \frac{x^2}{3}\,\mathrm{d}x = [x^2 - \frac{x^3}{9}]\Big|_0^6 = 36 - 6^3/9 = 12.$$

## 18. Substitution

1. (a) $\frac{\csc 3x}{3} + C.$
   (b) $-\frac{\tan 3x^2}{6} + C.$
   (c) $2\sqrt{x} + 3x - 2\ln x + C.$
   (d) $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$
2. Use trig identity: $2\sin 5x \cos 3x = \sin 8x + \sin 2x$. Then

$$\int_0^{\pi/6} \sin 8x + \sin 2x \,\mathrm{d}x = \left[-\frac{\cos 8x}{8} - \frac{\cos 2x}{2}\right]\Bigg|_0^{\pi/6} = 0.4375.$$

3. $\frac{1}{2}\tan^{-1} x^2 + C$. (Substitute $u = x^2$.)

## 19. Differential Equations

1. (a) $\int e^y \, \mathrm{d}y = \int e^t \, \mathrm{d}t$, so $e^y = e^t + C$ and $y = \ln(e^t + C)$.

   (b) $\int \frac{\mathrm{d}y}{y^2} = \int t \, \mathrm{d}t$, so $-\frac{1}{y} = \frac{1}{2}t^2 + \frac{C}{2}$ and $y = -\frac{2}{t^2+C}$.

   (c) $\int \sec^2 y \, \mathrm{d}y = \int \mathrm{d}t$, so $\tan y = t + C$ and $y = \tan^{-1}(t + C)$.

   (d) $\int \sin y \, \mathrm{d}y = \int -t \cos t \, \mathrm{d}t$, so $-\cos y = \cos t - t \sin t - C$ (by the hint) and $y = \cos^{-1}(t \sin t - \cos t + C)$.

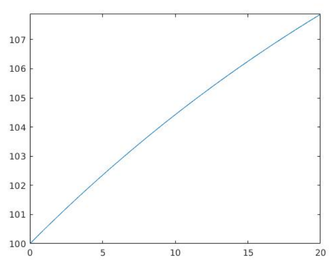
2. Using Newton's law of cooling, $\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_\infty)$ (where $T_\infty$ is the ambient temperature). Solving this differential equation, we find $\int \frac{1}{T - T_\infty} \, \mathrm{d}T = \int k \, \mathrm{d}t$ and so $T = T_0 e^{kt} + T_\infty$. We have $T_\infty = 30°$, and $T_0 = 100°$; also, at $t = 3$ we have $T = 70$ so $70 = 100e^{3k} + 30$; hence $k = \frac{\ln 0.4}{3} = -0.31$ and by direct substitution $T = 100e^{-0.31t} + 30$. Let $T = 31$; then $t = 14.86$ and so the temperature will drop to 31° after around fifteen minutes.

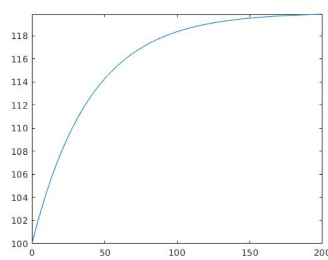
3. (a) We have $\frac{\mathrm{d}V}{\mathrm{d}t} = \text{rate in} - \text{rate out} = 3 - kV$. Hence $\int \frac{1}{3-kV} \, \mathrm{d}V = \int \mathrm{d}t$, so $-\frac{\ln(3-kV)}{k} = t + C$ and $V = \frac{3-Ke^{-kt}}{k}$. At $t = 0$, $V = 100$; so $100k = (3 - K)$. We also have $kV = 3$ where $V = 120$, so $k = 1/40 = 0.025$. Hnce $2.5 = 3 - K$ and $K = 0.5$. It immediately follows that

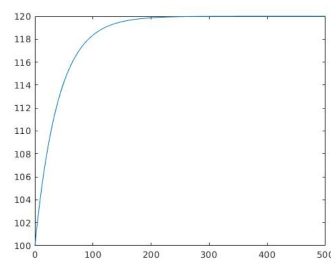
   $$V = \frac{3 - 0.5e^{-0.025t}}{0.025}$$

   and at $t = 10$, $V = 104$ litres.

