Q1 PART I

$$\boxed{x^2\frac{\partial^2 f}{\partial x^2}+x\frac{\partial f}{\partial x}=-\lambda f,\quad f(1)=f(2)=0}$$

(1) SHOW IT'S A REGULAR SL PROBLEM & PUT INTO SL FORM.

$x^2\frac{\partial^2 f}{\partial x^2}+x\frac{\partial f}{\partial x}+\lambda f=0$ $\qquad a(x)=x^2,\ b(x)=x,\ c(x)=\lambda$

$x\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}+\frac{\lambda}{x}f=0$ $\qquad p(x)=e^{\int\frac{1}{x}}=x$

$\frac{\partial}{\partial x}\left[x\frac{\partial f}{\partial x}\right]+\frac{\lambda}{x}f=0$ $\qquad r(x)=\frac{1}{x}$

$\boxed{\frac{\partial}{\partial x}\left[x\frac{\partial f}{\partial x}\right]=-\frac{\lambda}{x}f\quad \text{(SL FORM)}}$ $\qquad q(x)=\frac{1}{x}\lambda$

FURTHER IT'S REGULAR BECAUSE IT'S BOUNDED & $p(1)=1,\ p(2)=2,\ r(1)=1,\ r(2)=\frac{1}{2}$ SO $r(x)$ & $p(x)$ DO NOT VANISH AT THE BOUNDARIES

(2) FIND THE EIGENFCNS & EIGENVALUES

WE HAVE

$\frac{\partial}{\partial x}\left[x\frac{\partial f}{\partial x}\right]=-\frac{\lambda}{x}f$

$x\frac{\partial^2 f}{\partial x}+\frac{\partial f}{\partial x}+\frac{\lambda}{x}f=0$ (CAUCHY-EULER BY WOLFRAM)

SUBS $f(x)=x^a$ INTO GOV. EQN

$x\frac{\partial^2}{\partial x^2}(x^a)+\frac{\partial}{\partial x}(x^a)+\frac{\lambda}{x}x^a=0$ $\qquad \frac{\partial}{\partial x}(x^a)=ax^{a-1}$

$x\left[a(a-1)(x^{a-2})\right]+ax^{a-1}+\frac{\lambda}{x}x^a=0$ $\qquad \frac{\partial^2}{\partial x}(x^a)=a(a-1)(x^{a-2})$

$x\left[(a^2-a)(x^{a-2})\right]+ax^{a-1}+\lambda x^{a-1}=0$

$(a^2-a)x^2+ax+\lambda x=0$

$a^2x^a-ax^a+ax^a+\lambda x^a=0$

$\cancel{x^a}(a^2-\cancel{a}+\cancel{a}+\lambda)=0$

$a^2+\lambda=0\rightarrow a=\sqrt{-\lambda}=\pm i\sqrt{\lambda}\rightarrow f(x)=x^{\pm i\sqrt{\lambda}}=e^{\pm i\sqrt{\lambda}\ln(x)}=c_1e^{i\sqrt{\lambda}\ln(x)}+c_2e^{-i\sqrt{\lambda}\ln(x)}$

$=c_1\left[\cos\sqrt{\lambda}\ln x+i\sin\sqrt{\lambda}\ln x\right]+c_2\left[\cos\sqrt{\lambda}\ln x-i\sin\sqrt{\lambda}\ln x\right]$ (EULER'S)

$=A\cos\sqrt{\lambda}\ln x+B_n\sin\sqrt{\lambda}\ln x$ (BY WOLFRAM)

BCS:

- $f(1)=0$:

$0=A\cos(0)+B\sin(0)\rightarrow A=0\rightarrow f(x)=B\sin(\sqrt{\lambda}\ln x)$

- $f(2)=0$:

$0=B\sin(\sqrt{\lambda}\ln 2)\rightarrow n\pi=\sqrt{\lambda}\ln 2\rightarrow \lambda_n=\left(\frac{n\pi}{\ln 2}\right)^2$

(NOT NORMALIZED)

