

CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 3

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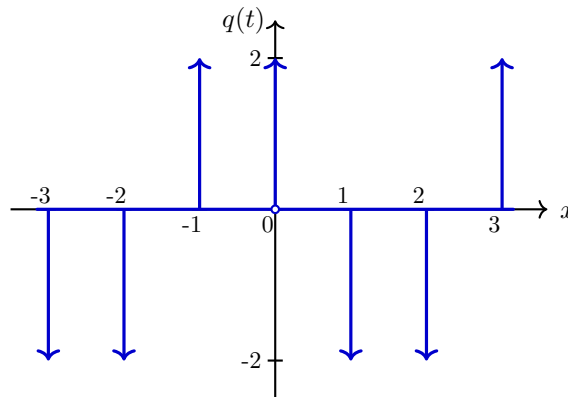
1.

$$\begin{aligned}\int_{-\infty}^t x(s)ds &= \int_{-\infty}^t \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 s} ds \\ \int_{-\infty}^t x(s)ds &= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^t e^{jk\omega_0 s} ds \\ \int_{-\infty}^t x(s)ds &= \sum_{k=-\infty}^{\infty} a_k \frac{e^{jk\omega_0 t}}{jk\omega_0}\end{aligned}$$

Equality above proves that $\int_{-\infty}^t x(s)ds$ has spectral coefficients $a_k \frac{1}{jk\omega_0}$ where $\omega_0 = 2\pi/T$

2. (a) $x(t)x(t) \longleftrightarrow \sum_{l=-\infty}^{\infty} a_l \cdot a_{k-l} = a_k * a_k$ (from multiplication property)
 (b) $R\{a_k\}$
 (c) $x(t+t_0) \longleftrightarrow e^{jk\omega_0 t_0} a_k$, $x(t-t_0) \longleftrightarrow e^{-jk\omega_0 t_0} a_k$ (from time-shift property)
 $x(t+t_0) + x(t-t_0) \longleftrightarrow a_k(e^{jk\omega_0 t_0} + e^{-jk\omega_0 t_0})$ (from linearity property)

3. Let $q(t) = \frac{dx(t)}{dt}$ and $z(t) = \sum_{k=-\infty}^{\infty} \delta(t-4k)$, then



$$q(t) = z(t) + z(t+1) - z(t-1) - z(t-2)$$

Fourier series of $z(t) \longleftrightarrow a_k = \frac{1}{4} \int_{-2}^2 \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{4}$, then

$$q(t) \longleftrightarrow b_k = \frac{1}{4} + e^{jk\omega_0} \frac{1}{4} - e^{-jk\omega_0} \frac{1}{4} - e^{-2jk\omega_0} \frac{1}{4}$$

$$\omega_0 = \frac{2\pi}{T} \text{ and } T = 4 \longrightarrow b_k = \frac{1}{4}(1 + e^{jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}} - e^{-jk\pi})$$

$$b_k = \frac{1}{4}(1 + \cos(k\frac{\pi}{2}) + j\sin(k\frac{\pi}{2}) - \cos(k\frac{\pi}{2}) + j\sin(k\frac{\pi}{2}) - \cos(k\pi) + j\sin(k\pi))$$

$$b_k = \frac{1}{4}(1 - \cos(k\pi) + 2j\sin(k\frac{\pi}{2}))$$

$$x(t) \longleftrightarrow c_k = \frac{b_k}{jk\omega_0} = \frac{1 - \cos(k\pi) + 2j\sin(k\frac{\pi}{2})}{2\pi jk}$$

4. (a) $x(t) = 1 + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{j(2\omega_0 t + \frac{\pi}{4})}{2}$

$$x(t) = 1 + e^{j\omega_0 t}(\frac{1}{2j} + 1) + e^{-j\omega_0 t}(\frac{-1}{2j} + 1) + e^{2j\omega_0 t}(\frac{e^{\frac{\pi}{4}}}{2}) + e^{-2j\omega_0 t}(\frac{e^{\frac{\pi}{4}}}{2})$$

$$a_0 = 1$$

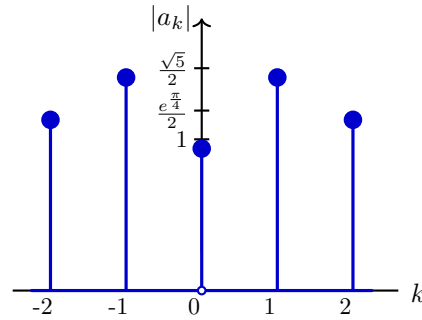
$$a_1 = \frac{1}{2j} + 1 = 1 - \frac{j}{2}$$

$$a_{-1} = \frac{-1}{2j} + 1 = 1 + \frac{j}{2}$$

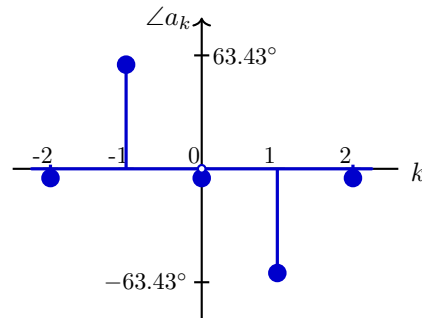
$$a_2 = \frac{e^{\frac{\pi}{4}}}{2}$$

$$a_{-2} = \frac{e^{\frac{\pi}{4}}}{2}$$

Magnitudes:



