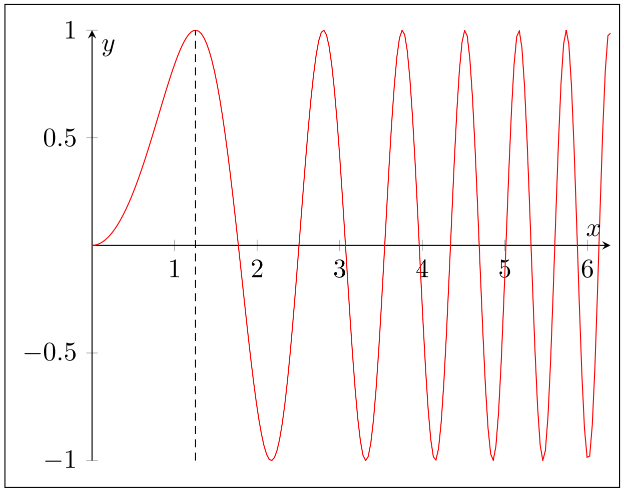
NCEA Level 3 Calculus (Differentiation)
4. The Chain Rule



Consider the function $x \mapsto \sin(x^2)$. This function is made up of two functions, applied one after the other:

$$x \overset{f}{\mapsto} x^2 \overset{g}{\mapsto} \sin(x^2).$$

We often notate this *function composition* as $g \circ f$ (note that we evaluate from the right, so $(g \circ f)(x) = g(f(x))$).

Obviously the derivative of $\sin(x^2)$ is not just $\cos(2x)$, since the former has a stationary point at $x = \sqrt{\dfrac{\pi}{2}}$ but $\cos(\sqrt{2\pi}) \neq 0$. This shows us that, in general, the derivative of a function composition is not simply the composition of the derivatives.

In fact, it turns out that the derivative of $f \circ g$ is $g'(f' \circ g)$; in other words,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = g'(x) f'(g(x)).$$

This is known as the *chain rule*, since we are "chaining" together functions.

Before proving the chain rule, let us convince ourselves that it is plausible. We can interpret the derivative $\frac{\mathrm{d}g}{\mathrm{d}x}$ as the rate of change of g with respect to x, and the derivative $\frac{\mathrm{d}f}{\mathrm{d}g}$ as the derivative of f with respect to small changes in g; it is intuitive that if g changes twice as fast as x at some point, and f changes five times as fast as g, then f changes $2 \times 5 = 10$ times as fast as x.

A rigorous proof is given below, and it matches our intuition reasonably well.

Example. The correct derivative of $\sin(x^2)$ is $2x\cos(x^2)$.

Example. If $f(r) = \sqrt{r^2 - 3}$, then $f'(r) = 2r\frac{1}{2}\left(r^2 - 3\right)^{-1/2} = \dfrac{r}{\sqrt{r^2 - 3}}$.

Example. If $g(x) = \sin((\sin^7 x^7 + 1)^7)$, then we compute:

$$g(x) = \sin\left(\left[\left(\sin x^7\right)^7 + 1\right]^7\right)$$

$$g'(x) = 7x^6 \cdot \cos x^7 \cdot 7\left(\sin x^7\right)^6 \cdot 7\left[\left(\sin x^7\right)^7 + 1\right] \cdot \cos\left(\left[\left(\sin x^7\right)^7 + 1\right]^7\right)$$

This result can probably be simplified, however the point is to evaluate the derivative chain from inside to outside in a systematic fashion.

Example. One of the main uses of the chain rule is in related rates problems. For example, consider a disc of radius r. The area of this disc is given by $A = \pi r^2$, and so $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$. But what if r is itself changing with respect to time, say at a rate of $3\,\mathrm{m\,s^{-1}}$? Then $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}(2\pi r) = 6\pi r = 6\pi(r_0 + 3t)$ (where r_0 is the radius at $t = 0$).

Proof of the chain rule (optional). The proof is a little fiddly, and comes in two parts. Recall that in the work on limits, we found that an alternative definition of the derivative of f at x was

$$f'(x) = \lim_{k \to x} \frac{f(x) - f(k)}{x - k}.$$

Now, suppose we wish to find the derivative of $f \circ g$ at x. In the first case, suppose that g is not constant around x (in other words, we can zoom in 'far enough' towards x so that for all k in the zoomed in area, $g(k) \neq g(x)$). Then:

$$\begin{aligned}
\lim_{k \to x} \frac{f(g(x)) - f(g(k))}{x - k} &= \lim_{k \to x} \frac{f(g(x)) - f(g(k))}{g(x) - g(k)} \cdot \frac{g(x) - g(k)}{x - k} \\
&= \lim_{k \to x} \frac{f(g(x)) - f(g(k))}{g(x) - g(k)} \cdot \lim_{k \to x} \frac{g(x) - g(k)}{x - k} \\
&= f'(g(x))g'(x),
\end{aligned}$$

(noting that as $k \to x$, $g(k) \to g(x)$). This calculation only works when g is not constant around x, because if g *is* constant around x then for all k sufficiently close to x, $g(x) - g(k) = 0$ and the limit does not exist.

