ECE 6320: Homework 2

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Sorry, I didn't get to making the figures... Maybe next time.

1 Problem 1

Consider the inverted pendulum in Figure 1 . The equation of motion can be written as

Put in a figure here...

Figure 1: Inverted Pendulum

$$ml^2\ddot{\theta} = mgl\sin\theta - b\dot{\theta} + T \tag{1}$$

where T denotes the applied torque at the base and g is the gravitational acceleration. For this system assume the input and output to the system are the signals u and y defined as $T = sat(u)$, $y = \theta$ where "sat" denotes the unit-slope saturation function that truncates u at $+1$ and -1.

(a) Linearize this system around the equilibrium point for which $\theta = 0$.

Ok, first, let's set out some definitions.

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= g(x,u) \end{aligned} \tag{2}$$

is our nonlinear system. Let our state be

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \tag{3}$$

and our equation of motion is given by equation 1 above. Taking the derivative of our state, we get

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \frac{g}{l}\sin(\theta) - \frac{b}{ml^2}\dot{\theta} + \frac{1}{ml^2}T(u) \end{bmatrix} = f(x,u). \tag{4}$$

In addition, our output equation is

$$y = \theta. \tag{5}$$

Now, when we do the linearization around an equilbrium point, we will obtain equations of the form

$$\begin{aligned} \dot{\delta x} &= A\delta x + B\delta u \\ \delta y &= C\delta x + D\delta u, \end{aligned} \tag{6}$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{(\overline{x},\overline{u})} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(\overline{x},\overline{u})} \qquad C = \left.\frac{\partial g}{\partial x}\right|_{(\overline{x},\overline{u})} \qquad D = \left.\frac{\partial g}{\partial u}\right|_{(\overline{x},\overline{u})}, \tag{7}$$

and $\overline{x} = x^{eq}$, and $\overline{u} = u^{eq}$. So for this system, we get

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_{(\overline{x},\overline{u})} = \left.\left[\left(\frac{\partial f_i}{\partial x_j}\right)_{ij}\right]\right|_{(\overline{x},\overline{u})} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos(\bar{\theta}) & \frac{-b}{ml^2} \end{bmatrix} \\ B &= \left.\frac{\partial f}{\partial u}\right|_{(\overline{x},\overline{u})} = \left.\left[\left(\frac{\partial f_i}{\partial u_j}\right)_{ij}\right]\right|_{(\overline{x},\overline{u})} = \begin{bmatrix} 0 \\ \frac{1}{ml^2}\frac{\partial T(u)}{\partial u} \end{bmatrix} \\ C &= \left.\frac{\partial g}{\partial x}\right|_{(\overline{x},\overline{u})} = \left.\left[\left(\frac{\partial g_i}{\partial x_j}\right)_{ij}\right]\right|_{(\overline{x},\overline{u})} = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ D &= \left.\frac{\partial g}{\partial u}\right|_{(\overline{x},\overline{u})} = \left.\left[\left(\frac{\partial g_i}{\partial u_j}\right)_{ij}\right]\right|_{(\overline{x},\overline{u})} = 0. \end{aligned} \tag{8}$$

The derivative of the $T(u)$ is

$$\frac{\partial T(u)}{\partial u} = \upsilon(u) = \begin{cases} 1 & -1 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Ok, now for $\bar{\theta} = 0$, We get the system

$$
\begin{aligned}
\dot{\delta x} &= \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & \frac{-b}{ml^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{v(u)}{ml^2} \end{bmatrix} \delta u \\
\delta y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x.
\end{aligned}
\tag{9}
$$

(b) Linearize this system around the equilibrium point for which $\theta = \pi$ (Assume that the pendulum is free to rotate all the way to this configuration without hitting the table).

Plugging in $\theta = \pi$ into equation 8, we get

$$
\begin{aligned}
\dot{\delta x} &= \begin{bmatrix} 0 & 1 \\ \frac{-g}{l} & \frac{-b}{ml^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{v(u)}{ml^2} \end{bmatrix} \delta u \\
\delta y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x.
\end{aligned}
\tag{10}
$$

(c) Linearize the system around the equilibrium point for which $\theta = \frac{\pi}{4}$.

And finally, pluggin in $\theta = \pi/4$ into equation 8, we obtain

$$
\begin{aligned}
\dot{\delta x} &= \begin{bmatrix} 0 & 1 \\ \frac{g\sqrt{2}}{2l} & \frac{-b}{ml^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{v(u)}{ml^2} \end{bmatrix} \delta u \\
\delta y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x.
\end{aligned}
\tag{11}
$$

2 Problem 2

A UAV flying in plane and constant altitude has linear velocity v and angular velocity ω can be modeled by the nonlinear system

$$
\begin{aligned}
\dot{p}_x &= v\cos\theta, \\
\dot{p}_y &= v\sin\theta, \\
\dot{\theta} &= \omega
\end{aligned}
$$

where (p_x, p_y) denote the position of UAV in the horizontal plane and θ is the UAVs orientation. Regard this a system with input $u := \begin{bmatrix} v & \omega \end{bmatrix}^T \in \mathbb{R}^2$.

