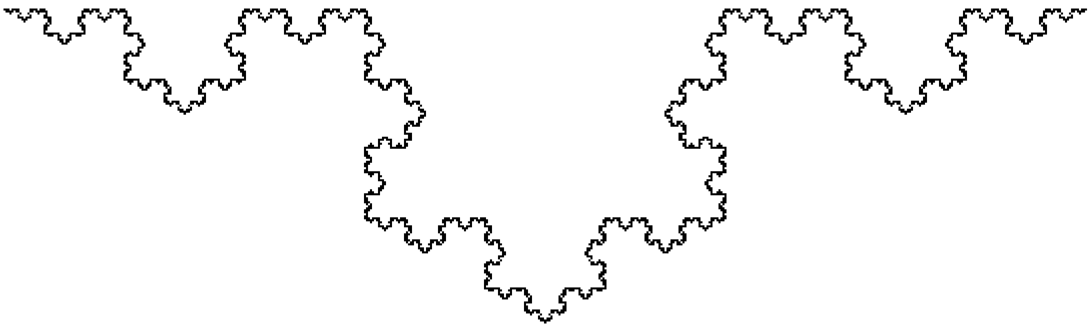
Level Two Mathematics



Alexander Elzenaar

https://github.com/aelzenaar/ncea-notes

Contents

Chapter 1

Preamble

NCEA Level 2 Mathematics
Preface

These notes present the NCEA Level 2 mathematics content from a mainly geometric standpoint, and so in some places are a little nonstandard. For example, the study of quadratic equations is approached by analysing quadratic graphs, and many of the exercises ask for intuitive and/or geometric explanations of many of the phenomena we study. This is because I see so many students entering L3 calculus with little to no ability to make use of the interplay between geometric and algebraic views of the same picture.

For example, I can cite the example of finding turning points of graphs: students remember very well that the way to solve this kind of exam problem is to "take the derivative and set it to zero", but most of them cannot explain why this works! This is foreign to me, because I don't remember not understanding this (and in fact I think the technique was taught to me geometrically first, which is really the only way to do it)! I find it hard to believe that teachers of L2 mathematics are so incompetent that they don't mention geometry at all when they introduce calculus, so maybe the problem is that when studying for exams the students just remember how to solve problems and neglect the "big picture": that memorising how to solve "types of problems" for an exam is both counterproductive and damaging because they don't remember the important ideas (about tangent lines, slopes, approximations, geometry) and then wonder why the "skills" they've learned (symbol pushing on an exam paper) are useless in the "real world"!

These notes also have another agenda: to introduce students in a calm way to mathematical proof. There is a definite increase in sophistication required for the later sections, but right from the first problem set students are asked to justify statements mathematically. I make no apology to those who want to use these notes but avoid forcing students to write proofs: it is simply how mathematics is done (and I don't think many of the exercises, if any, are out of the reach of the enthusiastic student).

I have tried to address many of these ideas in my student introduction as well.

Some regrets

It is my feeling that geometry is not given a prominent enough place in the school curriculum; nonetheless, I have (for the most part) resisted the temptation to include more Euclidean geometry beyond the standard coordinate geometry and trigonometry (even when it would make things easier, like problem 10 of section 3 on the centres of the triangle). I have also not included any material on the geometric meaning of the integral, as it is no longer in the Y12 curriculum.

Another omission is unfortunately shared by many other sets of notes: a dense subset of the results and theories described are not placed in their proper historical context. This is primarily for the sake of space, and the author respectfully submits that he is even less of a historian than a mathematician. A readable (even for high-school students) book which discusses the history of many of the topics is Kline's *Mathematics for the non-mathematician*, though I cannot vouch for its accuracy.

Guide to the bibliography

The bibliography is a mixture of further reading and additional problemsets. I have not included many drill-type problems (like "solve for x given $x^2 + 3x - 20$") because they are easily found for those sections that need them: in particular, Spiegel is a good source of algebra drill problems and Foerster is a good source of trigonometry drill problems. For calculus and coordinate geometry, I have included Thompson — although I know no good source that covers just the material in calculus needed for L2 and so it should be used with caution.

In terms of additional reading, most (all?) of the books are suitable for an enthusiastic Y12 student. I particularly recommend Lauwerier, Bóna, or any books on graph theory and the four-colour theorem for students interested in computer science and/or programming.

The two books by Ben Goldacre are also highly recommended for any students who will be going into the sciences or medicine.

Many of the titles are popular mathematics books (e.g. the two by Bellos) that cover the material we see this year at a slightly lower level, and put it in context (although some of the topics should be taken with a grain of salt: Bellos includes chapters on borderline crank topics like the Golden Ratio).

List of sections with the standards that they cover

Geometry

1. (2.1) Coordinate Geometry
2. (2.4) Arcs and Sectors of Circles
3. (2.4) Trigonometry

Algebra

4. (2.2) Functions
5. (2.2/2.6) Quadratic Modelling
6. (2.6/2.14) Simultaneous Equations
7. (2.6/2.14) Linear Inequations
8. (2.6) The Quadratic Formula
9. (2.2/2.6) Exponential and Logarithmic Functions
10. (2.2/2.6) Negative and Fractional Powers

Calculus

11. (2.7) Slopes and Differentiation
12. (2.7) Tangent Lines and Approximation
13. (2.7) Turning Points Optimisation
14. (2.7) Anti-differentiation
15. (2.7) Kinematics and Rates of Change

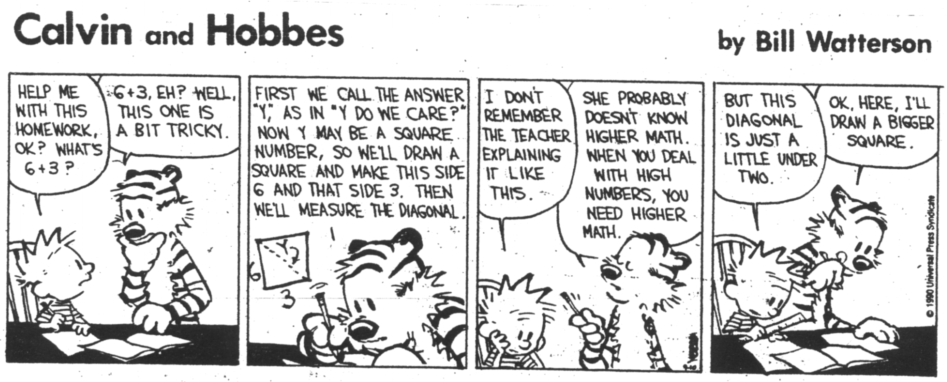
Combinatorics

16. (Stats?) Counting and Combinatorics
17. (2.3) Number Sequences and Fractals
18. (2.5) Graphs and Networks

Statistics

19. (2.10/2.11) The Statistical Enquiry Process
20. (2.9/2.11) Sampling
21. (2.9) Statistical Inference
22. (2.12/2.13) Probability and Risk
23. (2.12) Probability Distributions

NCEA Level 2 Mathematics
Introduction to the Notes



What is mathematical proof?

A proof, that is, a mathematical argument, is a work of fiction, a poem. Its goal is to satisfy. A beautiful proof should explain, and it should explain clearly, deeply, and elegantly. A well-written, well-crafted argument should feel like a splash of cool water, and be a beacon of light — it should refresh the spirit and illuminate the mind. And it should be charming.

— From *A Mathematician's Lament*, by Paul Lockhart.

Mathematics is fantastic. It is a subject where we do not have to take anyone's word or opinion. The truth is not determined by a higher authority who says 'because I say so', or because they saw it in a dream, the pixies at the bottom of their garden told them, or it came from some ancient mystical tradition. The truth is determined and justified with a mathematical proof.

A proof is an explanation of why a statement is true. More properly it is a convincing explanation of why the statement is true. By convincing I mean that it is convincing to a mathematician. (What that means is an important philosophical point which I am not going to get into; my interest is more in practical matters.)

Statements are usually proved by starting with some obvious statements, and proceeding by using small logical steps and applying definitions, axioms and previously established statements until the required statement results.

The mathematician's concept of proof is different to everyday usage. In everyday usage or in court for instance, proof is evidence that something is likely to be true. Mathematicians require more than this. We like to be 100% confident that a statement has been proved. We do not like to be 'almost certain'.

Having said that, how confident can we be that a theorem has been proved? Millions have seen a proof of Pythagoras' Theorem; we can be certain it is true. Proofs of newer results, however, may contain mistakes. I know from my own experience that some proofs given in books and research journals are in fact wrong.

— From *How to Think Like a Mathematician*, by Kevin Houston.

Why am I expected to prove things?

As a student, you are expected to (try to) learn to think like a mathematician — and that means to justify and prove things. Some students (hopefully not you) think that this is somehow too hard: that being expected to think creatively is something that is best left out of the mathematics classroom. For that student, I give a number of reasons for the fact (and it is a fact) that creative mathematical thinking is both possible and necessary for the secondary school student.

1. If we don't justify our statements, or understand what we are doing, and simply plug numbers into a formula we are doing then we are not only not doing mathematics, but we are wasting our time! If

mathematics was just about plugging numbers into formulae, we wouldn't bother teaching it to students — because computers are far more reliable, and complain far less. You should be aiming to understand *why*, not simply to memorise *how*: and the idea of a proof is to *explain the why.*

2. As Paul Lockhart puts it in his classic book *A Mathematician's Lament*, we believe that secondary students are mature enough to have sophisticated opinions on literature and creative enough to be allowed to paint, write, and create music; so it isn't really a stretch to believe that secondary students are capable of doing their own mathematics.

3. A good argument is aesthetic (as some of you, who may be considering going to university to study law, literature, or another art, will well be aware); mathematicians have a reasonably well-developed idea of an aesthetic mathematical argument or mathematical theory, and it is accessible to you. Mathematics is one of the oldest forms of human expression, and some of the most elegant proofs that you will meet this year date back at least as far as the Greeks — and over the coming years, you will meet modern proofs in subjects that have a different aesthetic nature to (say) geometry.

4. A (correct) mathematical proof is absolute and forever, in that if a mathematical statement is proved then it cannot be overturned in thirty years like a theory of chemistry or biology might be upon the discovery of hitherto unknown evidence. Mathematical reasoning is not empirical (based simply on observation and the collection of evidence), it is based on logical inference and argument.

5. Mathematical reasoning is important in other artistic disciplines (history, linguistics, philosophy) and in the scientific disciplines (physics, chemistry, biology) — and is particulary intertwined with computer science.

Homework expectations

For each topic there is a page of homework, usually a piece of reading or a video with a few questions. In addition, it will be highly adventageous for you to keep up with reading the notes as we go along. *It is your responsibility to do the homework and to keep up with the notes.* By this last statement I mean, if you have questions, or trouble, or you're getting things confused, or you forget things, *let me know and we will work through them.*

I am hesitant to recommend a specific length of time that you should be spending on the homework problems because this varies from person to person. You should be spending enough time that you have a good attempt at each problem — by which I mean, you have at least identified what kind of techniques might be applicable, and you have tried to apply them.

What you get out of the homework

- An opportunity to practice the material we cover.
- A different perspective on the subject (from the readings).
- A chance for you to think things through on your own.
- A way to test your own knowledge.
- And a baseline for the kinds of things you are expected to be able to do by the end of each topic.

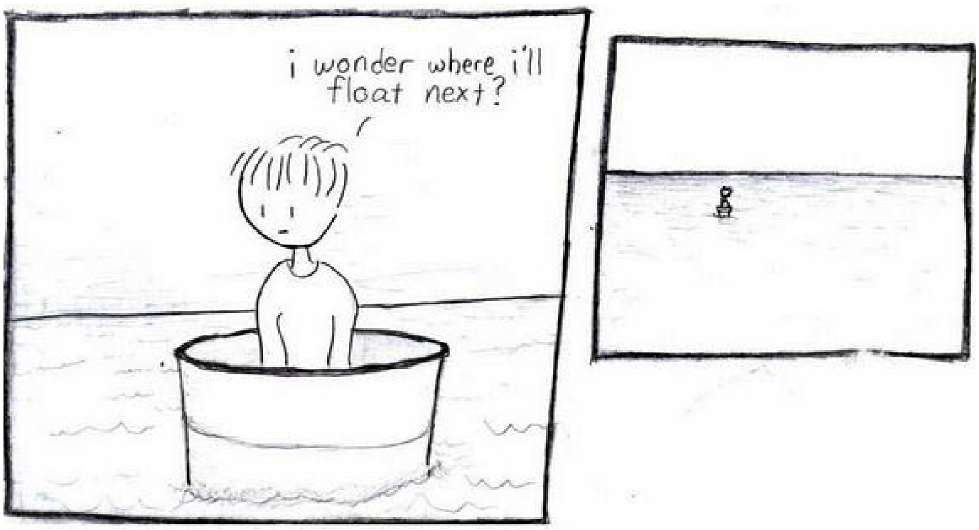
Some advice

- Draw pictures, even if you are not strictly doing "geometry".
- Take the time to write clearly and slowly, because a piece of mathematics that is not clear and transparent to the reader is not mathematics at all.

- Always ask 'why', because nothing is every arbitrary: there will be a reason for everything, even if it is not explicitly spelled out in the notes. Ask why things are true, ask why particular problems are in the notes (because most of them have a particular goal, or a particular idea to illustrate that may not be immediately obvious).
- If you don't understand something (and by 'understand' I mean *really understand*, in that you can 'see' it in your mind and it has become almost obvious), ask questions and try to see things from another perspective.
- If you try a problem and you can't see a way through it immediately, come back later; usually, there is not an 'answer' that is to be calculated, but a process to be discovered or an argument to be developed — and it sometimes takes time for your mind to think unconsciously.
- Spend time on problems you think you can't do, because usually there is only a small flash of inspiration required; and anyway, the more difficult problems are far more interesting than the 'punch some buttons on the calculator' exercises, because they have a deeper meaning and are far more satisfying to complete. You might even surprise yourself.
- Always try to generalise: sure, it works for right-angled triangles: but what about *all* triangles?

The key is not calculations, it is comprehension. Calculations about triangles are not interesting, but the fact that such calculations are possible is not only interesting but almost counterintuitive; solving a quadratic equation is not interesting, but the idea of completing the square is interesting because it almost shouldn't work; and taking derivatives is not interesting, but the notion of describing geometry in a smooth and continuous way is, because it highlights the completeness of our number system. Look for the counterintuitive, not the boring routine; and try to see the beauty behind the arguments.

Good luck.



— `https://www.xkcd.com/1/`

Acknowledgements

To the several years worth of students onto whom these notes have been forced: thank you for your patience. I would also like to thank several of my undergraduate mathematics collegues for their constructive comments and complaints.

NCEA Level 2 Mathematics
Year 11 diagnostics

These questions are at the level required for NCEA Level 1 mathematics. At a bare minimum, students starting Y12 should be comfortable with those questions in part A. Students aiming for a merit or excellence in Level 2 should also revise the questions in part B.

Part A

A calculator isn't needed.

1. Write $\frac{3}{4} - \frac{1}{3}$ as a single fraction.
2. If $a = 5$ and $b = a - 3$, what is the value of $3a - b$?
3. What possible values of x make the equation $3x - 8 = -4x + 34$ true?
4. What possible values of y make the equation $x^2 - x = 20$ true?
5. Write $\dfrac{x+1}{x-1} + \dfrac{x+1}{x-2}$ as a single fraction.
6. Give the equation of the line through the two points $(0, 1)$ and $(2, 3)$.
7. The two shorter sides of a right-angled triangle measure $2\,\mathrm{m}$ and $1\,\mathrm{m}$. What are the measures of the three internal angles of the triangle?
8. Calculate the mean value of the following data: $\{2, 3, 6, 7, 9\}$.

Part B

A calculator might be useful here.

1. Because of inflation, the price of all homes for sale in a new subdivision are to be increased by 5%. If the new price for each lot is to be \$450,000, what is the current price of a lot?

2. A rectangular plot of land has area $28\,\mathrm{km}^2$. If one side length is three kilometres more than the other side length, what are the dimensions of the plot?

3. Consider the following party trick.

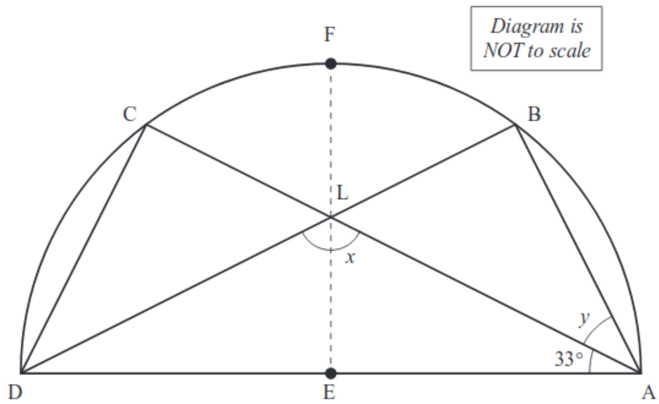
   Pick any number.
   Add 11.
   Divide by 2.
   Multiply by 6.
   Subtract 3 times your original number.

   The magician (perhaps too strong a word) hands you a sealed envelope containing your final number. Why are you not surprised to find she is correct?

4. A chemist is mixing one container of a 7% solution of a chemical and another container of a 4% solution to produce 5 litres of a 5% solution. Calculate the volume of each solution needed.

5. A flagpole is known to be 10 m high. At a certain time, the length of the shadow cast by the flagpole is 0.5 m. At what angle are light rays from the top of the flagpole hitting the ground?

6. The following construction is made from a semicircle and two triangles.

   The diameter of the semicircle is AD. The construction is symmetric about the line EF. Calculate the values of x and y.

7. Consider two lines. One passes through the points $(0,3)$ and $(2,5)$; the other passes through the points $(3,4)$ and $(5,6)$. Either find the point of intersection, or use precise mathematical reasoning to show that they do not intersect.

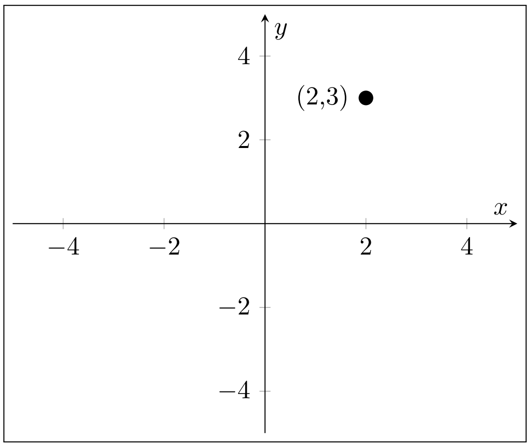
Chapter 2

Geometry

NCEA Level 2 Mathematics
1. Coordinate Geometry

The ancient Greeks were doing geometry at least as early as the 7th century BCE: Euclid's *Elements*, a mathematical book containing a collection of geometric propositions and proofs, is arguably the most influential mathematical work of all time and was regularly taught in schools until the 19th century. Of course, there have been some revolutions in geometry since the time of Euclid, and one of the most important was an idea due to René Descartes ("Renay Daycart") who first published the idea of using a coordinate system in 1637, in his book *La Géométrie*.

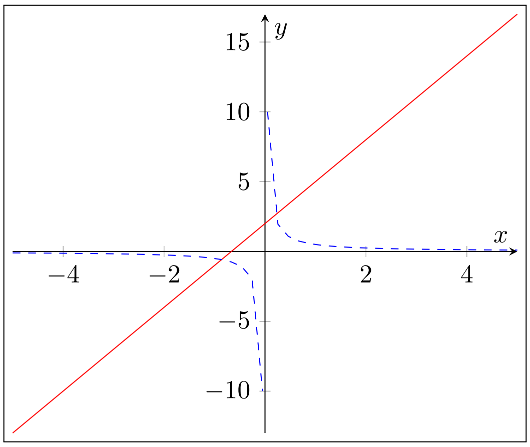
The idea is relatively simple: assign every point in the plane two numbers (coordinates) that describe its location with respect to the coordinate axes.



Traditionally, the two axes are the x- and y- axes; in three dimensions, we add a z-axis as well, at right angles to the first two.

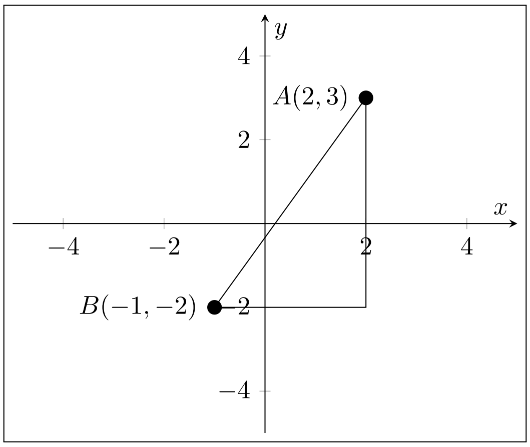
The reason this idea is so powerful is that now we can represent geometric objects algebraically, by writing down an equation so that the coordinates of every point of our object satisfies the equation.

For example, the equations $y = 3x + 2$ (red) and $y = \frac{1}{2x}$ (blue) are graphed here.



Points

One thing that we can do within a coordinate system is define the concept of "distance" between two points. We will do this by taking our two points and drawing a right-angled triangle, as follows:



Then the distance between the two points A and B is given by Pythagoras' theorem:-

$$d(A,B) = \sqrt{(3 - -2)^2 + (2 - -1)^2}.$$

More generally, the distance between two points $A(x_0, y_0)$ and $B(x_1, y_1)$ is given by

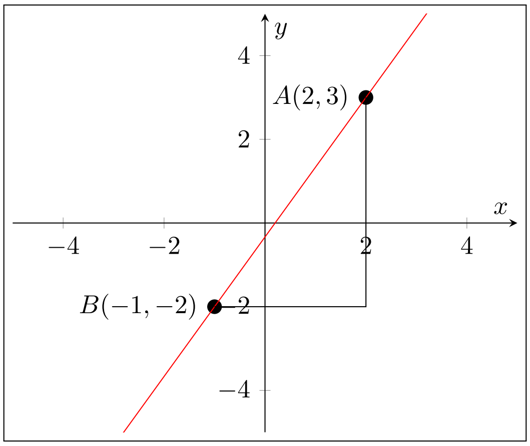
$$d(A,B) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

We can also find the midpoint of two points: the point precisely halfway between them. Clearly, if a point is halfway between two points on the plane then it is halfway between the two points on both coordinate axes; hence the point halfway between two points $A(x_0, y_0)$ and $B(x_1, y_1)$ is given by

$$m(A,B) = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right).$$

Lines

Turning to lines now, it is fairly obvious that every line (when we mention lines, we of course always mean *straight lines*) has a 'slope'. The slope (or 'gradient') of a line is a number which describes how 'steep' the line is. We define the 'slope' of a line to be the 'distance a point on the line travels upwards, as we move to the right by one unit length'. For example, consider the line in red here.



The slope of this line is $\frac{3--2}{2--1} = \frac{5}{3}$: every three steps to the right, we move five steps up — so if we move one step to the right, we move 5/3 of a step up.

We can use this idea to write down the equation of a line in general. Note that we have to take for granted one thing:

There is precisely one line through any two points.

We can't *prove* this within our conception of what it means to be a geometry, it is an axiom (something we assume without proof to be true). Note, however, that it is possible to define geometry in general in a different way so that this does become a theorem (something with a proof), but this requires other axioms beneath it that we cannot prove.

From our discussion above, we know that the slope of the line through the points (x_0, y_0) and (x_1, y_1) is given by

$$\frac{y_1 - y_0}{x_1 - x_0} \left(= \frac{\text{rise}}{\text{run}} \right).$$

We can now prove the following:

Theorem. *The unique line through the points* (x_0, y_0) *and* (x_1, y_1) *is given by the linear equation*

$$y - y_0 = \frac{y_1 - y_0}{x_0 - x_1}(x - x_0) = m(x - x_0),$$

where m *is the slope of the line.*

Proof. Let (x, y) be any point on the line. We need to find an expression for y in terms of x. However, note that the slope of the line between (x, y) and (x_0, y_0) is the same as the slope between (x_0, y_0) and (x_1, y_1) (because the line is straight). Therefore, we have

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_0 - x_1}$$

and rearranging this, we have

$$y - y_0 = \frac{y_1 - y_0}{x_0 - x_1}(x - x_0),$$

Q.E.D. □

For example, the line through $(2,1)$ and $(3,4)$ has slope $\frac{4-1}{3-2} = 3$, and equation

$$y - 1 = 3(x - 2).$$

On the other hand, we could swap around the two points and write the equation

$$y - 4 = 3(x - 3).$$

This is slightly annoying: both describe the same line, so why do we have two equations? Well, we can rearrange them:

$$\begin{aligned} y - 1 = 3(x-2) &\implies y = 3(x-2) + 1 \implies y = 3x - 6 + 1 \implies y = 3x - 5; \\ y - 4 = 3(x-3) &\implies y = 3(x-3) + 4 \implies y = 3x - 9 + 4 \implies y = 3x - 5. \end{aligned}$$

So the equations are the same if we carry out this process of putting them into the form

$$y = mx + c$$

where m is still the slope of the line, and c is the y-coordinate of the place where the line crosses the y-axis (since this point is the point on the line where $x = 0$ and so $y = m0 + c = c$).

Questions

1. Find the distance between the following pairs of points, drawing a diagram for each showing the geometric meaning of the thing you are calculating.
   (a) $(3,4)$ and $(6,8)$
   (b) $(3,0)$ and $(0,4)$
   (c) $(-1,1)$ and $(-4,-4)$
   (d) $(-4,-4)$ and $(-1,1)$
2. Justify the following statements mathematically, where A and B are points.
   (a) The distance from A to B is the same as the distance from B to A.
   (b) The distance from the midpoint of A and B ($m(A,B)$) to A is the same as the distance from $m(A,B)$ to B.
   (c) The equation of the line describing the y-axis is $x = 0$, and the equation of the line describing the x-axis is $y = 0$.
   (d) The slope of a horizontal line is zero.
   (e) A vertical line does not have a well-defined slope.
   (f) A line with a negative slope is sloping downwards.
3. Find the slope of the line through the following points.
   (a) $(5,7)$ and $(-2,1)$
   (b) $(-2,1)$ and $(5,7)$
   (c) $\left(\frac{2}{3},1\right)$ and $(4,2)$
   (d) $(2\pi,1)$ and $(0,\pi+1)$
4. Find the points where the line $y = -3x + 2$ crosses the x- and $y-$axes. These are called the x- and y-intercepts.
5. Find the point where the line $y = -x + 2$ crosses the line $y = x + 2$.

6. Justify intuitively the "triangle inequality", if A, B, and C are any three points:

$$d(A, B) \leq d(A, C) + d(C, B). \qquad \text{(Triangle inequality)}$$

7. Two lines are *perpendicular* if the angle between them is a right angle. Show that if the gradient of a line is m then the gradient of any perpendicular line is $-1/m$.

8. Two lines are *parallel* if they do not intersect. Show that two lines are parallel if and only if they have the same gradient.

9. Let AB be some line segment. The perpendicular bisector of AB is the (unique) line perpendicular to AB and passing through $m(A, B)$.
   (a) Find the equation of the perpendicular bisector of the segment joining $(3, 2)$ and $(2, 5)$.
   (b) In general, if $A = (A_x, A_y)$ and $B = (B_x, B_y)$, find the equation of the perpendicular bisector of AB.

10. Let $ABCD$ be a quadrilateral, such that $d(A, B) = d(B, C)$ and $d(C, D) = d(D, A)$. Show that the line AC is the perpendicular bisector of the segment BC.

11. In three dimensions, we can define the distance between two points (x_0, y_0, z_0) and (x_1, y_1, z_1) using the 3D version of Pythagoras' theorem:

$$d(A, B) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

   (We will prove this version of Pythagoras' theorem in three dimensions in a few weeks.)
   (a) Show that the distance between two points on the x, y-plane (the plane $z = 0$) reduces to the 2D version we gave above.
   (b) Write down a plausible definition for the midpoint of A and B in three dimensions.
   (c) Using your definition, find the midpoint of $A(2, -2, 5)$ and $B(6, 8, 9)$; is the distance between the midpoint and A half the distance between A and B?

12. Which of the following triangles ABC are isosceles or right-angled (or both)?
   (a) $A = (1, 3) \quad B = (4, -2) \quad C = (1, -2)$
   (b) $A = (1, -5) \quad B = (2, -1) \quad C = (9, -6)$
   (c) $A = (4, 0) \quad B = (8, 2) \quad C = (8, -2)$

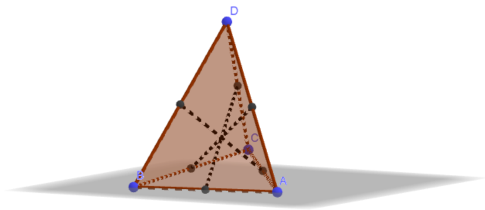
13. Consider the three points $(2, 4)$, $(-1, 8)$, and $(6, -3)$. Compute the equations for the perpendicular bisectors of each side, and show that all three bisectors intersect at a single point. Does this happen for every triangle?

14. Let $ABCD$ be a quadrilateral. Show that the quadrilateral $WXYZ$, where

$$\begin{aligned} W &= m(A, B) \\ X &= m(B, C) \\ Y &= m(C, D) \\ Z &= m(D, A), \end{aligned}$$

   is a parallelogram.

15. Challenge problem 1. Let $ABCD$ be a parallelogram. Draw a square on each side of the parallelogram, and choose the centre of each of the squares.
    (a) (Napoleon's theorem) Show that these centres form the vertices of a larger square.
    (b) Show that the diagonals of this square intersect at the same point as the diagonals of $ABCD$.
16. Challenge problem 2. Consider a tetrahedron (see the figure below and `tetrahedron.ggb`). For each pair of opposite sides, draw a line segment connecting their midpoints.



    (a) Show that all three such line segments intersect at a single point.
    (b) In what ratio does the point of intersection divide each segment?

NCEA Level 2 Mathematics (Homework)
1. Coordinate Geometry

Reading

Go and watch...

`https://www.youtube.com/watch?v=X1E7I7_r3Cw`

What's it good for?

People use coordinate geometry for...

- Computer graphics: coordinate geometry is a compact, efficient, and simple way to store information about and manipulate complicated graphical data. Vector graphics programs, like Inkscape and Adobe Illustrator, store all their graphics as coordinate geometric objects as they take up less storage space, are easier to edit, and easier to scale and otherwise modify than storing objects pixel-by-pixel.
- Coordinate systems are used by physicists, geographers, and other people who need to describe the positions of objects in space relative to each other.

Questions

[This is a sample Ministry of Education L2 assessment task for this standard.]

The *reflection* of a point A in a line L is the unique point A' such that (1) the line AA' is perpendicular to L, and (2) if I is the point of intersection between AA' and L, then $I = m(A, A')$. (In other words, A' is the unique point such that L is the perpendicular bisector of AA'.)

Develop a general method for finding the co-ordinates of the reflection of any point in the mirror line joining the points $(0, 9/4)$ and $(9/2, 0)$ by completing the following steps:

1. Reflect the point $(4, 1)$ in the mirror line joining the points $(0, 9/4)$ and $(9/2, 0)$ and find the co-ordinates of the reflection;

2. If (a, b) is any point not on the mirror line, reflect the point (a, b) in the mirror line and find the co-ordinates of the reflection.

The quality of your discussion and reasoning will determine the overall grade. Show your calculations. Use appropriate mathematical statements. Clearly communicate your strategy and method at each stage of the solution.

NCEA Level 2 Mathematics
2. Arcs and Sectors of Circles

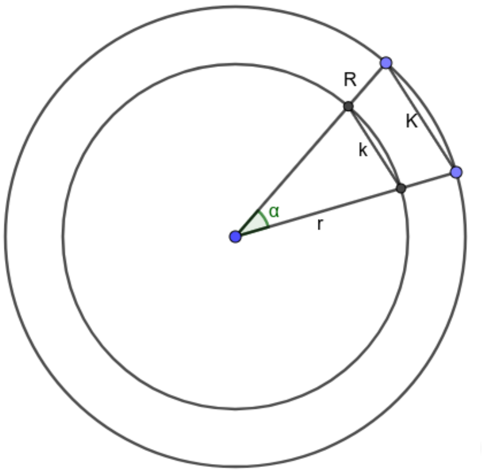
The Greeks studied two fundamental geometric objects: lines, and circles. Last week we looked at lines; this time, we will look at circles.

A circle is simply the set of all points that are at a distance r from a point (x_0, y_0): the point (x_0, y_0) is called the centre of the circle, and the distance r is the radius of the circle. If (x, y) is a point on the circle, then we have

$$d((x, y), (x_0, y_0)) = r \implies \sqrt{(x - x_0)^2 - (y - y_0)^2} = r$$

and the equation of the circle in cartesian coordinates is $(x - x_0)^2 + (y - y_0)^2 = r^2$.

Circumferii



Suppose the angle α is measured in degrees. Then the circumference of the inner circle is $c \approx \frac{360°}{\alpha}k$, and the circumference of the outer circle is $C \approx \frac{360°}{\alpha}K$. Now, the two triangles formed are similar since they have an identical angle and two sides with the same ratio of r/R; hence $k/K = r/R$. We can rewrite:

$$\frac{k}{K} = \frac{r}{R} \implies \frac{c\frac{\alpha}{360°}}{C\frac{\alpha}{360°}} \approx \frac{r}{R} \implies \frac{c}{C} \approx \frac{r}{R}.$$

As the size of the angle α becomes smaller and smaller, this approximation becomes exact: $\frac{c}{C} = \frac{r}{R}$, and so $\frac{c}{r} = \frac{C}{R}$. In other words, the ratio of the circumference of any circle to its radius is always the same. For historical reasons, we actually write this in terms of the diameter, and call the constant of proportionality π. We have therefore sketched a proof that

$$c = 2\pi r$$

for any circle with radius r and circumference c.

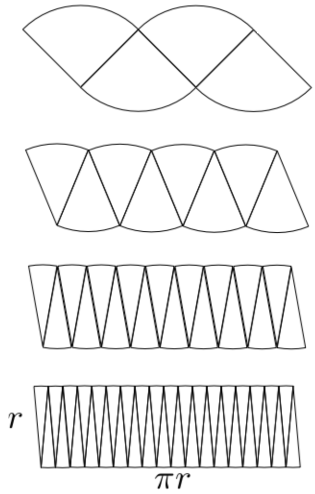
The number π is approximately equal to

$$3.14159265358979323846264 33...$$

and in one of the exercises below you will calculate a first approximation to this value: it isn't just a number that is plucked out of thin air!

Areas

The other main result we have for circles is the area; by slicing the circle into smaller and smaller pieces, we can approximate the area of a circle with the area of a rectangle:



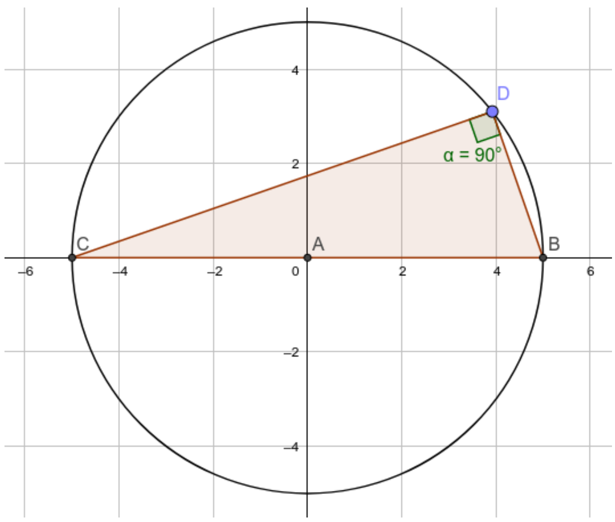
By means of this, we can intuitively justify that the area of a circle with radius r is

$$A = \pi r^2.$$

This idea of a limiting process will be made more clear next year (when you will be able to provide proper proofs of these facts), but hopefully these two results seem plausible.

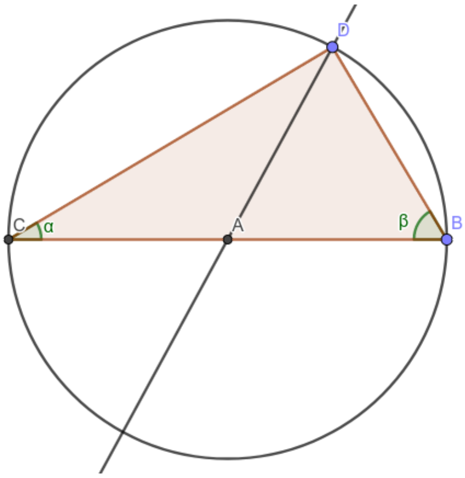
Thale's Theorem

As a taster for some of the other pretty theorems one can prove about circles, consider any circle; pick a diameter of the circle and any point on the circle itself; then the resulting triangle is always right-angled, as in the following diagram.



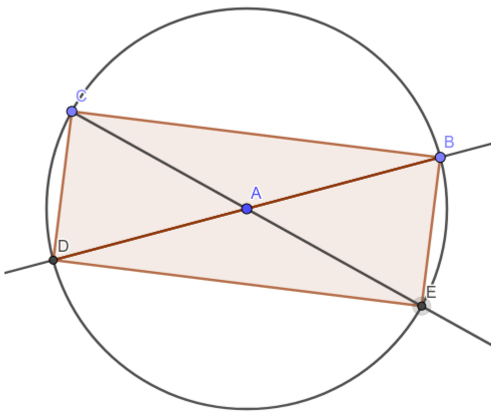
I actually know *two* proofs of this; we'll look at both!

Angle-y proof. The first proof is quite cute: we simply consider the next figure and do some angle pushing.



Clearly ACD and ABD are both isosceles, so $CDA = \alpha$ and $ADB = \beta$; hence $CDB = \alpha + \beta$, and so (using the fact that the internal angles of a triangle add to 180°) we have $180° = \alpha + \beta + \alpha + \beta$; so $CDB = \alpha + \beta = 180°/2 = 90°$. □

Rotate-y proof. The intuitive idea behind the second proof is that we take the triangle, 'rotate it around', and see that the resulting shape is a rectangle. In order to make this idea more precise, consider the following diagram.



Here, we took our initial triangle to be BCD, sitting on the diameter BD. Draw the line through C and A; clearly (since it passes through the centre) it is a diameter of the circle. Hence the two diagonals of the quadrilateral $BCDE$ are the same length, and it is therefore a rectangle (so in particular DCB is a right angle). □

Which proof do you prefer? Why?

Questions

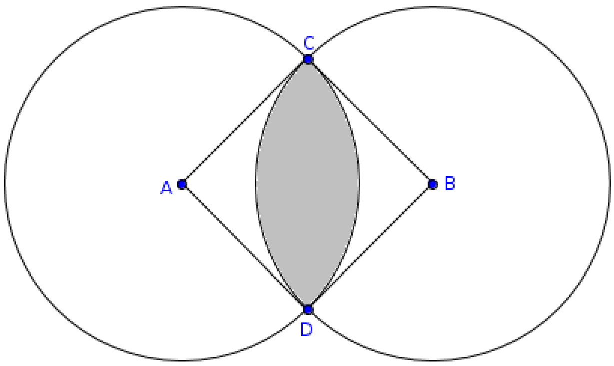
1. Suppose we take a circle of radius r, and cut out a slice with an angle α (like cutting a pizza). Draw a picture. What is the area of this slice, and what is the length of the arc of the circumference that is part of the slice? (Such a slice is called a *sector*.)

2. Notice that the formulae derived above all have ugly factors of 360°. Define one radian to be the angle such that the arc length of the sector defined by that angle is just r, the radius of the circle. Radians are, in many ways, a much more natural angle measurement unit.
   (a) Draw a picture to show this geometrically.
   (b) Show that one radian is precisely $\frac{\pi}{180}$ degrees.
   (c) How many radians are in a full circle?
   (d) Show that, in radians, the formulae derived in question 1 above simplify dramatically.

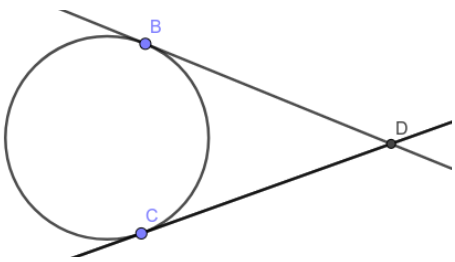
3. Suppose a circle has area 49π. What is the arc length of a sector of this circle with area 25π?

4. Show that if the angle subtended by a chord at the centre is 90° then $\ell = \sqrt{2}r$, where ℓ is the length of the chord.

5. In the following diagram, the two circles are centred at A and B and intersect at C and D. The quadrilateral $ABCD$ is a square. Consider the shaded area as a subset of one of the circles. What proportion of the circle's area is shaded?

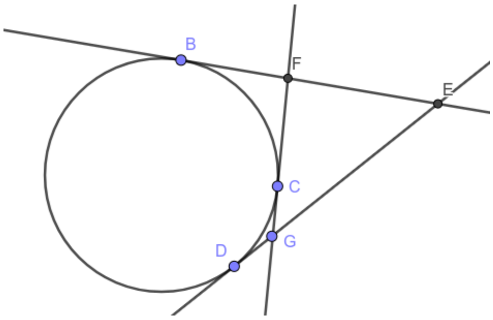


6. Suppose that two lines tangent to a circle at points B and C intersect at a point D, as shown. Show that the two segments BD and CD have equal lengths.



7. A simple definition of a circle is the set of all points $P = (x, y)$ such that $d(A, P) = r$ (where A is the centre of the circle and r is the radius of the circle). Use this definition to write an equation for the circle of radius r centred at (x_0, y_0).

8. In the following figure, all three lines are tangent to the circle. If the length of the segment BE is 5, what is the perimeter of the triangle FGE? [Hint: use result 6 above.]



9. (a) Find the area of the largest square that one can fit inside a circle of radius r.
   (b) Find the area of the smallest square that fits outside a circle of radius r.
   (c) Hence show that $2 < \pi < 4$.
   (d) How might you improve your estimate of π?

NCEA Level 2 Mathematics (Homework)
2. Arcs and Sectors of Circles

Reading

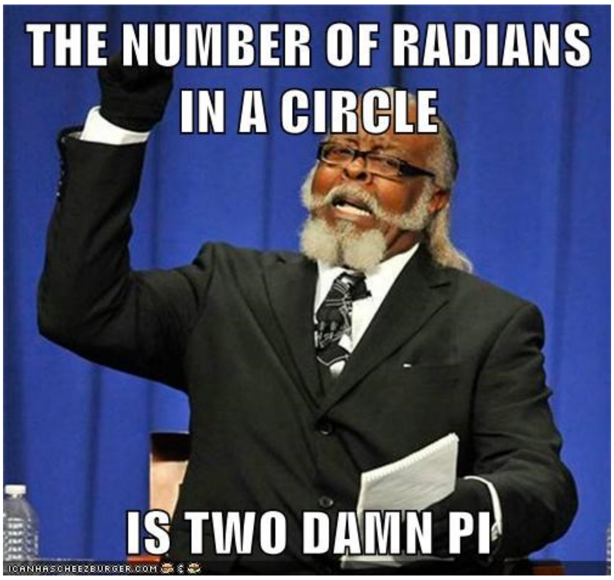
Go and watch...

`https://www.youtube.com/watch?v=QncgmzH6yQU`

What's it good for?

People use the geometry of circles for...

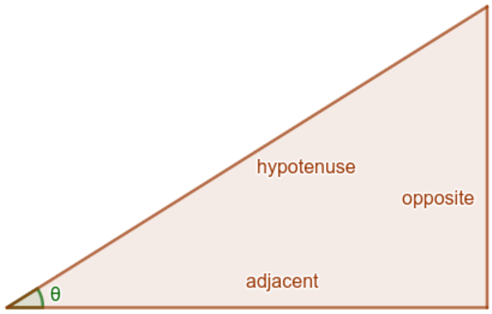
- Physics: physicists and engineers often want to model things that rotate, and proper definitions of rotational speed and acceleration require the use of the geometry of arc lengths and sector areas.
- Mathematics: the idea of a 'limiting process', where we take sums of things that we let become infinitesimally small, is a fundamental idea that underpins entire branches of mathematics and allows us to formally define the concepts of area and volume, and enables us to better understand things which are continuous.



