AM213B Midterm

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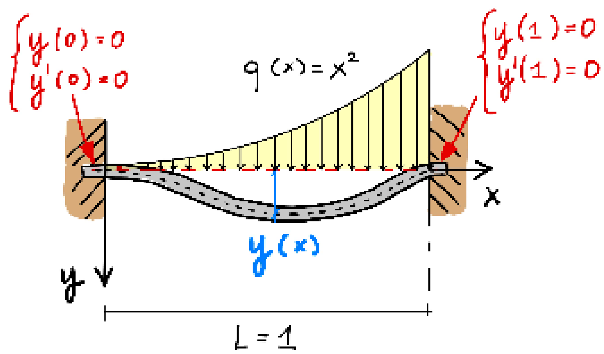
Q1.

Consider the fully clamped Euler-Bernoulli beam



The vertical displacement y(x) satisfies the following two-plot boundary value problem:

$$EI\frac{d^4y}{dx^4} = q(x)$$
$$y(0) = 0$$
$$y(1) = 0$$
$$\frac{dy(0)}{dx} = 0$$
$$\frac{dy(1)}{dx} = 0$$

Where EI = 1 and q(x) = x^2 (load).

Q1a.

Determine the analytical solution y(x) of the BVP.

Solution

First we integrate $EI\frac{d^4y}{dx^4} = q(x) = x^2$ to derive the general solution without enforcing our boundary conditions:

$$
\begin{aligned}
EI\frac{d^4y}{dx^4} &= q(x) \\
\frac{d^4y}{dx^4} &= x^2 \\
\int \frac{d^4y}{dx^4} &= \int x^2 \\
\frac{d^3y}{dx^3} &= \frac{x^3}{3} + c_1 \\
\int \frac{d^3y}{dx^3} &= \int \frac{x^3}{3} + c_1 \\
\frac{d^2y}{dx^2} &= \frac{x^4}{12} + c_1x + c_2 \\
\int \frac{d^2y}{dx^2} &= \int \frac{x^4}{12} + c_1x + c_2 \\
\frac{dy}{dx} &= \frac{x^5}{60} + \frac{c_1x^2}{2} + c_2x + c_3 \\
\int \frac{dy}{dx} &= \int \frac{x^5}{60} + \frac{c_1x^2}{2} + c_2x + c_3 \\
y &= \frac{x^6}{360} + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4
\end{aligned}
$$

Now we enforce our boundary conditions:

$$
\begin{aligned}
0 &= y(0) : \\
&\implies c_4 = 0 \\
0 &= y(1) : \\
0 &= \frac{1}{360} + \frac{c_1}{6} + \frac{c_2}{2} + c_3 \\
&\implies c_3 = 0 \\
y'(0) &= 0 : \\
&\implies c_3 = 0 \\
y'(1) &= 0 : \\
0 &= \frac{1}{60} + \frac{c_1}{2} + c_2
\end{aligned}
$$

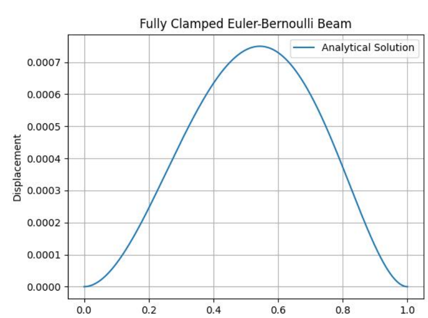
Now we have the system to solve for c_1 and c_2:

$$\begin{aligned}
0 &= \frac{1}{360} + \frac{c_1}{6} + \frac{c_2}{2} + c_3 \\
&= \frac{1}{60} + c_1 + 3c_2 + 6c_3 \\
0 &= \frac{1}{60} + \frac{c_1}{2} + c_2 \\
&= \frac{1}{10} + 3c_1 + 6c_2 \\
\implies \frac{1}{60} + c_1 + 3c_2 + 6c_3 &= \frac{1}{10} + 3c_1 + 6c_2 \\
\implies c_1 &= -4c_2 \\
0 &= \frac{1}{60} + c_1 + 3c_2 \\
\implies 0 &= \frac{1}{60} + -4c_2 + 3c_2 \\
0 &= \frac{1}{60} - c_2 \\
\implies c_2 &= \frac{1}{60} \\
c_1 &= -4c_2 \\
\implies c_1 &= \frac{-4}{60} \\
&= -\frac{1}{15} \\
\therefore (c_1, c_2) &= (-\frac{1}{15}, \frac{1}{60})
\end{aligned}$$

Thus we have the solution with our boundary conditions imposed:

$$y = \frac{x^6}{360} - \frac{x^3}{90} + \frac{x^2}{120}$$

We show the plot of the analytical solution:



Q1b.