$\phi_n(x)=B_n\sin\left(\frac{n\pi}{\ln 2}\ln(x)\right)$

$\lambda_n=\left(\frac{n\pi}{\ln 2}\right)$

(3) ORTHOGONALITY RELATIONSHIP $\qquad$ REF UNCW MATH463 "ODE-MAIN" p198

$$\int_1^2\sin\left(\frac{n\pi}{\ln 2}\ln x\right)\sin\left(\frac{n'\pi}{\ln 2}\ln x\right)\left(\frac{1}{x}\right)=\begin{cases}0 & n\neq n'\\ \int_1^2\sin^2\left(\frac{n\pi}{\ln 2}\ln x\right)\frac{1}{x} & n=n'\end{cases}$$

CASE 2: $n\neq n'$:

$\int_1^2\sin^2\left(\frac{n'\pi}{\ln 2}\ln(x)\right)\frac{1}{x}\,dx$ $\qquad u=\ln x\rightarrow du=\frac{1}{x}dx$

$\int_0^{\ln 2}\sin^2\left(\frac{n'\pi}{\ln 2}u\right)du$

$\int_0^{\ln 2}B^2\sin^2\left(\frac{n'\pi}{\ln 2}u\right)du=1$

$B^2\left(\frac{\ln 2}{2}\right)=1$ $\quad$ BY $\int_0^L\sin(n\pi x/L)\sin(n'\pi x/L)=\frac{L}{2}\delta_{nn'},\ L=\ln(2)$

$B=\sqrt{\frac{2}{\ln 2}}$

NORMALIZED EIGENFCN: $\phi_n(x)=\sqrt{\frac{2}{\ln 2}}\sin\left(\frac{n\pi}{\ln 2}\ln(x)\right)$

$$\boxed{\int_1^2\frac{2}{\ln 2}\sin\left(\frac{n\pi}{\ln 2}\ln x\right)\sin\left(\frac{n'\pi}{\ln 2}\ln x\right)\frac{1}{x}=\delta_{nn'}}$$

$\hookrightarrow \int_1^2\frac{2}{\ln 2}\sin^2\left(\frac{n\pi}{\ln 2}\ln x\right)\frac{1}{x}=1$ BY WOLFRAM :D

Q1 PART 2

$\mathcal{L}(\phi_n) = -\lambda_n \sigma \phi_n$

$$\boxed{\begin{gathered} x^2 \frac{\partial^2 G}{\partial x^2} + x \frac{\partial G}{\partial x} = \delta(x - x') \\ G(1,x') = G(2,x') = 0 \end{gathered}}$$

REF UNCW MATH463 "ODE-MAIN" p197

(1) FIND THE GREEN'S FCN W/ EIGENFCN EXPANSION

THM: $\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n$

BY PART 1 $\begin{cases} \phi_n(x) = \sqrt{\frac{2}{\ln 2}} \sin\left(\frac{n\pi}{\ln 2} \ln x\right) \\ \lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 \end{cases}$

$$u_n(x) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{\ln 2}} \sin\left(\frac{n\pi}{\ln 2} \ln x\right)$$

- SUBS. INTO ODE:

$$\mathcal{L}(u_n(x)) = \mathcal{L}(\Sigma c_n \phi_n) = \Sigma c_n \mathcal{L}(\phi_n) = \Sigma c_n(-\lambda_n \sigma \phi_n) = \Sigma c_n\left(-\lambda_n \tfrac{1}{x} \phi_n\right)$$

- PROJECT TO OBTAIN c_n:
  - ORTHOGONALITY CONDITION:

$$\int_1^2 \frac{2}{\ln 2} \sin\left(\frac{n\pi}{\ln 2} \ln x\right) \sin\left(\frac{n'\pi}{\ln 2} \ln x\right) \frac{1}{x} = \delta_{nn'}$$

$$\Sigma c_n(-\lambda_n) \tfrac{1}{x} \phi_n = F(x)$$

$$\Sigma c_n(-\lambda_n) \tfrac{1}{x} \sqrt{\tfrac{2}{\ln 2}} \sin\left(\tfrac{n\pi}{\ln 2} \ln x\right) = F(x)$$

$$\Sigma c_n(-\lambda_n) \tfrac{1}{x} \tfrac{2}{\ln 2} \sin\left(\tfrac{n\pi}{\ln 2} \ln x\right) \sin\left(\tfrac{n'\pi}{\ln 2} \ln x\right) = F(x) \sin\left(\tfrac{n'\pi}{\ln 2} \ln x\right) \sqrt{\tfrac{2}{\ln 2}}$$

