Topics in Geometry

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In another moment Alice was through the glass and had jumped lightly down into the Looking-glass room... Then she began looking about, and noticed that what could be seen from the old room was quite common and uninteresting, but that all the rest was as different as possible. [Car98, p. 127]

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Preface

These notes, which I wrote in 2019, are intended to be an introduction to geometry for younger students who have not learned the subject before. I hope that I have included a good balance of practical numeracy material (on measurement, and trigonometry), traditional Euclidean geometry, and some more modern topics (isometries and symmetries being the ‘core’ of this discussion).

The notes are structured as a series of loosely connected exercises, many of which require mathematical justification; I have consciously decided not to follow a purely axiomatic approach in the style of Euclid, but I think enough of the structure has been erected to allow an axiomatic study to be motivated by — or at the very least not precluded by — the material within. One good modern axiomatic treatment is given in [Lee17].

There are not many diagrams in these notes; this is by design, as I feel that any diagrams which I provide would be far inferior in terms of actual learning content than diagrams drawn by students themselves. The formatting of these notes, with a far wider margin than even LaTeX normally includes, is designed with copious note-taking in mind.

Needless to say, these notes are wholly and fundamentally unusable as self-study material at Y11. I would recommend a book like [LM83] or [BB59] for this purpose.

There is more material here than can reasonably be covered in the ten weeks or so that schools tend to spend on geometry; my intent is that the instructor picks and chooses material beyond the core material, and my hope is that students find it sufficiently interesting to continue glancing at the notes for their own amusement.

I also wish to note that a large proportion of these notes overlaps in content with my Level 2 notes and my Level 3 geometry notes; this is mainly because I usually feel the need to include a lot of extra content in higher level notes that would be better placed here.

Recommendations for suplementary and further reading

References are given throughout the notes when something of particular historical or mathematical interest is discussed; however, many of the references are intended more for the teacher than for the student. I would like to take this opportunity to highlight a few of my favourites.

If one is interested in classical geometry, then books like [Sve91] or [CG67] are highly recommended. For more modern geometry, especially for students who have the basics of calculus available, [Cox61] is a classic. There are a variety of problem books available: one such book is [PS17], though there are many others.

The author’s favourite geometry book, however, is [Ber87].

Chapter 0

Proof in mathematics

> *A mathematician does not want to be told something:*
> *they want to find it out for themself. [Saw82, p. 19]*

In this topic, we are going to prove things. This is a simple statement, but one it is worth spending a bit of time thinking about.

First of all, what is **mathematical proof**? The most simple definition is that a proof is an explanation of why a statement is true. More precisely, one starts from a set of accepted assumptions (the **hypotheses** of the proof), and one argues why certain conclusions *must* be true.

Proof is what sets mathematics apart from the sciences — in physics and chemistry, for example, one does not prove things simply by thought (a physical theory is incorrect if it disagrees with experiment, no matter how logically sound it is). In all fields of mathematics, things must be proved before they are used.

Here are some more words:

- A **theorem** is a true statement that has been deduced (via a mathematical proof) from some agreed assumptions. (For example, Pythagoras' theorem.)
- A **proposition** is a small theorem.
- An **axiom** or **postulate** is a basic assumption that is accepted without proof. (For example, 'all right angles are equal'.)

Chapter 1

Basic theory

The first known systematic treatment of geometry is Euclid's *Elements* (one particularly nice edition is [Euc10]). Euclid lived in Alexandria (in Greek Egypt) in around 300 BCE; his book is the oldest known treatment of a field of mathematics in a deductive way, that is by starting with a small set of 'obvious' (i.e. unproved) statements with the goal of deducing as much theory as possible using purely logical reasoning.

Euclid's development of geometry is rather lacking in a few ways, at least if it is judged against modern mathematical standards; however, the results proved by Euclid still form the basic foundation of the geometry which we will be studying here. For a commentary on Euclid from a mathematical perspective, see [Har00].

Rather than work axiomatically, we will take for granted that the things we discuss are well-defined and exist, and we will allow ourselves to think of the foundations of geometry in an intuitive fashion rather than a precise fashion. For a precise, axiomatic development of the foundations of geometry see [Lee17].

1.1 Terminology

We will take our **space** to be a set of **points**, which we will label with capital Roman letters A, B, C, and so on. Through every pair of points we will be allowed to draw a unique **line**; the line through A and B will be denoted by $\overline{AB}$. The portion of the line that lies *between* A and B will be called a **segment** and will be denoted by $[A, B]$. To each segment we assign a number, called the **length** of the segment $[A, B]$ or the **distance** between A and B, in the usual way (i.e. the length is 'whatever the ruler reads when you hold it up to the segment'). The length of $[A, B]$ will be denoted by $|AB|$.

Proposition 1. *If ℓ and m are distinct lines (we will use lowercase Roman letters to denote lines), then ℓ and m either do not intersect or intersect at precicely one point. There are no other options.*

Proof. Suppose ℓ and m intersect at more than one point: say they intersect at two different points, A and B. We need to show that ℓ and m cannot be distinct lines. But by our assumption above, there is a *unique* line between any two points; hence ℓ and m must both be this unique line, and must therefore be the same. ∎

Let ℓ and m be lines. Then ℓ and m are said to be **parallel** if they do not intersect in exactly one point; by the above proposition, this means that they must intersect either zero times or they must be the same line. If ℓ and m are parallel, we will write $\ell \parallel m$.

It is necessary for us to assume one thing about parallel lines without proof:

Axiom 2 (Parallel postulate)**.** *If ℓ is a line, and P is any point that does not lie on ℓ, then there is a unique line through P parallel to ℓ.*

This postulate was Euclid's fifth axiom, and is of a much more sophisticated nature than the other four (which stated things like 'it is possible to draw a unique line through two points' and 'all right angles are equal'); it was thought for over a thousand years that this postulate was so much more complex than the others that it must follow from them as a theorem. A later commentator, Proclus, wrote in fifth century Athens that "[the parallel postulate] ought to be struck from the postulates altogether. For it is a theorem — one that invites many questions, which Ptolemy proposed to resolve in one of his books — and requires for its demonstration a number of definitions as well as theorems" (quoted in [Har00, pp. 296-297]).

It turns out that proving the parallel postulate from Euclid's other four axioms is impossible, for a fairly simple reason: it is possible to exhibit a perfectly reasonable geometry that satisfies the first four but not the fifth. The most familiar example is a sphere. Dix a line of latitude and a point not on that line. Then all the lines through that point intersect the original line, so there are no parallel lines!

Exercise 3. Show that, if ℓ and m are lines, then:

1. $\ell \parallel \ell$;

2. if $\ell \parallel m$ then $m \parallel \ell$;

3. if $\ell \parallel m$ and $m \parallel n$ then $\ell \parallel n$ (where n is another line).

We say that being parallel is an **equivalence relation**.

Exercise 4. Suppose ℓ and m are parallel; let n be a line, and suppose it intersects line m at a point P. How are lines ℓ and n related?

A **polygon**, or more specifically an ***n*–gon** where n is some number, is a set of points $A_1, A_2, ..., A_n$ (all different) together with the line segments $[A_1, A_2]$, $[A_2, A_3]$, ..., $[A_n, A_1]$. The polygon will be denoted by $A_1A_2A_3 \cdots A_n$. The points defining the polygon are called the **vertices**, and the line segments are called the **edges** of the polygon. Two polygons are called **equal** if they share the same points and edges, up to relabelling. Note that we allow edges to cross each other; note also that the order of the points matters: $ABCD$ is not the same polygon as $ACBD$.

A polygon with three edges is called a **triangle**; a polygon with four edges is called a **quadrilateral** or a **quadrangle**. If all the sides of a polygon are the same length, it is called **equilateral**.

Exercise 5. Let A, B, C, and D be pairwise distinct points (i.e. no two are the same); how many different 4-gons can you form with these points as vertices?

The **circle** with **centre** O and **radius** r, where O is a point and r is a positive number, is the set of all points that lie at a distance r from O (i.e. a point X lies on the circle precisely when $|OX| = r$).

Exercise 6. Let $[A, B]$ be a line segment. Draw the circle with centre at A of radius $|AB|$, and the circle with centre B of radius $|AB|$. These two circles will intersect at two points, X and Y. What kinds of triangles are ABX and ABY?

Exercise 7. Let $[A, B]$ be a line segment. The **midpoint** of $[A, B]$ is the point M such that $|AM| = |BM|$. Can you find M by drawing only circles of a given centre and radius?

1.2 Angles

Let O, A, and B be three distinct points. A line eminating from O and passing through A indefinitely will be called the **ray** $\overrightarrow{OA}$; if we consider the two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ then together they split up the space into two regions, called the **angles** associated with the rays. The angle $\angle AOB$ is the angle which is found by starting at $\overrightarrow{OA}$, and then rotating anticlockwise to $\overrightarrow{OB}$. In other words, the notation 'walks' from A, to O, and to B, and we take the region on our left. The angle $\angle BOA$ is the angle found by starting at B, walking to A, and taking the region on our left. In all these cases, O is called the **vertex** of the angle.

If A and B are the same point, then we still say that AOB is an angle; however, we will determine from context in each case whether we mean the angle with no interior or the angle whose interior is the entire space.

If $\angle AOB$ is an angle, we may draw a circle at O; then the portion of the circle lying within each region is called an **arc** of the circle, and the portion of the inside of the circle is called a **sector** of the circle.

There are several different ways of assigning a measure to angles; we will use the so-called **degree** measure here. The idea is to associate with a full angle — that is, an angle $\angle OAB$ such that a circle at O has no arc cut off — a measure of $360°$.

If A, O, and B all lie on a line (if they are **colinear**), we will call the angle $\angle AOB$ the **straight angle**; it is clear that any circle around O is divided into two equal parts, and so the angle $\angle OAB$ must have measure of one-half a full turn; that is, $180°$. An angle whose measure is one-half a straight angle is called a **right angle**, and has measure $90°$.

