AM213A Homework 3

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Part 1: Numerical Coding Problems

Q2. Warming-Up

From matrix A, we see that our computed trace is 22.0 and is confirmed by our analytical solution:

$$\begin{aligned} Tr(A) &= \\ 22.0 &= 2.0 + 3.0 + 9.0 + 8.0 \end{aligned}$$

Similarly, for the column norms of A, we have:

$$\begin{aligned} Norm(A_0) &= 10.95 \\ \sqrt{120} &= \sqrt{2^2 + 4^2 + 8^2 + 6^2} \\ Norm(A_1) &= 10.39 \\ \sqrt{108} &= \sqrt{1^2 + 3^2 + 7^2 + 7^2} \\ Norm(A_2) &= 13.12 \\ \sqrt{172} &= \sqrt{1^2 + 3^2 + 9^2 + 9^2} \\ Norm(A_3) &= 9.49 \\ \sqrt{90} &= \sqrt{0^2 + 1^2 + 5^2 + 8^2} \end{aligned}$$

Q3. Gaussian Elimination with Partial Pivoting:

For Gaussian Elimination with partial pivoting, we have matrices A and B as our input, and by Gaussian Elimination is returned as an equivalent upper triangular matrix for A, and a matrix B with corresponding operations. Singularity is tested by checking if the diagonals are machine 0. We then perform back-substitution to find our solution matrix X. We check the accuracy of our solution by computing:

$$E = A_s X - B_s$$

and confirming our column norms are within machine accuracy.

Q4. LU Decomposition with partial pivoting:

For our LU Decomposition with partial pivoting, we input a square matrix and decompose it into it's LU decomposition, where the diagonal and upper diagonals correspond to the U matrix, and L corresponds to a matrix with elements in the lower diagonal and 1s on the diagonal. We also store a permutation vector s where it stores the swapping components performed during partial pivoting. Similarly to Gaussian Elimination, we perform back-substitution on the U elements in the LU decomposition, giving us a solution matrix X. We again check the accuracy of our solution by computing:

$$E = A_s X - B_s$$

and ensuring our column norms are within machine accuracy.

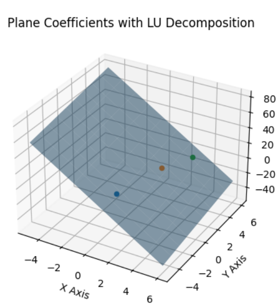
Q5. A very Basic Application:

To graph this plane, we used the LU decomposition to solve Ax = b with matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & 5 \\ \pi & \mathrm{e} & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Then we use the plane equation and plot z:

$$x_1 x + x_2 y + x_3 z = 1$$
$$z = (1 - x_1 x - x_2 y)/c$$



Part 2: Theory Questions

Definitions:

Definition 1 (Unitary).

- *An orthogonal matrix* $Q \in \mathbb{R}^{mxm}$ *satisfies* $Q^T = Q^{-1}$, *or equivalently* $Q^T Q = I$
- *A unitary matrix* $Q \in \mathbb{C}^{mxm}$ *satisfies* $Q^* = Q^{-1}$ *or equivalently,* $Q^* Q = I$

Definition 2 (Symmetry). *A matrix* A *is symmetric if* $A = A^T$

Definition 3 (Similarity Transformation). *Let* A *and* B *be two square matrices of the same dimension, then* A *is similar to* B *if there exists a nonsingular matrix* P *such that:*

$$B = P^{-1}AP$$

The operation $B = P^{-1}AP$ *is a similar transformation of* A. *For similar matrices* A *and* B, *we have that they have the exact same eigenvalues.*

Definition 4 (Positive-Definite). *A matrix* A *is positive-definite iff:*

$$x^T Ax > 0, \forall x \in \mathbb{C}^n / 0$$

Definition 5 (Hermitian). *A matrix* A *is Hermitian iff* $A = \bar{A}^T$

Properties:

- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$

Q1.

