Complex Analysis Problems and Solutions

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Contents

Chapter 1

Class

1.1 MATH 534

1.1.1 Chapter 1

1. (a) We wish to show the hypothesis of part (b) holds.

Proof. By the assumption of this problem, $|f(x)| \leq A$ for all $x \in [a,b]$. We have then $\displaystyle\int_a^b |f(x)|dx \leq \int_a^b |A|dx = \left|\int_a^b f(x)dx\right|$. We next show the reverse inequality. Let $\lambda = \dfrac{|\int_a^b f(x)dx|}{\int_a^b f(x)dx}$. If the denominator of λ is zero, then $\displaystyle\int_a^b f(x)dx = 0$ implies $0 = \left|\int_a^b f(x)dx\right| \geq \int_a^b |f(x)|dx$, so that $|f(x)| = 0$, by continuity, or that $f(x)$ is a constant identically zero. Since we require $\arg(f) \neq 0$, we do not consider this case. Now let λ be given as above. Then we have

$$\begin{aligned}
\left|\int_a^b f(x)dx\right| &= \lambda \int_a^b f(x)dx \\
&= \int_a^b \lambda f(x)dx \\
&= \int_a^b Re(\lambda f(x))dx \text{ (since this integral is equal to a real number)} \\
&\leq \int_a^b |\lambda f(x)|dx \text{ (by monotonicity of the integral)} \\
&= \int_a^b |f(x)|dx,
\end{aligned}$$

so that $\displaystyle\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$, from which we conclude $\displaystyle\int_a^b |f(x)|dx = \left|\int_a^b f(x)dx\right|$. Therefore by the conclusion of (b) we have $\arg(f)$ is constant. Next we prove $|f(x)|$ is constant over $[a,b]$, and it will follow that $f(x)$ is constant. By the Cauchy Schwarz inequality applied to 1, $f(x)$,

$$\begin{aligned}\left(\int_a^b |f(x)|dx\right)^2 &\leq \left(\int_a^b |f(x)|^2 dx\right)(b-a)\\ &\leq |A|(b-a)\int_a^b |f(x)|dx \text{ (by the problem hypothesis)}\\ &= \left|\int_a^b f(x)dx\right| \int_a^b |f(x)|dx\\ &= \left(\int_a^b |f(x)|dx\right)^2\end{aligned}$$

Therefore we actually have an equality in Cauchy-Schwarz so we know $|f(x)| = c \cdot 1$, for $c \in \mathbb{R}$, which, together with $\arg(f)$ being constant, implies that $f(x)$ is constant. □

(b) Assume $|A| = (1/(b-a)) \int_a^b |f(x)|dx$. Define $\lambda = \frac{|\int_a^b f(x)dx|}{\int_a^b f(x)dx}$. If the denominator of λ is zero, then $\int_a^b f = 0$ together with the hypothesis of the problem implies $0 = \int_a^b |f(x)|dx$, so that $|f(x)| = 0$, by continuity, or that $f(x)$ is a constant identically zero, and the result follows anyway.

Proof. Assume the denominator of λ is nonzero. By the hypothesis of this problem, $\int_a^b \lambda f(x)dx = \left|\int_a^b f(x)dx\right| = \int_a^b |f(x)|dx = \int_a^b |\lambda f(x)|dx$, which implies $\int_a^b Re(\lambda f(x))dx = \int_a^b |\lambda f(x)|dx$. Equivalently, $\int_a^b |\lambda f(x)| - Re(\lambda f(x))dx = 0$, but since the integrand is nonnegative and continuous, we have $Re(\lambda f(x)) = |\lambda f(x)|$, which tells us $\lambda f(x) \in \mathbb{R}_{\geq 0}$, or that for some nonnegative function $g(x)$, $\lambda f(x) = g(x)$. Solving for $f(x)$ yields $f(x) = \lambda^{-1} g(x)$, or that $f(x)$ is the product of a fixed complex number times a real valued function, as required. □

2. (a) Consider the equation $ax^3 + bx^2 + cx + d = 0$. We substitute $x = u + t$ and set the coefficient for u^2 equal to zero. Direct computation shows the coefficient of u^2 is $3at + b = 0 \Rightarrow t = \frac{-b}{3a}$.

   (b) If the coefficient of u is zero, then $a(u+t)^3 + b(u+t)^2 + c(u+t) + d = 0$ simplifies to $au^3 + at^3 + bt^2 + ct + d = 0$, and we can simply take the cube root to find $u = ((-1/a)(at^3 + bt^2 + ct + d))^{1/3}$.

   If the coefficient of u is nonzero, then we set $u = kv$ for some nonzero constant k and choose k so that $v^3 = 3v + r$ for some constant r. With this transformation, the cubic polynomial becomes $ak^3v^3 + 3at^2kv + at^3 + bkvt + bt^2 + ckv + ct + d = 0$. Equivalently, $ak^3v^3 = -kv(3at^2 + bt + c) - (at^3 + bt^2 + ct + d)$. After dividing by ak^3, we set $3 = \frac{-v(3at^2 + bt + c)}{ak^2}$ to obtain a quadratic in k^2 that corresponds to the choice of k we're interested in.

   (c) Set $v = z + 1/z$ so that $v^3 = 3v + r$ becomes $z^3 + 1/z^3 = r$ or that $z^6 - rz^3 + 1 = 0$. Introduce the variable $w = z^3$ to obtain a quadratic in w: $w^2 - rw + 1 = 0$. The quadratic formula yields roots $w = (1/2)(r \pm \sqrt{r^2 - 4})$, and after resubstituting $w = z^3$, we see $z = ((1/2)(r \pm \sqrt{r^2 - 4}))^{1/3}$

1.1.2 Chapter 2

1. - Let $z = Re^{i\theta}$ and $w = Se^{i\alpha}$. Then the equation $z^n = w$ implies $R^n e^{in\theta} = Se^{i\alpha}$. Therefore by equating polar coordinates, we have $R^n = S$ and $\alpha + 2\pi k = n\theta$, so that $z = S^{1/n}e^{i(\alpha+2\pi k)/n}$.
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2. Suppose $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. We first show $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\{z \in \mathbb{C} \mid |z| < 1\}$. We will show that the radius of convergence $R \geq 1$. Assume not, then by Theorem 2.2 (Root Test), we have $\liminf \frac{1}{|a_n|^{1/n}} < 1$. Therefore all but finitely many $n \in \mathbb{N}$ satisfy $\frac{1}{|a_n|^{1/n}} < 1 \Rightarrow 1 < |a_n|^2$. Summing, we obtain $\infty = \sum 1 < \sum |a_n|^2 < \infty$, a contradiction to the hypothesis of the problem. Therefore we have f analytic over $\{z \mid |z| < R\}$. We wish to compute $\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{n=0}^{\infty} a_n r^n e^{in\theta}\right)\left(\sum_{n=0}^{\infty} \overline{a_n} r^n e^{-in\theta}\right) d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k \overline{a_{n-k}} r^n e^{i(n-2k)\theta} d\theta$. Since f is analytic, and our limit is taken as $r \uparrow 1$, we know the power series for f evaluated at $z \in \{z \in \mathbb{C} \mid |z| < 1\}$ will converge uniformly to $f(z)$. Therefore integration along along a circle of fixed radius $r < 1$ may be switched with the infinite summation, to obtain $\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k \overline{a_{n-k}} r^n \int_0^{2\pi} e^{i(n-2k)\theta} d\theta$. But since $\int_0^{2\pi} e^{in\theta} d\theta = \begin{cases} 2\pi & \text{if } n = 0 \\ \left.\frac{e^{i(n+1)\theta}}{i(n+1)}\right|_0^{2\pi} = 0 & \text{otherwise.}\end{cases}$, we conclude that $\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k \overline{a_{n-k}} r^n \int_0^{2\pi} e^{i(n-2k)\theta} d\theta = \frac{1}{2\pi}\sum_{n=0}^{\infty} 2\pi |a_n|^2 r^{2n}$, and since for $r < 1$, the preceding infinite sum is finite and converges uniformly, $\lim_{r\uparrow 1} \frac{1}{2\pi}\sum_{n=0}^{\infty} 2\pi |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} |a_n|^2 = \lim_{r\uparrow 1}\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}$.

3. Let f have a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ expansion at 0 which converges in all of $\mathbb{C}$. Further suppose $\int_{\mathbb{C}} |f(x+iy)| dxdy = 0$. We wish to show $f(z)$ is a polynomial. Changing to polar coordinates, we have $\int_{\mathbb{C}} |f(x+iy)| dxdy = \int_0^{\infty}\int_0^{2\pi} |f(re^{i\theta})| r dr d\theta = \int_0^{\infty}\int_0^{2\pi}\sum_{n=0}^{\infty} a_n r^{n+1} e^{in\theta} d\theta dr$. For a fixed radius $R < \infty$, since f is entire, the power series for $f(z)$ converges uniformly on the disk $\{z \in \mathbb{C} \mid |z| \leq R\}$. Therefore we may switch the inner integral with the sum to obtain $\int_0^{\infty}\int_0^{2\pi}\sum_{n=0}^{\infty} a_n r^{n+1} e^{in\theta} d\theta dr = \int_0^{\infty}\sum_{n=0}^{\infty} a_n r^{n+1}\int_0^{2\pi} e^{in\theta} d\theta dr$. But as in the previous problem, $\int_0^{2\pi} e^{in\theta} d\theta = \begin{cases} 2\pi & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$. After integration, only one term remains in the sum and we are left with $\int_0^{\infty} 2\pi a_0 r dr$, but by assumption, this is equal to zero, and we conclude $a_0 = 0$. Now for $z \neq 0$, let $g(z) = f(z)/z$. g will have a power series $\sum_{n=0}^{\infty} a_{n+1} z^n$ that agrees with $g(z)$ for all $z \neq 0$. Now since $|f(z)/z| = |g(z)| \leq |f(z)|$ for $|z| \geq 1$, by monotonicity of integration, we have $0 \leq \int_1^{\infty}\int_0^{2\pi} |g(z)| \leq \int_1^{\infty}\int_0^{2\pi} |f(z)| \leq \int_0^{\infty}\int_0^{2\pi} |f(z)| dz = 0$. We repeat the above argument for

$g(z)$ to conclude $a_1 = 0$: for fixed $R > 1$, $g(z)$ is analytic in the annulus $1 \leq |z| \leq R$, so that we actually have uniform convergence and the inner integration and summation may be switched. Again by the properties of the integral of $e^{in\theta}$ over an arc of 2π, we conclude $a_1 = 0$. Inductively, we conclude that for all $n \in \mathbb{N}$, $a_n = 0$. Since $f(z)$ is entire and agrees with its power series on the whole complex plane, we have $f \equiv 0$.

4. Let f be analytic in a connected open set U such that for each $z \in U$, there exists $n \in \mathbb{N}$ so that $f^{(n)}(z) = 0$. Consider the collection $s_n = \{z \in U \mid f^{(n)}(z) = 0\}$. By the hypothesis of the problem, $\bigcup_n s_n = U$. Since U is uncountable, at least one of these sets must be uncountable. Choose k so that s_k is uncountable. Assume for contradiction that all $z \in s_k$ are isolated points. Since $\mathbb{C}$ is a Hausdorff space, we may separate distinct pairs of points $z_1, z_2 \in s_k$ by disjoint open sets $z_1 \in B_1, z_2 \in B_2$ with $B_1 \cap B_2 = \emptyset$. We can choose rational coordinates $(p_i, q_i) \in B_i$ corresponding to the open set containing z_i. This yields an injection from $s_k \hookrightarrow \mathbb{Q} \times \mathbb{Q}$. By Schoder-Bernstein, we arrive at a contradiction in the cardinality of s_k. Therefore there exists a point $z_0 \in s_k$ that is not an isolated point, and by Corollary 3.3, $f \equiv 0$ over U.

5. Let f be analytic in a region U containing the point $z = 0$. f therefore has a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that converges to $f(z)$ for each $z \in U$. Suppose $|f(1/n)| < e^{-n}$ for $n \geq n_0$. We therefore have $0 \leq \lim_{n\to\infty} |f(1/n)| < \lim_{n\to\infty} e^{-n} = 0$, and by continuity we conclude $f(0) = 0$ and therefore $a_0 = 0$. Now choose N so that a_N is the first nonzero coefficient of the power series for f. Over U, we may write $f(z) = \sum_{n=N}^{\infty} a_n z^n$. We can write $f(z) = z^N \sum_{n=0}^{\infty} a_{n+N} z^n$. Define $g(z) = f(z)/z^N$. Computing $|g(0)| = \lim_{n\to\infty} |g(1/n)| = \lim_{n\to\infty} \left| \dfrac{f(1/n)}{1/n^m} \right| < \lim_{n\to\infty} e^{-n} n^m = 0$, so that $a_N = 0$, a contradiction.

1.1.3 Chapter 3

1. Let Ω be an open set and let f be an analytic function one-to-one map of Ω onto $f(\Omega)$. If $z_n \in \Omega \to \partial\Omega$, we show that $f(z_n) \to \partial f(\Omega)$ in the sense that $f(z_n)$ eventually lies outside each compact subset of $f(\Omega)$. Since f is bijective onto its image $f(\Omega)$, there is an inverse f^{-1} and because f is analytic and an open mapping, we know that $f^{-1} : f(\Omega) \to \Omega$ is continuous. Therefore f is actually a homeomorphism between Ω and $f(\Omega)$. Now assume that $z_n \in \Omega \to \partial\Omega$ but $f(z_n)$ does not converge to $\partial f(\Omega)$. Therefore there exists some compact subset $A \subset f(\Omega)$ so that $f(z_n) \in A$ for all $n \in \mathbb{N}$. Because f^{-1} is continuous, we know $f^{-1}(A)$ is compact and we can conclude $z_n \in f^{-1}(A)$ for all $n \in \mathbb{N}$, which is a contradiction since z_n would not converge to $\partial\Omega$.

2. (a) Assume φ is an analytic one-to-one map of $\mathbb{D}$ onto $\mathbb{D}$. Since φ is one-to-one, there exists a unique point $a \in \mathbb{D}$ so that $\varphi(a) = 0$. By corollary 4.4, we may write $\varphi(z) = \left(\dfrac{z-a}{1-\bar{a}z} \right) g(z)$, where g is analytic in $\mathbb{D}$ and $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. We first prove $g(a) \neq 0$ by contradiction. Assume $g(a) = 0$, then we have $\varphi(z) - \varphi(a) = \sum_{m=n}^{\infty} a_m (z-a)^m$ with nonzero coefficient for a_2, which tells us by Corollary 3.3 that for each $\epsilon > 0$ there is a $\delta > 0$ so that $f(z) - w$ has 2 distinct roots in $\{z : 0 < |z-a| < \epsilon\}$, provided $0 < |w - f(z_0)| < \delta$, contradicting the injectivity of φ. What we know so far is that $|g(z)| \leq 1$ in $\mathbb{D}$, and we wish to now show $|g(z)| \geq 1$ in $\mathbb{D}$. We now examine $1/g$, an analytic function with no zeros in $\mathbb{D}$ by the previous argument and by injectivity of φ. Since $1/g = T_a(z)/\varphi(z)$, as $z_n \to \partial\mathbb{D}$, $|T_a(z)| \to 1$, as proved in the notes, and $|\varphi(z)| \to 1$ by the previous problem, so that $|1/g| \to 1$ as $z_n \to \partial\mathbb{D}$. By the maximum

modulus principle, we conclude $|1/g| \leq 1$ over $\mathbb{D}$, so that $|g| \equiv 1$, or that $g(z) = c$ for all $z \in \mathbb{D}$ with $|c| = 1$, as required.

Now assume $\varphi(z) = c\left(\frac{z-a}{1-\bar{a}z}\right)$, where $|c| = 1$ and $|a| < 1$. We show φ is analytic and one-to-one onto $\mathbb{D}$. As φ is the quotient of analytic functions, φ is analytic away from the zeros of $1 - \bar{a}z$, which is zero when $z = (\bar{a})^{-1}$, whose modulus is greater than 1. Therefore analyticity over $\mathbb{D}$ is established. Injectivity is established as follows. Assume $\varphi(z) = \varphi(w)$. We show $w = z$:

$$\begin{aligned} \frac{z-a}{1-\bar{a}z} &= \frac{w-a}{1-\bar{a}w} \\ \Leftrightarrow (1-\bar{a}w)(z-a) &= (1-\bar{a}z)(w-a) \\ \Leftrightarrow z + |a|^2 w &= w + |a|^2 z \\ \Leftrightarrow z &= w, \end{aligned}$$

where the cancellation of $(1-|a|^2)$ is valid since $|a| < 1$ by hypothesis. Surjectivity is easily established as well: given $x \in \mathbb{D}$, consider the equation $\varphi(z) = x$, where $|x| < 1$:

$$\begin{aligned} c\frac{z-a}{1-\bar{a}z} &= x \\ \Rightarrow cz - ca &= x - \bar{a}xz \\ \Rightarrow z &= \frac{x+ca}{c+\bar{a}x}, \end{aligned}$$

which actually gives us a formula for any $z \in \mathbb{D}$ for the inverse function $\varphi^{-1}(z) = \frac{z+ca}{c+\bar{a}z}$.

(b) Assume f is analytic in $\mathbb{D}$ and that $|f(z)| \to 1$ as $|z| \to 1$. By the maximum modulus principle, we have $|f(z)| \leq 1$ in $\mathbb{D}$. Since $|f(z)| \to 1$ as $|z| \to 1$, there is some $R > 0$ so that all the zeros of $f(z)$ are contained in the closed ball $B(0, R)$ of radius R. Therefore since $|z_k| \leq R$, we have by Corollary 4.5, $\infty > \sum_k (1 - |z_k|) \geq \sum_k (1 - R) = \infty$, which is a contradiction. Therefore there are only finitely many zeros of $f(z)$ in $\mathbb{D}$. By Corollary 4.4, we may write $f(z) = \prod_{k=1}^{n} \left(\frac{z - z_k}{1 - \overline{z_k}z}\right) g(z)$, where g is analytic in $\mathbb{D}$, $|g(z)| \leq 1$ on $\mathbb{D}$, and $z_1, z_2, \ldots, z_n$ are the zeros of $f(z)$ in $\mathbb{D}$. If $g(z)$ has any zeros, we may rewrite $f(z) = \prod_{i=1}^{n} \left(\frac{z - z_k}{1 - \overline{z_k}z}\right) \prod_{j=1}^{m} (z - a_j)^{n_j} h(z)$, where $a_1, \ldots, a_m$ are the zeros of $g(z)$ with multiplicities n_j, $1 \leq j \leq m$. What's left to show is that $|h(z)| = M$ for some constant M.

3. (a) Let $p(z)$ be a polynomial. By corollary 2.2, we may write $p(z) = c\prod_{i=1}^{n}(z - z_i)$, where $z_1, \ldots, z_n$ are the complex zeros of $p(z)$ and c is a complex constant. We have $p'(z) = c\sum_{i=1}^{n}\prod_{j \neq i}^{n}(z - z_j)$ so that $\frac{p'(z)}{p(z)} = \frac{\sum_{i=1}^{n}\prod_{j\neq i}^{n}(z - z_j)}{\prod_{i=1}^{n}(z - z_i)} = \sum_{i=1}^{n}\frac{1}{1 - z_i}$. Clearly if a is a zero of $p(z)$ and a zero of $p'(z)$, then $a \in \mathbb{H}$. Therefore assume a is a zero of $p'(z)$ but not a zero of $p(z)$. We therefore have $0 = \sum_{i=1}^{n}\frac{1}{a - z_i} = \sum_{i=1}^{n}\frac{\bar{a} - \overline{z_i}}{|a - z_i|^2}$. Splitting the numerator, moving to opposite sides, and taking

conjugates, we have $a\left(\sum_{i=1}^{n}\frac{1}{|a-z_i|^2}\right)=\sum_{i=1}^{n}\frac{z_i}{|a-z_i|^2}$. This tells us

$$a=\left(\sum_{i=1}^{n}\frac{1}{|a-z_i|^2}\right)^{-1}\sum_{i=1}^{n}\frac{z_i}{|a-z_i|^2},$$

so that we may write $a=\sum_{i=1}^{n}a_iz_i$, where

$$a_i=\frac{1}{|a-z_i|^2}\left(\sum_{i=1}^{n}\frac{1}{|a-z_i|^2}\right)^{-1},\quad 1\le i\le n$$

Clearly a_i is positive and $\sum_{i=1}^{n}a_i=1$. By assumption, $\operatorname{Im}(z_i)>0$ for $1\le i\le n$ and since $a_i>0$, $\operatorname{Im}(a_iz_i)=a_i\operatorname{Im}(z_i)>0$, from which we conclude $\operatorname{Im}(a)=\sum_{i=1}^{n}a_i\operatorname{Im}(z_i)>0$ so that $a\in\mathbb{H}$, as required.

(b) We really have an almost identical argument. For n points $z_1,\ldots,z_n$ in the plane, the convex hull $C=\{\sum_{i=1}^{n}\lambda_iz_i\mid\lambda_i\ge 0,\sum_{i=1}^{n}\lambda_i=1\}$ is the set of all convex combinations with appropriate weights λ_i. We write $a=\sum_{i=1}^{n}a_iz_i$ again as done in part (a), and since these a_i defined above satisfy $a_i\ge 0$ for each i and further satisfy $\sum_{i=1}^{n}a_i=1$, this is exactly a convex combination of the points z_i and thus a lies in the convex hull of the zeros of p.

4. We apply Schwarz' Lemma in the Invariant Form to obtain the inequality $\left|\frac{f(1/3)-f(0)}{1-\overline{f(0)}f(1/3)}\right|=\left|\frac{f(1/3)-1/2}{1-(1/2)f(1/3)}\right|\le\left|\frac{1/3-0}{1-0(1/3)}\right|=1/3$. We are therefore left to show that

$$\left|\frac{f(1/3)-1/2}{1-(1/2)f(1/3)}\right|\le 1/3$$

implies $|f(1/3)|\ge 1/5$. To this end, we square both sides and rearrange to obtain

$$(f(1/3)-1/2)(\overline{f(1/3)}-1/2)\le(1/9)(1-(1/2)f(1/3))(1-(1/2)\overline{f(1/3)}),$$

which can be simplified to

$$(35/36)|f(1/3)|^2-(8/9)\operatorname{Re}(f(1/3))+5/36\le 0,$$

so that we obtain

$$\frac{35|f(1/3)|^2+5}{32}\le\operatorname{Re}(f(1/3))\le|f(1/3)|,$$

giving us the quadratic $35|f(1/3)|^2-32|f(1/3)|+5\le 0$ in $|f(1/3)|$. This is an upward opening parabola with zeros $1/5,5/7$, so we conclude that if the original inequality from Schwarz Lemma holds, then we must have $1/5\le|f(1/3)|\le 5/7$, as required.

5. Suppose $f(z)$ is analytic in $\mathbb{D}$ and $|f(z)| \leq M$ on $\mathbb{D}$. We first show that $f(z)$ has only finitely many zeros on the disk of radius $1/4$. Assume there were infinitely many zeros, z_k satisfying $|z_k| \leq 1/4$. By Corollary 4.5, we have $\infty > \sum_k (1 - |z_k|) \geq \sum_k (1 - (1/4)) > \infty$, a contradiction, so there are only finitely many zeros of $f(z)$ in the disk of radius $1/4$. Let $g(z) = f(z)/M$ for $z \in \mathbb{D}$. Clearly $g(z)$ is analytic and by construction $|g(z)| \leq 1$. Therefore by Corollary 4.4, we may write $g(z) = \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \overline{z_j} z} \right) h(z)$, where $z_1, \ldots, z_n$ are the zeros of $g(z)$ in the disk of radius $\frac{1}{4}$ and $h(z)$ is analytic on $\mathbb{D}$ and $|h(z)| \leq 1$. We therefore have $|g(0)| = (\prod_{j=1}^{n} |z_j|)|h(0)| \leq \prod_{j=1}^{n} |z_j|$ so that $|f(0)| \leq M \prod_{j=1}^{n} |z_j| \leq M4^{-n}$, since $|z_j| \leq \frac{1}{4}$ for $1 \leq j \leq n$. We therefore have $|f(0)| \leq M4^{-n}$, and by solving we obtain $n \leq \frac{1}{\log 4} \log \left| \frac{M}{f(0)} \right|$.

