Metropolis Hastings Algorithm

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Metropolis Algorithm

Question

Suppose we have data $X_1, ..., X_n$ which we believe come from a normal distribution with mean θ and variance 1. Suppose we are uncertain about θ, and for us θ has the following Cauchy Distribution:

$$f(\theta) = \frac{1}{\pi(1+\theta^2)}, -\infty < \theta < \infty$$

This is a special form of the Cauchy called the standard Cauchy Distribution with parameters $x_0 = 0, \gamma = 1$.

We will write a Metropolis Hastings Algorithm whose limiting distribution is our posterior distribution:

$$f(x, \theta, \sigma^2 = 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}} \frac{1}{\pi(1+\theta^2)}$$
$$X_i \sim \text{Normal}(\theta, 1)$$
$$\theta \sim \text{Cauchy}(0, 1)$$

First lets simulate some data

```
set.seed(100)
# choose a cauchy sample for mean
theta_x <- rcauchy(1,location = 0, scale = 1)
theta_x
```

```
## [1] 1.449469
```

```
# generate samples
data_y <- rnorm(n = 1000, mean = theta_x, sd = 1)
```

Now we can create our metropolis algorithm:

The Metropolis Algorithm calls for θ_{prop} to be sampled from a symmetric proposal distribution centered at the current parameter value, θ_{curr}. For this task we will use $\theta_{prop} \sim \text{Normal}(\theta_{curr}, \sigma^2)$.

The proposal distribution is seperate and distinct from either the prior or posterior distribution for the parameter. The proposal distribution's sole purpose is to give candidate parameter values to *try* and potentially accept as a valid sample from the posterior distribution of θ.

- samples is the number of samples we want to draw from the posterior distribution and determines the length of the resulting MCMC chain
- theta_start gives us a θ to start the algorithm
- sd is the standard deviation of the proposal distribution

Within the function, we construct a for loop that repeatedly draws θ_{prop} from a standard normal proposal distribution (using rnorm). It then computes the ratio of Bayes' numerators and carries out the accept / reject logic. We store the results in a vector called posterior_thetas which are initialized to NA.

```r
metropolis_algo <- function(samples, theta_start, sd){
  # declarations
  theta_curr <- theta_start
  # vector of NAs to store sampled parameters
  posterior_thetas <- rep(NA, times = samples)

  for (i in 1:samples){
    # proposal distribution
    theta_prop <- rnorm(n = 1, mean = theta_curr, sd = sd)

    # if proposed parameter is outside range, set to current value. Else keep proposed value
    theta_prop <- ifelse((theta_prop < 0 | theta_prop > 1), theta_curr, theta_prop)

    # bayes numerators
    posterior_prop <- dcauchy(theta_prop, location = 0, scale = 1) *
      dnorm(data_y, mean = theta_prop, sd = 1)
    posterior_curr <- dcauchy(theta_curr, location = 0, scale = 1) *
      dnorm(data_y, mean = theta_curr, sd = 1)

    # calculate probability of accepting
    p_accept_theta_prop <- min(posterior_prop / posterior_curr, 1.0)

    rand_unif <- runif(n=1)

    # probabilistically accept proposed theta
    theta_select <- ifelse(p_accept_theta_prop > rand_unif, theta_prop, theta_curr)
    posterior_thetas[i] <- theta_select

    # reset theta_curr for the next iteration of the loop
    theta_curr <- theta_select
  }
  return(posterior_thetas)
}
```

