Weibull Distribution in Reliability Analysis

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Weibull Analysis for Reliability Engineering

Weibull analysis is the process of discovering the trends in product or system failure and using them to predict future failures in similar situations.

The primary advantage is that it can provide reasonably accurate analyses and failure forecasts with extremely small data samples.

The Weibull distribution can model early life failures, failures caused by chance causes and other degradation processes.

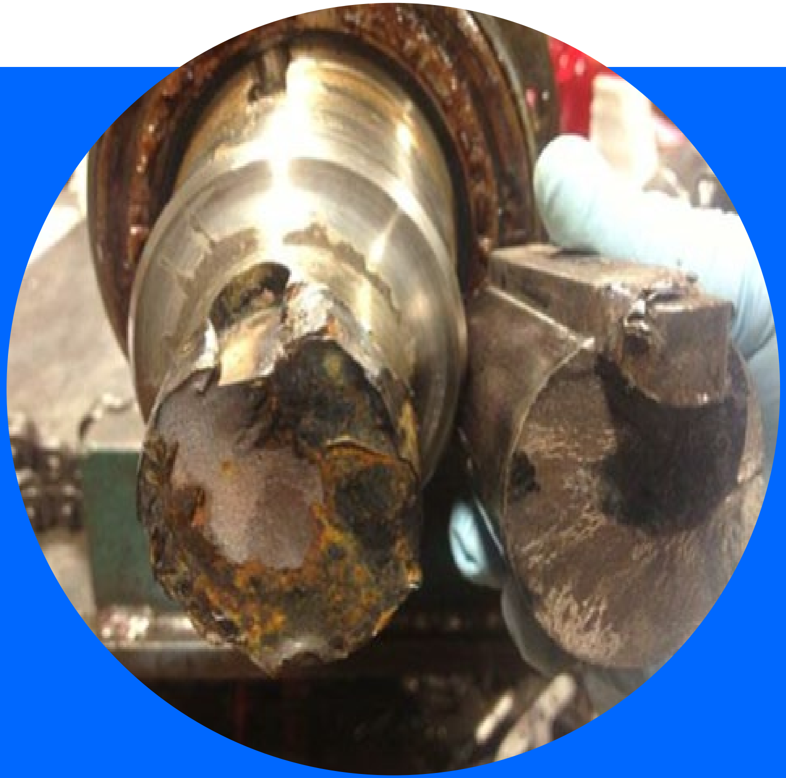
What to expect from Weibull Analysis?

What type of failure mechanism is the root cause?

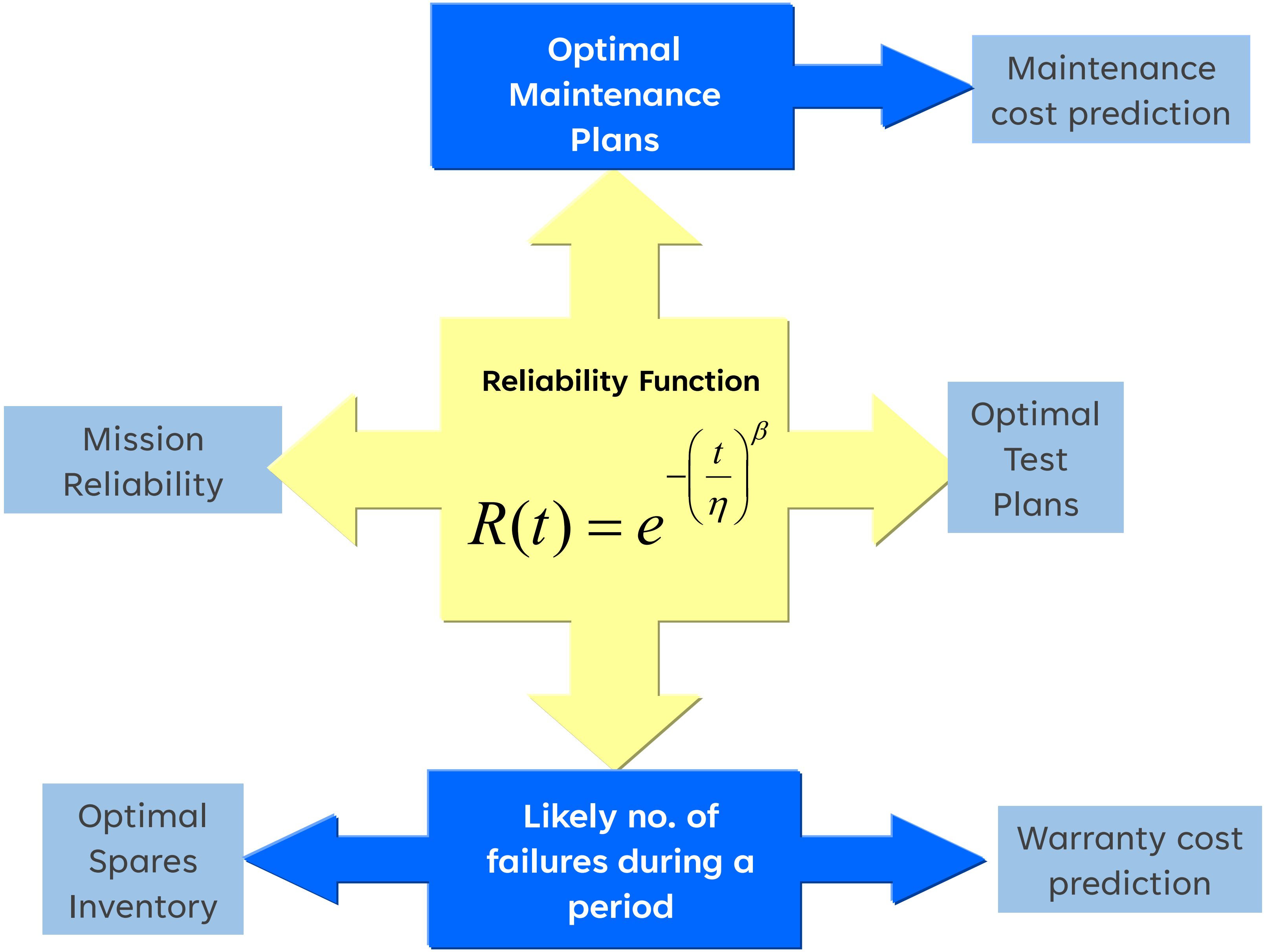
How many failures are expected?

