

NCEA Level 3 Calculus (Differentiation)

14. Differentiation Revision

Halfway there! Let's do a bit of revision.

Questions

1. **A** True or False:

- (a) If a function f is continuous around a point, then it is differentiable at that point.
- (b) If a function f is differentiable around a point, then it is continuous at that point.
- (c) If f and g are differentiable, then $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$.
- (d) If f and g are differentiable, then $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$.
- (e) $\frac{d}{dx}(x + 3)^2 = 2(x + 3)$.
- (f) $\frac{d}{dx}(x^2 + 3)^2 = 2(x^2 + 3)$.
- (g) $\frac{d}{dx} \tan^2 x = \frac{d}{dx} \sec^2 x$.
- (h) $\frac{d^2 y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.
- (i) If $A = \pi r^2$, then $dA\pi = 0$.

2. **A** Find $\frac{dy}{dt}$ in each case.

- | | |
|------------------------------------|---|
| (a) $y = t^2 + 3t$ | (g) $y = \sin^2 t$ |
| (b) $y = \frac{4-t}{3+t}$ | (h) $y = \sin t^2$ |
| (c) $y = (t^4 - 3t^2 + 5)^2$ | (i) $y = \cos \tan t$ |
| (d) $y = (t + 1)^{2017}$ | (j) $y = (27t + 3)^{2017}(t^2 - \sqrt{t})^{2020}$ |
| (e) $y = \frac{3t+4t^2}{\sqrt{t}}$ | (k) $y = \left(t + \frac{1}{t^2}\right)^{\sqrt{7}}$ |
| (f) $y = \sin 2t$ | |

3. **M** Find the equation of the tangent line to the curve $\sqrt{1 + 4\sin x}$ at the point $(0, 1)$.

4. **A** Find y'' in each case:

- (a) $y = 3x^3 + 2x + \sqrt{2x} + \frac{1}{x^2}$
- (b) $y = e^{2x}$
- (c) $y = \sqrt{4t + 1}$
- (d) $y = 4\sin^2 x$

5. **M** Find the n th derivative of e^{2x} (where n is a natural (counting) number).

6. **E** The height of a projectile after t seconds can be modelled by $h = 3t(t - 10)$. At what time is the height of the projectile at a maximum? Use the second-derivative test to prove that you have found a maximum.

7. M Find $f'(x)$ if:
- $f(x) = \frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + 3$
 - $f(x) = 2 \ln \left| \frac{x-2}{x} \right| + \frac{2}{x} + 7$
8. M In each case, find y' in terms of x and y :
- $x^2 + y^2 = 4$
 - $x^2 + 4xy + y^2 = 13$
 - $xy^4 + x^2y = x + 3y$
 - $x^2 \cos y + \sin 2y = xy$
9. M By differentiating the double-angle formula for cosine,
- $$\cos 2x = \cos^2 x - \sin^2 x,$$
- obtain the double-angle formula for the sine function.
10. M Find f' in terms of g' if $f(x) = x^2g(x)$.
11. M The volume of a cube is increasing at a rate of $10 \text{ cm}^3 \text{ min}^{-1}$. How fast is the surface area increasing when the length of an edge is 30 cm ?
12. M The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$.
- Find the rate of change of volume with respect to height if the radius is constant.
 - Find the rate of change of volume with respect to radius if the height is constant.
13. M A particle moves along a horizontal line such that its coordinate at time t is $x = \sqrt{b^2 + c^2 t^2}$ ($t \geq 0$), where b and c are positive constants. Find its velocity and acceleration functions.
14. M From first principles, show that:
- $\frac{d}{dx} x^2 = 2x$
 - $\frac{d}{dx} [2x^3 + 2x] = 6x^2 + 2$
15. E Find the derivative of $f(x) = \frac{4}{\sqrt{1-x}}$ from first principles.
16. E Each limit represents the derivative of a function f at a point a . Identify each function and point.
- $\lim_{h \rightarrow 0} \frac{\cos(\pi+h)+1}{h}$
 - $\lim_{t \rightarrow 1} \frac{t^4+t-2}{t-1}$
17. M Find the best linear approximation to $f(x) = \sqrt{25-x^2}$ near $x = 3$.
18. M A balloon is rising at a constant speed of 2 m s^{-1} . A girl is cycling along a straight road at a speed of 5 m s^{-1} . When she passes under the balloon, it is 15 m above her. How fast is the distance between the person and the balloon increasing 3 seconds later?
19. E On a straight shoreline there is a tree and exactly opposite it, 100 m away in the sea, stands a lighthouse. A strong and thin spotlight on its top revolves at the rate of one revolution per 4 seconds, its light creating a running light spot on the shore. You stand on the shore 100 m from the tree. How fast does this spot move when it goes past you?
20. M Find $\frac{dy}{dx}$ in terms of t for the following parametrically-defined curves.
- $t \mapsto (1 + e^{2t}, e^t)$

- (b) $x = \tan t$, $y = \sec 2t$
- (c) $x = \frac{t^2-10}{t^2+1}$, $y = tx$
21. E Find the lowest point on the curve $\gamma : t \mapsto (t^3 - 3t, t^2 + t + 1)$. Prove you have found a minimum.
 22. M Find the acceleration of a particle at time t if its displacement from the origin at time t is $-t^6 + 5t^4 + \sin t$.
 23. E A piece of wire 10 m long is cut into two pieces; one is bent into a square and the other into a circle. Where should the wire be cut to ensure the total area of the enclosed shapes is (a) a minimum and (b) a maximum?
 24. M A cone is made by cutting a sector out of a circle of paper of radius R and gluing together the edges of the cut. Find the maximum possible volume of the cone.
 25. E Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r .
 26. E At which points does the ‘bouncing wagon’ curve $2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$ have horizontal tangents?
 27. E Salt forms a cone as it falls from a conveyor belt. The slant of the cone forms an angle of 30° with the horizontal. The belt delivers salt at a rate of 2 m^3 per minute. When the radius of the cone is ten metres, what is the rate of increase of the slant height (measured along the surface of the cone)?
 28. S Suppose we take a circle of radius r , and inscribe within it a triangle such that each corner of the triangle is located on the circle. What is the maximum possible area of the triangle?
 29. E Take derivatives of the following functions with respect to x :
 - (a) $f(x) = 2^x$
 - (b) $g(x) = \log_{\log x} x$
 30. S Scholarship 2017: For $y = x^{(x^x)}$, find $\frac{dy}{dx}$ when $x = 2$.
 31. S Scholarship 2015 (adapted): A car is driving along a road shaped like a parabola at night. The parabola has a vertex at the origin, and the car starts at a point 100 m west and 100 m north of the origin.
 - (a) Write an equation modelling the road as a parabola.
 - (b) Find the general equation for the tangent line to the parabola at some point (x_0, y_0) , and substitute into it the parabola equation to obtain an equation only in x , x_0 , and y_0 .
 - (c) Suppose there is a statue of the Roman emperor Augustus located 100 m east and 50 m north of the origin. Write the equation for the tangent line of the parabola passing through the statue (so that it only depends on a value x on the parabola).
 - (d) Hence find the single point (x, y) on the road where the headlights of the car illuminate the statue.
 32. O (Difficult) Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.
 33. S (Difficult) A cone of radius r centimetres and height h centimetres is lowered point first at a rate of 1 cm^2 into a cylinder of radius R centimetres that is partially filled with water. How fast is the water level rising at the instant that the cone is fully submerged?