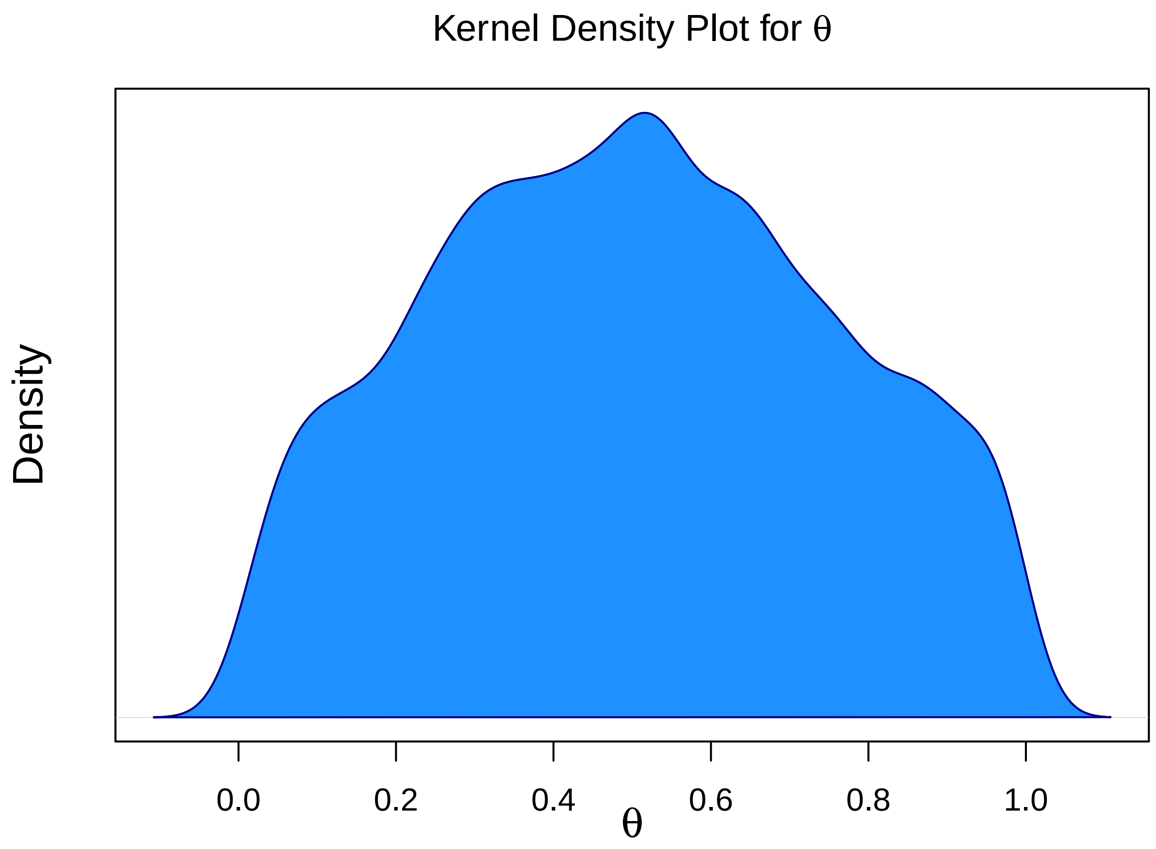
Now we can try 10,000 samples with a starting value of 0.9 and a sd for our normal proposal distribution of 0.05

```
set.seed(999)
posterior_thetas <- metropolis_algo(samples = 10000, theta_start = 0.9, sd = 0.05)
```

Lets take a look at the kernel density estimate of the posterior.

```
opar <- par()
par(mar = c(2.5, 3.5, 3, 2.1), mgp = c(1.7, 1, 0))

d <- density(posterior_thetas)
plot(d, main = expression(paste("Kernel Density Plot for ", theta)),
     xlab = expression(theta), ylab = "Density", yaxt = "n", cex.lab = 1.3)
polygon(d, col = 'dodgerblue1', border = 'dark blue')
```



Gibbs Sampler

Problem

Suppose we have a **Zero-Inflated Poisson Model**. In this model, random data $X_1, ..., X_n$ are of the form $X_i = R_i Y_i$ where $Y_i \sim \text{Poisson}(\lambda)$ and $R_i \sim \text{Bernoulli}(p)$, and our samples are iid.

If given an outcome $x = (x_1, ..., x_n)$, our goal is to estimate λ and p.

A zero-inflated Poisson model is used when we have a random event containing excess zero-count data in unit time. For example, the number of insurance claims within a population for a certain type of risk would be zero-inflated by those people who have not taken out insurance against the risk and thus are unable to claim.

Using a hierarchical Bayes model:

$p \sim \text{Uniform}(0, 1)$ | Prior for p

$(\lambda|p) \sim \text{Gamma}(a, b)$ | Prior for λ

$(r_i|p,\lambda) \sim$ Bernoulli(p) | Independently (from model above)

$(x_i|r,\lambda,p) \sim$ Poisson(λr_i) | Independently (from model above)

Given a, b, $r = (r_1, ..., r_n)$, we have the posterior:

$f(x,r,\lambda,p) = \frac{b^\alpha \lambda^{\alpha-1} e^{-b\lambda}}{\Gamma(\alpha)} \Pi_{i=1}^n \frac{e^{-\lambda r_i}(\lambda r_i)^{x_i}}{x!} p^{r_i}(1-p)^{1-r}$

We want to use Gibbs sampling to sample from our posterior pdf $f(\lambda, p, r|x)$. First though, we must learn the full conditional distributions for λ, p, and r_i.

Given $f(x,r,\lambda,p)$, the full conditional densities are all proportial to our joint density as functions of λ, p, r_i. For instance :

$\pi(\lambda|p,r,x) \propto \lambda^{\alpha-1}e^{-b\lambda}\Pi_{i=1}^n e^{-\lambda r_i}(\lambda)^{x_i} \propto \lambda^{\alpha-1}e^{-b\lambda}e^{\lambda\sum_i r_i}\lambda^{\sum_i x_i}$

Where only terms depending on λ are kept. Then

$\pi(\lambda|p,r,x) \propto \lambda^{\alpha-1+\sum_i x_i}e^{-\lambda[b+\sum_i r_i]}$. Therefore

$\lambda|p,r,x \sim$ Gamma$(a + \sum_i x_i, b + \sum_i r_i)$

Similarly for p:

$\pi(p|\lambda,r,x) \propto \Pi_{i=1}^n p^{r_i}(1-p)^{1-r_i} = p^{\sum_i r_i}(1-p)^{n-\sum_i r_i}$, leading to:

$p|\lambda,r,x \sim$ Beta$(1 + \sum_i r_i, n + 1 - \sum_i r_i)$.

