AM213B Midterm

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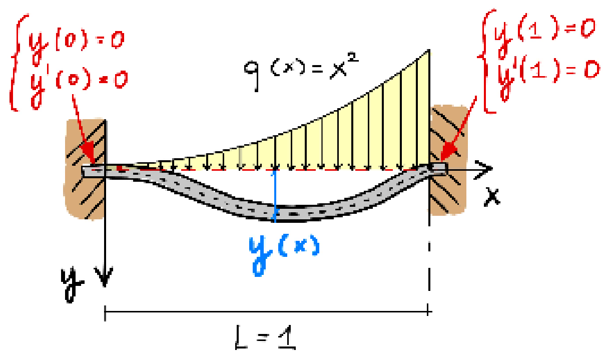
Q1.

Consider the fully clamped Euler-Bernoulli beam



The vertical displacement y(x) satisfies the following two-plot boundary value problem:

$$EI\frac{d^4y}{dx^4} = q(x)$$
$$y(0) = 0$$
$$y(1) = 0$$
$$\frac{dy(0)}{dx} = 0$$
$$\frac{dy(1)}{dx} = 0$$

Where EI = 1 and q(x) = x^2 (load).

Q1a.

Determine the analytical solution y(x) of the BVP.

Solution

First we integrate $EI\frac{d^4y}{dx^4} = q(x) = x^2$ to derive the general solution without enforcing our boundary conditions:

$$
\begin{aligned}
EI\frac{d^4y}{dx^4} &= q(x) \\
\frac{d^4y}{dx^4} &= x^2 \\
\int \frac{d^4y}{dx^4} &= \int x^2 \\
\frac{d^3y}{dx^3} &= \frac{x^3}{3} + c_1 \\
\int \frac{d^3y}{dx^3} &= \int \frac{x^3}{3} + c_1 \\
\frac{d^2y}{dx^2} &= \frac{x^4}{12} + c_1x + c_2 \\
\int \frac{d^2y}{dx^2} &= \int \frac{x^4}{12} + c_1x + c_2 \\
\frac{dy}{dx} &= \frac{x^5}{60} + \frac{c_1x^2}{2} + c_2x + c_3 \\
\int \frac{dy}{dx} &= \int \frac{x^5}{60} + \frac{c_1x^2}{2} + c_2x + c_3 \\
y &= \frac{x^6}{360} + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4
\end{aligned}
$$

Now we enforce our boundary conditions:

$$
\begin{aligned}
0 &= y(0) : \\
&\implies c_4 = 0 \\
0 &= y(1) : \\
0 &= \frac{1}{360} + \frac{c_1}{6} + \frac{c_2}{2} + c_3 \\
&\implies c_3 = 0 \\
y'(0) &= 0 : \\
&\implies c_3 = 0 \\
y'(1) &= 0 : \\
0 &= \frac{1}{60} + \frac{c_1}{2} + c_2
\end{aligned}
$$

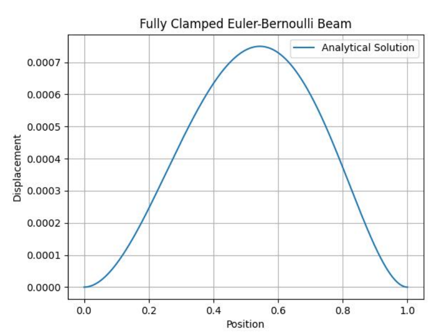
Now we have the system to solve for c_1 and c_2:

$$\begin{aligned}
0 &= \frac{1}{360} + \frac{c_1}{6} + \frac{c_2}{2} + c_3 \\
&= \frac{1}{60} + c_1 + 3c_2 + 6c_3 \\
0 &= \frac{1}{60} + \frac{c_1}{2} + c_2 \\
&= \frac{1}{10} + 3c_1 + 6c_2 \\
\implies \frac{1}{60} + c_1 + 3c_2 + 6c_3 &= \frac{1}{10} + 3c_1 + 6c_2 \\
\implies c_1 &= -4c_2 \\
0 &= \frac{1}{60} + c_1 + 3c_2 \\
\implies 0 &= \frac{1}{60} + -4c_2 + 3c_2 \\
0 &= \frac{1}{60} - c_2 \\
\implies c_2 &= \frac{1}{60} \\
c_1 &= -4c_2 \\
\implies c_1 &= \frac{-4}{60} \\
&= -\frac{1}{15} \\
\therefore (c_1, c_2) &= (-\frac{1}{15}, \frac{1}{60})
\end{aligned}$$

Thus we have the solution with our boundary conditions imposed:

$$y = \frac{x^6}{360} - \frac{x^3}{90} + \frac{x^2}{120}$$

We show the plot of the analytical solution:



Q1b.

