QECDT Topics Presentation: Complexity theory

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Complexity theory in a nutshell

"How hard can it be?", *Clarkson*

What is complexity theory?

Complexity theory is the study of how difficult it is to solve a problem with a computer.

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How do we measure difficulty?

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What is a problem?

What is complexity theory?

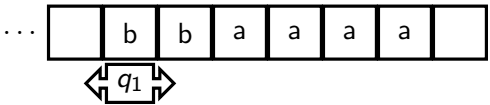
Complexity theory is the study of how difficult it is to solve a problem with a computer.

What is a computer?

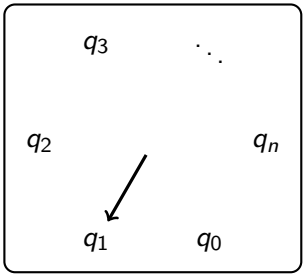
Structure of part one

- What is a computer?
- What is a problem?
- How do we measure difficulty?

Our model of a computer



Machine Tape (RAM) with head at letter b



Set of States (CPU) with current state q_0

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[1] Original version at `http://www.texample.net/tikz/examples/turing-machine-2/`

Formal definition of a computer

A Turing Machine (TM) is specified as a tuple of seven components $\langle Q, \Gamma, b, \Sigma, q_0, F, \delta \rangle$:

- Q is the set of all possible states
- Γ is tape alphabet
- $b \in \Gamma$ is the blank symbol for the tape
- $\Sigma = \Gamma / b$ is the input alphabet for the tape
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accepting states
- $\delta : Q \times \Gamma \rightarrow Q \times \Sigma \times \{L, R\}$ is the transition function

But how do we run it?

The majority of computation time is spent repeating the following loop. Note that T_h is the h-th cell of the tape.

$q = q_0$
$h = 0$
$T = w$ // Tape starts as just input, followed by blank cells
while $\delta(q, T_h)$ *is not undefined* **do**
    $(q, h, i) = \delta(q, T_h)$
    if $i = L$ **then**
        $h = h - 1$ // Move tape head to the left
    else
        $h = h + 1$ // Move tape head to the right
    end
end

What happens when it stops?

When we reach a point that $\delta(q, T_h)$ undefined, the machine has *halted*. What happens next depends on the state the machine stopped in.

if $q \in F$ **then**
  accept
  return T `// This is thought of as the output of` $M(w)$
else
  reject
end

What can we store in a machine's memory?

- Integers

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- Integers
- Rational numbers

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- Integers
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- Floating point numbers

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What can we store in a machine's memory?

- Integers
- Rational numbers
- Floating point numbers
- Boolean (True/False) statements
- Text
- Other machines

Universal Turing Machines

Turing showed in his PhD thesis that we could represent the current state of any *TM* – including current state, tape and position of tape head – as an integer.

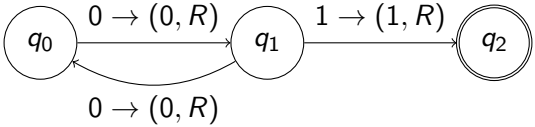
Not only that, but we could manipulate this integer such that it matched performing the next step of the computation.

This gave way to Universal Turing Machines; machines capable of running any *TM* given to them.

The Church-Turing Thesis

"[A]ll effectively calculable sequences are computable", *Turing*

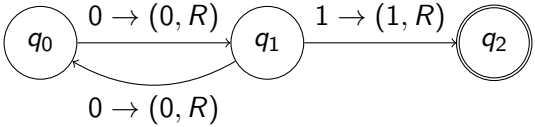
Quiz Time!



q_0 is the start state, and q_2 is the accept state.

Question: Does $M(01)$ **accept** or **reject**?

Quiz Time!



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Answer: $M(01)$ **accept**s!

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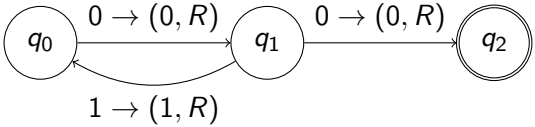

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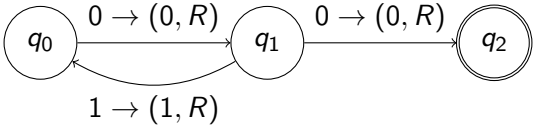

q_0 is the start state, and q_2 is the accept state.

Answer: $M(01)$ **reject**s!

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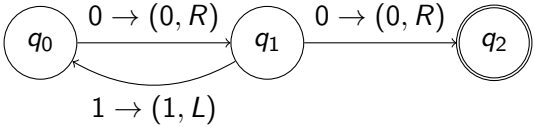

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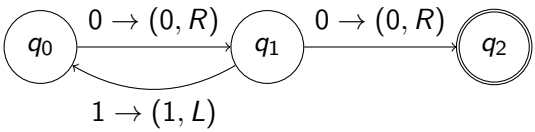

q_0 is the start state, and q_2 is the accept state.

Answer: $M(w)$ doesn't halt. This can be interpreted as rejection, depending on how the problem is phrased.

The halting problem

You might think it would be useful if we could tell when a machine was going to halt.

Formally, we would want a *TM* H, such that given $M \in TM$ and $w \in \Sigma_M^*$:

- $H(M, w)$ halts in the **accept** state if $M(w)$ halts and
- $H(M, w)$ halts in the **reject** state if $M(w)$ does not halt.

Sadly, Turing proved that such a machine is impossible.

