NCEA and Scholarship practice questions

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This document contains some further mixed practice questions for exams, other than those in the notes themselves.

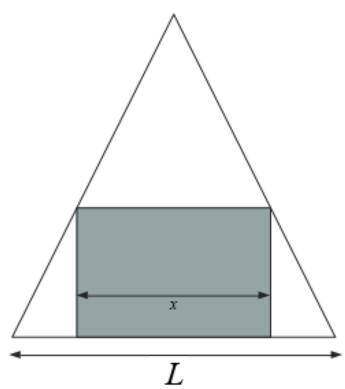
Differentiation

1. (a) Differentiate

$$y = x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2}.$$

   (b) A stone is dropped into a lake. The resulting circular ripple spreads at a constant speed of $0.5\,\mathrm{m\,s^{-1}}$. Find the rate of change of the area of the ripple after 10 seconds.

   (c) Consider an equilateral triangle with side length L. Find the maximum area of a rectangle sitting on the lowest edge of the triangle (as shown in the figure). *You need not prove that you have found a maximum.*

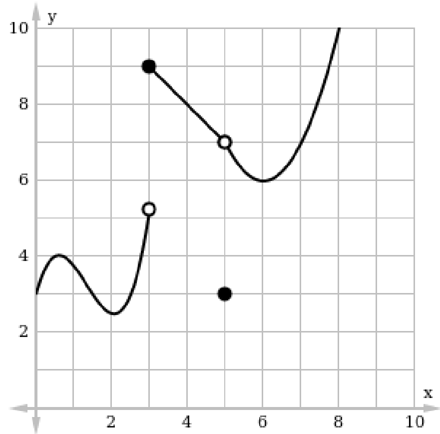


2. (a) Suppose f is a function defined by

$$f(x) = \cos(2x) + e^{x/2}.$$

   i. Find $f'(x)$ and $f''(x)$.
   ii. Determine the nature of the critical point of f at $(0, 0)$.

   (b) Consider the function g shown in the figure.



i. Find $g(3)$.

ii. Explain why $\lim_{x\to 3} g(x)$ does not exist.

iii. Find the value of $\lim_{x\to 5} g(x)$.

(c) From first principles, prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 - 5x) = 2x - 5.$$

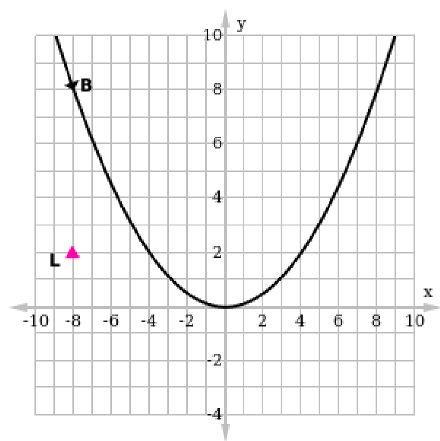
3. (a) Find $\frac{\mathrm{d}x}{\mathrm{d}t}$ if

$$x = \frac{\sec t}{1 + \tan t}.$$

(b) A boat follows a parabolic trajectory determined by the parametric equations

$$y(t) = 2t^2$$
$$x(t) = 4t$$

and shown in the figure.



i. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

ii. For which value of t is the distance between the lighthouse L at $(-8, 2)$ and the boat B at a minimum? *You need not prove that the value which you find is a minimum.*

(c) Show that if

$$y = \frac{1}{m}\sec^2(m\ln\theta)$$

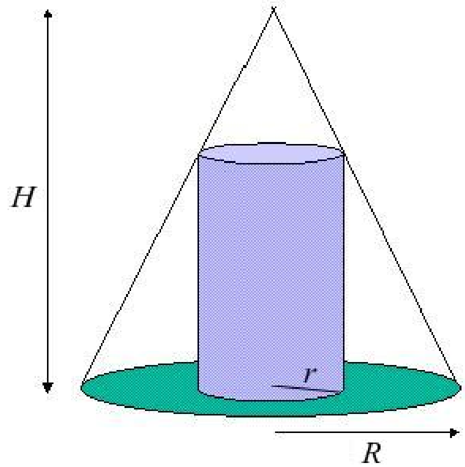
(where m is a constant) then the rate of change of y with respect to θ when $\theta = 1$ is 0.

