CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 2

Çavuşoğlu, Arda
e2448249@ceng.metu.edu.tr

Çolak, Ahmet Eren
e2587921@ceng.metu.edu.tr

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1. (a) $\dot{y}(t) = x(t) - 5y(t) \longrightarrow \dot{y}(t) + 5y(t) = x(t)$

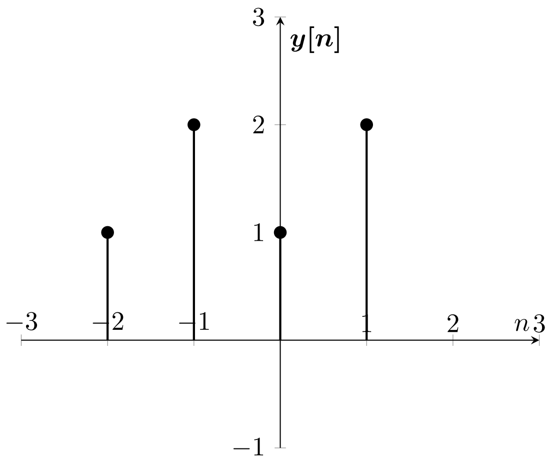
   (b) $x(t) = (e^{-t} + e^{-3t})u(t), y(t) = y_p(t) + y_h(t)$

   $y_p(t) = Kx(t) = K(e^{-t} + e^{-3t})u(t) \longrightarrow \frac{d(K(e^{-t}+e^{-3t})u(t)}{dt} + 5K(e^{-t} + e^{-3t})u(t) = (e^{-t} + e^{-3t})u(t)$
   $\frac{d(K(e^{-t}+e^{-3t})u(t)}{dt} = K(-e^{-t} - 3e^{-3t})u(t)$
   $Ku(t)(-e^{-t} - 3e^{-3t} + 5e^{-t} + 5e^{-3t}) = (e^{-t} + e^{-3t})u(t)$
   $K(4e^{-t} + 2e^{-3t}) = e^{-t} + e^{-3t}, K = \frac{e^{-t}+e^{-3t}}{4e^{-t}+2e^{-3t}} \longrightarrow y_p(t) = \frac{e^{-2t}+2e^{-4t}+e^{-6t}}{4e^{-t}+2e^{-3t}}$

   $y_h(t) = Ce^{\alpha t} \longrightarrow C\alpha e^{\alpha t} + 5Ce^{\alpha t} = 0$
   $C(\alpha + 5)e^{\alpha t} = 0 \longrightarrow \alpha = -5, y_h(t) = Ce^{-5t}$

   $y(t) = \frac{e^{-2t}+2e^{-4t}+e^{-6t}}{4e^{-t}+2e^{-3t}}u(t) + Ce^{-5t}$
   $y(0) = 0 \longrightarrow \frac{4}{6} + C = 0, C = -\frac{2}{3}$
   $y(t) = \frac{e^{-2t}+2e^{-4t}+e^{-6t}}{4e^{-t}+2e^{-3t}}u(t) - \frac{2}{3}e^{-5t}$

2. (a) $x[n] = 2\delta[n] + \delta[n+1], h[n] = \delta[n-1] + 2\delta[n+1]$
   $h_1[n] = \delta[n-1], h_2[n] = \delta[n+1] \longrightarrow h[n] = h_1[n] + h_2[n] + h_2[n]$
   $x[n] * h[n] = x[n](h_1[n] + h_2[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n] + x[n] * h_2[n]$ from distributivity.
   $x[n] * h_1[n] = x[n-1] = 2\delta[n-1] + \delta[n]$
   $x[n] * h_2[n] = x[n+1] = 2\delta[n+1] + \delta[n+2]$
   $y[n] = 2\delta[n-1] + \delta[n] + 2\delta[n+1] + \delta[n+2]$



(b) $x(t) = u(t-1) + u(t+1), h(t) = e^{-t}sin(t)u(t), y(t) = \frac{dx(t)}{dt} * h(t)$

$\dot{x}(t) = \delta(t-1) + \delta(t+1)$

$x_1(t) = \delta(t-1), x_2(t) = \delta(t+1) \longrightarrow \dot{x}(t) = x_1(t) + x_2(t)$

$\dot{x}(t) * h(t) = h(t) * \dot{x}$ (from commutativity), $h(t) * \dot{x}(t) = h(t) * x_1(t) + h(t) * x_2(t)$ (from distributivity)