We occasionally split a degree into sixty **minutes**, and one minute into sixty **seconds**; the notation $30°40'2''$ represents an angle with measure 30 degrees, 40 minutes, and 2 seconds (i.e. $30 + \frac{40}{60} + \frac{2}{60 \times 60} \approx 30.6672°$); this notation is really only used in navigation in the modern world, and you will hardly ever see it.

Angles that are measured anticlockwise are positive; angles that are measured clockwise are negative. Two angles are said to be **congruent** (we will often be flexible with language and call them equal) if they have the same measure.

Exercise 8. An angle 15° cuts out an arc a on a circle whose circumference is 5 cm. How long is the arc a?

Use a protractor to draw a picture and check.

Exercise 9. Wellington is around 41.3° south of the Equator. The circumference of the earth is around 40 000 km. How far away, around the Earth, is Wellington from the nearest point on the Equator?

Two angles which share a common ray are called **adjacent**. If $\angle AOB$ and $\angle BOC$ are adjacent, then $\angle AOC = \angle AOB + \angle BOC$.

The following terms are standard, but basically unimportant. We will use them only occasionally in these notes.

1. Two angles whose measure sum to 180° are called **supplementary**.

2. Two angles whose measure sum to 90° are called **complementary**.

3. If an angle has measure less than a right angle it is called **acute**.

4. If an angle has measure greater than a right angle but less than a straight angle it is called **obtuse**.

5. If an angle has measure greater than a straight angle it is called **reflex**.

Suppose two distinct lines intersect at a point O. Then these lines define four angles; suppose their measures are, clockwise, α, β, γ, and δ (we will always use Greek letters to denote angles). Then each pair of non-adjacent angles (α and γ, and β and δ) is called a pair of **opposite** angles.

Exercise 10. Suppose α, β, γ, and δ are measures of angles organised in this way; show that $\alpha = \gamma$, and $\beta = \delta$. (In other words, opposite angles have equal measure.)

Exercise 11. Suppose ℓ and m intersect at a point X, and one of the angles between the lines at X is a right angle. Show that all four are right angles.

Such lines are called **perpendicular**, and we write $\ell \perp m$.

Exercise 12. Let ℓ and m be parallel lines, let A and B are any two points on ℓ, draw the lines through A and B perpendicular to ℓ, and let A' and B' be the points of intersection of these perpendicular lines with m.

- Show that $|AA'| = |BB'|$. [You will need the parallel postulate.]

- Show that if two lines are parallel, and a third line is perpendicular to one of them, then it is perpendicular to both.

Note: it may be easier to prove these in the opposite order and/or at the same time as each other.

Exercise 13. Let ℓ and m be parallel lines, and suppose n is a third line which intersects them at A and B respectively. What are the relationships between the angles around A and the angles around B?

Suppose it is not known that ℓ and m are parallel. Could you use knowledge of the angles around A and B to check whether they are?

Exercise 14. Let $[A, B]$ be a segment. The unique line ℓ that passes through the midpoint of $[A, B]$ and is perpendicular to $\overline{AB}$ is called the **perpendicular bisector** of $[A, B]$.

Given such a segment, can you draw the perpendicular bisector using only a compass and a straightedge — i.e. without a protractor?

If $A_1A_2A_3 \cdots A_n$ is a polygon such that no edges cross (we will call such polygons **normal**), then to each vertex we may assign two angles. Draw a small circle around each vertex that doesn't include any other vertex; then the arc of the circle that lies inside the polygon is cut off by one of the angles at the vertex whose rays include the edges of the polygon, and this angle is called the **interior angle** of the polygon. The other angle is called the **exterior angle**. If a polygon's interior angles are all acute, then the polygon is called **convex**. If all the interior angles of a polygon are equal, the polygon is called **equiangular**. If a polygon is both equiangular and equilateral, it is called a **regular polygon**.

Exercise 15. The internal angles of a triangle add to $180°$. The internal angles of a normal quadrilateral add to $360°$. What do the internal angles of a normal n-gon add to?

Exercise 16. Draw examples of all the following.

A regular quadrilateral is usually called a **square**. By the previous exercise, the internal angles of a square are all $360°/4 = 90°$. A quadrilateral such that all internal angles are right angles is called a **rectangle**. A non-equiangular but equilateral quadrilateral is called a **rhombus**. A quadrilateral such that both pairs of opposite sides are parallel is called a **parallelogram**. A quadrilateral such that at least one pair of opposite sides is parallel is called a **trapezoid**.

A **chord** of a circle is a line segment with endpoints on the circle. A **diameter** is a chord passing through the centre.

Exercise 17 (Inscribed angle theorem)**.** Draw a circle with centre O; let AB be a chord of the circle, and pick a point X on the circle. Then the measure of $\angle AOB$ is twice the measure of $\angle AXB$. What if X is (a) inside, (b) outside the circle?

Exercise 18 (Thale's theorem)**.** Draw a circle with centre O; let AB be a diameter of the circle, and pick a point X on the circle. Then AXB is a right-angled triangle with right angle at X.

Thales is, according to classical sources, the mathematician credited with first introducing geometry to Greece (from ancient Egypt, where he measured the heights of the pyramids); according to Proclus, "[Thales] first went to Egypt and thence introduced this study into Greece. He discovered many propositions himself, and instructed his successors in the principles underlying many others." (quoted in [Hea81, pp. 128-129]).

1.3 Congruence and similarity

Suppose $\mathscr{F}$ and $\mathscr{G}$ are **figures** (complicated collections of points, like polygons or circles). It is fairly intuitive that if we can 'lift one up, and put it down exactly on the other one', then the two figures are the same shape and size. We would like to make this more precise.

Definition 19. An **isometry** Isom is a **transformation** – a way of moving every point X in our space to another point $\mathsf{Isom}(X)$ — such that whenever X and Y are points, the distance $\left|\mathsf{Isom}(X)\mathsf{Isom}(Y)\right|$ is equal to the distance $|XY|$.

A point which is not moved by a transformation is called a **fixed point**.

It is non-trivial to prove the following theorem, within which we will intermix some definitions.

Theorem 20 (Classification of plane isometries)**.** *There are precisely five kinds of isometry.*

1. *The **identity transformation** Id, which leaves everything where it is. (In other words, every point is a fixed point.)*

2. *The **translations** Tr_{XY} (one for each pair of distinct points X and Y) which send every point in the plane to the point a distance $|XY|$ away, in the direction parallel to $\overrightarrow{XY}$. (No points are fixed.)*

3. *The **rotations** $\mathsf{Rot}_{O,\theta}$ (one for each point O and angle θ between, but not including, zero and a full turn), which send every point X on the circle centred at O passing through X to the unique point Y such that $\angle XOY$ has measure θ. (The point O is fixed, and no others.)*

4. *The **reflections** Ref_{ℓ} (one for each line ℓ), which send every point X to the unique point X' such that ℓ is the perpendicular bisector of XX'. (The line ℓ is fixed and no other points are; every line perpendicular to ℓ is transformed to itself, but no other lines except ℓ are.)*

5. *The **glide reflections** $\mathsf{GlRef}_{\ell,x}$ which consist of a reflection across ℓ and then a translation along ℓ through a distance x. (The line ℓ is fixed, and no other points are fixed, and no other lines are transformed to themselves.)*

Further, every transformation is uniquely determined by which points and lines it fixes (i.e. transforms onto themselves).

Proof. See: [Art91, chapter 5], or [Cox61, chapter 3]. We will consider isometries in more detail later on. ∎

If we do one isometry after another, the total is still an isometry. Further, the action of undoing an isometry is also an isometry. (Isometries are said to form a **group**.)

Exercise 21.

1. Two successive rotations around different points form a translation.

2. Two successive reflections across lines which intersect at exactly one point form a single rotation about that point.

3. Two successive reflections across lines which do not intersect form a translation.

4. Two successive reflections across the same line form the identity.

Two figures $\mathscr{F}$ and $\mathscr{G}$ are called **congruent** if there is an isometry that maps one onto the other: more precisely, $\mathscr{F} \cong \mathscr{G}$ if and only if there is an isometry Isom such that whenever X is a point on $\mathscr{F}$ then $\mathsf{Isom}(X)$ is a point on $\mathscr{G}$, and whenever Y is a point on $\mathscr{G}$ then there is a point X on $\mathscr{F}$ such that $\mathsf{Isom}(X) = Y$. (In other words, if Isom is a one-to-one correspondence between the two figures.)

It turns out that for triangles we have a number of different ways to talk about congruence.

Proposition 22. *Two triangles ABC and DEF are congruent if one of the following is true (and therefore all are true):*

1. *(SSS) $|AB| = |DE|, |BC| = |EF|$, and $|CA| = |FD|$.*
2. *(SAS) The angles at A and D are congruent, and $|AB| = |DE|$ and $|CA| = |FD|$.*
3. *(ASA) The angles at A and D are congruent, the angles at B and E are congruent, and $|AB| = |DE|$.*

Proof. See [Lee17]. ∎

It is possible that two figures can be the same shape even if they are not the same size. For example, all circles centred at a given point can be transformed into each other by shrinking and stretching equally in all directions.

Definition 23. A **dilation** $\mathsf{Dil}_{O,\mu}$ is a transformation that sends each point X to the point on the ray $\overrightarrow{OX}$ that is a distance $\mu|OX|$ from O.

Two figures $\mathscr{F}$ and $\mathscr{G}$ are said to be **similar**, and we write $\mathscr{F} \sim \mathscr{G}$, if one can be transformed onto the other by means of a succession of dilations and isometries (i.e. rigid movements and shrinking and stretching equally in all directions.)

Exercise 24.

1. All circles are similar.
2. Two rectangles, of side lengths x_1 and y_1 and x_2 and y_2 respectively, are similar when $x_1/x_2 = y_1/y_2$.

There are a number of useful characterisations of similar triangles.

