1

Scholarship Calculus: trigonometry set 7.

1a. Suppose $f(\alpha) = f(\beta)$. Then $a\cos\alpha + b\sin\alpha = a\cos\beta + b\sin\beta$

$$\Rightarrow a(\cos\alpha - \cos\beta) = b(\sin\beta - \sin\alpha)$$

$$\Rightarrow -a\left(2\sin\frac{\alpha+\beta}{2}\cancel{\sin\frac{\alpha-\beta}{2}}\right) = 2b\left(\cos\frac{\beta+\alpha}{2}\cancel{\sin\frac{\beta-\alpha}{2}}\right)$$

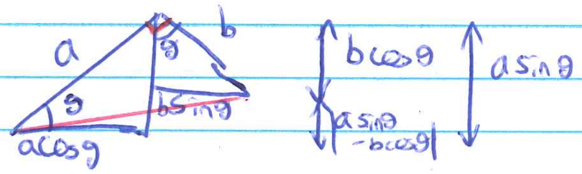
$$\Rightarrow \frac{b}{a} = \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}} = \tan\frac{\alpha+\beta}{2}.$$

b. $f(\alpha) = 0 \Rightarrow a\cos\alpha = -b\sin\alpha$

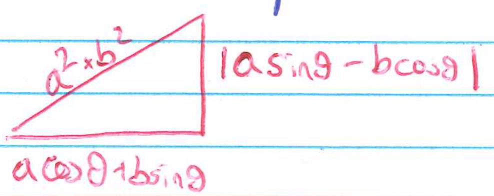
$$\Rightarrow \tan\alpha = -\frac{a}{b}.$$

So $\alpha = \arctan\left(-\frac{a}{b}\right) + n\pi \quad (\pi \in \mathbb{Z})$.

c. This is more interesting. Consider:



Consider the lower triangle:



Hence $(a\cos\theta + b\sin\theta)^2 + |a\sin\theta - b\cos\theta|^2 = a^2 + b^2$

$$(a\cos\theta + b\sin\theta)^2 \leq a^2 + b^2$$

$$f(\theta) = a\cos\theta + b\sin\theta \leq \sqrt{a^2 + b^2}$$

and equality is attained when $a\sin\theta - b\cos\theta = 0$

i.e. $\cancel{\tan b/a}$ $b/a = \tan\theta$.

2a. $\sin 7\theta = \sin(3\theta + 2\theta + 2\theta)$

$$= \sin 3\theta \cos 2\theta \cos 2\theta + 2\cos 3\theta \sin 2\theta \cos 2\theta - \sin 3\theta \sin^2 2\theta$$

$$= (3\sin\theta - 4\sin^3\theta)(1-2\sin^2\theta)^2 + 2(4\cos^3\theta - 3\cos\theta)\cdot 2\cos\theta\sin\theta\cdot(1-2\sin^2\theta) - (3\sin\theta - 4\sin^3\theta)(2\cos\theta\sin\theta)^2$$

$$= (3\sin\theta - 4\sin^3\theta)(1 - 4\sin^2\theta + 4\sin^4\theta) + 4\sin\theta\cos\theta(4\cos^3\theta - 3\cos\theta - 8\cos^3\theta\sin^2\theta + 6\sin^2\theta\cos\theta)$$
$$- 12\cos^2\theta\sin^3\theta + 16\cos^2\theta\sin^5\theta$$

$$= -16\sin^7\theta + 16\sin^5\theta\cos^2\theta + 28\sin^5\theta - 32\sin^3\theta\cos^4\theta + 12\sin^3\theta\cos^2\theta - 16\sin^3\theta + 16\sin\theta\cos^4\theta - 12\sin\theta\cos^2\theta + 3\sin\theta$$

$$= -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta$$

$\therefore\ 64\sin^7\theta + \sin 7\theta = 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta.$

* $\sin\theta = 0$ – one set of solns.

Let $x = \sin\theta$, and assume $x \neq 0$. Then $112x^5 - 56x^3 + 7x = 0$

$\Rightarrow 112x^4 - 56x^2 + 7 = 0.$

$$x^2 = \frac{56 \pm \sqrt{56^2 - 4\cdot 112\cdot 7}}{2\cdot 112}$$
$$= \frac{56}{224} \pm \frac{\sqrt{0}}{2\cdot 112}$$
$$= \frac{1}{4}.$$

So $x = \pm\frac{1}{2}$. $\Rightarrow \sin\theta = \pm\frac{1}{2}$.

$\arcsin \pm\frac{1}{2} = \pm\frac{\pi}{6}$.

