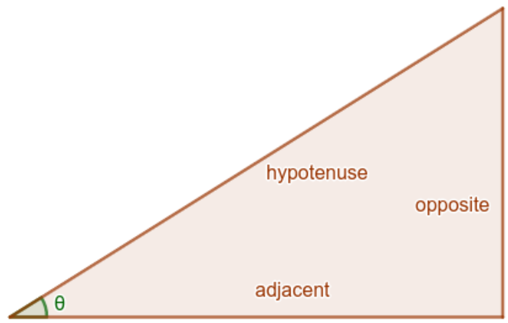
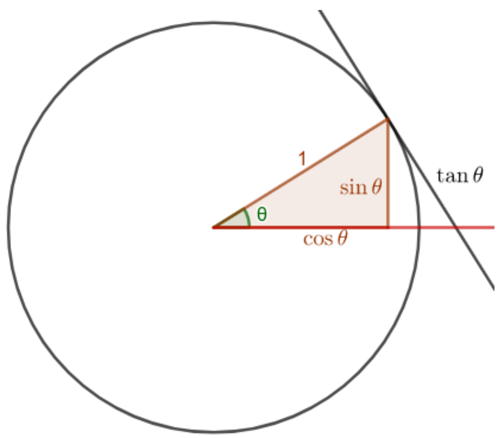
NCEA Level 2 Mathematics
3. Trigonometry



We are now going to look at triangles inside circles. Now, last year we learned that any triangles with two equal angles are similar; in particular, if we take ratios of sides, we obtain the same value. This means that if we have any right-angled triangle with angle θ like the one above, then the ratios $\frac{\text{opposite}}{\text{hypotenuse}}$, $\frac{\text{adjacent}}{\text{hypotenuse}}$, and $\frac{\text{opposite}}{\text{adjacent}}$ all depend only on the angle θ; we call them the sine, cosine, and tangent of the angle respectively:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}.$$

In particular, if we draw our triangle inside a unit circle then we can draw the following:



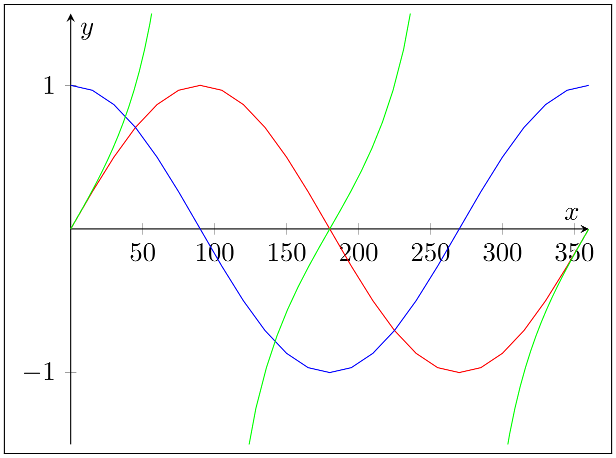
In fact, we can take this as our definition of sin and cos. To show that $\tan\theta$ is indeed the line segment marked there, first notice that since the large triangle is right-angled, the angle at the intersection of the horizontal line and the tangent line is $90° - \theta$; so the other non-right-angle in the smaller triangle is θ. Hence the hypotenuse of the triangle is $\frac{\text{adjacent}}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$, as proposed.

Note also that, from this diagram, we have

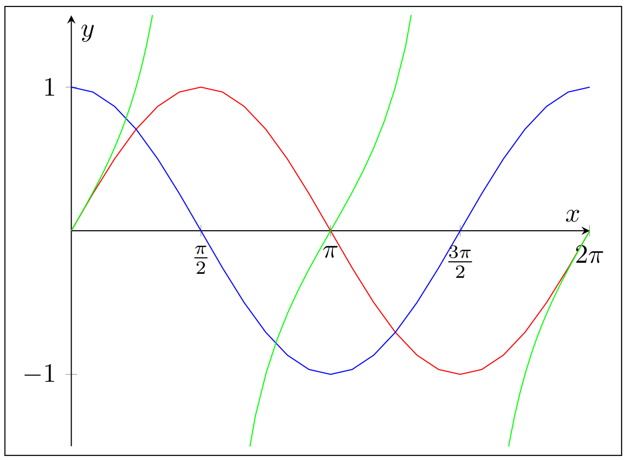
$$\sin^2\theta + \cos^2\theta = 1$$

for every angle θ.

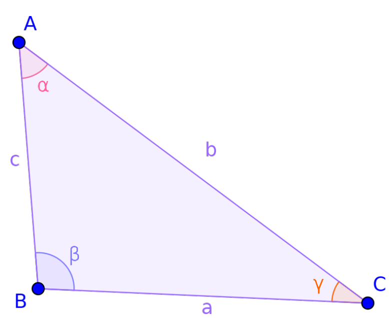
Since the sin of an angle is just the height of the point above the x-axis in the diagram above, we have that $-1 \leq \sin\theta \leq 1$; similarly, $-1 \leq \cos\theta \leq 1$. Note that when $\theta = 90°$, the tangent line becomes horizontal and so never intersects the x-axis: so $\tan 90°$ is undefined. We can even graph $\sin\theta$ (red), $\cos\theta$ (blue), and $\tan\theta$ (green):



If we graph them in radians, only the labels on the x-axis change:



Let us now begin to look at more general triangles:



Theorem (Sine rule)**.** *In any triangle, with the angles and sides labelled as above, we have*

$$\frac{\alpha}{\sin a} = \frac{\beta}{\sin b} = \frac{\gamma}{\sin c}.$$

Proof. Drop an altitude from B to AC, creating two new right-angled triangles. Then the length of this line can be calculated using both of the resulting right-angled triangles: so $c \sin \alpha = a \sin \gamma$ and $\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$. This proves the theorem. □

