Solutions

Alex Elzenaar

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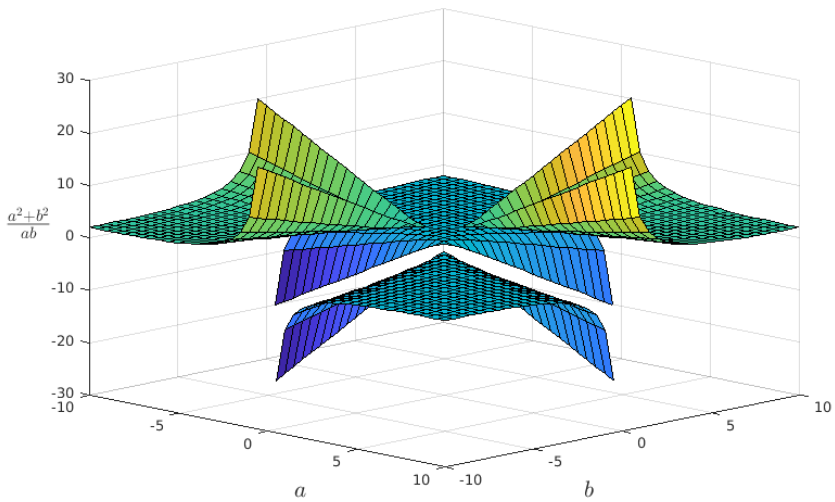


Contents

1 Introduction

What does it really mean to *solve* an equation? These notes attempt to briefly present an outline to Level 3 and Scholarship methods for both finding solutions to equations and interpreting those solutions.

We will begin with simple linear equations, and will work our way up to arbitrary polynomials. Geometric and algebraic interpretations will be presented, along with a number of examples and exercises.

Philosophically, we have made the decision to leave a number of important results (chiefly de Moirve's Theorem) as exercises. However, these results are required knowledge for the later problems — so the reader should at least glance at previous problems before attempting later ones.

Some exercises are much more difficult than others, and the difficulty does not always increase within a section (i.e. sometimes the first exercises can be quite hard). The number of stars by each problem is roughly indicative of a mixture of difficulty and time required. There is, however, no guarantee that the author's idea of a difficult question will match up with the reader's idea of a difficult question!

Note that there are fully worked answers for all problems, and often these solutions contain insights or additional information not in the main text. They are designed to be read in conjunction with the problems (but have a go yourself before reading the solutions). Additional information on the problems themselves, as well as a discussion of some notation, can be found in the introduction to the solutions.

The culmination of the text comes in sections 6 and 8; the former is a digression into more pure mathematics (developing a theory of the roots of unity), and the latter is a more applied section (developing the solution of the general cubic). Both should be easily within reach of the enthusiastic student.

2 Linear Equations

The following quote is known as *Diophantus' Riddle*.

> God gave him his boyhood one-sixth of his life,
> One twelfth more as youth while whiskers grew rife;
> And then yet one-seventh ere marriage begun;
> In five years there came a bouncing new son.
> Alas, the dear child of master and sage
> After attaining half the measure of his father's life chill fate took him.
> After consoling his fate by the science of numbers for four years,
> He ended his life.

The poem describes the length of life of the Greek mathematician and philosopher Diophantus, and can be expressed symbolically as:

$$D = \frac{1}{6}D + \frac{1}{12}D + \frac{1}{7}D + 5 + \frac{1}{2}D + 4$$

In order to find Diophantus' age, we must find some number D such that both sides of the equals sign are equal. It should be an easy exercise for the reader to rearrange the equation to obtain $D = 84$.

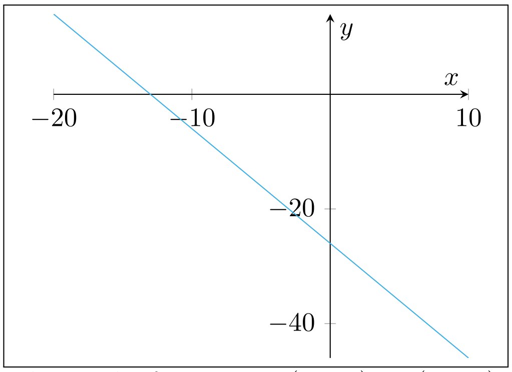
The equation above is known as a linear equation. A linear equation always has at most one *solution* — a value which can be substituted in for the unknown to make both sides of the equation equal.

All linear equations can be put into the standard form $ax + b = 0$ for some a and b. In general, to solve any linear equation we can rearrange it into this form and then take $x = -\frac{b}{a}$.

It is trivial, then, to solve linear equations algebraically. However, it is often useful to be able to visualise the solutions to equations. Take the equation $3x - 2(x + 4) = 3(x + 6)$ for example — it is difficult to see at a glance what the solution is without first rearranging it into the form $ax + b = 0$. One method that we can use is to subtract the right-hand side from the left-hand side, replace the resulting zero with y, and graph it.

$$\begin{aligned}
& 3x - 2(x + 4) = 3(x + 6) & \\
\implies & 3x - 2(x + 4) - 3(x + 6) = 0 & \text{(rearranging to make the RHS zero)} \\
& 3x - 2(x + 4) - 3(x + 6) = y & \text{(substituting } y \text{ for the RHS)}
\end{aligned}$$

After graphing this equation, we can look at the x value where $y = 0$ — this will be our solution.



The graph of $y = 3x - 2(x+4) - 3(x+6)$.

From the graph we can see that when $x = -11$, the function passes through the x-axis — and so $x = -11$ is the solution to our linear equation.

This is an important thing to note — for equations, we can graph the function, and the solutions will be the places where it cuts the x-axis (i.e. where it becomes zero). Because of this, we often call the solutions of an equation the *zeroes* of an equation, or the *roots* (because they are the places where the function grows from). In general, if we have some function f then we can talk about the roots or zeros of f, and the solutions of $f(x) = 0$.

Equations like $4x - 3x = x + 1$ do not touch the x-axis. These equations have no solutions since they are not satisfied by *any* value of x.

Exercises

1. Find some x such that $3 \times (2 \times x + 5) - 2 \times (x + 5) = 3 \times x + 14$.
2. *2. A bottle of orange juice costs \$20. The juice costs \$19 more than the empty bottle. How much does the empty bottle cost?
3. Rearrange, graph, and solve $4x + 2 = 45(x + 4(x - 8))$.
4. Does the function $f(x) = 3(1 + 2x) - 6x$ have any roots?
5. Suppose $f(x)$ is a function defined implicitly by $f(x) + x = 3f(x) - x$. Find the solutions of $f(x) = 0$.
6. *6. Solve $\frac{2x}{x+3} = \frac{3}{x-10} + 2$ for x.

3 Quadratic Equations

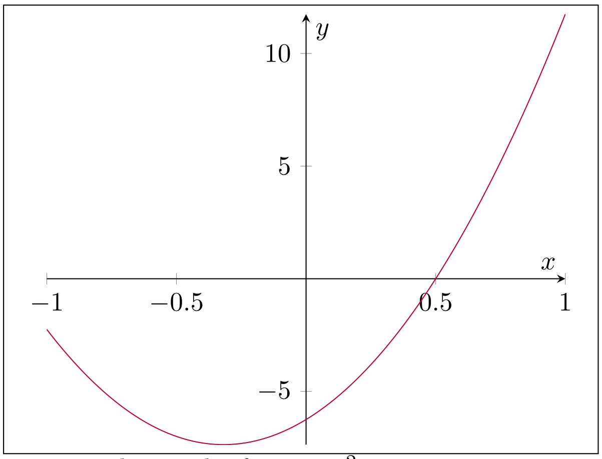
Several ancient clay tablets from the Old Babylonian period (1800 - 1600 BC) contain mathematical problems and solutions. Such a problem, given on the tablet illustrated, is (translated of course):[1]

> I have added up seven times the side of my square and eleven times the area, getting 6;15.

The number 6;15 is in sexagesimal notation; it means $6 + \frac{15}{60} = 6.25$. We can therefore rewrite this problem as follows, where x is the length of the side of the square and x^2 is the area:

$$11x^2 + 7x = 6.25$$

We can use a similar graphical approach to that above as we are simply finding the zeros of $11x^2 + 7x - 6.25 = y$.



The graph of $y = 11x^2 + 7x - 6.25$.

We find that the only positive solution to the equation is $x = 0.5$ — or 0;30 in Babylonian notation.

This problem is an example of a quadratic equation — an equation where the highest power of x is 2. All quadratic equations can be put into the form $ax^2 + bx + c = 0$, where a, b, and c are constants.

The Quadratic Formula

In fact, there is a general formula to solve *any* quadratic equation. The following proof of this illustrates the idea of *completing the square*, which is a way to reduce solving a general quadratic equation (a hard problem) to a simpler problem: that of solving $x^2 = m$ for x given a value for m. A second proof is given as an exercise at the end of the section.

[1] Translation from [15].

Tablet BM 13901, from `http://www.mat.uc.pt/~mat0703/PEZ/Matem%C3%A1tica%20na%20Babil%C3%B3nia.htm`

Theorem (Quadratic Formula)**.** *A quadratic equation* $ax^2 + bx + c = 0$ *(where* $a \neq 0$*) has at most two solutions, which are given by*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

Proof. We first simplify the equation by dividing through by a, obtaining that $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

The next step is to rearrange the formula to obtain an equation of the form $(x+p)^2 = q$. This can be done by noting that $(x+p)^2 - q = x^2 + 2px + p^2 - q$; matching coefficients, we find that $2p = \frac{b}{a}$, and therefore that $\frac{c}{a} = p^2 - q \Rightarrow q = (\frac{b}{2a})^2 - \frac{c}{a}$.