How reliable is the existing part compared to a possible new design?

When should we replace an existing part with a new one to minimize maintenance costs?



RELIABILITY ESTIMATION



Optimal Maintenance Plans
Maintenance cost prediction
Reliability Function
$R(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}}$
Mission Reliability
Optimal Test Plans
Optimal Spares Inventory
Likely no. of failures during a period
Warranty cost prediction

18 June 1887 –12 October 1979

Historical Background

- Swedish engineer, scientist, and mathematician
- His 1951 article “A Statistical Distribution Function of Wide Applicability” (J. of Applied Mechanics, vol. 18: 293–297) discusses a few applications.
- The US Air Force funded Weibull’s research to study applicability of this distribution.
- Many others have contributed to refine the applicability of Weibull Distribution in life data analysis (LDA)

Weibull Cumulative Distribution Function

The Cumulative distribution function (CDF) provides the probability of failure, F(t), up to time t.

The Reliability at time t, R(t) will be:

$$F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

$$R(t) = 1 - F(t)$$

$$R(t) = e^{-\left[\frac{t}{\eta}\right]^{\beta}}$$

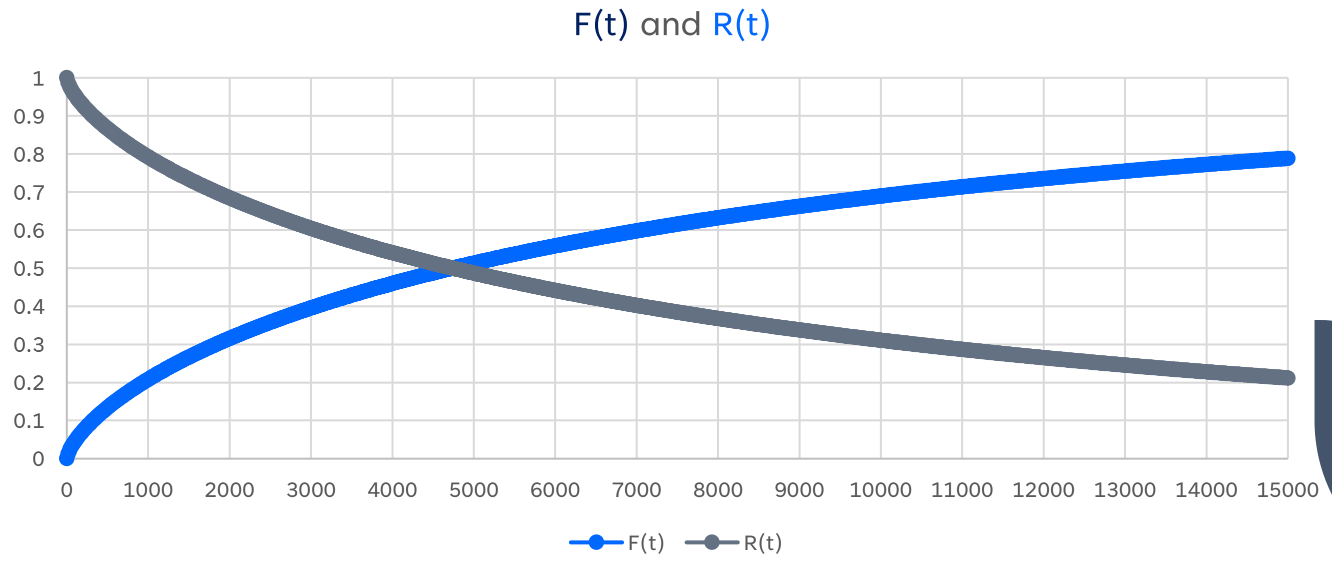
Where, β is the Shape Parameter,
η is the Scale Parameter.

Note: If β = 1, the Weibull reduces to the Exponential Distribution.

This is known as two-parameter Weibull Distribution. Occasionally, a third parameter is added when it is known as three-parameter Weibull.

Application Example

Suppose a company manufactures tires for superbikes. The company has. The company has supplied 12,000 tires to a motorcycle manufacturer. Now, we will see how the functions F(t) and R(t) change with various values of shape parameter β. The Scale parameter is assumed as 8000.



Weibull Shape Parameter: β

- The Weibull shape parameter, β, is also known as the Weibull slope
- The value of β is equal to the slope of the line in a probability plot
- Different values of the shape parameter can have marked effects on the behaviour of the distribution
- Some values of the shape parameter will cause the distribution equations to reduce to those of other distributions. For example, when $\beta = 1$, the pdf of the three-parameter Weibull reduces to that of the two-parameter exponential distribution.
- The parameter β is a pure number (i.e., it is dimensionless).

Application Example

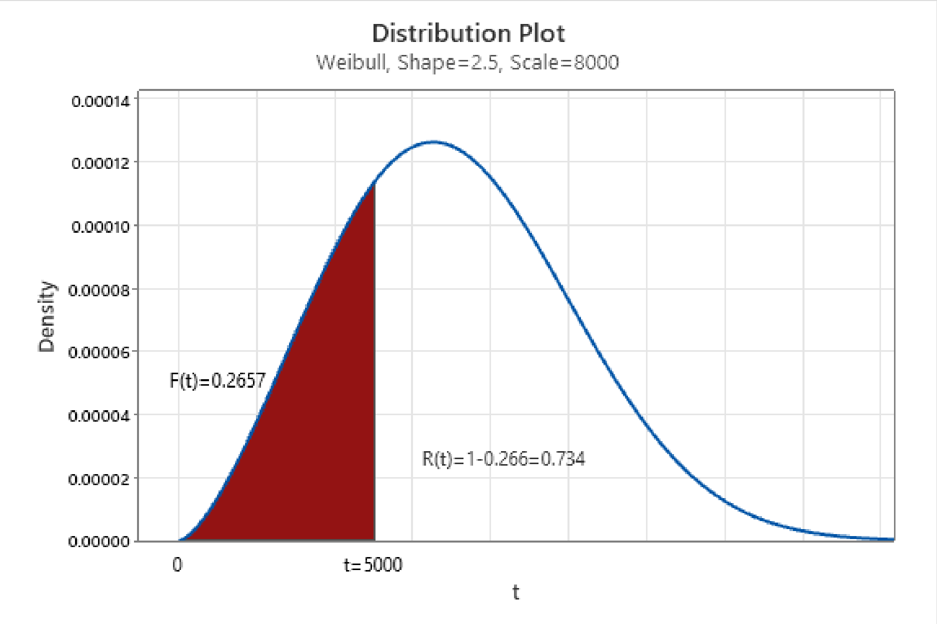
We will consider the same example of tires. The company has supplied 12,000 tires to a superbike manufacturer. What percent of tires are likely to fail within a warranty period of 5000 kilometers if ß=2.5? The Scale parameter is estimated at 8000 from the past field data.

