ECE 6320: Homework 8

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1 Problem 1

Consider a communication satellite of mass m orbiting around the earth. The altitude of the satellite is specified by $r(t)$, $\theta(t)$, and $\phi(t)$ as shown. The orbit can be controlled by three orthogonal thrusts; u_r , $u_\theta(t)$, and $u_\phi(t)$. If you remember you have already computed the exact nonlinear equations in homework 1. You can refresh your memory by looking equation (2.47) on page 36 in Chen. One solution which corresponds to a circular orbit is given by

$$x0(t) = \begin{bmatrix} r_0 & 0 & \omega_0 t & \omega_0 & 0 & 0 \end{bmatrix}.$$

Around this solution (use the linearized state-space).

1.1 The linearized system

The full nonlinear system equations are given (from hw 1) as

$$\begin{aligned}
\ddot{r} &= r\dot{\phi}^2 + r\cos^2(\phi)\dot{\theta}^2 - \frac{GM}{r^2} + \frac{u_r}{m} \\
\ddot{\phi} &= -2\frac{\dot{r}}{r}\dot{\phi} - \cos(\phi)\sin(\phi)\dot{\theta}^2 + \frac{u_\phi}{mr} \\
\ddot{\theta} &= 2\frac{\sin(\phi)}{\cos(\phi)}\dot{\phi}\dot{\theta} - 2\frac{\dot{r}}{r}\dot{\theta} + \frac{u_\theta}{mr\cos(\phi)}.
\end{aligned} \tag{1}$$

The linearized system around the given trajectory is given in Example 2.9 in the Checn book. It is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2\omega_0 r_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{-2\omega_0}{r_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_0^2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{mr_0} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{mr_0} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) Is the system controllable when all the thrusters can be used?

We first build the controllability matrix:

```
%}
Con = contMat(A,B);
if exist('prob1a1.txt','file')
  delete('prob1a1.txt');
end
if exist('prob1a0.txt','file')
  delete('prob1a0.txt');
end
rank1 = double(rank(Con));
if rank1 == 6
  fid = fopen('prob1a1.txt','wt');
else
  fid = fopen('prob1a0.txt','wt');
end
fprintf(fid,'%d',rank1);
fclose(fid);
disp(rank1)

%{
```

The rank of the controllability matrix is 6 . Thus, the system is controllable.

(b) Is the system controllable when thruster u_ϕ cannot be used?

```
%}
B1b = B;
B1b(:,3) = 0;
```

```
Con = contMat(A,B1b);
fname = 'prob1b';
if exist([fname '1.txt'],'file')
  delete([fname '1.txt']);
end
if exist([fname '0.txt'],'file')
  delete([fname '0.txt']);
end
rank1b = double(rank(Con));
if rank1b == size(A,1)
  fid = fopen([fname '1.txt'],'wt');
else
  fid = fopen([fname '0.txt'],'wt');
end
fprintf(fid,'%d',rank1b);
fclose(fid);
disp(rank1b)
%{
```

The rank of the controllability matrix is 4 . Thus, the system is *not* controllable.

(c) Is the system controllable when thruster u_θ cannot be used?

```
%}
B1c = B;
B1c(:,2) = 0;
Con = contMat(A,B1c);
fname = 'prob1c';
if exist([fname '1.txt'],'file')
  delete([fname '1.txt']);
end
if exist([fname '0.txt'],'file')
  delete([fname '0.txt']);
end
rank1c = double(rank(Con));
if rank1c == size(A,1)
  fid = fopen([fname '1.txt'],'wt');
else
  fid = fopen([fname '0.txt'],'wt');
end
fprintf(fid,'%d',rank1c);
fclose(fid);
disp(rank1c)
%{
```

The rank of the controllability matrix is 5 . Thus, the system is *not* controllable.