There are many other unsolvable problems as well, within the area of **computability theory**. We will not cover this area, but some reading on the subject is suggested at the end.

Nondeterminism: Spot the difference!

A Nondeterministic Turing Machine (NTM) is specified as a tuple of seven components $\langle Q, \Gamma, b, \Sigma, q_0, F, \delta \rangle$:

- Q is the set of all possible states
- Γ is tape alphabet
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- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accepting states
- $\delta : Q \times \Gamma \to (Q \times \Sigma \times \{L, R\})^*$ is the transition function

What, what's the difference?

*NTM*s are different because of the transition function.

In deterministic *TM*s, the transition goes from one machine setup to another.

In *NTM*s, the transition function goes from one to many setups.

These setups are run simultaneously, and the machine accepts if one setup halts in an accepting state, or rejects if all setups halt in the rejecting state.

Computation Tree

Power of Nondeterminism

Note that a *TM*s only transition from one machine setup to another, while *NTM*s transition from one to many.

From this, we can conclude that any *TM* is by definition also an *NTM*.

Question: Do *NTM*s violate the Church-Turing Thesis? Or put another way, is there any problem that can be solved by an *NTM* that cannot be solved by a *TM*?

Using *TM*s to simulate *NTM*s

Recall the computation tree:

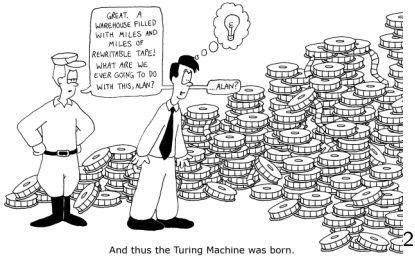
We can use a technique called Breadth-First Search to search every branch until we find one that halts in an **accept** state.

Note that this does not mean we can simulate NTMs easily...

Summary of part one

- What is a computer? *Deterministic Turing Machine, Non-Deterministic Turing Machine*
- What is a problem? *Deciding if a word is in a language, verifying that a word is in a language*
- How do we measure difficulty? *Upper bound of time for an input of length n*

End of part one



And thus the Turing Machine was born.

[2]

[2] http://www.cs.utah.edu/~draperg/cartoons/2005/turing.html

Structure of part two

- Putting it all together!
- ...only to get another (very difficult) problem.
- How might we try to solve this new problem?

Pop quiz!

Does $P = NP$?

The P versus NP problem

Arguably first proposed by Gödel in a letter to von Neumann in 1956.[3]

First stated formally by Cook in 1971.[4]

Solving the problem will earn you a million dollars, courtesy of the Clay Mathematics Institute.[5]

[3] https://ecommons.cornell.edu/bitstream/handle/1813/6910/89-994.pdf

[4] http://dl.acm.org/citation.cfm?coll=GUIDE&dl=GUIDE&id=805047

[5] http://www.claymath.org/millennium-problems/p-vs-np-problem

The easy side: $P \subseteq NP$

Recall that any TM is by definition non-deterministic.

Likewise, any polynomial-time TM is also non-deterministic.

Hence $P \subseteq NP$.

The easy side: $P \subseteq NP$

Alternative proof (using verification):

Let $TM\ M$ decide L in polynomial time. Define V as follows:

$V(w, c)$:
if $M(w)$ *accepts* **then**
 | **accept**
end
else
 | **reject**
end

V verifies L in polynomial time. Hence $P \subseteq NP$.

The harder side: Is $NP \subseteq P$

Another way to think of this problem: *If a problem can be easily verified, can it be easily solved?*

How might we answer this question?

Why not look at the hardest problems in NP?

If $P = NP$, then even the hardest problems in NP will be solvable in polynomial time.

And if $P \subset NP$, then these are the problems that won't have a polynomial time solution, as could be checked by lower-bound analysis.

But how can we determine the hardest problems in NP?

Summary of part two

- Putting it all together! P, NP
- ...only to get another (very difficult) problem. *Are easy to verify problems easy to solve?*
- How might we try to solve this new problem? *NP-Complete problems*

What else is there?

Recall our three questions from part one:

- What is a computer?
- What is a problem?
- How do we measure difficulty?

What if we answered these differently?

What is a computer?

Probabilistic Turing Machines: BPP, RP

Parallel Computing: NC

Talking to another, more powerful computer: MA, IP

Quantum computers: EQP, BQP

Time travel: P_{CTC}

What is a problem?

Computational problems: *NP-Hard*

Counting problems: $\#P$

Complementary problems: *co-NP*

How do we measure difficulty?

Exponential time: EXP

Linear time: LIN

Space complexity: $PSPACE$, $EXPSPACE$

Sublinear working space: L

This is only the beginning

There are many more complexity classes out there, and very quickly relating them in a simple equation like this:

$$P \subseteq NP$$

Becomes this:

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE = NPSPACE = IP = P_{CTC} \subseteq EXP \subseteq EXPSPACE$$

The end

PROOF:

$e^{i \cdot P_i} = -1$

And,

$P_i = P \cdot i$

So,

$e^{i \cdot P_i} = e^{P \cdot i \cdot i} = e^{-P}$

So,

$e^{-P} = -1$

Squaring both sides,

$e^{-2P} = 1$

Which leaves

$P = 0$

Thus,

$P = NP$

QED

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[6] http://www.smbc-comics.com/?id=3919

The end

[7]

[7] http://www.smbc-comics.com/?id=3919