   (b) The rate of water flow out is $kV = 3 - 0.5e^{-0.025t}$, which is always less than 3 (the rate in). In fact, as $t \to \infty$, the volume tends to 120 L and the rate in tends to equal the rate out.







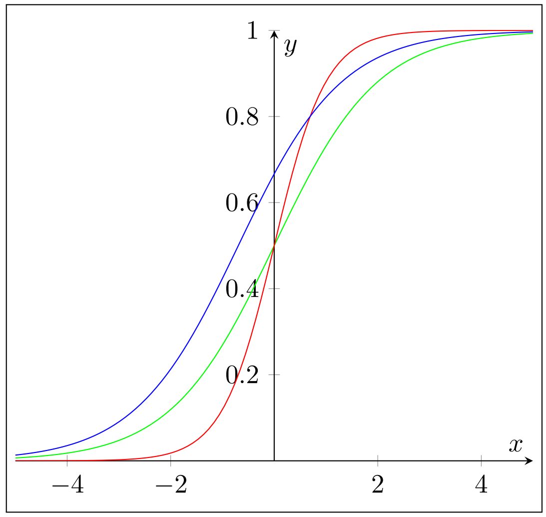

## 20. Partial Fractions

1. (a)

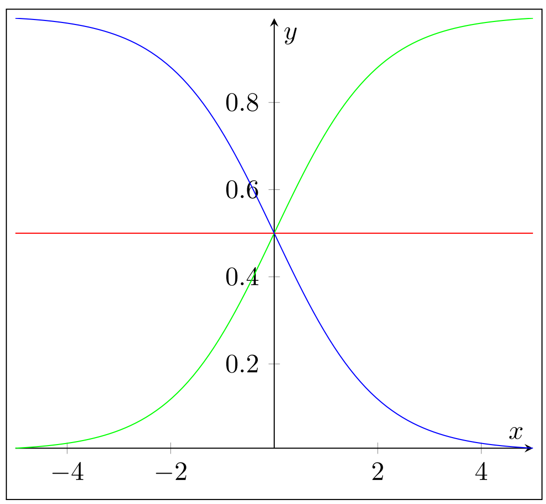
$$\begin{aligned}
\int r \, \mathrm{d}t &= \int \frac{\mathrm{d}P}{P(1-P)} = \\
&= \int \frac{1}{P} + \frac{1}{1-P} \, \mathrm{d}P \\
rt + C &= \ln \frac{P}{1-P} \\
Ke^{rt} &= \frac{P}{1-P} \\
\frac{Ke^{rt}}{1+Ke^{rt}} &= P.
\end{aligned}$$

(b) It should be clear that as $t \to \infty$, $P \to 1$. (If we look at $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{rP}{P_\infty}(P_\infty - P)$, $P \to P_\infty$.)

Green: $r = K = 1$; red: $r = 2$, $K = 1$; blue: $r = 1$, $K = 2$.




(c) $r$ lets us vary how fast the population gets to the maximum. Green: $r = K = 1$; red: $r = 0$; blue: $r = -1$.




(d) Write it yourself.

2. (a) Draw a triangle with angle $x/2$, hypotenuse $\sqrt{1+t^2}$, adjacent edge 1, and opposite edge $t$.

(b)

$$\sin x = 2\sin(x/2)\cos(x/2) = \frac{2t}{1+t^2}$$

$$\cos x = (\cos(x/2))^2 - (\sin(x/2))^2 = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 = \frac{1-t^2}{1+t^2}.$$

(c) We have $x = \tan^{-1} 2t$, so the result follows immediately.

(d) i. Let $t = \tan(x/2)$. Then, substituting, we have

$$\int \frac{1}{1 - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t = \int \frac{1}{t^2}\,\mathrm{d}t = -\frac{1}{t} + C = -\frac{1}{\tan\frac{x}{2}} + C.$$

ii. Similarly,

$$\int \frac{1}{3\frac{2t}{1+t^2} - 4\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t = \int \frac{1}{3t-2+2t^2}\,\mathrm{d}t = \int \frac{1}{(2t-1)(t+2)}\,\mathrm{d}t$$
$$= \frac{1}{5}\ln\frac{1-2t}{t+2} + C = \frac{1}{5}\ln\frac{1-2\tan\frac{x}{2}}{\tan\frac{x}{2}+2} + C.$$