Proof. Assume $f \in H(\mathbb{D})$. Compute $\lim_{r \uparrow 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}$. $f \in H(\mathbb{D})$ means $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $f(z)$ converges absolutely and uniformly on $|z| \leq r$. Since the partial sums $s_n(z) \to s(z)$ uniformly and absolutely, clearly $\overline{s_n(z)} \to \overline{s(z)}$ uniformly and absolutely, so that $|f|^2 = f\overline{f} = \left(\sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right) \left(\sum_{n=0}^{\infty} \overline{a_m} r^m e^{-im\theta} \right)$. Therefore

$$\int_0^{2\pi} \left(\sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right) \left(\sum_{m=0}^{\infty} \overline{a_m} r^m e^{-im\theta} \right) \frac{d\theta}{2\pi} = \int_0^{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i\theta(n-m)} \frac{d\theta}{2\pi},$$

and that we may interchange the integral and outer sum is as follows (keeping in mind r is fixed). First $\left| \sum_{m=0}^{\infty} \overline{a_m} r^m e^{-im\theta} \right| \leq \sum_{m=0}^{\infty} |a_m| r^m = C_r$ which is finite, by hypothesis. But this means the partial sums in the variable n [denoted $f_n(\theta)$] converge uniformly for $\theta \in [0, 2\pi]$ to the limit function, since we have terms like $f_n(\theta)g(\theta) \to f_n(\theta)g(\theta)$ uniformly, which is the case since $|g(\theta)| \leq C_r < \infty$ independent of θ. The convergence is uniform since we may simply find N so that $|f_n(\theta) - f(\theta)| < \epsilon / C_r$ for large enough n. At this point, we have

$$\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} a_n r^n \int_0^{2\pi} \sum_{m=0}^{\infty} \overline{a_m} r^m e^{i(n-m)\theta} \frac{d\theta}{2\pi},$$

and since $\overline{f(z)}$ converges uniformly and absolutely on compact subsets (in particular for fixed r and $\theta \in [0, 2\pi]$), we have

$$\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} a_n r^n \sum_{m=0}^{\infty} \overline{a_m} r^m \int_0^{2\pi} e^{i(n-m)\theta} \frac{d\theta}{2\pi},$$

and since $\int_0^{2\pi} e^{ik\theta} \frac{d\theta}{2\pi} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise,} \end{cases}$ the sum is only nonzero when $n = m$ and we reduce to

$$\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

Denote the right-hand member as $s(r)$ and its partial sums as $s_n(r)$. Since $s_k(r) \leq s(r) \leq \sum_{n=0}^{\infty} |a_n|^2$ (the first inequality follows by the partial sums increasing and the second since $r < 1$), we take $r \uparrow 1$ to obtain

$$s_k(1) = \lim_{r\uparrow 1} s_k(r) \le \lim_{r\uparrow 1} s(r) \le \sum_{n=0}^{\infty} |a_n|^2,$$

and since this inequality is independent of k, we may take k to infinity and obtain $\lim_{r\uparrow 1} s(r) = \sum_{n=0}^{\infty} |a_n|^2$. $\square$

1.1.4 Chapter 4

1. Define $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$, where $n^{-z} = e^{-z\log n}$. We show the series is absolutely convergent for $\operatorname{Re}(z) > 1$. Consider $|n^{-z}| = |e^{-z\log(n)}| = e^{-\operatorname{Re} z \log(n)} = e^{-\log n \operatorname{Re} z} = n^{-\operatorname{Re} z}$, so that the absolute series becomes $\sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re}(z)}}$. By the $p-$series test, this series converges if and only if $\operatorname{Re}(z) > 1$. Since the series converges absolutely for $\operatorname{Re}(z) > 1$, we have $\zeta(z)$ is analytic for $\operatorname{Re}(z) > 1$, as required.

   We now show $n^{-z} - \int_n^{n+1} x^{-z}dx = \int_n^{n+1}\int_n^x zt^{-z-1}dtdx$ by integrating the right-hand side. We have

$$\begin{aligned}
\int_n^{n+1}\int_n^x zt^{-z-1}dtdx &= z\int_n^{n+1}\int_n^x t^{-z-1}dtdx \\
&= z\int_n^{n+1}\left(\frac{t^{-z}}{-z}\bigg|_n^x\right)dx \\
&= \int_n^{n+1}(-x^{-z}+n^{-z})dx \\
&= n^{-z} - \int_n^{n+1} x^{-z}dx,
\end{aligned}$$

   as required.

   We have $(z-1)\zeta(z) = 1 + z(z-1)\sum_{n=1}^{\infty}\int_n^{n+1}\int_n^x t^{-z-1}dtdx$. Now we switch the integration to obtain $(z-1)\zeta(z) = 1 + z(z-1)\sum_{n=1}^{\infty}\int_n^{n+1}\int_t^{n+1} t^{-z-1}dxdt$. Since $n+1-t < 1$, we have

$$(z-1)\zeta(z) \le 1 + z(z-1)\sum_{n=1}^{\infty}\int_n^{n+1} t^{-z-1}dt = 1 + (1-z)\sum_{n=1}^{\infty}(n+1)^{-z} - n^{-z}.$$

   We first note that the partial sums $S_N = \sum_{n=1}^{N}(n+1)^{-z} - n^{-z} = (N+1)^{-z} - 1$ are telescoping. We now examine the modulus of the partial sums, we have as $N \to \infty$ $|(N+1)^{-z} - 1| \le 1 + |(N+1)^{-z}| = 1 + (N+1)^{-\operatorname{Re} z} \to 1$ whenever $\operatorname{Re} z > 0$.

2. For $r > 1/2$, by Cauchy's integral formula, we have $f'(z) = \int_{C_r(0)} \frac{f(\zeta)d\zeta}{(z-\zeta)^2}$, which is valid for $|z| < r$. Consider a parameterization $\gamma(t) = re^{it}$, $0 \le t \le 2\pi$. Then our integral becomes

$$f'(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(re^{it})rie^{it}dt}{(z-re^{it})^2}.$$

We integrate with respect to r the modulus of both sides of the above equation from $r = 3/4$ to $r = 1$ to obtain

$$\int_{3/4}^{1} |f'(z)|dr \leq \frac{1}{2\pi} \int_{3/4}^{1} \int_{0}^{2\pi} \frac{|f(re^{it})|rdtdr}{|z - re^{it}|^2}.$$

Since $|z| \leq 1/2$, we have $|z - re^{it}| \geq 1/4$, since the closest z and re^{it} can be is when $r = 3/4$ and z lies on the disk of radius $1/2$. Therefore we have $|z - re^{it}|^2 \geq 1/16$ and therefore our estimate becomes

$$|f'(z)| \leq \frac{32}{\pi} \int_{3/4}^{1} \int_{0}^{2\pi} |f(re^{it})|rdtdr \leq \frac{32}{\pi} \int_{0}^{1} \int_{0}^{2\pi} |f(re^{it})|rdtdr = \frac{32}{\pi} \int_{\mathbb{D}} |f(x+iy)|dxdy$$

3. Consider the three closed and bounded rectangles defined for each $n \in \mathbb{N}$ as $A_n := [1/n, n] \times [-n, n]$, $B_n := [-1/(n+1), 1/(n+1)] \times [-n, n]$, and $C_n := [-n, -1/n] \times [-n, n]$. For each n, we have $\mathbb{C} \setminus (A_n \cup B_n \cup C_n)$ is connected. The function $f_n(z) = \begin{cases} 1 & \text{if } z \in A_n \\ 0 & \text{if } z \in B_n \\ -1 & \text{if } z \in C_n \end{cases}$ is analytic on the union $A_n \cup B_n \cup C_n$ for each $n \in \mathbb{N}$, so by Runge's theorem and the fact that $\mathbb{C} \setminus (A_n \cup B_n \cup C_n)$ is connected, we may approximate $f_n(z)$ arbitrarily closely by polynomials $\{f_{nk}(z)\}_{k=1}^{\infty} \to f_n(z)$ uniformly. If we define $p_k(z) = f_{kk}(z)$ for each $k \in \mathbb{N}$, then $\lim_{k\to\infty} p_k(z) = \begin{cases} 1 & \text{if } \mathrm{Re}(z) > 0 \\ 0 & \text{if} \, \mathrm{Re}(z) = 0 \\ -1 & \text{if } \mathrm{Re}(z) < 0 \end{cases}$, as required.

4.

1.1.5 Chapter 5

- 6. Let Ω be a region and assume $p_n(z)$ is a sequence of polynomials converging uniformly to a function $f(z)$ on Ω. Consider $S^2 \setminus \pi(\Omega) = A_\infty \cup \bigcup_{\alpha \in J} A_\alpha$, the complement of Ω in the Riemann Sphere. We have one component, A_∞ containing the north pole, the point at infinity, and the latter union our collection of "holes" as in the notes. Let $\tilde{\Omega} = \Omega \cup \bigcup_{\alpha \in J} A_\alpha$. First note that $\Omega \subset \tilde{\Omega}$. $\tilde{\Omega}$ is open since it is the complement of the closed A_∞ and is connected since π is a homeomorphism and preserves connectedness. $\tilde{\Omega}$ is simply connected since the complement is connected in the Riemann sphere. I claim first that $\partial\tilde{\Omega} \subset \partial\Omega$. To see this, take $x \in \partial\tilde{\Omega}$. Then any neighborhood U of x satisfies $U \cap \tilde{\Omega} \neq \emptyset$ and $U \cap \tilde{\Omega}^C \neq \emptyset$. Since $\tilde{\Omega}^C \subset \Omega^C$, we have $U \cap \Omega^C \neq \emptyset$. What's left is to show $U \cap \Omega \neq \emptyset$. Assume the intersection is empty, then we know $U \subset \Omega^C$. Therefore U is in some component of Ω^C (note here we can say that U is actually contained entirely within one of the components of $S^2 \setminus \pi(\Omega)$ in the Riemann sphere). $U \not\subset A_\infty$ since then $U \cap \tilde{\Omega} = \emptyset$, so we must have $U \subset A_\alpha$ for some $\alpha \in J$. Since $A_\alpha \subset \tilde{\Omega}$, we have $U \cap \tilde{\Omega} \neq \emptyset$, a contradiction. Therefore $U \cap \Omega \neq \emptyset$, so that $x \in \partial\Omega$. Since we have uniform convergence of the sequence of polynomials $\{p_n(z)\}_{n=1}^{\infty}$ on Ω, we know that the sequence is Cauchy. Therefore for all $\epsilon > 0$ there exists $N > 0$ so that for all $n, m \geq N$, $|p_n(z) - p_m(z)| < \epsilon$ for $z \in \Omega$. Since the difference of any two polynomials is another polynomial and thus entire, by continuity we have $|p_n(z) - p_m(z)| \leq \epsilon$ on $\partial\Omega$, so that $|p_n(z) - p_m(z)| \leq \epsilon$ on $\partial\tilde{\Omega}$ since we are taking a supremum over a smaller set. By the maximum modulus principle applied to the difference of polynomials over the region $\tilde{\Omega}$, $|p_n(z) - p_m(z)| < \epsilon$ on $\tilde{\Omega}$ as well, so that the sequence uniformly converges on both Ω and $\tilde{\Omega}$, as required.
- 7.

1. $\dfrac{z^2+1}{e^z} \mapsto e^{-1/z}(z^{-2}+1) = \dfrac{z^2+1}{z^2}e^{-1/z} = (z^2+1)\displaystyle\sum_{n=0}^{\infty}\dfrac{(-1)^n}{z^{n+2}n!}$, so at $z=0$ there is an essential singularity at $z=0$.

2. $\dfrac{1}{e^{1/z}-1} - z \mapsto \dfrac{1}{e^z-1} - \dfrac{1}{z} = \dfrac{z+1-e^z}{z(e^z-1)} = \dfrac{z-(z+\frac{z^2}{2!}+\cdots)}{z(z+\frac{z^2}{2!}+\cdots)} = \dfrac{-z^2(\frac{1}{2}+\frac{z}{6}+\cdots)}{z^2(1+\frac{z}{2}+\cdots)}$, so at $z=0$, the function evaluates to $-1/2$. This expansion was valid for all $|z|<\infty$ from the Taylor series of $\exp(z)$. Therefore there is a removable singularity at $z=0$.

3. $e^{z/(1-z)} \mapsto e^{1/(z-1)}$. This function is analytic away from $z=1$. At $z=0$ the function is e^{-1}.

4. $ze^{1/z} \mapsto \dfrac{e^z}{z} = \displaystyle\sum_{n=0}^{\infty}\dfrac{z^{n-1}}{n!}$, which is valid for all $z \in \mathbb{C}\setminus\{0\}$. This expansion tells us we have a simple pole at $z=0$.

5. $z^2 - z \mapsto \dfrac{1}{z^2} - \dfrac{1}{z} = \dfrac{1-z}{z^2}$, which is a rational function with a pole of order 2 at $z=0$.

6. $\dfrac{1}{z^3}e^{1/z} \mapsto z^3e^z$, which is an entire function with a zero of order 3 at $z=0$.

- 8. $\dfrac{z}{(z^2+4)(z-3)^2(z-4)} = \dfrac{A}{z-4} + \dfrac{B}{z-3} + \dfrac{C}{(z-3)^2} + \dfrac{Dz+E}{z^2+4}$ by partial fraction decomposition. We deal with each term individually. For the first term, we write $\dfrac{A}{z-4} = \dfrac{A}{-4(1-z/4)} = -A\displaystyle\sum_{n=0}^{\infty}\dfrac{z^n}{4^{n+1}}$, which is valid for $|z|<4$, which contains our annulus. For the second term, we write $\dfrac{B}{z-3} = \dfrac{B}{z(1-3/z)} = B\displaystyle\sum_{n=0}^{\infty}\dfrac{3^n}{z^{n+1}}$, which is valid for $|z|>3$, which again contains our annulus. For the next term, we write $\dfrac{C}{(z-3)^2} = -C\dfrac{d}{dz}\left(\dfrac{1}{z-3}\right) = -C\displaystyle\sum_{n=0}^{\infty}\dfrac{d}{dz}\left(\dfrac{3^n}{z^{n+1}}\right) = C\sum_{n=0}^{\infty}\dfrac{(n+1)3^n}{z^{n+2}}$. Finally, for $\dfrac{Dz+E}{z^2+4}$, we write $\dfrac{Dz+E}{z^2(1+4/z^2)} = \dfrac{Dz+E}{z^2}\displaystyle\sum_{n=0}^{\infty}\dfrac{(-1)^n4^n}{z^n} = D\sum_{n=0}^{\infty}\dfrac{(-1)^n4^n}{z^{n+1}} + E\sum_{n=0}^{\infty}\dfrac{(-1)^n4^n}{z^{n+2}}$, which is valid for $2<|z|$, which again contains our annulus. Therefore

$$\frac{z}{(z^2+4)(z-3)^2(z-4)} = -A\sum_{n=0}^{\infty}\frac{z^n}{4^{n+1}} + B\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}} + C\sum_{n=0}^{\infty}\frac{(n+1)3^n}{z^{n+2}} + D\sum_{n=0}^{\infty}\frac{(-1)^n4^n}{z^{n+1}} + E\sum_{n=0}^{\infty}\frac{(-1)^n4^n}{z^{n+2}}$$

- 9. See looseleaf page.

- 10. Let $p(z) = 3z^5+21z^4+5z^3+6z+7$. Consider $|p(z)-21z^4|$ on $|z|=2$. We have $|p(z)-21z^4| \le 3(2^5)+5(2^3)+6(2)+7 = 155 < 336 = 21|z|^4$ on $|z|=2$. By Rouche, we know $p(z)$ and $21z^4$ have the same number of zeros in $\{z : |z|<2\}$, namely four zeros. Now consider $|z|=1$. We have $|p(z)-21z^4| \le 21 = 21|z|^4$, so we have a possible equality. From the triangle inequality, we have $|3z^5+5z^3+6z+7| \le |3z^5+5z^3| + |6z+7| \le |3z^5+5z^3| + 6|z| + 7 \le 3|z|^5 + 5|z|^4 + 6|z| + 7 = 21$. If there is an equality at the last step, there must be an equality at all steps, so we consider $|6z+7| = 6|z|+7 = 13$ on $|z|=1$. We have two solutions, namely $z=1$ and $z=-10/3$. The second solution does not satisfy our constraints, so we must have $z=1$ for equality at this step. But then we see $|p(z)-21z^4| < 21+p(1)$, since $p(1)=42>0$. Therefore $p(z)$ and $21z^4$ also share the same number of zeros on $|z|<1$, namely four. Therefore on $\mathbb{D}$ $p(z)$ has exactly four zeros (none of which are on the boundary), and no zeros inside the annulus $\{z : 1<|z|<2\}$.

- 11. Let $p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_0$. Let $f(z) = z^n$. We wish to compare $p(z)$ to $f(z)$. Let $R = \sqrt{1 + |c_0|^2 + \cdots + |c_{n-1}|^2}$, then on $|z| = R$, we have

$$\begin{aligned}
|p(z) - f(z)| &\le (|c_0|^2 + \cdots + |c_{n-1}|^2)^{1/2}(1 + |z|^2 + \cdots + |z|^{2n-2})^{1/2} \\
&= \sqrt{R^2 - 1}(1 + R^2 + \cdots + R^{2n-2}) \\
&= \sqrt{R^2 - 1}\sqrt{\frac{R^{2n} - 1}{R^2 - 1}} \\
&< \sqrt{R^{2n}} = |f(z)|,
\end{aligned}$$

so that by Rouche's theorem all the zeros of $p(z)$ lie within the disk centered at zero of radius $\sqrt{|c_0|^2 + \cdots + |c_{n-1}|^2 + 1}$, as required.

1.1.6 Chapter 6

1.1.7 Chapter 7

1.2 MATH 535

1.2.1 Chapter 8

5. Throughout, let v be a real-valued twice continuously differentiable function on a region Ω.

   $(i) \Rightarrow (iii)$. Suppose v is subharmonic. Let B be a disk such that $\overline{B} \subset \Omega$, and suppose u is harmonic on $\overline{B}$. It is easy to see from the definition of harmonic and subharmonic that $v - u$ subharmonic. By Theorem 1.3 (the Maximum Principle for Subharmonic Functions), we see that $v - u$ satisfies the maximum principle.

   $(iii) \Rightarrow (i)$. Since subharmonicity is a local property, it suffices to show that v is subharmonic at a point $z_0 \in \Omega$. Suppose that if B is a disk containing z_0 with $\overline{B} \subset \Omega$ and if u is harmonic on $\overline{B}$, then $v - u$ satisfies the maximum principle. Fix a disk B centered at z_0 with $\overline{B} \subset \Omega$. Let $P(z)$ be the Poisson integral of $v|_{\partial B}$ on B. By a version of Schwarz's Theorem that applies to disks that aren't necessarily the unit disk (whose proof mimics the proof of the Schwarz Theorem in the notes), we know that $P(z)$ is harmonic in B and that $\lim\limits_{z \to \zeta} P(z) = v(\zeta)$ for all $\zeta \in \partial B$. But, by hypothesis, $v - P$ satisfies the maximum principle, so we deduce that $v(z) - P(z) \le 0$, or $v(z) \le P(z)$, for $z \in B$. In particular, $v(z_0) \le P(z_0)$. This inequality holds for B (and corresponding Poisson integral P) of arbitrarily small radius containing z_0. This is exactly what it means for v to be subharmonic.

   $(ii) \Rightarrow (i)$. Suppose that $\Delta v \ge 0$ in Ω. Fix $z_0 \in \Omega$ and $r > 0$ so that the closure of the disk $D(z_0, r)$ centered at z_0 of radius r is contained in Ω. Put $u = 1$ in the version of Green's Theorem stated in Exercise 3. Then we obtain
$$\int_{\partial D(z_0,r)} \frac{\partial v}{\partial \eta}\,|dz| = \int_{D(z_0,r)} \Delta v \, dxdy.$$
   (Here, $\eta(\zeta)$ is the outward-facing unit normal at $\zeta \in \Omega$, just as in Exercise 3. We can parametrize $\partial D(z_0, r)$ by the circle $\gamma(\theta) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$. If we set $M(r) = \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\,d\theta$, then differentiating under the integral, we see that $M'(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta} v'(z_0 + re^{i\theta})\,d\theta$. We can

rewrite $\dfrac{\partial v}{\partial \eta}|dz|$ in another useful way. Note that $|dz| = r\,d\theta$. Also, using the relations $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$, and $\theta = \arctan(y/x)$, we easily find (the plug-and-chug part omitted) using the chain rule that

$$\begin{aligned}
\frac{\partial v}{\partial \eta} &= (v_x, v_y)\cdot(\cos\theta, \sin\theta) \\
&= v_x\cos\theta + v_y\sin\theta \\
&= (v_\theta\theta_x + v_r r_x)\cos\theta + (v_\theta\theta_y + v_r r_y)\sin\theta \\
&= \cdots \text{(substitute in partials and make cancellations)} \\
&= \frac{\partial v}{\partial r}
\end{aligned}$$

It follows now that

$$rM'(r) = \int_0^{2\pi} re^{i\theta}v'(z_0+re^{i\theta})\,d\theta = \int_{\partial D(z_0,r)} \frac{\partial v}{\partial \eta}|dz| = \int_{D(z_0,r)} \Delta v\,dxdy.$$

If we regard the radius $r > 0$ as varying subject to the constraing that $\overline{D}(z_0,r) \subset \Omega$, then since $\Delta v \geq 0$ by assumption, we see that $M'(r) \geq 0$, i.e., $M(r)$ is nondecreasing. Since $v(z_0) = 0$, we have, for all sufficiently small $r > 0$,

$$v(z_0) \leq M(r) = \frac{1}{2\pi}\int_0^{2\pi} v(z_0+re^{it})\,dt.$$

Hence, v is subharmonic at z_0. Since z_0 was arbitrary, v is subharmonic in Ω.

$(i) \Rightarrow (ii)$. The proof of this part is very similar to the last. Suppose v is subharmonic in Ω. Suppose for a contradiction that $\Delta v(z_0) < 0$ for some $z_0 \in \Omega$. By continuity of v, $\Delta v(z_0+re^{it}) < 0$ for all $r < \epsilon$, where $\epsilon > 0$ is taken to be sufficiently small. In the last part, we computed

$$rM'(r) = \int_{D(z_0,r)} \Delta v\,dxdy.$$

Hence, in our case, $M(r)$ is decreasing for $0 < r < \epsilon$. But then

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})\,dt < u(z_0)$$

for all $0 < r < \epsilon$, contradicting subharmonicity of v. We conclude that $\Delta v(z) \geq 0$ for all $z \in \Omega$.

6. Let u be a real-valued harmonic function in $\mathbb{D}$ with $|u| \leq 1$ and $u(0) = 0$. Let $f(z)$ be analytic in $\mathbb{D}$ such that $\operatorname{Re} f = u$ and $f(0) = 0$. Note that it is possible to find such a function f by Corollary 1.7 and since $u(0) = 0$.

   Let $g(z) = e^{i\pi f(z)/2}$. Since $|\operatorname{Re} f(z)| = |u(z)| \leq 1$, we see that $\frac{\pi|\operatorname{Re} f(z)|}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Hence, $\arg g(z) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, so $\operatorname{Re} g(z) \geq 0$. Geometrically, this means that $\operatorname{Re} g(z)$ is no farther from the point 1 than it is from the point -1. Hence, if we set

$$h(z) = \frac{g(z)-1}{g(z)+1} = \frac{e^{i\pi f(z)/2}-1}{e^{i\pi f(z)/2}+1},$$

   we see that $|h(z)| \leq 1$. Note that $h(0) = 0$ and that $h(z)$ is analytic in $\mathbb{D}$. Therefore, the ordinary Schwarz Lemma for analytic functions on the disk applies, and we conclude

$$\left|\frac{e^{i\pi f(z)/2}-1}{e^{i\pi f(z)/2}+1}\right| = |h(z)| \leq |z| \tag{1.1}$$

for all $z \in \mathbb{D}$. Schwarz's Lemma also tells us that equality holds for some nonzero $z \in \mathbb{D}$ if and only if $h(z)$ is a rotation of the disk fixing 0.

