Homework 3

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1 Problem 1

Consider a system with input u and output y. Three experiments are performed on the system using the input $u_1(t)$, $u_2(t)$, and $u_3(t)$ for $t \geq 0$. In each case, the initial state $x(0)$ at $t = 0$ is the same. The corresponding outputs are denoted by y_1, y_2 and y_3. Which one of the following statements are correct if $x(0) \neq 0$ and which are correct if $x(0) = 0$?

1. If $u_3 = u_1 + u_2$, then $y_3 = y_1 + y_2$.
2. If $u_3 = 0.5(u_1 + u_2)$ then $y_3 = 0.5(y_1 + y_2)$.
3. If $u_3 = u_1 - u_2$ then $y_3 = y_1 - y_2$.

My Answer It is unclear what type of system is being used in the question. If the system in question is a linear system, then both the zero-state responses and the zero-input reponse satisfy the superposition principle. By this, if the system is linear, that all three ((a), (b), and (c)) are true both when $x(0) = 0$. But when $x(0) \neq 0$ (i.e. there is some previous state), then the only one that is correct is (b), because the sum has twice the initial state dependence than the single output would have.

Now if the system is *not* linear, then all bets are off. The order order interactions may occur in all sort of manners, and the state may have interesting interactions with the input that cannot gaurantee that the linearity of the choices is conserved.

2 Problem 2

Consider a system described by

$$\ddot{y} + 2\dot{y} - 3y = \dot{u} - u \tag{1}$$

- Find the transfer function of the system.
- So, to find the transfer function, let's first take the Laplace transform of the system.

  $$Y(s)s^2 + 2Y(s)s - 3Y(s) - Y(0) = U(s)s - U(s).$$

  The transfer function is defined with $Y(0) = 0$, so that term will drop out. Next we'll gather terms on each side of the equation.

  $$Y(s)\left(s^2 + s - 3\right) = U(s)\left(s - 1\right)$$

Next we'll look at the ratio of output to input to finally give us our transfer function:

$$\begin{aligned}\frac{Y(s)}{U(s)} &= \frac{s-1}{s^2+2s-3} \\ &= \frac{(s-1)}{(3s^2-2s-1)} \\ &= \frac{(s-1)}{(3s+1)(s-1)} \\ &= \frac{1}{(3s+1)} = G(s).\end{aligned}$$

- Find the impulse response of the system.

  To find the inpulse response of the system, we need to take the inverse Laplace transform of the transfer function. Thus we get:

$$\begin{aligned}Y_{imp}(t) &= \mathcal{L}^{-1}\left\{G(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{3(s+\frac{1}{3})}\right\} \\ &= \frac{1}{3}\left(e^{-\frac{1}{3}t}u(t)\right) \\ &= \frac{1}{3}e^{-\frac{1}{3}t}u(t).\end{aligned}$$

3 Problem 3

Show that the following pair of systems are zero-state equivalent, but not algebraically equivalent.

-

A

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 01 & \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

B

$$\dot{\overline{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \overline{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$\overline{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \overline{x}$$

In order to show zero-state equivalence, we need to show that bothe systems realize the same *transfer function*.

$$\begin{aligned}G_A &= C(sI-A)^{-1}B + D \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{(s-1)^2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{(s-1)^2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 \\ 0 \end{bmatrix} \\ &= \frac{(s-1)}{(s-1)^2} \\ &= \frac{1}{(s-1)}\end{aligned}$$

$$\begin{aligned}G_B &= C(sI-A)^{-1} + D \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{(s-1)(s-2)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-2 & 0 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{(s-1)(s-2)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-2 \\ 0 \end{bmatrix} \\ &= \frac{(s-2)}{(s-1)(s-2)} \\ &= \frac{1}{(s-1)}\end{aligned}$$

Thus, the systems are zero-state equivalent. Now to show that these are *not* algebraicly equivalent, wee can take a look at the characteristic polynomial.

$$p_A(s) = (s-1)^2 \qquad\qquad P_B = (s-1)(s-2)$$

These are not the same, and thus these two systems are not algebraicly equivalent.

- $$\begin{array}{cc} \text{A} & \text{B} \\ \begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned} & \begin{aligned} \dot{\overline{x}} &= \overline{x} + u \\ \overline{y} &= \overline{x} \end{aligned} \end{array}$$

  Same deal for this one. Let's look at the transfer functions:

  $$\begin{aligned} G_A &= C(sI-A)^{-1}B + D \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{(s-1)^2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{(s-1)^2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 \\ 0 \end{bmatrix} \\ &= \frac{(s-1)}{(s-1)^2} \\ &= \frac{1}{(s-1)} \end{aligned} \qquad \begin{aligned} G_B &= C(sI-A)^{-1} + D \\ &= 1(s-1)^{-1}1 \\ &= \frac{1}{(s-1)} \end{aligned}$$

  So, we can say that these systems are zero-state equivalent. Now let's consider the characteristic polynomials:

  $$p_A(s) = (s-1)^2 \qquad\qquad p_B(s) = (s-1)$$

4 Problem 4

Consider the following two systems

$$\begin{array}{cc} \text{A} & \text{B} \\ \begin{aligned} \dot{x} &= \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x \end{aligned} & \begin{aligned} \dot{\overline{x}} &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \overline{x} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \overline{x} \end{aligned} \end{array}$$

- Are these systems zero-state equivalent?

