Level Three Trigonometry

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Contents

Preface

These notes present a brief overview of the trigonometry required for Level 3 Calculus. We mainly treat the subject from a functional viewpoint.

The definitions given are based on the intuitive geometric definitions of the functions on the unit circle, but since the intuition should be familiar to readers we make the extension from functions of acute angles to functions of real numbers very explicit. Because of this, these notes are likely inappropriate for students who are at all uncomfortable in their understanding of the geometric behaviour of the trigonometric functions; these students are directed to the Level 2 notes.

A large number of the standard identities have proofs which either follow directly from each other, or are very similar to each other. As such, most of the identities are stated as exercises rather than theorems — the reader is invited to make their own way through these, proving as many (or as few) as desired.

Section 1:   Introduction: The Pythagorean Theorem

1.1 Notation. We will denote points by uppercase letters; the line through the points X and Y is denoted XY, and the length of the segment between X and Y will be denoted $|XY|$. If O, A, and B are points, then $\angle AOB$ denotes the angle that starts at the ray OA and is measured anticlockwise to the ray OB. The area of a figure $\mathfrak{F}$ is denoted by $\mathcal{A}(\mathfrak{F})$.

Consider a right-angled triangle ABC. The two shorter edges, labelled a and b, are called the **legs** of the triangle, and the longest side c (opposite the right angle) is called the **hypotenuse**.
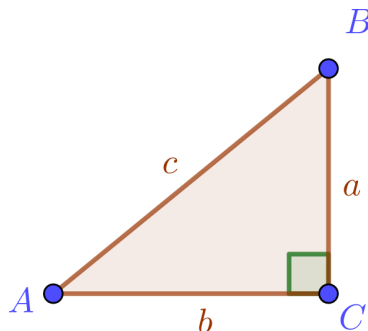
There is a fundamental relation between the three lengths, which was known to the ancient Greeks. Legend has it that the relation is due to the cult of Pythagoras, and that Pythagoras himself was so enthralled by it that he sacrificed an ox to give thanks for its discovery — although, as Kline points out, "if all the legends telling of Pythagoras sacrificing an ox were true, he could not have had time for mathematics".[1] In addition, there is evidence that the theorem was known (if not proved) before the time of Pythagoras by the Babylonians, the Indians, and the Chinese.

1.2 Theorem. *If ABC is a triangle as labelled above, then $c^2 = a^2 + b^2$.*

We give two proofs, both different to the four-triangle proof which is seen in most popular books. The first is as given by Euclid in around 300 BCE, and involves comparing areas.
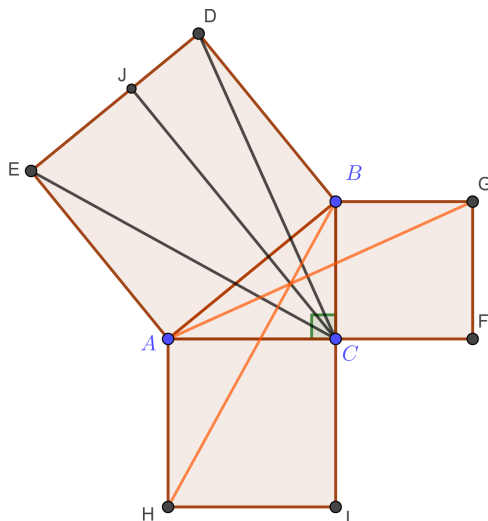

Proof I (Euclid). In the figure, we have:

$$\mathcal{A}(BE) = \mathcal{A}(JB) + \mathcal{A}(JA) \qquad \text{(Decomposing into rectangles.)}$$

$$= 2\mathcal{A}(BDC) + 2\mathcal{A}(AEC) \qquad \text{(Triangle } DBC \text{ has height } DJ \text{ and base } BD, \text{ similarly for } AEC.)$$

$$= 2\mathcal{A}(BGA) + 2\mathcal{A}(BHA) \qquad \text{(Triangles } BDC \text{ and } BGA \text{ share lengths } BG = BC \text{ and } DB = AB$$
$$\text{and angles at } B \text{ so are equal; similarly, } AEC \text{ and } BHA \text{ are equal.)}$$

$$= \mathcal{A}(BF) + \mathcal{A}(AI) \qquad \text{(Triangle } BGA \text{ has height } BC \text{ and base } BG, \text{ similarly for } BHA.)$$

$\square$

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[1]Morris Kline, *Mathematics for the Nonmathematician.* Dover Publications (1985). Page 68.

The second proof involves using similar triangles.
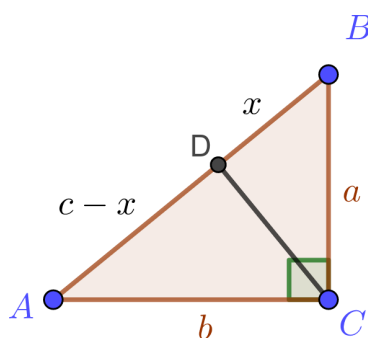
Proof II. First, note that the three triangles ABC, CBD, and ACD are similar. Thus, considering CBD and ABD, $c/a = a/x$; so $cx = a^2$. Similarly, considering ACD and ABC, $c/b = b/(c-x)$; so $c(c-x) = b^2$. Adding the two expressions we have obtained, we have $a^2 + b^2 = cx + c(c-x) = c^2$, which is the statement of the theorem. □

1.3 Exercise. This exercise asks you to prove a couple of useful theorems about triangles.

1. Show that the sum of the inscribed angles of any triangle is π radians. (You may use any facts which seem to you to be completely obvious; for example, the fact that a line crossing a pair of parallel lines cuts the lines at equal angles, if this fact is obvious to you.)

2. (Inscribed angle theorem.) If AB is a chord in a circle with centre O, and if C is any other point on the circle, then the measure of the angle BOA is twice the angle of the angle BCA. (For example, in the following diagram $\theta = 2\gamma$.)
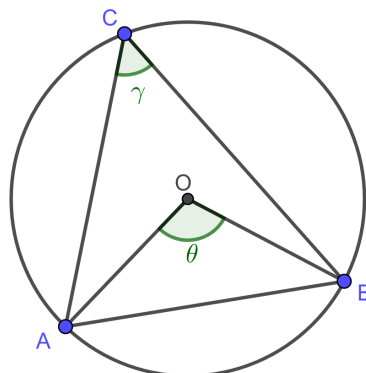
Hint: there is an obvious line missing in the above diagram. Draw it in, and do some angle-pushing.

Section 2: Definitions
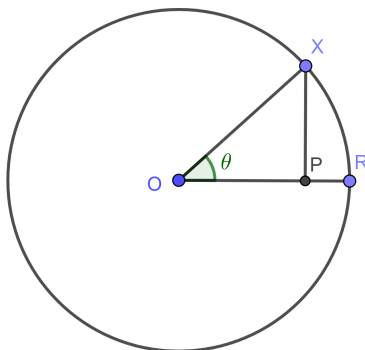
2.1 Definition. Consider a unit circle with centre O and radius OR. Let X be any point on the circle, such that the angle between OR and OX is θ (i.e. such that the proportion of the circumference between R and X is $\theta/2\pi$). Draw the line through X, perpendicular to the radius OR and cutting the latter at P. Then we define two functions of θ as follows, called the **sine** and **cosine** functions respectively, for all $0 \leq \theta < \pi/2$:-

$$\sin(\theta) = |PX| \tag{2.2}$$
$$\cos(\theta) = |OP| . \tag{2.3}$$

It is customary to drop the parentheses around the argument of trigonometric functions if it is a single term; for example, $\sin\theta$ simply means $\sin(\theta)$. We adopt this convention whenever it is unambiguous.
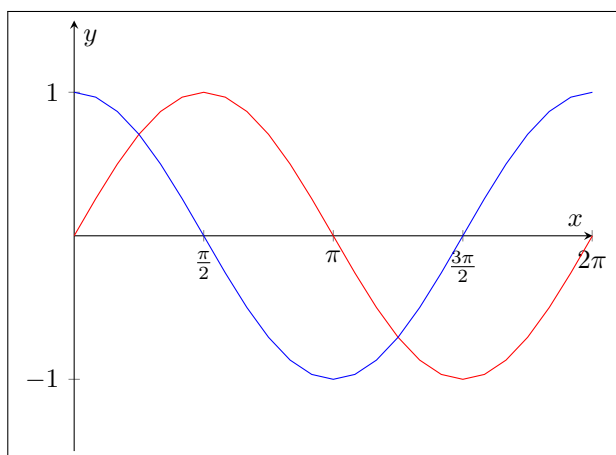
It is also customary to write $\sin^2\theta$ in place of $(\sin\theta)^2$, and in general $\sin^n\theta$ for $(\sin\theta)^n$ whenever n is a positive integer.