To deal with this case, assume that g *is* constant around x. Then $g'(x) = 0$, and also for all h close enough to zero we have $g(x + h) = g(x)$. Then

$$(f \circ g)'(x) = \lim_{h \to 0} \frac{f(g(x + h)) - f(g(x))}{h} = \lim_{h \to 0} \frac{f(g(x)) - f(g(x))}{h} = 0 = 0 \times f'(g(x)) = g'(x)f'(g(x)).$$

$\square$

Questions

1. $\boxed{\texttt{A}}$ Identify the inner and outer functions, but do not attempt to differentiate.
   (a) $\sqrt{\sin x}$
   (b) $\sin\cos\tan x$
   (c) $(2x+3)^{17}$
   (d) $97(x+2)^2$
   (e) $\ln\sin x$
   (f) $\frac{1}{\sqrt{23x-x^2}}$

2. $\boxed{\texttt{A}}$ Differentiate with respect to t:
   (a) $(2t+3)^{3000}$
   (b) $\sin\ln t$
   (c) $\sqrt{t^3+10t^2+3}$
   (d) $\csc e^t$
   (e) $\sin^3 t + 14\ln(3t)$
   (f) $\sin\sin\sin t$
   (g) $\cot(t+\sec t)$
   (h) $\sin^2((t+\sin t)^2)$
   (i) $\ln\sqrt{t+9}$
   (j) $\sqrt{t}+\frac{1}{\sqrt[3]{t^4}}$
   (k) $e^{\sec(t^2)}$
   (l) $\sin\sqrt{t+\tan t}$

3. $\boxed{\texttt{A}}$ The derivative of a function is $2\cos 2x$. What could the original function be?

4. $\boxed{\texttt{M}}$ Differentiate $y=\sin^2 x+\cos^2 x$, and hence prove that $\sin^2 x+\cos^2 x=1$.

5. $\boxed{\texttt{A}}$ Suppose that the displacement of a particle on a vibrating spring is given by $x(t)=5+\frac{1}{8}\sin(5\pi t)$, where x is measured in centimetres and t in seconds.
   (a) Find the velocity of the particle at time t.
   (b) At which times is the particle momentarily stationary?

6. The volume of a spherical balloon at a time t is given by $V(t)$, and its radius is given by $r(t)$.
   (a) $\boxed{\texttt{A}}$ What do the derivatives $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}r}$ represent?
   (b) $\boxed{\texttt{M}}$ The volume of a sphere of radius r is $V_{\text{sphere}}=\frac{4}{3}\pi r^3$. Find $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of $\frac{\mathrm{d}r}{\mathrm{d}t}$.

7. $\boxed{\texttt{M}}$ If $F(x)=f(3f(4f(x)))$, where $f(0)=0$ and $f'(0)=2$, find $F'(0)$.

8. $\boxed{\texttt{A}}$ Suppose $f(x)=g(x+g(a))$ for some differentiable function g and constant a. Find $f'(x)$.

9. The depth of water at the end of a jetty in a harbour varies with time due to the tides. The depth of the water is given by the formula
$$W=4.5-1.2\cos\frac{\pi t}{6}$$
   where W is the water depth in metres, and t is the time in hours after midnight.
   (a) $\boxed{\texttt{A}}$ What is the rate of change of water depth 5 hours after midnight?
   (b) $\boxed{\texttt{M}}$ When is the first time after $t=0$ that the tide changes direction?
   (c) $\boxed{\texttt{E}}$ At that time, is the water changing from rising to falling or from falling to rising?

10. Consider the function ψ given by

$$\psi(t) = \frac{e^t + e^{-t}}{2}$$

   (a) $\boxed{\texttt{A}}$ Compute ψ'.

   (b) $\boxed{\texttt{M}}$ Find $\psi^{(2017)}$ (the 2017th derivative of ψ).

   (c) $\boxed{\texttt{A}}$ Show that ψ satisfies the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}(\psi + \psi') = \psi + \psi'$.

11. The force F (in newtons) acting at an angle θ with the horizontal that is needed to drag a mass of W kilograms along a horizontal surface at a constant velocity is given by

$$F = \frac{\mu W}{\cos\theta + \mu \sin\theta}$$

   where μ is the coefficient of static friction (a constant).

   (a) $\boxed{\texttt{A}}$ If $W = 200\,\mathrm{kg}$ and $\mu = 0.2$, find $\frac{\mathrm{d}F}{\mathrm{d}\theta}$ when $\theta = \frac{\pi}{6}$ rad.

   (b) $\boxed{\texttt{M}}$ Suppose now that θ is a function of time, so that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.5\,\mathrm{rad/s}$. Find $\frac{\mathrm{d}F}{\mathrm{d}t}$.

12. $\boxed{\texttt{E}}$ Find the 73rd derivative of $\sin 6x$.

13. $\boxed{\texttt{E}}$ Recall that the *absolute value* of x, denoted $|x|$, is the value obtained by 'throwing away the sign' of x.

   (a) Prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}|x| = \frac{x}{|x|}.$$

   [*Hint: Write* $|x| = \sqrt{x^2}$.]

   (b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of both f and f'.