# Topics in Quantum Engineering Presentation: Complexity theory

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# Complexity theory in a nutshell

"How hard can it be?", *Clarkson*

# What is complexity theory?

Complexity theory is the study of how difficult it is to solve a problem with a computer.

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How do we measure difficulty?

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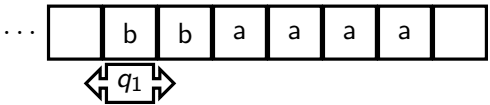
What is a computer?

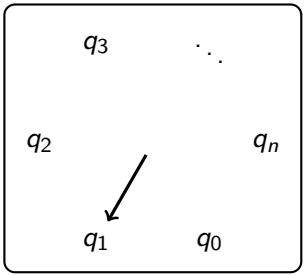
# Structure of part one

- What is a computer?
- What is a problem?
- How do we measure difficulty?

# Our model of a computer




Machine Tape (RAM) with head at letter b




Set of States (CPU) with current state $q_0$

1

[1] Original version at `http://www.texample.net/tikz/examples/turing-machine-2/`

# Formal definition of a computer

A Turing Machine ($TM$) is specified as a tuple of seven components $\langle Q, \Gamma, b, \Sigma, q_0, F, \delta \rangle$:

- $Q$ is the set of all possible states
- $\Gamma$ is tape alphabet
- $b \in \Gamma$ is the blank symbol for the tape
- $\Sigma = \Gamma / b$ is the input alphabet for the tape
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accepting states
- $\delta : Q \times \Gamma \rightarrow Q \times \Sigma \times \{L, R\}$ is the transition function

# But how do we run it?

The majority of computation time is spent repeating the following loop. Note that $T_h$ is the $h$-th cell of the tape.

$q = q_0$
$h = 0$
$T = w$ // Tape starts as just input, followed by blank cells
**while** $\delta(q, T_h)$ *is not undefined* **do**
    $(q, h, i) = \delta(q, T_h)$
    **if** $i = L$ **then**
        $h = h - 1$ // Move tape head to the left
    **else**
        $h = h + 1$ // Move tape head to the right
    **end**
**end**

# What happens when it stops?

When we reach a point that $\delta(q, T_h)$ undefined, the machine has *halted*. What happens next depends on the state the machine stopped in.

**if** $q \in F$ **then**
  **accept**
  **return** $T$ `// This is thought of as the output of` $M(w)$
**else**
  **reject**
**end**

# What can we store in a machine's memory?

- Integers

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- Integers
- Rational numbers

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- Integers
- Rational numbers
- Floating point numbers
- Boolean (True/False) statements
- Text
- Other machines

# Universal Turing Machines

Turing showed in his PhD thesis that we could represent any *TM* after any number of transitions – including current state, tape contents and position of tape head – as an integer.

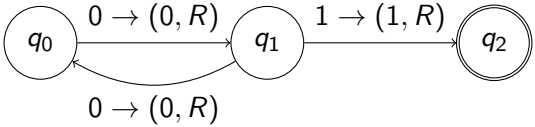
Not only that, but we could manipulate this integer such that it matched performing the next step of the computation.

This gave way to Universal Turing Machines; machines capable of running any *TM* given to them.

# The Church-Turing Thesis

"[A]ll effectively calculable sequences are computable", *Turing*

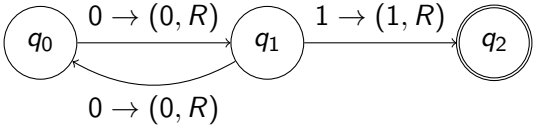
# Quiz Time!




$q_0$ is the start state, and $q_2$ is the accept state.

Question: Does $M(01)$ **accept** or **reject**?

# Quiz Time!

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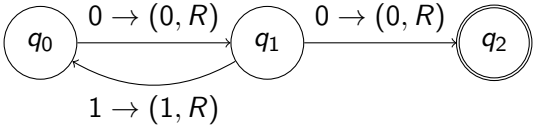


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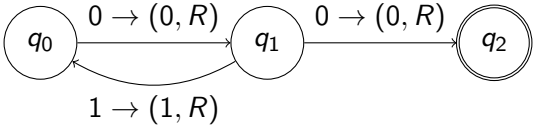
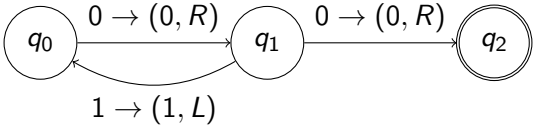



$q_0$ is the start state, and $q_2$ is the accept state.

