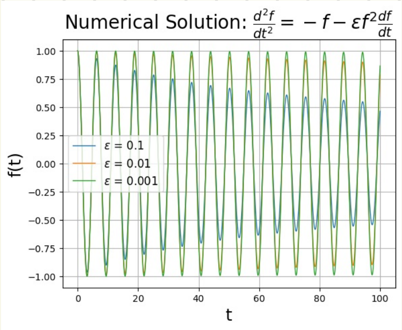
1A) $\boxed{\begin{array}{l}\frac{\partial^2 f}{\partial t^2} = -f - \varepsilon f^2\left(\frac{\partial f}{\partial t}\right)\\ f(0)=1\\ \frac{\partial f}{\partial t}\Big|_0 = 0\end{array}}$



- PLOTTING:

LET $f_1 = f$, $f_2 = \frac{\partial f}{\partial t} = \frac{\partial f_1}{\partial t}$

$\rightarrow \frac{\partial^2 f}{\partial t^2} = -f_1 - \varepsilon f_1^2 f_2$

BY OUR NUMERICAL PLOT, WE SEE ε PRIMARILY CHANGES THE AMPLITUDE & THUS CHANGING ONLY t WILL NOT WORK. WE REQUIRE OUR COEFS OF OUR SOLUTION TO VARY W/TIME & CHOOSE MULTIPLE TIMESCALES.

- BY BUSH p120, USE MULTIPLE SCALES W/ ODE FORM $\frac{\partial^2 u}{\partial t^2} + u = \varepsilon F\left(u, \frac{du}{dt}\right)$

- LET $t_s = \varepsilon t$ & $t_f = t$, THEN $f(t) = f(t_f, t_s)$

$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_s}\frac{\partial t_s}{\partial t} + \frac{\partial}{\partial t_f}\frac{\partial t_f}{\partial t}$

$= \frac{\partial}{\partial t_s}(\varepsilon) + \frac{\partial}{\partial t_f}(1)$

$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial t_s^2}\varepsilon^2 + \frac{\partial^2}{\partial t_f^2} + 2\varepsilon\frac{\partial^2}{\partial t_s \partial t_f}$

LET $f = f_0 + \varepsilon f_1 + \cdots$

SUBSTITUTE:

$\frac{\partial^2 f}{\partial t^2} = -f - \varepsilon f^2\left(\frac{\partial f}{\partial t}\right)$

$\left[\varepsilon^2\frac{\partial^2}{\partial t_s^2} + \frac{\partial^2}{\partial t_f^2} + 2\varepsilon\frac{\partial^2}{\partial t_s \partial t_f}\right](f_0 + \varepsilon f_1 + \cdots) = -(f_0 + \varepsilon f_1 + \cdots) - \varepsilon(f_0 + \cdots)^2\left[\varepsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}\right](f_0 + \cdots)$

$\left.\begin{array}{l} f_0(0) + \varepsilon f_1(0) + \cdots = 1 \\ f_0'(0) + \varepsilon f_1'(0) + \cdots = 0 \end{array}\right\}$ I.C.

NOW WE FIND ORDERS OF EPSILON.

$O(1)$: $\frac{\partial^2 f_0}{\partial t_f^2} = -f_0$, $f_0(0) = 1$, $f_0'\big|_0 = 0$

HAS GEN. SOLN $f_0 = A(t_s)\sin(t_f) + B(t_s)\cos(t_f) = A(t_s)e^{it_f} + A(t_s)^* e^{-it_f}$

$O(\varepsilon)$: $\frac{\partial^2 f_1}{\partial t_f^2} + 2\frac{\partial^2 f_0}{\partial t_f \partial t_s} = -f_1 - \left[f_0^2\frac{\partial f_0}{\partial t_f}\right]$

$\frac{\partial^2 f_1}{\partial t_f^2} + f_1 = -2\frac{\partial^2 f_0}{\partial t_s \partial t_f} - f_0^2\frac{\partial f_0}{\partial t_f}$

$= -2\left[iA'e^{it_f} - iA^{*\prime}e^{-it_f}\right] - \left[Ae^{it_f} + A^*e^{-it_f}\right]^2\left(iAe^{it_f} - iA^*e^{-it_f}\right)$

$= -2\left[iA'e^{it_f} - iA^{*\prime}e^{-it_f}\right] - \left[A^2e^{2it_f} + 2AA^* + A^{*2}e^{-2it_f}\right]\left(iAe^{it_f} - iA^*e^{-it_f}\right)$

$= -2iA'e^{it_f} + 2iA^{*\prime}e^{-it_f} - \left[iA^3e^{3it_f} + iA^2A^*e^{it_f} - iAA^{*2}e^{-it_f} - iA^{*3}e^{-3it_f}\right)$

$= -2iA'e^{it_f} + 2iA^{*\prime}e^{-it_f} - iA^3e^{3it_f} - iA^2A^*e^{it_f} + iAA^{*2}e^{-it_f} + iA^{*3}e^{-3it_f}$

$= e^{it_f}\{-2iA' - iA^2A^*\} + e^{-it_f}\{2iA^{*\prime} + iAA^{*2}\} - iA^3e^{3it_f} + iA^{*3}e^{-3it_f}$

NOW TO REMOVE SECULAR TERMS OF e^{it_f} & e^{-it_f}:

$\begin{cases} 0 = -2iA' - iA^2A^* = -2iA' - i|A|^2A \\ 0 = 2iA^{*\prime} + iAA^{*2} = 2iA^{*\prime} + i|A^*|^2A^* \end{cases}$

NOW WE SOLVE FOR A:

$0 = -2\cancel{i}A' - \cancel{i}|A|^2A$

$\frac{\partial A}{\partial t_s} = -\frac{1}{2}|A|^2A$

NOW LET $A = |A|e^{i\theta(t_s)}$

$\frac{\partial |A|e^{i\theta}}{\partial t_s} = -\frac{1}{2}|A|^2\left(|A|e^{i\theta}\right)$

$|A|i\cancel{e^{i\theta}}\frac{\partial\theta}{\partial t_s} + \frac{\partial|A|}{\partial t_s}\cancel{e^{i\theta}} = -\frac{1}{2}|A|^3\cancel{e^{i\theta}}$

NOW SEPARATE REAL & IM PART:

IM: $|A|\frac{\partial\theta}{\partial t_s} = 0 \rightarrow \theta = \theta_0$ (CONSTANT)

