AM129: Fortran Advection-Diffusion Report

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Abstract

In this script we aim to solve the linear diffusion and advection equations using finite difference schemes. The schemes utilized for the diffusion equation is the second-order central difference scheme, while for the advection equation we use the forward time backward space, forward time center space, and the lax-wendroff methods.

Methods

To use the code, refer to "/Project/README.md" Here we have the linear advection-diffusion partial-differential equation:

$$u_t + au_x = \kappa u_{xx} \tag{1}$$

For diffusion we use the finite difference method:

- Second-Order Central Difference Scheme:
  $u_i^{n+1} = u_i^n + \kappa \frac{\Delta t}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$

For advection we use the finite difference methods:

- Upwind for $a > 0$ (FTBS - Forward Time Backward Space):
  $u_i^{n+1} = u_i^n - \frac{a\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$
- Centered for any a (FTCS - Forward Time Center Space):
  $u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$
- Lax-Wendroff (LW for any a:
  $u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2}(\frac{a\Delta t}{\Delta x})^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$

1.1.1. 1D Diffusion

We let a = 0, then we have the classical heat equation (or diffusion equation) of the form:

$$u_t = \kappa u_{xx} \tag{2}$$

with $\kappa > 0$

Initial Condition

We have initial condition:

$$u_0(x) = \begin{cases} 0°\text{F}, & x \in [0, 1) \\ 100°\text{F}, & x = 1 \end{cases}$$

Boundary Conditions

- Temperature u at 0 at the left boundary, $u(0, t) = 0$
- Temperature u at 100 at the right boundary, $u(1, t) = 100$ for $t > 0$

Material Properties

Let $\kappa = 1.156 cm^2/sec$, the value for copper.

1.1.2. 1D Advection

We let $\kappa = 0$. then we have the linear advection equation:

$$u_t + a u_x = 0 \tag{3}$$

We use an advection velocity of $a = 1$

Initial Conditions

- Smooth continuous initial profile:

$$u(x, 0) = \sin(2\pi x) \tag{4}$$

- Discrete initial profile:

$$u_0(x) = \begin{cases} -1, & x \in [0, 1/3) \cup x \in (2/3, 1] \\ 1, & x \in [1/3, 2/3] \end{cases}$$

Boundary Conditions:

$$u_0^n = u_N^n \tag{5}$$
$$u_{N+1}^n = u_1^n \tag{6}$$

Diffusion Questions:

(a) Find $\mathbf{t_{max}}$

We find a time t_{max} at which the temperature of the material reaches a steady state. We define our tolerance as:

$$\|\frac{\Delta u^n}{\Delta t}\| \equiv \frac{1}{N\Delta t}\sum_{i=1}^{N}|u_i^n - u_i^{n-1}| < \epsilon = 10^{-6} \tag{7}$$

- N = 32, C = 1.0:
  For this case, we have $t_{max} \approx 1.8547$ achieved over 4,391 timesteps. We write data files for $t/t_{max} = 0, 0.2, 0.5, 0.8,$ and 1 in these respective files:

  diff_0_N=32.dat, diff_20_N=32.dat, diff_50_N=32.dat, diff_80_N=32.dat, diff_100_N=32.dat

- N = 128, C = 1.0:
  For this case, we have $t_{max} \approx 1.7753$ achieved over 67,249 timesteps. We write data files for $t/t_{max} = 0, 0.2, 0.5, 0.8,$ and 1 in these respective files:

  diff_0_N=128.dat, diff_20_N=128.dat, diff_50_N=128.dat, diff_80_N=128.dat, diff_100_N=128.dat

(b) Is there any difference in solution between the two different grid resolutions?

For these two grid resolutions, we see that a larger gridsize will require less time to reach steady state.

(c) What happens if your $\mathbf{\Delta t_{diff}}$ f fails to satisfy the CFL condition?

If we vary the CFL condition, our solution would diverge since it is necessary condition for stability.

(d) What is your value of $\mathbf{t_{max}}$(sec) for $\kappa = 1.156$? What happens when κ is increased or decreased by a factor of 10?

- For $\kappa = .1156$, we have $t_{max} = 15.7039$ found in 59,486 timesteps.
- For $\kappa = 11.56$, we have $t_{max} = 0.198$ found in 75,013 timesteps.

Thus for larger diffusion coefficients it follows that we reach steady state faster since heat is diffused more quickly over the space.

Advection Questions:

(e) Find a time analytically at which the solution returns to it's initial condition

We solve the PDE and find that total time is 1.

(f) Use two grid resolutions of 32 and 128. Identify which schemes work the best given the discontinuous initial condition.