Answer: $M(01)$ **reject**s!

# Quiz Time!

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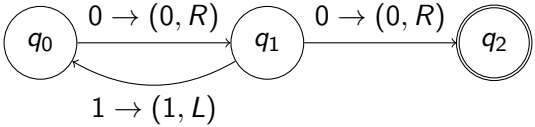


$q_0$ is the start state, and $q_2$ is the accept state.

Answer: $M(w)$ doesn't halt.

# The halting problem

You might think it would be useful if we could tell when a machine was going to halt.

Formally, we would want a *TM* $H$, such that given $M \in TM$ and $w \in \Sigma_M^*$:

- $H(M, w)$ halts in the **accept** state if $M(w)$ halts and
- $H(M, w)$ halts in the **reject** state if $M(w)$ does not halt.

Sadly, Turing proved that such a machine is impossible.

There are many other unsolvable problems as well, within the area of **computability theory**. We will not cover this area, but some reading on the subject is suggested at the end.

# Nondeterminism: Spot the difference!

A Nondeterministic Turing Machine ($NTM$) is specified as a tuple of seven components $\langle Q, \Gamma, b, \Sigma, q_0, F, \delta \rangle$:

- $Q$ is the set of all possible states
- $\Gamma$ is tape alphabet
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- $F \subseteq Q$ is the set of accepting states
- $\delta : Q \times \Gamma \to (Q \times \Sigma \times \{L, R\})^*$ is the transition function

# What, what's the difference?

*NTM*s are different because of the transition function.

In deterministic *TM*s, the transition goes from one machine setup to another.

In *NTM*s, the transition function goes from one to many setups.

These setups are run simultaneously, and the machine accepts if one setup halts in an accepting state, or rejects if all setups halt in the rejecting state.

# Computation Tree

# Power of Nondeterminism

Note that a *TM*s only transition from one machine setup to another, while *NTM*s transition from one to many.

From this, we can conclude that any *TM* is by definition also an *NTM*.

Question: Do *NTM*s violate the Church-Turing Thesis? Or put another way, is there any problem that can be solved by an *NTM* that cannot be solved by a *TM*?

# Using *TM*s to simulate *NTM*s

Recall the computation tree:

We can use a technique called Breadth-First Search to search every branch until we find one that halts in an **accept** state.

Note that this does not mean we can simulate NTMs easily...

# Describing problems as *languages*

A language $L$ is a (potentially infinite) set.

Example languages:

- Text strings that contain the word "Hello"
- Satisfiable boolean expressions
- Eulerian graphs
- Hamiltonian graphs
- The halting problem

# Decidable languages

A language $L$ is *decidable* if there exists a machine $M$ such that:

- $\forall w \in L, M(w)$ **accepts**
- $\forall w \notin L, M(w)$ **rejects**

# Verifiable languages

A language $L$ is *verifiable* if there exists a machine $V$ such that:

- $\forall w \in L, \exists c$ s.t. $V(w, c)$ **accepts**
- $\forall w \notin L, V(w, c)$ **rejects** $\forall c$

# Performance of a machine

Problem: We want to talk about how much time it takes for a machine to solve a problem.

Solution: Assume $\delta$ takes a constant amount of time to run and count the number of times we call $\delta$.

To remain general, we will focus on how the number of times we call $\delta$ scales as the input becomes larger.

We will focus on how the machine performs on the worst case input.

# Formal definitions of time complexity

Let $\mathrm{TIME}(M, w)$ be the number of times machine $M$ is called on input $w$.

