Cosmology 18/9-14

How do we describe how the Universe is evolving.

Robinson-Walker:

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dx^2}{1-\frac{kx^2}{R^2}} + x^2 d\Omega^2\right]$$ space time metric

Proper distance: $d_p(t) = a(t) f(x)$

↳ func of position

$$f(x) = \begin{cases} R\sinh^{-1}\left(\frac{x}{R}\right) & : \text{closed}, \ k=+1 \\ R\sin^{-1}\left(\frac{x}{R}\right) & : \text{open}, \ k=-1 \\ x & : \text{flat} \quad k=0 \end{cases}$$

proper motion

$$v_p(t) = \frac{\dot a}{a} d_p(t)$$

H

ds: space-time integral

$ds=0 \Rightarrow$ null-geodesic (light).

Redshift

Pseudo-distance.

↘ light - no motion in θ, φ

Robertson-Walker: $ds^2 = 0 = c^2 dt - a(t)^2 dr^2$

$$\Rightarrow \quad c\,dt = a(t)\,dr$$ comoving distance

For light

$c\,dt = \lambda \Leftrightarrow \delta t = \frac{\lambda}{c}$

$$\Rightarrow c\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_0^r dr$$

$$\Rightarrow c\int_{t_e+\frac{\lambda_e}{c}}^{t_o+\frac{\lambda_o}{c}} \frac{dt}{a(t)} = \int_0^r dr \Leftrightarrow c\int_{t_e}^{t_o}\frac{dt}{a(t)} = c\int_{t_e+\frac{\lambda_e}{c}}^{t_o+\frac{\lambda_o}{c}} \frac{dt}{a(t)}$$

e: emitted

.... some math

Cosmology 18/9-14

Redshift continued:

subtract by

$$c\int_{t_e}^{t_0}\frac{dt}{a(t)} - c\int_{t_e+\frac{\lambda_e}{c}}^{t_0}\frac{dt}{a(t)} = c\int_{t_e+\frac{\lambda_e}{c}}^{t_0+\frac{\lambda_0}{c}}\frac{dt}{a(t)} - c\int_{t_e+\frac{\lambda_e}{c}}^{t_0}\frac{dt}{a(t)}$$

Now limits of integrations change

$$\cancel{c}\int_{t_e}^{t_e+\frac{\lambda_e}{c}}\frac{dt}{a(t)} = \cancel{c}\int_{t_0}^{t_0+\frac{\lambda_0}{c}}\frac{dt}{a(t)}$$

$\frac{\cancel{1}}{a(t)}$ take $a(t)=a(t_e)$ and $a(t_0)$

$$\frac{1}{a(t_e)}\int_{t_e}^{t_e+\frac{\lambda_e}{c}}dt = \frac{1}{a(t_0)}\int_{t_0}^{t_0+\frac{\lambda_0}{c}}dt \Rightarrow \frac{1}{a(t_e)}\cdot\left[\left(\cancel{t_e}+\frac{\lambda_e}{c}\right)-\cancel{t_e}\right]$$

$$= \frac{1}{a(t_0)}\left[\left(\cancel{t_0}+\frac{\lambda_0}{c}\right)-\cancel{t_0}\right]$$

$$\Rightarrow \frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)}$$

$$\boxed{z=\frac{\lambda_0-\lambda_e}{\lambda_e}}$$ ← measurable. definition of z.

$$\Rightarrow \boxed{\lambda_0 = \lambda_e\frac{a(t_0)}{a(t_e)}}$$

Plug into def of redshift

$a(t_0)=1$

$$\Rightarrow z = \frac{\lambda_e\frac{a(t_0)}{a(t_e)}-\lambda_e}{\lambda_e} = \frac{a(t_0)}{a(t_e)}-1 \Rightarrow z+1 = \frac{a(t_0)}{a(t_e)}$$

$\Rightarrow$ ~~z~~ ~~z+1~~

$$\boxed{z+1=\frac{1}{a(t_e)}}$$

Determine value of $a(t)$ when light was emitted!

