CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 3

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1.

$$\int_{-\infty}^{t} x(s)ds = \int_{-\infty}^{t} \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 s} ds$$

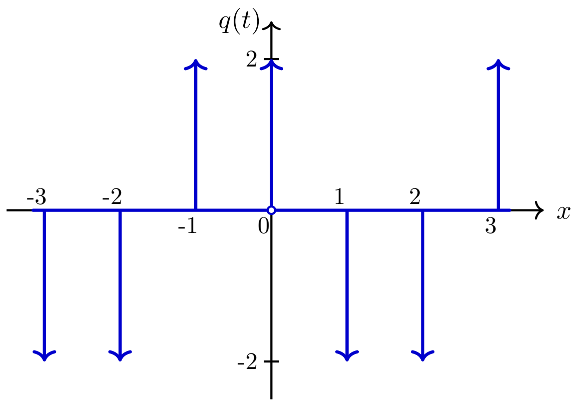
$$\int_{-\infty}^{t} x(s)ds = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{t} e^{jkw_0 s} ds$$

$$\int_{-\infty}^{t} x(s)ds = \sum_{k=-\infty}^{\infty} a_k \frac{e^{jkw_0 t}}{jkw_0}$$

Equality above proves that $\int_{-\infty}^{t} x(s)ds$ has spectral coefficients $a_k \frac{1}{jkw_0}$ where $w_0 = 2\pi/T$

2. (a) $x(t)x(t) \longleftrightarrow \sum_{l=-\infty}^{\infty} a_l \cdot a_{k-l} = a_k * a_k$ (from multiplication property)
   (b) $R\{a_k\}$
   (c) $x(t+t_0) \longleftrightarrow e^{jk\omega_0 t_0} a_k$, $x(t-t_0) \longleftrightarrow e^{-jk\omega_0 t_0} a_k$ (from time-shift property)
   $x(t+t_0) + x(t-t_0) \longleftrightarrow a_k(e^{jk\omega_0 t_0} + e^{-jk\omega_0 t_0})$ (from linearity property)

3. Let $q(t) = \frac{dx(t)}{dt}$ and $z(t) = \sum_{k=-\infty}^{\infty} \delta(t-4k)$, then



$q(t) = z(t) + z(t+1) - z(t-1) - z(t-2)$

Fourier series of $z(t) \longleftrightarrow a_k = \frac{1}{4}\int_{-2}^{2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{4}$, then

$q(t) \longleftrightarrow b_k = \frac{1}{4} + e^{jk\omega_0}\frac{1}{4} - e^{-jk\omega_0}\frac{1}{4} - e^{-2jk\omega_0}\frac{1}{4}$

$\omega_0 = \frac{2\pi}{T}$ and $T = 4 \longrightarrow b_k = \frac{1}{4}(1 + e^{jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}} - e^{-jk\pi})$

$b_k = \frac{1}{4}(1 + cos(k\frac{\pi}{2}) + jsin(k\frac{\pi}{2}) - cos(k\frac{\pi}{2}) + jsin(k\frac{\pi}{2}) - cos(k\pi) + jsin(k\pi))$

$b_k = \frac{1}{4}(1 - cos(k\pi) + 2jsin(k\frac{\pi}{2}))$

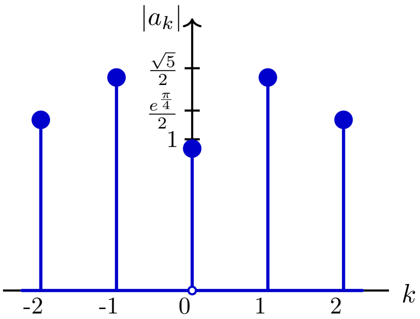
$x(t) \longleftrightarrow c_k = \frac{b_k}{jk\omega_0} = \frac{1 - cos(k\pi) + 2jsin(k\frac{\pi}{2})}{2\pi jk}$

4. (a) $x(t) = 1 + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{j(2\omega_0 t + \frac{\pi}{4})}{2}$

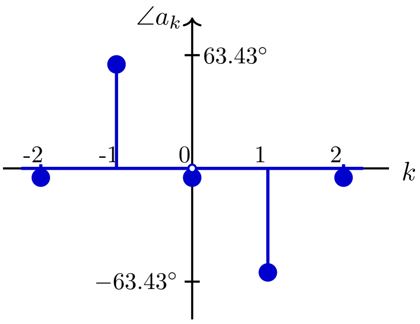
$x(t) = 1 + e^{j\omega_0 t}(\frac{1}{2j} + 1) + e^{-j\omega_0 t}(\frac{-1}{2j} + 1) + e^{2j\omega_0 t}(\frac{e^{\frac{\pi}{4}}}{2}) + e^{-2j\omega_0 t}(\frac{e^{\frac{\pi}{4}}}{2})$