Proposition 25. *Two triangles ABC and DEF are similar if one of the following is true (and therefore all are true):*

1. *The following relationship holds:*
$$\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|CA|}{|FD|} \; (= \mu) \tag{1.1}$$
2. *The angles at A and D are congruent, and $\frac{|AB|}{|DE|} = \frac{|CA|}{|FD|}$ $(= \mu)$.*
3. *There is a matching between two vertices of the two triangles such that matched vertices have congruent angles.*

4. *There is a matching between all three vertices of the two triangles such that matched vertices have congruent angles.*

Proof. See [Lee17, chapter 12]. ∎

Exercise 26. Let ABC be a triangle. Show that the following are equivalent (i.e. if one is true then the other is true):

1. The interior angles at B and C are congruent.

2. The lengths $|AB|$ and $|AC|$ are equal.

If one (and therefore both) hold, then ABC is called **isosceles**.

Exercise 27. Prove the following:

1. The opposite sides of a parallelogram have equal length. [Hint: consider a diagonal of the parallelogram.]

2. The opposite angles of a parallelogram are equal.

3. The diagonals of a parallelogram intersect each other.

4. If the opposite sides of a quadrilateral are equal, then the figure is a parallelogram. [Hint: consider a diagonal again.]

Exercise 28. Let ABC be a triangle, let M be the midpoint of $[A, B]$, and consider the lines through M parallel to the other two sides of the triangle. The large triangle ABC is now cut into two smaller triangles, and a quadrilateral. Show that the two smaller triangles are congruent to each other, and similar to ABC.

1.4 Perimeter and circumference

The **perimeter** of a polygon $\mathscr{F} = A_1 A_2 \cdots A_n$ is defined to be the sum

$$\mathcal{P}(\mathscr{F}) = |A_1A_2| + |A_2A_3| + \cdots + |A_{n-1}A_n| + |A_nA_1|. \tag{1.2}$$

Exercise 29. Let c be a circle, with centre O and radius r. Pick n points equally spaced about the circumference of the circle, $A_1, ..., A_n$; so $A_1 \cdots A_n$ is a regular n–gon. It is clear that, as n increases, the perimeter of the n–gon approaches the circumference of the circle. Call this perimeter p_n.

Let $\mathscr{C}$ be a second circle centred at O (two circles are **concentric** — i.e. they have the same centre); let A_1' be the point of intersection between $\overrightarrow{OA_1}$ and $\mathscr{C}$, A_2' the point of intersection between $\overrightarrow{OA_2}$ and $\mathscr{C}$, and so forth. Call the perimeter of the resulting regular n–gon P_n.

1. Show that OA_1A_2 is similar to $OA_1'A_2'$.

2. Use this to show for every n, $\frac{P_n}{R} = \frac{p_n}{r}$.

3. Since this holds for all n, we can conclude that if P is the circumference of $\mathscr{C}$ and p is the circumference of c, $P/R = p/r$. Hence show that the ratio of circumference to radius is the same for every circle.

For historical reasons, the number π is defined to be one-half of this ratio: $\pi = P/2R$. It turns out that

$$\pi = 3.14159265358979323846264338327950... \tag{1.3}$$

We have now got a formula for the circumference of a circle of radius r:

$$C = 2\pi r. \tag{1.4}$$

Exercise 30. We can exploit a fact about perimeters to show that $\pi < 4$.

1. What is the circumference of a circle with diameter 1?
2. Draw the square with side-edge 1 that just touches this circle at the centre of each of its sides. What is the perimeter of this square?
3. Using your picture, how are the two perimeters related? Hence show that $\pi < 4$.
4. How could you improve your upper estimate of π?

In the next section we will show that $\pi > 2$ (exercise 40).

1.5 Area

The **area** of a square $\mathscr{S}$ of side length x is defined to be $\mathcal{A}(\mathscr{S}) = x^2$.

We will now extend the idea of area to more complicated figures; the basic ideas we want are the following:

Axioms 31 (Area function)**.** Suppose we define some way of assigning a number to a collection of different figures (for example, we have just assigned a number — area — to every figure in the collection of squares). We will say that this method of assigning numbers is a **Jordan area method** if the following are true:

1. Every figure in our collection is assigned a number which is zero or positive.
2. If we assign a number to two figures, one of which lies inside the other, the inside figure has a smaller number.
3. If we can split up a figure $\mathscr{F}$ into a finite number of non-overlapping figures $\mathscr{F}_1, ..., \mathscr{F}_n$ which are in our collection, then the figure $\mathscr{F}$ is in our collection and we assign to it the sum of the numbers we assigned to $\mathscr{F}_1, ..., \mathscr{F}_n$.
4. The method assigns to every square $\mathscr{S}$ the number $\mathcal{A}(\mathscr{S})$ as already established.
5. The method assigns to every figure with no inside (e.g. a line, or a collection of lines, or a single point,...) the number 0.
6. If $\mathscr{F} \cong \mathscr{G}$, then the numbers assigned to $\mathscr{F}$ and $\mathscr{G}$ are equal.

It turns out that there is only one way of assigning area to all the things we are used to which satisfies these properties in a nice way (we do need to phrase them more precisely to prove this, and we won't do it here. See [LS14, chapter 8], or [Har00, chapter 5].) The basic idea is that we take all the figures we're interested in measuring, we fit into them lots of little squares, and then we add up the areas of the little squares. We will now have a look at this kind of technique in a superficial way that is, nonetheless, rather convincing; in fact, the methods we use here were those used by the Greeks.

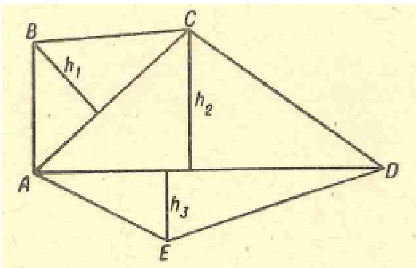
Exercise 32. Let $ABCD$ be a rectangle, such that $|AB| = |CD| = x$ and $|BC| = |DA| = y$. Show that $\mathcal{A}(ABCD) = xy$.

Exercise 33. Let ABC be a triangle such that $|BC| = b$. Let A' be the point on $\overline{BC}$ such that $\overline{AA'} \perp \overline{BC}$; let $\left|AA'\right| = h$. (The line $\overline{AA'}$ is called an **altitude** of the triangle.) Show that $\mathcal{A}(ABC) = \frac{1}{2}bh$.

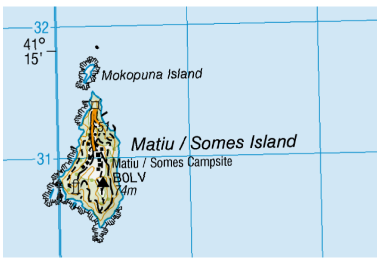
Exercise 34. In order to represent a piece of large land on a small flat map, we shrink every length equally according to a given ratio. We say a map is given to a scale of 1:1000 if one unit of measurement on the map represents 1000 such units in the real world. (So a length of 1 metre on the map would represent an actual length of 1 kilometre.)

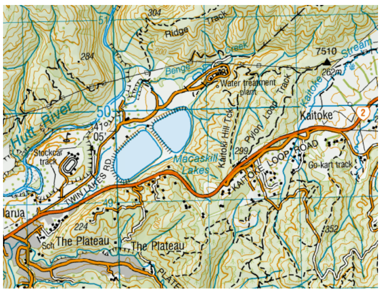
1. The plan of a plot of land has the shape of a square with side length 10.0 cm. If the plan is to a scale of 1:10 000, what is the area and perimeter of the plot of land?

2. The following figure ([KR78, p. 21]) presents the plan of a plot of land drawn to a scale of 1:1000. If $|AC| = 6$ cm, $|AD| = 7.6$ cm, $h_1 = 3$ cm, $h_2 = 4.8$ cm, and $h_3 = 3.2$ cm, compute the area of the plot.



3. On the extracts from the LINZ maps below, the side length of a square is 1 km. Calculate the area of (a) Somes Island, and (b) each of the Twin Lakes (the Macaskill Lakes).





Exercise 35. In exercise 16 we named several different types of quadrilaterals. Find the area of each.

Exercise 36. Let ABC be a triangle; let D and E be points on $[B, C]$. Then

$$\frac{\mathcal{A}(ABD)}{\mathcal{A}(ACE)} = \frac{|BD|}{|EC|}. \tag{1.5}$$

Using the theory of area we may prove the side-splitter theorem.

Proposition 37 (Side-splitter theorem)**.** *Let ABC be a triangle and let ℓ be a line parallel to $\overline{BC}$ intersecting $[A, B]$ at D and $[A, C]$ at E. Then the following proportions hold:*

$$\frac{|AD|}{|AB|} = \frac{|AE|}{|AC|} \text{ and } \frac{|AD|}{|DB|} = \frac{|AE|}{|EC|}. \tag{1.6}$$

Proof.

$$\begin{aligned}\frac{|AD|}{|AB|} = \frac{\mathcal{A}(AED)}{\mathcal{A}(AEB)} &= \frac{\mathcal{A}(AED)}{\mathcal{A}(AED) + \mathcal{A}(EBD)} \\ &= \frac{\mathcal{A}(AED)}{\mathcal{A}(AED) + \mathcal{A}(ECD)} = \frac{\mathcal{A}(ADE)}{\mathcal{A}(ADC)} = \frac{|AE|}{|AC|}.\end{aligned}$$

[Exercise: prove the second equality in (1.6).] ∎

Exercise 38. Find the area of a regular n–gon with side length x.
Hints:

- There is a point inside the polygon which is equidistant from every vertex. You may assume this without proof. Try rearranging the triangles you can form using this point, and apply exercise **??**.
- You should get a slightly different answer for odd n than for even n.

Compare with exercise 56

Exercise 39. By utilising the same 'approximation' trick as in exercise 29, show that the area of a circle of radius r is πr^2.

Exercise 40. Compare with exercise 30. Draw a circle of radius 1; pick four equally spaced points A, B, C, D on the circumference, forming a square within the circle. Let O be the centre of the circle; then (prove all these statements) the four angles at O are right angles; thus the area of the square is the sum of areas of four triangles with bases 1 and heights 1; hence $\pi > 2$.