Determine the numerical solution to the problem by using the shooting method with Newton's iteration. Use the explicit RK4 scheme defined by the following butcher array:

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

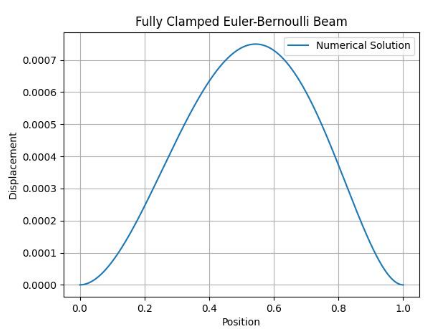
To solve the initial value corresponding to the shooting method. In particular, set $\Delta x = \frac{1}{N}$, and N = 60000.
Solution: We use the RK3 method defined by the butcher array:

$$\begin{aligned} K_1 &= f(x_k, y_k) \\ K_2 &= f(x_k + \frac{h}{2}, y_k + \frac{h}{2}K_1) \\ K_3 &= f(x_k + \frac{h}{2}, y_k + \frac{h}{2}K_2) \\ K_4 &= f(x_k + h, y_k + K_3) \\ u_{k+1} &= u_k + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \end{aligned}$$

We solve the initial value and find:

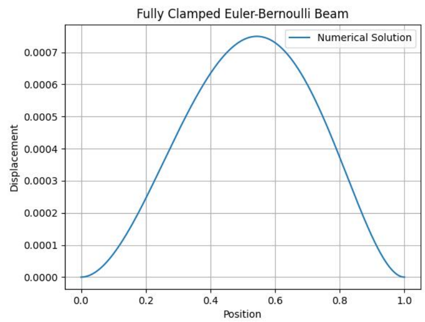
$$(v_0, v_1) = (0.01666667, -0.06666833)$$

Additionally we plot the numerical solution and find:



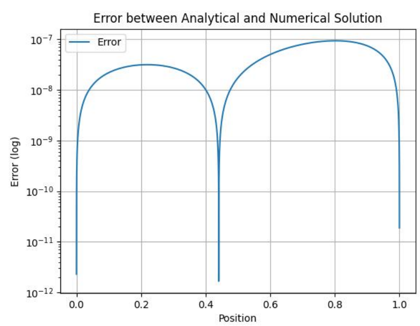
Q1c.

Plot the numerical solution obtained with the shooting method
Solution:
We plot the numerical solution:



Q1d.

Plot the error between the analytical and numerical solution using a logarithmic scale.
Solution:
We plot the error and find:



Q2.

Consider the implicit RK3 method defined by the Butcher array

$$
\begin{array}{c|ccc}
0 & & & \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \\
\frac{1}{2} & 0 & \frac{1}{2} & \\
1 & 0 & 1 & 0 \\
\hline
 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6}
\end{array}
$$

Q2a.

Prove that the method is convergent
Solution:

We prove convergence of the RK3 method defined by:

$$\begin{aligned}
K_1 &= f(u_k, t_k) \\
K_2 &= f(u_k + \frac{h}{4}K_1 + \frac{h}{4}K_2, t_k + \frac{h}{2}) \\
K_3 &= f(u_k + hK_1, t_k + h) \\
u_{k+1} &= u_k + h(\frac{1}{6}K_1 + \frac{2}{3}K_2 + \frac{1}{6}K_3)
\end{aligned}$$

We obtain:

$$\begin{aligned}
u_{k+1} &= u_k + h(\frac{1}{6}K_1 + \frac{2}{3}K_2 + \frac{1}{6}K_3) \\
u_{k+1} - u_k &= h(\frac{1}{6}K_1 + \frac{2}{3}K_2 + \frac{1}{6}K_3) \\
&\implies \alpha_0 = -1, \alpha_1 = 1
\end{aligned}$$

We find the first characteristic polynomial,

$$\begin{aligned}
&\rho(z) = z - 1 \\
&\implies \rho(1) = 0
\end{aligned}$$

Thus the method is zero-stable.