RE: $\frac{\partial|A|}{\partial t_s} = -\frac{1}{2}|A|^3 \xrightarrow{\text{BY WOLFRAM}} |A| = \frac{1}{\sqrt{A_0 + t_s}}$

NOW WE HAVE A:

$A = |A|e^{i\theta} = \frac{1}{\sqrt{A_0 + t_s}}e^{i\theta_0}$

NOW THAT WE HAVE A, WE CAN RECONSTRUCT f,

$$f = Ae^{it_f} + A^* e^{-it_f}$$

$$= \frac{1}{\sqrt{A_0+t_s}} e^{i\theta_0} e^{it_f} + \frac{1}{\sqrt{A_0+t_s}} e^{-i\theta_0} e^{-it_f}$$

$$= \frac{1}{\sqrt{A_0+t_s}} \left[e^{i(\theta_0+t_f)} + e^{-i(\theta_0+t_f)} \right]$$

$$= 2\frac{1}{\sqrt{A_0+t_s}} \cos(\theta_0+t_f)$$

$$2b\cos(x) = b(e^{-ix} + e^{ix})$$

NOW WE FIT I.CS: $\left(f(0)=1,\ \frac{\partial f}{\partial t}\Big|_0 = 0\right)$

$$\frac{\partial f_0}{\partial t_f} = \left(\frac{2}{\sqrt{A_0+t_s}}\right)\left(-\sin(\theta_0+t_f)\right)$$

$$= \frac{-2\sin(\theta_0+t_f)}{\sqrt{A_0+t_s}}$$

$f(0)=1$: $\qquad \frac{1}{2}\sqrt{A_0} = \cos(\theta_0)$

$$1 = 2\frac{1}{\sqrt{A_0}}\cos(\theta_0) \rightarrow \frac{1}{2}\sqrt{A_0} = \cos(\theta_0) \rightarrow \frac{1}{2}\sqrt{A_0} = 1 \rightarrow A_0 = 4$$

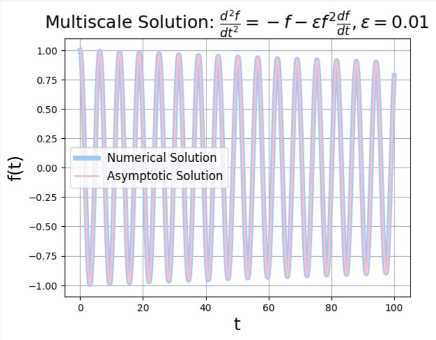
$f'(0)=0$:

$$0 = \frac{-2\sin(\theta_0)}{\sqrt{A_0}} \rightarrow \sin(\theta_0) = 0 \rightarrow \theta_0 = 0$$

$$\rightarrow f = \frac{2}{\sqrt{A_0+t_s}}\cos(\theta_0+t_f)$$

$$= \frac{2}{\sqrt{A_0+t_s}}\cos(t_f)$$

$$\boxed{f \doteq \frac{2}{\sqrt{4+\epsilon t}}\cos(t)}$$

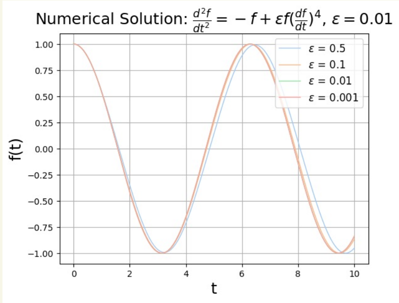


(18)

$$\boxed{\begin{aligned}&\frac{\partial^2 f}{\partial t^2} = -f + \epsilon f\left(\frac{\partial f}{\partial t}\right)^4,\\ &f(0)=1\\ &\frac{\partial f}{\partial t}\Big|_0 = 0\end{aligned}}$$

P97

#4



WE MAINLY SEE A CHANGE IN FREQUENCY IN OUR NUMERICAL SOLN, WE PROPOSE A CHANGE OF TIME TO USE LINDSTEDT-POINCARE METHOD.

$$\begin{cases} \text{LET } \tau = t(1 + a_0\epsilon + a_2\epsilon^2 + \cdots) \\ \frac{\partial}{\partial t} = (1+\epsilon a_0 + \cdots)\frac{\partial}{\partial \tau} \\ \frac{\partial^2}{\partial t^2} = (1+\epsilon a_0 + \cdots)^2 \frac{\partial^2}{\partial \tau^2} \\ f(\tau) = f_0(\tau) + \epsilon f_1(\tau) + \cdots \end{cases}$$

SUBSTITUTE:

$$(1+\epsilon a_0 + \cdots)^2 \frac{\partial^2}{\partial \tau^2}(f + \epsilon f_1 + \cdots) = -(f_0 + \epsilon f_1 + \cdots) + \epsilon(f_0 + \cdots)\left[(1+\epsilon a_0 + \cdots)\frac{\partial}{\partial \tau}(f_0 + \cdots)\right]^4$$

$$(1+\epsilon a_0 + \cdots)^2 (f_0'' + \epsilon f_1'' + \cdots) = -(f_0 + \epsilon f_1 + \cdots) + \epsilon(f_0 + \cdots)\left[(1+\cdots)(f_0' + \epsilon f_1' + \cdots)\right]^4$$

INITIAL COND:

$f(0) = 1$:

$$1 = f_0(0) + \epsilon f_1(0) + \cdots$$

$f'(0) = 0$:

$$0 = f'(0) + \epsilon f'(0) + \cdots$$

COLLECT ORDERS:

$\mathcal{O}(1)$:

$$\left.\begin{aligned} &f'' = -f_0,\ f_0(0) = 1,\ f_0'(0) = 0 \\ &f_0 = c_1 \sin(\tau) + c_2\cos(\tau) \\ &f_0(0) = 1 \rightarrow c_2 = 1 \\ &f_0'(0) = 0 \rightarrow 0 = c_1 \end{aligned}\right\} \Rightarrow f_0 = \cos(\tau)$$

$\mathcal{O}(\epsilon)$:

$$2a_0 f_0'' + f_1'' = -f_1 + f_0(f_0')^4$$

$$\begin{aligned} f_1'' + f_1 &= f_0(f_0')^4 - 2a_0 f_0'' \\ &= \cos(\tau)(\cancel{-}\sin\tau)^4 + 2a_0(+\cos(\tau)) \\ &= \tfrac{1}{16}(32a_0\cos(\tau) + 2\cos(\tau) - 3\cos(3\tau) + \cos(5\tau)) \quad \text{BY WOLFRAM} \\ &= 2a_0\cos\tau + \tfrac{1}{8}\cos(\tau) + \cdots = (2a_0 + \tfrac{1}{8})\cos\tau + \cdots \end{aligned}$$

