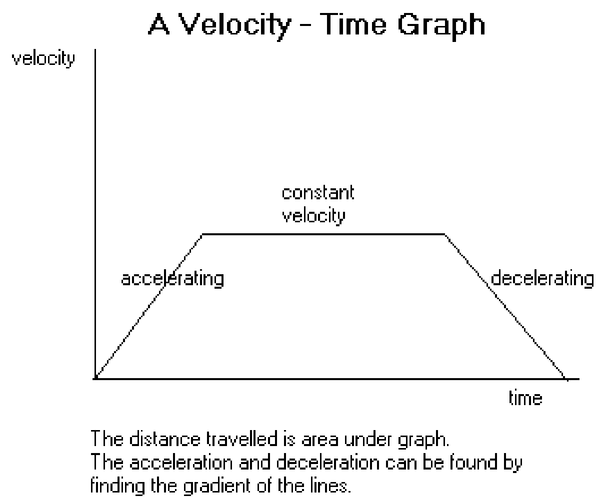
NCEA Level 3 Calculus (Differentiation)
12. Kinematics

This is an easy week, especially if you are doing physics (or did level 2 physics). Take a well-earned break! Calculus was independently developed by Sir Isaac Newton to describe motion in physics. This use is known as *kinematics* (from the Greek *kinein*, 'to move'). Suppose a particle moves from position x_0 to position x_1 over a time Δt. We call the ratio

$$\frac{x_1 - x_0}{\Delta t}$$

the *average velocity* of the particle; if we let $x_1 \to x_2$ (or let $t \to 0$), we obtain the derivative $\frac{\mathrm{d}x}{\mathrm{d}t} = v$, the *instantaneous velocity* of the particle at the point x (usually just abbreviated to velocity).

Similarly, the rate of change of velocity is called *acceleration*. We have $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$. (Out of interest, the third derivative of displacement is known as *jerk*, and the fourth is *jounce*.)



If we draw a velocity-time graph, like that above, the area under the graph gives us the distance travelled by the object during a time interval; this means that in some sense finding area is inverse to finding slope. Intuitively, this is a bit weird — the way to think about it is that if we increase the area under the velocity curve by a little bit, then the distance we move in that time also increases proportionally. On the other hand, if we move a certain distance then we increase the area under the velocity curve because we have a non-zero velocity.

Algebraically, we have rate $= \frac{\text{change in quantity}}{\text{time}}$ and so change in quantity = rate × time. In calculus, we are interested in finding the total change in a quantity over a longer period of time than just a single instant, by adding up infinitely many instantaneous rates.

We will explore this idea in more detail in a couple of weeks when we start talking about *integration*.

We can derive the following *kinematic equations* if acceleration is kept constant over a time period t:

$$v_f = v_i + a\Delta t$$
$$\Delta x = v_i \Delta t + \frac{1}{2} a \Delta t^2$$
$$v_f^2 = v_i^2 + 2a\Delta x$$
$$\Delta x = \frac{v_f + v_i}{2} \Delta t$$

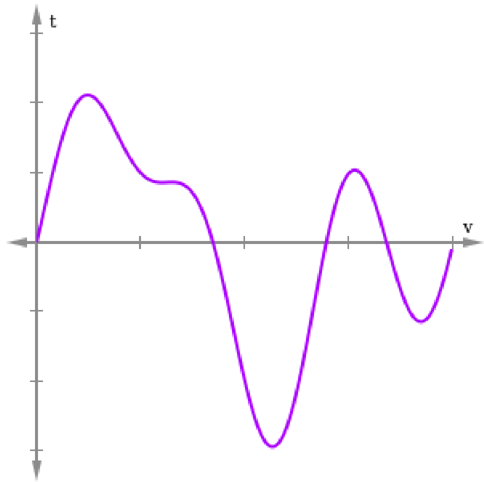
These can all be found from the definitions of acceleration and velocity, and by taking areas and slopes of graphs (if we remember that the velocity graph must be made up of straight lines since the acceleration is constant). The equations are not examinable in the L3 calculus exam. We mainly look at them for context.

Questions

All distances are given in m, and all times in s, unless otherwise stated.

1. A Describe the relationship between velocity and displacement.
2. A A particle moves from $x = 2\,\mathrm{m}$ to $x = 3\,\mathrm{m}$ over a time $3\,\mathrm{s}$. What is its average velocity over that time?
3. A A particle moves from $(3, 4)$ to $(12, -3)$ over a time $10\,\mathrm{s}$. If the displacements are measured in metres, what is the magnitude of its average velocity during that period?
4. A An object A has a positive acceleration a, and a second object B has a negative acceleration $-a$. Both are moving in the same direction. Which of the following is **not** true?
   (a) Object B is slowing down compared to object A.
   (b) Object B has a lower velocity than object A.
   (c) At some point, object B will reach a velocity of zero and then start moving in the opposite direction.
   (d) If object B is behind object A, the two will never cross paths.
5. A Suppose a particle has a velocity of $34\,\mathrm{m\,s^{-1}}$. How long does it take for the particle to travel $150\,\mathrm{m}$?
6. The velocity v of an object t seconds after it moves from the origin is given by
$$v(t) = 3t^2 - 6t - 24.$$
   (a) A Write down the formulae for the acceleration of the particle after t seconds.
   (b) A Work out the initial velocity and acceleration.
   (c) A When is the object at rest momentarily?
   (d) M When is the object travelling at minimum speed?
   (e) M When did the object return to the origin?
7. A A well-wrapped food parcel is dropped from an aeroplane flying at a height of $500\,\mathrm{m}$ above the ground. The constant acceleration due to gravity is $-9.81\,\mathrm{m\,s^{-2}}$. Air resistance is negligible.
   (a) How long does it take for the food parcel to hit the ground?
   (b) How fast is the food parcel moving when it hits the ground?
8. A A racing car travelling at $210\,\mathrm{km\,h^{-1}}$ skids for a distance of $150\,\mathrm{m}$ after its brakes are applied. The brakes provide a constant deceleration.
   (a) What is the deceleration in $\mathrm{m\,s^{-2}}$?
   (b) How long does it take for the car to stop?

9. $\boxed{\texttt{M}}$ The following is a graph of the instantaneous velocity of an object moving in one dimension over time.



(a) Draw the acceleration of the object over time.

(b) Draw the position of the object over time, if it was originally located at $x = 0$.

10. $\boxed{\texttt{E}}$ The displacement of an object moving in a straight line on either side of a fixed origin is given by

$$s(t) = 2t^3 - 12t^2 + 18t + 3.$$

Find the minimum velocity of the object. Prove that it is a minimum. Which position is it at at that time?

11. $\boxed{\texttt{M}}$ Derive the kinematic equations by looking at the velocity graph of an object with constant acceleration.