Clearly, we also have $\sin 0 = 0$ and $\cos 0 = 1$.

We then extend the domains of definition of sine and cosine to the entire range $0 \leq \theta < 2\pi$, as follows:

$$
\begin{array}{c|c|c|c}
 & \frac{\pi}{2} \leq \theta < \pi & \pi \leq \theta < \frac{3\pi}{2} & \frac{3\pi}{2} \leq \theta < 2\pi \\
\hline
\sin\theta = & \sin(\pi - \theta) & -\sin(\theta - \pi) & -\sin(2\pi - \theta) \\
\cos\theta = & -\cos(\pi - \theta) & -\cos(\theta - \pi) & \cos(2\pi - \theta).
\end{array}
\tag{2.4}
$$

Drawing the graphs of $y = \sin x$ (red) and $y = \cos x$ (blue) for $0 \leq x < 2\pi$, we obtain the following picture.

Finally, we extend the domains of definition to the entire real line by declaring that for all real θ, the identities

$$\sin\theta = \sin(\theta + 2n\pi) \text{ and } \cos\theta = \cos(\theta + 2n\pi) \tag{2.5}$$

hold (where n is any integer). This simply implies that the graphs of the two functions repeat every 2π; hence the graphs as drawn above can be extended as far as we like in either direction by just making shifted copies of the bit we already drew.
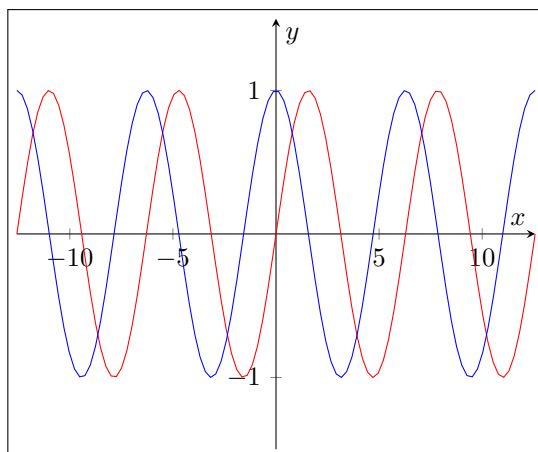
2.6 Exercise. Consider the unit circle again, call PX positive whenever X lies above the OR axis and negative whenever X lies below that axis. Then call OP positive whenever P lies on the segment OR and negative when P lies on the opposite side of R. Show that if we define $\sin\theta$ and $\cos\theta$ by PX and OP respectively for all $0 \le \theta < 2\pi$, keeping signs consistent, then the resulting functions agree with our definition above.

2.7 Proposition. *The following identities always hold.*

1. $\sin 2n\pi = 0$ *and* $\cos 2n\pi = 1$ *for all integers n.*

2. $\sin^2\theta + \cos^2\theta = 1$ *for all θ.*

3. $-1 \le \sin\theta \le 1$ *and* $-1 \le \cos\theta \le 1$ *for all θ.*

4. $\sin(-\theta) = -\sin\theta$ *and* $\cos(-\theta) = \cos\theta$ *for all θ.*

5. $\cos\theta = \sin(\theta + \pi/2)$.

6. $\sin(\pi - \theta) = \sin\theta$ *and* $\cos(\pi - \theta) = -\cos\theta$.

Proof.

1. $\sin 2n\pi = \sin(0 + 2n\pi) = \sin 0 = 0$. A similar calculation holds for cosine.

2. When $0 \le \theta < \pi/2$, the statement follows from an application of Pythagoras' theorem to the right triangle OPX. When $\pi/2 \le \theta < \pi$,

$$\sin^2\theta + \cos^2\theta = \sin^2(\pi - \theta) + (-\cos(\pi - \theta))^2 = 1$$

(since $0 \le \pi - \theta < \pi/2$); similar arguments show that the statement holds for all $0 \le \theta < 2\pi$. If θ is not in this range, then $\theta = \theta' + 2n\pi$ for some integer n and some θ' that does lie in that range; since $\sin(\theta' + 2n\pi) = \sin\theta'$, and $\cos(\theta' + 2n\pi) = \cos\theta'$, we can apply what we have already proved in this case as well.

3. For $0 \le \theta < \pi/2$, the two functions are defined using the lengths PX and OP which are positive and always less than 1. For all other values of θ, the functions are defined in terms of the values on this interval and are not scaled beyond a negative sign; so both functions are bounded above by 1 and below by -1.

   Alternatively, note that by (2) above, $\sin^2\theta \le 1$ and $\cos^2\theta \le 1$ for all θ.

4. Note that $\sin(-\theta) = \sin(2\pi - \theta) = -\sin\theta$, and $\cos(-\theta) = \cos(2\pi - \theta) = \cos\theta$ (by the final column of table 2.4).

5. Suppose $0 \le \theta < \pi/2$. Then, considering the following diagram, we perform some angle-pushing.
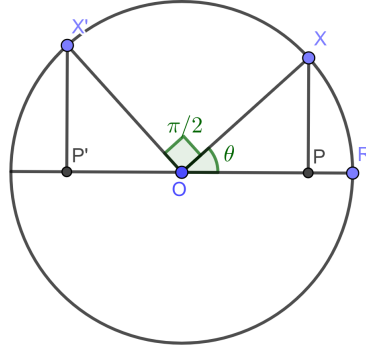
The angle $\angle X'OP'$ has magnitude $\frac{\pi}{2} - \theta$; hence the other angle in the right triangle $X'OP'$ has magnitude θ (since the angle-sum of triangles is π). The two triangles XOP and $X'OP'$ share all three of their angles and so are similar, and share an edge length (the radius of the circle) and so are equal, with lengths $|X'P'| = |OP|$ and $|OP'| = |XP|$. But $|X'P'| = \sin(\theta + \pi/2)$, and $|OP| = \cos\theta$; hence the proposition holds in this case.

If $\pi/2 \leq \theta < \pi$, then $\pi \leq \theta + \pi/2 < 3\pi/2$ and $\sin(\theta + \pi/2) = -\sin(\theta + \pi/2 - \pi)$; then (applying the case we have already proved) $-\sin(\theta - \pi/2) = -\cos(\theta - \pi) = \cos\theta$ as required. We extend the proof to all $0 \leq \theta < 2\pi$ using similar arguments, and then conclude it holds for all θ by periodicity.

6. Here, we apply table 2.4 in the reverse and simplify, if needed, using (1), (4), and periodicity.

$\square$

2.8 Theorem. *If ABC is a right triangle such that the right angle is at C, and α is the interior angle at A, then*

$$(2.9) \qquad \sin\alpha = \frac{|BC|}{|AB|} = \frac{opposite}{hypotenuse} \ and$$

$$(2.10) \qquad \cos\alpha = \frac{|AC|}{|AB|} = \frac{adjacent}{hypotenuse}.$$

Proof. Construct a similar triangle $A'B'C'$ by scaling every length of ABC by $1/|AB|$; drawing a circle with radius 1 and centre A, we see that $|B'C'| = \cos\alpha$ and $|A'C'| = \sin\alpha$ (taking the base radius to be $A'C'$ and applying our original definition 2.1). But $|B'C'| = |BC|/|AB|$, and $|A'C'| = |AC|/|AB|$, so we are done. $\square$

Philosophically, the above identities and theorems all follow directly from the geometric definition (exercise 2.6). This is simply because our functional definition of sine and cosine (based on geometry for a small interval, and then the use of symmetry and periodicity to extend it to the whole real line) was devised so that they behave in the way we expect them to based on our intuition from previous study.