TO REMOVE SECULAR TERMS, WE ELIMINATE $\cos(\tau)$ TERMS:

$$0 = 2a_0 + \frac{1}{8}$$

$$a_0 = -\frac{1}{16}$$

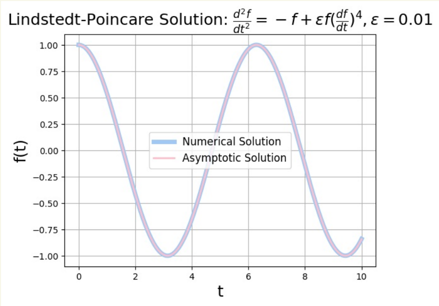
NOW $\tau = t(1 + a_0 + \cdots)$

$$= t(1 - \tfrac{1}{16}\epsilon + \cdots)$$

WE CONSTRUCT THE SOLUTION:

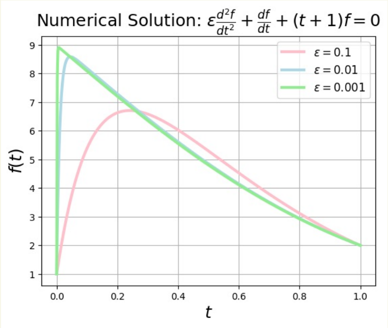
$$f = f_0(\tau) + \cdots$$

$$\boxed{f = \cos\left(t - \epsilon\frac{t}{16} + \cdots\right) + \cdots} \longrightarrow \text{MATCHES MULTISCALE SOLUTION :D}$$



1c)

$$\boxed{\begin{array}{l} \varepsilon \frac{\partial^2 f}{\partial t^2} + \frac{\partial f}{\partial t} + (t+1)f = 0 \\ f(0) = 1 \\ f(1) = 2 \end{array}}$$



PLOTTING:

LET $f_1 = f$, $f_2 = \frac{\partial f}{\partial t} = \frac{\partial f_1}{\partial t}$

$\frac{\partial^2 f}{\partial t^2} = \frac{1}{\varepsilon}\left[-\left(\frac{\partial f}{\partial t}\right) - (t+1)f\right]$

$= \frac{1}{\varepsilon}\left[-f_2 - (t+1)f_1\right]$

WE SEE A BOUNDARY LAYER ASSOCIATED W/ $t=0$, WE SOLVE FOR A SOLUTION IN & OUTSIDE THE LAYER.

- f_{OUT}: $\varepsilon = 0$

  $\frac{\partial f}{\partial t} + (t+1)f = 0$

  BY WOLFRAM, $f_{OUT} = c_1 e^{-\frac{1}{2}t(t+2)}$

  WE USE BC AT $t=1$:

  $2 = c_1 e^{-\frac{1}{2}(3)}$

  $2e^{\frac{3}{2}} = c_1$

  $f_{OUT} = 2e^{\frac{3}{2}} e^{-\frac{t}{2}(t+2)}$ *

- f_{IN}: LET $s = \frac{t}{\varepsilon^\alpha} \rightarrow t = s\varepsilon^\alpha$

  $\frac{\partial}{\partial t} = \frac{\partial}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial}{\partial s}\frac{1}{\varepsilon^\alpha}$

  $\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial s^2}\frac{1}{\varepsilon^{2\alpha}}$

  SUBSTITUTE & BALANCE:

  $\varepsilon \frac{1}{\varepsilon^{2\alpha}} \frac{\partial^2 f}{\partial s^2} + \frac{1}{\varepsilon^\alpha}\frac{\partial f}{\partial s} + (s\varepsilon^\alpha + 1) f = 0$

  $(s\varepsilon^\alpha + 1)$ IS SMALL COMPARED TO OTHER TERMS

  $\varepsilon^{1-2\alpha}\frac{\partial^2 f}{\partial s^2} + \varepsilon^{-\alpha}\frac{\partial f}{\partial s} = 0$

  $\rightarrow \alpha = 1$

  SOLVE ODE FOR f_{IN}

  $\cancel{\varepsilon^{-1}} \frac{\partial^2 f}{\partial s^2} + \cancel{\varepsilon^{-1}} \frac{\partial f}{\partial s} = 0$

  $f_{IN} = C_{IN} e^{-s} + D_{IN}$

  USE BC AT $t = 0 \rightarrow s = 0$, $f = 1$

  $1 = C_{IN} + D_{IN} \rightarrow C_{IN} = 1 - D_{IN}$

- MATCHING COND:

  $\lim_{s\to\infty} f_{IN} = \lim_{x\to 0} f_{OUT}$

  $\lim_{s\to\infty}(C_{IN}e^{-s} + D_{IN}) = \lim_{x\to 0} 2e^{\frac{3}{2}} e^{-\frac{t}{2}(t+2)}$

  $D_{IN} = 2e^{3/2}$

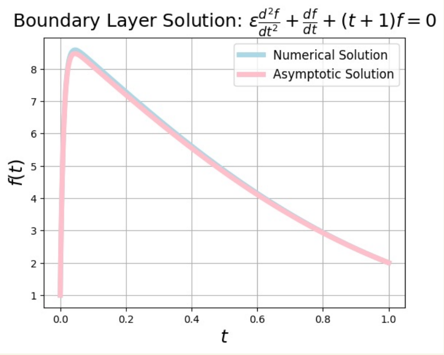
  $\rightarrow C_{IN} = 1 - 2e^{3/2}$

$$\rightarrow \begin{cases} f_{IN} = (1-2e^{3/2})e^{-s} + 2e^{3/2} \\ f_{OUT} = 2e^{3/2} e^{-\frac{t}{2}(t+2)} \\ L = 2e^{3/2} \end{cases}$$

- COMPOSITE FCN:

  $f = (1-2e^{3/2})e^{-s} + 2e^{3/2} e^{-\frac{t}{2}(t+2)}$

$$\boxed{f = (1-2e^{3/2})e^{-\left(\frac{t}{\varepsilon}\right)} + 2e^{3/2 - \frac{t}{2}(t+2)}}$$