(a) Construct a state-space model for this system with state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p_x \cos\theta + (p_y - 1)\sin\theta \\ -p_x \sin\theta + (p_y - 1)\cos\theta \\ \theta \end{bmatrix} \tag{12}$$

and output $y = \left[\begin{smallmatrix} x_1 & x_2 \end{smallmatrix}\right]^T \in \mathbb{R}^2$.

Well, we will start by computing the derivative of x

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} \dot{p}_x \cos(\theta) - p_x \sin(\theta)\dot{\theta} + \dot{p}_y \sin(\theta) + p_y \cos(\theta)\dot{\theta} + \cos(\theta)\dot{\theta} \\ -\dot{p}_x \sin(\theta) - p_x \cos(\theta)\dot{\theta} + \dot{p}_y \cos(\theta) - p_y \sin(\theta)\dot{\theta} + \sin(\theta)\dot{\theta} \\ \dot{\theta} \end{bmatrix} \\
&= \begin{bmatrix} v \cos(\theta)\cos(\theta) - p_x \sin(\theta)\omega + v \sin(\theta)\sin(\theta) + p_y \cos(\theta)\omega + \cos(\theta)\omega \\ -v \cos(\theta)\sin(\theta) - p_x \cos(\theta)\omega + v \sin(\theta)\cos(\theta) - p_y \sin(\theta)\omega + \sin(\theta)\omega \\ \omega \end{bmatrix} \\
&= \begin{bmatrix} v\left(\cos^2(\theta) + \sin^2(\theta)\right) + \left(-p_x \sin(\theta) + p_y \cos(\theta) + \cos(\theta)\right)\omega \\ -\left(p_x \cos(\theta) + p_y \sin(\theta) - \sin(\theta)\right)\omega \\ \omega \end{bmatrix} \\
&= \begin{bmatrix} v + x_2\omega \\ -x_1\omega \\ \omega \end{bmatrix} \\
&= f(x, u) \\
y &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= g(x, u).
\end{aligned} \tag{13}$$

We can then find the statespace as

$$\dot{x} = f(x,u) = \begin{bmatrix} v + x_2\omega \\ -x_1\omega \\ \omega \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{14}$$

$$y = g(x,u) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u = \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{15}$$

(b) Compute a local linearization for this system around the equilibrium point $x^{eq} = 0$ and $u^{eq} = \theta$.

A local linearization around a equilibrium point (x^{eq}, u^{eq}) can be found by

$$A = \left[\left(\frac{\partial f_i}{\partial x_j}\right)_{ij}\right|_{x^{eq},u^{eq}} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\Bigg|_{x^{eq},u^{eq}} \qquad B = \left[\left(\frac{\partial f_i}{\partial u_j}\right)_{ij}\right|_{x^{eq},u^{eq}} = \begin{bmatrix} 1 & x_2 \\ 0 & -x_1 \\ 0 & 1 \end{bmatrix}\Bigg|_{x^{eq},u^{eq}} \tag{16}$$

$$C = \left[\left(\frac{\partial g_i}{\partial x_j}\right)_{ij}\right|_{x^{eq},u^{eq}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\Bigg|_{x^{eq},u^{eq}} \qquad D = \left[\left(\frac{\partial g_i}{\partial x_j}\right)_{ij}\right|_{x^{eq},u^{eq}} = 0. \tag{17}$$

Thus, plugging in our values for ω, v, and x, we get

$$\dot{\delta x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u \tag{18}$$

$$\delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x. \tag{19}$$

(c) Show that $\omega(t) = v(t) = 1$, $p_x(t) = \sin t$, $p_y = 1 - \cos t$, and $\theta(t) = t, \forall t \geq 0$ is a solution to the system.

Plugging these values into our formula for x, we get

$$x = \begin{bmatrix} \sin(t)\cos(t) - \cos(t)\sin(t) \\ -\sin^2(t) - \cos^2(t) \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix}. \tag{20}$$

Then, taking the derivative of this with respect to t, we get

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{21}$$

Ok, now let's plug in the values for the trjectory into the equation 14 for $\dot{x}$ (from above)

$$\dot{x} = \begin{bmatrix} v + x_2\omega \\ -x_1\omega \\ \omega \end{bmatrix} = \begin{bmatrix} 0 + 0 \cdot 1 \\ -0 \cdot 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{22}$$

(d) Show that a local linearization of this system around this trajectory ($\omega(t) = v(t) = 1$, $p_x(t) = \sin t$, $p_y = 1 - \cos t$, and $\theta(t) = t$, $\forall\, t \geq 0$) results in an Linear Time Invariant (LTI) system.

