Student Information

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Answer 1

a)

X and Y are independent if and only if their joint density function can be factored into their marginal PDFs. From the part b, it is known that:

$$f(x) = \frac{2\sqrt{1-x^2}}{\pi}$$

$$f(y) = \frac{2\sqrt{1-y^2}}{\pi}$$

Since $f(x)f(y) \neq 1/\pi$, X and Y are not independent.

b)

$$f(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi}$$

$$f(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2\sqrt{1-y^2}}{\pi}$$

c)

$$E(X) = \int_{-1}^{1} \frac{2x\sqrt{1-x^2}}{\pi} dx = 0$$

d)

$$Var(X) = \int_{-1}^{1} \frac{2x^2\sqrt{1-x^2}}{\pi} dx = 0.25$$

Answer 2

a)

Since A and B are independent, product of their marginal PDFs is the joint density function.

$$f(t_A) = \frac{1}{100}$$

$$f(t_B) = \frac{1}{100}$$

Then $f(t_A, t_B) = f(t_A)f(t_B)$

$$f(t_A, t_B) = \frac{1}{100} \cdot \frac{1}{100} = 10^{-4}$$

b)

Probability that A pushes the button in the first 10 seconds is $P(0 < A < 10)$ and the probability that B pushed the button in the last 10 seconds is $P(90 < B < 100)$. Probability of both events occuring equals product of their individual probabilities.

$$P(0 < A < 10) = \frac{10-0}{100} = 0.1 \quad P(90 < B < 100) = \frac{100-90}{100} = 0.1$$

$$P(0 < A < 10 \cap 90 < B < 100) = 0.1 \cdot 0.1 = 0.01$$

c)

When A and B considered on a coordinate system, let t_A be the axis for the values of A and t_B be the axis for the values of B. Then z axis will be equal to joint density function $f(t_A, t_B)$. Bounded region between lines $t_B = t_A - 20$, $t_B = 100$, $t_A = 100$ and axises is the area where A pushes the button at most 20 seconds after B.

$$P(A - B < 20) = \int_0^{80} \int_0^{t_B+20} f(t_A, t_B) dt_A dt_B + \int_{80}^{100} \int_0^{100} f(t_A, t_B) dt_A dt_B$$

$$P(A - B < 20) = \int_0^{80} \int_0^{t_B+20} 10^{-4} \cdot dt_A dt_B + \int_{80}^{100} \int_0^{100} 10^{-4} \cdot dt_A dt_B = 0.68$$

d)

Probability that elapsed time is less than 30 seconds is $P(|A - B| < 30)$. Therefore bounded region between lines $t_A - t_B = 30$, $t_A - t_B = -30$, $t_A = 100$, $t_B = 100$ and axises is the area where elapsed time is less than 30 seconds.

$$P(|A - B| < 30) = \int_0^{100} \int_{t_A+30}^{t_A-30} f(t_A, t_B) dt_B dt_A - 2 \int_{70}^{100} \int_{100}^{t_A-30} f(t_A, t_B) dt_B dt_A$$

$$P(|A - B| < 30) = \int_0^{100} \int_{t_A+30}^{t_A-30} 10^{-4} \cdot dt_B dt_A - 2 \int_{70}^{100} \int_{100}^{t_A-30} 10^{-4} \cdot dt_B dt_A = 0.51$$

Answer 3

a)

CDF of T equals to $P(min(X_1, X_2, \ldots, X_n) \leq t)$ which is the area under the minimum of PDFs of Xs on the interval $(0, t)$.

$$\int_0^t min(\lambda_1 e^{-\lambda_1 t}, \lambda_2 e^{-\lambda_2 t}, \ldots, \lambda_n e^{-\lambda_n t})dt$$

b)

Let T be a random variable representing the time past until one of the computers fail. Then T must be equal to $min(C_1, C_2, \ldots, C_{10})$. First find the CDF of T using the equation found from the part a.

Intersection point of PDF of C_1 and C_{10} is $-ln(0.1)/0.9 = 2.558$. Therefore minimum of PDFs is $0.1e^{-0.1t}$ on the interval $(0, 2.558)$ and e^{-t} on the interval $(2.558, \infty)$. Then $F(t)$, CDF of T, for $t \in (0, 2.558)$ is:

$$\int_0^t 0.1e^{-0.1t}dt = 1 - e^{-0.1t}$$

And on for $t \in (2.558, \infty)$:

$$\int_0^t e^{-t}dt = 1 - e^{-0.1t} = 1 - e^{-t}$$

Then $f(t)$, PDF of T, is:

$$f(t) = 0.1e^{-0.1t} \quad , \quad t \in (0, 2.558)$$

$$f(t) = e^{-t} \quad , \quad t \in (2.558, \infty)$$

Now $E(T)$ can be calculated.

$$E(T) = \int_0^{2.558} 0.1te^{-0.1t}dt + \int_{2.558}^{\infty} te^{-t} \approx 0.55$$

Expected time before one of the computers fail is 0.55 years.

Answer 4

a)

The probability that at least 70% of participants being undergraduate students equals to binomial cdf $1 - P(X \leq 69)$ where $p = 0.74$ and $n = 100$. Since binomial distribution is summation of Bernouilli trials, central limit theorem can be used to approximate this calculation.

$$1 - P(X \leq 69) = 1 - P\left(\frac{S_n - \mu}{\sigma} \leq \frac{69.5 - \mu}{\sigma}\right)$$

Since binomial distribution is discrete, continuity correction must be applied. That's why 69.5 appears in the equation. When $\mu = np = 74$ and $\sigma = \sqrt{np(1-p)} = 4.386$ substitued in the equation above:

$$1 - P(X \leq 69) = 1 - P\left(\frac{S_n - 74}{4.386} \leq \frac{69.5 - 74}{4.386}\right)$$

$$1 - P(X \leq 69) = 1 - \phi(-1.0260) = 0.8476$$

b)

The probability that at most 5% of participants pursuing a doctoral degree equals to binomial cdf $P(X \leq 5)$ where where $p = 0.1$ and $n = 100$. As explained in the part a, central limit theorem can also be used to approximate this calculation.

$$P(X \leq 5) = P\left(\frac{S_n - \mu}{\sigma} \leq \frac{5.5 - \mu}{\sigma}\right)$$

When When $\mu = np = 10$ and $\sigma = \sqrt{np(1-p)} = 3$ substitued in the equation above:

$$P(X \leq 5) = P\left(\frac{S_n - 10}{3} \leq \frac{5.5 - 10}{3}\right)$$

$$P(X \leq 5) = \phi(-1.8333) = 0.0668$$