2.

$$\boxed{\begin{aligned} &\frac{\partial^2 f}{\partial x^2} + \lambda(x+1)^2 f = 0 \\ &f(1) = 0 \\ &f(2) = 0 \end{aligned}}$$

LET $\lambda = \frac{1}{\varepsilon^2}$

$$\begin{cases} x_f = \frac{g(x)}{\varepsilon},\ x_s = x \\ f = f_0 + \varepsilon f_1 + \cdots \\ \rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial x_s} + \frac{g'}{\varepsilon}\frac{\partial}{\partial x_f} \\ \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x_s^2} + \frac{g''}{\varepsilon}\frac{\partial}{\partial x_f} + 2\frac{g'}{\varepsilon}\frac{\partial^2}{\partial x_s \partial x_f} + \frac{g'^2}{\varepsilon^2}\frac{\partial^2}{\partial x_f^2} \end{cases}$$

SUBSTITUTE INTO OUR GOVERNING EQUATION & GATHER ORDERS OF ε:

USED (ASYMPTOTIC_SUBSTITUTION.IPYNB) TO CHECK ORDERS

- $O(\varepsilon^{-2}) =$ $\frac{\partial^2 f_0}{\partial x_f^2} g'^2 + f_0 t^2 + 2 f_0 t + f_0 = 0$

$$\frac{\partial^2 f_0}{\partial x_f^2} g'^2 = - f_0 (t+1)^2$$

$$g'^2 = (t+1)^2 \rightarrow g' = t+1 \rightarrow g = \frac{t^2}{2} + t$$

- $O(\varepsilon^{-1})$: $\frac{\partial f_0}{\partial x_f} g'' + \frac{\partial^2 f_1}{\partial x_f^2} g'^2 + 2\frac{\partial^2 f_0}{\partial x_f \partial x_s} g' + f_1 t^2 + 2 f_1 t + f_1 = 0$

$$\frac{\partial f_0}{\partial x_f} g'' + \frac{\partial^2 f_1}{\partial x_f^2} g'^2 + 2\frac{\partial^2 f_0}{\partial x_f \partial x_s} g' + f_1 (t+1)^2 = 0$$

$$\frac{\partial^2 f_1}{\partial x_f^2} g'^2 + f_1 (t+1)^2 = -\frac{\partial f_0}{\partial x_f} g'' - 2\frac{\partial^2 f_0}{\partial x_f \partial x_s} g'$$

$$= -g''\left[A\cos(x_f) - B\sin(x_f)\right] - 2g\left[A'\cos(x_f) - B'\sin(x_f)\right]$$

$$= \cos(x_f)(-g''A - 2g'A') + \sin(x_f)(g''B + B'2g')$$

- REMOVE SECULAR TERMS

$$0 = -g''A - 2g'A' \rightarrow \cancel{2g'}\frac{A'}{A} = \frac{-g''\cancel{A}}{2g'} \rightarrow \ln(A) = -\tfrac{1}{2}\ln(g') + C \quad \rightarrow A = g'^{-1/2} u_A$$

$$0 = g''B + B'2g'$$

$$\therefore \begin{cases} A = g'^{-1/2} u_A \\ B = g'^{-1/2} u_B \end{cases}$$

$$f = g'^{-1/2} u_A \sin(x_f) + g'^{-1/2} u_B \cos(x_f)$$

$$= \frac{1}{\sqrt{x+1}} u_A \sin\left(\frac{g}{\varepsilon}\right) + \frac{1}{\sqrt{t+1}} u_B \cos\left(\frac{g}{\varepsilon}\right)$$

$$f = \frac{1}{\sqrt{x+1}}\left[u_A \sin\left(\frac{\frac{x^2}{2}+x}{\varepsilon}\right) + u_B \cos\left(\frac{\frac{x^2}{2}+x}{\varepsilon}\right)\right]$$

$\rightarrow$ FOLLOWS SOLN OF $\frac{\partial^2 f}{\partial t^2} = -\frac{\omega^2(t)}{\varepsilon^2} f(t)$ (L18)

INITIAL COND:

$f(1) = 0 \quad f(2) = 0$

$f(1) = 0$:

$$0 = \cancel{\frac{1}{\sqrt{2}}}\left[u_A \sin\left(\frac{\frac{1}{2}+1}{\varepsilon}\right) + u_B \cos\left(\frac{\frac{1}{2}+1}{\varepsilon}\right)\right]$$

$$u_B = -\cancel{\frac{1}{\sqrt{2}}} u_A \tan\left(\frac{3}{2\varepsilon}\right)$$

$f(2) = 0$:

$$0 = \frac{1}{\sqrt{3}}\left[u_A \sin\left(\frac{\frac{4}{2}+2}{\varepsilon}\right) + u_B \cos\left(\frac{\frac{4}{2}+2}{\varepsilon}\right)\right]$$

$$= \cancel{\frac{1}{\sqrt{3}}}\left[u_A \sin\left(\frac{4}{\varepsilon}\right) + u_B \cos\left(\frac{4}{\varepsilon}\right)\right]$$

$$= \cancel{u_A} \sin\left(\frac{4}{\varepsilon}\right) + \left[-\cancel{u_A} \tan\left(\frac{3}{2\varepsilon}\right)\right]\cos\left(\frac{4}{\varepsilon}\right)$$

$$0 = \sin\left(\frac{4}{\varepsilon}\right) - \tan\left(\frac{3}{2\varepsilon}\right)\cos\left(\frac{4}{\varepsilon}\right)$$

BY WOLFRAM,

$$\boxed{\begin{aligned} &\varepsilon = \frac{5}{2\pi n} \rightarrow \lambda_n = \frac{1}{\varepsilon^2} = \left(\frac{2\pi n}{5}\right)^2 \\ &\phi_n = \frac{1}{\sqrt{x+1}}\left[u_A \sin\left(\left(\frac{x^2}{2}+x\right)\left(\frac{2\pi n}{5}\right)\right) + u_B \cos\left(\left(\frac{x^2}{2}+x\right)\frac{2\pi n}{5}\right)\right] \end{aligned}}$$

