Torrecelli's Leaky Bucket

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1 Methods

The preliminary model that we used to model the drainage rate of water from a small hole in a water bottle was based on Torrecelli's Law, where the height of the water level at a given time is given by

$$h = \left[\sqrt{h_0} \frac{a}{A} \sqrt{\frac{g}{2}} t \right]^2 , \tag{1}$$

where h is the height of water in the bottle, h_0 is the initial height, a is the area of the hole in the bottle, A is the cross sectional area of the bottle, and t is elapsed time. This, however, is a highly idealized model and may not accurately describe the actual behavior of how the water level varies with time.

Data collection was straightforward. The bottle was filled, with the hole at the bottom being plugged. The stopwatch was started the moment that the hole was unplugged, and the time elapsed between each full-centimeter decrease of water level was recorded. Four data runs were performed, and the average time was calculated for each height.

2 Results

Our data always starts when $h_0 = 12cm$. The data that we collected can be seen in table 1.

We estimated a to be around $.275cm^2$, and the cross sectional area to be $30.88cm^2$. We assumed g was the standard physics class value of $9.80cm/s^2$.

```
%}
T = nanmean(t);
tmod = 0:.01:T(end);

g = 9.81; % m/sec
```

height(cm)	time 1 (sec)	time 2 (sec)	time 3 (sec)	time 4 (sec)	time 5 (sec)
11	0.82	1.13	1.98	1.99	1.73
10	1.46	2.34	2.73	3.12	2.65
9	2.16	3.37	3.74	4.22	3.84
8	3.19	4.4	4.92	5.5	4.9
7	4.4	5.65	5.95	6.65	6.14
6	5.57	6.73	6.93	8.02	7.32
5	6.67	8.16	8.55	9.61	8.51
4	8.09	8.72	10.2	11.3	10.17
3	9.43	11.33	12.16	13.31	12.2
2	11.3	13.53	14.07	16.43	14.39
1	13.49	16.78	16.75	18.47	17.25

Table 1: The data (minus the first and last data points because of uncertainty) that was collected during 5 different runs of the experiment.

```
g = g*100; % cm/sec

A = 30.876; % Area of the crosssection of the bottle. cm^2

a = .275; % Area of the hole. cm^2

h0 = h(1);

tt = 0:0.1:20;
h_est = (sqrt(h0) - a/A*sqrt(g/2)*tt).^2;

% Graph the average of the data and the Toricelli's model
f1 = figure(4);
clf
plot(tt, h_est)

hold on, plot(T-T(1),h,'r','linewidth',2)
grid on
ylabel('height')
xlabel('time')
legend('Torricelli''s Law','Average data')
%{
```

In figure 1, we show the average of our data runs and what Torrecelli's Law would predict. As can be seen, there seems to be a decent discrepancy between the model and the measurements. So, maybe we can take a look at the model and find a better model.