Substituting, we have the following equation:

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

This simple equation can be solved for x by taking a single square root (remembering both positive and negative roots):

$$\begin{aligned} x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

□

The expression $\Delta_2 = b^2 - 4ac$ is known as the *discriminant* of the quadratic, and it determines the nature of the solutions; if $\Delta_2 = 0$, there is one repeated root, while if $\Delta_2 > 0$ there are two real roots. We discuss the case where $\Delta_2 < 0$ later on.

Factors

To create a quadratic equation with given solutions, we write down a linear expression for each solution that evaluates to zero when the solution is substituted in, and then multiply them. For example, if we wanted a quadratic equation with the solutions $x = 2$ and $x = 3$, we take the two expressions $x - 2$ and $x - 3$, multiply the left hand sides together (obtaining $(x-2)(x-3) = x^2 - 5x + 6$), and set it to zero.

This works because if we substitute in one of our original solutions (in our example, 2 or 3) then one of the two parts of the left hand side will become zero and so the entire left hand side becomes zero.

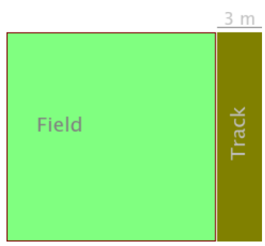
In general, this idea works for higher-degree equations (like cubics, quartics, and so on). We can make a cubic with the solutions $x = -3$, $x = 9$, and $x = 13$ by writing $0 = (x+3)(x-9)(x-13)$.

We can even multiply together higher-degree polynomials to create a polynomial that has the roots of all of them — if (for example) we take the equation $(x^2 - 4)(x^2 - 9) = 0$, it will have the four solutions $x = \pm 2$ and $x = \pm 3$.

The polynomials which multiply together together to form a larger polynomial are known as *factors* of the larger polynomial, and the process of splitting a polynomial into factors is known as *factorising*.

Exercises

1. Find an example of a **quadratic equation** with the solutions: (a) $x = 7$ and $x = 4$, (b) $x = 7$ and $x = -4$, (c) $x = -7$ and $x = -4$, and (d) only $x = 3$
2. Use the discriminant Δ_2 of the following quadratics to find the number of distinct real roots each one has, without explicitly calulating those roots.
   (a) $3x^2 + 6x + 3 = 0$
   (b) $x^2 + 10x + 1 = 0$
   (c) $x^2 + 5x + 9 = 0$
   (d) $x^2 - \frac{14x}{3} + \frac{49}{9} = 0$
3. Use the *difference of two squares* identity $x^2 - b^2 = (x - b)(x + b)$ to factorise and hence solve the following equations for x:
   (a) $x^2 - 9 = 0$
   (b) $x^2 - 7 = 0$
   (c) $x^2 - 15 = 1$
   (d) $x^2 - 2ab = a^2 + b^2$
4. Two cars start at the same point. One begins to travel due north at a constant speed of 50 kph; two hours later, the second begins to travel due east at a constant speed of 100 kph. How long after the first car begins to move are the two cars exactly 300 kilometres apart?
5. A dairy farmer owns a square cow field alongside a 3 m wide vehicle track. According to LINZ, the total area of the field and the track is 49 m^2; what are the dimensions of the field alone?



*6. Prove that $ax^2 + bx + c = Ax^2 + Bx + C$ implies that $a = A$, $b = B$, and $c = C$. This result allows us to *match coefficients*, an important tool which we can use to reason about the symmetries of polynomials.

7. Show that if α and β are the two solutions of $x^2 + bx + c = 0$, then we have $-b = \alpha + \beta$ and $c = \alpha\beta$.

8. Factorise $x^2 - 3x - 40$ by inspecting the coefficients and using the identity that a quadratic with the two solutions a and b is given by $(x-a)(x-b) = x^2 - (a+b)x + ab$ (note the change of sign in the factors).

9. For which values of k does the graph of the quadratic function $y = x^2 + (3k-1)x + (2k+10)$ not touch the x-axis?

10. Do the zeroes of a function uniquely identify that function? Why/why not?

*11. Solve the following equations in the real numbers: (a) $w^4 + 30w^2 + 29 = 0$, and (b) $3e^{2x} - 24e^x - 8 = 0$.

12. Write each of the following in the form $(x+p)^2 = q$ for some p and q, and hence find their solutions by completing the square.

    (a) $x^2 - 3x + 4 = 0$

    (b) $x^2 - 6x - 10 = 0$

    (c) $x^2 - 26x + 47 = 0$

    (d) $6x^2 - 12x + 13 = 0$

    (e) $-2x^2 + 3x + 5 = 0$

13. Suppose that $x^2 + bx + c = 0$ has two roots, α and β.

    (a) Show that $\alpha^2 + \beta^2 = b^2 - 2c$.

    (b) Show that $\Delta_2\left[x^2 + bx + c\right] = (\alpha - \beta)^2$.

*14. Flesh out the following alternative proof of the quadratic formula from [8]. Let α and β be the two roots of the equation $x^2 + bx + c = 0$.

    (a) Then $x = \frac{1}{2}\left((\alpha+\beta) + (\alpha-\beta)\right) = \frac{1}{2}\left((\alpha+\beta) + \sqrt{(\alpha-\beta)^2}\right)$.

    (b) Note that $\sqrt{(\alpha-\beta)^2}$ has two values and show that taking the negative value still gives a root.

    (c) But $\alpha + \beta = -b$ and $(\alpha - \beta)^2 = b^2 - 4c$.

    (d) So $x = \frac{1}{2}\left(-b \pm \sqrt{b^2 - 4c}\right)$, which is the quadratic formula.

4 Higher-degree Polynomials

Before we go any further, we must define the notion of a polynomial. A formal definition allows us to reason exactly about a mathematical object with more precision than our intuition.

Definition (Polynomial)**.** A *polynomial* is an expression that only has constant terms and terms involving powers of some variable. All polynomials can be written in the form $a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$, where x is the variable and $a_{n\ldots0}$ are the coefficients (and $a_n \neq 0$).

The value n is called the *degree* of the polynomial — i.e. the degree of a quadratic equation is 2. Given a polynomial p, we can write its degree as ∂p.

Factorising and Division

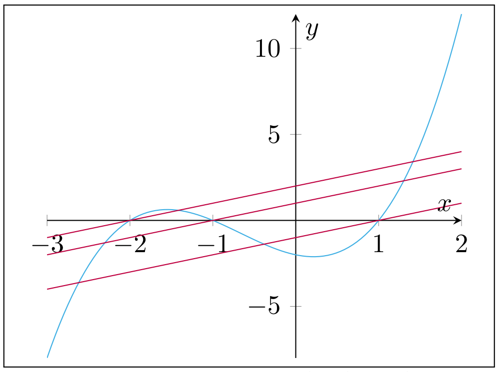
We can find solutions to general polynomial equations $p(x) = 0$ by factorising them and then finding the zeros of each factor. This is the reverse of the process outlined in the previous section for creating polynomials with an arbitrary number of solutions.

Solving a polynomial equation is equivalent to finding its factors.

For example, if we take the equation $x^2 + 3x + 2$ then we can see that two factors of this are $(x + 1)$ and $(x + 2)$ (because $x^2 + 3x + 2 = (x + 1)(x + 2)$). The two solutions are therefore $x = -1$ and $x = -2$.

Formally, we say that a polynomial $g(x)$ is a factor of a polynomial $p(x)$ if there is some polynomial $q(x)$ (the quotient) such that $p(x) = g(x) \cdot q(x)$. In this case, we say that $p(x)$ divided by $g(x)$ is $q(x)$. Even if $g(x)$ is not a factor of $p(x)$, it is possible to write that $p(x) = g(x) \cdot q(x) + r(x)$ for some remainder polynomial r if the degree of $g(x)$ is less than or equal to the degree of $p(x)$. This is similar to the division of integers: $11 = 2 \times 5 + 1$ where the factor is 2, the quotient is 5, and the remainder is 1.

Higher-degree polynomials are much harder to factorise and solve. However, once we find one factor we can divide it out to obtain a simpler polynomial. For example, take the cubic $x^3 + 2x^2 - x - 2 = 0$. We can see that one solution is $x = 1$ (it is usually a good idea to try simple solutions like 1 and 0 first), and therefore one factor of the equation is $(x - 1)$. If we divide out the cubic by this factor, we obtain that $x^3 + 2x^2 - x - 2 = (x - 1)(x^2 + 3x + 2)$, and we can solve the quadratic easily! The three solutions to this cubic are therefore $x = 1$, $x = -1$, and $x = -2$. The following graph shows both the cubic (in blue) and its factors (in purple), to illustrate this idea.



The solutions to an equation are exactly the solutions of its factors.

If we know that the solutions to a polynomial are integers, we need only try the factors of the constant term as this term is simply the product of the roots of the polynomial.