$a_0 = 1$
$a_1 = \frac{1}{2j} + 1 = 1 - \frac{j}{2}$
$a_{-1} = \frac{-1}{2j} + 1 = 1 + \frac{j}{2}$
$a_2 = \frac{e^{\frac{\pi}{4}}}{2}$
$a_{-2} = \frac{e^{\frac{\pi}{4}}}{2}$

Magnitudes:



Phases:



(b)

$$x(t) = e^{jwt}$$
$$y(t) = H(jw)e^{jwt}$$

Subsitute $y(t)$ and $x(t)$ in the differential equation:

$$H(jw)jwe^{jwt} + H(jw)e^{jwt} = e^{jwt}$$
$$H(jw) = \frac{1}{1 + jw}$$

(c)

$$x(t) = e^{j0t} + \frac{1}{2j}e^{jw_0 t} - \frac{1}{2j}e^{-jw_0 t} + e^{jw_0 t} + e^{-jw_0 t} + \frac{1}{2}e^{\pi/4}e^{j2w_0 t} + \frac{1}{2}e^{\pi/4}e^{-j2w_0 t}$$

$$y(t) = H(0)e^{j0t} + \frac{1}{2j}H(jw_0)e^{jw_0 t} - \frac{1}{2j}H(-jw_0)e^{-jw_0 t} + H(jw_0)e^{jw_0 t} + H(-jw_0)e^{-jw_0 t} + \frac{1}{2}H(j2w_0)e^{\pi/4}e^{j2w_0 t} +$$
$$\frac{1}{2}H(-j2w_0)e^{\pi/4}e^{-j2w_0 t}$$

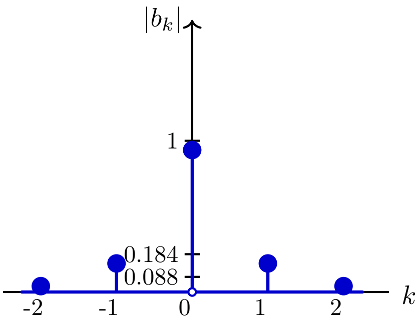
$$b_0 = 1$$
$$b_1 = H(jw_0)\left(1 + \frac{1}{2j}\right)$$
$$b_{-1} = H(-jw_0)\left(1 - \frac{1}{2j}\right)$$
$$b_2 = \frac{e^{\pi/4}}{2}H(j2w_0)$$

$$b_{-2} = \frac{e^{\pi/4}}{2} H(-j2w_0)$$

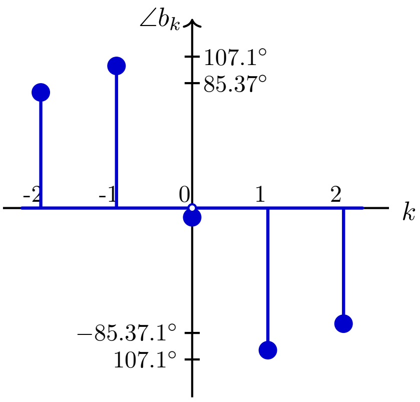
When $w_0 = 2\pi$:

$$b_0 = 1$$
$$b_1 = \frac{1-2\pi j}{1-4\pi^2}\left(1+\frac{1}{2j}\right), |b_1| = 0.184, \angle b_1 = -107.1°$$
$$b_{-1} = \frac{1+2\pi j}{1-4\pi^2}\left(1-\frac{1}{2j}\right), |b{-}1| = 0.184, \angle b_{-1} = 107.1°$$
$$b_2 = \frac{1-4\pi j}{1-16\pi^2}\frac{e^{\pi/4}}{2}, |b_2| = 0.088, \angle b_2 = -85.37°$$
$$b_2 = \frac{1+4\pi j}{1-16\pi^2}\frac{e^{\pi/4}}{2}, |b_2| = 0.088, \angle b_2 = 85.37°$$

Magnitudes:



Phases:



(d)

$$y(t) = \sum_{k=-2}^{2} b_k e^{jw_0kt}$$

5. (a) Period of $x[n] = T_x = 4$

$x[n] \longleftrightarrow a_k = \frac{1}{4}\sum_{n=1}^{4} x[n]e^{-jk\frac{\pi}{2}n}$

$= \frac{1}{4}(e^{-jk\frac{\pi}{2}} - e^{-jk\frac{3\pi}{2}})$

$= \frac{1}{4}(cos(k\frac{\pi}{2}) - jsin(k\frac{\pi}{2}) - cos(k\frac{3\pi}{2}) + jsin(\frac{3\pi}{2}))$

(b) Period of $y[n] = T_y = 4$

$y[n] \longleftrightarrow b_k = \frac{1}{4}\sum_{n=1}^{4} y[n]e^{-jk\frac{\pi}{2}n}$

