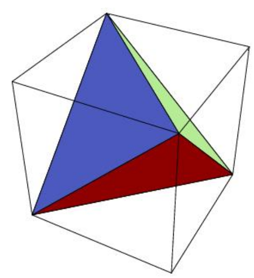
NCEA Level 3 Calculus (Integration)
26. More Interesting Problems

These problems do not just concern integration.

Questions

1. E Find the equation of the line through the point $(3, 5)$ which cuts off the least area from the first quadrant.
2. E The area of a square is increasing at a constant rate of k m^2 per second. A tetrahedron (equilateral triangular pyramid), has a side length that is the same as that of the square at each instant. The initial volume of the tetrahedron was $1\,\mathrm{m}^3$. In terms of k, what is the volume of the tetrahedron three seconds after that?
3. S Scholarship 2017: The hyperbolic functions $\sinh x$ and $\cosh x$ are defined as follows:
$$\begin{aligned}\sinh x &= \frac{1}{2}(e^x - e^{-x})\\ \cosh x &= \frac{1}{2}(e^x + e^{-x}).\end{aligned}$$
Show that $\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$.
4. S Consider the tetrahedron inscribed inside a cube, as in the figure.



The volume V of the cube at any instant t is increasing at a rate proportional to the value of V at that instant. The initial volume of the cube at $t = 0$ was 8 cubic units. What is the volume of the tetrahedron at time $t = 20$?
5. S If $x \sin \pi x = \int_0^{x^2} f(t)\,\mathrm{d}t$, where f is continuous, find $f(4)$. [*Hint: you need not perform any integration.*]
6. S If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.$$
Hence, or otherwise, evaluate
$$\lim_{x\to 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}.$$
7. (a) S Consider the differential equation
$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + 5\frac{\mathrm{d}\Phi}{\mathrm{d}t} + 6\Phi(t) = 0.$$
Let f and g be the functions defined by $f(x) = e^{-2x}$ and $g(x) = e^{-3x}$.

i. Show that all linear combinations of f and g are solutions to the differential equation.

ii. Find the (unique) solution passing through $(0, 1)$ and $(1, 1)$.

(b) $\boxed{\mathtt{O}}$ More generally, consider the differential equation $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$. Let the zeroes of the quadratic polynomial $p(D) = aD^2 + bD + c$ be α and β. Show that all the linear combinations of $e^{\alpha x}$ and $e^{\beta x}$ are solutions to the differential equation.

8. $\boxed{\mathtt{S}}$ Compute the following definite integral. [*Hint: begin with a substitution.*]

$$\int_0^{\pi/6} \sqrt{\tan\theta}\,\mathrm{d}\theta$$

9. (a) $\boxed{\mathtt{E}}$ Consider the two functions $p(x) = 3x^5 - 5x^3 + 2x$ and $q(x) = 3x^5$. Show that their ratio approaches 1 as $x \to \infty$.

(b) $\boxed{\mathtt{S}}$ Let $p(x)$ and $q(x) \neq 0$ be polynomials. Recall that the degree of a polynomial is the highest n such that x^n has a non-zero coefficient. Compute the limit

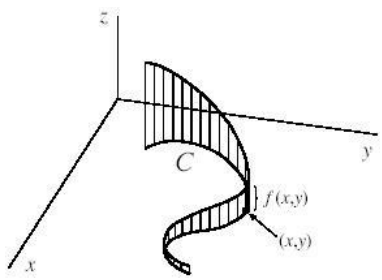
$$\lim_{x\to\infty} \frac{p(x)}{q(x)}$$

if:

i. the degree of $p(x)$ is less than that of $q(x)$.

ii. the degree of $p(x)$ is greater than that of $q(x)$.

10. $\boxed{\mathtt{S}}$ A definite integral calculates the between a curve and straight line, the x-axis. In a similar way, it is possible to use integration to calculate the area above a curved line and below a surface $z = f(x, y)$, like that in the figure.



If the curve C is defined parametrically, that is $C(t) = (x(t), y(t))$, then the integral along the line can be calculated with the formula

$$\int_a^b f(x(t), y(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Compute the line integral of the function $f(x, y) = 2 + x^2y$ around the upper half of the unit circle.

11. $\boxed{\mathtt{S}}$ The **sine integral** function is defined by

$$\mathrm{Si}(x) = \begin{cases} \int_0^x \frac{\sin t}{t}\,\mathrm{d}t, & \text{for } x \neq 0; \\ 1, & \text{for } x = 0. \end{cases}$$

(a) Recall that $\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a)$. Use this to show that $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f'(t)\,\mathrm{d}t = f'(x)$.

(b) Find the x-coordinate of the first local maximum of Si to the right of the origin. Carefully prove that you have found a maximum.

(c) Use the result in (a) to find an expression for the integral

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)\,\mathrm{d}t,$$

where f is continuous and g and h are differentiable.

12. E Minimise the function $f(x) = b\log_b N$ with respect to b, and show that the result is independent of the constant N.*

13. S We can calculate **improper integrals** (those where the bounds are infinite) as follows:

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x.$$

Decide whether each of the integrals given below exists. If so, calculate its value; if not, explain why.

(a) $\displaystyle\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ (b) $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ (c) $\displaystyle\int_1^\infty \sin x\,\mathrm{d}x$

14. (a) S Show that $F(x) = \tan^{-1} x$ is an anti-derivative of $f(x) = \frac{1}{1+x^2}$ in the following ways:

i. Differentiate $F(x)$ and simplify to give $f(x)$.

ii. Use the substitution $x = \tan\theta$ to integrate $f(x)$ and simplify to give $F(x)$.

(b) O Recall that $22/7$ is often given as a rough approximation to π. Consider the integral

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x,$$

and hence show that $22/7 > \pi$.†

15. S Consider the operator $\mathcal{L}$ defined by

$$\mathcal{L}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\left[f\left(e^x\right)\right].$$

(a) Show that $\mathcal{L}x^n = n$ and that $\mathcal{L}[u(x)]^n = n\mathcal{L}u(x)$.

(b) Find an expression for $\mathcal{L}[u(x)v(x)]$ and $\mathcal{L}[u(x)/v(x)]$.

(c) Find an expression for $\mathcal{L}[u(x)+v(x)]$.

(d) For which y is $\mathcal{L}y = y$?

16. S Compute the following indefinite integrals:

(a) $\displaystyle\int \frac{\sin\frac{1}{x}}{x^2}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{\ln x\cos x - \frac{\sin x}{x}}{(\ln x)^2}\,\mathrm{d}x$

* Dudley, *Mathematical Cranks*, p.52.

† Nahin, *Inside Interesting Integrals*, pp.23-4.

17. $\boxed{0}$ A while ago (when we talked about the product and quotient rules), I claimed that the radius of the circle best approximating a continuous curve around a point (x, y) is given by

$$\text{radius of curvature} = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|}.$$

Let us attempt to prove this.

(a) Let f be a continuous function at x such that the second derivative of f at x exists. By recalling our work on approximations, explain why knowing up to the second derivative of f should be enough to find the 'best circular approximation' of f at $(x, f(x))$.

(b) Consider the circle of radius r centred at (x_0, y_0). Suppose that this circle passes through the point (x_1, y_1); suppose further that the first derivative of the y-ordinate of the circle with respect to the x-ordinate is m, and that the second derivative is c. Write down expressions for r, x_0, and y_0 in terms of x_1, y_1, m, and c.

(c) Use part (b) to write down the radius of the unique circle passing through $(x, f(x))$ with matching first and second derivatives to f.