Now we show the method is consistent,

$$\begin{aligned}
\sum_{i=1}^{3} b_i &= \frac{1}{6} + \frac{2}{3} + \frac{1}{6} \\
&= 1
\end{aligned}$$

Thus the method is consistent.

Now we've shown the method is both consistent and zero stable, and thus is convergent.

Q2b.

Plot the region of absolute stability of the method.
Solution:
We find different regions depending on whether we use the explicit or implicit formula for the region of stability. We test both because as we will see in the later parts of the solutions for 2, the explicit formula leads to us finding the

correct Δt^*, which is shown by convergence in part d, but contradicts the fact this is an implicit method. This implies there is some error in either my RK3 method, or my region derived. Currently, I believe the explicit formulation is correct since it shows convergence of the RK3 methods, but this may be an error in my numerical solution.

We use the equation derived using Cramer's rule:

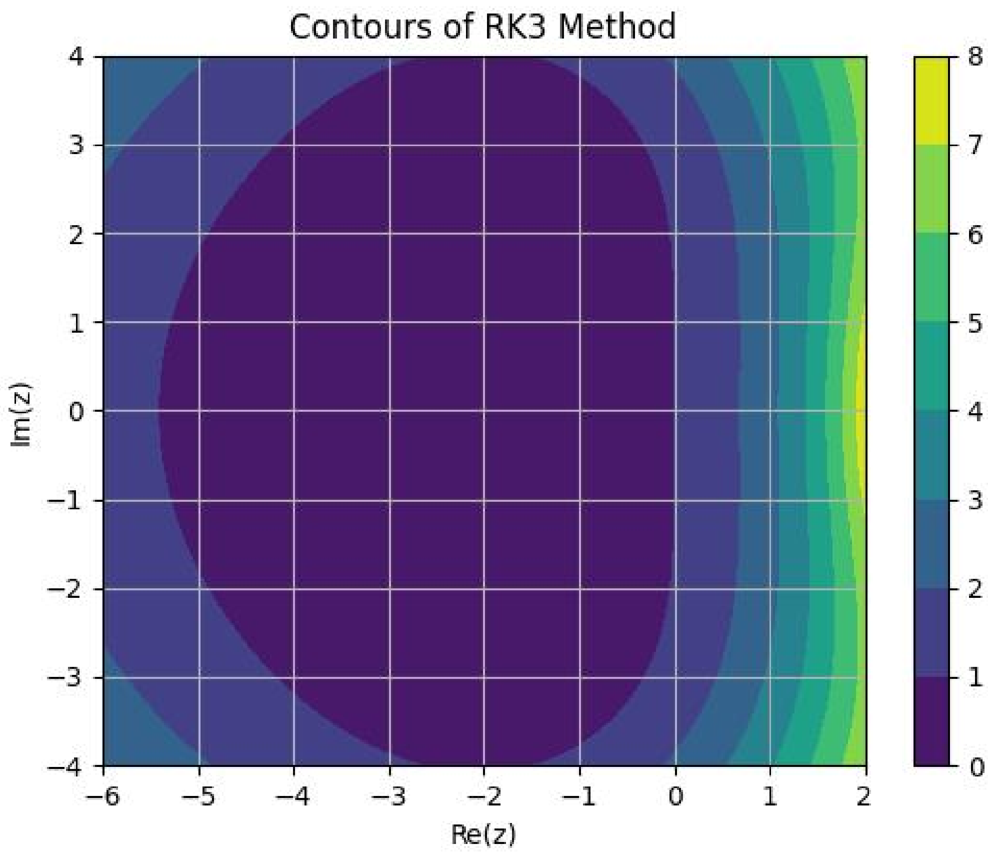
Implicit Formula:

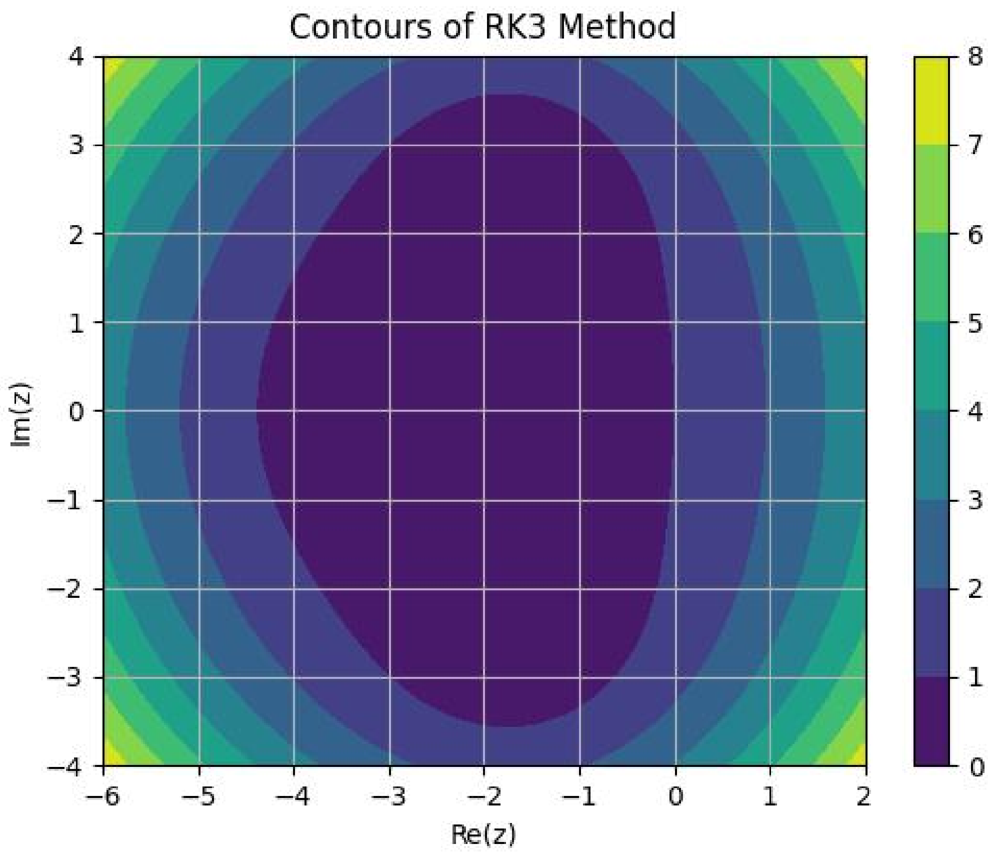
$$S(z) = \frac{det(I - zA + zhb^T)}{det(I - zA)}$$

Explicit Formula:

$$S(z) = \frac{det(I - zA + zhb^T)}{1}$$

The region of stability is defined at z $<$ 1 in the contour maps below, shown respective to the formulas used (implicit then explicit)


Contours of RK3 Method
Im(z)
Re(z)
4
3
2
1
0
−1
−2
−3
−4
−6
−5
−4
−3
−2
−1
0
1
2
8
7
6
5
4
3
2
1
0



Q2c.

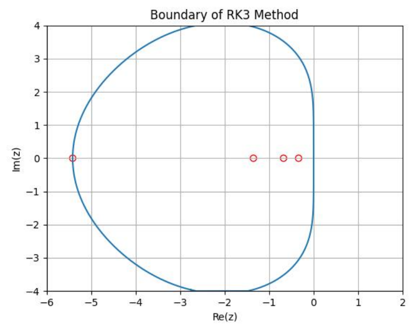
Find Δt^* for which the implicit RK3 defined in Q2 applied to $\frac{dy}{dt} = By$ where

$$B = \begin{bmatrix} -1 & 3 & -5 & 7 \\ 0 & -2 & 4 & -6 \\ 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -16 \end{bmatrix}$$
$$y_0 = [1, 1, 1, 1]^T$$

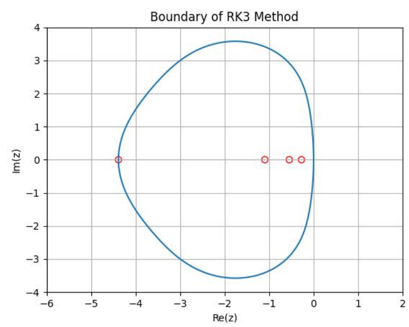
is absolutely stable.
Solution:
We find respective Δt^* derived for each region of stability. The Δt^* derived from the explicit formula corresponds to converging solutions, but contradicts the fact that the method is implicit. We note both below with a tolerance of

10^{-8} from the boundary, along with the plotted eigenvalues to shown that they are within the region of stability for the respective formulations.