$$\int_1^2 \Sigma c_n(-\lambda_n) \tfrac{1}{x} \tfrac{2}{\ln 2} \sin\left(\tfrac{n\pi}{\ln 2} \ln x\right) \sin\left(\tfrac{n'\pi}{\ln 2} \ln x\right) dx = \int_1^2 F(x) \sin\left(\tfrac{n'\pi}{\ln 2} \ln x\right) \sqrt{\tfrac{2}{\ln 2}}\, dx$$

$$-c_n \lambda_n \underbrace{\int_1^2 \tfrac{1}{x} \tfrac{2}{\ln 2} \sin\left(\tfrac{n\pi}{\ln 2} \ln x\right) \sin\left(\tfrac{n'\pi}{\ln 2} \ln x\right) dx}_{\delta_{nn'}} = \sqrt{\tfrac{2}{\ln x}} \int_1^2 F(x) \sin\left(\tfrac{n'\pi}{\ln 2} \ln x\right) dx$$

$$-c_n \lambda_n = \sqrt{\tfrac{2}{\ln 2}} \int_1^2 F(x) \sin\left(\tfrac{n'\pi}{\ln 2} \ln x\right) dx$$

$$c_n = -\frac{1}{\lambda_n} \left(\sqrt{\tfrac{2}{\ln 2}}\right) \int_1^2 F(x') \sin\left(\tfrac{n'\pi}{\ln 2} \ln x'\right) dx'$$

- CONSIDER u:

$$u(x) = \sum_{n=1} c_n \sqrt{\tfrac{2}{\ln 2}} \sin\left(\tfrac{n\pi}{\ln 2} \ln x\right)$$

$$= \sum_{n=1} \left[-\frac{1}{\lambda_n} \sqrt{\tfrac{2}{\ln 2}} \int_1^2 F(x') \sin\left(\tfrac{n'\pi}{\ln 2} \ln x'\right) dx' \right] \sqrt{\tfrac{2}{\ln 2}} \sin\left(\tfrac{n\pi}{\ln 2} \ln x\right)$$

$$= \sum_{n=1} -\frac{1}{\lambda_n} \frac{2}{\ln 2} \sin\left(\tfrac{n\pi}{\ln 2} \ln x\right) \int_1^2 F(x') \sin\left(\tfrac{n'\pi}{\ln 2} \ln x'\right) dx'$$

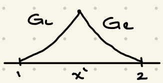
$$\boxed{G(x,x') = \sum_{n=1} -\frac{1}{\lambda_n} \frac{2}{\ln 2} \sin\left(\tfrac{n\pi}{\ln 2} \ln x'\right) \sin\left(\tfrac{n\pi}{\ln 2} \ln x\right) \qquad \lambda_n = \left(\tfrac{n\pi}{\ln 2}\right)^2}$$

(2) FIND $G(x,x')$ W/ THE PATCHING FCN

REF: DAMTP.CAM.AC.UK "GREENS METHOD"

$$x^2 \frac{\partial^2 G}{\partial x^2} + x \frac{\partial G}{\partial x} = F(x)$$

$$G(1,x') = 0 \qquad G(2,x') = 0$$



- GENERAL SOLN TO $x^2 \frac{\partial^2 G}{\partial x^2} + x \frac{\partial G}{\partial x}$

$\cancel{x^2}\left(\frac{A}{-1} \frac{1}{\cancel{x^2}}\right) + \cancel{x}\left(A \frac{1}{\cancel{x}}\right) = 0$ ✓

$G = A\log(x) + B$ BY WOLFRAM

BC: $G = \begin{cases} A_L \log(x) + B_L & x < x' \\ A_R \log(x) + B_R & x > x' \end{cases}$, $G_L(1) = 0 \rightarrow 0 = B_L$; ~~$G_R(2) = 0 \rightarrow 0 = A_R \log(2) + B_R$~~ $\rightarrow G = \begin{cases} A_L \log(x) \\ A_R \log(3-x) \end{cases}$

$\hookrightarrow A_R \log(3-x) + B_R$, $G_R(2) = 0 \rightarrow 0 = A_R \log(3-2) + B_R \rightarrow B_R = 0$