Similarly for r_i:

Note that only the x_i's for which $r_i = 1$ need to be simulated. We can write the full conditional density for r_i as such:

$\pi(r_i|\lambda,x,p) \propto \Pi_{i=1}^n e^{-r_i\lambda}(\lambda r_i)p^{r_i}(1-p)^{1-r_i} \propto \frac{pe^{-\lambda}}{pe^{-\lambda}+(1-p)I\{x_i=0\}}$. Then

$r_i|\lambda,p,x \sim$ Bernoulli$(\frac{pe^{-\lambda}}{pe^{-\lambda}+(1-p)I\{x_i=0\}})$.

Now, given our 3 full conditionals, we can run our gibbs sampler:

```r
# simulate x_i
n = 10^5
p = 0.3
lam = 2
r = as.integer(runif(n) < p)
x = rep(0, n)
x[r == 1] = rpois(sum(r == 1), lam)
```

```r
# Gibbs Sampler
a <- 2
b <- 2
T <- 10^4
lamb <- pe <- rep(0.5, T)

for (t in 2:T){
  r = (x == 0) * (runif(n) < 1 / (1 + (1 - pe[t-1]) / (pe[t-1]*exp(-lamb[t-1])))) + (x>0)
  lamb[t] = rgamma(1, a + sum(x), b + sum(r))
  pe[t] = rbeta(1, 1 + sum(r), n - sum(r) + 1)
}
```

```r
# plot
lambda_plot <- ggplot() + geom_histogram(aes(x = lamb, y = ..density..), color = "black", fill = "blue"

p_plot <- ggplot() + geom_histogram(aes(x = pe, y = ..density..), color = "black", fill = "green", alpha
  ggtitle("Given data with p = 0.3")

grid.arrange(lambda_plot, p_plot, ncol = 1)
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
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```
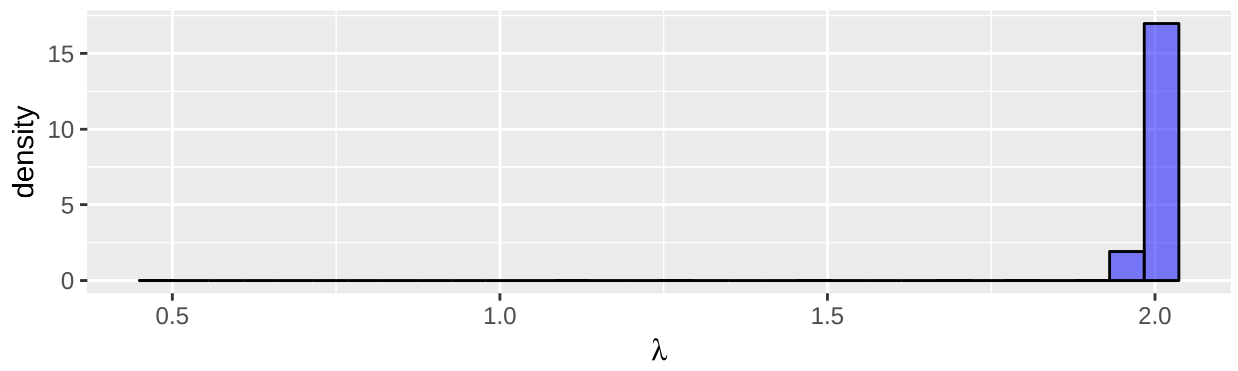
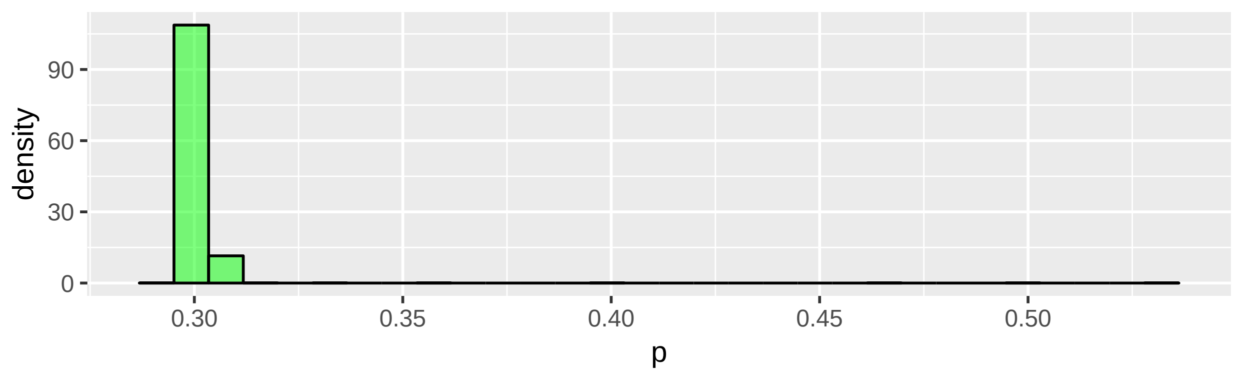

Given data with lambda = 2



Given data with p = 0.3


density
p