$= \frac{1}{4}(2e^{-jk2\pi} + e^{-jk\frac{\pi}{2}} + e^{-jk\frac{3\pi}{2}})$

$= \frac{1}{4}(2cos(k2\pi) - 2jsin(k2\pi) + cos(k\frac{\pi}{2}) - jsin(k\frac{\pi}{2}) + cos(k\frac{3\pi}{2}) - jsin(k\frac{3\pi}{2}))$

$= \frac{1}{4}(2 + cos(k\frac{\pi}{2}) - jsin(k\frac{\pi}{2}) + cos(k\frac{3\pi}{2}) - jsin(k\frac{3\pi}{2}))$, as $y[n]$ is a discrete function

(c) Period of $x[n] \times y[n] = 4$

$x[n] \times y[n] \longleftrightarrow c_k = \sum_{l=1}^{4} a_l b_{k-l} = \sum_{l=1}^{4} b_l a_{k-l}$

$= \frac{1}{2} a_{k-l} + \frac{1}{2} a_{k-3} + ak - 4$

$= \frac{1}{8}(sin(k\frac{\pi}{2}) + jcos(k\frac{\pi}{2}) + sin(k\frac{3\pi}{2}) + jcos(k\frac{3\pi}{2})) + \frac{1}{8}(-sin(k\frac{\pi}{2}) - jcos(k\frac{\pi}{2}) - sin(k\frac{3\pi}{2}) - jcos(k\frac{3\pi}{2})) + \frac{1}{4}(cos(k\frac{\pi}{2}) - jsin(k\frac{\pi}{2}) - cos(k\frac{3\pi}{2}) + jsin(\frac{3\pi}{2}))$

$= \frac{1}{4}(cos(k\frac{\pi}{2}) - jsin(k\frac{\pi}{2}) - cos(k\frac{3\pi}{2}) + jsin(\frac{3\pi}{2}))$

(d) $g[x] = x[n] \times y[n] = sin(\frac{\pi}{2}n) + sing(\frac{\pi}{2}n)cos(\frac{\pi}{2}n) = sin(\frac{\pi}{2}n) + \frac{1}{2}sin(\pi n) = sin(\frac{\pi}{2}n)$ and $T_g = 4$

$g[n] \longleftrightarrow d_k = \frac{1}{4}\sum_{n=1}^{4} g[n]e^{-jk\frac{\pi}{2}n} = \frac{1}{4}(e^{-jk\frac{\pi}{2}} - e^{-jk\frac{3\pi}{2}})$

$= \frac{1}{4}(cos(k\frac{\pi}{2}) - jsin(k\frac{\pi}{2}) - cos(k\frac{3\pi}{2}) + jsin(k\frac{3\pi}{2}))$

The Fourier coefficients found in part c are equal to the ones found in part d.

6. (a)

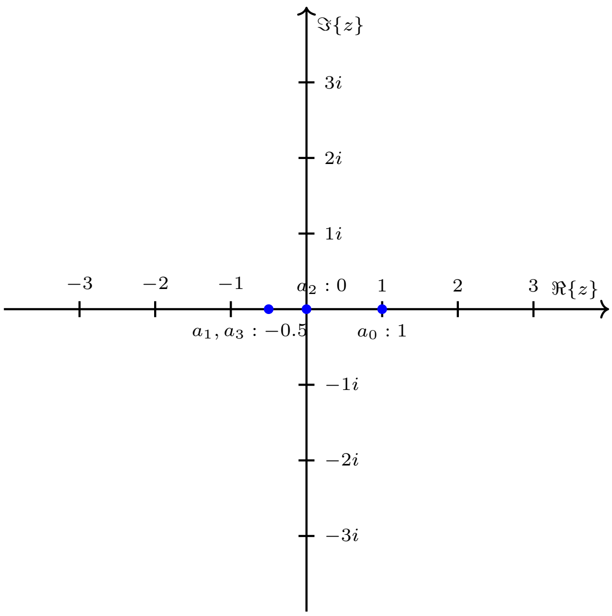
$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi/4kn}$$

$$a_0 = \frac{1}{4} \cdot 4 = 1$$

$$a_1 = \frac{1}{4} \cdot (0 - j - 2 + j) = \frac{-1}{2}$$

$$a_2 = \frac{1}{4} \cdot (0 - 1 + 2 - 1) = 0$$

$$a_3 = \frac{1}{4} \cdot (0 + j - 2 - j) = \frac{-1}{2}$$



(b)

$$y[n] = x[n] - \sum_{k=-\infty}^{\infty} \delta[n - (3 + 4k)]$$

Let spectral coefficients of $\sum_{k=-\infty}^{\infty} \delta[n - (3 + 4k)]$ be c_k