CONTINUITY: $G_L(x') = G_R(x')$

$$A_L \log(x') = A_R \log(3-x')$$

JUMP DISCONTINUITY:

$$\left.\frac{\partial G_R}{\partial x}\right|_{x'} - \left.\frac{\partial G_L}{\partial x}\right|_{x'} = \frac{1}{\alpha(x')} = \frac{1}{x'^2}$$

$$\left.\begin{aligned} \frac{\partial G_R}{\partial x} &= \frac{\partial}{\partial x}\left(A_R \log(3-x)\right) = \frac{-A_R}{3-x'} \\ \frac{\partial G_L}{\partial x} &= \frac{A_L}{x} \end{aligned}\right\} \rightarrow \frac{-A_R}{3-x'} - \frac{A_L}{x'} = \frac{1}{x'^2}$$

Q1 PART 2 CONT.

$$\begin{cases} \dfrac{-A_R}{3-x'} - \dfrac{A_L}{x'} = \dfrac{1}{x'^2} & \text{(JUMP COND.)} \\ A_L \ln(x') = A_R \ln(3-x') & \text{(CONTINUITY)} \end{cases} \rightarrow A_L = \frac{A_R \ln(3-x')}{\ln(x')}$$

SUBS $A_L \rightarrow$ JUMP TO OBTAIN A_R

$$\frac{-A_R}{3-x'} - \left(\frac{A_R \ln(3-x')}{x' \ln x'}\right) = \frac{1}{x'^2}$$

$$A_R\left(-\frac{1}{3-x'} - \frac{\ln(3-x')}{x' \ln x'}\right) = \frac{1}{x'^2}$$

$$A_R\left(-\frac{1}{(3-x')}\left(\frac{x' \ln x'}{x' \ln x'}\right) - \left(\frac{3-x'}{3-x'}\right)\left(\frac{\ln(3-x')}{x' \ln x'}\right)\right) = \frac{1}{x'^2}$$

$$A_R\left(-\frac{x' \ln x'}{(3-x') x' \ln x'} - \frac{(3-x')(\ln(3-x'))}{(3-x') x' \ln x'}\right) = \frac{1}{x'^2}$$

$$A_R\left(-\frac{x' \ln x' - (3-x') \ln(3-x')}{(3-x') x' \ln x'}\right) = \frac{1}{x'^2}$$

$$A_R = -\frac{1}{\cancel{x'}}\left(\frac{(3-x') \ln x' \cdot \cancel{x'}}{x' \ln x' + (3-x') \ln(3-x')}\right)\frac{1}{x'}$$

$$A_R = \frac{(x'-3) \ln x'}{x' \ln x' + (3-x') \ln(3-x')}\left(\frac{1}{x'}\right)$$

NOW SOLVING A_L:

$$A_L = \left[\frac{(x'-3)\cancel{\ln x'}}{x' \ln x' + (3-x') \ln(3-x')}\right] \frac{\ln(3-x')}{\cancel{\ln(x')}} \left(\frac{1}{x'}\right)$$

$$A_L = \frac{(x'-3) \ln(3-x')}{x' \ln x' + (3-x') \ln(3-x')}\left(\frac{1}{x'}\right)$$

$$\boxed{G(x,x') = \begin{cases} \dfrac{(x'-3) \ln(3-x')}{x' \ln x' + (3-x') \ln(3-x')} \ln(x) \left(\dfrac{1}{x'}\right) & x < x' \\ \dfrac{(x'-3) \ln(x')}{x' \ln x' + (3-x') \ln(3-x')} \dfrac{\ln(3-x)}{x'} & x > x' \end{cases}}$$

(3) PLOT $G(x, \frac{5}{4})$

METHOD 1:

$$G(x,x') = \sum_{n=1} -\frac{1}{\lambda_n} \frac{2}{\ln 2} \sin\left(\frac{n\pi}{\ln 2} \ln x'\right) \sin\left(\frac{n\pi}{\ln 2} \ln x\right) \qquad \lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2$$

METHOD 2:

$$G(x,x') = \begin{cases} \dfrac{(x'-3) \ln(3-x')}{x' \ln x' + (3-x') \ln(3-x')} \dfrac{\ln(x)}{x'} & x < x' \\ \dfrac{(x'-3) \ln(x')}{x' \ln x' + (3-x') \ln(3-x')} \dfrac{\ln(3-x)}{x'} & x > x' \end{cases}$$

