NCEA Level 3 Calculus (Integration)
17. The Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem)
Suppose f is a continuous function, and suppose F is any antiderivative of f (so $F' = f$). Then:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b$$

In other words, the definite integral of a function can be found by evaluating the indefinite integrals at the endpoints. This actually follows from a much more intuitive result:

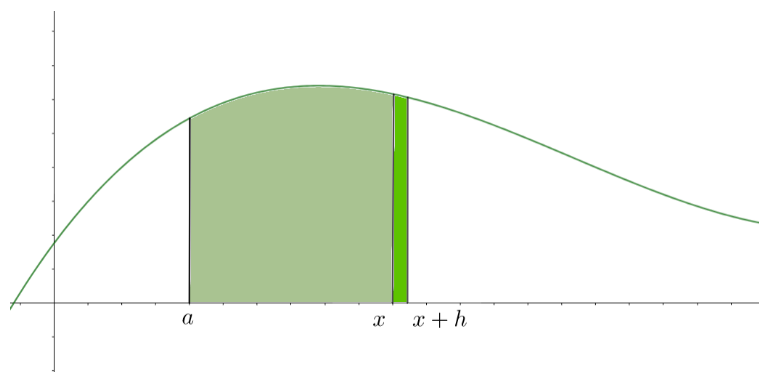
Theorem (Second Fundamental Theorem)
Suppose f is a continuous function. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

For some intuition, we can consider the following graph of $y = f(x)$. The shaded area is the value of $A(x) = \int_a^x f(t)\,\mathrm{d}t$; then we use the fact that the area of the darker shaded area is approximated by $hf(x)$ (base times height) to see that

$$\frac{A(x+h) - A(x)}{h} \approx \frac{hf(x)}{h} = f(x).$$

If we take limits, as we do in the proof of this theorem (see supplementary sheet), then this approximation becomes exact: the rate of change of the area under a curve is simply the height of the curve.



We also have the following theorem:

Theorem. *Suppose f, g are functions and λ is a real constant. Then:*

1. $\lambda \int_a^b f(x)\,\mathrm{d}x = \int_a^b \lambda f(x)\,\mathrm{d}x$.
2. $\int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x = \int_a^b f(x) + g(x)\,\mathrm{d}x$.
3. $\int_a^a f(x)\,\mathrm{d}x = 0$.
4. $\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$.

Note that the areas below a curve are assigned **negative area**!

Example. The definite integral $\int_0^\pi \cos(x)\,\mathrm{d}x$ is equal to 0; this is because the (negative) area under the $x-$axis exactly cancels the (positive) area above the $x-$axis. (Draw a picture.)

Questions

1. $\boxed{\texttt{A}}$ Compute the following definite integrals.
   (a) $\int_0^1 \mathrm{d}x$
   (b) $\int_{-1}^1 e^x \, \mathrm{d}x$
   (c) $\int_3^4 x^2 + 3x - 1 \, \mathrm{d}x$
   (d) $\int_0^1 x^n \, \mathrm{d}x$ for integer values of n.

2. $\boxed{\texttt{A}}$ Find the area underneath the given curves between the given bounds:
   (a) $y = 6x^2 + 4x + 9$ between $x = 0$ and $x = 4$
   (b) $y = \sin x$ between $x = 0$ and $x = \pi$
   (c) $y = \sin x$ between $x = -\pi$ and $x = \pi$
   (d) $y = \cos x$ between $x = -\pi$ and $x = \pi$
   (e) $y = \frac{1}{x}$ between $x = 1$ and $x = 2$

3. $\boxed{\texttt{A}}$ Find all the problems in the following working.

$$\int_1^{-1} \frac{\mathrm{d}x}{x} = \ln|-1| - \ln|1| = 0$$

4. $\boxed{\texttt{A}}$ Show that $\int \ln x \, \mathrm{d}x = x \ln x - x + C$.

5. $\boxed{\texttt{M}}$ Let f be a function such that for all x, $f(-x) = -f(x)$. Such a function is called *odd*. Show that for all a,

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = 0.$$

   What does this mean geometrically?