Because of this, we will simply view the *functions* sin and cos as black boxes from now on: they have the properties that we expect the geometric sine and cosine curves to have, and so we can use the two points of view interchangably. Thus, in what follows we will freely use both functional (i.e. depending on the fiddly details of the definition and the tables and the extensions from intervals to the real line) and geometric arguments to justify many of the statements we will make.

Instead of pushing the definitions further, we move on to defining some more trigonometric functions. Consider the following diagram.

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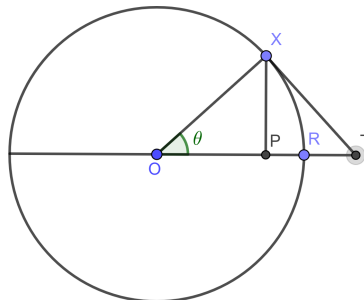
2.11 Theorem. *If OXP is drawn in a unit circle as in the diagram, and the tangent line to the circle at X is drawn to meet the line OR at some point T, then $|XT| = \frac{\sin\theta}{\cos\theta}$.*

Proof. I claim that XOT is similar to POX. This is proved by noticing that both triangles contain a right angle and share the angle of measure θ. Hence $\frac{|XT|}{|XO|} = \frac{|XP|}{|OP|}$; but $|XO| = 1, |XP| = \sin\theta$, and $|OP| = \cos\theta$, so we are done. $\qquad\square$
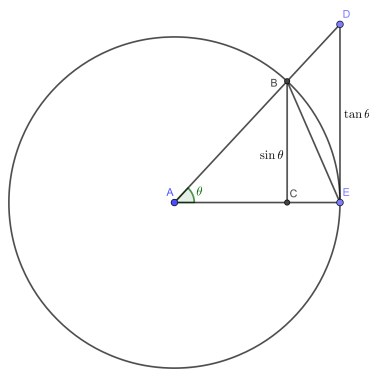
Motivated by this theorem, we make the following

2.12 Definition. The **tangent** function is the function tan that is defined by

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

for all $\theta \neq \frac{\pi}{2}n$ where n is an odd integer (since $\cos\frac{\pi}{2}n$ is zero for odd n).

The following theorem is useful in calculus.

2.13 Theorem. *For $-\pi/2 < \theta < \pi/2$, $1 < \frac{\sin x}{x} < \cos x$. In particular, as $x \to 0$, $\frac{\sin x}{x} \to 1$.*

Proof. For $\theta > 0$, we can use the diagram above. We have that $\mathcal{A}(\triangle ABC) = \frac{1}{2}\sin\theta$, that $\mathcal{A}(\bigtriangledown ABC) = \frac{\theta}{2}$, and that $\mathcal{A}(ADE) = \frac{1}{2}\tan\theta$. By comparison of these areas, we have $\frac{1}{2}\sin\theta \leq \frac{1}{2}\theta \leq \frac{1}{2}\tan\theta$ and hence $1 \leq \frac{\theta}{\sin\theta} \leq \cos\theta$.

If $\theta < 0$ then $-\theta > 0$; we then have $\frac{\theta}{\sin\theta} = \frac{-\theta}{\sin(-\theta)}$. Applying the part we have already proved, we have $1 \leq \frac{-\theta}{\sin(-\theta)} \leq \frac{1}{\cos(-\theta)}$. Replacing $\cos(-\theta)$ with $\cos\theta$, allowed because of proposition 2.7, we have the desired result for negative θ. $\qquad\square$

2.14 Exercise. Here are some revision problems from your earlier studies that should help you remember the mechanical aspects of applying trigonometry.

1. To measure the width BC of a straight river, a surveyor starts at C and walks along the bank to a point A. He finds that the distance between A and C is $50\,\text{m}$, and the angle at A is $20°$. What is the width of the river?

2. Before the advent of modern technology, it was still possible to make rough calculations of the circumference and radius of the earth. For example, the Alexandrian mathematician Eratosthenes calculated the circumference of the earth in around 200 BCE. Carry out the following programme to arrive at Eratosthenes' estimate.

   (a) First, note that the sun is far enough away for the light rays arriving at the earth to be almost parallel. At the summer solstice, light hits the town of Syene so that the bottom of a deep well is fully illuminated; conclude that the sun is immediately overhead.

   (b) In Alexandria, $800\,\text{km}$ further north, a $1\,\text{m}$ vertical stick casts a shadow of $13\,\text{cm}$. Compute the angle at which the sun's rays are hitting the stick.

   (c) Thus compute the angle at the centre of the earth between the radii of the planet at Alexandria and Syene. What is the circumference of the earth?

3. Alternatively, suppose an astronomer climbs Mt Kaukau (height: $445\,\text{m}$ above sea level) and uses an inclinometer to measure the angle between a vertical line and her line of sight to the horizon. Suppose that this angle turns out to be $89°$. What is the radius of the earth, and hence its circumference?

Section 3: Taxonomy of Functions

For the remainder of these notes, we only consider angles $0 \le \theta < 2\pi$ unless otherwise stated. This is because all of the interesting theory occurs in this case, and extending results to arbitrary θ is tedious and doesn't present any inherent complexity. The reader is invited to keep in mind our definitions from above, and to fill in on her own those places where a negative sign or some other modification may be needed to make a result hold in general.

Consider now the following diagram based on our definitions above, where the tangent line XT to the unit circle has been extended to meet the vertical line at B; we also drop a perpendicular line from X onto the vertical line at A. Our goal is to work out how most of the new lengths we can play with depend on the trigonometric functions we have already defined; the reader is encouraged to annotate the diagram with the names of the various functions, like tan, as they are defined. A complete diagram is given at the end of the section.
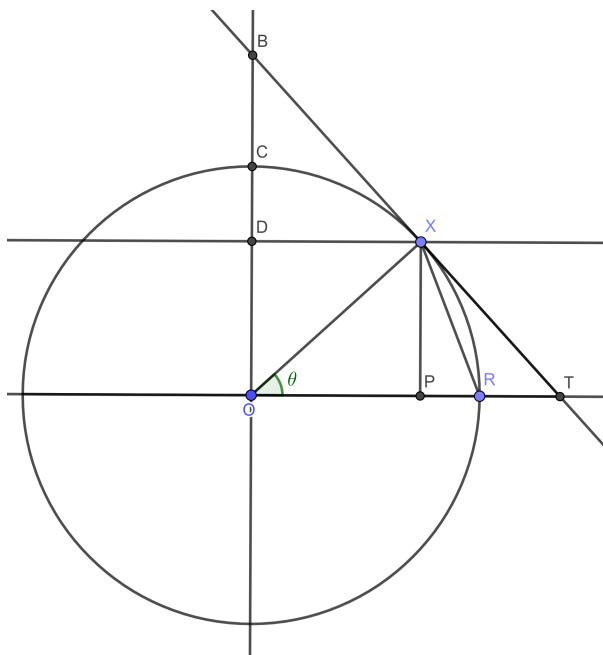
3.1   Reciprocal Functions

Firstly, we calculate the length of OT. Consider the triangle OXT; we can calculate its area in two different ways, recalling that the area of a triangle is one half of the base times the height.

$$\mathcal{A}(OXT) = \frac{1}{2}|XT||OX| = \frac{1}{2}|OT||PX|$$

However, $|XT| = \tan\theta, |OX| = 1,$ and $|PX| = \sin\theta$. Hence we have $\tan\theta = |OT|\sin\theta$; but $\tan\theta = \frac{\sin\theta}{\cos\theta}$, so $|OT| = \frac{1}{\cos\theta}$.

