

NCEA Level 2 Mathematics

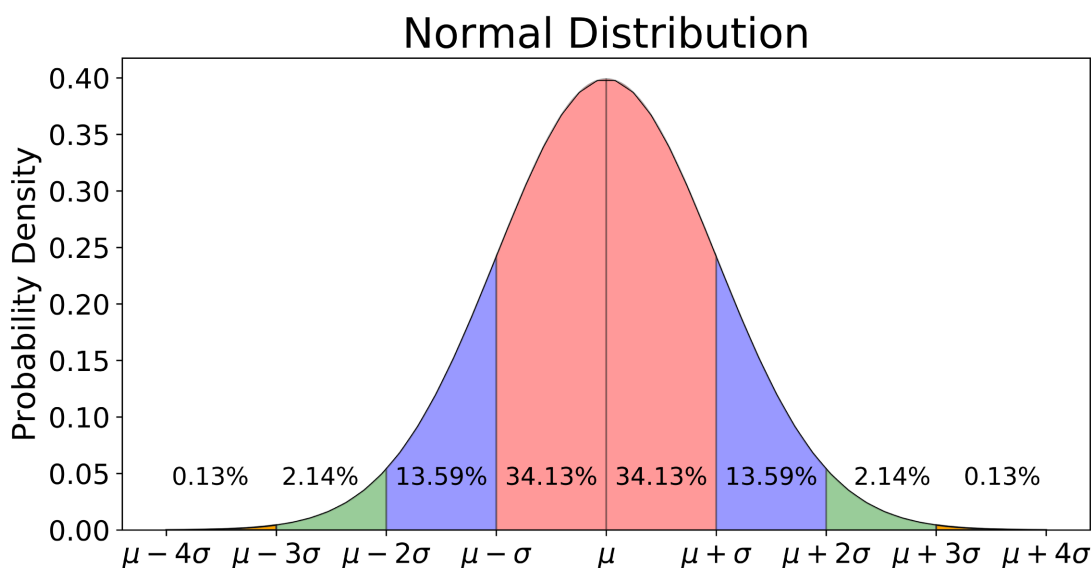
22. Probability Distributions

We have now looked at ‘discrete’ probabilities: probabilities relating to experiments in which there are a finite number of outcomes. There are six outcomes when a die is rolled, there are two possibilities for flipping a coin, and so on.

Now, we will look briefly at ‘continuous’ probabilities. These arise when we perform experiments involving real-world measurements: if we measure the heights of a thousand people, each one will have a slightly different reading.

Our question then becomes ‘what is the probability that our reading falls between x_0 and x_1 ’, rather than ‘what is the probability of a reading x ’.

It turns out that most ‘natural’ experiments produce probabilities fitting what is known as a *normal curve*, or *standard distribution*. Such a curve is determined by two numbers: a standard deviation σ that determines the spread of the curve (the width), and a mean value μ which tells us where the peak is.



The x -axis represents possible measurement values. The probability of a value lying between the measurements x_0 and x_1 is the area underneath the curve between the lines $x = x_0$ and $x = x_1$. The height of the curve at any given point has no meaning, only the area.

The *standardised* normal curve has mean $\mu = 0$ and standard deviation $\sigma = 1$. If our random variable is X , and we want to find the probability it lies between x_0 and x_1 given some mean μ and standard deviation σ , then we can transform our problem to one involving a random variable Z and the standardised curve: our probability will be the area under the standardised curve between $z_0 = (x_0 - \mu)/\sigma$ and $z_1 = (x_1 - \mu)/\sigma$. This can be summarised by writing

$$Z = \frac{X - \mu}{\sigma}.$$

The table given in the NCEA L2 external formula sheet gives the area under the standardised curve between 0 and z ; it is reproduced over the page.