Note that $e^{i\pi f(z)/2} = \dfrac{1+h(z)}{1-h(z)}$. Hence, $\tan|\dfrac{\pi u}{2}| = \tan|\arg\dfrac{1+h}{1-h}|$. Write $z = re^{i\theta}$, $h(re^{i\theta}) = se^{i\phi}$. Note that Schwarz's Lemma implies $s < r$. Also,

$$\frac{1+h}{1-h} = \frac{1+se^{i\phi}}{1-se^{i\phi}} = \frac{1-s^2+2is\sin\phi}{1+s^2-2s\cos\phi}.$$

The denominator of the right-hand side is purely real, and we deduce

$$\tan\frac{1+h}{1-h} = \frac{2s|\sin\phi|}{1-s^2} \le \frac{2s}{1-s^2}.$$

All this implies that

$$\frac{\pi}{2}|u| = \arctan(\tan|\arg\frac{1+h}{1-h}|) \le \arctan(\frac{2s}{1-s^2}) = 2\arctan s \le 2\arctan r = 2\arctan|z|.$$

Here, we have used the basic real trigonometric identity $\arctan\dfrac{2\alpha}{1-\alpha^2} = 2\arctan\alpha$ and the fact that arctan is increasing. Hence,

$$|u| \le \frac{4}{\pi}\arctan|z|, \tag{1.2}$$

as desired.

It is clear that if equality does not hold in (1.7) at any nonzero point $z \in \mathbb{D}$, then equality cannot hold in (1.2) at any nonzero point $z \in \mathbb{D}$. In fact, it is not much harder to see from the above manipulations between equations (1.7) and (1.2) that if equality holds in (1.2) at some nonzero $z \in \mathbb{D}$, then equality also holds in (1.7) at the same point, which in turn holds if and only if h is a rotation of the disk.

7. Let $u(z) = u(x,y)$ be a harmonic function in $\mathbb{D}$. Since $\mathbb{D}$ is simply connected, $u(z)$ is the real part of some function $g(z)$ analytic in $\mathbb{D}$. The function $g(z)$ has a power series $\sum_{n=0}^{\infty} c_n z^n$ based at 0 that converges at every point in $\mathbb{D}$, as $\mathbb{D}$ itself is the largest disk centered at 0 that is contained in the region of analyticity of g. Write $c_n = a_n + ib_n$, where a_n, b_n are real. For $z = x+iy \in \mathbb{D}$ (here, x, y are real), we have

$$g(z) = \sum_{n=0}^{\infty} c_n(x+iy)^n = \sum_{n=0}^{\infty} c_n\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}i^k y^k\right)$$

In this form, the real part of $g(z)$ is given by

$$\operatorname{Re} g(x+iy) = u(x,y) = \sum_{n=0}^{\infty} c_n\left(\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}x^{n-2k}(-1)^k y^{2k}\right).$$

Define $f(z) = 2u(\frac{z}{2},\frac{z}{2i}) - u(0,0)$ by formally replacing x by $\frac{z}{2}$ and y by $\frac{z}{2i}$. This substitution yields

$$\begin{aligned} f(z) &= 2c_0 - u(0,0) + 2\sum_{n=1}^{\infty} c_n\left(\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}\frac{z^{n-2k}}{2^{n-2k}}\frac{z^{2k}}{(2i)^{2k}}\right) \\ &= a_0 + 2ib_0 + \sum_{n=1}^{\infty} c_n z^n\left(\frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}\right) \\ &= a_0 + 2ib_0 + \sum_{n=1}^{\infty} c_n z^n, \end{aligned}$$

where we have used the well known identity $\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} = 2^{n-1}$, which is valid for integers $n \geq 1$.

Since g and f differ by a constant, we see that f is analytic in $\mathbb{D}$. In particular, since the power series representation above for f converges at all points $z \in \mathbb{D}$, f is indeed meaningfully defined in terms of the power series. Finally, it is now clear that $\operatorname{Re} f = u$.

8. Suppose u is harmonic in $\mathbb{C}$ and satisfies $|u(z)| \leq M|z|^k$ for some fixed k and all $|z| > R$, where $R > 0$ is some fixed large number. Since $\mathbb{C}$ is simply connected, there exists an entire function $f(z)$ such that $\operatorname{Re} f = u$. Moreover, we may choose f so that $\operatorname{Im} f(0) = 0$.

   Fix z_0 with $|z_0| > R$, and put $S = 2|z_0|$. By an analogous version of Corollary 1.7 to disks of radius not necessarily 1, the uniqueness assertion of the corollary gives

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Se^{it} + z}{Se^{it} - z} u(Se^{it})\, dt.$$

   Note that $|Se^{it} + z_0| \leq \frac{3S}{2}$ and that $|Se^{it} - z_0| \geq \frac{S}{2}$. It follows that

$$\begin{aligned} |f(z_0)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{Se^{it} + z}{Se^{it} - z} u(Se^{it}) \right| \cdot |u(Se^{it})|\, dt \\ &\leq \frac{M \cdot S^k}{2\pi} \cdot \frac{3S/2}{S/2} \int_0^{2\pi} dt \\ &= 3 \cdot 2^k M |z_0|^k. \end{aligned}$$

   This bound is independent of R, so it holds for z_0 provided $|z_0| > R$. We conclude that the analytic function f satisfies $|f(z)| \leq A|z|^k$ for some constant A and all z with $|z| > R$. By an exercise from long ago, we know that this implies that f is a polynomial of degree at most k. Hence, u is the real part of a polynomial of degree at most k.

9. Suppose u is harmonic in $\mathbb{C}$ and $\liminf_{r\to\infty} M(r)/\log r \leq 0$, where $M(r) = \sup_{|z|=r} u(z)$.

   Fix $\delta > 0$. Fix $r_0 > 0$ small so that $M(r_0) < u(0) + \delta$, which is possible by continuity of u. Choose $\epsilon > 0$ so small that $M(r_0) - \epsilon \log r_0 < u(0) + \delta$. Choose $R > r_0 > 0$ large so that $M(R)/\log R \leq \epsilon$, which is possible by the assumption that $\liminf_{r\to\infty} M(r)/\log r \leq 0$. It will be important later on to note that there exist arbitrarily large R satisfying this inequality.

   The function $u(z) - \epsilon \log |z|$ defined in the annulus $A = \{z : r_0 \leq |z| \leq R\}$ is harmonic, being the difference of two harmonic functions. Hence, by the maximum principle, it takes its maximum on ∂A. Letting $M = \max\{0, u(0) + \delta\}$, it follows that $u(z) - \epsilon \log |z| \leq M$. Note in particular that the bound M depends only on u and δ. Since $\epsilon > 0$ can be made arbitrarily small, we conclude that $u(z) \leq M$ for all $z \in A$. In fact, we can conclude that $u(z) \leq M$ for all $z \in \mathbb{C}$, we can choose $r_0 > 0$ as small as we want and since we can always find R as large as we want so that $M(R)/\log R \leq \epsilon$ for fixed $\epsilon > 0$.

   We have shown so far that $u(z)$, which is harmonic everywhere, is bounded by M. Hence, $M - u(z)$ is a positive function that is harmonic in $\mathbb{C}$. We will show that $M - u(z)$ must be constant. One can prove the following version of Harnack's Inequality by mimicking the proof of the version of Harnack's Inequality in the notes: if $s(z)$ is a positive harmonic function on the open disk of radius $R > 0$, then for all $|z| < R$,

$$\frac{R - |z|}{R + |z|} s(0) \leq s(z) \leq \frac{R + |z|}{R - |z|} s(0).$$

We apply this version of Harnack's Theorem to the present case. Fix $z_0 \in \mathbb{C}$, and fix $R > |z_0|$, and consider the function $v(z)$ defined on $\{z : |z| < R\}$ given by $v(z) = M - u(z)$. Then

$$\frac{R - |z_0|}{R + |z_0|} v(0) \leq v(z_0) \leq \frac{R + |z_0|}{R - |z_0|} v(0),$$

and letting $R \to \infty$, we conclude $v(z_0) = v(0)$. Hence, v is constant, hence so is $u(z)$. This completes the proof.

10. Suppose v is subharmonic in $\mathbb{C}$. Fix $\rho_0 > 1$ and let $r > \rho_0$. Let C_1 and C_2 be the circles centered at 0 of radii ρ_0 and r, respectively. Orient both C_1 and C_2 in the positive direction. The cycle $C_2 - C_1$ bounds an annulus A in $\mathbb{C}$. Applying Green's Theorem to the functions $\log|z|$ and v on A, we see that

$$\int_A (\log|z|\,\Delta v - v\Delta u)\,dxdy = \int_{C_2 - C_1} \left(\log|z|\frac{\partial v}{\partial \eta} - v\frac{\partial}{\partial \eta}\log|z|\right)|dz|.$$

In fact, since $\log|z|$ is harmonic on A, we know that $\Delta u = 0$. Also, by exercise 5, $\Delta v \geq 0$, as v is subharmonic on A, so that $\log|z|\,\Delta v \geq 0$ everywhere in A. We now see that

$$\int_A (\log|z|\,\Delta v - v\Delta u)\,dxdy = \int_A \log|z|\,\Delta v\,dxdy$$

is nonnegative and increases with r (keeping ρ_0 fixed), and hence the same is true of

$$\int_{C_2 - C_1} \left(\log|z|\frac{\partial v}{\partial \eta} - v\frac{\partial}{\partial \eta}\log|z|\right)|dz|.$$

If we put $K = \int_{C_1} \left(\log|z|\frac{\partial v}{\partial \eta} - v\frac{\partial}{\partial \eta}\log|z|\right)|dz|$, which is a constant independent of r, we we see that

$$\int_{C(r)} \left(\log|z|\frac{\partial v}{\partial \eta} - v\frac{\partial}{\partial \eta}\log|z|\right)|dz| - K \geq 0, \tag{1.3}$$

where $C(r)$ is the circle of radius r centered at 0 oriented counterclockwise, and that, in fact, the left-hand side increases with r.

Since $C(r)$ is a circle, we have that $\frac{\partial v}{\partial \eta}|dz| = r\,\frac{\partial v}{\partial r}\,d\theta$, as was computed in Exercise 5. Similaly, $\frac{\partial}{\partial \eta}(\log r)\,|dz| = r\,\frac{\partial}{\partial r}(\log r)\,d\theta = d\theta$. Hence, parametrizing $C(r)$ by the curve $\gamma(t) = re^{i\theta}$, $\theta \in [0, 2\pi]$, we see that (1.3) can be rewritten as

$$\int_0^{2\pi} \left(r\log r\frac{\partial v}{\partial r} - v\right)d\theta - K = r\,\log r\int_0^{2\pi}\frac{\partial v}{\partial r}\,d\theta - 2\pi M_1(r) - K, \tag{1.4}$$

where $M_1(r) = \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\,d\theta$. Note that

$$M_1'(r) = \frac{\partial}{\partial r}\left[\frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\,d\theta\right] = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial v}{\partial r}\,d\theta. \tag{1.5}$$

It follows from (1.3), (1.4), and (1.5) that

$$M_1'(r)\log r - \frac{M_1(r)}{r} - \frac{K}{r} \geq 0.$$

If we trace back through the logic, we see that equality holds for all $r > \rho_0$ in the above line if and only if $\Delta v(z) = 0$ for all $|z| > \rho_0$. If $\Delta v \neq 0$ at some point z with $|z| > \rho_0$, then by continuity $\Delta v \neq 0$ in some small neighborhood, and we see that there exists $\delta > 0$ such that for all sufficiently large r,

$$M_1'(r) \log r - \frac{M_1(r)}{r} - \frac{K}{r} > \delta > 0.$$

In this case, since K is constant, it is straightforward to see that for sufficiently large r,

$$M_1'(r) \log r - \frac{M_1(r)}{r} \geq 0.$$

In either case ($\Delta v \equiv 0$ or $\Delta v \not\equiv 0$), we deduce that

$$\frac{d}{dr}\left(\frac{M_1(r)}{\log r}\right) = \frac{(\log r) M_1'(r) - \frac{1}{r} M_1(r)}{(\log r)^2}$$

is nonnegative for sufficiently large r, hence

$$\frac{M_1(r)}{\log r}$$

is increasing for sufficiently large r, hence has a limit (possibly infinite).

1.2.2 Chapter 9

3. We wish to find a conformal map from the upper half-plane $\mathbb{H}$ onto the region $\{x + iy : x^2 - y^2 > 1 \text{ and } x > 0\}$. We will build up the map in intermediate stages.

   Put $\Omega_1 = \mathbb{H}$. Let $\varphi_1(z) = \sqrt{z} = e^{\frac{1}{2}\log z}$, where $\log z$ is defined on the simply connected region $\mathbb{C} \setminus (-\infty, 0]$ and is chosen so that $\log 1 = 0$. Then, as we have seen many times over the course, φ_1 maps $\mathbb{H}$ conformally onto the first quadrant $\Omega_2 := \{z : x > 0, y > 0\}$.

   Let $\varphi_2(z) = \frac{1+z}{1-z}$. Since φ_2 is an LFT, it takes disks to disks and their boundary circles to boundary circles. (Here, "disk" can mean half-plane, as in Chapter VI of the notes.) The first quadrant Ω_2 is the intersection of two disks, namely, the upper half-plane and the right half-plane. The map φ_2 takes the real line to the real line. Hence, φ_2 will map the upper half-plane to either the upper or lower half-plane. But, $\varphi_2(1) = \infty$, $\varphi_2(0) = 1$, and $\varphi_2(-1) = 0$. As we run from -1 to 0 to 1, the upper half-plane lies to our left. Hence, since φ_2 is orientation-preserving, as we run from $\varphi_2(-1) = 0$ to $\varphi_2(0) = 1$ to $\varphi_2(1) = \infty$, the image $\varphi_2(\mathbb{H})$ better lie to our left. We conclude that $\varphi_2(\mathbb{H}) = \mathbb{H}$. Now we argue that φ_2 takes the right half-plane to the unit disk $\mathbb{D}$. Each point z in the right half-plane is closer to 1 in distance than to -1, so $|1 - z| < |1 + z|$ for each $z \in \{z : x > 0\}$. Hence, $\varphi_2(z) > 1$ for each $z \in \{z : x > 0\}$, and, of course, $\varphi_2(z) = 1$ if $\operatorname{Re} z = 0$. Since φ_2 takes disks to disks and their bounding circles to bounding circles, we conclude that $\varphi_2(\{z : \operatorname{Re} z > 0\}) = \mathbb{C} \setminus \overline{\mathbb{D}}$. By injectivity of φ_2, it follows that $\varphi_2(\Omega_2) = \varphi_2(\mathbb{H} \cap \{z : \operatorname{Re} z > 0\}) = \mathbb{H} \cap \{z : |z| > 1\}$. Set $\Omega_3 := \mathbb{H} \cap \{z : |z| > 1\}$.

   Let $\varphi_3(z) = z^{1/4} = e^{\frac{1}{4}\log z}$, where $\log z$ is defined on $\mathbb{C} \setminus (-\infty, 0]$ and $\log 1 = 0$. If $z = re^{i\theta}$, where $r > 0$ and $0 < \theta < \pi$, then $\varphi_3(z) = r^{1/4} e^{i\theta/4}$. From this, it is easy to see that φ_3 maps Ω_3 onto the region $\Omega_4 := \{z : |z| > 1 \text{ and } 0 < \operatorname{Im} z < \operatorname{Re} z\}$. The map is conformal, as well, as we have seen before.

   Let $\varphi_4(z) = \frac{1}{\sqrt{2}}(z + \frac{1}{z})$. Note that $\varphi_4([1, \infty)) = [\sqrt{2}, \infty)$. Analogous to the behavior of $\frac{1}{2}(z + \frac{1}{z})$, we know from Chapter VI that φ_4 maps the unit circle to the line segment $[-\sqrt{2}, \sqrt{2}]$. In fact, the arc

$\{e^{i\theta} : 0 \le \theta \le \frac{\pi}{4}\}$ is mapped under φ_4 to the segment $[1, \sqrt{2}]$. Now we examine how the half-line $\{z : |z| \ge 1, \operatorname{Re} z = \operatorname{Im} z\}$ is mapped under φ_4. Let $z = a + ia$, where $a \ge \frac{1}{\sqrt{2}}$. We find

$$\frac{1}{\sqrt{2}}(z + \frac{1}{z}) = u + iv,$$

where $u = \frac{1}{\sqrt{2}}(a + \frac{1}{2a})$ and $v = \frac{1}{\sqrt{2}}(a - \frac{1}{2a})$. Note that $u^2 - v^2 = 1$. Moreover, if $a = \frac{1}{\sqrt{2}}$, $\varphi_4(a + ia) = 1$, and we see that $\varphi_4(\{z : |z| \ge 1, \operatorname{Re} z = \operatorname{Im} z\}$ is in fact the top half of the branch of the hyperbola $u^2 - v^2 = 1$, $u > 0$.

Up to this point, we have determined that $\varphi_4(z)$ maps $\partial\Omega_4$ onto $\{u + iv : u^2 - v^2 = 1, u > 0, v \ge 0\} \cup [1, \infty)$. It is now easy to see, e.g., by evaluating φ_4 at a point of Ω_4, that, in fact, φ_4 maps into the "interior" of the half-branch of the hyperbola, i.e., φ_4 maps Ω_4 conformally onto $\Omega_5 := \{u + iv : u > 0, v > 0, u^2 - v^2 > 1\}$.

We have a conformal map $\varphi := \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ from $\mathbb{H}$ onto Ω_5. We would like a conformal map from $\mathbb{H}$ onto the whole interior of the branch of $u^2 - v^2 = 1$ lying in the half-plane $\{u \ge 0\}$. To obtain such a map, we will need the Schwarz Reflection Principle.

It is straightforward to trace through the conformal mappings to determine that φ maps $(-\infty, 1] \subset \partial\mathbb{H}$ one-to-one onto $[1, \infty) \subset \partial\Omega_5$. By a variant of the Schwarz Reflection Principle (Corollary IX.1.3), we can extend φ to a one-to-one analytic function $\widetilde{\varphi}$ on $\widetilde{\Omega} := \mathbb{H} \cup \mathbb{H}^- \cup (-\infty, 1)$ (where $\mathbb{H}^-$ is the lower half-plane) that maps onto the branch of the hyperbola $\{u + iv : u^2 - v^2 > 1, u > 0\}$. In fact,

$$\widetilde{\varphi}(z) = \begin{cases} \varphi(z) & \text{if } z \in \mathbb{H}, \\ \overline{\varphi(\overline{z})} & \text{if } z \in \mathbb{H}^-, \\ \lim\limits_{\substack{w \to z \\ w \in \mathbb{H}}} \varphi(w) & \text{if } z \in (-\infty, 1). \end{cases}$$

Finally, if we define $\tau : \mathbb{H} \to \mathbb{C} \setminus [1, \infty)$ by $\tau(z) = 1 + z^2$, we see that $\varphi \circ \tau$ maps $\mathbb{H}$ conformally onto $\{u + iv : u^2 - v^2 > 1, u > 0\}$, as desired.

4. Let $f : 0 \le t \le \pi$ be a given real-valued continuous function. Define a function $\widetilde{f}$ on $\partial\mathbb{D}$ by

$$\widetilde{f}(e^{i\theta}) = \begin{cases} f(\theta) & \text{if } 0 \le \theta \le \pi, \\ f(2\pi - \theta) & \text{if } \pi \le \theta \le 2\pi. \end{cases}$$

Using the fact that f is continuous, it is easy to see that $\widetilde{f}$ is continuous on $\partial\mathbb{D}$. Define

$$G(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \widetilde{f}(e^{i\theta})\, d\theta,$$

the Herglotz integral of $\widetilde{f}$ on $\mathbb{D}$. Let $u(z) = \operatorname{Re} G(z)$. By Schwarz's Theorem, $u(z)$ is harmonic in $\mathbb{D}$, and we can extend u to be continuous on $\partial\mathbb{D}$ so that u and $\widetilde{f}$ agree on $\partial\mathbb{D}$. As we have seen, another way to write u is $u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \widetilde{f}(e^{i\theta})\, d\theta$.

It remains to check that $u_y = 0$ on $(-1, 1)$. First, note that for all $z \in \mathbb{D}$, $|e^{i\theta} - \overline{z}| = |e^{-i\theta} - z|$, and

$\widetilde{f}(e^{-i\theta}) = \widetilde{f}(e^{i\theta})$ for all θ, so that

$$\begin{aligned} u(\overline{z}) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|\overline{z}|^2}{|e^{i\theta}-\overline{z}|^2}\widetilde{f}(e^{i\theta})\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{-i\theta}-z|^2}\widetilde{f}(e^{-i\theta})\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\widetilde{f}(e^{it})\,dt \\ &= u(z) \end{aligned}$$

where we have made the change of variables $t = -\theta$ to obtain the second to last equality.

Since

$$\frac{\partial u}{\partial y}(x+iy) = iu'(x+iy) \quad \text{and} \quad \frac{\partial u}{\partial y}(x-iy) = -iu'(x-iy),$$

we see that if $x+iy$ is on $(-1,1)$, i.e., if $y=0$, then

$$\frac{\partial u}{\partial y}(x) = iu'(x) = -iu'(x)$$

which implies $\dfrac{\partial u}{\partial y}(x) = 0$.

We have produced a function u that satisfies the desired criteria (and more): u is harmonic on $\mathbb{D}$, $\dfrac{\partial u}{\partial y} = 0$ on $(-1,1)$, u is continuous on $\partial\mathbb{D}$, and $u = f$ on $\partial\mathbb{D} \cap \{\operatorname{Im} z > 0\}$.

5. Let g be continuous on $\partial\mathbb{D}$ and suppose $\displaystyle\int_0^{2\pi} g(e^{i\theta})\,d\theta = 0$. Define $\widetilde{g}(e^{i\theta})$ on $\partial\mathbb{D}$ by $\widetilde{g}(e^{i\theta}) = \displaystyle\int_0^{\theta} g(e^{it})\,dt$. The function $\widetilde{g}$ is continuous at all points in $\partial\mathbb{D}$, including at 1, by the condition $\displaystyle\int_0^{2\pi} g(e^{i\theta})\,d\theta = 0$. Now define

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\widetilde{g}(e^{i\theta})\,d\theta,$$

which by Schwarz's Theorem is analytic in $\mathbb{D}$. Also, $\operatorname{Re} f$ tends to $\widetilde{g}$ on the boundary of the disk.

Let $v = \operatorname{Re} F$ and $u = -\operatorname{Im} F$. Inside $\mathbb{D}$, we know that $u_x = -v_y$ and $u_y = v_x$ by the Cauchy-Riemann equations. Also, the directional derivative of v in the direction of the tangent vector at a point $re^{i\theta}$ with $0 < r < 1$ is given by

$$\begin{aligned} \frac{\partial v}{\partial s} &= v_x(-\sin\theta) + v_y\cos\theta \\ &= u_y(-\sin\theta) + (-u_x)\cos\theta \\ &= \frac{\partial u}{\partial \eta} \end{aligned}$$

where η is the inward pointing unit normal at $re^{i\theta}$. But, on $|z| = r$, we also have

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial \theta}.$$

(We did a similar computation showing $\frac{\partial u}{\partial \eta} = -\frac{\partial u}{\partial r}$ on last homework, so we omit the similar calculation here.)

Using the integral representation for v (and hence for v_θ) and a difference quotient argument applied to $|v_\theta(z) - g(e^{it_0})|$ as $z \to e^{it_0}$, as in the proof of the boundary condition in Schwarz's Theorem, one can show that

$$\lim_{z\to\zeta} \frac{\partial v}{\partial \theta}(z) = \frac{d}{d\theta}\widetilde{g}(\zeta) = g(\zeta)$$

for all $\zeta \in \mathbb{D}$. Hence,

$$\lim_{z\to\zeta} \frac{\partial u}{\partial \eta}(z) = \lim_{z\to\zeta} \frac{\partial v}{\partial \theta}(z) = g(\zeta)$$

for $\zeta \in \mathbb{D}$, so the function u satisfies the desired criteria.

6. Define $\widetilde{g}$ on the real line by

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{if } x \in (-1,1), \\ 0 & \text{else.} \end{cases}$$

and put $h(x) = \int_{-\infty}^{x} \widetilde{g}(t)\, dt$. Note that h is continuous on the real line. Using the Poisson kernel for the upper half-plane, define for $z = x + iy$ with $\operatorname{Im} z > 0$:

$$v(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} h(t)\, dt.$$

Then v is harmonic in $\mathbb{H}$ with boundary values $h(z)$, since h is continuous on the whole real line, by an analog of Schwarz's Theorem for the upper half-plane Poisson kernel. Using the integral representation of v (and hence that of v_y) and a difference quotient argument as in the proof of Schwarz's Theorem, one can show that

$$\lim_{y\to 0^+} \frac{\partial v}{\partial y}(x+iy) = \frac{d}{dx}h(x) = \widetilde{g}(x),$$

which is equal to g on $(-1,1)$.