$$c_k = \frac{1}{4}\sum_{n=0}^{3} \delta[n-3]e^{-j\pi/4kn}$$

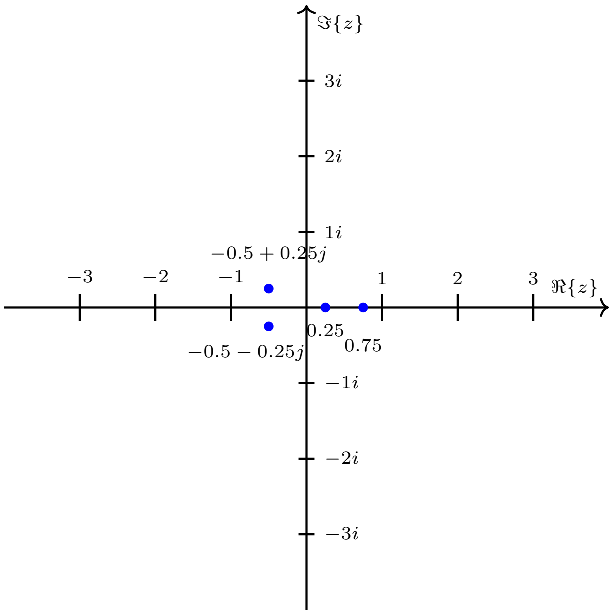
$$c_0 = 1/4$$

$$c_1 = j/4$$

$$c_2 = -1/4$$
$$c_3 = -j/4$$

Then spectral coefficients of $y[n]$, b_k equals to $a_k - c_k$.

$$b_0 = 3/4$$
$$b_1 = \frac{-1}{2} - j/4$$
$$b_2 = 1/4$$
$$b_3 = \frac{-1}{2} + j/4$$



7. (a)

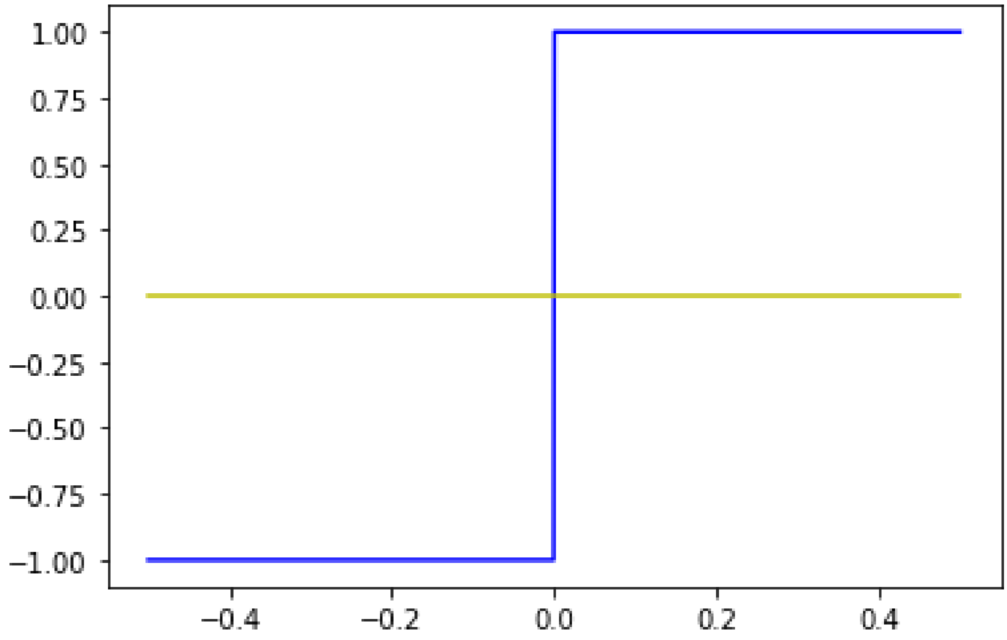
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$
$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jkw_0) e^{jkw_0 t}$$

Since $w_0 = 2\pi/(\pi/K) = 2K$, coefficient a_k where the value of $|k \cdot 2K|$ is greater than 80, is zero.

(b) If $y(t) \neq x(t)$, then there must be at least one value of a_k, where $|k \cdot 2K|$ is greater than 80 and non-zero.

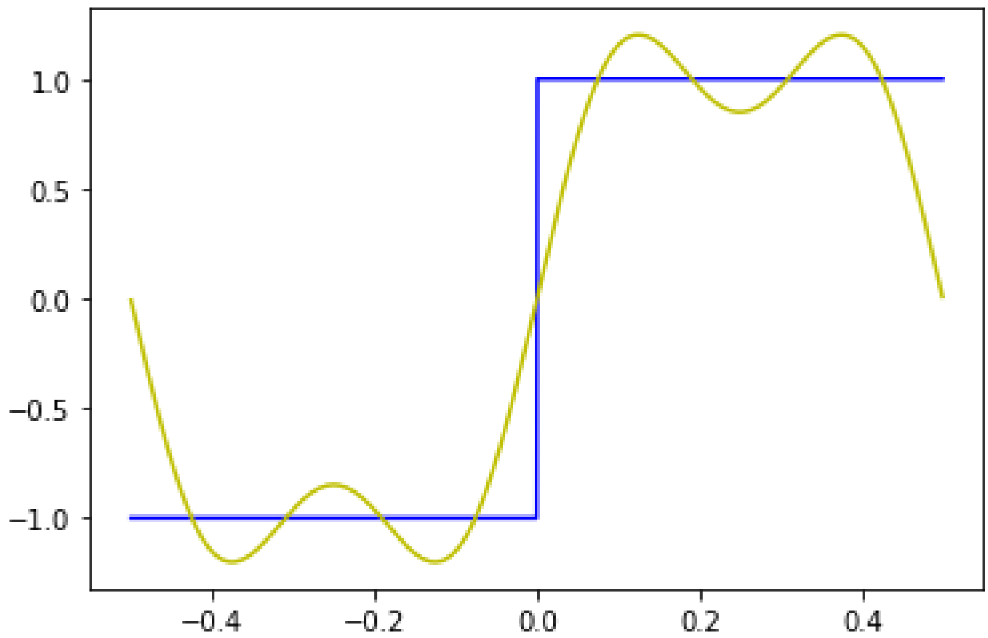
8.