$h(t) * x_1(t) = h(t-1), h(t) * x_2(t) = h(t+1)$

$y(t) = h(t-1) + h(t+1) = e^{1-t}sin(t-1)u(t-1) + e^{-1-t}sin(t+1)u(t+1)$

3. (a) $h(t) = e^{-2t}u(t), x(t) = e^{-t}y(t) = x(t) * h(t)$

$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$

$= \int_0^t e^{\tau-t}.e^{-2\tau}d\tau$

$= e^{-t}\int_0^t e^{-\tau}d\tau$

$= e^{-t}(-e^{-T})|_0^t$

$e^{-t}(1-e^{-t})u(t)$

(b) $h(t) = e^{3t}u(t), x(t) = u(t) - u(t-1)y(t) = x(t) * h(t)$

$x(t) = u(t) - u(t-1) = \delta(t)$

$h(t) * x(t) = h(t) * \delta(t) = h(t) = e^{3t}u(t)$

4. (a) Guess $y[n] = Cz^n$:

$$Cz^n - Cz^{n-1} - Cz^{n-2} = 0$$
$$Kz^{n-2}(z^2 - z - 1) = 0$$
$$(z^2 - z - 1) = 0$$
$$z_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$y[n]$ is the linear combination of solutions for z_1 and z_2.

$$y[n] = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

By the initial conditions $y[0] = 1$ and $y[1] = 1$, coefficients C_1 and C_2 can be calculated.

$$C_1 = \frac{1+\sqrt{5}}{10}, \quad C_2 = \frac{9-\sqrt{5}}{2}$$
$$y[n] = \frac{1+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{9-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

(b) Assume $y(t) = Ce^{\alpha t}$, then characteristic equation will be:

$$\alpha^3 - 6\alpha^2 + 13\alpha - 10 = 0$$
$$\alpha_1 = 2, \quad \alpha_{2,3} = 2 \pm j$$

For the imaginary part, there are 2 unique solutions.

$$y_1'(t) = e^{2t}e^{jt} = cos(t) + jsin(t)$$
$$y_2'(t) = e^{2t}e^{-jt} = cos(t) - jsin(t)$$

$$y_1(t) = \frac{1}{2}y_1'(t) + \frac{1}{2}y_2'(t) = e^{2t}cos(t)$$
$$y_2(t) = \frac{1}{2j}y_1'(t) - \frac{1}{2j}y_2'(t) = e^{2t}sin(t)$$

Solution for the imaginary part is the linear combination of these 2 unique solutions.

$$C_2y_1(t) + C_3y_2(t)$$

$$y(t) = C_1e^{2t} + C_2e^{2t}cos(t) + C_3e^{2t}sin(t)$$

By the initial conditions $y''(0) = 3$, $y'(0) = 1.5$, $y(0) = 1$, solution is:

$$y(t) = 2e^{2t} - 1e^{2t}cos(t) - 0.5e^{2t}sin(t)$$

5. (a) Guess $y_p(t) = H(\lambda)cos(5t) = H(\lambda)(0.5e^{5jt} + 0.5e^{-5jt})$. Using the linearity property solutions is in the form of:

$$y_p(t) = 0.5H(5j)e^{5jt} + 0.5H(-5j)e^{-5jt}$$

where $H(\lambda)$ corresponds to the transfer function. To find the transfer function assume, input is $x(t) = e^{\lambda t}$. Then output will be in the form of $H(\lambda)e^{\lambda t}$

$$H(\lambda)e^{\lambda t}(\lambda^2 + 5\lambda + 6) = e^{\lambda t}$$

$$H(\lambda) = \frac{1}{\lambda^2 + 5\lambda + 6}$$

$$H(5j) = \frac{1}{-19 + 25j}, \quad H(-5j) = \frac{1}{-19 - 25j}$$

$$y_p(t) = \frac{1}{-38 + 50j}e^{5jt} + \frac{1}{-38 - 50j}e^{-5jt}$$

(b) Characteristic equation of the systems is:

$$\alpha^2 + 5\alpha + 6 = 0$$

$$\alpha_1 = -3, \quad \alpha_2 = -2$$

Homogenious solution is $y_h(t) = C_1e^{-3t} + C_2e^{-2t}$

(c) Since the system is rest initially, the homogenious solution is 0. Then the general solution equals to:

$$y(t) = \frac{1}{-38 + 50j}e^{5jt} + \frac{1}{-38 - 50j}e^{-5jt}$$

6. (a) $x[n] = \delta[n]$

$$w[n] = \delta[n] + \frac{1}{2}w[n-1]$$

$$w[0] = \delta[0] + \frac{1}{2}w[-1] = 1$$

$$w[1] = \delta[1] + \frac{1}{2}w[0] = \frac{1}{2}$$

$$w[2] = \delta[2] + \frac{1}{2}w[1] = \frac{1}{4}$$

$$h_0[n] = w[n] = \left(\frac{1}{2}\right)^n u[n]$$

(b)

$$h[n] = h_0[n] * h_0[n]$$

$$h[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k] \left(\frac{1}{2}\right)^{n-k} u[n-k]$$

$$h[n] = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

$$h[n] = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{-k}$$

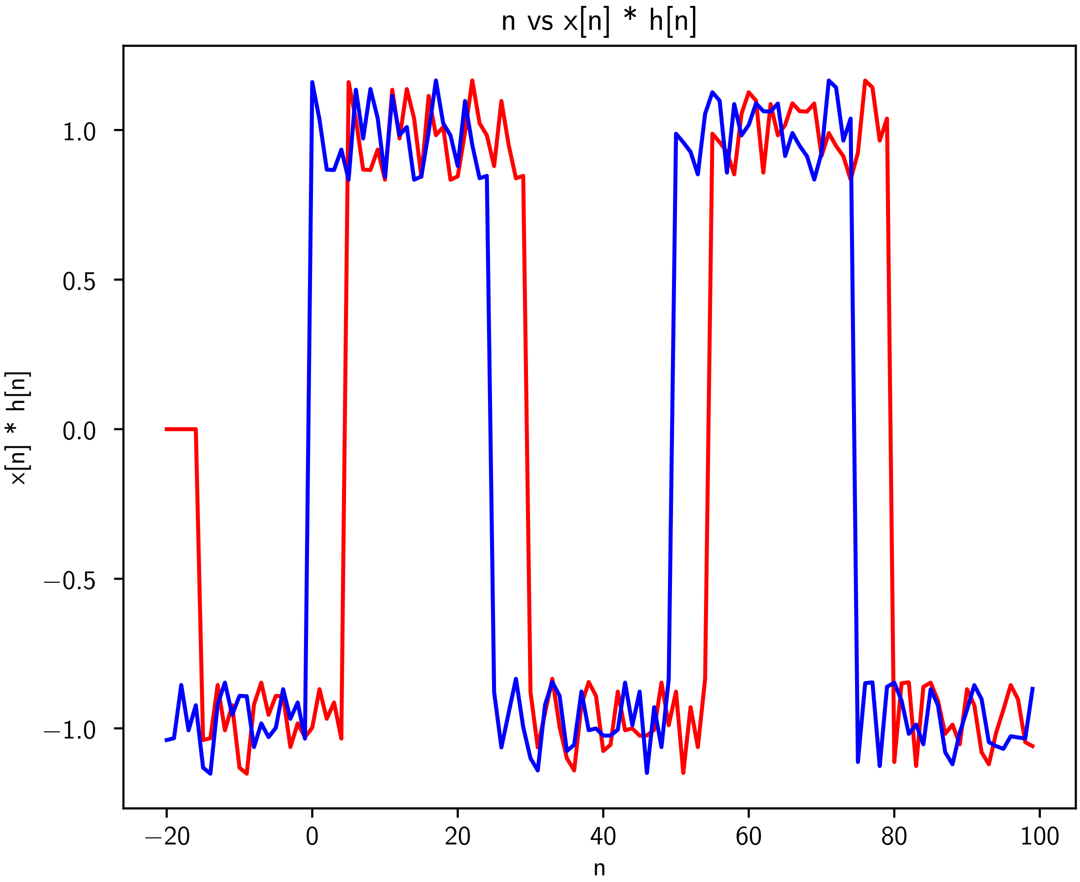
$$h[n] = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 1 = \left(\frac{1}{2}\right)^n u[n]n$$

(c)

