# Calculus of Happiness

Question 1: Equations of Fun

Take the following system of equations:

$$9^{2x+y} - 9^x \times 3^y = 6$$
$$\log_{x+1}(y+3) + \log_{x+1}(y+x+4) = 3$$

(a) i. Express $9^{2x+y} - 9^x \times 3^y = 6$ in the form $u^2 - u - 6 = 0$ where $u = 3^{2x+y}$.

ii. Solve the quadratic equation for $u$, and hence find possible real values for $2x + y$.

(b) i. Express the relation $\log_{x+1}(y+3) + \log_{x+1}(y+x+4) = 3$ in the form $p(x, y) = (x+1)^3$ for some polynomial $p$.

ii. Expand the new equation into the form $q(x, y) = 0$ for some other polynomial $q$.

iii. Substitute in the expression linking $x$ to $y$ from part (a) above, and hence solve the system of simultaneous equations for real values of $x$ and $y$.

Question 2: Parabola of Joy

A car is driving along a road shaped like a parabola at night. The parabola has a vertex at the origin, and the car starts at a point 100 m west and 100 m north of the origin.

(a) Write an equation modelling the road as a parabola.

(b) Find the general equation for the tangent line to the parabola at some point $(x_0, y_0)$, and substitute into it the parabola equation to obtain an equation only in $x$, $x_0$, and $y_0$.

(c) Suppose there is a statue of the Roman emperor Augustus located 100 m east and 50 m north of the origin. Write the equation for the tangent line of the parabola passing through the statue (so that it only depends on a value $x$ on the parabola).

(d) Solve this equation for $x$, and hence find the single point $(x, y)$ on the road where the headlights of the car illuminate the statue.

Question 3: Limits of Excitement

A function $f(x)$ is *continuous* at some point $(x_0, f(x_0))$ if and only if $\lim_{x \to x_0} f(x)$ exists and is equal to $f(x_0)$.

(a) Find a value of $k$ such that $F(x)$ is continuous at $x = -3$ where

$$F(x) = \begin{cases} \frac{x^2-9}{x+3} & \text{if } x \neq -3 \\ k & \text{if } x = -3 \end{cases}$$

(b) Show whether or not $g(x)$ is continuous at 2, 3, and 4 where

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leq 2 \\ 2 - x & \text{if } 2 < x \leq 3 \\ x - 4 & \text{if } 3 < x \leq 4 \\ \pi & \text{if } x \geq 4 \end{cases}$$

(c) Find all values of $\alpha$ such that $\Phi$ is continuous on $\mathbb{R}$ where

$$\Phi(x) = \begin{cases} x + 1 & \text{if } x \leq \alpha \\ x^2 & \text{if } x > \alpha \end{cases}$$