## 21. Integration by Parts

1. (a)

$$\int x\cos 5x\,\mathrm{d}x = \frac{1}{5}x\sin 5x - \int \frac{1}{5}\sin 5x\,\mathrm{d}x = \frac{1}{5}(x\sin 5x + \cos 5x) + C.$$

(b)

$$\int \cos x \ln\sin x\,\mathrm{d}x = \sin x\ln\sin x - \int \sin x\frac{\cos x}{\sin x}\,\mathrm{d}x$$
$$= \sin x\ln\sin x - \int \cos x\,\mathrm{d}x = \sin x(\ln\sin x - 1) + C.$$

(c) Let $u = \sqrt{x}$, so $\mathrm{d}x = 2u\,\mathrm{d}u$ and our integral becomes

$$\int 2u\cos u\,\mathrm{d}u = 2u\sin u - \int 2\sin u\,\mathrm{d}u = 2u\sin u + 2\cos u + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$$

2. (a) Let $u = \theta^2$, so our integral becomes $\frac{1}{2}\int_{\pi/2}^{\pi} u\cos u\,\mathrm{d}u$. From 1(c) above, we know that $\int u\cos u\,\mathrm{d}u = u\sin u + \cos u + C$. Hence the required result is

$$\frac{1}{2}\int\limits_{\pi/2}^{\pi} u\cos u\,\mathrm{d}u = \frac{1}{2}\left[u\sin u + \cos u\right]\Big|_{u=\pi/2}^{\pi} = -\frac{1}{2} - \frac{\pi}{4}.$$

(b) We use integration by parts twice.

$$\int (x^2+1)e^{-x}\,\mathrm{d}x = -e^{-x}(x^2+1) + \int 2xe^{-x}\,\mathrm{d}x$$
$$= -e^{-x}(x^2+1) - 2xe^{-x} + \int 2e^{-x}\,\mathrm{d}x$$
$$= -e^{-x}(x^2+1) - 2xe^{-x} - 2e^{-x} + C.$$

Hence the result we are looking for is $3 - 6e^{-1}$.

3. (a) Apply integration by parts to $\int 1\cdot(\ln x)^n\,\mathrm{d}x$ by integrating 1 and differentiating $(\ln x)^n$.

(b) Applying (a), we find

$$\int (\ln x)^3\,\mathrm{d}x = x(\ln x)^3 - \int (\ln x)^2\,\mathrm{d}x$$
$$= x(\ln x)^3 - (x(\ln x)^2 - \int \ln x\,\mathrm{d}x)$$
$$= x(\ln x)^3 - (x(\ln x)^2 - (x\ln x - x))$$
$$= x(\ln x)^3 - x(\ln x)^2 + x\ln x - x.$$

## 22. Lengths, Volumes, and Areas

1. We simply calculate the relevant integral:

$$\pi \int_1^2 x^{-2}\,\mathrm{d}x = \pi[(-2^{-1}) - (-1^{-1})] = \frac{\pi}{2}.$$

2. Calculating the surface area:

$$\begin{aligned} 2\pi \int_0^\pi \sin x \sqrt{1-\cos^2 x}\,\mathrm{d}x &= 2\pi \int_0^\pi \sin^2 x\,\mathrm{d}x \\ &= \pi\left[x - \sin 2x\right]\Big|_0^\pi \\ &= \pi^2 \end{aligned}$$

So the radius of the equivalent circle is $\sqrt{\pi}$.

3. Summing along the axis from base to point, each slice has an area $\left(\frac{L}{H}x\right)^2 = \frac{L^2}{H^2}x^2$; hence the total volume is

$$V = \int_0^H \frac{L^2}{H^2}x^2\,\mathrm{d}x = \frac{1}{3}L^2 H.$$

4. We have $r = a(1-\cos\theta)$ so $\frac{\mathrm{d}r}{\mathrm{d}\theta} = a\sin\theta$. Hence:

$$\begin{aligned} S &= \int_0^{2\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \sqrt{a^2 - 2a^2\cos\theta + a^2(\sin^2\theta + \cos^2\theta)}\,\mathrm{d}\theta \\ &= a\sqrt{2}\int_0^{2\pi} \sqrt{1-\cos\theta}\,\mathrm{d}\theta. \end{aligned}$$