2. $\boxed{\frac{\partial^2 f}{\partial x^2} + \lambda(x+1)^2 f = 0}$

LET $\lambda = \frac{1}{\varepsilon}$

$\frac{\partial^2 f}{\partial x^2} + \frac{1}{\varepsilon}(x+1)^2 f = 0$

LET $x_s = t$ & $x_f = \frac{g(x_s)}{\varepsilon^{1/2}}$

LET $f = f_0 + \sqrt{\varepsilon} f_1 + \cdots$

WE COMPUTE $\frac{d}{dx}$ & $\frac{d^2}{dx^2}$:

$$\frac{d}{dx} = \frac{d}{dx_s}\frac{dx_s}{dx} + \frac{d}{dx_f}\frac{dx_f}{dx}$$
$$= \frac{d}{dx_s}(1) + \frac{d}{dx_f}\left(\frac{1}{\varepsilon^{1/2}}\right) g'(x_s)$$
$$\frac{d^2}{dx^2} = \left(\frac{d}{dx_s} + g'(x_s)\frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial x_f}\right)^2$$
$$= \frac{d^2}{dx_s^2} + g'(x_s)^2\frac{1}{\varepsilon}\frac{\partial^2}{\partial x_f^2} + \frac{d}{dx_s}\left(g'(x_s)\frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial x_f}\right) + g'(x_s)\frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial x_f}\frac{\partial}{\partial x_s}$$
$$= \frac{d^2}{dx_s^2} + g'^2\frac{1}{\varepsilon}\frac{\partial^2}{\partial x_f^2} + \frac{1}{\sqrt{\varepsilon}}\left[\cancel{g'\frac{\partial^2}{\partial x_s\partial x_f}} + \frac{\partial}{\partial x_f}g''\right] + \frac{g'}{\sqrt{\varepsilon}}\frac{\partial^2}{\partial x_s\partial x_f}\ (2)$$

$$\begin{cases} \frac{\partial}{\partial x} = \frac{\partial}{\partial x_s} + \frac{g'}{\sqrt{\varepsilon}}\frac{\partial}{\partial x_f} \\ \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x_s^2} + \frac{g'^2}{\varepsilon}\frac{\partial^2}{\partial x_f^2} + \frac{g''}{\sqrt{\varepsilon}}\frac{\partial}{\partial x_f} + \frac{2g'}{\sqrt{\varepsilon}}\frac{\partial}{\partial x_s\partial x_f} \end{cases}$$

SUBSTITUTE INTO EIGENVALUE PROBLEM:

$$\left[\frac{\partial^2}{\partial x_s^2} + \frac{g'^2}{\varepsilon}\frac{\partial^2}{\partial x_f^2} + \frac{g''}{\sqrt{\varepsilon}}\frac{\partial}{\partial x_f} + \frac{2g'}{\sqrt{\varepsilon}}\frac{\partial}{\partial x_s\partial x_f}\right](f_0 + \sqrt{\varepsilon}f_1 + \cdots) + \frac{1}{\varepsilon}(x^2+2x+1)(f_0 + \sqrt{\varepsilon}f_1 + \cdots) = 0$$

COLLECT ORDERS OF ε:

$O(\varepsilon^{-1})$:

$$g'^2\frac{\partial^2 f_0}{\partial x_f^2} + (x^2+2x+1)f_0 = 0$$
$$g'^2\frac{\partial^2 f_0}{\partial x_f^2} = -f_0(x+1)^2$$

$\therefore$ WE CHOOSE $g'^2 = (x+1)^2$ TO OBTAIN $f_0 = A(x_s)\sin(x_f) + B(x_s)\sin(x_f)$

$\therefore g' = (x+1) \rightarrow g = \int_0^x x'+1\,dx' = \frac{x^2}{2} + x$

$O(\varepsilon^{-1/2})$:

$$g'^2\frac{\partial^2 f_1}{\partial x_f^2} + g''\frac{\partial f_0}{\partial x_f} + 2g'\frac{\partial f_0}{\partial x_s\partial x_f} + f_1(x+1)^2 = 0$$
$$g'^2\frac{\partial^2 f_1}{\partial x_f^2} - f_1(x+1)^2 = -g''\frac{\partial f_0}{\partial x_f} - 2g'\frac{\partial f_0}{\partial x_s\partial x_f}$$
$$= -g''\left[A\cos(x_f) - B\sin(x_f)\right] - 2g'\left[A'\cos(x_f) - B'\sin(x_f)\right]$$
$$= \sin(x_f)\{g''B + 2g'B'\} + \cos(x_f)\{-g''A - 2g'A'\}$$

NOW WE ELIMINATE SECULAR TERMS:

$\begin{cases} 0 = g''B + 2g'B' \\ 0 = -g''A + 2g'A' \end{cases}$ $\rightarrow$ SOLVE FOR A OR B

$$0 = g''B + 2g'B'$$
$$-2g'B' = g''B$$
$$\frac{B'}{B} = -\frac{1}{2}\frac{g''}{g'}$$
$$\ln(B) = -\frac{1}{2}\ln(g') + C_B$$
$$B = e^{\ln(g')^{-1/2}}k_B$$

$= g^{-1/2}k_B$ & SIMILARLY FOR A, $A = g^{-1/2}k_A$.

NOW WE CONSTRUCT THE FULL SOLN:

$f_0 = A(x_s)\sin(x_f) + B(x_s)\cos(x_f)$:

$$= \frac{k_A}{\sqrt{g'}}\sin(x_f) + \frac{k_B}{\sqrt{g'}}\cos(x_f)$$
$$= \frac{k_A}{\sqrt{x+1}}\sin\left(\frac{1}{\sqrt{\varepsilon}}\left(\frac{x^2}{2}+x\right)\right) + \frac{k_B}{\sqrt{x+1}}\cos\left(\frac{1}{\sqrt{\varepsilon}}\left(\frac{x^2}{2}+x\right)\right)$$

NOW WE USE THE BOUNDARY CONDITIONS TO SOLVE FOR ε: $f(1)=0$, $f(2)=0$

$f(1)=0$:

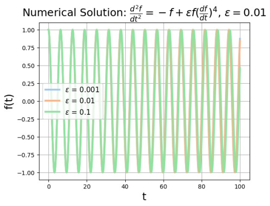
$$0 = \frac{k_A}{\cancel{\sqrt{2}}}\sin\left(\frac{1}{\sqrt{\varepsilon}}\left(\frac{1}{2}+1\right)\right) + \frac{k_B}{\cancel{\sqrt{2}}}\cos\left(\frac{1}{\varepsilon}\left(\frac{1}{2}+1\right)\right)$$
$$k_B = -k_A\tan\left(\frac{1}{\sqrt{\varepsilon}}\left(\frac{3}{2}\right)\right)$$