Phases:



(b)

$$x(t) = e^{j\omega t}$$

$$y(t) = H(j\omega)e^{j\omega t}$$

Substitute $y(t)$ and $x(t)$ in the differential equation:

$$H(j\omega)j\omega e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$H(j\omega) = \frac{1}{1 + j\omega}$$

(c)

$$x(t) = e^{j0t} + \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{1}{2}e^{\pi/4}e^{j2\omega_0 t} + \frac{1}{2}e^{\pi/4}e^{-j2\omega_0 t}$$

$$y(t) = H(0)e^{j0t} + \frac{1}{2j}H(j\omega_0)e^{j\omega_0 t} - \frac{1}{2j}H(-j\omega_0)e^{-j\omega_0 t} + H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t} + \frac{1}{2}H(j2\omega_0)e^{\pi/4}e^{j2\omega_0 t} +$$

$$\frac{1}{2}H(-j2\omega_0)e^{\pi/4}e^{-j2\omega_0 t}$$

$$b_0 = 1$$

$$b_1 = H(j\omega_0) \left(1 + \frac{1}{2j}\right)$$

$$b_{-1} = H(-jw_0) \left(1 - \frac{1}{2j}\right)$$

$$b_2 = \frac{e^{\pi/4}}{2} H(j2w_0)$$

$$b_{-2} = \frac{e^{\pi/4}}{2} H(-j2w_0)$$

When $w_0 = 2\pi$:

$$b_0 = 1$$

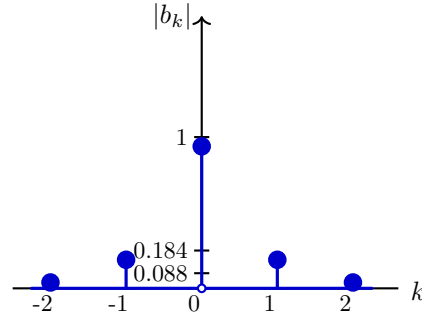
$$b_1 = \frac{1 - 2\pi j}{1 - 4\pi^2} \left(1 + \frac{1}{2j}\right), |b_1| = 0.184, \angle b_1 = -107.1^\circ$$

$$b_{-1} = \frac{1 + 2\pi j}{1 - 4\pi^2} \left(1 - \frac{1}{2j}\right), |b_{-1}| = 0.184, \angle b_{-1} = 107.1^\circ$$

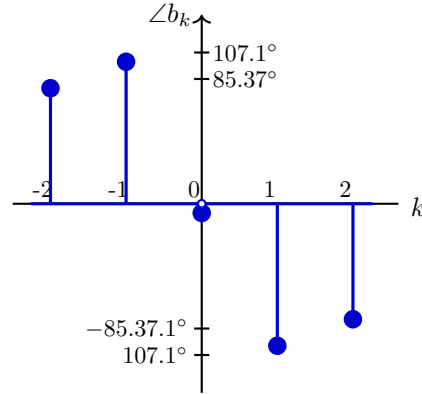
$$b_2 = \frac{1 - 4\pi j}{1 - 16\pi^2} \frac{e^{\pi/4}}{2}, |b_2| = 0.088, \angle b_2 = -85.37^\circ$$

$$b_{-2} = \frac{1 + 4\pi j}{1 - 16\pi^2} \frac{e^{\pi/4}}{2}, |b_{-2}| = 0.088, \angle b_{-2} = 85.37^\circ$$

Magnitudes:



Phases:



(d)

$$y(t) = \sum_{k=-2}^2 b_k e^{jw_0 k t}$$

5. (a) Period of $x[n] = T_x = 4$

$$\begin{aligned} x[n] &\longleftrightarrow a_k = \frac{1}{4} \sum_{n=1}^4 x[n] e^{-jk \frac{\pi}{2} n} \\ &= \frac{1}{4} (e^{-jk \frac{\pi}{2}} - e^{-jk \frac{3\pi}{2}}) \\ &= \frac{1}{4} (\cos(k \frac{\pi}{2}) - j \sin(k \frac{\pi}{2}) - \cos(k \frac{3\pi}{2}) + j \sin(k \frac{3\pi}{2})) \end{aligned}$$