4. (a) Find $\frac{\mathrm{d}D}{\mathrm{d}t}$ and $\frac{\mathrm{d}D}{\mathrm{d}x}$ if $D = A\sin(kx - \omega t + \phi_0)$.

   (b) Find the rate of change of y with respect to x at the point $(0, \frac{\pi}{4})$ if

$$\tan y = e^x$$

   (c) Consider a cylinder of radius r within a right-angled circular cone of radius R and height H (see figure). Show that the volume of the cylinder is maximised when $r = \frac{2}{3}R$. *Show any derivatives you require, and carefully justify that the volume is a maximum.*



5. (a) Compute the derivatives of the following functions.

      i.
$$y = \frac{e^{-x} + \sin x}{x}$$

      ii.
$$y = \sqrt{x}\left(x^2 + 5x + \frac{1}{x}\right)$$

   (b) i. Write down the expression for the derivative of x^3 from first principles. *You need not evaluate the limit.*

      ii. Hence, or otherwise, compute the following limit:

$$\lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$$

   (c) Find the interval(s) on which the function f defined below is concave up.

$$f(x) = \frac{x^5}{20} - \frac{2x^3}{3} + 16x + 9$$

6. (a) Show that the rate of change of g with respect to t at $t = 2$ is $\frac{3\pi}{2}$ if

$$g(t) = \sin(\pi\sqrt{3t^2 + 4}).$$

   (b) Find the equation to the normal line of the curve $y = x^4 - 3x^3 - 2x^2 + 2x - 1$ at the point $(0, -1)$.

   (c) Sand is being poured into a hole at a rate of $\frac{\mathrm{d}S}{\mathrm{d}t} = 3t + 4$, and the depth of the hole is given by $h = 12 - \sqrt{S}$. Find $\frac{\mathrm{d}h}{\mathrm{d}t}$ in terms of t and S, and show that h has no maxima or minima at any time after $t = 0$.

(d) Show that $y = 2x^3 - 18x^2 + 90x + 3$ has no tangent line with a slope of 3.

7. (a) For i. and ii., find the derivatives of the given functions with respect to x.

   i. $f(x) = e^{-\sqrt{x}}$

   ii. $g(x) = \dfrac{2\sqrt{x+1}}{\ln x}$

   (b) The path through space of a particle can be modelled by the following 2D parametric equation.

$$x(t) = \sin(2t)$$
$$y(t) = \sin(t + \sin(2t))$$

   i. Calculate the velocity of the particle as it passes through the origin at $t = 0$. Give your answer in the form $(v_x(t), v_y(t))$, where $v_x(t)$ and $v_y(t)$ are the x and y components of the velocity respectively.

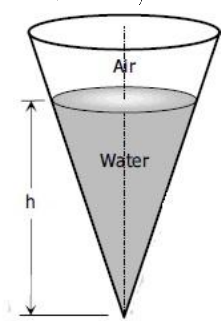
   ii. Write down an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$, and explain its geometric interpretation.

8. (a) Find the equation of the tangent line to the curve

$$y^5 + 4x^2y - x^3 + 2x^3y - 1 = 0$$

   at the point $(-5, 1)$. *Write your answer in the form $y = mx + c$.*

   (b) Amanda is dreaming that water is leaking out of a tank shaped like an upside-down cone, as in the figure. The radius of the base of the cone is $R = 2\,\mathrm{m}$, and the height of the cone is $H = 5\,\mathrm{m}$.



   i. Water is leaking out of the tank at a constant rate of $0.2\,\mathrm{m}^3$ per second. What is the rate-of-change of the depth of the water at the instant that the depth of the water is $h = 3\,\mathrm{m}$?

   ii. Amanda notices that the tank has less water in it than she would prefer, and dreams that she begins to pump water back in at a constant rate **without plugging the original leak**. When the height of the water is $h = 2\,\mathrm{m}$, the water level is rising at a rate of $0.1\,\mathrm{m\,s^{-1}}$. Find the rate at which water is being pumped back in.

9. (a) Let $\varphi(x) = \sqrt{x}$. Explain why the limit $\lim_{x\to 0} \varphi(x)$ does not exist, even though $\varphi(x)$ has a value at $x = 0$.

