NCEA Level 2 Mathematics
2. Arcs and Sectors of Circles

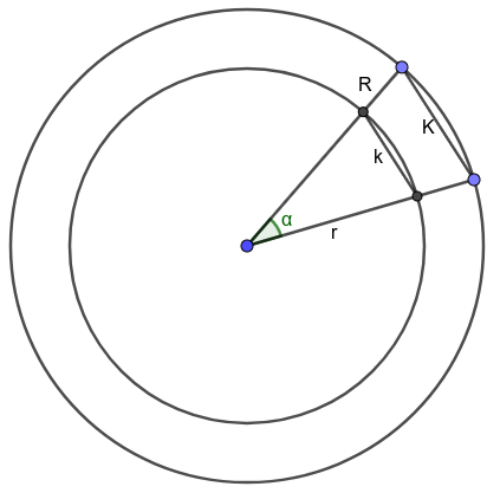
The Greeks studied two fundamental geometric objects: lines, and circles. Last week we looked at lines; this time, we will look at circles.

A circle is simply the set of all points that are at a distance r from a point (x_0, y_0): the point (x_0, y_0) is called the centre of the circle, and the distance r is the radius of the circle. If (x, y) is a point on the circle, then we have

$$d((x, y), (x_0, y_0)) = r \implies \sqrt{(x - x_0)^2 - (y - y_0)^2} = r$$

and the equation of the circle in cartesian coordinates is $(x - x_0)^2 + (y - y_0)^2 = r^2$.

Circumferii



Suppose the angle α is measured in degrees. Then the circumference of the inner circle is $c \approx \frac{180^\circ}{\alpha} k$, and the circumference of the outer circle is $C \approx \frac{180^\circ}{\alpha} K$. Now, the two triangles formed are similar since they have an identical angle and two sides with the same ratio of r/R; hence $k/K = r/R$. We can rewrite:

$$\frac{k}{K} = \frac{r}{R} \implies \frac{c\frac{\alpha}{180^\circ}}{C\frac{\alpha}{180^\circ}} \approx \frac{r}{R} \implies \frac{c}{C} \approx \frac{r}{R}.$$

As the size of the angle α becomes smaller and smaller, this approximation becomes exact: $\frac{c}{C} = \frac{r}{R}$, and so $\frac{c}{r} = \frac{C}{R}$. In other words, the ratio of the circumference of any circle to its radius is always the same. For historical reasons, we actually write this in terms of the diameter, and call the constant of proportionality π. We have therefore sketched a proof that

$$c = 2\pi r$$

for any circle with radius r and circumference c.

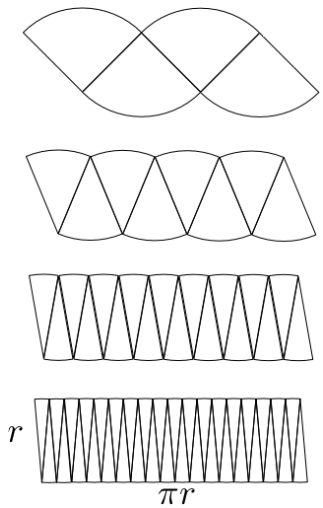
The number π is approximately equal to

$$3.14159265358979323846264 33...$$

and in one of the exercises below you will calculate a first approximation to this value: it isn't just a number that is plucked out of thin air!

Areas

The other main result we have for circles is the area; by slicing the circle into smaller and smaller pieces, we can approximate the area of a circle with the area of a rectangle:



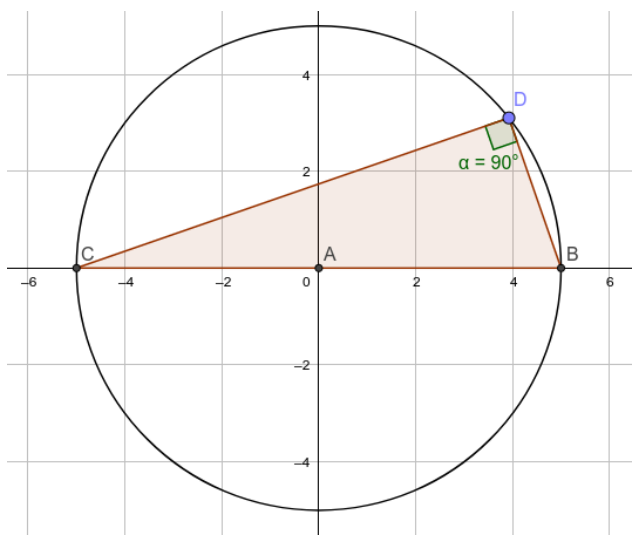
By means of this, we can intuitively justify that the area of a circle with radius r is

$$A = \pi r^2.$$

This idea of a limiting process will be made more clear next year (when you will be able to provide proper proofs of these facts), but hopefully these two results seem plausible.

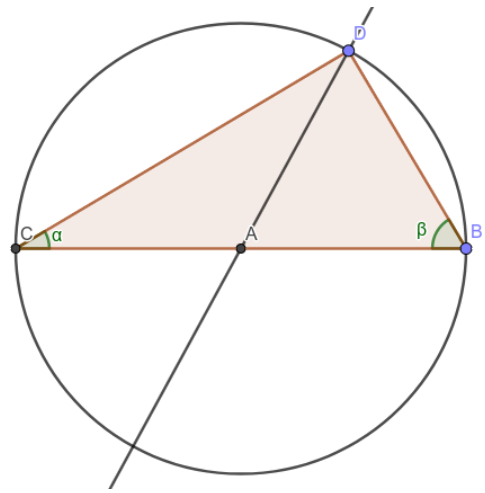
Thale's Theorem

As a taster for some of the other pretty theorems one can prove about circles, consider any circle; pick a diameter of the circle and any point on the circle itself; then the resulting triangle is always right-angled, as in the following diagram.