One algorithm to divide one polynomial by another is long division, such as the following calculation which provides us with an example of the division of a polynomial by another that is not a factor.

$$
\begin{array}{r|l}
 & \phantom{-x^4+{}} x^2 + 1 \\ \hline
x^2 - 1 & \phantom{-{}} x^4 + x - 3 \\
 & -x^4 + x^2 \\ \hline
 & \phantom{-x^4+{}} x^2 + x - 3 \\
 & \phantom{-x^4+{}} -x^2 + 1 \\ \hline
 & \phantom{-x^4+x^2+{}} x - 2
\end{array}
$$

Here, we have divided $p(x) = x^4 + x - 3$ by $g(x) = x^2 - 1$ to obtain the quotient $q(x) = x^2 + 1$ and the remainder $r(x) = x - 2$ — i.e. we can write $x^4 + x - 3 = (x^2 - 1)(x^2 + 1) + x - 2$.

The Remainder Theorem

One important theorem that follows from the idea of polynomial division is the remainder theorem.

Theorem (Remainder Theorem)**.** *If we can write* $p(x) = (x - a)q(x) + r$*, then* $p(a) = r$*.*

Here we are dividing a polynomial $p(x)$ by some other linear equation $(x - a)$, to get a quotient $q(x)$ and a remainder r (which is just a number, not a polynomial, in this case). It follows that the remainder is just the value of $p(x)$ evaluated at a.

Proof. If we substitute a into the statement above, it simplifies and we obtain that $p(a) = (a - a)q(a) + r = 0 \cdot q(a) + r = r$. □

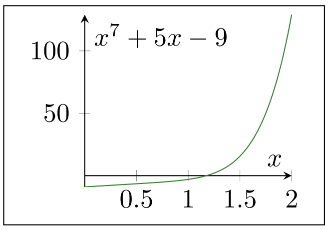
An important consequence of this theorem is that if a is a solution to the polynomial, then $r = p(a) = 0$ and so $(x - a)$ is a factor. As we expect, the solutions to an equation are the same as the solutions to the factors of the equation.

If we can divide out a polynomial multiple times by a factor, we call the root(s) corresponding to that factor 'repeated.' The *multiplicity* of some root α of $p(x) = 0$ is the number of times which we can divide out the factor $(x - \alpha)$ from $p(x)$ — in other words, the number of times that the root is repeated.

A second application of the remainder theorem is less obvious — we can use it to evaluate polynomials. For example, we can use the remainder theorem to find $f(4)$ if $f(x) = 4x^{17} - 4x^{16} + 3x^4 - 6x + 12$ by dividing $f(x)$ by $(x - 4)$ and taking the remainder.

Exercises

1. Find three different polynomials with variable x that have the two roots $x = 2$ and $x = 3$.
2. Show that $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ divided by $x^3 + 7$ gives a quotient of $x^3 + x^2 + x - 6$ and a remainder of $-6x^2 - 6x - 41$ by expanding and simplifying $(x^3 + 7)(x^3 + x^2 + x - 6) + (-6x^2 - 6x - 41)$.
3. Divide, finding the quotient and remainder polynomials: (a) $x^2 - 4$ by $x - 2$, (b) $x^2 - 4$ by $x - 3$, and (c) $t^7 - t^3 + 5$ by $t^3 + 7$.
4. If $x = 3$ is one zero of $x^3 - 3x^2 - 4x + 12$, find the other two.
5. Solve $x^3 - x^2 - 3x + 3 = 0$.
6. Find the roots of $x^4 - x^3 - 43x^2 + 85x - 42$.
7. How many distinct solutions does
$$(x^2 - 2x - 24)(x^2 + 5x) = (x^2 - 2x - 24)(4x + 12)$$
have?
8. Show that $t = 4$ is a zero of $t^4 - (6 + \sqrt{3})t^3 + 6\sqrt{3}t^2 + 32t - 32\sqrt{3}$.
9. Find the remainder after dividing $x^7 + 5x - 9$ by $(x - 6)$.



10. Find a polynomial $p(x)$ with integer coefficients such that: (a) $p\left(\frac{1}{2}\right) = 0$; (b) $p\left(\frac{1}{2} + \frac{1}{2}\sqrt{3}\right) = 0$; (c) $p\left(2i - \sqrt{2}\right) = 0$; and (d) $p\left(\sqrt{i} + \frac{1}{\sqrt[3]{2}}\right) = 0$.

*11. If $x^2 + bx + c$ and $x^2 + dx + e$ have a common factor of $(x - p)$, show that $\frac{e-c}{b-d} = p$.

12. Let $p(x) = (x^2 - 25)^5$. One root of $p(x)$ is $x = 5$. What is the multiplicity of this root?

13. Is $(x+3)$ a factor of $2x^3 + x^2 - 5x + 7$?

14. Use the remainder theorem to compute $f(3)$ for $f(x) = x^4 + x - 10$.

15. Show that if α and β are roots of $x^n - x = 0$ (for $n > 1$), then α^{-1} and $\alpha\beta$ are also roots. Why does this not imply that $x^2 - x = 0$ and $x^3 - x = 0$ have at least four roots?

*16. Elliptic curves are a form of cubic; they are equations of the form $y^2 = x^3 + ax + b$.

    (a) Find the x-intercepts of $y^2 = x^3 - 2x$.

    (b) Find the z-intercepts of $y^2 = x^3 - \frac{4}{3}x - \frac{16}{27}$, given that $z = x - \frac{1}{3}$.

    (c) Consider an elliptic curve $\mathcal{E}$, and let P and Q be two rational points (i.e. points whose coordinates are rational) which are lying on the curve. Let $\mathcal{L}$ be the chord line uniquely determined by P and Q. Show that if $\mathcal{L}$ and $\mathcal{E}$ intersect at a third point R, then this third point is rational.

**17. The polynomial $x^3 + px - 1$ has three real non-zero roots, α, β, and γ.

    (a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ in terms of p, and hence show that p is negative.

    (b) Find the cubic polynomial with coefficients in terms of p with the roots α^2, β^2, and γ^2.

18. Take the general cubic, $at^3 + bt^2 + ct + d$. Show that the substitution $t = y - \frac{b}{3a}$ will give a cubic in y with no quadratic term (this is known as a Tschirnhaus substitution and is often the first step to create a general formula to solve the cubic).

19. Show that $\sqrt{2} + \sqrt{3} = \sqrt{5 + \sqrt{6}}$.

**20. Prove the following identity.[2]

$$\sqrt[3]{-18 + \sqrt{325}} + \sqrt[3]{-18 - \sqrt{325}} = -3$$

21. Let $w = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are rational. Find rational numbers p, q, r and s such that

$$w = p + q(\sqrt{2} + \sqrt{3}) + r(\sqrt{2} + \sqrt{3})^2 + s(\sqrt{2} + \sqrt{3})^3.$$

[2] See chapter 1 of [14] for historical context, but note that the identity as stated there is in error (and not listed in errata).

5 Complex Numbers

Now that we have somewhat developed the theory of polynomial equations, a sensible question to ask ourselves is the following.

When do polynomial equations have solutions?

The answer to this simple question is actually quite nuanced, and occupied mathematicians in Europe for centuries.

Consider, for example, the simple quadratic equation

$$x^2 - 2 = 0.$$

Notice that the coefficients of this equation are integers, but the solutions are not — in fact, the solutions ($\pm\sqrt{2}$) are not even rational! Notice also that the number of solutions is 2, the same as the degree of the polynomial.

Let us take a look at another example,

$$x^3 - x = 0.$$

Again, the degree of this polynomial is three — as is the number of solutions that we obtain from it.

These two examples, as well as our previous work, suggest that the number of solutions of a polynomial is the same as its degree. There is one problem with this, illustrated by another simple equation:

$$x^2 + 1 = 0.$$

If we try to solve this equation, we end up with a seemingly nonsense result: that $x = \pm\sqrt{-1}$. Since there is no real number with a negative square, it seems like we need to throw out our pretty result. However, remember that we have already seen an example of an equation where the solution is out of reach if we only look for solutions that are the 'same kind' as the coefficients.

Just as we extend the natural numbers to the integers, the integers to the rationals, and the rationals to the reals, we can extend our number system further so that the equation $x^2 + 1 = 0$ has a solution. We begin by defining i to be a square root of -1 (and setting $-i$ to be the other square root); then the set of all numbers of the form $a + bi$ (for any real numbers a and b) is called the set of *complex numbers*; this system of writing them is called *rectangular form*.

The complex numbers are generally considered to have been formally born in Girolamo Cardano's *Ars Magna* in 1545, where he introduces them only to dismiss them as 'as subtle as they are useless'. Despite this, modern mathematics has fully accepted the existence of complex numbers for two main reasons: firstly they do not introduce any contradictions into basic arithmetic or any theory that we care about, and secondly they allow us to state the following historic theorem.

Theorem (Fundamental Theorem of Algebra)**.** *Let $p(x)$ be a polynomial with complex coefficients. Then, counting repeated roots, there are exactly ∂p complex roots of $p(x)$.*

The proof of the Fundamental Theorem unfortunately requires concepts and techniques far beyond the scope of this book (Stewart gives a proof in [14], and Artin gives two(!) proofs in [1] §9.9). All proofs (that I am aware of) require the use of analysis (i.e. there is no pure-algebra proof).