(b) Period of $y[n] = T_y = 4$

$$\begin{aligned}
 y[n] &\longleftrightarrow b_k = \frac{1}{4} \sum_{n=1}^4 y[n] e^{-jk\frac{\pi}{2}n} \\
 &= \frac{1}{4} (2e^{-jk2\pi} + e^{-jk\frac{\pi}{2}} + e^{-jk\frac{3\pi}{2}}) \\
 &= \frac{1}{4} (2\cos(k2\pi) - 2j\sin(k2\pi) + \cos(k\frac{\pi}{2}) - j\sin(k\frac{\pi}{2}) + \cos(k\frac{3\pi}{2}) - j\sin(k\frac{3\pi}{2})) \\
 &= \frac{1}{4} (2 + \cos(k\frac{\pi}{2}) - j\sin(k\frac{\pi}{2}) + \cos(k\frac{3\pi}{2}) - j\sin(k\frac{3\pi}{2})), \text{ as } y[n] \text{ is a discrete function}
 \end{aligned}$$

(c) Period of $x[n] \times y[n] = 4$

$$\begin{aligned}
 x[n] \times y[n] &\longleftrightarrow c_k = \sum_{l=1}^4 a_l b_{k-l} = \sum_{l=1}^4 b_l a_{k-l} \\
 &= \frac{1}{2} a_{k-l} + \frac{1}{2} a_{k-3} + ak - 4 \\
 &= \frac{1}{8} (\sin(k\frac{\pi}{2}) + j\cos(k\frac{\pi}{2}) + \sin(k\frac{3\pi}{2}) + j\cos(k\frac{3\pi}{2})) + \frac{1}{8} (-\sin(k\frac{\pi}{2}) - j\cos(k\frac{\pi}{2}) - \sin(k\frac{3\pi}{2}) - j\cos(k\frac{3\pi}{2})) + \frac{1}{4} (\cos(k\frac{\pi}{2}) - j\sin(k\frac{\pi}{2}) - \cos(k\frac{3\pi}{2}) + j\sin(k\frac{3\pi}{2})) \\
 &= \frac{1}{4} (\cos(k\frac{\pi}{2}) - j\sin(k\frac{\pi}{2}) - \cos(k\frac{3\pi}{2}) + j\sin(k\frac{3\pi}{2}))
 \end{aligned}$$

(d) $g[x] = x[n] \times y[n] = \sin(\frac{\pi}{2}n) + \sin(\frac{\pi}{2}n)\cos(\frac{\pi}{2}n) = \sin(\frac{\pi}{2}n) + \frac{1}{2}\sin(\pi n) = \sin(\frac{\pi}{2}n)$ and $T_g = 4$

$$\begin{aligned}
 g[n] &\longleftrightarrow d_k = \frac{1}{4} \sum_{n=1}^4 g[n] e^{-jk\frac{\pi}{2}n} = \frac{1}{4} (e^{-jk\frac{\pi}{2}} - e^{-jk\frac{3\pi}{2}}) \\
 &= \frac{1}{4} (\cos(k\frac{\pi}{2}) - j\sin(k\frac{\pi}{2}) - \cos(k\frac{3\pi}{2}) + j\sin(k\frac{3\pi}{2}))
 \end{aligned}$$

The Fourier coefficients found in part c are equal to the ones found in part d.

6. (a)

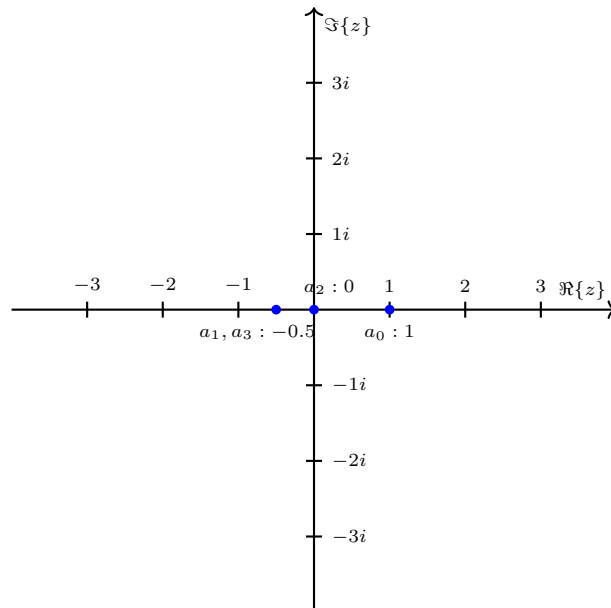
$$a_k = \frac{1}{4} \sum_{n=0}^3 x[n] e^{-j\pi/4kn}$$

$$a_0 = \frac{1}{4} \cdot 4 = 1$$

$$a_1 = \frac{1}{4} \cdot (0 - j - 2 + j) = \frac{-1}{2}$$

$$a_2 = \frac{1}{4} \cdot (0 - 1 + 2 - 1) = 0$$

$$a_3 = \frac{1}{4} \cdot (0 + j - 2 - j) = \frac{-1}{2}$$



(b)

$$y[n] = x[n] - \sum_{k=-\infty}^{\infty} \delta[n - (3 + 4k)]$$

Let spectral coefficients of $\sum_{k=-\infty}^{\infty} \delta[n - (3 + 4k)]$ be c_k

