# ECE 6320: Homework 1

Andrew Pound

September 5, 2014

## 1 Problem 1

A communication satellite of mass m orbiting around the earth is shown in Figure 1. The altitude of the satellite is specified by $r(t)$, $\theta(t)$, and $\phi(t)$ as shown. The orbit can be controlled by three orthogonal thrusts; $ur$ , $u_\theta(t)$, and $u_\phi(t)$. Use Lagrange's equation to derive the satellite's equations of motion.

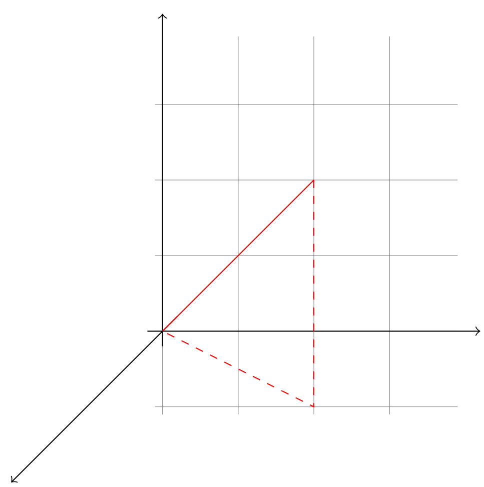

Figure 1: Satellite figure

### 1.1 Working the problem

We will choose the generalized coordinates $q = \begin{bmatrix} r & \phi & \theta \end{bmatrix}^T$. Using these coordinates, we know that the velocity is given by

$$v = \begin{bmatrix} \dot{r} \\ r\dot{\phi} \\ r\cos(\phi)\dot{\theta} \end{bmatrix}. \tag{1}$$

Utilizing this we can readily calculate the Kinetic energy of the system as

$$\begin{aligned} T &= \frac{1}{2} m v^T v \\ &= \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\phi}^2 + r^2 \cos^2(\phi) \dot{\theta}^2 \right). \end{aligned} \tag{2}$$

Likewise, the potential energy is easily calculated as

$$V = -\frac{GMm}{r}. \tag{3}$$

Thus, the Lagrangian is given as

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\phi}^2 + r^2 \cos^2(\phi) \dot{\theta}^2 \right) + \frac{GMm}{r}. \end{aligned} \tag{4}$$

Then, to find the equations of motion, we now need to calculate the derivatives to find the equations of motion from the equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}_i} \right) - \frac{\partial L}{\partial z_i} = F \quad \forall \; i. \tag{5}$$

**In the $r$-direction**

$$\frac{\partial L}{\partial \dot{r}} = \frac{2}{2} m \dot{r} \tag{6}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = m \ddot{r} \tag{7}$$

$$\frac{\partial L}{\partial r} = m r \dot{\phi}^2 + m r \cos^2(\phi) \dot{\theta}^2 - \frac{GMm}{r^2}. \tag{8}$$

Thus we get the equation

$$m \ddot{r} - m r \dot{\phi}^2 - m r \cos^2(\phi) \dot{\theta}^2 + \frac{GMm}{r^2} = u_r. \tag{9}$$

$$\ddot{r} = r \dot{\phi}^2 + r \cos^2(\phi) \dot{\theta}^2 - \frac{GM}{r^2} + \frac{u_r}{m} \tag{10}$$

**In the $\phi$-direction**

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{2}{2} m r^2 \dot{\phi} \tag{11}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = 2 m r \dot{r} \dot{\phi} + m r^2 \ddot{\phi} \tag{12}$$

$$\frac{\partial L}{\partial \phi} = -\frac{2}{2} m r^2 \cos(\phi)\sin(\phi)\dot{\theta}^2 \tag{13}$$

Thus, we get the equation

$$m r^2 \ddot{\phi} + 2 m r \dot{r} \dot{\phi} + m r^2 \cos(\phi)\sin(\phi)\dot{\theta}^2 = r u_\phi . \tag{14}$$

$$\ddot{\phi} = -2\frac{\dot{r}}{r}\dot{\phi} - \cos(\phi)\sin(\phi)\dot{\theta}^2 + \frac{u_\phi}{mr} \tag{15}$$