Simlar to above with the linearization around the equilibrium point, we take derivatives to find the linearization around the trajectory specified. We obtain the matrices

$$A = \left[\left(\frac{\partial f_i}{\partial x_j}\right)_{ij}\right]\Bigg|_{x^{sol},u^{sol}} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\Bigg|_{x^{sol},u^{sol}} \qquad B = \left[\left(\frac{\partial f_i}{\partial u_j}\right)_{ij}\right]\Bigg|_{x^{sol},u^{sol}} = \begin{bmatrix} 1 & x_2 \\ 0 & -x_1 \\ 0 & 1 \end{bmatrix}\Bigg|_{x^{sol},u^{sol}} \tag{23}$$

$$C = \left[\left(\frac{\partial g_i}{\partial x_j}\right)_{ij}\right]\Bigg|_{x^{sol},u^{sol}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\Bigg|_{x^{sol},u^{sol}} \qquad D = \left[\left(\frac{\partial g_i}{\partial x_j}\right)_{ij}\right]\Bigg|_{x^{sol},u^{sol}} = 0. \tag{24}$$

Now we can plug in the trajectory and we will get the system

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u \tag{25}$$

$$\delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x, \tag{26}$$

which is an LTI, because the equations are linear and all the coefficient matrices are not time varying.

3 Problem 3

Consider the inverted pendulum in Figure 1

(a) Assume that you can control the system in torque, i.e., that the control input is $u = T$.

Design a feedback linearization controller (after feedback linearization use a simple PID controller) to derive the pendulum to the up right position. Use the following values for the parameters: $l = 1m, m = 1kg, b = 0.1Nm^{-1}s^{-1}$, and $g = 9.8ms^{-2}$. Modify *InvertedPendulum.zip* to show that your controller works. Remember that the *InvertedPendulum.zip* is a simulation for a inverted pendulum on a cart and you can still use it for a inverted pendulum by just giving zero input to the cart. You can also create your own simulator in Matlab-Simulink to show your work.

The equation of motion is

$$ml^2\ddot{\theta} = mgl\sin(\theta) + b\dot{\theta} + T, \tag{27}$$

which gives

$$\ddot{\theta} = \frac{g}{l}\sin(\theta) - \frac{b}{ml^2}\dot{\theta} + \frac{1}{ml^2}T. \tag{28}$$

We want $\frac{1}{ml^2}T$ to be able to cancel out all the terms of on the right hand of the equation above. Thus, our T wil be

$$T = ml^2\left(\frac{-g}{l}\sin(\theta) + \frac{b}{ml^2}\dot{\theta} + u_{lin}\right). \tag{29}$$

Now if we use a PID controller for the linear input portion of T, then we can stick that in and we'll get

$$T = ml^2\left(\frac{-g}{l}\sin(\theta) + \frac{b}{ml^2}\dot{\theta} - k_p\theta - k_d\dot{\theta}\right). \tag{30}$$

I set this up in the *InvertedPendulum* files. The parameters given wer in the file

```

clear all
close all

P.x0=[0.2; 0.05];% y theta ydot thetadot
P.g=9.8;
P.M=1; % cart mass kg
P.m=1; % pendulum bob weight
P.l=1; % length of pendulum rod meters
P.b=0.1;
```

I also found it necessary to modify the pendulum.m file in the following ways:

```
function [sys,x0,str,ts,simStateCompliance]=mdlInitializeSizes(P)
```

```
sizes.NumContStates  = 2;
sizes.NumDiscStates  = 0;
sizes.NumOutputs     = 2;
```

```
%
%=============================================================================
% mdlDerivatives
% Return the derivatives for the continuous states.
%=============================================================================
%
function sys=mdlDerivatives(t,x,u,P)
xdot=zeros(2,1);
theta=x(1);
thetadot=x(2);

m=P.m;
l=P.l;
g=P.g;
b=P.b;

xdot(1)=x(2);
xdot(2)=g*sin(theta)/l - b*thetadot/(m*l^2) + u/(m*l^2);
sys = xdot;
```

I also modified the simulink model, by adding in a PID controller with the PID parameters in paralell with the controller specified in the file controller.m:

```
function out=controller(in,P)
m=P.m;
g=P.g;
l=P.l;
b=P.b;

theta=in(1);
thetadot=in(2);


out=m*l^2*(-g/l*sin(theta) + b/(m*l^2)*thetadot);
```

I had difficulties getting the drawPendulum to work, so I just dumped the output to the workspace and plotted θ from there. The plot is shown in figure 2.