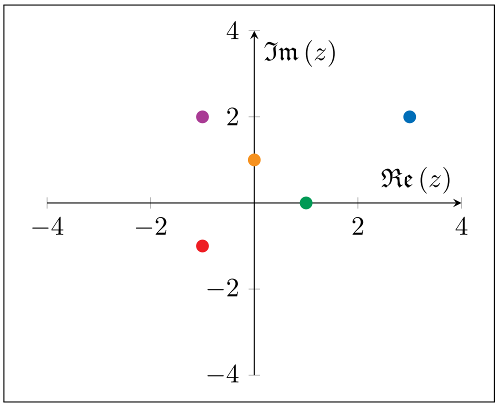
The first rigorous proof of this theorem was published by French mathematician Jean-Robert Argand in 1814; the name is somewhat incorrect in the modern era as the study of algebra is no longer purely devoted to the properties of the complex number field. An analogue of the theorem in a more modern context is Kronecker's Theorem, which states that a polynomial with coefficients in set of numbers in which we can add, subtract, multiply, and divide has at least one solution in a 'bigger' set of numbers that contains the original set.

Given a complex number $z = a+bi$, we call a the *real part* of z and b the *imaginary part* of z (writing $\mathfrak{Re}\,(z)$ and $\mathfrak{Im}\,(z)$ respectively). The term 'imaginary' is purely historical — the complex numbers exist in exactly the same way that a number like $\pi = 3.14159...$ or $e = 2.718...$ exists.

We say that two complex numbers are equal **if and only if both the real and imaginary parts of those numbers are equal**, and we perform arithmetic on complex numbers in the same way that we perform arithmetic on the reals, remembering that $i^2 = -1$.

Complex numbers also have a geometric interpretation. If we set up a mapping $a + bi \mapsto (a, b)$ it is easy to see that we have an exact correspondence between complex numbers and points on a Cartesian plane. This kind of diagram is called an *Argand diagram*, named after French mathematician Jean-Robert Argand.

For convenience, we also often associate with each complex number the unique *vector* pointing from the origin to the location of the point on the Argand plane. Given a complex number $z = a + bi$, the length of its associated vector is called its *modulus*; this is written $|z|$, and (by the Pythagorean theorem) we have the simple result that $|z| = \sqrt{a^2 + b^2}$. The meaning of this is exactly the same as that of the absolute value of a real number (and in fact the modulus of a real number is exactly its absolute value) — despite this, note that while there is a natural ordering on the real numbers (for example, $-2 < 0 < \pi < \sqrt{17}$) there can be no such ordering on the complex numbers. It is completely nonsensical to talk about any complex number being 'bigger' than any other.

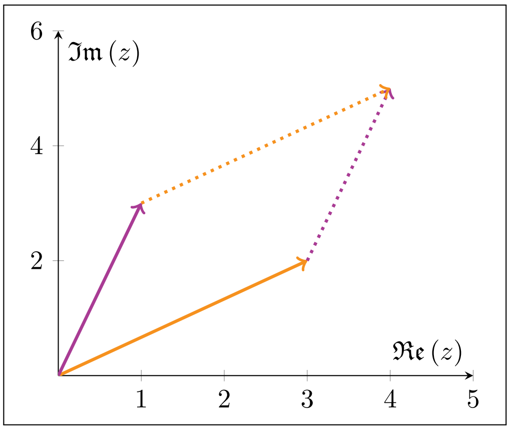


The five points marked on this Argand diagram are i, 1, $3+2i$, $-1-i$, and $2i-1$. See if you can label each point on the diagram itself.

Due to the way we constructed the complex numbers, there are natural rules for arithmetic. Suppose we have two complex numbers, $u = a + bi$ and $v = c + di$. We have the following rules for addition and subtraction by collecting the real and imaginary terms together:

$$u + v = (a + c) + (b + d)i, \text{ and}$$
$$u - v = (a - c) + (b - d)i.$$

Geometrically, we are simply taking the vectors associated with each point and adding them.

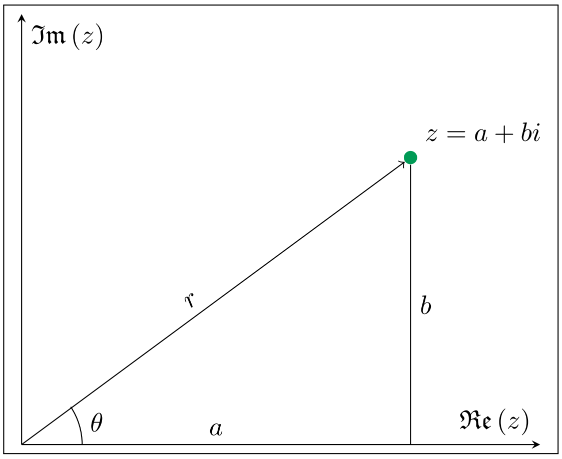


Before we discuss multiplication of complex numbers, we will discuss one more simple but important operation: that of complex conjugation. Our intent is to take the geometric property of reflection (across the real axis), and give it an algebraic meaning; in particular, the *complex conjugate* of $z = a + bi$ is $\overline{z} = a - bi$. An initial application of this concept is the simple result that if a polynomial has real coefficents, any complex roots must come in conjugate pairs (you are asked to prove this in the special case of quadratic polynomials as exercise 27, and in the general case as exercise 9.35).

There is a rule for multiplying complex numbers that is similar to the rules for addition; we simply collect like terms and remember that $i^2 = -1$ in order to obtain the result

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

This way of writing the result is a little esoteric and has no nice geometric meaning. In order to remedy this problem, we rewrite our complex numbers in *polar form*. Instead of writing the number in the form $z = a + bi$, where a and b are the distances from the two axes, we write $z = r \operatorname{cis} \theta$, where r is the distance of the number from the origin (simply the modulus $|z|$ again) and θ is the angle that number's vector makes with the x-axis (known as the *argument* of z, and written as $\arg(z)$). The function $\operatorname{cis} \theta$ is defined to be $\cos \theta + i \sin \theta$.



It is simple to see that $\theta = \tan^{-1} \frac{b}{a}$; we have already got a formula for r.

This new notation simplifies multiplication significantly, because you will show as an exercise that

$$(r \operatorname{cis} \theta)(t \operatorname{cis} \varphi) = (rt) \operatorname{cis}(\theta + \varphi).$$

For example, if we were to multiply $3 \operatorname{cis} \pi$ by $4 \operatorname{cis} \pi$, we would obtain $12 \operatorname{cis} 2\pi = 12$. This is as expected, as $p \operatorname{cis} \pi = -p$, and $-3 \times -4 = 12$.

Note that any complex number has an infinite number of different representations in polar form, simply by adding 2π (an entire rotation around the origin) to its argument. This fact comes into play later on when we try to solve equations involving powers of complex numbers.

It is possible to make a sensible definition for complex exponents using *Euler's formula*,

$$\operatorname{cis} \theta = e^{i\theta}.$$

From this we obtain the following equation, which relates five fundamental mathematical constants in one expression and which some have called the most beautiful equation in mathematics:

$$e^{i\pi} + 1 = 0. \qquad \text{(Euler's Identity)}$$

From `http://brownsharpie.courtneygibbons.org/?p=848`

A simple proof of Euler's formula, using calculus, is presented here.

Proof. We wish to show that $\frac{e^{i\theta}}{\cos\theta + i\sin\theta} = 1$. Note that $\operatorname{cis}\theta \neq 0$ for all θ.

Suppose f is a function of θ, defined by $f(\theta) = \frac{e^{i\theta}}{\cos\theta + i\sin\theta}$. Then, taking the derivative, we have

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}\theta} &= \frac{(\cos\theta + i\sin\theta)\cdot ie^{i\theta} - e^{i\theta}\cdot(-\sin\theta + i\cos\theta)}{\operatorname{cis}^2\theta} \\
&= \frac{ie^{i\theta}\cos\theta - e^{i\theta}\sin\theta + e^{i\theta}\sin\theta - ie^{i\theta}\cos\theta}{\operatorname{cis}^2\theta} \\
&= 0.
\end{aligned}$$

So f is a constant function, and so has the same value for all θ. But $f(0) = \frac{e^{i\cdot 0}}{\cos 0 + i\sin 0} = 1$; so $\frac{e^{i\theta}}{\cos\theta + i\sin\theta} = 1$ for all θ. □

As a short aside, we have already seen that we can view the complex numbers as points in the plane, $\mathbb{R}^2$. This means that we can carry out geometric operations algebraically using complex numbers. For example, the length of the line joining two points z_1 and z_2 is simply $|z_1 - z_2|$.

We define the *locus* of an equation to be the set of all points satisfying that equation; for example, the locus of $x^2 + y^2 = 1$ is the set of all points making up the unit circle. We can use our correspondence between points on a plane and complex numbers to reason about the locus of a complex equation; for example, the locus of the equation $|z| = 1$ is the set of all points at a unit distance from the origin: it is another way of talking about the unit circle. (If you write $z = x + iy$, then $|z| = x^2 + y^2$ and the comparison becomes even more obvious.)

Exercises

1. Evaluate the following expressions, and plot the answers on an Argand diagram:

   (a) $(3 + 2i) + (6 - 2i)$

   (b) $24 - (6 + 2i)$

   (c) $2(2 + i) + 6i - 7$

2. If we add two real numbers, can we obtain an imaginary number? If we add two imaginary numbers, can we obtain a real number?