$$T_M(n) = \max_{w \in \Sigma^n} (\mathrm{TIME}(M, w))$$

# Problem: $T_M(n)$ might be complicated to work out

```
while ((1 << (i + 1)) < m_max) {
    matcher->rows[i].row_size = 1 << i;
    lookup_size = 0;
    for (j = 0; j < num_patterns; j++) {
        if ((periods[j] > num_patterns) && (m[j] > num_patterns << 1) && (m[j] - num_patterns > matcher->rows[i].row_size <
            set_fingerprint(matcher->printer, &P[j][matcher->rows[i].row_size], matcher->rows[i].row_size, matcher->tmp);
            fingerprint_concat(matcher->printer, old_patterns[prev_row[j]], matcher->tmp, patterns[lookup_size]);
            prev_row[j] = lookup_size;
            if (m[j] - num_patterns <= (matcher->rows[i].row_size << 2)) {
                end_pattern[lookup_size] = num_progressions;
                set_fingerprint(matcher->printer, &P[j][matcher->rows[i].row_size << 1], m[j] - num_patterns - (matcher->ro
                fingerprint_concat(matcher->printer, patterns[lookup_size], matcher->tmp, prefix[num_prefixes]);
                progression_index[num_prefixes] = num_progressions;
                prefix_length[num_prefixes] = m[j] - num_patterns;
                num_suffixes[num_prefixes] = 1;
                set_fingerprint(matcher->printer, &P[j][m[j] - num_patterns], num_patterns, suffixes[num_prefixes][0]);
                for (k = 0; k < num_prefixes; k++) {
                    if (fingerprint_equals(prefix[k], prefix[num_prefixes])) {
                        end_pattern[lookup_size] = progression_index[k];
                        for (l = 0; l < num_suffixes[k]; l++) {
                            if (fingerprint_equals(suffixes[k][l], suffixes[num_prefixes][0])) {
                                break;
                            }
                        }
                        if (l == num_suffixes[k]) {
                            fingerprint_assign(suffixes[num_prefixes][0], suffixes[k][l]);
                            num_suffixes[k]++;
                        }
                        break;
                    }
                }
                if (k == num_prefixes) num_prefixes++;
            } else {
                end_pattern[lookup_size] = -1;
            }
            for (k = 0; k < lookup_size; k++) {
                if (fingerprint_equals(patterns[k], patterns[lookup_size])) {
                    prev_row[j] = k;
                    if ((end_pattern[k] == -1) && (end_pattern[lookup_size] != -1)) {
                        end_pattern[k] = end_pattern[lookup_size];
                        num_progressions++;
                    }
                    else if (end_pattern[lookup_size] != -1) progression_index[num_prefixes - 1] = end_pattern[k];
                    break;
                }
            }
            if (k == lookup_size) {
                lookup_size++;
                if (end_pattern[lookup_size -1] != -1) num_progressions++;
```

2

[2] https://github.com/djmylt/dict_matching/blob/master/dict_matching.h

# Solution: Approximate!

Let $f$ and $g$ be functions over the real numbers. We say that:

$$f(n) \in O(g(n)) \text{ iff } \exists c > 0, n_0 \geq 0 \text{ s.t. } \forall n \geq n_0, f(n) \leq c \cdot g(n)$$

$$f(n) \in \Omega(g(n)) \text{ iff } \exists c > 0, n_0 \geq 0 \text{ s.t. } \forall n \geq n_0, f(n) \geq c \cdot g(n)$$

$$f(n) \in \Theta(g(n)) \text{ iff } \exists c_1, c_2 > 0, n_0 \geq 0 \text{ s.t. } \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

# Examples

Prove each of the following statements:

$$n \in O(n^2)$$
$$2^n \in \Omega(n^2)$$
$$n \in \Theta(2n)$$

# Examples

Prove each of the following statements:

$$n \in O(n^2) : c = 1, n_0 = 0$$

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# Examples

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# Examples

Prove each of the following statements:

$$n \in O(n^2) : c = 1, n_0 = 0$$

$$2^n \in \Omega(n^2) : c = 1, n_0 = 1$$

$$n \in \Theta(2n) : c_1 = 0.5, c_2 = 1, n_0 = 0$$

# Polynomial Time

We are particularly interested in machines that scale efficiently as the size of the problem increases.

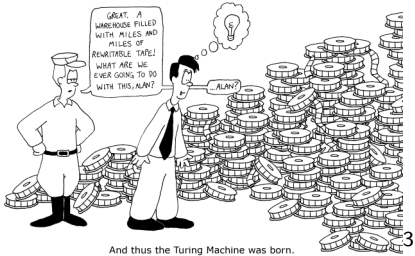
For this, we are going to focus on polynomial time complexity:

$$\mathrm{POLY}(n) = \bigcup_{k=0}^{\inf} O(n^k)$$

# Summary of part one

- What is a computer? *Deterministic Turing Machine, Non-Deterministic Turing Machine*
- What is a problem? *Deciding if a word is in a language, verifying that a word is in a language*
- How do we measure difficulty? *Upper bound of time for an input of length $n$*

# End of part one




And thus the Turing Machine was born.