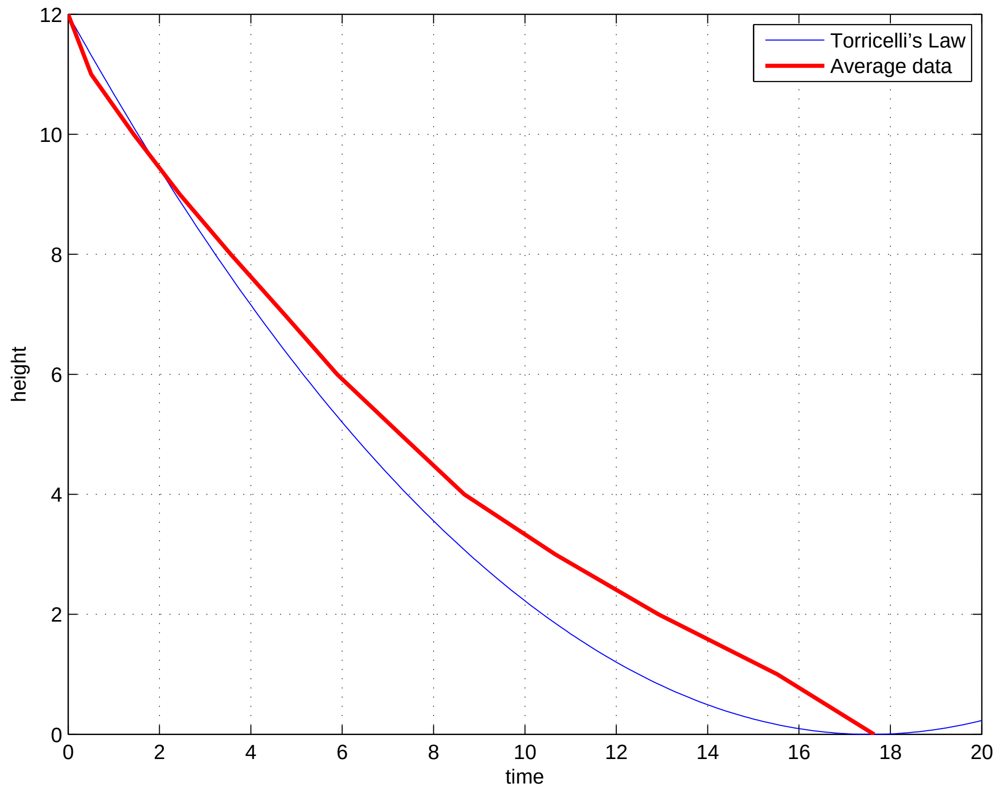

Figure 1: The comparison to the average of our runs and what Torrecelli's law predicts.

3 Dimensional Analysis

So, enter dimensional analysis. We first identify our parameters that we are working with and form the *dimension matrix*. We end up with the following

$$\begin{array}{c} \overbrace{\begin{array}{cccccc} h_0 & h & a & A & t & g \end{array}}^{\text{parameters}} \\ \mathbf{A} = \left.\begin{bmatrix} 1 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}\begin{array}{c} L \\ T \end{array}\right\} \text{dimensions} \end{array} \tag{2}$$

The rank of $\mathbf{A}$ is 2, thus we have 4 independent dimensionless quantities (Π_i) that we can use. In order to make things easier, we kept the constant parameters all in their own Π_i's. Thus, we found the dimensionless quantities

$$\Pi_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{h}{h_0} \qquad \Pi_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} = \frac{a}{A^2} \qquad \Pi_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \frac{t^2 g}{h_0} \qquad \Pi_4 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} = \frac{h_0^2 a}{A^2}. \tag{3}$$

We will assume

$$f(\Pi_1) = 0, \quad \text{where} \quad \Pi_1 = g\left(\Pi_2, \Pi_3, \Pi_4\right). \tag{4}$$

Then, let us say that $g(\dot{)}$ can be written as

$$g\left(\Pi_2, \Pi_3, \Pi_4\right) = G\left(\Pi_2\right) + H\left(\Pi_3\right) + I\left(Pi_4\right), \tag{5}$$

This leads us to the equation

$$\begin{aligned} \Pi_1 &= G\left(\Pi_2\right) + H\left(\Pi_3\right) + I\left(\Pi_4\right), \\ h &= h_o\left(G\left(\Pi_2\right) + H\left(\Pi_3\right) + I\left(\Pi_4\right)\right) \\ &= G'\left(\Pi_2\right) + H'\left(\Pi_3\right) + I'\left(\Pi_4\right) \\ &= \underbrace{G'\left(\frac{a}{A^2}\right)}_{\text{constant}} + H'\left(\frac{t^2 g}{h_0}\right) + \underbrace{I'\left(\frac{h_0^2 a}{A^2}\right)}_{\text{constant}} \\ &= C + H'\left(\frac{t^2 g}{h_0}\right). \end{aligned} \tag{6}$$

The other dimensionless quantities only include the hole area a, the cross dimensional area A, and the initial height h_0, and since those quantities are all fixed as far as this experiment is concerned, we "wrapped" them up into the constants of the equations above so that we only effectively had to deal with one dimensional quantity, $\Pi_3 = t^2 \frac{g}{h_0}$.