Chapter 2

Trigonometry

A **right-angled triangle** is, as the name suggests, a triangle with a right angle. The side of the triangle opposite the right angle is called the **hypotenuse**, and the other two sides are called the **legs**.

2.1 Pythagoras' theorem

The main goal of this section is to present a number of proofs of the following well-known theorem.

Theorem 41 (Pythagoras)**.** *Let us consider a right-angled triangle with leg lengths a and b, and hypotenuse length c. Then*

$$a^2 + b^2 = c^2. \tag{2.1}$$

Equivalently, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the two other sides.

Pythagoras was a Greek philosopher active in the middle of the 6th century BCE; it is likely that he was born on the island of Samos in the Aegean sea — this is at least according to Ovid, [Ovi08, book XV, from line 60] — and his followers, the Pythagoreans, were a rather strange religous sect. See [Kli85, pp. 58-60] for a lighthearted discussion of the Pythagoreans.

It is believed that the theorem was also known by the ancient Egyptian, Indian, and Chinese civilisations (although it is unknown whether they had deductive proofs in the same sense as the Greeks).

The Pythagoran theorem is useful because it allows us to find distances between two points if we know the horizontal and vertical distances between them. This in turn allows us to assign coordinates to points in the normal (Cartesian) way, with no problems.

Applications and consequences of Pythagoras' theorem

Exercise 42.

1. A rectangle has side lengths 3 and 4. What is the length of the diagonal?

2. A rectangle has diagonal length 13 and one side length 5. What is the length of the remaining side?

3. A rectangular box has side lengths 2, 3, 7. What is the longest stick that fits in the box?

4. Show that the two diagonals of a rectangle are the same length.

5. A right triangle has leg lengths in the ratio $3 : 5$ and area 20. How long is the hypotenuse?

6. A ship travels 5 km south, 2 km east, 1 km north, and 6 km west. How far away is it from its starting point as the crow flies?

7. Let $ABCD$ be a normal quadrilateral such that the internal angles at A and B are right angles. Find the length $|AB|$ if the opposite edge has length 20 and the two adjacent sides have lengths of 8 and 12.

Exercise 43.

1. A square has area A. What is the length of its diagonal?

2. A rectangle has area A. Do you have enough information to find the length of its diagonal? If so, find the diagonal length. If not, what other information might you need?

Exercise 44. Suppose a right triangle has leg lengths 8 and 15.

1. Find the length of the hypotenuse.

2. What happens to the length of the hypotenuse if:

   - Both the leg lengths are doubled?
   - Both the leg lengths are tripled?
   - Both the leg lengths are multiplied by a number μ?

Exercise 45. Let ABC be a right triangle with right angle at C. What is the relationship between the areas of the semicircles with diameter $|AB|, |AC|$, and $|BC|$?

Exercise 46. Recall that an **integer** is a number of the form $..., -2, -1, 0, 1, 2, ...$

A **rational number** is a number which can be written in the form a/b, where a and b are integers. The question is, are all numbers rational? (Clearly all integers are rational: if z is an integer, then $z = z/1$ and 1 is an integer.)

It turns out that the answer is no, and one simple example of an **irrational** number is the hypotenuse of the right-angled triangle with side length 1: $\sqrt{1^2 + 1^2} = \sqrt{2}$.

The (undoubtably false, though often repeated) story goes that the Pythagoreans were so upset at this result — and the loss of their philosophy that all nature reduced to whole numbers or fractions — that they threw the discoverer off a boat and into the sea, and vowed never to reveal the discovery. (See [Kli85, §4-3].)

1. Justify why every rational number can be written in the form a/b where one of a or b is odd.

2. Show that if a is an integer, then a^2 is odd exactly when a is odd and a^2 is even exactly when a is even.

3. Suppose $\sqrt{2} = a/b$, where a and b are integers. Suppose we have written it in the form of (1); that is, either a or b (or both) is odd. Show that $a^2 = 2b^2$.

4. Using the previous result, show that a is even. Hence $a = 2a'$ for some integer a'.

5. Thus $(2a')^2 = 2b^2$.

6. Thus $b^2 = 2a'^2$, and hence b is even.

7. Use (4) and (6) to arrive at an absurdity.

Exercise 47. Let ABC be a triangle, and let D be the foot of the altitude from A onto BC. Show that, if E is any point on $[A, D]$, then

$$|AC|^2 - |CE|^2 = |AB|^2 - |EB|^2 . \tag{2.2}$$

What if:

1. E lies on the ray $\overrightarrow{AD}$?

2. E lies on the ray $\overrightarrow{DA}$?

[PS17, problem 3-1]

Proofs of Pythagoras' theorem

Exercise 48 (An area pushing proof)**.** Suppose a, b, and c are sides of a right angled triangle as specified in the theorem statement.

Let $ABCD$ be a square of side length $a + b$ such that each side is divided into segments of length a and length b, and the division alternates around the square. Call the points of division E, F, G, H so that E is on the segment $[A, B]$, F is on $[B, C]$, G is in $[C, D]$, and H is on $[D, A]$.

1. Show that $EFGH$ is a square of side length c.

2. Thus the area of the square $ABCD$ can be written as the sum of the areas of four right-angled triangles and the area of $EFGH$. Do so.

3. But the area can also be written as $(x + y)^2$. Set these two different expressions for the area equal to each other.

4. Prove Pythagoras' theorem.

Exercise 49 (A proof via similar triangles)**.** Let ABC be a triangle with right angle at C, hypotenuse length c, and leg lengths a (opposite A) and b (opposite B). Let H be the foot of the altitude of ABC from C to $\overline{AB}$. Let α be the measure of the angle at A and β be the measure of the angle at B. Let $x = |BH|$ so $c - x = |AH|$.

1. Show that $ABC \sim ACH$ and $ABC \sim CBH$.
2. Conclude that $\frac{b}{c-x} = \frac{c}{b}$ and $\frac{a}{x} = \frac{c}{a}$.
3. Prove Pythagoras' theorem.

For Euclid's proof via area dissection, see Theorem 1.2 of my L3 trigonometry notes [Elz18].

For a proof via the 'power of a point with respect to a circle', see exercise 69.

A large collection of other proofs can be found in [Loo27].

2.2 Triangle ratios

Proposition 50. *Let a right angled triangle have side lengths a, o, and h, where h is the length of the hypotenuse. Then the ratios o/h, a/h, and o/a depend only on the angle θ at the vertex opposite o.*

Proof. This is a special case of exercise 25.3. ∎

Because the angles depend only on θ, we need only specify θ to identify them. We call the length a the **adjacent leg**, and the length o the **opposite leg**. We then defined the **sine**, **cosine**, and **tangent** functions by

$$\sin\theta = \frac{o}{h} = \frac{\text{opposite}}{\text{hypotenuse}}, \tag{2.3}$$

$$\cos\theta = \frac{a}{h} = \frac{\text{adjacent}}{\text{hypotenuse}}, \text{ and} \tag{2.4}$$

$$\tan\theta = \frac{o}{a} = \frac{\text{opposite}}{\text{adjacent}}. \tag{2.5}$$

Together these three functions are called the **trigonometric functions**.

Exercise 51 (Finding lengths of triangles)**.** The majority of these (purely computational) problems are taken from [Foe77].

1. Draw (accurately) a right angled triangle with one leg 8 cm long and one acute angle of measure 34° with the 8 cm leg as its adjacent side. Calculate the lengths of the other leg and the hypotenuse using the relevant trigonometric functions; use a ruler to measure the lengths and check your results agree to within one millimetre.

2. You must order a new rope for a flagpole. To find out what length of rope you need, you observe that the pole casts a shadow of 11.6 m long on the ground. The angle of elevation of the sun is $36°50'0''$. How tall is the pole?

3. A cat is trapped on a tree branch 6.5 m above the ground. Your ladder is only 6.7 m long. If the very end of the ladder is leant on the branch, what angle will the ladder make with the ground?

4. Air New Zealand's domestic jet flights travel at a maximum altitude of around 8000 m. They start descending when they are quite far away from the airport, so that they will not have to dive at a steep angle.

   (a) If the pilot wants the plane's path to make an angle of $3°$ with the ground, how far away from the destination airport must they start descending?

   (b) If the pilot begins a descent 150 km away, what angle will the plane's path make with the horizontal?

   (c) Generally, jet flights travel at around 10 000 m above the ground: an extra 2 km above the height travelled by domestic flights in New Zealand. Why do you think domestic flights here travel lower than might be expected (especially as planes are often more fuel-efficient at their normal height of ten kilometres)? (Useful fact: the distance between Wellington and Christchurch airports is approximately 300 km).

5. Baldwin Street in Dunedin is often cited as the 'steepest street in the world' (for example, by Guinness World Records). It has a slope of $19°$; if you walk along the street so you have travelled one metre horizontally, how far have you travelled vertically? How far do you have to travel horizontally in order to rise by one metre?

6. The James Webb Space Telescope, the replacement for the Hubble Telescope due to launch in 2021, has a resolution of $0°0'0.32''$ — when the lines drawn between two stars and the telescope make this angle or greater, then it will be able to distinguish between them.[1]

   One light year is the distance that a beam of light will travel in one earth year; it is 9.461×10^{15} m.

   (a) Alpha Centauri, the closest star system to the Sun, is actually made up of three stars orbiting each other (named Rigil Kentaurus, Toliman, and Proxima Centauri). The naked eye sees one single blob of light. The distance between the stars is approximately 0.21 ly, and the system lies around 4.37 ly away from the Sun. Will the telescope see three seperate stars, or one blob of light?

   (b) How far apart do two stars in the Andromeda Galaxy (2.537 million light years away) need to be for the telescope to be able to distingush between them? Given that the Andromeda Galaxy is roughly circular and has a radius of 110 000 ly, will the telescope be able to distinguish between *any* stars in that galaxy?