Determine the numerical solution to the problem by using the shooting method with Newton's iteration. Use the explicit RK4 scheme defined by the following butcher array:

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

To solve the initial value corresponding to the shooting method. In particular, set $\Delta x = \frac{1}{N}$, and N = 60000.

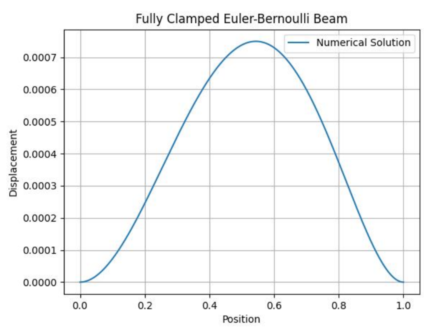
Solution: We use the RK3 method defined by the butcher array:

$$\begin{aligned} K_1 &= f(x_k, y_k) \\ K_2 &= f(x_k + \frac{h}{2}, y_k + \frac{h}{2}K_1) \\ K_3 &= f(x_k + \frac{h}{2}, y_k + \frac{h}{2}K_2) \\ K_4 &= f(x_k + h, y_k + K_3) \\ u_{k+1} &= u_k + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \end{aligned}$$

We solve the initial value and find:

$$(v_0, v_1) = (0.01666667, -0.06666833)$$

Additionally we plot the numerical solution and find:

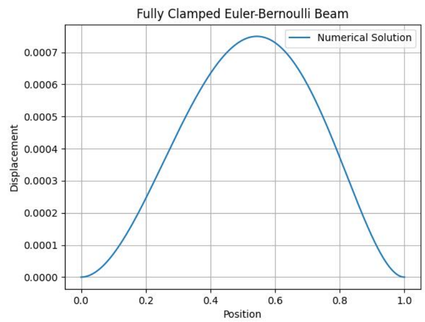


Q1c.

Plot the numerical solution obtained with the shooting method

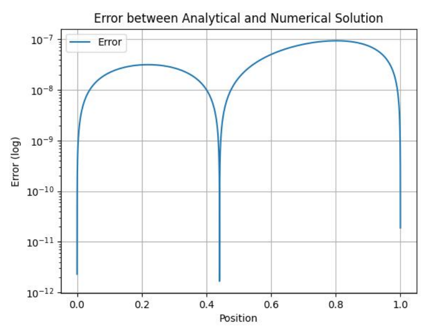
Solution:

We plot the numerical solution:



Q1d.

Plot the error between the analytical and numerical solution using a logarithmic scale.
Solution:
We plot the error and find:



Q2.

Consider the implicit RK3 method defined by the Butcher array

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \\ \frac{1}{2} & 0 & \frac{1}{2} & \\ 1 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Q2a.

Prove that the method is convergent
Solution:

We prove convergence of the RK3 method defined by:

$$\begin{aligned}
K_1 &= f(u_k, t_k) \\
K_2 &= f(u_k + \frac{h}{4}K_1 + \frac{h}{4}K_2, t_k + \frac{h}{2}) \\
K_3 &= f(u_k + hK_1, t_k + h) \\
u_{k+1} &= u_k + h(\frac{1}{6}K_1 + \frac{2}{3}K_2 + \frac{1}{6}K_3)
\end{aligned}$$

We obtain:

$$\begin{aligned}
u_{k+1} &= u_k + h(\frac{1}{6}K_1 + \frac{2}{3}K_2 + \frac{1}{6}K_3) \\
u_{k+1} - u_k &= h(\frac{1}{6}K_1 + \frac{2}{3}K_2 + \frac{1}{6}K_3) \\
&\implies \alpha_0 = -1, \alpha_1 = 1
\end{aligned}$$

We find the first characteristic polynomial,

$$\begin{aligned}
&\rho(z) = z - 1 \\
&\implies \rho(1) = 0
\end{aligned}$$

Thus the method is zero-stable.

Now we show the method is consistent,

$$\begin{aligned}
\sum_{i=1}^{3} b_i &= \frac{1}{6} + \frac{2}{3} + \frac{1}{6} \\
&= 1
\end{aligned}$$

Thus the method is consistent.

