Sarah Peerson

Cosmology 16/9-14

Error in homeworkset

#7 : minus signs missing

Hyperbolic surfaces: ~~not~~ not euclidic $\Rightarrow ds^2 = dx^2 + dy^2 - dz^2$

$x^2 + y^2 - z^2 = R^2$

Homework: we'll add phi-dimension

Today: space-time metrics

Similar to before: (4) new

$$\underbrace{ds^2}_{\text{invariant}} = \sum_{\mu,\nu=0}^{N} g_{\mu\nu} dx^\mu dy^\nu$$

math is the same, just adding a time dimension.

$dx^0 = c\,dt$
$dx^1 = dx$
$dx^2 = dy$
$dx^3 = dz$

Minkowski: flat

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

used in special relativity

$ds^2 < 0$ timelike
$ds^2 > 0$ spacelike
$ds^2 = 0$ lightlike

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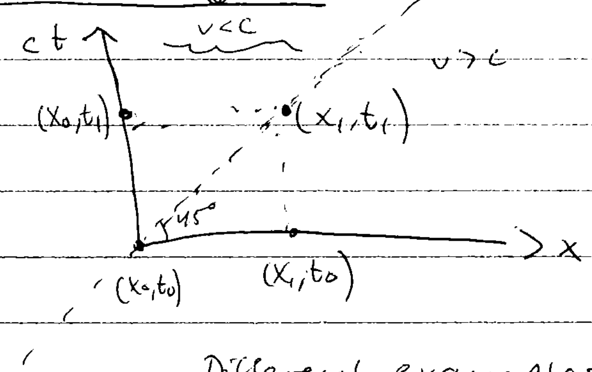
Space-time diagram



Flat static space, one dim in space

$$\boxed{ds^2 = -c^2dt^2 + dx^2}$$

Integrate ds^2 twice

$$\Delta s^2 = -c^2 dt^2 + \Delta x^2$$

Different examples:

case #1: $\Delta s^2 = -c^2(t_0 - t_0)^2 + (x_1 - x_0)^2 = \Delta x^2$ } (x_0, t_0) and (x_1, t_0)

$\Delta s^2 > 0 \Rightarrow$ space like separation

Case #2: $\Delta s^2 = -c^2(t_1 - t_0)^2 + (x_1 - x_0)^2 = 0$ } (x_0, t_0) (x_1, t_1)

$\Rightarrow \Delta s^2 = 0 \Rightarrow$ lightlike-separation

$C = \dfrac{\Delta x}{\Delta t}$ } slope of line $\Rightarrow$ speed of light

$\div$ sign from minkowski space

null-geodesic

Case #3: $ds^2 = -c^2(t_1 - t_0)^2 + (x_0 - x_0)^2 = -c^2\Delta t^2$

(x_0, t_0) (x_0, t_1) $ds^2 < 0 \Rightarrow$ time-like separation

Transformations: Galilean transformations: $dx' = dx - v\,dt$

$dt' = dt$

Lorentz transformations: $$\boxed{\begin{aligned} dx' &= \gamma(dx - v\,dt) \\ dt' &= \gamma\left(dt - \frac{v}{c^2}dx\right) \end{aligned}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Minkowski-space is Lorentz invariant

$$(ds')^2 = -c^2(dt')^2 + (dx')^2 = -c^2\gamma^2\left(dt - \frac{v}{c^2}dx\right)^2 + \gamma^2(dx - v\,dt)^2$$

Space-time

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Minkowski: Lorentz invariant $\Rightarrow (ds')^2 = ds^2$ } $\Rightarrow$ inertial frame

start from simple space time metric $\Rightarrow$ special relativity will hold.

Robinson-Walker **not** ~~true~~ true that it is Lorentz invariant

$$\boxed{ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dx^2}{1-\frac{kx^2}{R^2}} + x^2 d\Omega^2\right]}$$

$\boxed{d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2}$

k: curvature constant

null-geodesic

$ds = 0$ } lightlike

$$\Rightarrow c^2dt^2 = a(t)^2\left[\frac{dx^2}{1-\frac{kx^2}{R^2}}\right]$$

turning off θ, φ
only radial motion
for homework: use Ω.

"proper time" "cosmic time"

$a(t)$ is scalefactor

$a(t=0) = 1$ by convention $t=0 \Rightarrow$ today!

scalefactor: very small number in early Universe

Lorentz inv?