6. $\boxed{\texttt{M}}$ Let f be an odd function with period 2 such that $\int_0^1 f(x) \, \mathrm{d}x = k$. Compute:
   (a) $\int_{-1}^1 f(x) \, \mathrm{d}x$
   (b) $\int_0^{-1} f(x) \, \mathrm{d}x$

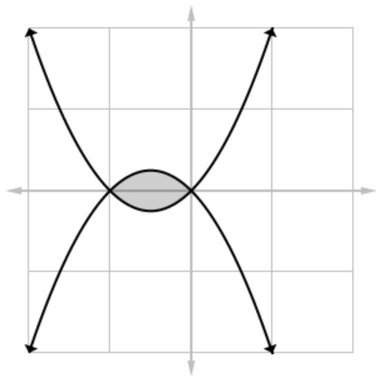
7. $\boxed{\texttt{A}}$ Let f be a function such that for all x, $f(-x) = f(x)$. Such a function is called *even*. Show that for all a,

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = 2 \int_0^a f(x) \, \mathrm{d}x.$$

   What does this mean geometrically?

8. $\boxed{\texttt{M}}$ If $\int_{-2}^1 f(x) \, \mathrm{d}x = 2$ and $\int_1^3 f(x) \, \mathrm{d}x = -6$, what is the value of $\int_{-2}^3 f(x) \, \mathrm{d}x$?

9. $\boxed{\texttt{M}}$ Find the area between the curves $y = x^2 + x$ and $y = -x^2 - x$ shaded here.

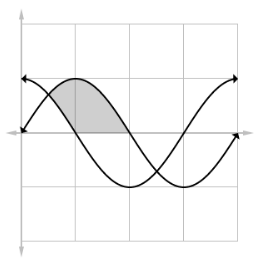


10. $\boxed{\texttt{M}}$ Find the area between the two curves $y = 1 + x^2$ and $y = 3 + x$.

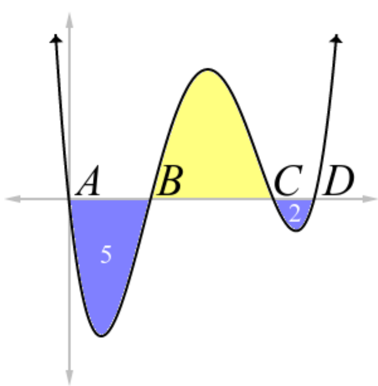
11. $\boxed{\texttt{M}}$ Find the area of the region bounded by $f(x) = 4$, $g(x) = \frac{e^x}{5}$, and $x = 0$.

12. $\boxed{\texttt{M}}$ What is the area of the region between the graphs of $f(x) = 2x^2 + 5x$ and $g(x) = -x^2 - 6x + 4$ from $x = -4$ to $x = 0$?

13. $\boxed{\texttt{E}}$ Find the area bounded by the curves $y = \sin x$ and $y = \cos x$ and the $x-$axis graphed here.



14. $\boxed{\texttt{A}}$ Consider the function f graphed below; the total **unsigned** area between the curve and the $x-$axis is 10 square units. Find $\int_A^D f(x)\,\mathrm{d}x$.



15. $\boxed{\texttt{M}}$ (a) Sketch the graph of $y = |\sin x|$.

(b) Compute $\int_0^{\pi/2} y\,\mathrm{d}x$ using the FTC.

(c) Hence, without doing any anti-differentiation, compute $\int_0^{2\pi} y\,\mathrm{d}x$.

16. $\boxed{\texttt{M}}$ Define $F(x)$ by

$$F(x) = \int_{\frac{\pi}{4}}^{x} \cos(2t)\,\mathrm{d}t.$$

(a) Use the Second Fundamental Theorem of Calculus to find $F'(x)$.

(b) Verify part (a) by integration and differentiation.

17. $\boxed{\texttt{M}}$ Compute $\frac{\mathrm{d}}{\mathrm{d}x} \int_2^x t^t\,\mathrm{d}t$.