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Q. 1

$$
\begin{aligned}
(A \cup B) \setminus (A \cap B) &= \{x \mid x \in (A \cup B) \land x \notin (A \cap B)\} && \text{by definiton of set difference} \\
&= \{x \mid x \in (A \cup B) \land \neg(x \in (A \cap B))\} && \text{by definiton of } \notin \\
&= \{x \mid (x \in A \lor x \in B) \land \neg(x \in A \land x \in B))\} && \text{by definition of set union} \\
&= \{x \mid (x \in A \lor x \in B) \land (\neg(x \in A) \lor \neg(x \in B))\} && \text{by De Morgan's law} \\
&= \{x \mid ((x \in A) \land (\neg(x \in A) \lor \neg(x \in B))) \\
&\quad \lor ((x \in B) \land (\neg(x \in A) \lor \neg(x \in B)))\} && \text{by distributive laws} \\
&= \{x \mid (((x \in A) \land \neg(x \in A)) \lor ((x \in A) \land \neg(x \in B))) \\
&\quad \lor (((x \in B) \land \neg(x \in A)) \lor ((x \in B) \land \neg(x \in B)))\} && \text{by distributive laws} \\
&= \{x \mid (F \lor ((x \in A) \land \neg(x \in B))) \\
&\quad \lor (((x \in B) \land \neg(x \in A)) \lor F)\} && \text{by complement laws} \\
&= \{x \mid ((x \in A) \land \neg(x \in B)) \lor ((x \in B) \land \neg(x \in A))\} && \text{by identity laws} \\
&= \{x \mid ((x \in A) \land (x \notin B)) \lor ((x \in B) \land (x \notin A))\} && \text{by definiton of } \notin \\
&= \{x \mid x \in (A \setminus B) \lor x \in (B \setminus A)\} && \text{by definiton of set difference} \\
&= (A \setminus B) \cup (B \setminus A) && \text{by definition of set union}
\end{aligned}
$$

Q. 2

$$A = \{f \mid f \subseteq \mathbb{N} \times \{0,1\}\}$$
$$B = \{f \mid f : \{0,1\} \to \mathbb{N}, \text{ f is a function}\}$$

Set A can be represented as:

$$A = \mathcal{P}(\{(0,0),(1,0),(2,0)\ldots,(0,1),(1,1),(2,1),\ldots\})$$

For an arbitrary value of x in the domain of a function f, let $(x, f(x))$ represent function f. Then set B can be expressed as:

$$B = \mathcal{P}\{(0,n_1),(1,n_2)\} \qquad n_1, n_2 \in \mathbb{N}$$
$$B = \{\{(0,n_1),(1,n_2)\},\{(0,n_1)\},\{(1,n_2)\},\emptyset\} \qquad n_1, n_2 \in \mathbb{N}$$

When common elements in A and B considered, there are 9 of them:

$$\{(0,0),(1,0)\},\{(0,0),(1,1)\},\{(0,1),(1,0)\},\{(0,1),(1,1)\},\{(0,0)\},\{(0,1)\},\{(1,0)\},\{(1,1)\},\emptyset$$

To prove $A \setminus B$ is uncountable, let $s_1, s_2, s_3, \ldots$ denote the elements of set B, letter N denote absence of the element in the subset and letter Y denote existence of the element in the subset.

Let's represent the first 9 elements of the set A as the 9 common elements of set A and B. These 9 elements can be written as below:

$$
\begin{array}{rccccccc}
 & (0,0) & (0,1) & (1,0) & (1,1) & \dots & & \\
s_1 = & \text{Y}, & \text{N}, & \text{Y}, & \text{N}, & \text{N}, & \text{N}, & \dots \\
s_2 = & \text{Y}, & \text{N}, & \text{N}, & \text{Y}, & \text{N}, & \text{N}, & \dots \\
s_3 = & \text{N}, & \text{Y}, & \text{Y}, & \text{N}, & \text{N}, & \text{N}, & \dots \\
s_4 = & \text{N}, & \text{Y}, & \text{N}, & \text{Y}, & \text{N}, & \text{N}, & \dots \\
s_5 = & \text{Y}, & \text{N}, & \text{N}, & \text{N}, & \text{N}, & \text{N}, & \dots \\
s_6 = & \text{N}, & \text{Y}, & \text{N}, & \text{N}, & \text{N}, & \text{N}, & \dots \\
s_7 = & \text{N}, & \text{N}, & \text{Y}, & \text{N}, & \text{N}, & \text{N}, & \dots \\
s_8 = & \text{N}, & \text{N}, & \text{N}, & \text{Y}, & \text{N}, & \text{N}, & \dots \\
s_9 = & \text{N}, & \text{N}, & \text{N}, & \text{N}, & \text{N}, & \text{N}, & \dots
\end{array}
$$