Exercise 52. Design and construct a piece of equipment to allow you to measure the angle of elevation of a tall structure (i.e. the angle made between the horizontal ground and the line joining the point of observation to the top of the building). Use your device to measure the height of a building.

[1] *Tweet Chat with John Mather* (James Webb Space Telescope project scientist), retrieved from `https://jwst.nasa.gov/faq_tweetchat1.html` on 21 May 2019.

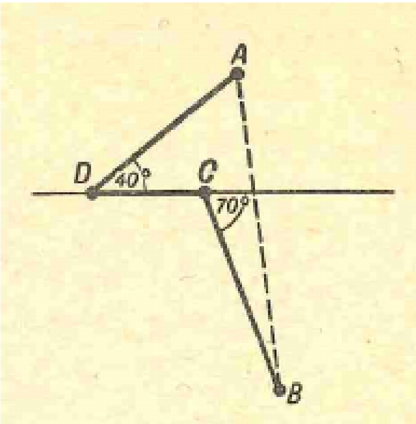
Exercise 53. Research how trigonometry can be used to measure areas and create maps. Some possible search terms could include:

- Theodolite
- Triangulation station
- Great Indian trigonometric survey

Exercise 54 (Finding more lengths and areas)**.** The majority of these (purely computational) problems are taken from [KR78].

1. How long will a (taut) transmission belt need to be when the two pulleys are 12 cm and 34 cm in diameter and their centres are one metre apart?

2. Two points A and B lie on opposite sides of a road. In order to get from A to B it is necessary to drive 3.5 km along a side road that joins the main road at an angle of 40°, then drive 2.5 km along the main road and turn right onto another side road which makes an angle 70° with the main road, and drive another 4 km. All sections of roads traversed are straight. By how much will the distance from A to B be shortened if a straight road is built between them?



3. At 7 o'clock in the morning a passenger plane took off from a town A and after thirty minutes stay in the town B it took off again at 8.10, turned 35° to the right, and landed in town C at 9.00. Determine the distance as-the-crow-flies between towns A and C if the average speed of the plane was 320 km h^{-1}.

4. Two sides of a triangle are equal to 5 cm and 6 cm, and its area to 5.28 cm^2. Find the third side.

Exercise 55.

1. A circle has circumference 25.12 m. What is the area of the largest equilateral triangle that can be drawn in the circle?

2. A circle has circumference C. What is the area of the largest equilateral triangle that can be drawn in the circle?

Exercise 56. Find the area of a regular n–gon $\mathscr{P}$ with side length x by taking the centre of the polygon (you may assume that the centre — a point equidistant from each vertex — exists), cutting the polygon into $2n$ right angled triangles, and finding the area of each of these.

You should find that

$$\mathcal{A}(\mathscr{P}) = \frac{x^2}{4\tan\frac{180^\circ}{n}}. \tag{2.6}$$

Compare this with exercise 38.

2.3 Analytic geometry

Let us now fix a point O, and a second point 1. Draw a line through $O1$ (this will be the x**–axis**, or the **primary axis**); then draw a point 2 such that $|12| = |O1|$, draw a point 3 such that $|23| = |O1|$, and so forth. We can add negative numbers in the natural fashion as well. (It is customary to pick the line and points such that the primary axis is horizonal and 1 is to the right of O.)

Draw the line through O perpendicular to the primary axis; this will be the **secondary axis**, or y**–axis**. Pick a point on the line a distance $|O1|$ from O (it is customary to draw it above the point O) and call it 1; then draw 2, 3, -72, and so forth on this secondary axis in the normal way.

It is clear that we can add all the real numbers onto our two number lines in a natural way so that if $a < b < c$ then the point corresponding to b lies between a and c, and such that $|ac| = (c-a)|O1|$. Once we have chosen all these points, we have obtained what is known as a **Cartesian coordinate system**. To every point P on the plane we can now assign two numbers: the x**–coordinate** (or **abscissa**) and the y**–coordinate** (or **ordinate**). This is done by drawing perpendicular lines from P to the x–axis and y–axis respectively, and choosing the number on the axis assigned to the intersection point as the relevant coordinate. If P has coordinates x and y, we write $P = (x, y)$.

Given any equation relating two variables, say $x^2 + y^2 = 1$, we can now colour in all the points (x, y) such that the equation becomes true; this is called the **graph** of the equation.

Exercise 57. Plot graphs of the following equations:

1. $x^2 + y^2 = 1$
2. $x^2 + y^2 = 4$
3. $x^2 - y^1 = 1$
4. $x = y$
5. $x = -y$
6. $x = (1/2)y$
7. $y = x^3$
8. $x = y^3$

9. $y = (x-3)(x-2)(x+1)$ (remember there are invisible $\times$ signs between the brackets)

10. $x = (y-3)(y-2)(y+1)$

11. $0 = (x+y)(x-y)$

This relation between algebraic quantities (namely equations) and geometry was first developed by René Descartes (pictured) in the early 1600s, and was one of the great early breakthroughs in thought of modern European mathematics.

Another method of depicting geometric objects in the coordinate plane is the so-called **method of locii**. We define a set of points P satisfying certain restrictions; the resulting figure is called the **locus** of the point P.

Exercise 58. Consider the locii of all the points P such that:

1. For some fixed point $O, |OP| = k$ for some fixed number k. (The circle.)

2. For two fixed points A and $B, |PA| + |PB| = k$ for some fixed number k. (An **ellipse**.)

3. For two fixed points A and $B, |PA| - |PB| = k$ for some fixed number k. (A **hyperbola**.)

4. For two fixed points A and $B, |PA| = |PB|$.

5. For two fixed points A and B, and some number $0 < \rho < 1, |PA| = \rho|PB|$. (If A and B are held fixed but ρ is changed, we obtain a family of circles called the **circles of Appolonius**.)

2.4 Origami

We want to consider a square piece of paper (with side length 1, say). We will allow ourselves to fold only creases that are reproducable: that is, we will not allow ourselves to carry out a folding procedure that does not produce consistent results.

We therefore must start with either a side-to-side fold, or a corner-to-corner fold; the first will give us the midpoints of two opposite sides (and the line joining them), and the second will give us a diagonal of the square.

To be consistent, we will label our paper $ABCD$ anticlockwise from the top left.

Exercise 59.

1. Using only reproducable folds, fold C to the midpoint, E of $[A, B]$.
2. Let F be the point marked off from $[C, D]$. Show that DEF is a right triangle with side lengths in the ratio $3 : 4 : 5$.
3. Prove **Haga's first theorem** [Hag08, p. 7]: if the folds are performed in this way then:
   (a) The edge $[C, D]$ is divided in the ratio $3 : 5$.
   (b) The edge $[A, B]$ is divided in the ratios $2 : 1$ and $7 : 1$.
   (c) The edge $[B, C]$ is divided in the ratio $3 : 5$.

Exercise 60. Prove **Haga's second theorem**

Chapter 3

Tilings and wallpaper patterns

Early in our studies we defined the concept of a **transformation**: a way of moving points around. We defined a **isometry** to be a transformation that preserved distance, and theorem 20 told us that there were only five kinds.

3.1 Symmetries

We call a figure **symmetric** if there is an isometry that transforms the figure to itself. If we can draw a line through a figure such that reflection through the line leaves the figure unchanged, then the line is called a **line of symmetry**.

Exercise 61. Find lines of symmetry for all the capital letters:

A B C D E F G H I J K L M

N O P Q R S T U V X Y Z

How many letters have no lines of symmetry? Which has the most?

Exercise 62. The word BEE is made up of letters which are symmetric, and the word itself has a line of symmetry (horizontally through the middle). The word MUM has a vertical line of symmetry through the middle. Explain the differences between BEE, MUM, TOO, and EYE: all three are made up of symmetric letters, but not all have the same symmetries as words.

Exercise 63. A figure is said to have a **centre of symmetry** O if we can rotate the figure around the point O by some angle (less than a full turn) so that the figure seems unchanged.

Which letters have centres of symmetry, and through which angles can we rotate them?

Imagine that the following sequence of letters goes on to infinity in both directions.

K K K K K K K K K K K K K

If we translate the image left or right by a length which is a multiple of the distance between K's then the image appears unchanged. This is an example of a **translational symmetry**.

Exercise 64. Draw a picture which possesses **glide symmetry**: it is mapped onto itself by a glide reflection.

3.2 Tilings of the plane

Definition 65. A figure $\mathscr{F}$ is said to **tile** the plane if an arrangement of copies of the figure can be found such that:

1. each point in the plane lies within one of the copies, and
2. no two copies overlap.

It turns out that there are only seventeen different 'types' of tilings of the plane. The following illustrations are from [Ber87, pp. 13,19], but were cited there as originating from [Y. Brossard. *Rosaces, Frises et Pavages*. CEDIC, Paris, 1977].

