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MATHEMATICAL INTRODUCTION TO
CELESTIAL MECHANICS

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To A. K. P.

Preface

In recent years there has been a strong revival of interest in celestial mechanics, but not much of it has been reflected in the offerings of mathematics departments. The recent work of Kolmogoroff, Arnold, and J. Moser shows that it is a field very much alive mathematically and deserves restoration to the mathematics curriculum. The main purpose in writing this book is to make available the basic mathematics underlying the subject, in a manner suitable to this century. A secondary purpose is to lay the groundwork for a sequel of a more advanced character.

The selection of material is based on several years of experience with a one-term course offered to students with a background in vector analysis, partial differentiation, and ordinary differential equations. I have found that the first two chapters cover the major part of the term. The remainder can be filled out by either Chapter 3 or Chapter 4, which have deliberately been made independent of one another. Ideally, perturbation theory should be combined with Hamilton-Jacobi theory, and their separation here may be a just cause for complaint. But I believe that a thorough grounding in each should precede their union.

I wish to make these acknowledgements: to the Air Force Office of Scientific Research, for a grant which enabled me to begin; to the Argonne National Laboratory for a grant which enabled me to finish; to Miss Grace M. Krause of the Argonne National Laboratory for her superb preparation of the manuscript; to Mr. Kerry M. Krafthefer of the Argonne National Laboratory for his distinctive drawings and table.

HARRY POLLARD

NOTE ON THE USE OF THIS BOOK

1. Vectors are printed in bold-face. Where possible the length of a vector is indicated by the same letter in italic. For example the length of $\mathbf{v}$ is v. When this cannot be done, the length is indicated by the customary absolute-value symbol. Thus the length of $\mathbf{a} \times \mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$.

2. Starred exercises are not necessarily difficult. The star indicates an important final result or a result to be used in the sequel. Therefore starred exercises should not be omitted.

3. All references to formulas and exercises are to the same chapter where they occur, unless otherwise stated.

4. The dependence of chapters is indicated by the following diagram.

$$\begin{tikzcd} & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 4 \end{tikzcd}$$

Contents

Chapter One

Chapter Two

Chapter Three

Chapter Four

MATHEMATICAL INTRODUCTION TO

CELESTIAL MECHANICS

Chapter One

THE CENTRAL FORCE PROBLEM

1. FORMULATION OF THE PROBLEM

Celestial mechanics begins with the central force problem: to describe the motion of a particle of mass m which is attracted to a fixed center O by a force $mf(r)$ which is proportional to the mass and depends only on the distance r between the particle and O. The function f will be called a *law of attraction.* It is assumed to be continuous for $0 < r < \infty$.

Mathematically, the problem is easy to formulate. Indicate the position of the mass by the vector $\mathbf{r}$ directed from O. According to Newton's second law, the motion of the particle is governed by the equation

$$m\ddot{\mathbf{r}} = -\, mf(r)r^{-1}\mathbf{r},$$

where $r^{-1}\mathbf{r}$ is a unit vector directed to the position of the particle. If $\mathbf{v}$ denotes the velocity vector $\dot{\mathbf{r}}$, the equation can be written as the pair

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = -f(r)r^{-1}\mathbf{r}. \tag{1.1}$$

Observe that the value of m is irrelevant to the equations of motion. The problem is now this: to study the properties of pairs of vector-valued functions $\mathbf{r}(t)$, $\mathbf{v}(t)$ which simultaneously satisfy the Eqs. (1.1) over an interval of time.

The special case when the law of attraction is Newton's law of gravitation is the most important. In this case $f(r) = \mu r^{-2}$, where μ is a positive constant depending only on the units chosen and on the particular source of attraction. The Eqs. (1.1) become

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = -\, \mu r^{-3}\mathbf{r}. \tag{1.2}$$

