NCEA Level 2 Mathematics
5. Quadratic Modelling

This topic is primarily revision from Level 1.

A linear function is one of the form

$$f(x) = mx + c;$$

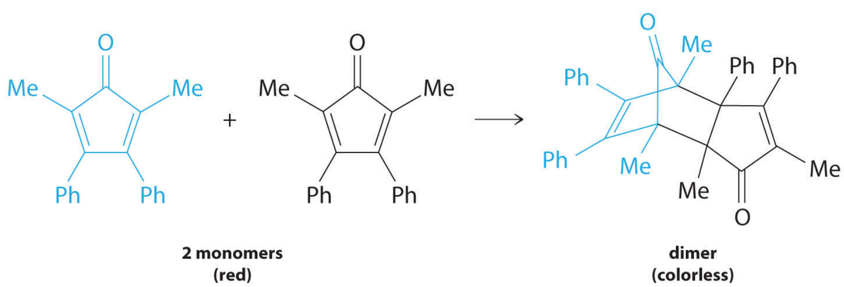
we have already seen that the graph of such a function is a straight line.

The natural next step is to consider quadratic functions: those of the form

$$f(x) = ax^2 + bx + c.$$

Last year, we saw that the graphs of quadratic functions are parabolae; so we can use quadratic equations to model situations which are vaguely parabolic.

Example.



The following table gives the instantaneous rates of reaction for the dimerization reaction above.

Time (min)	**Concentration of reactant** (M)	**Instantaneous reaction rate** (M/min)
0	0.0054	
10	0.0044	8.0×10^{-5}
26	0.0034	5.0×10^{-5}
44	0.0027	3.1×10^{-5}
70	0.0020	1.8×10^{-5}
120	0.0014	8.0×10^{-6}

It is known that this reaction is second-order: that is, the reaction rate is modelled by a quadratic function of the reaction concentration. Let's call the reaction rate at a particular concentration $R(C)$; so

$$R(C) = XC^2 + YC + Z.$$

By using the values in the table above, we have that

$$\begin{aligned} 8.0 \times 10^{-5} &= X \cdot (0.0044)^2 + Y(0.0044) + Z \\ 5.0 \times 10^{-5} &= X \cdot (0.0034)^2 + Y(0.0034) + Z \\ 3.1 \times 10^{-5} &= X \cdot (0.0027)^2 + Y(0.0027) + Z. \end{aligned}$$

Using `matlab` to find X, Y, and Z we have

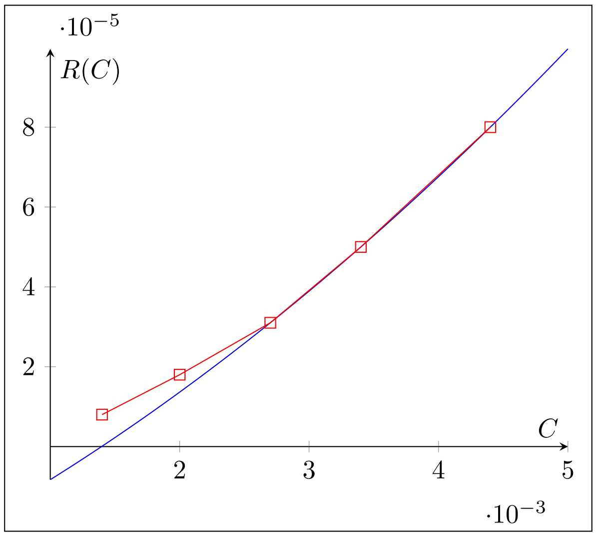
```
>> syms X Y Z;
>> eqn1 = 8.0e-5 == X * (0.0044)^2 + Y * (0.0044) + Z;
>> eqn2 = 5.0e-5 == X * (0.0034)^2 + Y * (0.0034) + Z;
>> eqn3 = 3.1e-5 == X * (0.0027)^2 + Y * (0.0027) + Z;
>> [A,B] = equationsToMatrix([eqn1,eqn2,eqn3],[X,Y,Z]);
>> linsolve(A,B);
```

```
ans =

                                    6148914691236521356/3658604241285734375
    3647826973392469608438645938122 77/2159658755391560033062155395792896 0
```

So our model is
$$R(C) \approx 1.6807C^2 + 0.01689C - 2.686 \times 10^{-5}.$$

Graphing our model (blue) next to the original data (red), we see that the match is reasonable.



This is useful because it allows us to predict the reaction rate for concentrations that we have not measured; alternatively, we can predict the reactant concentration given the reaction rate (which can be easily measured).

Some situations require us to find a minimum or a maximum value. Suppose, for example, that we are asked the following.

Example. Find the dimensions of a rectangle with perimeter 1000 m such that the area is maximised.