Now we look for a function w that is harmonic on $\mathbb{D}^+$ such that $w = f - v$ on $\partial\mathbb{D}^+ \cap \{\operatorname{Im} z > 0\}$ and $\frac{\partial w}{\partial y} = 0$ on $(-1,1)$. This is essentially Problem #4, so we simply mention the solution here. Let $\widetilde{f}$ be defined as in Problem #4 and set

$$w(z) = \operatorname{Re} \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}(\widetilde{f}(e^{i\theta}) - v(e^{i\theta}))\, d\theta.$$

Then $w(z)$ is continuous on $\overline{\mathbb{D}^+} \setminus \{\pm 1\}$, $w = f - v$ on $\partial\mathbb{D}^+ \cap \{\operatorname{Im} z > 0\}$, w is harmonic on $\mathbb{D}^+$, and $\frac{\partial w}{\partial y} = 0$ on $(-1,1)$.

Now, set $u(z) = w(z) + v(z)$ (where sensible). Then u is harmonic in $\mathbb{D}^+$, continuous on $\overline{\mathbb{D}^+} \setminus \{\pm 1\}$, $u = f$ on $\partial\mathbb{D}^+ \cap \{\operatorname{Im} z > 0\}$, and $\frac{\partial u}{\partial y} = g$ on $(-1,1)$, as desired.

8. The geodesic algorithm constructs a conformal map of $\mathbb{H}$ onto a simply connected region Ω_c whose computed boundary consists of curves $\gamma_1, \ldots, \gamma_n$, where γ_1 has endpoints z_1 and z_2, γ_2 has endpoints z_2 and z_3, and so on, with the last curve γ_n having endpoints z_n and z_1. By Theorem IX.3.4, we know that the computed boundary $\gamma_1 \cup \cdots \cup \gamma_n$ is C^1, and in particular, that γ_j and γ_{j+1}

meet at an angle of π at z_{j+1}. As noted in the proof of Theorem IX.3.4, the first arc γ_1 is a chord of D_1^+ and of D_1^-, and hence γ_1 lies in L_1, except for at endpoints, and so γ_1 is not tangent to either D_1^+ or D_1^-. The angle at z_2 between γ_1 and γ_2 is π, and by the tangency of ∂D_1^+ and ∂D_2^-, as well as the tangency of ∂D_1^- and ∂D_2^+, we see that γ_2 must enter $L_2 = D_2^+ \cap D_2^-$. As in Theorem IX.3.4, we know that γ_2 is a hyperbolic geodesic in $\mathbb{C}^* \setminus \gamma_1$. But, D_2^+ does not intersect L_1, so D_2^+ cannot intersect γ_1. Hence, we can apply Jorgensen's Lemma to deduce that γ_2 remains in D_2^+, except at the endpoints z_2 and z_3, where it exits. Also, γ_2 is not tangent to ∂D_2^+. In the same way, γ_2 enters and exits D_2^- at z_2 and z_3, remaining inside D_2^- between, and is not tangent to ∂D_2^-. Hence, γ_2 lies in $L_2 \cup \{z_2, z_3\}$, and because γ_3 and γ_2 form an angle of π at z_3, we see by the tangency condition on the lenses that γ_3 enters $D_3^+ \cap D_3^-$, and we use the fact that γ_3 is a hyperbolic geodesic in $\mathbb{C}^* \setminus (\gamma_1 \cup \gamma_2)$ and Jorgensen's Lemma, as before, to conclude that γ_3 stays inside L_3 between its endpoints. Continuing inductively in this fashion, we conclude that $\gamma_1 \cup \cdots \cup \gamma_n$ is contained in $\bigcup \overline{L_i}$.

1.2.3 Chapter 10

1. (IX.12) In order to construct a conformal map of $\mathbb{D}$ onto the interior of an ellipse, we will start by constructing a conformal map of $\mathbb{D}^+ = \mathbb{D} \cap \{\operatorname{Im} z > 0\}$ onto the upper half of the interior of an ellipse. As per the sketch on the next page, we are also interested in following what this conformal map does to the interval $(-1, 1) \subset \partial\mathbb{D}^+ \cap \mathbb{R}$.

   Let $\varphi_1 : \mathbb{D}^+ \to \mathbb{H}$ be defined by $\varphi_1(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$. We have seen before that this map is analytic, one-to-one, and onto $\mathbb{H}$. Note that φ_1 takes the interval $(-1, 1) \in \partial\mathbb{D}^+$ onto $(-\infty, -1) \cup (1, \infty) \in \partial\mathbb{H}$.

   We will determine a conformal map $\varphi_2 : \mathbb{H} \to \mathbb{H}$ after we determine a conformal map $\varphi_3 : \mathbb{H} \to R$, where R is defined in the next paragraph.

   Let R be the open rectangle with sides parallel to the coordinate axes and with top-left vertex 0 and bottom-right vertex $\pi - i$ on its boundary ∂R. Let $v_1 = 0$, $v_2 = -i$, $v_3 = \pi - i$, and $v_4 = \pi$ be the vertices of R, listed in counterclockwise order. By the Schwarz-Christoffel Theorem, there exists a conformal map of $\mathbb{H}$ onto R of the form

   $$\varphi_3(z) = A \int_0^z \frac{1}{[(\zeta - x_1)(\zeta - x_2)(\zeta - x_3)(\zeta - x_4)]^{1/2}}\, d\zeta + B,$$

   where x_1, x_2, x_3, x_4 are some prevertices on the real line satisfying $-\infty < x_1 < x_2 < x_3 < x_4 < \infty$ and A, B are constants. (Here, the branch $\sqrt{\zeta - x_j}$, $j = 1, 2, 3, 4$, that we are talking about is the branch defined on a slit plane where the slit runs along a vertical ray from x_j in the downward direction to ∞, chosen so that $\sqrt{\zeta - x_j}$ is positive for $\zeta > x_j$.) Moreover, φ_3 maps $(-\infty, x_1) \cup (x_4, \infty) \cup \{\infty\}$ onto the top edge $(v_1, v_4) = (0, \pi)$ of the rectangle and (x_1, x_2) and (x_3, x_4) onto the vertical sides.

   Now we go back and deal with φ_2. Let φ_2 be a conformal map of $\mathbb{H}$ onto itself by sending $\infty \mapsto \infty$, $-1 \mapsto x_2$, and $1 \mapsto x_3$.

   Now let $\varphi_4 : R \to A^+$ be given by $\varphi_4(z) = e^{iz}$, where A^+ is the top half of the annulus $\{z : 1 < |z| < e, \operatorname{Im} z > 0\}$. We have seen before that φ_4 is a conformal map from R onto A^+.

   Let $\varphi_5 : A^+ \to E^+$ be defined by $\varphi_5(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$, where $E^+ = \varphi_5(A^+)$. As we have seen before when studying the map $\frac{1}{2}\left(z+\frac{1}{z}\right)$, E^+ is in fact the top half of the interior of an ellipse centered at 0 with major and minor axes parallel to the x- and y-axes.

The composition $\varphi = \varphi_5 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ gives a conformal map of $\mathbb{D}^+$ onto E^+. If we consider the sketches, we see that as z tends to the interval $(-1,1)$, $\operatorname{Im}\varphi(z)$ tends to 0. By the Schwarz Reflection Principle, we can reflect to obtain a conformal map which we also denote by φ from the disk $\mathbb{D}$ onto the full ellipse E obtained by unioning E^+ with the reflection of E^+ through the real line, as well as the obvious points on the real line.

2. (IX.13) We wish to find a conformal map of $\mathbb{D}$ onto a regular n-gon. If we travel along the boundary of the n-gon and "turn" at a vertex, we turn an angle of $\frac{2\pi}{n}$. For $j = 1, 2, \ldots, n$, let $\zeta_j = e^{2\pi i j/n}$, so $\zeta_1, \ldots, \zeta_n$ are precisely the roots of $\zeta^n - 1$. For convenience, put $\zeta_0 = \zeta_n$. We will show that the Schwarz-Christoffel map on $\mathbb{D}$

$$F(z) = \int_1^z \frac{d\zeta}{(\zeta^n - 1)^{2/n}}$$

maps the unit disk conformally onto a regular n-gon.

By the Schwarz-Christoffel Theorem, we know $F(z)$ maps $\mathbb{D}$ conformally onto a polygon with n vertices, one vertex corresponding to each of the prevertices $\zeta_1, \ldots, \zeta_n$.

Since

$$(\zeta^n - 1)^{2/n} = (\zeta - \zeta_1)^{2/n} \cdots (\zeta - \zeta_n)^{2/n},$$

we see (as we saw in the proof of Schwarz-Christoffel) that the interior angle at each vertex of the image polygon will indeed be $(1 - 2/n)\pi = \frac{n-2}{n}\pi$, which is necessary for the polygon to be regular. All we need to check now is that the sides of the image polygon all have the same length. The length of the polygonal side with prevertices ζ_j and ζ_{j+1} is given by

$$\left| \int_{\zeta_j}^{\zeta_{j+1}} \frac{d\zeta}{(\zeta - \zeta_1)^{2/n} \cdots (\zeta - \zeta_n)^{2/n}} \right|.$$

We can parametrize this integral by $\zeta(t) = e^{it}$, where t runs from $2\pi j/n$ to $2\pi(j+1)/n$, and the integral becomes

$$\begin{aligned}
\int_{2\pi j/n}^{2\pi(j+1)/n} \frac{ie^{it}\,dt}{(e^{it} - \zeta_1) \cdots (e^{it} - \zeta_n)} &= \int_0^{2\pi/n} \frac{ie^{it}e^{2\pi i j/n}\,dt}{(e^{it}e^{2\pi i j/n} - \zeta_1) \cdots (e^{it}e^{2\pi i j/n} - \zeta_n)} \\
&= \int_0^{2\pi/n} \frac{ie^{it}\,dt}{(e^{it} - \zeta_1) \cdots (e^{it} - \zeta_n)} \\
&= \int_{\zeta_0}^{\zeta_1} \frac{d\zeta}{(\zeta - \zeta_1)^{2/n} \cdots (\zeta - \zeta_n)^{2/n}}
\end{aligned}$$

where we have obtained the middle equality by dividing the top and bottom of the expression

$$\frac{ie^{it}e^{2\pi i j/n}}{(e^{it}e^{2\pi i j/n} - \zeta_1) \cdots (e^{it}e^{2\pi i j/n} - \zeta_n)}$$

by $e^{2\pi i j/n}$ and noting that dividing $\zeta_1, \ldots, \zeta_n$ by $e^{2\pi i j/n}$ permutes the ζ_j's. From the above computation, we see that integrating from ζ_j to ζ_{j+1} gives a value that is independent of j. Hence, the sides of the image polygon all have the same length.

We have shown that $F(z)$ maps $\mathbb{D}$ conformally onto a regular n-gon.

3. (IX.14) We want to produce a version of Schwarz-Christoffel from the upper half plane $\mathbb{H}$ onto a polygonal region P that is unbounded. In this case, we will regard ∞ as a vertex. Also, as in the case when the polygonal region is bounded, the region P is allowed to contain "slits", i.e., if we traverse the boundary and arrive at a vertex v, we may turn away from v, head to the next vertex w, do a full turn of 2π, and head back towards v.

   Let $v_1, v_2, \ldots, v_n$ be the vertices of the polygonal region P, as seen as we travel along P in positive order. Note that ∞ may be represented in this list, possibly more than once (if there are slits emanating from ∞). Suppose we traverse the vertices in order. We start at $v_1 \neq \infty$, and when we arrive at v_2, we make a turn of $\beta_2\pi$, where $-1 < \beta_2 \leq 1$, provided $v_2 \neq \infty$. Then we arrive at v_3 and turn at an angle of $\beta_3\pi$, provided $v_3 \neq \infty$. When we arrive at an infinite vertex, we stipulate that we make a turn of π, so $\beta_j = -1$ in this case. If we come from ∞, arrive at a vertex v_j, and head back to ∞, then we stipulate that $\beta_j = 1$. Also, if we start from a vertex v_j, go to $v_{j+1} = \infty$, and come back to v_j immediately, we stipulate that $\beta_{j+1} = -1$.

   Now that we have the setup, the proof is nearly identical to the proof of the Schwarz-Christoffel Theorem from $\mathbb{H}$ onto bounded polygonal regions. In particular, in the bounded case, we were allowed to have two-sided arcs (corresponding to "slits" in our region), and this is still a possibility in the unbounded case, treated the exact same way. By the Schwarz Reflection principle, if φ is a conformal map of P onto $\mathbb{D}$, then φ extends analytically and one-to-one across the interior each boundary segment. (If one of these segments is a two-sided arc, the extensions may not agree, but that is fine.) If B_j is a small ball centered at $v_j \neq \infty$, then the map $(z - v_j)^{1/(1-\beta_j)}$ is one-to-one and analytic in $P \cap B_j$. If $v_j = \infty$, the map $(z - v_j)^{-1}$ is analytic in a neighborhood B_j of ∞ not containing any other v_k's. In either case, φ maps $\partial P \cap B_j$ onto a straight line segment. By the Schwarz Reflection principle, the inverse of this map composed with φ then extends to be analytic and one-to-one in a neighborhood of 0, hence φ extends to be one-to-one and continuous from $\overline{P}$ onto $\mathbb{D}$, from which it follows that if $f(z)$ is conformal from $\mathbb{H}$ onto P, then $f(z)$ extends to be one-to-one and continuous on $\overline{\mathbb{H}}$ and analytic at all $x_j = f^{-1}(v_j)$, where the prevertices can be assumed to be in order $-\infty < x_1 < \cdots < x_n < \infty$. Writing out the definition of $f'(z)$ as $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, we see as in the bounded case that $f'(x)$ points in the right direction, as $\arg f'(x)$ is given by the direction of the line segment from v_j to v_{j+1}. Note that the correct change in direction is made at a prevertex corresponding to a vertex of ∞. Since $f'(z) \neq 0$ on $\mathbb{H}$, we can define $\log f'(z)$ to be analytic on $\mathbb{P}$, hence $\arg f'(z)$ is a bounded harmonic function on $\mathbb{H}$ which is continuous at all boundary points except the finitely many prevertices. The function $\pi - \arg(z - a) = 0$ for $z < a$ and equals π for $z > a$, $a, z \in \mathbb{R}$, and an application of the Lindelöf Maximum Principle shows that $\arg f'(z) = c_0 + \sum_{j=1}^{n} \beta_j(\pi - \arg(z - x_1))$, so $f'(z) = A \prod_{j=1}^{n} (z - x_j)^{-\beta_j}$. The singularity at a prevertex x_j corresponding to ∞ is not integrable, but we take the integral to mean that the large value of the integral over a small neighborhood $(x_j - h, x_j)$ cancels with the integral over the interval $(x_j, x_j + h)$.

4. (X.6) Suppose $b_1, b_2, \ldots \to \infty$ with the b_k pairwise distinct, and let $a_1, a_2, \ldots$ be given with $|a_k| \leq M < \infty$ for all k. Fix $R > 0$. The sum

$$\sum_{k=1}^{\infty} \left(\frac{a_k}{z - b_k} - \left(\frac{a_k}{-b_k} \right) \sum_{j=0}^{k} \left(\frac{z}{b_k} \right)^j \right) \tag{1.6}$$

   can be split into two sums

$$\sum_{k:|b_k|<2R} \left(\frac{a_k}{z - b_k} - \left(\frac{a_k}{-b_k} \right) \sum_{j=0}^{k} \left(\frac{z}{b_k} \right)^j \right) + \sum_{k:|b_k|\geq 2R} \left(\frac{a_k}{z - b_k} - \left(\frac{a_k}{-b_k} \right) \sum_{j=0}^{k} \left(\frac{z}{b_k} \right)^j \right). \tag{1.7}$$

Since $b_k \to \infty$, there are only finitely many b_k with $|b_k| < 2R$. Therefore, the left-hand sum in (1.7) is finite, hence meromorphic in $|z| \leq R$. We will show that the right-hand sum in (1.7) is analytic in $|z| \leq R$. Note that each summand in the right-hand sum is analytic in $|z| \leq R$. Since there are only finitely many b_k with $|b_k| < 2R$, we can fix N so that $|b_k| \geq 2R$ whenever $k \geq N$. Consider the tail-end of the right-hand sum starting from $k = N$. Since $|a_k| \leq M$ and $|b_k|$ and using the fact that $\dfrac{a_k}{z - b_k} = \left(\dfrac{a_k}{-b_k}\right)\left(\dfrac{1}{1 - z/b_k}\right) = \dfrac{a_k}{-b_k}(1 + (z/b_k) + (z/b_k)^2 + \cdots)$ for $|z| \leq R$ when $k \geq N$, we compute

$$\begin{aligned}
\sum_{k \geq N} \left| \frac{a_k}{z - b_k} - \left(\frac{a_k}{-b_k}\right) \sum_{j=0}^{k} \left(\frac{z}{b_k}\right)^j \right| &\leq \sum_{k \geq N} \left(\left|\frac{a_k}{b_k}\right| \cdot \sum_{j=k+1}^{\infty} \left|\frac{z}{b_k}\right|^j \right) \\
&\leq \sum_{k \geq N} \left(\frac{M}{2R} \sum_{j=k+1}^{\infty} \left(\frac{R}{2R}\right)^j \right) \\
&\leq \frac{M}{2R} \sum_{k \geq N} 2^{-k} \\
&\leq \frac{M}{2R}.
\end{aligned}$$

This shows that the right-hand sum in (1.7) converges absolutely and, in fact, uniformly (by the above computation) on $|z| \leq R$. Hence, the series (1.6) converges to a meromorphic function on $|z| \leq R$. Since R is arbitrary, (1.6) converges to a meromorphic function in $\mathbb{C}$.

By construction, the partial sums of (1.6) are analytic at all z such that $z \neq b_k$ for all k. Since the partial sums converge uniformly on compact subsets containing z, we conclude (1.6) is analytic at z, by Weierstrass's Theorem.

If $z = b_{k_0}$ for some k_0, then break the sum (1.6) into two parts: the part consisting of the one summand corresponding to $k = k_0$, and the sum over all $k \neq k_0$. The first part ($k = k_0$) clearly has a simple pole at $z = b_{k_0}$. We reason as above that the second part (the sum over all $k \not\geq k_0$ in (1.6)) is analytic at $z = b_{k_0}$ since the b_k's do not cluster at b_{k_0}. Hence, the series (1.6) is the sum of an analytic function at $z = b_{k_0}$ and a meromorphic function with a simple pole at $z = b_{k_0}$, hence the series (1.6) has a simple pole at $z = b_{k_0}$. This completes the proof.

5. (X.7) We wish to find an explicit entire function g with $g(n \log n) = n^{\pi}$ for $n = 1, 2, 3, \ldots$. We will follow the method given in Corollary X.2.10 for constructing an interpolating function. Note that the conditions of Corollary X.2.10 are indeed satisfied: $n \log n \to \infty$ as $n \to \infty$.

   Note that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n \log n)^2} < \infty$. Hence, Theorem X.2.7 guarantees that the function

$$\widehat{G}(z) = \prod_{n=2}^{\infty} \left(1 - \frac{z}{n \log n}\right) e^{\frac{z}{n \log n}}$$

   represents an entire function with simple zeroes at $n \log n$ ($n \geq 2$) and no other zeroes. Hence, the function

$$G(z) := z\widehat{G}(z) = z \prod_{n=2}^{\infty} \left(1 - \frac{z}{n \log n}\right) e^{\frac{z}{n \log n}}$$

   represents an entire function with simple zeroes at $n \log n$ ($n \geq 1$) and no other zeroes.

   Now, let

$$d_n = G'(n \log n).$$

In fact, we can give a product representation for each d_n as follows. For $N \geq 2$, put

$$G_N(z) = z \prod_{n=2}^{N} \left(1 - \frac{z}{n \log n}\right) e^{\frac{z}{n \log n}},$$

the Nth partial product of $G(z)$. If we denote by $F_n(z)$ the term $\left(1 - \frac{z}{n \log n}\right) e^{\frac{z}{n \log n}}$, then $G_N(z) = z \prod_{n=2}^{N} F_n(z)$, so

$$G_N'(z) = \frac{1}{z} \prod_{m=2}^{N} F_m(z) + \sum_{n=2}^{N} F_n'(z) \prod_{\substack{m=2 \\ m \neq n}}^{N} F_m(z)$$

but evaluating at $n \log n$ for fixed $2 \leq n \leq N$, we see that most of these terms vanish and that

$$G_N'(n \log n) = -e \prod_{\substack{m=2 \\ m \neq n}}^{N} \left(1 - \frac{n \log n}{m \log m}\right) e^{\frac{n \log n}{m \log m}}.$$

Since the $G_N(z)$ converge uniformly on compact subsets of $\mathbb{C}$ to $G(z)$, we know by Weierstrass's Theorem that the $G_N'(z)$ converge uniformly on compact subsets to $G_N(z)$, so $G_N'(n \log n) \to G'(n \log n)$ for $n \geq 2$.

Still following the plan laid out in Corollary X.2.10 for finding the desired function g, we wish to find a meromorphic function F on $\mathbb{C}$ with singular part

$$S_n(z) = \frac{n^\pi / G'(n \log n)}{z - n \log n}$$

at $n \log n$ and no other poles in $\mathbb{C}$. The existence of such a function $F(z)$ is guaranteed by Mittag-Leffler's Theorem. Moreover, once we have such $F(z)$, the desired function $g(z)$ we want will be $g(z) = F(z)G(z)$, as proved in Corollary X.2.10.

It would be nice to give $F(z)$ in more explicit form. Put $r_n = n^\pi / G'(n \log n)$ and $p_n = n \log n$. In order to produce $F(z)$, we wish to show that the sum (which we will take to be $F(z)$)

$$\sum_{n=2}^{\infty} \left(\frac{r_n}{z - p_n} - \frac{r_n}{-p_n} \sum_{j=0}^{n} \left(\frac{z}{p_n}\right)^j \right) \tag{1.8}$$

converges in, say, $|z| \leq R$. If we examine the proof of Exercise X.6 above, we see that the crucial part is controlling

$$\left| \frac{r_n}{p_n} \right|$$

in a disk of radius R. Unfortunately, I haven't been able to estimate this quantity sufficiently well. If I were able to prove that this quantity (or a similar ratio, possibly with exponents in the denominator and/or numerator) could be controlled appropriately in such a way that the sum (1.8) was convergent, then I would have an explicit representation for the function $F(z)$ whose singular parts are $S_n(z)$ at the points $n \log n$ and with no other singularities. Then, as aforementioned, the function $g(z) = F(z)G(z)$ would be an entire function with $g(n \log n) = n^\pi$ for $n = 1, 2, \ldots$.

6. (X.8) In order to find an entire function of least possible genus g, we need to find the smallest integer g for which

$$\sum_{m,n} \frac{1}{|m + in|^{g+1}} \tag{1.9}$$

converges, where the sum is taken over all integers m, n such that $(m, n) \neq (0,0)$. The ordinary Euclidean norm ($|x+iy| = \sqrt{x^2+y^2}$) is equivalent to the taxi cab norm $|x+iy|_{\text{taxi}} := |x|+|y|$ in the sense that there exists a constant $c > 1$ such that

$$c^{-1}|x+iy|_{\text{taxi}} \leq |x+iy| \leq c|x+iy|_{\text{taxi}}$$

for all points $x+iy$ in $\mathbb{C}$. By this equivalence, it follows that the sum (1.9) converges (absolutely and independent of enumeration, since all terms are nonnegative) if and only if the sum

$$\sum_{m,n} \frac{1}{(|m|+|n|)^{g+1}} \tag{1.10}$$

converges, where the sum is taken over all integer pairs $(m,n) \neq (0,0)$.

Consider the square in $\mathbb{C}$ centered at 0 of side length $2N$, with sides parallel to the axes. Here, N is a positive integer, and by "square" we mean the boundary of the square, and not any part of the interior.) This square contains exactly $8N$ Gaussian integers. Moreover, each Gaussian integer $m+in$ on this square satisfies $N \leq |m|+|n| \leq 2N$. (The first inqeuality follows since no point on the square can be closer to $(0,0)$ than $(N,0)$ is with respect to the taxi cab metric, and the second inequality is clear since $|m|,|n| \leq N$.) Reordering the sum (1.10), we find

$$\sum_{N\geq 1} \frac{8N}{(2N)^{g+1}} \leq \sum_{m,n} \frac{1}{(|m|+|n|)^{g+1}} \leq \sum_{N \geq 1} \frac{8N}{N^{g+1}}.$$

The outer sums converge if and only if $g \geq 2$. It follows that the genus g of the sum (1.9) is equal to 2.