$$c_k = \frac{1}{4} \sum_{n=0}^3 \delta[n - 3] e^{-j\pi/4 kn}$$

$$c_0 = 1/4$$

$$c_1 = j/4$$

$$c_2 = -1/4$$

$$c_3 = -j/4$$

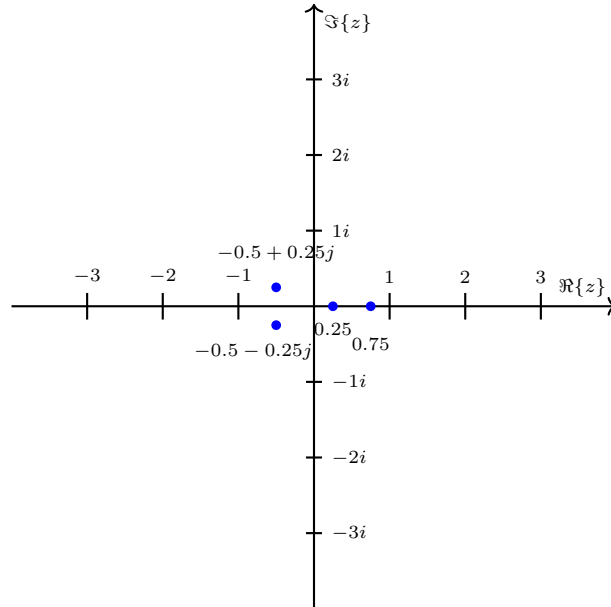
Then spectral coefficients of $y[n]$, b_k equals to $a_k - c_k$.

$$b_0 = 3/4$$

$$b_1 = \frac{-1}{2} - j/4$$

$$b_2 = 1/4$$

$$b_3 = \frac{-1}{2} + j/4$$



7. (a)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk w_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk w_0) e^{jk w_0 t}$$

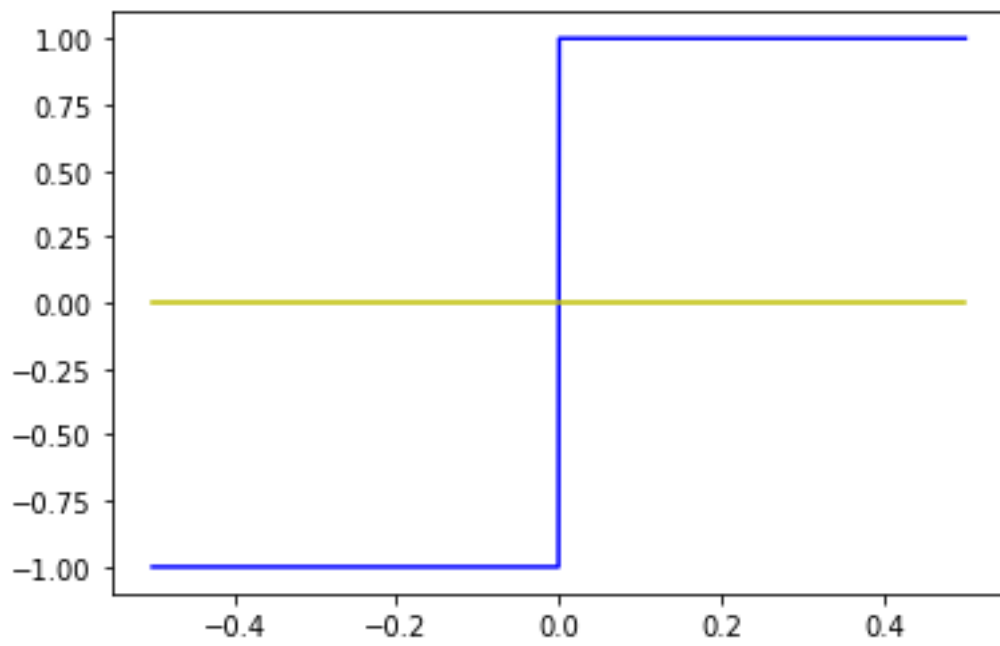
Since $w_0 = 2\pi/(\pi/K) = 2K$, coefficient a_k where the value of $|k \cdot 2K|$ is greater than 80, is zero.

(b) If $y(t) \neq x(t)$, then there must be at least one value of a_k , where $|k \cdot 2K|$ is greater than 80 and non-zero.

