# AM213B Midterm

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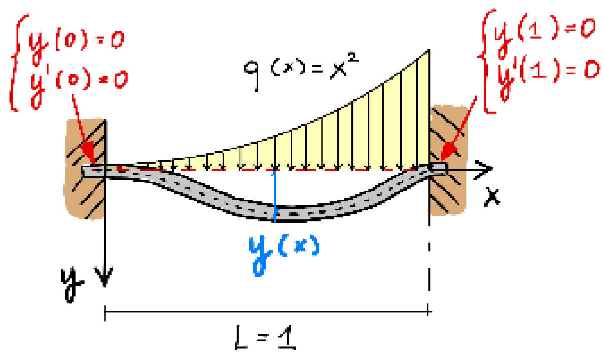
Spring 2024

## Q1.

Consider the fully clamped Euler-Bernoulli beam




The vertical displacement y(x) satisfies the following two-plot boundary value problem:

$$EI\frac{d^4y}{dx^4} = q(x)$$
$$y(0) = 0$$
$$y(1) = 0$$
$$\frac{dy(0)}{dx} = 0$$
$$\frac{dy(1)}{dx} = 0$$

Where EI = 1 and q(x) = $x^2$ (load).

## Q1a.

Determine the analytical solution y(x) of the BVP.

**Solution**

First we integrate $EI\frac{d^4y}{dx^4} = q(x) = x^2$ to derive the general solution without enforcing our boundary conditions:

$$\begin{aligned}
EI\frac{d^4y}{dx^4} &= q(x) \\
\frac{d^4y}{dx^4} &= x^2 \\
\int \frac{d^4y}{dx^4} &= \int x^2 \\
\frac{d^3y}{dx^3} &= \frac{x^3}{3} + c_1 \\
\int \frac{d^3y}{dx^3} &= \int \frac{x^3}{3} + c_1 \\
\frac{d^2y}{dx^2} &= \frac{x^4}{12} + c_1x + c_2 \\
\int \frac{d^2y}{dx^2} &= \int \frac{x^4}{12} + c_1x + c_2 \\
\frac{dy}{dx} &= \frac{x^5}{60} + \frac{c_1x^2}{2} + c_2x + c_3 \\
\int \frac{dy}{dx} &= \int \frac{x^5}{60} + \frac{c_1x^2}{2} + c_2x + c_3 \\
y &= \frac{x^6}{360} + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4
\end{aligned}$$

Now we enforce our boundary conditions:

$$\begin{aligned}
0 &= y(0): \\
&\implies c_4 = 0 \\
0 &= y(1): \\
0 &= \frac{1}{360} + \frac{c_1}{6} + \frac{c_2}{2} + c_3 \\
&\implies c_3 = 0 \\
y'(0) &= 0: \\
&\implies c_3 = 0 \\
y'(1) &= 0: \\
0 &= \frac{1}{60} + \frac{c_1}{2} + c_2
\end{aligned}$$

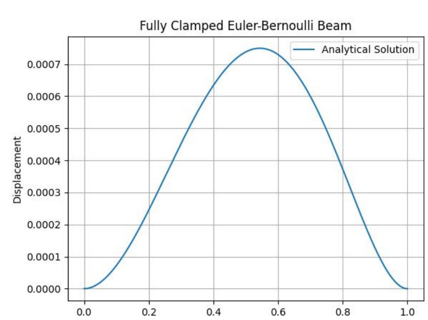
Now we have the system to solve for $c_1$ and $c_2$:

$$
\begin{aligned}
0 &= \frac{1}{360} + \frac{c_1}{6} + \frac{c_2}{2} + c_3 \\
&= \frac{1}{60} + c_1 + 3c_2 + 6c_3 \\
0 &= \frac{1}{60} + \frac{c_1}{2} + c_2 \\
&= \frac{1}{10} + 3c_1 + 6c_2 \\
\implies \frac{1}{60} + c_1 + 3c_2 + 6c_3 &= \frac{1}{10} + 3c_1 + 6c_2 \\
\implies c_1 &= -4c_2 \\
0 &= \frac{1}{60} + c_1 + 3c_2 \\
\implies 0 &= \frac{1}{60} + -4c_2 + 3c_2 \\
0 &= \frac{1}{60} - c_2 \\
\implies c_2 &= \frac{1}{60} \\
c_1 &= -4c_2 \\
\implies c_1 &= \frac{-4}{60} \\
&= -\frac{1}{15} \\
\therefore (c_1, c_2) &= (-\frac{1}{15}, \frac{1}{60})
\end{aligned}
$$

Thus we have the solution with our boundary conditions imposed:

$$
y = \frac{x^6}{360} - \frac{x^3}{90} + \frac{x^2}{120}
$$

We show the plot of the analytical solution:



## Q1b.

Determine the numerical solution to the problem by using the shooting method with Newton's iteration. Use the explicit RK4 scheme defined by the following butcher array:

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

To solve the initial value corresponding to the shooting method. In particular, set $\Delta x = \frac{1}{N}$, and N = 60000.