3.1 A fit

Thus, we postulate that we can use a model that looks like

$$h = c_1 + c_2\sqrt{\frac{t^2 g}{h_0}} \tag{7}$$

or even more generally,

$$h = c_1 + c_2\left(\frac{t^2 g}{h_0}\right)^{c_3}. \tag{8}$$

```
%}

tor = (sqrt(h0)-a/A*sqrt(g/2).*tmod).^2; % torrecelli's model

% height = constant1 + constant2*sqrt(tmod.^2*g/h0): constant1 = c(1),
% constant 2 = c(2);

F = @(c,xdata) c(1)*h0 + c(2)*(xdata.^2*g/h0).^c(3);
c0 = [5,-1/16,1/2];

c = lsqcurvefit(F,c0,T-T(1),h);


fig = figure(1); % plotting everything
clf
hold on
plot(T-T(1),h,'ro',tmod,tor,'b--',tmod,F(c,tmod),'g-')
title('Models of the Leaky Bucket')
grid on
xlabel('Time (s)')
ylabel('Height (cm)')
legend('Data (mean of runs)','Torricelli''s Model','Model 1')
%{
```

Figure 2 shows the fit of this model compared to Torrecelli's Law and the mean of the data collected. We can see that this fit actually does better at predicting the data. The coefficients found are shown in table 2.

c_1	c_2	c_3
1.01754	-0.463616	0.32414

Table 2: The coefficients from model 1, as found in MATLAB .

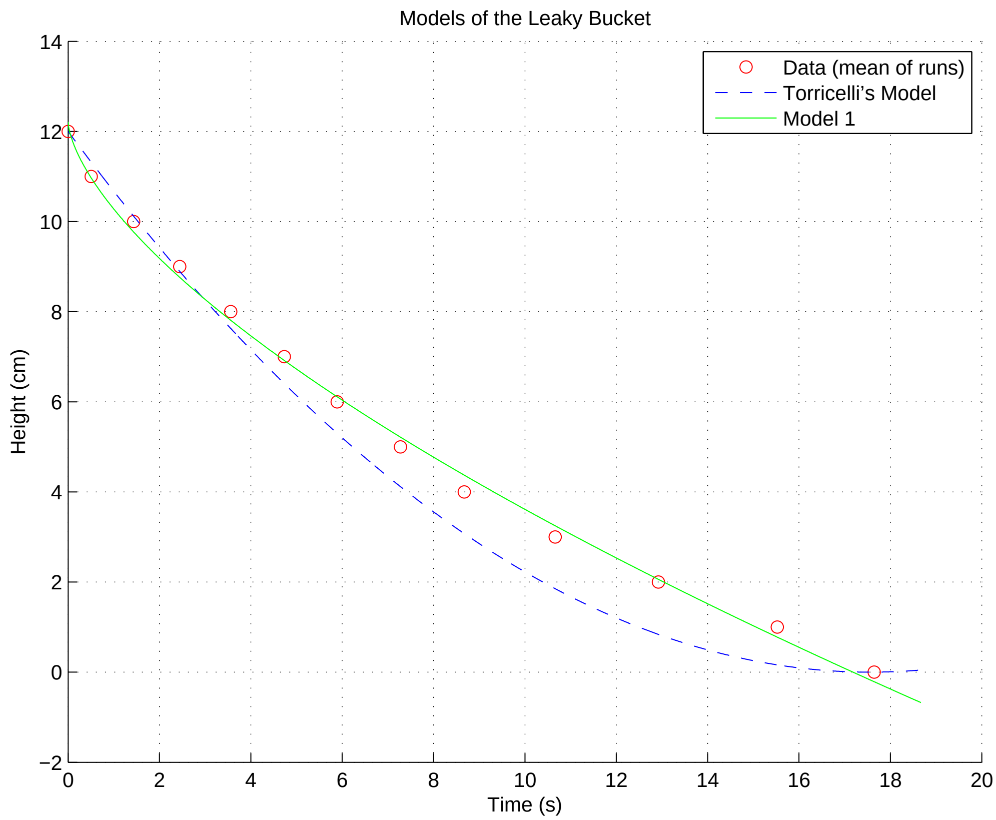

Figure 2: The fit of model 1 to the data compared to Torrecelli's Law. The mean of the collected data is shown also.