METHOD 1:

$$u(x,x') = \sum \frac{-1}{\lambda_n} \frac{2}{\ln 2} \sin \frac{n\pi}{\ln 2} \ln x \int_1^2 \sin\left(\frac{n\pi}{\ln 2} \ln x'\right) F(x') \qquad F(x') = 1$$

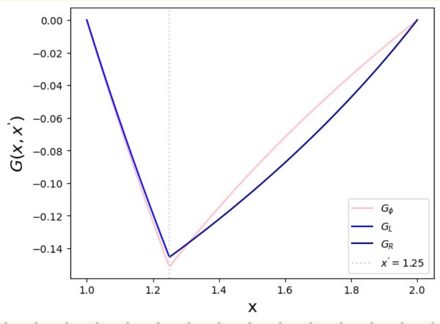
$$\int_1^2 \sin\left(\frac{n\pi}{\ln 2} \ln x'\right) x' dx = \frac{\ln 2(n\cdot\pi - 2\pi n \cos(n\pi) + \log 4 \sin(n\pi))}{(n\pi)^2 + \log 2^2}$$

$$u(x,x') = \sum_{n=1} \frac{-1}{\lambda_n} \frac{2}{\ln 2} \sin\left(\frac{n\pi}{\ln 2} \ln x\right) \cdot \underbrace{\frac{\ln 2(n\cdot\pi - 2\pi n \cos(n\pi) + \log 4 \sin(n\pi))}{(n\pi)^2 + \log 2^2}}$$

METHOD 2:

$$u(x,x') = \int_1^x \frac{(x'-3) \ln(3-x')}{x' \ln x' + (3-x') \ln(3-x')} \ln(x) F(x') + \int_x^2 \frac{(x'-3) \ln(x')}{x' \ln x' + (3-x') \ln(3-x')} \frac{\ln(3-x)}{x'}$$

$$= \ln x \int_1^x \frac{(x'-3) \ln(3-x')}{x' \ln x' + (3-x') \ln(3-x')} dx' + \ln(3-x) \int_x^2 \frac{(x'-3) \ln x'}{x' \ln x' + (3-x') \ln(3-x')} \frac{1}{x'}$$

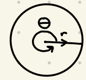


Q3) $\boxed{\dfrac{\partial^2 f}{\partial t^2} = c^2\nabla^2 f \text{ IN A DISK W/ RADIUS } a \longrightarrow \dfrac{\partial^2 f}{\partial t^2} = c^2\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}\right] \quad \text{I.C: } f(r,\theta,0)=h(r)\cos(5\theta) \quad \text{W/ } f(a,\theta,t)=0 \text{ \& REG. COND AT THE ORIGIN}}$

(MOSTLY COPIED FROM MY SUG. HW SOLN)

(1) 2D EIGENFCN SOLNS:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right] \qquad \text{LET } f(r,\theta,t)=u(r,\theta,t)$$



LET $u = R(r)A(\theta)T(t)$ & SUBS INTO GOV. EQN

$$RAT'' = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rR'AT\right)+\frac{1}{r^2}RA''T\right]$$

$$\frac{1}{c^2}\frac{T''}{T} = \cancel{c^2}\left[\frac{1}{r}\frac{1}{R}\frac{\partial}{\partial r}(rR')+\frac{1}{r^2}\frac{A''}{A}\right] = -\lambda \longrightarrow T'' = -\lambda c^2 T$$

NOW ISOLATE RADIAL TERMS:

$$\frac{1}{r}\frac{1}{R}\left(\frac{\partial}{\partial r}(rR')+\frac{1}{r^2}\frac{A''}{A}\right) = -\lambda$$

$$r\frac{1}{R}\left(\frac{\partial}{\partial r}(rR')+\frac{A''}{A}\right) = -\lambda r^2$$

$$r\frac{1}{R}\left(\frac{\partial}{\partial r}(rR')+\lambda r^2 = -\frac{A''}{A} = \lambda_1 \rightarrow \begin{cases}\frac{r}{R}\left(\frac{\partial}{\partial r}(rR')+\lambda r^2 = \lambda_1 \rightarrow r\left(\frac{\partial}{\partial r}(rR')\right)+\lambda r^2 R = \lambda_1 R \\ A'' = -\lambda_1 A\end{cases}$$

