CENG 382 - Analysis of Dynamic Systems
20221
Take Home Exam 1 Solutions

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1. (a) linear, time invariant, forced
   (b) linear, time variant, unforced
   (c) non-linear, time variant, forced
2. (a)

$$\frac{dx}{dt} = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ x_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

To simplify the problem, transformation below is applied.

$$u = x + A^{-1}b$$
$$\frac{du}{dt} = \frac{dx}{dt}$$
$$\frac{du}{dt} = Au$$

Then fundamental solution matrix can be calculated as:

$$X(t) = e^{At}$$

Matrix A, should be diagonalized to calculate e^{At}.

$$A = PDP^{-1}$$

$$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \quad \text{where } v_1 \text{ and } v_2 \text{ are eigen vectors}$$
$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{where } \lambda_1 \text{ and } \lambda_2 \text{ are eigen velues}$$

Characteristic equation of A is: $(\lambda - 4)(\lambda + 3) = 0$. Therefore $\lambda_1 = 4$ and $\lambda_2 = -3$.

Solution for v_1:

$$(A - 4I)v_1 = 0$$

$$\begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$v_1 = v_1$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution for v_2:

$$(A - (-3)I)v_1 = 0$$

$$\begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$5v_1 = -2v_2$$

$$v_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{7} \cdot \begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{7}$$

$$e^{At} = \begin{bmatrix} \frac{5e^{4t} + 2e^{-3t}}{7} & \frac{2e^{4t} - 2e^{-3t}}{7} \\ \frac{5e^{4t} - 5e^{-3t}}{7} & \frac{2e^{4t} + 5e^{-3t}}{7} \end{bmatrix}$$

State transition matrix $\phi(t, l)$, equals to $X(t)X^{-1}(l)$ where $X(t) = e^{At}$. Then $\phi(t, 0)u(0)$ is the solution $u(t)$, with the initial condition $u(0)$. Since $X^{-1}(0) = I$, $u(t) = X(t)u(0)$ where $u(0)$ is $x(0) + A^{-1}b$

$$A^{-1}b = \begin{bmatrix} \frac{5}{12} \\ \frac{1}{12} \end{bmatrix}$$

$$u(0) = x(0) + A^{-1}b = \begin{bmatrix} \frac{-7}{12} \\ \frac{25}{12} \end{bmatrix}$$

$$u(t) = X(t)u(0) = \begin{bmatrix} \frac{5e^{4t}}{28} - \frac{16e^{-3t}}{21} \\ \frac{5e^{4t}}{28} - \frac{40e^{-3t}}{21} \end{bmatrix}$$

$$x(t) = u(t) - A^{-1}b = \begin{bmatrix} \frac{5e^{4t}}{28} - \frac{16e^{-3t}}{21} - \frac{5}{12} \\ \frac{5e^{4t}}{28} - \frac{40e^{-3t}}{21} - \frac{1}{12} \end{bmatrix}$$

(b) System diverges as t goes to ∞ because of the terms including e^{4t}.

3. Fixed point of the system is where $\dot{x} = -7x + 5 = 0$. Therefore, $x = 5/7$ is the fixed point of the system.

   Apply transformation $u = x - 5/7$. Then equation becomes $\dot{u} = -7u$.

$$u(t) = u(0)e^{-7t}$$
$$u(0) = x(0) - \frac{5}{7}$$
$$x(t) = (x(0) - \frac{5}{7})e^{-7t} + \frac{5}{7}$$

   As t goes to ∞, $x(t)$ is going to converge to $5/7$ because the term with e^{-7t} will converge to 0.

4. Define y_1, y_2, y_3 as:

$$y_1 = x$$
$$y_2 = \frac{dx}{dt}$$
$$y_3 = \frac{d^2x}{dt^2}$$

   Then derivatives of y_1, y_2, y_3 are:

$$\frac{dy_1}{dt} = y_2$$
$$\frac{dy_2}{dt} = y_3$$
$$\frac{dy_3}{dt} = -t^3y_3 - (t+1)y_2 + y_1 + 2t + 1$$

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(t+1) & -t^3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2t+1 \end{bmatrix}$$

5. (a) Solution for the initial value $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$:

$$x(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ x(2) = \begin{bmatrix} 1 \\ 5 \end{bmatrix},\ x(3) = \begin{bmatrix} 1 \\ 9 \end{bmatrix},\ x(4) = \begin{bmatrix} 1 \\ 14 \end{bmatrix},\ \ldots$$

   Pattern above is following the equation: $\begin{bmatrix} 1 \\ \frac{(k+1)(k+2)-2}{2} \end{bmatrix}$. Hence solution for this initial condition is $\begin{bmatrix} 1 \\ \frac{(k+1)(k+2)-2}{2} \end{bmatrix}$.

   Solution for the initial value $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$x(k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   $x(k)$ is constant and equals to its initial value.

   Fundamental solution matrix is therefore equals to $X(k) = \begin{bmatrix} 1 & 0 \\ \frac{(k+1)(k+2)-2}{2} & 1 \end{bmatrix}$

(b)

$$\phi(k,0) = X(k)X^{-1}(0) = X(k)I = X(k) = \begin{bmatrix} 1 & 0 \\ \frac{(k+1)(k+2)-2}{2} & 1 \end{bmatrix}$$

(c)

$$\tilde{x} = \begin{bmatrix} 1 & 0 \\ k+2 & 1 \end{bmatrix} \tilde{x}$$

$$\tilde{x_1} = \tilde{x_1}$$
$$\tilde{x_2} = \tilde{x_1}(k+2) + \tilde{x_2}$$
$$\tilde{x_1}(k+2) = 0$$

Equations above only satisfied when $\tilde{x_1} = 0$ and any $\tilde{x_2}$. Hence, $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ is the set of fixed points of the system.

$$x(k) = \phi(k,0)x(0) = \begin{bmatrix} x_1(0) \\ x_1(0)\frac{(k+1)(k+2)-2}{2} + x_2(0) \end{bmatrix}$$

System will only converge to the fixed point, if $x_1(0) = 0$. The fixed point is not stable because system will not converge to it unless its initial condition is one of the fixed points.