$$F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

At 5000 kilometers, with β=2.5 and η=8000,

$$F(t) = 1 - e^{-\left(\frac{5000}{8000}\right)^{2.5}}$$

= 0.266 or 26.6 percent tires are expected to fail



As the batch size is 12000, 26.6% of 12000 that is 3188 tires are likely to fail

Weibull Probability Density Function

We need to differentiate the cumulative function F(t) to get the probability density function:

$$\frac{d}{dx}F(t) = f(t)$$

From calculus we have learnt,

$$\frac{d}{dx}(1) = 0, \frac{d}{dx}\left(e^{x}\right) = e^{x}$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$\frac{d}{dx}\left(e^{ax^{k}}\right) = ax^{k-1}e^{ax^{k}}$$

We now differentiate the CDF:

CDF $$F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

PDF $$f(t) = \frac{dF(t)}{dt} = 0 - \frac{d}{dx}\left[e^{-\left[\frac{t}{\eta}\right]^{\beta}}\right]$$

$$= \frac{d}{dx}\left[-e^{-\overbrace{\left[\frac{1}{\eta}\right]^{\beta}}^{a}[t]^{\beta}}\right] = -\left[\frac{1}{\eta}\right]^{\beta} t^{\beta-1}.\beta.e^{-\left[\frac{t}{\eta}\right]^{\beta}} \qquad \boxed{= \frac{\beta}{\eta^{\beta}} t^{\beta-1} e^{-\left(\frac{t}{\eta}\right)^{\beta}}}$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

Hazard Rate Function for Weibull Distribution

The hazard rate function is given by: $h(t) = \frac{f(t)}{R(t)}$

The probability density function of Weibull Distribution is given by

$$f(t) = \frac{\beta}{\eta^{\beta}} t^{\beta-1} e^{-\left[\frac{t}{\eta}\right]^{\beta}}$$

$$R(t) = e^{-\left[\frac{t}{\eta}\right]^{\beta}}$$

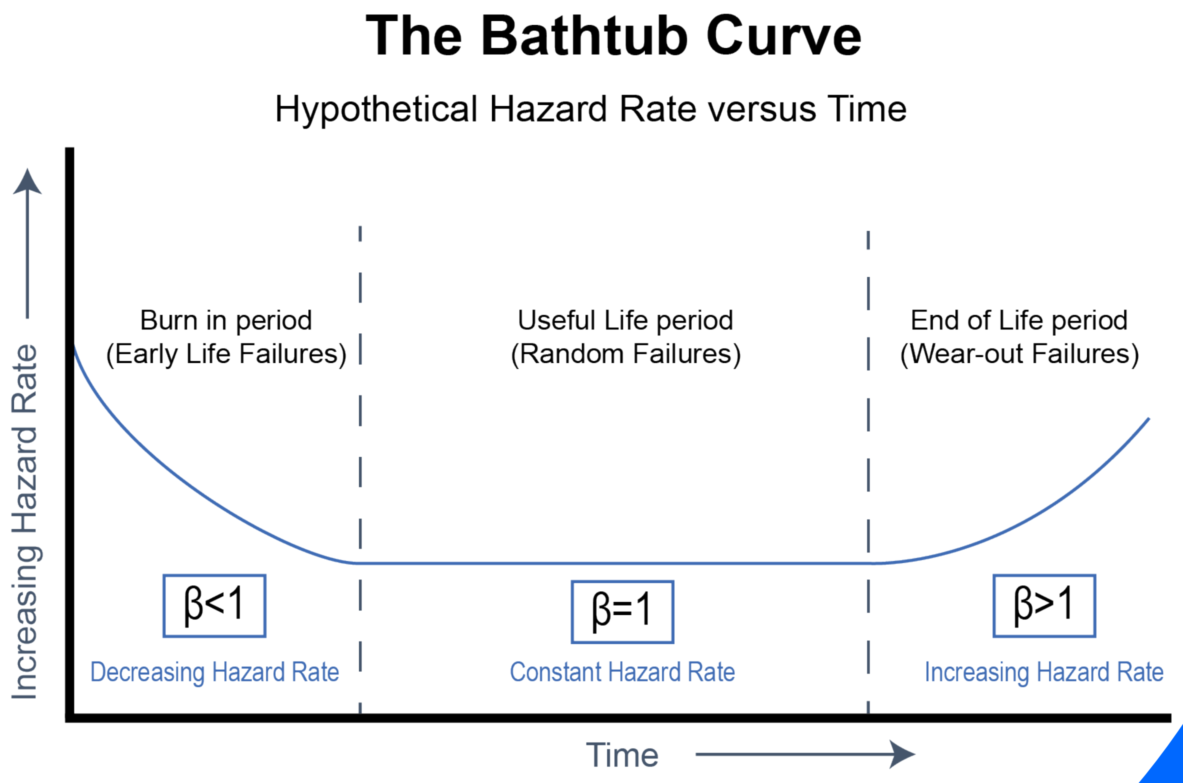
$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{\beta}{\eta^{\beta}} t^{\beta-1} e^{-\left[\frac{t}{\eta}\right]^{\beta}}}{e^{-\left[\frac{t}{\eta}\right]^{\beta}}} = \boxed{\frac{\beta}{\eta^{\beta}} t^{\beta-1}}$$

Thus, hazard rate for Weibull distribution varies with time.

Observing Hazard Rate Function

Observe in this equation that,

- Early failures ($\beta<1$) : When $\beta<1$, the hazard rate will reduce as t increases.
- Random failures ($\beta=1$) : When $\beta=1$, not a function of time, the hazard rate will be constant
- Wear-out failures ($\beta>1$) : Failure rates increasing, signs of wear and aging. When $\beta>1$, the hazard rate will increase as t increases.



