Homework 4: Numerical Coding

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(1) Cholesky Solution of the least-squares problem

- Cholesky decomposition and back-substitution.
- Fit data with a 3rd degree polynomial, using single-precision floating-point arithmetic. Report the polynomial coefficients, the Frobenius norm error on the fit.
- Fit data with a 5th degree polynomial, using single-precision floating-point arithmetic, Report the polynomial coefficients and the Frobenius norm error on the fit.
- Discuss what should be the maximum degree of polynomials for the given data? Explain briefly what happens when you try to fit the data with a higher-degree polynomial, and why.
- Discuss at what point the algorithm fails (single-precision). Or does the algorithm always work?

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a) To implement the Cholesky factorization, we take a Hermitian positive-definite matrix and find the LU factorization:

$$A = L * U$$

Since A is hermitian and positive-definite, then additionally we have:

$$\begin{aligned} A &= U^*U \\ &= LL^* \end{aligned}$$

Where U is upper-triangular and L is lower-triangular such that $u_{jj} = l_{jj} > 0$. We show that this factorization only exists for positive definite matrices:

$$x^*Ax = x^*U^*Ux$$

$$
\begin{aligned}
&= (Ux)^*(Ux) \\
&> 0
\end{aligned}
$$

In our algorithm, we decompose our input A and have an output with L in the lower-diagonal and diagonal, and the original elements of A in the upper-diagonal.
Next in our substitution algorithm we take our lower-triangular matrix and find y such that:

$$Ly = b$$

using forward-substitution where b is known.

Next, we use backwards-substitution to solve:

$$L^*x = y$$

Then we have a solution x such that:

$$Ax = b$$

where A is Hermitian and positive-definite using the Cholesky factorization method.

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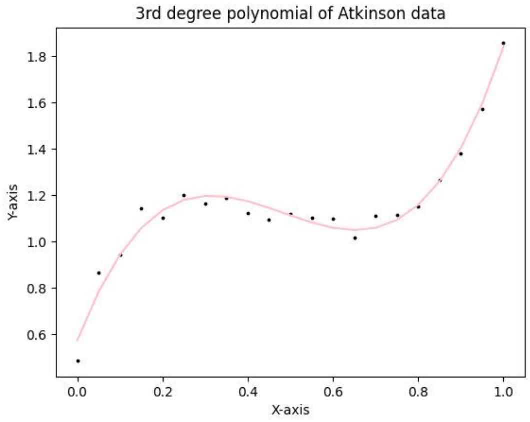
b) To use the Cholesky algorithm on the "Atkinson.dat" matrix, we find the Vandermonde matrix of the x coordinates with degree 3 denoted V and solve the system:

$$V^TVx = V^Ty$$

We find the coefficients by finding the solution of Cholesky with A = V^TV and b = V^Ty. The resulting coefficients are:

$$
\begin{aligned}
x_0 &= 0.57465867 \\
x_1 &= 4.72586144 \\
x_2 &= -11.12821778 \\
x_3 &= 7.66867762
\end{aligned}
$$



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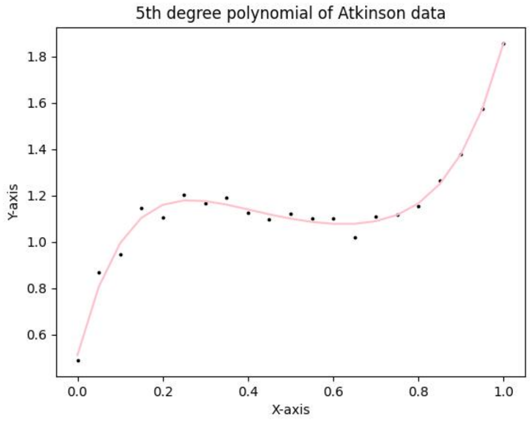
c) Similar with a fifth-degree polynomial we find:

$$
\begin{aligned}
x_0 &= 0.5096216 \\
x_1 &= 7.2032946 \\
x_2 &= -28.40831092 \\
x_3 &= 51.94483317 \\
x_4 &= -47.48833103 \\
x_5 &= 18.0983464
\end{aligned}
$$



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d)