   (b) Recall that an isoceles triangle is a triangle with two equal sides. Find the side-lengths of the isoceles triangle of greatest area with perimeter P.

   (c) Peter Pan is opening a lemonade stand in Taumarunui. Each lemon costs \$5, but only half a lemon is needed for each glass; there is also an initial fixed cost of \$5 for the glasses, sugar, and other equipment. After conducting a survey, Peter finds that if he sells the lemonade at a price c per glass, the expected demand is $D = 30e^{-c/2}$. What price should he sell a glass for in order to maximise his profit?

Integration

1. (a) Compute the following indefinite integrals.

   i.
$$\int \frac{3t^2 + 2t}{\sqrt{t}}\,\mathrm{d}t$$

   ii.
$$\int 2\sin 2x \sin(\cos 2x)\,\mathrm{d}x$$

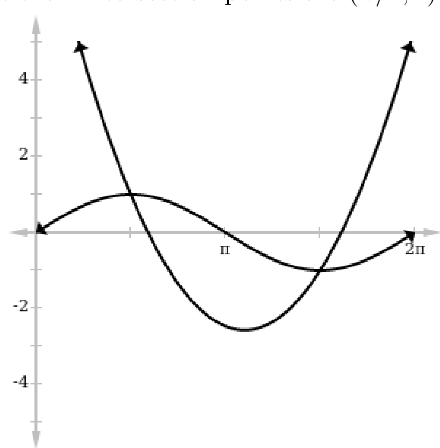
   (b) Suppose that the derivative of y is
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\ln(2)(x+2)}.$$
   If $y = 3$ when $x = 0$, find y when $x = -1$.

   (c) Suppose f is a continuous function. If $0 < x < y < 10$, $\int_x^{10} f(t)\,\mathrm{d}t = 3$, $\int_0^y \mathrm{d}t = 4$, and $\int_y^x f(t)\,\mathrm{d}t = 2$, find $\int_0^{10} f(t)\,\mathrm{d}t$.

   (d) The base of a solid is a square with vertices located at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. Each cross-section perpendicular to the x-axis is a semicircle (so the semicircle on the x-axis itself has radius 1). Form a definite integral and calculate the volume of the solid.

2. (a) Compute the following definite integral.
$$\int_{\pi/4}^{\pi/3} \csc^2\theta\,\mathrm{d}\theta$$

   (b) Find the area (to 1 decimal place) between the graphs of $y = \sin x$ and $y = x^2 - \frac{2(1+\pi^2)}{\pi}x + \frac{3\pi^2}{4} + 2$ shown in the figure, given that their intersection points are $(\pi/2, 1)$ and $(3\pi/2, -1)$.



   (c) The rate of change of a particular animal population over t days is given by
$$\frac{\mathrm{d}P}{\mathrm{d}t} = 1 - \frac{P}{M},$$
   where M is a constant.

   i. Find an explicit formula for P in terms of M and t, given that $P(0) = 100$.

   ii. If $\frac{\mathrm{d}P}{\mathrm{d}t} = 1$ when $t = 0$, find M. Hence, find the population after 100 days.

3. (a) The following table gives values of a function $D(t)$, modelling the data over a network in gigabits per hour. Use Simpson's rule with $n = 6$ to approximate $\int_0^6 D(t)\,\mathrm{d}t$.

t	$D(t)$
0	3.2
1	2.7
2	1.9
3	1.7
4	1.3
5	1.0
6	1.1

(b) Evaluate the following indefinite integral.

$$\int e^x(15+e^x)^{2017}+3\,\mathrm{d}x$$

(c) Suppose n is an integer constant. If y is defined implicitly as a function of x as follows, and $y=0$ when $x=0$, find y when $x=\frac{1}{n}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-(y+2)(n\pi\sin(xn\pi))$$

(d) Using the substitution $x=\sin\theta$, or otherwise, compute the following indefinite integral.

$$\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

4. (a) Compute the following definite integrals.

   i. $\displaystyle\int_{\pi/4}^{\pi/2} 2\csc 2x\cot 2x\,\mathrm{d}x$

   ii. $\displaystyle\int_{1}^{4} t\left(\frac{1.5}{\sqrt{t}}+12\right)\mathrm{d}t$

   (b) A balloon is being pumped up slowly such that its volume changes at a rate inversely proportional to the time t (in minutes); the rate can be modelled by $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{k}{t+1}$ for some constant k. The initial volume of the balloon was $0.5\,\mathrm{m}^3$, and after three minutes the volume had doubled. After what length of time will the volume of the balloon exceed $2\,\mathrm{m}^3$?

