NCEA Level 3 Trigonometry (exercise set)
6. Inverse Functions

Goal To continue studying the relationships between the trig functions in the form of identities.

1. (a) Show that $\arcsin x = 2 \arctan \left(\dfrac{x}{1 + \sqrt{1 + x^2}} \right)$ for $-1 \leq x \leq 1$.

   (b) Show that $\operatorname{arcversin} y = \arctan \left(\dfrac{\sqrt{2y - y^2}}{1 - y} \right)$ for $0 \leq y \leq 2$.

2. (a) Use corollary 4.3.3 to show that $\arctan a + \arctan b = \arctan \big[(a+b)/(1-ab) \big]$, and indicate any necessary restrictions on a and b.

   (b) Show that $\arctan 1 + \arctan 2 + \arctan 3 = \pi$.

   (c) Show that if $\tau > 0$ then $\arctan \tau + \arctan 1/\tau = \pi/2$. What if $\tau < 0$?

3. Suppose $p = b/a$ and $q = y/x$ are rational numbers in simplest form (i.e. a, b, x, y are integers, a and x are nonzero, and the fractions cannot be simplified further).

   Define a new operation $\otimes$ by
   $$p \otimes q = \frac{ay + bx}{ax - by}.$$

   (a) Suppose p and q are rational. Show that $p \otimes q = q \otimes p$.

   (b) Is there a rational number ϑ such that $\vartheta \otimes k = \vartheta$ for all rational numbers k?

   (c) Is there a rational number $\digamma$ such that $\digamma \otimes k = k$ for all rational numbers k?[1] (Hint: yes.)

   (d) Fix some rational number k; does there exist a rational number k' such that $k \otimes k' = \digamma$?

   (e) Show that
   $$\arctan (p \otimes q) = \arctan (p) + \arctan (q) \,.$$

   Compare with the identity $\log ab = \log a + \log b$. (Note: you may wish to complete 2(a) above first.)

4. Find an expression analogous to those in theorem 6.3 for $\tan(\operatorname{arcsec} x)$.

5. Find a such that $\arcsin a = 2 \arccos a$.

Additional reading Hobson chapter III.

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[1] See also `https://www.youtube.com/watch?v=GFLkou8NvJo`.