Problem Set #1 for G6010: Physical Cosmology

Fall Semester 2014

Problem #1

Given $ds^2 = dx^2 + dy^2 + dz^2$ and the constraint $x^2 + y^2 + z^2 = -R^2$, show first that

$$ds^2 = \frac{dr^2}{1 + r^2/R^2} + r^2 d\theta^2$$

and then that

$$ds^2 = d\rho^2 + R^2 \sinh^2(\rho/R)\ d\theta^2$$

for an open, two-dimensional curved space with radius of curvature, R.

Problem #2

Given $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$ and the constraint $x^2 + y^2 + z^2 + w^2 = R^2/\kappa$, derive the Robertson-Walker metric for three dimensions. Your final answer should be in the following form:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - \kappa r^2/R^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right].$$

Problem #3

When a cosmological constant is introduced, the Friedman equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3}.$$

Show that in this limit, the acceleration equation becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\,(\varepsilon + 3P) + \frac{\Lambda}{3},$$

and explain why the fluid equation remains

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0.$$

Problem #4

Reproduce the plot in Figure 1, which shows the scale factor, a, for various single-component universes. Note that the horizontal axis is $H_0(t - t_0)$ rather than just t.

Problem #5

Compute $a(t)$ for the multi-component universe where the Friedman equation is:

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \tag{1}$$

Assume $\Omega_{r,0} = 8.4 \times 10^{-5}$, $\Omega_{m,0} = 0.3$, and $\Omega_{\Lambda,0} = 0.7$, and note that by convention $a(t_0) = 1$. Your solution should look like the plot in Figure 2.

