Solutions to L3 Calculus Integration Exam 3

Alexander Elzenaar

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Question One

Part (a)

i. Note that $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{1}{2\sqrt{x}}$, so we must have

$$\int \frac{1}{\sqrt{2x+2}}\,\mathrm{d}x = \int \frac{2}{2\sqrt{2x+2}}\,\mathrm{d}x = \sqrt{2x+2} + C.$$

ii. Let $u = \tan x$ so $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$ and our integral becomes

$$\int \sec(u)\tan(u)\,\mathrm{d}u = \sec u + C = \sec\tan x + C.$$

Part (b)

First, we find the left-hand integration limit: $1/x = 1/x^2$ so $1 = 1/x$ and $x = 1$.

$$\begin{aligned}\int_1^2 \frac{1}{x} - \frac{1}{x^2}\,\mathrm{d}x &= \left[\ln x + \frac{1}{x}\right]\Bigg|_1^2 \\ &= (\ln 2 - \ln 1) + \left(\frac{1}{2} - 1\right) \\ &= \ln\frac{1}{2} - \frac{1}{2}.\end{aligned}$$

Part (c)

Separation of variables.

$$\begin{aligned}\int \frac{\cos y}{\sin y}\,\mathrm{d}y &= \int \frac{\sin x}{\cos x}\,\mathrm{d}x \\ \ln\sin y &= -\ln\cos x + C \\ \sin y &= Ke^{\ln\sec x} = K\sec x \\ y &= \sin^{-1}(K\sec x).\end{aligned}$$

(where $K = e^C$.)

Question Two

Part (a)

x	$(x^2e^{-x})^2$
1.0	**0.1353**
1.5	0.2520
2.0	**0.2931**
2.5	0.2632
3.0	**0.2008**
3.5	0.1368
4.0	0.0859
4.5	0.0506
5.0	0.0284

Note that $n = 8$ (number of intervals is one less than numbers of points), $a = 1.0$, $b = 5.0$. Hence:

$$\begin{aligned}\int_1^5 (x^2e^{-x})^2\,\mathrm{d}x \approx \frac{5-1}{8}&[0.1353 + 0.0284 + 4(0.2520 + 0.2632 + 0.1368 + 0.0506) +\\ &+ 2(0.2931 + 0.2008 + 0.0859)]\\ &= 2.0669\end{aligned}$$

and the required value is 6.4932.

Part (b)

We have $5 = [\frac{1}{2}(\ln x)^2]\Big|_1^k = \frac{1}{2}(\ln k)^2$, so $k = \exp(\sqrt{10})$.

Part (c)

Let $B(t)$ be the bank balance after t years. Then:

$$\frac{\mathrm{d}B}{\mathrm{d}t} = 0.04B,$$

so $\ln B = 0.04t + C \Rightarrow B = B_0e^{0.04t}$ $(B_0 = e^C)$. We have $t = 4$ and $B_0 = 2500$, so $B(4) = 2500e^{0.16} = \$2933.78$.

Question Three

Part (a)

Computing:

$$\begin{aligned}
\frac{\int_0^{12} 20 + 8\sin\left(\frac{\pi t}{12}\right)\,\mathrm{d}t}{12} &= \frac{1}{12}\left[20t - \frac{96}{\pi}\cos\left(\frac{\pi t}{12}\right)\right]\Bigg|_0^{12} \\
&= \frac{1}{12}\left[\left(240 + \frac{96}{\pi}\right) + \frac{96}{\pi}\right] \\
&= 25\,^{\circ}\mathrm{C}.
\end{aligned}$$

Part (b)

The area of the cross-section at height x is $A = (3 - 0.1x)(4 - 0.2x) = 0.02x^2 - x + 12$; hence the total volume is:

$$\int_0^{18} 0.02x^2 - x + 12\,\mathrm{d}x = \frac{0.02 \times 18^3}{3} - \frac{18^2}{2} + 12 \times 18 = 92.88\,\mathrm{m}^3.$$

Part (c)

$$\begin{aligned}
1 &= \int_m^{2m} x\cos(mx^2)\,\mathrm{d}x \\
&= \left[\frac{1}{2m}\sin(mx^2)\right]\Bigg|_m^{2m} \\
&= \frac{1}{2m}\left(\sin 4m^3 - \sin m^3\right) \\
&= \frac{1}{2m} 2\cos\frac{5m^3}{2}\sin\frac{3m^3}{2} \\
m &= \cos\frac{5m^3}{2}\sin\frac{3m^3}{2}.
\end{aligned}$$

Since $\forall x$ we have $\cos x \leq 1$ and $\sin x \leq 1$, it follows that $\cos\frac{5m^3}{2}\sin\frac{3m^3}{2} \leq 1$ and so $m \leq 1$.