Theorem 1 (The Schur Decomposition Theorem). *If* $A \in \mathbb{C}^{mxm}$ $\implies$ *there exists a unitary matrix Q and an upper triangular matrix U s.t.* $A = QUQ^{-1}$.

Use the Schur decomposition theorem to show that a real symmetric matrix A is diagonalizable by an orthogonal matrix, i.e. $\exists$ an orthogonal matrix Q s.t. $Q^T AQ = D$, where D is a diagonal matrix with its eigenvalues in the diagonal.

Let us consider $A \in \mathbb{R}^{mxm}$, then by theorem 1, there exists an orthogonal matrix Q and an upper triangular matrix U such that $A = QUQ^{-1}$. Thus we have:

$$\begin{aligned}
A &= QUQ^{-1} \\
&= (QUQ^{-1})^T \\
&= (Q^{-1})^T (QU)^T \\
&= (Q^{-1})^T U^T Q^T \\
&= (Q^T)^{-1} U^T Q^T \\
&= (Q^{-1})^{-1} U^T Q^T \\
&= QU^T Q^T \\
&= QU^T Q^{-1} \\
AQ &= QU^T \\
Q^{-1} AQ &= U^T \\
Q^T AQ &= U^T
\end{aligned}$$

Now let us consider A without applying the transpose:

$$\begin{aligned}
A &= QUQ^{-1} \\
AQ &= QU \\
Q^{-1} AQ &= U \\
Q^T AQ &= U
\end{aligned}$$

Now we have shown that $U = U^T$, this implies that the upper diagonal matrix U has the same entries as the transpose, i.e. a lower triangular matrix and is only true if U is diagonal. This implies A is diagonalizable. Now, by definition 3, it follows that A and U have the exact same eigenvalues. So let us consider the eigenvalues of U denoted by λ, with corresponding eigenvectors V:

$$(U - \lambda I)V = 0$$

We may consider the case where $(U - \lambda I) = 0$, since U and λI are both diagonal, this implies that $U = \lambda I$. Therefore we have shown that A is diagonalizable by an orthogonal matrix Q such that $Q^T AQ = U$, where U is a diagonal matrix with eigenvalues of the diagonal. ∎

Q2.

Consider the following system: $\begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Multiply the last row of the matrix and the right-hand side vector by a large constant c such that $c\epsilon >> 1$.
Perform Gaussian elimination with partial pivoting to the modified row-scaled system and discuss what happens. If solving the resulting system has numerical issues, identify the issues and discuss how to improve the method.

We perform Gaussian elimination with partial-pivoting and back-substitution.

$$\begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ c\epsilon >> 1 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ c \end{bmatrix}$$

$$\begin{bmatrix} c\epsilon & c \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ 2 \end{bmatrix} (R_1 \text{ swapped with } R_2)$$

$$\begin{bmatrix} c\epsilon & c \\ 0 & 1 - \frac{c}{c\epsilon} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ 2 - \frac{c}{c\epsilon} \end{bmatrix} (R_2 = R_2 - \frac{R_1}{c\epsilon})$$

$$\begin{bmatrix} c\epsilon & c \\ 0 & 1 - \frac{1}{\epsilon} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ 2 - \frac{1}{\epsilon} \end{bmatrix}$$

$$\begin{bmatrix} c\epsilon & c \\ 0 & -\frac{1}{\epsilon} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ -\frac{1}{\epsilon} \end{bmatrix} (\text{Since } \frac{1}{\epsilon} \text{ is large, we consider the constants negligible})$$

Now we perform back-substitution:

$$-\frac{1}{\epsilon} y = -\frac{1}{\epsilon}$$
$$y = 1$$
$$c\epsilon x + cy = c$$
$$c\epsilon x = c(1 - y)$$
$$x = \frac{c(1-y)}{c\epsilon}$$
$$x = \frac{c(1-1)}{c\epsilon}$$
$$x = 0$$

Thus we have found that (x,y)=(0,1). We confirm our solution:

$$\begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies 0 + 1 = 2 \text{ and } 1 = 1 \lightning$$

We see that we have a contradiction and have reached an incorrect solution due to division by ϵ causing us to disregard our constant terms. To remedy this, we perform the implicit pivoting algorithm instead where we first scale each row by the largest entry in absolute value, this prevents the negligence of constant terms.