Now we've shown the method is both consistent and zero stable, and thus is convergent.

Q2b.

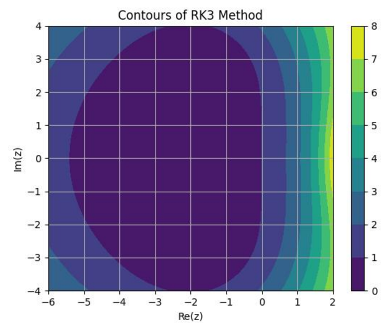
Plot the region of absolute stability of the method.
Solution:

We use the equation derived using Cramer's rule:

Implicit Formula:

$$S(z) = \frac{det(I - zA + zhb^T)}{det(I - zA)}$$

The region of stability is defined at z < 1 in the contour map below

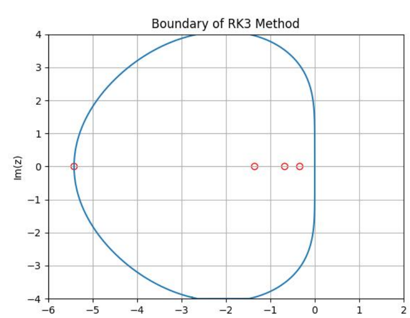


Q2c.

Find Δt^* for which the implicit RK3 defined in Q2 applied to $\frac{dy}{dt} = By$ where

$$B = \begin{bmatrix} -1 & 3 & -5 & 7 \\ 0 & -2 & 4 & -6 \\ 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -16 \end{bmatrix}$$
$$y_0 = [1, 1, 1, 1]^T$$

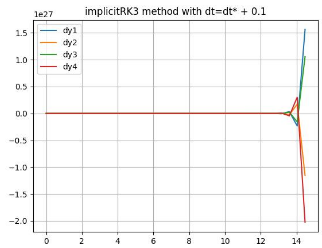
is absolutely stable.
Solution:
We use a tolerance of 10^{-8}, i.e. we find the largest timestep such that the largest eigenvalue is within 10^{-8} of the boundary.
We find $\Delta t^* \cdot$ =0.339

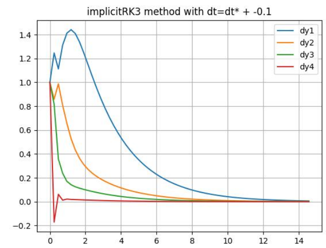


Q2d.

Verify your predictions numerically for values slightly larger and smaller Δt, and show the solution converges to zero or diverges to infinity.
Solution:

Next we show our scheme using $\Delta t^* \cdot = 0.339 \pm 0.1$





We find the Δt^* gives us the outcomes as expected. If we decrease our timestep by 0.1, we see faster convergence. If we increase our timestep to outside the stability region by a value of 0.1, we see that our method diverges. This implies that $\Delta t^* = \cdot = 0.339$ is correct.

Q3.

Consider the linear multi-step method:

$$u_{k+3} - \frac{1}{3}(u_{k+2} + u_{k+1} + u_k) = \frac{h}{12}(23f_{k+2} - 2f_{k+1} + 3f_k)$$

Q3a.

Show that the method is convergent and determine the convergence order.
Solution:

$$u_{k+3} - \frac{1}{3}(u_{k+2} + u_{k+1} + u_k) = \frac{h}{12}(23f_{k+2} - 2f_{k+1} + 3f_k)$$

For this scheme we have:

$$\begin{aligned}
\alpha_0 &= -\frac{1}{3} \\
\alpha_1 &= -\frac{1}{3} \\
\alpha_2 &= -\frac{1}{3} \\
\alpha_3 &= 1 \\
\beta_0 &= \frac{1}{4} \\
\beta_1 &= -\frac{1}{6} \\
\beta_2 &= \frac{23}{12} \\
\beta_3 &= 0
\end{aligned}$$

We compute the order of consistency:

$$\begin{aligned}
\rho(z) &= -\frac{1}{3} - \frac{z}{3} - \frac{z^2}{3} + z^3 \\
\rho'(z) &= -\frac{1}{3} - \frac{2}{3}z + 3z^2 \\
\sigma(z) &= \frac{1}{4} - \frac{1}{6}z + \frac{23}{12}z^2
\end{aligned}$$