Questions

1. A car tire has diameter 53 cm. A detector measures that a particular point on the tire tread rotates past 1061 times per second.
   (a) What speed is the car travelling at?
   (b) Is this setup practical and/or useful? Explain.
2. What is the radius of a circle such that the sector of area $\frac{\pi}{3}$ has arc length $\frac{\pi}{3}$?

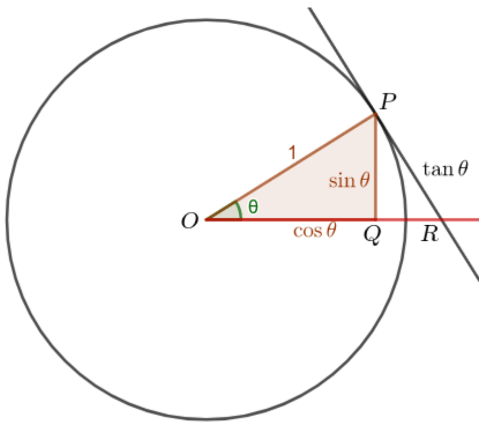
NCEA Level 2 Mathematics
3. Trigonometry



We are now going to look at triangles inside circles. Now, last year we learned that any triangles with two equal angles are similar; in particular, if we take ratios of sides, we obtain the same value. This means that if we have any right-angled triangle with angle θ like the one above, then the ratios $\frac{\text{opposite}}{\text{hypotenuse}}$, $\frac{\text{adjacent}}{\text{hypotenuse}}$, and $\frac{\text{opposite}}{\text{adjacent}}$ all depend only on the angle θ; we call them the sine, cosine, and tangent of the angle respectively:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}.$$

In particular, if we draw our triangle inside a unit circle then we can draw the following:



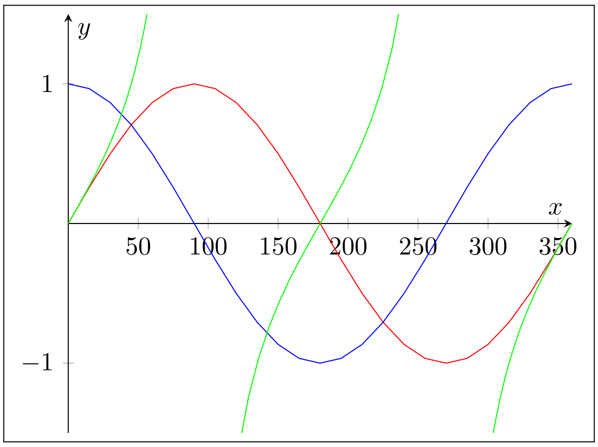
In fact, we can take this as our definition of sin and cos. To show that $\tan\theta$ is indeed the line segment marked, first notice that since the triangle OPR is right-angled, the angle at the intersection of the horizontal line and the tangent line is $90° - \theta$; so the other non-right-angle in the triangle PQR is θ. Hence the hypotenuse of PQR is $\frac{\text{adjacent}}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$, as proposed.

Note also that, from this diagram, we have

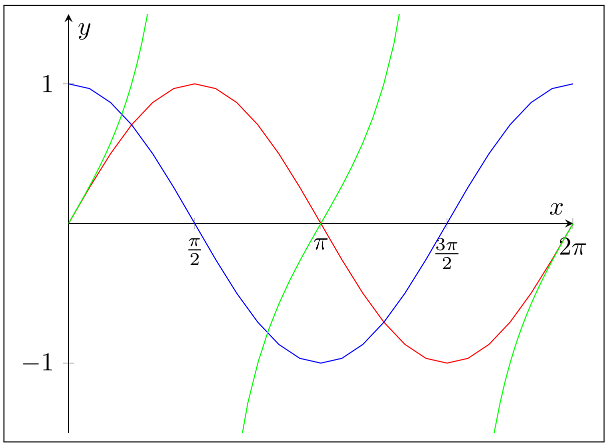
$$\sin^2\theta + \cos^2\theta = 1$$

for every angle θ.

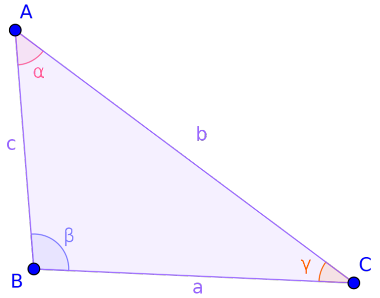
Since the sin of an angle is just the height of the point above the x-axis in the diagram above, we have that $-1 \leq \sin\theta \leq 1$; similarly, $-1 \leq \cos\theta \leq 1$. Note that when $\theta = 90°$, the tangent line becomes horizontal and so never intersects the x-axis: so $\tan 90°$ is undefined. We can even graph $\sin\theta$ (red), $\cos\theta$ (blue), and $\tan\theta$ (green):



If we graph them in radians, only the labels on the x-axis change:



Let us now begin to look at more general triangles:



Theorem (Sine rule)**.** *In any triangle, with the angles and sides labelled as above, we have*

$$\frac{\alpha}{\sin a} = \frac{\beta}{\sin b} = \frac{\gamma}{\sin c}.$$

Proof. Drop an altitude from B to AC, creating two new right-angled triangles. Then the length of this line can be calculated using both of the resulting right-angled triangles: so $c \sin \alpha = a \sin \gamma$ and $\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$. This proves the theorem. □

Theorem (Cosine rule)**.** *In any triangle, with the angles and sides labelled as above, we have*

$$a^2 = b^2 + c^2 - bc \cos \alpha.$$

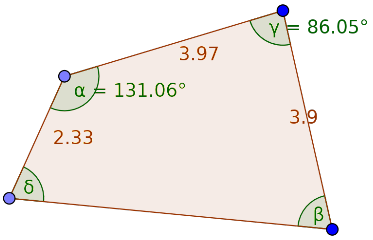
Proof. Drop an altitude from B to AC, creating two new right-angled triangles. Then the length b can be split into two lengths, $c\cos\alpha$ and $b - c\cos\alpha$; the length of the altitude is $c\sin\alpha$. Now, apply the Pythagorean theorem to the triangle including the angle γ:

$$a^2 = (b - c\cos\alpha)^2 + c^2\sin^2\alpha = b^2 - 2bc\cos\alpha + c^2\cos^2\alpha + c^2\sin^2\alpha = b^2 + c^2 - 2bc\cos\alpha.$$

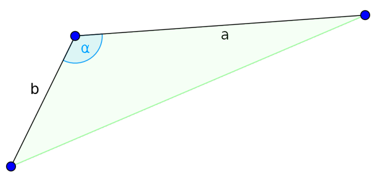
□

Questions

1. A beam of gamma rays is to be used to treat a tumor known to be 5.7 cm beneath the patient's skin. To avoid damaging a vital organ, the radiologist moves the source over by 8.3 cm.
   (a) At what angle to the patient's skin must the radiologist aim the gamma ray source to hit the tumor?
   (b) How far will the beam have to travel through the patient's body before reaching the tumor?
2. A surveyor measures the three sides of a triangular field and finds that they are 117, 165, and 257 metres long.
   (a) What is the measure of the largest angle of the field?
   (b) What is the area of the field?
3. A field has the shape of a quadrilateral (four-sided shape) that is *not* a rectangle. Three sides measure 2.33, 3.97, and 3.9 kilometres, and two angles measure $\alpha = 131.06°$ and $\gamma = 86.05°$ (as in the figure).



   (a) By dividing the quadrilateral into two triangles, find its area.
   (b) Find the length of the fourth side.
   (c) Find the measures of the other two angles, β and δ.
4. A surveyor is standing on top of a peak. She can see two prominent peaks ahead of her, and from previous measurements she knows that one of them is 8 km away from her and the other is 11 km away. She measures the angle between them to be 120°. How far apart are the two peaks (measured *along the ground*):
   (a) If that they have same height?
   (b) If the surveyor and the closer peak are at the same height, but the peak which is further away is 200 m higher?
5. Consider a triangle with sides of 5, 7, and 10 kilometres.
   (a) Find the measure of the largest angle of this triangle.
   (b) Find the area of the triangle.
6. Find the area of the triangle below.



7. For each item below, decide whether or not such a triangle exists. If at least one does, how many exist?
   (a) Exactly one angle greater than $90°$.
   (b) Two angles greater than $\pi/2$.
   (c) Two sides of length 200,000.
   (d) Three sides of length 200,000.
   (e) Sides of length 90, 30, and 30.
8. Prove that, if a quadrilateral has equal diagonals, then it is a rectangle. (We used this fact last week!)
9. Let A and B be points in three dimensional space. Show that

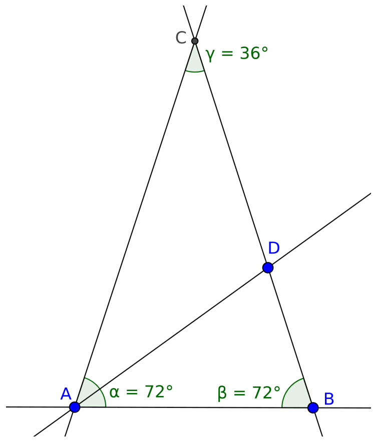
$$d(A, B) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

10. Let ABC be a triangle.*
    (a) Let $X = m(A, B)$, $Y = m(B, C)$, and $Z = (C, A)$ be the midpoints of the sides; then the lines CX, AY, and BZ are called the *medians* of the triangle. Show that the three medians always intersect at a single point N (the *centroid*).
    (b) Let k be the perpendicular bisector of AB, ℓ be the perpendicular bisector of BC, and m be the perpendicular bisector of CA. Show that k, ℓ, and m intersect at a single point O (the *circumcentre*). Show that O is the centre of the circle passing through A, B, and C.
    (c) Let λ be the line passing through A that bisects the angle of the triangle at A. Define μ and ν similarly as the angle bisectors at B and C. Show that λ, μ, and ν intersect at a single point P (the *incentre*). Show that P is the centre of the circle which is tangent to the three sides of the triangle.
    (d) Let ρ be the line through A perpendicular to BC; define σ and τ similarly to be lines through B and C. These lines are known as the *altitudes* of the triangle. Show that ρ, σ, and τ intersect at a single point (the *orthocentre*).
    (e) Show that for any triangle, the circumcentre, centroid, and orthocentre all lie on a single line (known as the *Euler line* of the triangle).
    (f) Show that for an isoceles triangle, the Euler line is the line of symmetry.
    (g) Show that the incentre lies on the Euler line exactly when the triangle is isoceles.
    (h) Show that for an equilateral triangle, the orthocentre, centroid, incentre, and circumcentre coincide. (We may simply call this point the centre.)
11. This question requires you to find exact values for trig functions *without* using a calculator. [Schol 1999]
    (a) Sketch an equilateral triangle of side 2 units. Hence, or otherwise, show that $\cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, and $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.
    (b) Sketch a right angled isosceles triangle whose equal sides enclose the right angle and are of length 1. Hence, or otherwise, show that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

*This question may be done fairly easily with coordinate geometry. However, infinitely more elegant solutions may be found by using the machinery of Euclidean geometry. See, for example, sections 1.4 to 1.6 of Coxeter's *Intro. to Geometry.*

12. Consider a regular tetrahedron (a four-sided shape such that each side is an equilateral triangle). For each vertex, draw the line joining it to the centre of the opposite side. By a problem in the section on coordinate geometry, the four lines must meet at a single point O. Show, by an exact calculation, that the angle at the point O between any line and any other line is around $109.5°$. (This is the bond angle between adjacent hydrogens in methane, CH_4.)

13. Consider the 75-75-36 triangle ABC given in the figure. The angle α has been bisected into two angles, and the resulting line meets the triangle at D.



(a) Show that ABC and ABD are similar triangles.

(b) Hence, or otherwise, show that $\frac{AB}{BD} = \frac{AB+BD}{AB}$.

(c) Show that the ratio of the long side of the triangle to the short side of the triangle is $\frac{AB}{BD} = \frac{1+\sqrt{5}}{2} = \phi$.

(d) Show that $\cos 72° = \frac{1}{2\phi}$.

(e) Find $\sin 36°$ and $\sin 72°$.

NCEA Level 2 Mathematics (Homework)
3. Trigonometry

Reading

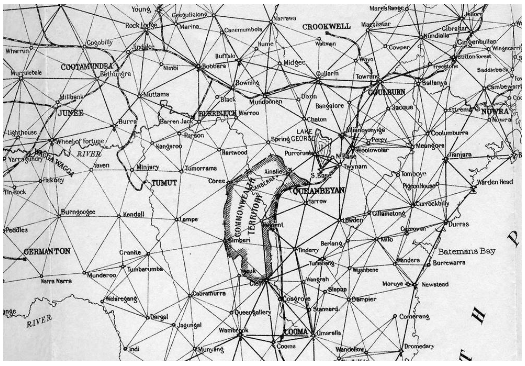
Go and watch...

`https://www.youtube.com/watch?v=o6KlpIWhbcw`

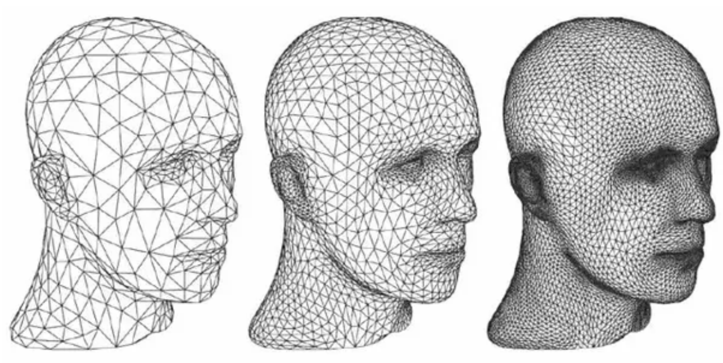
What's it good for?

People use the geometry of triangles for...

- Physics: physicists and engineers use trigonometry to calculate how forces and stresses are transmitted in systems that involve angles.
- Geography: triangulation is used to measure the areas and topography of large areas of land.

- Computer graphics: complicated 3D shapes are usually modelled with a large number of triangles or other polygons, to simplify and speed up shading and other algorithms.



Questions

[This is a sample Ministry of Education L2 assessment task for this standard.]

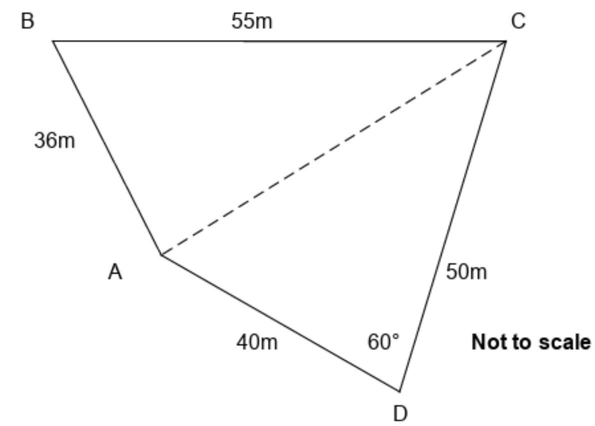
Your school is selling some unused land to raise money for a new gymnasium. The buyer will only purchase the land if the school can demonstrate that the land can be subdivided into four sections of at least $400\,\mathrm{m}^2$ each. The buyer will not purchase the land if all of the sections are triangular. This assessment activity requires you to determine the dimensions of appropriate subdivisions so that the sale can proceed.

Given the land diagram below,

1. Calculate the length of the pipeline running through the land.
2. Demonstrate that the land can be divided into four sections each of more than $400\,\mathrm{m}^2$ such that not all are triangular.
3. Show one possible way of dividing the land into four sections each of more than $400\,\mathrm{m}^2$ such that not all are triangular.

The quality of your reasoning, using a range of methods, and how well you link this context to your solutions will determine the overall grade. Clearly communicate your method using appropriate mathematical statements so that the new owner can easily verify the dimensions of the sections.

Resource 1: Land Diagram



The unused land bounded by ABCD needs to be split into four sections.
The pipeline running through the land is between points A and C.

Chapter 3

Algebra

NCEA Level 2 Mathematics
4. Functions

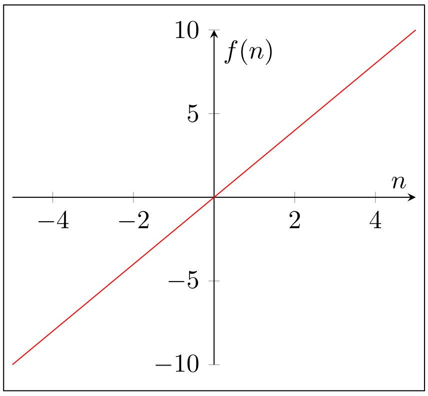
One of the most fundamental concepts in mathematics is that of a function. A function is a relationship between two sets of things, called the *range* and the *domain*, such that everything in the range is related to exactly one thing in the domain. You can think of a function as a rule: it could be given by a formula, or by a list of inputs and outputs, or in any other way that one likes.

If f is a function, and it relates x to y, then we write $f(x) = y$. In this notation, x is the *argument* or *input* and y is the *result* or *output*.

The above expression $f(x) = y$ is suggestive: we can graph functions by graphing every pair of numbers (x, y) which satisfies this equation.

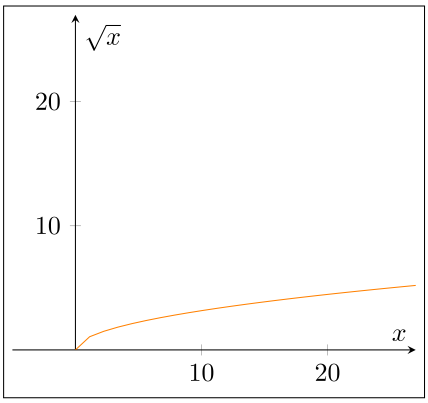
Example.

1. Suppose for every number n we associate its double, $2n$. This is a perfectly good function, which we can call f: then $f(n) = 2n$, and $f(2) = 2 \cdot 2 = 4$.

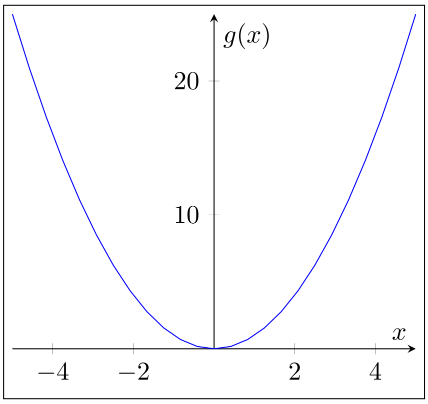


2. Suppose for every number x we associate the number which, when squared, gives x. This is *not* a well-defined function: for example, what number do we associate with 4: 2, or -2? What number do we associate with -1?

3. On the other hand, suppose for every positive number x we associate its positive square root. This time we do have a well-defined function. Note that its domain and range are both the positive numbers, and for every number in the domain there is precisely one number in the range.



4. Let $g(x) = x^2$. This is also a perfectly good function; every number has exactly one square. Note that $g(-2) = g(2) = 4$; this is allowed, but if a function f has the property that if x and y are different then $f(x) \neq f(y)$ then f is called one-to-one. The functions in (1) and (3) above are both one-to-one.



Notice that this is a square root on its side... can you explain why?

One-to-one functions are useful, because they have an *inverse.* That is, given any number in the *range* we can find the number in the *domain* that maps to it. If f is a one-to-one function, then its inverse is written as f^{-1}.

Example.

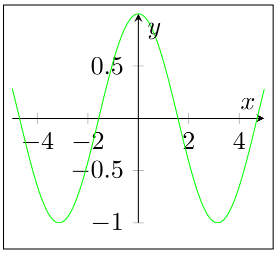
1. If $f(x) = 2x$, then $f^{-1}(x) = \frac{1}{2}x$.
2. If $g(x) = x^2$, then g does not have an inverse over all the numbers; but if we restrict the domain of g to the positive numbers, then it does have an inverse $g^{-1}(x) = +\sqrt{x}$.
3. If $h(x) = \sin x$ (and the range of x is restricted to $0 < x \leq 2\pi$) then h is a perfectly good function with range $-1 \leq h(x) \leq 1$; its inverse is $\sin^{-1}$, which takes a triangle ratio and returns the appropriate angle.

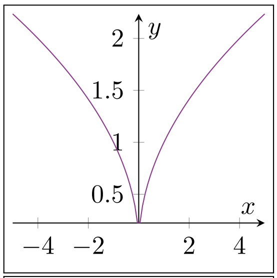
Questions

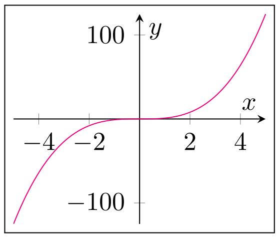
1. Justify the following statements with mathematical reasoning.
   (a) The graph of a function passes the vertical line test: any vertical line drawn on the graph crosses the function in at most one point.
   (b) The graph of a one-to-one function passes the horizontal line test: any horizontal line drawn on the graph crosses the function in at most one point.
   (c) The graph of the inverse of a function is the reflection of the graph of the function across the line $x = y$.
   (d) If we define f by
   $$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$
   then f is a function but it is not one-to-one.
   (e) If f and g are functions, and the range of g is the same as the domain of f, and we define a new function $(f \circ g)$ by $(f \circ g)(x) = f(g(x))$, then $(f \circ g)$ is a function with the same range as f and the same domain as g.

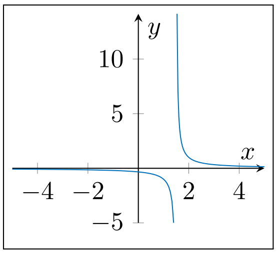
2. The following define functions only if they have a domain which is restricted. What is the largest possible domain for each, so that they are still functions? What is the range of each?
   (a) $f(x) = 1/x$
   (b) $g(\theta) = \tan\theta$
   (c) $h(x) = \left(\dfrac{x}{x-2}\right)^2$

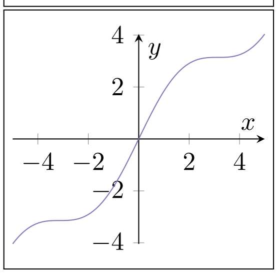
3. Which of the following graphs are graphs of functions? Of those, which are one-to-one?

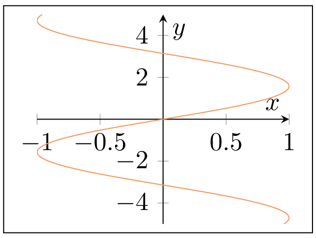
(a) 

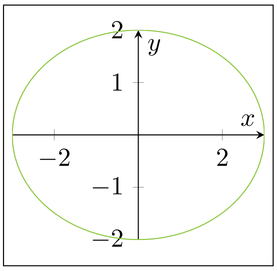
(b) 

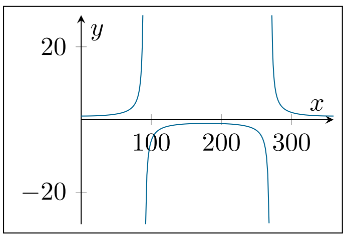
(c) 

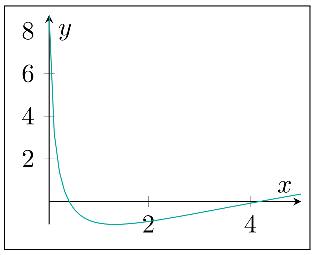
(d) 

(e) 

(f) 

(g) 

(h) 

(i) 

4. We will now look at shifting graphs around.
   (a) Graph f, if $f(x) = x^2$.
   (b) On the same paper, graph $f(x-1)$ and $f(x+1)$.
   (c) Graph $f(x) - 1$ and $f(x) + 1$.
   (d) Graph $2f(x)$ and $f(2x)$.
   (e) Make a conjecture about the relationship between the following graphs, if f is now any function and a, b, c, and d are any numbers:
      i. $y = f(x)$
      ii. $y = f(x+a)$
      iii. $y = f(x) + b$
      iv. $y = cf(x)$
      v. $y = f(dx)$
   (f) Explain intuitively why, when we add numbers inside the function argument, the graph shifts 'the wrong way'.
5. Graph the following functions.
   (a) $y = 2^{x+1} - 3$
   (b) $y = (x-3)^2 + 2$
   (c) $y = \cos(x - \pi/2) + 1$
   (d) $y = (x-1)(x-2)(x-3)$
   (e) $y = 3\sin(\frac{1}{2}x + 1) - 2$
6. Is it possible to find K such that the function

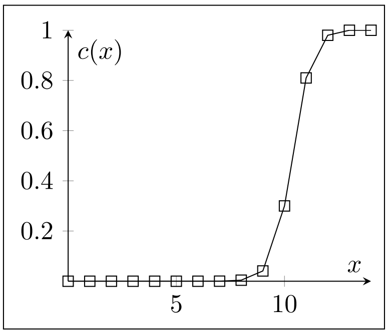
$$f(x) = \begin{cases} x^3 - Kx + K & x \leq 0 \\ \sin x & x \geq 0 \end{cases}$$

   is well-defined at 0?
7. Is it possible to find K such that the function

$$f(x) = \begin{cases} x^3 - Kx + 3 & x \leq 0 \\ K\cos x & x \geq 0 \end{cases}$$

   is well-defined at 0?
8. What is the equation of the straight line through $(-2, -1)$ that meets $y = (x-3)(x-2)$ at its y–intercept?
9. If I take $y = x^2$ and I want to shift and stretch (equally in each direction) its graph so that the vertex is sitting at $(-2, 3)$ and it goes through $(1, 1)$, can I do that? If so, what is the equation of the function that gives me the new graph?
10. If I take $y = 2^x$ and I want to shift and stretch (equally in each direction) its graph so that it passes through both $(-2, 3)$ and $(1, 1)$, can I do that? If so, what is the equation of the function that gives me the new graph?
11. Find A such that the graph of $f(x) = A \cdot 3^{x-1} + 2$ passes through the given point, or explain why no such A can be found. (a) $(0, 4)$; (b) $(0, 2)$; (c) $(0, -2)$; (d) $(0, -4)$.
12. Let $y = 3x^2 + bx + 9$. Find b such that the graph of the equation intersects $y = 2x^2$ at exactly one point.

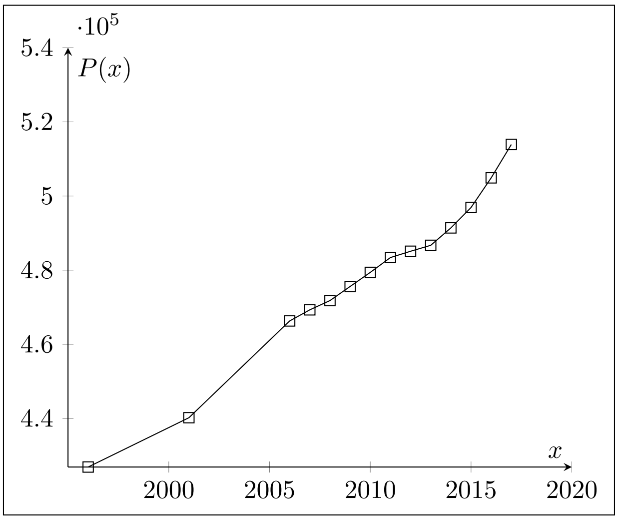
13. The following graph is of the function c, where $c(x)$ is the proportion of a particular chemical species in solution at pH x (pH is, roughly speaking, a measure of acidity). *



(a) Is this function invertible? That is, if I measure the concentration of this species at any point, can I identify precisely what the pH of the solution is?

(b) Explain why knowing the proportion of a chemical species in water at a given pH might be useful; why is mathematical modelling useful in such a situation?

14. The following graph is of the function P, where $P(x)$ is the population of the Greater Wellington region in the year x.† Note that the y-axis does *not* start at zero, and is given in units of hundreds of thousands.



(a) Explain why extrapolating using this graph to find $P(2020)$ is probably justified.

(b) Give an example of a function where the values of the function above any point cannot be extrapolated from knowing the values of the function before the point. Do you think that extrapolation is normally possible for mathematical models of the natural world?

(c) Describe how the rate of population growth changes as we move forwards in time from 1996 to 2017.

(d) Given the real-world context of this graph, how would you expect the slope of the graph to change:

i. in the medium term (up to, say, 2040), and

ii. in the long term (say around the year 2100).

*From *Quantitative Chemical Analysis*, 7th edition, by Daniel Harris (page 233).

†Statistics are the Statistics NZ 'Subnational population estimates (RC, AU), by age and sex, at 30 June 1996, 2001, 2006-17 (2017 boundaries)' (retrieved 19 June 2018).

NCEA Level 2 Mathematics (Homework)
4. Functions

Reading

Go and watch...

```
https://www.youtube.com/watch?v=PtLnwvH4kuE
```

What's it good for?

People use functions and mathematical modelling for...

- Statistics, engineering, and the sciences: extrapolation from a set of data and prediction of results from future experiments and other situations is a key part of the scientific method.
- Theoretical physics: in particular, the mathematics behind quantum mechanics is heavily dependent on linear transformations, a particular class of function.
- Mathematics: as I mentioned in the notes, functions are a key idea in mathematics because they allow us to describe things like curves in space, rates of change, and relationships between objects. Graphs and diagrams of functions are often another way to view a concept and let us gain more clarity. Normally, we don't talk about functions in general, but restrict ourselves to subclasses of functions with nice properties (preserving distance, or more generally 'closeness'; preserving algebraic properties like addition or multiplication; etc.).

Questions

[This is from a sample Ministry of Education L2 assessment task for this standard.]

Place cones at the following co-ordinates, in metres, with the positive y axis pointing due north:

Cone	Location	Cone	Location	Cone	Location
A	$(-14, 1)$	D	$(7, 6)$	G	$(7, 0)$
B	$(-16, 6)$	E	$(7, 10)$	H	$(17, -3)$
C	$(3, 6)$	F	$(10, 10)$	I	$(7, 1)$

Give equations for each of the following curves:

1. Start from a point one metre to the north of cone A. Ride in a straight line to a point two metres to the north of cone B.
2. Starting from the end of line 1, weave around cones B, C, and D, such that the maximum distance south of cone C is the same as the maximum distance north of cones B and D, passing through the point $(5, 6)$ following a curve of the form $f(x) = A\sin(x - 5) + C$.

As revision from L1, expand and simplify the following:

1. $(x - 2)^2(x + 8)$
2. $(5x - 4)(x + 2)(x + 1)$

NCEA Level 2 Mathematics
5. Quadratic Modelling

This topic is primarily revision from Level 1.

A linear function is one of the form

$$f(x) = mx + c;$$

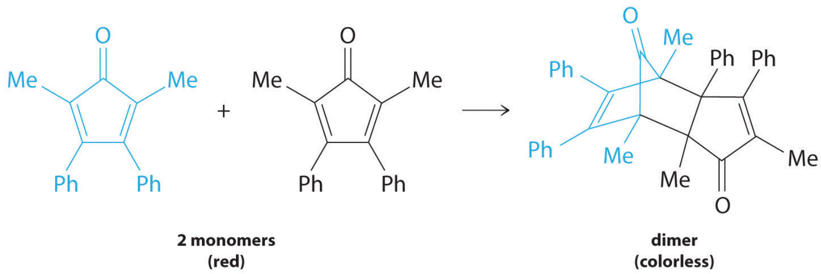
we have already seen that the graph of such a function is a straight line.

The natural next step is to consider quadratic functions: those of the form

$$f(x) = ax^2 + bx + c.$$

Last year, we saw that the graphs of quadratic functions are parabolae; so we can use quadratic equations to model situations which are vaguely parabolic.

Example.



The following table gives the instantaneous rates of reaction for the dimerization reaction above.

Time (min)	**Concentration of reactant** (M)	**Instantaneous reaction rate** (M/min)
0	0.0054	
10	0.0044	8.0×10^{-5}
26	0.0034	5.0×10^{-5}
44	0.0027	3.1×10^{-5}
70	0.0020	1.8×10^{-5}
120	0.0014	8.0×10^{-6}

It is known that this reaction is second-order: that is, the reaction rate is modelled by a quadratic function of the reaction concentration. Let's call the reaction rate at a particular concentration $R(C)$; so

$$R(C) = XC^2 + YC + Z.$$

By using the values in the table above, we have that

$$\begin{aligned} 8.0 \times 10^{-5} &= X \cdot (0.0044)^2 + Y(0.0044) + Z \\ 5.0 \times 10^{-5} &= X \cdot (0.0034)^2 + Y(0.0034) + Z \\ 3.1 \times 10^{-5} &= X \cdot (0.0027)^2 + Y(0.0027) + Z. \end{aligned}$$

Using `matlab` to find X, Y, and Z we have

```
>> syms X Y Z;
>> eqn1 = 8.0e-5 == X * (0.0044)^2 + Y * (0.0044) + Z;
>> eqn2 = 5.0e-5 == X * (0.0034)^2 + Y * (0.0034) + Z;
>> eqn3 = 3.1e-5 == X * (0.0027)^2 + Y * (0.0027) + Z;
>> [A,B] = equationsToMatrix([eqn1,eqn2,eqn3],[X,Y,Z]);
>> linsolve(A,B);
```

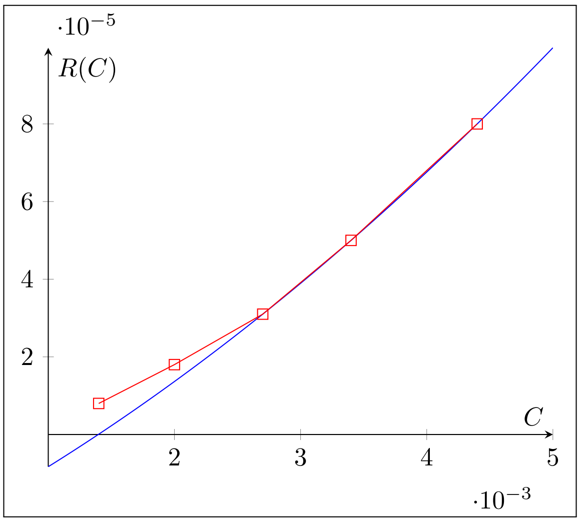
```
ans =

                                         6148914691236521356/3658604241285734375
   364782697339246960843864593812277/21596587553915600330621553957928960
-453142685283050860204790059017259/16872334026496562758298089029632000000
```

So our model is

$$R(C) \approx 1.6807C^2 + 0.01689C - 2.686 \times 10^{-5}.$$

Graphing our model (blue) next to the original data (red), we see that the match is reasonable.



This is useful because it allows us to predict the reaction rate for concentrations that we have not measured; alternatively, we can predict the reactant concentration given the reaction rate (which can be easily measured).

Some situations require us to find a minimum or a maximum value. Suppose, for example, that we are asked the following.

Example. Find the dimensions of a rectangle with perimeter 1000 m such that the area is maximised.

Solution. Later this year, we will learn a systematic way to solve optimisation problems like this. However, using calculus here would be like using a machine gun to kill a fly (although for some reason it is the favoured method of economics students)!

Let us call the two side lengths x and y; so we have that $1000 = 2x + 2y$ (so $500 = x + y$), and we want to minimise $A = xy$. Substituting, we have $A = x(500 - x) = 500x - x^2$. The maximum value will be the vertex of this parabola, so we need to put this formula into the form $A = -(x - b)^2 + c$; if we expand this, then $A = -x^2 + 2xb - b^2 + c$ and so we want $2b = 500$ and $c - b^2 = 0$. From these, we see that $b = 250$ and $c = (250)^2$. Hence $A = -(x - 250)^2 + (250)^2$; the vertex is at $x = 250$, and so $y = 250$: the area will be maximised when the rectangle is a square.

Here we are using the vertex form of the parabola, which we learned to find last year. The idea is that every parabola $y = ax^2 + bx + c$ is just a shifted version of the parabola $y = ax^2$; in particular, there exist numbers x_0 and y_0 such that

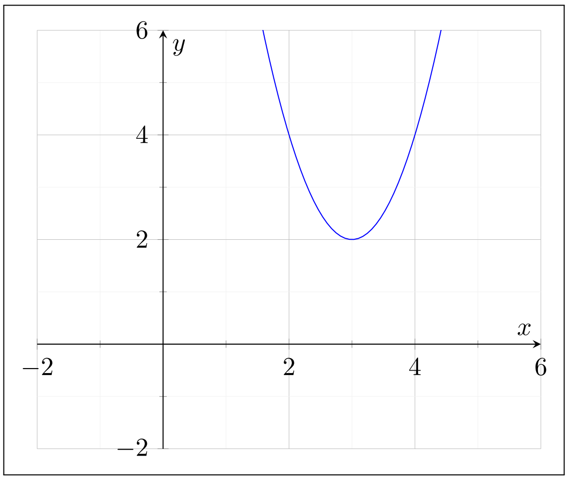
$$ax^2 + bx + c = a(x - x_0)^2 + y_0;$$

and the equation on the right, when graphed, has a vertex at (x_0, y_0) (why?).

Questions

You might want to use a calculator or a computer to help you with some of the calculations: for example, solving systems of equations or quadratic equations.

1. Give the equation of the following parabola:



2. (a) Find the vertices of the parabolae $y = x^2 + 4x - 9$ and $y = -4x^2 + 2x - 1$.
   (b) More generally, show that the vertex of the parabola $y = x^2 + bx + c$ is
$$\left(-\frac{b}{2}, c - \frac{b^2}{4}\right).$$

3. (a) Find the equation of the parabola with vertex $(2, 3)$ that passes through $(9, 1)$.
   (b) More generally, show that the equation of the parabola with vertex $V(x_0, y_0)$ passing through $A(x_1, y_1)$ is
$$y = \frac{y_1 - y_0}{(x_1 - x_0)^2}(x - x_0)^2 + y_0.$$

4. Show that the equation $y + 3 = \sqrt{x^2 + (y - 1)^2}$ describes a parabola, and find its vertex.

5. One application of mathematical modelling is in chemical spectroscopy: it is possible to measure the absorbance of light by substances, which is proportional to the amount of the substance present.

Amount of protein (µg)	**Absorbance**
0	0.099
5.0	0.185
10.0	0.282
15.0	0.345
20.0	0.425
25.0	0.483

   (a) Use the values for 0, 5.0, and 25.0 micrograms to write down an equation for a parabolic model of this data.
   (b) How accurately does this model predict the absorbance for 15 micrograms?

(c) A better model, taking into account all six data points, is found by computer to be

$$\text{amount of protein} = -0.000131429A^2 + 0.0188686A + 0.0982286.$$

If the absorbance is measured experimentally to be 0.3, how much protein is present in the sample?

6. Prove that the x–coordinate of a parabola's vertex is always halfway between the x–coordinates of its x–intercepts.

7. Let ℓ represent the line $y = -d$ for some real number d. If $P = (x, y)$ is any point, the distance $d(P, \ell)$ is defined to be $y + d$ (i.e. the length of the line segment from P to ℓ that is perpendicular to ℓ).

   (a) Let $O = (x_0, y_0)$ be some fixed point, and let R be some fixed number. Show that the set of all points P such that $d(P, O) = R = d(P, \ell)$ is a parabola.

   (b) On the other hand, suppose $y = ax^2 + bx + c$ describes a parabola. Show that there exists:

   - a real number d,
   - a real number R, and
   - a point $O = (x_0, y_0)$

   such that the parabola is the set of points described in (a).

NCEA Level 2 Mathematics (Homework)
5. Quadratic Modelling

Reading

Go and watch...

https://www.youtube.com/watch?v=hoh4TmPzu1w

What's it good for?

People use parabolae for...

- Engineering, and the sciences: modelling situations where a quantity is roughly proportional to the square of another. In physics, a perfect projectile follows the path of a parabola.
- Mathematics: a parabola is a specific case of a conic section (the others are the hyperbola, the circle, and the ellipse). Conic sections are geometric figures that were systematically studied by the Greeks.

Questions

1. Show that if $y = ax^2 + bx + c$, then
$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$
(You might want to start with the complicated equation and simplify it, rather than going the other way.)
2. Explain why this shows that changing b and c only shifts the parabola around the plane rather than changing the size. In other words, it is possible for us to shift any parabola anywhere we want without affecting the side.
3. In fact, we have shown that every quadratic can be written in the form $y = a(x - \alpha)^2 + \beta$; since changing α and β only shifts the parabola around, the only parameter that can change the size is a. Therefore, if we have a parabola $y = ax^2 + bx + c$ then it has the same shape as $y = ax^2$. Show that if (x_1, y_1) is on the parabola $y = ax^2$, then $(\frac{b}{a}x_1, \frac{b}{a}y_1)$ is on the parabola $y = bx^2$ (where a and b are any nonzero numbers).
4. Conclude that any parabola can be mapped onto any other parabola by shifting it around the plane (changing the location of the vertex) and scaling it in both directions by an appropriate constant.

NCEA Level 2 Mathematics
6. Systems of Equations

Let's go back to last week, where we had the following example of a quadratic model.

Example. The following table gives the instantaneous rates of reaction for the a particular chemical reaction.

Time (min)	**Concentration of reactant** (M)	**Instantaneous reaction rate** (M/min)
0	0.0054	
10	0.0044	8.0×10^{-5}
26	0.0034	5.0×10^{-5}
44	0.0027	3.1×10^{-5}
70	0.0020	1.8×10^{-5}
120	0.0014	8.0×10^{-6}

It is known that the reaction rate is modelled by a quadratic function of the reaction concentration. Let's call the reaction rate at a particular concentration $R(C)$; so

$$R(C) = XC^2 + YC + Z.$$

By using the values in the table above, we have that

$$\begin{aligned} 8.0 \times 10^{-5} &= X \cdot (0.0044)^2 + Y(0.0044) + Z \\ 5.0 \times 10^{-5} &= X \cdot (0.0034)^2 + Y(0.0034) + Z \\ 3.1 \times 10^{-5} &= X \cdot (0.0027)^2 + Y(0.0027) + Z. \end{aligned}$$

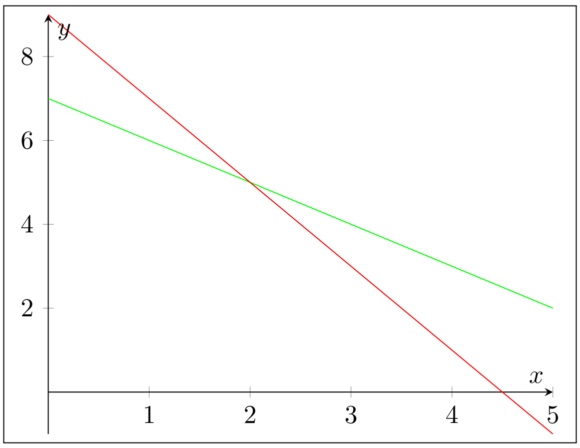
I'll leave off the end, because the point is we used a computer to solve this system of equations. This week, we'll learn a *systematic* method for solving such systems: not because it's easier than using a computer, but because it's interesting to see what's going on geometrically. (In my Y13 notes, there's a lot more detail — and proofs, which I'll skip this year.)

The main idea to get your head around this year is that a system of equations is *geometric*.

Example. Consider the system of equations

$$\begin{cases} x + y = 7 \\ 5x + 10y = 45. \end{cases}$$

If we plot all the values (x, y) which satisfy these equations, we obtain:



The key point here is that the point which satisfies both solutions is simply the geometric point of intersection.

In order to find this point algebraically, we can solve the first equation for y: $y = 7 - x$. We can then substitute this into the other: $5x + 10(7 - x) = 45$. Simplifying, we have

$$\begin{aligned} 5x + 70 - 10x &= 45 \\ 25 &= 5x \\ 5 &= x. \end{aligned}$$

Let's think about this a little more explicitly: since the same pair of values (x, y) makes both equations true, then if we can find y in terms of x using one equation then we can substitute it straight into the other one *because the symbol y represents the same thing in both.*

In another situation, we might have a line and a parabola.