Theorem (Cosine rule)**.** *In any triangle, with the angles and sides labelled as above, we have*

$$a^2 = b^2 + c^2 - bc \cos \alpha.$$

Proof. Drop an altitude from B to AC, creating two new right-angled triangles. Then the length b can be split into two lengths, $c\cos\alpha$ and $b-c\cos\alpha$; the length of the altitude is $c\sin\alpha$. Now, apply the Pythagorean theorem to the triangle including the angle γ:

$$a^2 = (b-c\cos\alpha)^2 + c^2\sin^2\alpha = b^2 - 2bc\cos\alpha + c^2\cos^2\alpha + c^2\sin^2\alpha = b^2 + c^2 - 2bc\cos\alpha.$$

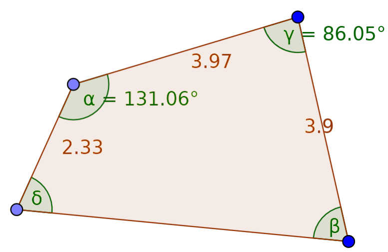
□

Questions

1. A beam of gamma rays is to be used to treat a tumor known to be 5.7 cm beneath the patient's skin. To avoid damaging a vital organ, the radiologist moves the source over by 8.3 cm.
   (a) At what angle to the patient's skin must the radiologist aim the gamma ray source to hit the tumor?
   (b) How far will the beam have to travel through the patient's body before reaching the tumor?

2. A surveyor measures the three sides of a triangular field and finds that they are 117, 165, and 257 metres long.
   (a) What is the measure of the largest angle of the field?
   (b) What is the area of the field?

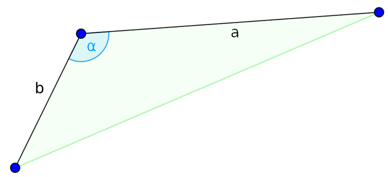
3. A field has the shape of a quadrilateral (four-sided shape) that is *not* a rectangle. Three sides measure 2.33, 3.97, and 3.9 kilometres, and two angles measure $\alpha = 131.06°$ and $\gamma = 86.05°$ (as in the figure).



   (a) By dividing the quadrilateral into two triangles, find its area.
   (b) Find the length of the fourth side.
   (c) Find the measures of the other two angles, β and δ.

4. Suppose that the country of Parah has just launched two satellites. The government of Noya sends aloft its most reliable astronaut, Ivan Advantage, to observe the satellites.
   (a) As Ivan approaches the two satellites, he finds that one of them is 8 km away from him and the other is at 11 km. He measures the angle between them to be 120°. How far apart are the satellites?
   (b) Three ships are assigned to rescue Ivan as his spacecraft plunges into the ocean. The ships are at the vertices of a triangle with sides of 5, 7, and 10 kilometres.
      i. Find the measure of the largest angle of this triangle.
      ii. Find the area of ocean in the triangular region bounded by the three ships.
   (c) To welcome their returning hero, the Noyans give Ivan a parade. The parade goes between the three cities of Triy, Kwe, and Stin. These cities are at the vertices of an equilateral triangle. The roads connecting them are straight, level, and direct, and the parade goes at a constant speed with no stops. From Triy to Kwe takes 80 min and from Kwe to Stin takes 80 min, but from Stin back to Triy along the third side of the triangle takes one hour and 20 minutes. How do you explain this discrepancy in times?

5. Find the area of the triangle below.



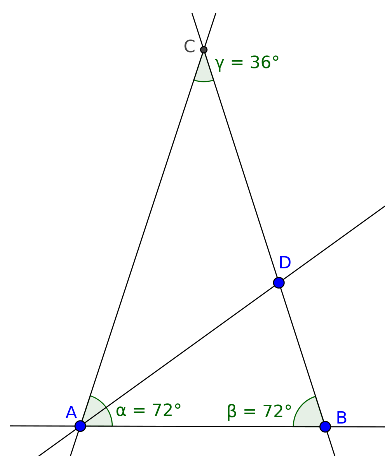
6. For each item below, decide whether or not such a triangle exists. If at least one does, how many exist?
   (a) Exactly one angle greater than 90°.
   (b) Two angles greater than $\pi/2$.
   (c) Two sides of length 200,000.
   (d) Three sides of length 200,000.
   (e) Sides of length 90, 30, and 30.

7. Prove that, if a quadrilateral has equal diagonals, then it is a rectangle. (We used this fact last week!)

8. This question requires you to find exact values for trig functions *without* using a calculator. [Schol 1999]
   (a) Sketch an equilateral triangle of side 2 units. Hence, or otherwise, show that
      - $\cos\frac{\pi}{3} = \frac{1}{2}$,
      - $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, and
      - $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

   (b) Sketch a right angled isosceles triangle whose equal sides enclose the right angle and are of length 1. Hence, or otherwise, show that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

9. Consider the 75-75-36 triangle ABC given in the figure. The angle α has been bisected into two angles, and the resulting line meets the triangle at D.



   (a) Show that ABC and ABD are similar triangles.
   (b) Hence, or otherwise, show that $\frac{AB}{BD} = \frac{AB+BD}{AB}$.
   (c) Show that the ratio of the long side of the triangle to the short side of the triangle is $\frac{AB}{BD} = \frac{1+\sqrt{5}}{2} = \phi$.
   (d) Show that $\cos 72° = \frac{1}{2\phi}$.
   (e) Find $\sin 36°$ and $\sin 72°$.