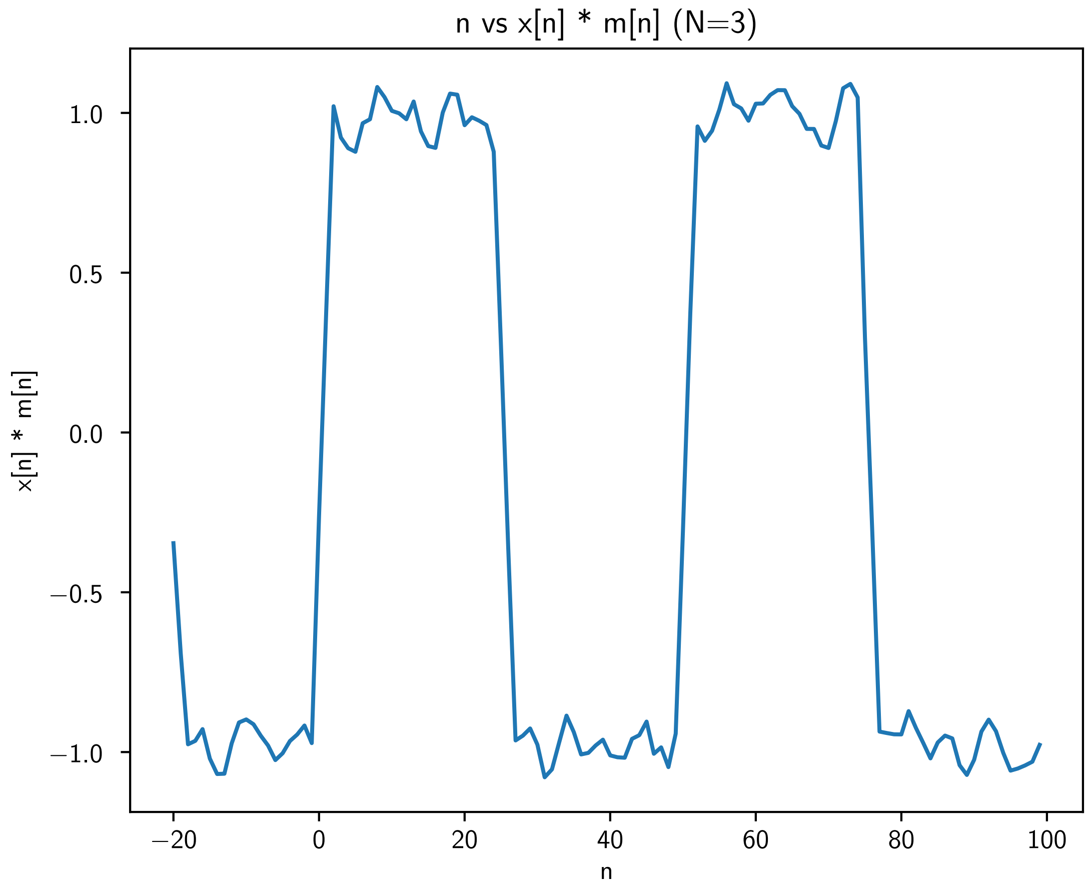
$$y[n] = h[n] * x[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k k u[k] x[n-k]$$