For part C:
N = 1:



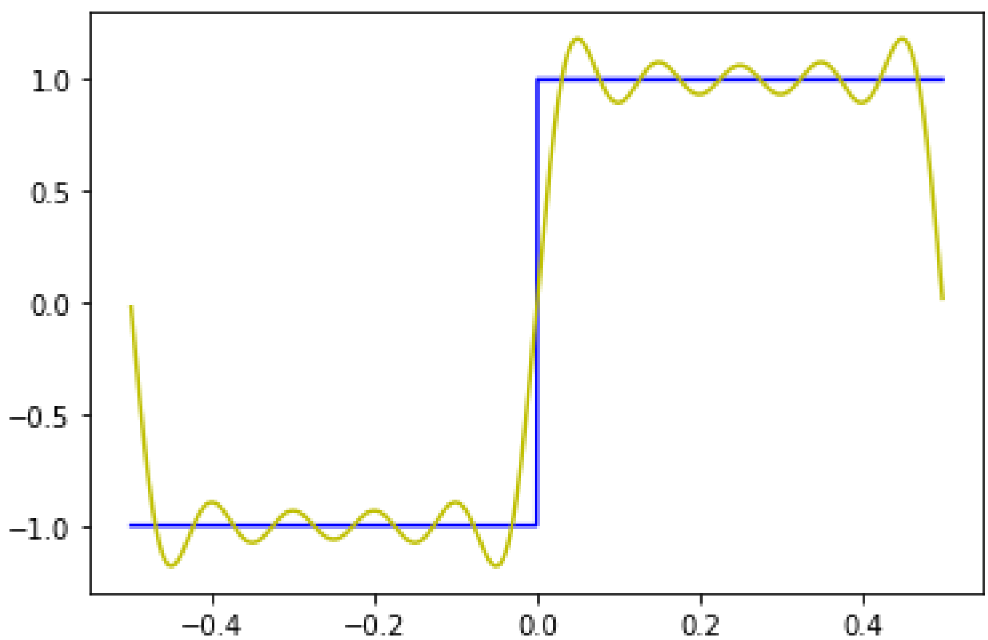
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−0.4
−0.2
0.0
0.2
0.4

N = 5:



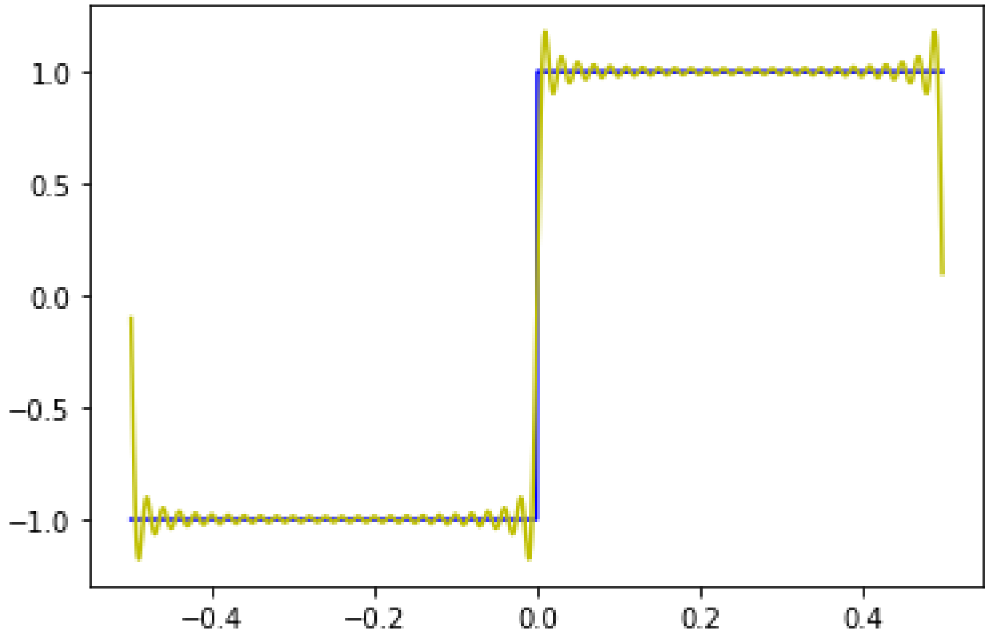
1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4

