NCEA Level 3 Calculus (Integration)
25. Integration Revision

Questions

1. True or false: A
   (a) $\int_a^b \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = y(b) - y(a)$
   (b) $\int_0^1 f(x)\,\mathrm{d}x + \int_0^1 g(x)\,\mathrm{d}x = \int_0^1 f(x) + g(x)\,\mathrm{d}x$
   (c) $\int_0^2 f(x)\,\mathrm{d}x + \int_0^1 f(x)\,\mathrm{d}x = \int_1^2 f(x)\,\mathrm{d}x$
   (d) $\int \sin(x)\,\mathrm{d}x \int \cos(x)\,\mathrm{d}x = \int \sin(x)\cos(x)\,\mathrm{d}x$
   (e) $\int \frac{1}{u}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x$
   (f) A definite integral always represents the area under a curve.
   (g) If $u = 2x$ then $\int \sqrt{2x}\,\mathrm{d}x = \int \sqrt{u}\,\mathrm{d}u$.
   (h) The indefinite integral of $\ln x$ is just $\ln x$.

2. Compute the following indefinite integrals: A
   (a) $\int \sin x\,\mathrm{d}x$
   (b) $\int \sec 3x \tan 3x\,\mathrm{d}x$
   (c) $\int \frac{2x^4 - x^2}{x^3}\,\mathrm{d}x$
   (d) $\int \sin x \cos x\,\mathrm{d}x$
   (e) $\int \sin^2 x\,\mathrm{d}x$
   (f) $\int \frac{1}{2u}\,\mathrm{d}u$
   (g) $\int \frac{2x + \sec^2 x}{2\sqrt{x^2 + \tan x}}\,\mathrm{d}x$
   (h) $\int \frac{t^4 - 1}{t - 1}\,\mathrm{d}t$
   (i) $\int \frac{t^{2017} + \sqrt{t^{2017}} + \sqrt[3]{t^{2017}}}{\sqrt[2017]{t^2}}\,\mathrm{d}t$
   (j) $\int \sec\theta\,\mathrm{d}\theta$ (hint: multiply through by $\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$)
   (k) $\int \sec x \tan x + \sec^2 x \tan x\,\mathrm{d}x$

3. Compute the definite integrals: A
   (a) $\int_0^1 x^2\,\mathrm{d}x$
   (b) $\int_0^{\pi/2} \tan(x/2)\,\mathrm{d}x$
   (c) $\int_0^4 x^3 + x^2 + \sqrt{x}\,\mathrm{d}x$
   (d) $\int_1^e \frac{1}{t}\,\mathrm{d}t$

4. Suppose $f'(x) + \frac{f(x)}{x} = 0$, and $f(1) = 1$. Find $f(x)$ explicitly. M

5. If $\int_A^B f(x)\,\mathrm{d}x = 3$ and $\int_A^C f(x)\,\mathrm{d}x = 4$, find $\int_B^C f(x)\,\mathrm{d}x$. M

6. A function ϕ, whose graph passes through the origin, is such that the slope of ϕ at any given point x is exactly $2\phi(x)$. Find ϕ exactly. M

7. Find A such that $\int_0^A \sin x\,\mathrm{d}x$ is maximised. E

8. Find B such that $\int_B^{B^2} x^2 - 4x - 4 \, \mathrm{d}x$ is maximised. E

(Note on the above two questions: the problem of finding a function $q(t)$ such that the integral of some expression involving $q(t)$ and its derivative is minimised over a given interval is a fundamental problem in physics, surprisingly!)

9. Find the area of the region bounded by $y = 1 + x^2$, $y = -1 - x^2$, $x = 1$, and $x = -1$. M

10. Use integration to find the area enclosed between the curve $y = e^{2x} - \frac{1}{e^{3x}}$ and the lines $y = 0$, $x = 0$, and $x = 1.2$. M

11. A function f is *even* if $f(-x) = f(x)$ for all x in its domain. M
    (a) What geometric property does the graph of an even function have?
    (b) Suppose f is an even function with $\int_0^7 f(x) \, \mathrm{d}x = 20$. Find:
        i. $\int_{-7}^7 f(x) \, \mathrm{d}x$
        ii. $\int_0^7 3f(x) + 2 \, \mathrm{d}x$

12. A function y is implicitly defined in terms of x in each case; find $y(x)$ explicitly. M
    (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2$, $y(1) = 1$.
    (b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{3y}$, $y(\pi/6) = 1$.