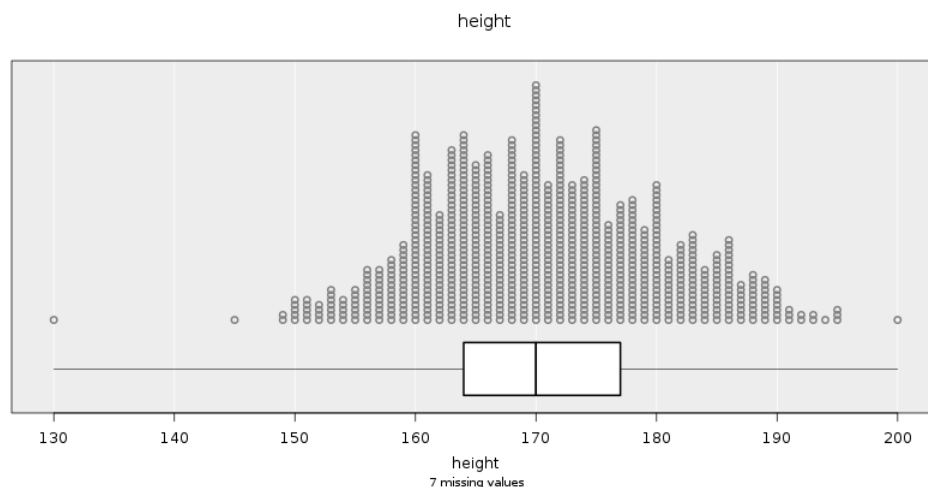
Note. The formula for the normal distribution curve with mean μ and standard deviation σ is

$$y = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right).$$

| z | | | | | | | | | | | Differences | | | | | | | | |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------------|---|----|----|----|----|----|----|----|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0.0 | .0000 | .0040 | .0080 | .0120 | .0160 | .0199 | .0239 | .0279 | .0319 | .0359 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 0.1 | .0398 | .0438 | .0478 | .0517 | .0557 | .0596 | .0636 | .0675 | .0714 | .0754 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 0.2 | .0793 | .0832 | .0871 | .0910 | .0948 | .0987 | .1026 | .1064 | .1103 | .1141 | 4 | 8 | 12 | 15 | 19 | 22 | 27 | 31 | 35 |
| 0.3 | .1179 | .1217 | .1255 | .1293 | .1331 | .1368 | .1406 | .1443 | .1480 | .1517 | 4 | 8 | 11 | 15 | 19 | 22 | 26 | 30 | 34 |
| 0.4 | .1554 | .1591 | .1628 | .1664 | .1700 | .1736 | .1772 | .1808 | .1844 | .1879 | 4 | 7 | 11 | 14 | 18 | 22 | 25 | 29 | 32 |
| 0.5 | .1915 | .1950 | .1985 | .2019 | .2054 | .2088 | .2123 | .2157 | .2190 | .2224 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 27 | 31 |
| 0.6 | .2258 | .2291 | .2324 | .2357 | .2389 | .2422 | .2454 | .2486 | .2518 | .2549 | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| 0.7 | .2580 | .2612 | .2642 | .2673 | .2704 | .2734 | .2764 | .2794 | .2823 | .2852 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 0.8 | .2881 | .2910 | .2939 | .2967 | .2996 | .3023 | .3051 | .3078 | .3106 | .3133 | 3 | 6 | 8 | 11 | 14 | 17 | 19 | 22 | 25 |
| 0.9 | .3159 | .3186 | .3212 | .3238 | .3264 | .3289 | .3315 | .3340 | .3365 | .3389 | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | 23 |
| 1.0 | .3413 | .3438 | .3461 | .3485 | .3508 | .3531 | .3554 | .3577 | .3599 | .3621 | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 18 | 21 |
| 1.1 | .3643 | .3665 | .3686 | .3708 | .3729 | .3749 | .3770 | .3790 | .3810 | .3830 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 19 |
| 1.2 | .3849 | .3869 | .3888 | .3907 | .3925 | .3944 | .3962 | .3980 | .3997 | .4015 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 15 | 16 |
| 1.3 | .4032 | .4049 | .4066 | .4082 | .4099 | .4115 | .4131 | .4147 | .4162 | .4177 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 14 |
| 1.4 | .4192 | .4207 | .4222 | .4236 | .4251 | .4265 | .4279 | .4292 | .4306 | .4319 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| 1.5 | .4332 | .4345 | .4357 | .4370 | .4382 | .4394 | .4406 | .4418 | .4429 | .4441 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 1.6 | .4452 | .4463 | .4474 | .4484 | .4495 | .4505 | .4515 | .4525 | .4535 | .4545 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1.7 | .4554 | .4564 | .4573 | .4582 | .4591 | .4599 | .4608 | .4616 | .4625 | .4633 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1.8 | .4641 | .4649 | .4656 | .4664 | .4671 | .4678 | .4686 | .4693 | .4699 | .4706 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 1.9 | .4713 | .4719 | .4726 | .4732 | .4738 | .4744 | .4750 | .4756 | .4761 | .4767 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 2.0 | .4772 | .4778 | .4783 | .4788 | .4793 | .4798 | .4803 | .4808 | .4812 | .4817 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 2.1 | .4821 | .4826 | .4830 | .4834 | .4838 | .4842 | .4846 | .4850 | .4854 | .4857 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| 2.2 | .4861 | .4864 | .4868 | .4871 | .4875 | .4878 | .4881 | .4884 | .4887 | .4890 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 2.3 | .4893 | .4896 | .4898 | .4901 | .4904 | .4906 | .4909 | .4911 | .4913 | .4916 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 2.4 | .4918 | .4920 | .4922 | .4925 | .4927 | .4929 | .4931 | .4932 | .4934 | .4936 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| 2.5 | .4938 | .4940 | .4941 | .4943 | .4945 | .4946 | .4948 | .4949 | .4951 | .4952 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2.6 | .4953 | .4955 | .4956 | .4957 | .4959 | .4960 | .4961 | .4962 | .4963 | .4964 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2.7 | .4965 | .4966 | .4967 | .4968 | .4969 | .4970 | .4971 | .4972 | .4973 | .4974 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 2.8 | .4974 | .4975 | .4976 | .4977 | .4977 | .4978 | .4979 | .4979 | .4980 | .4981 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2.9 | .4981 | .4982 | .4982 | .4983 | .4984 | .4984 | .4985 | .4985 | .4986 | .4986 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 3.0 | .4987 | .4987 | .4987 | .4988 | .4988 | .4989 | .4989 | .4989 | .4990 | .4990 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.1 | .4990 | .4991 | .4991 | .4991 | .4992 | .4992 | .4992 | .4992 | .4993 | .4993 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.2 | .4993 | .4993 | .4994 | .4994 | .4994 | .4994 | .4994 | .4995 | .4995 | .4995 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.3 | .4995 | .4995 | .4995 | .4996 | .4996 | .4996 | .4996 | .4996 | .4996 | .4997 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.4 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4998 | .4998 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.5 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.6 | .4998 | .4998 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.7 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.8 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .5000 | .5000 | .5000 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.9 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Questions

