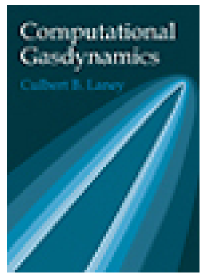

Computational Gasdynamics

Culbert B. Laney

Book DOI: http://dx.doi.org/10.1017/CBO9780511605604

Online ISBN: 9780511605604

Hardback ISBN: 9780521570695

Paperback ISBN: 9780521625586

Chapter

Chapter 24 - A Brief Introduction to Multidimensions pp. 597-604

Chapter DOI: http://dx.doi.org/10.1017/CBO9780511605604.030

Cambridge University Press

A Brief Introduction to Multidimensions

24.0 Introduction

This chapter briefly introduces computational gasdynamics in multidimensions. A full discussion of computational gasdynamics in multidimensions would require repeating the entire book point by point; this chapter only has space for multidimensional versions of Chapters 2, 3, and 11, and then only in a terse "just the facts" form. However, even this should be enough to clearly demonstrate how the one-dimensional concepts from the rest of the book carry over directly to multidimensional approximations.

24.1 Governing Equations

This section presents a brief multidimensional version of Chapter 2. In three dimensions, consider an arbitrary control volume R with surface S and surface normal $\hat{n} = (n_x, n_y, n_z)$. Also consider an arbitrary time interval $[t_1, t_2]$. Let $\mathbf{V} = (u, v, w)$ be the velocity vector and $e_{\mathrm{T}} = e + \frac{1}{2}V^2$ be the total energy per unit mass. Then the integral form of the three-dimensional (3D) Euler equations can be written as

$$\iiint_R [\mathbf{u}(\mathbf{x}, t_2) - \mathbf{u}(\mathbf{x}, t_1)]\, dR = -\int_{t_1}^{t_2} \iint_S (\mathbf{f} n_x + \mathbf{g} n_y + \mathbf{h} n_z)\, dS\, dt.$$

Also, the conservative differential form of the 3D Euler equations can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial y} + \frac{\partial \mathbf{h}}{\partial z} = 0,$$

where

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho \mathbf{V} \\ \rho e_{\mathrm{T}} \end{bmatrix} = \text{vector of conserved quantities}$$

and

$$\mathbf{f} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ (\rho e_{\mathrm{T}} + p) u \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ (\rho e_{\mathrm{T}} + p) v \end{bmatrix}, \qquad \mathbf{h} = \begin{bmatrix} \rho v \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ (\rho e_{\mathrm{T}} + p) w \end{bmatrix}.$$

Define flux Jacobian matrices as follows:

$$A = \frac{d\mathbf{f}}{d\mathbf{u}}, \qquad B = \frac{d\mathbf{g}}{d\mathbf{u}}, \qquad C = \frac{d\mathbf{h}}{d\mathbf{u}}.$$

Then the conservative differential form of the 3D Euler equations can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} + B\frac{\partial \mathbf{u}}{\partial y} + C\frac{\partial \mathbf{u}}{\partial z} = 0,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -u^2 + \frac{\gamma-1}{2}\mathbf{V}\cdot\mathbf{V} & (3-\gamma)u & -(\gamma-1)v & -(\gamma-1)w & \gamma-1 \\ -uv & v & u & 0 & 0 \\ -uw & w & 0 & u & 0 \\ -(\gamma e_{\mathrm{T}} - (\gamma-1)\mathbf{V}\cdot\mathbf{V})u & \gamma e_{\mathrm{T}} - \frac{\gamma-1}{2}(2u^2 + \mathbf{V}\cdot\mathbf{V}) & -(\gamma-1)uv & -(\gamma-1)uw & \gamma u \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -uv & v & u & 0 & 0 \\ -v^2 + \frac{\gamma-1}{2}\mathbf{V}\cdot\mathbf{V} & -(\gamma-1)u & (3-\gamma)v & -(\gamma-1)w & \gamma-1 \\ -vw & 0 & w & v & 0 \\ -(\gamma e_{\mathrm{T}} - (\gamma-1)\mathbf{V}\cdot\mathbf{V})v & -(\gamma-1)uv & \gamma e_{\mathrm{T}} - \frac{\gamma-1}{2}(2v^2 + \mathbf{V}\cdot\mathbf{V}) & -(\gamma-1)vw & \gamma v \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -uw & w & 0 & u & 0 \\ -vw & 0 & w & v & 0 \\ -w^2 + \frac{\gamma-1}{2}\mathbf{V}\cdot\mathbf{V} & -(\gamma-1)u & -(\gamma-1)v & (3-\gamma)w & \gamma-1 \\ -(\gamma e_{\mathrm{T}} - (\gamma-1)\mathbf{V}\cdot\mathbf{V})w & -(\gamma-1)uw & -(\gamma-1)vw & \gamma e_{\mathrm{T}} - \frac{\gamma-1}{2}(2w^2 + \mathbf{V}\cdot\mathbf{V}) & \gamma w \end{bmatrix}.$$