3.1 Definition. The **secant** function is defined by $\sec\theta = |OT| = \frac{1}{\cos\theta}$.

We next calculate $|BX|$. Note that DBX and PXT are similar triangles, and so $\frac{|BX|}{|DX|} = \frac{|XT|}{|PT|}$. Substituting in what we know, this becomes

$$(3.2) \qquad \frac{|BX|}{\cos\theta} = \frac{\tan\theta}{|PT|};$$

but PXT is a right triangle and so, applying the Pythagorean theorem,

$$|PT| = \sqrt{\tan^2\theta - \sin^2\theta} = \sqrt{\frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta} = \sin\theta\sqrt{\frac{1}{\cos^2\theta} - 1},$$

8

or (substituting into equation 3.2)

$$\frac{|BX|}{\cos\theta} = \frac{\tan\theta}{\sin\theta\sqrt{\frac{1}{\cos^2\theta} - 1}}.$$

Cancelling $\tan\theta = \sin\theta/\cos\theta$,

$$|BX| = \frac{1}{\sqrt{\frac{1}{\cos^2\theta} - 1}}.$$

To simplify further, recall that $\sin^2\theta + \cos^2\theta = 1$, and so (dividing by $\cos^2\theta$)

$$(3.3) \qquad\qquad \tan^2\theta + 1 = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

Hence $|BX| = \frac{1}{\sqrt{\tan^2\theta}} = \frac{1}{\tan\theta}$.

3.4 Definition. The **cotangent** function is defined by $\cot\theta = |BX| = \frac{1}{\tan\theta}$.

Our next trick is to calculate $|OB|$. Again, we use similar triangles: OBT is similar to PXT. In particular, $\frac{|OB|}{|OT|} = \frac{|PX|}{|PT|}$. But $|OT| = \sec\theta$, $|PX| = \sin\theta$, and $|PT| = \sin\theta\sqrt{\sec^2\theta - 1} = \sin\theta\tan\theta = \frac{\sin^2\theta}{\cos\theta}$. Hence

$$|OB| = |OT|\frac{|PX|}{|PT|} = \sec\theta\frac{\sin\theta}{\sin^2\theta/\cos\theta} = \frac{1}{\sin\theta}.$$

3.5 Definition. The **cosecant** function is defined by $\csc\theta = |OB| = \frac{1}{\sin\theta}$.

3.6 Exercise. Show the pretty identity that $\sec^2\theta\csc^2\theta = \sec^2\theta\csc^2\theta$. (Hint: apply Pythagoras to OBT.)

3.2 Historical Functions

3.7 Exercise.

1. By applying Pythagoras' theorem to BOT, show that $\cot^2\theta + \tan^2\theta + 2 = \csc^2\theta + \sec^2\theta$, and conclude by equation 3.3 that $\cot^2\theta + 1 = \csc^2\theta$.

2. By imitating the proof of equation 3.3 (i.e. using the identity $\sin^2\theta + \cos^2\theta = 1$), show that $\cot^2\theta + 1 = \csc^2\theta$.

A few easy lengths follow, which only have names for historical reasons.

3.8 Definition.

1. The **versine** function is defined by $\operatorname{versin}\theta = |PR| = 1 - \cos\theta$.

2. The **coversine** function is defined by $\operatorname{cvs}\theta = |CD| = 1 - \sin\theta$.

3. The **exsecant** function is defined by $\operatorname{exsec}\theta = |RT| = \sec\theta - 1$.

4. The **excosecant** function is defined by $\operatorname{excsc}\theta = |BC| = \csc\theta - 1$.

3.9 Exercise. Taking inspiration from our proofs above showing that the reciprocal trig functions correspond to lengths, show the following identities involving the historical trig functions.

1. $\operatorname{exsec}\theta = \operatorname{excsc}\theta\tan\theta$

2. $(\operatorname{versin}\theta)(\operatorname{cvs}\theta) = \sin^2\theta$
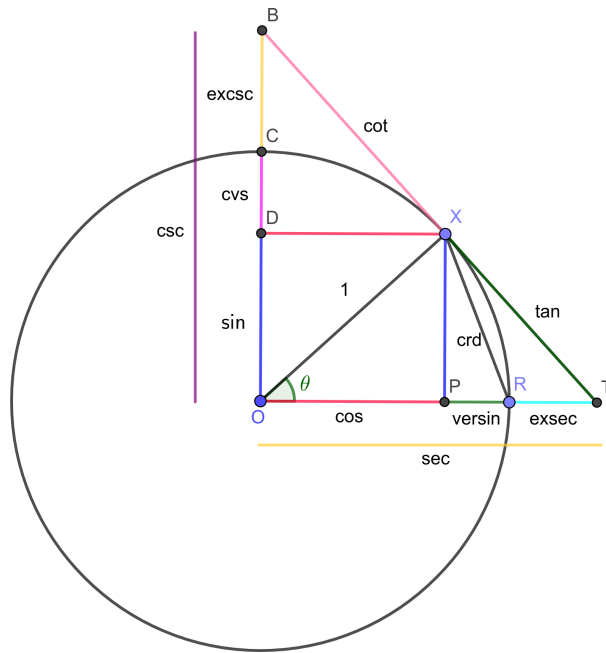
3.3 Chords

Finally, we compute $|XR|$. We have

$$|XR|^2 = \sin^2\theta + \operatorname{versin}^2\theta = \sin^2\theta + (1 - \cos\theta)^2 = \sin^2\theta + 1 - 2\cos\theta + \cos^2\theta = 2 - 2\cos\theta.$$

3.10 Definition. The **chord** function is defined by $\operatorname{crd}\theta = |XR| = \sqrt{2 - 2\cos\theta}$.

3.4   Summary of the Taxonomy

We have the following definition, which summarises all the definitions we have made.

3.11 Definition.

1. **Basic Functions**

   (a) $\sin\theta = |PX|$ (sine)
   (b) $\cos\theta = |OP|$ (cosine)
   (c) $\tan\theta = |XT| = \frac{\sin\theta}{\cos\theta}$ (tangent)

2. **Reciprocal Functions**

   (a) $\csc\theta = |OB| = \frac{1}{\sin\theta}$ (cosecant)
   (b) $\sec\theta = |OT| = \frac{1}{\cos\theta}$ (secant)
   (c) $\cot\theta = |BX| = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$ (cotangent)

3. **Historical Functions**

   (a) $\operatorname{versin}\theta = |PR| = 1 - \cos\theta$ (versine)
   (b) $\operatorname{cvs}\theta = |CD| = 1 - \sin\theta$ (coversine)
   (c) $\operatorname{exsec}\theta = |RT| = \sec\theta - 1$ (exsecant)
   (d) $\operatorname{excsc}\theta = |BC| = \csc\theta - 1$ (excosecant)

4. **Other Functions**

   (a) $\operatorname{crd}\theta = |RX| = \sqrt{2 - 2\cos\theta}$ (chord)

Of these, the basic functions and the reciprocal functions are the most useful and the only ones worth learning.

We also restate the following identities for convenience. Since the second two follow immediately from the first, and the first is a trivial application of Pythagoras' theorem, the reader is advised not to try to memorise them.

3.12 Theorem. *For all angles θ,*

1. $\sin^2\theta + \cos^2\theta = 1$,

2. $\tan^2\theta + 1 = \sec^2\theta$, *and*

3. $1 + \cot^2\theta = \csc^2\theta$.

Section 4:   Identity Fishing

Recall that if we consider the function exp which maps $x \mapsto e^x$, then we have the addition rule $\exp(x+y) = \exp(x)\exp(y)$. Our goal in this section is to derive similar rules for the basic trigonometric functions.

4.1 Theorem. *Suppose α and β are angles. Then:*

1. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

2. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Proof. We prove the formulae for angles in the range $0 \le \alpha < \pi/2$ and $0 \le \beta\pi/2$, and this result can be generalised to all θ using the same methods we have used before.

