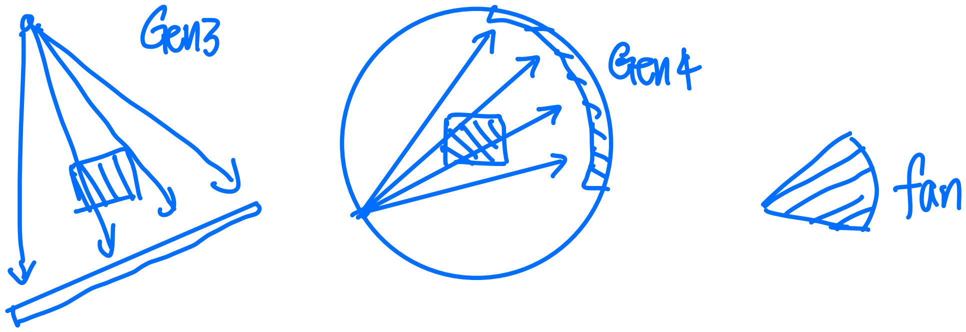
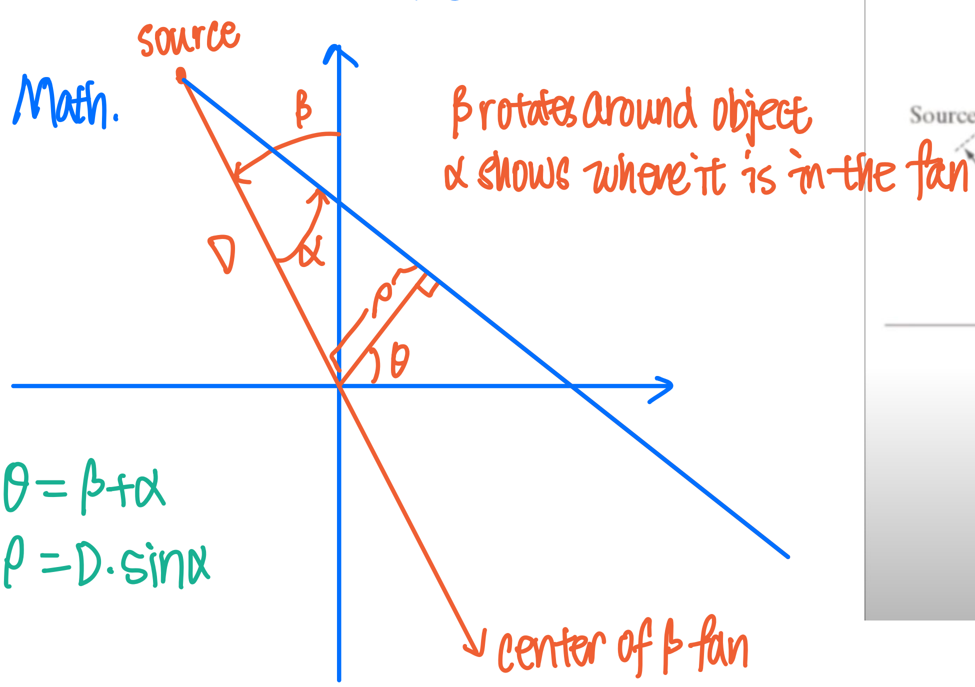
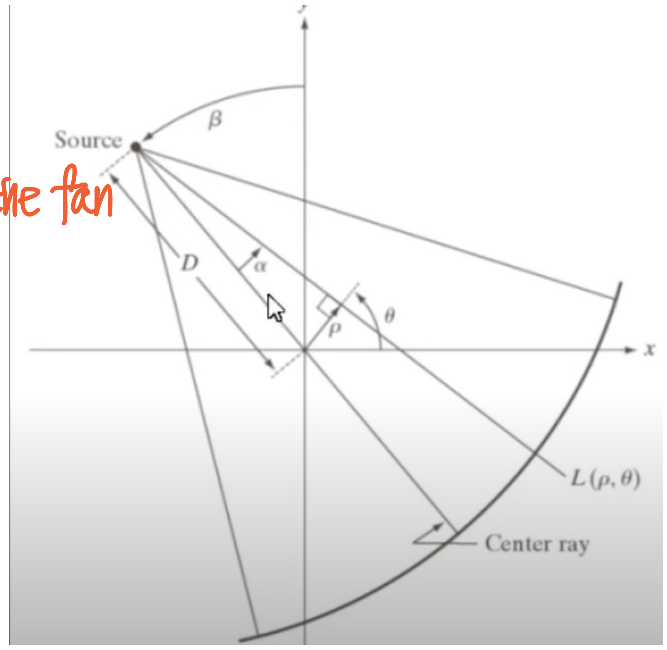
Fanbeam Reconstruction



CT scanning
computer tomography

Math.