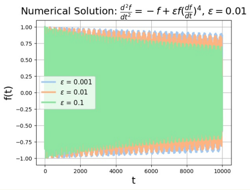
$f(2)=0$:

$$0 = \frac{k_A}{\cancel{\sqrt{3}}}\sin\left(\frac{1}{\sqrt{\varepsilon}}\left(\frac{2^2}{2}+2\right)\right) + \frac{k_B}{\cancel{\sqrt{3}}}\cos\left(\frac{1}{\sqrt{\varepsilon}}\left(\frac{2^2}{2}+2\right)\right)$$
$$= k_A\sin\left(\frac{4}{\sqrt{\varepsilon}}\right) + k_B\cos\left(\frac{4}{\sqrt{\varepsilon}}\right)$$
$$= \cancel{k_A}\sin\left(\frac{4}{\sqrt{\varepsilon}}\right) + \left[-\cancel{k_A}\tan\left(\frac{3}{2\sqrt{\varepsilon}}\right)\right]\cos\left(\frac{4}{\sqrt{\varepsilon}}\right)$$
$$0 = \sin\left(\frac{4}{\sqrt{\varepsilon}}\right) - \tan\left(\frac{3}{2\sqrt{\varepsilon}}\right)\cos\left(\frac{4}{\sqrt{\varepsilon}}\right)$$

EIGENVALS ARE ROOTS OF ↑

(B) $\boxed{\begin{array}{l} \frac{\partial^2 f}{\partial t^2} = -f + \varepsilon f\left(\frac{\partial f}{\partial t}\right)^4, \\ f(0)=1 \\ \left.\frac{\partial f}{\partial t}\right|_0 = 0 \end{array}}$





- PLOTTING:

  LET $f_1 = f$, $f_2 = \frac{\partial f}{\partial t} = \frac{\partial f_1}{\partial t}$

  $\frac{\partial^2 f}{\partial t^2} = -f_1 + \varepsilon(f_1)(f_2)^4$

- IF WE PLOT A SHORT TIME, WE SEE AMPLITUDE & FREQUENCY STAY CONSTANT, BUT IF WE USE A LONG TIME, IT SHOWS AMPLITUDE DECAYS AT A SMALL RATE, SO WE USE MULTISCALE METHOD
  THIS OBTAINS COEFS. DEPENDING ON TIME THAT CONTROLS CHANGE IN AMPLITUDE

- LET $t_s = t\varepsilon$ & $t_f = t$, $f = f_0 + \varepsilon f_1 + \cdots$

  $$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_s}\varepsilon + \frac{\partial}{\partial t_f}$$
  $$\frac{\partial^2}{\partial t^2} = \varepsilon^2\frac{\partial^2}{\partial t_s^2} + 2\varepsilon\frac{\partial^2}{\partial t_s \partial t_f} + \frac{\partial^2}{\partial t_f^2}$$

- SUBSTITUTE:

  $$\left[\varepsilon^2\frac{\partial^2}{\partial t_s^2} + 2\varepsilon\frac{\partial^2}{\partial t_s\partial t_f} + \frac{\partial^2}{\partial t_f^2}\right](f_0 + \varepsilon f_1 + \cdots) = -(f_0 + \varepsilon f_1 + \cdots) + \varepsilon(f_0 + \cdots)\left(\left(\varepsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}\right)(f_0 + \cdots)\right)^4$$

- ORDERS OF ε:

  $O(\varepsilon^0)$: $\frac{\partial^2 f_0}{\partial t_f^2} = -f_0 \rightarrow f_0 = A(t_s)\sin(t_f) + B(t_s)\cos(t_f) = Ae^{it_f} + A^* e^{-it_f}$

  $O(\varepsilon)$: $2\frac{\partial^2 f_0}{\partial t_s \partial t_f} + \frac{\partial^2 f_1}{\partial t_f^2} = -f_1 + f_0\left(\frac{\partial f_0}{\partial t_f}\right)^4$

  $$\frac{\partial^2 f_1}{\partial t_f^2} + f_1 = -2\frac{\partial^2 f_0}{\partial t_s\partial t_f} + f_0\left(\frac{\partial f_0}{\partial t_f}\right)^4$$

  (BY WOLFRAM)

  $$= -2\left[iA'e^{it_f} - iA^{*\prime}e^{-it_f}\right] + A^5e^{5it_f} - 3A^4A^*e^{3it_f} + 2A^3A^{*2}e^{it_f} + 2A^2A^{*3}e^{-it_f} - 3AA^{*4}e^{-3it_f} + A^{*5}e^{-5it_f}$$
  $$= e^{it_f}\left\{-2iA' + 2A^3A^{*2}\right\} + e^{-it_f}\left\{2iA^{*\prime} + 2A^2A^{*3}\right\} + \text{MORE TERMS}$$

- WE REMOVE e^{it_f} & e^{-it_f} TO ELIMINATE SECULAR TERMS:

  $$\left.\begin{array}{l} 0 = -2iA' + 2A^3A^{*2} \\ 0 = 2iA^{*\prime} + 2A^2A^{*3} \end{array}\right\} \text{CHOOSE 1}$$

  $2iA' = 2A^3A^{*2}$

  $2iA' = 2|A|^4A$ $\leftarrow A = |A|e^{i\theta}$

  $2i\frac{\partial}{\partial t_s}(|A|e^{i\theta}) = 2|A|^4|A|e^{i\theta}$

  $2i\left[|A|i\cancel{e^{i\theta}}\frac{\partial\theta}{\partial t_s} + \cancel{e^{i\theta}}\frac{\partial|A|}{\partial t_s}\right] = 2|A|^5\cancel{e^{i\theta}}$

  $-2|A|\frac{\partial\theta}{\partial t_s} + 2i\frac{\partial|A|}{\partial t_s} = 2|A|^5$

  SEPARATE RE & IM:

  RE: $-\cancel{2}\cancel{|A|}\frac{\partial\theta}{\partial t_s} = \cancel{2}|A|^{\cancel{5}4}$ $\rightarrow$ $\frac{\partial\theta}{\partial t_s} = -A_0^4 \rightarrow \cancel{\theta = -A_0^4t_s + \theta_0}$