We compute the coefficients,

$$\begin{aligned}
C_0 &= \rho(1) \\
&= -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} + 1 \\
&= 0 \\
C_1 &= \rho'(1) - \sigma(1) \\
&= (-\frac{1}{3} - \frac{2}{3}1 + 3 * 1^2) - (\frac{1}{4} - \frac{1}{6}1 + \frac{23}{12} * 1^2) \\
&= -\frac{1}{3} - \frac{2}{3} + 3 - \frac{1}{4} + \frac{1}{6} - \frac{23}{12} \\
&= -\frac{4}{12} - \frac{8}{12} + \frac{36}{12} - \frac{3}{12} + \frac{2}{12} - \frac{23}{12} \\
&= 0
\end{aligned}$$

Therefore the scheme is consistent. Now we continue to compute the order of consistency.

$$
\begin{aligned}
C_2 &= \frac{1}{2!}\sum_{j=0}^{3}(j^2\alpha_j - 2j\beta_j) \\
&= \frac{1}{2!}(0 + 0 - 9 + 9) \\
&= 0 \implies \text{Not order 1} \\
C_3 &= \frac{1}{3!}\sum_{j=0}^{3}(j^3\alpha_j - 3j^2\beta_j) \\
&= \frac{1}{3!}(0 + 0 - 9 + 9) \\
&= 0 \implies \text{Not order 2} \\
C_3 &= \frac{1}{3!}\sum_{j=0}^{3}(j^3\alpha_j - 3j^2\beta_j) \\
&= \frac{1}{3!}(0 + \frac{1}{6} - \frac{154}{6} + 27) \\
&= \frac{1}{3!}(\frac{1}{6})(1 - 154 + 162) \\
&= \frac{1}{3!}\frac{1}{6}(9) \\
&\neq 0 \implies \text{Consistent with order 2!}
\end{aligned}
$$

Now we show it's zero-stable:

$$
\begin{aligned}
\rho(z) &= -\frac{1}{3} - \frac{z}{3} - \frac{z^2}{3} + z^3 \\
\rho(1) &= -\frac{1}{3} - \frac{1}{3} - \frac{1^2}{3} + 1^3 \\
&= 0 \implies \text{zero stable}
\end{aligned}
$$

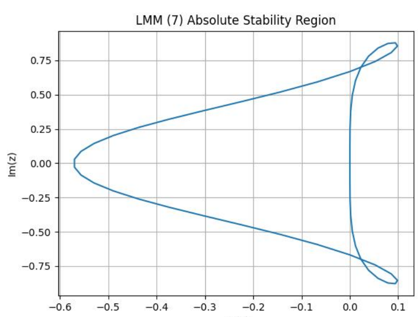
Now we've shown the scheme is both consistent with order 2 and zero stable, and is thus convergent with order 2.

Q3b.

Plot the region of stability. Is the LMM A-stable?
Solution: We first compute the region of stability,

$$
\begin{aligned}
\lambda_j \Delta t &= \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \\
\rho(z) &= -\frac{1}{3} - \frac{1}{3}z - \frac{1}{3}z^2 + z^3 \\
\rho(e^{i\theta}) &= -\frac{1}{3} - \frac{1}{3}e^{i\theta} - \frac{1}{3}(e^{i\theta})^2 + (e^{i\theta})^3 \\
&= \frac{1}{3}(-1 - e^{i\theta} - e^{2i\theta} + 3e^{3i\theta}) \\
\sigma(z) &= \frac{1}{4} - \frac{1}{6}z + \frac{23}{12}z^2 \\
\sigma(e^{i\theta}) &= \frac{1}{4} - \frac{1}{6}e^{i\theta} + \frac{23}{12}(e^{i\theta})^2 \\
&= \frac{1}{12}(3 - 2e^{i\theta} + 23e^{2i\theta}) \\
\implies \lambda_j \Delta t &= \frac{\frac{1}{3}(-1 - e^{i\theta} - e^{2i\theta} + 3e^{3i\theta})}{\frac{1}{12}(3 - 2e^{i\theta} + 23e^{2i\theta})} \\
&= 4\frac{-1 - e^{i\theta} - e^{2i\theta} + 3e^{3i\theta}}{3 - 2e^{i\theta} + 23e^{2i\theta}}
\end{aligned}
$$

We show the plot below:



By course note 5 (theorem 2), we see that the inside is the region of stability. Additionally we can conclude by the definition of A-stable, that the region shown is not A-stable, more explicitly since the scheme is conditionally absolutely stable, it cannot be A-stable.