Now that we have the genus, the rest is a direct application of Theorem X.2.7. Theorem X.2.7 gives us that

$$\prod_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0\}} \left(1-\frac{z}{m+in}\right) \exp\left(\frac{z}{m+in}+\frac{1}{2}\left(\frac{z}{m+in}\right)^2\right)$$

represents an entire function with simple zeroes at the Gaussian integers $m+in$ with $(m,n) \neq (0,0)$ and no other zeroes. In order to ensure a simple zero at 0, we tack on a factor of z, so the function

$$z \prod_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0\}} \left(1-\frac{z}{m+in}\right) \exp\left(\frac{z}{m+in}+\frac{1}{2}\left(\frac{z}{m+in}\right)^2\right)$$

represents an entire function with simple zeroes precisely at the Gaussian integers and no other zeroes.

1.2.4 Chapter 11

X.9 Let Ω be a region and $H(\Omega)$ the complex algebra of analytic functions on Ω. Suppose $g_1, g_2 \in H(\Omega)$ with no common zeroes. Let $a_1, a_2, \ldots$ be the zeroes of g_1, with each zero repeated as many times as its multiplicity. Also let $b_1, b_2, \ldots$ be the zeroes of g_2, with each zero repeated as many times as its multiplicity. Note that the sequence $a_1, b_1, a_2, b_2, \ldots$ tends to $\partial\Omega$. Indeed, if $a_1, b_1, a_2, b_2, \ldots$ had a subsequence converging to an accumulation point in Ω, then either infinitely many points in this subsequence would be a_i's or infinitely many would be b_j's, which can't happen since the zeroes of g_1, g_2 do not accumulate in Ω. Put $z_{2n-1} = a_n$ and $z_{2n} = b_n$. By Theorem 2.6 and Corollary X.2.10, there exists an analytic function $h(z)$ such that

$$h(z_n) = \begin{cases} 1 & \text{if } n \text{ even;} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

and so that the multiplicity of the zero z_{2n} of $h(z)$ is the multiplicity of the zero of the root b_n of g_2, and so that the multiplicity of the root z_{2n-1} of $1-h(z)$ is the multiplicity of the root a_n of g_1. Put $f_1 = \dfrac{h}{g_1}$ and $f_2 = \dfrac{1-h}{g_2}$. By construction, f_1 is analytic in Ω since a zero of g_1 of order m is also a zero of order m of h (so in fact f_1 has a removeable singularity at all zeroes of g_1). Similarly, f_2 is analytic in Ω. Finally,

$$f_1 g_1 + f_2 g_2 = \frac{h}{g_1} g_1 + \frac{1-h}{g_2} g_2 = 1.$$

X.10 a. Let $\Psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$. Recall our definition of the Gamma function:

$$\Gamma(z) = \frac{1}{zG(z)e^{\gamma z}}$$

where $G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$. Put $g_n(z) = \left(1 + \frac{z}{n}\right) e^{-z/n}$ and $G_N(z) = \prod_{n=1}^{N} g_n(z)$. Fix a compact set $K \subset \mathbb{C}$ not containing any of the nonpositive integers (which are the zeroes of Γ). We know that $G_N(z)$ converges uniformly in K to $G(z)$ and by Weierstrass's Theorem that $G_N'(z)$ converges uniformly in K to $G'(z)$. Since $G_N(z)$ is uniformly bounded from below in K, we conclude that $G_N'(z)/G_N(z) \to G'(z)/G(z)$ uniformly in K. It follows by Weierstrass's Theorem that

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right)$$

uniformly on compact subsets not containing the nonpositive integers. (This also shows that $\Psi(z)$ is meromorphic in the plane.) We can differentiate this series term by term to obtain

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} = \sum_{n=0}^{\infty} \frac{4}{(2z+2n)^2}$$

on compact subsets not containing the nonpositive integers. We also have, if $z + \frac{1}{2}$ is not a nonpositive integer, that

$$\Psi(z + \tfrac{1}{2}) = \sum_{n=0}^{\infty} \frac{1}{(z + \frac{1}{2} + n)^2} = \sum_{n=0}^{\infty} \frac{4}{(2z+1+2n)^2}.$$

Combining the preceding two lines, we obtain

$$4\Psi'(2z) = \Psi'(z) + \Psi'(z + \tfrac{1}{2})$$

if z is not a nonpositive integer.

b. If we integrate the preceding formula once, we obtain

$$4\Psi(2z) = \Psi(z) + \Psi(z + \tfrac{1}{2}) + a'$$

for some constant a'. Now let z be a complex number such that $z, z + \frac{1}{2}$, and $2z$ are all nonpositive integers. By considering a picture of the plane punctured at the nonpositive integers punctured, it is clear that there is a simply connected region Ω missing the nonpositive integers and containing $z, z + \frac{1}{2}$, and $2z$. On Ω, we can define $\log \Gamma(z) = \displaystyle\int_{z_0} \frac{\Gamma'(z)}{\Gamma(z)}\, dz$, and we can integrate the previous equation to obtain

$$2 \log \Gamma(2z) = \log \Gamma(z) + \log \Gamma(z + \tfrac{1}{2}) + a'z + b'.$$

Here we should be careful: the constant b' may depend on the simply connected region Ω and is only unique up to a factor of $2\pi i$, and $\log\Gamma(z)$ may not agree across different simply connected sets. However, when we exponentiate, this problem goes away and we obtain (after combining constants and renaming)

$$\Gamma(2z) = \Gamma(z)\Gamma(z+\tfrac{1}{2})e^{az+b}$$

for all z such that $z, z+\frac{1}{2}$, and $2z$ are nonpositive integers.

c. To find a and b, first set $z = \frac{1}{2}$ in the above relation. Then we obtain

$$\Gamma(1) = \Gamma(\tfrac{1}{2})\Gamma(1)e^{a/2+b}$$

which implies (since $\Gamma(1) \neq 0$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$) that

$$-\tfrac{1}{2}\log\pi = \tfrac{a}{2} + b \tag{1.11}$$

Now let $z = \frac{1}{4}$. We obtain

$$\Gamma(\tfrac{1}{2}) = \Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4})e^{a/4+b}.$$

But by the relation $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin\pi z}$ applied when $z = \frac{1}{4}$, and using again the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we find

$$\sqrt{\pi} = \frac{\pi}{\sin\frac{\pi}{4}}e^{a/4+b},$$

or

$$-\tfrac{1}{2}\log\pi - \tfrac{1}{2}\log 2 = \tfrac{a}{4} + b. \tag{1.12}$$

Equations (1.11) and (1.12) have the unique solution $a = 2\log 2$ and $b = -\log 2 - \frac{1}{2}\log\pi$.

X.11 a. Suppose $x > \frac{1}{2}$. We substitute $t = (\sqrt{x}+v)^2$ in the integral formula

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt.$$

With this substitution, we have $dt = 2(\sqrt{x}+v)\,dv$. Also, as t runs from 0 to ∞, v runs from $-\sqrt{x}$ to ∞. Hence,

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}\,dt \\ &= \int_{-\sqrt{x}}^\infty (\sqrt{x}+v)^{2x-2}e^{-x}e^{-2\sqrt{x}v}e^{-v^2}\,dv.\end{aligned}$$

Multiplying both sides by $e^x(\sqrt{x})^{1-2x}$, we obtain

$$\frac{\Gamma(x)e^x\sqrt{x}}{x^x} = 2\int_{-\sqrt{x}}^\infty e^{-2\sqrt{x}v}\left(1+\frac{v}{\sqrt{x}}\right)^{2x-1}e^{-v^2}\,dv.$$

Letting

$$\varphi_x(v) = \begin{cases} 0 & \text{if } v \leq -\sqrt{x}, \\ e^{-2v\sqrt{x}}\left(1+\dfrac{v}{\sqrt{x}}\right)^{2x-1} & \text{if } v \geq -\sqrt{x} \end{cases}$$

we obtain

$$\frac{\Gamma(x)e^x\sqrt{x}}{x^x} = 2\int_{-\infty}^\infty \varphi_x(v)e^{-v^2}\,dv,$$

as desired.

b. In order to prove the desired formula, we first need to get good control over

$$\varphi_x(v) = e^{\log \varphi_x(v)} = e^{-2v\sqrt{x}+(2x-1)\log(1+\frac{v}{\sqrt{(x)}})}.$$

Now fix x. We have the Taylor series expansion

$$x\log(1+\frac{v}{\sqrt{x}}) = \sum_{n=1}^{\infty}(-1)^{n-1}x\frac{v^n}{(\sqrt{x})^n} = \sqrt{x}v - \frac{v^2}{2} + \frac{v^3}{3\sqrt{x}} - \cdots$$

which is valid for fixed v whenever x is large enough (specifically, when $|v| < \sqrt{x}$). Hence, for fixed v, if we let x tend to ∞, we obtain the pointwise convergence

$$x\log(1+\frac{v}{\sqrt{x}}) - v\sqrt{x} \to -\frac{v^2}{2}$$

and $\varphi_x(v) \to e^{-v^2}$. Also, considering $\varphi_x(v)$ as a function of v, we see that $\varphi_x(v)$ has one critical point, which is where a global maximum occurs (as is easy to verify by considering first and second derivatives). The critical point is $v = -\dfrac{1}{2\sqrt{x}}$, so $\varphi_x(v) \le e\cdot\left(1-\dfrac{1}{2x}\right)^{2x-1}$ for all v. But $\left(1-\dfrac{1}{2x}\right)^{2x-1} \to \frac{1}{e}$ as $x\to\infty$, which means $\varphi_x(v)$ is bounded as $x\to\infty$, which means $\varphi_x(v)e^{-v^2}$ is absolutely integrable. Hence, we can apply the Lebesgue Dominated Convergence Theorem to obtain that

$$\begin{aligned}
\lim_{x\to\infty}\frac{\Gamma(x)e^x\sqrt{x}}{x^x} &= \lim_{x\to\infty} 2\int_{-\infty}^{\infty}\varphi_x(v)e^{-v^2}\,dv \\
&= \lim_{x\to\infty} 2\int_{-\infty}^{\infty} e^{-2v\sqrt{x}+(2x-1)\log(1+\frac{v}{\sqrt{x}})}e^{-v^2}\,dv \\
&= 2\int_{-\infty}^{\infty} e^{-2v^2}\,dv \\
&= \sqrt{2\pi},
\end{aligned}$$

as desired.

c. Recall $\Gamma(n) = (n-1)!$. Therefore, by the previous part,

$$\lim_{n\to\infty}\frac{\Gamma(n+1)e^{n+1}\sqrt{n+1}}{(n+1)^{n+1}} = \sqrt{2\pi}.$$

Hence for large n,

$$n! \approx \frac{\sqrt{2\pi}(n+1)^{n+1}e^{-(n+1)}}{\sqrt{n+1}}.$$

XI.4 We wish to compute $\displaystyle\int_{-\infty}^{\infty}\frac{x^3}{(x^2+1)^2}e^{i\lambda x}\,dx$ for $\lambda > 0$. We will follow closely the Fourier transform example given in the notes.

Let $A, B > 0$, and consider the closed rectangular contour consisting of the following four line segments: $\gamma_1 = [-A, B]$, $\gamma_2 = \{B+iy : 0 \le y \le A+B\}$, $\gamma_3 = \{x+i(A+B) : B \ge x \ge -A\}$, and $\gamma_4 = \{-A+iy : A+B \ge y \ge 0\}$. We orient the closed contour as indicated in the figure. For $|z| \ge 3$, note that

$$\left|2+\frac{1}{z^2}\right| \le 2+\frac{1}{|z|^2} \le \frac{1}{2}|z|^2$$

which implies

$$\frac{1}{2}|z| \leq |z| - \frac{1}{|z|}\left(\left|2+\frac{1}{z^2}\right|\right) = |z| - \left|\frac{2}{z} - \frac{1}{z^3}\right| \leq \left|z + \frac{2}{z} + \frac{1}{z^3}\right|.$$

Hence, for $|z| \geq 3$,

$$\left|\frac{z^3}{(z^2+1)^2}\right| = \left|\frac{1}{z+\frac{2}{z}+\frac{1}{z^3}}\right| \leq \frac{2}{|z|}.$$

Using this estimate, we find on γ_3 for $A, B \geq 3$ that

$$\left|\int_{\gamma_3} \frac{z^3 e^{i\lambda z}}{(z^2+1)^2}\, dz\right| \leq \int_{-A}^{B} \frac{2}{A+B} e^{-\lambda(A+B)}\, dx \leq \frac{2e^{-\lambda(A+B)}}{A+B} \cdot (A+B) \to 0$$

as A, B tend independently to ∞.

On γ_2, when $B \geq 3$, also using the above estimate, we find

$$\left|\int_{\gamma_2} \frac{z^3 e^{i\lambda z}}{(z^2+1)^2}\, dz\right| \leq \int_{0}^{A+B} \frac{2}{B} e^{-\lambda y}\, dy \leq \frac{2}{B}\left(\frac{1-e^{-\lambda(A+B)}}{\lambda}\right) \to 0$$

as $B \to \infty$. A nearly identical calculation shows that

$$\left|\int_{\gamma_4} \frac{z^3 e^{i\lambda z}}{(z^2+1)^2}\, dz\right| \to 0$$

as $B \to \infty$.

By the Residue Theorem, the integral of $\dfrac{z^3 e^{i\lambda z}}{(z^2+1)^2}$ around the complete closed contour (for A, B large) is $2\pi i$ times the sum of the residues inside. But the integrals along $\gamma_2, \gamma_3, \gamma_4$ tend to 0 as $A, B \to \infty$, and the integrand has only one residue inside the large box (at i) so

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x^3 e^{i\lambda x}}{(x^2+1)^2}\, dx &= \lim_{A,B\to\infty} \int_{-A}^{B} \frac{x^3 e^{i\lambda x}}{(x^2+1)^2}\, dx \\ &= 2\pi i \operatorname{Res}_i \frac{z^3 e^i \lambda z}{(z^2+1)^2}.\end{aligned}$$

It remains to compute the residue of the integrand at i. We will follow Example XI.2 in the notes. Write $G(z) = \dfrac{z^3 e^{i\lambda z}}{(z+i)^2}$. Then $G(z)$ is analytic in the contour $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, and $\dfrac{z^3 e^{i\lambda z}}{(z^2+1)^2} = \dfrac{G(z)}{(z-i)^2}$. Hence, by the Example in the notes,

$$\operatorname{Res}_i \frac{z^3 e^{i\lambda z}}{(z^2+1)^2} = G'(i).$$

It is straightforward to compute via the quotient rule that $G'(i) = \dfrac{2e^{-\lambda} - \lambda e^{-\lambda}}{4}$. We have proved

$$\int_{-\infty}^{\infty} \frac{x^3 e^{i\lambda x}}{(x^2+1)^2}\, dx = 2\pi i \left(\frac{2e^{-\lambda} - \lambda e^{-\lambda}}{4}\right).$$

XI.5 We wish to compute $\int_0^\infty \frac{x^\alpha}{x(x+1)}\,dx$ for $0<\alpha<1$. We will follow the Mellin transform example on p. 78 of the notes. We will use the exact same keyhole contour as in the example. Fix $\epsilon>0$ and $0<\delta<R$. Consider the closed keyhole contour centered at 0 which consists of a portion C_R of a circle of radius $R>0$ and a portion C_δ of a circle of radius $\delta>0$, along with the two straight line segments $\gamma_1=\{x+i\epsilon : \delta<x<R\}$ and $\gamma_2=\{x-i\epsilon : \delta<x<R\}$. So, orienting these segments as in the figure, the closed curve is $\gamma := \gamma_1+C_R+\gamma_2+C_\delta$. We define $\log z$ on $\mathbb{C}\setminus[0,\infty)$ so that $\log z=\log|z|+i\arg z$ with $0<\arg z<2\pi$. With this definition of $\log z$, we also define $z^\alpha=e^{\alpha\log z}$. With these definitions, $\log z$ and z^α are analytic inside and on the closed contour γ.

Put $f(z)=\frac{1}{z(z+1)}$. By the Residue Theorem, for $R>1$ and $0<\delta<1$, we have

$$\int_\gamma z^\alpha f(z)\,dz = 2\pi i \operatorname{Res}_{-1} z^\alpha f(z) = 2\pi i\cdot\lim_{z\to-1}(z+1)z^\alpha f(z) = -2\pi i e^{i\alpha\pi}$$

since -1 is a simple pole of $z^\alpha f(z)$.

For $R>1$,

$$\left|\int_{C_R} z^\alpha f(z)\,dz\right| \le \int_0^{2\pi}\frac{R^\alpha}{R(R-1)}\cdot R\,d\theta \to 0$$

as $R\to\infty$. Similarly, for $\delta<1$,

$$\left|\int_{C_\delta} z^\alpha f(z)\,dz\right| \le \int_0^{2\pi}\frac{\delta^\alpha}{\delta(\delta-1)}\cdot \delta\,d\theta \to 0$$

as $\delta\to 0$.

In computing the integrals over the horizontal line segments γ_1 and γ_2, first note the two different limits we obtain as we approach the positive real axis from the two different sides:

$$\lim_{\epsilon\to0}(x+i\epsilon)^\alpha f(x+i\epsilon) = e^{\alpha\log|x|}f(x)$$

and

$$\lim_{\epsilon\to0}(x-i\epsilon)^\alpha f(x-i\epsilon) = e^{2\pi i\alpha}e^{\alpha\log|x|}f(x).$$

Hence, letting $\epsilon\to0$ while keeping R,δ fixed, we obtain

$$\begin{aligned}\int_\delta^R \frac{x^\alpha}{x(x+1)}\,dx &= \lim_{\epsilon\to0}\left(\int_{\gamma_1}+\int_{\gamma_2}\frac{z^\alpha}{z(z+1)}\,dz\right)\\ &= (1-e^{2\pi i\alpha})\int_\delta^R x^\alpha f(x)\,dx.\end{aligned}$$

So, now letting $\delta\to0$ and $R\to\infty$, we obtain

$$\int_0^\infty \frac{x^\alpha}{x(x+1)}\,dx = \frac{-2\pi i e^{i\alpha\pi}}{1-e^{2i\alpha\pi}} = \frac{-2\pi i}{e^{-i\alpha\pi}-e^{i\alpha\pi}} = \frac{\pi}{\sin\pi\alpha}.$$

XI.6 We want to find the inverse Laplace transform of $F(z)=\frac{3z^2+12z+8}{(z+2)^2(z+4)(z-1)}$. The function $F(z)$ is analytic in $\{\operatorname{Re} z>1\}$. We first need to show that the integral defining $f(t)$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^\infty F(b+iy)e^{(b+iy)t}\,dy$$

converges and that the integral defining $\mathcal{L}(f)$ converges absolutely in $\operatorname{Re} z > 1$. Since $F(z)$ is a rational function with numerator of degree 2 and denominator of degree 4, for fixed $1 < \alpha < 2$, there is a constant C such that for all sufficiently large $|z|$,

$$|F(z)| \leq \frac{C}{|z|^\alpha},$$

and for $1 < p < \alpha$ and all sufficiently large z,

$$|F(z)| \leq \frac{C}{|z|^\alpha} \leq \frac{C'}{(1+|z|)^p}.$$

This is enough to deduce convergence of $F(z)$, and from this, absolutely convergence of $\mathcal{L}(f)$ is obtained exactly as in the proof of Theorem 1.3. We conclude as in Theorem XI.1.3 that the inverse Laplace transform f of F is independent of $b > 1$.

The function $F(z)e^{zt}$ is meromorphic in the plane with simple poles at $z = -4$ and $z = 1$, and a pole of order 2 at $z = -2$. Fix $b > 1$ and $R > 0$. Consider the contour consisting of the vertical line segment $\sigma_R = \{b + iy : -R \leq y \leq R\}$ and the semicircle C_R centered at b of radius R that lies to the left of the vertical line $\{\operatorname{Re} z = b\}$. See the figure below. We orient the closed curve $\sigma_R + C_R$ as in the figure. For R sufficiently large, the contour encloses all residues of $F(z)e^{zt}$, and by the Residue Theorem,

$$\int_{\sigma_R + C_R} F(z)e^{zt}\,dz = 2\pi i \Big(\sum_{a \in \{-4,-2,1\}} \operatorname{Res}_a F(z)e^{zt} \Big).$$

We estimate the integral along C_R first. We can parametrize C_R by $z = b + Re^{i\theta}$ where $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. If $b > 8$ and $R > 2b$, and if $z = b + Re^{i\theta}$, we have the basic estimates

$$|z| \leq b + R < 2R$$

and for $|a| \leq 4$,

$$|z + a| = |b + Re^{i\theta} + a| \geq R - b - |a| \geq \frac{R}{4}.$$

Using these estimates, we find

$$\begin{aligned}
\left| \int_{C_R} \frac{3z^2 + 12z + 8}{(z+2)^2(z+4)(z-1)} e^{zt}\,dz \right| &\leq \int_{\pi/2}^{3\pi/2} \frac{3(2R)^2 + 12(2R) + 8}{(R-2)^2(R-4)(R-1)} R\,d\theta \\
&\leq \int_{\pi/2}^{3\pi/2} \frac{44R^2}{(R/4)^4} R\,d\theta
\end{aligned}$$

where the last inequality comes from the above estimates plus the estimate $3(2R)^2 + 12(2R) + 8 \leq 44R^2$ since $R > 16$. Hence, as $R \to \infty$,

$$\left| \int_{C_R} \frac{3z^2 + 12z + 8}{(z+2)^2(z+4)(z-1)} e^{zt}\,dz \right| \to 0.$$

Now we calculate the residues we need. First, since -4 and 1 are simple poles of $F(z)e^{zt}$, we have

$$\operatorname{Res}_1 F(z)e^{zt} = \lim_{z \to 1} (z-1)F(z)e^{zt} = \frac{23}{45}e^t$$

and

$$\operatorname{Res}_{-4} F(z)e^{zt} = \lim_{z \to -4} (z-(-4))F(z)e^{zt} = -\frac{2}{5}e^{-4t}.$$

For the pole $z = -2$ of order 2, put $G(z) = F(z)e^{zt}(z+2)^2$, so that $G(z)$ is analytic near -2. By Example 2 of Chapter XI, we know

$$\operatorname{Res}_{-2} F(z)e^{zt} = G'(-2) = \frac{(6t-1)e^{-2t}}{9}.$$

Therefore, letting $R \to \infty$ and applying the Residue Theorem to $F(z)e^{zt}$ over the contour $\sigma_R + C_R$, we obtain

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(b+iy)e^{(b+iy)t}\,dy \\ &= \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\sigma_R} F(z)e^{zt}\,dz \\ &= \frac{23}{45}e^{t} - \frac{2}{5}e^{-4t} + \frac{6t-1}{9}e^{-2t} \end{aligned}$$

XI.7 In order to compute $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}$, we will consider two similar but different contour integrals. In both cases, the contour will be the following: let N be a positive integer, and consider the square contour S_N with bottom left vertex $(N+\frac{1}{2})(-1-i)$ and top right vertex $(N+\frac{1}{2})(1+i)$ whose sides are parallel to the coordinate axes. We will first consider

$$\int_{S_N}\frac{\pi\cot\pi z}{(z+\frac{1}{4})^3}\,dz.$$

As in the example on p. 81 of the notes, we note that

$$\cot\pi z = i\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1}.$$

Along the right-hand side of the square S_N, where $z = (N+\frac{1}{2}) + iy$, $-N - \frac{\leq}{y} \leq N + \frac{1}{2}$, we find by the reverse triangle inequality that

$$|1-e^{2\pi i z}| = |1-e^{2\pi i(N+\frac{1}{2})}e^{-2\pi y}| \geq 1 - |e^{2\pi i(N+\frac{1}{2})}|\cdot|e^{-2\pi y}| = 1-|e^{-2\pi y}| > 1-e^{-2\pi N} \geq 1-e^{-2\pi}.$$