Example (Extract from the sample L2 assessment for this topic.)**.** The student committee is planning the upcoming Performing Arts Showcase at your school this year. They are trying to determine how much they should make the price of adult tickets this year.

Determine the price they should set the adult tickets in 2012 if they want to make a profit of $3210 from ticket sales, given that:

- The cost of a child ticket is planned to be $5 and the cost of an adult ticket is not determined yet.
- The relationship between the expected number of child tickets and adult tickets to be sold can be modelled by $200x + y^2 = 80000$, where x represents the number of child tickets and y represents the number of adult tickets.
- The planning committee wants to make $3210 from ticket sales.
- There needs to be only one possibility for the number of child and adult tickets to be sold.

Solution. The total profit that will be made is $5x + my$, where m is the cost of an adult ticket (yet to be determined); we therefore have three important pieces of information: $5x + my = 3210$, $200x + y^2 = 80000$, and the curves of these two equations intersect at only one point. Solving the first for x, we have $x = \frac{3210 - my}{5}$; substituting, we have

$$\begin{aligned} 200\left(\frac{3210 - my}{5}\right) + y^2 &= 80000 \\ 128400 - 40my + y^2 &= 80000 \\ 48400 + 40my + y^2 &= 0 \end{aligned}$$

When we learned about parabolae, we learned that we could transform them into vertex form. Let's do that here, because our goal is to find a value m such that the vertex of $48400 + 40my + y^2 = x$ is sitting exacty on the y-axis — our parabola is flipped sideways, but the idea is the same. In fact, to make it clearer let's look at the 'graphy' form:*

$$(y + 20m)^2 + 48400 - 400m^2 = x.$$

This parabola is sitting on the y-axis when $48400 - 400m^2$ is zero (because this is the x-shift); hence $m = 11$, and the committee should sell the adult tickets for $11 each.

*The idea of switching from looking at something in an 'equationy' sense and turning something into a parameter is actually quite a powerful idea that's applicable in many different situations.

Questions

1. Graph and solve the system of equations
$$\begin{cases} 1 = x + y, \\ 2 = 2x - y. \end{cases}$$

2. Source: J. A. H. Hunter, *Mathematical Brain-teasers.* Dover (1976). Page 18, adapted.

   That croc was a freak,
   not at all the right shape,
   With a tail thrice as long
   as its head.

   Its body was short,
   far too short for the rest;
   Half as long as its tail,
   so they said.

   The body and tail,
   measured four metres point one:
   An ugly great brute,
   you'll agree.

   But I never discovered,
   the length of its head.
   Just how long do you think
   it would be?

3. Let us explore how many solutions we can obtain for simple systems. If
$$\begin{cases} a = bx + cy, \\ p = qx + ry \end{cases}$$
is a system of simultaneous equations in x and y then depending on the constants we choose there are three possible situations:

   - We can have no pairs (x, y) that satisfy both equations.
   - We can have precisely one pair (x, y) that satisfies both equations.
   - We can have infinitely many pairs (x, y) that satisfy both equations.

   (We will prove this next year.)

   (a) Draw a diagram showing each situation geometrically.

   (b) Give an example of a system for each case, taking care to show that your systems have the desired number of solutions.

4. Graph (using a computer or a calculator) and solve the system of equations
$$\begin{cases} y = 2x + 3, \\ x^2 + 2xy - 1 = 0. \end{cases}$$

5. [Extract from the sample L2 assessment for this topic.] For the performing arts showcase example above, you must also provide the planning committee with the change in the adult tickets sales in 2011 compared to 2010. You are given that:

- **2010 Performing Arts Showcase**
  - The total number of tickets sold was 400.
  - The relationship between the number of child tickets and adult tickets sold can be modelled by $x^2 + y = 22750$, where x represents the number of child tickets and y represents the number of adult tickets.
- **2011 Performing Arts Showcase**
  - The cost of a child ticket was \$5 and the cost of an adult ticket was \$10.
  - The relationship between the number of child tickets and adult tickets sold can be modelled by $xy = 1000 + 100x$, where x represents the number of child tickets and y represents the number of adult tickets.
  - More than 300 tickets were sold.
  - The money generated from ticket sales was \$2050.

6. For which values of c does the system of equations

$$\begin{cases} y^2 = 2x^2 + xy \\ y = cx - 2 \end{cases}$$

have precisely two solutions?

7. Find the parabola passing through the three points $(0, 0)$, $(1, 1)$, and $(3, 0)$.

8. Does there exist a parabola through the three points $(1, 1)$, $(2, -1)$, and $(3, 3)$? If so, find its equation, and calculate its vertex and x–intercepts.

9. The graph of the equation $xy = a$ (where a is a number) is called a *hyperbola*.
   (a) Show that $xy = a$ and $xy = b$ never intersect if $a \neq b$.
   (b) Find the point of intersection between the curves

$$\begin{cases} x(y + 1) = 4 \\ y = 2x - 3. \end{cases}$$

10. The line $y = 2x - 3$ intersects the circle $x^2 - 6x + y^2 = 0$ precisely twice.
    (a) Find the coordinates of the centre of the circle.
    (b) Find the points of intersection.
    (c) How could the constant term 3 of the line be changed such that the line becomes tangent with the circle?

NCEA Level 2 Mathematics (Homework)
6. Systems of Equations

Reading

Go and watch...

`https://www.youtube.com/watch?v=aOT_bG-vWyg`

What's it good for?

People use systems of equations for...

- Engineering, and the sciences: if a set of unknown quantities are all interrelated (for example, concentrations in a chemical equilibrium).
- Mathematics: solving systems of linear equations is the motivating example for *linear algebra*, which forms the algebraic basis of geometry and is itself used in engineering and the sciences for modelling systems.
- Computer graphics: finding intersection points of curves is often a problem that needs to be solved in computer graphics systems (e.g. working out where a ray of light hits a surface).

Questions

1. Find all the solutions to the system of equations

$$\begin{cases} 4x^2 + 16x + y^2 + 15 = 4xy + 8y \\ y = 2x + 3. \end{cases}$$

2. Graph the above system. If we replace the linear equation with $y = cx + 3$, for which values of c does the system have no solutions? Is it possible to pick a value of c such that the system has exactly one solution?

NCEA Level 2 Mathematics
7. Linear Inequalities

An equation is a statement which says that two quantities are identical. If we don't want to be so precise, we can talk about inequalities: statements which tell us about the *relative size* of two quantities. More precisely, if a and b are two quantities then:

$a = b$	a is identical to b
$a \neq b$	a is not identical to b
$a \leq b$	either a is identical to b, or a is smaller than b
$a < b$	a is not identical to b and a is smaller than b

This allows us to impose an *ordering* structure onto the integers, as well as the algebraic structure that they already had. We will look at the interplay between the two in the exercises.

Questions

1. Justify the following statements with mathematical reasoning (where a, b, and c are quantities):
   (a) Precisely one of $a < b$, $a = b$, or $a > b$ is true.
   (b) If $a \leq b$ and $b \leq c$ then $a \leq c$.
   (c) If $a \leq b$ and $b \leq a$ then $a = b$.
2. Using number lines, explain why
   (a) $(-3) + (-4) = -7$;
   (b) $(-2) + 4 = 2$;
   (c) $(-10) + 4 = -6$.
3. Consider the following multiplication table.

a		b	ab
2	$\times$	5	10
2	$\times$	4	8
2	$\times$	3	6
2	$\times$	2	4
2	$\times$	1	2
2	$\times$	0	0
2	$\times$	-1	
2	$\times$	-2	
2	$\times$	-10	

   (a) What is the pattern in the final column?
   (b) Fill in the final three lines of the table by continuing the pattern.
   (c) Based on this table, is it more reasonable for the product of a **negative by a positive** to be positive or negative?
   (d) Using this definition, fill in the first five lines of the next table:

a		b	ab
-3	$\times$	4	
-3	$\times$	3	
-3	$\times$	2	
-3	$\times$	1	
-3	$\times$	0	
-3	$\times$	-1	
-3	$\times$	-2	
-3	$\times$	-10	

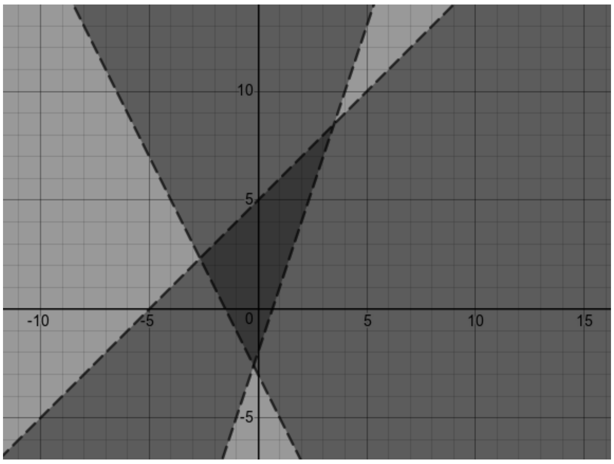
(e) Again using the pattern we see as we move down the final column, fill in the last three rows.

(f) Based on this table, is it more reasonable for the product of a **negative by a negative** to be positive or negative?

4. Justify the following statements with mathematical reasoning (where a, b, c, and d are quantities) — you may want to draw number lines, it makes things easier to visualise:
   (a) If $a \leq b$ and c is *positive* then $ac \leq bc$.
   (b) If $a \leq b$ and c is *negative* then $ac \geq bc$.
   (c) If $a \leq b$ and c is any quantity then $a + c \leq b + c$.
   (d) If $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.
   (e) If $a \leq b$ and $c \leq d$ then we cannot make any statement about the relative values of $a + d$ and $b + c$. [Hint: consider $1 \leq 2$ and $1 \leq 1$ as $a \leq b$ and $c \leq d$ respectively, then swap them around.]

5. We will now look at inequalities which involve variables.
   (a) For each of the following inequations, graph all the possible values of x and y that satisfy it.
      i. $4 + x < 3$
      ii. $3x + 2 \geq 2$
      iii. $x \geq y$
      iv. $x \leq y$
      v. $3x + 9y \leq 1$
      vi. $2x + y \geq 0$
   (b) Graph all possible values of x and y satisfying each of the following *sets* of inequalities. (The resulting region of the plane is called the *feasible region* of the system.)
      i. $x < y$, $x > y$, and $x < 2y$
      ii. $x \leq 2$, $x \geq -1$, $y \leq x$, $y \geq x - 3$

6. Consider the following graphed system of inequalities.



(a) Explicitly write down the three inequalities that have been graphed.

(b) What are the coordinates of the three intersection points?

7. (a) Show that no point simultaneously satisfies both of $y \geq 2x + 1$ and $y \leq 2x - 3$.

(b) Show that if $y \leq Ax + B$ is any linear inequality in x and y, then the feasible region of this inequality overlaps with at least one of the inequalities in (a).

8. The **arithmetic mean** of two numbers a and b is $\frac{a+b}{2}$; the **geometric mean** of a and b is $\sqrt{ab}$.

(a) Calculate the arithmetic and geometric means for several pairs of numbers. Make a conjecture about the relative order of the two means (is one always bigger than the other)?

(b) Show that

$$\left(\frac{a+b}{2}\right)^2 - ab = \left(\frac{a-b}{2}\right)^2.$$

(c) Suppose that a and b are positive numbers. Using part (b), or otherwise, show that the geometric mean of a and b is always less than or equal to their arithmetic mean. When are the two equal?

(d) Investigate the cases where a and b are both negative, or one is negative and one is positive. (Hint: one of these cases makes no sense.)

NCEA Level 2 Mathematics (Homework)
7. Linear Inequalities

Reading

Go and watch...

https://www.youtube.com/watch?v=ij-EK-MZv2Q

What's it good for?

People use systems of inequalities for...
- Economics: the setting of prices and production outputs in companies can often be modelled by a set of inequalities and an equation to maximise or minimise within the feasible region (this is called *linear programming*).
- Engineering: engineering problems like fluid flow through a network can also be modelled with linear programming.

Questions

1. Find the simplest set of inequalities that describes the same feasible region as the following system.

$$\begin{cases} 3x \leq 2 \\ x \leq -y \\ 3x \geq -1 - 3y \\ 8x \geq -2 \\ y \leq 5 \\ x \geq y + 2. \end{cases}$$

2. Consider the following system, the feasible region of which is a square.

$$\begin{cases} x \leq 1 \\ x \geq -1 \\ y \leq 1 \\ y \geq -1 \end{cases}$$

   At what point(s) within the feasible region is $x + 2y$ largest?

NCEA Level 2 Mathematics
8. The Quadratic Formula

Solving Quadratics

Recall that a quadratic equation is one of the form $y = ax^2 + bx + c$ (where $a \neq 0$). A couple of weeks ago, we saw that we can always rearrange such an equation into vertex form; we do this by trying to rewrite it as a square plus a constant. This process is known as *completing the square*.

Our goal is to end up with something that looks like

$$y = \alpha(x + \beta)^2 + \gamma$$

where $(-\beta, \gamma)$ are the coordinates of the vertex of the parabola and α (as we have seen) is the 'scaling factor' that gives us the shape. If α is negative then the parabola opens downwards, and if α is positive then the parabola opens upwards.

If we expand the parabola equation, we obtain

$$\begin{aligned} y &= \alpha(x^2 + \beta x + \beta^2) + \gamma \\ &= \alpha x^2 + 2\alpha\beta x + \alpha\beta^2 + \gamma. \end{aligned}$$

By comparing coefficients, we see that:

$$\begin{aligned} a &= \alpha \\ b &= 2\alpha\beta = 2a\beta \\ c &= \alpha\beta^2 + \gamma = a\beta^2 + \gamma. \end{aligned}$$

Clearly, then, we have $\beta = b/2a$. Substituting this into the third equation, we have $c = a(b/2a)^2 + \gamma$ and so $\gamma = c - \frac{b^2}{4a}$.

Reasoning thusly, we see that

$$y = ax^2 + bx + c = \alpha(x + \beta)^2 + \gamma = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a},$$

which is what I asked you to prove in the homework when we looked at parabolae.

Example. Suppose we are given a rectangular plot of land and are told that the area of the land is $32\,\mathrm{km}^2$ and that one side of the land is 8 km longer than the other. In order to find the dimensions of the land, we have a quadratic equation which we can simplify:

$$x(x + 8) = 32 \implies x^2 + 8x - 32 = 0.$$

Completing the square, we have that

$$0 = (x + 4)^2 - 16 + (-32) = (x + 4)^2 - 48$$

and so $x = \sqrt{48} - 4 \approx 2.9\,\mathrm{km}$.

This example suggests that we can write down a formula for the value of x in any quadratic equation

$0 = ax^2 + bx + c$ by rewriting that equation in vertex form and solving for x:

$$\begin{aligned} 0 = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\ \frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2 \\ \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = x + \frac{b}{2a} \\ -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}} = x \\ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = x. \end{aligned}$$

We have therefore proved the following

Theorem (Quadratic formula)**.** *If $ax^2 + bx + c = 0$, then there are at most two distinct values for x. These values are given by*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

when they exist.

These values are called the *solutions*, the *roots*, or the *zeroes* of the equation.

Example. The width of a canal at ground level is 16 m. The sides of the canal can be modelled by a quadratic expression that would give a maximum depth of 16 m. However, the base of the canal is flat and has a width of 12 m. What is the depth of the canal?

Solution. Model the canal with $y = a(x + 8)(x - 8)$. This parabola passes through $(0, -16)$, so we have $-16 = a(+8)(-8)$ and hence $a = \frac{16}{64} = \frac{1}{4}$; so $y = \frac{1}{4}(x + 8)(x - 8)$.

The width of the parabola is 12 at the y–value corresponding to the x–values ± 6; at $x = \pm 6$, $y = -7$ and so the depth of the canal is 7 m.

Forms of a Quadratic

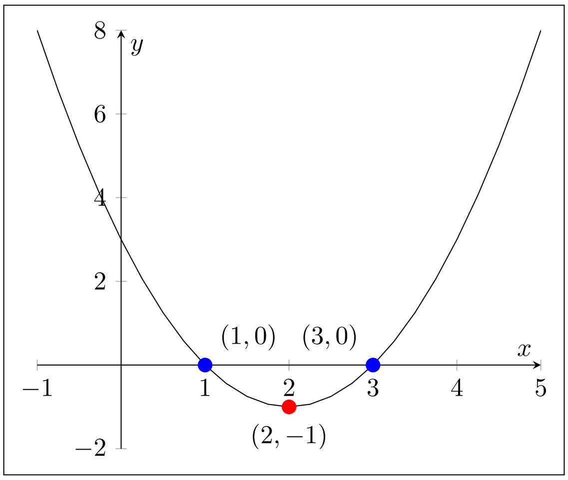
We have seen that there are three complementary ways of viewing the equation $y = ax^2 + bx + c$, each of which exhibits one particular characteristic of the function:

Form	**Exhibits**	**Example**
Expanded		$y = ax^2 + bx + c$
Factorised	roots/zeroes are α and β	$y = a(x - \alpha)(x - \beta)$
Completed square	vertex is at (x_0, y_0)	$y = a(x - x_0)^2 + y_0$

For example, consider the following equation:

$$y = (x - 1)(x - 3) = (x - 2)^2 - 1 = x^2 - 4x + 3.$$

The function is graphed below, so that we can see graphically that each coloured form is an important geometric feature of the parabola described by the equation.



You need to be comfortable transforming between these three forms.

Note that some quadratics, like $x^2 + 1$, cannot be transformed into the factorised form: there are *no* real numbers α and β such that $x^2 + 1 = (x - \alpha)(x - \beta)$.

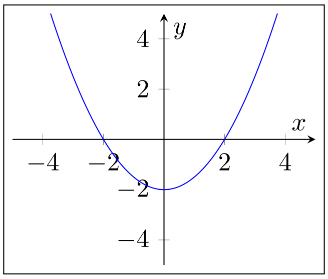
Classifying Roots

Let us look again at the vertex form of the general quadratic equation,

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

Solving the equation $0 = ax^2 + bx + c$ is equivalent to finding the x-intercepts of this parabola. The *number* of x-intercepts, and hence the number of solutions, must be at most two (because of the shape of the parabola), and can only be changed by shifting it up and down (changing the y-shift, $c - \frac{b^2}{4a}$).

Case I: two x-intercepts

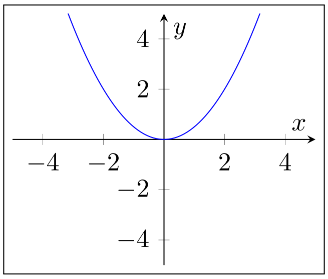


This happens in two situations:

- a is positive and $c - \frac{b^2}{4a}$ is less than zero. Hence $c < \frac{b^2}{4a}$, $4ac < b^2$, and $b^2 - 4ac > 0$.
- a is negative and $c - \frac{b^2}{4a}$ is greater than zero. Hence $c > \frac{b^2}{4a}$, $4ac < b^2$, and $b^2 - 4ac > 0$.

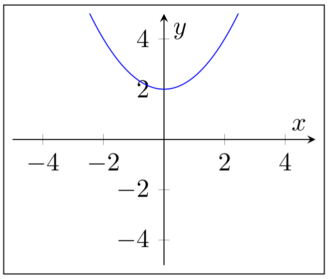
In either case, $b^2 - 4ac > 0$.

Case II: one x-intercept



This happens precisely when the vertex is sitting on the x-axis, so $c - \frac{b^2}{4a} = 0$ and $b^2 - 4ac = 0$.

Case III: no x-intercepts



This happens in two situations:

- a is positive and $c - \frac{b^2}{4a}$ is greater than zero. Hence $c > \frac{b^2}{4a}$, $4ac > b^2$, and $b^2 - 4ac < 0$.
- a is negative and $c - \frac{b^2}{4a}$ is less than zero. Hence $c < \frac{b^2}{4a}$, $4ac > b^2$, and $b^2 - 4ac < 0$.

In either case, $b^2 - 4ac < 0$.

Notice that the quantity $b^2 - 4ac$ tells us the nature of the roots in every case; it is known as the *discriminant* of the quadratic (and I denote it by Δ_2). We have therefore proved the following

Theorem. *Suppose $f(x) = ax^2 + bx + c$. Then:*

- *If $b^2 - 4ac < 0$, then $f(x) = 0$ has no solutions.*
- *If $b^2 - 4ac = 0$, then $f(x) = 0$ has precisely one solution.*
- *If $b^2 - 4ac > 0$, then $f(x) = 0$ has precisely two solutions.*

If we look at the quadratic equation again,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

notice that the discriminant appears underneath the square root sign and so it doesn't need to be memorised seperately.

Questions

Solving Quadratics

1. For each quadatic equation,
   - rewrite it into vertex form by completing the square if required;
   - graph the parabola it describes; and
   - calculate the x-intercept(s) of the parabola, if it has any.

   (a) $y = x^2 + 1$
   (b) $y = x^2 + x$
   (c) $y = x^2 - 4x + 4$
   (d) $y = x^2 + 2x + 3$
   (e) $y = -x^2 + 4x - 2$
   (f) $y = 2x^2 + 2x + 2$
2. Show that if $x^2 - bx + c = 0$, then b is the sum of the solutions of the equation.
3. This question is revision from Level 1.
   (a) Justify, with mathematical reasoning, the following statement: the roots of the equation $(x - \alpha)(x - \beta) = 0$ are α and β.
   (b) Give a quadratic equation with roots -1 and 6.
4. Find all the y-intercepts of $-(x^2 + 2x - 3)(4x^2 - 6x + 2) = y$.
5. Factorise and solve $5x^2 - 9x - 2 = 0$.
6. Consider the quadratic equation $x^2 + bx + c = 0$.
   (a) Calculate b and c such that the quadratic equation has solutions -1 and 3.

(b) Find the location of the vertex of the corresponding parabola, $y = x^2 + bx + c$.

7. Solve $\frac{x^2+5x+2}{x+2} = 3$.
8. Talia used timber to form the exterior sides of her rectangular garden. The length of the garden is x metres, and its area is $50\,\text{m}^2$.
   (a) Show that the perimeter of the garden is given by $2x + \frac{100}{x}$.
   (b) If she uses 33 m of timber to build the sides, find the dimensions of the garden.
9. David and Sione are competing in a cycle race of 150 km. Sione cycles on average 4 km per hour faster than David, and finishes half an hour earlier than David. Find David's average speed. *You MUST use algebra to solve this problem. Note that average speed* $= \frac{distance}{time}$.
10. Simplify fully $\dfrac{2x^2 - 8}{x^2 - 2x - 8}$.
11. The equation $(x + 2) - 3\sqrt{x + 2} - 4 = 0$ has only one real solution. Find the value of x.
12. Check, by direct substitution, that both

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

    are solutions of $ax^2 + bx + c = 0$.
13. (a) Suppose that it is known that one solution of $x^2 + bx + c$ is four times the other (i.e. the two solutions are α and 4α). Show that $c = 4b^2/25$.
    (b) Conversely, show that one solution of $x^2 + bx + \frac{4b^2}{25}$ is four times the other, no matter the value of b.
    (c) Show that one solution of $3x^2 + 15x + 12$ is four times the other.
14. (Challenge problem.) Let AB be the diameter of a circle centred at O. Draw the circles with diameters AO and OB; draw a third circle centred at T, tangent to all three existing circles. If the radius of the circle at T is 8, what is the length $d(A, B)$?

Classifying Roots

15. Without explicitly computing them, how many solutions does each quadratic equation have? Don't use the discriminant to decide for all four.
    (a) $0 = x^2 + 2$
    (b) $3 = 3x^2 + 3x$
    (c) $1 = -x^2 - 2x$
    (d) $0 = 2x^2 - 12x + 18$
16. Find k such that $x^2 + 3kx - 2 = 0$ has precisely one solution.
17. The equation $(2x - 3)(x + 4) = k$ has only one real solution; find the value of k.
18. Find all t such that the parabolae described by $y = tx^2 + x + 1$ and $y = -2x^2 - tx + 1$ meet at precisely one point.
19. By considering the quadratic formula, give another proof that the discriminant 'encodes' the nature of the roots of the quadratic.

20. The quadratic equation $mx^2 - (m+2)x + 2 = 0$ has two positive real roots. Find the possible value(s) of m, and the roots of the equation.

21. For what values of k does the parabola described by

$$y = x^2 + (3x - 1)x + (2k + 10)$$

never touch the x-axis?

22. Find the possible values of d if one or more real solutions exist for $x^2 + 5x - 1 = d(x^2 + 1)$. Interpret your answer geometrically.

23. How many real roots does $x^3 - 4x^2 + 7x - 4$ have?

24. Find expressions in terms of m and n for the roots of the equation

$$\frac{x - m}{x - n} = \frac{2(x + m)}{x + n}.$$

Give an inequality, in terms of m and n, so that the equation has two distinct roots.

25. (a) Two positive numbers have sum 25 and product 136. What are the two numbers?
    (b) For which numbers S and P is it possible to find at least one pair of real numbers with sum S and product P?

26. Let α and β be the roots of $x^2 + bx + c$.
    (a) Show that $\alpha^2 + \beta^2 = (-b)^2 - 2c$.
    (b) Conclude that $\Delta_2[x^2 + bx + c] = (\alpha - \beta)^2$.

27. Suppose α and β are the two solutions of $x^2 + ax + b = 0$. Write $(\alpha + \beta)^3$ in terms of a and b.

28. Let ρ be positive. Suppose P and Q are points such that $|PQ| \leq 2\rho$; show that the two circles of radius ρ with centres P and Q intersect; show that they intersect in exactly one place if and only if $|PQ| = 2\rho$.

29. (Baby algebraic geometry!) The equation $(x - 2)^2 + y^2 = 4$ describes a circle $\mathcal{C}$.
    (a) Show that $y = mx$ always intersects $\mathcal{C}$ in precisely two points, no matter the value of m.
    (b) Find all t such that $y = t(x - 3)$ intersects the circle $\mathcal{C}$ precisely once.
    (c) We will now study the intersections of $\mathcal{C}$ with the family of curves $\mathcal{V}_b$ defined by the equation $y^2 = x^2 + bx$ (where b is a real number).
        i. Show that $\mathcal{C}$ and $\mathcal{V}_b$ always have an intersection at $(0, 0)$, no matter what value b is given.
        ii. Show that if (x_0, y_0) is an intersection point of the two curves, then $(x_0, -y_0)$ is as well.
        iii. Show that there is a real number B_1 such that:
            - When $b < B_1$, $(0, 0)$ is the only intersection point;
            - When $b = B_1$, the two curves have exactly two intersection points; and
            - When $B_1 < b$, the two curves have exactly three intersection points.
        iv. Show that the equation of $\mathcal{V}_b$ can be written in the form

$$\frac{\left(x - \frac{b}{2}\right)^2}{(b/2)^2} - \frac{y^2}{(b/2)^2} = 1.$$

    (d) Show that if $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (with $b \leq a$) describes a curve $\mathcal{E}$ then the two points

$$O_1 = (+\sqrt{a^2 - b^2}, 0) \qquad O_2 = (-\sqrt{a^2 - b^2}, 0)$$

have the property that for all points P, Q on the curve $\mathcal{E}$, $d(P, O_1) + d(P, O_2) = d(Q, O_1) + d(Q, O_2)$. (Compare with the final problem in section 5.)

NCEA Level 2 Mathematics (Homework)
8. The Quadratic Formula

Reading

Go and watch...

`https://www.youtube.com/watch?v=v-pyuaThp-c`

What's it good for?

People use quadratic equations for...

- Engineering, economics, and the sciences: modelling situations (a perfect projectile follows a parabolic arc, a parabolic mirror reflects all light from the focus into parallel rays and is used in telescopes and radio dishes...)
- Mathematics: every polynomial is a product of quadratic and linear factors, and the fact that the quadratic formula even exists is actually quite surprising (there is no equivalent to 'completing the square' for cubics or anything more complicated).

Questions

1. Find the nature of the roots of the equation $x^2 + 3kx - 28$, if:
   (a) $k < 0$,
   (b) $k = 0$, and
   (c) $k > 0$.
2. Find the values of m for which one root of the equation $4x^2 = mx - 5$ is thrice the other root.
3. Suppose the quadratic equation $x^2 + bx + c = 0$ has the two roots α and β. Show that $bc = -\alpha^2\beta - \alpha\beta^2$.
4. The lengths of three sides of a right-angled triangle are $x - 2$, $2x$, and $x + 6$. If it is known that the length of the longest side is $x + 6$, compute x and give the length of the shortest side explicitly.

NCEA Level 2 Mathematics
9. Exponential and Logarithmic Functions

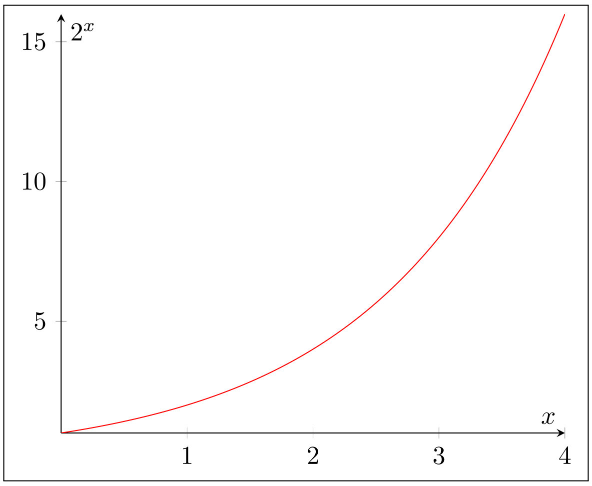
Exponentials

A particular species of bacteria reproduces by splitting in two every hour; if we start with one bacterium, after one hour we will have two; after two hourse, we will have four; after three hours, eight; and after n hours, we will have

$$2^n := \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}}$$

bacteria.

In general, equations of the form $y = a^x$ are called *exponential* equations.



Last year, we learned that exponents have the following properties:

1. $a^b \times a^c = \underbrace{a \times a \times \cdots \times a}_{b \text{ times}} \times \underbrace{a \times a \times \cdots \times a}_{c \text{ times}} = \underbrace{a \times a \times \cdots \times a}_{(b+c) \text{ times}} = a^{b+c}$.
2. $a^b \div a^c = \underbrace{a \times a \times \cdots \times a}_{b \text{ times}} \div \underbrace{a \div a \div \cdots \div a}_{c \text{ times}} = \underbrace{a \times a \times \cdots \times a}_{(b-c) \text{ times}} = a^{b-c}$.
3. $\left(a^b\right)^c = \underbrace{a^b \times a^b \times \cdots \times a^b}_{c \text{ times}} = \underbrace{a \times a \times \cdots \times a}_{bc \text{ times}} = a^{bc}$.
4. $a^1 = a$.
5. $a = a^1 = a^{0+1} = a^0 a^1 = a^0 a$, so $a^0 = 1$.

Note that we have some danger hiding in the background with these proofs: namely, if the powers are not whole numbers (or zero), they become meaningless! What does it mean to take 2 multiplied by itself π times? The solution, which we will look at briefly next week, is to define the function $x \mapsto a^x$ in a series of steps; we have already defined it when x is a natural number (or zero), and next week we will properly define it when x is an integer or rational number in general. Unfortunately, we won't have the necessary machinery to define it for any real number until next year.*

* For future reference, this is exercise 11 on the second L3 calculus worksheet.

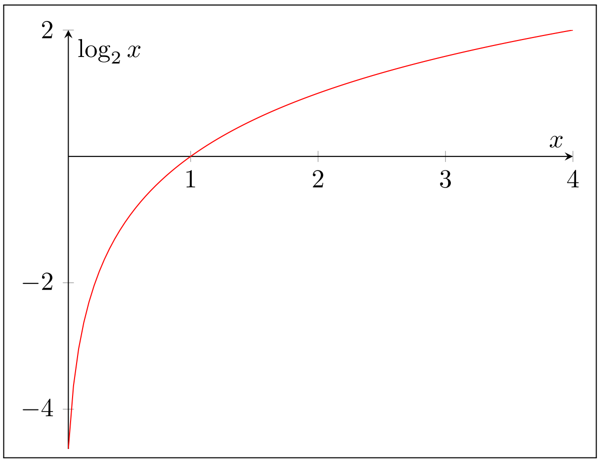
Logarithms

Suppose, on the other hand, we wish to know after how many hours we will have 1024 bacteria: we wish to find x such that $2^x = 1024$. This value is called the *logarithm* of 1024 with respect to 2, and we write $x = \log_2 1024$. In general, we have (as a definition),

$$y = a^x \iff x = \log_a y.$$

The quantity a is called the *base*, and is always positive.

Note that the function $x \mapsto \log_a x$ is the inverse of the function $x \mapsto a^x$.



If the base of a logarithm is 10, then we often don't write the base: so $\log 1000 = 3$, because $10^3 = 1000$.

The following logarithm laws can be derived from the exponent laws:

1. $\log_a x + \log_a y = \log_a xy$
2. $\log_a x - \log_a y = \log_a \frac{x}{y}$
3. $\log_a x^n = n \log_a x$
4. $\log_a 1 = 0$
5. $\log_a a = 1$
6. $\log_b x = \dfrac{\log_a x}{\log_a b}$ (change-of-base)

Example. Some elementary examples:

1. $\log_2 x = 10$ implies that $2^{10} = x$ and $x = 1024$.
2. $\log_x 49 = 2$ implies that $x^2 = 49$ and so $x = 7$.

Most applications of exponential and logarithmic equations outside of mathematics itself are to do with rates of change and rates of growth. This is because the rate of change of an exponential function is itself exponential, and so the exponential function will show up anywhere that a rate of change of a quantity is related directly to the amount of the quantity.

Example. A computer depreciates continuously in value from \$4699 to \$1500 over a period of 4.25 years. The value in dollars, y, of the computer t years after its value was \$4699 can be modelled by a function of the form

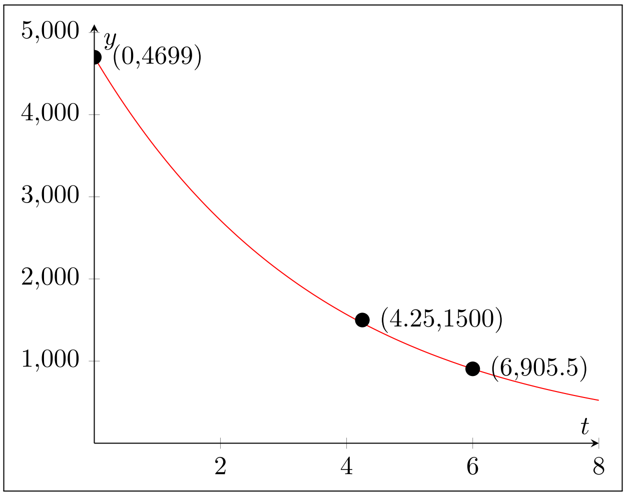
$$y = Ar^t,$$

where r is a constant. What is the value of the computer after six years?

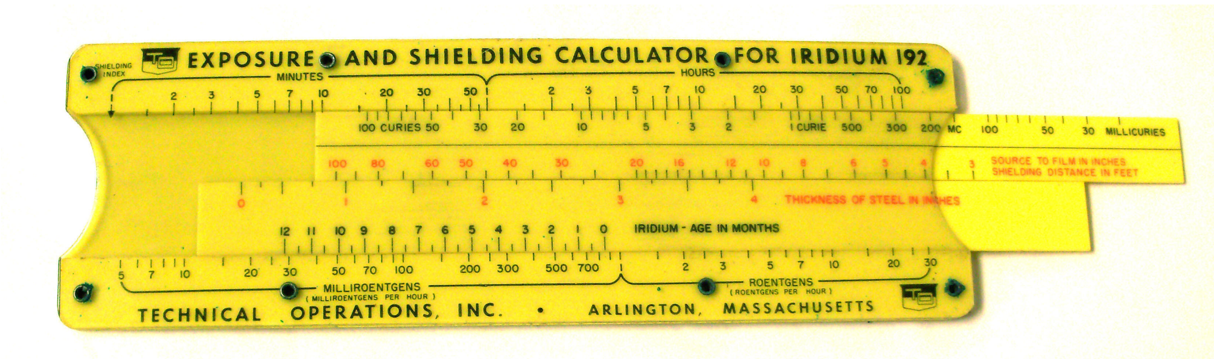
Solution. At $t = 0$, $y = 4699$ and so $4699 = Ar^0 = A$. On the other hand, we have $1500 = Ar^{4.25} = 4699 \cdot r^{4.25}$. Hence

$$\begin{aligned}
\frac{1500}{4699} &= r^{4.25} \\
\log \frac{1500}{4699} &= \log r^{4.25} \\
\log \frac{1500}{4699} &= 4.25 \log r \\
r &= 10^{\left(\frac{1}{4.25} \log \frac{1500}{4699}\right)} \approx 0.76.
\end{aligned}$$

Here, we used log base 10 because it happens to be on the calculator (we could have used any base); we then plugged the numbers into the calculator without worrying too much what powers that are fractions 'mean' (we'll discuss them next week). In the end, we found that a model for the value of the computer after t years is $y = 4699 \cdot 0.76^t$, and so after six years the computer is worth only around \$905.5.



Prior to the invention of the electronic calculator, mechanical devices called *slide rules* (see picture) were used by those who needed to make computations with large numbers. These devices consisted of two logarithmic scales next to each other, labelled with numbers; then the multiplication of a and b could be done by finding the lengths $\log a$ and $\log b$ on the slide rule, adding the two lengths together, and then using the rule $\log a + \log b = \log(ab)$ to read off the answer.



Joke. The water receded and the Ark came to rest upon the land. Noah opened the doors and commanded the animals, "Go forth and multiply." The animals slowly departed the Ark except for two snakes that remained in the back. Again Noah proclaimed ,"Go forth and multiply" yet the two snakes did not move. Noah walked to the back of the Ark and asked, "Why have you not followed my command?". The snakes answered, "Noah, we cannot, because we are adders."

Noah then went out upon the land and felled several large trees; from these trees he made a four legged platform. He then went inside the Ark and carried the snakes outside and upon placing them on the platform, his words became true.

As everyone knows, adders can multiply using log tables.†

Questions

1. Intuitively justify the following statements.
   (a) Multiplication ($n \times x$) is repeated addition ($+n$).
   (b) Exponentiation (n^x) is repeated multiplication ($\times n$).
   (c) Division ($x \div n$) counts 'how many' ($+n$)s.
   (d) Logarithms ($\log_n x$) count 'how many' ($\times n$)s.

2. This question is a list of mechanical exercises. It is important to be fluent with the mechanical use of exponentials and logarithms; however, in order to get anything more than a low achieved it is not enough to just focus on this kind of problem. In particular, if you plan to continue with any kind of mathematical subject next year (calculus, statistics, physics, chemistry) then you *must* be doing a significant number of other problems.
   (a) Evaluate $\log_2 32$ and $\log_3 1/9$.
   (b) Write $2\log 3 - 3\log 2$ as the log of a single number.
   (c) Solve $8^{x+1} = 4^{2x-5}$ for x.
   (d) Write $\log_2 \sqrt[3]{\frac{\sqrt[3]{15}\,5^4}{3^3\,\sqrt[3]{9}}}$ as a single number.
   (e) Simplify $\frac{4\log u^3}{\log u}$.
   (f) Solve $2\log x = \log 16$ for x.
   (g) Solve $\log_{x-1}(4x-4) = 2$ for x.
   (h) Express $\log \dfrac{U^3V^2}{W^5}$ as an algebraic sum of logarithms.
   (i) Solve $4^{2x-1} = 5^{x+2}$ for x.
   (j) Solve for x if $210 = (10^x)^3$.
   (k) Find x if $a^x = 5^{x-1}$ (where a is some constant).
   (l) Solve $\log x = 2\log mx$ for x in terms of m.
   (m) If a sequence of numbers is given by $a_n = 2^n + 3$, show that the difference between the $(n-1)$th and nth terms of the sequence is $a_n - 3$.

3. How many words with 4 letters of a 26 letter alphabet are possible?

4. Solve for y, if $\log_2(y^{-6}) = (\log_2 y)^2 + 8$.

5. Find all values of x satisfying $6(\log_8 x)^2 + 2\log_8 x - 4 = 0$.

6. Luka says that the equation $\log_x(4x+12) = 2$ has only one solution. Is he correct?

7. If the formula $P = A(0.75)^t$ models the amount P of a drug (in milligrams) in the bloodstream t hours after it is ingested, and the initial amount ingested is 500 mg, how long does it take for the amount of drug in the bloodstream to reduce by half?

†Attribution: `https://mathoverflow.net/a/1946`, although variations of this joke are widespread and numerous.

8. The graph of $y = a + b\log x + c(\log x)^2$ passes through $(1, 0)$, $(10, 7)$, and $(100, 13)$. What is the value of y when x is $1/10$?

9. (a) A function f is defined by the following rule:

$$f(x) = \begin{cases} 20 & 0 \le x \le 5 \\ ax^2 - 10ax + (20 + 25a) & x > 5. \end{cases}$$

   i. What is the domain of f?
   ii. If $f(9) = 52$, what is the value of a?
   iii. What is the range of f?

   (b) A function g is defined by the following rule:

$$g(x) = b + c\log_3 x.$$

   i. Choose b and c so that both the following are true:
      - The graphs of f and g meet at the point $(9, 52)$, and
      - $g(81) = 100$.
   ii. Find all the points (x, y) that lie on the graphs of both f and g.

10. Many population models are exponential.

Year	**World population (billion)**
1804	1
1927	2
1963	3
1974	4
1987	5
1999	6
2011	7

   (a) Assume the world population in the year t (CE) can be modelled with an exponential equation of the form

$$P = P_0 r^{ct}.$$

   Find P_0, r, and c using the data from 1804, 1927, and 1963 (the three earliest years given above).

   (b) Write another model, using the three *latest* years above. Compare the two models.

   (c) Using the second model, calculate a projected terrestrial population in 2024 and in 2100.

   (d) How accurate do you think an exponential model will be in the long term?

11. Radioactive substances slowly decompose into lead. The rate at which the decomposition occurs is usually measured in terms of the *half-life*: the time taken for an amount of the substance to decay by a factor of $\frac{1}{2}$. It can be shown (using calculus) that if A_0 is the amount of material initially present, then the material present after t years can be modelled by

$$A(t) = A_0(2.72)^{-kt}$$

   where k is a constant.

   (a) Show that $k = \frac{\log_{2.72} 2}{\tau}$, where τ is the half-life in years.

   (b) Radium has a half-life of 1590 yr. How long will it take for six grams of radium to decay to one gram?

   (c) A new radioactive substance decays from 1 gram to 0.98 grammes in one year. What is the half-life of the substance?

(d) One of the most useful applications of this model is in dating ancient materials. Because of the action of cosmic rays in the atmosphere, there is always a certain percentage of a particular radioactive form of carbon in the atmosphere (namely, carbon-14). Since living matter is always interchanging material with the environment, the living matter also contains the same percentage of carbon-14. Upon death, the exchange with the environment ceases and so no new carbon-14 enters cells; the remaining carbon-14 continues to decay, and the amount of remaining carbon-14 can be detected using instrumentation. The model can then be used to estimate the rough date of death.

The half-life of carbon-14 is 5570 yr.

In a sample taken from the bone structure of a recently discovered mummy, it is found that only 1/10 of the original amount of carbon-14 remains. How old is the mummy?

12. Prove the logarithm laws, using the exponent laws. For example,

$$\log_a x + \log_a y = \log_a(a^{\log_a x + \log_a y}) = \log_a(a^{\log_a x} a^{\log_a y}) = \log_a xy = \log_a(a^{\log_a xy}) = \log_a xy.$$

13. When a bank loans money, the interest is *compounded*; that is, you earn interest on the interest you have already been charged.

(a) Show that adding x% to a debt is equivalent to multiplying the debt by $(1 + \frac{x}{100})$.