Some interesting values of h(t)

$$h(t) = \frac{\beta}{\eta^{\beta}} t^{\beta-1}$$

At $\beta = 1$, the function $h(t)$ reduces to

$$h(t) = \frac{1}{\eta^{1}} t^{1-1} = \frac{1}{\eta}$$ which is a constant hazard rate

Thus, for β=1, Weibull Distribution becomes a special case of Exponential Distribution

At, $\beta = 2$, the function $h(t)$

$$h(t) = \frac{1}{\eta^{1}} t^{1-1} = \frac{1}{\eta}$$ which is a straight line

Equations that Define the Failure Data (2 parameters)

Probability Density Function

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{-\left(\frac{t}{\eta}\right)^{\beta}}, t > 0$$

Cumulative Distribution Function

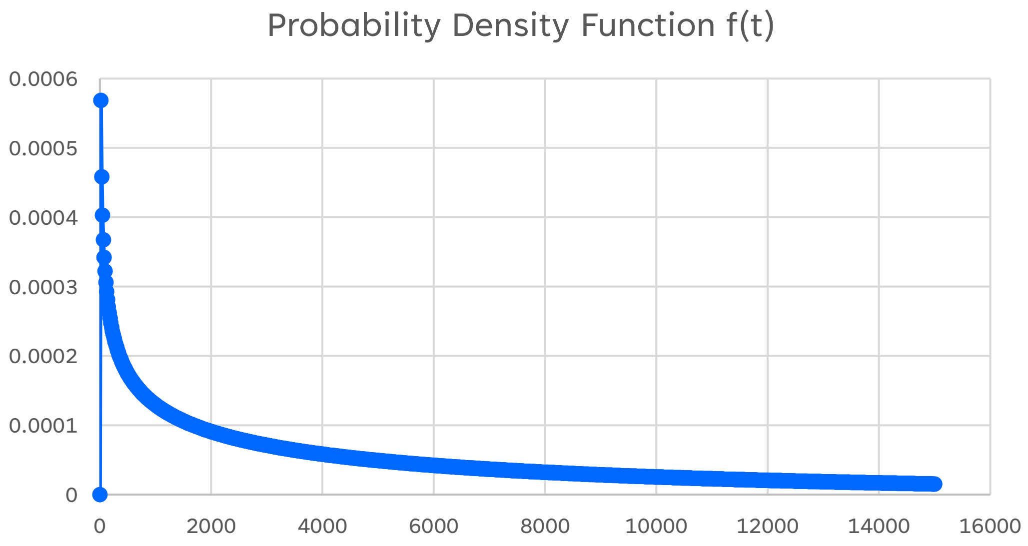
$$F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

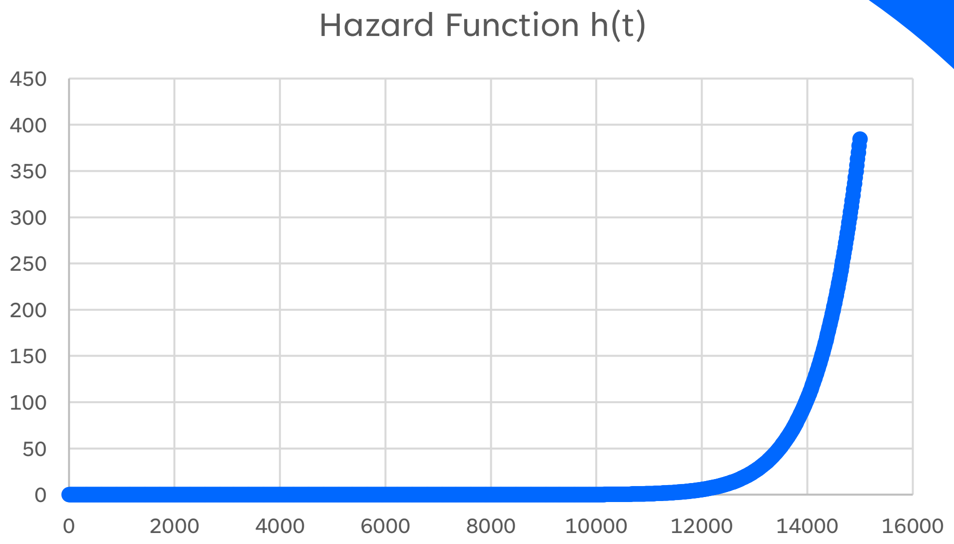
Failure Rate or Hazard Function

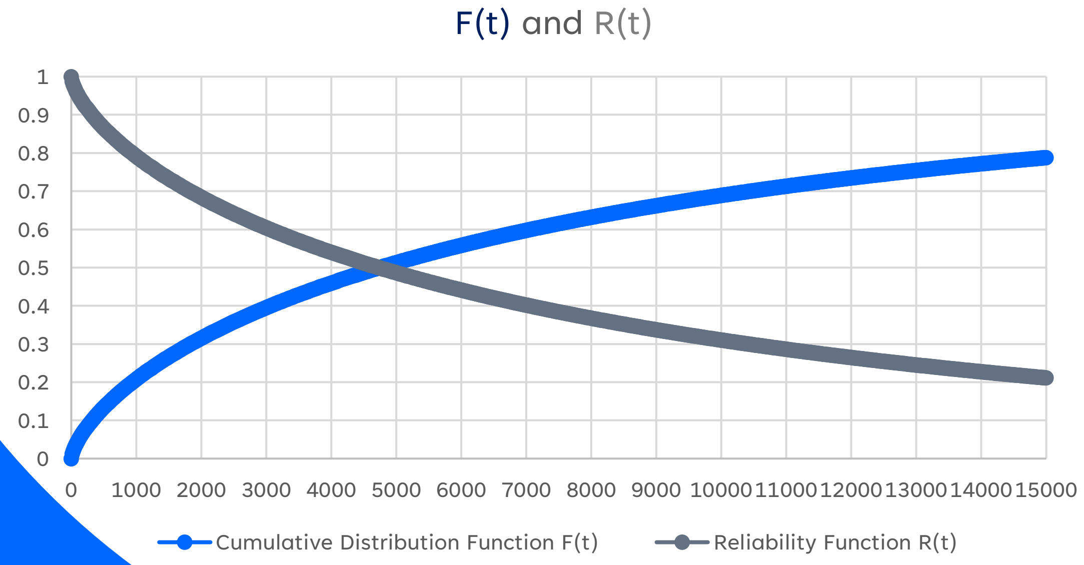
$$h(t) = \frac{\beta}{\eta}\left[\frac{t}{\eta}\right]^{\beta-1}$$