For both the continuous and discrete case we find that Lax-Wendroff is the best method, followed by the Forward-Time-Backward-Space Method, while the Forward-Time-Center-Space diverges.

(g) using $\mathbf{\Delta t_{adv} = 0.9\Delta x/a}$ and $\mathbf{\Delta t_{adv} = 1.2\Delta x/a}$

We see that with a large Δt_{adv}

1.2. Python Implementation

Plots for Q(a)

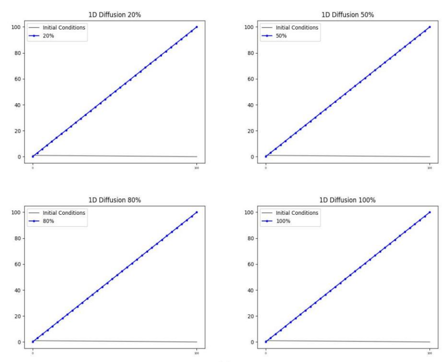

Figure 1: Solutions of Diffusion Equation with Nx = 32 and CFL C = 1.0

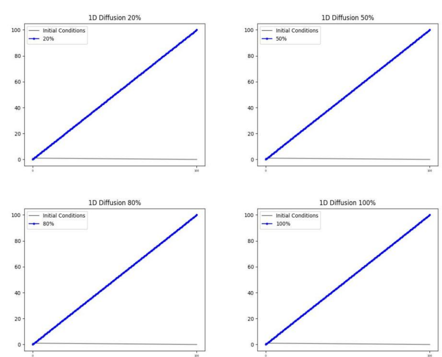

Figure 2: Solutions of Diffusion Equation with Nx = 128 and CFL C = 1.0

Plots for Q(f):

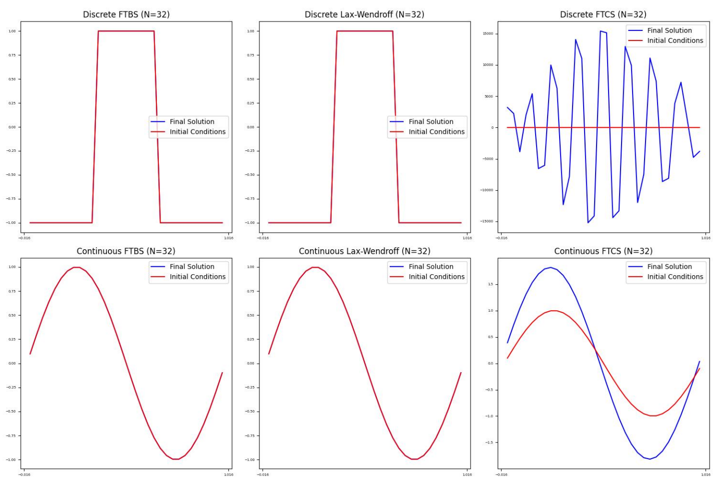

Figure 3: Advection Finite Difference Schemes for N = 32, C = 1.0

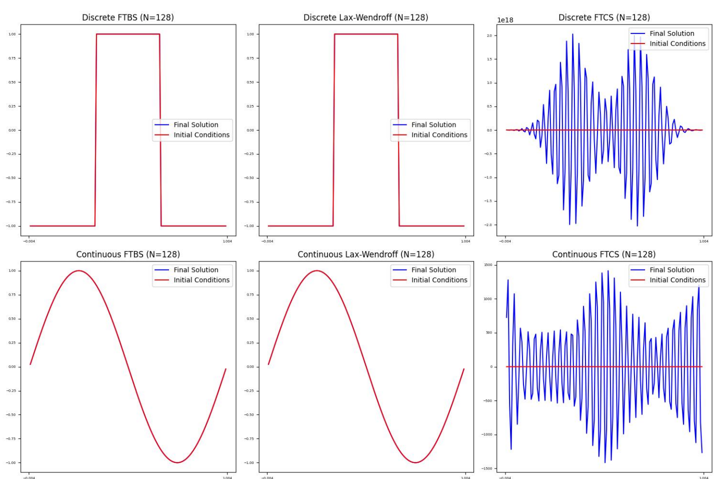

Figure 4: Advection Finite Difference Schemes for N = 128, C = 1.0

Plots for Q(g):