[3]

[3] http://www.cs.utah.edu/~draperg/cartoons/2005/turing.html

# Structure of part two

- Putting it all together!
- ...only to get another (very difficult) problem.
- How might we try to solve this new problem?

# Complexity Classes

Now that we have provided our definitions for a computer, a problem and performance, we can look at what problems can be solved under these restrictions.

These are called *complexity classes*.

# Deterministic polynomial time

A language $L$ can be solved in deterministic polynomial time iff $\exists$ *TM* $M$ such that:

- $M$ decides $L$
- and $T_M(n) \in \mathrm{POLY}(n)$

We shall refer to the set of these problems as $P$.

# Non-deterministic polynomial time

A language $L$ can be solved in non-deterministic polynomial time iff $\exists NTM\ M$ such that:

- $M$ decides $L$
- and $T_M(n) \in \mathrm{POLY}(n)$

We shall refer to the set of these problems as $NP$.

# Deterministic verification

Another definition for NP can be written based off of verification:

A language $L$ can be verified in deterministic polynomial time iff $\exists TM\ V$ such that:

- $V$ decides $L$
- and $T_V(n) \in \mathrm{POLY}(n)$
- and $|c| \in O(p(n))$ for some polynomial function $p$

# Non-determinism to verification

Recall the computation tree for an $NTM$.

We label each branch with some integer.

Our certificate $c$ is now the polynomial length sequence of integers that lead to the accepting state.

# Verification to non-determinism

We use the $NTM$ to brute-force guess the certificate. Since the certificate is polynomial-length this will take a polynomial number of steps.

We run $V(w, c)$ for every possible $c$ and accept if one instance of $V$ accepts. Since $V$ runs in polynomial time, this also takes a polynomial number of steps.

# Exercise Left For the Student

Does $P = NP$?

# The $P$ versus $NP$ problem

Arguably first proposed by Gödel in a letter to von Neumann in 1956.[4]

First stated formally by Cook in 1971.[5]

Solving the problem will earn you a million dollars, courtesy of the Clay Mathematics Institute.[6]

[4] https://ecommons.cornell.edu/bitstream/handle/1813/6910/89-994.pdf

[5] http://dl.acm.org/citation.cfm?coll=GUIDE&dl=GUIDE&id=805047

[6] http://www.claymath.org/millennium-problems/p-vs-np-problem

# The easy side: $P \subseteq NP$

Recall that any $TM$ is by definition non-deterministic.

Likewise, any polynomial-time $TM$ is also non-deterministic.

Hence $P \subseteq NP$.

# The easy side: $P \subseteq NP$

Alternative proof (using verification):

Let $TM$ $M$ decide $L$ in polynomial time. Define $V$ as follows:

$V(w, c)$ :
**if** $M(w)$ *accepts* **then**
 | **accept**
**end**
**else**
 | **reject**
**end**

$V$ verifies $L$ in polynomial time. Hence $P \subseteq NP$.

# The harder side: Is $NP \subseteq P$

Another way to think of this problem: *If a problem can be easily verified, can it be easily solved?*

# How might we answer this question?

Why not look at the hardest problems in $NP$?

If $P = NP$, then even the hardest problems in $NP$ will be solvable in polynomial time.

And if $P \subset NP$, then these are the problems that won't have a polynomial time solution, as could be checked by lower-bound analysis.

But how can we determine the hardest problems in $NP$?

# Polynomial time reducible

Take two languages $A$ and $B$. We say that $A \leq_p B$ iff $\exists TM\ M$ such that:

- $\forall w \in A, M(w) \in B$
- and $\forall w \notin A, M(w) \notin B$
- and $T_M(n) \in \mathrm{POLY}(n)$

Note that this property is transitive: $A \leq_p B$ and $B \leq_p C \rightarrow A \leq_p C$

# $NP$-Complete

A language $A$ is $NP$-Complete iff:

- $A \in NP$
- and $\forall B \in NP, B \leq_p A$

If one $NP$-Complete language is proven to be in $P$, then every $NP$ problem is also in $P$.

# Problem: There are lots of $NP$ problems

Solution: Generalise!

We know that any problem in $NP$ can be decided in polynomial time by an $NTM$.

So can we convert an $NTM$ to some other problem?

# Satisfiable boolean formulae

Take a boolean formula $f = a \vee b \wedge \bar{c}$.