Q3.

What can you say about the diagonal entries of a symmetric positive definite matrix? Justify your assertion.

What can you say about the diagonal entries of a symmetric positive definite matrix A? Justify your answer by proving or disproving it.

Let us assume A is of size mxm. Given that A is symmetric positive definite, then there exists A Cholesky factorization by definition **??** with a lower triangular matrix L of size mxm such that:

$$\begin{aligned} A &= LL^* \\ &= \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ l_{m1} & l_{m2} & \dots & l_{m,m-1} & l_{mm} \end{bmatrix} \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ l_{m1} & l_{m2} & \dots & l_{m,m-1} & l_{mm} \end{bmatrix}^* \\ &= \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ l_{m1} & l_{m2} & \dots & l_{m,m-1} & l_{mm} \end{bmatrix} \begin{bmatrix} l_{11}^* & l_{21}^* & l_{31}^* & \dots & l_{m1}^* \\ 0 & l_{22}^* & l_{32}^* & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & l_{mm}^* \end{bmatrix} \end{aligned}$$

Let us consider the diagonal elements of A, ie a_{ii} $i \in \mathbb{N}$ $[1, m]$:

$$\begin{aligned} a_{11} &= l_{11} l_{11}^* \\ a_{22} &= l_{21} l_{21}^* + l_{22} l_{22}^* \\ &\dots \end{aligned}$$

We can generalize the diagonal elements of A as:

$$a_{ii} = \sum_{c=1}^{i} l_{ic} l_{ic}^*$$

In lecture 9 - pg 2, we saw that the elements of our lower diagonal matrix are real, implying that:

$$a_{ii} = \sum_{c=1}^{i} l_{ic}^2$$

Thus, since the diagonal elements is the sum of squares, we can conclude that they are positive

In conclusion, we have shown that the diagonal elements of a symmetric positive definite matrix A are positive.

Q4.

Suppose $A \in \mathbb{C}^{mxm}$ is written in the block form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Where $A_{11} \in \mathbb{C}^{nxn}$ and $A_{22} \in \mathbb{C}^{(m-n)x(m-n)}$. Assume that A satisfies the condition:
A has an LU decomposition iff the upper-left kxk block matrix $A_{1:k,1:k}$ is non-singular $\forall k \in [1, m]$.

a)

Verify the formula:

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

which eliminates the block A_{21} from A. The matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is known as the Schur complement of A_{11} in A_1, denoted as A/A_{11}.

b)

Suppose that after applying n steps of Gaussian elimination on the matrix A in (2), A_{21} is eliminated row by row, resulting in a matrix:

$$\begin{bmatrix} A_{11} & C \\ 0 & D \end{bmatrix}$$

Show that the bottom-right $(m-n)x(m-n)$ block matrix D is again $A_{22} - A_{21}A_{11}^{-1}A_{12}$.

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a)

We perform matrix multiplication to verify the formula:

$$\begin{aligned}
&\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\
&= \begin{bmatrix} A_{11} & A_{12} \\ -A_{21}A_{11}^{-1}A_{11} + A_{21} & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{bmatrix} \\
&= \begin{bmatrix} A_{11} & A_{12} \\ -A_{21} + A_{21} & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{bmatrix} \\
&= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}
\end{aligned}$$

Thus we have verified that:

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

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b)

We consider the Gaussian elimination process on our matrix:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Since row operations can be denoted by matrix multiplication, let us consider n steps of Gaussian multiplication as n elimination matrices, denoted $E_i \in \mathbb{C}^{mxm}$ for the ith row elimination. Then we have:

$$E_nE_{n-1}...E_1\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & C \\ 0 & D \end{bmatrix}$$