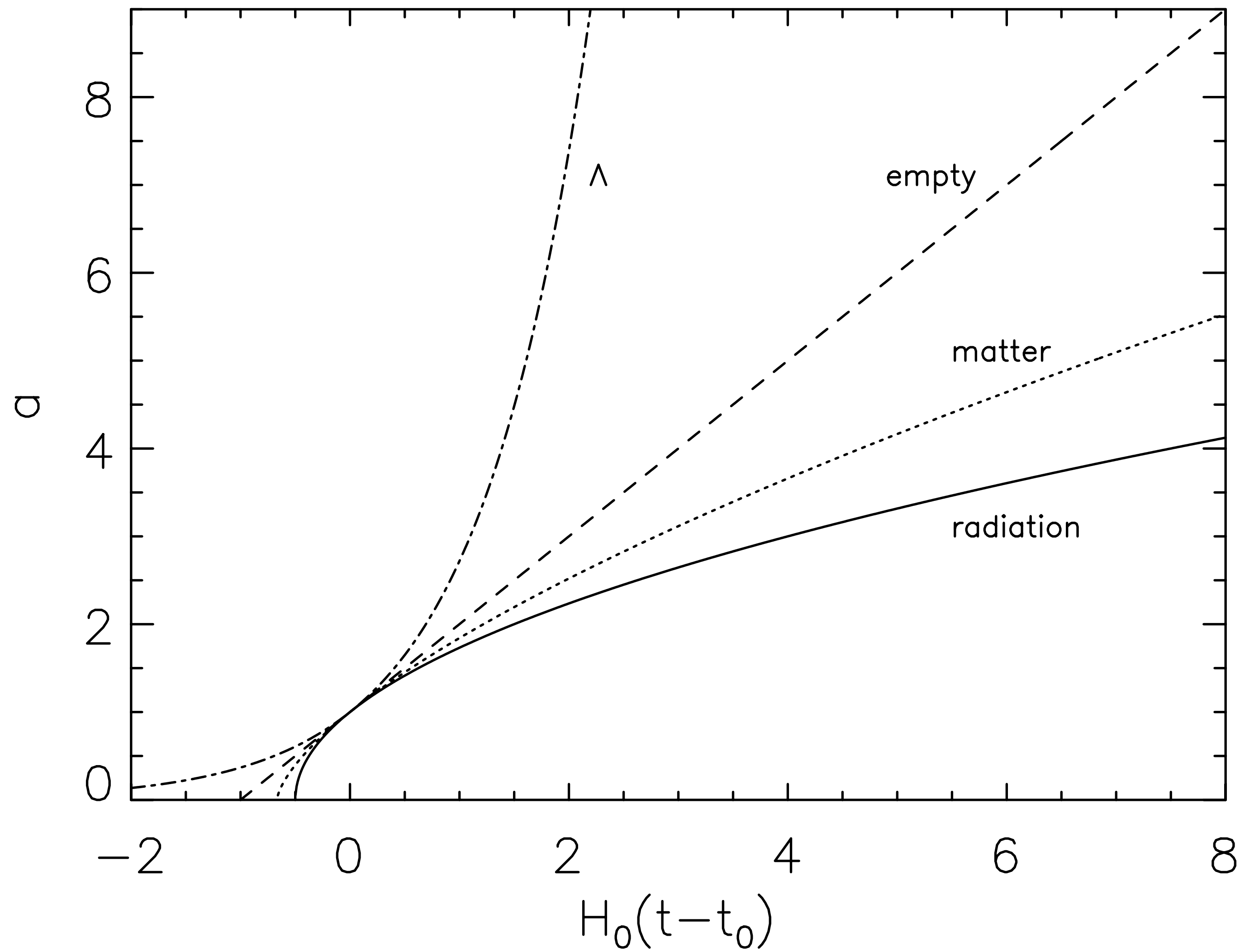

Figure 1: The scale factor as a function of time for an expanding, empty universe (dashed), a flat, matter-dominated universe (dotted), a flat, radiation-dominated universe (solid), and a flat, Λ-dominated universe (dot-dash).

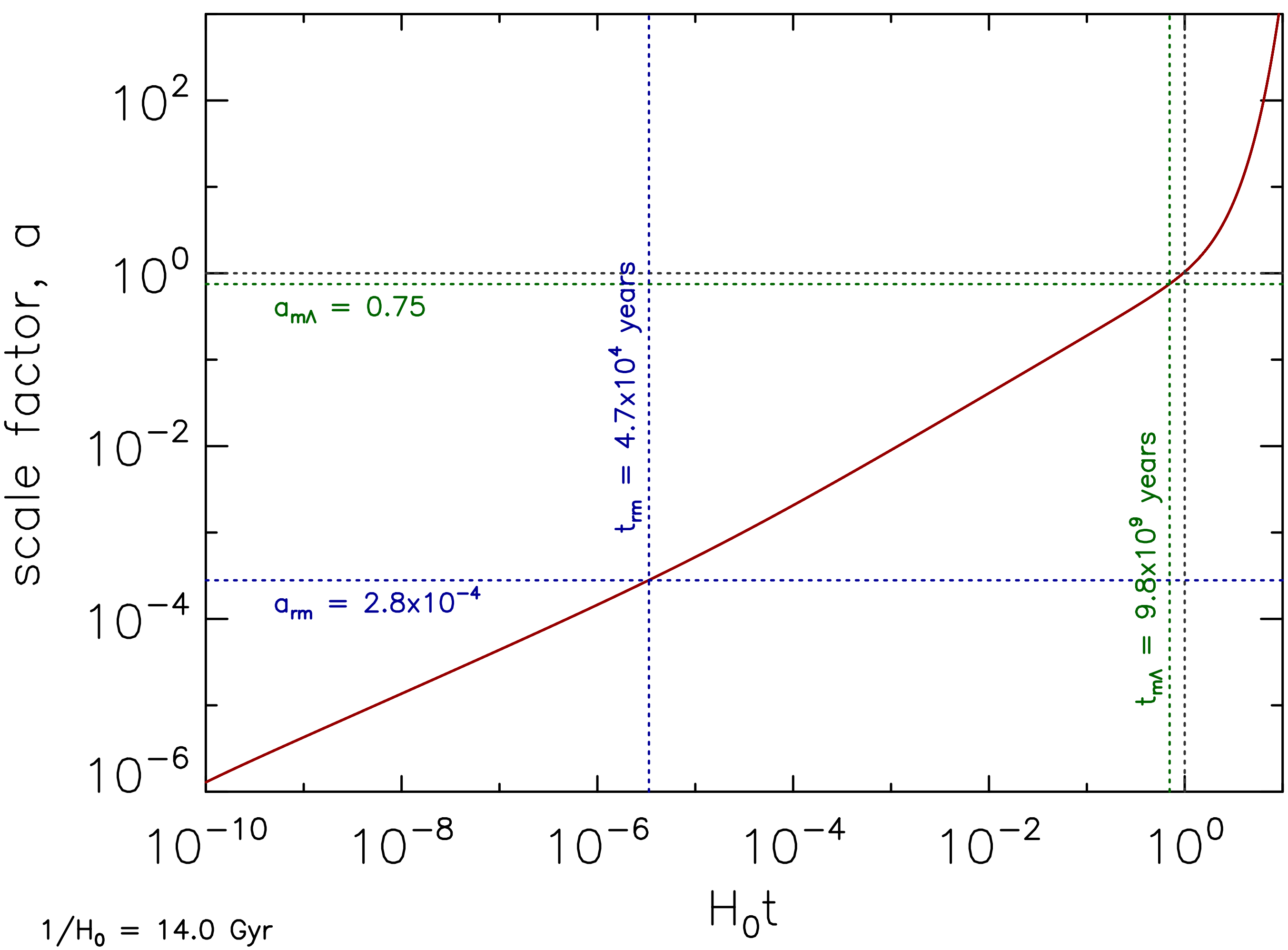

Figure 2: The scale factor as a function of time for the multi-component universe described in Problem #5.