(b) Suppose interest is calculated after each year (that is to say, it is compounded annually). If the initial debt is \$100, and the annual compound interest is 20%, what is the amount owed by the end of the first three years?

(c) Now, to simplify matters, we will say that our initial loan is \$1, and the interest rate is \$100 per annum. After one year, the value of the loan will double to \$2.

i. If the bank calculates compound interest every six months instead of every year (so our interest is 50% per six months), show that we owe an extra \$0.25 after one year.

ii. If the bank compounds interest every month, show that our total owed is now \$2.6130 after one year.

iii. How much will we owe if the bank compounds interest every day?

iv. In general, show that if we divide the annual percentage rate by n and compound it n times then the end-of-year balance of the loan is $(1 + \frac{1}{n})^n$.

(d) From your working in part (c) above, we can conclude that as n increases (that is, if the bank compounds interest with shorter and shorter time intervals) then the total owed after one year climbs closer and closer to \$2.7182818.... This number, which is fundamental in mathematics, is known as Euler's constant, or e. Show that if a bank compounds interest continuously, with an interest rate of x% per annum and initial loan L, then after t years the total owed is

$$L \times e^{xt/100}.$$

Leonhard Euler (1707–1783) was a Swiss mathematician who made a vast number of contributions to mathematics: the Wikipedia page *List of things named after Leonhard Euler* contains over one hundred entries. We will meet the number e in particular a few more times this year, and we will also see a number of different problems associated with Euler.

NCEA Level 2 Mathematics (Homework)
9. Exponential and Logarithmic Functions

Reading

Go and watch...

https://www.youtube.com/watch?v=N-7tcTIrers

What's it good for?

People use exponential and logarithmic equations for...

- Chemistry, physics, engineering: whenever the rate of growth or rate of decline of a quantity is proportional to (or inversely proportional to) the amount of quantity present, the quantity is an exponential or logarithmic function of time. (This includes rates of chemical reaction, rates of capacitor charge/discharge, the position of a damped spring over time, and many other examples.)

Questions

1. Thirty minutes after a patient is administered his first dose of a medication, the amount of medication in his bloodstream reaches 224 mg. The amount of the medication in the bloodstream decreases continuously by 20% each hour. The amount of the medication M mg in the patient's bloodstream after it is administered can be modelled by the function

$$M = 224 \times 0.8^{t-0.5}$$

   where t is the time in hours since the drug was first administered.

   (a) Explain what 0.8 means in this function.

   (b) Give the initial amount of medication administered.

   (c) A second dose of the medication can be administered some time later, and again the amount of the medication in the patient's bloodstream from the second dose can be modelled by the same function as that for the first. The total amount of the drug in the blood stream must never exceed 300 mg. How long after administering the first dose can the second dose be administered?

2. Here are some revision questions on topics we have already covered.

   (a) Rearrange the following formula to make x the subject: $\frac{4x}{5} = \frac{y(x+3)}{2}$.

   (b) Show that the solutions of $x^2 + x - 56 = 0$ are four times those of $4x^2 + x - 14 = 0$.

   (c) Find the relationship between the solutions of the equations $dx^2 + ex + f = 0$ and $x^2 + ex + df = 0$ where d, e, and f are real numbers.

   (d) Consider the equation $(3x + 1)^2 = -7$.

      i. Explain why it has no real solutions; explain what this means graphically.

      ii. Compute the discriminant of the equation, and explain why this further supports your answer to (i).

NCEA Level 2 Mathematics
10. Negative and Fractional Powers

Last week we defined the exponential function for powers which were whole numbers or zero, by defining $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$. We can make this definition more precise by making the following definition:*

Definition. If a is a number, then:

1. a^0 is defined to be 1.
2. a^n is defined to be $a \times a^{n-1}$, for integers $n > 0$.

One might easily ask if there is a way to extend this definition for non-whole-number powers; in fact, last week we implicitly used the fact that such an extension exists in solving some logarithmic equations (but relying on a calculator to 'know the definition' for us). Let us take inspiration from our recursive definition above, and try to 'pull ourselves up by our bootstraps' in steps: we will begin with negative powers.

So suppose we want to define what the value of a^{-n} is (where n is a positive integer). We can try to work out a plausible definition using the rules we want such a value to follow — for example, we want such a definition to obey the rule $a^b a^c = a^{b+c}$. In particular,

$$a^{-n} \times a^n = a^{(-n)+n} = a^0 = 1.$$

Hence a plausible definition for a^{-n} is $1/(a^n)$. This plausible definition also follows (to take another example) the rule $(a^b)^c = a^{bc}$, because $(a^{-n})^x = \left(\frac{1}{a^n}\right)^x = \frac{1}{a^{nx}} = a^{-nx}$ as we would expect.

So now we have a definition for all a^x, where x is an integer. The obvious next step is to look at rational powers; recall, a rational number is any number r that can be written in the form $r = \frac{p}{q}$, where p and q are both integers. As an aside, the following theorem is quite deep and perfectly accessible:-

Theorem. *There are real numbers which are not rational.*

Proof. In particular, we will show that any number x such that $x^2 = 2$ is irrational; for suppose that such an x can be written in the form $x = \frac{p}{q}$ where p and q are both positive integers. Then $2 = x^2 = \frac{p^2}{q^2}$, and hence $2q^2 = p^2$. But this implies that p^2 is even, and so p is itself even (because the squares of odd numbers are odd). Therefore, there is an integer n such that $p = 2n$. Substituting, we have $2q^2 = (2n)^2 = 4n^2$, and hence $q^2 = 2n^2$. But this means that q is even, and hence there is an integer m such that $q = 2m$; substituting, we have $(2m)^2 = 2n^2$, and hence $2m^2 = n^2$ and $2 = \frac{n^2}{m^2}$.

Notice, though, that $\frac{p^2}{q^2} = \frac{n^2}{m^2}$, but n and m were smaller than p and q respectively. Since we didn't say what p and q were to start with, this implies that for any pair of positive integers p and q such that $x = p/q$, there exist smaller positive integers n and m satisfying the same equation; and so we can repeat the whole process, finding two positive integers smaller than n and m, and so on *ad infinitum*.

But this is absurd: given any positive integer, there are only finitely many positive integers smaller than it! Thus our original assumption, that such integers p and q existed in the first place, must be false; so any number x such that $x^2 = 2$ cannot be rational. □

Real numbers which are not rational are (rather unimaginatively) called *irrational*. Other numbers which are irrational include π, the square root of any prime number, and e.

Returning to our main theme, we want to define a^r, where $r = \frac{p}{q}$ is a rational number. Let us again work out a plausible definition using the rules we want such a number to follow; this time, we will use the 'power multiplication' rule:

$$\left(a^{p/q}\right)^q = a^{((p/q)\cdot q)} = a^p.$$

So we can define $a^{p/q}$ to be $\sqrt[q]{a^p}$. (If there's any confusion, we will more precisely define it to be the *positive* root; also, we require our rational number p/q to be written so that q is positive so that we don't have to worry about defining negative roots).

Our full definition so far looks like:

*By 'more precise' I mean 'we make it clearer what we mean by $\cdots$'.

Definition. If a is a number, then:

1. a^0 is defined to be 1.
2. a^n is defined to be $a \times a^{n-1}$, for integers $n > 1$.
3. a^{-n} is defined to be $\frac{1}{a^n}$, for integers $n > 1$.
4. $a^{p/q}$ is defined to be $\sqrt[q]{a^p}$, for rational numbers p/q such that $q > 0$.

Our final trick will be to define a^x for any real number x. Since we don't have the necessary machinery to do it properly this year, our definition will be vague. We use the fact that we want a^x to be continuous: that is, we want it to 'have no gaps' and 'not jump around unexpectedly'. Since x is real, we can always write it in decimal expansion: say

$$x = x_0 + 0.x_1x_2x_3 \dots x_n \dots = x_0 + \frac{x_1}{10} + \frac{x_2}{100} + \dots + \frac{x_n}{10^n} + \dots$$

(where the notation $x_0 + 0.x_1x_2\dots$ means that x_0 is the 'integer part' of x and x_1, x_2 and so on are the digits of the decimal expansion). In particular, we have

$$a^x = a^{\left(x_0 + \frac{x_1}{10} + \frac{x_2}{100} + \dots + \frac{x_n}{10^n} + \dots\right)} = a^{x_0} \times a^{x_1/10} \times \dots \times a^{x_n/10^n} \times \dots ,$$

where we have already defined all the terms on the right — so we can define a^x to be 'the real number which we get closest to if we keep adding the terms on the right until infinity'. *This is obviously not precise, but just take my word for it that (a) it is possible to make the notion precise with a little more work, and (b) real powers are well-defined (that is, such a number always exists).*

Example.

1. $2^{3/2} = \sqrt[2]{2^3} = \sqrt{8}$.
2. $4^{-1/2} = \frac{1}{4^{1/2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$.
3. $27^{5/3} = \left(27^{1/3}\right)^5 = \left(\sqrt[3]{27}\right)^5 = 3^5 = 243$.
4. $2^\pi \approx 2^3 \times 2^{1/10} \times 2^{4/100} \times 2^{1/1000} \times 2^{5/10000} \approx 8.8244$. (my calculator tells me that $2^\pi \approx 8.8249$, so this approximation isn't even that bad!)

Questions

1. Graph the equation $y = \vartheta^x$ for different values of ϑ:

   $$\vartheta = 10 \quad 2 \quad 1 \quad 1/2 \quad 1/10 \quad 0 \quad -1/10 \quad -1/2 \quad -1 \quad -2 \quad -10$$

   (a) What do you notice? Compare and contrast the different curves. Is there any point which all 11 curves pass through?

   (b) When ϑ is negative, the curve is an *exponential decay* curve; when ϑ is positive, the curve is an *exponential growth* curve. Conjecture some situations where an exponential decay or growth curve might be a good model for some situation.

2. Make a conjecture about the value of 0^0: should it be zero (because $0^n = 0$ for all n), or one (because $n^0 = 1$ for all n)? It might be helpful to graph $y = x^x$ for very small positive and negative values of x.

3. Justify the following statements with mathematical reasoning:

   (a) $\sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$ (where p and $q > 0$ are integers).

(b) If r and s are rational numbers, then $a^r \times a^s = a^{(r+s)}$ (recall we only proved this rule last week for integer powers).

4. Evaluate $\sqrt{27^{-2/3}} + 5^{2/3} \cdot 5^{1/3}$.

5. A student was asked to evaluate $x + 2y + \sqrt{(x-2y)^2}$ for $(x, y) = (2, 4)$. They wrote

$$x + 2y + \sqrt{(x-2y)^2} = x + 2y + x - 2y = 2x$$

and thus obtained the value $2x = 2 \cdot 2 = 4$ for their answer. Were they correct?

6. Simplify the following, writing your answer with positive exponents:

(a) $\frac{(4a^3)^2}{b^3} \times \frac{2b^2}{(2a)^2}$

(b) $\frac{5x^2y}{2} \div \frac{10x}{y^2}$

(c) $(2a^7 \times 50a^3)^{-1/2}$

(d) $\frac{6m^5}{\sqrt{9m^{16}}}$

(e) $\sqrt{\frac{\left(16a^{(2/3)}\right)^{(3/2)}}{a^{-1/2}}}$

7. Evaluate $\log_{1/4} 16$, $\log_8 4$, and $\log \sqrt[4]{10}$.

8. Verify that the multiplication terms further to the right in the expression

$$a^{x_0} \times a^{x_1/10} \times a^{x_2/100} \times \cdots \times a^{x_n/10^n} \times \cdots$$

get closer and closer to 1. (Hint: each x_i, for $i > 0$, is a single digit and thus less than 10.) Hence justify why only taking a few of the first terms usually gives a good approximation to the 'real value' of $a^{x_0 + 0.x_1x_2\cdots}$.

9. A graph with Cartesian equation of the form $y = a(x - x_0)^{-1} + c$ is a *hyperbola*.

(a) Suppose a hyperbola passes through the points $(-1, 0)$, $(0, -1)$, and $(3, 2)$. Find the constants a, x_0, and c and give the equation of the hyperbola.

(b) Show that there is some value μ such that the hyperbola does not touch the line $x = \mu$. This line is called the *vertical asymptote* of the hyperbola.

(c) Show that there is some value λ such that the hyperbola does not touch the line $y = \lambda$. This line is called the *horizontal asymptote* of the hyperbola.

(d) Graph the hyperbola, using your graphing device of choice; describe the behaviour of the graph *around* the two asymptote lines.

(e) Graph the equation $y = x^{-n}$ for different values of n; what do you notice?

(f) Show that the hyperbola with vertical asymptote 'at infinity' is just a straight line $y = c$. (Hint: notice that in the hyperbola equation, $x = x_0$ is the vertical asymptote and 'substitute' $x_0 = \infty$ into the equation.) Is this what you expect intuitively?

10. Challenge question. Consider the equation $6^{2x} + m \cdot 6^x + n = 0$, where $n \leq 0$.

(a) Prove that the equation has precisely two solutions for 6^x.

(b) Show that only one of these solutions is valid for finding a solution for x if m is positive.

NCEA Level 2 Mathematics (Homework)
10. Negative and Fractional Powers

Reading

Editorial note: rather than boring you with more 'applications of exponents' (I listed most of the interesting ones last week), here's a little history.

In the May 1690 issue of the *Acta Eroditorum*, Jakob Bernoulli, the discoverer of e, revisited a question that had been puzzling mathematicians for a century. What is the correct geometry of the shape made by a piece of string when it is hanging between two points? This curve — called the 'catenary', from the Latin *catena*, chain — is produced when material is suspended by its own weight.

The curve whose identity Jakob Bernoulli so ardently sought turned out to have a hidden ingredient, e, the number he had uncovered in a different context [compound interest].

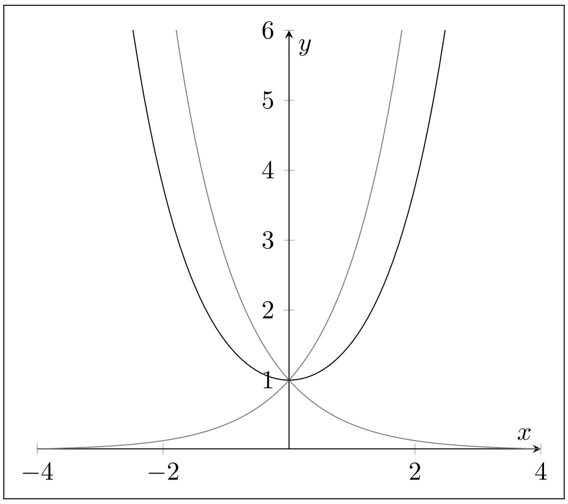
In modern notation, the equation for the catenary is

$$y = \frac{e^{ax} + e^{-ax}}{2a},$$

where a is a constant that changes the scale of the curve. The bigger a is, the further apart the two ends of the hanging string are.

Look closely at the equation. The term e^{ax} represents pure exponential growth, and the term e^{-ax} pure exponential decay. The equation adds them together, and then divides by two, which is a familiar arithmetical operation: adding two values and then halving the result is what we do when we want to find their average. In other words, the catenary is the average of the curves of exponential growth and decay, as illustrated below. Every point on the U falls exactly halfway between the two exponential curves.

Whenever we see a circle, we see π, the ratio of the circumference to the diameter. And whenever we see a hanging chain, a dangling spider's thread or the dip of an empty washing line, we see e.



Adapted from *Alex Through the Looking-Glass*, by Alex Bellos (pp.150-2).

Questions

1. Expand and simplify, writing with only positive exponents.

   (a) $\dfrac{(3+x^{3/2})(3-x^{3/2})}{x^{-4}}$

   (b) $\dfrac{x(y^3+\sqrt{y})+y^3}{y} - xy^{-(1/2)}$

2. Find all solutions to $8x^3 + 64 = 16\sqrt{8}x^{3/2}$.

3. Challenge question: we have seen that $\sqrt{2}$ is irrational; that is, $2^{1/2}$ is irrational and so it is possible for a^b to be irrational when both a and b are rational. Is it possible for a^b to be *rational* when both a and b are *irrational*?

Chapter 4

Calculus

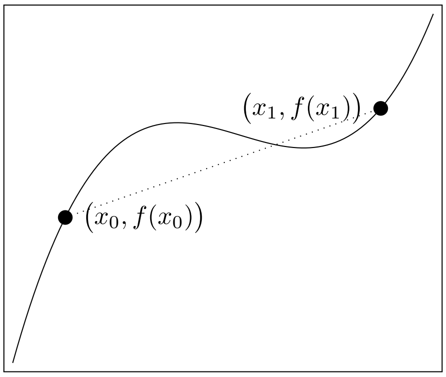
NCEA Level 2 Mathematics
11. Slopes and Differentiation

The main theme this year so far has been geometry and modelling the real world. This week, we move from modelling the stationary world of fields, fences, and compound interest to the world of continuity and motion.

We have already seen that if we model a changing quantity with a linear equation of the form $y = mx + c$, then it is possible to talk about the rate of change of that quantity by means of the slope of the graph of the equation.

Example. Suppose that the amount of sand in a pile (in kilograms, perhaps) is modelled by the equation mass $= 3(\text{time}) + 1$. Then the slope of the graph of mass versus time is 3, and so it makes sense to say that the rate of change of mass is three kilograms per unit of time.

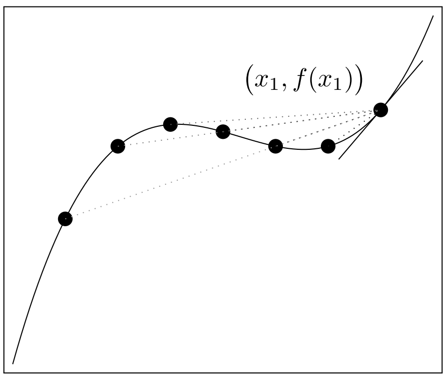
Our goal this week is to describe the slopes of functions which are not necessarily lines. Let us look at some arbitrary function f, and let us draw the graph $y = f(x)$. Then, recalling that we originally defined slope to be $m = \frac{\text{change in } y}{\text{change in } x}$, we can look at the slope between $y_0 = f(x_0)$ and $y_1 = f(x_1)$:



The average slope betwen $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is just the slope of the dotted line, which is simply

$$\frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

If we move the point (x_0, y_0) closer to (x_1, y_1) along the curve, then it makes sense that the slope of the dotted line will match the slope of $y = f(x)$ at the point (x_1, y_1) better and better (the dotted lines in the diagram below). If we imagine moving the point (x_0, y_0) until it coincides with the point (x_1, y_1), then the 'average' slope (indicated by the solid line below) will in face be the actual slope of the curve at x_1.



Example. Suppose we look at the curve $y = x^2$. The two points $(0, 0)$ and $(2, 4)$ are on this curve; the average slope of the curve between $x = 0$ and $x = 2$ is therefore $\frac{4-0}{2-0} = 2$.

Now, suppose we want to find the actual slope of the parabola at $x = 2$. We will do this in a slightly tricky way: finding the average slope between $x = 2$ and $x = 2 + h$, and then setting h to zero. The coordinates of the point on the parabola at $x = 2 + h$ will be $(2 + h, (2 + h)^2) = (2 + h, 4 + 4h + h^2)$ and so the average slope that we are looking for is

$$\frac{\text{rise}}{\text{run}} = \frac{(4 + 4h + h^2) - 4}{(2 + h) - 2} = \frac{4h + h^2}{h} = 4 + h;$$

hence the average slope between $x = 2$ and $x = 2 + h$ when $h = 0$ (i.e. when the two points are the same), and thus the actual slope of $y = x^2$ at $x = 2$, is 4.

More generally, suppose we are to find the slope of the parabola at (x_0, x_0^2). To do this, we will find the average slope between this point and the point $(x_0 + h, (x_0 + h)^2) = (x_0 + h, x_0^2 + 2x_0h + h^2)$. The calculation runs as follows:

$$\frac{\text{rise}}{\text{run}} = \frac{(x_0^2 + 2x_0h + h^2) - x_0^2}{(x_0 + h) - x_0} = \frac{2x_0h + h^2}{h} = 2x_0.$$

Does this function give us the slope we expect? For $x < 0$, the parabola is downward-sloping (if we move to the right, we move down) and so we would expect the slope to be negative; and for $x > 0$, the parabola is upwards-sloping and so we would expect the slope to be positive. The function $x \mapsto 2x$ satisfies both these criteria, and so this is reassuring: our algebraic slope-finding seems to have given a reasonable answer.

Hence, given the function $f : x \mapsto x^2$, we can write down another function $f' : x \mapsto 2x$ that gives the slope of the graph $y = f(x)$ at any point we choose.

It is, in fact, possible with most functions f we have met to write down another function f' such that the second function (called the *derivative* of f) gives the slope of $y = f(x)$ at any point we choose.

Example. Let's try $f(x) = x^3$ now; we'll go a bit quicker this time.

$$\frac{\text{rise}}{\text{run}} = \frac{(x + h)^3 - x^3}{(x + h) - h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = 3x^2 + 3xh + h^2$$

and so (setting h to zero) it seems that the derivative of f is $f' : x \mapsto 3x^2$.

Does this make sense? Well, $y = x^3$ is always sloping upwards, and $3x^2$ is never negative, so yes — this does make sense.

With the basic examples out of the way, I will state (without proof, but the proof is similar to the ideas above) the following theorem.

Theorem (Power rule)**.** *If f is a function defined by $f(x) = ax^r$, where r is any real number, then the derivative of f is the function defined by $f'(x) = rax^{r-1}$.*

This matches our example above: for x^2, we obtained $2x = 2x^{2-1}$ and for x^3 we obtained $3x^2 = 3x^{3-1}$. Note though that this theorem holds when r is any real number. Hence:

Example.

1. The derivative of $3x^{2018}$ is $6054x^{2017}$.
2. The derivative of $9x^{\pi}$ is $9\pi x^{\pi-1}$.
3. The derivative of 97 is 0.
4. The derivative of $\sqrt{x}$ is $\frac{1}{2}x^{-1/2}$.
5. The derivative of $1/x$ is $-\frac{1}{x^2}$.

We also have the following theorem which allows us to combine derivatives:

Theorem. *If f and g are functions, and λ is any number, then*

1. $(\lambda)' = 0$ — *the derivative of a flat line/constant is zero.*
2. $(f + g)' = f' + g'$ — *the derivative of the sum of two functions is the sum of the derivatives.*
3. $(\lambda f)' = \lambda f'$ — *the derivative of a number times a function is the number times the derivative.*

Hence,

Example.

1. The derivative of $2x^2 + 3x$ is $4x + 3$.
2. The derivative of $19x^{\sqrt{2}} - 3$ is $19\sqrt{2}x^{\sqrt{2}-1}$.

Finally, we can obviously take derivatives with respect to variables that are not x; obviously the two functions $x \mapsto f(x)$ and $y \mapsto f(y)$ are the same function, and the derivative of $3y^2$ with respect to y is $6y$.

If we want to make the variable of differentiation clear, we can use the alternative Leibniz notation: if $y = f(x)$, then the derivative is

$$\text{either} \qquad f' \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x}.$$

If we take the derivative of the derivative, the result is called the second derivative and is notated by

$$\text{either} \qquad f'' \qquad \text{or} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

In general, the nth derivative is notated by

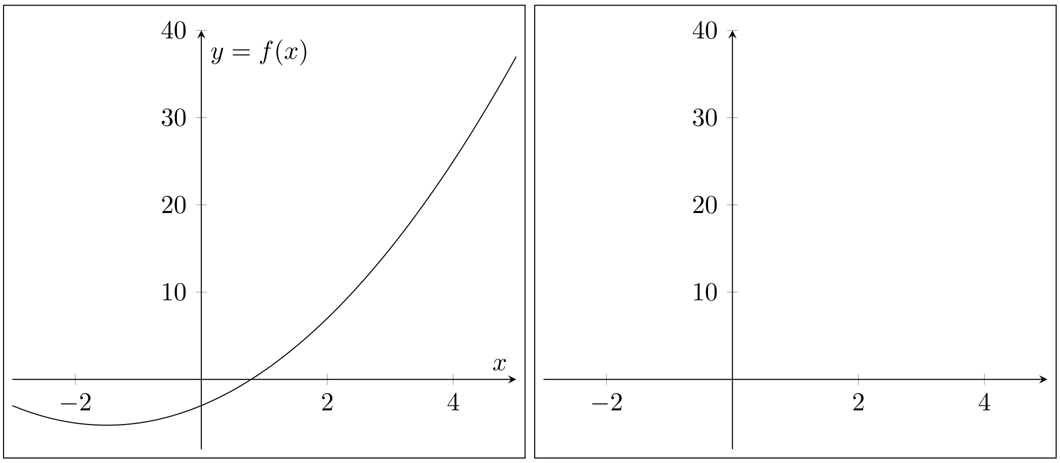
$$\text{either} \qquad f^{(n)} \qquad \text{or} \qquad \frac{\mathrm{d}^ny}{\mathrm{d}x^n}.$$

Maria Agnesi (1718–1799), an Italian mathematician, was the author of one of the first influential textbooks on calculus, and was also the first woman appointed as a mathematics professor at a university.

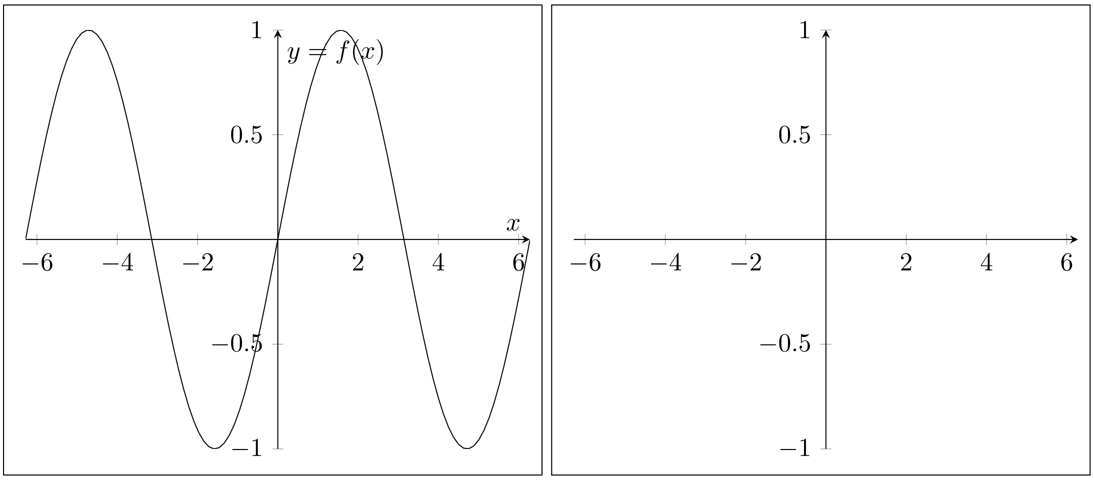
Questions

1. Draw the graph of the slope function (derivative) for each of the following graphed functions.

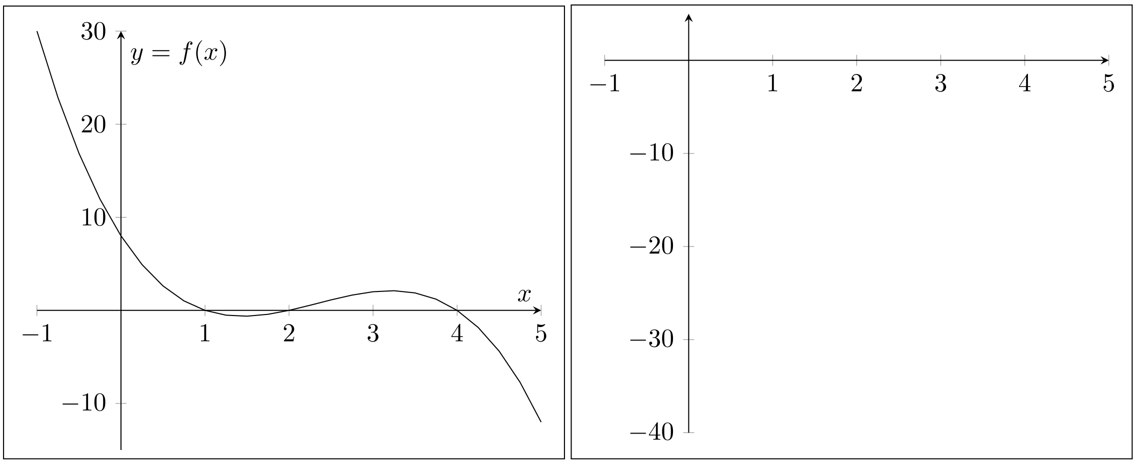
(a)

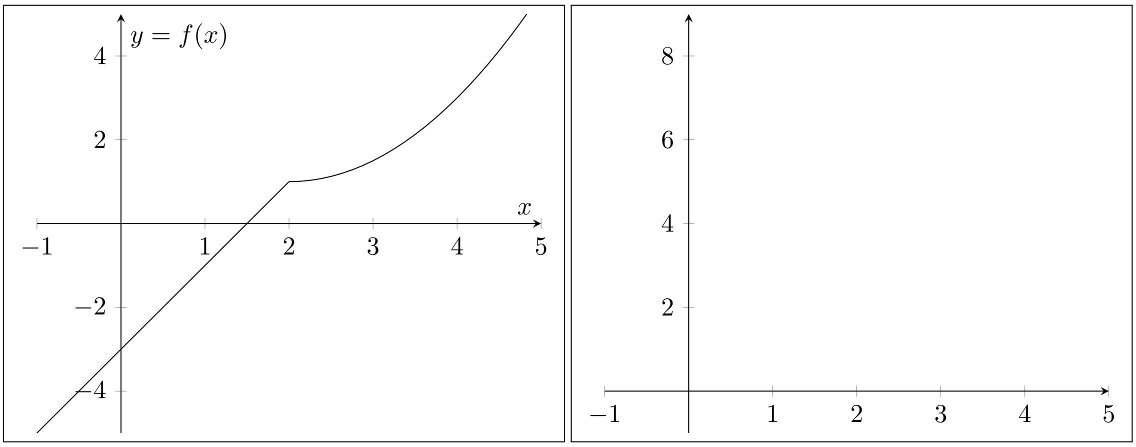


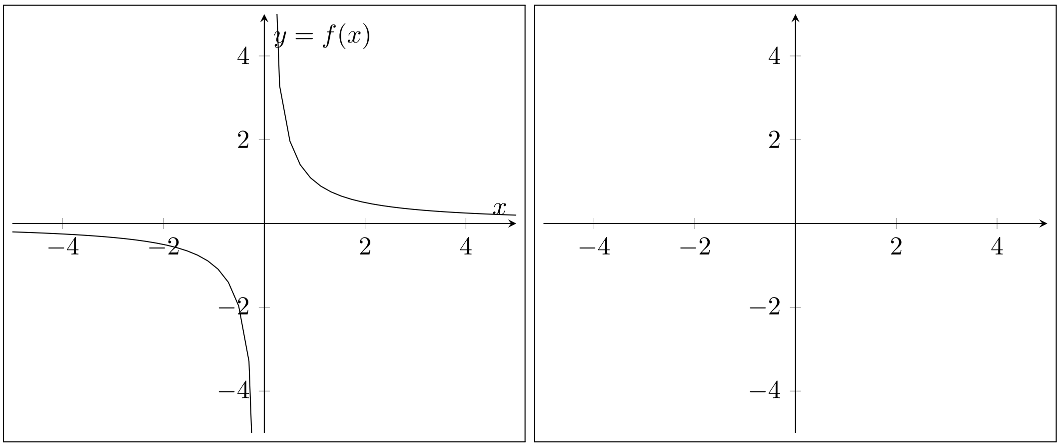
(b)



(c)



(d) 

(e) 

2. Differentiate the function, with respect to x or the stated variable.
   (a) $a(x) = 186.5$
   (b) $b(x) = \sqrt{30}$
   (c) $c(x) = 5x - 1$
   (d) $z = -4x^{10}$
   (e) $e(x) = x^3 - 4x + 6$
   (f) $f(t) = \frac{1}{2}t^6 - 3t^4 + t$ with respect to t
   (g) $g(x) = x^2(1 - 2x)$
   (h) $h(x) = (x - 2)(2x + 3)$
   (i) $y = x^{-2/5}$
   (j) $B(y) = cy^{-6}$ with respect to y

3. Since the sum rule for derivatives holds (i.e. $(f + g)' = f' + g'$), one might be tempted to guess that the product rule $(fg)' = f'g'$ also holds. Unfortunately, this is not the case.
   (a) Differentiate $f(x) = 4x^3$, $g(x) = 2x^2$, and $h(x) = 8x^5$.
   (b) Notice that $fg = h$, but $f'g' \neq h'$.
   (c) There is in fact a product rule for derivatives, but this year you won't need it. Find the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = (3x + 2)(x^2 + 1)$.

4. The following part-questions require you to find higher derivatives of functions.

(a) Find $\frac{d^2x}{dt^2}$ if $x = 3t^2 + 4$.

(b) Suppose $f(x) = 9x^2 + \frac{1}{x^3} + \sqrt{x}$. Find $f''(x)$.

(c) Find $\frac{d^3y}{dt^3}$ if $y = 3x^{-1}$.

(d) Find $f''(x)$ if $f'(x) = 9x^2 + 3x^{-2}$.

(e) Find $a(t) = \frac{d^2s}{dt^2}$ if $s(t) = 3t - 4t^{-1}$.

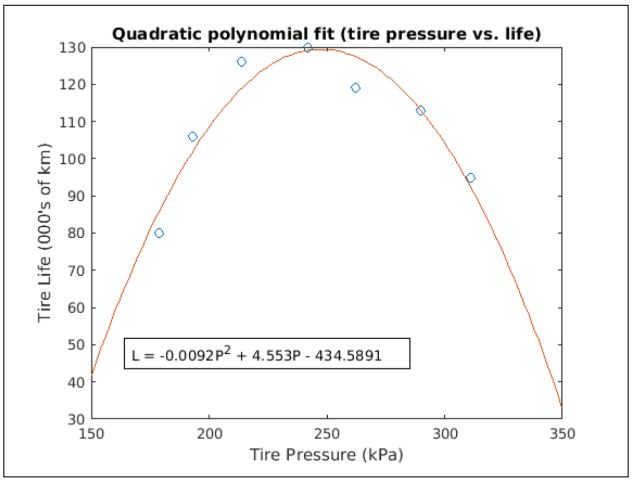
5. If we 'zoom out' from the graph of any polynomial, and ignore the 'wiggly bits' in the middle, the result always 'looks like' either the graph of $y = \pm x^2$, or the graph of $y = \pm x^3$.

   (a) Check this by zooming out from $y = 3x^6 + 2x^3 - 17x^2 + 3$ and $y = -2x^9 + 30x^2 - x$ on a graphing calculator.

   (b) Intuitively justify why the derivative of an odd polynomial (one which has an odd highest power of x) is an even polynomial, and the derivative of an even polynomial is an odd polynomial.

6. Where is the graph of $y = \frac{x^2}{2} - \frac{x^3}{3}$ increasing?

7. Car tires need to be inflated properly because overinflation or underinflation can cause premature treadwear. The graph shows tire life L (in thousands of kilometres) for a certain type of tire at various pressures P (in kPa), as well as a quadratic function that models the tire life.



   Use the model to estimate $\frac{dL}{dP}$ when $P = 200$ and when $P = 300$. What is the meaning of the derivative? What is the significance of the sign of the derivatives?

8. A function f is given by $f(x) = 2 - 4x + 4x^2 + ax^3$. The gradient of the graph at the point where $x = 1$ is 3. Find the value of a.

9. Let f be a function of x defined by $f(x) = 3x^2 + 6x + 6$. Show that f is a solution of the *differential equation* $f(x) - f'(x) = 3x^2$.

10. Show that the function f of y defined by $f(y) = \frac{3}{y^2} + 2y$ is not differentiable at $y = 0$.

11. It is natural to ask if there is any function such that the function is its own derivative.

    (a) An obvious candidate is the function K defined by $K(x) = 0$. Show that $\frac{dK}{dx} = K(x)$.

    (b) For a more interesting example, we will look at the exponential functions $y = a^x$. Draw the gradient function of $y = 2^x$; explain why looking at the exponential functions is probably a productive way to answer our question.

(c) Suppose we try to use the same technique as we used with the parabola. Show that the average slope between (x, a^x) and $(x+h, a^{x+h})$ is given by

$$a^x \left(\frac{a^h - 1}{h} \right).$$

(d) For a^x to be its own slope, we want $\frac{a^h-1}{h}$ to get closer and closer to 1 as h gets closer and closer to zero. Unfortunately we can't just substitute $h = 0$ straight in. Why?

(e) Let $a = (1+h)^{1/h}$ (where we take h very small), and show that $\frac{a^h-1}{h} = 1$. This suggests that a works as a base for our exponential-which-is-its-own-derivative if we let h get closer to zero here.

(f) Again, we can't let $h = 0$ in a — so let $n = 1/h$, and show that $a = \left(1 + \frac{1}{n}\right)^n$.

(g) If $n \to 0$, then $h \to \infty$. Therefore, our base we want is just $\left(1 + \frac{1}{n}\right)^n$ if we let n get infinitely large: the notation we use is

$$\lim_n \left(1 + \frac{1}{n}\right)^n,$$

which you can read as the *limit* with respect to n. Calculate this base for some large value of n. Is the number you see familiar?

NCEA Level 2 Mathematics (Homework)
11. Slopes and Differentiation

Reading

Go and watch...

`https://www.youtube.com/watch?v=axZTv5YJssA`

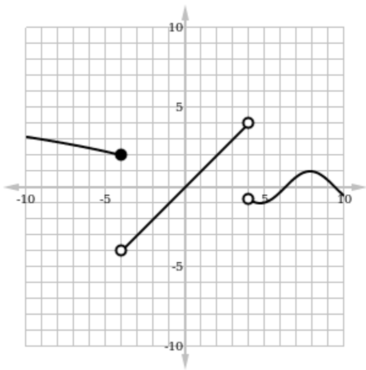
What's it good for?

People use calculus for...

- Engineering and physics: calculus is the natural language of any system which moves or changes over time (the movement of a vehicle or a robot arm, or the current in an electrical system).
- Chemistry and biology: rates of reaction and fluid pressure, population growth, and molecular kinetics are all examples of systems that are best understood using either calculus or further applications of calculus.
- Mathematics: the subjects which grow out of calculus (analysis and topology, for example) are fundamental geometric theories of space, distance, and transformation.

Questions

1. A function f is given by $f(x) = 2x^3 - 10x + 5$. Find the gradient of the graph of f at the point where $x = 2$.
2. Find the coordinates of the point on the curve $y = \frac{4}{x^2}$ where the gradient is 1.
3. Answer the following questions about this graph. Open circles denote locations where the function *is not* defined, while filled circles denote locations at the end of a segment where the function *is* defined.



   (a) What is the slope of the graph at $x = 8$?
   (b) Does the function have a derivative everywhere (i.e. can you guess the slope of the graph at every point)? If not, where does it fail to be differentiable?
   (c) At $x = -5$, is the derivative positive or negative?
   (d) What is the slope of the curve around $x = 0$?

NCEA Level 2 Mathematics
12. Tangent Lines and Approximations

One application of calculus is to find approximations to curves. Our goal is to write down a linear equation that approximates any given curve (around a given point). This is useful if (for example) we are given a complicated function like

$$f(x) = [\sin(x^{100})]^{[\cos(\sin x^2)]}.$$

This function is so weird that the graphing software I use cannot even graph it properly. The value of this function at $x = 0$ is very easy to calculate:

$$f(0) = [\sin(0^{100})]^{[\cos(\sin 0^2)]} = 0^{\cos(0)} = 0$$

However, as soon as we try to calculate other values it becomes difficult.

Suppose we want to know the value of a function near a point that it's easy to find the value of the function at — for example, $f(0.001)$. Our goal is to to draw the line through the easy point, with the same slope as the function at that point, and then work out what our desired x-value maps to using this easy function.

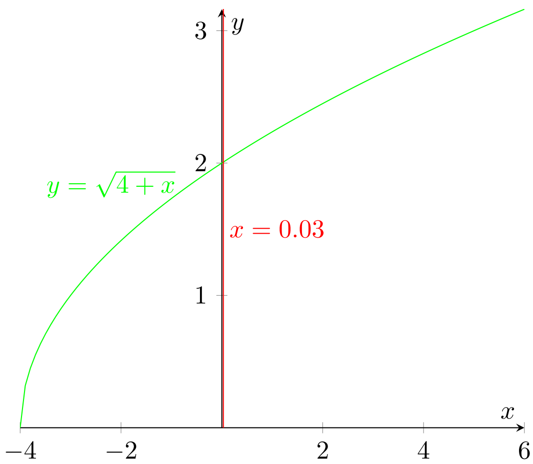
Definition. Let $y = f(x)$ be a curve, and let f be differentiable at some x-value x_0. Then the *tangent line* to the curve at x_0 is simply the line through the point $(x_0, f(x_0))$ that has the same slope as the curve at that point.

From our work on coordinate geometry, we know that the equation of this line is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Let's take an example.

Example. Let us calculate $\sqrt{4.03}$ by hand(!). If we consider the function $f(x) = \sqrt{4+x}$, then $\sqrt{4.03} = f(0.03)$. Let's draw the situation out:



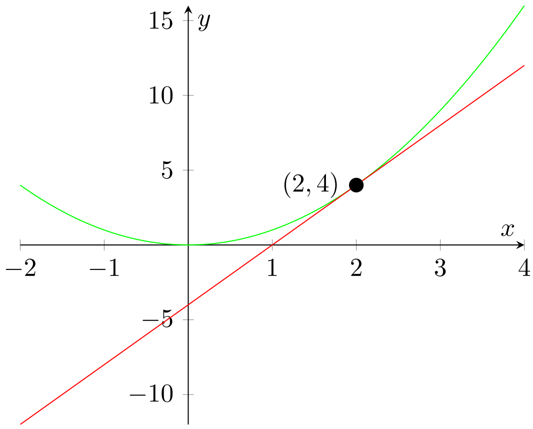
So we want the tangent line to f at the point $x = 0$. We have that $f'(x) = \frac{1}{2\sqrt{4+x}}$ (why? we can easily take the derivative of $\sqrt{x}$, it's just $\frac{1}{2\sqrt{x}}$ — and the graph of f is just the graph of $\sqrt{x}$ but shifted four units to the left, so we just shift the slope function itself four units to the left to match up), and so $f'(0) = \frac{1}{4}$. The tangent line is the line through $(0, f(0)) = (0, 2)$ with gradient $\frac{1}{4}$, which has equation

$$y - 2 = \frac{1}{4}x.$$

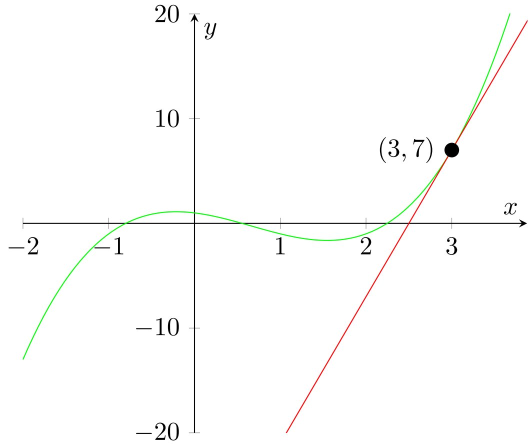
Hence $\sqrt{4.03} \approx \frac{1}{4} \times 0.03 + 2 = 2.0075$ — and, as promised, all of these calculations can be done without a calculator. (According to my calculator, $\sqrt{4.03} \approx 2.007486$ and so we are not far off at all.)

Now, with all the motivation out of the way, we will just look at some more simple examples.