- Implicit:
  We find $\Delta t^* \cdot =0.338$

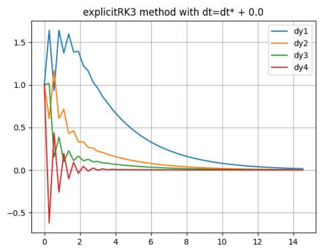


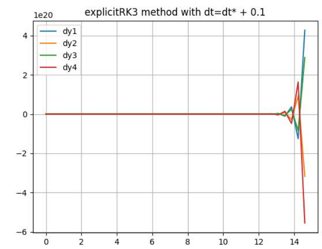
- Explicit:
  We find $\Delta t^* \cdot =0.274$

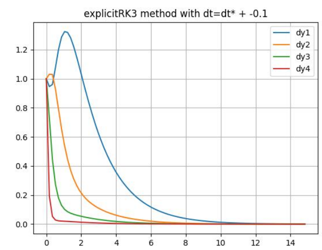


Q2d.

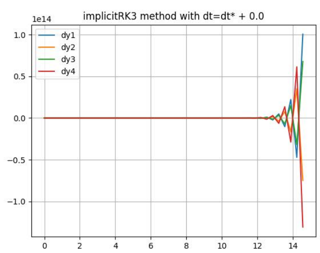
Verify your predictions numerically for values slightly larger and smaller Δt, and show the solution converges to zero or diverges to infinity.
Solution: First we show our scheme with our Δt^* found using the stability region assuming the scheme is explicit, i.e $\Delta t \cdot =0.274$,

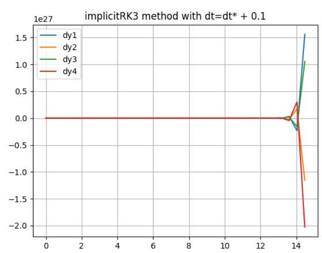


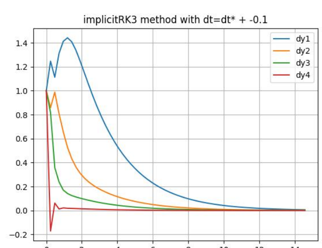




Next we show our scheme using the Δt^* using the formulation of the stability region when we assume the scheme is implicit, i.e. $\Delta t \cdot =$0.339







We find the Δt^* corresponding to the explicit formulation of the stability region gives us the outcomes as expected. At our maximum timestep that scales our eigenvalues to within the stability region, we see that the method converges. If we decrease our timestep, we see faster convergence. If we increase our timestep to outside the stability region, we see that our method diverges. This implies that $\Delta t^* = \cdot =$0.274 is correct. Whereas the Δt^* derived using the

stability region corresponding to the implicit formulation causes the solution to diverge, and is thus not within the stability region.
However, this contradicts the fact that the scheme is implicit, and may be the result of an incorrect RK3 implementation.

Q3.

Consider the linear multi-step method:

$$u_{k+3} - \frac{1}{3}(u_{k+2} + u_{k+1} + u_k) = \frac{h}{12}(23f_{k+2} - 2f_{k+1} + 3f_k)$$

Q3a.

Show that the method is convergent and determine the convergence order.
Solution:

$$u_{k+3} - \frac{1}{3}(u_{k+2} + u_{k+1} + u_k) = \frac{h}{12}(23f_{k+2} - 2f_{k+1} + 3f_k)$$

For this scheme we have:

$$\begin{aligned}
\alpha_0 &= -\frac{1}{3} \\
\alpha_1 &= -\frac{1}{3} \\
\alpha_2 &= -\frac{1}{3} \\
\alpha_3 &= 1 \\
\beta_0 &= \frac{1}{4} \\
\beta_1 &= -\frac{1}{6} \\
\beta_2 &= \frac{23}{12} \\
\beta_3 &= 0
\end{aligned}$$

We compute the order of consistency:

$$\begin{aligned}
\rho(z) &= -\frac{1}{3} - \frac{z}{3} - \frac{z^2}{3} + z^3 \\
\rho'(z) &= -\frac{1}{3} - \frac{2}{3}z + 3z^2 \\
\sigma(z) &= \frac{1}{4} - \frac{1}{6}z + \frac{23}{12}z^2
\end{aligned}$$

