Homework Assignments
ASEN 6062: Celestial Mechanics
Spring 2026

HW#1: Problems 1 - 7, Due Friday, February 6

1. Read Chapter 1 of Pollard (available on the Canvas site). Do all of the starred problems.

2. Establish the following identities related to conservation of linear and angular momentum, and energy:

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{m_i m_j}{\|\boldsymbol{r}_{ij}\|^3}\boldsymbol{r}_{ji} &= 0 \\
\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{m_i m_j}{\|\boldsymbol{r}_{ij}\|^3}\left(\boldsymbol{r}_{ji}\times\boldsymbol{r}_{ji}\right) &= 0 \\
\mathcal{U} = -\frac{\mathcal{G}}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{m_i m_j}{\|\boldsymbol{r}_{ij}\|} &= -\mathcal{G}\sum_{1\leq i<j\leq N}\frac{m_i m_j}{\|\boldsymbol{r}_{ij}\|} \\
\frac{d}{dt}\left(T+\mathcal{U}\right) &= 0
\end{aligned}$$

where $T = \frac{1}{2\sum_{i=1}^{N} m_i}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} m_i m_j \dot{\boldsymbol{r}}_{ij}\cdot\dot{\boldsymbol{r}}_{ij}$ and $\mathcal{U}$ is as above.

3. A homogenous function $F(\boldsymbol{x})$ of order n is a function such that $F(\alpha\boldsymbol{x}) = \alpha^n F(\boldsymbol{x})$. From Euler's Theorem on Homogeneous functions, a homogeneous function of order n has the property:

$$\boldsymbol{x}\cdot\frac{\partial F}{\partial\boldsymbol{x}} = nF$$

   - Show that $\mathcal{U}$ is a homogeneous function of order -1, where $\mathcal{U} = -\sum_{1\leq i<j\leq N}\frac{\mathcal{G}m_i m_j}{\|\boldsymbol{r}_{ij}\|}$.
   - Show that T and I_P are homogeneous functions of order 2, where $T = \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i$ and $I_P = \sum_{i=1}^{N} m_i\boldsymbol{r}_i\cdot\boldsymbol{r}_i$.
   - Show that $I_P\,\mathcal{U}^2$ is homogeneous of order 0.

4. Cauchy's Inequality states the following:

$$C^2 \leq AB$$

where $C = \sum_k a_k b_k$, $A = \sum_k a_k^2$ and $B = \sum_k b_k^2$. Apply this to the angular momentum integral $\boldsymbol{H} = \sum_{i=1}^{N} m_i\boldsymbol{r}_i\times\dot{\boldsymbol{r}}_i$ to find "Sundman's Inequality": $H^2 \leq 2TI_P$.

5. Lagrange's Identity states the following:

$$\left(\sum_{k=1}^{N} a_k^2\right)\left(\sum_{k=1}^{N} b_k^2\right) - \left(\sum_{k=1}^{N} a_k b_k\right)^2 = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(a_i b_j - a_j b_i\right)^2$$

- Show that this can be used to find an alternate form of the polar moment of inertia, $I_P = \sum_{i=1}^{N} m_i r_i^2$:

$$I_P = \frac{1}{2M}\sum_{i=1}^{N}\sum_{j=1}^{N} m_i m_j \boldsymbol{r}_{ij}\cdot\boldsymbol{r}_{ij} + M\boldsymbol{r}_C\cdot\boldsymbol{r}_C$$

  where $M = \sum_{i=1}^{N} m_i$ and $\boldsymbol{r}_C = \frac{1}{M}\sum_{i=1}^{N} m_i\boldsymbol{r}_i$.

- Use the identity to also show that the Kinetic Energy ($T = \frac{1}{2}\sum_{i=1}^{N} v_i^2$) can be expressed as:

$$T = \frac{1}{2M}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} m_i m_j v_{ij}^2 + \frac{1}{2}Mv_C^2$$

6. Verify the following identities:

$$\sum_{i=1}^{N} m_i\widetilde{\boldsymbol{r}}_i\cdot\boldsymbol{v}_i = \frac{1}{M}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} m_i m_j\widetilde{\boldsymbol{r}}_{ij}\cdot\boldsymbol{v}_{ij}$$

$$\sum_{i=1}^{N} m_i\left[r_i^2\boldsymbol{U} - \boldsymbol{r}_i\boldsymbol{r}_i\right] = \frac{1}{M}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} m_i m_j\left[r_{ij}^2\boldsymbol{U} - \boldsymbol{r}_{ij}\boldsymbol{r}_{ij}\right]$$

where $\widetilde{\boldsymbol{a}}$ is the cross product dyad, $\boldsymbol{U}$ is the unity dyad, and $\boldsymbol{rr}$ is a dyadic (see the dyad handout notes for more details).

7. Derive the Jacobi Equations of Motion for a 3-body system, using Lagrange's Equations. To do this, first explicitly write out the kinetic energy in terms of the first two Jacobi coordinate vector velocities and then evaluate all of the relevant partial derivatives, etc.

$$\frac{m_1 m_2}{m_1+m_2}\ddot{\boldsymbol{R}} = -\frac{\mathcal{G}m_1 m_2}{R^3}\boldsymbol{R} - \frac{\mathcal{G}m_1 m_2 m_3}{m_1+m_2}\left[\frac{\boldsymbol{\rho}+\frac{m_2}{m_1+m_2}\boldsymbol{R}}{|\boldsymbol{\rho}+\frac{m_2}{m_1+m_2}\boldsymbol{R}|^3} - \frac{\boldsymbol{\rho}-\frac{m_1}{m_1+m_2}\boldsymbol{R}}{|\boldsymbol{\rho}-\frac{m_1}{m_1+m_2}\boldsymbol{R}|^3}\right]$$

$$\frac{m_3(m_1+m_2)}{m_1+m_2+m_3}\ddot{\boldsymbol{\rho}} = -\mathcal{G}m_2 m_3\frac{\boldsymbol{\rho}-\frac{m_1}{m_1+m_2}\boldsymbol{R}}{|\boldsymbol{\rho}-\frac{m_1}{m_1+m_2}\boldsymbol{R}|^3} - \mathcal{G}m_3 m_1\frac{\boldsymbol{\rho}+\frac{m_2}{m_1+m_2}\boldsymbol{R}}{|\boldsymbol{\rho}+\frac{m_2}{m_1+m_2}\boldsymbol{R}|^3}$$

Starting from your previous results, derive the Jacobi coordinates for $N = 4$ and $N = 5$.