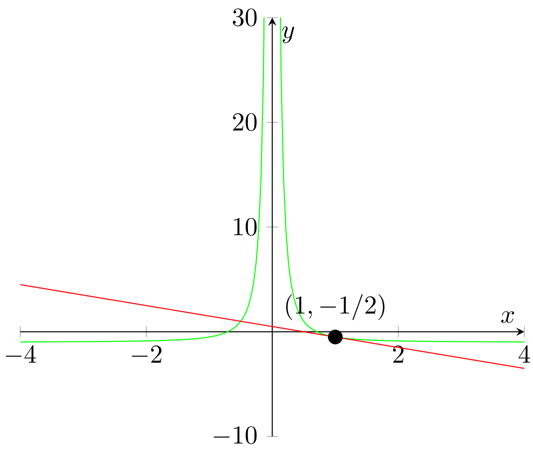
Example. Consider $y = x^2$. At $(2, 4)$, the tangent line has slope 4 and hence equation $y - 4 = 4(x - 2)$:



Example. Consider $y = x^3 - 2x^2 - x + 1$. At $(3, 7)$, the tangent line has slope 14 and hence equation $y - 7 = 14(x - 3)$:



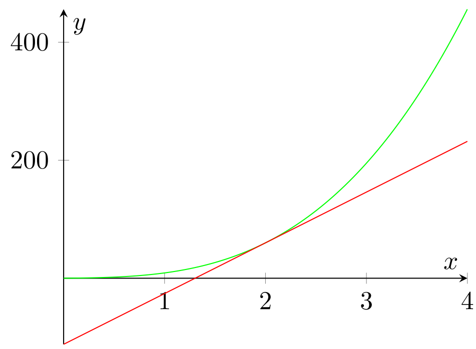
Example. Consider $y = \frac{1}{2x^2} - 1$. At $(1, -1/2)$, the tangent line has slope -1 and hence equation $y + 1/2 = -(x - 1)$:



Questions

1. Why is there no tangent line to $y = x^2$ at the point $(0, -1)$?

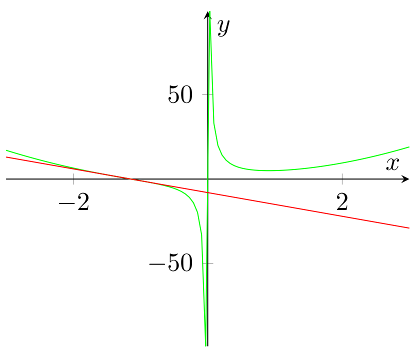
2. Consider the function $f(x) = 7x^3 + 2x$.



   (a) What is the slope of the graph of $y = f(x)$ around $x = 2$?

   (b) Give the equation of the tangent line to the graph at $x = 2$.

3. Consider the function $g(x) = 2x^2 + \frac{3}{x}$.



   (a) What is the slope of the graph of $y = g(x)$ around $x = -1$?

   (b) Give the equation of the tangent line to the graph at $x = -1$.

   (c) The normal line to a graph at a point is the line going through that point that lies at right angles to the graph (and hence to the tangent line to the graph).

      i. Consider the line with slope m going through (x_0, y_0); it has equation $(y - y_0) = m(x - x_0)$. What is the slope of the line at right angles to it going through the same point?

      ii. Give the equation of the normal line to the graph of $y = g(x)$ at $x = -1$.

4. (a) Find the slope function of $y = x^3 + 8x^2 + 22x - 21$ in two different ways:

      i. By rewriting the function as $y = (x-3)^3 + (x-3)^2 + (x-3)$ and then differentiating $z^3 + z^2 + z$;

      ii. By simply taking the derivative of the original function in its expanded form.

   (b) Hence explain how the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is related geometrically to the derivative $\frac{\mathrm{d}}{\mathrm{d}z}[z^3 + z^2 + z]$.

   (c) Show that there are no points where the graph of y versus x has a horizontal tangent line.

5. To expand slightly on the previous question, consider now the graph $y = (x^2 + 1)^2 + (x^2 + 1)$.

   (a) By expanding the brackets, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   (b) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 2(x^2 + 1) + 1$. (Where did this right-hand function come from?)

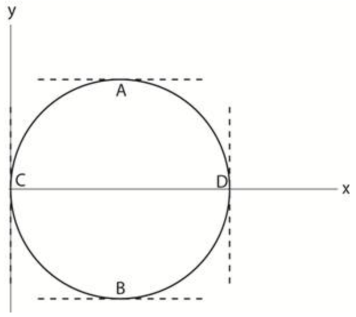
(c) Can you explain why our argument about 'shifting functions' does not work here? Hint: if we transform x to $x-3$, there is no shrinking or stretching going on — but this is not always the case.

6. A function f is differentiable at a point x if the value $f'(x)$ is well-defined.
   (a) Give some examples of functions which are *not* differentiable at some point.
   (b) Can you define differentiability in a different way, using tangent lines?
   (c) Is it ever possible for a function to have a horizontal normal line at any point? Explain how your answer is related to the idea of differentiability.
7. Consider the hyperbola $y = 1/x$.
   (a) Explain why the hyperbola has no tangent line at $x = 0$.
   (b) Show that the tangent lines to the hyperbola at $(-1, -1)$ and $(1, 1)$ are parallel.
   (c) More generally, show that the tangent lines to the hyperbola at $(-x, -1/x)$ and $(x, 1/x)$ are always parallel.
   (d) Are there any points on the hyperbola which share a common normal line (not simply a normal line with the same slope, but the same line full stop)? What about tangent lines? Hint: yes, and no.
8. Finally, here are some functions and points to find tangent lines at. If there is no tangent line at the point given, carefully explain why. Draw some graphs out as well.
   (a) $y = 3x^2 + 3x + 1$ at $(0, 1)$.
   (b) $y = \sqrt{1 - x^2}$ at $(1, 0)$.
   (c) $y = 1/x^2$ at $(1, 1)$.
   (d) $y = 1/x^2$ at $(2, 1/4)$.
   (e) $y = \sqrt[4]{x^3}$ at $(2, \sqrt[4]{8})$.
   (f) $y = \sqrt[3]{x^2 + 2x + 1}$ at $(0, 1)$. (Hint: complete the square.)

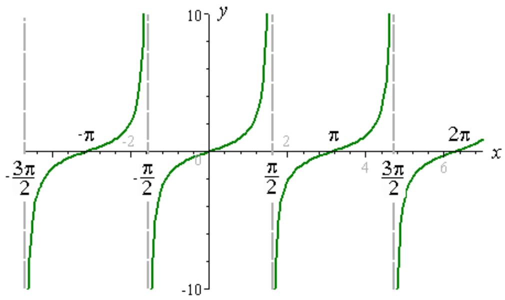
NCEA Level 2 Mathematics (Homework)
12. Tangent Lines and Approximations

Reading

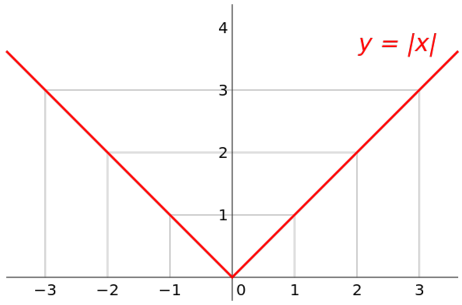
So far, we have only looked at 'nice' functions. However, it is possible to find functions which are not so nice, in the sense that the derivative does not exist at some point. What does this mean? It means that at some point, the graph of the function doesn't have a well-defined slope. For example, the function could become vertical (what is the slope of a vertical line?), or it could jump from one place to another without passing any of the points in between. Note that a function can be differentiable everywhere except one point, for example the absolute value function! Here are some examples of some non-differentiable functions.



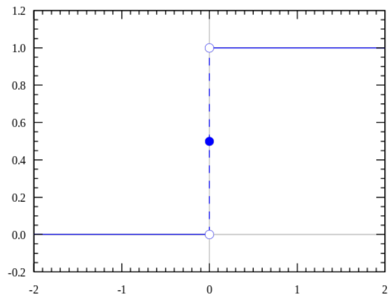
The circle is not differentiable at C or D because the tangent lines are vertical.



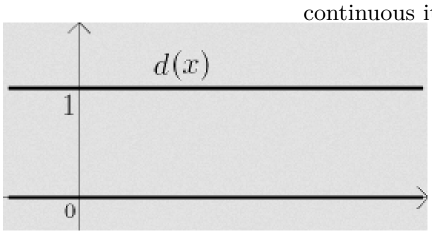
The tan function is not differentiable when the angle is an odd multiple of $\frac{\pi}{2}$ because the function is undefined there.



The absolute value function is not differentiable at $x = 0$ because it has no tangent line (try to draw one in!).



The Heaviside step function is the derivative of the absolute value (can you see this geometrically?) with $H(0) = \frac{1}{2}$, plugging the hole. However, because it is not continuous it is not differentiable at $x = 0$.



The Dirichlet function $d(x)$ takes the value 1 when x is irrational and 0 when x is rational, and so is continuous nowhere (there is a rational number between any two irrationals and vice versa, so the function jumps between 0 and 1 infinitely often). As you might expect, it is differentiable nowhere.

Questions

1. A function f is given by
$$f(x) = 2 - 4x + 5x^2 + ax^3. \tag{1}$$
The gradient of the graph at the point where $x = 1$ is 3. Find the value of a.

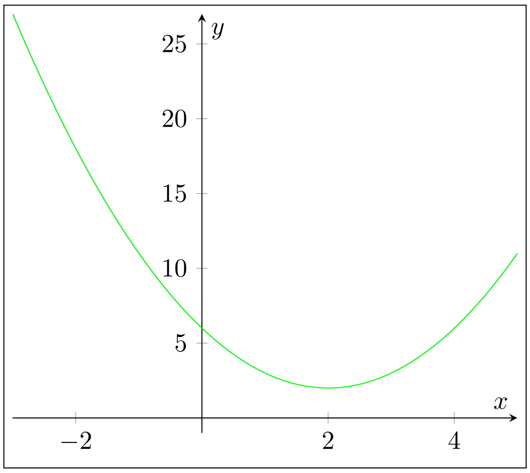
2. Give the equation of the tangent line to the curve
$$y = 3x^3 - \sqrt{x} + \frac{1}{x^3}$$
at the point $(1, 3)$.

3. Use calculus to show that the line $y = 15x - 12$ is tangent to the graph of the function $f(x) = 4x^2 - x + 4$.

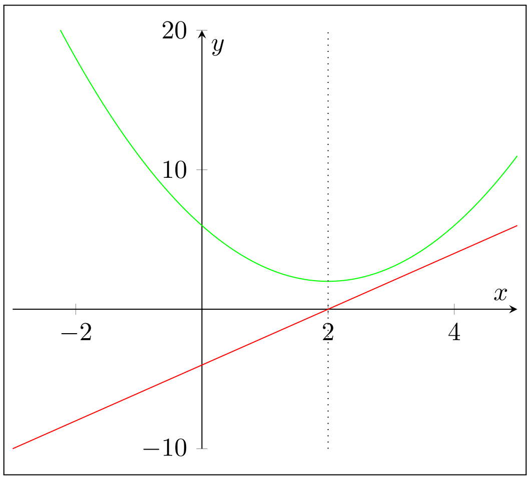
NCEA Level 2 Mathematics
13. Turning Points and Optimisation

Many problems reduce down to finding points where the graph of a function changes direction from increasing to decreasing, or vice-versa. These points are called turning points, or extreme points, of the graph. We can further classify turning points into local minima and local maxima.

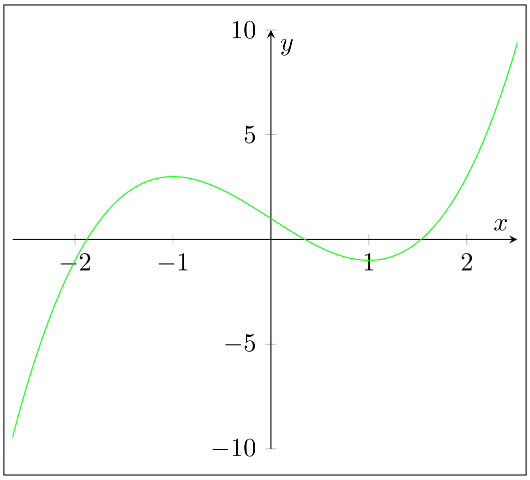
Example. Consider $y = (x-2)^2 + 2$, graphed here.



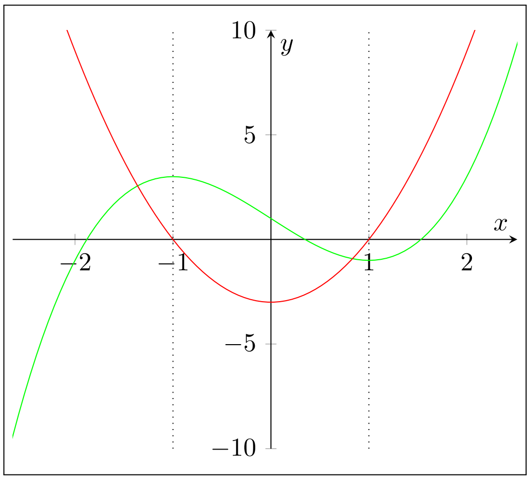
This function has a minimum at $(2, 2)$, because the function changes from decreasing to increasing there. We can rephrase this by saying that the derivative (in red) changes from negative to positive there, and in particular the derivative is exactly zero there.



Example. Consider $y = x^3 - 3x + 1$, graphed here.



This function has a minimum at $(1, -1)$, and a maximum at $(-1, 3)$. Again, if we plot the derivative $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 3$ in the same plot we see that the derivative is zero at the turning points.



Based on these two examples, we can state the following theorem (but again, we won't prove it here):

Theorem. *If the graph of $y = f(x)$ has a turning point (maximum or minimum) at $x = x_0$, then $f'(x_0) = 0$ or f is not differentiable at x_0.**

We can look at this from a tangent line perspective as well: at a maximum or a minimum, the graph is 'flat' and so the tangent line is horizontal there — so the slope of the tangent line and of the function is zero there.

On the other hand, note that $f'(x_0) = 0$ does not necessarily imply that f has a turning point at x_0. For example, consider $y = x^3$: this function has no turning points, but $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$ is zero at $(0, 0)$. Places where a function has a zero (or nonexistent) derivative are called critical points.

If the derivative at a critical point exists, but the point is not a maximum or a minimum, then the point is known as an inflection point. If we look at an entire interval, there may be multiple maxima or minima; the largest maximum is known as the global maximum of the function on the interval, and the smallest minimum is known as the global minimum of the function on that interval. Maxima and minima which are not global maxima or minima are known as local maxima or minima; a function might have no global maxima (take

*For an example of this second case, consider the graph of $y = |x|$.

$y = x^2$ for example), but on any given closed interval (i.e. an interval which includes its endpoints) it always attains a local maximum or minimum.

Example. Let us find the critical points of the graph of $y = 4x^3 - 6x^2 + 2x - 1$ and classify them. The critical points are precisely those places where the slope of the graph is zero — i.e. the roots of the derivative,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 - 12x + 2.$$

Setting the derivative to zero, we can use the quadratic formula to find the two roots are

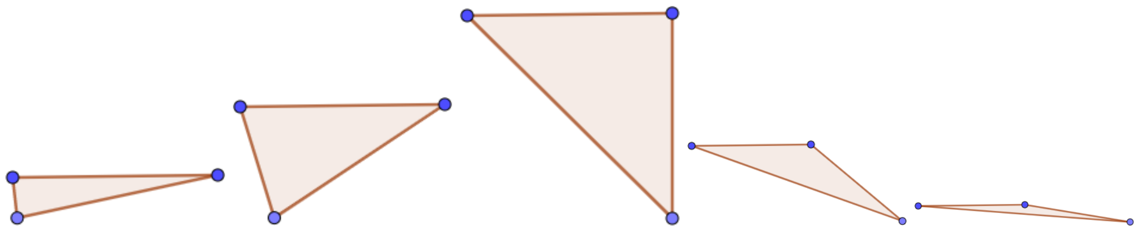
$$x = \frac{12 \pm \sqrt{144 - 4 \cdot 12 \cdot 2}}{2 \cdot 12} = \frac{12 \pm \sqrt{48}}{24} = \frac{1}{2} \pm \frac{\sqrt{12}}{6}.$$

Since the cubic has a positive leading coefficient, we know that it will go from $-\infty$ in the bottom left to $+\infty$ in the top right, and hence the two turning points will be a maximum and a minimum respectively.

Alternatively, to see this one could note that the derivative is positive to the left of the left-hand turning point, negative between the two, and positive again to the right of the right-hand turning point: so the graph changes from increasing, to decreasing, and back to increasing, and so we must have a maximum and a minimum in that order.

Let us look at one final geometric example.

Example. Suppose we have an isosceles triangle ABC, where $|AB| = |AC| = x$ and $|BC| = y$. Clearly if we spread the edges apart, the area increases and then decreases:



The area of the isosceles triangle is given by

$$A = \frac{1}{2} \times y \times \sqrt{x^2 - \frac{y^2}{4}}.$$

Now, we can't actually take the derivative of this function this year — but area is always positive, so the minimum of the area squared will be at the same place as the minimum of the area. Hence we want to minimise

$$A^2 = \frac{1}{4}y^2 \left(x^2 - \frac{y^2}{4}\right) = \frac{1}{4}x^2y^2 - \frac{1}{16}y^4$$

with respect to y (since x is fixed).

We have that $\frac{\mathrm{d}A^2}{\mathrm{d}y} = (1/2)x^2y - (1/4)y^3$; setting this to zero, we have that $x^2y = (1/2)y^3$. Since we know that $y \neq 0$ (that corresponds to a minimum, not a maximum) we can divide through, and so $x^2 = (1/2)y^2$ — so the minimum occurs when $y = x\sqrt{2}$.

Of course, this corresponds to the situation where the angle between the two equal sides is a right angle (for then $\sqrt{x^2 + x^2} = x\sqrt{2}$, as we just found is the case) — this is what we would expect from the symmetry of the triangle and from playing around with different pictures of an isoceles triangle. If we draw the triangle inside a circle, it is the halfway point between the two zero-area possibilities.

Questions

1. Consider the function graphed below.

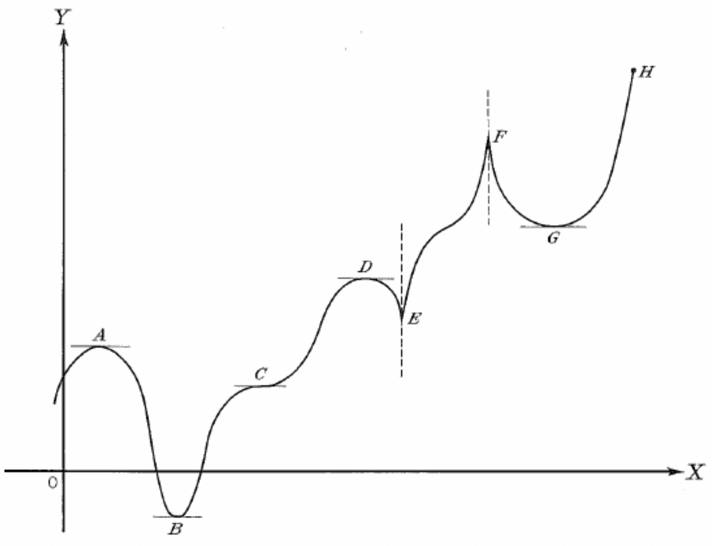


Figure 4-5

For each point A to H:

- Explain why the point is a critical point of the function by either giving the slope of the function at that point, or explaining why the derivative does not exist at that point.
- Label the point as an inflection point, a maximum, or a minimum.
- If the point is a maximum or a minimum, label it as either local or global.

2. Sketch graphs with the following properties:
   (a) A maximum at $x = 3$ and a minimum at $x = 6$.
   (b) No maximums or minimums, but critical points at $x = 2$ and $x = 4$.
   (c) Maximums at $x = 2$ and $x = 4$ but no minimums.

3. Find the maximum and minimum points of the function g defined by $g(x) = 2x^3 + x^2 + 2x$.

4. The function F, where $F(x) = x^{\frac{4}{5}}(x-4)^2$, has critical points at $x = 0$, $x = \frac{8}{7}$, and $x = 4$. Classify each one as a maximum, a minimum, or neither.

5. Show that the turning points of $y = x^4 - x^2$ are, alternately, minimum, maximum, and minimum.

6. Find the extreme values (if any) of the following functions of x:

   (a) $y = x^5$
   (b) $y = \frac{1}{x}$
   (c) $y = x^2 - 1$
   (d) $y = 2x^3 - 21x^2 + 72x + 18$
   (e) $f(x) = x^{10} - 4$
   (f) $y = \frac{1}{\sqrt{x}} + x^2$
   (g) $y = x^3 - x - 1$
   (h) $y = x^3 - x^2 + x - 1$
   (i) $f(x) = \frac{1}{x} + x - x^2$
   (j) $y = 16$
   (k) $y = \frac{x^{-2}+x^2}{2x}$
   (l) $x = \frac{y-2}{x+3}$

7. Find the maximum value of the derivative of $2x^2 - x^3$.

8. Prove that the function φ given by $\varphi(x) = \frac{x^{101}}{101} + \frac{x^{51}}{51} + x + 1$ has no extreme values.

9. Show that:
   (a) $y = x^4 + 3x^3 - x^2 - x + 20$ does not pass through the x-axis.
   (b) $x^3 - 5x + 100 = 0$ has only one real solution.
   (c) $x^3 - x^2 - x + 1$ has exactly two roots.

10. A farmer needs to create a rectangular field with a fence. He has 500 m of fencing available, and a building is on one side of the field (so that side does not need fencing). Determine the dimensions of the field to enclose the greatest area.

11. Suppose that a rectangle has perimeter p. Show that if the rectangle is to have the greatest possible area then it must be a square.

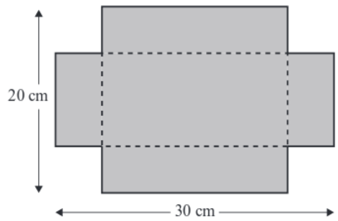
12. Find the maximum value of $y = -x^2 + 6x - 5$, both
    (a) *without* calculus; and
    (b) *with* calculus.

13. Find k so that $y = x^3 + kx^2 + x + 1$ has (a) two, (b) one, and (c) zero turning points.

14. The graph of $f(x) = x^3 + ax^2 + bx + 2$ has turning points when $x = -1$ and $x = 3$. Find the values of a and b.

15. Prove that the graph of $y = x^3(3 - x)$ has a local maximum when $x = \frac{9}{4}$. Carefully justify that the turning point *is* a local maximum.

16. Find the maximum volume of an open box (i.e. a box with base and sides, but no lid) that can be made from a rectangular piece of cardboard measuring 20 cm by 30 cm by removing the corner squares and folding along the dotted lines. Carefully justify that this is the maximum volume.



17. In an area surrounding a farming airstrip there is a height restriction for fireworks of 50 m. The height h metres above the ground reached by a firework t seconds after it is fired can be modelled by the function

$$h = 20t - 5t^2. \qquad (1)$$

    Will the firework break the 50 m limit?

18. By noting that the derivative changes from positive to negative at a maximum, and from negative to positive at a minimum, suggest a criteria for classifying turning points using the second derivative.

NCEA Level 2 Mathematics (Homework)
13. Turning Points and Optimisation

Reading

Go and watch...

`https://www.youtube.com/watch?v=F5RyVWI4Onk`

Polya's four-step approach to problem solving

1. *Preparation:* Understand the problem
   - Learn the necessary underlying mathematical concepts
   - Consider the terminology and notation used in the problem:
     (a) What sort of a problem is it?
     (b) What is being asked?
     (c) What do the terms mean?
     (d) Is there enough information or is more information needed?
     (e) What is known or unknown?
   - Rephrase the problem in your own words.
   - Write down specific examples of the conditions given in the problem.
2. *Thinking Time:* Devise a plan
   - You must start somewhere so try something.
   - How are you going to attack the problem?
   - Possible strategies: (i.e. reach into your bag of tricks.)
     (a) Draw pictures
     (b) Use a variable and choose helpful names for variables or unknowns.
     (c) Be systematic.
     (d) Solve a simpler version of the problem.
     (e) Guess and check. Trial and error. Guess and test. (Guessing is OK as long as you can back it up later.)
     (f) Look for a pattern or patterns.
     (g) Make a list.
   - Once you understand what the problem is, if you are stumped or stuck, set the problem aside for a while. Your subconscious mind may keep working on it.
   - Moving on to think about other things may help you stay relaxed, flexible, and creative rather than becoming tense, frustrated, and forced in your efforts to solve the problem.
3. *Insight:* Carry out the plan
   - Once you have an idea for a new approach, jot it down immediately. When you have time, try it out and see if it leads to a solution.
   - If the plan does not seem to be working, then start over and try another approach. Often the first approach does not work. Do not worry, just because an approach does not work, it does not mean you did it wrong. You actually accomplished something, knowing a way does not work is part of the process of elimination.

- Once you have thought about a problem or returned to it enough times, you will often have a flash of insight: a new idea to try or a new perspective on how to approach solving the problem.
- The key is to ***keep trying until something works***.

4. *Verification:* Look back

- Once you have a potential solution, check to see if it works.
  (a) Dd you answer the question?
  (b) Is your result reasonable?
  (c) Double check to make sure that all of the conditions related to the problem are satisfied.
  (d) Double check any computations involved in finding your solution.
- If you find that your solution does not work, there may only be a simple mistake. Try to fix or modify your current attempt before scrapping it. Remember what you tried — it is likely that at least part of it will end up being useful.
- Is there another way of doing the problem which may be simpler? (You need to become flexible in your thinking. There usually is not one right way.)
- Can the problem or method be generalized so as to be useful for future problems?

Questions

1. A cylindrical tube (with open ends) is to be made from a sheet of paper with area $25\,\mathrm{cm}^2$. What should the dimensions of the tube be in order to maximise the volume of the tube? Justify that you have found a maximum.

2. A function f is given by $f(x) = 2x^3 - 3ax^2 + 6bx - 2$. The function has two turning points, at $x = 2$ and at $x = 3$. What are the values of a and b?

3. Suppose a wire of length ℓ is cut into two pieces, one of length x and one of length $\ell - x$. One piece is used to form the circumference of a circle, and the other is used to form the perimeter of a square. How long should the length x be in order to ensure that the total area of the circle and the square is minimised? Carefully justify that you have found a minimum.

NCEA Level 2 Mathematics
14. Anti-differentiation

The final mathematical topic that we will look at with regard to calculus is the inverse operation to differentiation: given a slope function (a derivative), we will find the original function.

Example. Suppose that we know that the derivative of some function is the function f' given by $f'(x) = 3x^2 + 2x+1$. By reversing the power rule for derivatives, we can see that if we take f_0 to be $f_0(x) = x^3+x^2+x$ then we get the desired derivative (i.e. $f_0' = f'$). On the other hand, any function f_C such that $f_C(x) = x^3+x^2+x+C$ also has the same derivative.

From this example, we make two main observations:

Observation 1 (Inverse power rule)**.** *If we know a differentiation rule, like $ax^r \mapsto arx^{r-1}$, then we can reverse it and say that if the derivative looks like ax^r then the original function looks like $\frac{a}{r+1}x^{r+1}$.*

Observation 2. *Given any derivative f', we have infinitely many functions that have f' as their derivative: if one is f, then $f + C$ for any constant C also has f' as its derivative, because*

$$(f + C)' = f' + C' = f' + 0 = f'.$$

The set of antiderivatives of some function f' is denoted by

$$\int f'(x)\,\mathrm{d}x = f(x) + C.$$

The notation is unfortunate, but the best way to view it this year is as a pair of brackets: $\int$ and $\mathrm{d}x$. The mathematical operation of taking an antiderivative is called indefinite integration — if taking a function and finding its slope is splitting the function up into infinitely many pieces, then taking the antiderivative is packing all those infinitely many pieces back together into one integrated whole.

Example. Suppose that it is known that the function f

- has a derivative $f'(x) = 3x^5 + 4x^3 + 2x + 1$, and
- has a graph which passes through the point $(2, 1)$.

Then we know that all the possible candidates for f are given by

$$\int 3x^5 + 4x^3 + 2x + 1\,\mathrm{d}x = \frac{3}{6}x^6 + \frac{4}{4}x^4 + \frac{2}{2}x^2 + \frac{1}{1}x^1 + C = \frac{x^6}{2} + x^4 + x^2 + x + C.$$

We also know that $f(2) = 1$, so:

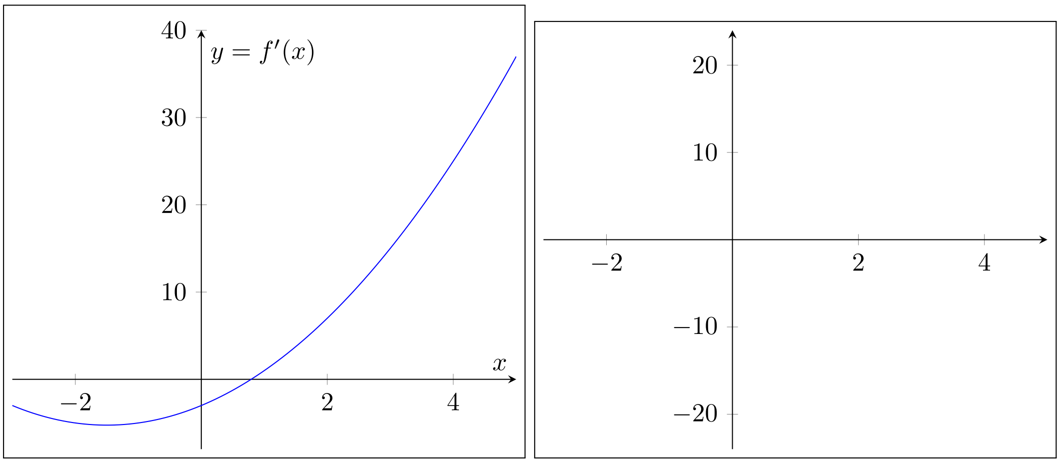
$$\begin{gathered} 1 = \frac{2^6}{2} + 2^4 + 2^2 + 2 + C \\ 1 = 2^5 + 2^4 + 2^2 + 2 + C = 32 + 16 + 4 + 2 + C = 54 + C \\ C = 1 - 54 = -53 \end{gathered}$$

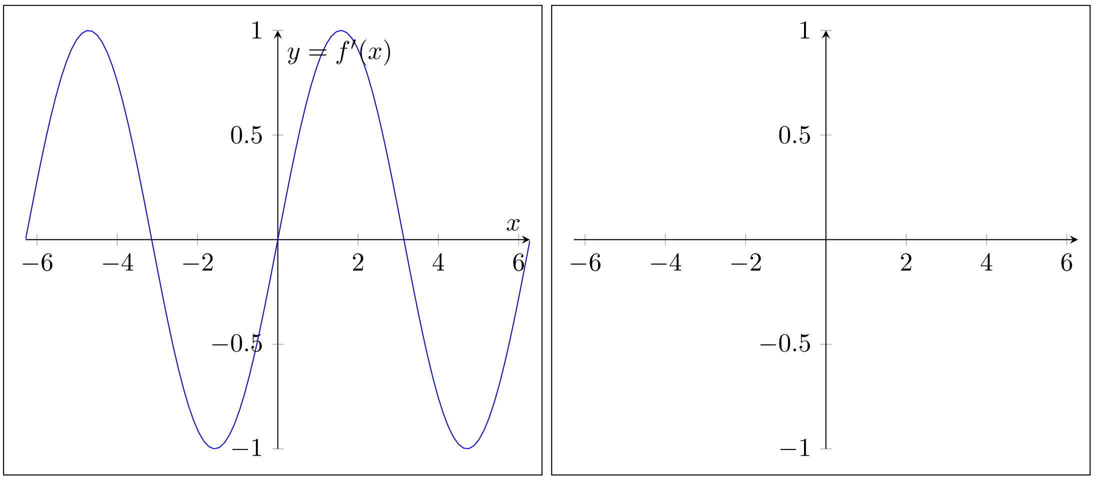
and hence $f(x) = \frac{x^6}{2} + x^4 + x^2 + x - 53$.

There is not much more to say about integration at this stage, because at Level 2 the geometric meaning of integration is no longer examinable. Suffice it to say, the operation of integration (as you will learn next year) is far deeper and more interesting than it first appears. The final problems in this week's problem set are an indication of this.

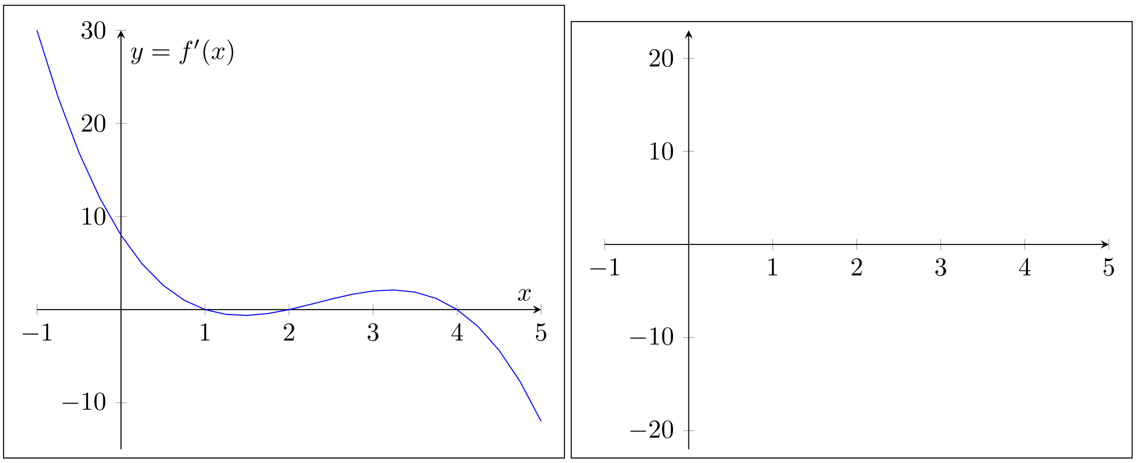
Questions

1. Find three functions that have derivatives equal to $x^2 - x$.
2. Find an antiderivative of $f(x) = 25x^4 + 12x^3 - x^{-2}$.
3. Evaluate $\int 12z^3 + 18z^{-4}\,\mathrm{d}z$.
4. Show that $x^3 + 3x + C$ is an antiderivative of $3x^2 + 3$.
5. Show that
$$\int 470x^4 + 2x + 1 + 6x^{-3}\,\mathrm{d}x = 94x^5 + x^2 + x - 3x^{-2} + C. \qquad (1)$$
6. Evaluate the following indefinite integral:
$$\int -x^6 - \frac{1}{3\sqrt{x}} + \frac{x^{19}}{47} - \frac{2}{x^{-\frac{4}{5}}}\,\mathrm{d}x$$
7. Given the slope functions below, draw the antiderivative of each which passes through $(0,0)$.

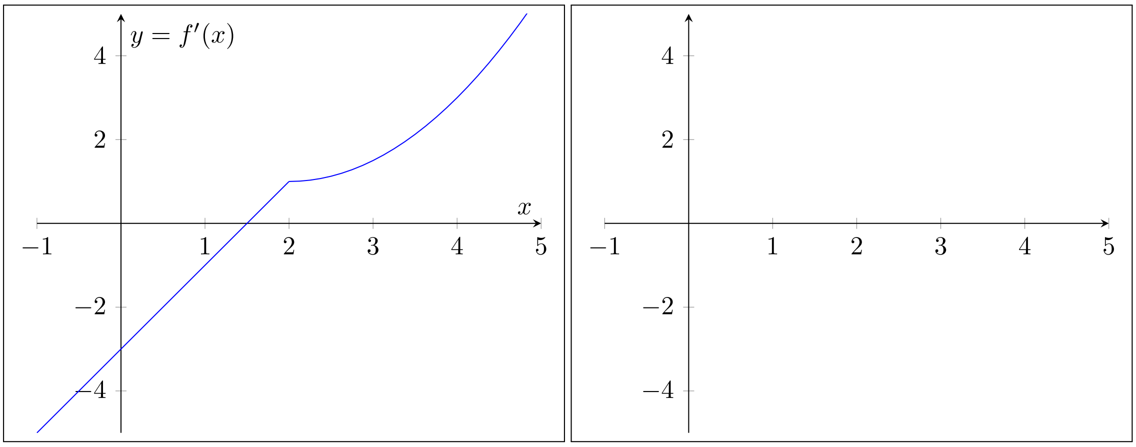
(a)


(b)


(c)



(d)



8. (a) By drawing graphs, show that it is plausible that $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$. (You may assume for the remainder of this question that this derivative is correct.)
   (b) Find all possible functions ψ such that

$$\psi'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

   (c) Suppose that we know that $\psi(0) = -8$. Find ψ.

9. Find g if $g'(x) = x\sqrt{x}$ and $g(1) = 2$.

10. Suppose that θ is a function of x such that

$$\theta'(x) = 8x^3 + 3x^2 + ax,$$

where a is a constant. Given that $\theta(0) = 9$ and $\theta(-1) = 14$, find θ and a.

11. Find $f(x)$ if $f''(x) = -2 + 12x - 12x^2$, $f(0) = 4$, and $f'(0) = 12$.

12. Suppose that f is a function of x given by $f(x) = x^3 - Ax^2 + 3x - B$, where A and B are real constants, which passes through $(0, -4)$ and has a critical point at $x = 1$. Find $f(x)$ exactly.

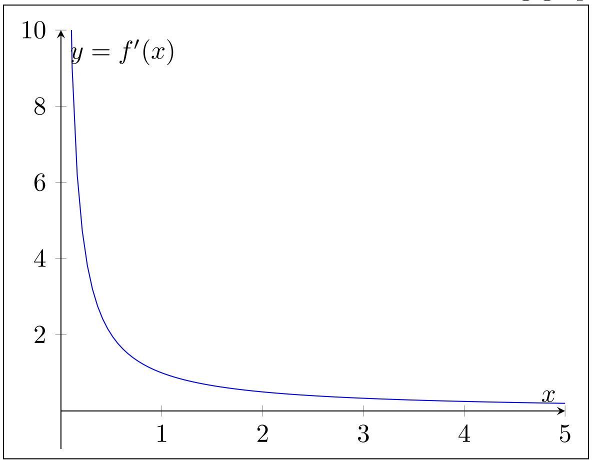
13. Suppose that y is a function of x given by

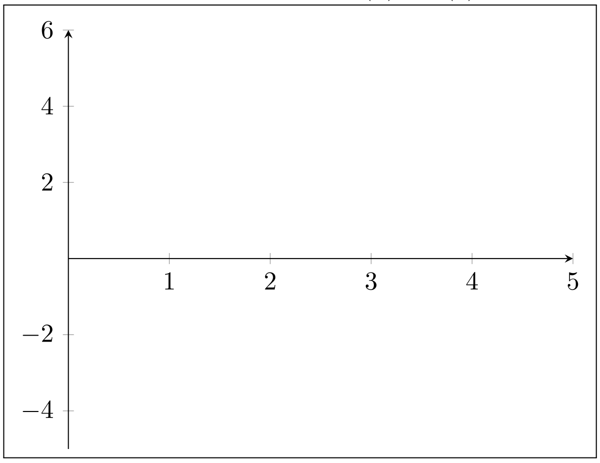
$$y = x^3 + Bx^2 + Cx + 1,$$

and that the graph of y has a minimum at $(3, 0)$.

Find B and C.

14. Every function which is made up of bits of the form ax^r added up can be differentiated using the power rule. However, not every function of that form has an anti-derivative that can be calculated using the inverse power rule.
    (a) Show that no such function f differentiates to give $f'(x) = \frac{1}{x}$. (Hint: try to integrate using the reverse power rule).
    (b) It is a consequence of a theorem of analysis that a function f exists so that $f'(x) = \frac{1}{x}$, even though it is not of the form above. Given the following graph of f', draw the graph of $y = f(x)$ if $f(1) = 0$.





    (c) The function which you graphed in part (b) is given the special name ln; so you have drawn the graph of $y = \ln(x)$. Let us consider the inverse function of ln, which we will call nl for the time being (because it's like ln backwards). Given the rule that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$, we can find the derivative of nl from the derivative of ln.
        i. Write $x = \ln y$, and give $\frac{\mathrm{d}x}{\mathrm{d}y}$.
        ii. Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y$.
        iii. Conclude that, since $x = \ln y \iff y = \mathrm{nl}\, x$, it must be the case that $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{nl}\, x = \mathrm{nl}\, x$.
    (d) So nl is its own derivative — a situation we already looked at! Clearly nl is not the zero function; we saw that there was only one other function which was its own derivative. What was it? (Hint: check the end of sheet 11.)
    (e) If ln is the inverse of the type of function that you found nl to be, what kind of function is it?
    (f) The letters ln stand for *logarithme naturel* (French). Why do you think this is?

15. (Advertisement for Level 3!)
    (a) Draw the line $y = 2t + 1$ and use geometry to find the area under this line, above the t-axis, and between the vertical lines $t = 1$ and $t = 3$.
    (b) If $x > 1$, let $A(x)$ be the area of the region that lies under the line $y = 2t + 1$ between $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$.
    (c) Find $A'(x)$. What do you notice?

NCEA Level 2 Mathematics (Homework)
14. Anti-differentiation

Reading

Go and watch...

`https://www.youtube.com/watch?v=j4hW7AwETZA`

What's it good for?

As I kind of hinted in the notes and in the final problem in the problemset, the geometric meaning of integration is to do with area. In fact, integration is the general way to find volumes, lengths, areas, and in general any kind of extent in space. People use integration for...

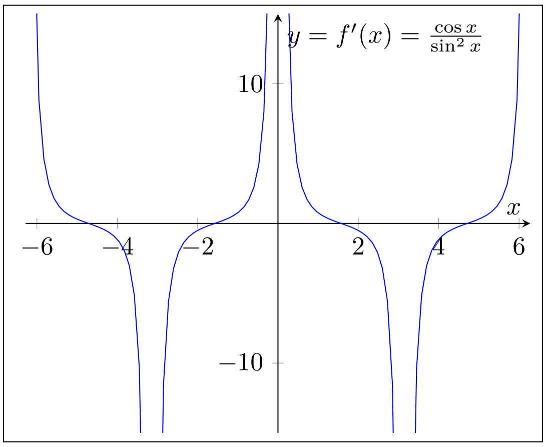
- The sciences and engineering: integration, both in its guise as "undoing slope-finding" and in its guise as an area-finding device, is heavily used in all the sciences (either explicitly, as in physics, or implicitly, as in chemistry and biology). Differential equations, equations involving functions and their derivatives, are also often found in these subjects as rates of change play an important role in engineering and in science; integrals are used to solve differential equations. In physics especially, multi-dimensional integrals and derivatives play an important role in the theories of the universe.

- Mathematics: the study of integrals can be done on two levels — as "recreational mathematics", where people try to solve hard integrals for fun (like solving a sudoku puzzle or a crossword), or as a deep subject known as measure theory — the theory of functions which measure things, and the different varieties of integrals that apply those functions onto space.

Questions

1. The following graph shows $f'(x)$, the derivative of a function f. Use the graph of the derivative to recreate the graph of the original function.

   Hint: Where must the original function be decreasing or increasing? Where will it have maximums and minimums? How fast does it change?

2. It is known that Φ is a function such that:

   - Φ has a stationary point at $x = 0$.
   - $\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} = 60x^4 - 180x^2 + 48$.
   - $\Phi(2) = 0$.

   (a) Describe the nature of the stationary point at $x = 0$.

   (b) Find the locations of the other stationary points.

   (c) Find an exact expression for $\Phi(x)$.

NCEA Level 2 Mathematics
15. Kinematics and Rates of Change

The original development of calculus was as a tool for physicists to describe motion. The application itself is entirely natural, and there is no new mathematics involved, so this week should be nice and easy.

