7–5 DYADIC NOTATION

Dyadic notation is an alternate notation which is convenient when using a vectorial approach to the dynamical equations, particularly the rotational equations of rigid bodies. The inertia dyadic $\mathbf{I}$ contains the same information concerning the moments and products of inertia of a given rigid body as does the inertia matrix $[I]$, but it also includes a pair of unit vectors with each of these terms. It will be seen that the nine components of an inertia dyadic transform in the same manner as the elements of the corresponding inertia matrix in a rotation of axes. Hence the dyadic $\mathbf{I}$ also represents a tensor of rank two.

Definition of a Dyadic

A *dyad* consists of a pair of vectors such as $\mathbf{ab}$, where the first vector $\mathbf{a}$ is called the *antecedent* and the second vector $\mathbf{b}$ is the *consequent.* A *dyadic* is just the sum of dyads. Suppose, for example, that we consider the dyad $\mathbf{ab}$. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are expressed in terms of a single set of independent unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, then one obtains

$$
\begin{aligned}
\mathbf{a} &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \\
\mathbf{b} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3
\end{aligned} \tag{7–57}
$$

and $\mathbf{ab}$ becomes the dyadic

$$
\mathbf{A} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\mathbf{e}_i\mathbf{e}_j \tag{7–58}
$$

where

$$
A_{ij} = a_i b_j \tag{7–59}
$$

Although these results are valid for $\mathbf{e}$'s which are not necessarily mutually orthogonal, we shall assume a right-handed system of orthogonal unit vectors when dyadic notation is used. Frequently, the Cartesian unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ will be employed.

A *conjugate dyadic* $\mathbf{A}^T$ is formed by interchanging the vectors of each dyad. If the dyadic $\mathbf{A}$ is equal to its conjugate, then $\mathbf{A}$ is *symmetric* and $A_{ij} = A_{ji}$. An example of a symmetric dyadic is the inertia dyadic

$$
\begin{aligned}
\mathbf{I} = {} & I_{xx}\mathbf{ii} + I_{xy}\mathbf{ij} + I_{xz}\mathbf{ik} \\
& + I_{yx}\mathbf{ji} + I_{yy}\mathbf{jj} + I_{yz}\mathbf{jk} \\
& + I_{zx}\mathbf{ki} + I_{zy}\mathbf{kj} + I_{zz}\mathbf{kk}
\end{aligned} \tag{7–60}
$$

If $\mathbf{A}$ is equal to the negative of its conjugate, then $\mathbf{A}$ is *skew-symmetric* and $A_{ij} = -A_{ji}$. The symmetry of a dyadic, or the lack of it, is a characteristic which is independent of the choice of the orthogonal unit vector set used in its description. If it is symmetric in one orthogonal coordinate system, it is symmetric in all such systems.

Dyadic Operations

The sum of dyadics is a dyadic obtained by adding corresponding elements. For example,

$$\mathbf{A} + \mathbf{B} = \mathbf{C} \tag{7-61}$$

where

$$C_{ij} = A_{ij} + B_{ij} \tag{7-62}$$

The *dot product* of a dyadic and a vector is a vector which, in general, differs in magnitude and direction from the original vector. For example, if $\mathbf{A} = \mathbf{ab}$ and $\mathbf{c}$ is a vector, then

$$\mathbf{A} \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \tag{7-63}$$

which is a vector having the direction of $\mathbf{a}$. On the other hand,

$$\mathbf{c} \cdot \mathbf{A} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} \tag{7-64}$$

which has the direction of $\mathbf{b}$. Hence we see that, in general, postmultiplying a dyadic by a vector gives a different result than premultiplying by the same vector.

For a *symmetric dyadic,* however, the order of taking the dot product with a vector does not matter. As an example, consider

$$\begin{aligned} \mathbf{I} \cdot \boldsymbol{\omega} = {} & (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\mathbf{i} \\ & + (I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\mathbf{j} \\ & + (I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)\mathbf{k} \end{aligned} \tag{7-65}$$

which we recognize as the angular momentum $\mathbf{H}$. Thus, we see that the dyadic equation

$$\mathbf{H} = \mathbf{I} \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \mathbf{I} \tag{7-66}$$

is equivalent to the matrix equation (7–38) or the algebraic equations (7–14).

Similarly, one can write a dyadic form of the equation for the rotational kinetic energy of a rigid body, namely,

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} \tag{7-67}$$

which is equivalent to Eq. (7–53) or Eq. (7–55).

We have seen that the dot product of a dyadic and a vector is a new vector, and thus the dyadic may be considered as an *operator* acting on the vector. A particularly simple symmetric dyadic which, in fact, leaves an arbitrary vector unchanged upon dot multiplication is the *unit dyadic*

$$\mathbf{U} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} \tag{7-68}$$

Thus

$$\mathbf{U} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{U} = \mathbf{a} \tag{7-69}$$

Using the unit dyadic, one can express the inertia dyadic $\mathbf{I}$ of Eq. (7–60) in the relatively simple integral form

$$\mathbf{I} = \int_V [(\boldsymbol{\rho} \cdot \boldsymbol{\rho})\mathbf{U} - \boldsymbol{\rho}\boldsymbol{\rho}]\rho \, dV \tag{7–70}$$

where the position vector

$$\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and the scalar ρ is the mass density.