Figure 1.7.4.1

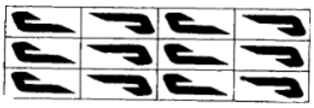


Figure 1.7.4.2

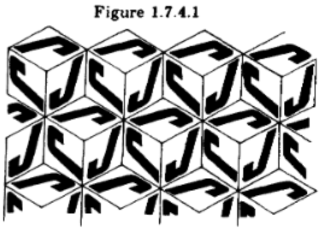


Figure 1.7.4.3

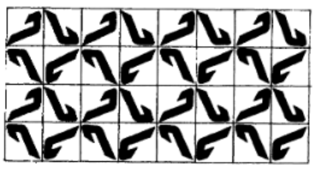


Figure 1.7.4.4

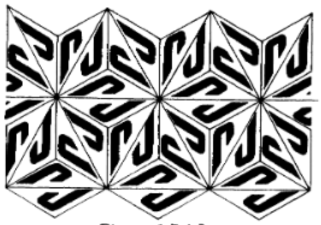


Figure 1.7.4.5

Figure 1.7.6.1

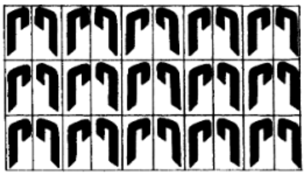


Figure 1.7.6.2

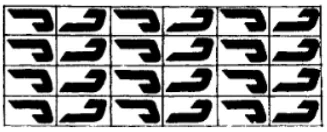


Figure 1.7.6.3

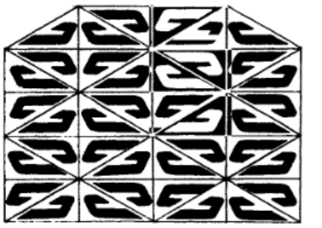


Figure 1.7.6.4

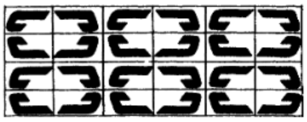


Figure 1.7.6.5

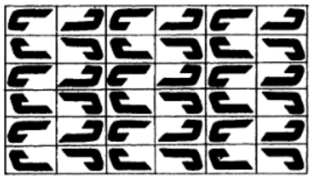


Figure 1.7.6.6

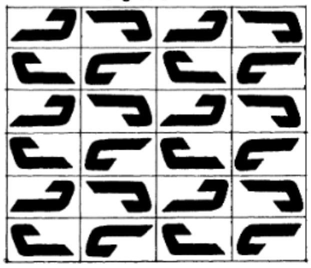


Figure 1.7.6.7

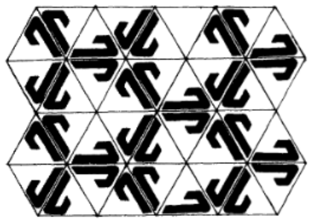


Figure 1.7.6.8

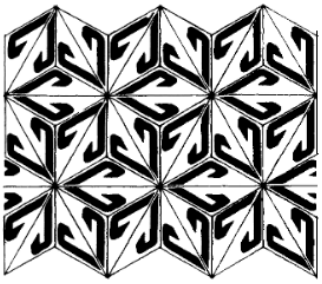


Figure 1.7.6.9

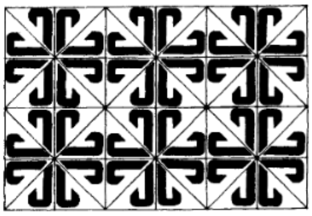
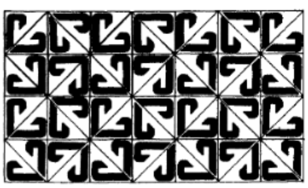
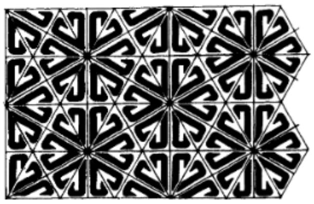


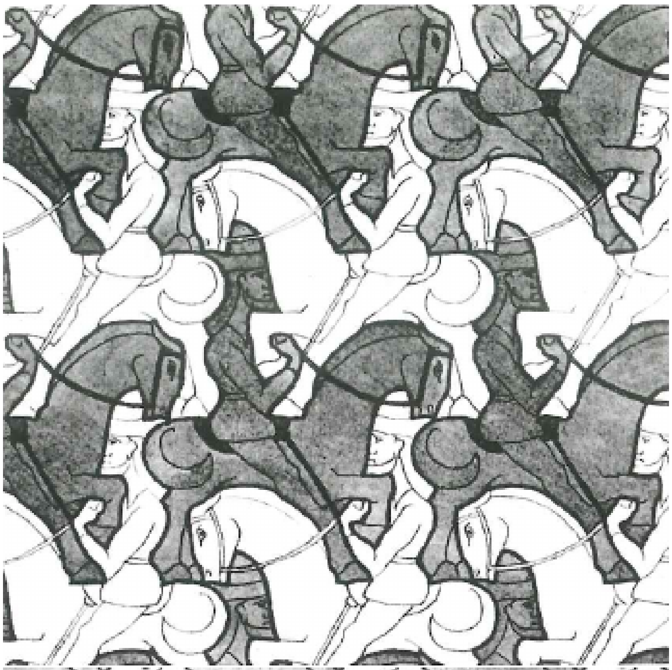
Figure 1.7.6.10

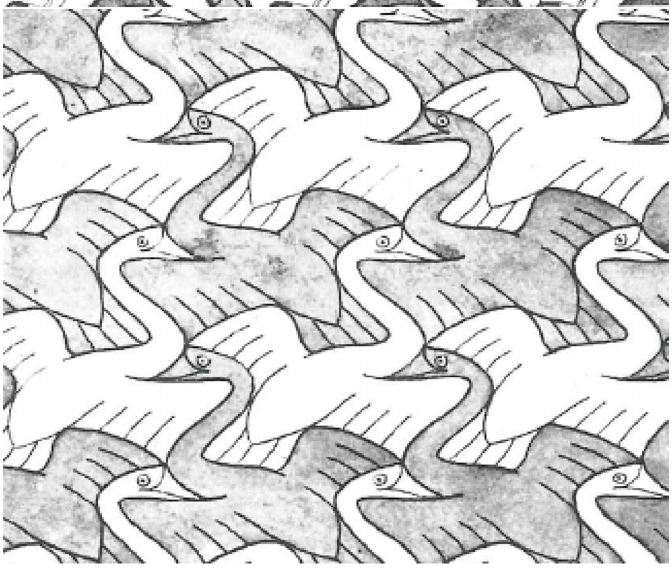


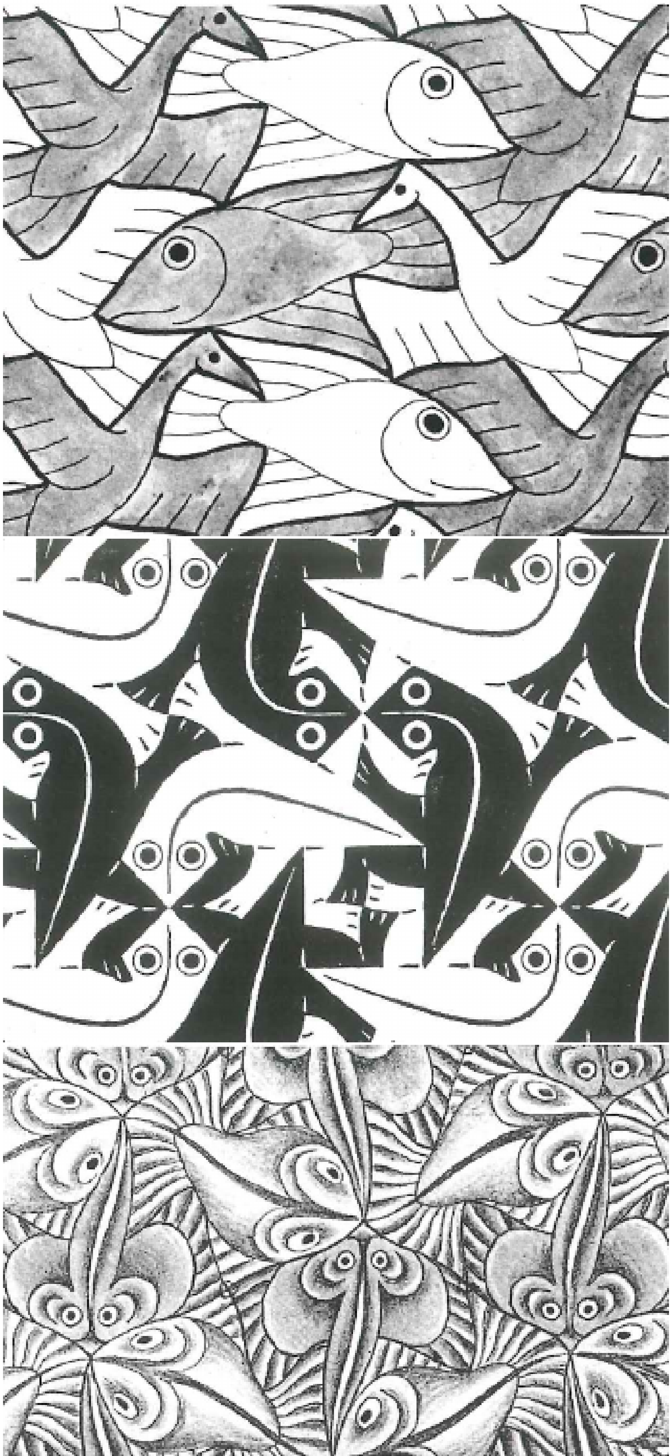


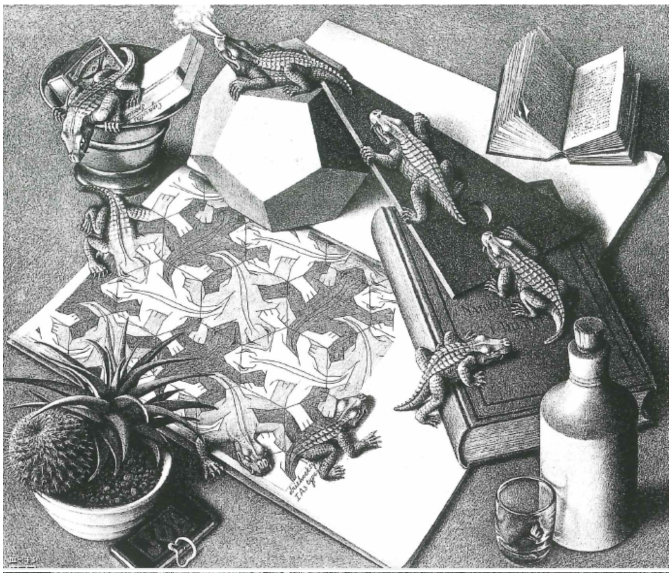
Exercise 66. Imagining each tiling to continue forever in each direction, what symmetries does each possess?