To represent the elements of $A \setminus B$, elements of set A can be written starting from s_{10}. By Cantor's diagonal argument, it can be concluded that $A \setminus B$ is uncountably infinite.

$$
\begin{array}{rcccccccc}
 & (0,0) & (0,1) & (1,0) & (1,1) & (2,1) & (0,2) & (3,1) & \dots \\
s_{10} = & \textbf{Y}, & \text{Y}, & \text{Y}, & \text{Y}, & \text{Y}, & \text{Y}, & \text{Y}, & \dots \\
s_{11} = & \text{N}, & \textbf{N}, & \text{N}, & \text{Y}, & \text{Y}, & \text{N}, & \text{N}, & \dots \\
s_{12} = & \text{Y}, & \text{N}, & \textbf{Y}, & \text{N}, & \text{Y}, & \text{N}, & \text{Y}, & \dots \\
s_{13} = & \text{N}, & \text{Y}, & \text{N}, & \textbf{Y}, & \text{N}, & \text{Y}, & \text{N}, & \dots \\
s_{14} = & \text{Y}, & \text{N}, & \text{N}, & \text{Y}, & \textbf{Y}, & \text{N}, & \text{Y}, & \dots \\
s_{15} = & \text{Y}, & \text{Y}, & \text{N}, & \text{Y}, & \text{Y}, & \textbf{N}, & \text{Y}, & \dots \\
s_{16} = & \text{N}, & \text{N}, & \text{N}, & \text{Y}, & \text{N}, & \text{Y}, & \textbf{N}, & \dots \\
\vdots & & & & & & & &
\end{array}
$$

Q. 3

Let's assume that the function $f(n) = 4^n + 5n^2logn$ is $O(2^n)$. Then, there must be a pair of witnesses C and k such that $f(n) = 4^n + 5n^2logn \leq C2^n$ whenever $n > k$.

$$
\begin{aligned}
4^n + 5n^2logn &\leq C(2^n) \\
2^n + \frac{5n^2logn}{2^n} &\leq C \qquad \text{(both sides divided by } 2^n\text{)}
\end{aligned}
$$

Since left hand side of the inequality is always increasing, there can not be a constant C, whatever k is. Therefore $f(n)$ is not $O(2^n)$.

Q. 4

$$
\begin{aligned}
(2x-1)^n - x^2 &\equiv -x-1 \mod (x-1) \\
(2x-1)^n - x + 2 &\equiv x^2-2x+1 \mod (x-1) && \text{by adding } x^2-x+2 \text{ to both sides} \\
(2x-1)^n - x + 2 &\equiv (x-1)^2 \mod (x-1) && \text{by rewriting left hand side as a square} \\
(2(x-1)+1)^n - x + 2 &\equiv (x-1)^2 \mod (x-1) && \text{by algebraic manipulations} \\
(2(x-1)+1)^n - x + 2 - (x-1)^2 &\mod (x-1) = 0 && \text{by definition of congruence modulo}
\end{aligned}
$$

Using the equality: $a+b \mod m = ((a \mod m)+(b \mod m)) \mod m$, the last equation can be written as:

$$((2(x-1)+1)^n \mod (x-1)) + (-(x-1)+1 \mod (x-1)) - ((x-1)^2 \mod (x-1)) = 0$$

Using the equality: $ab \mod m = (a \mod m)(b \mod m) \mod m$, the first term, $(2(x-1)+1)^n \mod (x-1)$, can be written as $(2(x-1)+1 \mod (x-1))^n$. Since $(2(x-1)+1) \mod (x-1)$ is equal to 1, first term is equal to $1^n = 1$.

Second term, $-(x-1)+1 \mod (x-1)$, is equal to 1 and last term, $((x-1)^2 \mod (x-1))$, is equal to 0. Hence $(1+1+0) \mod (x-1) = 0$, it can be written as:

$$2 = (x-1)\cdot k + 0 \qquad k \in \mathbb{Z}$$

To leave x alone, divide both sides by k and add 1 to both sides:

$$x = \frac{2+k}{k}$$

$k=1$, is the only situation where $x>2$ and $x \in \mathbb{Z}$. When $k=1$, $x = 3$**.**