$(ds')^2 = -c^2(dt')^2 + a^2(dx')^2$ } we now have scale factor also

$(ds')^2 = -c^2\gamma^2(dt - \frac{v}{c^2}dx)^2 + a^2\gamma^2(dx - vdt)^2$

if $a=1$ today it is Lorentz inv.

$\ldots$

$= \gamma^2(a^2v^2 - c^2)dt + \gamma^2\left(a^2 - \frac{v^2}{c^2}\right)dx^2 + 2\gamma^2v(1-a^2)\,dxdt \neq ds^2$

- concepts from special-rel are not necessarily valid in cosmology.

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We can now calculate distances

- proper distance: eg. how far away is galaxy which is moving away from us.

→ $d\theta, d\varphi, d\chi = 0$ and $dt = 0$. assume this fixed time

Robinson-Walker: $\boxed{ds = a(t)\,dr}$

$$d_p(t) = a(t)\int_0^{r} dr = a(t)\cdot r$$

Robinson Walker $\boxed{ds^2 = a(t)^2 \cdot \frac{dx^2}{1-\frac{kx^2}{R^2}}}$ Now we need curvature.

$$d_p(t) = a(t)\cdot\int_0^x \frac{dx}{\sqrt{1-\frac{k}{R^2}x^2}}$$

proper distance

Flat $k=0 \Rightarrow d_p(t) = a(t)\,x$

closed $k=1 \Rightarrow d_p(t) = a(t)\int_0^x \frac{dx}{\sqrt{1-\frac{x^2}{R^2}}} = a(t)\int \frac{R\,dy}{\sqrt{1-y^2}} = a(t)\cdot R\sin^{-1}\left(\frac{x}{R}\right)$

Open $k=-1$ $\quad d_p(t) = a(t) R\sinh^{-1}\left(\frac{x}{R}\right)$

$$\boxed{d_p(t) = a(t)\int\frac{R\,dy}{\sqrt{1+y^2}} = a(t)R\sinh^{-1}\left(\frac{x}{R}\right)}$$

Imp. for measurements

We really need to know scale factor.

Sarah Pearson

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We know universe is expanding, and we know distance to something: we can find change in distance

depends on curvature

$$\frac{d_p(t)}{dt} = \frac{d}{dt}\left[a(t)\, f(x)\right] = \dot{a}\, f(x)$$

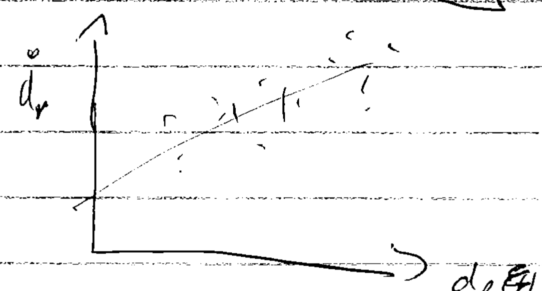
$$\left\{ f(x) \right. \quad R\sin^{-1}\left(\frac{x}{R}\right), \quad x, \quad R\sinh^{-1}\left(\frac{x}{R}\right)$$

$$f(x) = \frac{d_p(t)}{a(t)}$$

$$\Rightarrow \boxed{\frac{\dot{d}_p(t)}{dt} = \frac{\dot{a}}{a} d_p}$$

Expansion velocity of Universe. We don't know what $\dot{a}$ or a is yet, but we know proportionality!

$\Rightarrow$ Hubble's-plot:



$$\boxed{H_0 = \frac{\dot{a}}{a}} \quad \text{if } t = t_0 \text{ today}$$

Local Universe H_0 is a constant.

- slope of Hubble flow

Recession speed ↑ as d_p ↑

↳ we'll get to point where recession speed is more than speed of light

$$\dot{d}_p = \frac{\dot{a}}{a} d_p = v_p \qquad \text{if} \quad v_p(t_0) = c \quad \text{then} \quad \boxed{d_p(t_0) = \frac{c}{H_0}}$$

Hubble distance

it's okay that points move away faster than speed of light. Robertson-Walker is not Lorentz invariant.

$$H_0 = 70 \pm 7 \ \text{km/s/Mpc}$$

$$d_H(t_0) = \frac{3 \cdot 10^8 \text{ km/s}}{70.000 \text{ m/s/Mpc}} = 4300 \text{ Mpc} = 14 \text{ Gly}$$