For cosine, consider the following diagram.[2]
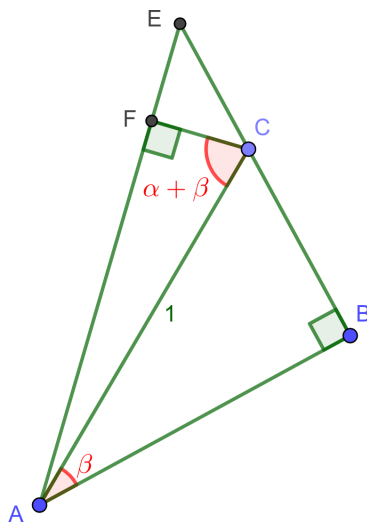
We have two triangles, where ABC has an angle β and hypotenuse $|AC| = 1$. From C we measure an angle $\alpha + \beta$, and extend this line to meet a line from C at a right angle (so both interior angles at F are right). Hence $|AB| = \cos\beta$, $|BC| = \sin\beta$, and $|FC| = \cos(\alpha + \beta)$. I claim that the interior angle at E is simply α; this can be seen by writing the interior angle of EFC at C as $\pi - (\alpha + \beta) + (\pi/2 - \beta) = \pi/2 - \alpha$ and then using the fact that the angle sum of a triangle is π. Using this fact and the length of FC, we have that
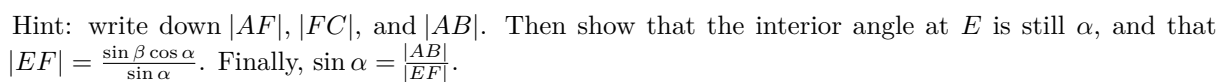
$$|EC| = \frac{\cos(\alpha + \beta)}{\sin\alpha}.$$

Then by taking side ratios of the large triangle ABE we have $\frac{|AB|}{|BE|} = \frac{\sin\alpha}{\cos\alpha}$, and hence

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\cos\beta}{\frac{\cos(\alpha+\beta)}{\sin\alpha} + \sin\beta},$$

which is rearranged to obtain $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ as required. $\qquad\square$

4.2 Exercise. Use the following diagram and an analagous argument to derive the sum formula for sine.

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[2]Both this proof and the one given in the exercise below are due to Leonard M. Smiley, University of Alaska Anchorage, `http://math.uaa.alaska.edu/~smiley/trigproofs.html` (retrieved via Archive.org on 14 Sept 2018).

Hint: write down $|AF|$, $|FC|$, and $|AB|$. Then show that the interior angle at E is still α, and that $|EF| = \frac{\sin\beta\cos\alpha}{\sin\alpha}$. Finally, $\sin\alpha = \frac{|AB|}{|EF|}$.

4.3 Corollary. *We also have the following identities:-*

1. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

2. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

3. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

4. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

5. $\sin(2\alpha) = 2\sin\alpha\cos\alpha$

6. $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1.$

7. $\tan(2\alpha) = \frac{2\tan\alpha}{1 - \tan^2\alpha}.$

Proof. For (1) and (2), simply replace β with $-\beta$ in theorem 4.1 and apply part (4) of proposition 2.7.

For (3), write $\tan(\alpha + \beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$, apply parts (1) and (2), and simplify. Then (4) follows in a similar way to (1) and (2) (noting that $\tan(-\theta) = -\tan\theta$).

For (5) and (6), set $\beta = \alpha$ and apply theorem 4.1; (7) is done in a similar way. $\qquad\square$

4.4 Exercise. Show the following product identities hold.

1. $2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$

2. $2\cos\alpha\sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$

3. $2\cos\alpha\cos\beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$

4. $2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$ (note reversed $\pm$)

Find an expression for $\tan\alpha\tan\beta$ in terms of cos.

4.5 Exercise. Show the following half-angle identities hold.

1. $\sin^2\left(\frac{\alpha}{2}\right) = \frac{1 - \cos\alpha}{2}$

2. $\cos^2\left(\frac{\alpha}{2}\right) = \frac{1 + \cos\alpha}{2}$

3. $\tan^2\left(\frac{\alpha}{2}\right) = \frac{1 - \cos\alpha}{1 + \cos\alpha}$

To remove the power of two on the left, we must take the square root of the right in each case. This root can have two signs: it may be positive, or negative, depending on the value of α. Classify all the relevant cases. (For which α is $\sin \alpha/2$ positive, and for which α is it negative? Repeat for all three.)

4.6 Exercise. Show the following sum identities hold.

1. $\sin \alpha + \sin \beta = 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$

2. $\sin \alpha - \sin \beta = 2 \cos \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}$

3. $\cos \alpha + \cos \beta = 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$

4. $\cos \alpha - \cos \beta = -2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}$ (note negative)

Finally, we prove the following aesthetic identity.

4.7 Theorem. *If $x + y + z = \pi$, then $\sin 2x + \sin 2y + \sin 2z = 4 \sin x \sin y \sin z$.*

Proof. First, note that since $z = \pi - x - y$, we have $\sin z = \sin(x + y)$ and $\cos z = -\cos(x + y)$. This follows from the facts $\sin \theta = \sin(\pi - \theta)$ and $\cos \theta = -\cos(\pi - \theta)$ which can be easily seen from symmetry in the case $0 \leq \theta < \pi/2$ using the unit circle (figure), and then for all other θ by applying table 2.4 and proposition 2.7. It also follows algebraically from the definition (see proposition 2.7).
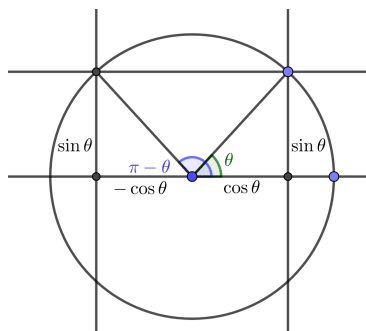
Then we argue as follows.

$$\sin 2x + \sin 2y + \sin 2x = 2 \sin(x + y) \cos(x - y) + 2 \sin z \cos z$$
$$= 2 \sin z \cos(x - y) - 2 \sin z \cos(x + y)$$
$$= 2 \sin z [\cos(x - y) - \cos(x + y)]$$
$$= 2 \sin z [\cos x \cos y + \sin x \sin y - \cos x \cos y + \sin x \sin y]$$
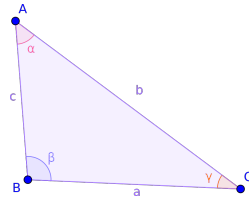$$= 4 \sin x \sin y \sin z.$$

$\square$

4.8 Exercise. Prove the following identities.

1. $\sin(\alpha + \beta + \gamma) = \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \sin \gamma - \sin \alpha \sin \beta \sin \gamma$

2. $\tan 3\alpha = \dfrac{3 \sin \alpha - 4 \sin^3 \alpha}{4 \cos^3 \alpha - 3 \cos \alpha}$

3. $\dfrac{1}{2} (\cos 2\beta - \cos 2\alpha) = \sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta$

4. $\sin A + \sin B + \sin C - \sin(A + B + C) = 4 \sin \frac{A+B}{2} \sin \frac{B+C}{2} \sin \frac{C+A}{2}$

5. $\dfrac{\sin \dfrac{B-C}{2}}{\sin \dfrac{B+C}{2}} + \dfrac{\sin \dfrac{C-A}{2}}{\sin \dfrac{C+A}{2}} + \dfrac{\sin \dfrac{A-B}{2}}{\sin \dfrac{A+B}{2}} + \dfrac{\sin \dfrac{B-C}{2}}{\sin \dfrac{B+C}{2}} \dfrac{\sin \dfrac{C-A}{2}}{\sin \dfrac{C+A}{2}} \dfrac{\sin \dfrac{A-B}{2}}{\sin \dfrac{A+B}{2}} = 0$

Section 5:   The Sine and Cosine Laws

We can calculate the chord function from definition 3.10 in a different way using the cosine rule which we learned last year.

5.1 Theorem (Law of cosines)**.** *If ABC is a triangle, labeled as in the picture, then $c^2 = a^2 + b^2 - 2ab\cos\gamma$.*

Rather than repeat the standard proof, I will give one based on the following intersecting chord theorem.