We turn our attention, then, to the integral $\int \sqrt{1-\cos\theta}\,\mathrm{d}\theta$. Let $u = 1-\cos\theta$; then $\mathrm{d}u = \sin\theta\,\mathrm{d}\theta$; but $\sin\cos^{-1}(1-u) = \sqrt{2u-u^2}$ (this can be verified by drawing a suitable triangle). Hence $\mathrm{d}u = \sqrt{2u-u^2}\,\mathrm{d}\theta$, and

$$\begin{aligned} \int \sqrt{1-\cos\theta}\,\mathrm{d}\theta &= \int \frac{\sqrt{u}}{\sqrt{2u-u^2}}\,\mathrm{d}u \\ &= \int \frac{1}{\sqrt{2-u}} \\ &= -2\sqrt{2-u} + C \\ &= -2\sqrt{1+\cos\theta} + C. \end{aligned}$$

Therefore (and changing our integral to double the integral from 0 to $\pi$ to avoid the problem of having a closed loop),

$$\begin{aligned}
S &= 2a\sqrt{2}\int\limits_0^{\pi} \sqrt{1-\cos\theta}\,\mathrm{d}\theta \\
&= 2a\sqrt{2}\left[-2\sqrt{1+\cos\theta}\right]\Big|_0^{\pi} \\
&= 2a\sqrt{2}\left[(-2\sqrt{1+\cos\pi}) - (-2\sqrt{1+\cos 0})\right] \\
&= 2a\sqrt{2}\left[(-2\sqrt{0}) - (-2\sqrt{2})\right] \\
&= 2a\sqrt{2}\times 2\sqrt{2} = 8a.
\end{aligned}$$

## 23. Trigonometric Substitution

*These ones are tedious and can be checked by the computer, so I have not written full answers for all of them.*

1. Let $x = 2\tan\theta$, so $\mathrm{d}x = 2\sec^2\theta$:

$$\int \frac{2\sec^2\theta}{\sqrt{x^2+4}}\,\mathrm{d}x = \int \frac{\sec^2\theta}{\sqrt{1+\tan^2 x}}\,\mathrm{d}x = \int \sec\theta\,\mathrm{d}x = \ln\left(\sqrt{\left(\frac{x}{2}\right)^2+1}+\frac{x}{2}\right)+C.$$

2. First, let $u = x^7$ so $\mathrm{d}u = 7x^6\,\mathrm{d}x$ and our integral becomes

$$\frac{1}{7}\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \frac{1}{14}\ln\frac{1+u}{1-u}+C = \frac{1}{14}\ln\frac{1+x^7}{1-x^7}+C.$$

3. Use $x = \frac{2}{5}\sec\theta$.
4. Use $x = \frac{2}{3}\sin\theta$.
5. Use $x = \frac{1}{6}\tan\theta$ and simplify.
6. Use integration by parts; the resulting integral $-\frac{\ln x}{4x^4} + \int \frac{\mathrm{d}x}{4x^5}$ is much simpler.
7. Use partial fractions.

## 24. Kinematics

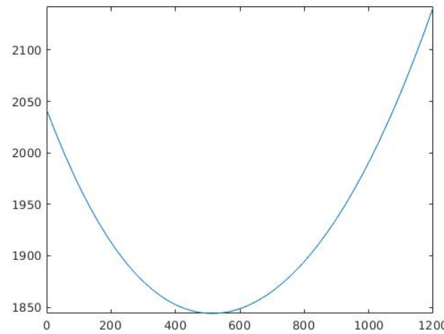
1. (a) $v = \frac{\mathrm{d}h}{\mathrm{d}t} = 122.5 - 9.8t$, so the initial velocity of the flare is $122.5\,\mathrm{m\,s^{-1}}$.
   (b) Zero.
   (c) When $v = 0$, $t = 12$ and the height at this time is around 764 metres.
2. Let $x$ be the distance from the point on the beach directly away from B. Then the total distance travelled is simply $D = \sqrt{600^2+x^2} + \sqrt{800^2+(1200-x)^2}$; taking the derivative:

$$\frac{\mathrm{d}D}{\mathrm{d}x} = \frac{x}{\sqrt{600^2+x^2}} - \frac{1200-x}{\sqrt{800^2+(1200-x)^2}}$$