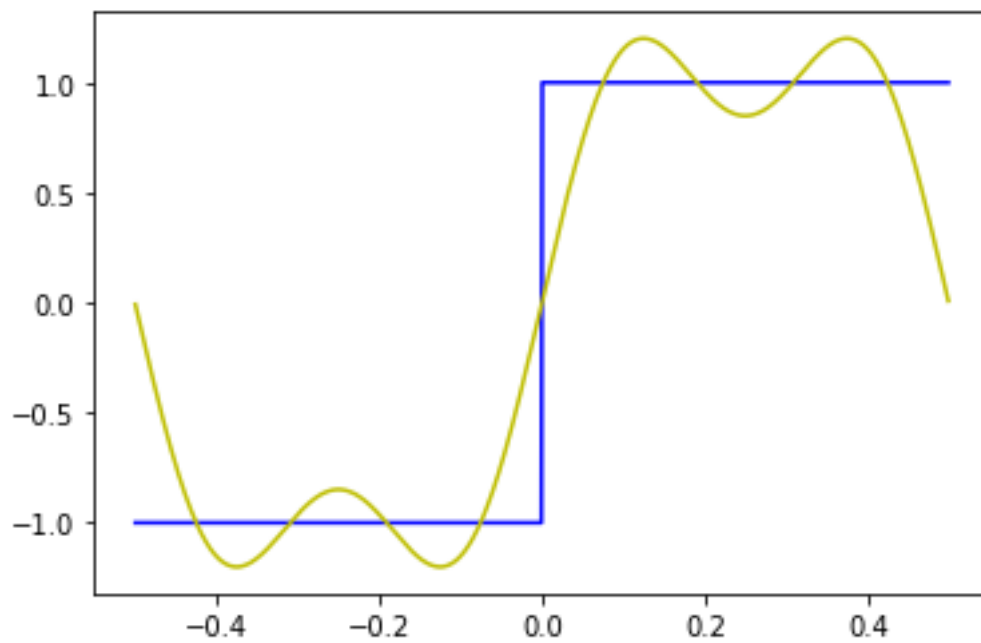
8.

For part C:

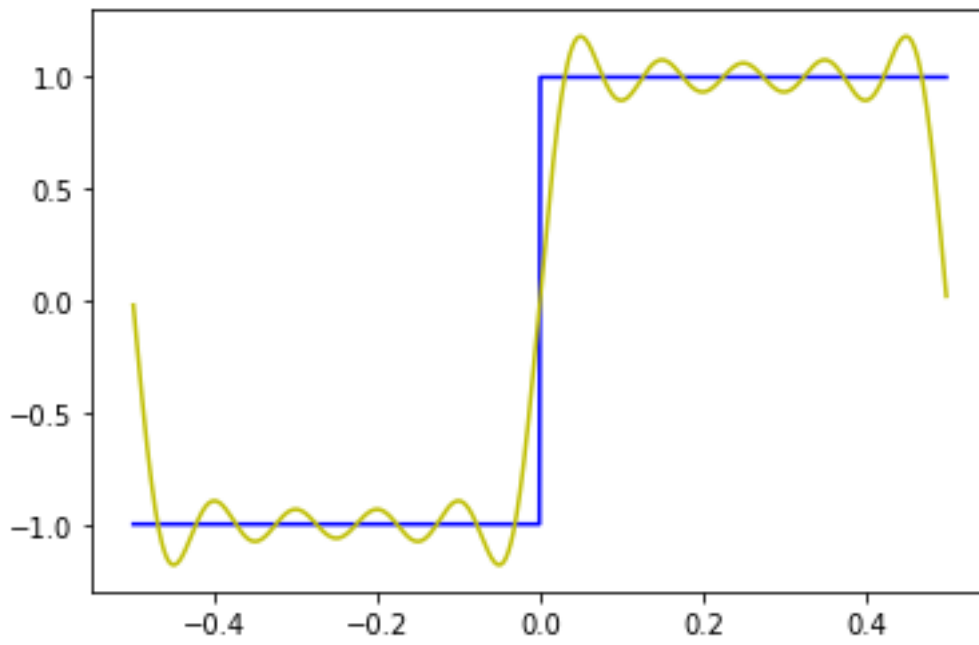
$N = 1$:



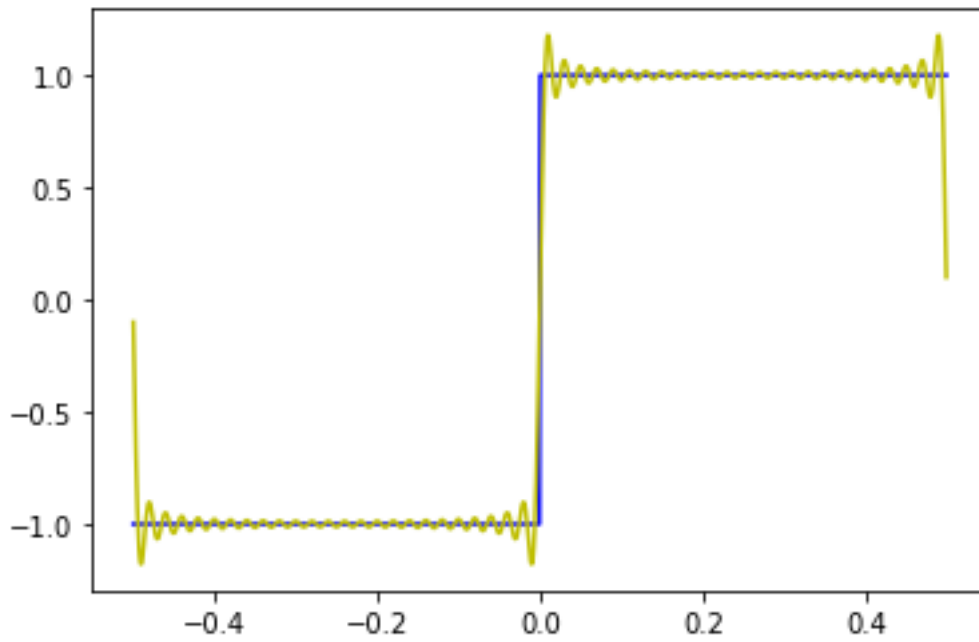
$N = 5$:



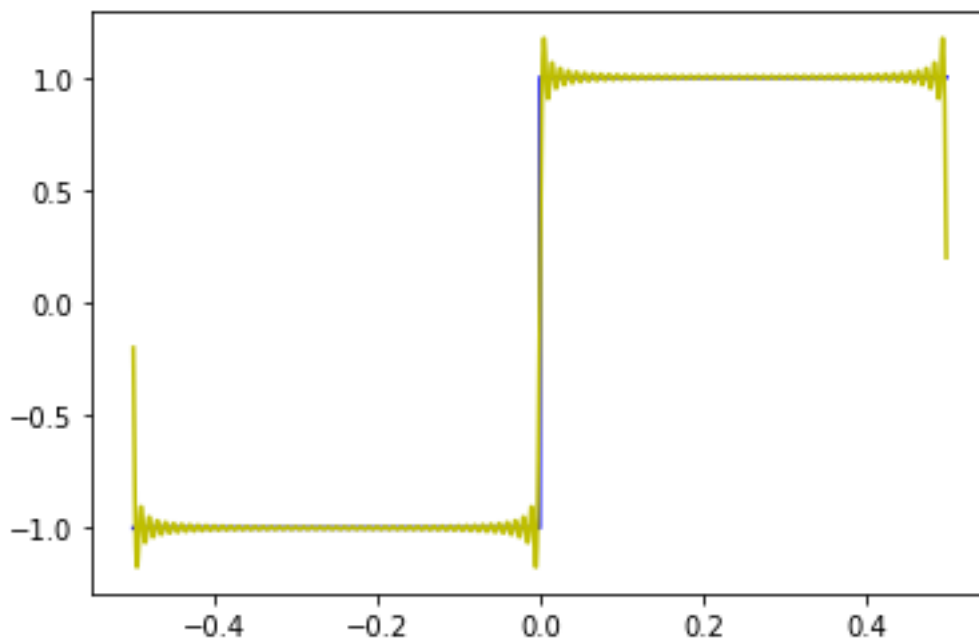
$N = 10$:



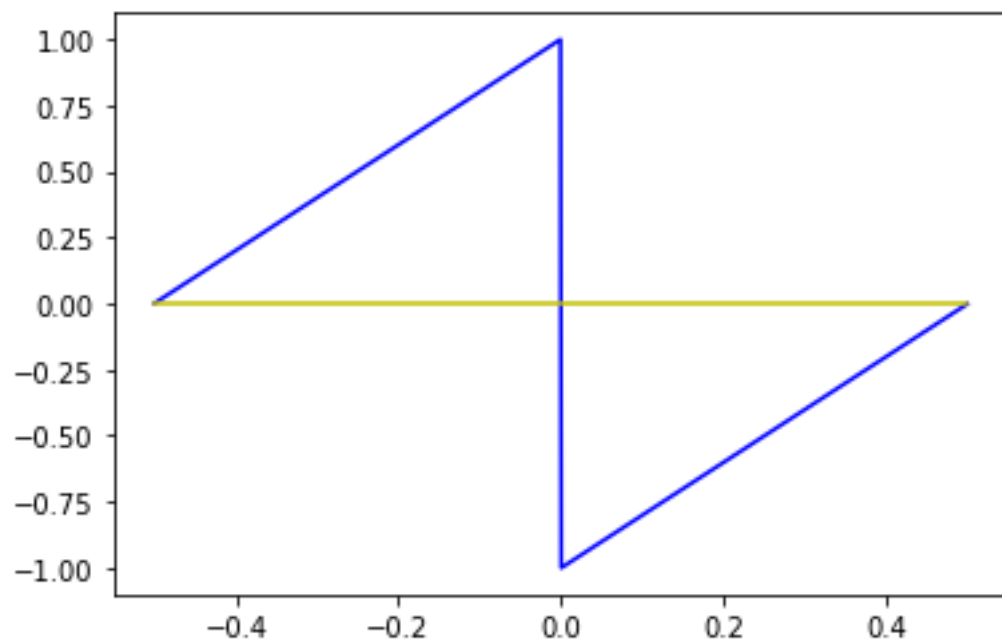
$N = 50$:



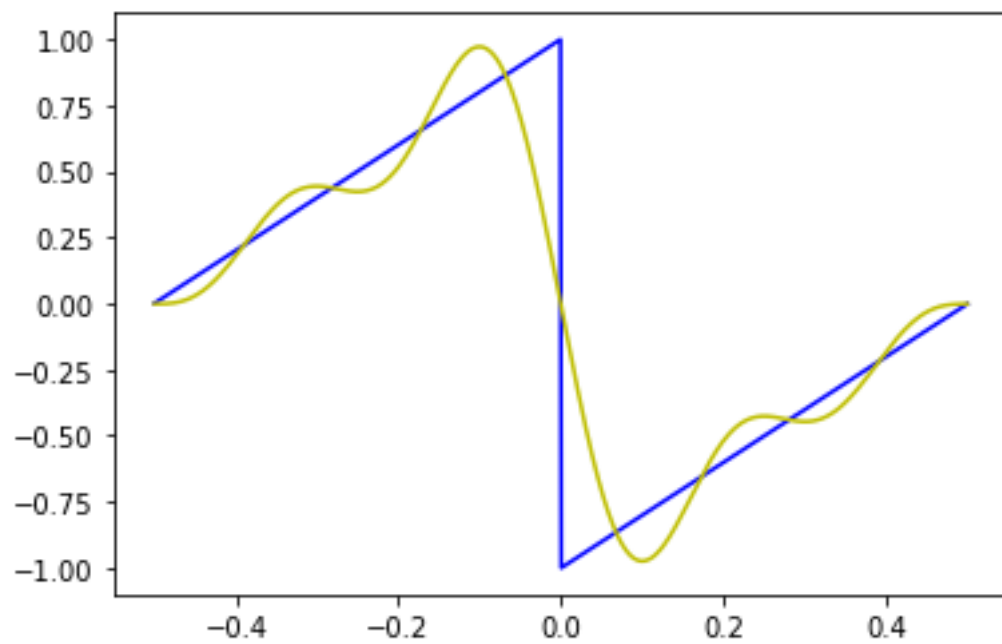
$N = 100$:



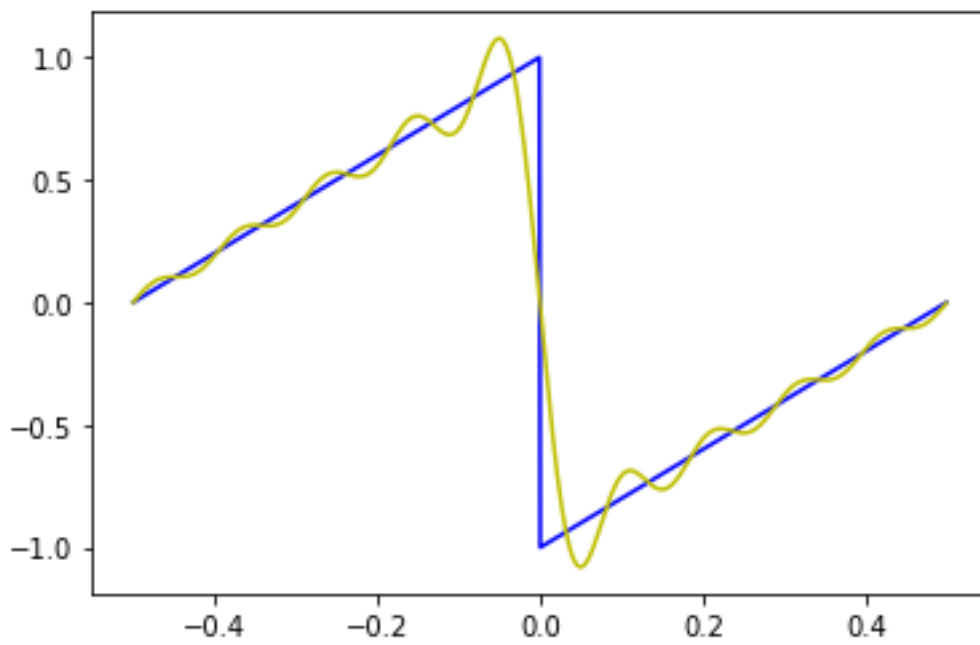
For part D:
 $N = 1$:



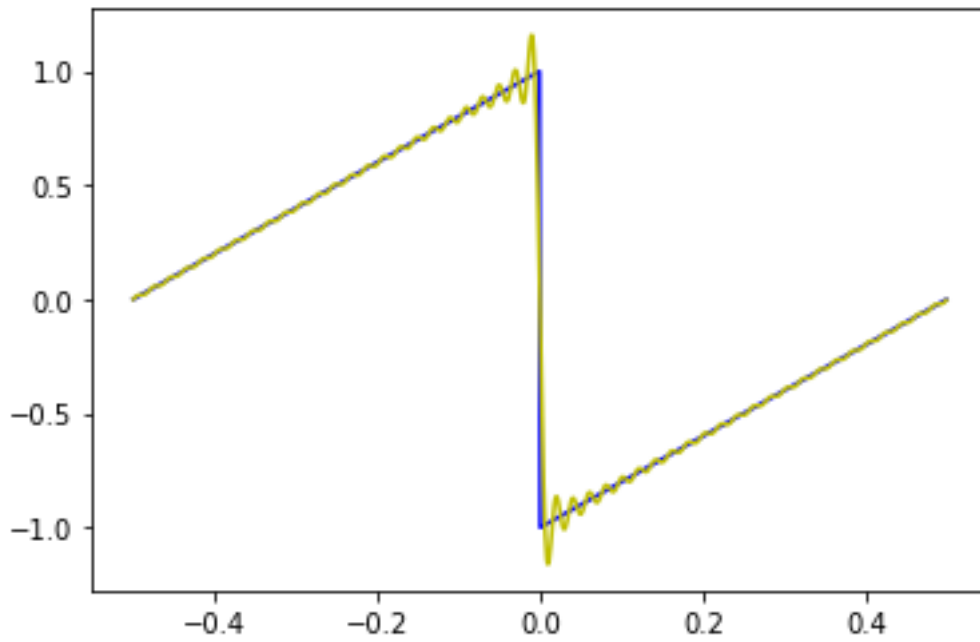
$N = 5$:



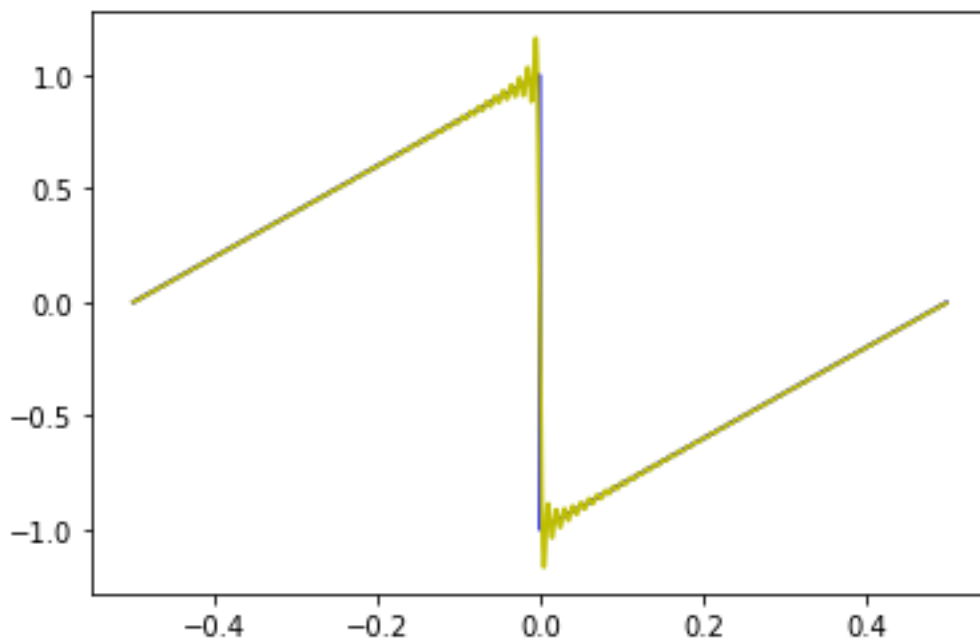
$N = 10$:



$N = 50$:



$N = 100$:



As n increases approximation becomes closer to the original function but on sharp corners our approximation does not converge to the original function.

Code:

```
import matplotlib.pyplot as plt
import numpy as np

def square_wave(n):
    return 1 if n > 0 else -1

def sawtooth_wave(n):
    return -1 + 2*n if n > 0 else 1 + 2*n

time = np.arange(-0.5, 0.5, 1/1000)
sq = np.array([square_wave(n) for n in time])
sw = np.array([sawtooth_wave(n) for n in time])

def fourier_coeffs(signal, period, num):
    coeffs = []
    for k in range(num):
        ak = signal*np.exp(-1j*(2*np.pi / period)*time*k)
        coeffs.append(ak.sum() / time.size)
    return coeffs

def inverse_fourier(coeffs, period, n):
    signal = np.array([2*a*np.exp(1j*(2*np.pi / period)*k*n) for k,a in enumerate(coeffs)])
    return signal.sum()

n = [1, 5, 10, 50, 100]
for num in n:
    a = fourier_coeffs(sq, 1, num)
    sq2 = np.array([inverse_fourier(a, 1, n) for n in time])

    plt.plot(time, sq, "b")
    plt.plot(time, sq2, "y")
    plt.show()

for num in n:
    a = fourier_coeffs(sw, 1, num)
    sw2 = np.array([inverse_fourier(a, 1, n) for n in time])

    plt.plot(time, sw, "b")
    plt.plot(time, sw2, "y")
    plt.show()
```