NOW WE HAVE 3 ODES:

$$\begin{cases} T''(t)+\lambda c^2 T(t) = 0 \rightarrow T(t) = \alpha_1\cos(c\sqrt{\lambda}t)+\alpha_2\sin(c\sqrt{\lambda}t) \\ r\left(\frac{\partial}{\partial r}(rR'(r))\right)+\lambda r^2R(r)-\lambda_1R(r)=0 \\ A''(\theta)+\lambda_1A(\theta)0 \rightarrow A(\theta) = \alpha_3\cos(\sqrt{\lambda_1}\theta)+\alpha_4\sin(\sqrt{\lambda_1}\theta)\end{cases}$$

SOLVING RADIAL ODE:

$$r\left(\frac{\partial}{\partial r}(rR'(r))\right)+\lambda r^2R(r)-\lambda_1R(r)=0$$

$$\frac{\partial}{\partial r}(rR'(r))+\lambda rR-\frac{1}{r}\lambda_1R(r)=0 \rightarrow w(r)=r$$

$$r\left(rR''(r)+R'(r)\right)+\left(\lambda r^2-\lambda_1\right)R(r)=0$$

BESSEL FCN (WIKI):

$$x^2\frac{\partial^2 y}{\partial x^2}+x\frac{\partial y}{\partial x}+(x^2-\alpha^2)y=0$$

$r^2R''+rR'+(\lambda r^2-\lambda_1)R=0$ PUT INTO BESSEL FORM:

LET $\lambda r^2 = x^2$ & $\alpha^2=\lambda_1 \rightarrow x=\sqrt{\lambda}r,\ r=\frac{x}{\sqrt{\lambda}}$

$$\frac{x^2}{\lambda}\frac{\partial^2 R}{\partial(x/\sqrt{\lambda})^2}+\frac{x}{\sqrt{\lambda}}\frac{\partial R}{\partial(x/\sqrt{\lambda})}+(x^2-\alpha)R=0$$

$$x^2\frac{\partial^2R}{\partial x^2}+x\frac{\partial R}{\partial x}+(x^2-\alpha^2)R=0$$

WE HAVE A REGULARITY CONDITION AT THE ORIGIN $\therefore R(x)=J_\alpha(x) \rightarrow R(r)=J_{\sqrt{\lambda_1}}(\sqrt{\lambda}r)$

NOW, $u(r,\theta,t)=J_{\sqrt{\lambda_1}}(\sqrt{\lambda}r)\left[\alpha_3\cos\sqrt{\lambda_1}\theta+\alpha_4\sin\sqrt{\lambda_1}\theta\right]\left[\alpha_1\cos(c\sqrt{\lambda}t)+\alpha_2\sin(c\sqrt{\lambda}t)\right]$

NOW WE FIT THE BOUNDARY CONDITION:

$u(a,\theta,t)=0 \rightarrow R(a)=0$:

$0=J_{\sqrt{\lambda_1}}\sqrt{\lambda}a \rightarrow z_n=\sqrt{\lambda}a \rightarrow \lambda_n=\left(\frac{z_n}{a}\right)^2$

NOW FIT INITIAL CONDITION: $u(r,\theta,0)=h(r)\cos(5\theta)$

$$u(r,\theta,0)=J_{\sqrt{\lambda_1}}\left(\frac{z_n}{a}r\right)\left[\alpha_3\cos\sqrt{\lambda_1}\theta+\alpha_4\sin\sqrt{\lambda_1}\theta\right]\left[\alpha_1\right]$$

$$h(r)\cos(5\theta)=J_{\sqrt{\lambda_1}}\left(\frac{z_n}{a}r\right)\left[\alpha_3\cos\sqrt{\lambda_1}\theta+\alpha_4\sin\sqrt{\lambda_1}\theta\right]\left[\alpha_1\right] \rightarrow \sqrt{\lambda_1}=5,\ \alpha_4=0$$

SPATIAL SOLN/ EIGENFCNS	$A(\theta)$: $\alpha_3\cos(5\theta)$, $\lambda_1=25$
	$R(r)$: $J_5\left(\frac{z_n}{a}r\right)c_n, \lambda_n=\left(\frac{z_n}{a}\right)^2$
TEMPORAL SOLN	$T(t)=\alpha_1\cos\left(c\frac{z_n}{a}t\right)+\alpha_2\sin\left(c\frac{z_n}{a}t\right)$, $\lambda_n=\left(\frac{z_n}{a}\right)^2$

(2) WE REQUIRE EIGENFCNS PROPORTIONAL TO $\cos(5\theta)$ TO FIT THE INITIAL STATE OF OUR WAVE PROBLEM. BECAUSE THESE WAVES ARE PROPOGATED OUT, THE WAVES THAT FOLLOW MUST BE PROPORTIONAL TO THE INITIAL CONDITION.