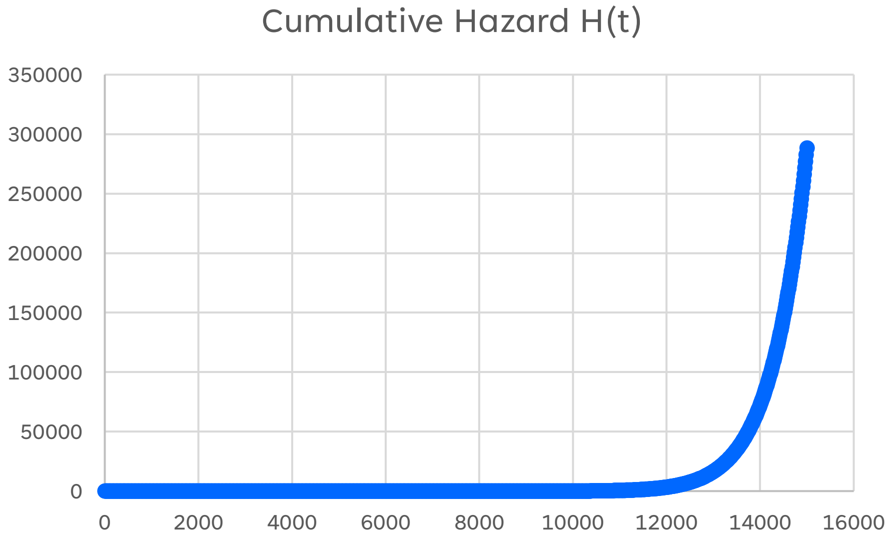
Cumulative Hazard Function

$$H(t) = \int_0^T h(t)\,dt = \left(\frac{t}{\eta}\right)^{\beta}$$


Probability Density Function f(t)
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0
0
2000
4000
6000
8000
10000
12000
14000
16000


Hazard Function h(t)
450
400
350
300
250
200
150
100
50
0
0
2000
4000
6000
8000
10000
12000
14000
16000


F(t) and R(t)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
11000
12000
13000
14000
15000
Cumulative Distribution Function F(t)
Reliability Function R(t)

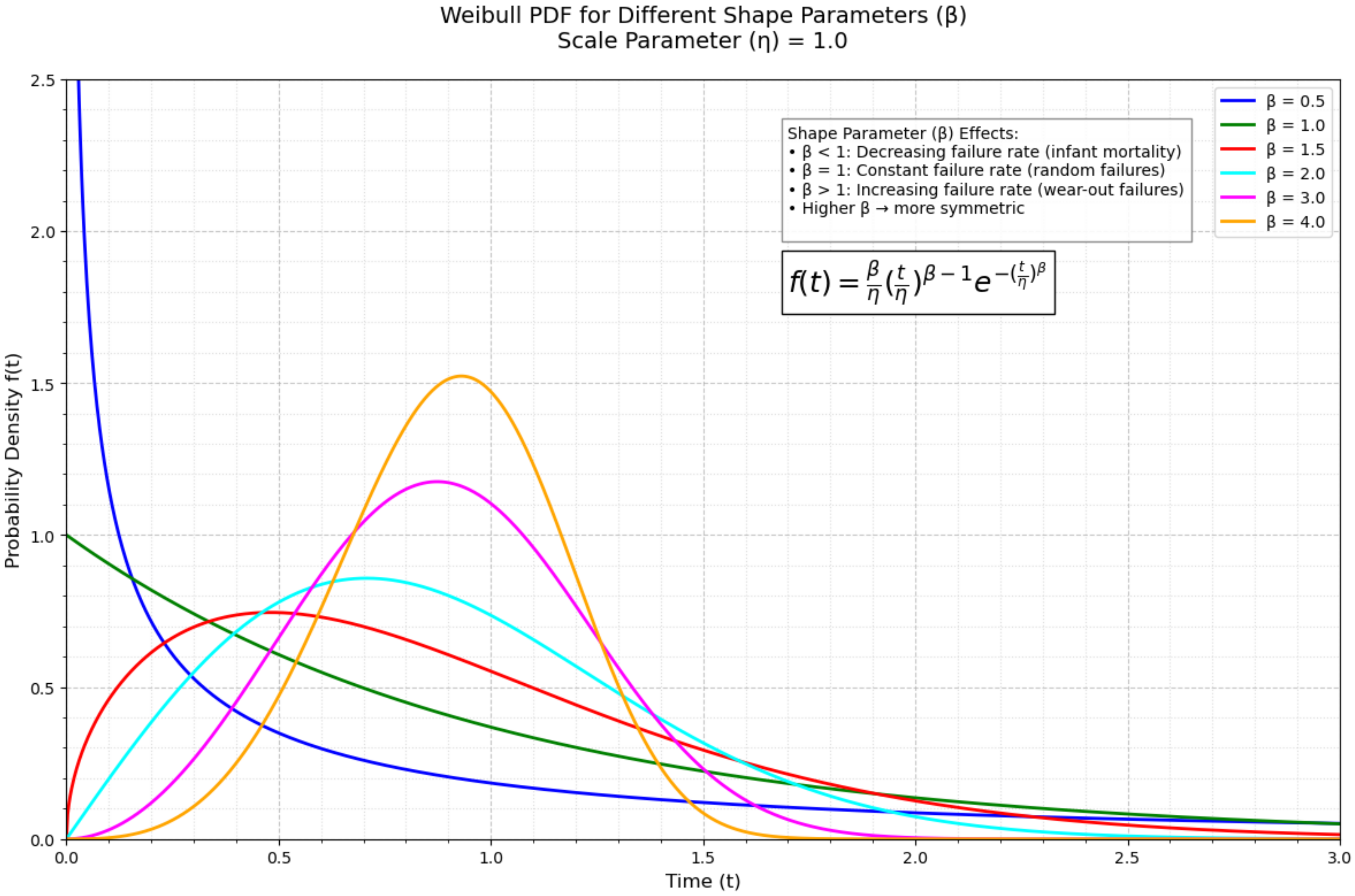

Cumulative Hazard H(t)
350000
300000
250000
200000
150000
100000
50000
0
0
2000
4000
6000
8000
10000
12000
14000
16000