5.2 Theorem. *Suppose that A, B, C, and D lie on a circle, and the lines AB and CD intersect at some point X inside the circle. Then $|AX||XB| = |CX||XD|$.*
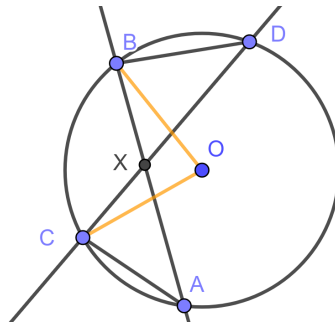
Proof. I claim that BXD and CXA are similar. Clearly the two angles $\angle DXB$ and $\angle CXA$ are equal, because they are opposite. By the inscribed angle theorem, $\angle BOC = 2\angle BDC = 2\angle BAC$ because all three are subtended by the chord BC. Hence both triangles have two equal angles, and are similar. The result follows from taking the side ratios $\frac{AX}{CX} = \frac{XD}{XB}$. $\square$

The law of cosines now follows immediately from the following diagram, due to Kung.[3]
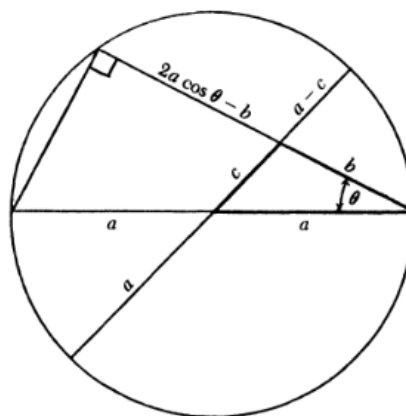
$$(2a\cos\theta - b)b = (a - c)(c + a)$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Then $\operatorname{crd}\theta = \sqrt{1^2 + 1^2 - 2\cdot 1\cdot 1\cdot\cos\theta} = \sqrt{2 - 2\cos\theta}$ as before.

[3]"Proof without Words: The law of cosines via Ptolemy's Theorem." *Mathematics Magazine*, 65(2), p. 103. doi:10.1080/0025570X.1992.11995990

5.3 Exercise. Let S be the area of the triangle ABC, and let p be its half-perimeter: that is, $p = \frac{1}{2}(a + b + c)$. Then draw the circle inside the triangle which is tangent to the three sides (the **inscribed circle**), and call its radius r.

1. Show that $S = \frac{1}{2}bc\sin\alpha$.

2. Let α be the centre of the inscribed circle. Divide the triangle into three by drawing segments from the three vertices to α. Using this, show that $S = pr$.

3. Show that $\sin\alpha = \frac{2}{bc}\sqrt{p(p-a)(p-b)(p-c)}$. (Hint: this looks like the cosine rule.)

4. Conclude that
$$S = \sqrt{p(p-a)(p-b)(p-c)}.$$

(This formula is attributed to Hero of Alexandria.) Thus we have a formula for the area of a triangle which relies only on the lengths of the sides, and not on the height.

5.4 Exercise (Law of sines)**.** As you will recall from last year, there is also a sine rule; we actually state here a slightly stronger form of it. Let ABC be a triangle labelled as in the statement for the law of cosines; draw the circle passing through the three vertices of the triangle (the so-called **circumcircle** of the triangle), and call its radius R. Prove that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R.$$

Hint: try to use the inscribed angle theorem.

Section 6:   Inverse Functions

Suppose we draw a right triangle ABC. It is clear that, if we know the three lengths, then the three angles are determined: there is only one such triangle with the three given lengths.

In fact, since we know that one angle is a right angle, if we know the lengths of only two sides together with their position in relation to the right angle (whether each is opposite, hypotenuse, or adjacent), then we know the length of the third side together with the angle.

This fact, which we have phrased here rather philosophically, simply means that given the ratios we found in 2.8,

$$(6.1) \qquad \sin \alpha = \frac{|BC|}{|AB|} = \frac{\text{opposite}}{\text{hypotenuse}} \text{ and } \cos \alpha = \frac{|AC|}{|AB|} = \frac{\text{adjacent}}{\text{hypotenuse}},$$

there is a unique α in the range $0 \le \alpha < \pi/2$ which will produce them.

More formally, we say that a function $f(x)$ is invertible if for every possible output y in a given range, then there is precisely one input x that will produce that output (i.e. there is a unique x such that $y = f(x)$). If a function is invertible, then we can define a second function g such that $g(f(x)) = x = f(g(y))$ for every input x and every output y. We have geometrically justified, then, that $\sin \alpha$ and $\cos \alpha$ are invertible on the range $0 \le \alpha < \pi/2$ — and in fact, by the way we defined the extensions of these functions into other intervals, they are invertible on the wider range $-\pi/2 \le \alpha < \pi/2$ (in the case of sin) and $0 \le \alpha < \pi$ (in the case of cos).

6.2 Definition. Suppose that we are given some real number x, such that $-1 \le x \le 1$. Draw a right triangle ABC with hypotenuse $|AB| = 1$ and side $|BC| = x$ (i.e. the right angle is at C). Then we define two functions as follows:

1. The function arcsin, by writing $\arcsin x = \alpha$ (where α is the interior angle at A), and

2. The function arccos, by writing $\arccos x = \beta$ (where β is the interior angle at B).

In other words, $\arcsin x = \alpha$ if and only if $-\pi/2 \le \alpha < \pi/2$ and $\sin \alpha = x$, and $\arccos x = \beta$ if and only if $0 \le \alpha < \pi$ and $\cos \beta = x$.

Further, suppose we are given some real number y. Define $\arctan y = \theta$ if and only if $-\pi/2 < \theta < \pi/2$, and $\tan \theta = x$.

You will also see these functions written as $\arcsin x = \sin^{-1} x$, $\arccos x = \cos^{-1} x$, and $\arctan x = \tan^{-1} x$; I will avoid this notation here as it clashes with the notation $\sin^n x = (\sin x)^n$ which we have already introduced (although philosophically speaking, if we are to drop any notation then we should drop the latter as it is unique to trig functions, while writing a superscript -1 for inverses is very standard across all of mathematics).

These new inverse functions are of mainly practical importance, as they allow us to reverse geometric calculations and move from lengths to angles. We graph them here, with $y = \arcsin x$ in red, $y = \arccos x$ in blue, and $y = \arctan x$ in dotted green.

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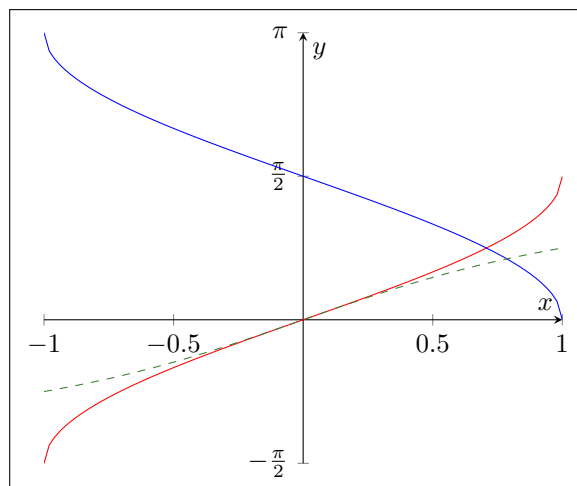
By definition, $\sin(\arcsin x) = \arcsin(\sin x) = 1$ (and analagous statements hold for cos and tan). However, the definitions do not say anything about combinations like $\sin(\arccos x)$ or $\arctan(\sin x)$. It turns out that expressions like these show up in calculus reasonably often, so we will look at them carefully.

Note that the first class of this kind of expression, those of the form inverse(trig(x)), can be turned into 'normal' equations: if $y = \arctan(\sin x)$, then $\tan y = \sin x$ and we can solve for y by studying the nature of solutions of trigonometric equations. We will do this in the next section.

For now, then, we will consider only expressions consisting of an inverse function inside a normal trigonometric function.