   (c) Find the area between the curves $y=-\frac{1}{2}x^2+\frac{3}{2}x$ and $y=\frac{1}{2}x^2-\frac{3}{2}x+2$ given that they intersect at $(1,1)$ and $(2,1)$.

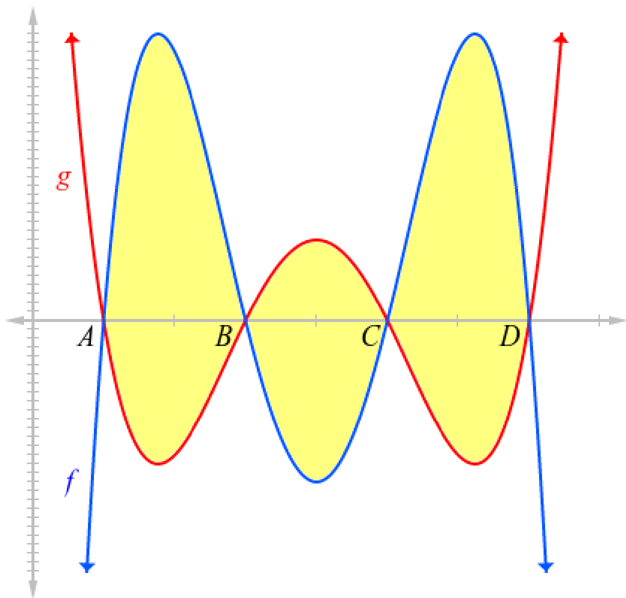
5. (a) Compute the following indefinite integrals.

   i. $\displaystyle\int\frac{2x+1}{x^2+x}\,\mathrm{d}x$

   ii. $\displaystyle\int\cos^4\theta\,\mathrm{d}\theta$

   (b) Consider the functions f and g graphed in the figure. Given that the shaded region has a total area of 32 units squared, $\int_A^B f(x)-g(x)\,\mathrm{d}x=2$, and $\int_C^D f(x)-g(x)\,\mathrm{d}x=10$, compute

$$\int_B^C f(x)-g(x)\,\mathrm{d}x.$$

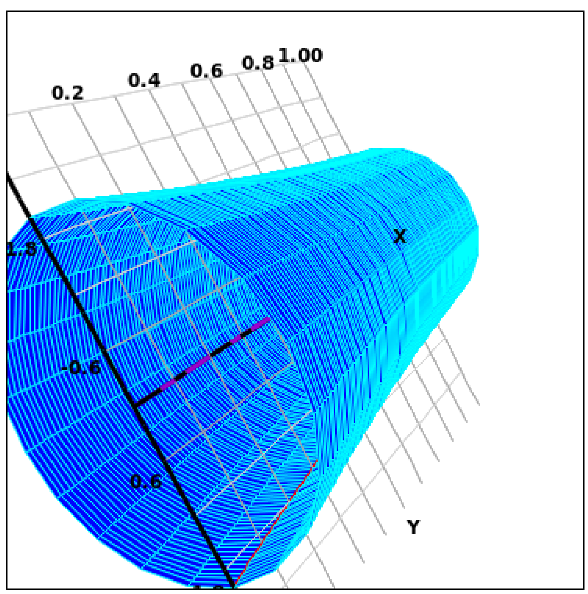


(c) Suppose that $\frac{\mathrm{d}y}{\mathrm{d}t} = (y+1)(\sin 2\pi t)$, and $y(0) = 1$. Find $y(0.5)$.

6. (a) The *volume of revolution* of a curve $y = f(x)$ between two bounds $x = a$ and $x = b$ is given by $\pi \int_a^b y^2 \, \mathrm{d}x$. Find the volume of revolution of $y = e^{-x} + 1$ between $x = 0$ and $x = 1$ by computing

$$\pi \int_0^1 (e^{-x} + 1)^2 \, \mathrm{d}x.$$

The volume is visualised in the figure.



(b) Suppose that the acceleration of a particle is described at time t by the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 12t + 12.$$

If the particle is initially located at $x = 3$, and is initially stationary, find the position of the particle at time $t = 10$.