In two dimensions, consider an arbitrary control volume A with perimeter C and perimeter normal $\hat{n} = (n_x, n_y)$. Also consider an arbitrary time interval $[t_1, t_2]$. Let $\mathbf{V} = (u, v)$ be the velocity vector and $e_{\mathrm{T}} = e + \frac{1}{2}V^2$ be the total energy per unit mass. Then the integral form of the two-dimensional (2D) Euler equations can be written as

$$\iint_A [\mathbf{u}(\mathbf{x}, t_2) - \mathbf{u}(\mathbf{x}, t_1)]\, dR = -\int_{t_1}^{t_2} \int_C (\mathbf{f}n_x + \mathbf{g}n_y)\, dS\, dt.$$

Also, the conservative differential form of the 2D Euler equations can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial y} = 0,$$

where

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho\mathbf{V} \\ \rho e_{\mathrm{T}} \end{bmatrix} = \text{vector of conserved quantities}$$

and

$$\mathbf{f} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (\rho e_{\mathrm{T}} + p)u \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (\rho e_{\mathrm{T}} + p)v \end{bmatrix}.$$

Define flux Jacobian matrices as follows:

$$A = \frac{d\mathbf{f}}{d\mathbf{u}}, \qquad B = \frac{d\mathbf{g}}{d\mathbf{u}}.$$

Then the conservative differential form of the 2D Euler equations can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} + B\frac{\partial \mathbf{u}}{\partial y} = 0,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -u^2 + \frac{\gamma-1}{2}\mathbf{V}\cdot\mathbf{V} & (3-\gamma)u & -(\gamma-1)v & \gamma - 1 \\ -uv & v & u & 0 \\ -(\gamma e_{\mathrm{T}} - (\gamma-1)\mathbf{V}\cdot\mathbf{V})u & \gamma e_{\mathrm{T}} - \frac{\gamma-1}{2}(2u^2 + \mathbf{V}\cdot\mathbf{V}) & -(\gamma-1)uv & \gamma u \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -uv & v & u & 0 \\ -v^2 + \frac{\gamma-1}{2}\mathbf{V}\cdot\mathbf{V} & -(\gamma-1)u & (3-\gamma)v & \gamma - 1 \\ -(\gamma e_{\mathrm{T}} - (\gamma-1)\mathbf{V}\cdot\mathbf{V})v & -(\gamma-1)uv & \gamma e_{\mathrm{T}} - \frac{\gamma-1}{2}(2v^2 + \mathbf{V}\cdot\mathbf{V}) & \gamma v \end{bmatrix}.$$

24.2 Waves

This section is a brief multidimensional version of Chapter 3. Waves behave much differently in multidimensions than they do in one dimension. Some of the complicating factors include:

- If the spatial dimension is N, waves may have any dimension between 1 and N in the $\mathbf{x}$–t plane.
- Waves may travel in an infinite number of directions. For example, two-dimensional waves may travel at any angle between 0° and 360°.
- Characteristic variables may not be constant along characteristics.
- The characteristic form is not unique. In other words, there are infinitely many wave descriptions of a given multidimensional flow.