(b) Assume now that the pendulum is mounted on a cart and that you can control the cart's jerk, which is the derivative of its acceleration a. In this case, $T = -mla\cos(\theta), \dot{a} = u$. Design the feedback lineariza-

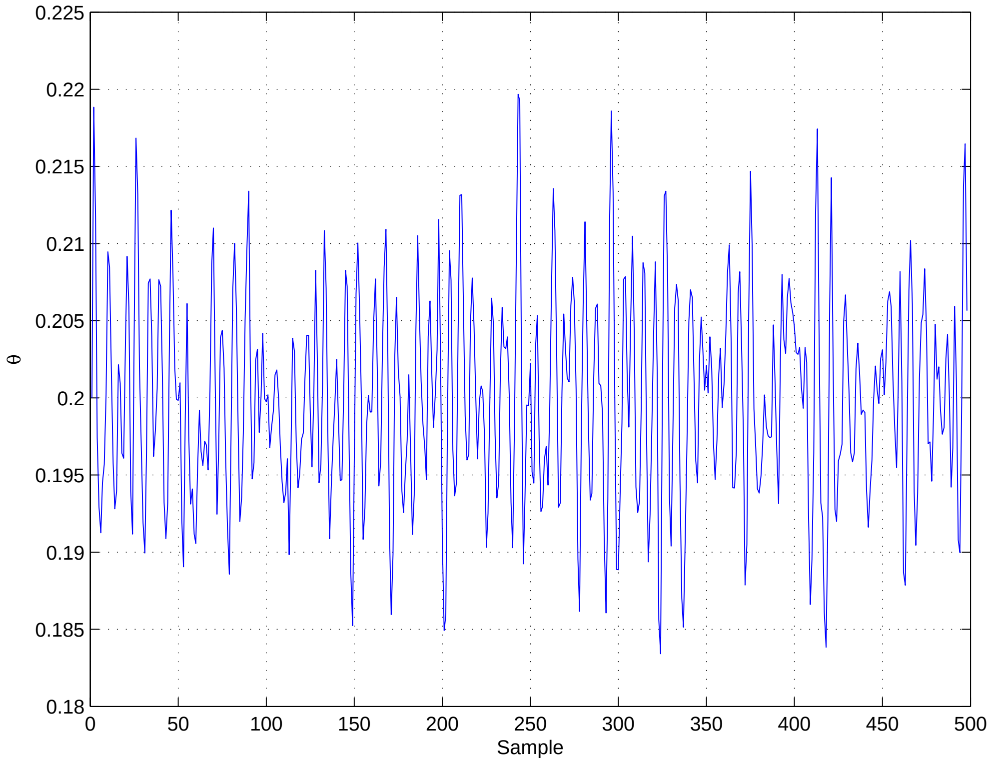

Figure 2: The θ measurement as output from the simulation.

tion controller for the new system.Show your work in using *InvertedPendulum.zip* (use parameters in *param.m*).

We begin by substituting the above expression for T into the equation of motion to get

$$\begin{aligned}\ddot{\theta} &= \frac{g}{l}\sin(\theta) - \frac{b}{ml^2}\dot{\theta} + \frac{T}{ml^2} \\ &= \frac{g}{l}\sin(\theta) - \frac{b}{ml^2}\dot{\theta} + \frac{mla\cos(\theta)}{ml^2} \\ &= \frac{g}{l}\sin(\theta) - \frac{b}{ml^2}\dot{\theta} + \frac{a}{l}\cos(\theta).\end{aligned} \tag{31}$$

Now, let's define a state vector and let $v = \ddot{\theta}$, and thus, $\dot{v} = \dddot{\theta}$. Taking a derivative of v to get $\dot{v}$, we obtain

$$\dot{v} = \frac{g}{l}\sin(\theta)\dot{\theta} - \frac{b}{ml^2}\ddot{\theta} - \frac{\dot{a}}{l}\cos(\theta) + \frac{a}{l}\sin(\theta)\dot{\theta} \tag{32}$$

Now, we are given that we can control the jerk, i.e. we can control $\dot{a}$. So let $u = \dot{a}$, and then compensating for all the "nonlinear" terms in the equations of motion, we get an $\dot{a}$ that looks like

$$u = \dot{a} = g\tan(\theta)\dot{\theta} - \frac{b}{ml}\frac{\ddot{\theta}}{\cos(\theta)} + a\tan(\theta)\dot{\theta} - \frac{l}{\cos(\theta)}u_{lin}. \tag{33}$$

The first three terms cancel out everything else in equation 32, leaving

$$\dot{v} = u_{lin}. \tag{34}$$