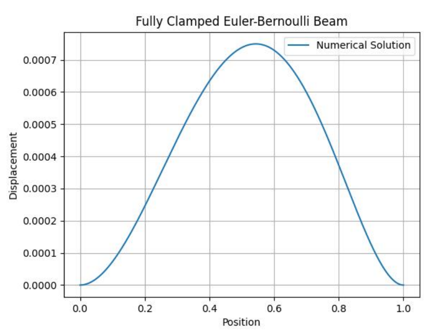
**Solution:** We use the RK3 method defined by the butcher array:

$$\begin{aligned} K_1 &= f(x_k, y_k) \\ K_2 &= f(x_k + \frac{h}{2}, y_k + \frac{h}{2}K_1) \\ K_3 &= f(x_k + \frac{h}{2}, y_k + \frac{h}{2}K_2) \\ K_4 &= f(x_k + h, y_k + K_3) \\ u_{k+1} &= u_k + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \end{aligned}$$

We solve the initial value and find:

$$(v_0, v_1) = (0.01666667, -0.06666833)$$

Additionally we plot the numerical solution and find:

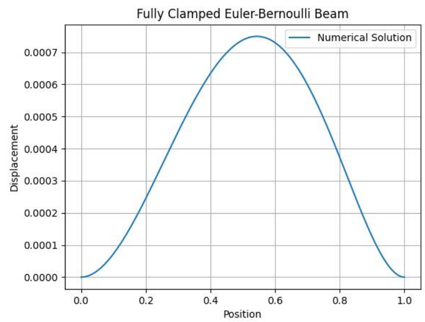



## Q1c.

Plot the numerical solution obtained with the shooting method

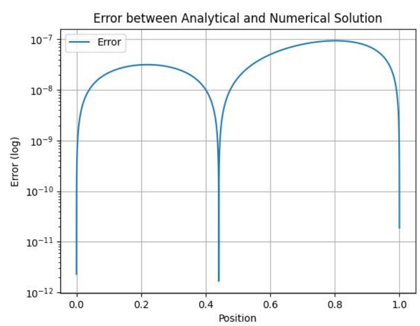
**Solution:**

We plot the numerical solution:




## Q1d.

Plot the error between the analytical and numerical solution using a logarithmic scale.
**Solution:**
We plot the error and find:




## Q2.

Consider the implicit RK3 method defined by the Butcher array

$$
\begin{array}{c|ccc}
0 & & & \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \\
\frac{1}{2} & 0 & \frac{1}{2} & \\
1 & 0 & 1 & 0 \\
\hline
 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6}
\end{array}
$$

## Q2a.

Prove that the method is convergent
**Solution:**

We prove convergence of the RK3 method defined by:

$$
\begin{aligned}
K_1 &= f(u_k, t_k) \\
K_2 &= f(u_k + \frac{h}{4}K_1 + \frac{h}{4}K_2, t_k + \frac{h}{2}) \\
K_3 &= f(u_k + hK_1, t_k + h) \\
u_{k+1} &= u_k + h(\frac{1}{6}K_1 + \frac{2}{3}K_2 + \frac{1}{6}K_3)
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
u_{k+1} &= u_k + h(\frac{1}{6}K_1 + \frac{2}{3}K_2 + \frac{1}{6}K_3) \\
u_{k+1} - u_k &= h(\frac{1}{6}K_1 + \frac{2}{3}K_2 + \frac{1}{6}K_3) \\
&\implies \alpha_0 = -1, \alpha_1 = 1
\end{aligned}
$$

We find the first characteristic polynomial,

$$
\begin{aligned}
&\rho(z) = z - 1 \\
&\implies \rho(1) = 0
\end{aligned}
$$

Thus the method is zero-stable.

