NCEA Level 3 Calculus (Differentiation)
3. Derivatives of Common Functions

Now that we understand the purpose and form of the derivative, we can begin to calculate the derivatives of some common functions. We list them here without proof, but it is easy to find the proofs yourself in a textbook or on the internet if you are interested.

Function	**Derivative**	**Notes**
x^n	nx^{n-1}	
e^x	e^x	Here, $e \approx 2.71828$ is Euler's number.
$\ln\lvert x\rvert$	$\frac{1}{x}$	Here, $\ln = \log_e$.
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\csc x$	$-\csc x \cot x$	
$\sec x$	$\sec x \tan x$	
$\cot x$	$-\csc^2 x$	

We also have several rules for finding the derivative of a more complicated function in terms of the derivatives of component functions.

Theorem. *Let f and g be functions, and let λ be a real constant. Then:*

1. $(\lambda)' = 0$*: The slope of a constant function is zero.*
2. $(\lambda f)' = \lambda f'$*: The slope of a scaled function is the scaled slope of the function.*
3. $(f+g)' = f' + g'$*: The slope of the sum of two functions is the sum of the individual slopes.*

These formulae can all be proved using the limit definition of the derivative, and it is a simple exercise to check them all. For example,

$$\begin{aligned}(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h} &= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= f'(x) + g'(x).\end{aligned}$$

Note that the obvious product rule, $(fg)' = f'g'$ does **not** hold. We will discuss this further soon, although it should be noted that Leibniz initially believed this rule to be true! A counterexample is outlined in the exercises for you to work out.

Examples.

1. $\frac{\mathrm{d}}{\mathrm{d}x}[\sin x + \cos x] = \cos x - \sin x$.
2. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{3x^2+2x+1}{x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[3x+2+x^{-1}\right] = 3 + 0 + (-1)x^{-2} = 3 - \frac{1}{x^2}$.
3. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{\mathrm{d}}{\mathrm{d}x}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

Applications. Many phenomena in physics can be modelled with sine waves; for example, if a particle on the end of a spring is moving with simple harmonic motion, then it has position $x = A\sin(\omega t + \phi)$; taking derivatives, we find that it has velocity $v = \frac{\mathrm{d}x}{\mathrm{d}t} = A\omega\cos(\omega t + \phi)$ and acceleration $a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -A\omega^2\sin(\omega t+\phi)$. In other words, it is always accelerating in the opposite direction to its movement!

Questions

1. Find the derivatives of $3x^3$, $2x^2$, and $6x^5$. Conclude that $(fg)' \neq f'g'$ in general. A

2. Find the derivatives of the following functions with respect to t: A
   (a) $y = 2t^3 + 3t^2$
   (b) $y = \sqrt{t}$
   (c) $y = (2t+1)(t-4)$
   (d) $g(t) = 4\sec t + 9\tan t$
   (e) $h(t) = \sqrt[5]{t} + 2\csc t - \ln t^3$
   (f) $\phi'(t) = \csc x + 12x^{1273} + 9$
   (g) $y = 2017t^{2016} + (t+2)^2$
   (h) $y = 940\sin t + \frac{1}{2}e^{t+2}$

3. Where is the function $x \mapsto x^3 - 2x^2 - x + 1$ increasing? A

4. Find the velocity v of a particle at time $t = 2\pi$ if its position function for $t > 0$ is $x = e^t - \sin t$. A

5. Find the slope of the tangent line to $y = x + \tan x$ at (π, π). A

6. It is **not** true that the derivative of $f(g(x))$ is $f'(g'(x))$. A
   (a) For a counterexample, consider $f(x) = x^2$ and $g(x) = x$; show that $f'(g'(x)) = 2$, but $\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = 2x$.
   (b) Compute the derivative of $\ln x^2$.

7. Suppose the derivative of a function is $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - x - 4$. What could the original function be? A

8. Find the 64th derivative of $\sin x$. M

9. Find the nth derivative of x^n. M

10. If $y = 2\sin 3x \cos 2x$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. (Hint: use an identity to rewrite this as a sum of functions.) M

11. For which values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent? M

12. Show that $y = 6x^3 + 5x - 3$ has no tangent line with a slope of 4. E

13. Find real values of α and β such that, if $y = \alpha\sin x + \beta\cos x$, then $y'' + y' - 2y = \sin x$. E

14. Consider a $12\,\mathrm{m}$ long ladder leaning against a wall such that the top of the ladder makes an angle θ with the wall. If this angle θ is varied, the distance D between the bottom of the ladder and the wall also changes. If $\theta = \pi/3$, what is the rate of change of D with respect to θ? E

15. Prove that the function φ given by $\varphi(x) = \frac{x^{101}}{101} + \frac{x^{51}}{51} + x + 1$ has no extreme values. E

16. The derivative is primarily a geometric concept, not an algebraic one. M
    (a) The area of a circle of radius r is $A = \pi r^2$. Find $\frac{\mathrm{d}A}{\mathrm{d}r}$. What do you notice?
    (b) Explain part (a) geometrically.
    (c) The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. Find an expression for the surface area.

17. We have the derivative of $\log_e$, but not for any other log base. Calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = \log_{10} x$. E

18. Prove the remaining two differentiation rules using the limit definition of the derivative. E

19. In this exercise, we will calculate the value of e based on solving the differential equation $f(x) = f'(x)$. Clearly one solution is $f(x) = 0$, but by drawing some pictures you should see that a reasonable guess for a more interesting solution would be an exponential function, $f(x) = a^x$. We just need to pick the right base. S

(a) Show that the derivative of a^x is given by

$$\lim_{h \to 0} a^x \left(\frac{a^h - 1}{h} \right).$$

(b) For a^x to be a solution to our differential equation, we need $\lim_{h\to 0} \frac{a^h - 1}{h}$ to be 1. Show that

$$a = \lim_{h \to 0} (1 + h)^{1/h}$$

works for this purpose.

(c) The expression in (b) is not the standard way to write this limit. Show that if $n = 1/h$, then

$$a = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n.$$

(d) By calculating the value of $\left(1 + \frac{1}{n}\right)^n$ for a suitably large value of n, obtain a numerical approximation for Euler's number, e.