Solution. Later this year, we will learn a systematic way to solve optimisation problems like this. However, using calculus here would be like using a machine gun to kill a fly (although for some reason it is the favoured method of economics students)!

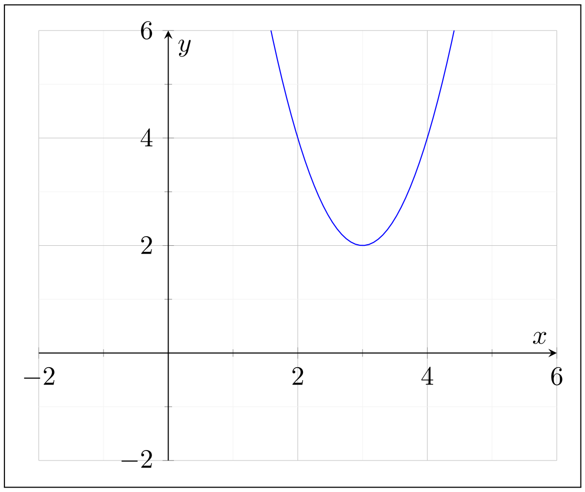
Let us call the two side lengths x and y; so we have that $1000 = 2x + 2y$ (so $500 = x + y$), and we want to minimise $A = xy$. Substituting, we have $A = x(500 - x) = 500x - x^2$. The maximum value will be the vertex of this parabola, so we need to put this formula into the form $A = -(x - b)^2 + c$; if we expand this, then $A = -x^2 + 2xb - b^2 + c$ and so we want $2b = 500$ and $c - b^2 = 0$. From these, we see that $b = 250$ and $c = (250)^2$. Hence $A = -(x - 250)^2 + (250)^2$; the vertex is at $x = 250$, and so $y = 250$: the area will be maximised when the rectangle is a square.

Here we are using the vertex form of the parabola, which we learned to find last year. The idea is that every parabola $y = ax^2 + bx + c$ is just a shifted version of the parabola $y = ax^2$; in particular, there exist numbers x_0 and y_0 such that
$$ax^2 + bx + c = a(x - x_0)^2 + y_0;$$
and the equation on the right, when graphed, has a vertex at (x_0, y_0) (why?).

Questions

You might want to use a calculator or a computer to help you with some of the calculations: for example, solving systems of equations or quadratic equations.

1. Give the equation of the following parabola:



2. (a) Find the vertices of the parabolae $y = x^2 + 4x - 9$ and $y = -4x^2 + 2x - 1$.
   (b) More generally, show that the vertex of the parabola $y = x^2 + bx + c$ is
$$\left(-\frac{b}{2}, c - \frac{b^2}{4}\right).$$

3. (a) Find the equation of the parabola with vertex $(2, 3)$ that passes through $(9, 1)$.
   (b) More generally, show that the equation of the parabola with vertex $V(x_0, y_0)$ passing through $A(x_1, y_1)$ is
$$y = \frac{y_1 - y_0}{(x_1 - x_0)^2}(x - x_0)^2 + y_0.$$

4. Show that the equation $y + 3 = \sqrt{x^2 + (y-1)^2}$ describes a parabola, and find its vertex.

5. One application of mathematical modelling is in chemical spectroscopy: it is possible to measure the absorbance of light by substances, which is proportional to the amount of the substance present.

Amount of protein (µg)	**Absorbance**
0	0.099
5.0	0.185
10.0	0.282
15.0	0.345
20.0	0.425
25.0	0.483

   (a) Use the values for 0, 5.0, and 25.0 micrograms to write down an equation for a parabolic model of this data.
   (b) How accurately does this model predict the absorbance for 15 micrograms?

(c) A better model, taking into account all six data points, is found by computer to be

$$\text{amount of protein} = -0.000131429A^2 + 0.0188686A + 0.0982286.$$

If the absorbance is measured experimentally to be 0.3, how much protein is present in the sample?

6. Prove that the x–coordinate of a parabola's vertex is always halfway between the x–coordinates of its x–intercepts.

7. Let ℓ represent the line $y = -d$ for some real number d. If $P = (x, y)$ is any point, the distance $d(P, \ell)$ is defined to be $y + d$ (i.e. the length of the line segment from P to ℓ that is perpendicular to ℓ).

   (a) Let $O = (x_0, y_0)$ be some fixed point, and let R be some fixed number. Show that the set of all points P such that $d(P, O) = R = d(P, \ell)$ is a parabola.

   (b) On the other hand, suppose $y = ax^2 + bx + c$ describes a parabola. Show that there exists:

   - a real number d,
   - a real number R, and
   - a point $O = (x_0, y_0)$

   such that the parabola is the set of points described in (a).