Effect of Shape parameter Beta

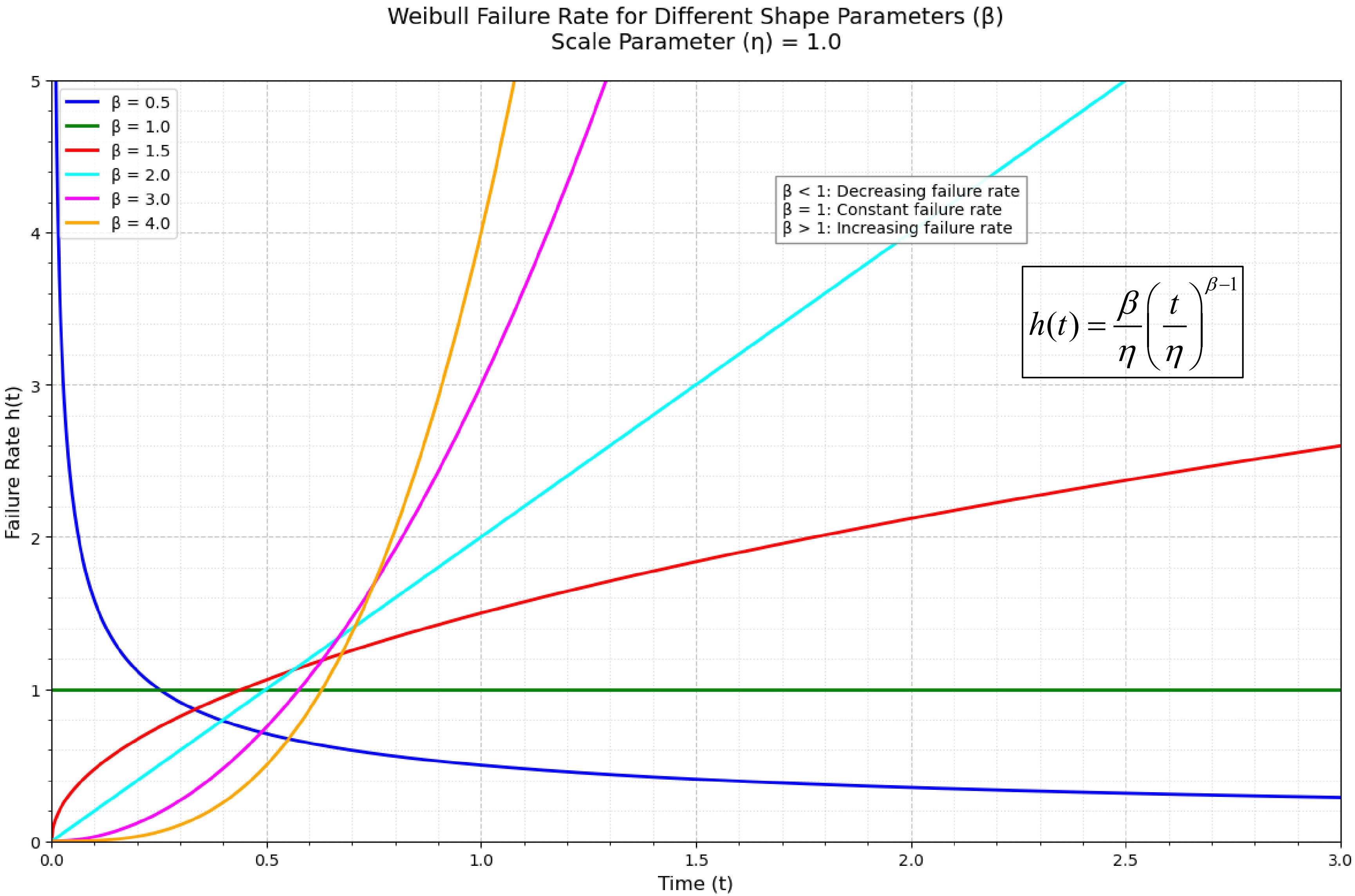
Depending upon the value of ß, the Weibull distribution function can take the form of the following distributions:

- β = 1.0: identical to the exponential distribution
- β = 2.0: identical to the Rayleigh distribution
- β = 2.5: approximates the lognormal distribution
- β = 3.6: approximates the normal distribution
- β = 5.0: approximates the peaked normal distribution

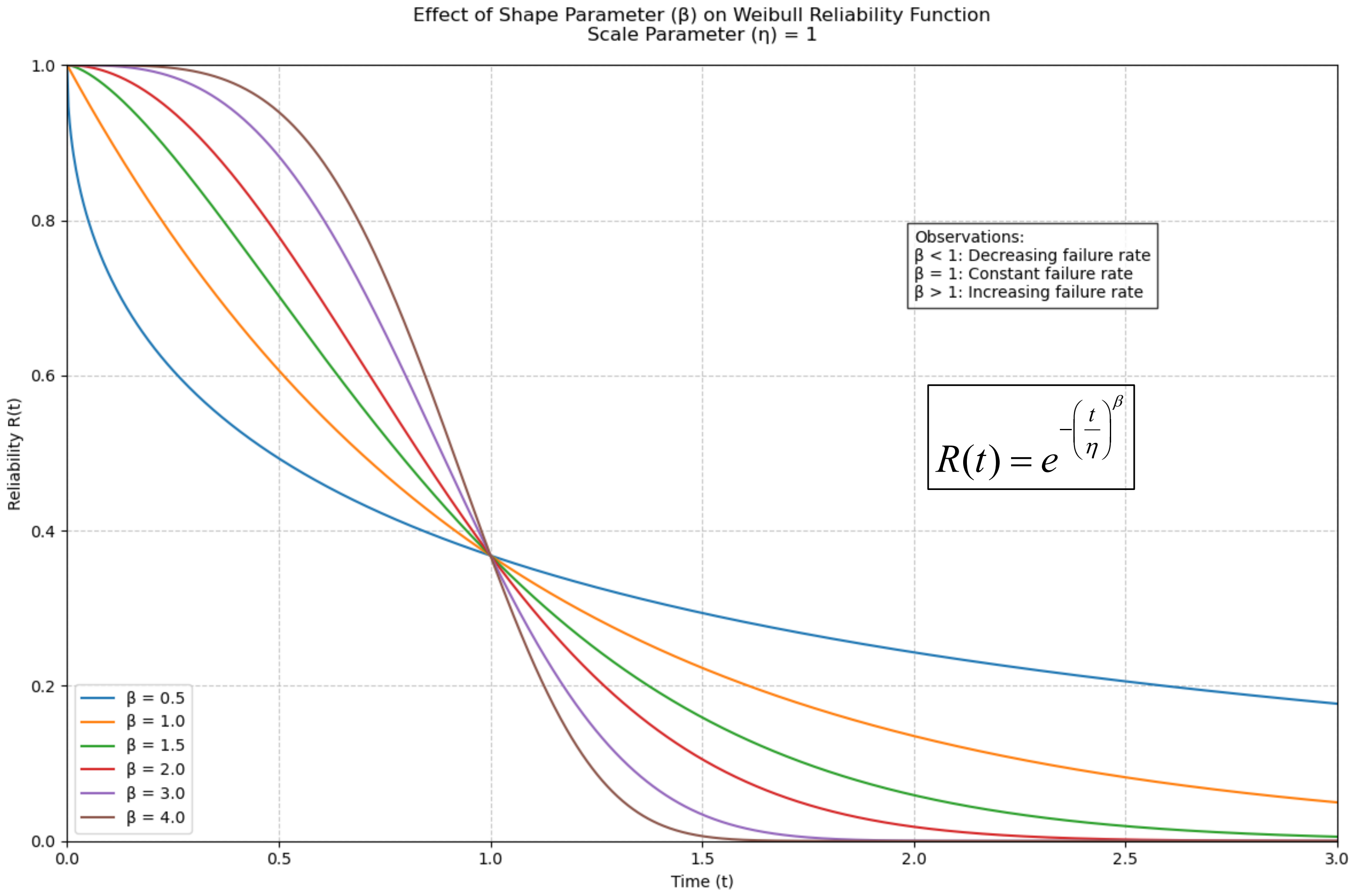
Effect of Shape Parameter (β)