(c) Newton's law of cooling states that the rate of heat loss of an object is directly proportional to the difference in the temperatures between the object and its surroundings. Symbolically, if the environmental temperature is T_0, then

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_0).$$

A loaf of bread is removed from an oven at a temperature of 230 °C, and is left to cool in an area with a temperature of 18 °C. How long, in terms of k, will it take for the bread to cool to 30 °C?

(d) Explain why $\int_0^{2\pi} \tan x \, \mathrm{d}x$ does not exist.

7. (a) For i. and ii., find the indefinite integrals. **Do not forget the constant of integration.**

i.

$$\int \frac{1}{\sqrt{2x+2}} \, \mathrm{d}x$$

ii.

$$\int \sec^2(x) \sec(\tan(x)) \tan(\tan(x)) \, \mathrm{d}x$$

(b) Find the area of the region bounded by $y = 1/x$, $y = 1/x^2$, and $x = 2$.

(c) Find the general solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan y \tan x$.

8. (a) Consider the following integral.

$$\pi \int_1^5 (x^2 e^{-x})^2 \, \mathrm{d}x.$$

Complete the following table, and use Simpson's rule with an interval length of 0.5 to approximate the value of the definite integral.

x	$(x^2e^{-x})^2$
1.0	
1.5	0.2520
2.0	
2.5	0.2632
3.0	
3.5	0.1368
4.0	0.0859
4.5	0.0506
5.0	0.0284

(b) Find k such that $\int_1^k \frac{\ln x}{x} \, \mathrm{d}x = 5$.

(c) The Bank of Money advertises that a particular savings account package has continuously compounding interest at a rate of 4% per annum. Express this as a differential equation, and hence find the accumulated interest on an initial deposit of \$2500 after four years.

9. (a) The average value of a function f on the interval $a \leq x \leq b$ is defined to be

$$f_{\text{avg}} = \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}.$$

The temperature T (in °C) of Napier t hours after 9 am on a particular day was modelled by

$$T(t) = 20 + 8\sin\left(\frac{\pi t}{12}\right).$$

Find the average temperature during the period from 9 am to 9 pm.

(b) The height of an obelisk, like that pictured in the figure, is 18 m. A horizontal cross-section at a distance x metres from the bottom is a rectancle of side lengths $(3 - 0.1x)$ and $(4 - 0.2x)$. Use integration to find the volume of the obelisk.

(c) Suppose that

$$1 = \int_m^{2m} x\cos(mx^2)\,\mathrm{d}x.$$

Show that

$$m = \cos\frac{5m^3}{2}\sin\frac{3m^3}{2},$$

and conclude that $m \leq 1$.

Algebra

Note: primitive roots of unity are not examinable at L3, but may show up for scholarship.

1. (a) i. Convert $w = -1 - i\sqrt{3}$ into polar form.
   ii. Hence, or otherwise, find w^{2016} **exactly** in rectangular form.

   (b) Show that a complex number z is real if and only if $z = \bar{z}$.

2. (a) i. Find all solutions to the equation $3z^3 - 1 = 0$ in polar form.
   ii. Convert them to rectangular form and plot them on an Argand diagram.

   (b) Find all the primitive sixth roots of $-i$.

3. (a) Suppose that $\frac{a}{b} = \frac{a+b}{a}$. Find $\frac{a}{b}$ exactly.

   (b) i. Consider the polynomial $p(x) = x^2 - 5x - 2k$, where k is a real constant. Suppose that $p(x)$ is known to have exactly one real root. Find k.
   ii. Find the real root.

4. (a) Find the remainder when $x^3 + 3kx^2 + 3k^2x + k^3$ is divided by $(x + k)$.

(b) Consider $x^3 + px^2 + q$, where p and q are real constants. Suppose that this polynomial has three real roots α, β, and γ. Show that if $\beta = \gamma$, then $\alpha = -2\beta$.

(c) Show that $z^n + z^{n-1} + \cdots + z + 1$ has at most one real root for $n \in \mathbb{N}$.

5. (a) i. Suppose that z is a complex number, and n is a real number. Show that $|z^n| = |z|^n$.

   ii. Hence, or otherwise, show that if $x + iy = (s + it)^n$, then $x^2 + y^2 = (s^2 + t^2)^n$ (where x, y, s, t, and n are real numbers).