Let $x(t)$ be a function describing the position of some object at some time t. Then the velocity of the object, measured by the rate of change of position, is just $v(t) = x'(t)$. The acceleration of the object, which is just the rate of change of velocity, is $a(t) = v'(t)$. This is summed up in the following table:

Displacement, x		$\int v(t)\,\mathrm{d}t$
Velocity, v	$\frac{\mathrm{d}x}{\mathrm{d}t}$	$\int a(t)\,\mathrm{d}t$
Acceleration, a	$\frac{\mathrm{d}v}{\mathrm{d}t}$	

There is only one fundamental concept in this topic that you must remember: the derivative is just a rate of change. Velocity is rate of change of position, and acceleration is rate of change of velocity. If you slow down faster, your acceleration is more negative.

Do not try to memorise the above table, understand what it is saying physically.

Note that if you are taking physics, physicists sometimes have an annoying habit of writing $\dot{x}$ instead of $\frac{\mathrm{d}x}{\mathrm{d}t}$; so dotted variables may indicate time derivatives. While this notation is horrible and ugly, it is often forced upon us. I will not use it myself, and you should not expect to ever see it in a mathematics paper again; but be aware that this notation does exist.

A note about L3 physics

This topic appears in L2 mathematics for two main reasons: firstly, as an easy historical application of calculus; and secondly, so that you are well-prepared for level three physics. While calculus is not actually required for physics until university, a solid understanding of this year's calculus topics will enable you to make connections next year that may be otherwise obscure. If you are planning to do scholarship physics next year (and I urge you to consider it if you enjoy physics), then knowledge of calculus is a definite advantage as it can simplify some of the problems!

Questions

All distances are given in m, and all times in s, unless otherwise stated.

1. If a ball is thrown into the air with a velocity of $10\,\mathrm{m\,s^{-1}}$, its height y in metres after t seconds is given by
$$y = 10t - 4.9t^2. \tag{1}$$
Find the vertical velocity of the ball when $t = 2$.

2. The velocity v $\mathrm{m\,s^{-1}}$ of an object t seconds after it passes a fixed point can be modelled by the function
$$v(t) = 4t^3 - t^2 + 2t.$$
Find the equation for the acceleration of the object.

3. A tank is being filled with water. The height of the water, $h$cm, in the tank at any time t minutes after it began filling is given by $h = t^2 + 2t$. Find the rate that the height of the water is changing at three minutes after the tank begins to fill.

4. A balloon has an initial volume $5\,\mathrm{cm}^3$, and is being inflated at a rate given by $\frac{\mathrm{d}V}{\mathrm{d}t} = 4t$, where V is the volume of the balloon in cubic centimetres and t is the time in seconds since the balloon began to inflate. Give the volume of the balloon after ten seconds.

5. A projectile follows a path through space modelled by $y = 4x - x^2$. At what distance along the ground is it at its maximum height, and what is that height?

6. The distance, s, of a moving point from a given point P at a time t is given by

$$s(t) = \frac{1}{3}t^3 - 2t^2 - 12t.$$

   Fully describe the acceleration of the object beginning from time $t = 0$.

7. A car begins to roll slowly down a sloping driveway, with velocity given by $v = 0.4t\,\mathrm{m\,s^{-1}}$ (where t is the time from the beginning of the car's roll). It is five seconds until someone notices the movement; in this time, how far does the car travel?

8. A child moves a Buzzy Bee™ toy forwards and backwards along a straight line. At time t, where $0 \leq t \leq 10$, the toy's position is modelled by $x = 3t - 1.3t^2 + 0.1t^3$.
   (a) At which time(s) is the toy stationary?
   (b) What is the acceleration of the toy at $t = 3$?
   (c) What displacement is the toy from the origin when the velocity of the toy is most negative?

9. The area of a circle varies if the radius r of the circle is changed. In this way, the area A of the circle is a function of r given by $A(r) = \pi r^2$.
   (a) What is the rate of change of the area of the circle with respect to the radius?
   (b) A piece of computer graphics software is drawing a circle so that the radius changes with time at a rate $\frac{\mathrm{d}r}{\mathrm{d}t} = 3t$. The initial radius of the circle is $2\,\mathrm{cm}$.
      i. Give the radius of the circle after three seconds.
      ii. After how long will the area of the circle reach $10\,\mathrm{cm}^2$?

10. A particle is moving through space along an axis. Its displacement from the origin at any time $t > 0$ is given by $s(t) = t^5 - 38t^4 + 560t^3 - 3982t^2 + 13599t - 17820$.
    (a) Find an expression for the velocity of the particle at time t, $v(t)$.
    (b) At what time is the particle moving with the most speed towards the origin, and how fast will it be moving at that time?
    (c) What is the acceleration of the particle at that time?
    (d) How many times does the particle change direction after $t = 0$?

11. An object is moving on a straight line; the point P lies on this line. Initially, the object is at point P and has a velocity of $4\,\mathrm{m\,s^{-1}}$. The acceleration of the object at a time t seconds after it leaves P is given by the function $a(t) = 2 - 6t$. How far from P is the object after three seconds?

12. Look at the following table of trigonometric derivatives.

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$

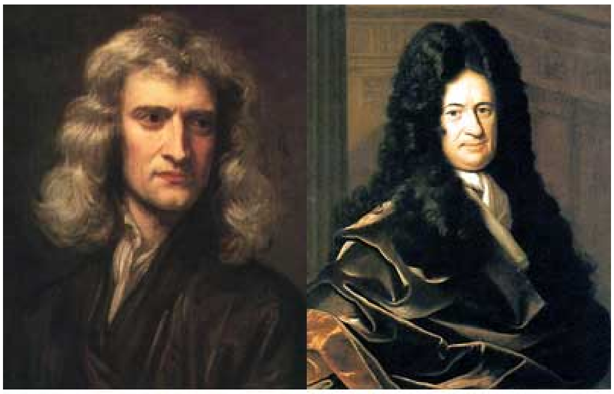
    A spring oscillates such that the position of its end after a length of time t is given by $x = 2\sin(t)$. What is the approximate acceleration of its end at $t = 5$?

NCEA Level 2 Mathematics (Homework)
15. Kinematics and Rates of Change

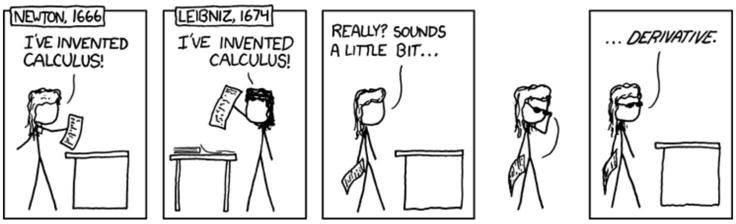
Reading

Go and watch...

https://www.youtube.com/watch?v=pI62ANEGK6Q



Sir Isaac Newton (left) and Gottfried Wilhelm von Leibniz (right)



https://xkcd.com/626/

The ancient Greeks used some of the ideas of calculus to calculate areas and volumes of objects: for example, Archimedes (c. 287 – 212 BC) used a limiting process to calculate the area of a circle. The Greeks also considered some of the philosophical problems with these kinds of limiting processes, some of which were not solved until the time of Cauchy in the 18th and 19th centuries.

The modern development of calculus is usually credited to Gottfried Leibniz (1946 – 1716) and Isaac Newton (1642 – 1727), who independently developed coherent theories of calculus at the turn of the 18th century. Both claimed scientific priority: Newton claimed to have begun development of his theory in 1666 as part of his investigations into the laws of motion, but did not publish it until after Leibniz began publishing in 1684.

Some argue that the resulting fallout between the followers of the Englishman, Newton, and the followers of the German, Leibniz, set English mathematics back by decades compared to Europe.

Questions

All distances are given in m, and all times in s, unless otherwise stated.

1. A particle moves in space along a single axis, with velocity function $v(t) = t^2 + t - 12$ (where t is measured from some arbitrary starting point).
   (a) What is the acceleration of the particle at $t = 10\,\mathrm{s}$?
   (b) The particle is closest to the origin at $t = 3\,\mathrm{s}$.
      i. By considering $v(t)$, show that $t = 3$ is indeed a turning point for the graph of the position function $x(t)$ of the particle.
      ii. If the minimum distance between the particle and the origin is $300\,\mathrm{m}$, calculate the distance from the particle to the origin at $t = 10\,\mathrm{s}$.
2. A cubic equation is a polynomial of degree three — that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. Recall that a critical point is a point x where $f'(x) = 0$, or $f'(x)$ is undefined.
   i. Sketch examples of a cubic function with zero, one, and two critical points.
   ii. Prove that a cubic function can have a maximum of two critical points.

Chapter 5

Counting

NCEA Level 2 Mathematics
16. Counting and Combinatorics

We move from calculus, the study of the continuous, to combinatorics, the study of the discrete. We begin with a simple question: how many ways are there of picking a committee of three people, given a group of six people? Our first attempt at this is relatively naive: there are six choices for the first person in the committee, five for the second, and four for the third — so $6 \times 5 \times 4 = 120$ altogether. The problem with this is that we have overcounted: suppose our six people are labelled from A to F; then we have (for example) counted ABC and BAC separately (because in one we picked A first, and in the other we picked B first) despite the simple fact that they are the same committee!

In order to solve this problem, we need to count cleverer. Because the solution in this case works in general, we'll work it out in general to save time. So suppose we have a group of n people, and we want to pick a committee of r of them. Then, if we count different orderings separately, we obtain

$$n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

different committees (where the notation $n!$, read n bang or n factorial, denotes $n(n-1)\cdots 3 \cdot 2 \cdot 1$). This is the problem we solved above (and is called the number of permutations of r things out of n in total, or ${}^nP_r$); we just need to divide through by the number of times we counted each group. There are $r!$ different orderings for a set of r people, and so we counted each committee of r people $r!$ times in total — once for each ordering. Hence the total number of committees possible, up to ordering, is

$$\frac{n!}{(n-r)!r!} = \binom{n}{r} = {}^nC_r$$

which is read 'n choose r'. We have therefore proved:

Theorem. *The number of ways to choose r objects from a set of n, if we don't care about order, is given by $\binom{n}{r}$. If we care about order, the number of choices for the smaller set is given by ${}^nP_r$.*

There is an interesting pattern that we can make with these 'choice constants', known as *Pascal's triangle* after French mathematician Blaise Pascal.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
\cdots
\end{array}$$

The row number (starting from zero) is n, and the column number (from the left and starting from zero) is r. Let's look at some of the patterns here.

Observation 1. *The first and last column of each row is 1. In our counting notation, this is the observation that $\binom{n}{0} = \binom{n}{n} = 1$ for all sets of size n.*

Proof. We need to check that each observation is true for the whole table, not just the portion I gave above. This proof is left to you! □

Observation 2. *The second and $r-1$th column of each row is just n (except the first). In our counting notation, this is the observation that $\binom{n}{1} = \binom{n}{n-1} = n$.*

Proof. We need to check that each observation is true for the whole table, not just the portion I gave above. This proof is left to you! This one can be done either by using the factorial formula, or by counting. □

Observation 3. *The triangle is symmetric about the middle axis. In our counting notation, this is the observation that* $\binom{n}{r} = \binom{n}{n-r}$.

Proof 1: less satisfying.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$$

□

Proof 2: counting. Suppose we pick a subset of size r out of our set of size n. This also uniquely identifies a subset of size $n-r$: the elements we didn't take! Since there is a one-to-one correspondence between picking sets of size r and sets of size $n-r$, the number of choices for both sizes of subset must be the same. □

Observation 4. *Each number is given by the sum of the two numbers diagonally above from it. In our counting notation, this is the observation that* $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$.

Proof. Pick some object x in our larger set of size n. Then we are left with some set of size $n-1$ left over. If we want to pick a subset of size r from the set of size n, then we have two cases: either the subset we pick contains x, or it does not. If it contains x, we need to pick another $r-1$ things from the remaining $n-1$ in order to fill our set up; if it does not, then we need to pick up a full r things from the remaining $n-1$. In the first case, there are $\binom{n-1}{r-1}$ choices; in the second case, there are $\binom{n-1}{r}$ choices. □

Observation 5. *The sum of all the numbers in a row is* 2^n*:*

$$\begin{aligned}
2^0 &= 1 \\
2^1 &= 1+1 \\
2^2 &= 1+2+1 \\
2^3 &= 1+3+3+1 \\
2^4 &= 1+4+6+4+1 \\
2^5 &= 1+5+10+10+5+1
\end{aligned}$$

Proof. The sum of all the numbers in a row is the total number of ways we can choose subsets of size zero, plus subsets of size one, all the way up to subsets of size n: in short, the sum is just the total number of subsets of the larger set. But there are 2^n different subsets of the larger set (for each element, we either keep it or not: n choices, with two possible outcomes for each). □

I will only give one application of Pascal's triangle and the counting numbers here, but rest assured there are many others.

Example. Recognise this pattern?

1. $(x+y)^1 = \mathbf{1}x + \mathbf{1}y$
2. $(x+y)^2 = \mathbf{1}x^2 + \mathbf{2}xy + \mathbf{1}y^2$
3. $(x+y)^3 = \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3$
4. $(x+y)^4 = \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4$

Theorem (Binomial theorem)**.**

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

Proof. Recall how expanding binomials work: we write

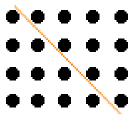
$$(x+y)^n = (x+y)(x+y)\cdots(x+y)$$

and then expand using the distributive law. Each term in the result consists of either x or y from the first bracket, either x or y from the second, and so on until we pick either x or y from the final bracket; so each term contains exactly n x's and y's, and there is one term for each possible combination of picking things from brackets. In fact, there will be $\binom{n}{r}$ terms with r x's — because that is the number of different ways to pick r x's out of the n brackets. The remaining $n-r$ things we picked for each of those terms must be y's; and we when we add up the terms where we pick 0 x's, 1 x, 2 x's, and so on, we obtain the desired result. □

Because of this proof, the counting coefficients $\binom{n}{r}$ are more properly called *binomial coefficients*.

Questions

1. Given that $n!$ is the number of orderings of n objects, explain why $0! = 1$ is a reasonable definition.
2. I have a bucket of 10 numbered balls, and a bucket of 10 green balls that all look the same.
   (a) Suppose I pick a set of five balls from each. How many different ways are there of doing this if I count two sets the same when I can't tell them apart?
   (b) How many different ways are there of picking a set of ten balls, but this time I pick randomly from both buckets?
3. How many words of exactly ten characters are possible with an alphabet of 26 characters if:
   (a) Repetitions are allowed.
   (b) Repetitions are not allowed at all.
   (c) Repetitions are allowed, but adjacent characters must be different.
4. Repeat question 3, but now for words of ten characters or less.
5. Verify the binomial theorem for $(2+3)^4$, by both doing the exponent and by expanding out. (This allows us to split a large exponential operation into several smaller ones.)
6. (a) Add up $1+2+\cdots+99+100$.
   (b) Add up $1+2+\cdots+(n-1)+n$. How many different proofs of the resulting formula can you think of? Hint:



7. Justify observations 1 and 2 by counting cleverly.
8. Give another proof for observation 5 by expanding $(1+1)^n$ using the binomial theorem.
9. Prove that, if we alternately add $+$'s and $-$'s between the members of each row of Pascal's triangle, then the resulting sum is zero. For example, $1-4+6-4+1=0$. (Hint: $(1-1)^n = 0$.) Rephrase this in terms of binomial coefficients.
10. A company once had a "matching picture" contest in which the object was to match the pictures of four celebrities with their baby pictures. Contestents were able to enter as many times as they wished, and the first prize was \$10,000.
    (a) How many different entries would you need to send in to be sure of having all four pictures matched correctly?

(b) Does it seem reasonable to think that you would win \$10,000 if you sent in an entry with all the pictures correctly matched?

11. Let us calculate the number of ways there are to pick a team of seven out of a pool of twenty people, such that one of the members of the team is captain.
    (a) We can do this two ways; calculate both counts, and check they agree.
        i. Pick one captain out of twenty, and then six remaining players out of the 19 left.
        ii. Pick seven players out of twenty, and then one captain out of the seven.
    (b) Prove the following generalisation in two different ways:

$$\binom{n}{b}\binom{n-b}{c} = \binom{n}{b+c}\binom{b+c}{b}$$

        i. As in (a), double-count the number of ways to pick a team of $b+c$ players from n, where b players are 'special'.
        ii. Algebraically, with the factorial definition of binomial coefficients.
    (c) Notice that the actual quantity in part (b)i is the number ways of picking *three* subsets, of size x, y, and z, from a set of n, and the subsets *partition* the set: use up all the elements without overcounting, so $x+y+z=n$.
        i. Show that the number of ways to partition a set of size n using m smaller subsets, with sizes $r_1, r_2, ..., r_m$, is given by the *multinomial coefficient*

$$\left\{ \begin{matrix} n \\ r_1, r_2, \ldots, r_m \end{matrix} \right\} = \frac{n!}{r_1! r_2! \cdots r_m!}.$$

        Hint: order n, then chop off the first r_1 elements, then the next r_2 elements, and so on, taking account of ordering.
        ii. Show that the binomial coefficients are special cases of the multinomial coefficients, but that

$$\binom{n}{r} \neq \left\{ \begin{matrix} n \\ r \end{matrix} \right\}$$

        in general (hence why I use different brackets).

12. We have defined the number e to be $\lim_n \left(1+\frac{1}{n}\right)^n$.
    (a) Show that the ith term in the expansion of $\left(1+\frac{1}{n}\right)^n$ for finite n is given by

$$\frac{n!}{(n-i!)i!n^i}.$$

    (b) Suppose we hold i fixed, and send n off to infinity. Show that

$$\frac{n!}{(n-i)!} = n(n-1)(n-2)\cdots(n-i+1)$$

    for all i, and conclude that

$$\frac{n!}{(n-i)!n^i}$$

    tends towards 1. (Hint: as n tends to infinity, but x remains finite, $n-x \approx n$.)
    (c) Hence, by expanding $\left(1+\frac{1}{n}\right)^n$ using the binomial theorem, show that

$$e = \lim_n \left(1+\frac{1}{n}\right)^n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{i!} + \cdots,$$

    and check that the two values roughly agree for $n = 10$.

NCEA Level 2 Mathematics (Homework)
16. Counting and Combinatorics

Reading

Go and watch...

`https://www.youtube.com/watch?v=aSsCU0mT-Bk`

What's it good for?

People use combinatorics for...

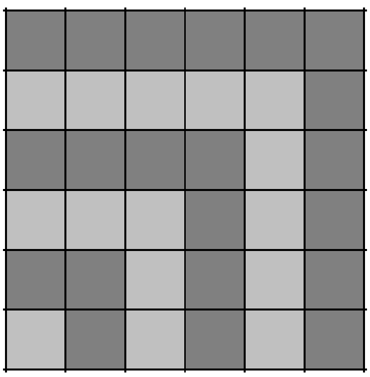
- Computer science: combinatorics (both the counting we did this week, and the work we'll do over the next few weeks) is a foundation of computer science as it allows the efficiency of various algorithms to be measured.
- Statistics and probability: if you want to measure probabilities, you need to be able to count how many ways each possibility can occur!
- Mathematics: combinatorics is one of the fastest-growing areas of modern mathematics, as many modern problems are phrased in terms of the discrete (that is, individual pieces and finite sets) rather than the continuous (infinite sets, as studied in calculus).

Questions

1. How many different ways are there to pick seven numbered tennis balls from a bucket of ten, and then order them?
2. Check that

$$\begin{gathered} 1 = 1 \\ 1 + 3 = 4 \\ 1 + 3 + 5 = 9 \\ 1 + 3 + 5 + 7 = 16 \\ 1 + 3 + 5 + 7 + 9 = 25 \\ \cdots \end{gathered}$$

Generalise (i.e. state and prove some result for all sums of odd numbers). Hint:



3. The first few powers of 11 are rows of Pascal's triangle (11, 121, 1331, ...). Is this a coincidence? (Hint: no.)

4. Extension question. If you have n pieces of spaghetti (each arbitrarily long), and you cross them such that every piece crosses every other piece exactly once, how many crossing points are there?

NCEA Level 2 Mathematics
17. Number Sequences and Fractals

A number sequence is an arrangement of numbers in which each successive number follows the last according to a uniform rule. More precisely, a number sequence is a correspondence between the counting numbers and a set of numbers: $(a_n) = a_1, a_2, a_3, \ldots$.

Arithmetic Sequences

Arithmetic sequences are the simplest kind of interesting sequence.

Definition. An arithmetic sequence is a number sequence in which each successive term may be found by adding the same number; formally, a sequence is arithmetic if $a_{n+1} = a_n + k$ for every $n > 1$ and for some constant k.

Example. In the following sequence, $a_1 = 2$ and $a_{n+1} = a_n + 3$.

$$2, 5, 8, 14, 17, \cdots$$

For arithmetic sequences, if we know the initial value and the constant difference then we can find each number in the sequence easily.

Theorem. *If $a_1, a_2, \cdots$ is an arithmetic sequence with constant difference k, then $a_n = a_1 + (n-1)k$. (This is called the* general term *of the sequence.)*

Proof. $a_n = a_{(n-1)} + k = a_{(n-2)} + 2k = \cdots = a_{(n-(n-1))} + (n-1)k$. □

Suppose we want to find the *sum* of the first n values of some sequence $a_1, a_2, \ldots$. For a simple example, we turn to a problem from last week.

Example. We will find the sum of the first n counting numbers. Behold:

$$\begin{array}{ccccccccccccccccc} 1 & + & 2 & + & 3 & + & 4 & + & \cdots & + & (n-3) & + & (n-2) & + & (n-1) & + & n \\ n & + & (n-1) & + & (n-2) & + & (n-3) & + & \cdots & + & 4 & + & 3 & + & 2 & + & 1 \end{array}$$

Adding the two rows together and dividing by two, we obtain the result that

$$1 + \cdots + n = \frac{n(n+1)}{2}.$$

(This is called the nth partial sum of the series $1 + 2 + 3 + \cdots + n$.)

We now turn our attention to finding the nth partial sum of the series $a_1 + a_2 + a_3 + \cdots + a_n$. By the theorem above, we can rewrite this as

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_n &= [a_1 + (1-1)k] + [a_1 + (2-1)k] + [a_1 + (3-1)k] + \cdots + [a_1 + (n-1)k] \\ &= [a_1 + \cdots + a_1] + [0k + 1k + 2k + \cdots + (n-1)k] \\ &= na_1 + k[0 + 1 + 2 + \cdots + (n-1)] \\ &= na_1 + k\frac{n(n-1)}{2}. \end{aligned}$$

Hence, we have proved the following

Theorem. *The nth partial sum of the series $a_1 + a_2 + a_3 + \cdots + a_n$ is given by*

$$na_1 + k\frac{n(n-1)}{2}$$

(but you should memorise the idea of the proof, not the formula.)

Geometric Sequences

The next simplest kind of sequence after arithmetic sequences (where you add a constant term) is a geometric sequence (where you multiply by a constant term).

Definition. A geometric sequence is a number sequence in which each successive term may be found by multiplying by the same number; formally, a sequence is geometric if $a_{n+1} = ka_n$ for every $n > 1$ and for some constant k.

Example.

1. $a_1 = 1$, $a_n = 2a_{n-1}$: $1, 2, 4, 6, 8, ...$ (the binary sequence).
2. $a_1 = 100$, $a_n = \frac{1}{10}a_{n-1}$: $100, 10, 1, 0.1, 0.01, ...$
3. $a_1 = 1$, $a_n = -1a_{n-1}$: $1, -1, 1, -1, ...$

Theorem. *If $a_1, a_2, \cdots$ is an geometric sequence with constant ratio k, then:*

1. *the general term of the sequence is $a_n = a_1 k^{n-1}$.*
2. *the nth partial sum of the sequence is $s_n = a_1 \frac{1-k^n}{1-k}$.*

Proof.

1. Exercise.
2. This proof uses a little trick:

$$\begin{aligned} s_n &= a_1k^0 + a_1k^1 + a_1k^2 + \cdots + a_1k^{n-1} = a_1^n(1 + k + k^2 + \cdots + k^{n-1}) \\ (1-k)s_n &= a_1(1-k)(1 + k + k^2 + \cdots + k^{n-1}) \\ &= a_1[(1 + k + k^2 + \cdots + k^{n-1}) - k(1 + k + k^2 + \cdots + k^{n-1})] \\ &= a_1[(1 + k + k^2 + \cdots + k^{n-1}) - (k + k^2 + k^3 + \cdots + k^n)] \\ &= a_1[1 - k^n] \\ s_n &= a_1\frac{1-k^n}{1-k}. \end{aligned}$$

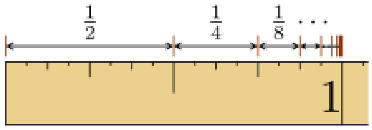
□

Geometric sequences are slightly more interesting than arithmetic sequences; if we add up all the terms of an arithmetic sequence, the resulting partial sums always grow towards $\pm\infty$. On the other hand, it is possible for the sum of all the terms of a geometric sequence to tend to some finite value. One case in which this happens is the following example.

Example. Consider the geometric sequence given by $a_n = \frac{1}{2}^n$:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

The sum $1 + 1/2 + 1/4 + \cdots$ converges to 1, as shown graphically here.



On the other hand, it is very easy for series to diverge (not tend towards any finite value): just take $k > 1$. More interestingly, we can consider the following sequence:

Example. Consider the geometric sequence given by $a_n = (-1)^n$:

$$1, -1, +1, -1, +1, \ldots$$

The sum $1 - 1 + 1 - 1 + \cdots$ does not converge as it jumps between $+1$ and -1 infinitely often.

Let us try to work out under which conditions the geometric series *does* converge. We want to look at the behaviour of the quantity $a_1 \frac{1-k^n}{1-k}$ as n grows. We make the following observations:

1. If $k = 1$, then the quantity is undefined: but then, the sequence looks like $a_1 + a_1 + \cdots$, which grows arbitrarily large and so the sum diverges.
2. If $k > 1$, then the sum also grows arbitrarily large and the series diverges.
3. If $k = -1$, then the sum diverges but now oscillates around zero instead of growing to infinity.
4. If $k < -1$, then the sum grows arbitrarily large and oscillates between being positive and negative, so diverges.
5. If $-1 < k < 1$, then k^n gets smaller as n increases — so tends to zero, and the sum of the series as $n \to \infty$ is given by $\lim a_n = a_1 \frac{1}{1-k}$.

In summary, we have seen that:

- The general term of an arithmetic sequence is $a_n = a_1 + (n-1)k$.
- The nth partial sum of an arithmetic series is $s_n = na_1 + k\frac{n(n-1)}{2}$.
- The general term of a geometric sequence is $a_n = a_1 k^{n-1}$.
- The nth partial sum of a geometric series is $s_n = a_1 \frac{1-k^n}{1-k}$.
- If $-1 < k < 1$ is the ratio of a geometric series, then the series converges to $\lim a_n = a_1 \frac{1}{1-k}$

These five facts are the important things to remember.

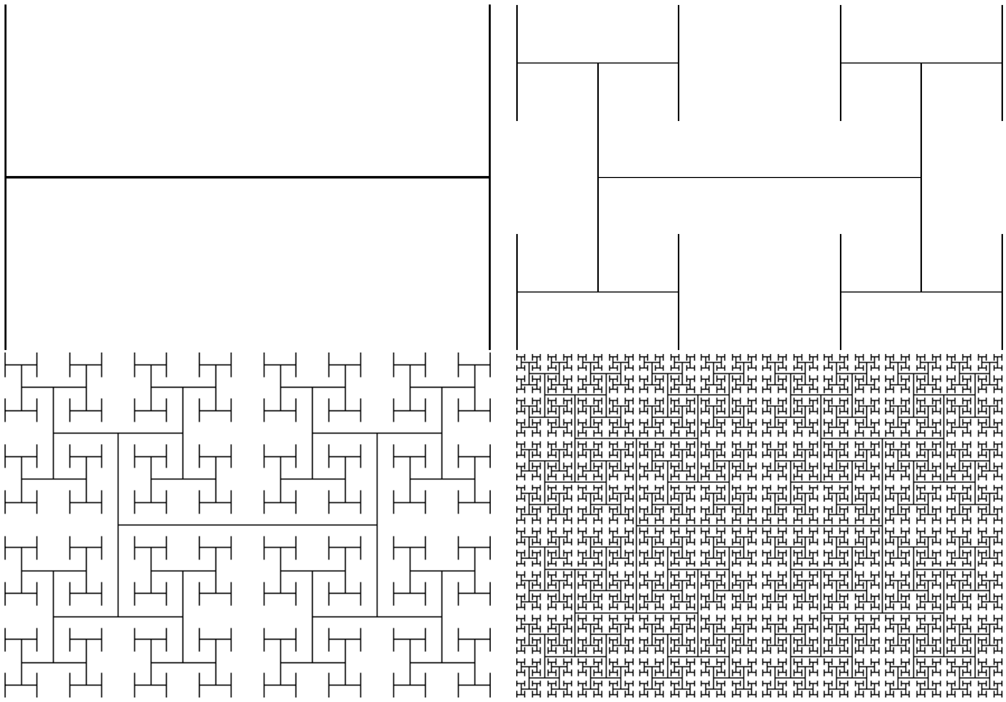
Fractals

We now move from sequences of numbers to sequences of geometric objects, with an very brief overview of fractals. Informally, a fractal is a geometric figure with the property of self-similarity: if you zoom in, it looks 'the same'. Many examples of fractal geometry can be found in nature; the traditional example is a coastline:

As we zoom in, the coastline reveals new 'wiggles' that we couldn't see at a larger scale — but it always looks wiggly in the same way, no matter how far you zoom in.

Construction 1 (H-fractal)**.** Begin at step one with a line segment of length 1. Then at each step n, add a line segment of length $\left(\frac{1}{\sqrt{2}}\right)^{n-1}$ to the endpoints of each line segment created at step $n-1$. (The lengths are purely aesthetic.)

The curve produced is the H-fractal; a Python script to draw it (`hcurve.py`) is given in the appendices. Below we have the curve after the second, fourth, eighth, and twelfth steps.



How many segments are present at the nth step? At the first step we have one, at the second three, at the third seven, at the fourth fifteen, and so on: the number seems to be $2^n - 1$ segments. How do we prove this? Well, suppose we have k endpoints at the $n-1$th step. Then at the nth step, we add k lines and hence $2k$ endpoints — so the number of endpoints doubles each time. Initially we have two endpoints, so the number of endpoints added at each step follows a geometric sequence with initial term 2 and ratio 2: at step n, we add 2^n endpoints. The sum of all these endpoints is simply $2\frac{1-2^n}{1-2} = 2(2^n - 1)$ (using our formula for the partial sum of a geometric series); and every line has two endpoints, so we must divide by two.

An interesting property of this curve is that, as well as having 'infinite length', it is space-filling — when we complete the construction at infinity, the curve will cover every point in the rectangle that it is bounded by.*

Other methods of producing fractals involve removing portions of a figure rather than adding portions.

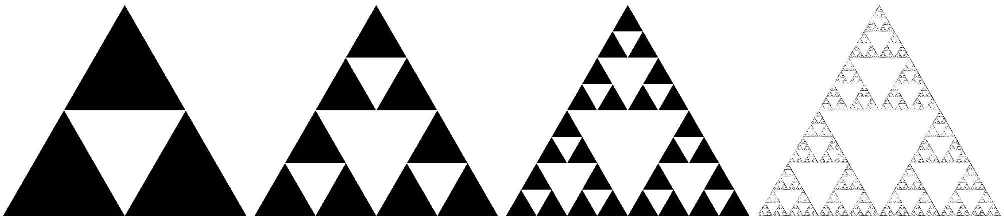
Construction 2 (Cantor set)**.** At step 1, begin with a unit segment; then at step n, remove the middle third of each of the segments remaining from the previous step.

This construction, pictured below, was first discovered by Henry John Stephen Smith in 1874 and entered mainstream mathematical knowledge in 1883 due to Georg Cantor (the father of set theory). The Cantor set itself, produced by continuing the construction to infinity, still has infinitely many points but has (in a precise sense) zero length. A Python script to draw it (`cantor.py`) is given in the appendices.

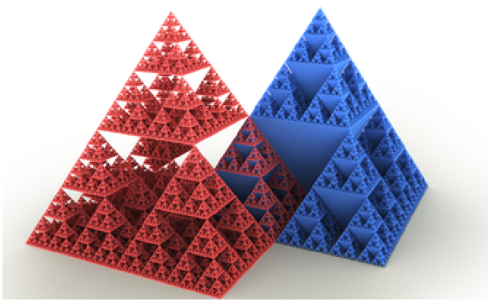
Our initial length is 1; at each step we remove precisely 1/3 of the remaining length, or equivalently we keep 2/3 of the remaining length; so the length of the set at step n is given by the geometric sequence with general term $\left(\frac{2}{3}\right)^{n-1}$. In particular, since 2/3 is less than 1, if we take $n \to \infty$ the sequence goes to zero.

Construction 3 (Sierpinski triangle)**.** At step 1, we start with a filled equilateral triangle. At the nth step, split each filled triangle into four equilateral triangles and remove the central triangle.

If this construction is carried on forever, the result is the Sierpinski triangle. This fractal was first described by a Polish mathematician, Waclaw Sierpinski, in 1916, and is a generalisation of the Cantor construction to two dimensions. Similarly to the Cantor set, the measure of the Sierpinski triangle (in this case the area) is zero but it still contains infinitely many points! A Python script to draw it (`sierp.py`) is given in the appendices; below, we have the triangle after two, three, four, and nine iterations.



One possible generalisation to three dimensions is the Sierpinski tetrahedron, produced by removing pyramids from a pyramid; two are pictured below.†



*More precisely, we can pick some number N such that after the Nth step, the curve comes within any distance δ of any point within the rectangle that we want. Obviously if δ is small then we need N to be (very) large, but the point is that it's theoretically possible!

Questions

Sequences and Series

1. The following are snippets of sequences that are either arithmetic or geometric. Give the general term of each. (In each case, $a_1 = 1$.)
   (a) $\ldots, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \cdots$
   (b) $\ldots, 5, 7, 9, \ldots$
   (c) $\ldots, -10, 100, -1000, \ldots$
   (d) $\ldots, 0.02116, 0.0097336, 0.004477456, \ldots$

2. Prove that the general term of a geometric sequence (a_n) with constant ratio k is $a_1 k^{n-1}$.

3. Compare the formulae that give the general term of an arithmetic sequence and a geometric sequence.

4. The Fibonacci sequence is the sequence defined by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-2} + f_{n-1}$. The first few numbers in the sequence are $1, 1, 2, 3, 5, 8, 13, 21, \cdots$.
   (a) The Fibonacci sequence was studied by Leonardo of Pisa (Fibonacci) in the middle ages, in his book *Liber Abaci* ("Book of Calculation"). In this book appeared the following problem:

   A pair of rabbits one month old are too young to produce more rabbits, but suppose that in their second month and every month thereafter they produce a new pair. If each new pair of rabbits does the same, and none of the rabbits die, how many pairs of rabbits will there be at the beginning of each month?

   Show that the solution is given by the Fibonacci sequence.
   (b) Prove that the Fibonacci sequence is neither an arithmetic nor a geometric sequence.
   (c) It turns out that, despite not being a geometric sequence, the ratios of adjacent Fibonacci numbers tend to a constant value. Taking this to be true without proof, we will calculate what this ratio is.
      i. Justify why, if n is very large, $\frac{f_{n+1}}{f_n} \approx \frac{f_n}{f_{n-1}}$.
      ii. Show that, if $x = \frac{f_n + 1}{f_n}$, then $x \approx 1 + \frac{1}{x}$.
      iii. Show that $x = \frac{1}{2} + \frac{\sqrt{5}}{2}$. (This value is usually called φ, the golden ratio.)
      iv. Experimentally verify that this is the approximate ratio between adjacent values for the first few values of the Fibonacci series.
      v. Explain why we have *not* proved that this is the eventual ratio between adjacent values of the Fibonacci sequence.

5. According to legend, the game of chess was invented by an ancient Indian minister for his ruler; the ruler was impressed, and asked the minister what reward he wanted, and the minister requested that the ruler take a chessboard and give him one grain of wheat on the first square, two grains on the second, four on the third, eight on the fourth, and so on. The ruler laughed it off as a meager prize for such a brilliant invention.
   (a) How much wheat did the minister ask for?
   (b) The weight of a grain of wheat is around 65 mg. Compare the weight of the wheat on the first half of the chessboard to the weight on the second half.

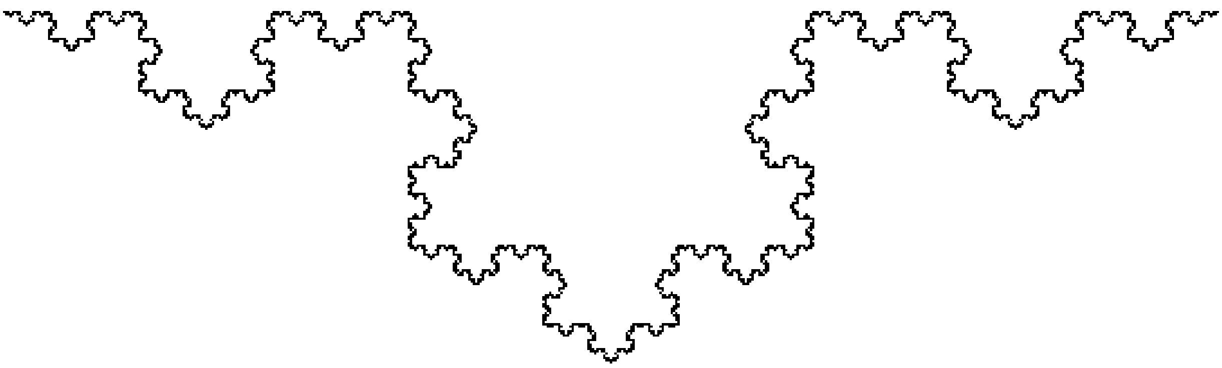
Fractals

6. What is the total length of the H-fractal after n steps?

7. Consider the figure given of the Cantor set, where each iteration is added on below the previous terms in order to form a fractal figure. Supposing that we continue this on for n steps (so the figure above is of step 6 of this process), and the height of each term is constant at 1 unit.
   (a) What is the total area of the fractal after n steps?
   (b) What happens to the area as n tends towards infinity? Does the area converge to some finite number?
8. Show that the area of the Sierpinski triangle is zero, by calculating the total area of triangles that are removed.
9. The Sierpinski triangle is very amenable to generalisation.
   (a) Give a construction for a fractal produced in a similar way to the Sierpinski triangle, but instead of subdividing a triangle into triangles subdivide a square into squares. (The resulting fractal is known as the Sierpinski carpet.) What is the area of this figure after n steps?
   (b) Generalise part (a) to three dimensions, by removing cubes from a cube. (The resulting cube is known as the Menger sponge.) Find the volume after n steps.
10. The Koch curve is produced by continuing the following construction to infinity:

    Construction 4. At step 1, we have a unit segment. At the nth step, we replace the middle third of each segment from the previous step with the upright sides of an equilateral triangle, so the segment is replaced by four segments with total length 4/3 of the segment length.

    (a) Below is the seventh iteration of the Koch curve (see `koch.py` in the appendices). Draw the first few iterations.



    (b) Show that the length of the curve as $n \to \infty$ becomes infinite.
    (c) This curve is an example of a **continuous** curve that has **no tangent line anywhere**. Justify both bolded claims.
    (d) Show that, if we start with an equilateral triangle and then add our equilateral triangles always on the outside, then the area enclosed by the final figure is precisely 8/5 of the original triangle area. The resulting figure, the Koch snowflake, is therefore a finite area bounded by a curve of infinite length.

H-fractal: `hcurve.py`

```
import tkinter as tk
from math import sqrt

def hcurve(x, y, length, scale, horizontal, iterations, canvas):
  if(iterations == 0):
    return

  if horizontal == True:
    x0 = x - length/2
    x1 = x + length/2
    y0 = y1 = y
  else:
    y0 = y - length/2
    y1 = y + length/2
    x0 = x1 = x

  canvas.create_line(x0, y0, x1, y1)

  hcurve(x0, y0, length*scale, scale, not(horizontal), iterations - 1, canvas)
  hcurve(x1, y1, length*scale, scale, not(horizontal), iterations - 1, canvas)

w = tk.Canvas(width=500,height=500)
hcurve(250, 250, 220, 1/sqrt(2), True, 12, w)
w.grid()
w.mainloop()
```

Cantor set: `cantor.py`

```
import tkinter as tk
from math import sqrt

def comb(x, y, length, height, iterations, canvas):
  print(length)
  if(iterations == 0):
    return

  x0 = x - length/2
  x1 = x + length/2 + 1
  y0 = y - height/2
  y1 = y + height/2 + 1

  w.create_rectangle(x0, y0, x1, y1, fill='black')

  comb(x - length/3, y + height + 10, length/3, height, iterations - 1, canvas)
  comb(x + length/3, y + height + 10, length/3, height, iterations - 1, canvas)


w = tk.Canvas(width=1400,height=500)
comb(700, 20, 1300, 20, 6, w)
w.grid()
w.mainloop()
```

Sierpinski triangle: `sierp.py`

```
import tkinter as tk
from math import sqrt

def sierp(x, y, length, iterations, canvas):
  if(iterations == 0):
    return

  xleft = x - length/2
  xright = x + length/2
  yleft = yright = y + length*sqrt(3)/2

  canvas.create_polygon(x, y, xleft, yleft, xright, yright, fill = 'black')
  canvas.create_polygon((x+xleft)/2, (y+yleft)/2,
                        x, yleft, (x+xright)/2, (y+yright)/2, fill = 'white')

  sierp((x+xleft)/2, (y+yleft)/2, length/2, iterations - 1, canvas)
  sierp((x+xright)/2, (y+yright)/2, length/2, iterations - 1, canvas)
  sierp(x,y, length/2, iterations - 1, canvas)

w = tk.Canvas(width=600,height=600)
sierp(300, 20, 550, 5, w)
w.grid()
w.mainloop()
```

Koch curve: `koch.py`

```
import tkinter as tk
from math import sqrt, atan2, cos, sin

def koch(coords, where, iterations):
  x0 = coords[where][0]
  y0 = coords[where][1]
  x1 = coords[where + 1][0]
  y1 = coords[where + 1][1]
  length = sqrt((x1 - x0)**2 + (y1 - y0)**2)
  r = length*1/((sqrt(3)))
  theta = atan2(y1 - y0, x1 - x0)
  phi = atan2(1, sqrt(3))
  xnew1 = x0 + length/3 * cos(theta)
  ynew1 = y0 + length/3 * sin(theta)
  xnew2 = x0 + r * cos(theta + phi)
  ynew2 = y0 + r * sin(theta + phi)
  xnew3 = x0 + 2*length/3 * cos(theta)
  ynew3 = y0 + 2*length/3 * sin(theta)

  coords.insert(where + 1, [xnew1, ynew1])
  coords.insert(where + 2, [xnew2, ynew2])
  coords.insert(where + 3, [xnew3, ynew3])

  if(iterations > 1):
    coords = koch(coords, where, iterations - 1)
    coords = koch(coords, where + 4**(iterations-1), iterations - 1)
    coords = koch(coords, where + 2*4**(iterations-1), iterations - 1)
    coords = koch(coords, where + 3*4**(iterations-1), iterations - 1)
  return coords

w = tk.Canvas(width=600,height=600)
coords = koch([[30, 300],[600 - 30, 300]], 0, 6)
for i in range(0, len(coords) - 1):
  w.create_line(coords[i][0], coords[i][1], coords[i + 1][0], coords[i + 1][1])
w.grid()
w.mainloop()
```

NCEA Level 2 Mathematics (Homework)
17. Number Sequences and Fractals

Reading

Go and watch...

Series of three videos:
`https://www.youtube.com/watch?v=ahXIMUkSXX0`
`https://www.youtube.com/watch?v=lOIP_Z_-0Hs`
`https://www.youtube.com/watch?v=14-NdQwKz9w`

What's it good for?