In three dimensions, suppose that the characteristic surfaces for all five families have a common unit normal $\hat{\mathbf{n}}$ in the $\mathbf{x}$ plane. You can choose $\hat{\mathbf{n}}$ however you like. The characteristic form of the 3D Euler equations written in terms of primitive variables is

$$c_i'\left(\frac{\partial\rho}{\partial t}+\mathbf{V}\cdot\nabla\rho\right)+\left(\hat{\mathbf{n}}\times\mathbf{b}_i'\right)\cdot\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}+\frac{\nabla p}{\rho}\right)$$
$$-\frac{1}{a^2}c_i'\left(\frac{\partial p}{\partial t}+\mathbf{V}\cdot\nabla p\right)=0 \qquad (i=1,2,3),$$
$$\hat{\mathbf{n}}\cdot\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)+a\nabla\cdot\mathbf{V}+\frac{1}{\rho a}\left(\frac{\partial p}{\partial t}+(\mathbf{V}+a\hat{\mathbf{n}})\cdot\nabla p\right)=0,$$
$$-\hat{\mathbf{n}}\cdot\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)+a\nabla\cdot\mathbf{V}+\frac{1}{\rho a}\left(\frac{\partial p}{\partial t}+(\mathbf{V}-a\hat{\mathbf{n}})\cdot\nabla p\right)=0,$$

where a is the speed of sound, $\mathbf{c}'=(c_1',c_2',c_3')$ is an arbitrary vector, and $B'=[\mathbf{b}_1'\,|\,\mathbf{b}_2'\,|\,\mathbf{b}_3']$ is an arbitrary set of vectors.

The 3D characteristic variables are defined as follows:

$$dv_i=\begin{bmatrix}c_i'\\ \hat{\mathbf{n}}\times\mathbf{b}_i'\\ -\frac{1}{a^2}c_i'\end{bmatrix}\cdot\begin{bmatrix}d\rho\\ d\mathbf{V}\\ dp\end{bmatrix}=c_i'd\rho+\left(\hat{\mathbf{n}}\times\mathbf{b}_i'\right)\cdot d\mathbf{V}-\frac{1}{a^2}c_i'\,dp \qquad (i=1,2,3),$$
$$dv_4=e\begin{bmatrix}0\\ \hat{\mathbf{n}}\\ \frac{1}{\rho a}\end{bmatrix}\cdot\begin{bmatrix}d\rho\\ d\mathbf{V}\\ dp\end{bmatrix}=e\hat{\mathbf{n}}\cdot d\mathbf{V}+\frac{e}{\rho a}dp,$$
$$dv_5=f\begin{bmatrix}0\\ -\hat{\mathbf{n}}\\ \frac{1}{\rho a}\end{bmatrix}\cdot\begin{bmatrix}d\rho\\ d\mathbf{V}\\ dp\end{bmatrix}=-f\hat{\mathbf{n}}\cdot d\mathbf{V}+\frac{f}{\rho a}dp,$$

where e and f are arbitrary. The characteristic form of the 3D Euler equations written in terms of characteristic variables is

$$\frac{\partial v_1}{\partial t}+\mathbf{V}\cdot\nabla v_1+\frac{a}{2}\left(\hat{\mathbf{n}}\times\mathbf{b}_1'\right)\cdot\nabla\left(\frac{1}{e}v_4+\frac{1}{f}v_5\right)=0,$$
$$\frac{\partial v_2}{\partial t}+\mathbf{V}\cdot\nabla v_2+\frac{a}{2}\left(\hat{\mathbf{n}}\times\mathbf{b}_2'\right)\cdot\nabla\left(\frac{1}{e}v_4+\frac{1}{f}v_5\right)=0,$$
$$\frac{\partial v_3}{\partial t}+\mathbf{V}\cdot\nabla v_3+\frac{a}{2}\left(\hat{\mathbf{n}}\times\mathbf{b}_3'\right)\cdot\nabla\left(\frac{1}{e}v_4+\frac{1}{f}v_5\right)=0,$$
$$\frac{\partial v_4}{\partial t}+(\mathbf{V}+a\hat{\mathbf{n}})\cdot\nabla v_4+ea(\hat{\mathbf{n}}\times\mathbf{b}_1)\cdot\nabla v_1$$
$$+ea(\hat{\mathbf{n}}\times\mathbf{b}_2)\cdot\nabla v_2+ea(\hat{\mathbf{n}}\times\mathbf{b}_3)\cdot\nabla v_3=0,$$
$$\frac{\partial v_5}{\partial t}+(\mathbf{V}-a\hat{\mathbf{n}})\cdot\nabla v_5+fa(\hat{\mathbf{n}}\times\mathbf{b}_1)\cdot\nabla v_1$$
$$+fa(\hat{\mathbf{n}}\times\mathbf{b}_2)\cdot\nabla v_2+fa(\hat{\mathbf{n}}\times\mathbf{b}_3)\cdot\nabla v_3=0.$$