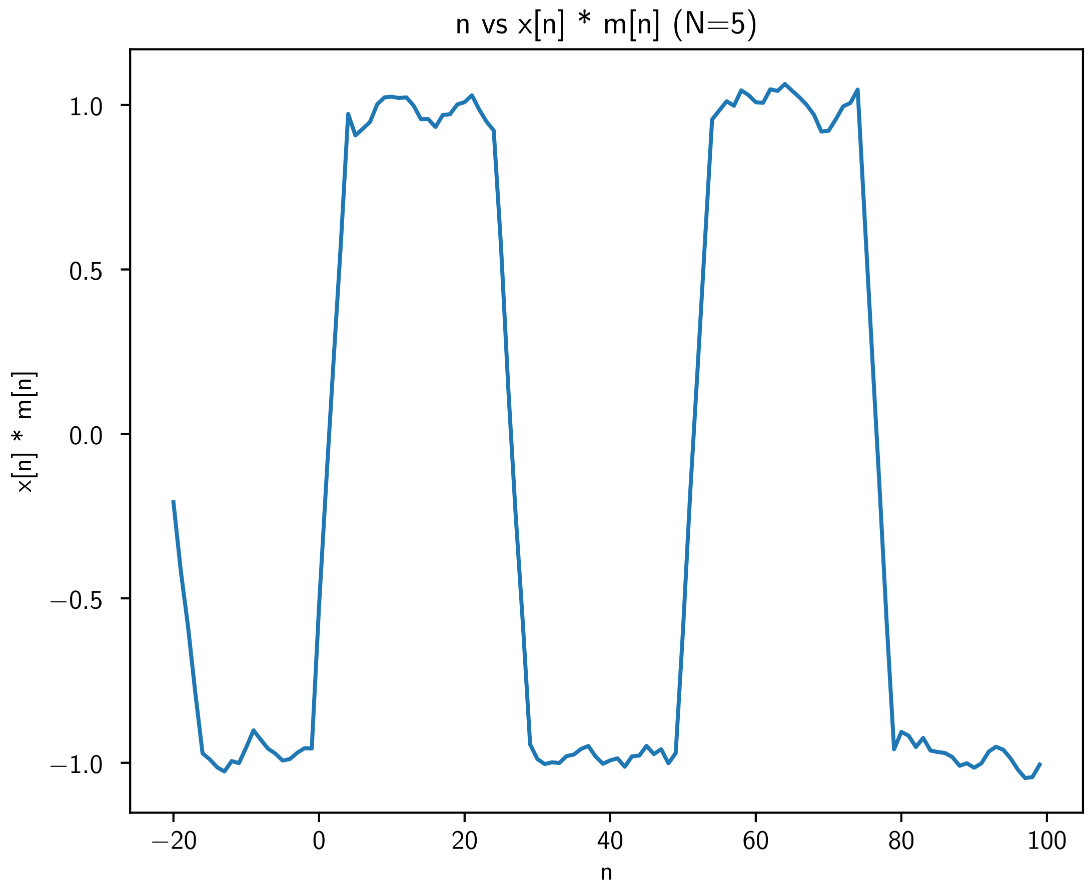
$$y[n] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k k x[n-k]$$