(d) Is the system controllable when thruster u_r cannot be used?

```
%}
B1d = B;
B1d(:,1) = 0;
Con = contMat(A,B1d);
fname = 'prob1d';
if exist([fname '1.txt'],'file')
  delete([fname '1.txt']);
end
if exist([fname '0.txt'],'file')
  delete([fname '0.txt']);
end
rank1d = double(rank(Con));
if rank1d == size(A,1)
  fid = fopen([fname '1.txt'],'wt');
else
  fid = fopen([fname '0.txt'],'wt');
end
fprintf(fid,'%d',rank1d);
fclose(fid);
disp(rank1d)

%{
```

The rank of the controllability matrix is 6 . Thus, the system is controllable.

(e) Is the system controllable when thruster u_ϕ and u_θ cannot be used?

```
%}
B1e = B;
B1e(:,2:3) = 0;
Con = contMat(A,B1e);
fname = 'prob1e';
if exist([fname '1.txt'],'file')
  delete([fname '1.txt']);
end
if exist([fname '0.txt'],'file')
  delete([fname '0.txt']);
end
rank1e = double(rank(Con));
if rank1e == size(A,1)
  fid = fopen([fname '1.txt'],'wt');
else
  fid = fopen([fname '0.txt'],'wt');
end
fprintf(fid,'%d',rank1e);
fclose(fid);
disp(rank1e)

%{
```

The rank of the controllability matrix is 3 . Thus, the system is *not* controllable.

2 Problem 2

The cart carrying the inverted pendulum is driven by an electric motor. Assume that the motor drives one pair of the wheels of the cart, so that the whole cart, pendulum and all, becomes the load on the motor. The differential equations of this system are written as

$$\begin{aligned} \ddot{x} + \frac{k^2}{Mr^2R}\dot{x} + \frac{mg}{M}\theta &= \frac{k}{MRr}e \\ \ddot{\theta} - \left(\frac{M+m}{Ml}\right)g\theta - \frac{k^2}{Mr^2Rl}\dot{x} &= -\frac{k}{MRrl}e \end{aligned}$$

where k is the motor torque constant, R is the motor resistance, r is the ratio of motor torque to linear force applied to the cart ($\tau = rf$), and e is the voltage applied to the motor. Let the state vector and input be defined $\mathbf{x} = [x, \dot{x}, \theta, \dot{\theta}]^T$ and $u = e$.

(a) Find the A and B matrices of the state space

The first thing is to rearrange the equations as

$$\begin{aligned} \ddot{x} &= -\frac{k^2}{Mr^2R}\dot{x} - \frac{mg}{M}\theta + \frac{k}{MRr}e \\ \ddot{\theta} &= \left(\frac{M+m}{Ml}\right)g\theta + \frac{k^2}{Mr^2Rl}\dot{x} - \frac{k}{MRrl}e. \end{aligned}$$

The we can stack these to form

$$\mathbf{x} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-k^2}{Mr^2R} & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{k^2}{Mr^2Rl} & \frac{M+m}{Ml}g & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k}{MRr} \\ 0 \\ -\frac{k}{MRrl} \end{bmatrix} e.$$

(b) Is the system controllable?

```
%}
%% Problem 2.
disp(['Problem 2...'])
m = 0.1;
M = 1;
```

```
l = 1;
g = 9.8;
k = 1;
R = 100;
r = 0.02;

syms m M l g k R r;

contMat = @(A,B) [B A*B A^2*B A^3*B];

A = [ 0 1 0 0;...
      0 -k^2./(M*r^2*R) -m*g/(M) 0;...
      0 0 0 1;
      0 k^2/(M*r^2*R*l) (M+m)/(M*l)*g 0];

B = [ 0; k/(M*R*r); 0; -k/(M*R*r*l)];

Con = contMat(A,B);
if exist('prob2b1.txt','file')
   delete('prob2b1.txt');
end
if exist('prob2b0.txt','file')
   delete('prob2b0.txt');
end
rank2 = double(rank(Con));
if rank2 == size(A,1)
   fid = fopen('prob2b1.txt','wt');
else
   fid = fopen('prob2b0.txt','wt');
end
fprintf(fid,'%d',rank2);
fclose(fid);
disp(rank2)

%{
```

The rank of the controllability matrix is 4 . Thus the system is controllable.