In two dimensions, suppose that the characteristic curves for all four families have a common unit normal $\hat{\mathbf{n}}$ in the $\mathbf{x}$ plane. You can choose $\hat{\mathbf{n}}$ however you like. The characteristic form of the 2D Euler equations written in terms of primitive variables is

$$c_1'\left(\frac{\partial\rho}{\partial t}+\mathbf{V}\cdot\nabla\rho\right)+b_1'(n_y,-n_x)\cdot\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}+\frac{\nabla p}{\rho}\right)-\frac{1}{a^2}c_1'\left(\frac{\partial p}{\partial t}+\mathbf{V}\cdot\nabla p\right)=0,$$

$$c_2'\left(\frac{\partial\rho}{\partial t}+\mathbf{V}\cdot\nabla\rho\right)+b_2'(n_y,-n_x)\cdot\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}+\frac{\nabla p}{\rho}\right)-\frac{1}{a^2}c_2'\left(\frac{\partial p}{\partial t}+\mathbf{V}\cdot\nabla p\right)=0,$$

$$\hat{\mathbf{n}}\cdot\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)+a\nabla\cdot\mathbf{V}+\frac{1}{\rho a}\left(\frac{\partial p}{\partial t}+(\mathbf{V}+a\hat{\mathbf{n}})\cdot\nabla p\right)=0,$$

$$-\hat{\mathbf{n}}\cdot\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)+a\nabla\cdot\mathbf{V}+\frac{1}{\rho a}\left(\frac{\partial p}{\partial t}+(\mathbf{V}-a\hat{\mathbf{n}})\cdot\nabla p\right)=0,$$

where $\mathbf{b}'=(b_1',b_2')$ and $\mathbf{c}'=(c_1',c_2')$ are arbitrary.

The 2D characteristic variables are defined as follows:

$$\begin{aligned}
dv_1 &= c_1'd\rho+b_1'(n_y,-n_x)\cdot d\mathbf{V}-\frac{1}{a^2}c_1'\,dp,\\
dv_2 &= c_2'd\rho+b_2'(n_y,-n_x)\cdot d\mathbf{V}-\frac{1}{a^2}c_2'\,dp,\\
dv_3 &= e\hat{\mathbf{n}}\cdot d\mathbf{V}+\frac{e}{\rho a}\,dp,\\
dv_4 &= -f\hat{\mathbf{n}}\cdot d\mathbf{V}+\frac{f}{\rho a}\,dp,
\end{aligned}$$

where e and f are arbitrary. The characteristic form of the 2D Euler equations written in terms of characteristic variables is

$$\begin{aligned}
&\frac{\partial v_1}{\partial t}+\mathbf{V}\cdot\nabla v_1+\frac{a}{2}b_1'(n_y,-n_x)\cdot\nabla\left(\frac{1}{e}v_3+\frac{1}{f}v_4\right)=0,\\
&\frac{\partial v_2}{\partial t}+\mathbf{V}\cdot\nabla v_2+\frac{a}{2}b_2'(n_y,-n_x)\cdot\nabla\left(\frac{1}{e}v_3+\frac{1}{f}v_4\right)=0,\\
&\frac{\partial v_3}{\partial t}+(\mathbf{V}+a\mathbf{n})\cdot\nabla v_3+ab_1(n_y,-n_x)\cdot\nabla v_1+ab_2(n_y,-n_x)\cdot\nabla v_2=0,\\
&\frac{\partial v_4}{\partial t}+(\mathbf{V}-a\mathbf{n})\cdot\nabla v_4+ab_1(n_y,-n_x)\cdot\nabla v_1+ab_2(n_y,-n_x)\cdot\nabla v_2=0.
\end{aligned}$$

This section has written the characteristic forms in the most general way possible. Other references usually arbitrarily choose some or all of the free parameters b, c, e, and f.