3. Let $v = 3 - 7i$ and $w = -4 + 6i$.

   (a) Find the real numbers p and q such that $pv + qw = 6.5 - 11i$.

   (b) Show that any complex number z can be written as $z = pv + qw$ for some real p and q.

4. Solve the quadratic equation $x^2 + 4 = 0$.

5. Prove that $z + \overline{z} = 2 \cdot \mathfrak{Re}\,(z)$ and $z - \overline{z} = 2i \cdot \mathfrak{Im}\,(z)$.

6. Verify the following properties of conjugation.

   (a) $\overline{\overline{z}} = z$

   (b) $\overline{w} + \overline{z} = \overline{w + z}$

   (c) $\overline{w}\overline{z} = \overline{wz}$

7. Find i^{957}.

8. Show that $|a + bi| \geq |a|$ and $|a + bi| \geq |b|$.

9. Find $(3 + 2i)(6 + 8i)$ in rectangular form.

10. (a) Convert $1 + i$ into polar form.

    (b) Find $(1 + i)(\sqrt{2}\operatorname{cis}\frac{3\pi}{4})$ in both polar form and rectangular form.

11. Compute $(6\operatorname{cis}\frac{23\pi}{24})(9\operatorname{cis}\frac{14\pi}{17})$, leaving your answer in polar form.

12. (a) Prove that $(r\operatorname{cis}\theta)(t\operatorname{cis}\varphi) = (rt)\operatorname{cis}(\theta + \varphi)$.

    (b) Describe the geometric meaning of the multiplication of complex numbers.

13. Let $u = 2\operatorname{cis}\frac{\pi}{2}$ and $v = 3\operatorname{cis}\frac{3\pi}{2}$. Plot u, v, and uv on an Argand diagram.

14. Prove **de Moivre's Theorem**: $(r\operatorname{cis}\theta)^n = (r^n)\operatorname{cis}(n\theta)$.

15. Show that if $u = r\operatorname{cis}\theta$ and $v = t\operatorname{cis}\varphi$ then $\frac{u}{v} = \frac{r}{t}\operatorname{cis}(\theta - \varphi)$.

16. Using de Moirve's Theorem, prove that for complex numbers w and m and integers n and m, (a) $w^n w^m = w^{n+m}$, and (b) $(w^n)^m = w^{nm}$.

17. Convert $w = 1 + \sqrt{3}i$ into polar form, and calculate w^3.

18. Show that for any complex number z, the product $z\overline{z}$ is both real and non-negative. Hence show that $(x - z)(x - \overline{z})$ has only real coeffients.

19. Let x be a real number. Show that, for all integers n, $\operatorname{cis}\frac{2x\pi}{n} = \operatorname{cis}\frac{2(x+n)\pi}{n}$.

20. For which complex numbers is z^2 real? What about z^3?

21. Transform $\frac{a+bi}{c+di}$ so that the only imaginary part is in the numerator.

22. Find $(a + bi)^{-1}$ in rectangular form.

23. Write the complex number $\left(\frac{4i^7 - i}{1+2i}\right)^2$ in the form $a + bi$, where a and b are real numbers.

24. (a) Prove that a number z is real if and only if $\overline{\overline{z}} = z$.

    (b) Hence, or otherwise, show that $z\overline{w} + w\overline{z}$ is always real.

    (c) Show that $z\overline{w} + w\overline{z} \leq 2|w||z|$.

25. Show that if $z = a + ib$ then $\sqrt{z\overline{z}} = |z|$.

*26. If $\zeta = \sqrt{\frac{1}{2}(a + \sqrt{a^2 + b^2})} + i\sqrt{\frac{1}{2}(-a + \sqrt{a^2 + b^2})}$ is a complex number, find ζ^2 in the form $p + iq$ (where a, b, p, and q are real).

*27. If $z = x + iy$, and $az^2 + bz + c = 0$, show that $a\overline{z}^2 + b\overline{z} + c = 0$ if a, b, and c are real. (This exercise is generalised in 9.35.)

*28. Use Euler's identity to find $\ln(-1)$, and hence $\ln(-x)$ for real x.

*29. Prove that for every positive integer n, $(-1+\sqrt{3}i)^{3n} + (-1-\sqrt{3}i)^{3n} = 2^{3n+1}$.

30. Show that $y_1(x) = e^{ix} + e^{-ix}$ and $y_2(x) = 2\cos x$ are both solutions of the differential equation
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0$$
with initial conditions $y(0) = 2$ and $y'(0) = 0$. (Also see 32 below.)

31. Find $\sqrt{i}$ in rectangular form.

32. (a) Show that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and that $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

    (b) If $x + x^{-1} = 2\cos\theta$, find $x^n + x^{-n}$ in terms of n and θ.

33. (a) Show that $(2 \pm i)^3 = 2 \pm 11i$.

    (b) Simplify fully $\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$.

    (c) Show that (b) is a root of the cubic equation $t^3 - 15t - 4 = 0$, and hence find all three solutions.

34. Show that, if a and b are fixed complex numbers, then $|z - a| = |z - b|$ describes a line.

*35. Let $v = 1 + i$ and $z = x + iy$ for any real numbers x and y.

    (a) Show that the equation $|z - v| = |vz|$ represents a circle, and find its centre and radius.

    (b) Find the point of intersection of the circle with the straight line $|z - v| = |z + v|$.

36. You do not need the fundamental theorem of algebra for this exercise.

    (a) Prove that all cubic equations with real coefficients must have exactly three roots in the complex numbers.

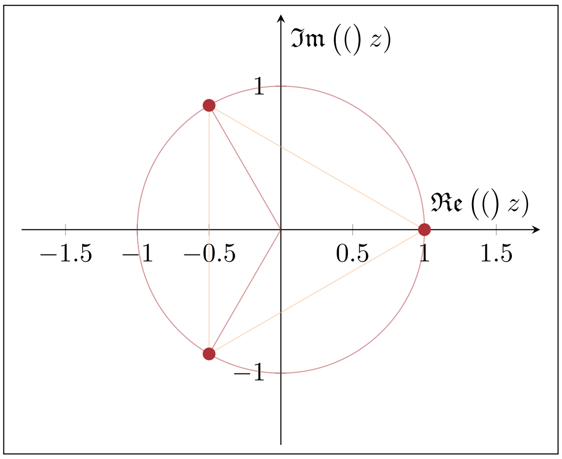
    (b) Let $p(x)$ be a polynomial of degree n such that there exists some complex number ζ such that $p(\zeta) = 0$. Show that $p(x) = 0$ has exactly n solutions (counting repeated roots).

6 Roots of Unity

Let us take the equation $z^3 = 1$. We know that this equation has exactly three complex roots, and of these we already know that the only real root is $z = 1$. How can we find the two non-real roots?

Noting that $1 = 1 \operatorname{cis} 2k\pi$ for all natural numbers k, we can apply de Moivre's Theorem to show that $1^{(\frac{1}{3})} = \operatorname{cis} \frac{2k\pi}{3}$.

We can then set n to 0, 1, and 2 to obtain our three roots of the original equation: $z = 1$, $z = \operatorname{cis} \frac{2\pi}{3}$, and $z = \operatorname{cis} \frac{4\pi}{3}$ respectively. Note that if we set k to any higher number (3, for example) we obtain one of the roots we already have (e.g. $\operatorname{cis} \frac{6\pi}{3} = 1$), since we have gone 'around the circle'.



If we look at the roots geometrically (on an Argand diagram like that above), we see that the nth roots of 1 will be arranged in a circle of radius 1 centred on the origin, and the angles between them will be exactly $\frac{2\pi}{n}$. If n is even, both 1 and -1 will both be real roots, but if n is odd then 1 will be the only real root. In general, the sum of all n nth roots of unity is zero (see exercise 6).

Any integer power of an nth root of unity is also an nth root of unity. However, not all nth roots will 'generate' all the other ones in this way; for example, $\operatorname{cis} \frac{2\pi}{3}$ is a sixth root of unity but will miss out every second sixth root if we raise it to integer powers. Roots which *do* generate all the others are called *primitive roots of unity.*

We formally define a primitive nth root of unity to be a complex number ω such that $\omega^n = 1$ but $\omega^k \neq 1$ for all $k < n$. In other words, if we list all the roots of unity in order for $n = 1,\ 2,\ 3$ and so on, then a root is only primitive for that value of n for which it first appears.

For example, $\mu = \operatorname{cis} \frac{2\pi}{3}$ is a primitive third root of unity (since the smallest nonzero n such that $\mu^n = 1$ is $n = 3$).

We now prove that, as we claimed, the primitive nth roots of unity 'generate' all the other nth roots of unity:

Theorem. *Given the polynomial $z^n = 1$, with primitive root ω, all solutions are given by ω^k for $0 \leq k \leq (n-1)$. In other words, the integer powers of a primitive nth root of unity must be* all *the nth roots of unity.*

Proof. We first prove that all the powers ω^k for k defined above are distinct. Suppose that there are two values for k, say $k = a$ and $k = b$, such that $\omega^a = \omega^b$ and $a \neq b$. Then we have that $\omega^{a-b} = 1$ and therefore $a - b = 0$ (the other possibility here would be that $a - b$ were equal to some non-zero multiple of n, none of which are possible values of k) so $a = b$.