**In the $\theta$-direction**

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{2}{2} m r^2 \cos^2(\phi)\dot{\theta} \tag{16}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2 m r \dot{r} \cos^2(\phi)\dot{\theta} - 2 m r^2 \cos(\phi)\sin(\phi)\dot{\phi}\dot{\theta} + m r^2 \cos^2(\phi)\ddot{\theta} \tag{17}$$

$$\frac{\partial L}{\partial \theta} = 0 \tag{18}$$

Thus, we get the equation

$$m r^2 \cos^2(\phi)\ddot{\theta} + 2 m r \dot{r} \cos^2(\phi)\dot{\theta} - 2 m r^2 \cos(\phi)\sin(\phi)\dot{\phi}\dot{\theta} = r\cos(\phi) u_\theta \tag{19}$$

$$\ddot{\theta} = 2\frac{\sin(\phi)}{\cos(\phi)}\dot{\phi}\dot{\theta} - 2\frac{\dot{r}}{r}\dot{\theta} + \frac{u_\theta}{mr\cos(\phi)} \tag{20}$$

**Full equations of motion**

$$\begin{aligned}
\ddot{r} &= r\dot{\phi}^2 + r\cos^2(\phi)\dot{\theta}^2 - \frac{GM}{r^2} + \frac{u_r}{m} \\
\ddot{\phi} &= -2\frac{\dot{r}}{r}\dot{\phi} - \cos(\phi)\sin(\phi)\dot{\theta}^2 + \frac{u_\phi}{mr} \\
\ddot{\theta} &= 2\frac{\sin(\phi)}{\cos(\phi)}\dot{\phi}\dot{\theta} - 2\frac{\dot{r}}{r}\dot{\theta} + \frac{u_\theta}{mr\cos(\phi)}
\end{aligned} \tag{21}$$

# 2 Problem 2

Using equations of motion derived in Question 1, develop a simulator using SIMULINK (s-function). Verify using the simulator that for $x_0(t) = \begin{bmatrix} r_0 & 0 & \omega_0 t & \omega_0 & 0 & 0 \end{bmatrix}$ the satellite will move in a circular orbit.

- Use *SatelliteDummy.zip* folder to access the dummy m-files.
- Use *param.m* to see various parameters to be used in the simulator.
- Use *satelliteModel.m* (s-function) for satellite's equations of motion. This function should take input

$u = \begin{bmatrix} u_r = 0 & u_\theta = 0 & u_\phi = 0 \end{bmatrix}^T$ and output will be $\begin{bmatrix} r & \theta & \phi \end{bmatrix}^T$.

- Use *polar2cart.m* to convert polar coordinates to cartesian coordinates for plotting
- Use *drawSpacecraft.m* as it is to draw the satellite orbiting around the earth (Please fee free to modify it to make it better looking satellite).
- *SatelliteSim.slx* (simulink-model file) to run the simulation
- Submit all your working files as zip folder on canvas. The name of the folder should (*satelliteLastnameofStudent.zip*). Example *satelliteSharma.zip*.
- Show your working simulation to TA.

## 2.1 Solution

I was able to get the model coded up correctly with the right parameters. The relevant lines of code are:

```
function xdot=mdlDerivatives(t,x,u,P)
r= x(1);
rdot= x(2);
theta= x(3);
thetadot= x(4);
phi=x(5);
phidot=x(6);

ur=u(1);
utheta=u(2);
uphi=u(3);
m=P.m;
k=P.k;
xdot(1)=rdot;
xdot(2)=r*phidot^2 + r*(cos(phi))^2*thetadot^2 - k/r^2 + ur/m;
xdot(3)=thetadot;
xdot(4)=2*sin(phi)*phidot*thetadot/cos(phi) -2*rdot*thetadot/r ...
          + utheta/(m*r*cos(phi));
xdot(5)=phidot;
xdot(6)=-2*rdot*phidot/r - cos(phi)*sin(phi)*thetadot^2 + uphi/(m*r);
%sys = xdot;
```

In addition to the lines above, We also needed to modify the file *polar2cart.m*. The contents of that file are reproduced below.

```
function X=polar2cart(uu)
 r= uu(1);
 theta= uu(2);
```

```

phi=uu(3);

X=zeros(3,1);
X(1)=r*sin(theta)*cos(phi);
X(2)=r*cos(theta)*cos(phi);
X(3)=r*sin(phi);
```

I had some difficulties in getting the simulation to work. Some of it was that the simulation was built in a newer MATLAB thatn I had on my machine. I had to go and use a computer in a lab in order to get it to work. Also, I ended up outputing the timeseries data from the model (the data that was passed into the *drawSpacecraft* routine) in order to plot it myself. I never did figure out how to speed up the stepping of the simulation, thus my simulation took about 40 mins to complete one revolution.