6.3 Theorem. *In general, the following identities hold:*

1. $\sin(\arccos x) = \sqrt{1 - x^2}$
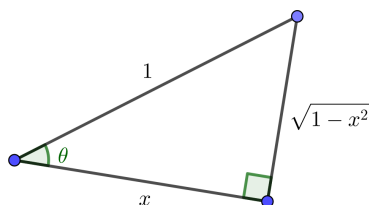
2. $\sin(\arctan x) = \frac{x}{\sqrt{x^2+1}}$

3. $\cos(\arcsin x) = \sqrt{1 - x^2}$

4. $\cos(\arctan x) = \frac{1}{x^2+1}$

5. $\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$

6. $\tan(\arccos x) = \frac{\sqrt{1-x^2}}{x}$

Proof. We prove (1) and (6) here, leaving the remainder as an exercise. Consider the triangle given below.

Then $\theta = \arccos x$ (since $\cos\theta = \frac{x}{1}$). Calculating $\sin\theta$ and $\tan\theta$ we arrive at the required identities. $\square$

6.4 Exercise.

1. Prove that the identity $\arctan x = 2\arctan(\csc\arctan x - \tan\arccot x)$ holds.

2. Show that, if $\arccos x + \arccos y + \arccos z = \pi$, then $x^2 + y^2 + z^2 + 2xyz = 1$.

Section 7:   Trigonometric Equations

The trigonometric functions are not very well-behaved with respect to the rational numbers — they tend to map simple inputs to complicated real numbers. However, there are a few special values which we can compute, corresponding to triangles with simple sides.

7.1 Exercise.

1. **Standard special triangles.** Show, by drawing appropriate triangles, the following identities. This is not year twelve any more, so in particular you must prove that if you construct a triangle with given angles then it has the correct lengths (or vice versa, if you pick a triangle with given lengths then it has the given angles).

   (a) $\sin \pi/4 = \cos \pi/4 = 1/\sqrt{2}$ and $\tan \pi/4 = 1$ (45-45-90)
   (b) $\sin \pi/6 = 1/2$, $\cos \pi/6 = \sqrt{3}/2$, and $\tan \pi/6 = 1/\sqrt{3}$ (60-30-90)
   (c) $\sin \pi/3 = \sqrt{3}/2$, $\cos \pi/3 = 1/2$, and $\tan \pi/6 = \sqrt{3}$ (60-30-90)

2. **Magical exotic triangles.**

   (a) Draw an isoceles triangle with angles $\pi/5$, $2\pi/5$, and $2\pi/5$ (36-72-72), and shortest length 1.[4] By bisecting one of the angles with measure $2\pi/5$ and working with ratios of lengths of similar triangles, show that the two longer lengths ϕ satisfy the equation $1 + \phi = \phi^2$. (The measure ϕ is known as the golden ratio.) Conclude that:

       i. $\sin \pi/5 = \frac{\sqrt{\phi^2 - 1/4}}{\phi}$, $\cos \pi/10 = \frac{1}{2\phi}$, and $\tan \pi/10 = 2\sqrt{\phi^2 - 1/4}$.
       ii. $\sin \pi/10 = \frac{1}{2\phi}$, $\cos \pi/10 = \frac{\sqrt{\phi^2 - 1/4}}{\phi}$, and $\tan \pi/10 = \frac{1}{2\sqrt{\phi^2 - 1/4}}$.

Remark: To remember (1a) and (1b) above, cut an equilateral triangle in half.

We can also easily compute the zeroes of the trigonometric functions.

7.2 Theorem.

 1. $\sin \theta = 0$ *precisely when $\theta = n\pi$ for arbitrary integers n.*

 2. $\cos \theta = 0$ *precisely when $\theta = n\frac{\pi}{2}$ for odd integers n.*

 3. $\tan \theta = 0$ *precisely when $\theta = n\pi$ for arbitrary integers n.*

Proof. We give the proof for cos carefully, leaving the proof for sin to the reader. (Note that the result for tan follows directly from the result for sin.)

From the unit circle definitions, we see that the line segment that defines $\cos \theta$ is zero precisely when $\theta = \pi/2$ or $\theta = 3\pi/2$. We then have, by periodicity, that $\cos(2\pi m + \pi/2) = 0$ and $\cos(2\pi m + 3\pi/2)$ are all of the zeroes of cos for arbitrary integers m. But $2\pi n + \pi/2 = (4m+1)\pi/2$, and $2\pi m + 3\pi/2 = (4m+3)\pi/2$; and every odd number n is either of the form $n = 4m + 1$ or $n = 4m + 3$. The result follows. $\qquad\square$

Before we can start solving trigonometric equations, we also need to take account of the periodicity of the functions. For example, note that $\sin \pi/4 = \sin 3\pi/4$.

7.3 Theorem. *In all of the following, n is an arbitrary integer.*

 1. *Suppose that $\sin x = \sin y$. Then every $y = n\pi + (-1)^n x$ is a solution, and every solution for y is of this form.*

 2. *Suppose that $\cos x = \cos y$. Then every $y = 2n\pi \pm x$ is a solution, and every solution for y is of this form.*

 3. *Suppose that $\tan x = \tan y$. Then every $y = n\pi + x$ is a solution, and every solution for y is of this form.*

When looking for particular solutions, it is often useful to sketch the graphs of the relevant functions; this gives far better intuition than simply applying the formulae given here and hoping.

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[4]This triangle is known as the **golden triangle**.

Proof. We prove (1) and leave the other two as exercises. We need to check two things: (a) that $\sin x = \sin(n\pi + (-1)^n x)$ for any integer n, and (b) that if $\sin x = \sin y$ then y is of the form given.

For (a), we have two cases: either $n = 2k$, or $n = 2k + 1$ for some integer k (i.e. n is either even or odd). If n is even, then $\sin(n\pi + (-1)^n x) = \sin(2k\pi + (-1)^{2k}x) = \sin x$; if n is odd, then $\sin(n\pi + (-1)^n x) = \sin((2k+1)\pi + (-1)^{2k+1}x) = \sin(2k\pi + \pi - x) = \sin(\pi - x) = \sin x$.

For (b), note from our definitions and proposition 2.7 that $\sin x = \sin y$ precisely when $y = (\pi - \theta) + 2k\pi$ or $y = \theta + 2k\pi$. In the first case, $y = (2k + 1)\pi - (-1)^{2k+1}\theta$ and we let $n = 2k + 1$, and in the second case we let $n = 2k$. $\qquad\square$

We will now solve some trigonometric equations.

7.4 Example. Consider $\sin x = \cos x$. Then $\tan x = 1$, and thus (by exercise 7.1 and the above theorem) $x = \pi/4 + n\pi$ for arbitrary integers n.

7.5 Example. A more difficult example: consider the equation $\sin 2x \sec 4x + \cos 2x = \cos 6x$.

$$\sin 2x \sec 4x + \cos 2x - \cos 6x = 0$$
$$\sin 2x \sec 4x + 2\sin 4x \sin 2x = 0 \qquad \text{(By identity for } \cos\alpha - \cos\beta\text{.)}$$
$$\sin 2x(\sec 4x + 2\sin 4x) = 0 \qquad \text{(So one possibility is } \sin 2x = 0\text{.)}$$
$$\sec 4x + 2\sin 4x = 0 \qquad \text{(Assume not, and divide through to find more solutions.)}$$
$$1 + 2\sin 4x \cos 4x = 0 \qquad \text{(Multiply through by } \cos 4x\text{.)}$$
$$1 + \sin 8x = 0 \qquad \text{(Double angle formula.)}$$

Hence we have two possibilities: $\sin 2x = 0$, and $\sin 8x = -1$. In the first case, one solution is $2x = 0$ and so by the theorem above all the solutions are given by $2x = n\pi$ (where n is an arbitrary integer); in the second case, one solution is $8x = 3\pi/2$, and so all of them are $8x = n\pi + (-1)^n 3\pi/2$ (again, n is arbitrary).