$\theta = \beta + \alpha$

$\rho = D \cdot \sin\alpha$

each fan beam $p(\alpha, \beta)$ corresponds to some parallel beam $L(\rho, \theta)$

use change of coordinate

original im

radar transform

$$F(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} g(\rho,\theta)\, s(x\cos\theta + y\sin\theta - \rho)\, d\rho\, d\theta$$

↑ 1D spatial convolution with radar function for parallel beams

$$= \frac{1}{2}\int_0^{2\pi}\int_{-T}^{T} g(\rho,\theta)\, s(x\cos\theta + y\sin\theta - \rho)\, d\rho\, d\theta$$

= object with certain width, T is ρ_{max}

switch to polar, $(x,y)\to(r,\varphi)$

$x=r\cos\varphi,\ y=r\sin\varphi \qquad x\cos\theta+y\sin\theta=r\cos(\theta-\varphi)$

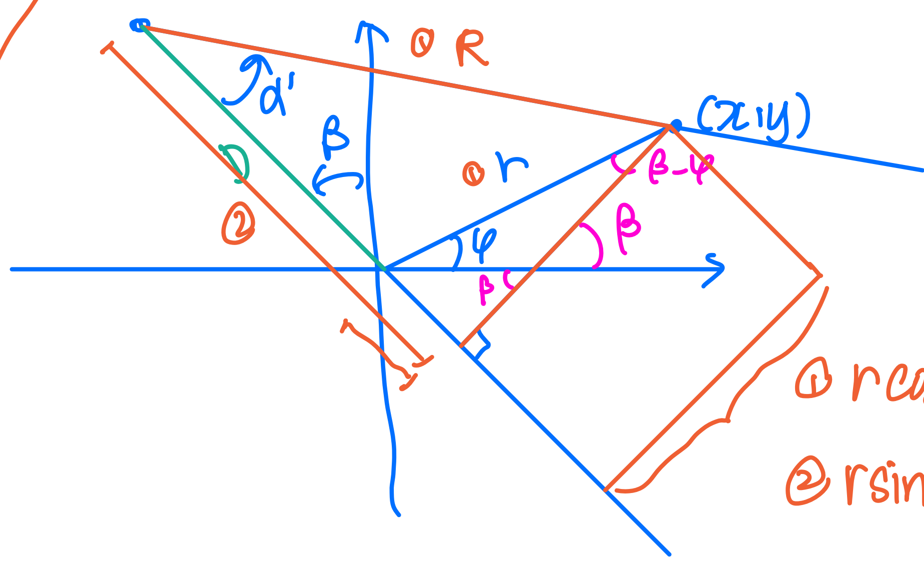
$$F(r,\varphi)=\frac{1}{2}\int_0^{2\pi}\int_{-T}^{T} g(\rho,\theta)\,\delta(r\cos(\theta-\varphi)-\rho)\,d\rho\,d\theta$$

Find $d\rho\,d\theta=\left|\det\begin{bmatrix}1 & 1\\ D\cos\alpha & 0\end{bmatrix}\right|\cdot d\alpha\,d\beta = D\cos\alpha\,d\alpha\,d\beta$

$$F(r,\varphi)=\frac{1}{2}\int_{-\sin^{-1}\left(\frac{T}{D}\right)}^{\sin^{-1}\left(\frac{T}{D}\right)}\int_{-\alpha}^{2\pi-\alpha}\underbrace{g(D\sin\alpha,\alpha+\beta)}_{P}\,\delta(r\cos(\alpha+\beta-\varphi)-\underbrace{D\sin\alpha}_{\rho})\cdot D\cos\alpha\,d\beta\,d\alpha$$

rename/symplify use new terms

$$=\frac{1}{2}\int_{-\alpha_m}^{\alpha_m}\int_0^{2\pi}P(\alpha,\beta)\,\delta(r\cos(\alpha+\beta-\varphi)-D\sin\alpha)\,D\cos\alpha\,d\beta\,d\alpha$$



represent (x,y) using α' with fixed β

① $r\cos(\beta-\varphi)=R\cdot\sin\alpha'$

② $r\sin(\beta-\varphi)+D=R\cos\alpha'$

$$^{*}\ r\cos(\alpha+\beta-\varphi)-D\sin\alpha = r\cos(\beta-\varphi)\cos\alpha - r\sin(\beta-\varphi)\sin\alpha - D\sin\alpha$$

$$=\underline{r\cos(\beta-\varphi)\cos\alpha-(r\sin(\beta-\varphi)+D)\sin\alpha}$$