I also did not know how to make the movie. So, I opted to output the time series and plot it myself after the fact. I hope that this is satisfactory enough. Below is the code to plot the timeseries data output from the simulation. The plot produced is shown in Figure 2.

```
%}

% Setup up the same parameters:
param

% Load in the data from the simulation
load output.mat
d = simout.data;
d = d(1:100:end,:);

% Set the radius of the Earth
R = P.r0 - 1500;
[xs,ys,zs] =  sphere;


f1 = figure;
% Plot the Earth
surf(R*xs,R*ys,R*zs)
hold on
% Plot the trajectory of the Satellite:
plot3(d(:,1),d(:,2),d(:,3),'linewidth',2)
grid on

% Output to .eps and .pdf:
print(f1, '-depsc2', 'orbit.eps')
system('ps2pdf -dEPSCrop orbit.eps')

%{
```

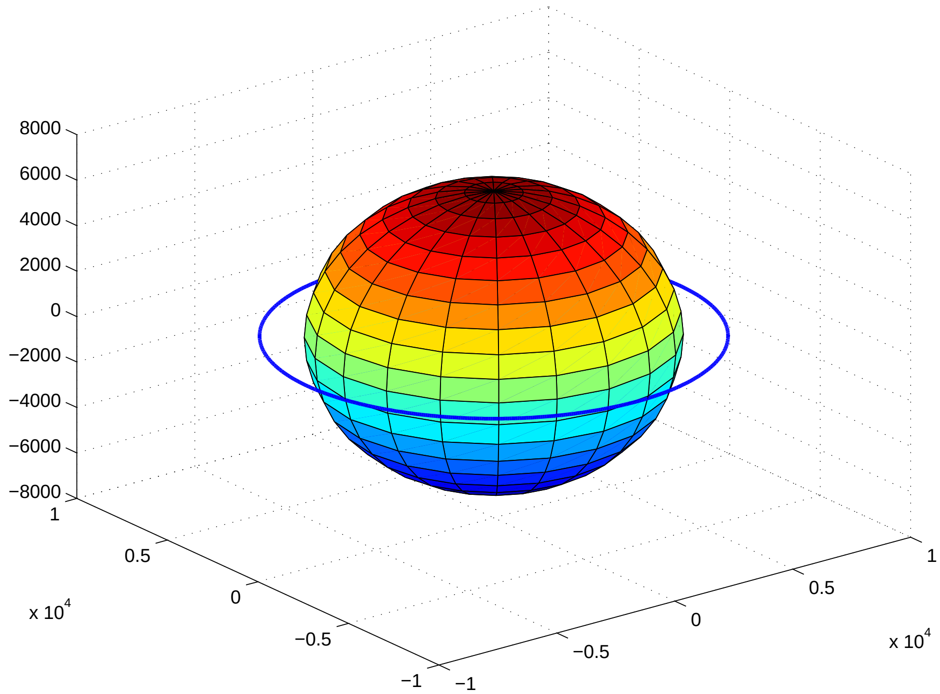


Figure 2: The orbit that was output from the simulation.

# 3 Problem 3

Find state-space equations to describe the pendulum system in Figure 3. The system is useful to model a two link robotic manipulators. To find the state-space equations consider $\theta_1$ and $\theta_2$ very small.