Effect of β on Failure Rate Function



Effect of β on Reliability Function



Weibull Scale Parameter: η

At time = η, reliability will be:

$$R(t) = e^{-\left[\frac{t}{\eta}\right]^{\beta}} = e^{-\left[\frac{\eta}{\eta}\right]^{\beta}} = e^{(-1)^{\beta}} = e^{(-1)} = 0.368$$

$$\therefore F(\eta) = 1 - R(\eta) = 1 - 0.368 = 0.632$$

When $t = \eta$, $F(t) = 0.632$

Thus, at time equal to characteristic life η, reliability is 0.368 and probability of failure is 0.632. This means, 63.2% parts are expected to fail by characteristic life. So, the survival rate is 37%

Note that this is a fixed value and does not depend on value of β.

Weibull Distribution Application Example

Leakage at front oil seal in a system exhibits Weibull distribution with characteristic life of 1500 hours. The shape parameter is 0.75. What is its reliability at 1500 hours?

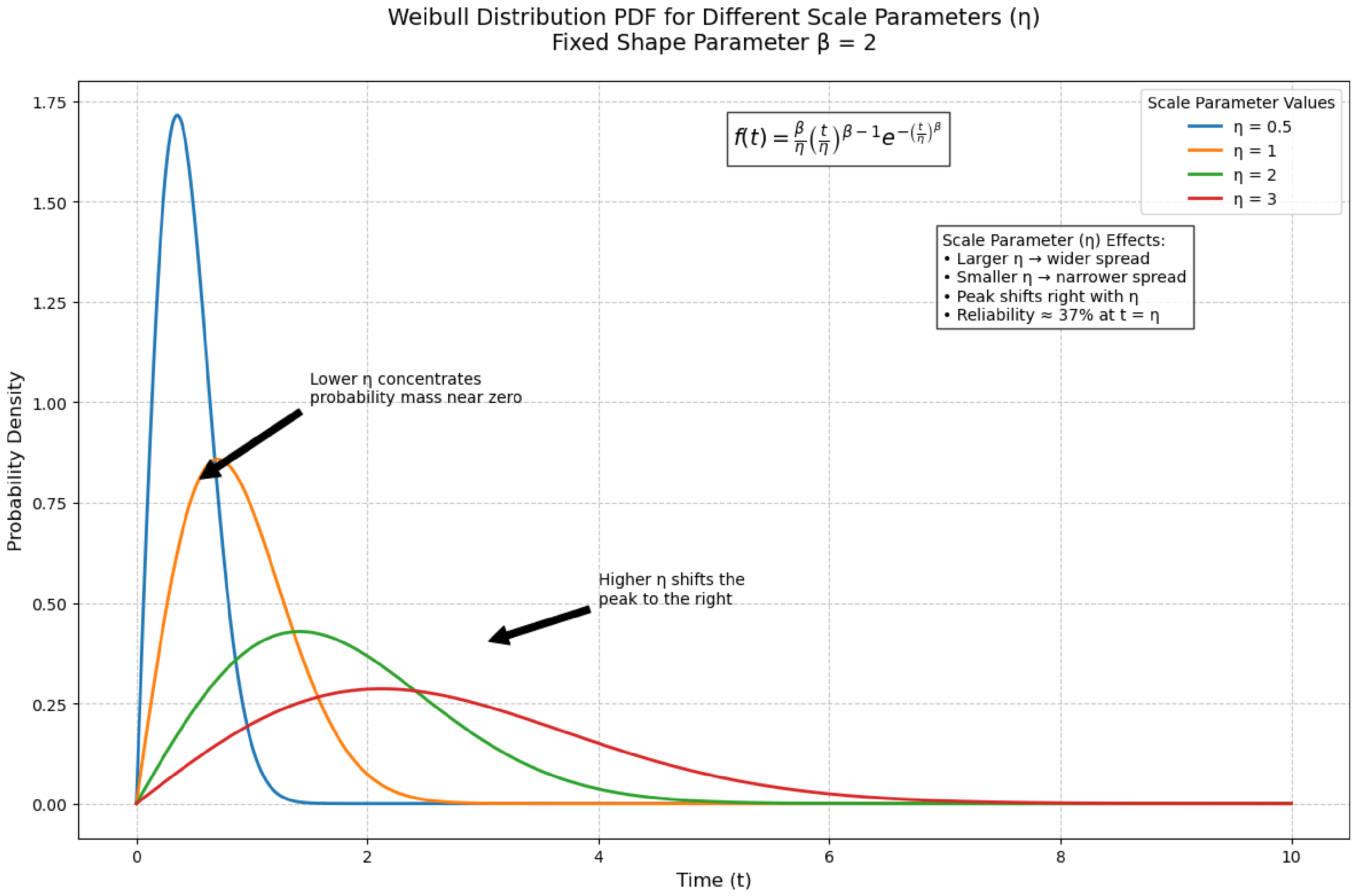
$$\eta = 1500, \beta = 0.75$$

For $t = 1500$ hours,

$$R(1500) = e^{-\left(\frac{t}{\eta}\right)^{\beta}} = e^{-\left(\frac{1500}{1500}\right)^{0.75}} = e^{-1} = 0.368$$

$$F(1500) = 1 - R(1500) = 1 - 0.368 = 0.632$$

Effect of varying Scale parameter (η)



B_{10} and B_x life

- The B_{10} life is time by which 10% failures occur.
- In general, B_x life is time by which x% failures occur.
- Some literature uses L_{10} or L_x instead of B_{10} or B_x .