   (b) Show that the locus of $|z + a| - |z - b| = 0$ is a straight line (where a and b are complex constants).

6. (a) i. Show that $(4 - 3i)^2 = 7 - 24i$.

   ii. Solve $z^2 - (2 + i)z + (-1 + 7i) = 0$ for z.

   (b) Find **all four** roots of $x^4 + 6x^3 + x^2 - 6x - 2$ exactly. *Hint: two of the roots are small positive or negative integers.*

7. (a) Find all three cube roots of unity in rectangular form.

   (b) Suppose that a is a complex number such that $a^2 + a + \frac{1}{a} + \frac{1}{a^2} + 1 = 0$. Evaluate $a^{2m} + a^m + \frac{1}{a^m} + \frac{1}{a^{2m}}$.

   (c) Solve $x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$, and hence show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. *Hint: multiply the polynomial by* $(x - 2)$.

8. (a) Rewrite $\frac{24 - i^7}{3 - 4i}$ in the form $a + bi$, where a and b are integers.

   (b) Solve $\sqrt{x - 2} + \sqrt{x + 3} = 10$.

   (c) Consider $p(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$.

   i. Find the remainder upon division of $p(x)$ by $(x - (1 + i))$.

   ii. Hence factorise $p(x)$ completely into linear factors.

   (d) Suppose $x^3 + px + q = 0$ has three complex roots: α, β, and γ. Find some cubic polynomial with coefficients in terms of p and q and roots α^2, β^2, and γ^2.

9. (a) Find all the square roots of i in rectangular form.

   (b) i. Compute $(1 - i)^2$.

   ii. Hence, or otherwise, find all zeros of $z^2 - (5 + 5i)z + 13i$.

   (c) Find all the solutions to $0 = x^4 - 2x^3 - 5x^2 + 6x$.

10. (a) Rewrite $\alpha = \frac{4}{\sqrt{2}} - \frac{4}{\sqrt{2}}i$ in polar form, and calculate α^4 (keeping your answer in polar form).

    (b) Describe the locus of all complex numbers z such that $|z| = 2(1 + \mathfrak{Re}(z) + \mathfrak{Im}(z))$.

    (c) Show that $\zeta = \operatorname{cis}(6\pi/283)$ is a **primitive** 283rd root of unity. (Hint: 283 is prime.)

Scholarship

1. (a) Consider the sequence of functions

$$f_n(x) = nx(1 - x^2)^n \qquad (0 \leq x \leq 1,\ n = 1, 2, 3, \ldots).$$

Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x.$$

   (b) Compute the following definite integral.

$$\int_0^1 \sin^3 x \cos^4 x + \sin^4 x \cos^3 x\,\mathrm{d}x.$$

(c) Suppose that f is a function satisfying

$$\begin{cases} \dfrac{f(x)}{2f'(x)} = 3(x^3 - 2x^2 - x + 2), \\ f(3) = 1. \end{cases}$$

Find $f(x)$ explicitly. You need only calculate any constants of integration to three decimal places.

2. (a) (10 points) Consider the cubic equation $p(x) = x^3 + px + q$ (where p and q are real).
   i. Let the three roots of $p(x)$ be α, β, and γ. Show that

$$(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = -4p^3 - 27q^2.$$

   ii. Find the nature of the roots of $p(x)$ in the cases where $-4p^3 - 27q^2$ is greater than, less than, or equal to zero.