1. A manufacturing company finds the life of a battery for a product to be normally distributed, with mean 4 years and standard deviation 1 year.
 - (a) Given a randomly chosen battery, what is the chance it lasts between three and five years?
 - (b) Given 10 000 batteries, how many should be expected to have an abnormally short lifespan (less than two years)?
 - (c) What is the minimum length of life for the longest-lived 10%?
2. In the 2017 Census at School, 866 Y12 students answered the following question: 'What is your height (nearest cm), without shoes on?'. The mean of this sample was 170.6 cm, and the standard deviation was 9.476 cm.
 - (a) Draw a standard deviation fitting these parameters.
 - (b) What is the probability that a random individual from this sample has a height:
 - i. Between 160 and 180 centimetres?
 - ii. Less than 150 centimetres?
 - (c) What is the minimum height of the tallest 10% of people? (In other words, which height x is such that there is a 0.1 probability that a sampled individual has a height greater than x ?)
 - (d) A histogram for the responses is given below.

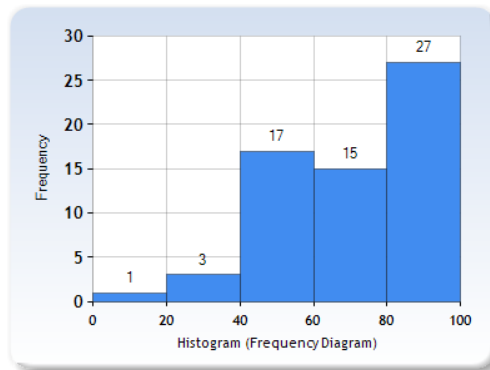


Compare the normal distribution and the histogram that resulted from the sample results.

In your answer you should consider the shape, centre, and spread of both distributions, and should provide numerical evidence where appropriate.

- (e) In a given school, 10% of year 12 students were taller than 175.0 cm, and the mean height of the students was 167.2 cm.
 - i. What is the standard deviation of this smaller sample of students?
 - ii. In this school, what was the maximum height of the shortest 20% of students?
 - iii. What is the probability that, given two randomly selected Y12 students at this school, both have heights greater than 180 cm? (You may assume that all heights are independent — e.g. there are no identical twins.)
3. The grades of students in a certain course have a mean of 71.04% and a standard deviation of 21.168 percentage points.
 - (a) Assuming the grades match a standard distribution, what is the probability that a randomly chosen student passed (had a grade of greater than 50%)?

(b) The following histogram depicts the actual grade distribution.



Discuss whether a normal distribution would actually be a good fit for this data.

(c) Using the histogram, compute the probability that a randomly chosen student passed.