  IM: $2\frac{\partial|A|}{\partial t_s} = 0 \rightarrow |A| = A_0$ (CONSTANT)

  $f = A_0e^{-iA_0^4\theta_0}e^{it_f} + A_0e^{iA_0^4\theta_0}e^{-it_f}$

  $= A_0\left[e^{i(-A_0^4\theta_0 + t)} + e^{-i(-A_0^4\theta_0 + t)}\right]$

  $= A_0\left[2\cos(-A_0^4\theta_0 + t)\right]$

  INITIAL COND:

  $f(0) = 1$:

  $1 = A_0 2\cos(-A_0^4\theta_0)$

  $\frac{\partial f}{\partial t} = A_0(-2\sin(-A_0^4\theta_0 + t))$

  $\left.\frac{\partial f}{\partial t}\right|_0 = 0$:

  $0 = A_0(-2\sin(-A_0^4\theta_0))$

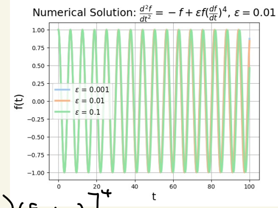
  $\rightarrow \theta_0 = 0$

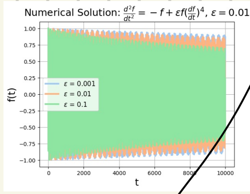
  $1 = A_0 2\cancel{\cos(0)} \rightarrow A_0 = \frac{1}{2}$

  FULL SOLN:

  $f_0 = \cos(t)$

1B) $\boxed{\begin{aligned} &\frac{\partial^2 f}{\partial t^2} = -f + \varepsilon f \left(\frac{\partial f}{\partial t}\right)^4, \\ &f(0)=1 \\ &\left.\frac{\partial f}{\partial t}\right|_0 = 0 \end{aligned}}$





#2

$$\left(\frac{\partial}{\partial t_f}\frac{\partial t_f}{\partial t} + \frac{\partial}{\partial t_s}\frac{\partial t_s}{\partial t}\right)^2 f = -(f_0 + \varepsilon f_1 + \cdots) + \varepsilon(f_0 + \cdots)\left[\left(\frac{\partial}{\partial t_f}(1) + \frac{\partial}{\partial t_s}\varepsilon\right)(f_0 + \cdots)\right]^4$$

$$\left(\frac{\partial^2}{\partial t_f^2} + \frac{\partial^2}{\partial t_s^2}\varepsilon^2 + 2\varepsilon\frac{\partial^2}{\partial t_s \partial t_f}\right)(f) = -(f_0 + \varepsilon f_1 + \cdots) + \varepsilon(f_0 + \cdots)\left[\frac{\partial f_0}{\partial t_f} + \frac{\partial f_0}{\partial t_s}\varepsilon\right]^4$$

$O(1)$: $\frac{\partial^2 f_0}{\partial t_f^2} = -f_0 \rightarrow f_0 = A(t_s)\sin(t_f) + B(t_s)\sin(t_f)$

$$= Ae^{it_f} + A^* e^{-it_f}$$

$O(\varepsilon)$: $\frac{\partial^2 f_1}{\partial t_f^2} + 2\frac{\partial^2 f_0}{\partial t_s \partial t_f} = -f_1 + f_0\left(\frac{\partial f_0}{\partial t_f}\right)^4$

$$\frac{\partial^2 f_1}{\partial t_f^2} + f_1 = -2\frac{\partial^2 f_0}{\partial t_s \partial t_f} + f_0\left(\frac{\partial f_0}{\partial t_f}\right)^4$$

$$= -2\left[iA'e^{it_f} - iA^{*\prime}e^{-it_f}\right] + \left(Ae^{it_f} + A^*e^{-it_f}\right)\left[iAe^{it_f} - Bie^{-it_f}\right]^4$$

$$= -2iA'e^{it_f} + 2iA^{*\prime}e^{-it_f} + A^5e^{5ix_f} - 3A^4A^*e^{3ix_f} + 2A^3A^{*2}e^{ix_f} + 2A^2A^{*3}e^{-ix_f} - 3AA^{*4}e^{-3ix_f} + A^{*5}e^{-5ix_f}$$

$$= e^{it_f}\{-2iA' + 2A^3A^{*2}\} + e^{-it_f}\{2iA^{*\prime} + 2A^2A^{*3}\} + \cdots$$

$$\rightarrow \begin{cases} 2iA' = 2A^3A^{*2} \\ -2iA^{*\prime} = 2A^2A^{*3} \end{cases}$$

$\cancel{2}iA' = \cancel{2}A^3A^{*2}$

$iA' = A|A|^4$

$i\frac{\partial}{\partial t_s}(|A|e^{i\theta}) = |A|e^{i\theta}\ |A|^4$

$i\left[|A|\cancel{e^{i\theta}}\frac{\partial\theta}{\partial t_s} + \frac{\partial|A|}{\partial t_s}\cancel{e^{i\theta}}\right] = |A|^5\cancel{e^{i\theta}}$

$-|A|\frac{\partial\theta}{\partial t_s} + i\frac{\partial|A|}{\partial t_s} = |A|^5$

R_e: $-|A|\frac{\partial\theta}{\partial t_s} = |A|^5 \rightarrow \frac{\partial\theta}{\partial t_s} = -|A|^4 \rightarrow \frac{\partial\theta}{\partial t_s} = -A_0^4 \rightarrow \theta = \theta_0 - A_0^4 t_s$ BY WOLFRAM

I_m: $\frac{\partial|A|}{\partial t_s} = 0 \rightarrow |A| = A_0$ (CONSTANT)

$$\Bigg\} \quad A = |A|e^{i\theta} = A_0 e^{i(\theta_0 - A_0^4 t_s)}$$

i IS SQUARED AWAY IMPLYING $|A|$ IS CONSTANT → NO AMPL. CHANGE? ε

FULL SOLN:

$$f_0 = Ae^{it_f} + A^*e^{-it_f}$$

$$= A_0e^{i(\theta_0 - A_0^4 t_s)}e^{it_f} + A_0e^{-i(\theta_0 - A_0^4 t_s)}e^{-it_f}$$

$$= A_0\left[e^{i(\theta_0 - A_0^4 t_s + t_f)} + e^{-i(\theta_0 - A_0^4 t_s + t_f)}\right]$$

$$= 2A_0\cos(\theta_0 - A_0^4 t_s + t_f)$$

$$= 2A_0\cos(\theta_0 - A_0^4 \varepsilon t + t)$$

$1 = 2A_0\cos(\theta_0)$

$\frac{1}{2} = A_0\cos(\theta_0)$

$\frac{\partial f}{\partial t_f} = -2A_0\sin(\theta_0 - A_0^4\varepsilon t + t)$

$\left.\frac{\partial f}{\partial t}\right|_0 = -2A_0\sin(\theta_0)$

$0 = -2A_0\sin(\theta_0) \rightarrow \theta_0 = 0$

$1 = 2A_0\cos(\theta_0)$

$= 2A_0 \rightarrow A_0 = \frac{1}{2}$

$f = \frac{1}{2}(2)\cos\left(0 - \left(\frac{1}{2}\right)^4\varepsilon t + t\right)$

$= \cos\left(-\frac{1}{16}\varepsilon t + t\right)$

↓

CHANGE OF FREQ???!