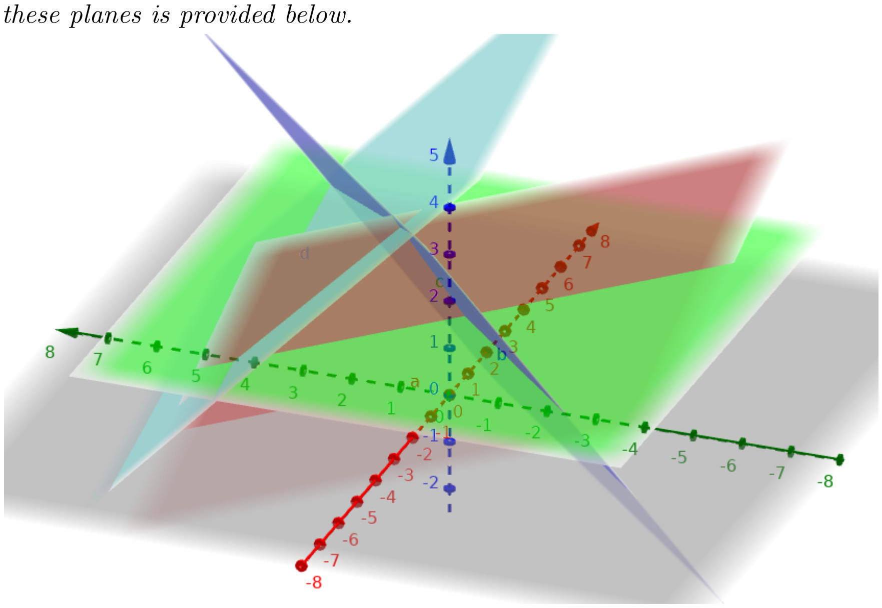
Answer ONE of (b) and (c).

(b) Find a formula for $\binom{n-1}{k} - \binom{n-1}{k-1}$ in terms of $\binom{n}{k}$.

(c) Minimise $F(x) = x^3 + y^4 + z^5$ with respect to the variables x, y, and z subject to:

$$\begin{gathered} x + y + z \geq 1 \\ x - y + z \leq 2 \\ z \geq 2 \\ 0.75x - 5.87y - 5.78z \geq -31.74. \end{gathered}$$

A graph of these planes is provided below.

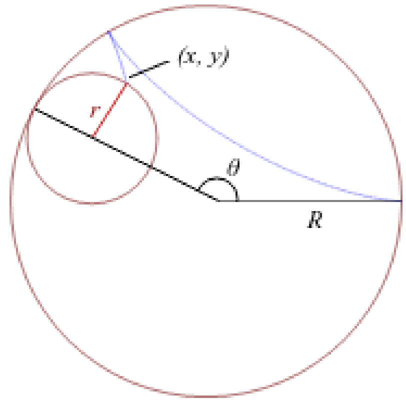


3. (a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x)(\sec^{-1} x)$ in terms of x only.

(b) Consider a circle of radius r that is rolling around the inside of a circle of radius R ($R > r$). Let $p = (x, y)$ be the point on the inner circle that is initially touching the outer circle. Show that

$$x = (R - r)\cos\theta + \cos\left(\frac{R - r}{r}\theta\right),$$
$$y = (R - r)\sin\theta - \sin\left(\frac{R - r}{r}\theta\right).$$

A diagram is provided below.



(c) The Steiner inellipse is the unique ellipse inscribed in a triangle and tangent to the midpoint of each side. Find the coordinates of the two focii of the Steiner inellipse inscribed in the triangle with vertices $(1, 7)$, $(7, 5)$, and $(3, 1)$.

4. (a) i. By differentiating $f(x)g(x)$, show that

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x.$$

ii. Find an antiderivative of

$$\frac{\sqrt{4x^2 - 9}}{x^2}.$$

(b) Discuss the significance of the fundamental theorem of calculus. You should write approximately half a page.

(c) An **homomorphism** is a function f such that $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for *all* real numbers a and b. Show that the only homomorphisms which are polynomials are $f(x) = 0$ and $f(x) = x$.

5. You may use the following theorem in answering this question. **Do not attempt to prove this theorem.**

> **Mean Value Theorem**
> Let a and b be real numbers such that $a < b$, and let f be a function satisfying two hypotheses:
>
> 1. f is continuous at all x such that $a < x < b$.
> 2. f is differentiable at all x such that $a \leq x \leq b$.
>
> Then there exists some number c such that
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(a) Verify that the function f defined by

$$f(x) = \frac{x}{x + 2}$$

satisfies the hypotheses of the mean value theorem on the interval $1 < x < 4$. Find all points c that satisfy the conclusion of the mean value theorem.

(b) A point a is called a **fixed point** of a function f if $f(a) = a$. Suppose that $f'(x) \neq 1$ for all real numbers x; show that f has at most one fixed point.

(c) Find a function f such that $f'(-1) = \frac{1}{2}$, $f'(0) = 0$, and $f''(x) > 0$ for all x, or prove that such a function cannot exist.