Since there are n possible values for k, there are n distinct powers of ω which are all roots of the polynomial. However, the polynomial is of degree n and so has exactly n roots — thus, the powers of ω are exactly the roots of the polynomial. $\square$

Roots of unity allow us to find all the nth roots of any number easily. Suppose $a^n = x$; then the nth roots of x will be $a = \omega^k \sqrt[n]{x}$ $(0 \leq k < n)$ where ω is a primitive nth root of unity and $\sqrt[n]{x}$ is any nth root of x.

Exercises

For the exercises marked †, it may be useful to use the result that for all integers a and b there exist integers m and n such that $am + bn = \gcd(a, b)$ (where the gcd of two numbers is the largest integer that divides into both of them). Two integers a and b are coprime if they share no divisors (i.e. if $\gcd(a, b) = 1$).

1. Let $p(z) = z^5 - 1$.
   (a) Find exactly each of the roots of $p(z)$.
   (b) Let α be the root of p with the smallest non-zero positive argument. Show explicitly that the roots can be written as 1, α, α^2, α^3, and α^4.
2. Find all solutions of $z^3 + n = 0$, where n is a positive real number, in exact form in terms of n.
3. Solve for z if $(z - 3)^7 = 1$.
4. Find the fifth roots of $4 + 4i$ in polar form, and draw them on an Argand diagram. Hence find integers p and q such that $(p + qi)^5 = (4 + 4i)$.
5. Write down all of the primitive sixth roots of unity. What about the primitive fifth roots of unity?

*6. (a) Let α be a complex root of $x^3 = 1$. Show by computation that $\alpha^2 + \alpha + 1 = 0$.
   (b) In general, prove that the sum of all n nth roots of unity is zero (for $n > 1$).

*7. Find the product of all n nth roots of unity.

*8. Solve $(z+1)^3 = 8$ for z and show that the sum of the solutions is -3.

*9. Given that $a = b + kn$ for some integer k, show that $z^a = z^b$ where z is a primitive nth root of unity.

10. Prove that the product of an ath root of unity by a bth root of unity is an abth root of unity.

†*11. Prove the following: Let a and b be coprime integers. Then all the abth roots of unity can be obtained as products of ath roots of unity and bth roots of unity.

*12. The theorem stated in this section requires ω to be a **primitive** nth root of unity in order for all the nth roots of unity to be powers of ω. Why do we need this restriction?

13. Prove the following: Let a and b be coprime integers. Then $x^a - 1 = 0$ and $x^b - 1 = 0$ have only the trivial root $x = 1$ in common.

14. (a) Prove the converse of the theorem in the text: i.e. show that if ζ generates all the kth roots of unity then it is a primitive kth root of unity.

    (b) Let ζ be the root of $p(x) = x^k - 1$ with smallest positive argument. Show that ζ is a primitive kth root of unity.

†15. Let α be the kth root of unity with smallest positive argument. Show that the primitive kth roots of unity are precisely α^n where $0 < n < k$ and $\gcd(n, k) = 1$.

16. The fifth-degree polynomial $p(x)$, where $p(k) = 0$, has as its roots the vertices of a regular pentagon centred around $(\frac{1}{2}k, 0)$. Give $p(x)$ such that all coefficients are real.

**17. Show that all solutions to $(z+1)^n = z^n$ lie on the line $\mathfrak{Re}\,(z) = -\frac{1}{2}$.

18. Find all the third roots of 2.

19. A *group* is a set G together with some operation $\cdot$ satisfying the following:

    (a) For all a, b in G, $a \cdot b$ is in G.

    (b) For all a, b, c in G, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

    (c) There is some element e in G such that for all a in G, $a \cdot e = a$.

    (d) For every element a in G there is some b in G such that $a \cdot b = e$.

    Show that the set of all nth roots of unity form a group under multiplication.

20. Let U be the set of all complex numbers $u = a + bi$ such that $a^2 + b^2 = 1$.

    (a) Describe all elements of the form $(3 + 4i)u$ for some u in U.

    (b) Describe all elements of the form $(c + di)u$ for some u in U.

21. Suppose $\sigma(n)$ is the function that sends n to the sum of its divisors. For example, the divisors of 4 are 1, 2, and 4; so $\sigma(4) = 1 + 2 + 4 = 7$. Prove that if p is prime then $\sigma(p^n) = \frac{p^{n+1}-1}{p-1}$.

7 The Double-Triangle Problem

As an application of our work on primitive roots, we solve the *double-triangle problem* which appeared in the 2009 New Zealand Scholarship examination.

(b) Six points are shown in the Argand diagram in Figure 2. They are the roots of $p(x)$, a degree 6 polynomial with real coefficients.

The points lie on two concentric circles centred at the origin, and are the vertices of equilateral triangles, as shown in the figure.

The positive real root of $p(x)$ is k.

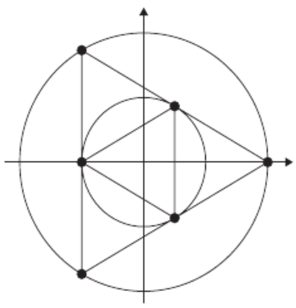


Figure 2: Argand diagram showing the roots of $p(x)$.

(i) List the **exact** roots of $p(x) = 0$. Hence or otherwise write $p(x)$.
It need not be expanded, but should not contain complex terms.

The question states that the positive real solution is $z_1 = k = k \operatorname{cis} 0$. We can therefore see that the other two solutions on the outer circle will be given by $z_2 = k \operatorname{cis} \frac{2\pi}{3}$ and $z_3 = k \operatorname{cis} \frac{-2\pi}{3}$.

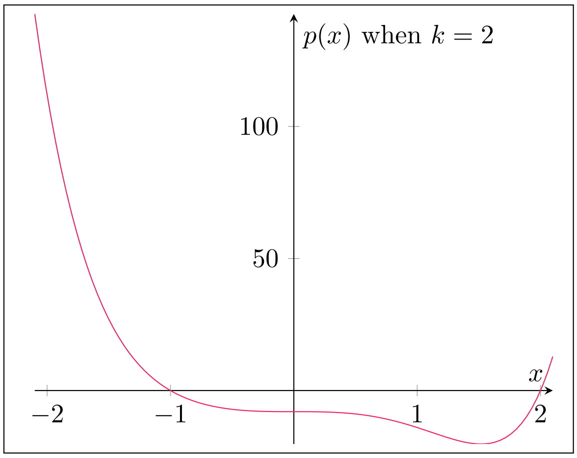
The negative real solution will have the same real part as the two complex solutions on the outer circle; we can calculate this by taking $z_4 = k \cos \frac{2\pi}{3} = -\frac{k}{2}$. We can find the other two solutions on the inside circle by rotating our negative real solution by $\frac{2\pi}{3}$, obtaining $z_5 = \frac{k}{2} \operatorname{cis}(\pi - \frac{2\pi}{3}) = \frac{k}{2} \operatorname{cis} \frac{\pi}{3}$ and $z_6 = \frac{k}{2} \operatorname{cis} \frac{-\pi}{3}$.

Root	**Polar form**	**Rectangular form**
z_1	$k \operatorname{cis} 0$	k
z_2	$k \operatorname{cis} \frac{2\pi}{3}$	$-\frac{k}{2} + i\frac{k\sqrt{3}}{2}$
z_3	$k \operatorname{cis} \frac{-2\pi}{3}$	$-\frac{k}{2} - i\frac{k\sqrt{3}}{2}$
z_4	$\frac{k}{2} \operatorname{cis} \pi$	$-\frac{k}{2}$
z_5	$\frac{k}{2} \operatorname{cis} \frac{\pi}{3}$	$\frac{k}{4} + i\frac{k\sqrt{3}}{4}$
z_6	$\frac{k}{2} \operatorname{cis} \frac{-\pi}{3}$	$\frac{k}{4} - i\frac{k\sqrt{3}}{4}$

We now generate our polynomial using the techniques described in the section on quadratic equations:

$$
\begin{aligned}
f(x) &= (x - z_1)(x - z_4) = (x - k)\left(x + \frac{k}{2}\right) \\
g(x) &= (x - z_2)(x - z_3) = \left(x - \left(-\frac{k}{2} + i\frac{k\sqrt{3}}{2}\right)\right)\left(x - \left(-\frac{k}{2} - i\frac{k\sqrt{3}}{2}\right)\right) \\
&= x^2 - \frac{k}{2}x + \frac{k^2}{4} \\
h(x) &= (x - z_5)(x - z_6) = \left(x - \left(\frac{k}{4} + i\frac{k\sqrt{3}}{4}\right)\right)\left(x - \left(\frac{k}{4} - i\frac{k\sqrt{3}}{4}\right)\right) \\
&= x^2 + kx + k^2 \\
\therefore\ p(x) &= f(x)g(x)h(x) = (x - k)\left(x + \frac{k}{2}\right)\left(x^2 - \frac{k}{2}x + \frac{k^2}{4}\right)\left(x^2 + kx + k^2\right)
\end{aligned}
$$

Note that because all of the complex roots come in conjugate pairs the imaginary parts cancel leaving us with a sextic polynomial with real coefficients.



Exercises

Work through the problem, writing out all working clearly. Attempt the problem without looking at the solution. Describe the main ideas used in each step of the solution, and make sure you understand why each step is taken and why the solution is correct.