Given the complexity and numerous free parameters found in multidimensional characteristic analysis, no one has yet found a way to usefully exploit it for numerical method construction in the same way as one-dimensional characteristic analysis, except occasionally as part of the boundary treatment. The only popular wave-inspired numerical techniques – Riemann solvers and flux vector splitting – are strictly one dimensional. Since the 1980s, various researchers such as Roe, Van Leer, Colella, and Hirsch have proposed numerous ingenious "truly multidimensional" numerical methods based on multidimensional characteristics. However, none of them has yet proven suitable for practical computations. Instead, most successful multidimensional numerical methods are based on strictly one-dimensional techniques, as described in the next section.

24.3 Conservation and Other Numerical Principles

This section is a brief two-dimensional version of Chapter 11. For simplicity, this section only considers rectangular discretization of a two-dimensional spatial domain. Other discretizations, such as triangles, involve numerous additional complications having to do with indexing and tracking cell-edge normals. There are two simple techniques for extending one-dimensional methods to two dimensions. The first technique yields conservative numerical methods of the form:

$$\bar{u}_{ij}^{n+1} = \bar{u}_{ij}^{n} - \frac{\Delta t}{\Delta x_i}\left(\hat{f}^{n}_{i+1/2,j} - \hat{f}^{n}_{i-1/2,j}\right) - \frac{\Delta t}{\Delta y_j}\left(\hat{g}^{n}_{i,j+1/2} - \hat{g}^{n}_{i,j-1/2}\right),$$

where $\hat{f}_{i+1/2,j}$ and $\hat{g}_{i,j+1/2}$ are constructed exactly like $\hat{f}_{i+1/2}$ in one dimension. The only real difference is that $\hat{f}_{i+1/2,j}$ and $\hat{g}_{i,j+1/2}$ have four components in two dimensions, as opposed to the three components of $\hat{f}_{i+1/2}$ in one dimension. This is sometimes called a *dimension-by-dimension approximation.*

For example, in one dimension, FTCS has the following conservative numerical flux:

$$\hat{\mathbf{f}}^{n}_{i+1/2} = \frac{\mathbf{f}(\mathbf{u}^n_{i+1}) + \mathbf{f}(\mathbf{u}^n_i)}{2} = \frac{1}{2}\begin{bmatrix} (\rho u)_{i+1} + (\rho u)_i \\ (\rho u^2)_{i+1} + (\rho u^2)_i + p_{i+1} + p_i \\ (\rho u e_{\mathrm{T}})_{i+1} + (\rho u e_{\mathrm{T}})_i + (pu)_{i+1} + (pu)_i \end{bmatrix},$$

as seen in Chapter 11. Then, for a rectangular grid, a two-dimensional version of FTCS has the following conservative numerical fluxes:

$$\hat{\mathbf{f}}^{n}_{i+1/2,j} = \frac{\mathbf{f}(\mathbf{u}^n_{i+1,j}) + \mathbf{f}(\mathbf{u}^n_{ij})}{2} = \frac{1}{2}\begin{bmatrix} (\rho u)_{i+1,j} + (\rho u)_{ij} \\ (\rho u^2)_{i+1,j} + (\rho u^2)_{ij} + p_{i+1,j} + p_{ij} \\ (\rho uv)_{i+1,j} + (\rho uv)_{ij} \\ (\rho u e_{\mathrm{T}})_{i+1,j} + (\rho u e_{\mathrm{T}})_i + (pu)_{i+1,j} + (pu)_{ij} \end{bmatrix},$$

$$\hat{\mathbf{g}}^{n}_{i,j+1/2} = \frac{\mathbf{g}(\mathbf{u}^n_{i,j+1}) + \mathbf{g}(\mathbf{u}^n_{ij})}{2} = \frac{1}{2}\begin{bmatrix} (\rho v)_{i,j+1} + (\rho v)_{ij} \\ (\rho uv)_{i,j+1} + (\rho uv)_{ij} \\ (\rho v^2)_{i,j+1} + (\rho v^2)_{ij} + p_{i,j+1} + p_{ij} \\ (\rho v e_{\mathrm{T}})_{i,j+1} + (\rho v e_{\mathrm{T}})_{ij} + (pv)_{i,j+1} + (pv)_{ij} \end{bmatrix}.$$