Let us denote the product of the elimination matrices as E and show E as a block matrix where the size of each block corresponds to the size of the associated A block:

$$\begin{aligned}
E\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &= \begin{bmatrix} A_{11} & C \\ 0 & D \end{bmatrix} \\
\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &= \begin{bmatrix} A_{11} & C \\ 0 & D \end{bmatrix}
\end{aligned}$$

Now let us consider a system of equations:

$$\begin{aligned}
E_{11}A_{11} + E_{12}A_{21} &= A_{11} \\
E_{21}A_{11} + E_{22}A_{21} &= 0 \\
E_{11}A_{11} + E_{12}A_{22} &= C
\end{aligned}$$

$$E_{21}A_{12} + E_{22}A_{22} = D$$

Let us try:

$$E = \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}$$

Then substituting into our system:

$$\begin{aligned} IA_{11} + 0A_{21} &= A_{11} \\ -A_{21}A_{11}^{-1}A_{11} + IA_{21} &= 0 \\ IA_{11} + 0A_{22} &= C \\ -A_{21}A_{11}^{-1}A_{12} + IA_{22} &= D \end{aligned}$$

Thus:

$$D = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

Q5.

Consider solving Ax = b, with A and b are complex valued of order m, i.e. $A \in \mathbb{C}^{mxm}$ and $b \in \mathbb{C}^m$

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a)

Modify this problem to a problem where you only solve a real square system of order 2m.
Hint: Decompose A = $A_1 + iA_2$ where $A_1 = Re(A)$ and $A_2 = Im(A)$ and similarly for b and x. Determine equations to be satisfies by $x_1 = Re(x)$ and $x_2 = Im(x)$

Let's consider the suggested substitutions:

$$\begin{aligned} Ax &= b \\ (A_1 + iA_2)(x_1 + ix_2) &= (b_1 + ib_2) \\ A_1x_1 + iA_1x_2 + iA_2x_2 - A_2x_2 &= b_1 + ib_2 \\ (A_1x_1 - A_2x_2) + i(A_1x_2 + A_2x_2) &= b_1 + i(b_2) \\ &\implies \\ A_1x_1 - A_2x_2 &= b_1 \text{ For real part} \\ A_1x_2 + A_2x_2 &= b_2 \text{ For imaginary part} \\ &\implies \end{aligned}$$

$$Re(A)x_1 - Im(A)x_2 = Re(b) \text{ For real part}$$
$$Re(A)x_2 + Im(A)x_2 = Im(b) \text{ For imaginary part :)}$$
$$\begin{bmatrix} Re(A) & -Im(A) \\ Re(A) & Im(A) \end{bmatrix} \begin{bmatrix} Re(x) \\ Im(x) \end{bmatrix} = \begin{bmatrix} Re(b) \\ Im(b) \end{bmatrix}$$

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b)

Determine the storage requirement and the number of floating-point operations for the real-valued method in (a) of solving the original complex-valued system Ax = b.
Compare these results with those based on directly solving the original complex-values system using Gaussian elimination without pivoting and complex arithmetic.
Use the fact that operation count of Gaussian elimination is $\mathcal{O}(m^3/3)$ for an mxm real-valued system with one right hand vector. Pay close attention to the greater expense of complex arithmetic operations. Make your conclusion by quantifying the storage requirement and the operating expense of complex arithmetic operations. Make your conclusion by quantifying the storage requirement and the opening expense of each method. Draw your conclusion on which method is computationally advantageous.

- Real Method
  Since our matrix for part A has a dimension 2mx2m, we can conclude the storage requirement is $4m^2$. For the right-hand side, we have 2m. Thus our total storage is $4m^2$ + 2m.
  For Gaussian elimination of a real matrix, we have storage size of $m^2 + m$, with an associated operation count of $m^3/3$ for the elimination, since we're solving a matrix of 4 times the size, we have an operation count of $\frac{4m^3}{3}$

References

- AM213A Lecture Notes
- AM213A Textbook
- ChatGPT 3.5 (primarily for typesetting)