The *cross product* of a dyadic and a vector is a dyadic. Once again, the order of multiplication is important, that is,

$$\mathbf{c} \times \mathbf{A} \neq \mathbf{A} \times \mathbf{c}$$

As an example, consider the cross-product $\boldsymbol{\omega} \times \mathbf{I}$, where we use the compact notation

$$\mathbf{I} = \sum_i \sum_j I_{ij}\mathbf{e}_i\mathbf{e}_j \tag{7–71}$$

and the $\mathbf{e}$'s form an orthogonal set of unit vectors. Then

$$\boldsymbol{\omega} \times \mathbf{I} = \sum_i \sum_j I_{ij}(\boldsymbol{\omega} \times \mathbf{e}_i)\mathbf{e}_j \tag{7–72}$$

which is a dyadic. Now let us postmultiply by $\boldsymbol{\omega}$, using the dot product, and obtain

$$\boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \times \mathbf{H} = \sum_i \sum_j I_{ij}\omega_j(\boldsymbol{\omega} \times \mathbf{e}_i) \tag{7–73}$$

This is identical with the result obtained by performing vector cross-multiplication of $\boldsymbol{\omega}$ with the expression

$$\mathbf{H} = \sum_i \sum_j I_{ij}\omega_j\mathbf{e}_i \tag{7–74}$$

given previously in Eq. (7–15).

The time derivative of a dyadic is illustrated by differentiating Eq. (7–71) with the result

$$\dot{\mathbf{I}} = \sum_i \sum_j (\dot{I}_{ij}\mathbf{e}_i\mathbf{e}_j + I_{ij}\dot{\mathbf{e}}_i\mathbf{e}_j + I_{ij}\mathbf{e}_i\dot{\mathbf{e}}_j) \tag{7–75}$$

Now assume a body-fixed coordinate system in which the moments and products of inertia are constant, and note that

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i, \qquad \dot{\mathbf{e}}_j = -\mathbf{e}_j \times \boldsymbol{\omega} \tag{7–76}$$

Then we obtain

$$\dot{\mathbf{I}} = \boldsymbol{\omega} \times \mathbf{I} - \mathbf{I} \times \boldsymbol{\omega} \tag{7–77}$$

Finally, let us consider the rate of change of the angular momentum of a rigid body about a reference point fixed in the body. Differentiating Eq. (7–66), we obtain

$$\begin{aligned}\dot{\mathbf{H}} &= \mathbf{I}\cdot\dot{\boldsymbol{\omega}} + \dot{\mathbf{I}}\cdot\boldsymbol{\omega}\\ &= \mathbf{I}\cdot\dot{\boldsymbol{\omega}} + (\boldsymbol{\omega}\times\mathbf{I} - \mathbf{I}\times\boldsymbol{\omega})\cdot\boldsymbol{\omega}\\ &= \mathbf{I}\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathbf{I}\cdot\boldsymbol{\omega}\end{aligned} \tag{7–78}$$

where we note that

$$\mathbf{I}\times\boldsymbol{\omega}\cdot\boldsymbol{\omega} = 0 \tag{7–79}$$

because the second vector in each dyad of $\mathbf{I}\times\boldsymbol{\omega}$ is normal to $\boldsymbol{\omega}$, and therefore its dot product with $\boldsymbol{\omega}$ vanishes.

7–6 TRANSLATION OF COORDINATE AXES

The defining equations for the moments and products of inertia, as given by Eqs. (7–10) and (7–11), do not require that the origin of the Cartesian coordinate system be taken at the center of mass. So let us calculate the moments and products of inertia for a given body with respect to a set of parallel axes that do not pass through the center of mass. Consider the body shown in Fig. 7-3. The center of mass is located at the origin O' of the primed system and at the point (x_c, y_c, z_c) in the unprimed system. Let us take an infinitesimal volume element dV which is located at (x, y, z) in the unprimed system and at (x', y', z') in the primed system. These coordinates are related by the equations:

$$\begin{aligned}x &= x' + x_c\\ y &= y' + y_c\\ z &= z' + z_c\end{aligned} \tag{7–80}$$

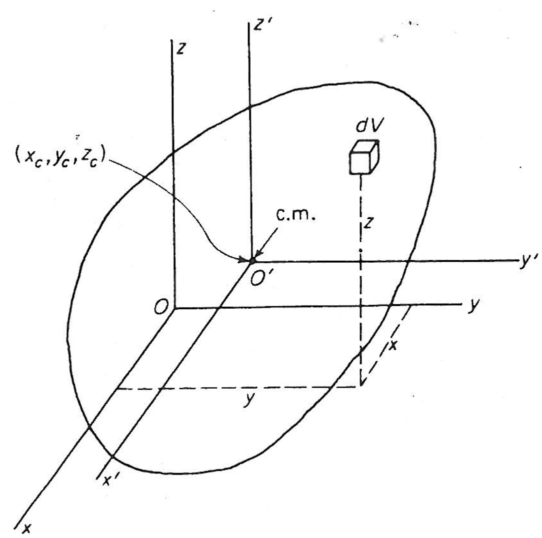

Figure 7-3 A rigid body, showing the location of parallel coordinate axes and a typical volume element.

D. Scheeres
scheeres@umich.edu

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second edition

Donald T. Greenwood

Professor, Department of Aerospace Engineering
The University of Michigan

Prentice Hall, Upper Saddle River, New Jersey 07458