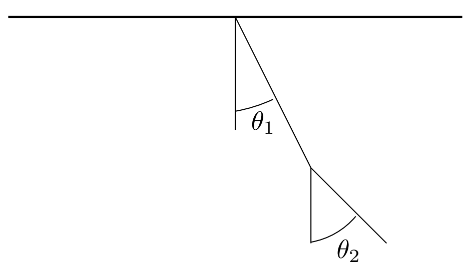


Figure 3: Satellite figure

## 3.1 Solution

First, let's define the position of mass 1, $m_1$, as $(x_1, y_1)$ and we let

$$x_1 = l_1 \sin(\theta_1) \qquad \dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1 \tag{22}$$

$$y_1 = l_1 \cos(\theta_1) \qquad \dot{y}_1 = -l_1 \sin(\theta_1)\dot{\theta}_1 \tag{23}$$

And now for mass 2, $m_2$ we obtain

$$x_2 = l_2 \sin(\theta_2) + l_1 \sin(\theta_1) \qquad \dot{x}_1 = l_2 \cos(\theta_2)\dot{\theta}_2 + l_1 \cos(\theta_1)\dot{\theta}_1 \tag{24}$$

$$y_2 = l_2 \cos(\theta_2) + l_1 \cos(\theta_1) \qquad \dot{y}_1 = -l_2 \sin(\theta_2)\dot{\theta}_2 - l_1 \sin(\theta_1)\dot{\theta}_1 \tag{25}$$

We will need the squared terms of the above velocities, so let's calculate those:

$$\begin{aligned}
\dot{x}_1^2 &= l_1^2 \cos^2(\theta_1)\dot{\theta}_1^2 \\
\dot{y}_1^2 &= l_1^2 \sin^2(\theta_1)\dot{\theta}_1^2 \\
\dot{x}_2^2 &= l_2^2 \cos^2(\theta_2)\dot{\theta}_2^2 + l_1 l_2 \cos(\theta_1)\cos(\theta_2)\dot{\theta}_1\dot{\theta}_2 + l_1^2 \cos^2(\theta_1)\dot{\theta}_1^2 \\
\dot{y}_2^2 &= l_2^2 \sin^2(\theta_2)\dot{\theta}_2^2 + l_1 l_2 \sin(\theta_1)\sin(\theta_2)\dot{\theta}_1\dot{\theta}_2 + l_1^2 \sin^2(\theta_1)\dot{\theta}_1^2
\end{aligned} \tag{26}$$

### 3.1.1 Solving for $T$ and $V$

Now let's solve for the Kinetic Energy

$$\begin{aligned}
T &= \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) \\
&= \frac{1}{2}m_1\left(l_1^2\cos^2(\theta_1)\dot{\theta}_1^2 + l_1^2\sin^2(\theta_1)\dot{\theta}_1^2\right) \\
&\quad + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\underbrace{\left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right)}_{\cos(\theta_1-\theta_2)}\right) \\
&= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right) \\
&= \frac{1}{2}\left(m_1+m_2\right)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2).
\end{aligned} \tag{27}$$

Ok, let's solve for the Potential Energy now

$$
\begin{aligned}
V &= -m_1 g y_1 - m g y_2 \\
&= -m_1 g l_1 \cos(\theta_1) - m_2 g \left(l_1 \cos(\theta_1) + l_2 \cos(\theta_2)\right) \\
&= -(m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2).
\end{aligned}
\tag{28}
$$

### 3.1.2 The Lagrangian $L$

Sticking these together, we get the Lagrangian

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}\left(m_1 + m_2\right) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_2).
\end{aligned}
\tag{29}
$$

We now need to calculate the derivatives to find the equations of motion from the equations

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_i}\right) - \frac{\partial L}{\partial z_i} = F \quad \forall \ i. \tag{30}
$$

**First equation of motion** So for the first, we are looking for the derivative with respect to $\dot{\theta}_1$. We get

$$
\frac{\partial L}{\partial \dot{\theta}_1} = \frac{2}{2}\left(m_1 + m_2\right) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \tag{31}
$$

Then taking the time derivative, we get

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \left(m_1 + m_2\right) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)\right). \tag{32}
$$

Ok, now we need the derivative with respect to $\theta_1$ to get

$$
\frac{\partial L}{\partial \theta_1} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - \left(m_1 + m_2\right) g l_1 \sin(\theta_1). \tag{33}
$$

