CENG 382 - Analysis of Dynamic Systems
20221
Take Home Exam 2 Solutions

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1. (a)

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

(b) Probability that a randomly chosen grandson of an unskilled laborer is a professional man can be found in the entries of the second power of the state transition matrix, P.

$$P^2 = \begin{bmatrix} 0.54 & 0.3 & 0.16 \\ 0.28 & 0.48 & 0.24 \\ 0.2 & 0.46 & 0.34 \end{bmatrix}$$

Unskilled labors are the third state and the professionals are the first state. Therefore probability of a randomly chosen grandson of an unskilled labor being a professional is the 0.2.

(c) Probability of a randomly chosen grandson of a professional being a professional is 0.54 as it can be found in the first entry of second power of the state transition matrix, P^2.

(d) After 100 generations Markov chain is going to converge to a fixed state. If $p(k)$ is the state of the Markov chain after k steps, $p(k+1)$ can be expressed as:

$$p(k+1) = p(k) \cdot P$$

As $k \to \infty$, system satisfies the equation below.

$$p(k+1) = p(k), \quad p(k) = p(k) \cdot P$$

$$\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

$$\begin{aligned} -0.3p_1 + 0.2p_2 + 0.1p_3 &= 0 \\ 0.2p_1 - 0.4p_2 + 0.4p_3 &= 0 \\ 0.1p_1 + 0.2p_2 - 0.5p_3 &= 0 \end{aligned}$$

When system of equations above is solved $p_1 = 1.5p_3$, $p_2 = 1.75p_3$ and p_3 is found free. Sum of p_1, p_2 and p_3 must be equal to 1.

$$1.5 \cdot p_3 + 1.75 \cdot p_3 + p_3 = 4.25 \cdot p_3 = 1$$

$$p_3 = \frac{100}{425}$$
$$p_1 = 1.5 \cdot p_3 = \frac{150}{425}$$
$$p_2 = 1.75 \cdot p_3 = \frac{175}{425}$$

$$p(100) \approx \begin{bmatrix} \frac{150}{425} & \frac{175}{425} & \frac{100}{425} \end{bmatrix}$$

2. (a) If controllability matrix M, has rank 3, then system is controllable.

$$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$
$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Since $rank(M) = 3$, system is controllable.

(b)

$$x(1) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} u_0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+u_0 \\ -3 \\ 1 \end{bmatrix}$$
$$x(2) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1+u_0 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+u_1 \\ -3-2u_0 \\ -3 \end{bmatrix}$$
$$x(3) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1+u_1 \\ -3-2u_0 \\ -3 \end{bmatrix} + \begin{bmatrix} u_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3+u_2 \\ 1-2u_1 \\ -3-2u_0 \end{bmatrix}$$
$$x(3) = \begin{bmatrix} -3+u_2 \\ 1-2u_1 \\ -3-2u_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$u_0 = -7/2$$
$$u_1 = -3/2$$
$$u_2 = 7$$

3. (a) Let's assume that the initial state of the system is $x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$. Then sequence of outputs and following states are:

$$x(k+1) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(k) = \begin{bmatrix} x_3(k) \\ -2x_1(k) - x_3(k) \\ x_2(k) \end{bmatrix}$$
$$x(k+2) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(k+1) = \begin{bmatrix} x_2(k) \\ -2x_3(k) - x_2(k) \\ -2x_1(k) - x_3(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} x(k) = -2x_2(k) - 4x_3(k)$$

$$y(k+1) = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} x(k+1) = 4x_1(k) - 4x_2(k) + 2x_3(k)$$

$$y(k+2) = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} x(k+2) = 8x_1(k) - 2x_2(k) + 8x_3(k)$$

The systems of output equations can represented in matrix form:

$$\left[\begin{array}{ccc|c} 0 & 2 & -4 & y(k) \\ 4 & -4 & 2 & y(k+1) \\ 8 & 2 & 8 & y(k+2) \end{array}\right]$$

Since matrix above is non-singular there is a solution for $x_1(k)$, $x_2(k)$ and $x_3(k)$. It is possible the determine the initial state by looking at most 3 consecutive outputs. Therefore system is observable.

(b) Observability matrix M equals to $\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$.

$$M = \begin{bmatrix} 0 & 2 & -4 \\ 4 & -4 & 2 \\ 8 & 2 & 8 \end{bmatrix}$$

Since $rank(M) = 3$, system is observable.

4.

$$\dot{x} = f(x) = 3x^2 - 3x^3$$

Fixed points are the values of x which make dx/dt zero.

$$\dot{x} = 3x^2 - 3x^3 = 0$$
$$3x^2 \cdot (1 - x) = 0$$

$$x_1 = 0, \;\; x_2 = 1$$

Linearization of $f(x)$ around the first fixed point $x_1 = 0$, can be written in the equation below.

$$f(x) \cong \frac{d}{dx} f(x_1)(x - x_1) + f(x_1)$$
$$\frac{df}{dx} = 6x - 9x^2, \;\; \frac{d}{dx} f(0) = 0$$
$$f(x) \cong 0 \cdot (x - 0) + f(0) = f(0) = 0$$

Stability test fails because coefficient of the x is zero. Problem requires further analysis. When x is slightly larger than $x_1 = 0$, $f'(x) = 6x - 9x^2 > 0$. Therefore $f(x)$ will move away from $x_1 = 0$ which shows that $x_1 = 0$ is unstable.

Linearization of $f(x)$ around the second fixed point $x_2 = 1$, can be written in the equation below.

$$f(x) \cong \frac{d}{dx} f(x_2)(x - x_2) + f(x_2)$$
$$\frac{d}{dx} f(1) = -3$$
$$f(x) \cong -3 \cdot (x - 1) + f(1) = -3 \cdot (x - 1) + 0$$

Since coefficient of x is negative, fixed point $x_2 = 1$ is stable.