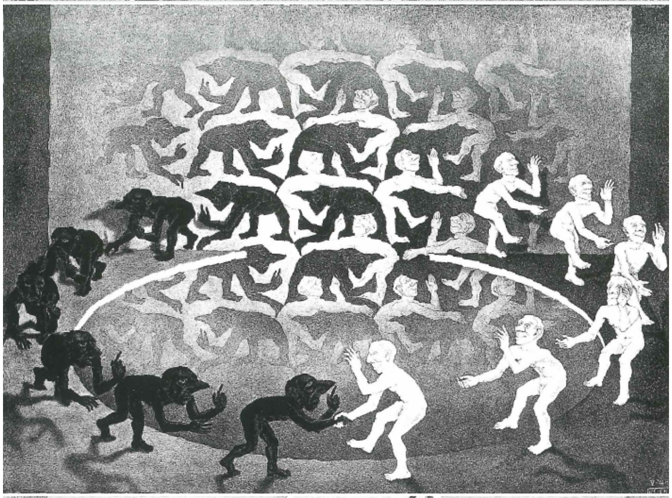
Exercise 67. Here are some tilings of the plane — or parts of tilings — all due to M. C. Escher [Esc06]. Explore the symmetries used.

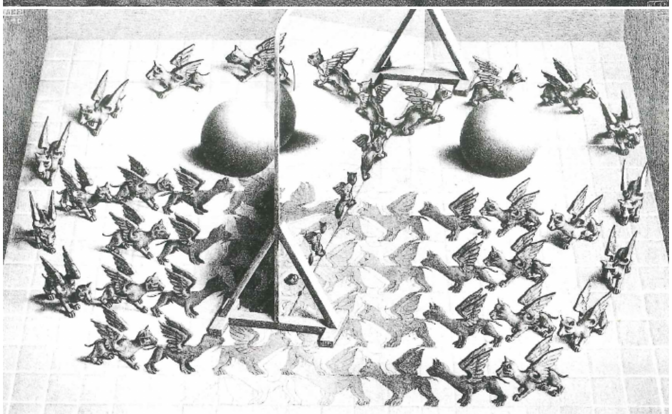


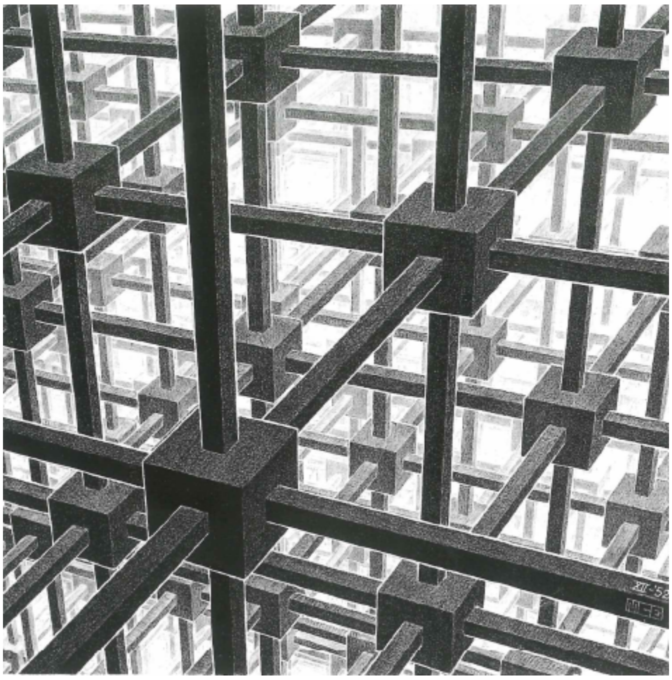












Exercise 68. Design your own Escher-style tilings, and analyse their symmetries.
Recall that we have (at least):

- Translations
- Rotations
- Reflections
- Glide reflections
- Dilations

Chapter 4

Circle inversion

Reflection is an important transformation in geometry; but why do we limit ourselves to reflection across lines?

4.1 The power of a point

See [Sve91, chapter 1].

Exercise 69. Suppose $\mathscr{C}$ is a circle.

1. Suppose A, B, C, and D are points on $\mathscr{C}$ that form a normal quadrilateral $ABCD$. Show that the opposite internal angles of $ABCD$ add to 180°.
2. Suppose P is a point outside the circle, and let ℓ and m be any two lines through P that intersect $\mathscr{C}$ at A_ℓ and B_ℓ, and A_m and B_m respectively. Show that $|PA_\ell||PB_\ell| = |PA_m||PB_m|$. So this quantity is independent of which line we choose: it only depends on P and $\mathscr{C}$. This quantity is called the **power of the point** P with respect to $\mathscr{C}$, and we will write $\operatorname{Pow}_{\mathscr{C}} P$ for this number.[1]
3. Let r be the radius of $\mathscr{C}$, and let O be its centre. Pick a line through P that intersects the circle at a single point T, and pick the point of the circle X that lies on $[P, O]$; let $d = |PX|$. Show that $\operatorname{Pow}_{\mathscr{C}} P = |PT|^2 = d^2 - r^2$.
4. Prove Pythagoras' theorem.

Exercise 70. Let P be a point *inside* a circle $\mathscr{C}$. Show that, if $\operatorname{Pow}_{\mathscr{C}} P$ is still defined by $d^2 - r^2$ where d is the distance from P to $\mathscr{C}$, then $\operatorname{Pow}_{\mathscr{C}} P$ is negative or zero (when?). Further, $|\operatorname{Pow}_{\mathscr{C}} P|$ is the product $|PA_1||PA_2|$ where $[A_1, A_2]$ is any chord of the circle passing through P.

Exercise 71. Pick two points A and B. Consider the family of circles which pass through both points (there are infinitely many such circles, all with $[A, B]$ as a common chord; we have a so-called **coaxial family of circles**).

1. Show that the centres of the family all lie on one line.

[1] According to [Cox61, p. 81], this terminology is due to Jacob Steiner (1796–1863).

2. Show that, if P is *any* point on $\overline{AB}$, then for any two circles $\mathscr{C}$ and $\mathscr{D}$ in the family, $\operatorname{Pow}_{\mathscr{C}} P = \operatorname{Pow}_{\mathscr{D}} P$. The line $\overline{AB}$ is the **radical axis of the coaxial family of circles**.

3. What if the two points A and B are the same (i.e. we consider the family of circles which are tangent to each other at some point $A = B$).

Exercise 72.

1. Do two circles $\mathscr{C}_1$ and $\mathscr{C}_2$ that never intersect have a radical axis — a line, all of whose points have equal power with respect to the two circles? (Hint: yes. Further, you can show that *all* the points of equal power must lie on this line.)

2. Can you find a third circle $\mathscr{C}_3$ that shares the same radical axis? (Hint: yes.)

3. Can you find infinitely many such circles? (Hint: what do you think?)

Exercise 73. Let $[A, B]$ be the chord of a radical family of circles (the **elliptical family** $\mathscr{C}_1$, $\mathscr{C}_2$,...; let P be a point on the radical axis $\overline{AB}$. The power of P with respect to all the elliptical family is constant, and so the lengths of the tangents $[P, T_1]$, $[P, T_2]$, ... to the elliptical family are constant. Thus the points T_1, T_2, and so on lie on some circle. We can draw such a circle for each point P, and we obtain a second family of circles, the **hyperbolic family**. Here are some facts:

- The circles of the two families cut each other at right angles (they are **orthogonal**).
- The line joining the centres of the circles in the elliptical family is the radical axis of the hyperbolic family.
- These circles are precisely the **circles of Appolonius**!!!

4.2 Inversion proper

The transformation we will now discuss, which is a combination of dilation and isometry, is the so-called **circle inversion**; it was (according to [Cox61, p. 77]) invented by one L. J. Magnus in 1831. Like the reflection, if we do it twice then we get the identity transformation; but unlike reflection, we fix a circle and not a line!' Our theory broadly follows chapter 6 of that reference.

Definition 74. Let O be the centre of a circle $\mathscr{C}$ of radius k. Given any point P distinct from O we define the **inverse** of P (with respect to $\mathscr{C}$) to be the point $P' = \operatorname{Inv}_{\mathscr{C}} P$ such that $|OP||OP'| = k^2$.

Exercise 75. Here are some basic properties of inversion:

1. If $P' = \operatorname{Inv}_{\mathscr{C}} P$ then $P = \operatorname{Inv}_{\mathscr{C}} P'$.

2. If P lies on the circle $\mathscr{C}$ then $P = \operatorname{Inv}_{\mathscr{C}} P$.

3. If P lies inside the circle then $\operatorname{Inv}_{\mathscr{C}} P$ lies outside the circle.

4. If P lies outside the circle then $\operatorname{Inv}_{\mathscr{C}} P$ lies inside the circle.

Exercise 76. Show that if $\mathscr{C}$ and $\mathscr{D}$ are concentric circles with radii k and k' then inversion through $\mathscr{C}$ followed by inversion through $\mathscr{D}$ is just a dilation through the centre of the circles of magnification $\mu = (k/k')^2$.

Exercise 77 (Euclid III.35)**.** Let two chords $[P, P']$ and $[Q, Q']$ of a circle intersect at a point O (not necessarily the centre). Then the rectangle contained by the segments of one is the rectangle contained by the segments of the other (more precisely, $|OP|\left|OP'\right| = |OQ|\left|OQ'\right|$).

Exercise 78. The inverse of a given point P in a circle $\mathscr{C}$ is the second intersection of any two circles through P orthogonal to the $\mathscr{C}$. (Hint: use the previous exercise.)

Exercise 79. Through a given point P, draw a circle orthogonal to two given circles.

Exercise 80. Fix a circle of inversion centred at O with radius k.

1. Any line through O inverts to itself.
2. Any line not through O inverts to a circle through O.
3. Any circle through O inverts to a line not through O.

Exercise 81. Fix a circle of inversion $\mathscr{C}$ centred at O with radius k; fix another circle $\mathscr{K}$, not through O, with centre C. Let $\overline{OP}$ meet the circle $\mathscr{K}$ at the second point Q. Let p be the power of O with respect to $\mathscr{K}$.