Similar computations show that this bound holds on all sides of the square S_N. But $\frac{\zeta+1}{\zeta-1}$ is bounded by some constant C in the region $|\zeta-1| > 1-e^{-2\pi}$. This implies $\pi\cot\pi z$ is bounded uniformly on S_N by C, independent of N. It follows that

$$\left|\int_{S_N}\frac{\pi\cot\pi z}{(z+\frac{1}{4})^2}\,dz\right| \leq 4\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}}\frac{C}{((N+\frac{1}{2})-\frac{1}{4})^3}\,ds \to 0$$

as $N \to \infty$. But, inside the square S_N, $\frac{\pi\cot\pi z}{(z+\frac{1}{4})^3}$ has simple poles at the integers $-N, -N+1, \ldots, N, N+1$ and a pole of order 3 at $-\frac{1}{4}$. Since $\pi\cot\pi z$ has residue 1 at every integer (a fact we've seen in the notes and in class), the residue of $\frac{\pi\cot\pi z}{(z+\frac{1}{4})^3}$ at each integer n is $\frac{1}{(n+\frac{1}{4})^3}$. By Example 2 in the notes, the residue of $\frac{\pi\cot\pi z}{(z+\frac{1}{4})^3}$ at $-\frac{1}{4}$ is given by

$$\frac{1}{2!}\left[\frac{d^2}{dz^2}(\pi\cot\pi z)\right]_{z=-\frac{1}{4}} = -2\pi^3,$$

by a simple computation. Putting all the pieces together and letting $N \to \infty$, the Residue Theorem gives us that

$$2\pi^3 = -\operatorname{Res}_{-1/4} \frac{\pi \cot \pi z}{(z+\frac{1}{4})^3} = \sum_{n=-\infty}^{\infty} \operatorname{Res}_n \frac{\pi \cot \pi z}{(z+\frac{1}{4})^3} = \sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{4})^3},$$

or

$$\frac{\pi^3}{32} = \sum_{n=-\infty}^{\infty} \frac{1}{(4n+1)^3}. \tag{1.13}$$

Now, a completely analogous argument with the function $\frac{\pi \cot \pi z}{(z-\frac{1}{4})^3}$ instead of $\frac{\pi \cot \pi z}{(z+\frac{1}{4})^3}$ shows that $\operatorname{Res}_{1/4} \frac{\pi \cot \pi z}{(z-\frac{1}{4})^3} = 2\pi^3$, which gives the slightly different sum

$$-\frac{\pi^3}{32} = \sum_{n=-\infty}^{\infty} \frac{1}{(4n-1)^3}. \tag{1.14}$$

Subtracting (1.14) from (1.13) and then halving the result (which is legal since the resulting double sum is invariant under $n \mapsto -n$), we find (after reindexing)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

XI.8 We wish to compute $\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6}$. We will use the same square contours S_N as in the previous problem, and we will consider this time the integral over S_N of the function $\frac{\pi \cot \pi z}{z^6}$. This function has simple poles at exactly the nonzero integers, and a pole of order 7 at $z = 0$. Since $\pi \cot \pi z$ has residue 1 at all integers, we see that for integers $n \neq 0$, $\operatorname{Res}_n \frac{\pi \cot \pi z}{z^6} = \frac{1}{n^6}$. The hard part is computing $\operatorname{Res}_0 \frac{\pi \cot \pi z}{z^6}$. For this, we note that there is a Laurent series about 0 that starts out like

$$\frac{\pi \cot \pi z}{z^6} = \frac{b_{-7}}{z^7} + \frac{b_{-6}}{z^6} + \frac{b_{-5}}{z^5} + \cdots.$$

We remark that $\frac{\pi \cot \pi z}{z^6}$ is an odd function, so $b_{-6} = b_{-4} = b_{-2} = \cdots = 0$. Using this observation and rearranging a bit gives

$$\pi \cos \pi z = \sin \pi z \left(\frac{b_{-7}}{z} + b_{-5} z + b_{-3} z^3 + \cdots \right).$$

Using the first few terms of each of the power series for $\sin \pi z$ and $\cos \pi z$, we find

$$\pi \left(1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \cdots \right) = \left(\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \cdots \right) \left(\frac{b_{-7}}{z} + b_{-5} z + b_{-3} z^3 + \cdots \right).$$

Multiplying out the first few terms and equating coefficients yields the four equations

$$\begin{aligned}
\pi &= \pi b_{-7} \\
-\frac{\pi^3}{2!} &= \pi b_{-5} - \frac{\pi^3}{3!} b_{-7} \\
\frac{\pi^5}{4!} &= \pi b_{-3} - \frac{\pi^3}{3!} b_{-5} + \frac{\pi^5}{5!} b_{-7} \\
-\frac{\pi^7}{6!} &= \pi b_{-1} - \frac{\pi^3}{3!} b_{-3} + \frac{\pi^5}{5!} b_{-5} - \frac{\pi^7}{7!} b_{-7}
\end{aligned}$$

It is straightforward to solve these equations from top to bottom. One finds $b_{-7} = 1$, $b_{-5} = -\dfrac{\pi^2}{3}$, $b_{-3} = -\dfrac{\pi^4}{45}$, and $b_{-1} = -\dfrac{2\pi^6}{945}$. Since $\displaystyle\int_{S_N} \frac{\pi \cot \pi z}{z^6}\, dz \to 0$ as $N \to \infty$, we conclude by the Residue Theorem that

$$-\operatorname{Res}_0 \frac{\pi \cot \pi z}{z^6} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \operatorname{Res}_n \frac{\pi \cot \pi z}{z^6}$$

hence

$$\frac{2\pi^6}{945} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^6}.$$

Since the right-hand sum is invariant under the transformation $n \mapsto -n$, we conclude

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

XI.9 Define $\log z$ on $\mathbb{C} \setminus [0, \infty)$ by $\log z = \log |z| + i \arg z$ with $0 < \arg z < 2\pi$. We consider the same keyhole contour as in Exercise XI.5 above, and the notation $R, \delta, \epsilon, C_R, C_\delta, \gamma_1, \gamma_2$ will have the same meaning in this problem. We have redrawn the contour below for convenience.

When $R > \sqrt{2}$,

$$\frac{R\sqrt{R}\log R}{R^2 - 1} \leq \frac{2R\sqrt{R}\log R}{R^2} = \frac{2\log R}{R^{1/2}}.$$

Also, $\dfrac{\log R}{R^{1/2}} \to 0$ as $R \to \infty$ by L'Hôpital's rule. Hence,

$$\left| \int_{C_R} \frac{\sqrt{z}\log z}{(1+z^2)}\, dz \right| \leq \int_0^{2\pi} \frac{\sqrt{R}(\log R + 2\pi)}{R^2 - 1} R\, d\theta \to 0$$

as $R \to \infty$.

To estimate the integral over C_δ, note that

$$\delta^{3/2} \log \delta = \frac{\log \delta}{1/\delta^{3/2}} \to 0$$

as $\delta \to 0$ by L'Hôpital's rule. Hence,

$$\left| \int_{C_\delta} \frac{\sqrt{z}\log z}{(1+z^2)}\, dz \right| \leq \int_0^{2\pi} \frac{\sqrt{\delta}(\log \delta + 2\pi)}{\delta^2 - 1} \delta\, d\theta \to 0$$

as $\delta \to 0$.

Now we compute the integral along the straight line segments γ_1 and γ_2. As $\epsilon \to 0$, we obtain different limits for $\dfrac{\sqrt{x+i\epsilon}\log(x+i\epsilon)}{1+(x+i\epsilon)^2}$ and $\dfrac{\sqrt{x-i\epsilon}\log(x-i\epsilon)}{1+(x-i\epsilon)^2}$ because of our definition of $\log z$. For fixed R, δ, we find

$$\lim_{\epsilon \to 0} \int_{\gamma_1} \frac{\sqrt{x+i\epsilon}\log(x+i\epsilon)}{1+(x+i\epsilon)^2}\, dx = \int_\delta^R \frac{\sqrt{x}\log x}{1+x^2}$$

whereas (the minus sign is because we are traversing γ_2 in the opposite direction in the equality)

$$-\lim_{\epsilon \to 0} \int_{\gamma_2} \frac{\sqrt{x+i\epsilon}\log(x+i\epsilon)}{1+(x+i\epsilon)^2}\, dx = \int_\delta^R \frac{(\sqrt{x}e^{2\pi i/2})(\log x + 2\pi i)}{1+x^2}\, dx$$

hence

$$\lim_{\epsilon\to 0}\int_{\gamma_2}\frac{\sqrt{x+i\epsilon}\log(x+i\epsilon)}{1+(x+i\epsilon)^2}\,dx=\int_\delta^R\frac{\sqrt{x}(\log x+2\pi i)}{1+x^2}\,dx.$$

Now letting $\delta\to 0$ and $R\to\infty$ gives the improper integrals $\int_0^\infty$.

The function $\dfrac{\sqrt{z}\log z}{1+z^2}$ has residues $\pm i$ in the contour $\gamma_1+C_R+\gamma_2+C_\delta$, provided $R>1$ and $\delta<0$. Each of $\pm i$ is a simple pole of this function, so

$$\operatorname{Res}_i\frac{\sqrt{z}\log z}{1+z^2}=\lim_{z\to i}(z-i)\frac{\sqrt{z}\log z}{1+z^2}=\frac{\sqrt{i}\log i}{2i}=-\frac{\pi i e^{i\pi/4}}{4}$$

and

$$\operatorname{Res}_{-i}\frac{\sqrt{z}\log z}{1+z^2}=\lim_{z\to -i}(z+i)\frac{\sqrt{z}\log z}{1+z^2}=\frac{\sqrt{-i}\log(-i)}{-2i}=\frac{3\pi i e^{3i\pi/4}}{4}.$$

Putting everything together, the Residue Theorem gives us in the limit, as $\epsilon\to 0$ and then as $R\to\infty$ and $\delta\to 0$, that

$$2\pi i\left(-\frac{\pi i e^{i\pi/4}}{4}+\frac{3\pi i e^{3i\pi/4}}{4}\right)=2\int_0^\infty\frac{\sqrt{x}\log x}{1+x^2}\,dx+2\pi i\int_0^\infty\frac{\sqrt{x}}{1+x^2}\,dx.$$

The left-hand side is equal to $\dfrac{\pi^2}{\sqrt{2}}+\sqrt{2}\pi^2 i$, and the right-hand side is the sum of a real integral and $2\pi i$ times another real integral. Equating real parts of both sides and dividing by 2, we conclude

$$\int_0^\infty\frac{\sqrt{x}\log x}{1+x^2}\,dx=\frac{\pi^2}{2\sqrt{2}}.$$

XI.10 We will compute $\displaystyle\int_0^\infty\frac{x^3+8}{x^5+1}\,dx$ by considering a contour integral of $\dfrac{z^3+8}{z^5+1}\log z$. We define $\log z$ in $\mathbb{C}\setminus[0,\infty)$ by $\log z=\log|z|+i\arg z$, where $0<\arg z<2\pi$. We consider the same keyhole contour as in the last problem, and the notation $R,\delta,\epsilon,C_R,C_\delta,\gamma_1,\gamma_2$ will keep its meaning. We have redrawn the contour below for convenience.

We first note that when $R\geq 4$,

$$R\cdot\frac{R^3+8}{R^5-1}\log R\leq 4R\cdot\frac{R^3}{R^5}\log R=4\frac{\log R}{R}$$

and that, by L'Hôpital's rule,

$$\lim_{R\to\infty}\frac{\log R}{R}=0.$$

Hence, as $R\to\infty$,

$$\left|\int_{C_R}\frac{z^3+8}{z^5+1}\log z\,dz\right|\leq\int_0^{2\pi}\frac{R^3+8}{R^5-1}(\log R+2\pi)R\,d\theta\to 0.$$

Also, by L'Hôpital's rule, $\delta\log\delta=\dfrac{\log\delta}{1/\delta}\to 0$ as $\delta\to 0$, and $\dfrac{\delta^3+8}{\delta^5-1}\to -8$ as $\delta\to 0$, so

$$\left|\int_{C_\delta}\frac{z^3+8}{z^5+1}\log z\,dz\right|\leq\int_0^{2\pi}\frac{\delta^3+8}{\delta^5-1}(\log\delta+2\pi)\delta\,d\theta\to 0$$

as $\delta \to 0$.

If we let ϵ tend to 0 (while keeping R, δ fixed), we find

$$\lim_{\epsilon\to 0}\int_{\gamma_1}\frac{(x+i\epsilon)^3+8}{(x+i\epsilon)^5+1}\log(x+i\epsilon)\,dx = \int_\delta^R \frac{x^3+8}{x^5+1}\log x\,dx$$

whereas (the negative sign since we're traversing γ_2 in the "wrong" direction)

$$-\lim_{\epsilon\to 0}\int_{\gamma_2}\frac{(x-i\epsilon)^3+8}{(x-i\epsilon)^5+1}\log(x-i\epsilon)\,dx = \int_\delta^R \frac{x^3+8}{x^5+1}(\log x+2\pi i)\,dx.$$

Putting all this together with the Residue Theorem, as $\epsilon \to 0$ first and then $R \to \infty$ and $\delta \to 0$, we have

$$\begin{aligned}2\pi i\sum_a \operatorname{Res}_a \frac{z^3+8}{z^5+1}\log z &= \int_0^\infty \frac{x^3+8}{x^5+1}\log x\,dx - \int_0^\infty \frac{x^3+8}{x^5+1}(\log x+2\pi i)\,dx\\ &= -2\pi i\int_0^\infty \frac{x^3+8}{x^5+1}\,dx,\end{aligned}$$

where the sum is over all residues of $\dfrac{z^3+8}{z^5-1}\log z$ contained in $\gamma_1+C_R+\gamma_2+C_\delta$ (this sum is independent of R, δ, ϵ, provided R is larger than 1 and δ and ϵ are smaller than 1). Put $\zeta = e^{\pi i/5}$. The residues occur at $\zeta, \zeta^3, \zeta^5, \zeta^7, \zeta^9$. Each of these is a simple pole of $\dfrac{z^3+8}{z^5-1}\log z$, so the corresponding residue is

$$\begin{aligned}\operatorname{Res}_{\zeta^k}\frac{z^3+8}{z^5+1}\log z &= \lim_{z\to\zeta^k}(z-\zeta^k)\frac{z^3+8}{z^5-1}\log z\\ &= \lim_{z\to\zeta^k}\frac{z^3+8\log z}{(z^5-(-1))/(z-\zeta^k)}\\ &= (\zeta^{3k}+8)\log\zeta^k\left[\frac{1}{\frac{d}{dz}z^5}\right]_{z=\zeta^k}\\ &= \frac{1}{5}\zeta^{-4k}(\zeta^{3k}+8)\log\zeta^k\\ &= \frac{1}{5}\zeta^{-4k}(\zeta^{3k}+8)\frac{k\pi}{5}i\end{aligned}$$

for $k = 1,3,5,7,9$. Hence,

$$\int_0^\infty \frac{x^3+8}{x^5+1}\,dx = -\frac{1}{25}\sum_{k\in\{1,3,5,7,9\}} k\pi i\zeta^{-4k}(\zeta^{3k}+8).$$

Mathematica simplifies this to $\dfrac{9\pi}{5}\sqrt{2+\frac{2}{\sqrt5}} \approx 9.621$.

XI.11 Consider the dog bone contour in the figure, which consists of a portion C_0 of a circle of radius δ about 0, a portion C_1 of a circle of radius δ about 1, and the horizontal line segments γ_+ and γ_-, which are part of the lines $\{\operatorname{Im} z = \epsilon\}$ and $\{\operatorname{Im} z = -\epsilon\}$, respectively, such that the horizontal line segments connect the two circles as indicated in the figure below:

We define $\sqrt{z}$ and $\sqrt{1-z}$ via different branches of the logarithm as follows. We define $\sqrt{z} = e^{\frac12\log z}$ on $\mathbb{C}\setminus(-\infty,0]$ so that $\log 1 = 0$. We define $\sqrt{1-z} = e^{\frac12\log z}$ on $\mathbb{C}\setminus(-\infty,1]$ so that $\log 2 = \ln 2$

(technically, this is a different branch of log from the one in the last sentence!). To avoid further confusion, we will not refer to log from now on. Instead, we will refer only to $\sqrt{z}$ and $\sqrt{1-z}$.

Defining $\sqrt{z}$ and $\sqrt{1-z}$ in this way gives rise to a function $\sqrt{z(1-z)}$ which is well defined and continuous on $\mathbb{C} \setminus [0,1]$. How? For clarity, put $f(z) = \sqrt{z}$ and $g(z) = \sqrt{1-z}$, and let $h(z)$ be the function we are trying to define on $\mathbb{C} \setminus [0,1]$. There is no issue in defining $h(z)$ when $z \in \mathbb{C} \setminus (-\infty, 1]$: in this case, take $h(z) = f(z)g(z)$, and $h(z)$ will be analytic at z. If $x < 0$, $f(x)g(x)$ can now be made meaningful: if we approach the real axis from above and separately from below, the two limiting values of $f(x)$ differ by a multiplicative factor of -1, as do the two limiting values of $g(x)$, but when we take their product, these multiplicative factors cause no confusion. Thus, we can define $h(x)$ to be the limiting value of $f(x)$ from above the real line times the limiting value of $g(x)$ from above the real line, and h will be continuous at x. By this definition of $h(x)$, it is clear now that if $R \subset \mathbb{C} \setminus [0,1]$ is any rectangle about $x < 0$ with sides parallel to the coordinate axes, then $\int_{\partial R} h(z)\, dz = 0$ (since the integral along opposite sides will cancel completely). Hence, by Morera's Theorem, $h(z)$ is analytic on $\mathbb{C} \setminus [0,1]$.

We estimate the integral of $h(z)$ along C_0 and C_1 first. On C_0, $z = \delta e^{i\theta}$ (for suitable θ so that the point is really in C_0). Hence, by the reverse triangle inequality,

$$\left| \frac{1}{\sqrt{z(1-z)}} \right| = \left| \frac{1}{\delta e^{i\theta} - \delta^2 e^{2i\theta}} \right| \leq \frac{1}{\sqrt{\delta - \delta^2}}$$

so that

$$\left| \int_{C_0} h(z)\, dz \right| \leq \int_0^{2\pi} \frac{\delta}{\sqrt{\delta - \delta^2}}\, d\theta.$$

By L'Hôpital's rule, $\frac{\delta}{\sqrt{\delta - \delta^2}} \to 0$ as $\delta \to 0$. Hence, first letting $\epsilon \to 0$, then letting $\delta \to 0$ yields $\int_{C_0} h(z)\, dz \to 0$. which tends to 0 as $\delta \to 0$ by L'Hôpital's rule. A similar analysis shows that $\int_{C_1} h(z)\, dz \to 0$ as $\epsilon \to 0$ then $\delta \to 0$, as well.

Along the horizontal line segments, we find

$$\lim_{\epsilon \to 0} \int_{\gamma_+} h(x + i\epsilon)\, dx = -\int_0^1 \frac{1}{\sqrt{x(1-x)}}\, dx$$

(where the negative sign is because we are traversing γ_+ in the "wrong" direction) and

$$\lim_{\epsilon \to 0} \int_{\gamma_-} h(x - i\epsilon)\, dx = -\int_0^1 \frac{1}{\sqrt{x(1-x)}}\, dx$$

(the negative is introduced because as we approach a point $x \in (0,1)$ from above and below the real axis, we get the same answer for the limiting value $f(x)$ but opposite-signed answers for the limiting value $g(x)$).

In order to compute $\int_{\gamma_- + C_1 + \gamma_+ C_0} h(z)\, dz$, we can't use the Residue Theorem since there are infinitely many points in the dog bone at which $h(z)$ is not analytic. But, $h(z)$ is analytic everywhere outside the dog bone. Since $h(z) \to 0$ as $z \to \infty$, $h(z)$ can be extended to be analytic at ∞ by the Riemann Removeable Singularity Theorem (with $h(\infty) = 0$). Hence, $h(z)$ has a valid Laurent series $\sum_{n=-\infty}^{-1} a_n z^n$

in the annulus $1+\delta < |z| < \infty$. About any positively oriented circle C_R of radius $R > 1+\delta$ centered at 0, we have $a_{-1} = \dfrac{1}{2\pi i}\displaystyle\int_{|z|=R} f(z)\,dz$ by Cauchy's Integral Formula. But, by Cauchy's Theorem, $\displaystyle\int_{C_R} h(z)\,dz - \int_{\gamma_- + C_1 + \gamma_+ + C_0} h(z)\,dz = 0$, since $C_R - (\gamma_- + C_1 + \gamma_+ + C_0) \sim 0$ (the cycle winds around no points outside the enclosed region). But,

$$a_{-1} = \lim_{z\to\infty} z \cdot \frac{1}{\sqrt{z(1-z)}}$$

and

$$z \cdot \frac{1}{\sqrt{z(1-z)}} = \frac{\sqrt{z}}{\sqrt{1-z}} = \frac{1}{1/z-1} \to \sqrt{-1}$$

so either $a_{-1} = i$ or $a_{-1} = -i$. But, up to this point, we know

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx = -a_{-1}\pi i.$$

Since the integral is certainly nonnegative, we deduce a_{-1} is i and not $-i$. Hence,

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx = \pi.$$

We can evaluate this integral another way by making the substitution $w = 1/x$. Then $dw = -\dfrac{dx}{x^2}$, or $dx = -\dfrac{dw}{w^2}$. Also, the limits of integration under this substitution are $w : \infty \to 0$ since $x : 0 \to 1$. Hence,

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx &= \int_\infty^1 \frac{1}{\sqrt{\frac{1}{w}(1-\frac{1}{w})}} \cdot \frac{-dw}{w^2} \\ &= \int_1^\infty \frac{dw}{w\sqrt{w-1}}\end{aligned}$$

Now make the substitution $w = \sec^2\theta$, so θ runs from 0 to $\frac{\pi}{2}$ as w runs from 1 to ∞. Also, $dw = 2\sec^2\theta\tan\theta$, and we find

$$\begin{aligned}\int_1^\infty \frac{dw}{w\sqrt{w-1}} &= \int_0^{\pi/2} \frac{2\sec^2\theta\tan\theta}{\sec^2\theta\sqrt{\tan^2\theta}}\,d\theta \\ &= \int_0^{\pi/2} 2\,d\theta \\ &= \pi,\end{aligned}$$

so we have computed in another way that $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi$.

Chapter 2

UW Prelims

2.1 UW 2012

1.

2.

3.

4. Montel says that a family $\mathcal{F}$ is normal if and only if the family is locally uniformly bounded. Let $A = \sup_{f \in \mathcal{F}} |f'(1/2)|$. By the definition of $\mathcal{F}$, the functions are actually uniformly bounded by 1, so the family is normal. By Cauchy's theorem, the derivatives at $z = 1/2$ are uniformly bounded as well. Therefore if $\{f_n(z)\} \subset \mathcal{F}$ is any sequence so that so that $|f_n'(1/2)| \to \sup_{f \in \mathcal{F}} |f'(1/2)|$, then normality implies there is a subsequence that converges normally on D to some analytic $F(z)$. Now since each $|f_n(z)| \leq 1$, it follows that $|F(z)| \leq 1$, so that $F(z) \in \mathcal{F}$ and $|F'(1/2)| = A$. Next, Schwarz-Pick says that any holomorphic function on $\mathbb{D}$ satisfies the following inequality:

$$\frac{|F(z_1) - F(z_2)|}{|1 - \overline{F(z_1)}F(z_2)|} \leq \frac{|z_1 - z_2|}{|1 - \overline{z_1}z_2|},$$

for any $z_1, z_2 \in \mathbb{D}$. Now fix $z_1 = 1/2$ and let $z_2 \to 1/2$ to obtain the inequality

$$|F'(1/2)| \leq \frac{4}{3}(1 - |F(1/2)|^2).$$

Evidently to sharpen this inequality we want $F(1/2) = 0$, so we reduce to those $f \in \mathcal{F}$ with the property that $f(1/2) = 0$. Now we obtain $|F'(1/2)| \leq 4/3$. We know by Schwarz' lemma that if there is equality then F is simply a rotation of the disk, i.e, the functions $F(z) = e^{i\theta}\dfrac{z - 1/2}{1 - z/2}$ are all those members of $\mathcal{F}$ with the property that $|F'(1/2)| = 4/3 = A$.

5.

6. Answer is $\dfrac{e^{-\pi/2}\pi}{1 + e^{-\pi}}$. Take a rectangular contour. Zeroes are when $z = \dfrac{\pi(2k+1)i}{2}$. Rectangular contour with $x + i\pi$ as the upper line and starting/ending at $x = -N$ to $x = M$. Only the pole at $z = ipi/2$ is enclosed in the contour.