Now we show the method is consistent,

$$
\begin{aligned}
\sum_{i=1}^{3} b_i &= \frac{1}{6} + \frac{2}{3} + \frac{1}{6} \\
&= 1
\end{aligned}
$$

Thus the method is consistent.

Now we've shown the method is both consistent and zero stable, and thus is convergent.

## Q2b.

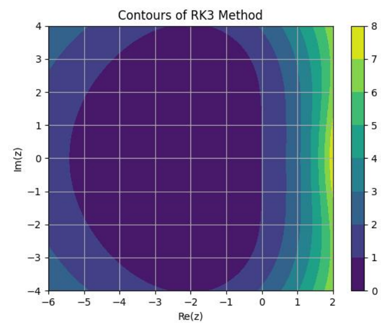
Plot the region of absolute stability of the method.
**Solution:**

We use the equation derived using Cramer's rule:

Implicit Formula:

$$S(z) = \frac{det(I - zA + zhb^T)}{det(I - zA)}$$

The region of stability is defined at z < 1 in the contour map below

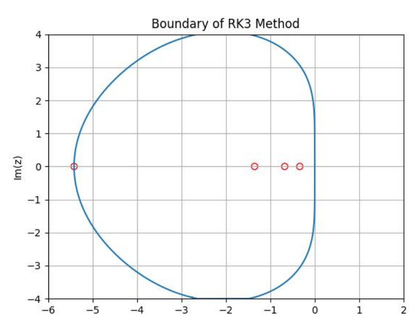



## Q2c.

Find $\Delta t^*$ for which the implicit RK3 defined in Q2 applied to $\frac{dy}{dt} = By$ where

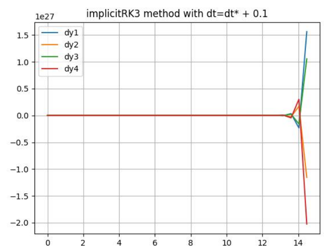
$$B = \begin{bmatrix} -1 & 3 & -5 & 7 \\ 0 & -2 & 4 & -6 \\ 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -16 \end{bmatrix}$$
$$y_0 = [1, 1, 1, 1]^T$$

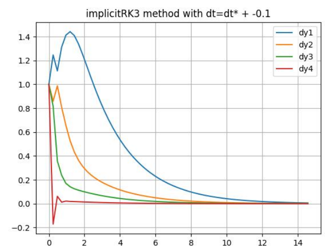
is absolutely stable.
**Solution:**
We use a tolerance of $10^{-8}$, i.e. we find the largest timestep such that the largest eigenvalue is within $10^{-8}$ of the boundary.
We find $\Delta t^* \cdot$ =0.339



## Q2d.

Verify your predictions numerically for values slightly larger and smaller $\Delta t$, and show the solution converges to zero or diverges to infinity.
**Solution**:
Next we show our scheme using $\Delta t^* \cdot =0.339 \pm 0.1$




We find the $\Delta t^*$ gives us the outcomes as expected. If we decrease our timestep by 0.1, we see faster convergence. If we increase our timestep to outside the stability region by a value of 0.1, we see that our method diverges. This implies that $\Delta t^* = \cdot =0.339$ is correct.

## Q3.

Consider the linear multi-step method:

$$u_{k+3} - \frac{1}{3}(u_{k+2} + u_{k+1} + u_k) = \frac{h}{12}(23f_{k+2} - 2f_{k+1} + 3f_k)$$

## Q3a.

Show that the method is convergent and determine the convergence order.
**Solution:**

$$u_{k+3} - \frac{1}{3}(u_{k+2} + u_{k+1} + u_k) = \frac{h}{12}(23f_{k+2} - 2f_{k+1} + 3f_k)$$

For this scheme we have:

$$\begin{aligned}
\alpha_0 &= -\frac{1}{3} \\
\alpha_1 &= -\frac{1}{3} \\
\alpha_2 &= -\frac{1}{3} \\
\alpha_3 &= 1 \\
\beta_0 &= \frac{1}{4} \\
\beta_1 &= -\frac{1}{6} \\
\beta_2 &= \frac{23}{12} \\
\beta_3 &= 0
\end{aligned}$$

We compute the order of consistency:

$$\begin{aligned}
\rho(z) &= -\frac{1}{3} - \frac{z}{3} - \frac{z^2}{3} + z^3 \\
\rho'(z) &= -\frac{1}{3} - \frac{2}{3}z + 3z^2 \\
\sigma(z) &= \frac{1}{4} - \frac{1}{6}z + \frac{23}{12}z^2
\end{aligned}$$