2. THE CONSERVATION OF ANGULAR MOMENTUM: KEPLER'S SECOND LAW

Let us now assume that (1.1) is satisfied for some interval of time by the pair of functions $\mathbf{r}(t)$, $\mathbf{v}(t)$ which we write simply as $\mathbf{r}$, $\mathbf{v}$. From the second equation of the pair we conclude that

$$\mathbf{r} \times \dot{\mathbf{v}} = -f(r)r^{-1}(\mathbf{r} \times \mathbf{r}) = 0,$$

since the cross-product of a vector with itself is zero. Therefore, the derivative of the vector $\mathbf{r} \times \mathbf{v}$, which is $\mathbf{r} \times \dot{\mathbf{v}} + \mathbf{v} \times \mathbf{v}$, vanishes identically. Hence,

$$\mathbf{r} \times \mathbf{v} = \mathbf{c}, \tag{2.1}$$

where $\mathbf{c}$ is a constant vector. The vector $m\mathbf{c}$ is called the *moment of momentum* and its length mc the *angular momentum* of the particle. We ignore these refinements and refer to either $\mathbf{c}$ or c as the angular momentum. The assertion (2.1) is known as *the conservation of angular momentum.*

An important consequence of the principle can be deduced immediately. According to (2.1) we have $\mathbf{c}\cdot\mathbf{r} = 0$. If $c \neq 0$, this means that $\mathbf{r}$ is always perpendicular to the fixed vector $\mathbf{c}$. Consequently, *if $c \neq 0$, all the motion takes place in a fixed plane through the origin perpendicular to* $\mathbf{c}$.

If $c = 0$, a little more subtlety is needed. Let $\mathbf{u}$ be a differentiable vector function of time and u its length. Since $u^2 = \mathbf{u}\cdot\mathbf{u}$, it follows that $u\dot{u} = \mathbf{u}\cdot\dot{\mathbf{u}}$. Therefore, if $u \neq 0$, we have

$$\begin{aligned}\frac{d}{dt}\frac{\mathbf{u}}{u} &= \frac{u\dot{\mathbf{u}} - \dot{u}\mathbf{u}}{u^2}\\ &= \frac{(\mathbf{u}\cdot\mathbf{u})\dot{\mathbf{u}} - (\mathbf{u}\cdot\dot{\mathbf{u}})\mathbf{u}}{u^3},\end{aligned}$$

or

$$\frac{d}{dt}\frac{\mathbf{u}}{u} = \frac{(\mathbf{u} \times \dot{\mathbf{u}}) \times \mathbf{u}}{u^3}, \tag{2.2}$$

according to the vector formula

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}.$$

As an application of (2.2), let $\mathbf{u} = \mathbf{r}$. Then (2.2) becomes

$$\frac{d}{dt}\frac{\mathbf{r}}{r} = \frac{(\mathbf{r} \times \mathbf{v}) \times \mathbf{r}}{r^3} = \frac{\mathbf{c} \times \mathbf{r}}{r^3}, \tag{2.3}$$

by (2.1). Therefore, if $c = 0$, the vector $\mathbf{r}/r$ is a constant, and *the motion takes place along a fixed straight line through the origin.*

In case $c \neq 0$, another important consequence can be deduced from (2.1). Introduce into the plane of motion a polar coordinate system centered at O and forming a right-handed system with the vector $\mathbf{c}$. (See Fig. 1.) Then $\mathbf{r} = [r\cos\theta, r\sin\theta, 0]$ and $\mathbf{c} = [0, 0, c]$. A simple computation shows that (2.1) yields $r^2\dot{\theta} = c$. According to the calculus, the rate at which area is swept out by a radius vector from O is just $\frac{1}{2}r^2\dot{\theta}$. Therefore *the particle sweeps out area at the constant rate $c/2$.* This fact is Kepler's *second law.*

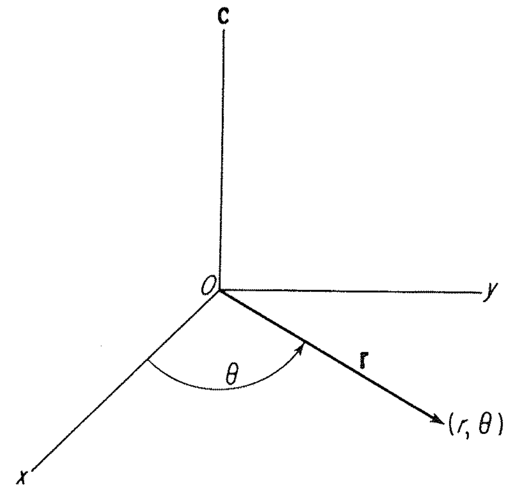

Figure 1

EXERCISE 2.1. Set up the equations of motion of a particle moving subject to two distinct centers of attraction, each with its own law of attraction.