Once again we need to see if these systems realize the same transfer function. So for system A, we get

$$
\begin{aligned}
G_A &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} s-2 & -1 & -2 \\ 0 & s-2 & -2 \\ 0 & 0 & s-1 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{\det(\bar{A})} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} (s-2)(s-1) & 0 & 0 \\ (s-1) & (s-2)(s-1) & 0 \\ 2+2(s-2) & 2(s-2) & (s-2)^2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(s-2)} & \frac{1}{(s-2)^2} & \frac{2}{(s-2)(s-1)} \\ 0 & \frac{1}{(s-2)} & \frac{2}{(s-2)(s-1)} \\ 0 & 0 & (s-2)^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad (2) \\
&= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(s-2)} + \frac{1}{(s-2)^2} \\ \frac{1}{(s-2)} \\ 0 \end{bmatrix} \\
&= \frac{1}{(s-2)} + \frac{1}{(s-2)^2} - \frac{1}{(s-2)} \\
&= \frac{1}{(s-2)^2}.
\end{aligned}
$$

Now for system B, let's do the same thing.

$$
\begin{aligned}
G_B &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} s-2 & -1 & -1 \\ 0 & s-2 & -1 \\ 0 & 0 & s+1 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{\det(\bar{A})} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} (s-2)(s+1) & 0 & 0 \\ (s+1) & (s-2)(s+1) & 0 \\ 1+(s-2) & (s-2) & (s-2)^2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(s-2)} & \frac{1}{(s-2)^2} & \frac{(s-1)}{(s-2)^2(s+1)} \\ 0 & \frac{1}{(s-2)} & \frac{1}{(s-2)(s+1)} \\ 0 & 0 & (s+1) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad (3) \\
&= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(s-2)} + \frac{1}{(s-2)^2} \\ \frac{1}{(s-2)} \\ 0 \end{bmatrix} \\
&= \frac{1}{(s-2)} + \frac{1}{(s-2)^2} - \frac{1}{(s-2)} \\
&= \frac{1}{(s-2)^2}.
\end{aligned}
$$

Thus, they are zero-state equivalent.

- Are they algebraically equivalent? Let's look at the characteristic polynomials to find this out.

$$p_A(s) = (s-2)^2(s-1) \qquad\qquad p_B(s) = (s-2)^2(s+1)$$

And hence, they are not algebraicly equivalent.

5 Problem 5

Find state space realization for the transfer function matrix

$$\hat{G}(s) = \begin{bmatrix} \frac{-(12s+6)}{3s+34} & \frac{22s+23}{3s+34} \end{bmatrix} \tag{4}$$

The first thing that we need to do is to get $\hat{G}_{sp}(s)$ from $\hat{G}(s)$. We get that by subtracting off the D component. We get D by taking the limit

$$\begin{aligned} D &= \lim_{s\to\infty} \hat{G}(s) \\ &= \frac{1}{s+34/3} \begin{bmatrix} -4s-2 & 22/3s+23/3 \end{bmatrix} \Big|_{s\to\infty} \\ &= \begin{bmatrix} -4 & 22/3 \end{bmatrix} \end{aligned} \tag{5}$$

Now, we can subtract this off of the transfer function to obtain the strictly proper portion,

$$\hat{G}(s) = \frac{1}{s+34/3} \begin{bmatrix} -4s - \frac{6}{3} + 4s + \frac{136}{3} & \frac{22}{3}s + \frac{69}{9} - \frac{22}{3}s - \frac{506}{9} \end{bmatrix} = \frac{1}{s+34/3} \begin{bmatrix} \frac{130}{3} & \frac{-437}{9} \end{bmatrix}. \tag{6}$$

Matching up these terms with the formulas we went over in class, we can see that our matrices will have the following dimensions

$$A \in \mathbb{R}^{2\times 2} \qquad B \in \mathbb{R}^{2\times 2} \tag{7}$$

$$C \in \mathbb{R}^{1\times 2} \qquad D \in \mathbb{R}^{1\times 2}. \tag{8}$$

We can identify the terms of $\hat{G}(s)$ as follows

$$\alpha_1 = 34/3 \qquad N_1 = \begin{bmatrix} \frac{130}{3} & \frac{-437}{9} \end{bmatrix}. \tag{9}$$

Using this and equation (4.34) from the book (3rd ed.) we get the following state space representation in controllable canonical form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1 I_p \end{bmatrix} \mathbf{x} + \begin{bmatrix} I_p \end{bmatrix} \mathbf{u} \qquad \Rightarrow \qquad \dot{\mathbf{x}} = \begin{bmatrix} -\frac{34}{3} & 0 \\ 0 & -\frac{34}{3} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} \tag{10}$$

$$y = \begin{bmatrix} N_1 \end{bmatrix} \mathbf{x} + D\mathbf{u} \qquad \qquad y = \begin{bmatrix} \frac{130}{3} & -\frac{437}{9} \end{bmatrix} \mathbf{x} + \begin{bmatrix} -4 & \frac{22}{3} \end{bmatrix} \mathbf{u}. \tag{11}$$