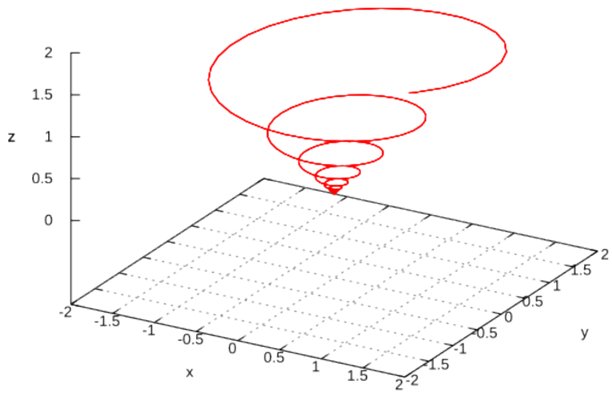
NCEA Level 3 Calculus (Differentiation)
12. Parametric Functions

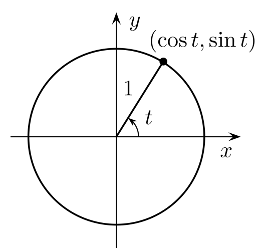


Some curves cannot be described simply with a function; for example, the above track of a particle is too complicated to analyse using any of the techniques which we have studied so far. One strategy which does work is to split the x, y, and z components apart and study them seperately. For example, we can **parameterise** the above curve as:

$$\begin{aligned} x(t) &= e^{-t}\cos(10t) \\ y(t) &= e^{-t}\sin(10t) \\ z(t) &= e^{-t}. \end{aligned}$$

With this example as our initial motivation, we move from discussing functions $\mathbb{R} \to \mathbb{R}$, and turn our attention to functions $\mathbb{R} \to \mathbb{R}^n$. This type of function is often called a **curve**; a set of component functions, like the one given above for the spiral, is called a **parameterisation**.

For a simpler example, consider the unit circle $x^2 + y^2 = 1$. By recalling the definitions of the trigonometric functions, we can parameterise the circle as $(x, y) = (\cos\theta, \sin\theta)$ for $0 \leq \theta < 2\pi$. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = -\cos\theta = -x$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \sin\theta = y$, so by the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{x}{y}$ — a much simpler calculation than taking the derivative of the square root required by working directly with the circle formula.



In order to justify this in general, we note that the output of a curve is a set of ordered pairs and is therefore (locally) a function from the first coordinate to the second coordinate. Suppose that we have some curve that is parametrised by $(x, y) = (f(t), g(t))$. Then $y = g(f^{-1}(x))$ and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{f'(f^{-1}(x))} \cdot g'(f^{-1}(x)) = \frac{1}{f'(t)} \cdot g'(t) = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}.$$

In order to find the second derivative, we replace y with $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}\frac{\mathrm{d}y}{\mathrm{d}x}}{\mathrm{d}x} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\mathrm{d}y}{\mathrm{d}x}\right) \cdot \frac{\mathrm{d}t}{\mathrm{d}x}\,.$$

Questions

1. In each case find $\frac{dy}{dx}$. M
   (a) $x = t\sin t$, $y = t^2 + t$
   (b) $x = 2\sec\theta$, $y = 3\tan\theta$
   (c) $x = \cos\theta$, $y = \cos 3\theta$
   (d) $x = e^{\sin\theta}$, $y = e^{\cos\theta}$

2. Find the equation of the chord joining the two points $t = 2$ and $t = 4$ on the curve $(x, y) = (2t - 3, t^3 + 6)$. A

3. Determine the point(s) of intersection of the curves γ and δ: M

$$\gamma : t \mapsto (t^2 - 2, t - 1)$$
$$\delta : t \mapsto (t, 2/t)$$

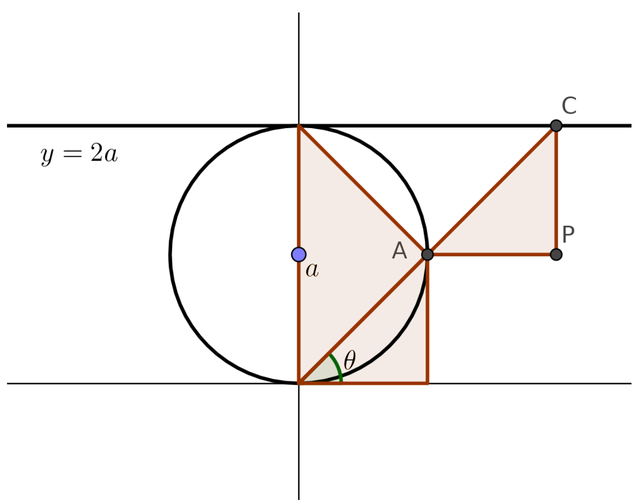
4. (a) If $y = 2t$ and $x = 4t^2$ define a curve, what is the gradient $\frac{dy}{dx}$ in terms of t? M
   (b) Show that this curve is a parabola. E

5. A curve has parametric equations $x = t^2 + 1$ and $y = t^3 + 2$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. M

6. Find the equation of the tangent to the curve $t \mapsto (2x^2 + 1, t^3 - 1)$ at $t = 2$. M

7. If $t \mapsto (x, y)$ is a parametric curve, find an expression for $\frac{d^3y}{dx^3}$ analagous to that found for the second derivative. E

8. A curve, called a *witch of Maria Agnesi*, consists of all possible positions of the point P in the diagram below. Show that the curve is given by $(x, y) = (2a\cot\theta, 2a\sin^2\theta)$ and find the derivative $\frac{dy}{dx}$. S



9. A particle moves through space over time; the position of the particle at time t is given by $(3\sin t, 2\cos t)$ $(0 \le t < 2\pi)$.
   (a) What is the component of the acceleration of the particle in the x direction at $t = \pi/4$? A
   (b) At what times is the particle stationary in the x direction? A
   (c) Is the particle ever momentarily totally stationary? M

10. Find the rightmost point on the curve $x = t - t^6$, $y = e^t$. E

11. For which values of t is the curve $x = \cos 2t$, $y = 3\cos t$ concave up? E

12. Show that the curve $\gamma : t \mapsto (\cos t, \sin t \cos t)$ has two tangents at (0,0) and find their equations. S

13. Scholarship 2000: The piriform is the curve defined by the equation $16y^2 = x^3(8 - x)$ where $x \geq 0$. S
    (a) Show that
$$\begin{cases} x = 4(1 + \sin\theta) \\ y = 4(1 + \sin\theta)\cos\theta. \end{cases}$$
    are parametric equations for the piriform.
    (b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ, and show that $\theta = \frac{\pi}{6}$ is a stationary point of the curve.

14. We define a surface C parametrically in terms of two parameters, t and θ: O
$$(x, y, z) = (t, t\cos\theta, t\sin\theta).$$
    (a) Show that the Cartesian equation for this surface is $x^2 = y^2 + z^2$. (This is a cone.)
    (b) Show that the intersection between C and the plane $z = 2$ is a hyperbola.
    (c) For what angle α does the intersection between C and the plane parametrically defined by
$$(x, y, z) = (u\tan\alpha + 1, u, v)$$
    (for parameters u and v) become a parabola? (Hint: $x = y\tan\alpha + 1$, and z is arbitrary.)