There is a second simple approach for extending a method from one to two dimensions. For specificity, consider finite-volume methods; the same concept applies to finite-difference methods. Finite-volume methods approximate the integral form of the governing equations as follows:

$$\begin{aligned}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}} &(\mathbf{u}(x,y,t^{n+1}) - \mathbf{u}(x,y,t^n))\,dx\,dy \\ &= -\int_{t^n}^{t^{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}} (\mathbf{f}(x_{i+1/2},y,t) - \mathbf{f}(x_{i-1/2},y,t))\,dy\,dt \\ &\quad - \int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} (\mathbf{g}(x,y_{j+1/2},t) - \mathbf{g}(x,y_{j-1/2},t))\,dx\,dt.\end{aligned}$$

Now consider the following one-dimensional governing equation:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} (\mathbf{u}(x, y, t^{n+1}) - \mathbf{u}(x, y, t^n))\,dx = -\int_{t^n}^{t^{n+1}} (\mathbf{f}(x_{i+1/2}, y, t) - \mathbf{f}(x_{i-1/2}, y, t))\,dt.$$

Suppose the numerical approximation to this one-dimensional governing equation is written in the following operator form:

$$\overline{\mathbf{u}}^{n+1} = L_x \overline{\mathbf{u}}^n.$$

Similarly, consider the following one-dimensional governing equation:

$$\int_{y_{j-1/2}}^{y_{j+1/2}} (\mathbf{u}(x, y, t^{n+1}) - \mathbf{u}(x, y, t^n))\,dy = -\int_{t^n}^{t^{n+1}} (\mathbf{g}(x, y_{j+1/2}, t) - \mathbf{g}(x, y_{j+1/2}, t))\,dt.$$

And suppose the numerical approximation to this one-dimensional governing equation is also written in the following operator form:

$$\overline{\mathbf{u}}^{n+1} = L_y \overline{\mathbf{u}}^n.$$

Now suppose that the two one-dimensional methods are applied in succession so that

$$\overline{\mathbf{u}}^{n+1} = L_x L_y \overline{\mathbf{u}}^n \qquad \text{or} \qquad \overline{\mathbf{u}}^{n+1} = L_y L_x \overline{\mathbf{u}}^n.$$

This is an approximation to the original two-dimensional governing equation. Unfortunately, this numerical approximation is at most first-order accurate. However, the following approximation may yield up to second-order accuracy:

$$\overline{\mathbf{u}}^{n+2} = L_y L_x L_x L_y \overline{\mathbf{u}}^n$$

or

$$\overline{\mathbf{u}}^{n+2} = L_x L_y L_y L_x \overline{\mathbf{u}}^n.$$

This is known as *dimensional splitting*, *Strang time splitting*, or *time splitting*.

For example, dimensional splitting gives the following two-dimensional version of FTCS:

$$\overline{\mathbf{u}}_{ij}^{n+2} = L_y L_x L_x L_y \overline{\mathbf{u}}_{ij}^n,$$

where

$$(L_x \overline{\mathbf{u}}^n)_{ij} = \overline{\mathbf{u}}_{ij}^n - \frac{\Delta t}{2\Delta x_i}\left(\mathbf{f}\left(\overline{\mathbf{u}}_{i+1,j}^n\right) - \mathbf{f}\left(\overline{\mathbf{u}}_{i-1,j}^n\right)\right),$$

$$(L_y \overline{\mathbf{u}}^n)_{ij} = \overline{\mathbf{u}}_{ij}^n - \frac{\Delta t}{2\Delta y_j}\left(\mathbf{g}\left(\overline{\mathbf{u}}_{i,j+1}^n\right) - \mathbf{g}\left(\overline{\mathbf{u}}_{i,j-1}^n\right)\right).$$

Of course, this version of FTCS is quite a bit different from that given by dimension-by-dimension splitting. Most practical methods use dimension-by-dimension splitting. Whereas one-dimensional methods easily extend to multidimensions in the interior, boundary treatments are a different story; Chapter 19 makes several references to some of the relevant issues, and includes an extensive list of references for anyone who wishes to research this difficult topic for themselves. This marks the end of a long journey. I hope you are not too sore.