The following numerical data can be used if would rather use numbers then letters: $m = 0.1kg$, $M = 1.0kg$, $l = 1.0m$, $g = 9.8ms^{-2}$, $k = 1V$, $R = 100\Omega$, and $r = 0.02m$.

3 Problem 3

State-space equations of a double-effect evaporator is given as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & -.00156 & -.0711 & 0 & 0 \\ 0 & -.1419 & .0711 & 0 & 0 \\ 0 & -.00875 & -1.102 & 0 & 0 \\ 0 & -.00128 & -.1489 & 0 & -.0013 \\ 0 & .0605 & .1489 & 0 & -.0591 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & -.143 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & .392 \\ 0 & .108 & -.0592 \\ 0 & -.0486 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Determine whether or not the evaporator is controllable from each of the following combinations of inputs:

3.1 My Answer

First, let's put them into Matlab .

```
%}
%% Problem 3
disp(['Problem 3...'])
A = [...
      0 -.00156 -.0711 0 0;...
      0 -.1419   .0711  0 0;...
      0 -.00875 -1.102 0 0;...
      0 -.00128 -.1489 0 -.0013;...
      0 .0605    .1489  0 -.0591];

B = [...
      0 -.143   0;  ...
      0 0       0;...
      .392  0   0;...
      0 .108    -.0592;...
      0 -.0486  0];

contMat = @(A,B) [B A*B A^2*B A^3*B A^4*B];
%{
```

(a) u_1 only;

```
%}
%% Problem 3a
B3a = B;
B3a(:,2:3) = 0;
Con = contMat(A,B3a);
if exist('prob3a1.txt','file')
```

```
  delete('prob3a1.txt');
end
if exist('prob3a0.txt','file')
  delete('prob3a0.txt');
end
rank3a = double(rank(Con));
if rank3a == size(A,1)
  fid = fopen('prob3a1.txt','wt');
else
  fid = fopen('prob3a0.txt','wt');
end
fprintf(fid,'%d',rank3a);
fclose(fid);
disp(rank3a)
%{
```

The rank of the controllability matrix is 4 . Thus the system is *not* controllable.

(b) u_1 and u_2

```
%}
%% Problem 3b
B3b = B;
B3b(:,3) = 0;
Con = contMat(A,B3b);
if exist('prob3b1.txt','file')
  delete('prob3b1.txt');
end
if exist('prob3b0.txt','file')
  delete('prob3b0.txt');
end
rank3b = double(rank(Con));
if rank3b == size(A,1)
  fid = fopen('prob3b1.txt','wt');
else
  fid = fopen('prob3b0.txt','wt');
end
fprintf(fid,'%d',rank3b);
fclose(fid);
disp(rank3b)
%{
```

The rank of the controllability matrix is 5 . Thus the system is controllable.

(c) u_1 and u_3 ;

```
%}
%% Problem 3c
```

```
B3c = B;
B3c(:,2) = 0;
Con = contMat(A,B3c);
if exist('prob3c1.txt','file')
  delete('prob3c1.txt');
end
if exist('prob3c0.txt','file')
  delete('prob3c0.txt');
end
rank3c = double(rank(Con));
if rank3c == size(A,1)
  fid = fopen('prob3c1.txt','wt');
else
  fid = fopen('prob3c0.txt','wt');
end
fprintf(fid,'%d',rank3c);
fclose(fid);
disp(rank3c)
%{
```

The rank of the controllability matrix is 5 . Thus the system is controllable.

(d) u_2 and u_3.

```
%}
%% Problem 3a
B3d = B;
B3d(:,1) = 0;
Con = contMat(A,B3d);
if exist('prob3d1.txt','file')
  delete('prob3d1.txt');
end
if exist('prob3d0.txt','file')
  delete('prob3d0.txt');
end
rank3d = double(rank(Con));
if rank3d == size(A,1)
  fid = fopen('prob3d1.txt','wt');
else
  fid = fopen('prob3d0.txt','wt');
end
fprintf(fid,'%d',rank3d);
fclose(fid);
disp(rank3d)
%{
```

The rank of the controllability matrix is 3 . Thus the system is *not* controllable.