People use sequences, series, and fractals for...

- Science: the study of fractals and chaotic patterns are increasingly important in modern science. According to Wikipedia,[a] phenomena known to have fractal features include:
  - River networks
  - Fault lines
  - Mountain ranges
  - Craters
  - Lightning bolts
  - Coastlines
  - Mountain goat horns
  - Trees
  - Algae
  - Geometrical optics
  - Animal coloration patterns
  - Romanesco broccoli
  - Pineapple
  - Heart rates
  - Heart sounds
  - Earthquakes
  - Snowflakes
  - Psychological subjective perception
  - Crystals
  - Blood vessels and pulmonary vessels
  - Ocean waves
  - DNA
  - Soil pores
  - Rings of Saturn
  - Proteins
  - Surfaces in turbulent flows

[b]

- Mathematics: The behaviour of finite sequences and series is connected with combinatorics (like we saw last week and will see next week), while the behaviour of infinite sequences and series is connected with calculus.

[a] `https://en.wikipedia.org/wiki/Fractal#Natural_phenomena_with_fractal_features`
[b] By Jon Sullivan, `http://pdphoto.org/PictureDetail.php?mat=pdef&pg=8232`.

Questions

[This is a sample Ministry of Education L2 assessment task for this standard.]

This assessment activity requires you to create a fractal and use sequences and series to investigate features of the shape. Features of fractals include such things as length, area, number of items, volume.

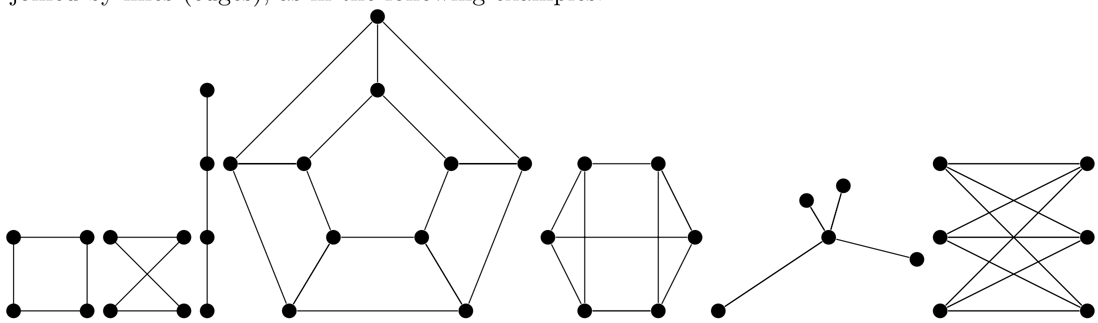
Create your own fractal. Include:

- Details of how the fractal is created, i.e. the initial unit segment or shape, and how your fractals are formed, including diagrams.
- The values generated for at least three stages (after the initial stage) of the fractal for at least two of the features of the fractal.
- The totals for at least two features of the fractal for any given stage.
- Describe what will happen to the values and totals for each feature as the number of iterations increases.
- For your chosen features, will there be a point where the next iteration makes no significant difference to the feature? Describe the conditions under which this might happen.

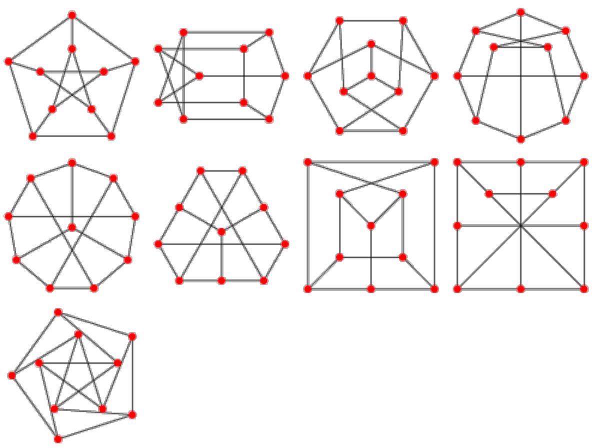
The quality of your reasoning and how well you link this context to generalisations of arithmetic and geometric sequences will determine the overall grade. Include calculations, diagrams or formulae, as appropriate. Clearly communicate your method using correct mathematical statements where appropriate.

NCEA Level 2 Mathematics
18. Graphs and Networks

The final topic that we will look at is the theory of graphs. A graph is a set of points (called vertices) that are joined by lines (edges), as in the following examples:



As we see at the left, the same graph can be drawn in different ways. The following nine drawings are all of the same graph, the Petersen graph.



In this topic, we will only consider graphs that are *finite*.

Definition.

- Two vertices are *adjacent* if there is an edge joining them.
- The *order* of a vertex is the number of edges incident to it.
- If we label the vertices, then each edge can be identified by its endpoints: if an edge joins a and b, we call the edge ab.
- A *path* on a graph between two vertices a and b is an ordered set of edges $av_1, v_1v_2, \ldots, v_nb$ such that no edge is repeated.
- A *cycle* is a path on the graph between a and itself.
- If every two vertices are connected by some path, then the graph is called *connected*.
- If the graph has no cycles, it is called a *tree*.

Traversibility

Questions about paths and traversibility have been asked about graphs for hundreds of years. In 1736, Leonhard Euler solved the following problem:

The city of Königsberg in Prussia was set on a river, and included two large islands connected by bridges as in the following diagram.

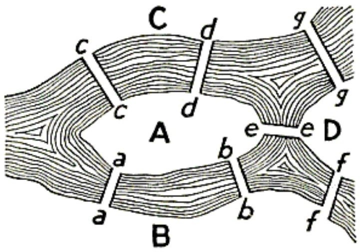
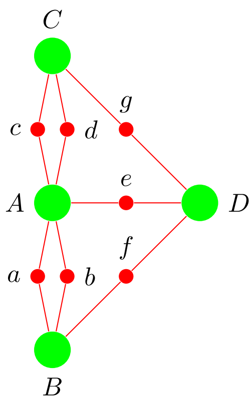


Figure 98. *Geographic Map: The Königsberg Bridges.*

Is it possible to walk a path in the city such that every bridge is crossed precisely once?

The answer, as we will see, is no. We begin by drawing a graph of the situation in order to eliminate all of the extraneous details beyond the connectedness of the city.



A path that traverses every edge of a graph exactly once is called an Eulerian path (and an Eulerian path that is also a cycle is called an Eulerian cycle).

Theorem (Euler)**.** *A connected graph has an Eulerian path if and only if it has either zero or two vertices with odd order.*

We will prove the 'only if' half here: that a graph has an Eulerian path only if it has zero or two vertices with odd order. The 'only if' part was proved by Carl Hierholzer in 1873.

Proof of necessity. Suppose a graph has an Eulerian path. Consider some vertex v that the path passes through. Then:

1. If the path does not have an endpoint on v, then v has even order because every edge at the vertex is an edge of the path, and each time the path enters the vertex it leaves (so all the edges at v can be paired up).

2. If the path has precisely one endpoint on v, it has odd order, because all the edges but the endpoint edge can be paired up as in (1).

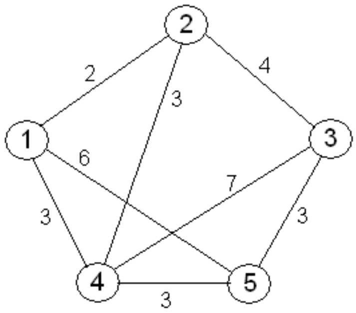
3. If the path has two endpoints at v, then v has even order, because we can pair the two endpoints together and then pair the other edges as in (1).

Every vertex has odd order, except if a vertex falls under case (2). But this can happen at most twice (a path has only two endpoints), and if both endpoints are at different vertices then case (2) applies to both. Hence a graph with an Eulerian path has either zero or two vertices of odd order. □

Hence the Königsberg challenge cannot be solved: there are more than two vertices of odd order.

Weighted graphs

Graphs can be used to model situations in subjects including computer science, scheduling, linguistics, and biology. For example, suppose some company has five distribution centres and wants to find an economical shipping pattern. Let's draw a graph with five vertices and label the edges with the cost of sending a truck (in hundreds of dollars) between the two joined centres:



(If we label edges like this, the graph becomes a *weighted graph.* The numbers are referred to as costs, weights, or distances.)

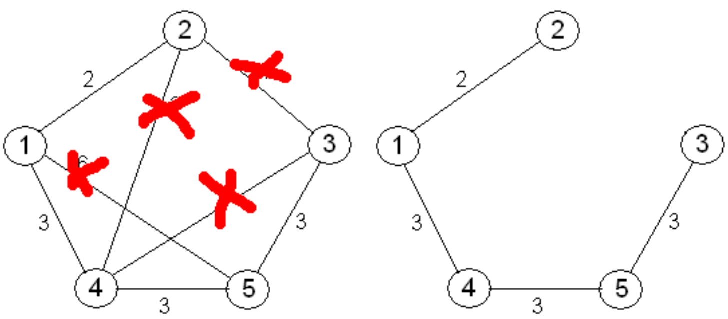
Suppose the company wants to send packages between every pair of distribution centrs, but does not want to pay to run all of the routes graphed. What are the edges that we can safely delete? In practice, we want to find a minimum spanning tree for this graph: a subgraph, without cycles, so that the total cost of the subgraph is a minimum. (Finding this graph would allow this company to minimise expendiature while not losing any connections.) One algorithm to find the minimum spanning tree is the reverse-delete algorithm:

Algorithm (Reverse-delete)**.**

1. List all of the edges in descending weight-order.
2. Examine each edge in order, starting with the most expensive. If deleting this edge disconnects the graph, do not remove the edge; otherwise, remove the edge.

The remaining graph is a minimum spanning tree.

For example, in this case we obtain the following minimum spanning tree:



On the other hand, suppose the company does not run all its routes constantly and instead wants to find the cheapest route for a particular package from one centre to another, utilising any of the eight edges above. This can be done with Dijkstra's algorithm:

Algorithm (Dijkstra)**.** Suppose we want to find the shortest path from some vertex a to some vertex b. In fact, this algorithm will give us the shortest path from a to *any* other vertex!

Label vertex a with zero and every other vertex as ∞; then set a as the current vertex, and:

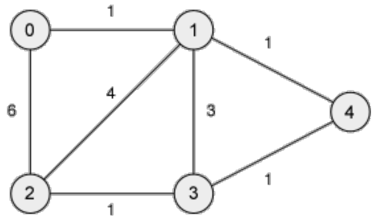
1. Consider every unvisited vertex v directly adjacent to the current vertex, and update the label of v to the smallest value out of:
   - Either the current value on v, or
   - The sum of the current value on the current vertex and the distance from the current vertex to v.
2. If the vertex v has been relabeled by step (1), then mark with an arrow the edge joining the current vertex with v and unmark any other edges on v that have been marked in previous steps.
3. Label the current vertex as visited.
4. If every vertex is visited, then we halt; otherwise, set the current vertex to the vertex with minimal label and repeat from step (1).

Each vertex v is now labeled with the minimum distance from a to v, and the shortest path from a to v is marked by the arrows.

Questions

1. Justify, with mathematical reasoning, the following statements.
   (a) A graph that is not connected has no Eulerian paths.
   (b) If a graph has no vertices of odd order, all Eulerian paths are circuits; if there are two vertices with odd order, all Eulerian paths begin at one and end at the other.
   (c) The sum of all the orders of all the vertices must be even.
2. Consider the Königsberg bridge graph.
   (a) Show that it is possible to exhibit a Eulerian path on the graph resulting from adding a single extra bridge.
   (b) Show that, no matter where the bridge is placed, there will still be no Eulerian circuit.
3. Suppose there are three houses on a flat plane (or plain), and each needs to be connected to water, gas, and electricity.
   (a) Show that this is impossible without two connections crossing, without using a third dimension or running a connection through a house.
   (b) Show that, on a torus (a doughnut), it *is* possible.
4. Legend has it that an ancient Indian lord had five sons, and he allowed them to split his land up after his death between them, on the proviso that the land of each son must be in one piece, and must share a boundary with the land of all four other sons. Why is this funny?

5. Consider the following weighted graph G.

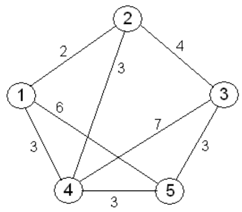


(a) Give an example of a graph with more than one minimal spanning tree.

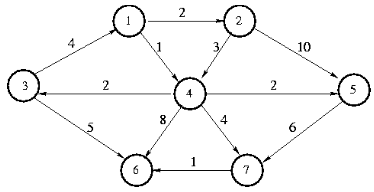
(b) Show that G has precisely one minimal spanning tree.

(c) Find a vertex v of G such that the minimal spanning tree of the graph also gives the shortest paths from v to every other vertex.

6. Find the cheapest shipping path between distribution centres 1 and 3 in the example of a weighted graph from above (reproduced here).



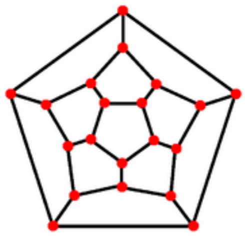
7. Suppose that, in addition to weights on edges of a graph, we assign a direction: perhaps we are modelling the direction and velocity of some fluid flow. Consider the following directed graph:



(a) Call a directed graph connected if we can give a directed path from every vertex a to every other vertex b (i.e. a path with arrows from a to b). Explain why this directed graph is not connected.

(b) Find the shortest path from vertex 4 to every other vertex.

8. A **Hamiltonian path** is a path on a graph such that every vertex appears on the path exactly once. If the two endpoints of the path are adjacent, then the path is called a **Hamiltonian cycle**. Find a Hamiltonian cycle on the dodecahedron graph:



The existence of Hamiltonian paths on graphs is much more difficult than the existence of Eulerian paths. One recent result (from 2005) is that a graph with n vertices has a Hamiltonian path if, for every pair of non-adjacent vertices, the sum of their degrees and the distance between them is greater than n.

NCEA Level 2 Mathematics (Homework)
18. Graphs and Networks

Reading

Go and watch...

https://www.youtube.com/watch?v=CruQylWSfoU

What's it good for?

People use graphs and networks for...

- Computer science: modelling computer networks, modelling connections between objects (Google's search algorithm is based on weighted graphs).
- Engineering: electrical circuits can be modelled with weighted graphs.
- Linguistics: syntax trees and semantic networks are used when modelling language and the changes of languages over time.
- Physics and chemistry: graphs can be used to model crystal structures or the structures of molecules (especially in organic chemistry).
- Mathematics: graph theory has tight links with topology and knot theory. The four-colour theorem (the vertices of any graph that can be drawn with no edge crossings can be coloured with at most four colours so no adjacent vertices have the same colour) was only proved in the latter half of the 20th century, and was the first major proof to involve considerable work by computers.

Questions

[This is a sample internal assessment task for this standard, based on those provided by the Ministry of Education.]

This assessment activity requires you to apply graphs and networks to a real-world situation. Read the entire activity before beginning work.

The Lower North Island Logistics Company (LNILC) wants to streamline its business by redesigning its logistics network. In Resource A, a list of town/city pairs in the current network and travel times by truck between them are given.

1. Draw a weighted graph modelling the information in Resource A.
2. Find the shortest route (in terms of travel time) between Carterton and Marton.
3. The cost of providing a truck between two of the city pairs is directly proportional to the travel time.
   (a) Determine any links that will (i) be in *every* minimal spanning tree, and (ii) will be in *no* minimal spanning tree.
   (b) Determine the smallest possible network that covers all the destinations, and costs the least to maintain.
4. LNILC is considering signing a deal with KiwiRail that provides rail services between the following city pairs:
   - Wellington to Palmerston North: 135 minutes
   - Palmerston North to Whanganui: 50 minutes
   - Whanganui to New Plymouth: 50 minutes
   - Palmerston North to Hastings: 110 minutes
   - Wellington to Carterton: 80 minutes
   - Carterton to Masterton: 18 minutes
   - Masterton to Hastings: 140 minutes
   - Masterton to Palmerston North: 65 minutes

   How do your answers above change, incorporating these new links?
5. Produce a final report, incorporating your findings above, that could recommend:
   - If any truck links should not be maintained;
   - Which, if any, rail links should be included in the future network;
   - Any other changes that could be made to improve the cost-effectiveness of the entire network, and minimise the time taken for packages to travel between particular major towns and cities. (For example, you may recommend that a particular link be kept despite not appearing in your minimum spanning tree if it significantly reduces the distance between two key destinations.) Resource B may be useful for this.

The quality of your reasoning and how well you link this context to graph and network methods will determine the overall grade. Include calculations, diagrams or formulae, as appropriate. Clearly communicate your method using correct mathematical statements where appropriate.

Resource A: town-city pairs with times

All links are bi-directional and all travel times are in minutes.

City 1	City 2	Travel time by road
Wellington	Porirua	27
Wellington	Lower Hutt	22
Lower Hutt	Wainuiomata	12
Lower Hutt	Upper Hutt	22
Porirua	Upper Hutt	28
Porirua	Kapiti	30
Kapiti	Upper Hutt	68
Upper Hutt	Martinborough	49
Upper Hutt	Carterton	51
Carterton	Masterton	15
Martinborough	Masterton	43
Masterton	Dannevirke	77
Masterton	Palmerston North	79
Kapiti	Otaki	20
Otaki	Levin	18
Levin	Palmerston North	40
Levin	Whanganui	74
Palmerston North	Whanganui	56
Palmerston North	Fielding	16
Fielding	Marton	25
Marton	Whanganui	29
Palmerston North	Dannevirke	54
Dannevirke	Napier	89
Dannevirke	Hastings	74
Napier	Hastings	24
Whanganui	Hawera	68
Hawera	Stratford	26
Stratford	New Plymouth	33
Hawera	New Plymouth	90

Resource B: major towns and cities in the Lower North Island

The following urban areas served by LNILC have populations over 10,000:

- Wellington (207,900)*
- Napier-Hastings (133,000)*
- Lower Hutt (104,700)
- Palmerston North (85,300)
- New Plymouth (57,500)*
- Porirua (55,900)
- Upper Hutt (43,200)
- Kapiti (42,300)
- Whanganui (40,300)*
- Masterton (21,800)
- Levin (20,900)
- Fielding (16,550)

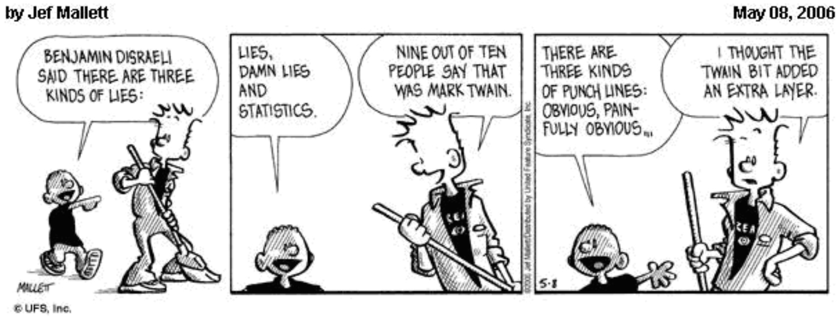
(*Urban area served by a port.)

Chapter 6

Statistics

NCEA Level 2 Mathematics
19. The Statistical Enquiry Process

Statistics is one of the most important tools in experimental science, and medicine. Even in the everyday world we are bombarded with statistics given to us by the media, politicians, and the internet. Because of this, being able to interpret statistics and judging whether or not they support a given argument, result, or point of view is an important skill in the modern world.



The statistical inquiry cycle is something you should already know about from as far back as intermediate school. Here it is, in a slightly expanded form:

1. We begin with a question we want to answer, and perhaps a hypothesis backed up by some theory.
2. We decide what data we need to collect to answer our question.
3. We plan how we will collect our data, and how we will ensure it is accurate and precise.
4. We plan how we will process our data.
5. We collect our data. (This is intentionally *after* we plan how we will use it.)
6. We process our data in accordance with our plan.
7. We answer our question.
8. We decide how reliable our process was.
9. We write a report, detailing *all* of the above steps, such that another person could pick it up and try to replicate our findings.

Writing a question, and deciding on data to collect

A question should be written clearly and precisely, and there should be a straightforward way to answer it.

Some good questions:

- Are high school students in Wellington City more likely to cycle to school than students in Upper Hutt?
- Do male Y12 students tend to be taller than female Y12 students?
- Does this drug lower the risk of heart failure after a stroke?
- With what speed does a 0.5 kg weight hit the ground after being dropped 10 m?

Some bad questions (why?):

- Does this drug work?
- Will I crash if I drink and drive?
- Does the average person support the attempt by the USA to bring freedom and democracy to other places in the world?
- Any question you write after you've already collected your data!*

When you write your question, you should come up with a good idea as to what kind of data you will need to gather to answer it. This could be something like a set of measurements, or responses to a questionnaire, or something else.

Planning how to collect data

Usually, we want to collect data to answer a question about a whole population. Unfortunately, it's not cheap or easy to measure or survey every individual in a population, so we have to make do with examining a smaller sample; we then need to process our data to make an educated inference about the state of the full population. In order to do this, we need a reliable sample that reflects the trends in the population.

In general, we need two things from a sample:

- A sample needs to be *large*.
- A sample needs to be *random*.

Both of these statements are harder to unpack than they might seem right now:- they are what we will discuss in the next topic.

At this stage, we also decide how we are going to actually collect the data: will we need to write a questionnaire? Do we need any specialised equipment? How are we going to ensure the accuracy and precision of our data? Do we need a control group?

Planning how to process data

We then need to decide what we are going to do to our massive spreadsheet of data after we collect it. If we want to work out the average value of some measurement, we need to decide how to do this (median, or mean, or something else?). If we want to work out whether there is a difference in some value between two populations (whether one group is taller than another group, for example), then we need to decide how big a difference will be 'significant' and not likely to be due to random chance.

Collecting and processing the data

This is where we apply the plan we've come up with so far. If we deviate from the plan in any way, we carefully specify why and how we are doing so, and we make a careful note that this needs to be stated in our report. Further, we *only* peform the analysis we planned to do. (This is for two reasons: firstly, as I mentioned above, if you measure a bunch of variables then some of them will end up significant simply due to random chance; and secondly, the data we planned to collect is more likely to be reliable than the data we collected along the way with no planning.)

---

*Try googling 'data dredging': "Data dredging (also data fishing, data snooping, data butchery, and p-hacking) is the misuse of data analysis to find patterns in data that can be presented as statistically significant when in fact there is no real underlying effect. This is done by performing many statistical tests on the data and only paying attention to those that come back with significant results, instead of stating a single hypothesis about an underlying effect before the analysis and then conducting a single test for it.

"The process of data dredging involves automatically testing huge numbers of hypotheses about a single data set by exhaustively searching—perhaps for combinations of variables that might show a correlation, and perhaps for groups of cases or observations that show differences in their mean or in their breakdown by some other variable." (Wikipedia contributors. (2019, January 13). Data dredging. In Wikipedia, The Free Encyclopedia. Retrieved 22:29, January 20, 2019, from `https://en.wikipedia.org/w/index.php?title=Data_dredging&oldid=878260765`) See also `https://xkcd.com/882/`.

Answering the question

Now, we look at our data processing and we give a simple answer to our question together with a rough idea of how likely we are to be correct:

According to our analysis, our hypothesis was incorrect: in our sample, girls tended to be taller than boys. We found that the mean height of the girls in our sample was 1.8 m*, while the mean height of the boys was* 1.6 m*. The difference between these means is large enough that it is unlikely to be due to random chance, and so it is likely that in the general population of the class the average height of girls is greater than the average height of boys.*

Evaluating our process

The final step we perform before writing down our report is to evaluate our findings.

- What difficulties did we encounter when trying to measure our results?
- Did we have to change any of our planning while performing our experiment?
- Did anything go wrong that might affect our results?
- Is there anything we would do differently? Why?

Why do we do this? Because it allows people who read our report to decide for themselves how reliable our process and results are, and it allows people who want to reproduce our findings to be aware of why we chose to do things a particular way and what challenges we faced when performing our experiment.

Questions

This week, we will work through the planning for a sample task provided by the Ministry of Education (resource 2.10A).

It has been claimed that teenagers are becoming addicted to energy drinks (which contain high amounts of caffeine) without knowledge of the short-term or long-term effects. People are also concerned that teenagers feel the need for mind or body-altering substances like energy drinks (see the web links below).

- `http://www.stuff.co.nz/the-press/news/2820097/Caffeine-drinks-may-be-hurting-teenagers`
- `http://kidshealth.org/teen/food_fitness/nutrition/caffeine.html`
- `http://www.foodsafety.govt.nz/elibrary/industry/Caffeine_Intake-Confirms_Advice.htm`.

Select an experimental situation, based on the effects of drinking caffeine, to investigate. Identify the variables you think are important, and write a question to investigate.

The following situations are suggested as a basis for your experiment:

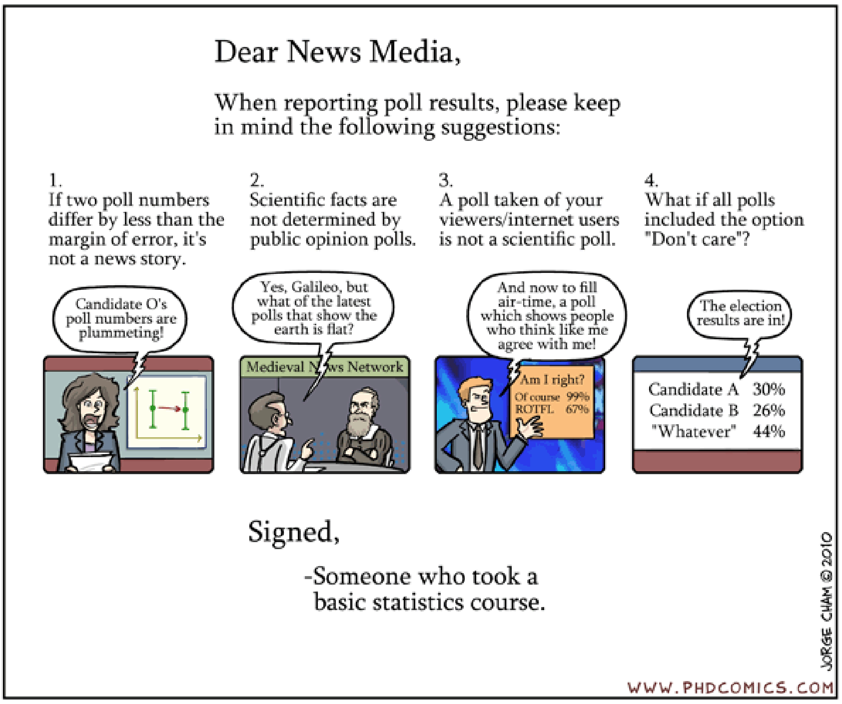
- the effect of drinking caffeine on heart rate;
- the effect of drinking caffeine on reaction times;
- the effect of drinking caffeine on memory.

Write a plan for the experiment. The plan should:

- describe the variables and measures you have chosen and why you have chosen them;
- explain how you will collect your data and record your results;
- link to relevant knowledge about the situation;
- describe any related variables and the possible effects of these;
- describe the experimental method.

NCEA Level 2 Mathematics (Homework)
19. The Statistical Enquiry Process

Reading



(PHD Comics)

Statisticians often use the method of comparison. They want to know the effect of a treatment, like a vaccine, on a response, like catching a virus. To find out, they compare the responses of a treatment group, which gets the treatment, with those of a control group, which doesn't. Usually, it is hard to judge the effect of a treatment properly without comparing it to something else.

If the treatment group is just like the control group, apart from the treatment, then an observed diffrence in the responses of the two groups is likely to be due to the effect of the treatment. However, if the treatment group is different from the control group with respect to these other factors as well, the observed difference may be due in part to these other factors.

The best way to make sure that the treatment group is like the control group is to assign subjects to treatment or control at random. This kind of experiment is called a randomised controlled experiment.

Whenever possible, in a well-designed experiment the control group is given a placebo, which is neutral but which resembles the treatment. This is to make sure the response is to the treatment itself rather than to the idea of treatment.

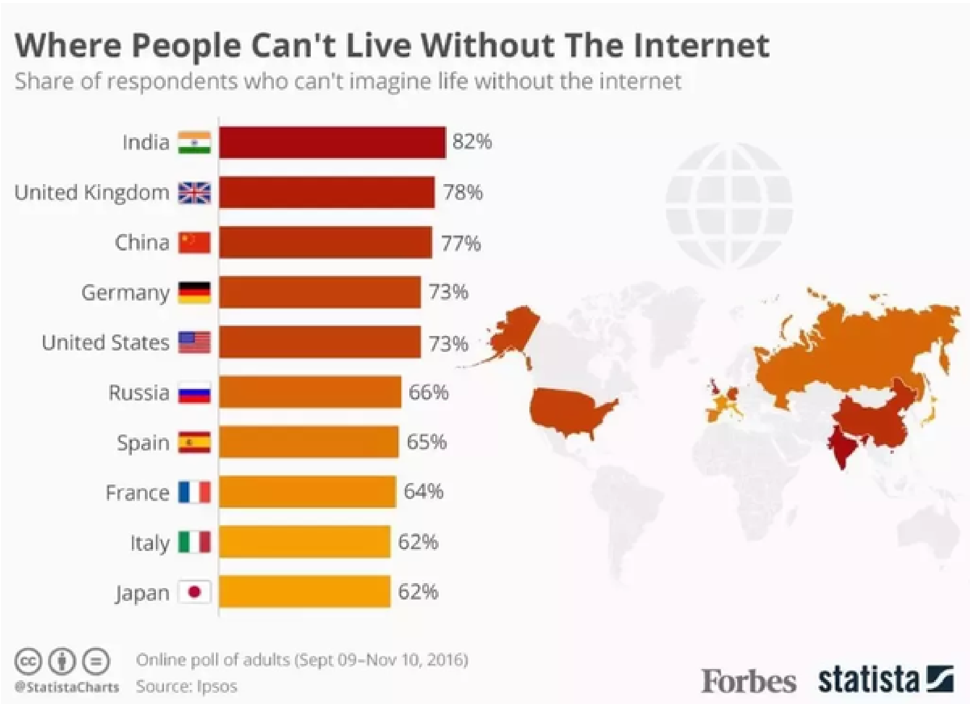
A well-designed experiment is run double-blind whenever possible. The subjects do not know whether they are in treatment or in control. Neither do thos who evaluate the responses. This guards against bias, either in the results or in the evaluations.

Adapted from *Statistics*, by Freedman et al. (p. 9).

Questions

1. In October, 1976, a nationwide (in the USA) vaccination programme was started against swine flu. The first shots were given to the group most at risk — the elderly and infirm. During the first week of the programme, 24,000 persons aged 65 and over were given shots, and three of these persons died. As a result, eight states suspended their vaccination programme.* What would a statistician say? (Note: you might find it useful to know that the national average death rate in the US for persons aged 65 to 74 at the time was around 80 per 100,000 per week, for all causes).

2. Discuss the following results.



*Freedman et al. (p. 18, ex. 6).

NCEA Level 2 Mathematics
20. Sampling

Last time, we mainly looked at the broad picture: what we need to think about, in general, when we try to answer a statistical question. This time we will begin to think about some of the practical issues we need to overcome.

As we've already discussed, it is usually impractical to measure an entire population. Our goal is therefore to measure a smaller sample and then extrapolate our findings. This process, known as *statistical inference*, requires us to have a good method for choosing our sample so that it is representative.

We will look at several examples of bad methods of sampling to begin with.

The examples

1. I asked all my friends whether they own a car, and none of them do.
2. A survey of high-school students samples all the Y13 students at a particular school, and concludes that only 7% of students use illegal drugs.
3. In 1936, a US presidential election poll posted questionnaires to ten million people selected from telephone books and club membership lists, and got 2.4 million responses. Based on these, they predicted a decisive victory for one candidate (57% of the popular vote). In reality, the other candidate won by a landslide (62%).*
4. A psychiatrist finds that practically everyone is neurotic.
5. A drug trial is performed; the patients were analysed according to the treatment they actually took, rather than the treatment they were assigned at the randomisation stage of the trial.
6. A majority of people attending a public meeting on a new cycleway are strongly against it. However, a combination of online survey, door knocking surveys, and paper surveys performed by the council found that 75% of the population of the area was either in favour or strongly in favour of the new cycleway.
7. The council conducted a survey on a new residential development. The survey was conducted by posting a survey to all those on the list of a local resident's association. 90% of respondents were against the new development.

The problems

1. Some people aren't my friends. In addition, most of my friends live in urban areas with frequent public transport, and tend to be more affluent.

*Freedman et al., *Statistics*. Section 19.2.

2. This doesn't include students who drop out of school, or are homeschooled. It also only measures students at a particular school, which might be more or less affluent than average and thus drug use by its students might be more or less probable.

3. Despite the large sample size, the sampling method used tended to screen out the poor (who didn't belong to clubs, or have telephones) who were more likely to vote for the other candidate.

4. The psychiatrist's patients are far from a sample of the population.

5. This might seem reasonable (if 30% of participants drop out, they didn't receive the benefit of the treatment and so shouldn't be part of the 'participated in treatment' group during analysis). However, the problem is that the question that should be being answered is 'is this treatment effective?' rather than 'out of the people who chose to take our tablets, is the treatment effective?'. After all, if people don't end up taking the medication after being given it, this is philosophically and medically the same as if the medication was ineffective.[†]

6. There is a bias in the sample of people attending the public meeting: for example, they are likely to have strong opinions on the cycleway and are likely to be more involved in local politics and resident groups. The council survey reaches a broader spectrum of residents and thus has less selection bias.

7. By now, you should be able to come up with your own explanation as to why this sample was not representative.

From studying these examples, we have the following broad guidelines:

- When a selection procedure is biased, taking a larger sample with the same bias doesn't help.
- If a large proportion of people don't respond, the results are likely to be biased.
- Picking a sample from a certain group of people within a population is likely to be biased.
- Allowing people to choose whether or not to respond introduces bias.

There are various methods of producing a sample, based on these lessons. They include:

Simple random sampling
Taking a full list of all the people in the population, and picking some proportion of them entirely at random.

Quota sampling
Giving each interviewer a quota of subjects to interview, and interviewing a fixed proportion of particular categories within that quota (e.g. 50% non-male, 20% below the age of thirty, and so forth) which match the proportion of the categories in the overall population.

In general, simple random sampling is the best method as long as it is carried out correctly:

- The list of possible interviewees must be a full list of the entire population to be surveyed.
- The sampling method should be truly random (in other words, 'choosing every nth person' is not a good idea).

The problem of non-response bias remains; it is therefore a very good idea to conduct surveys using many different methods (postal surveys, physical meetings, door-knocking, phone surveys, and so forth) in order to minimise this.

[†]See Ben Goldacre, *Bad Pharma*, pages 200-1.

Questions

1. For each of the following situations, discuss whether or not the sampling plan is 'good statistics.'
   (a) A restaurant leaves comment cards on all of its tables to encourage diners to participate in a brief survey to learn about their overall experience.
   (b) A school wants feedback on tuck shop options. Each student has a student identification number; the deans use a computer to generate fifty random identification numbers and those students are asked to take a survey.
   (c) A council wants feedback on a proposal for new play equipment at a park. They send someone out to stand in the park and interview everyone who shows up for five hours one day.
   (d) A local company is losing business to the internet; they interview their remaining customers to ask what other services that they could provide to draw people in.
   (e) A physicist is measuring the power output of a lamp. She makes ten measurements. After entering them into her spreadsheet, she finds that a couple are much lower than she would expect, and throws them out.
   (f) A student is interested in whether the number of days a year a fellow student cycles to school is correlated with the average income of people in the area in which they live. (This data is freely available from the NZ census, and is reliable.) They interview fifty students at their school (randomly chosen from the quad one lunchtime), asking them roughly how many times they cycled to school in the last year, and which street they live on.

2. Come up with a good sampling plan for the following research questions.
   (a) You want to predict the outcome of the upcoming local mayoral election.
   (b) You want to know whether people who cycle to school tend to live in 'richer' areas.
   (c) You want to compare the voting preferences of below-the-voting-age high-schoolers with those of the general voting population.
   (d) You want to know whether students living in cities are more or less worried about 'sustainability' than high school students in the country.
   (e) You want to know whether or not a cycleway would be generally approved of on a particular stretch of road.

NCEA Level 2 Mathematics (Homework)
20. Sampling

Reading

Please read the following article on political polls in New Zealand prior to the last General Election:

`https://www.stuff.co.nz/national/politics/96793252/qa-what-should-we-make-of-political-poll-results`

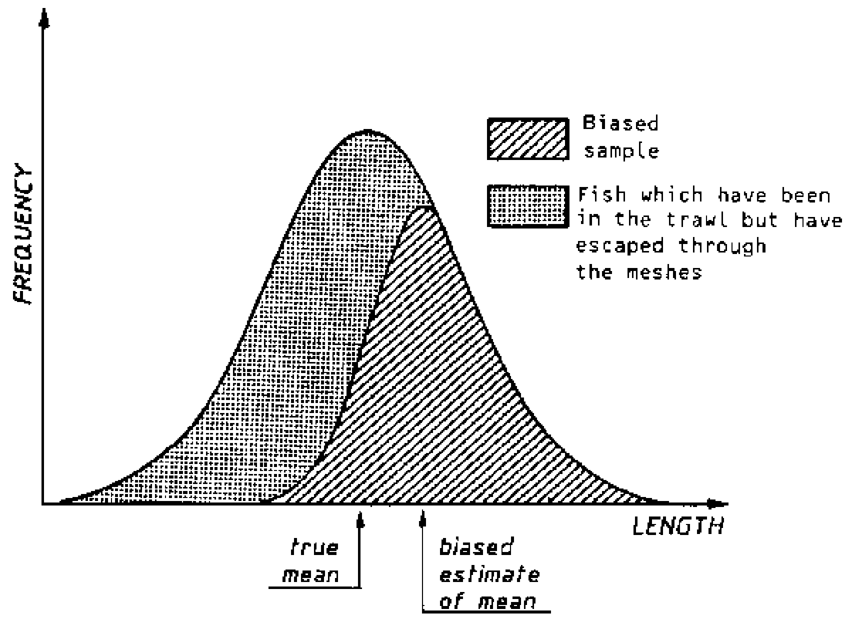
Think about how reasonable the sample methods used are. If you were given a couple of million dollars and asked to produce your own poll based on a sample of the population, how would you go about this?

Optional: Another interesting blog post on the problems with self-selected samples, and correcting for selection bias, can be found here:

`http://www.goodmath.org/blog/2016/09/29/polls-and-sampling-errors-in-the-presidental-debate-results/`

Questions

1. Discuss how the data in the following graph might affect your interpretation of a survey, performed by a student in your biology class, which finds the average size of a particular species of fish in a given area.



2. Watch the following clip from the television series *Yes Prime Minister*: `https://www.youtube.com/watch?v=GOZZJXw4MTA`. An upmarket toothpaste manufacturer has asked you to run a survey for a marketing campaign, comparing how many people use their toothpaste versus the product of a competitor who produces a cheaper toothpaste.
   (a) Design a sampling method and survey questions that are likely to cast the products of the manufacturer in a good light.
   (b) Design a sampling method and survey questions that are likely to cast the products of their competitor in a bad light.
   (c) Design a sampling method and survey questions to fairly compare the products usage.

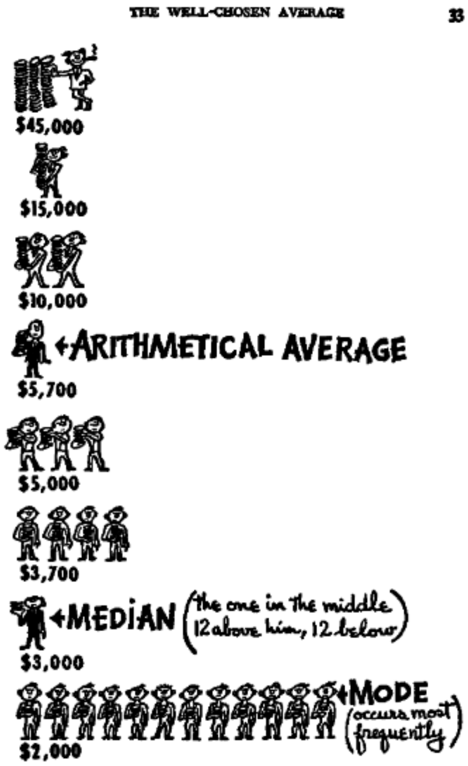
NCEA Level 2 Mathematics
21. Statistical Inference

Unless we are very lucky, the statistics we gather from our small sample will not exactly match the parameters of the population. We can't exactly predict what the population looks like!

Summarising data

We first need to take our data and work out what it is telling us about the *sample*.

We have a number of statistics that we could calculate. First, we look at the *measures of central tendancy*; these tell us, in some sense, what the typical value for a measurement in our sample is.



From *How to lie with statistics*, by Darrell Huff (p.33).

Arithmetic mean

This is the 'usual' average value; it tells us where the 'centre of mass' of the data lies. However, it might not be typical of any actual data value; see the diagram above, or consider that the mean of $\{13, 15, 16, 80\}$ is 31, but no actual data value is anywhere near 31.

Median and quartiles

If we list out all the data values for our measurement in order and then pick the three quartiles (the *lower quartile* is the value that 25% of the values lie below, the *median* is the value that 50% of the values lie below, and the *upper quartile* is the value that 75% of values lie below), then we have some information about the 'shape' of the data. If we have two populations to compare (female versus non-female.)

Mode

The *mode* is the most common value for the measurement.

We also have measures of spread, which tell us how likely we are to find points far away from our central point.

Range and inter-quartile range

These values, the difference between the maximum and minimum and between the two quartiles respectively, give us a rough idea of how spread out our data values are. However, they don't tell us whether the 'typical' data value is close to or far away from the centre — only how far apart the furthest data points are.

Variance and standard deviation

The *variance* s^2 of a set of data is the mean of the squares of the distances of each point to the mean.

$$s^2 = \frac{\sum(x - \bar{x})^2}{n}.$$

The *standard deviation* of the set is s, the square root of the variance.

We will use the central points and the spread measurements of our sample to estimate the same values for the population as a whole.

Drawing an inference

We then need to take our *statistics* and work out how sure we can be about the *parameters* of the population.

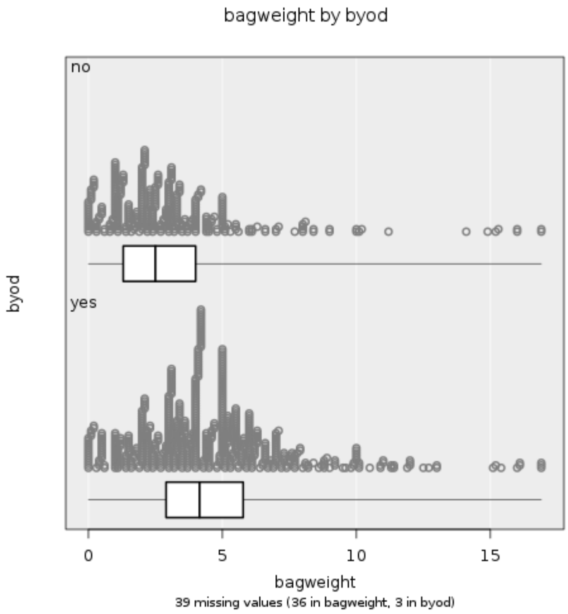
At this level, we won't go into the details too far. What we will do is assume that our median value for our sample is a reasonable guess for the median value of our population, and then write down an *informal confidence interval*:

$$\text{sample median} \pm 1.5 \cdot \frac{\text{IQR}}{\sqrt{n}}$$

(where IQR is the inter-quartile range of the sample). Note: bigger sample size, smaller informal confidence interval. (Why?)