Estimating the B_{10} life for Weibull Distribution

B_{10} life is defined as time till 10% failures or 90% reliability

$$R(t) = 0.9 = e^{\left(-\frac{t}{\eta}\right)^{\beta}} = e^{\left(-\frac{t}{1500}\right)^{0.75}}$$

Taking logarithms, we get

$$\ln(0.9) = -\left(\frac{t}{1500}\right)^{0.75}$$

$$-\ln(0.9)\times 1500^{0.75} = t^{0.75}$$

Solving this equation, we get $t = 75$ hours

Thus, we can expect that about 10% seals will leak by 75 hours.

Three parameter Weibull Distribution

A third parameter is sometimes added to the equation.

γ is the 'failure free' life

$$f(t) = \frac{\beta}{\eta^{\beta}} (t - \gamma)^{\beta - 1} e^{-\left[\frac{t-\gamma}{\eta}\right]^{\beta}}$$

$$R(t) = e^{-\left[\frac{t-\gamma}{\eta}\right]^{\beta}}$$

Examples of failure free life are batteries, printer cartridges, tires, ball bearings etc.

Location Parameter: γ

γ which is the 'failure free' life is observed on components such as:

Bearings as these cannot fail immediately due to fatigue or spalling and many rotations are required for damage to occur

Tires may not fail at zero kilometers as it takes many kilometers before these are worn out and fail

Application Example

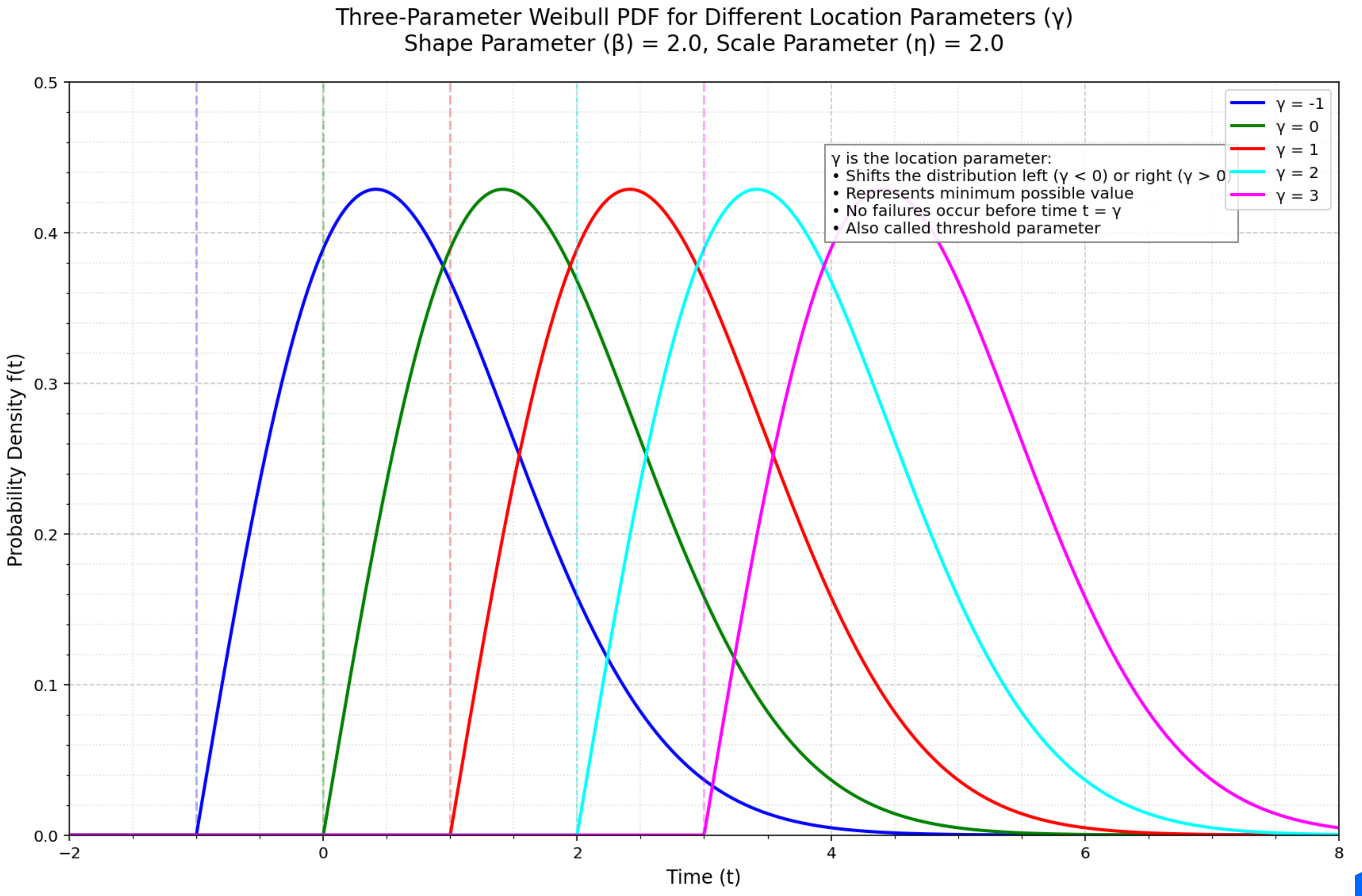
A component has characteristic life of 300 hours. The shape parameter is 2.5. What is its reliability at 126 hours if the location parameter is 50?

Scale η=300, shape β=2.5, location γ=50

$$R(126) = e^{-\left[\frac{t-\gamma}{\eta}\right]^{\beta}} = e^{-\left[\frac{126-50}{300}\right]^{2.5}} = 0.968$$

Note that shape parameter of 2.5 indicates wear-out period.

Effect of Location Parameter (γ)



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