1B) $\frac{\partial^2 f}{\partial t^2} = -f + \varepsilon f\left(\frac{\partial f}{\partial t}\right)^4$, $f(0)=1$ $f'|_0 = 0$

#3

TIMESCALE ?

$\left[f'' = -f + \varepsilon f (f')^4\right]$

$\frac{F}{T^2} = -\cancel{F} + \frac{F^2}{T^4}$

$\frac{F}{T^2} = \frac{F}{T^2}$ ✓

LET $t_s = \varepsilon t$, $t_f = t$

$\frac{d}{dt} = \frac{d}{dt_s}\varepsilon + \frac{d}{dt_f}$

$\frac{d^2}{dt^2} = \varepsilon^2 \frac{d^2}{dt_s^2} + 2\varepsilon \frac{d}{dt_s dt_f} + \frac{d^2}{dt_f^2}$

$\rightarrow \left(\frac{d^2}{dt_f^2} + 2\varepsilon\frac{d^2}{dt_s dt_f} + \varepsilon^2\frac{d^2}{dt_s^2}\right)(f_0 + \varepsilon f_1 + \cdots) = -(f_0 + \cdots) + \varepsilon\left[(f_0+\cdots)\left(\frac{\partial}{\partial t_s}\varepsilon + \frac{d}{dt_f}\right)\right]^4$

$+ \varepsilon\,(f_0 + \cdots)\left(\frac{\partial f_0}{\partial t_f}^4 + \cdots\right)$

$O(\varepsilon^0)$: $\frac{d^2 f_0}{dt_f^2} = -f_0$

$f_0 = A(t_s)e^{it_f} + A^*(t_f)e^{-it_f}$ LET $B = A^*$

$O(\varepsilon)$: $\frac{d^2 f_1}{dt_f^2} + 2\frac{\partial^2}{\partial t_s \partial t_f} f_0 = -f_1 + f_0\left[\frac{\partial f_0}{\partial t_f}\right]^4$

$\frac{\partial^2 f_1}{\partial t_f^2} + f_1 = -2\frac{\partial^2 f_0}{\partial t_s \partial t_f} + f_0\left[\frac{\partial f_0}{\partial t_f}\right]^4$

$= -2\left[iA'e^{it_f} - iB'e^{-it_f}\right] + \left[Ae^{it_f} + Be^{-it_f}\right]\left(\cancel{i}Ae^{it_f} - \cancel{i}Be^{-it_f}\right)^4 \cancel{(i)^4}$

$= -2i\left[A'e^{it_f} - B'e^{-it_f}\right] + \left[Ae^{it_f} + Be^{-if}\right]\left[A^4e^{4ix_f} - 4A^3Be^{2ix_f} + 6A^2B^2 - 4AA^{*3}e^{-2ix_f} + B^4e^{-4ix_f}\right]$

$= -2iA'e^{it_f} + 2iB'e^{-it_f} + A^5e^{5ix} - 3A^4A^*e^{3ix} + 2A^3B^2e^{ix} + 2A^2B^3e^{-ix} - 3AB^4e^{-3ix} + B^5e^{-5ix}$

$= e^{it_f}\{-2iA' + 2A^3B^2\} + e^{-it_f}\{2iB' + 2A^2B^3\}$

$0 = -2iA' + 2A^3A^{*2}$

$2iA' = 2A^3A^{*2}$

$2iA' = 2A(\cancel{A^2A^{*2}})|A|^4$

$2iA'$

$2i\frac{\partial}{\partial t_s}\left(|A(t_s)|e^{i\theta(t_s)}\right) = 2\left(|A|e^{i\theta(t_s)}\right)|A|^4$

$\cancel{2}i\left\{\left(\frac{\partial|A|}{\partial t_s} + i|A|\frac{\partial\theta}{\partial t_s}\right)\cancel{e^{i\theta}}\right\} = \cancel{2}|A|^5\cancel{e^{i\theta}}$

IM: $\frac{\partial|A|}{\partial t_s} = 0 \rightarrow |A| = A_0$

RE: $-|A|\frac{\partial\theta}{\partial t_s} = |A|^5$

$\frac{\partial\theta}{\partial t_s} = -|A(t_s)|^4$

$\theta = -\int|A(t_s)|^4 dt_s = -A_0^4 x_s + \theta_0$

$A = A_0e^{i(-A_0^4x_s + \theta_0)}$

$f = A_0e^{i(-A_0^4x_s+\theta_0)}e^{ix_f}$

$+ A_0e^{-i(-A_0^4x_s+\theta_0)}e^{-ix_f}$

$= A_0e^{-iA_0^4 + \theta_0 i + ix_f}$

$+ A_0e^{iA_0^4 - \theta_0 i - ix_f}$

$= A_0\{e^{i(-A_0^4x_s + \theta_0 + x_f)}$

$+ e^{-i(-A_0^4x_s + \theta_0 + x_f)}$

$= 2A_0\cos(-A_0^4x_s + \theta_0 + x_f)$

$\frac{f}{dx_f} = -2A_0\sin(-A_0^4x_s + \theta_0 + x_f)$

$f(0) = 1$

$1 = 2A_0\cos(+\theta_0)$

$\frac{1}{2} = A_0\cos(+\theta_0)$

$f'(0) = 0$

$0 = -2A_0\sin(-\theta_0)$

$\theta_0 = 0$

$A_0 = \frac{1}{2}$

$\cos\left(\frac{1}{16}\varepsilon x - x\right)$?