Thus the solutions of the original equation are $x = n\pi/2$ and $x = n\pi/8 + (-1)^n 3\pi/16$ for all integers n.

7.6 Exercise. Solve the following equations for θ:

1. $\sin 2\theta = \cos\theta$

2. $\sin\theta + 2\cos\theta = 1$

3. $\sec 4\theta - \sec 2\theta = 2$

4. $\tan(\pi/4 + \phi) = 3\tan(\pi/4 - \phi)$

5. $\cos 5\theta + 5\cos 3\theta + 10\cos\theta = \frac{1}{2}$

6. $\tan 3\theta - \tan 2\theta - \tan\theta = 0$

Section 8:   Periodic Models

The trigonometric functions are very useful for modelling periodic phenomena.

8.1 Example. Suppose we have a coil of wire rotating in a uniform magnetic field, like in the figure below. The induced current will oscillate from some maximum value $I_{\max}$, down to zero, and then reverse until it reaches $-I_{\max}$. This device, which tranforms mechanical (kinetic) energy into an oscillating voltage, is an AC generator and larger versions are used in most power stations worldwide (the only major power source which does not use such a generator is solar power).
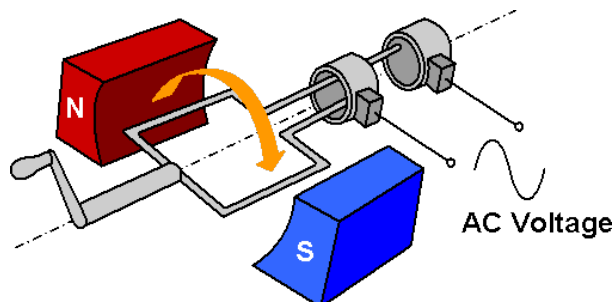

Since the current is generated by circular rotation, one might guess that trigonometric functions would be a good choice for a mathematical model of an alternating current; this guess turns out to be correct.

The most general sine curve is given by

(8.2)
$$f(t) = A\sin(\omega t - \phi) + y_0.$$

The various constants are as follows:

- The **amplitude**, A, is the difference between the maximum and minimum values of f.

- The **y-shift**, y_0, is the average of the maximum and minimum values of f.

- The **angular frequency**, ω, is $2\pi/T$ where T is the period of f.

- The **phase shift**, ϕ, is the time at which the cycle begins.
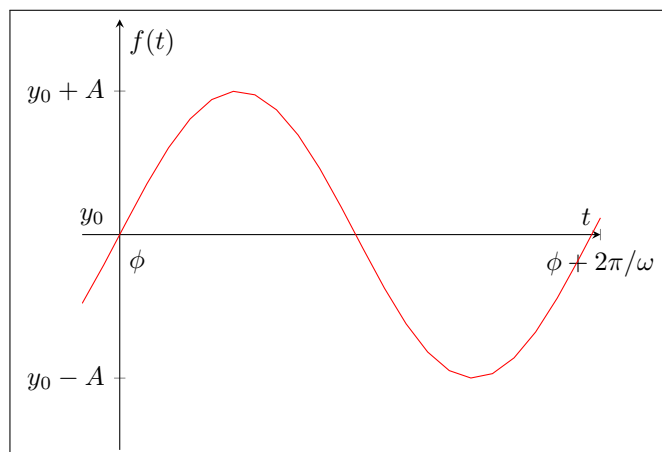
If we replace sin with any trig function, we get a similar picture; further, if we replace sin with an arbitrary function f, then these parameters behave in precisely the same way: if we write

(8.3)
$$y = Af(\omega t - \phi) + y_0.$$

then A behaves as a stretch in the y-direction, ω acts as a stretch in the x-direction, and ϕ and y_0 give x- and y-shifts respectively. Thus there is not much to say about the modelling potential of trig functions in general, except to note that they behave particularly nicely with respect to these parameters due to being bounded (in the case of sin and cos) and periodic (in the case of all trig functions).

Section 9: Epilogue: Hyperbolic Trigonometric Functions

This section requires some knowledge of complex algebra and conic sections.

We will now consider a hyperbola in place of the circle we have already studied.

9.1 Definition. The functions **hyperbolic sine** and **hyperbolic cosine** are respectively defined by

$$\sinh u = \frac{e^u - e^{-u}}{2}$$

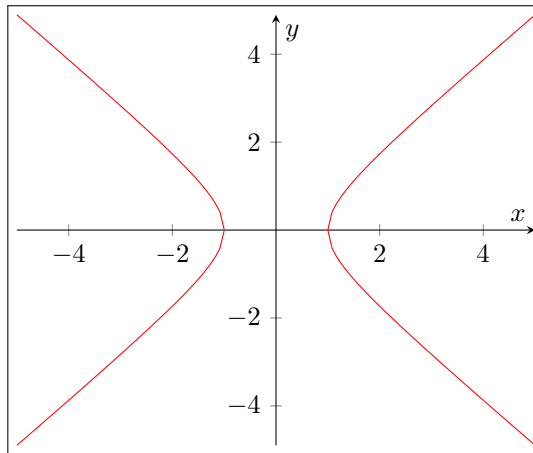$$\cosh u = \frac{e^u + e^{-u}}{2}.$$

9.2 Exercise.

1. With reference to the polar form of complex numbers: show that $\cos u = \frac{e^{iu} + e^{-iu}}{2}$ and that $\sin u = \frac{e^{iu} - e^{-iu}}{2i}$.

2. Hence show that $\sinh u = -i \sin iu$ and $\cosh u = \cos iu$.

3. Conclude that $\sinh$ and $\cosh$ have imaginary periods of $2i\pi$.

9.3 Exercise. Define tanh, sech, csch, and coth analagously to the circular functions. Then show that

1. $\cosh^2 u - \sinh^2 u = 1$,

2. $\operatorname{sech}^2 u + \tanh^2 u = 1$, and

3. $\coth^2 u - \operatorname{csch}^2 u = 1$.

9.4 Exercise. Postulate and prove formulae for $\cosh(u \pm v)$ and $\sinh(u \pm v)$.

Let us now draw the hyperbola $x^2 - y^2 = 1$.

By exercise 9.3 above, if we let $x = \cosh u$ and $y = \sinh u$ then these values satisfy the hyperbola equation. In fact, we can obtain every value on the hyperbola in this way; for example, if $y = \sinh u = (e^u - e^{-u})/2$ then $0 = e^{2u} - 2ye^u - 1$ and this quadratic has nonnegative discriminant for all y.

9.5 Exercise.

1. Show that the equation above has solutions in u for all y — in particular, that the roots of the quadratic are always positive (if some root e^u was nonpositive then there would be no such u).

2. Show that $x = \cosh u$ has solutions for all x such that $|x| \geq 1$.

Hence the hyperbolic trigonometric functions play the same role for the hyperbola as the usual functions play for the circle.

One might now ask why it is the case that the circular functions and hyperbolic functions seem to be related by 'passing through $\mathbb{C}$'. The answer is actually quite simple, but requires some study of the geometry of the complex exponential function which is far beyond what we need for these notes.[5]

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[5]See, for example, A.I. Markushevich, *Theory of Functions of a Real Variable*. Revised English Edition (trans. by Richard A. Silverman). Chelsea Publishing Company (1977). Vol. 1, pages 146 to 159 (§40-2).

Further Reading

See references in text, as well as:

- Marcel Berger, *Geometry I* (trans. by M. Cole and S. Levy). Springer-Verlag (1994).[6]

- H.S.M. Coxeter, *Introduction to Geometry.* John Wiley & Sons (1961).

- Paul Foerster, *Trigonometry: functions and applications.* Second Edition. Addison-Wesley Publishing Co. (1977).

- E.W. Hobson, *A Treatise on Plane Trigonometry.* Second Edition. Cambridge University Press (1897).

- Serge Lang and Gene Murrow, *Geometry: A high-school course* Springer-Verlag (1983).

- Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger, $A = B$. A K Peters, Ltd. (1996).

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[6]Particularly chapter 10.