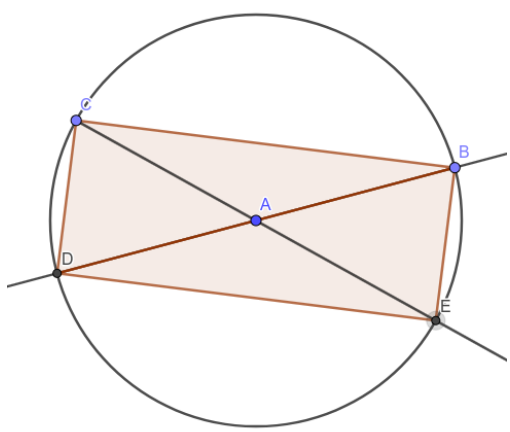
I actually know *two* proofs of this; we'll look at both!

Angle-y proof. The first proof is quite cute: we simply consider the next figure and do some angle pushing.



Clearly ACD and ABD are both isosceles, so $CDA = \alpha$ and $ADB = \beta$; hence $CDB = \alpha + \beta$, and so (using the fact that the internal angles of a triangle add to $180°$) we have $180° = \alpha + \beta + \alpha + \beta$; so $CDB = \alpha + \beta = 180°/2 = 90°$. $\square$

Rotate-y proof. The intuitive idea behind the second proof is that we take the triangle, 'rotate it around', and see that the resulting shape is a rectangle. In order to make this idea more precise, consider the following diagram.



Here, we took our initial triangle to be BCD, sitting on the diameter BD. Draw the line through C and A; clearly (since it passes through the centre) it is a diameter of the triangle. Hence the two diagonals of the quadrilateral $BCDE$ are the same length, and it is therefore a rectangle (so in particular DCB is a right angle). $\square$

Which proof do you prefer? Why?

Questions

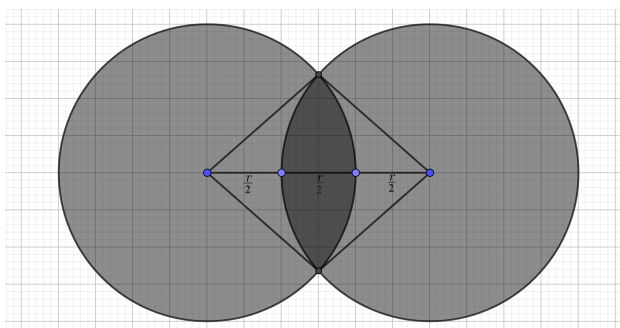
1. Suppose we take a circle of radius r, and cut out a slice with an angle α (like cutting a pizza). Draw a picture. What is the area of this slice, and what is the length of the arc of the circumference that is part of the slice? (Such a slice is called a *sector*.)

2. Notice that the formulae derived above all have ugly factors of $360°$. Define one radian to be the angle such that the arc length of the sector defined by that angle is just r, the radius of the circle. Radians are, in many ways, a much more natural angle measurement unit.
   (a) Draw a picture to show this geometrically.
   (b) Show that one radian is precisely $\frac{\pi}{180}$ degrees.
   (c) How many radians are in a full circle?
   (d) Show that, in radians, the formulae derived in question 1 above simplify dramatically.

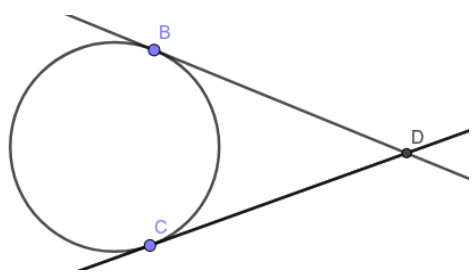
3. Suppose a circle has area 49π. What is the arc length of a sector of this circle with area 25π?

4. Show that if the angle subtended by a chord at the centre is $90°$ then $\ell = \sqrt{2}r$, where ℓ is the length of the chord.

5. Find the area of the shaded region in the diagram below.

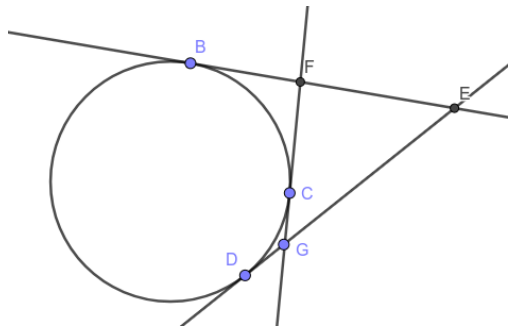


6. Suppose that two lines tanget to a circle at points B and C intersect at a point D, as shown. Show that the two segments BD and CD have equal lengths.



7. A simple definition of a circle is the set of all points $P = (x, y)$ such that $d(A, P) = r$ (where A is the centre of the circle and r is the radius of the circle). Use this definition to write an equation for the circle of radius r centred at (x_0, y_0).

8. In the following figure, all three lines are tangent to the circle. If the length of the segment BE is 5, what is the perimeter of the triangle FGE? [Hint: use the previous result above.]



9. (a) Find the area of the largest square that one can fit inside a circle of radius r.
   (b) Find the area of the smallest square that fits outside a circle of radius r.
   (c) Hence show that $2 < \pi < 4$.
   (d) How might you improve your estimate of π?