STAT/BIOSTAT 571: Homework 9

To be handed in on Weds March 9th, in class. Please see 'Chapter 0' of the slides for a summary of how to answer questions appropriately, and the guidelines from 570. Where solutions require use of `R`, summarize your findings in a written answer, and append your **annotated** code, to show what you did. For each question, write up your solution on your own, using **full sentences**.

1. **[Induced marginal models]** Suppose you have paired binary observations Y_{ij}, for $i = 1, ..., n$ and $j = 0, 1$. There is one binary covariate, which by design has values $\{x_{i0}, x_{i1}\} = \{0, 1\}$ in every pair. We assume a model where the random effects distribution supports only two values;

$$
\begin{aligned}
Y_{ij}|b_i, x_{ij} &\overset{indept}{\sim} Bern(\mu_{ij}), \text{ for } 1 \le i \le n, \text{ and } j = 0, 1 \\
\mu_{ij} &= \text{expit}(\beta_0 + b_i + \beta_1 x_{ij}), \\
\text{where } \beta_0 &= 0 \\
\text{and } b_i &= \begin{cases} 0 & \text{with probability } 1/2 \\ -\beta_1 & \text{with probability } 1/2 \end{cases}
\end{aligned}
$$

   (a) Calculate the marginal mean model, i.e. $\mathbb{E}[\,Y_{ij}|X_{ij} = x\,]$

   (b) Assuming $\beta_0 = 0$ is known, derive the likelihood for the whole dataset, writing it as a function of β_1. Give a formula for its MLE $\hat{\beta}_1$. Hint: review HW1 Q2 first

   (c) Is the MLE in b) consistent for β_1, under the stated intercept and random effects distribution? Would the MLE in b) still be consistent if this distribution had been mis-specified, and were actually some distribution that was free of β_1 but otherwise unknown?

2. **[Conditional likelihood and GLMM]** This question also considers paired binary observations, with a binary covariate where $\{x_{i0}, x_{i1}\} = \{0, 1\}$. A mixed model for the data assumes

$$
\begin{aligned}
Y_{ij}|b_i, x_{ij} &\overset{indept}{\sim} Bern(\mu_{ij}) \\
\mu_{ij} &= \text{expit}(\beta_0 + b_i + x_{ij}\beta_1) \\
b_i &\overset{i.i.d.}{\sim} H,
\end{aligned}
$$

   for $1 \le i \le n$ and $j = 0, 1$, and where H has mean zero.

   (a) Briefly describe how, laying out the data for each pair in its own contingency table, the row and column totals for each cluster's data can take only three possible values. Give formulae describing the probabilities that each possible value actually occurs, in any particular cluster

   (b) Using results from slides to help you, state the conditional likelihood that could be used in this setting, for inference on β_1

   (c) For $\beta_1 = 0$, but no further restrictions on β_0 and/or H, show that the three probabilities in a) are bounded – i.e. that they do not cover the unit simplex. (The unit simplex is the space of points $\{p_1, p_2, p_3\}$ such that $p_1 + p_2 + p_3 = 1$ and $0 \le p_1, p_2, p_3 \le 1$.) Show the bounds on a graph

   (d) Show the same bounds for $\beta_1 = 2$ and $\beta_1 = -2$. Compare your answer with c), and briefly comment on the implications of your findings for conditional likelihood analysis suggested in b)

   (e) An alternative analysis uses MLEs from the mixed model likelihood where H is $N(0, \sigma^2)$ for unknown σ^2. Using any numerical method you find convenient, find the MLE for $\{\beta_0, \beta_1, \sigma\}$ under this model, for the two datasets on the class site. Compare your estimates of β_1 to the cMLE, and explain what you find.

3. **[Exchangeability]**

   (a) Consider the random effects distribution in Q1, interpreted as a prior on parameter b_i – a prior stated conditional on the value of β_1. In a Bayesian analysis that also assumes $\beta_1 \sim N(0,1)$, make a graph illustrating the implicit prior on $\{\mu_{i0}, \mu_{i1}\}$, i.e. the prior on the probabilities of obtaining $Y_{ij} = 1$ for the two observations in a single cluster. Are these parameters exchangeable?

   (b) Compare a) to a Bayesian version of Q2, in which the prior states that $\beta_0 = 0, b_i \sim N(0,1)$ and $\beta_1 \sim N(0,1)$, all independently. Illustrate the implicit prior on $\{\mu_{i0}, \mu_{i1}\}$, and state whether the parameters are exchangeable

   Note: to plot priors for bivariate parameters, you may find the `contour()` and/or `image()` functions useful.

4. **[Review]** For the material in each section in chapters 8 and 9 of Jon's book (i.e. §8.1—8.10 and 9.1–9.21) give a range of slides where this course covered the material. If a topic was not covered you should indicate this too.