7. Let $\tau \in \mathbb{C} \setminus \mathbb{R}$ and $\Lambda = \{a + b\tau \mid a, b \in \mathbb{Z}\}$. Let f be a non constant meromorphic function with the property that $f(z + \omega) = f(z)$ for all $z \in \mathbb{C}$ and $\omega \in \Lambda$. For $a \in \mathbb{C}$, denote

$$P_a = \{a + t + s\tau : 0 \leq t \leq 1, 0 \leq s \leq 1\}.$$

   - Note $f'(z)$ is also doubly periodic with same period. Therefore $\int_{P_a} f'/f dz = 0$ since the opposite sides cancel. This is also the number of zeros times the number of poles, so they are equal. If there are zeros on the fundamental parallelogram, then since they are isolated, we can remove a bump or just move the parallelogram up. Need more details here.
   - Degree zero makes no sense. If the degree is one, $f(z)$ has a simple pole in P_a. On the one hand, $\int_{P_a} f(z)dz = 0$ as we know from the previous part, but by the residue theorem this is integral also gives $2\pi i$ times the residue at the pole in P_a. Since the pole is simple, this residue is nonzero. Therefore degree 1 cannot happen as well.

8.

2.2 UW 2011

1. Omitted

2.

3. Put $g(z) = \left(\overline{f(\bar{z})}\right)^{-1}$. Then $g(z)$ is a meromorphic function in the plane; moreover, for real $z, g(z) = f(z)$. Therefore, as meromorphic functions, $g(z) \equiv f(z)$ by the identity principle. But whenever $g(z)$ has poles, i.e. $f(\bar{z}) = 0$, the identity states that $f(z)$ will have a pole. Since $f(z)$ is entire, there can therefore be no poles of $g(z)$, equivalently zeros of $f(z)$. Therefore $f(z)$ is a non vanishing entire function, so we can define a logarithm of $g(z)$ to be analytic on $\mathbb{C}$, whence $f(z) = e^{g(z)}$, as required.

4. Jordan Curve Theorem!

5. Fix $0 < r < R$, let $\epsilon > 0$, and put $u_\epsilon(z) = u(z) - U(z) + \epsilon \log \left|\frac{z - z_0}{r}\right|$, where $U(z) = PI_{|z-z_0|=r}(u(z))$. $u_\epsilon(z)$ is subharmonic in $D'(z_0, r)$, $u_\epsilon(z) \leq 2M$, and $u_\epsilon \to -\infty$ as $z \to z_0$. Since $\limsup_{z \in D'(z_0;r) \to \zeta \in |z-z_0|=r} u_\epsilon(z) \leq 0$, we have $\limsup_{z \to \zeta \in \partial D'(z_0,r)} u_\epsilon(z) \leq 0$. By the maximum principle, $u_\epsilon(z) \leq 0$ persists in $D'(z_0;r)$. Fix $z \in D'(z_0;r)$ and let $\epsilon \to 0$ to obtain $u(z) \leq U(z)$. The same analysis applied to $v_\epsilon(z) = U(z) - u(z) + \epsilon \log \left|\frac{z - z_0}{r}\right|$ allows us to conclude $u(z) = U(z)$. But by Schwarz' theorem, the right-hand member is harmonic at z_0, so we can extend $u(z)$ to be $U(z_0)$ at $z = z_0$ and obtain a harmonic function in $D(z_0;r)$. Letting $r \to R$ proves the claim.

6. Let $a_{n_k} = \left(1 - \frac{1}{n^3}\right) \exp\left\{2\pi i \frac{k}{n}\right\}, 0 \leq k < n$. Then $\sum_{n,k} 1 - |a_{n_k}| < \infty$, so that $B(z) = \prod_{n,k} B(a_n, z)$ where $B(a_n, z) = \frac{|a|}{a} \frac{a - z}{1 - \bar{a}z}$ has the property that $B(z) \in H^\infty(\mathbb{D})$ and furthermore every point on the boundary is a limit point of the zeros of $B(z)$. Indeed, the zeros are scaled nth roots of unity which form a dense

7.

8.

2.3 UW 2010

1.

2.

3.

4.

5. Define $-\pi \arg z < \pi$. With this choice we have a square root defined on the right-half plane. Therefore we can view $g(z^2)$ as a map from the right-half disk

6.

7. We show that $\mathcal{F}$ is locally uniformly bounded in $\mathbb{D}$. Fix a disk $B = B(z_0, R_0)$ so that $\overline{B} \subset \mathbb{D}$. Then

$$\left| \iint_B f^2 dxdy \right| \leq \iint_B |f|^2 dxdy \leq \iint_{\mathbb{D}} |f|^2 dxdy \leq 1,$$

and since f^2 is analytic, we can change variables to r, θ and the left-most member in the above line becomes $\left| \int_0^{R_0} 2\pi f^2(z_0) r dr \right| = 2\pi^2 R_0^2 |f(z_0)|^2 \leq 1$, so that $|f(z_0)| \leq \dfrac{1}{\pi^2 R^2}$. Now, since $R < 1 - |z_0|$, we may apply this procedure to disks where R increases to $1 - |z_0|^2$, we obtain, after taking square roots, the inequality $|f(z_0)| \leq \dfrac{1}{\pi(1 - |z_0|)}$. Now fix any disk Δ with $\overline{\Delta} \subset \mathbb{D}$. Then for each $z \in \Delta$, $|f(z)| \leq \dfrac{1}{\pi(1 - |z|)}$. Since these disks are properly within Δ, $1 - |z| > 0$, so that $|f(z)| \leq C$ for some appropriately chosen C. Therefore $\mathcal{F}$ is locally uniformly bounded, and by Montel's theorem $\mathcal{F}$ is a normal family.

8. $F(z) = \int_0^\infty x^{z-1} e^{-x^2} dx.$

   - Prove that F is an analytic function on the region $\operatorname{Re} z > 0$.
   - Prove that F extends to a meromorphic function on the whole complex plane.
   - Find all the poles of F and find the singular parts of F at these poles.

   Proof. Let S denote an arbitrary square contained in $\operatorname{Re} z > 0$ whose sides are parallel to the coordinate axes. We show $\int_S F(z) dz = 0$ for all such S, and from Morera's theorem it will follow that $F(z)$ is analytic in $t = \operatorname{Re} z > 0$. We wish to switch the order of integration, so we show $\int_S \int_0^\infty x^{t-1} e^{-x^2} dx |dz| < \infty$. Split the inner integral into $\int_0^1 x^{t-1} e^{-x^2} dx + \int_1^\infty x^{t-1} e^{-x^2} dx$. The first integrand is bounded above by $\int_0^1 x^{t-1} dx$ and is finite by direct integration. For the latter integral, since $x \geq 1$, we may work instead with $\int_1^\infty x^t e^{-x^2} dx$. Since $e^{x^2} = 1 + x^2 + x^4/2! + \cdots$, choose n_0 so that $t - 2n_0 < -1$. Then $x^{2n_0} \leq e^{x^2}$ so that $\int_1^\infty x^t e^{-x^2} dx \leq \int_1^\infty x^{t-2n_0} dx < \infty$ (functions like $1/x^{1+\epsilon}$ are integrable at infinity.) We may actually choose n_0 so that this relationship holds for all t in the square by choosing the largest of the n_0's. Integrating the outer integral, we simply use $|\int_S F(z)dz| \le \ell(S) \sup_{\operatorname{Re} z = t \in S} |F(z)|$, which by our previous argument the right-hand term will be finite. Therefore we may switch the integrations to obtain $\int_S F(z)dz = \int_0^\infty \int_S x^{z-1}e^{-x^2} dzdx$, and for fixed $x > 0$ the inner integral is zero by Cauchy's theorem, as $x^{z-1} = e^{(z-1)\log x}$ is an analytic function within the square. Actually to even apply Morera's theorem, we need $F(z)$ to be a continuous function of z. This follows from the dominated convergence theorem: Let t_{max} denote the maximum value of t in S. Then $|x^{z-1}e^{-x^2}| \le x^{t_{max}-1}e^{-x^2}$, and we know already that the right-hand side is $L^1(\mathbb{R}, dx)$. Therefore continuity is established. $\square$

2.4 UW 2009

1.

2.

3. Let $A_n = \{z \in \mathbb{C} : f^{(n)}(z) = 0\}$. By the hypothesis of the problem, $\cup_{n\ge 0} A_n = \mathbb{C}$. By isolated zeros, we know that each A_n is a discrete set and furthermore each A_n is closed since $f(z)$ is continuous. Therefore we have written $\mathbb{C}$ as the countable union of nowhere-dense closed sets, which contradicts the Baire Category theorem. Therefore there is some A_{n_0} with nonempty interior, which by isolated zeros implies $f^{(n_0)}(z) \equiv 0$, and we deduce $f(z)$ is a polynomial.

4.

5.

6.

7.

8.
   - To show $\prod_{k=1}^{\infty} |1 + \frac{i}{k}|$ converges, look at $\sum_{k=1}^{\infty} \log|1+\frac{i}{k}| = \sum_{k=1}^{\infty} \frac{1}{2}\log(1+\frac{1}{k^2}) \le \sum_{k=1}^{\infty} \frac{1}{2}(1/k^2) < \infty$, as required. Now if $\prod_{k=1}^{\infty}(1+i/k)$ converged, which is to say $\sum_{k=1}^{\infty}\log(1+i/k) = \sum_{k=1}^{\infty} \log|1+i/k| + i\arg(1+i/k) = \sum_{k=1}^{\infty} \log|1+i/k| + i\arctan(1/k)$ converged, then we would have $\sum_{k=1}^{\infty} i\arctan(1/k)$ converging. This follows from the fact that $\lim(a_n+b_n) - \lim(b_n) = \lim(a_n) < \infty$ if the left-hand members are both convergent. But since $\lim_{x\to 0}\frac{\tan x}{x} = 1$, we can find x small enough so that $\frac{\tan x}{x} < 2$, which implies $\arctan(1/k) > 1/(2k)$ for k large enough, which implies $\sum_{k=1}^{\infty} 1/k < \infty$, a contradiction.
   -

2.5 UW 2008

1.

2. We use Jensen's theorem to derive the inequality $n(r)\log 2 \le \log M(2r)$

3. Otherwise $\forall r > 0 \exists f$ such that $f(\mathbb{D}) \not\supset \mathbb{D}_r$. In particular select $r_n = 1/n$ and find $f_n(z)$ so that $z_n \in \mathbb{D}_{1/n}$ is omitted from $f_n(\mathbb{D})$. Put $g_n(z) = f_n(z) - z_n$. Then $\{f_n\} \subset \mathcal{F}$ so it has a subsequence that converges normally to some analytic f by Weierstrass. Evidently $g_n(z) \to f$. By Hurwitz, $f(z)$ is either identically zero or nonzero on the compact. But by the assumption on the family, if f is identically zero then the derivative condition is violated and if $f(z)$ is nonzero then zero at zero fails.

4. Assume $f(U) \neq \mathbb{D}$. By the maximum principle, $f(U) \subset \mathbb{D}$. By assumption, find $z_0 \in \mathbb{D}$ so that $f(z) \neq z_0$ for any $z \in U$. The function $T(z) = \dfrac{f(z) - z_0}{1 - \overline{z_0} f(z)} : U \to \mathbb{D}$ is nonvanishing and analytic. When $z \in \partial U$, $|T(z)| = 1$ since $|f(z)| = 1$. By the maximum and minimum modulus principles, $|T(z)| = 1$ throughout U, so that $T(z)$ is a unimodular constant. Rearranging, we obtain $f(z)$ is constant as well.

5.

6.

7.

8.

2.6 UW 2007

1. Answer is $\dfrac{\pi(a+1)}{4e^a}$. Semicircular contour. Very straight-forward.

2.

3. Let $p(z) = az^4 + bz + 1, a \in [1, \pi], b \in [2\pi - 2, 7]$. Then $p(z)$ and bz have the same number of roots in $|z| < 1$ and $p(z)$ and az^4 have the same number of roots in $|z| < 2$. Therefore there are at most 3 roots in $1 < |z| < 2$, but there might be fewer if some of the three roots lie on $|z| = 1$.

4. Assume there is a univalent map $f(z)$ from the annulus $\Omega_1 = \{z : 1/2 < |z| < 1\}$ onto the punctured disk $\Omega_2 = \{z : 0 < |z| < 1\}$. Then $g(z) = f^{-1}(z)$ is analytic and is a map from the punctured disk to the annulus. Evidently $g(z)$ is bounded in a neighborhood of zero, and by Riemann's theorem on removable singularities, we can extend $g(z)$ to be defined at 0 to have $g(z) \in H(\mathbb{D})$. Since one-to-one analytic maps are proper; that is, boundaries must map to boundaries, we see that $|g(0)|$ is either 1 or 1/2. If $|g(0)| = 1$, then by open mapping $g(z)$ maps a neighborhood of zero to a neighborhood of a point on $|z| = 1$, which cannot happen. If $|g(0)| = 1/2$, the same thing happens as a neighborhood of a point on $|z| = 1/2$ will contain points whose modulus is smaller than 1/2, contradicting the hypothesis. Therefore no such map exists.

5.

6. If $\sum_{n\geq 0} \dfrac{M_n z^n}{n!}$ converges in $\mathbb{D}$, then $\mathcal{F} \ni f(z) = \sum_{n\geq 0} a_n z^n$ has a convergent power series centered at zero. Fix a compact subset of $\mathbb{D}$ and find $R < 1$ so that $|z| < R$ contains the compact K. Such R can be chosen since K is a compact subset of $\mathbb{D}$ and thus $d(K, \partial\mathbb{D})$ is positive. Choose ϵ small enough and consider a ball of radius $1 - \epsilon$. Then for $z \in K$, $|f(z)| \leq \sum_{n\geq 0} \dfrac{M_n}{n!} |z|^n \leq \sum_{n\geq 0} \dfrac{M_n}{n!} (1-\epsilon)^n < \infty = C_K < \infty$, so by Montel's theorem $\mathcal{F}$ is locally uniformly bounded and thus a normal family.
Conversely, suppose $\mathcal{F}$ is a normal family.

7. Firstly, $f(0) = 0$, so zero is attained at least once. For $z \neq 0$, the solving equation $f(z) = 0$ is equivalent to $g(z) = \sin z/z^2 = 1$. Since $g(z)$ has an essential singularity at infinity, Great Picard says that, in a deleted neighborhood of infinity, $g(z)$ attains every complex number, infinitely many times, with at most one possible exception. If there was no $z \in \mathbb{C}$ so that $g(z) = 1$, then since $g(z)$ is odd, $g(-z) = -1$ would have no solution, which contradicts the theorem.

8.

2.7 UW 2006

1. Let $p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_0$. Let $f(z) = z^n$. We wish to compare $p(z)$ to $f(z)$. Let $R = \sqrt{1 + |c_0|^2 + \cdots + |c_{n-1}|^2}$, then on $|z| = R$, we have

$$\begin{aligned}|p(z) - f(z)| &\leq (|c_0|^2 + \cdots + |c_{n-1}|^2)^{1/2}(1 + |z|^2 + \cdots + |z|^{2n-2})^{1/2} \\ &= \sqrt{R^2 - 1}(1 + R^2 + \cdots + R^{2n-2}) \\ &= \sqrt{R^2 - 1}\sqrt{\frac{R^{2n} - 1}{R^2 - 1}} \\ &< \sqrt{R^{2n}} = |f(z)|,\end{aligned}$$

so that by Rouche's theorem all the zeros of $p(z)$ lie within the disk centered at zero of radius $\sqrt{|c_0|^2 + \cdots + |c_{n-1}|^2 + 1}$, as required.

2.

3. Let r be the smallest disk centered at zero omitted by the map $f(z)$ and select z_0 in this disk. Then the function $g_1(z) = \dfrac{f(z) - z_0}{1 - \overline{z_0} f(z)} : \mathbb{D} \to \mathbb{D}$ and is non vanishing on a simply connected domain. Hence we can define $g_2(z) = \sqrt{g_1(z)}$ to be analytic on $\mathbb{D}$. The function $h(z) = \dfrac{g_2(z) - g_2(0)}{1 - \overline{g_2(0)} g_2(z)} \in H(\mathbb{D})$ and furthermore $h(0) = 0$ and $|h(z)| \leq 1$. By Schwarz' Lemma, $|h'(0)| < 1$, and a computation shows, if we define $b = \sqrt{-z_0}$, that for this to happen we must have $1 + |b^2| < 14|b|$, which has solutions whenever $|b| < 7 - 4\sqrt{3}$ (the other solution is outside the disk). Therefore as long as $|b| < 7 - 4\sqrt{3}$

4. *Proof.* Assume not. Then there exists a $z_0 \in \Omega$ and a radius $r > 0$ so that

$$|f(z_0)| = \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta / 2\pi,$$

and by the Cauchy integral formula for $f(z_0)$, we obtain

$$|\int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta| = \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta,$$

which implies $f(z) = \alpha g(z)$ on $|z - z_0| = r$, where $g(z)$ is real valued and $|\alpha| = 1$ (i.e., f has constant argument on the circle). Define $g(z) = f(z)/\alpha$ in Ω. Then $g(z)$ is evidently analytic and real valued on $|z - z_0| = r$. By the Poisson integral for $\operatorname{Im} g(z)$ for points $z \in D(z_0; r)$, we obtain $\operatorname{Im} g(z) \equiv 0$ within $\overline{D}(z_0; r)$. Therefore by open mapping (the image of the disk D is sent to the real line, which isn't an open set), $g(z)$ is a real constant throughout $\overline{D}$, and since αg and f agree on a set with a limit point, we obtain f is identically constant on $\overline{D}$, and by isolated zeros f is identically constant throughout Ω, a contradiction. $\square$

5. Let $f \in H(\mathbb{C})$ with only finitely many zeros. Assume $m(r) = \min_{|z|=r} |f(z)|$ does not tend to 0 as $r \to \infty$. We negate the statement $(\forall \epsilon > 0)(\exists \delta > 0).(R > \delta \Rightarrow m(r) < \epsilon)$ to get $(\exists \epsilon > 0)(\forall \delta > 0).(R > \delta \Rightarrow m(r) \geq \epsilon)$. For $\delta = 1, 2, \ldots$, we may select $R_1, R_2, \ldots$ and form a sequence of positive real numbers (we may further assume increasing by passing to a subsequence) with the property $m(R_n) \geq \epsilon$. Since $f(z)$ has only finitely many zeros, we may write $f(z) = \dfrac{(z-z_1)\cdots(z-z_n)}{g(z)}$, where $1/g \in H(\mathbb{C})$ and is non vanishing. For large enough R_n we may enclose all zeros of $f(z)$ in $|z| \leq R_n$. Now fix z with $R_n < |z| < R_{n+1}$. We have $|g(z)| \leq \dfrac{|(z-z_1)\cdots(z-z_n)|}{\epsilon} \leq C|z|^n$. Indeed, $f(z)$ is nonzero and analytic in the annular region $R_n < |z| < R_{n+1}$ and thus satisfies the minimum modulus principle. By Cauchy's estimates $g(z)$ is necessarily a polynomial. But since $1/g \in H(\mathbb{C})$, we arrive at a contradiction by the fundamental theorem of algebra.

6. Let $\Omega = \{z : |z| \leq 2\}$ and $[0,1]$ be the line segment from 0 to 1.

   - We apply Morera's theorem and show that $\int_S f(z)dz = 0$ for all rectangles S contained in Ω whose sides are parallel to the coordinate axes (where the integral is taken with positive orientation). By the analyticity of f on $\Omega \setminus [0,1]$, the only difficulty arises when the rectangles meet $[0,1]$. There are evidently two problems: the case when a subset of $[0,1]$ is completely within the of the rectangle except for possibly two points, and the case where a subset of $[0,1]$ is one of the sides of the rectangle. By writing the former case as the integral of the sum of two rectangles in the latter case, it is enough to prove that $\int_S f(z)dz = 0$ for rectangles S described by the latter case. Let S be such a rectangle. Consider the function $f_\epsilon(z) = f(z + i\epsilon)$, where $\epsilon > 0$. Since $\int_S f(z+i\epsilon) = 0$ and since $f_\epsilon(z) \to f(z)$ uniformly for $z \in S$ (which follows since S is compact), we may conclude $\int_S f(z)dz = \lim_{\epsilon \to 0} \int_S f_\epsilon(z)dz = 0$. By Morera's theorem, $f(z)$ extends to be analytic in $|z| < 2$, as required.

   - Define $f(z) = \sqrt{z}\sqrt{z-1}$, where $0 < \arg z < 2\pi$ and $0 < \arg(z-1) < 2\pi$. Clearly $f \in H^\infty(\Omega \setminus [0,2))$. The doubt arises across $(0,2)$, where there are branch cuts. Tending to points $x \in (0,1)$ from above, the product tends to $i\sqrt{x}\sqrt{x-1}$. Tending from below we obtain $-i\sqrt{x}\sqrt{x-1}$, so analyticity fails on $(0,1)$. Now we just need to show analyticity works on $(1,2)$. Indeed, tending to $x \in (1,2)$ above yields $\sqrt{x}\sqrt{x-1}$, and tending from below gives $(-\sqrt{x})(-\sqrt{x-1}) = \sqrt{x}\sqrt{x-1}$, which is great! By the previous part, this function extends to be continuous on $(1,2)$. Therefore $f \in H^\infty(\Omega \setminus [0,1])$ and cannot be extended to an analytic function in Ω.

7.

8.

2.8 UW 2005

1. We have $z^n = -1 = exp\{i\pi + i2k\pi\}$, so that if $z = re^{i\theta}$, then $n\theta = \pi(2k+1)$, so that $\theta = \pi\dfrac{2k+1}{n}$. Now n is an even integer, and we repeat whenever $2k = 1 \geq n$, so evidently $k \in \{0, 1, \ldots, n/2 - 1\}$.

2. Partial fractions yields $f(z) = \dfrac{1/(a-b)}{z-a} + \dfrac{1/(b-a)}{z-b}$. There are three regions to consider: $|z| <$

$|a|, |a| < |z| < |b|$,and $|b| < |z|$. For the first case, we expand $\frac{1}{z-a} = \frac{1}{-a(1-z/a)} = \frac{-1}{a}\sum_{n=0}^{\infty}(\frac{z}{a})^n$, which is valid whenever $|z| < |a|$. Similarly, we write $\frac{1}{z-b} = \frac{-1}{b}\sum_{n=0}^{\infty}(\frac{z}{b})^n$, which is valid whenever $|z| < |b|$ and is in particular valid in the first region. Therefore we can say $f(z) = \frac{-1}{a-b}\sum_{n=0}^{\infty}\frac{z^n}{a^{n+1}} + \frac{-1}{b-a}\sum_{n=0}^{\infty}\frac{z^n}{b^{n+1}}$.

3. By Montel's theorem, $g_n(z)$ is a normal family. Since the family $f_n - g_n$ omits at least three values, by Marty's theorem it is normal in the chordal metric. The limit function therefore is either analytic or identically infinity. But since $|g_n| \leq 1$ and $f_n(z)$ converges for each z, it follows that $|f_n - g_n| \leq 1 + |f_n(z)|$ so that the limit function is point wise bounded and thus the convergence is normal in the Euclidean metric. Therefore $f_n = g_n + (f_n - g_n)$ is a normal family as well. This follows more generally from the fact that $\{a_n\}$ and $\{b_n\}$ being normal families on a domain G implies their sum is a normal family. Fix a compact K and consider $\{f_{n_k}\} \to f$ uniformly on G. By considering the subsequence $\{g_{n_k}\} \subset \{g_n\}$ we may pass to a further subsequence $g_{n_{k_j}}$ and have this sequence converging on K. But since f_{n_k} converges, any subsequence converges as well, so that $f_{n_{k_j}} + g_{n_{k_j}}$ converges on K, which is the condition of being normal. Now we show the sequence f_n converges normally on G. Fix a compact set K. If f_n does not converge normally on K, then the family is not uniformly Cauchy. Therefore there exists ϵ so that for all $N \in \mathbb{N}$ and $n, m \geq N$ $|f_n(z) - f_m(z)| > \epsilon$. Therefore we obtain two subsequences f_{m_1}, f_{n_1} (where we choose $m_1, n_1 \geq 1 = N$ and so forth. Thus the inequality reads for all $k \in \mathbb{N}, |f_{n_k} - f_{m_k}| > \epsilon$. By normality, there is a subsequence of $\{f_{n_k}\}$ that converges normally on K and similarly for $\{f_{m_k}\}$. Call the limit functions f, g. By Weierstrass, both are analytic functions as they are the normal limit of analytic functions. Moreover, since $f_n(z)$ converges point wise for each $z \in G$, we have $f(z) = g(z)$ for each $z \in K$ so that $f \equiv g$ by isolated zeros. But at this point we arrive at a contradiction, since (relabel and pass to the subsequences) $\epsilon < |f_n - f_m| \leq |f_n - f| + |f_m - g|$, and the right-most member can be made arbitrarily small. Therefore f_n are uniformly Cauchy and thus f_n converges normally on G.