3.2 A different fit

The model above matches very well with the data, but doesn't seem to capture the end result and slow down all that well. We tried a different fit to see if we could come up with something better. This time we postulated the model as

$$H = c_1 h_0 e^{-c_2\left(\frac{t^2 g}{h_0}\right)^{c_3}} \tag{9}$$

Following the same procedure as before, we obtain the fit shown in figure 3.

```
%}

tor = (sqrt(h0)-a/A*sqrt(g/2).*tmod).^2; % torrecelli's model

% height = constant1 + constant2*sqrt(tmod.^2*g/h0): constant1 = c(1),
% constant 2 = c(2);

F2 = @(c,xdata) c(1)*h0.*exp(-c(2)*(xdata.^2*g/h0).^c(3));
% c02 = [5,-1/16,1/2];
c02 = [ 1.0180, 0.0025, 0.6833]; % A lucky first guess... ;)
```

```
%c2 = lsqcurvefit(F2,c02,T-T(1),h);
c2 = nlinfit(T-T(1), h, F2, c02);

fig = figure(1); % plotting everything
clf
hold on
plot(T-T(1),h,'ro',tmod,tor,'b--',tmod,F(c,tmod),'g-',tmod,F2(c2,tmod),'m-')
title('Models of the Leaky Bucket')
grid on
xlabel('Time (s)')
ylabel('Height (cm)')
legend('Data (mean of runs)','Torricelli''s Model','Model 1','Model 2')
%{
```

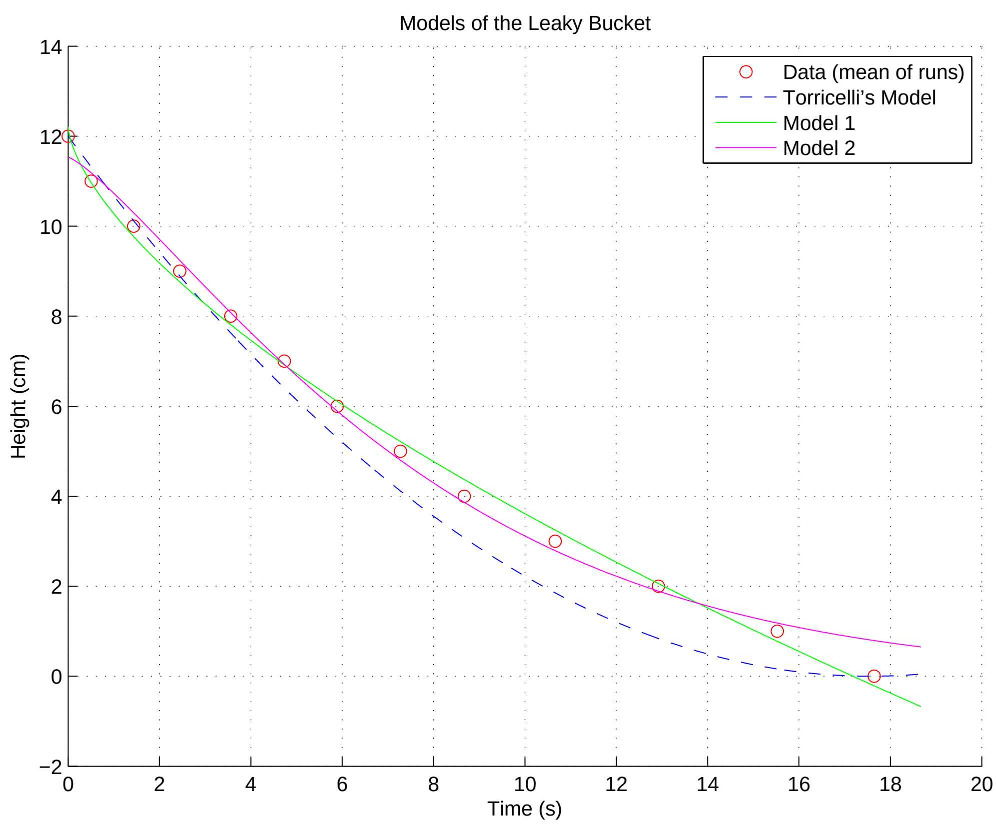

Figure 3: The fit of Models 1 and 2 to the data compared to Torrecelli's Law. The mean of the collected data is shown also.

The coefficients for the second model can be seen in table 3.

The first model fits the data points the best, but also it dips below zero for its end behavior, which is obviously not correct.

The second model also seems to fit the data well and has better end behavior than the first model.

4 Conclusion

Both of the proposed models fit the data better than Torrecelli's Law, suggesting that the are better for the modeling of this experiment. Thus, dimensional analysis proves to be a valid method to discover a different model that better fits the measured data in this experiment.

c_1	c_2	c_3
0.960885	0.00446806	0.630452

Table 3: The coefficients from model 1, as found in MATLAB .