Are there any other possible ways of completing the problem? Is this the only possible solution?

8 Solving the Cubic

This section presents a general solution for the cubic equation similar to the solution of the quadratic equation given in exercise 3.14. Be sure to complete that exercise before reading this section. The general outline of this proof is given in §14 of [8].

Historical Background

Niccolo Fontana (Tartaglia)

Évariste Galois

This particular method of solution was presented first by the French mathematician Alexandre-Théophile Vandermonde in 1771. However, it is believed that the first person to solve the general cubic equation was the Italian Scipio de Ferro who passed on at least part of his method to his student Antonio Fior. A solution was independently discovered at around the same time (in 1535) by Niccolo Fontana (also known as Tartaglia, the Stammerer), who was conviced to pass them on to another Italian, Girolamo Cardano. Cardano later (in 1545) published the solution in his book, *Ars Magna*, which also included a solution to the general quartic by Ludovico Ferrari.

Notation and Preliminary Results

Suppose we have some polynomial $t^3 - \sigma_1 t^2 + \sigma_2 t - \sigma_3$ (note the signs on the coefficients) with the three complex roots (not necessarily distinct) x, y, and z. So $\sigma_1 = x + y + z$, $\sigma_2 = xy + xz + yz$, and $\sigma_3 = xyz$, where σ_n is known as the nth elementary symmetric polynomial in x, y, and z.

We note the following three identities:

$$\begin{aligned} x^2z + xy^2 + yz^2 + x^2y + xz^2 + y^2z &= \sigma_1\sigma_2 - 3\sigma_3 \\ x^3 + y^3 + z^3 &= \sigma_1^3 - 3(\sigma_1\sigma_2 - 3\sigma_3) - 6\sigma_3 \\ x^2 + y^2 + z^2 &= \sigma_1^2 - 2\sigma_2 \end{aligned}$$

Finally, let α be a primitive cube root of 1.

Computations for the Solution

Note first that

$$\begin{aligned}
x &= \frac{1}{3}\Big((x+y+z)+(x+\alpha y+\alpha^2 z)+(x+\alpha^2 y+\alpha z)\Big)\\
&= \frac{1}{3}\Big((x+y+z)+\sqrt[3]{(x+\alpha y+\alpha^2 z)^3}+\sqrt[3]{(x+\alpha^2 y+\alpha z)^3}\Big)\\
&= \frac{1}{3}\Big(\sigma_1+\sqrt[3]{(x+\alpha y+\alpha^2 z)^3}+\sqrt[3]{(x+\alpha^2 y+\alpha z)^3}\Big)
\end{aligned}$$

(remembering that the three cube roots of 1 add to 0).

Now, we must find expressions for $u=(x+\alpha y+\alpha^2 z)^3$ and $v=(x+\alpha^2 y+\alpha z)^3$ in terms of the coefficients of the polynomial, σ_n. Suppose that we can find uv and $u+v$ in terms of the elementary symmetric polynomials. Then we can find u and v using the quadratic formula.

So what are uv and $u+v$?

Expanding u and v individually, we find

$$\begin{aligned}
u &= 3(x^2z+xy^2+yz^2)\alpha^2+3(x^2y+xz^2+y^2z)\alpha+x^3+y^3+z^3+6xyz, \text{ and}\\
v &= 3(x^2z+xy^2+yz^2)\alpha+3(x^2y+xz^2+y^2z)\alpha^2+x^3+y^3+z^3+6xyz.
\end{aligned}$$

Adding these together, we have

$$\begin{aligned}
u+v = \quad & 3\alpha(x^2z+xy^2+yz^2+x^2y+xz^2+y^2z)\\
& +3\alpha^2(x^2z+xy^2+yz^2+x^2y+xz^2+y^2z)\\
& +12xyz+2(x^3+y^3+z^3)\\
= \quad & 3(\alpha+\alpha^2)(\sigma_1\sigma_2-3\sigma_3)+12\sigma_3+2(\sigma_1^3-3(\sigma_1\sigma_2-3\sigma_3)-6\sigma_3)\\
= \quad & -3(\sigma_1\sigma_2-3\sigma_3)+12\sigma_3+2(\sigma_1^3-3(\sigma_1\sigma_2-3\sigma_3)-6\sigma_3)\\
= \quad & 2\sigma_1^3-9\sigma_1\sigma_2+27\sigma_3.
\end{aligned}$$

To find uv, we proceed as follows:

$$\begin{aligned}
(x+\alpha y+\alpha^2 z)(x+\alpha^2 y+\alpha z) &= (xy+xz+yz)\alpha^2+(xy+xz+yz)\alpha+x^2+y^2+z^2\\
&= (\alpha+\alpha^2)(\sigma_2)+\sigma_1^2-2\sigma_2\\
&= \sigma_1^2-3\sigma_2\\
&\Downarrow\\
uv &= (\sigma_1^2-3\sigma_2)^3.
\end{aligned}$$

Solution

So, to find the solutions of a cubic equation $t^3-\sigma_1 t^2+\sigma_2 t-\sigma_3=0$:

1. Calculate:

$$\begin{aligned}
u+v &= 2\sigma_1^3-9\sigma_1\sigma_2+27\sigma_3\\
uv &= (\sigma_1^2-3\sigma_2)^3
\end{aligned}$$

2. Then calculate:

$$u, v = \frac{(u+v) \pm \sqrt{(u+v)^2 - 4uv}}{2}$$

3. Hence, we have nine possible solutions (one for each choice of cube root), of which three will work in the original equation (trial and error must be used at this point):

$$x = \frac{1}{3}\left(\sigma_1 + \sqrt[3]{u} + \sqrt[3]{v}\right)$$

   Remember that the three cube roots of a number will be $\sqrt[3]{u}$, $\alpha\sqrt[3]{u}$, and $\alpha^2\sqrt[3]{u}$ where α is a complex cube root of unity.

A variation of this method also solves quartic equations. However, no general solution to the quintic equation (or any higher degree equation) exists in terms of radicals (terms under a $\sqrt{}$). The lack of a general solution for any polynomial with $\delta > 5$ was originally proved by Ruffini and Abel in the early 1800s, and a general study of the symmetries of the roots of polynomials (the beginnings of Galois theory) was first published (after being rejected twice) by the French Academy of Sciences in 1843 after their late author Évariste Galois was killed in a duel over a girl in 1832 (at the age of 20). For more historical details, see the introductory chapter of [14].

Clearing Up Some Technical Points

How did we know that uv and $u+v$ could be expressed in terms of the elementary symmetric polynomials (the coefficients)? Well, it so happens that uv and $u+v$ are symmetric in x, y, and z (this is not hard to check), and we have a theorem due to Sir Isaac Newton which states that:

Theorem (Fundamental Theorem on Symmetric Polynomials)**.** *Suppose that r_1, r_2, ..., r_n are the roots of some polynomial. Then every symmetric polynomial in r_1, r_2, ..., r_n can be expressed (uniquely) as a polynomial in the elementary symmetric functions σ_1, σ_2, ..., σ_n.*

Note that the elementary symmetric functions of order n are simply the coefficients of the polynomial $(x-a_1)(x-a_2)\cdots(x-a_n)$. For example, the second-order elementary symmetric functions are just $\sigma_1 = a_1 + a_2$ and $\sigma_2 = a_1a_2$; the third-order elementary symmetric functions are just $\sigma_1 = a_1 + a_2 + a_3$, $\sigma_2 = a_1a_2 + a_1a_3 + a_2a_3$, and $\sigma_3 = a_1a_2a_3$.

Basically, the theorem implies that if we have a symmetric expression in the roots of some polynomial, then we can write that expression in terms of the coefficients of the polynomial.

One example is the discriminant of a polynomial. We have already met the discriminant of a quadratic, $\Delta_2 = b^2 - 4ac$; we can more generally define the discriminant of an nth degree polynomial to be

$$\Delta_n[(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)] = \prod_{i<j}(\alpha_i - \alpha_j)^2$$

which is obviously symmetric in each α_i.

Exercises

**1. Check the author's algebra.

2. Solve $t^3 + t^2 - 89t + 231 = 0$.

3. Solve $t^3 + 21t^2 - 32t + 3510 = 0$.

4. Solve $2t^3 + 4it^2 + 58t - 84i = 0$.

5. In 1225, Leonardo of Pisa (Fibonacci) was asked by Holy Roman Emperor Frederick II to solve the cubic equation $x^3 + 2x^2 + 10x = 20$. His solution was
$$x = 1 + \frac{22}{60} + \frac{7}{60^2} + \frac{42}{60^3} + \frac{33}{60^4} + \frac{4}{60^5} + \frac{40}{60^6}.$$
   (a) Show that the equation has exactly one real root.

   (b) Use the method outlined in this section to find numerical approximations to the three roots of the polynomial.

6. Solve $t^3 - 15t - 4 = 0$ using the methods outlined in this section. See exercise 33 from the section on complex numbers.

7. Verify that uv and $u + v$ are symmetric in x, y, and z.

8. Read the historical introduction of Ian Stewart's *Galois Theory* [14].

9. The discriminant of the general quartic equation $q(x) = A(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$ is given by the formula
$$\Delta_4[q(x)] = (\alpha - \beta)^2(\alpha - \gamma)^2(\alpha - \delta)^2(\beta - \gamma)^2(\beta - \delta)^2(\gamma - \delta)^2.$$
Suppose that for a particular quartic $Q(x)$ with real coefficients, $\Delta_4[Q(x)] > 0$. What can you say about the number of real roots?