Hence the totality of solns are:- ~~$\theta \in \{\pi n,\ \pi/6 + 2\pi n,\ -\pi/6 + 2\pi n\}$ $n \in \mathbb{Z}$.~~

$$\theta \in \begin{cases} n\pi, \text{ or} \\ n\pi + (-1)^n \pi/6. \end{cases}$$

2b. $\cot^2\theta - 3\cot\theta + 2 = 0$

$\Rightarrow \cot\theta = 1$ or $\cot\theta = -2$

$\Rightarrow \tan\theta = 1$ or $\tan\theta = -\frac{1}{2}$

So $\theta = n\pi + \pi/4$ or $\theta = -\arctan(\frac{1}{2}) + n\pi$ $(n \in \mathbb{Z})$, 2c. $\cos 3\theta \cos\theta + \sin 3\theta \sin\theta = \cos 2\theta$

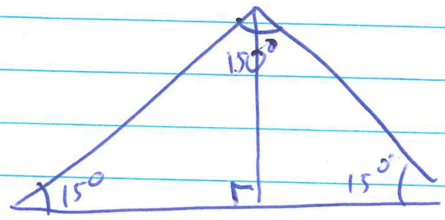
$(4\cos^3\theta - 3\cos\theta)\cos\theta + (3\sin\theta - 4\sin^3\theta)\sin\theta = 1 - 2\sin^2\theta$

$4\cos^4\theta - 3\cos^2\theta + 3\sin^2\theta - 4\sin^4\theta = 1 - 2\sin^2\theta$
$4(1-\sin^2\theta)^2 - 3(1-\sin^2\theta) + 3\sin^2\theta - 4\sin^4\theta = 1 - 2\sin^2\theta$

$\Rightarrow -\sin^2\theta = 0$.

Here we have solutions exactly when $\sin\theta$ has roots:-
i.e. $\theta = \pi n$ $(n \in \mathbb{Z})$.

3.



$15° = \frac{1}{2}\, 30°$.

$\sin(15°) = \sqrt{\frac{1-\cos 30°}{2}}$
$= \sqrt{\frac{1-\sqrt{3}/2}{2}}$
$= \frac{\sqrt{2-\sqrt{3}}}{2}$.

$\cos(15°) = \sqrt{\frac{1+\sqrt{3}/2}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

$\sin(150°) = \sin(180° - 30°)$
$= \sin(30°) = 1/2$

$\cos(150°) = \cos(180° - 30°)$
$= -\cos(30°) = -\sqrt{3}/2$.

4. From prob set 6, $4\arctan 1/5 - \arctan 1/239 =$
$= 2\arctan\left(\frac{5}{12}\right) - \arctan 1/239 =$
$= \arctan\left(120/119\right) - \arctan(1/239)$
$= \arctan(1)$
$= \pi/4$.

5a. $\frac{\sin x}{x}$ "=" $\frac{1}{x}(x-n\pi)(x+n\pi)\cdots \qquad (n\in\mathbb{Z})$

"=" $\frac{1}{x}\cdot(x^2-n^2\pi^2)\cdots$

"=" $\left(x-\frac{n^2\pi^2}{x}\right)\cdots \qquad (n\neq 0)$

"=" $\left(1-\frac{x^2}{n^2\pi^2}\right) \qquad$ (since $x-\frac{n^2\pi^2}{x}=0 \Leftrightarrow 1-\frac{x^2}{n^2\pi^2}=0$).

b. $$\frac{\sin \pi/2}{\pi/2} = \left(1-\frac{(\pi/2)^2}{\pi^2}\right)\left(1-\frac{(\pi/2)^2}{2^2\pi^2}\right)\cdots$$

$$\frac{2}{\pi} = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^2\cdot 2^2}\right)\left(1-\frac{\cancel{(\pi/2)^2}\,1}{3^2\cdot 2^2}\right)\cdots$$

$$= \left(1-\tfrac12\right)\left(1+\tfrac12\right)\left(1-\tfrac14\right)\left(1+\tfrac14\right)\left(1-\tfrac16\right)\left(1+\tfrac16\right)\cdots$$

$$= \frac12\cdot\frac32\cdot\frac34\cdot\frac54\cdot\frac56\cdot\frac76\cdots$$

$$\frac{\pi}{2} = \frac21\cdot\frac23\cdot\frac43\cdot\frac45\cdot\frac65\cdot\frac67\cdots$$

c. $\pi \approx 2\times\left(\frac21\cdot\frac23\cdot\frac43\cdot\frac45\cdot\frac65\cdot\frac67\cdots\frac{14}{15}\right)$

$= 3.038\ldots$

Convergence is slow!!