CENG 382 - Analysis of Dynamic Systems
20221
Take Home Exam 3 Solutions

Çolak, Ahmet Eren
e2587921@ceng.metu.edu.tr

January 8, 2023

---

1. (a) Jacobian matrix of the system F is:

$$F(x_1, x_2) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 2x_2 \\ 6x_1 & -2 \end{bmatrix}$$

$$F(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Characteristic equation of the Jacobian matrix at fixed point $x = (0, 0)$ is:

$$(\lambda + 1)(\lambda + 2) = 0$$

Eigen values of the matrix are $\lambda_1 = -1$ and $\lambda_2 = -2$. Since all eigen values are less than 0, fixed point is stable.

(b) If a Lyapunov function exists for the fixed point $x = (0, 0)$, then the fixed point is stable. Since Lyapunov function $V = x_1^2/2 + x_2^2/4$ is polynomial it is continous on $\mathbb{R}^2$. First condition of Lyapunov function is satisfied.

To find the minimum value of the Lyapunov function, let's check its partial derivatives.

$$\frac{\partial V(x_1, x_2)}{\partial x_1} = x_1 = 0$$

$$\frac{\partial V(x_1, x_2)}{\partial x_2} = \frac{x_2}{2} = 0$$

Minimum value of the Lyapunov functions is at $x_1 = 0, x_2 = 0$. Therefore the second condition is satisfied.

Lastly, if the Lyapunov function $V(x_1, x_2)$ is decreasing on a spherical region centered at the fixed point for every trajectory of the system, then fixed point $x = (0, 0)$ is stable.

$$\frac{dV(x_1, x_2)}{dt} = \frac{\partial V(x_1, x_2)}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V(x_1, x_2)}{\partial x_2}\frac{dx_2}{dt}$$

$$\frac{dV(x_1, x_2)}{dt} = x_1(x_2^2 - x_1) + \frac{x_2}{2}(3x_1^2 - 2x_2)$$

$$\frac{dV(x_1,x_2)}{dt} = x_2^2(x_1-1) + x_1^2(3x_2/2-1)$$

When $x_1 < 1$ and $x_2 < 2/3$ condition is satisfied. Spherical region is centered at $x = (0,0)$ and its radius is $2/3$. Since all conditions are satisfied, Lyapunov function V exists and fixed point is stable.

2. Let's choose Lyapunov function as the $V(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2$. Since the function is polynomial it is continous on $\mathbb{R}^3$.

   Let's check where the minimum of the function is.

$$\frac{\partial V}{\partial x_1} = 2x_1 = 0, \quad x_1 = 0$$
$$\frac{\partial V}{\partial x_2} = 2x_2 = 0, \quad x_2 = 0$$
$$\frac{\partial V}{\partial x_3} = 2x_3 = 0, \quad x_3 = 0$$

   Minimum of the function is at $(0,0,0)$, so the second condition is satisfied.

   Let's check $V(x(k+1))) - V(x(k))$ to determine $V(x(k))$ is non-increasing.

$$V(x(k+1)) = \frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{4}$$

$$V(x(k+1)) - V(x(k)) = \frac{-x_1^2}{2} - \frac{x_2^2}{2} - \frac{3x_3^2}{4}$$

   Since x_1^2, x_2^2 and x_3^2 are always non negative, it can be concluded that $V(x(k+1)) - V(x(k)) \leq 0$. Since Lyapunov function is always non increasing fixed point is stable.

3. Let's first determine the fixed points of the system.

$$\frac{dx}{dt} = 0$$

$$x_1 + x_2 - 4x - 1(x_1^2 + x_2^2) = 0$$
$$-x_1 + x_2 - 4x - 2(x_1^2 + x_2^2) = 0$$
$$\frac{x_1 + x_2}{x_1} = 4(x_1^2 + x_2^2)$$
$$\frac{x_2 - x_2}{x_2} = 4(x_1^2 + x_2^2)$$
$$1 + \frac{x_2}{x_1} = 1 - \frac{x_1}{x_2}$$
$$x_2^2 = -x_1^2$$