1. The dilation centred at O with magnification k^2/p transforms $\mathscr{K}$ and the radius $[C, Q]$ into another circle $\mathscr{L}$ with centre D and (parallel) radius $[D, R]$, so that
$$\frac{|OR|}{|OQ|} = \frac{|OD|}{|OC|} = \frac{k^2}{p}. \tag{4.1}$$
2. Show that R is the inverse of P with respect to the circle of inversion $\mathscr{C}$.
3. Show that $\mathscr{K}$ and $\mathscr{L}$ are inverses with respect to $\mathscr{C}$.

One incredibly beautiful and striking application of inversion is **Steiner's porism** [Cox61, p. 87].

Exercise 82 (Steiner's porism)**.** If we have two nonconcentric circles, one inside the other, and we draw circles successively touching them and one another, it may happen that the ring of circles closes: that is, the last circle in the ring exactly touches the first.

Claim: if this happens once, it will always happen, whatever the position of the first circle.

Proof: invert the two original circles into concentric circles.

Chapter 5

Three-dimensional geometry

Solid geometry was also studied by the Greeks, and indeed it has often been said (sometimes even jokingly) that Euclid's goal was not to write a book on elementary geometry; it was to write an admittedly longwinded book on the platonic solids [Cox63, p. 13].

5.1 Spheres, cylinders, and cones

The most basic kinds of solids are the **prisms**: we take a polygon and stretch it "upwards", out of the plane it lies in. A rectangular prism is a brick shape, which is obtained by taking a rectangle (one of the ends) and stretching it to create depth.

Exercise 83. Draw triangular and pentagonal prisms.

In three dimensions, there are a variety of analogues for circles. They are all familiar objects in everyday life:

Definition 84.

1. If O is a fixed point and ρ is a fixed positive number, the locus of all points X in three-dimensional space such that $|OX| = \rho$ is called the **sphere** centred at O with radius ρ. The sphere together with its interior is called a **ball**.

2. If C is a circle with centre O and radius ρ, and P is a point on the line through O perpendicular to the plane containing C, then the **right circular cylinder** with height $[O, P]$ and base C is the set of all points that lie on circles with radius ρ centred on $[O, P]$.

3. if C is a circle with centre O and radius ρ, and P is a point that does not lie in the plane containing C, then the **circular cone** with base C and apex P is the set of all points that lie on segments $[P, X]$ where X is a point inside C.

Exercise 85. Draw pictures of all three types of figure from the above definition. What do you get if you intersect them with each other?

Our main goal is to calculate the **volume** enclosed by each of the three objects in definition 84. A formal definition of volume would go somewhat like the definition of area in 31; I will not write down the full definition here. Rather than being too precise, we will use informal reasoning like in the following proposition.

Proposition 86. *A rectangular prism $\mathcal{RP}$ with side lengths x, y, and z has volume $\mathcal{V}(\mathcal{RP}) = xyz$.*

"Proof". The base of the rectangular prism has area xy. If the height was 1, then the volume would be $1 \cdot xy$; but we have a height of z, and so we stretch the short prism by z; since we have stretched in one direction we need to multiply the volume by the stretch factor, and the new volume is $z \cdot 1 \cdot xy = xyz$. ∎

(Note we often abuse language and say that an object *has* a volume, rather than the more correct statement that the object *encloses* a volume.)

Exercise 87. Use similar reasoning to justify the formula for the volume of a cylinder $\mathcal{CY}$ with height h and radius ρ:

$$\mathcal{V}(\mathcal{CY}) = \pi r^2 h. \tag{5.1}$$

We will require the following principle.

Theorem 88 (Cavalieri's principle)**.** *If a family of parallel planes gives equal cross-sectional areas when slicing two different solids, then the two solids have equal area.*

Proof. See a future course on calculus [Ste12, page 362, exercise 5.2.63]. ∎

To illustrate the above principle, it is productive to consider a stack of coins forming a cylinder; then Cavalieri's principle simply states that if you push your stack to form a slanted cylinder then the volume stays the same.

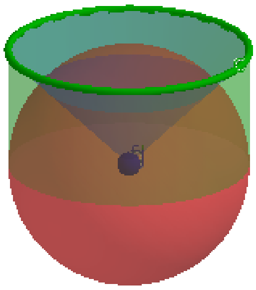
Exercise 89. We will now calculate the volume of a cone. You will need to use Cavalieri's principle several times in this exercise.

1. Show that a cube of side length u can be cut into six pyramids, with square bases of area u^2 and heights $u/2$. Thus the volume of such a pyramid is $\frac{1}{3}\frac{u}{2}u^2$. Conclude that a pyramid with square base of area u^2 and height h is $\frac{1}{3}hu^2$.
2. Consider now a cone $\mathcal{CO}$ with height h and radius ρ. Apply Cavalieri's principle to this cone and to the square pyramid with base $\rho\sqrt{\pi}$ and height h to conclude that the volume of the cone is

$$\mathcal{V}(\mathcal{CO}) = \frac{\pi}{3} h \rho^2. \tag{5.2}$$

Now we give the classical Greek proof for the volume of a sphere.

Exercise 90. Consider a sphere $\mathcal{S}$ of radius ρ. Slice it in half, forming a **hemisphere** with radius ρ and height ρ. Fit about this hemisphere a cylinder of radius ρ and height ρ, and form a cone whose base is the end of the cylinder and whose apex is the centre of the hemisphere's base.



Prove that for any horizontal slice, the areas of the hemisphere cross-section H, the cone cross-section N, and the cylinder cross-section L satisfy the relation $H^2 = L^2 - N^2$.

Hence apply your knowledge of the volumes for a cylinder and a cone, and Cavalieri's principle, to conclude that

$$\mathcal{V}(S) = \frac{4}{3}\pi R^3. \tag{5.3}$$

Exercise 91. Show that a sphere takes up two-thirds of the volume of the smallest cylinder enclosing it.

(A sketch of the resulting object was requested by Archimedes, the Greek mathematician who was the first to write arguments like those in this section, to be drawn on his tombstone.)

A classic piece of mathematical fiction that asks how we can think about higher dimensions beyond our familiar three is [Abb18].

5.2 The Platonic solids

Recall that a **regular polygon** is a polygon which is both equiangular and equilateral. There are infinitely many regular polygons: for every whole number n, draw n equally spaced points around a circle and join adjacent points with segments. It will turn out that in three dimensions the equivalent notion to a regular polygon is much more rare.

Definition 92. A **polyhedron** (plural: polyhedra) is an object (living in three dimensions) that is made up of a finite connected set of polygons (each lying in a plane called a **bounding plane** for the polyhedron) such that:

1. every side of each polygon belongs to just one other polygon;

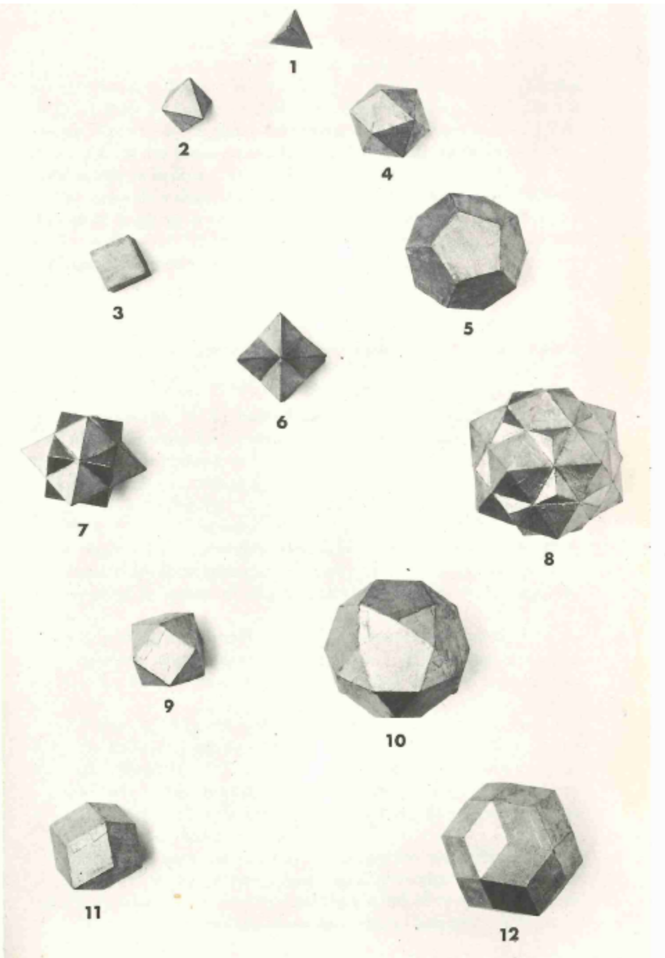
2. and at any vertex, you can walk from the inside of one polygon to the inside of any other polygon at that vertex by crossing edges and interiors of polygons that touch at that vertex and without crossing the vertex itself.[1]

The polygons are called the **faces** of the polyhedron, and the sides of the polygons are called (unsurprisingly) the **sides** of the polyhedron.

[1] This second condition can be more concisely stated as "the polygons at each vertex form a single circuit", and its purpose is simply to prevent things like a pair of pyramids joined only at their tips from being called a polyhedron.

Using the above definition one can see that a polyhedron must separate space into two pieces, one infinite (the **exterior** of the polyhedron) and the other bounded (the **interior** of the polyhedron). A polyhedron is called **convex** if none of the bounding planes intersect the interior.

Example 93. The following figure, from [Cox63, plate I], displays five polyhedra.



They are named as follows:

1. tetrahedron;
2. octahedron;
3. cube;
4. icosahedron;
5. pentagonal dodecahedron;
6. two tetrahedra cutting each other;
7. a cube and an octahedron cutting each other;
8. an icosahedron and a dodecahedron cutting each other;
9. cuboctahedron;

10. icosidodecahedron;
11. rhombic dodecahedron;
12. tricontahedron.

A very nice reference for this subject is [Cox63, §§1.2-1.3]

5.3 Möbius bands and Klein bottles

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