Setting to zero, we have

$$\begin{aligned} x\sqrt{800^2+(1200-x)^2} &= (1200-x)\sqrt{600^2+x^2} \\ 800^2x^2+(1200-x)^2x^2 &= (1200-x)^2(600^2+x^2) \\ 0 &= 1200^2 600^2 - 600^2 2400x + (600^2-800^2)x^2 \\ x &\in \{-3600, 3600/7\}. \end{aligned}$$

Since $x \geq 0$, $x = 3600/7 \approx 514$. The total distance travelled is therefore around 1844 metres. By graphing $D$ versus $x$, we see that this is indeed the required medium:




## 25. Integration Revision

1. (a) $\int_1^2 \sin x \,\mathrm{d}x = [-\cos x]\Big|_1^2 = \cos 1 - \cos 2.$

   (b) $\int \frac{u^2+1}{u^3+3u} = \frac{1}{3}\ln(u^3+3u)+C.$

   (c) $\int_0^{\pi/6} \tan x\,\mathrm{d}x = [\ln \sec x]\Big|_0^{\pi/6} \approx 0.1438.$

2. We have $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2+4x-4}{2y-4}$, so $\int 2y-4\,\mathrm{d}y = \int 3x^2+4x-4\,\mathrm{d}x$. Hence $y^2-4y = x^3+2x^2-4x+C$; we also have $C=-2$, so $y^2-4y = x^3+2x-4x-2$. We are trying to find $y$ if $x=2$; so $y^2-4y = 8+4-4-2 = 6$. Solving $y^2-4y-6=0$, we find $y = \frac{4\pm\sqrt{38}}{2}$.

3. (a) Let $t = a\tan\theta$. Then:

$$\begin{aligned} \int \frac{a^3}{t^2+a^2}\,\mathrm{d}t &= \int \frac{a^4\sec^2\theta}{a^2\tan^2+a^2}\,\mathrm{d}\theta \\ &= \int \frac{a^2\sec^2\theta}{\sec^2\theta}\,\mathrm{d}\theta \\ &= a^2\theta = a^2\tan^{-1}\left(\frac{t}{a}\right); \end{aligned}$$

   hence $\omega(a,x) = a^2\tan^{-1}\left(\frac{x}{a}\right)$.

   (b) It follows that $\omega(2,2) = 4\tan^{-1}1 = \pi$.

   (c) We wish to find $x$ such that $\pi = 3\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$; in other words, $x = \sqrt{3}\tan\left(\frac{\pi}{3}\right) = 3$.

4. Note first that $\int_{-\pi/2}^{\pi/2} \sin^5 x \, \mathrm{d}x = 0$ since $\sin^5$ is odd. Then, we argue as follows:

$$\begin{aligned}
\int \cos^5 x \, \mathrm{d}x &= \int \cos x (1 - \sin^2 x)^2 \, \mathrm{d}x \\
&= \int (1 - t^2)^2 \, \mathrm{d}t \qquad (t = \sin x) \\
&= \int 1 - 2t^2 + t^4 \, \mathrm{d}t \\
&= t - \frac{2}{3} t^3 + \frac{t^5}{5} + C \\
&= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C.
\end{aligned}$$

Hence

$$\int_{-\pi/2}^{\pi/2} \sin^5 x \, \mathrm{d}x = (\sin \frac{\pi}{2} - \frac{2}{3} \sin^3 \frac{\pi}{2} + \frac{1}{5} \sin^5 \frac{\pi}{2}) - (\sin \frac{-\pi}{2} - \frac{2}{3} \sin^3 \frac{-\pi}{2} + \frac{1}{5} \sin^5 \frac{-\pi}{2}).$$

But $\sin(\pi/2) = 1$; so we have $(1 - \frac{2}{3} + \frac{1}{5}) - (-1 + \frac{2}{3} - \frac{1}{5}) = 2(1 - \frac{2}{3} + \frac{1}{5}) = \frac{16}{15}$.