9 Final Exercises

> In mathematics you don't understand things. You just get used to them. (John von Neumann)

These exercises broadly cover the content of the book, and their difficulty varies! However, in general it is a good idea to look at the exercises in each section first as they often include some content themselves.

1. Is $(x-15)$ a factor of $(x^3-19x-30)$? Is (x^2+5x+6) a factor?
2. Factor completely $9x^4-13x^2+4$.
3. Solve $x^3+9x^2=60-8x$.
4. Find k such that $(x-4)$ is a factor of $x^3+7x^2-14x+k$.
5. Find a value of $k \neq 0$ such that $kx^2-6x+1=0$ will have just one root.
6. Find all sixth roots of i.
7. Find k such that $8-x+2\sqrt{2x+k}=0$ has exactly one real root.
8. Solve $(\alpha^2+2\alpha-4)(\alpha^7+1)=0$.
9. Solve $x^4+x^2+1=0$ for x.

*10. Solve $\beta^2+\beta+1=0$ for x if $\beta=x^2+x+1$.

*11. Let α, β, and γ be the three roots of $ax^3+bx^2+cx+d=0$. Prove that: (a) $\alpha+\beta+\gamma=\frac{-b}{a}$, (b) $\alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a}$, and (c) $\alpha\beta\gamma=\frac{-d}{a}$. Hence show that $\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2=\frac{bd}{a^2}$.

12. Solve $(z+1)^3=8(z-1)^3$ for z. Give exact answers in the form $a+ib$.
13. Graph the equation $|z|=3$ in the complex plane.
14. If $z=1+i$ and $w=\frac{1}{z}+i$, find the argument of w.

*15. If $\frac{z+2i}{z-2i}$ is purely imaginary, describe the possible values of z.

16. If $|z-1+2i|=|z+1|$ and $z=x+yi$, find an expression for y in terms of x (i.e. find the locus of z).
17. Sketch the region satisfied by $\mathfrak{Re}\,(z-i\overline{z})>2$.
18. If $x=2$ and $x=6$ are solutions of $p(x)=Ax^2+Bx+C$ and $p(0)=-4$, find A, B, and C.
19. If $w=2-3i$ is a zero of $3w^3-14w^2+Aw-26$ (where A is real), find A and the remaining two roots.

*20. Use de Moivre's Theorem to show that

   (a) $\sin 2\theta=2\sin\theta\cos\theta$ and $\cos 2\theta=\cos^2\theta-\sin^2\theta$; and

   (b) $\sin 3\theta=3\sin\theta-4\sin^3\theta$ and $\cos 3\theta=4\cos^3\theta-3\cos\theta$.

21. Use Euler's formula to prove that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$, and that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$.

**22. Show that $\arctan a + \arctan b = \arctan \frac{a+b}{1-ab}$.

*23. Suppose that $|z+w| = |z-w|$. Show that $\arg z - \arg w = \pm\frac{\pi}{2}$.

*24. If $3z^3 + (2-3ai)z^2 + (6+2bi)z + 4$ has exactly one real root, what value must the quotient b/a take if both a and b are real? Find the real root.

*25. As we have already seen, complex numbers can be used to solve problems in geometry. For example, multiplication by i rotates a point by $\frac{\pi}{2}$ around the origin. A generalisation of this allows us to rotate a point z around an arbitrary point a by that angle: $z' = a + i(z-a)$.

A pirate named Kim Dotcom has hidden his sick tunes on a desert island tax haven in order to evade American FBI agents. You have been hired to find the data stick containing his mp3 files so that his extradition case can be resolved, but all you can find is a treasure map with the following instructions:

> From the statue of Richard Seddon, go to the kauri tree (counting your steps), and then turn exactly 90°left and walk the same number of steps to the point g'. Returning to the statue, walk to the beech tree (again counting your steps). Turn right by 90°, and walk the same number of steps to point g''. The treasure is buried exactly at the midpoint of the line joining g' and g''.
> (Kim Dotcom's Amazing Totally Secret Treasure Map™)

Given that the kauri tree is at $(0,0)$, the beech tree is at $(10,0)$, and the statue is somewhere on the line $y = 2009$, find the location of the treasure using the geometry of complex numbers and rid New Zealand of Kim Dotcom forever.[3]

*26. A line in $\mathbb{C}^2$ (the plane with complex coordinates) is defined to be the locus of a linear equation $ax + by + c = 0$ where a, b, and c are complex constants. Prove that, given two distinct points (x_0, y_0) and (x_1, y_1) in $\mathbb{C}^2$, there is a **unique** line through those two points. *Hint: it is certainly **not** true that there is a unique linear equation whose graph includes both points.*

*27. Find all possible values for θ if $\operatorname{cis}^2\theta + \operatorname{cis}\theta + 1 = 0$.

28. Graph the locus of $\arg w = |w|$. What about $\arg w + |w| = 1$?

29. Given a quadratic equation $x^2 + px + q$ and a root α, show that the other root β is given by $\beta = -p - \alpha$ and find a similar expression for finding two roots of a cubic given the third.

30. Suppose p is a quadratic (i.e. $p(x) = ax^2 + bx + c$ for some a, b, and c). Suppose further that $p(0) = 9$, and $p(3) = 0$. How many distinct roots does p have?

*31. Use the identity $x^2 + y^2 = (x - iy)(x + iy)$ to prove that if m and n are integers that can be written as the sum of two squares, then their product mn can also be written as a sum of two squares.

[3] Problem taken from [12], section 5.1.

32. Solve the following system of equations:

$$x^2 + 4xy + y^2 = 2$$
$$x^2 - 2xy + y^2 = -4$$

33. Suppose ω is a primitive cube root of unity. Show that

$$y_2 = \omega\sqrt[3]{\frac{1}{2}(-1+\sqrt{5})} + \omega^2\sqrt[3]{\frac{1}{2}(-1-\sqrt{5})}$$

and

$$y_3 = \omega^2\sqrt[3]{\frac{1}{2}(-1+\sqrt{5})} + \omega\sqrt[3]{\frac{1}{2}(-1-\sqrt{5})}$$

are complex conjugates.

34. Find all solutions to $x^{n-1} + x^{n-2} + \cdots + 1 = 0$, where n is a natural number.

35. (a) Let $p(x) = \sum_{r=0}^{n} p_r x^r$ be a polynomial with real coefficients. If $p(z) = 0$, then $p(\overline{z}) = 0$. (This is a generalisation of 5.27 to arbitrary degree polynomials.)

    (b) Let $p(x) = \sum_{r=0}^{n} p_r x^{2r}$ be a polynomial with real coefficients and only even powers of x. If $p(a+bi) = 0$, then $p(\pm a \pm bi) = 0$ for all possible combinations of $\pm$.

*36. Under which conditions will the equation $x^2 + a(1+i)x + b(1+i) = 0$ have one or more real solutions if both a and b are real?

37. Find the complex number z which satisfies $\arg(z-1-i) = -\frac{\pi}{6}$ and $\arg(z-1+i) = \frac{\pi}{6}$.

**38. Find all integer values of a and b such that $\frac{a^2+b^2}{ab}$ is an integer.

**39. Let w and z be complex numbers, and let $u = w+z$ and $v = w^2+z^2$. Prove that w and z are real **if and only if** u and v are real and $u^2 \leq 2v$.

*40. Suppose $x + \frac{1}{x} = 1$.

    (a) Show, without calculating x, that we must necessarily have

$$x^7 + \frac{1}{x^7} = 1$$

    (b) Calculate the possible values of x and verify this fact.

**41. Calculate i^i.

42. Let $\mathbb{R}[\epsilon]$ be the real numbers together with some new element $\epsilon \neq 0$ such that $\epsilon^2 = 0$.

    (a) Does there exist β in $\mathbb{R}[\epsilon]$ such that $\beta\epsilon = 1$ (i.e. does ϵ have a multiplicative inverse)?

    (b) When does $(a+b\epsilon)^{-1}$ exist?

    (c) Solve $x^2 - 1 = 2\epsilon$ in $\mathbb{R}[\epsilon]$.

43. Take the polynomial $x^2 = 4$ in the integers modulo 6 (i.e. the integers 0, 1, 2, 3, 4, 5 such that $5 + 1 = 0$). Solve for all possible values of x.

44. **Bonus exercise I:** In general, what numbers will work in the place of 3 and 9 in the XKCD comic at the bottom of the bibliography?

45. **Bonus exercise II:** Notice that $\frac{3}{16} - \frac{3}{19} = \frac{3}{16} \cdot \frac{3}{19}$. For which values of a, b, and d is the identity $\frac{a}{b} - \frac{a}{d} = \frac{a}{b} \cdot \frac{a}{d}$ true?

10 Bibliography and Further Reading

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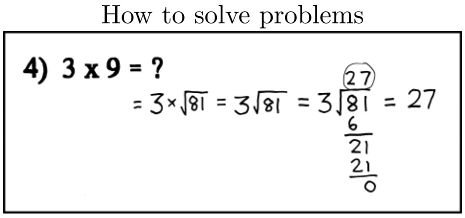
With thanks to Heydin Leeet for highlighting a typo in an exercise.

How to solve problems



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