13. Use integration to find the area enclosed between $y = 1 - 0.2x^4$ and $y = 0.4x^4$. M

14. Mr Leibniz has a continer of oil and places it in the garage. Unfortunately, he puts the container on top of a sharp nail and it begins to leak. The rate of decrease of the volume of oil in the container is given by the differential equation $\frac{\mathrm{d}V}{\mathrm{d}t} = -kVt$, where V is the remaining volume of oil remaining after t hours have passed. The volume of oil in the container when it was placed in the garage was 3000 mL; after twenty hours have passed, the volume remaining is 2400 mL. How much (if any) oil will remain in the container after 96 hours have passed? E

15. An object has acceleration (in one dimension) $a(t) = 0.2t + 0.3\sqrt{t}$ for $0 \leq t \leq 10$. At $t = 4$, the velocity of the object is $5 \, \mathrm{m\,s^{-1}}$. How far has the object travelled after nine seconds? M

16. The formula for integration by parts is $\int f(x)g'(x) \, \mathrm{d}x = f(x)g(x) - \int f'(x)g(x) \, \mathrm{d}x$. S
    (a) Find $\int \ln x \, \mathrm{d}x$ explicitly.
    (b) Hence, or otherwise, find $\int_1^2 4x \ln x \, \mathrm{d}x$.

17. (a) i. Show that $\frac{3}{x+1} - \frac{4}{x-2} = \frac{-x-10}{x^2-x-2}$. E
        ii. Hence, or otherwise, compute $\int_0^1 \frac{-x-10}{x^2-x-2} \, \mathrm{d}x$. E
    (b) Find β such that $\int_\beta^{2\beta} \frac{1}{x^2+5x+6} \, \mathrm{d}x = 1$. S

18. Use the trapezium rule to approximate the value of $\int_1^4 x^x \, \mathrm{d}x$. This function cannot be integrated in terms of elementary functions. A

19. The indefinite integral $\int e^{-x^2} \, \mathrm{d}x$ cannot be integrated in terms of elementary functions, but it can be shown that the definite integral $\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x$ has the value $\sqrt{\pi}$. Most of the curve lies within the bounds $-2 \leq x \leq 2$. Use Simpson's rule to approximate the value of $\int_{-2}^{2} e^{-x^2}$, and compare this with the expected value. M

20. Consider the function implicitly defined by $y(x) = x + \int_0^{\sqrt{2}} y(x) \, \mathrm{d}x$. If the graph of the function includes the point $(0, 1)$, find $y(x)$ explicitly. M

21. An object is at $4°$ in a refrigerator. It is removed and placed on a shelf with an ambient temperature of $20°$. After two minutes, the object has warmed to $5°$. How long will it take for the object to reach $10°$? [*Use Newton's law of cooling.*] E

22. A property owner assumes that the rate of increase of the value of his property at any time is proportional to the value, $\$V$, of the property at that time. E

    (a) Write the differential equation that expresses this statement.

    (b) The property was valued at \$365 000 in May 2012, and at \$382 000 in November 2013. Solve the differential equation from (a) to find the price that the owner paid for the property in May 2007 when he purchased the property, given his assumption is accurate.

23. Find x such that $\int_1^x \frac{\ln u}{u}\,\mathrm{d}u = 1$. M

24. The formula for surface area of the volume of revolution of $y = f(x)$ is $2\pi\int_a^b f(x)\sqrt{f'(x)+1}\,\mathrm{d}x$. Find the area of the surface obtained by rotating about the x-axis the part of the curve $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y$ that lies between $y = 1$ and $y = e$. E

25. The base of a solid is the region bounded by the parabolae $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid, if cross-sections perpendicular to the x-axis are squares with one side lying along the base. E

26. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. S

27. Scholarship 2016: Consider $I_n = \int_0^{\pi/2} \frac{\sin 2nx}{\sin x}\,\mathrm{d}x$, where $n \geq 0$. Show that $I_n - I_{n-1} = \frac{2(-1)^{n-1}}{2n-1}$. O

28. Scholarship 2010: A flower pattern is constructed by using a sinusoidal function $r(\theta)$ to define the distance from the origin to the curve at a radial angle θ. The x and y coordinates of a point on the curve are given by the following equations, where $0 \leq b \leq a$ and n is a positive integer (the number of petals). S

$$r(\theta) = a + b\sin(n\theta) \qquad x(\theta) = r(\theta)\cos(\theta) \qquad y(\theta) = r(\theta)\sin(\theta)$$

    (a) The area inside such a function is given by

$$A = \frac{1}{2}\int_0^{2\pi} (r(\theta))^2\,\mathrm{d}\theta$$

    when $r(\theta) \geq 0$.
    Show that the area of a flower pattern is $\pi\left(a^2 + \frac{1}{2}b^2\right)$.

    (b) A lemon squeezer with base radius a_0 and height H is made to the following specifications.
    At a height h (where $0 \leq h \leq H$) the cross-section is a flower pattern with

$$a(h) = \frac{H-h}{H}a_0 \text{ and } b(h) = \frac{h}{H}a(h).$$

    Use integration with respect to h to show that the volume of a lemon squeezer is **exactly** 5% **greater than** the volume of a cone with the same base and height.

29. Evaluate the following: O

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^x \left(\int_0^{\sin t} \sqrt{1+u^4}\,\mathrm{d}u\right)\mathrm{d}t$$

[*Hint: no integration is required. Use the FTC and the (differentiation) chain rule.*]