observe $z \rightarrow$ get $a(t)$

Cosmology 18/9-14

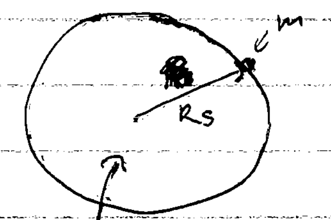
Now to physics... we need GR

↳ but we'll first introduce equations they derive these

Friedman:

- We'll derive it from Newtonian physics.

s: sphere



$$R_s(t) = a(t)\cdot r_s$$

↑ co-moving

$$M_s = \rho V = \rho(t)\cdot\frac{4}{3}\pi(R_s)^3$$

$$M_s = \rho(t)\frac{4}{3}\pi\left(a(t)\, r_s\right)^3 \quad \Big\}\text{ mass of sphere}$$

Acceleration on surface is important.

we dropped m

$$\text{First note: } ma = F = -\frac{GM_s m}{R_s(t)^2} \Rightarrow a = -\frac{GM_s}{R_s(t)^2} = \underbrace{\frac{d^2R_s}{dt^2}}_{= a}$$

mult by these two terms

$$\Rightarrow \left(\frac{dR_s}{dt}\right)\frac{d}{dt}\left(\frac{dR_s}{dt}\right) = -\frac{GM_s}{R_s(t)^2}\left(\frac{dR_s}{dt}\right)$$

trick: mult by dt.

$$\Rightarrow \int\left(\frac{dR_s}{dt}\right) d\left(\frac{dR_s}{dt}\right) = \int -\frac{GM_s}{R_s(t)^2}\cdot dR_s$$

$$\Rightarrow \underbrace{\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2}_{\text{kinetic energy / mass}} = \underbrace{+\frac{GM_s}{R_s(t)}}_{\text{gravitational pot energy per unit mass}} + U \leftarrow [\text{energy/mass}]. \qquad (*)$$

18/9-14

Freedman

We know that $\frac{dR_s}{dt} = \dot{a} r_s$ & $R_s = a r_s$

We know M_s from before: $M_s = \rho(t) \frac{4}{3}\pi a^3(t) r_s^3$

$$\Rightarrow (*) \quad \frac{1}{2}\dot{a}^2 r_s^2 = G \frac{\rho(t) \frac{4}{3}\pi a(t)^3 \cdot r_s^3}{a(t) r_s} + u$$

$$\Rightarrow \frac{1}{2}\dot{a}^2 r_s^2 = \frac{4}{3}\pi G \rho(t) a(t)^2 r_s^2 + u$$

$$\Rightarrow \dot{a}^2 = \frac{8}{3}\pi G \rho(t) a(t)^2 + \frac{2u}{r_s^2}$$

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G \rho(t) + \frac{2u}{r_s^2 a(t)^2}}$$

We don't know $\rho(t)$ nor $a(t)$.

Freedman

Testcases:

* $u > 0$: Right-hand-side is positive $\boxed{\text{open}}$
  - $\Rightarrow \dot{a}$ is also positive
  - $\Rightarrow$ expansion never stops.

* $u < 0$: Right-hand-side might start positive and evolve into being negative
  - expansion $\rightarrow$ contraction $\boxed{\text{closed}}$

~~$\dot{a}=0$~~ if $\dot{a} = 0$ at one point then

$$2G\frac{M_s}{a^3 r_s^3} = \frac{2u}{r_s^2 a^2} \Rightarrow \boxed{a_{max} = \frac{M_s G}{u r_s}}$$

Cosmology

18/9-14

Freedman

*if $u = 0$ Right hand side always positive

$\boxed{\text{Flat}}$ $\Rightarrow$ Universe expands forever.

Newtonian form

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G \rho(t) + \frac{2u}{r_s^2 a(t)^2}}$$

GR-form of Freedman:

$\boxed{\rho = \frac{\varepsilon}{c}}$

$\rho(t)$ before

$2u$ before.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varepsilon(t) - \frac{kc^2}{R^2}\frac{1}{a^2}}$$

k: curvature constant.