We compute the coefficients,

$$
\begin{aligned}
C_0 &= \rho(1) \\
&= -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} + 1 \\
&= 0 \\
C_1 &= \rho'(1) - \sigma(1) \\
&= (-\frac{1}{3} - \frac{2}{3}1 + 3 * 1^2) - (\frac{1}{4} - \frac{1}{6}1 + \frac{23}{12} * 1^2) \\
&= -\frac{1}{3} - \frac{2}{3} + 3 - \frac{1}{4} + \frac{1}{6} - \frac{23}{12} \\
&= -\frac{4}{12} - \frac{8}{12} + \frac{36}{12} - \frac{3}{12} + \frac{2}{12} - \frac{23}{12} \\
&= 0
\end{aligned}
$$

Therefore the scheme is consistent. Now we continue to compute the order of consistency.

$$
\begin{aligned}
C_2 &= \frac{1}{2!}\sum_{j=0}^{3}(j^2\alpha_j - 2j\beta_j) \\
&= \frac{1}{2!}(0 + 0 - 9 + 9) \\
&= 0 \implies \text{Not order 1} \\
C_3 &= \frac{1}{3!}\sum_{j=0}^{3}(j^3\alpha_j - 3j^2\beta_j) \\
&= \frac{1}{3!}(0 + 0 - 9 + 9) \\
&= 0 \implies \text{Not order 2} \\
C_3 &= \frac{1}{3!}\sum_{j=0}^{3}(j^3\alpha_j - 3j^2\beta_j) \\
&= \frac{1}{3!}(0 + \frac{1}{6} - \frac{154}{6} + 27) \\
&= \frac{1}{3!}(\frac{1}{6})(1 - 154 + 162) \\
&= \frac{1}{3!}\frac{1}{6}(9) \\
&\neq 0 \implies \text{Consistent with order 2!}
\end{aligned}
$$

Now we show it's zero-stable:

$$\begin{aligned}\rho(z) &= -\frac{1}{3} - \frac{z}{3} - \frac{z^2}{3} + z^3 \\ \rho(1) &= -\frac{1}{3} - \frac{1}{3} - \frac{1^2}{3} + 1^3 \\ &= 0 \implies \text{zero stable}\end{aligned}$$

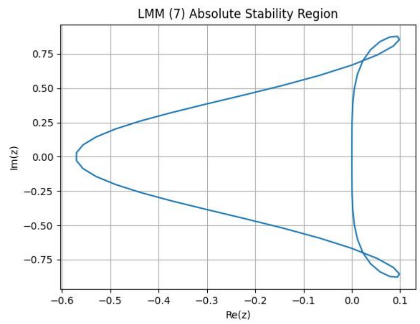
Now we've shown the scheme is both consistent with order 2 and zero stable, and is thus convergent with order 2.

Q3b.

Plot the region of stability. Is the LMM A-stable?
Solution: We first compute the region of stability,

$$\begin{aligned}\lambda_j \Delta t &= \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \\ \rho(z) &= -\frac{1}{3} - \frac{1}{3}z - \frac{1}{3}z^2 + z^3 \\ \rho(e^{i\theta}) &= -\frac{1}{3} - \frac{1}{3}e^{i\theta} - \frac{1}{3}(e^{i\theta})^2 + (e^{i\theta})^3 \\ &= \frac{1}{3}(-1 - e^{i\theta} - e^{2i\theta} + 3e^{3i\theta}) \\ \sigma(z) &= \frac{1}{4} - \frac{1}{6}z + \frac{23}{12}z^2 \\ \sigma(e^{i\theta}) &= \frac{1}{4} - \frac{1}{6}e^{i\theta} + \frac{23}{12}(e^{i\theta})^2 \\ &= \frac{1}{12}(3 - 2e^{i\theta} + 23e^{2i\theta}) \\ \implies \lambda_j \Delta t &= \frac{\frac{1}{3}(-1 - e^{i\theta} - e^{2i\theta} + 3e^{3i\theta})}{\frac{1}{12}(3 - 2e^{i\theta} + 23e^{2i\theta})} \\ &= 4\frac{-1 - e^{i\theta} - e^{2i\theta} + 3e^{3i\theta}}{3 - 2e^{i\theta} + 23e^{2i\theta}}\end{aligned}$$

We show the plot below:



By course note 5 (theorem 2), we see that the inside is the region of stability. Additionally we can conclude by the definition of A-stable, that the region shown is not A-stable.