N = 10:

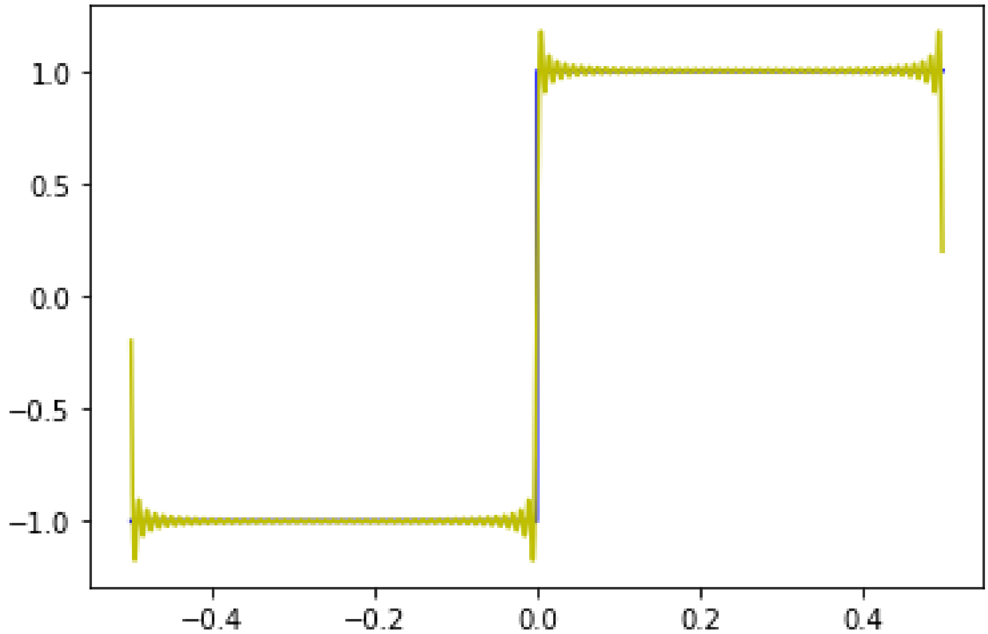


1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4

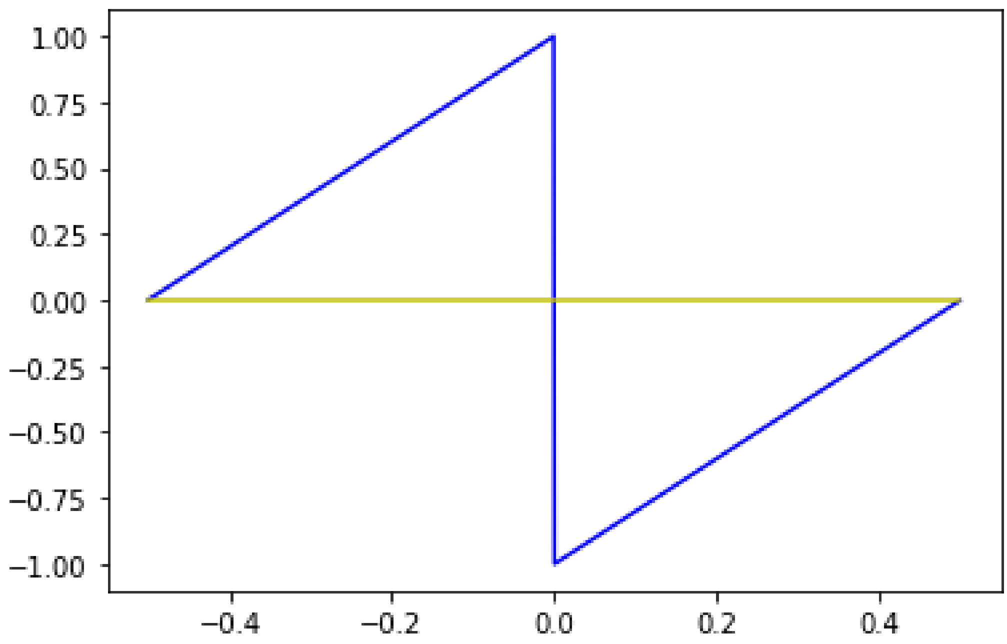
N = 50:



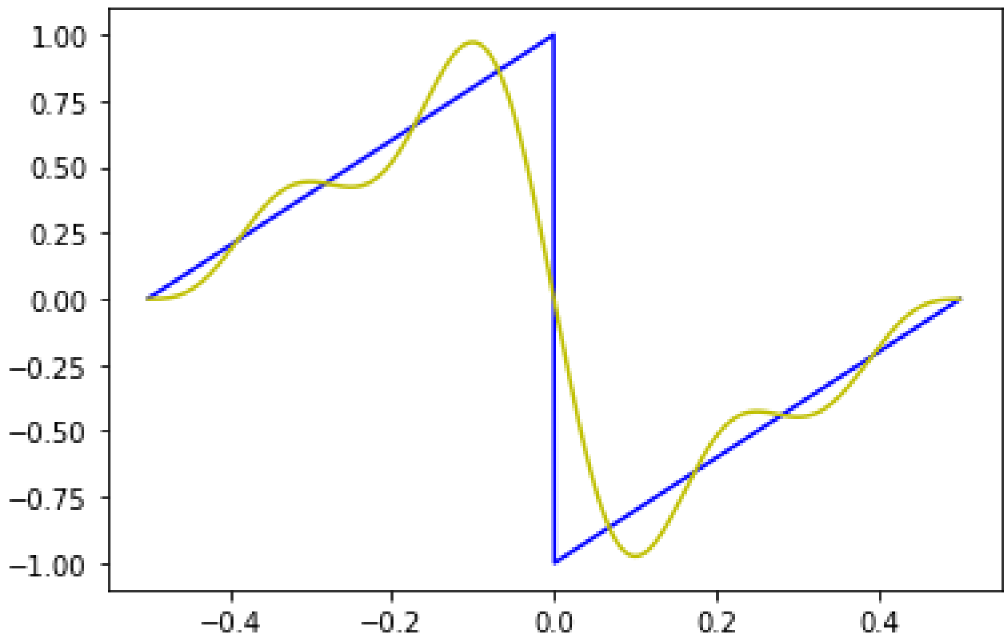
N = 100:



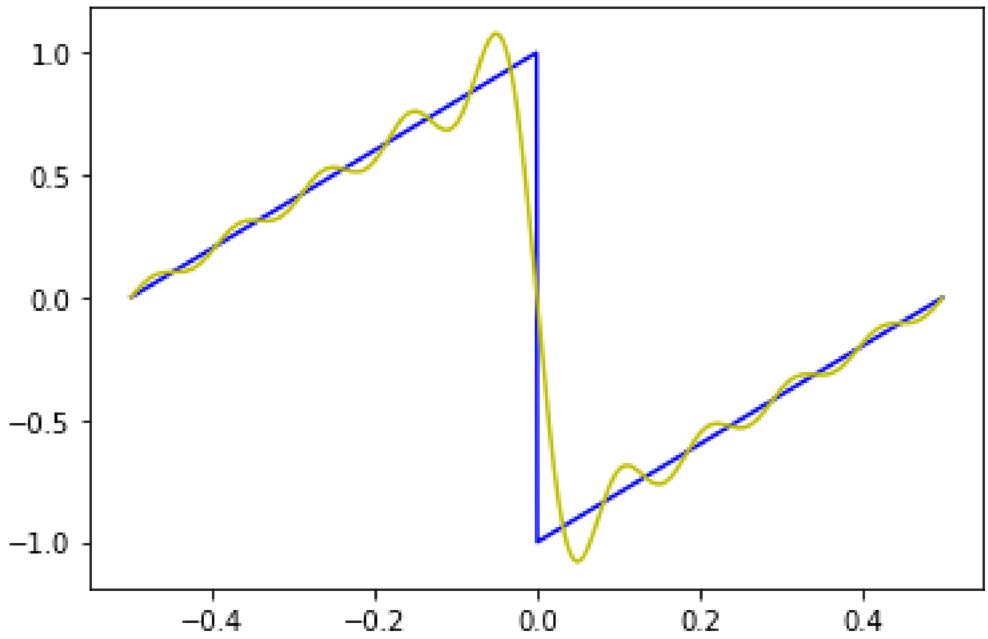
For part D:
N = 1:


1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−0.4
−0.2
0.0
0.2
0.4

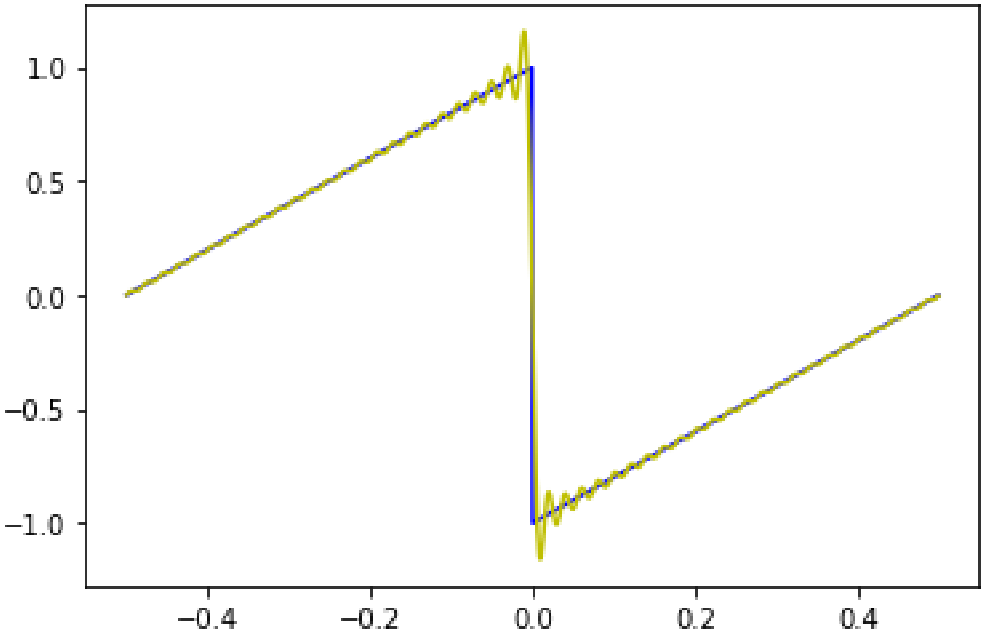
N = 5:


1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−0.4
−0.2
0.0
0.2
0.4

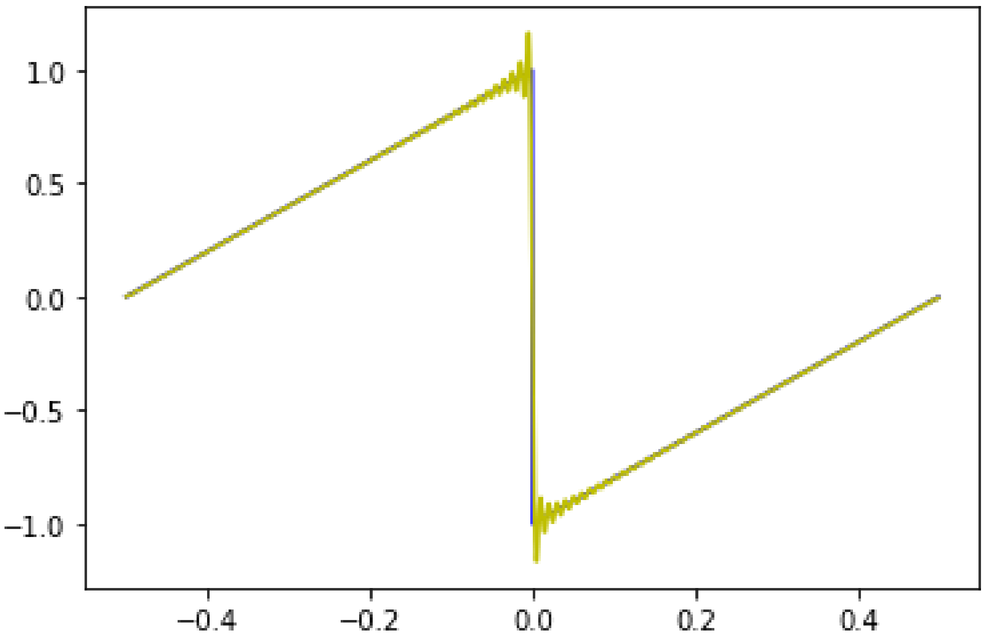
N = 10:


1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4

N = 50:



N = 100:



As n increases approximation becomes closer to the original function but on sharp corners our approximation does not converge to the original function.

Code:

```
import matplotlib.pyplot as plt
import numpy as np

def square_wave(n):
    return 1 if n > 0 else -1

def sawtooth_wave(n):
    return -1 + 2*n if n > 0 else 1 + 2*n

time = np.arange(-0.5, 0.5, 1/1000)
sq = np.array([square_wave(n) for n in time])
sw = np.array([sawtooth_wave(n) for n in time])

def fourier_coeffs(signal, period, num):
    coeffs = []
    for k in range(num):
```

```
        ak = signal*np.exp(-1j*(2*np.pi / period)*time*k)
        coeffs.append(ak.sum() / time.size)
    return coeffs

def inverse_fourier(coeffs, period, n):
    signal = np.array([2*a*np.exp(1j*(2*np.pi / period)*k*n) for k,a in enumerate(coeffs)])
    return signal.sum()

n = [1, 5, 10, 50, 100]
for num in n:
    a = fourier_coeffs(sq, 1, num)
    sq2 = np.array([inverse_fourier(a, 1, n) for n in time])

    plt.plot(time, sq, "b")
    plt.plot(time, sq2, "y")
    plt.show()

for num in n:
    a = fourier_coeffs(sw, 1, num)
    sw2 = np.array([inverse_fourier(a, 1, n) for n in time])

    plt.plot(time, sw, "b")
    plt.plot(time, sw2, "y")
    plt.show()
```