(3) PROJECT THE INITIAL CONDITION TO FIND THE FORMAL SOLN.

$$u(r,\theta,t)=\sum_{n=1} J_5\left(\frac{z_n}{a}r\right)\cos 5\theta\left[\alpha_1\cos\left(c\frac{z_n}{a}t\right)+\alpha_2\sin\left(c\frac{z_n}{a}t\right)\right]$$

ORTHOG. CONDITIONS:

$R(r)$: $\int_0^a J_5\left(\frac{z_n}{a}r\right)J_5\left(\frac{z_{n'}}{a}r\right)r\,dr=0$ IF $n=n'$

$A(\theta)$: $\int_0^{2\pi}\cos(5\theta)\cos(5\theta')d\theta=0$ IF $\theta=\theta'$

$$u(r,\theta,t)=\sum_{n=1}A_nJ_5\left(\frac{z_n}{a}r\right)\cos5\theta\cos\left(c\frac{z_n}{a}t\right)+\sum_{n=1}^{\infty}B_nJ_5\left(\frac{z_n}{a}r\right)\cos5\theta\sin\left(c\frac{z_n}{a}t\right)$$

$$u(r,\theta,t) = \sum_{n=1} A_n J_5\left(\frac{z_n}{a}r\right)\cos 5\theta \cos\left(c\frac{z_n}{a}t\right) + \sum_{n=1}^{\infty} B_n J_5\left(\frac{z_n}{a}r\right)\cos 5\theta \sin\left(c\frac{z_n}{a}t\right)$$

INITIAL CONDITION:

$$u(r,\theta,0) = h(r)\cos 5\theta$$

$$h(r)\cos 5\theta = \sum A_n J_5\left(\frac{z_n}{a}r\right)\cos 5\theta \cancelto{1}{\cos\left(c\frac{z_n}{a}0\right)} + \cancel{\sum_{n=1}^{\infty} 0}$$

ORTHOGONALITY CONDITION:

$$\int_0^a\int J_5\left(\frac{z_n}{a}r\right) J_5\left(\frac{z_{n'}}{a}r\right) r \cos 5\theta \cos 5\theta' \, dr\, d\theta = 0 \quad \text{IF } n \neq n' \ \& \ \theta \neq \theta'$$

MULTIPLY BY $J_5\left(\frac{z_{n'}}{a}r\right) r\cos 5\theta'$ & INTEGRATE

$$\int_0^a\int_0^{2\pi} h(r)\cos 5\theta \, J_5\left(\frac{z_{n'}}{a}r\right) r\cos 5\theta' \, dr\, d\theta = A_n \int_0^a\int_0^{2\pi} J_5\left(\frac{z_n}{a}r\right)\cos 5\theta \ r\, J_5\left(\frac{z_{n'}}{a}r\right)\cos 5\theta' \, dr\, d\theta$$

$$\boxed{\begin{gathered} u(r,\theta,t) = \sum_{n=1} A_n J_5\left(\frac{z_n}{a}r\right)\cos 5\theta \cos\left(c\frac{z_n}{a}t\right) + \sum_{n=1}^{\infty} B_n J_5\left(\frac{z_n}{a}r\right)\cos 5\theta \sin\left(c\frac{z_n}{a}t\right) \\ A_n = \frac{\int_0^a\int_0^{2\pi} h(r)\cos 5\theta\, r J_5\left(\frac{z_n}{a}r\right)\cos(5\theta)\, dr\, d\theta}{\int_0^a\int_0^{2\pi} J_5^2\left(\frac{z_n}{a}r\right)\cos^2 5\theta \, r\, dr\, d\theta} \end{gathered}}$$