

NCEA Level 3 Calculus (Integration)

20. Partial Fractions

Goal for this week

To learn to integrate ratios of polynomials.

Definition (Rational Function). A **rational function** is a function f which can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

for suitable polynomials p and q (where $q \neq 0$).

We can already integrate some rational functions; in particular, those of the form $f(x) = p'(x)/p(x)$:

$$\int \frac{p'(x)}{p(x)} dx = \ln|p(x)|.$$

This week we will learn a technique that, in theory, allows us to integrate *all* rational functions. To understand the idea, note that we can easily integrate all functions of the form

$$f(x) = \frac{A}{(ax + b)^n}$$

for real constants A , a , b , and n .

Our task is simply to ‘deconstruct’ arbitrary fractions into this form.

Example.

$$\frac{2}{x-4} + \frac{3}{x+1} = \frac{5x-10}{x^2-3x-4}$$

so

$$\int \frac{5x-10}{x^2-3x-4} dx = \int \frac{2}{x-4} + \frac{3}{x+1} dx = 2\ln|x-4| + 3\ln|x+1|.$$

In effect, the technique of partial fractions is the reverse of this: we *decompose* the more complex rational function into two or more functions which are easier to integrate.

Let $f(x) = \frac{p(x)}{q(x)}$. Then we have four cases

1. $q(x)$ is the product of distinct linear factors.
2. $q(x)$ is the product of linear factors, some of which are repeated.
3. $q(x)$ is the product of distinct factors, some of which are irreducible quadratics.
4. $q(x)$ contains a repeated irreducible quadratic factor.

Note: the degree of p must be less than the degree of q , so you may need to use long division before applying the technique of partial fractions.

We consider only the first two cases here. See Stewart §7.4 for the others.

Type I: Distinct linear factors

Suppose that $q(x) = (\alpha_1 x + \beta_1) + \cdots + (\alpha_n x + \beta_n)$. Then the partial fraction decomposition is of the form

$$\sum_{i=1}^n \frac{A_i}{\alpha_i x + \beta_i}.$$

Example.

$$\frac{11x - 2}{6x^2 + x - 1} = \frac{11x - 2}{(2x + 1)(3x - 1)} = \frac{A}{2x + 1} + \frac{B}{3x - 1}.$$

So $11x - 2 = A(3x - 1) + B(2x + 1)$. Let $x = 1/3$, so $B = \frac{11/3 - 2}{2/3 + 1} = 1$; then let $x = -1/2$, so $A = \frac{11/2 + 2}{3/2 + 1} = 3$.

Hence

$$\int \frac{11x - 2}{6x^2 + x - 1} dx = \int \frac{3}{2x + 1} + \frac{1}{3x - 1} dx = \frac{3}{2} \ln|2x + 1| + \frac{1}{2} \ln|3x - 1| + C.$$

Type II: Repeated linear factors

Suppose some factor $(\alpha_i x + \beta_i)^r$ appears in the factorisation of $Q(x)$. Then the partial fraction decomposition will include

$$\sum_{j=1}^r \frac{A_{ij}}{(\alpha_i x + \beta_i)^j}.$$

Example. Consider $\int \frac{2x+4}{x^3-2x^2} dx$. We wish to find a partial fraction expansion:

$$\begin{aligned} \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} &= \frac{2x+4}{x^3-2x^2} \iff 2x+4 = Ax(x-2) + B(x-2) + Cx^2 \\ &\iff 2x+4 = (A+C)x^2 + (B-2A)x - 2B \end{aligned}$$

Matching coefficients, we find $B = -2$, $A = -2$, and $C = 2$. Then:

$$\begin{aligned} \int \frac{2x+4}{x^3-2x^2} dx &= \int \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2} dx \\ &= -2 \ln|x| + \frac{2}{x} + 2 \ln|x-2| + C \\ &= 2 \ln \left| \frac{x-2}{x} \right| + \frac{2}{x} + C. \end{aligned}$$

Questions

1. Find $\int \frac{A}{(ax+b)^n} dx$ if A , a , b , and n are real constants.

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2. Evaluate, using partial fractions:

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(a) $\int \frac{3x-1}{(x-3)(x+4)} dx$

(b) $\int \frac{1}{x^2-3x-4} dx$

(c) $\int \frac{1}{x^2-6x-7} dx$

(d) $\int \frac{11x+17}{2x^2+7x-4} dx$

(e) $\int \frac{5x-10}{x^2-3x-4} dx$

(f) $\int \frac{x+7}{x^2-x-6} dx$

(g) $\int \frac{1}{x^2+5x+6} dx$

(h) $\int \frac{2x^2+3}{x(x-1)^2} dx$

3. Some more interesting problems:

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(a) Rewrite in the form $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$ and integrate:

$$\int \frac{4x}{x^3-x^2-x+1} dx.$$

(b) Use the obvious substitution and divide through:

$$\int \frac{\sqrt{x+1}}{x} dx.$$

4. Use appropriate substitutions to evaluate:

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(a) $\int \frac{\cos \theta}{\sin^2 \theta + 4 \sin \theta - 5} d\theta$

(b) $\int \frac{e^{3x}}{e^{2x}+4} dt$

(c) $\int \frac{5+2 \ln x}{x(1+\ln x)^2} dx$

5. Don't use a sledgehammer to kill a fly, and compute the following:

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(a) $\int \frac{x^2(5x^2 + 4x - 3)}{x^5 + x^4 - x^3 + 1} dx$,

(b) $\int \frac{x^2 + 1}{x(x^2 + 3)} dx$

6. We have already computed $\int \sec x dx$ via a bit of a trick, but we could also use partial fractions.

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(a) Show that $\sec x = \frac{\cos x}{1 - \sin^2 x}$.

(b) Hence, or otherwise, compute $\int \sec x dx$.

7. Solve the following differential equation for $y(x)$:

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$$\frac{dy}{dx} = \frac{y^2 - a^4}{x^2 - a^2}$$

8. Scholarship 2008:

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(a) $\frac{A}{x} + \frac{B}{P-x} = \frac{1}{x(P-x)}$ where x is a variable and P is a constant. Find A and B in terms of P .

(b) When a rumour about a teacher is started at a school of size P students, it spreads at a rate (in students per day) that is proportional to the product of the number of students who know the rumour, N , and those who do not. Find an expression for the number of students N who know the rumour after t days.

(c) For a particular rumour about a teacher, 0.5% of students know the rumour initially. The principal will need to act to stop the rumour once more than half the school's students know it. When $\frac{1}{5}$ of the students know the rumour, the number who know the rumour is increasing at a rate of $0.08P$ students per day. How long will it be before the principal must act?

9. Scholarship 2015: The rate of spread of a rumour at a particular school is proportional to both the number of students who know a rumour, S , and the number of students who do not. If N is the total number of students in the school, then $\frac{dS}{dt} = kS(N - S)$. Initially, two students knew the rumour. Show

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that the number of students who know the rumour at time t is $S(t) = \frac{N}{1 + \frac{1}{2}e^{-kNt}(N-2)}$.

10. Recall that $\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2+1}$.

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(a) Find a partial expansion of the given rational function as follows:

$$\frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} = \frac{A}{3x - 1} + \frac{Bx + C}{x^2 + 1}$$

(b) Hence (or otherwise) compute:

$$\int_{\pi/4}^{\pi/3} \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} dx.$$

11. (Revenge of the limits.) Compute the following series. [Hint: this sheet is on partial fractions.]

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$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \left(\frac{1}{n(n+1)} \right)$$