4.

5. $f(z)$ extends to be continuous on $i\mathbb{R}$, and by Schwarz reflection principle, we reflect about the imaginary axis and define a function in the whole plane by the identity $F(z) = \begin{cases} f(z) & \text{if } \operatorname{Re} z > 0, \\ 0 & \text{if } \operatorname{Re} z = 0, \\ \overline{f(-\overline{z})} & \text{otherwise.} \end{cases}$

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7.

8. Hi

   - $u(z) = \frac{\log |z|/r}{\log R/r}$.

2.9 UW 2004

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2.10 UW 2003

1. $\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)d\zeta}{\zeta - z} = \begin{cases} f(z) & \text{if } |z|<1 \\ 0 & \text{if } |z|>1 \end{cases}$. The integral is not defined if $|z| = 1$. This follows form Cauchy's theorem applied the holomorphic function $F(\zeta) = \frac{f(\zeta)-f(z)}{\zeta - z}$. $F(z)$ extends to be holomorphic at $\zeta = z$ by Riemann's theorem (define $F(z) = f'(z)$).

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6. The angle of the two sectors to the right is $\pi/6$ by taking inverse tangents. Set $\alpha = \pi/4 + \pi/6$. Then define $z^{\pi/(2\alpha)}$ to be analytic on $\mathbb{D} \setminus \operatorname{Re} z \le 0$. Then we get to the right-half disk, and the map $-i\frac{z-i}{z+i}$ maps to the upper-half plane, and a Cayley finishes the job.

7.

8.

Chapter 3

TAMU Prelims

3.1 TAMU January 2013

1. Put $f(z) = \dfrac{z}{1-z-z^2}$. Then $f(z)$ has a taylor development at zero, say, $f(z) = \sum_{n\geq 0} a_n z^n$. Where the series converges in the largest open disk centered at zero that $f(z)$ is analytic. Note that $f(z)$ has poles at $z = -\dfrac{1}{2} \pm \dfrac{\sqrt{5}}{2}$. Therefore $z = (1-z-z^2)\sum_{n\geq 0} a_n z^n$ and equating coefficients on both sides we obtain $a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$. Furthermore, $f(0) = 0 = a_0$ and $1 = a_1 - a_0$ by equating coefficients (aka uniqueness of power series coefficients on both sides). Here we are only speaking of those z where $f(z)$ has this Taylor series. Here we can say $(1-z-z^2)\sum_{n\geq 0} a_n z^n = \sum_{n\geq 0}(a_n - a_{n+1} - a_{n+2})z^n$ since the series converges absolutely.

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4. Can only get an estimate for $|f'(0)| < 4/\pi$.

5. Just use Cauchy estimate and trap a compact set in a disk and do the integration over a slightly bigger disk. Just use the fact that this series converges! So yes it is normal.

6. Assume $|\operatorname{Re} f(z)| \leq M$ for $z \in \mathbb{D}'$. Composing with a map φ sending $|\operatorname{Re} z| \leq M$ to $\mathbb{D}$, we obtain a map $g(z) = (\varphi \circ f)(z) : \mathbb{D}' \to \mathbb{D}$ which extends to be analytic at zero by Riemann's theorem. Therefore since φ is a conformal map, $f(z)$ extends to be analytic at zero by setting $\varphi^{-1} \circ g(0) = f(0)$.

7. Write $f(z) = (g \circ h)(z)$ where $g(w) = w^3 + w^2$ and $h(z) = e^z$. Then the composition is surjective since polynomials are surjective and so is the exponential function (except for zero). Therefore the function is surjective away from zero. But the function is obviously zero too so yeah.

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9. Done in a UW prelim.

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3.2 TAMU August 2012

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3.3 TAMU January 2009

hey

Let Ω be the half-strip defined by $z = x + iy, x > 0, 0 < y < \pi$ and let f be analytic on Ω, continuous on $\overline{\Omega}$ (the closure of Ω in $\mathbb{C}$ - not the closure on the Riemann sphere), real on $\partial\Omega$ and suppose $f(z)e^{-z} \to 1$ uniformly in y as $x \to +\infty$. Prove f maps Ω one-to-one and conformally onto the upper half plane $\mathbb{H}$. (Hint: compare f to a conformal map of Ω onto $\mathbb{H}$.)

Proof. Let $\varphi(z) = \frac{1}{2}(e^z + e^{-z})$ be a conformal map of Ω to $\mathbb{H}$. Then $f/\varphi = 2f/(e^z + e^{-z}) \to 2$ uniformly in y as $x \to +\infty$ on Ω since $|e^{-2z}| = e^{-2x} \to 0$ as $x \to +\infty$. Therefore for each ϵ_n, there exists R_n so that on $B_n = \{(x,y) : R_n < x < \infty, 0 < y < \pi/2\}$, $f/\varphi < 2 + R_n$. On $A_n = \Omega \setminus B_n$, f/φ is also bounded. This is true by the maximum principle since f and φ are continuous on $\overline{\Omega}$ and therefore bounded on both upper and lower line segments of A_n (note here we use that $\varphi \neq 0$ on this set. The only problem is on the line segment $x = 0$, where $\varphi(z) = cos(y)$ has a zero at $y = \pi/2$. But by Lindelof's maximum principle, we may still conclude f/φ is bounded inside A_n. □

Determine all entire functions f satisfying $|f| = 1$ on $\partial\mathbb{D}$.

Proof. Consider the number of zeros of f in $\mathbb{D}$. If there are countably many in the disc, then they must accumulate. If they accumulate within the disc, then f is identically zero. If they accumulate on the boundary, then the hypothesis is violated. (Actually one can just say $\overline{\mathbb{D}} \subset \mathbb{C}$ and f entire implies there are only finitely many zeros in $\overline{\mathbb{D}}$ and they must all lie in the interior by the hypothesis.) Oops. Will need that other argument for the rational f to be presented next. Now there are only finitely many zeros. List them as $a_1, \ldots, a_n$ (possibly repeated). Let $B_{a_i}(z)$ denote the Blaschke factor with zero at a_i, i.e. the automorphism of the disc sending a_i to zero. Then the quotient $g = f/\Pi B_{a_i}$ is analytic in $\mathbb{D}$ with removable singularity at each a_i. Now g is a nonzero analytic function in $\mathbb{D}$ satisfying $|g| = 1$ on $\partial\mathbb{D}$, so the minimum and maximum modulus principle applied to g implies, $g \equiv \lambda$ for unimodular λ. Since f is entire and agrees with $\lambda \Pi B_{a_i}(z)$ on a set with an accumulation point, isolated zeros implies $f \equiv \lambda \Pi B_{a_i}(z)$. The fact that $1 - \overline{a}_i z = 0$ is zero in $|z| > 1$ implies f has a pole. Therefore either f has finitely many zeros all of whom are at $a_i = 0$, or f is nonzero. Therefore $f = \lambda z^n, n \geq 0, |\lambda| = 1$. □

Suppose $f \in H(\mathbb{D})$ and assume that $|f(z)| \to 1$ as $|z| \to 1$. Show that f is rational.

Proof. We take cases on the zeros of f. Assume f has infinitely many zeros. If they accumulate to the boundary, then the hypothesis fails for that sequence of zeros $|z_j|$. If they accumulate inside, then f is identically zero by isolated zeros. Here we used that an infinite subset of the compact set $\overline{\mathbb{D}}$ has an accumulation point (this is an equivalent form of compactness for metric spaces). Therefore there are only finitely many zeros. If there are NO zeros, then the limsup form of the maximum principle implies $1 = \limsup_{z \to \partial \mathbb{D}} |f(z)| = \sup_{z \in \Omega} |f(z)|$ and similarly the minimum modulus principle applied to $1/f \in H(\mathbb{D})$ implies f is constant and so rational. Now if there are finitely many zeros, divide by the Blaschke factors to form a non vanishing analytic function in the disc. Again by maximum and minimum modulus, this quotient is a unimodular constant so that f is a finite blaschke product. You might ask, well why don't we just factor out $(z - a_i)$ where a_i are the zeros? The problem is that the Blaschke factors tend to 1 as $|z| \to 1$ so we can still apply the hypothesis of the problem and the maximum principle; without it, we have no control on the analytic function we form. □

Gamelin IX.1.8. Let $f \in H(\mathbb{D})$ with $f(0) = 0, |f'(0)| < 1$. Show that there is a c_R so that on $|z| \leq r, |f(z)| < c|z|$.

Proof. Let R be given. Since $|f(z)| < 1$ in $\mathbb{D}$, it holds in particular for $|z| \leq R$. Define $g(z) = f(z)/z$. $g(z)$ has a removable singularity at 0 and we can define its value to be $f'(0)$, so $g \in H(D(0;R))$. Schwarz' lemma implies $|f(z)| \leq |z|$ for $|z| < 1$. On $|z| = R$, we obtain $|f(z)| \leq R$. Let C denote the maximum of $|f(z)|$ on $|z| = R$, which exists by continuity and compactness. Evidently, $C < R$ by the lemma, strict since otherwise $f = \lambda z$ (lambda unimodular), which violates the hypothesis on the derivative. Therefore $|g(z)| \leq C/R$, where $C/R < 1$. Evidently the nth iterate satisfies $|f_n(z)| \leq c^n |z|$, so that $\lim |f_n(z)| \to 0$ and we deduce for disks $\overline{D}(0;R)$, $f_n(z) \to 0$ uniformly (this bound was independent of $z \in \overline{D}(0;R)$. Since this holds for disks, the convergence is normal. Indeed, any compact set $K \subset \mathbb{D}$ can be covered by disks whose closure lies completely within $\mathbb{D}$. The convergence is normal on these disks and we may reduce to a finite sub cover by disks whose closure lies in $\mathbb{D}$. Therefore the functions converge normally to 0. □

TAMU Jan 2009 #10

Proof. The boundedness and connectedness of Ω implies that one of lies within the other. The Jordan curve theorem says they both separate into bounded components, and one bounded must intersect the other unbounded giving a bounded (clean this up). By relabeling, assume WLOG γ_1 is the interior curve. Let $f \in H(\Omega)$. We show f has an analytic antiderivative if and only if $\int_{\gamma_1} f dz = 0$. Note first that $\int_{\gamma_1} f = \int_{\gamma_2} f$. This follows since the curve $\gamma_2 - \gamma_1 \sim 0$. Indeed, points z within the "hole" created by γ_1 have zero winding number since the ray connecting a point z to ∞ is zero (a negative 1 is picked up from γ_1 and canceled by the +1 from γ_2). Points outside clearly lie in the unbounded component and their winding number is evidently zero. Therefore by Cauchy's theorem $\int_{\gamma_2 - \gamma_1} f = 0$. If f has an analytic antiderivative, F, then the fundamental theorem of calculus implies $\int_{\gamma_1^\epsilon} f dz = 0$ for this curve whose distance is uniformly ϵ away from γ_1 and entirely contained within Ω. Such an $\epsilon > 0$ can be chosen by considering $\inf_{s,t} d(\gamma_1(s), \gamma_2(t)) \neq 0$. Since the distance function is continuous in s, t and $s, t \in [a_1, b_1] \times [a_2, b_2]$ is compact, this value is attained. If it were zero, then disjointness would fail. Taking ϵ smaller than this infimum yields a sequence of curves entirely within Ω converging to something I'm making up. I would like to rigorize this. Or at least use uniform continuity like Hart said. Instead of integrating over curves tending to γ_1, integrate instead over γ_1 of the function a little bit away from γ_1. Uniform continuity implies $|\int_{\gamma_1} f_\epsilon - f dz| \leq \ell(\gamma_1) \sup_z |f_\epsilon - f|$ which can be made arbitrarily small. $\square$

Chapter 4

Other Prelims

4.1 Temple January 2012

1. • Compute Laplacian in polar coordinates:

$$\Delta u(r,\theta) = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) u(r,\theta),$$

   which holds for all $r \neq 0$.

   • Since $\Omega = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 0\}$ is simply connected and does not contain 0, we may define $\log z^2$ so as to be analytic on Ω. Moreover, $\log z^2 = \log |z|^2 + i \arg z^2$, where $0 < \arg z < 2\pi$. Now $|z|^2 = x^2 + y^2$, so that $u(x,y) = \operatorname{Re} \log z^2$, and it follows that $v(x,y) := \arg z^2$ is a harmonic conjugate of $u(x,y)$ in Ω.

   • Let $\tilde{v}(x,y)$ be the alleged harmonic conjugate in $\mathbb{C} \setminus \{0\}$. Denote by f the analytic function $u + i\tilde{v}$ in $\mathbb{C} \setminus \{0\}$. Then $\log z^2 - f(z)$ is purely imaginary in Ω. By open mapping, $\log z^2 - f(z) \equiv i\alpha$, an imaginary constant. Therefore $H(\mathbb{C} \setminus \{0\}) \ni f(z) = \log z^2 + i\alpha$ in Ω, therefore $\tilde{v}(x,y) = log|z|^2 + i\alpha$ off a line. But by the formula we can then extend $\arg z^2$ to be $\tilde{v} - i\alpha$ which is analytic off 0, which is a contradiction. ($\arg z^2$ will approach $4\pi i$ as you tend below and 0 as you tend above).

2. $f \in H(\mathbb{C})$.

   • Assume this inequality holds for $|z| > R'$. Fix $z_0 \in \mathbb{C} \setminus \{|z| \leq R'\}$. Choose R large enough so that $z_0 \in R' < |z| < R$. Let C_{2R} be the curve $2Re^{it}, 0 \leq t \leq 2\pi$. Then by Cauchy's integral formula, $|f^{(3)}(z_0)| \leq \dfrac{2\pi(2R)\left(\frac{(2R)^4}{1+(2R)^2}\right)}{2\pi R^4}$, and taking R to infinity yields $|f^{(3)}(z_0)| = 0$. This argument works for arbitrary $R' < |z_0| \in \mathbb{C}$, so that $f^{(3)}(z) \equiv 0$ by isolated zeros. By the same argument it follows that for $n \geq 3$, $f^{(n)}(z) \equiv 0$, so that $f(z)$ is a polynomial of degree less than or equal to 2.

   • Now assume the inequality holds for all $z \in \mathbb{C}$. From the previous part, $f(z)$ is at most a degree two polynomial. Let $f(z) = az^2 + bz + c$, where not all of a, b, c are zero. The inequality implies $f(0) = 0$, so we conclude $c = 0$. Therefore we may write $f(z) = zg(z)$, where $g(z) \in H(\mathbb{C})$. Now $|g(z)| \leq \dfrac{|z|^3}{1+|z|^2}$ whenever $z \neq 0$, but by continuity as $|z| \to 0$,

$|g(z)| \to 0$, so that $b = 0$. it follows that $f(z) = az^2$, and the inequality reads $|a|(\frac{1}{|z|^2} + 1) \leq 1$ for all $z \neq 0$, which is absurd. (can also note $|a| \leq 1/2$ by evaluating at 1.)

3. $f, g \in H(D'(z_0; r))$, $r > 0$.
   -

4. asdf
   -

5.

6. - Want to show given $\epsilon > 0$ we can find N so that $n \geq N$ implies $|f_n(z) - f(z)| < \epsilon$ for all $z \in \overline{D}(0; 1)$. We have the result already for $|z| = 1$. Consider some $z \in \mathbb{D}$. By Cauchy's integral formula, we have $|f_n(z) - f(z)|$
   - Weierstrass' Theorem:

7. -
   - Choose $r > 0$ and $n \geq 0$. We directly show there must exist z on $|z| = r$ such that $|p_n(z)| = |e^z|$.

4.2 Temple August 2012

1.

2.

3. - Consider $g(z) = e^{-f(z)} \in H(\mathbb{C})$. $|g(z)| = e^{-Ref(z)} < 1$, and by Liouville's theorem $g(z)$ is a constant. Now assume $f(z)$ is non constant. Then there are two points $z, w \in \mathbb{C}$ so that $f(z) \neq f(w)$. Since $f(\mathbb{C})$ is connected, there necessarily exists an accumulation point in $f(\mathbb{C})$ (else the set is discrete and thus disconnected). Therefore $\exp(z)$ is identically a constant on $f(\mathbb{C})$ by the identity theorem and we can further say identically constant on $\mathbb{C}$, a contradiction. Alternatively, if f is non constant then $f(\mathbb{C})$ is open so $\exp(f(\mathbb{C}))$ is constant on an open set, which implies exp is a constant, a contradiction.
   - Let u_1 and u_2 be two harmonic functions in $\mathbb{C}$. Assume that $u_1 - u_2$ is non constant and $u_1 \neq u_2$ for all $z \in \mathbb{C}$. Then $u_1 - u_2$ is harmonic and is either strictly positive or strictly negative by continuity, and since $\mathbb{C}$ is simply connected, $u_1 - u_2$ is the real part of some analytic function $f(z)$. By the previous part $f(z)$ is a constant so $u_1 - u_2 = \operatorname{Re} f$ is constant, a contradiction.

4.

4.3 Stanford Problems

Let $f \in H(D(0; 2))$, show $\int_0^1 f(x)dx = \frac{1}{2\pi i} \oint_{|z|=1} f(z) \log z dz$.

4.3.1 Fall 1998

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4.3.2 Spring 1998

1.

2. (a) Use the identity $f(z) - 1 = \frac{z}{2} f(z/4)$. Since $f(z)$ has an essential singularity, near the singularity $f(z)$ attains every value infinitely often, with one possible exception. If zero was that exception, then this identity says $f(z/4)$ would also attain 0 only finitely often, and thus $z/2f(z/4)$ would as well. Therefore $f(z)$ would attain 1 only finitely many times as well, which is a contradiction.

   (b) Assume $f(z)$ is not a polynomial. We can factor out the zeros to write $f(z) = e(z)e^{g(z)}$, where $e(z) = (z - z_1)(z + z_1) \cdots (z - z_n)(z + z_n)$. By evenness, we must have $e^{g(z)}$ even .. we want to divide the $e(z)$ on both sides, but we can't. For sure it's even away from the finitely many zeros, and since e^g is entire and by continuity it's even there too! Now taking moduli and applying the hypothesis, we get a bound on $\operatorname{Re} g(z) \le |z|$ for large enough $|z|$. Therefore $g(z)$ is linear, whence this shit aint even unless $g(z)$ is constant, which is a contradiction.

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6.

4.4 Wisconsin Problems

Compute $\int_{|z|=2} \frac{z^5}{z^7 + 3z - 10} dz$. Substitute $z = 1/w$, then $w^2 dw = -dz$. The integral we obtain is $\int_{|w|=1/2} \frac{w^4}{1 + 3w^6 - 10w^7} dw$. By Rouche's theorem, let $g - h = 10w^7 - 3w^6 - 1$ and $h = 1$, then $|g| = |10w^7 - 3w^6| < 1 = |h|$ so that no zeros of the denominator are within the circle of radius $1/2$. Therefore the integral is zero since the integrand is zero.

4.5 Columbia 1997

1. Let f be holomorphic on some open set U of $\mathbb{C}$. Let $z_0 \in U$ such that $f'(z_0) \neq 0$. Show that if C is a circle of center z_0 and small enough radius, then:

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{1}{f(z) - f(z_0)} dz.$$

2. Suppose $f(x, z)$ is a continuous function on $\mathbb{R} \times \mathbb{C}$ such that, for each $x \in \mathbb{R}$, the function $z \mapsto f(x, z)$ is holomorphic. Show that the following function is holomorphic:

$$F(z) = \int_0^1 f(x, z) dx.$$

3. Show that if $a > 0$ then

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + 1} dx = \pi e^{-a}.$$

4. Let f be holomorphic on some open set U containing a closed disk D_R of center 0 and radius R. Let $z_1, z_2, \ldots, z_N$ be the zeros of f in the open disk, each zero being repeated according to its multiplicity. Prove that

$$|f(0)| \leq \frac{|z_1 \cdots z_N|}{R^N} \sup_{|z|=R} |f(z)|$$

5.

6.

4.6 Columbia Analysis

1. The largest disk we can apply this formula to is a disk of radius $R = d(\partial\Omega, w)$, so we obtain $|f(w)|^2 \leq \frac{1}{\pi^2 d(\partial\Omega, w)^2} ||f||^2_{L^2(\Omega)}$. But since $d(\partial\Omega, w) \geq d(\partial\Omega, K)$, we obtain $|f(w)|^2 \leq \frac{1}{\pi^2 d(\partial\Omega, K)^2} ||f||^2_{L^2(\Omega)}$, taking sups on both sides gives what we wanted. ACTUALLY SUPREMUM IS ATTAINED. Take a disk about the sup, say it's attained at z_0.

Chapter 5

Miscellaneous Problems

Prove that if $u(z)$ is a bounded harmonic function in $D'(z_0; R)$ then $u(z)$ extends to be harmonic in the full disc $D(z_0; R)$. Prove that if $f(z) \in H(D'(z_0; R))$ and $\operatorname{Re} f(z)$ has a removable singularity at z_0, then $f(z)$ extends to be analytic in $D(z_0; R)$.
TAMU August 2008 #9 Show that the range of $z^2 + \cos(z)$ is all of $\mathbb{C}$.
TAMU January 2009 #9 Let $f(z) \in H(\mathbb{D})$ be univalent and $f(0) = 0$. Prove there exists $g(z) \in H(\mathbb{D})$ such that $g(z)^2 = f(z^2)$ for all $z \in \mathbb{D}$.
Fresnel Integrals
integrate $\log \sin \theta$
Show that if $f \in H(D'(z_0; R))$ and if f'/f has a simple pole at z_0, then $f(z)$ has either a zero or a pole at z_0.

Proof. By composing with a linear map, assume $z_0 = 0$. First note that there is some deleted neighborhood about 0 where $f(z) \neq 0$. This follows from the fact that if $f(w) = 0$, then $f(z) = (z-w)^k g(z), k \geq 1$ and g analytic. Then $f'/f = k/(z-w) + g'/g$, whence f'/f has a pole at w. Since the poles are isolated, there must exist some deleted neighborhood about 0 where $f(z) \neq 0$. Choose such a neighborhood. We may define $\log f$ so as to be analytic off imaginary axis and 0. Then $\log f(z) - \log f(z_1) = \int_{z_1}^{z} f'/f dz = \alpha \int_{z_1}^{z} dz/z + H(z)$, where H is analytic. The right-hand side follows from the fact that, near zero, $f'/f = \alpha/z + h(z)$, with $h(z)$ analytic and we take any polygonal path connecting z_1 to z, and the integral is independent of the path by simple connectedness. Exponentiating both sides, we obtain $f(z) = \dfrac{f(z_1)}{z_1^\alpha} z^\alpha e^{H(z)}$. Now α is a nonzero integer since ? $\square$

5.1 Notes to myself

Any nonzero entire $f \in H(\mathbb{C})$ is of the form $f = \exp(g)$. f'/f analytic in the whole plane, is the derivative of some $g(z)$. Therefore $f \exp(-g)$ has zero derivative so that $f = \exp(g)$ but the constant multiple is absorbed in $g(z)$.
$\lim_{z \to 0} \dfrac{\log(1+z)}{z} = 1.$
$\tan(\theta) \approx \theta, \sin\theta \approx \theta, \cos\theta \approx 1 - \theta^2/2$
$\log x \leq x - 1$
NOTE THAT THE RIEMANN SPHERE IS COMPACT! If I have an analytic function on the sphere, there can be only finitely many zeros by sequential compactness and the identity theorem. Moreover, the singularity at infinity is removable since we're analytic there too. Therefore if f has a power series

centered at zero, to have a removable singularity at infinity means that there can be no powers of z^{-1} in the expansion of $f(1/z)$, which implies f is a constant.

Let's talk about meromorphic functions on the sphere. There can be only finitely many zeros and poles by sequential compactness. Therefore it must be rational! After factoring out, we'd have a rational function times the quotient of nonzero analytic functions on the sphere, which we've just argued are constants.