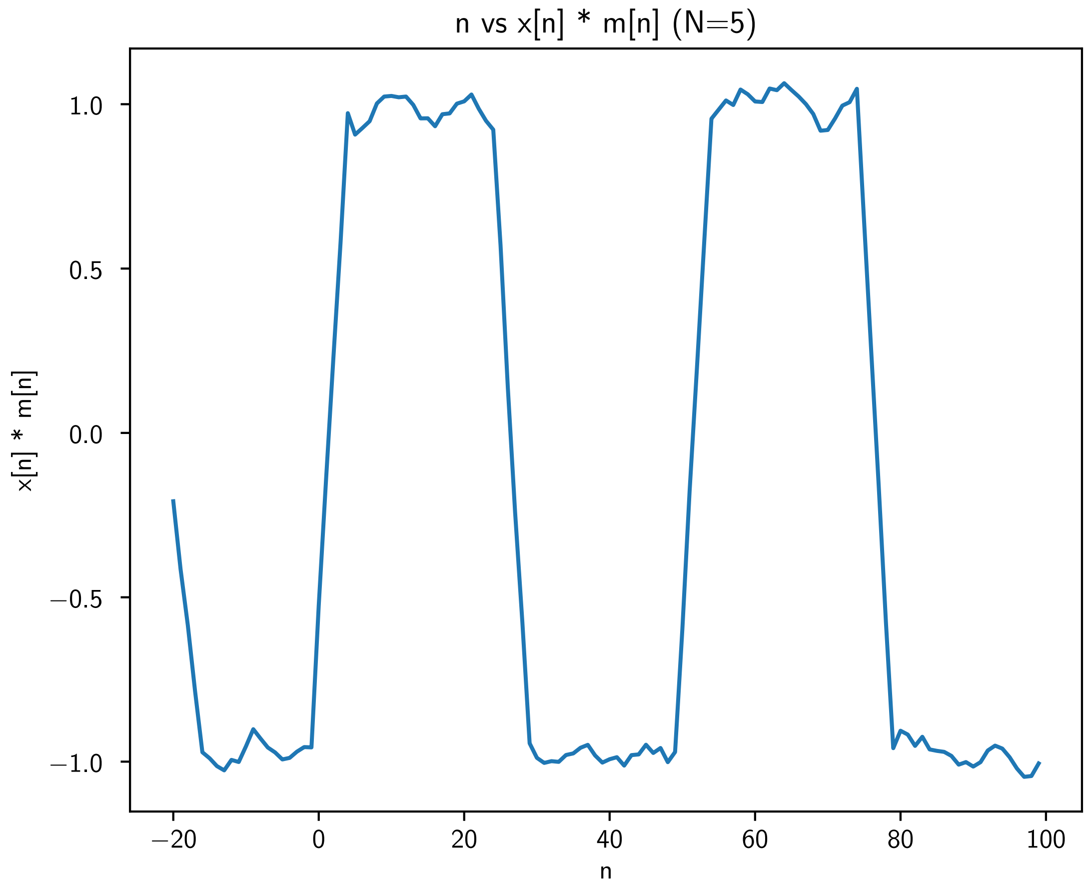


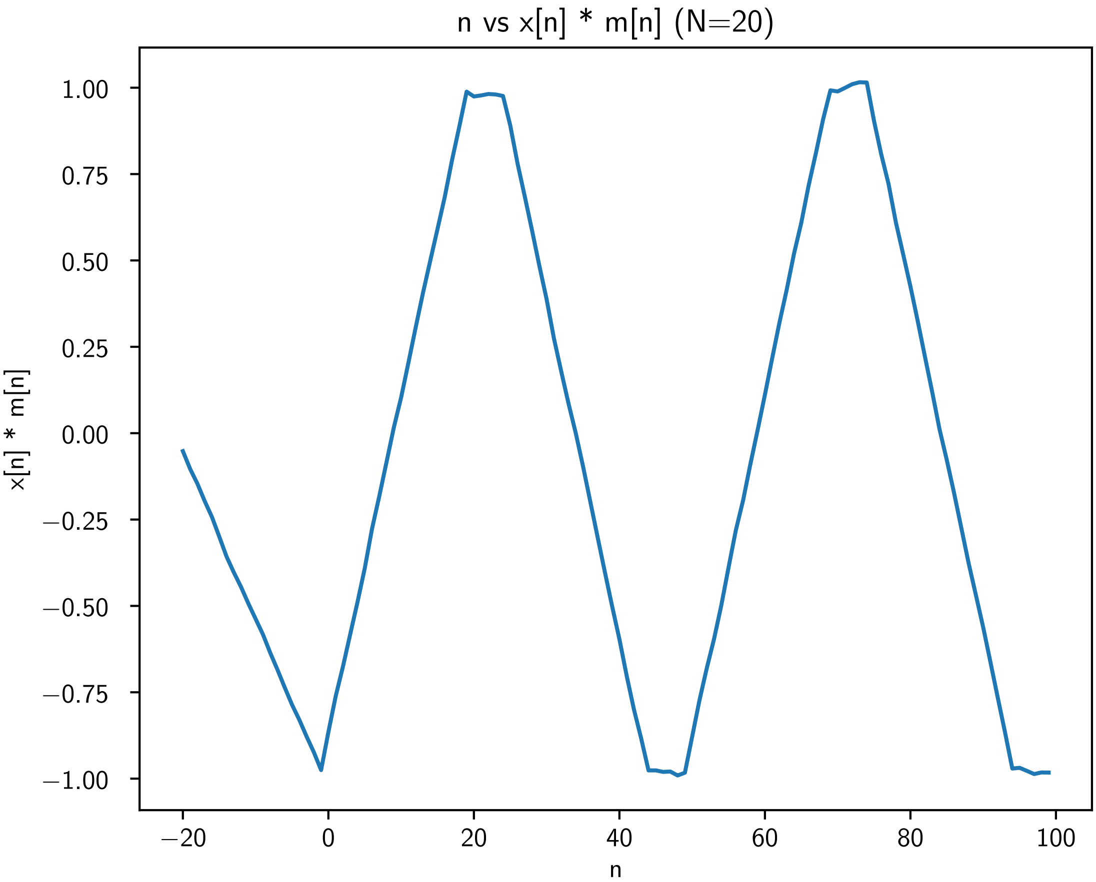
7. (a) Convolving with the $\delta[n-5]$ shifted the signal 5 units to the right.


n vs x[n] * m[n] (N=3)
1.0
0.5
0.0
−0.5
−1.0
x[n] * m[n]
−20
0
20
40
60
80
100
n

(b)


n vs x[n] * m[n] (N=5)
1.0
0.5
0.0
−0.5
−1.0
x[n] * m[n]
−20
0
20
40
60
80
100
n





Convolving with the N-point moving average filter reduces the noise of the signal. Noise is reduced more as N increases.