We compute the coefficients,

$$\begin{aligned}
C_0 &= \rho(1) \\
&= -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} + 1 \\
&= 0 \\
C_1 &= \rho'(1) - \sigma(1) \\
&= (-\frac{1}{3} - \frac{2}{3}1 + 3 * 1^2) - (\frac{1}{4} - \frac{1}{6}1 + \frac{23}{12} * 1^2) \\
&= -\frac{1}{3} - \frac{2}{3} + 3 - \frac{1}{4} + \frac{1}{6} - \frac{23}{12} \\
&= -\frac{4}{12} - \frac{8}{12} + \frac{36}{12} - \frac{3}{12} + \frac{2}{12} - \frac{23}{12} \\
&= 0
\end{aligned}$$

Therefore the scheme is consistent. Now we continue to compute the order of consistency.

$$\begin{aligned}
C_2 &= \frac{1}{2!}\sum_{j=0}^{3}(j^2\alpha_j - 2j\beta_j) \\
&= \frac{1}{2!}(0+0-9+9) \\
&= 0 \implies \text{Not order 1} \\
C_3 &= \frac{1}{3!}\sum_{j=0}^{3}(j^3\alpha_j - 3j^2\beta_j) \\
&= \frac{1}{3!}(0+0-9+9) \\
&= 0 \implies \text{Not order 2} \\
C_3 &= \frac{1}{3!}\sum_{j=0}^{3}(j^3\alpha_j - 3j^2\beta_j) \\
&= \frac{1}{3!}(0+\frac{1}{6}-\frac{154}{6}+27) \\
&= \frac{1}{3!}(\frac{1}{6})(1-154+162) \\
&= \frac{1}{3!}\frac{1}{6}(9) \\
&\neq 0 \implies \text{Consistent with order 2!}
\end{aligned}$$

Now we show it's zero-stable:

$$\begin{aligned}
\rho(z) &= -\frac{1}{3}-\frac{z}{3}-\frac{z^2}{3}+z^3 \\
\rho(1) &= -\frac{1}{3}-\frac{1}{3}-\frac{1^2}{3}+1^3 \\
&= 0 \implies \text{zero stable}
\end{aligned}$$

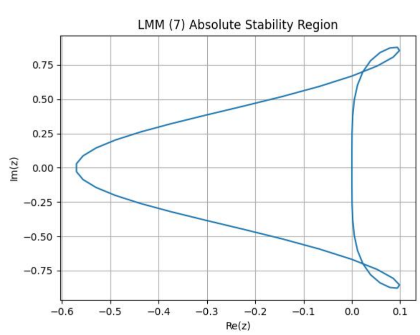
Now we've shown the scheme is both consistent with order 2 and zero stable, and is thus convergent with order 2.

## Q3b.

Plot the region of stability. Is the LMM A-stable?
**Solution:** We first compute the region of stability,

$$
\begin{aligned}
\lambda_j \Delta t &= \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \\
\rho(z) &= -\frac{1}{3} - \frac{1}{3}z - \frac{1}{3}z^2 + z^3 \\
\rho(e^{i\theta}) &= -\frac{1}{3} - \frac{1}{3}e^{i\theta} - \frac{1}{3}(e^{i\theta})^2 + (e^{i\theta})^3 \\
&= \frac{1}{3}(-1 - e^{i\theta} - e^{2i\theta} + 3e^{3i\theta}) \\
\sigma(z) &= \frac{1}{4} - \frac{1}{6}z + \frac{23}{12}z^2 \\
\sigma(e^{i\theta}) &= \frac{1}{4} - \frac{1}{6}e^{i\theta} + \frac{23}{12}(e^{i\theta})^2 \\
&= \frac{1}{12}(3 - 2e^{i\theta} + 23e^{2i\theta}) \\
\implies \lambda_j \Delta t &= \frac{\frac{1}{3}(-1 - e^{i\theta} - e^{2i\theta} + 3e^{3i\theta})}{\frac{1}{12}(3 - 2e^{i\theta} + 23e^{2i\theta})} \\
&= 4\frac{-1 - e^{i\theta} - e^{2i\theta} + 3e^{3i\theta}}{3 - 2e^{i\theta} + 23e^{2i\theta}}
\end{aligned}
$$

We show the plot below:




By course note 5 (theorem 2), we see that the inside is the region of stability. Additionally we can conclude by the definition of A-stable, that the region shown is not A-stable.