The population median will lie within this range with 90% confidence: if we sample ten times, calculating this value each time, then on average nine of these intervals will contain the population median.*

In order to compare two subsets of a population, it is useful to use side-by-side box and whisker graphs to visualise the shapes of the data sets. For example:



*Next year, we will learn how to make this idea more useful by re-sampling the population a bunch of times to make the confidence intervals we obtain smaller.

Questions

1. Consider a data set with at least three values. Suppose we increase the highest value by 10, and decrease the lowest by 5. Do the mean or the median change? Is it possible for the mode to change?

2. Repeat (1), but now decrease the lowest value by 10 rather than 5 (so the increase of the highest value is the same as the decrease of the lowest).

3. If a data set has an even number of points, is the median ever equal to a value in the set?

4. Consider the numbers $\{2, 3, 5, 5, 5, 7, 9\}$. Which measure(s) of central tendancy (mean, median, or mode) makes sense in the following situations:
   (a) If the numbers represent colours of T-shirts ordered from a website?
   (b) If the numbers represent distances from a point to given destinations?
   (c) If the numbers are survey responses on a scale of 1–10?

5. A set of five data values has a mean of 39. A new data point with value 17 is added to the set; what is the new mean?

6. Compare and contrast the inter-quartile range and the standard deviation as measures of spread. Discuss the following statement: "mean is to median as standard deviation is to range".

7. One important use of the standard deviation is Chebyshev's theorem: for any set of data, and for any constant $k > 1$, the proportion of the data that lies within k standard deviations on either side of the mean is at least $1 - 1/k^2$.
   (a) If a set of data has mean μ and standard deviation σ, at least what proportion of the data lies within the range from $\mu - 3\sigma$ to $\mu + 3\sigma$?
   (b) Show that if a set of data has mean μ and standard deviation σ, then (i) at least 75% of the data lies in the range from $\mu - 2\sigma$ to $\mu + 2\sigma$; (ii) at least 93.8% of the data lies in the range from $\mu - 4\sigma$ to $\mu + 4\sigma$.

8. Consider the following data from the 2017 Census at School. (The question answered was 'How much water did you drink yesterday?'.) Is there any significant difference between the Auckland and Wellington medians?

```
=====================================================================================
                               iNZight Summary
-------------------------------------------------------------------------------------
        Primary variable of interest: region (categorical)
                  Secondary variable: drinkwater (numeric)

        Total number of observations: 1000
   Number omitted due to missingness: 68 (68 in drinkwater)
   Total number of observations used: 932
=====================================================================================

Summary of drinkwater by region:
--------------------------------

                     Min   25%   Median   75%    Max    Mean      SD   Sample Size
    Auckland Region    0   250      700  1000   4000   799.1   724.7           471
  Wellington Region    0   250      600  1000   4000   774.6   702.1           461


=====================================================================================
```

9. The following table shows the data collected from a sample of 1000 Census at School respondents (2017 survey), answering the question 'In how many languages can you hold a conversation about a lot of every day things?'. The notation $[a, b)$ means 'all the numbers between a and b, including a but not including b'.

	Female	**Male**	**Totals**	**Percentages**
[1,2)	413	247	660	66%
[2,3)	179	83	262	26.2%
[3,4)	41	18	59	5.9%
[4,5)	10	5	15	1.5%
[5,6)	3	1	4	0.4%
[6,7)	0	0	0	0%
[7,8)	0	0	0	0%
[8,9)	0	0	0	0%
[9,10]	0	0	0	0%
Totals	646	354	1000	100%

(a) Calculate the statistics for the total sample. Hence write down a guess for the population measurement, and a confidence interval for that guess. Is it more or less than you would expect?

(b) Calculate the statistics for male and female respondents from the sample separately. Is there a significant difference between your sample medians? Consider the male and female populations of respondents. Can you conclude that, in general, one gender tends to be comfortable with more languages than the other? (Two populations are likely to have different population medians if the informal confidence intervals for the median of each don't overlap.)

10. The following tables show the data collected from a sample of 1000 Census at School respondents answering the question 'What is the main way you usually get to school?'. The first table is sampled from the 2009 data set, and the second from the 2017 data set. Pick a mode of transport, make a hypothesis (with reasoning) as to whether there is likely to be any significant change in usage of your chosen mode over the eight years between the two surveys, and test your hypothesis.

	Totals	**Percentages**
Bike	51	5.1%
Bus	300	30%
Motor	348	34.8%
Other	8	0.8%
Train	27	2.7%
Walk	266	26.6%
Totals	1000	100%

	Totals	Percentages
Bike	40	4%
Boat	9	0.9%
Bus	298	29.8%
Motor	368	36.8%
Other	18	1.8%
Train	14	1.4%
Walk	253	25.3%
Totals	1000	100%

NCEA Level 2 Mathematics (Homework)
21. Statistical Inference

Reading

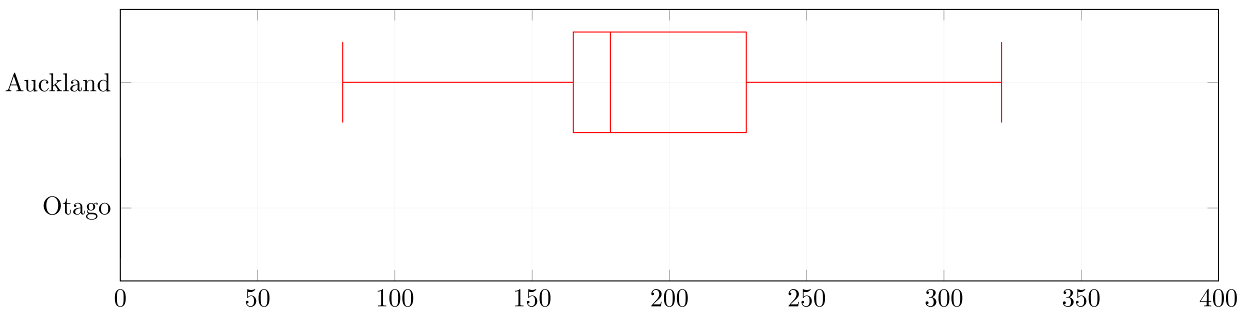
Every year, thousands of New Zealand high school students try to decide which of the eight universities in New Zealand they should attend. One of the tools which they can use to help inform their decision is the idea of a 'university ranking', a list of universities ordered by how 'good' they are according to some set of criteria.

The following table shows the global rankings given by eight different websites for both the University of Auckland and the University of Otago.

	The University of Auckland	**The University of Otago**
topuniversities.com	81	169
CWUR	252	354
Times	165	225
ARWU	175	350
CWTS Leiden	204	267
university-list.net	182	270
4icu	165	*No rank*
RankPro	321	351

Here are some calculated values and a box-and-whisker graph for the University of Auckland data:

	Auckland	**Otago**
Mean ($\bar{x}$)	193.125	
Minimum	81	
Lower quartile	165	
Median	178.5	
Upper quartile	228	
Maximum	321	



Questions

1. Fill in the University of Otago column in the table using the data given, and draw a box-and-whisker graph for it in the space underneath the University of Auckland graph.

2. Write a couple of paragraphs comparing the two graphs (half a page at most) and discussing the pros and cons of university ranking surveys. You might want to discuss the questions below:
   (a) How do you think the ranking websites might have conducted their surveys (who did they talk to and/or what data might they have gathered)? What might the pros and cons of your predicted surveying method be?
   (b) Is one university generally ranked higher than the other?
   (c) Looking at the graphs, can you see any visible skew in the data?
   (d) Do you think that the higher ranked university is necessarily better, or could there be some other reason for its high ranking — in other words, does the ranking of the university actually reflect how 'good' it is, or could there be some other force(s) at play here?

NCEA Level 2 Mathematics
22. Probability and Risk

Basic probability

Suppose we are performing an experiment where the outcome is uncertain. Some examples of such experiments include:-

- Flipping a coin.
- Measuring rainfall.
- Determining the sex of a newborn child.
- Checking the outcome of a sporting event.

The set of possible outcomes of an experiment is called the *sample space*. The sample space of the first experiment above, that of flipping a coin, is

$$S_{\text{coin flip}} = \{H, T\}.$$

The *probability* of a given outcome is the proportion of times that we would expect an experiment to give a particular outcome if we run the experiment multiple times. We write $P(\text{outcome})$ for the probability of a particular outcome.

For example, $P(H) = 0.5$ because, if we were to flip a coin many times, we would expect half the tosses to result in an outcome of H.

A probability must lie between 0 and 1 inclusive (because it makes no sense for an outcome to happen in 200% of the experimental runs).

There are a couple of ways to determine the probability of something. We either find a probability experimentally, by doing an experiment a bunch of times, or we do it theoretically, by counting outcomes.

- To determine a probability experimentally — based on empirical evidence:
$$P(A) = \frac{\text{number of times } A \text{ happened}}{\text{number of times the experiment ran}}.$$
- If there are n ways for an outcome A to occur, and each way is equally likely, then
$$P(A) = \frac{\text{number of ways } A \text{ could happen}}{\text{number of ways any outcome could happen}}.$$

We have some rules to calculate with probabilities. If A and B are two possible outcomes, I will write $P(A \text{ or } B)$ for the probability that A *or* B happens,[*] and $P(A \text{ and } B)$ for the probability that A *and* B happens.

If the total sample space consists of n outcomes, $S = \{A_1, A_2, ..., A_n\}$, then

$$P(A_1 \text{ or } A_2 \text{ or } \cdots \text{ or } A_n) = 1.$$

In other words, there is a probability of 1 — absolute certainty — that there will be an outcome. Either heads or tails must be flipped.[†]

Two outcomes A and B are said to be *mutually exclusive* if they can never happen at the same time; in other words, if $P(A \text{ and } B) = 0$. To take the example of flipping a coin, heads and tails are mutually exclusive outcomes.

[*] By 'A or B happens', I mean that one of A, B, or both A and B happens.

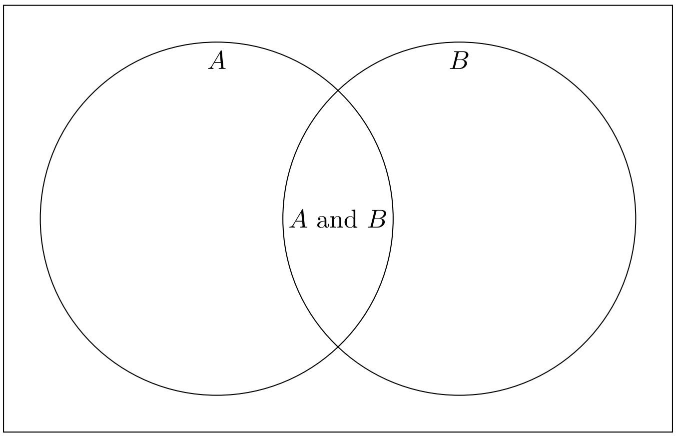
[†] Yes, it is possible for a coin to land on its rim. I am assuming an infinitely thin coin, or maybe a counter.

If two events A and B are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$: the number of times that A or B happens is the sum of the number of times that A or B happens.

Suppose A is an outcome. Then 'not A' is an outcome as well. Further, A and not A are mutually exclusive; and they make up an entire sample space for our experiment because if we do the experiment, then either A is the outcome or it isn't. Hence $P(A) + P(\text{not } A) = 1$, and thus $P(\text{not } A) = 1 - P(A)$. To save ~~chalk~~ typing, I will write $\neg A$ for 'not A'.

Now, suppose two events A and B are *not* mutually exclusive: so it is possible for A and B to be an outcome of an experiment. We want to show that $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Proof 1: via Venn diagram. If we draw a Venn diagram showing the $P(A \text{ and } B)$, we end up with:



So we see that $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. □

Proof 2: algebraic. First, see that A or B = A or (B and $\neg A$), where these events are mutually exclusive. (Either A happens — which could include A and B happening — or B happens but not A). We can apply the rule from above: $P(A \text{ or } B) = P(A) + P(B \text{ and } \neg A)$.

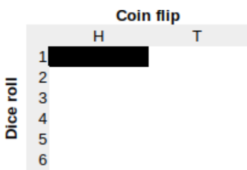
Now, note that B = (B and A) or (B and $\neg A$) (B happens whenever B and A both happen, or when B happens without A), where again the events are mutually exclusive. Hence $P(B) = P(A \text{ and } B) + P(B \text{ and } \neg A)$. So we have both of the following:

$$P(A \text{ or } B) = P(A) + P(B \text{ and } \neg A)$$
$$P(B) = P(A \text{ and } B) + P(B \text{ and } \neg A).$$

From the second, $P(B \text{ and } \neg A) = P(B) - P(A \text{ and } B)$; and substituting into the first, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. □

Two outcomes A and B are called *independent* if the the occurrence or nonoccurrence of one does not affect the likelihood of occurrence or nonoccurence of the other. For example, the events 'it will rain tomorrow in Brazil' and 'the sun will rise at 5:30 am in New Zealand' are independent.[‡]

Suppose I flip a coin and roll a die. By drawing a table (see below) and counting the outcomes, I find that there is a 1 in 12 chance ($P = \frac{1}{12}$) that I flip a heads and roll a 6.



[‡]One hopes. It is possible to imagine scenarios where they turn out not to be independent — for example, suppose a crazed supervillain from Dunedin decides he will blow up the country if it rains in Brazil tomorrow.

Further experimentation should convince you that, if A and B are independent, then $P(A \text{ and } B) = P(A)P(B)$.

If two events are *not* independent, we can't apply this rule. We can, however, discuss the probability that A occurs given that we know B has happened; we write $P(A|B)$, read *the probability of A given B*, for this.

Formally, what we want to know is the proportion of times that A happens, out of all the times that B happens; writing this down, we have

$$P(A|B) = \frac{P(B \text{ and } A)}{P(B)}.$$

Instead of our sample space being the entire set of possibilities, we restrict our sample space to just being the set of times B has occurred, and then we calculate this probability in the usual way.

It is often easier to work out relative probabilities using either a table or a tree diagram.

Example (NZQA, 2012)**.** BigGen power company keeps track of complaints made to the company by its customers.

- 12% of customers have made a complaint in the last 12 months.
- If a customer made a complaint, there was a 0.7 chance that they left BigGen.
- The percentage of customers leaving BigGen over the last 12 months was 10%.

Calculate the following:

1. The proportion of customers that complained and left BigGen.
2. The number of customers that would be expected to complain, but stay with BigGen, given that there were 250,000 customers in total at the start of the year.
3. The probability that a customer who didn't complain left BigGen.
4. If a customer left BigGen, the probability that they complained.

Solution. Consider (1) first. The probability that a customer complained and left can be calculated directly, since

$$P(\text{complained and left}) = P(\text{left}|\text{complained}) \cdot P(\text{complained}) = 0.7 \cdot 0.12 = 0.084.$$

On the other hand, consider the tree diagram drawn below; the probability $P(\text{complained and left})$ can be calculated from it by just multiplying down the branch.

For (2), we need the probability $P(\text{complained and not left})$; we can fill in the probability $P(\text{not left}|\text{complained})$, as it will be $1 - P(\text{left}|\text{complained}) = 1 - 0.7 = 0.3$. Then $P(\text{complained and not left}) = 0.3 \cdot 0.12 = 0.036$.

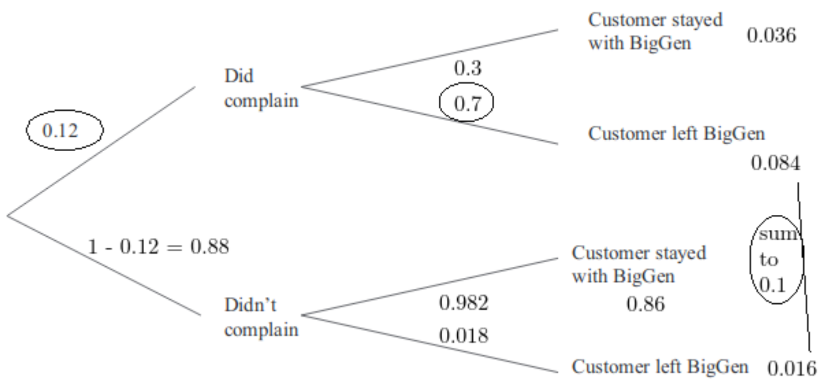
For (3), we need $P(\text{left}|\text{not complained})$. We know that $P(\text{left}) = 0.1$ (it was given); and

$$P(\text{left and not complained}) + P(\text{left and complained}) = P(\text{left})$$

so $P(\text{left and not complained}) = 0.1 - 0.084 = 0.016$; then

$$P(\text{left}|\text{not complained}) = 0.016/P(\text{not complained}) = 0.016/0.88 = 0.018.$$

Finally, for (4) we want $P(\text{complained}|\text{left})$. This is given by $P(\text{complained and left})/P(\text{left})$; filling this in, $P(\text{complained}|\text{left}) = 0.084/0.1 = 0.84$.



All this information is summarised in the tree diagram above; the circled pieces of information are the ones given, and they are sufficient to reproduce the rest. Cover up the example above, and attempt to reproduce the tree diagram yourself using only the information given and the probability laws.

Note that $P(A|B) \neq P(B|A)$ in general: the probability that someone complained given that they left was 0.84, while the probability that someone left given that they complained was 0.7.

Relative risk

Suppose we are conducting a medical trial, and we want to examine the differences in outcomes between two groups. For example, suppose we survey 1000 people and we split them up into two groups according to whether they smoke. We find that 200 people are smokers, and 800 are non-smokers. We then track them for the rest of their lives, and find that twenty of the smokers developed lung cancer, and four of the non-smokers did.

	Smoker	**Non-smoker**	**Total**
Cancer	20	4	24
Non-cancer	180	796	976
Total	200	800	1000

We can't directly compare the actual numbers of cancer sufferers, 4 versus 20, to show that lung cancer and smoking are linked because the sizes of the cohorts are different: it makes no sense to say that smokers are five times (20/4) more likely to get cancer on the basis of these figures. To solve this problem, we want to calculate the risks (probabilities) of lung cancer in the presence of smoking versus non-smoking, and then compare these.

We learn that, given a person is a smoker, they have a $20/200 = 0.1$ probability of developing lung cancer. On the other hand, if a person does not smoke, they have a $4/800 = 0.005$ probability of developing lung cancer. This means that the *relative risk* of lung cancer in the presence of smoking is $0.1/0.005 = 20$: smokers are twenty times more likely to get lung cancer than non-smokers.

The 2017 NCEA L2 probability exam cites another example; the following table shows data from a sample of 2500 New Zealanders aged between 15 and 24:

	Obese	**Non-obese**	**Total**
Male	222	983	1205
Female	285	1010	1295
Total	507	1993	2500

The risk of being obese, given that an individual in the sample is male, is $222/1205 = 0.184$; the risk of being obese given that an individual is female is $285/1295 = 0.220$. Thus the relative risk is $0.220/0.184 = 1.196$: if an individual from the sample is female, they are almost 20% more likely to be obese than a male from the sample.

Reports like this often generate sensational headlines in the media, and so it is important to understand how they are calculated and the assumptions made.

Questions

Basic probability

1. A newspaper reports the probability that a certain airline's flights arrive on time.
   (a) Only one of these numbers could represent this probability; which one, and why?
   $$-6.86, -0.686, 0.686, \text{or } 6.86$$
   (b) Was the probability based on equally likely outcomes, a long-run set of observed outcomes, or subjective assessment?
2. Suppose a fair 12-sided die is rolled. What is the probability that:
   (a) An even number is rolled.
   (b) A number divisible by three is rolled.
   (c) A number greater than (but not equal to) seven is rolled.
   (d) The number 12 is rolled.
   (e) The number 14 is rolled.
   (f) On two consecutive (independent) rolls, a six and then a five are rolled.
   (g) On two consecutive rolls, a five is not rolled either time.
3. A factory makes three models of car, and each car can be one of three colours. The following table provides some information about the cars manufactured in a particular week.

	Silly Sedan	**Horrible Hatchback**	**Vast Van**	**Total**
Yellow	7			23
Black		16		34
Green	3	8	2	13
Total	20		14	

   (a) Complete the table.
   (b) Suppose a random car is chosen for inspection. What is the probability that the car is either a yellow car, or a sedan, or both?
4. In the ANA online Health and Safety Survey of 2001, several thousand nurses reported the usual length of the shift they worked at their main job. What is the probability that one of these nurses worked a 12-hour shift?

Less than 8 hours	5.0%
8 hours	47.0%
10 hours	20.0%
12 hours	?%
More than 12 hours	22.0%

5. A statistics teacher sets up a two-way table of counts for all 2,000 students in her history of teaching, keeping track of gender and whether or not a student received an E. The table shows that 1,200 were female, and that 500 out of all the students received an E. Overall, 300 female students received an E. Complete the table. Suppose a student were male; what is the chance that they did not receive an E?

	E	not E	Total
Female			
Male			
Total			

6. (NCEA 2017) A survey of young adults produced the following results:

	Obese	**Not Obese**	**Total**
Current smoker	103	317	420
Non-smoker	404	1676	2080
Total	507	1993	2500

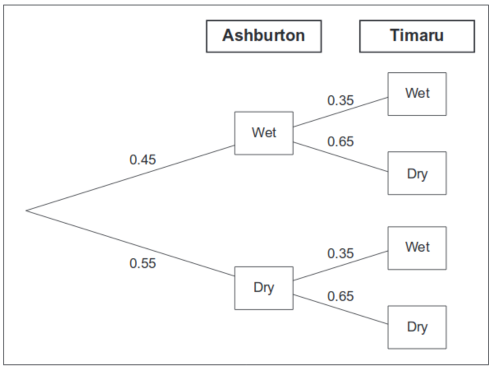
Would it be correct to claim that young adult smokers are more at risk of being obese than young adult non-smokers? (In other words, is the probability that a young adult is obese given that they are a smoker greater than the probability of obesity given that they are not a smoker?)

7. (NCEA 2018)
   (a) Nancy finds some data from NIWA on weather in Ashburton and in Timaru over the past seven years. She analyses the data and finds that:
   - It was wet on 45% of days in Ashburton.
   - If it was wet in Ashburton, the probability that it was wet on the same day in Timaru was 63%.
   - If it was dry in Ashburton, the probability that it was dry on the same day in Timaru was 88%.

   i. Find the probability that it was dry in both Ashburton and Timaru on a randomly chosen day.
   ii. Find the probability that only one of the towns was wet on a given day.
   iii. If it was a dry day in Timaru, what is the probability that it was also dry in Ashburton on the same day?

   (b) Nancy's friend Teri uses the NIWA data for the past seven years to find out that:
   - 45% of days in Ashburton were wet
   - 35% of days in Timaru were wet.

   Teri constructs the tree diagram below.



   Elain why Teri's tree diagram would not give a correct answer to the probability that it is dry in both towns on the same day. (Note: the distance between Ashburton and Timaru is around 75 km. Both towns lie on the Canterbury plains.)

Relative risk

8. A study is conducted of 1500 randomly selected candidates for an international examination to investigate whether Y12 candidates are more successful than Y13 candidates. The findings are summarised in the table below.

	Y12	**Y13**	**Total**
Pass	347	853	1200
Fail	33	267	300
Total	380	1120	1500

(a) What proportion of candidates passed the exam?

(b) What proportion of candidates who failed were in year 12?

(c) There were about 52 500 candidates from year 12 and year 13 who attempted the exam. Based on the results of the study, how many candidates would be expected to be in year 13 and pass the exam?

(d) It is claimed that year 13 candidates are four more times more likely to fail the exam than year 12 candidates. Discuss your agreement or disagreement with this statement.

9. Polygraph (lie-detector) tests are often routinely administered to government employees or prospective employees in sensitive positions. A study performed in the US in 2002 found that lie detector results are "better than chance, but well below perfection". Typically, the test will conclude someone is a spy 80% of the time when he or she actually is a spy; but 16% of the time, the test will conclude someone is a spy when he or she is not.

   (a) Assuming that 10 out of every 10 000 employees are actual spies, compute the following:
      i. The probability that an employee is a spy and is "detected" to be one.
      ii. The probability that an employee is **not** a spy and is "detected" to be one.
      iii. The overall probability that the polygraph "detects" that an employee is a spy.

   (b) On the basis of the given information, and the probabilities that you have calculated, is the use of the polygraph "worth it"?

10. A survey was conducted in 2000, asking respondents about various driving habits. This table classifies 1 086 participants according to type of car driven, and whether or not they were in the habit of making insulting gestures at other drivers.

	Economy	**Family**	**Luxury**	**Sports**	**Truck**	**Utility**	**Van**	**Total**
Gestures	79	65	16	58	42	32	8	300
No Gestures	281	170	45	95	77	79	39	786
Total	360	235	61	153	119	111	47	1086

   (a) Before calculating any conditional probabilities, identify the types of cars whose owners you would suspect to have a tendancy to make insulting gestures at other drivers.

   (b) For each type of car, find the (conditional) probability that surveyed drivers of that type of car make insulting gestures (e.g. given that a person drives a truck, what is the probability that they make insulting gestures).

   (c) Comment on whether your suspicions in part (a) were correct.

   (d) To three decimal places each, find (using the table):
      i. The probability of driving a van ($P(V)$).
      ii. The probability of making insulting gestures ($P(G)$).
      iii. The probability of driving a van *and* making insulting gestures ($P(V \text{ and } G)$).

   (e) Check if $P(V \text{ and } G) = P(V) \times P(G)$ to see if the events V and G are independent. Explain the outcome.

   (f) You found $P(G|V)$ in part (b). Check if $P(V \text{ and } G) = P(V) \times P(G|V)$, and explain the outcome.

   (g) Find the overall probability of driving an economy car, a family car, a luxury car, or a van.

   (h) Find the probability of making insulting gestures, given that someone drives an economy car, a family car, a luxury car, or a van.

   (i) Are drivers of economy cars, family cars, luxury cars, or vans less likely in general to make insulting gestures?

NCEA Level 2 Mathematics (Homework)
22. Probability and Risk

Reading

Go and watch... (basic probability)

`https://www.youtube.com/watch?v=fwD98HiQSJc`

Go and watch... (relative risk)

`https://www.youtube.com/watch?v=felIAwyaGFM`

Questions

1. Watch the first video above. I have six tetrahedral dice. I roll them 1000 times. How many times should I expect to get six of the same number (six 1's, six 2's, six 3's, or six 4's) in a roll?

2. Watch the second video above. Consider the following two tables, which show data from two groups of New Zealander males over the age of fifty.

A	**Heart disease**	**None**	**Total**
Overweight	24	142	166
Not overwt	14	1706	184
Total	38	312	350

B	**Heart disease**	**None**	**Total**
Smoker	12	65	77
Non-smoker	9	84	93
Total	21	149	170

   (a) Discuss the following statement (is it correct? why?): The *absolute risk* of heart disease in group A was $38/350 = 0.109$, while the absolute risk of heart disease in group B was much greater: 0.124. Thus, smoking is more problematic than obesity when it comes to risks of heart disease.

   (b) **Important lesson:** it is not the *risk* that matters when comparing probabilities, it is the *relative risk*. In other words, we don't care about the probability that a person gets heart disease given that they are overweight — we care about whether this probability is higher than the probability that they get heart disease given that they are not overweight.

      i. Suppose a person from group A is known to be overweight. What is the risk that they develop a heart disease?

      ii. Suppose a person from group A is known to be not overweight. What is the risk that they develop a heart disease?

      iii. How much more likely is a person from group A to get heart disease if they are overweight?

   (c) *According to this study, a 50-year-old male is 60% more likely to develop heart disease if they are a smoker than if they are a non-smoker.* Does the evidence support this statement?

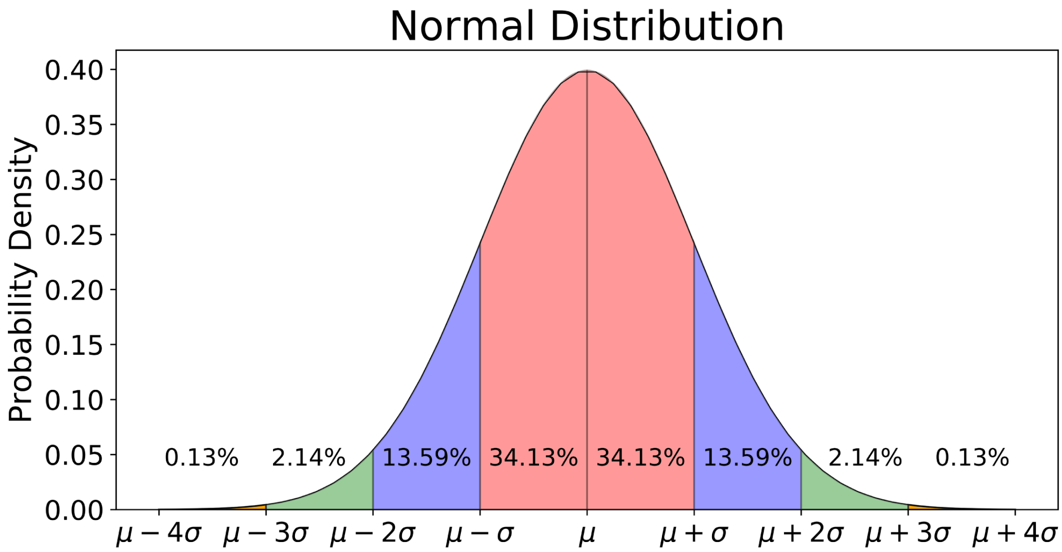
NCEA Level 2 Mathematics
22. Probability Distributions

We have now looked at 'discrete' probabilities: probabilities relating to experiments in which there are a finite number of outcomes. There are six outcomes when a die is rolled, there are two possibilities for flipping a coin, and so on.

Now, we will look briefly at 'continuous' probabilities. These arise when we perform experiments involving real-world measurements: if we measure the heights of a thousand people, each one will have a slightly different reading.

Our question then becomes 'what is the probability that our reading falls between x_0 and x_1', rather than 'what is the probability of a reading x'.

It turns out that most 'natural' experiments produce probabilities fitting what is known as a *normal curve*, or *standard distribution*. Such a curve is determined by two numbers: a standard deviation σ that determines the spread of the curve (the width), and a mean value μ which tells us where the peak is.



The x-axis represents possible measurement values. The probability of a value lying between the measurements x_0 and x_1 is the area underneath the curve between the lines $x = x_0$ and $x = x_1$. The height of the curve at any given point has no meaning, only the area.

The *standardised* normal curve has mean $\mu = 0$ and standard deviation $\sigma = 1$. If our random variable is X, and we want to find the probability it lies between x_0 and x_1 given some mean μ and standard deviation σ, then we can transform our problem to one involving a random variable Z and the standardised curve: our probability will be the area under the standardised curve between $z_0 = (x_0 - \mu)/\sigma$ and $z_1 = (x_1 - \mu)/\sigma$. This can be summarised by writing

$$Z = \frac{X - \mu}{\sigma}.$$

The table given in the NCEA L2 external formula sheet gives the area under the standardised curve between 0 and z; it is reproduced over the page.

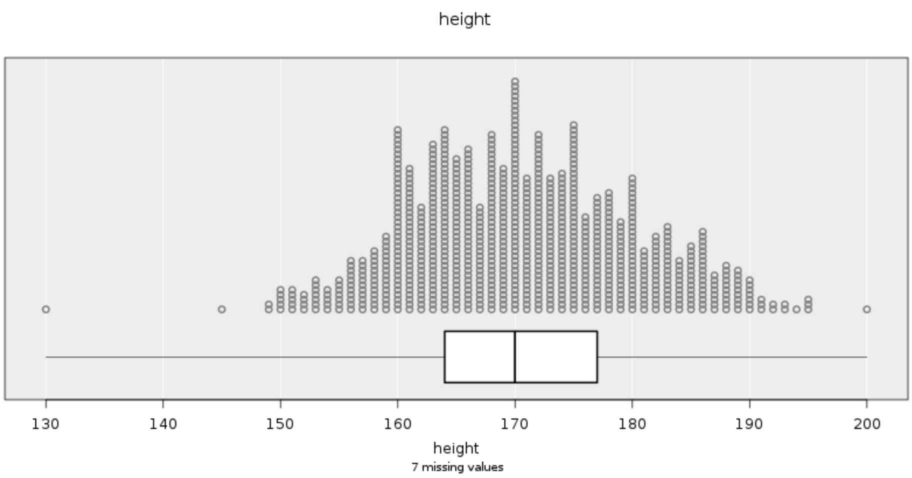
Note. The formula for the normal distribution curve with mean μ and standard deviation σ is

$$y = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right).$$

											Differences								
z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359	4	8	12	16	20	24	28	32	36
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754	4	8	12	16	20	24	28	32	36
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141	4	8	12	15	19	22	27	31	35
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517	4	8	11	15	19	22	26	30	34
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879	4	7	11	14	18	22	25	29	32
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224	3	7	10	14	17	21	24	27	31
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549	3	6	10	13	16	19	23	26	29
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852	3	6	9	12	15	18	21	24	27
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133	3	6	8	11	14	17	19	22	25
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389	3	5	8	10	13	15	18	20	23
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621	2	5	7	9	12	14	16	18	21
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830	2	4	6	8	10	12	14	16	19
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015	2	4	5	7	9	11	13	15	16
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177	2	3	5	6	8	10	11	13	14
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319	1	3	4	6	7	8	10	11	13
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441	1	2	4	5	6	7	8	10	11
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545	1	2	3	4	5	6	7	8	9
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633	1	2	3	3	4	5	6	7	8
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706	1	1	2	3	4	4	5	6	6
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767	1	1	2	2	3	4	4	5	5
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817	0	1	1	2	2	3	3	4	4
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857	0	1	1	2	2	2	3	3	4
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890	0	1	1	1	2	2	2	3	3
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916	0	0	1	1	1	2	2	2	2
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936	0	0	1	1	1	1	1	2	2
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952	0	0	0	1	1	1	1	1	1
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964	0	0	0	0	1	1	1	1	1
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974	0	0	0	0	0	1	1	1	1
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981	0	0	0	0	0	0	0	0	1
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986	0	0	0	0	0	0	0	0	1
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990	0	0	0	0	0	0	0	0	0
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993	0	0	0	0	0	0	0	0	0
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995	0	0	0	0	0	0	0	0	0
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997	0	0	0	0	0	0	0	0	0
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998	.4998	0	0	0	0	0	0	0	0	0
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	0	0	0	0	0	0	0	0	0
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.5000	.5000	.5000	0	0	0	0	0	0	0	0	0
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	0	0	0	0	0	0	0	0	0

Questions

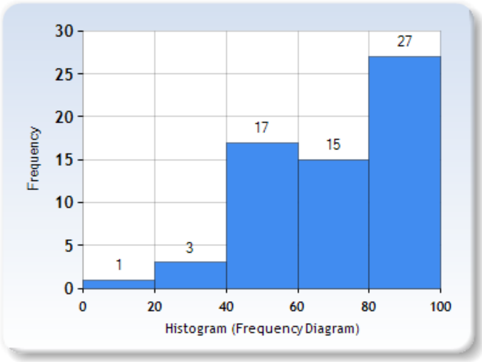
1. A manufacturing company finds the life of a battery for a product to be normally distributed, with mean 4 years and standard deviation 1 year.
   (a) Given a randomly chosen battery, what is the chance it lasts between three and five years?
   (b) Given 10 000 batteries, how many should be expected to have an abnormally short lifespan (less than two years)?
   (c) What is the minimum length of life for the longest-lived 10%?
2. In the 2017 Census at School, 866 Y12 students answered the following question: 'What is your height (nearest cm), without shoes on?'. The mean of this sample was 170.6 cm, and the standard deviation was 9.476 cm.
   (a) Draw a standard deviation fitting these parameters.
   (b) What is the probability that a random individual from this sample has a height:
      i. Between 160 and 180 centimetres?
      ii. Less than 150 centimetres?
   (c) What is the minimum height of the tallest 10% of people? (In other words, which height x is such that there is a 0.1 probability that a sampled individual has a height greater than x?)
   (d) A histogram for the responses is given below.



   Compare the normal distribution and the histogram that resulted from the sample results.

   In your answer you should consider the shape, centre, and spread of both distributions, and should provide numerical evidence where appropriate.
   (e) In a given school, 10% of year 12 students were taller than 175.0 cm, and the mean height of the students was 167.2 cm.
      i. What is the standard deviation of this smaller sample of students?
      ii. In this school, what was the maximum height of the shortest 20% of students?
      iii. What is the probability that, given two randomly selected Y12 students at this school, both have heights greater than 180 cm? (You may assume that all heights are independent — e.g. there are no identical twins.)
3. The grades of students in a certain course have a mean of 71.04% and a standard deviation of 21.168 percentage points.
   (a) Assuming the grades match a standard distribution, what is the probability that a randomly chosen student passed (had a grade of greater than 50%)?

(b) The following histogram depicts the actual grade distribution.



Discuss whether a normal distribution would actually be a good fit for this data.

(c) Using the histogram, compute the probability that a randomly chosen student passed.

NCEA Level 2 Mathematics (Homework)
23. Probability Distributions

Reading

Go and watch...

`https://www.youtube.com/watch?v=UCmPmkHqHXk`

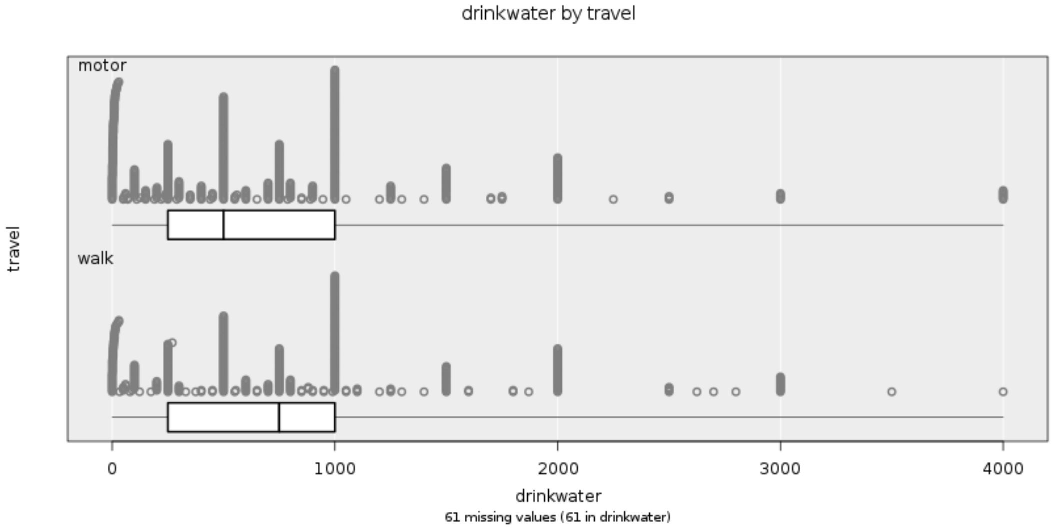
`https://www.smbc-comics.com/comic/stats-gang`

Questions

1. How are probability distributions related to the kind of data analysis we began probability with? Discuss. (Write around half a page, and do some research.)

2. In 2017, one of the Census at School questions asked for the amount of water, in millilitres, drunk in the last day. The following graphs and statistical information shows the data for this question for students

who drove to school, and for students who walked.




```
===========================================================================
                           iNZight Summary
---------------------------------------------------------------------------
        Primary variable of interest: travel (categorical)
                  Secondary variable: drinkwater (numeric)

        Total number of observations: 1000
   Number omitted due to missingness: 61 (61 in drinkwater)
   Total number of observations used: 939
===========================================================================

Summary of drinkwater by travel:
--------------------------------

          Min   25%   Median    75%    Max    Mean      SD   Sample Size
   motor    0   250      500   1000   4000   728.5   707.1           530
    walk    0   250      750   1000   4000   825.9   733.1           409


===========================================================================
```

(a) Compare and contrast the shapes of the statistics for the two groups.

(b) Is there any meaningful difference between the volume of water drunk in the last day between the two groups?

(c) Calculate the probability that a randomly chosen person from each group drank more than 800 mililitres in the last day. What is the relative risk — is a walker more likely to have drunk more than 800 mililitres compared to a motorist?

(d) Suppose that at a particular school the mean value for water drunk in the last day (over all students) was 712 mL. The survey also shows that only 10% of students drank less than 400 mL.

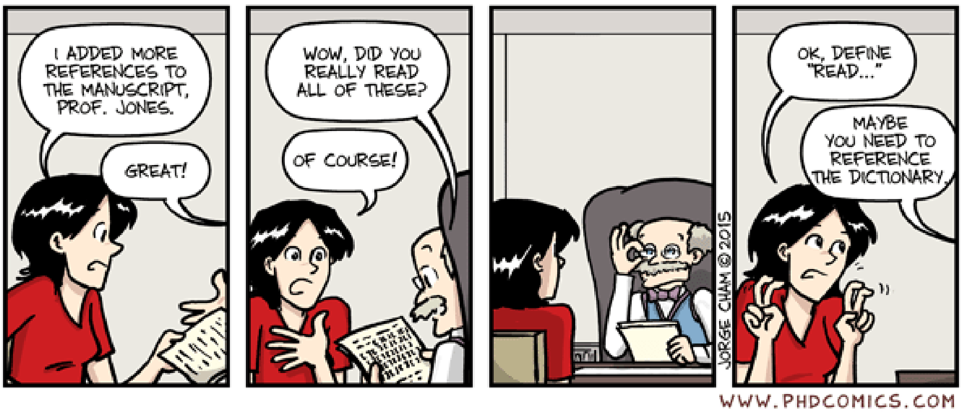
  i. What is the standard deviation of the sample in this case?

  ii. What is the minimum amount of water for a student to have drunk in the last day in order to be in the top 5% of water drinkers, at this school?

Chapter 7

Addendum

NCEA Level 2 Mathematics Bibliography



See the preface for a guide to the bibliography.

- Bellos, Alex — *Alex's adventures in numberland* (2010, Bloomsbury)
- Bellos, Alex — *Alex through the looking-glass* (2014, Bloomsbury)
- Bóna, Miklós — *A walk through combinatorics (3e)* (2011, World Scientific Publishing)
- Cipra, Barry — *Misteaks and how to find them before the teacher does (3e)* (2000, Academic Press)
- Dudley, Underwood — *A budget of trisections* (1987, Springer-Verlag)
- Foerster, Paul — *Trigonometry: functions and applications (2e)* (1977, Addison-Wesley Publishing Co.)
- Freedman, David; Pisani, Robert; and Purve, Roger — *Statistics* (1978, W. W. Norton and Co.)
- Fritsch, Rudolf and Gerda — *The four-color theorem* (1998, Springer)
- Goldacre, Ben — *Bad pharma* (2012, Fourth Estate)
- Goldacre, Ben — *Bad science (new ed.)* (2009, Harper)
- Huff, Darrell — *How to lie with statistics* (1993, W. W. Norton and Co.)
- Kline, Morris — *Mathematics for the non-mathematician* (1985, Dover)
- Lang, Serge; and Murrow, Gene — *Geometry: A high-school course* (1983, Springer-Verlag)
- Lauwerier, Hendrik — *Fractals: endlessly repeated geometric figures* (1991, Priceton University Press)
- Lockhart, Paul — *A mathematician's lament* (2009, Bellevue)
- Mandelbrot, Benoit — *The fractal geometry of nature* (1983, W.H.Freeman)
- Pfenning, Nancy — *Elementary statistics: looking at the big picture* (2011, Brooks/Cole)
- Pólya, George — *How to solve it: a new aspect of mathematical method (2e)* (1971, Princeton University Press)
- Posamentier, Alfred S. and Salkind, Charles T. — *Challenging problems in geometry* (1996, Dover)
- Spiegel, Murray — *Theory and problems of college algebra* (1956, Schaum Publishing Co.)
- Steenrod, Norman — *How to write mathematics* (1973, AMS)
- Thompson, Silvanus; and Gardner, Martin — *Calculus made easy* (1998, St Martin's Press)
- Wilson, Robert — *Graphs, colourings, and the four-colour theorem* (2002, Oxford University Press)
- Wilson, Robin — *Introduction to graph theory (4e)* (1996, Longman)