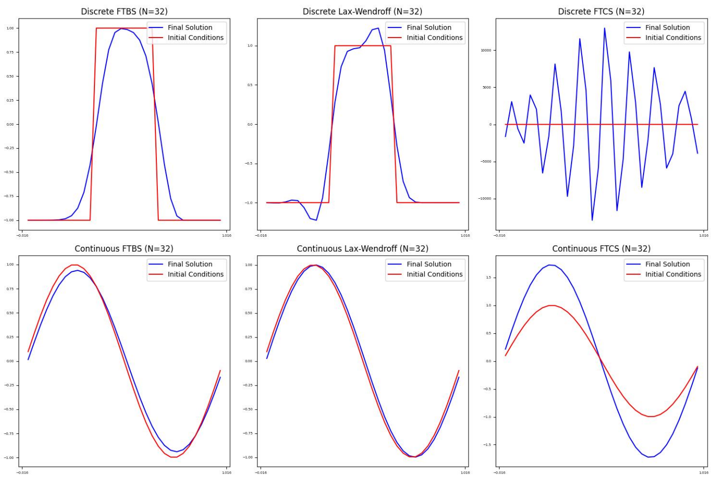

Figure 5: Schemes for N = 32, CFL = 0.9

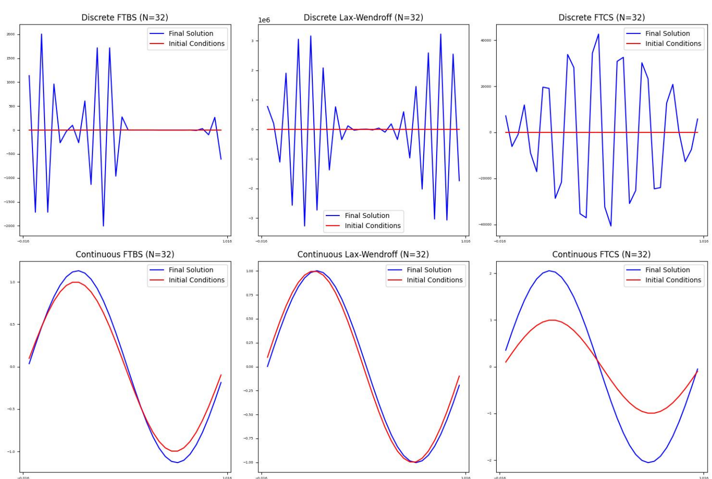

Figure 6: Schemes for N = 32, CFL = 1.2

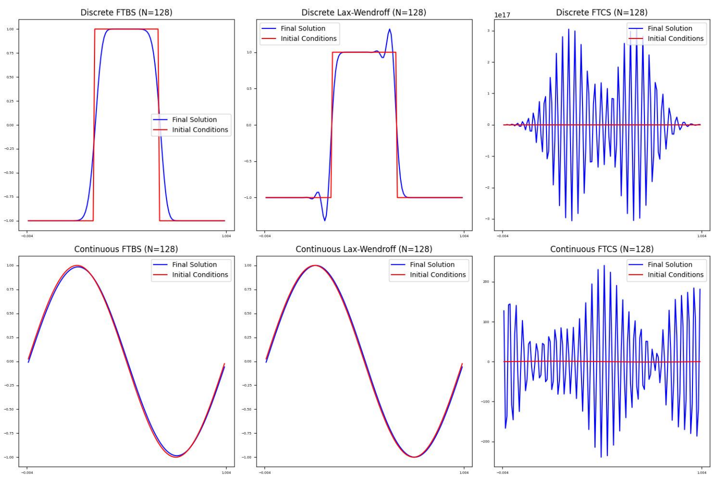

Figure 7: Schemes for N = 128, CFL=0.9

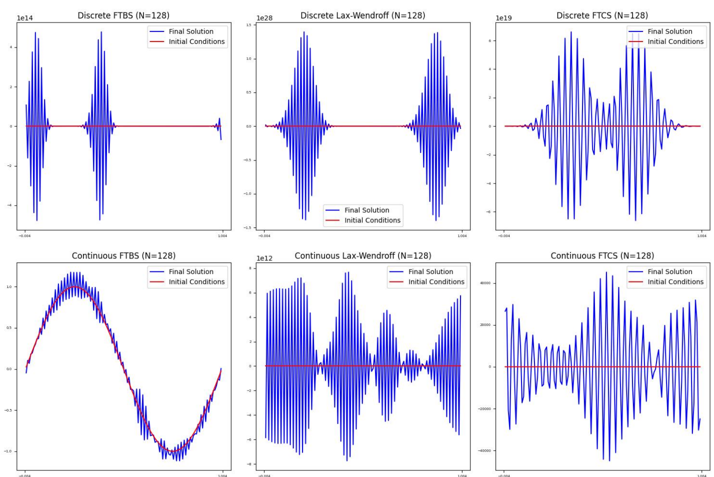

Figure 8: Schemes for N = 128, CFL=1.2