   The only values that satisfy the equation $x_2^2 = -x_1^2$ are $x_1 = 0$ and $x_2 = 0$. $(0,0)$ is the only fixed point of the system. Then we check the stability of the fixed point via linearization. Jacobian matrix of the system is:

$$F(x_1,x_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 - 4x_2^2 - 12x_1^2 & -8x_1x_2 + 1 \\ -1 - 8x_2x_1 & 1 - 12x_2 - 4x_1^2 \end{bmatrix}$$

$$F(0,0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Characteristic equation of the matrix above is:

$$\lambda^2 - 2\lambda + 2 = 0$$

Eigen values of the matrix are $\lambda_{1,2} = 1 \pm i$. Since real part of one of the eigen values is positive, fixed point is unstable.

To check if the system has a periodic limit cycle, we should find where the Lyapunov function is decreasing and increasing.

$$\frac{dV(x_1,x_2)}{dt} = x_1(x_1 + x_2 - 4x_1(x_1^2 + x_2^2)) + x_2(-x_1 + x_2 - 4x_2(x_1^2 + x_2^2))$$

$$\frac{dV(x_1,x_2)}{dt} = x_1^2 + x_2^2 - 4x_1^4 - 8x_1^2x_2^{2-}4x_2^4$$

$$\frac{dV(x_1,x_2)}{dt} = x_1^2 + x_2^2 - 4(x_1^2 + x_2^2)^2 = (x_1^2 + x_2^2)(1 - 4(x_1^2 + x_2^2))$$

When $0 < x_1^2 + x_2^2 < 4$, $V(x_1,x_2)$ is increasing and when $x_1^2 + x_2^2 > 4$, $V(x_1,x_2)$ is decreasing. Therefore the system has a periodic limit cycle on the circle $x_1^2 + x_2^2 = 4$.

4. (a) Fixed points satisfy the equation in discrete systems $x(k+1) = x(k)$.

$$x(k) = 3 - x^2(k)$$

$$x^2(k) + x(k) - 3 = 0$$

$$\tilde{x}_{1,2} = \frac{-1 \pm \sqrt{13}}{2}$$

(b) Periodic points of prime period 2, satisfy the equation $f^2(x) = x$ where $f(x) = 3 - x^2$.

$$f^2(x) = f(3 - x^2) = -4x^4 + 6x^2 - 6 = x$$

$$x^4 - 6x^2 + x + 6 = 0$$

Roots of the equation $x^2 + x - 3 = 0$ are also the roots of the equation $x^4 - 6x^2 + x + 6 = 0$. In other words if the prime period of a value is 1 (which means it is a fixed point), then 2 is also its period. Because if a value is repeated in every step, it means it also repeats in every 2 step. Therefore we can divide $x^4 - 6x^2 + x + 6$ by $x^2 + x - 3$ without any remainders. When polynomial division is applied, $x^4 - 6x^2 + x + 6 = (x^2 + x - 3)(x^2 - x - 2)$ is obtained. Roots of the $x^2 - x - 2$ are the values of prime period 2. This property of fixed points helps with finding roots.

$$x^2 - x - 2 = (x-2)(x+1) = 0$$

$$x_1 = 2, \quad x_2 = -1$$

Therefore periodic points of prime period 2 are -1 and 2.

(c) We can test the stability of periodic points via linearization. If the coefficient of x in the linearized form of $f^2(x)$ is negative, then periodic point is stable.

Linearization for periodic point $x = 2$:

$$\frac{df^2(x)}{dx} = -4x^3 + 12x, \quad \frac{df^2}{dx}(2) = -8$$

$$f^2(x) \cong f^2(2) + (-8)(x-2)$$

Since the coefficient of x (-8), is negative, periodic point is stable.

Linearization for periodic point $x = -1$:

$$\frac{df^2(x)}{dx} = -4x^3 + 12x, \quad \frac{df^2}{dx}(-1) = -8$$

$$f^2(x) \cong f^{(}-1) + (-8)(x+1)$$

Since the coefficient of x (-8), is negative, this periodic point is also stable.