

NCEA Level 3 Calculus (Integration)

26. More Interesting Problems

These problems do not just concern integration.

Questions

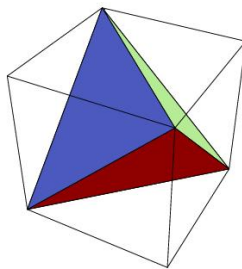
1. **E** Find the equation of the line through the point $(3, 5)$ which cuts off the least area from the first quadrant.
2. **E** The area of a square is increasing at a constant rate of $k \text{ m}^2$ per second. A tetrahedron (equilateral triangular pyramid), has a side length that is the same as that of the square at each instant. The initial volume of the tetrahedron was 1 m^3 . In terms of k , what is the volume of the tetrahedron three seconds after that?
3. **S** Scholarship 2017: The hyperbolic functions $\sinh x$ and $\cosh x$ are defined as follows:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad (1)$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}). \quad (2)$$

Show that $\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$.

4. **S** Consider the tetrahedron inscribed inside a cube, as in the figure.



The volume V of the cube at any instant t is increasing at a rate proportional to the value of V at that instant. The initial volume of the cube at $t = 0$ was 8 cubic units. What is the volume of the tetrahedron at time $t = 20$?

5. **S** If $x \sin \pi x = \int_0^{x^2} f(t) dt$, where f is continuous, find $f(4)$. [*Hint: you need not perform any integration.*]
6. **S** If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.$$

Hence, or otherwise, evaluate

$$\lim_{x \rightarrow 0} \frac{\sin(a+2x) - 2\sin(a+x) + \sin a}{x^2}.$$

7. (a) **S** Consider the differential equation

$$\frac{d^2\Phi}{dt^2} + 5\frac{d\Phi}{dt} + 6\Phi(t) = 0.$$

Let f and g be the functions defined by $f(x) = e^{-2x}$ and $g(x) = e^{-3x}$.

- i. Show that all linear combinations of f and g are solutions to the differential equation.
- ii. Find the (unique) solution passing through $(0, 1)$ and $(1, 1)$.
- (b) O More generally, consider the differential equation $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$. Let the zeroes of the quadratic polynomial $p(D) = aD^2 + bD + c$ be α and β . Show that all the linear combinations of $e^{\alpha x}$ and $e^{\beta x}$ are solutions to the differential equation.
8. S Compute the following definite integral. [*Hint: begin with a substitution.*]

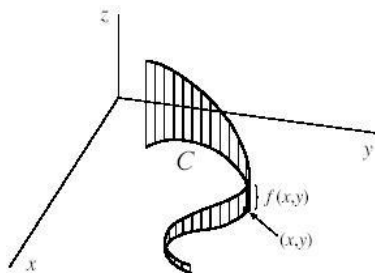
$$\int_0^{\pi/6} \sqrt{\tan \theta} d\theta$$

9. (a) E Consider the two functions $p(x) = 3x^5 - 5x^3 + 2x$ and $q(x) = 3x^5$. Show that their ratio approaches 1 as $x \rightarrow \infty$.
- (b) S Let $p(x)$ and $q(x) \neq 0$ be polynomials. Recall that the degree of a polynomial is the highest n such that x^n has a non-zero coefficient. Compute the limit

$$\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)}$$

if:

- i. the degree of $p(x)$ is less than that of $q(x)$.
- ii. the degree of $p(x)$ is greater than that of $q(x)$.
10. S A definite integral calculates the between a curve and straight line, the x -axis. In a similar way, it is possible to use integration to calculate the area above a curved line and below a surface $z = f(x, y)$, like that in the figure.



If the curve C is defined parametrically, that is $C(t) = (x(t), y(t))$, then the integral along the line can be calculated with the formula

$$\int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Compute the line integral of the function $f(x, y) = 2 + x^2 y$ around the upper half of the unit circle.

11. S The **sine integral** function is defined by

$$\text{Si}(x) = \begin{cases} \int_0^x \frac{\sin t}{t} dt, & \text{for } x \neq 0; \\ 1, & \text{for } x = 0. \end{cases}$$

- (a) Recall that $\int_a^b f'(t) dt = f(b) - f(a)$. Use this to show that $\frac{d}{dx} \int_0^x f'(t) dt = f'(x)$.

- (b) Find the x -coordinate of the first local maximum of Si to the right of the origin. Carefully prove that you have found a maximum.
- (c) Use the result in (a) to find an expression for the integral

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt,$$

where f is continuous and g and h are differentiable.

12. **E** Minimise the function $f(x) = b \log_b N$ with respect to b , and show that the result is independent of the constant N .*
13. **S** We can calculate **improper integrals** (those where the bounds are infinite) as follows:

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

Decide whether each of the integrals given below exists. If so, calculate its value; if not, explain why.

(a) $\int_1^\infty \frac{1}{x} dx$

(b) $\int_1^\infty \frac{1}{x^2} dx$

(c) $\int_1^\infty \sin x dx$

14. (a) **S** Show that $F(x) = \tan^{-1} x$ is an anti-derivative of $f(x) = \frac{1}{1+x^2}$ in the following ways:
- Differentiate $F(x)$ and simplify to give $f(x)$.
 - Use the substitution $x = \tan \theta$ to integrate $f(x)$ and simplify to give $F(x)$.
- (b) **O** Recall that $22/7$ is often given as a rough approximation to π . Consider the integral

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx,$$

and hence show that $22/7 > \pi$.†

15. **S** Consider the operator \mathcal{L} defined by

$$\mathcal{L}f(x) = \frac{d}{dx} \ln [f(e^x)].$$

- Show that $\mathcal{L}x^n = n$ and that $\mathcal{L}[u(x)]^n = n\mathcal{L}u(x)$.
 - Find an expression for $\mathcal{L}[u(x)v(x)]$ and $\mathcal{L}[u(x)/v(x)]$.
 - Find an expression for $\mathcal{L}[u(x) + v(x)]$.
 - For which y is $\mathcal{L}y = y$?
16. **S** Compute the following indefinite integrals:

(a) $\int \frac{\sin \frac{1}{x}}{x^2} dx$

(b) $\int \frac{\ln x \cos x - \frac{\sin x}{x}}{(\ln x)^2} dx$

* Dudley, *Mathematical Cranks*, p.52.

† Nahin, *Inside Interesting Integrals*, pp.23-4.