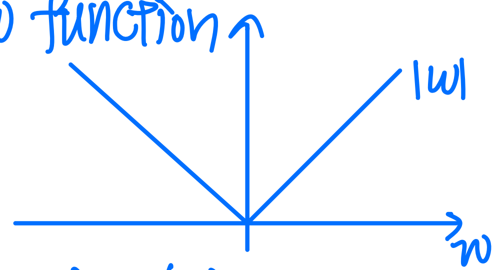
$$= R\sin\alpha'\cos\alpha - R\cos\alpha'\sin\alpha$$

$$= R\sin(\alpha'-\alpha)$$ and $R(r,\varphi,\beta)$, $\alpha(r,\varphi,\beta)$

Original $F(r,\varphi)$

* s is inverse F of the window function

$$s(t) = \int_{-\infty}^{\infty} |w| e^{i2\pi wt}\, dw$$



$$\therefore s(R\sin(\alpha'-\alpha)) = \int_{-\infty}^{\infty} |w| e^{i2\pi w(R\sin(\alpha'-\alpha))}\, dw$$

$$w' = \frac{wR\sin\alpha'-\alpha}{\cancel{\sin}\,\alpha'-\alpha} \qquad \therefore dw' = \frac{R\sin(\alpha'-\alpha)}{\cancel{\sin}\,(\alpha'-\alpha)} \cdot dw$$

$$\therefore s(R\sin(\alpha'-\alpha)) = \int_{-\infty}^{\infty} |w'| \cdot \left(\frac{\alpha'-\alpha}{R\sin(\alpha'-\alpha)}\right)^2 e^{i2\pi w'(\alpha'-\alpha)}\, dw'$$

$$= \left(\frac{\alpha'-\alpha}{R\sin(\alpha'-\alpha)}\right)^2 s(\alpha'-\alpha)$$

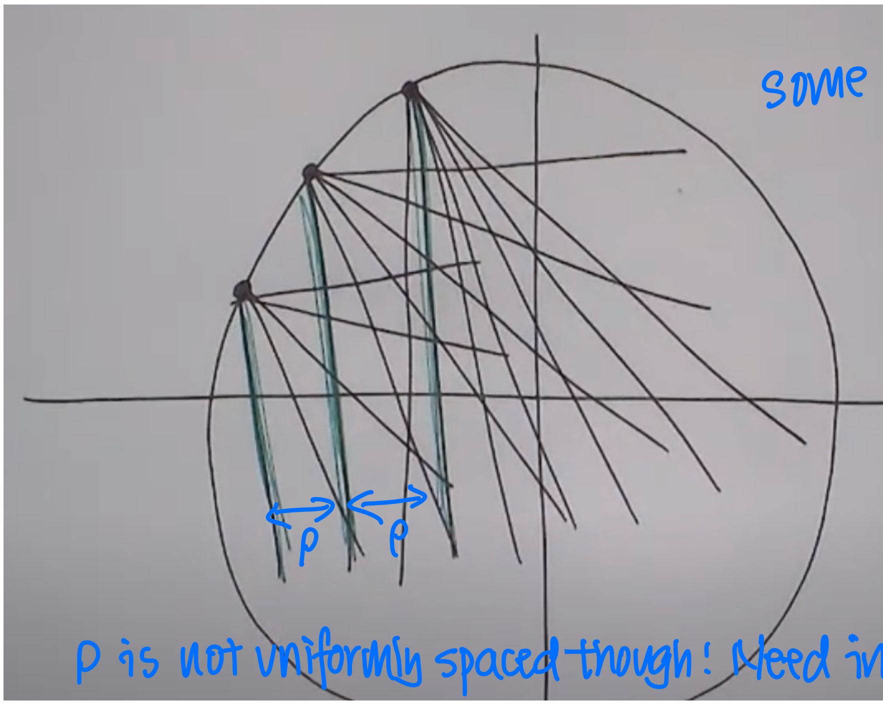
$$F(r,\varphi) = \int_0^{2\pi} \frac{1}{R^2}\left[\int_{-\alpha_m}^{\alpha_m} q(\alpha,\beta)\cdot h(\alpha'-\alpha)\, \underbrace{d\alpha}_{\text{dummy}}\right] d\beta$$

where $h(\alpha) = \frac{1}{2}\left(\frac{\alpha}{\sin\alpha}\right)^2 \cdot s(\alpha)$

$q(\alpha,\beta) = p(\alpha,\beta)\, D\cos(\alpha)$

convolution for a fixed β and a window function h like a filter!

Approximation



some of them are parallel

ρ is not uniformly spaced though! Need interpolation

resort fan beams from diff β into collection of beams

$P(\alpha, \beta) = g(\rho, \theta) = g(D \sin\alpha, \alpha+\beta)$

if a fan beams is equally spaced, $\Delta\alpha = \Delta\beta = \gamma$

$P(n\gamma, m\gamma) = g(D \sin n\gamma, (n+m)\gamma)$

→ nth ray in mth projection = nth ray in $(n+m)^{th}$ parallel projection

para2Fan() and Fan2para()

Fanbeam() radar transform