R: radius of curvature.

energy density
(consist of rad+matter+ .... dark energy)...

Now we play with Freedman eq ☺

← For Andrew with:

$v_p(t) = \frac{\dot{a}}{a} d_p(t)$ $\quad H(t) = \frac{\dot{a}}{a}$ $\quad H(t_0) = 70$ km/s/Mpc

Find Hubble distance with Freedman

$$H(t_0)^2 = H_0^2 = \frac{8\pi G}{3}\varepsilon_0 - \frac{kc^2}{R^2}$$

if $\varepsilon_0 = 0$

$\Rightarrow H_0^2 = -\frac{kc^2}{R^2}$

$\Rightarrow R^2 = -\frac{kc^2}{H_0^2}$ will only work for $k < 0 = -1$

Hubble distance with Friedman

If $k=-1$

$$\Rightarrow \quad R = \frac{c}{H_0} = d_H(t_0) =$$

* compute energy density ε_0:

$$H(t)^2 = \frac{8\pi G}{3}\varepsilon(t) - \frac{kc^2}{R^2 a^2}$$

if $k=0 \Rightarrow$ flat uni $\Rightarrow$ critical energy dens.

$$H(t)^2 = \frac{8\pi G}{3}\varepsilon(t) \Rightarrow \boxed{\varepsilon_c(t) = H(t)^2 \frac{3c^2}{8\pi G}}$$

if $k=0$ we get this.

Hence if $\varepsilon(t)$ is this exact thing our Universe is flat.

1) if $\varepsilon(t) > \varepsilon_c(t) \Rightarrow k=+1$ } positive curv.

2) if $\varepsilon(t) < \varepsilon_c(t) \Rightarrow k=-1$ } negative curve

3) if $\varepsilon(t) = \varepsilon_c(t) \Rightarrow k=0$ Flat uni!

Hint: imp. to leave in curvature but our observations imply that $k=0$... kind of weird coincidence.

We leave in k for now for sake of completeness of equations

Cosmo 18/9-14

we can compute $\boxed{\underset{\text{today}}{\varepsilon_c(t_0)} = H_0^2 \frac{3c^2}{8\pi G}}$

Hence we know what it should be today.

$\varepsilon_c(t_0) = 5200 \frac{MeV}{m^3}$: Critical energy density today.

$\rho_c(t_0) = \frac{\varepsilon_c(t_0)}{c} = 9{,}2 \cdot 10^{-27} \frac{kg}{m^3} = \underbrace{1{,}4 \cdot 10^{11} \frac{M_\odot}{Mpc^2}}$

density today if $k = 0$.

We now define Ω:

$$\boxed{\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)}}$$

$\boxed{\text{if } k = 0}$

$\varepsilon_c(t) = H(t)^2 \frac{3c^2}{8\pi G}$

$\Rightarrow \varepsilon(t) = \Omega(t) \cdot \varepsilon_c(t)$

$\Rightarrow \boxed{\varepsilon(t) = \Omega(t) \cdot H(t)^2 \frac{3c^2}{8\pi G}}$ Plug this into Freedman eq to get Freedman (Ω).

$$H(t)^2 = \cancel{\frac{8\pi G}{3c^2}} \left(\Omega(t) \cdot H(t)^2\right) \cdot \cancel{\frac{3c^2}{8\pi G}} - \frac{kc^2}{R^2 a^2}$$

$\Rightarrow \left(1 - \Omega(t)\right) H(t)^2 = -\frac{kc^2}{R^2 a^2} \Leftrightarrow \frac{k}{R^2} = \overbrace{\frac{a(t)^2 H(t)}{c^2} \left(\Omega(t) - 1\right)}^{\text{Current}}$

~~...~~ and $t = t_0 \Rightarrow$

$$\boxed{\frac{k}{R^2} = \frac{H_0^2}{c^2}\left(\Omega_0 - 1\right)}$$

if $k = 0$ then uni is flat and $\Omega_0 = 1$