Putting this all together, we get that

$$
(m_1+m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2^2(\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) - \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)\right) + (m_1+m_2) g l_1 \sin(\theta_1) = 0 \tag{34}
$$

**Second equation of motion** We need to take the derivative with respect to $\dot{\theta}_2$. This gives us the following equation

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{2}{2} m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2). \tag{35}$$

Then we can take the time derivative,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \left(\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 \sin(\theta_1 - \theta_2)\left(\dot{\theta}_1 - \dot{\theta}_2\right)\right). \tag{36}$$

Now we calculate the derivative with respect to $\theta_2$

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin(\theta_2). \tag{37}$$

Putting this together, we will obtain the equation

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \left(\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 \sin(\theta_1 - \theta_2)\left(\dot{\theta}_1 - \dot{\theta}_2\right)\right) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = u. \tag{38}$$

**Small Angle Approximations** Let's us a small angle approximation for both $\theta_1$ and $\theta_2$. The approximation is

$$\begin{aligned} \sin(\theta_i) &\approx \theta_i \\ \cos(\theta_i) &\approx 1 \\ \sin(\theta_1 - \theta_2) &\approx 0 \\ \cos(\theta_1 - \theta_2) &\approx 1. \end{aligned} \tag{39}$$

Then we can simplify Equation 34 to get

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 = -\left(m_1 + m_2\right) g \theta_1. \tag{40}$$

And for Equation 38, we get

$$l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 = \frac{u}{m_2} - g\theta_2. \tag{41}$$

**Matrix Formulation** Formulating the previous equations into a matrix form, we get

$$\begin{bmatrix} (m_1+m_2)l_1 & m_2l_2 \\ l_2 & l_1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = -g \begin{bmatrix} (m_1+m_2) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_2} \end{bmatrix} u. \tag{42}$$

Which leads us to

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} (m_1+m_2)l_1 & m_2l_2 \\ l_2 & l_1 \end{bmatrix}^{-1} \left( -g \begin{bmatrix} (m_1+m_2) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_2} \end{bmatrix} u \right). \tag{43}$$

Let $M = (m_1 + m_2)$ and $D = Ml_1^2 - m_2l_2^2$, then

$$\begin{bmatrix} Ml_1 & m_2l_2 \\ l_2 & l_1 \end{bmatrix}^{-1} = \frac{1}{D} \begin{bmatrix} l_1 & -m_2l_2 \\ -l_2 & Ml_1 \end{bmatrix}, \tag{44}$$

and we get the folowing

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \frac{-g}{D} \begin{bmatrix} Ml_1 & -m_2^2l_2 \\ -Ml_2 & Mm_2l_1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \frac{1}{D} \begin{bmatrix} -l_2 \\ \frac{M}{m_2}l_2 \end{bmatrix} u. \tag{45}$$

**State Space Representation** Ok, letting the state be $\mathbf{x} = \begin{bmatrix} \theta_1 & \dot{\theta}_1 & \theta_2 & \dot{\theta}_2 \end{bmatrix}^T$, we want something in the form

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{gMl_1}{D} & 0 & \frac{gm_2^2l_2}{D} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{gMl_2}{D} & 0 & -\frac{gMm_2l_1}{D} & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-l_2}{D} \\ 0 \\ \frac{Ml_2}{Dm_2} \end{bmatrix} u. \end{aligned} \tag{46}$$

The output equation would be something like

$$\mathbf{y} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \mathbf{0}u. \tag{47}$$

**The solution** The state space representation is given as

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u},\end{aligned} \tag{48}$$

where the matrices are

$$\begin{aligned}\begin{bmatrix}\dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2\end{bmatrix} &= \begin{bmatrix}0 & 1 & 0 & 0 \\ -\frac{gMl_1}{D} & 0 & \frac{gm_2^2l_2}{D} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{gMl_2}{D} & 0 & -\frac{gMm_2l_1}{D} & 0\end{bmatrix}\begin{bmatrix}\theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2\end{bmatrix} + \begin{bmatrix}0 \\ \frac{-l_2}{D} \\ 0 \\ \frac{Ml_2}{Dm_2}\end{bmatrix}u \\ \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix} &= \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2\end{bmatrix} + \mathbf{0}u.\end{aligned} \tag{49}$$