We say that $f$ is *satisfiable* if we can assign 0 or 1 to each variable such that $f = 1$.

Examples:

- $f$ is satisfiable: $(a = 1, b = 1, c = 0)$
- but $f' = x \wedge \bar{x}$ is not satisfiable

We call $SAT$ the language of all satisfiable boolean formulae.

# $SAT \in NP$

$SAT$ can be verified in polynomial time:

- Make the certificate the values we assign to each variable.
- Have V substitute the value for each variable into the formula.
- **accept** if the formula evaluates to 1, otherwise **reject**.

$SAT \in NP$-Complete

# Proving other *NP*-Complete problems

Now that we have one *NP*-Complete problem, proving others is a lot easier.

Recall that polynomial-time reducibility is transitive.

So if $A$ is *NP*-Complete, then $B$ is *NP*-Complete if:

- $B \in NP$
- and $A \leq_p B$

# Other *NP*-Complete problems

- $3SAT$

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- Hamiltonian graphs

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# Other *NP*-Complete problems

- 3*SAT*
- Hamiltonian graphs
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- Super Mario Bros
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- Candy Crush Saga

# Other *NP*-Complete problems

- *3SAT*
- Hamiltonian graphs
- Cliques
- Knapsack
- Longest path
- Tetris
- Lemmings
- Super Mario Bros
- Bejeweled
- Candy Crush Saga
- Flood-It

# Summary of part two

- Putting it all together! *Complexity classes, $P$, $NP$*
- ...only to get another (very difficult) problem. *Are easy to verify problems easy to solve?*
- How might we try to solve this new problem? *NP-Complete problems*

# What else is there?

Recall our three questions from part one:

- What is a computer?
- What is a problem?
- How do we measure difficulty?

What if we answered these differently?

# How do we measure difficulty?

Exponential time: $EXP$

Linear time: $LIN$

Space complexity: $PSPACE$, $EXPSPACE$

Sublinear working space: $L$

# What is a problem?

Computational problems: $NP$-Hard

Counting problems: $\#P$

Complementary problems: co-$NP$

# What is a computer?

Probabilistic Turing Machines: $BPP, RP$

Postselection: Post-$BPP$

Parallel Computing: $NC$

Talking to another, more powerful computer: $MA, IP$

Oracles: $\Delta_i P, \Pi_i P, \Sigma_i P, PH$

Advice: $P/poly, P/log$

Quantum computers: $EQP, BQP$

Time travel: $P_{CTC}$

# This is only the beginning

There are many more complexity classes out there, and very quickly relating them in a simple equation like this:

$$P \subseteq NP$$

Becomes this:

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE = NPSPACE = IP = P_{CTC} \subseteq EXP \subseteq EXPSPACE$$

# Open problems in complexity theory

- Does $P = NP$?

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- Does $P = NP$?
- Does $L = NL$?

# Open problems in complexity theory

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# Open problems in complexity theory

- Does $P = NP$?
- Does $L = NL$?
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- Does $NC = P$?

# Open problems in complexity theory

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- And lots more...

# The end

PROOF:

$e^{i \cdot P_i} = -1$

And,

$P_i = P \cdot i$

So,

$e^{i \cdot P_i} = e^{P \cdot i \cdot i} = e^{-P}$

So,

$e^{-P} = -1$

Squaring both sides,

$e^{-2P} = 1$

Which leaves

$P = 0$

Thus,

$P = NP$

QED

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[7] http://www.smbc-comics.com/?id=3919

# The end


[8]

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# Suggested books

- *Quantum Computing Since Democritus* by Scott Aaronson offers a broad overview of many complexity classes
- *Introduction to the Theory of Computation* by Michael Sipser is the recommended textbook for most computability and complexity theory courses

# Suggested papers

- *On Computable Numbers, with an Application to the Entscheidungsproblem* by Alan Turing is Turing's PhD thesis, which provides the original definition of Turing Machines, a formal definition of the Universal Turing Machine, and a proof that the Halting problem is undecidable
- *The Complexity of Theorem Proving Procedures* by Stephen Cook provides the proof that $SAT \in NP$-Complete
- *Reducibility Among Combinatorial Problems* by Richard Karp provides 21 $NP$-Complete problems

# Suggested websites

- `https://complexityzoo.uwaterloo.ca/` is a Wiki originally developed by Scott Aaronson, which provides a list of every complexity class ever stated