EXERCISE 2.2. Suppose that a particle subject to attraction by a fixed center starts from rest, i. e., that at some instant $t = 0$ we have $v = 0$. Then by (2.1) $c = 0$ and the motion is linear. Suppose, moreover, that $f(r)$ is positive for $0 < r < \infty$. Prove that the particle must collide with the center of force in a finite length of time t_0.

EXERCISE 2.3. In the preceding problem, can you tell where the particle well be at each instant of time between 0 and t_0? First try the case $f(r) = \mu r^{-3}$ (inverse cube law), then $f(r) = \mu r^{-2}$ (inverse square law).

3. THE CONSERVATION OF ENERGY

So far we have found a vector $\mathbf{c}$ which remains constant throughout a particular motion. There is another constant of the motion which is of major importance, this time a scalar quantity called the *energy*. To find it, start with the second of Eqs. (1.1) and take the dot product of each side with $\mathbf{v}$. We obtain

$$\begin{aligned}\dot{\mathbf{v}}\cdot\mathbf{v} &= -f(r)r^{-1}(\mathbf{r}\cdot\mathbf{v})\\ &= -f(r)r^{-1}r\dot{r}\\ &= -f(r)\frac{dr}{dt}.\end{aligned}$$

Integration of both sides yields

(3.1) $$\tfrac{1}{2}v^2 = f_1(r) + h,$$

where $f_1(r)$ is a function whose derivative is $-f(r)$ and h is a constant. The function $f_1(r)$ is determined conventionally this way:

$$f_1(r) = \int_r^a f(x)dx$$

where (i) a is chosen as $+\infty$ if the integral converges; (ii) a is chosen to be 0 if the first choice leads to a divergent integral but the second does not; (iii) a is chosen to be 1 if the first two choices fail. Thus, if $f(r)$ is of the form $f(r) = \mu r^{-p}$, then $a = \infty$ if $p > 1$; $a = 0$ if $p < 1$; $a = 1$ if $p = 1$. The most important case is that of Newton:

$$f(r) = \mu r^{-2}, \qquad f_1(r) = \mu r^{-1}.$$

With the above convention the function $-mf_1(r)$ is known as the *potential energy* and is denoted by the symbol $-U$. The quantity $T = mv^2/2$ is called the *kinetic energy*, and $h_1 = mh$ the *energy*. The statement (3.1) becomes

(3.2) $$T = U + h_1,$$

and is known as the *principle of conservation of energy*.

EXERCISE 3.1. Show that if $f(r) = \mu r^{-p}$, where $p > 1$, then a particle moving with negative energy cannot move indefinitely far from O.

EXERCISE 3.2. Show that if $f(r) = \mu r^{-p}$, then $f_1(r) = \mu(p-1)^{-1}r^{1-p}$ if $p \neq 1$ and $f_1(r) = \mu \log\frac{1}{r}$ if $p = 1$.

*EXERCISE 3.3. Let $\mathbf{a} = \mathbf{r}$, $\mathbf{b} = \mathbf{v}$ in the standard vector formula

$$(\mathbf{a}\cdot\mathbf{b})^2 + (\mathbf{a}\times\mathbf{b})^2 = a^2b^2.$$

Conclude that

$$v^2 = \dot{r}^2 + c^2r^{-2}.$$

What is the physical meaning of the components $\dot{r}$ and c/r of $\mathbf{v}$? Show that the law of conservation of energy can be written

$$r^2\dot{r}^2 + c^2 = 2r^2[f_1(r) + h].$$

4. THE INVERSE SQUARE LAW: KEPLER'S FIRST LAW

In this section we shall assume that the particle is moving according to Newton's law of gravitation. The governing equations are then (2.1), which we repeat here for convenience as

(4.1) $$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = -\mu r^{-3}\mathbf{r}.$$

It turns out that, in addition to the vector $\mathbf{c}$, there is another important vector which remains constant throughout the motion. It does not have a name in astronomical literature. We shall call it the *eccentric axis* and denote it by the symbol $\mathbf{e}$. To derive it, start with the formula (2.3) and multiply both sides by $-\mu$. Then

$$-\mu\frac{d}{dt}\frac{\mathbf{r}}{r} = \mathbf{c}\times(-\mu r^{-3}\mathbf{r}).$$

According to the second of Eqs. (4.1), this becomes

$$\mu\frac{d}{dt}\frac{\mathbf{r}}{r} = \dot{\mathbf{v}}\times\mathbf{c}.$$

Integration of both sides yields

(4.2) $$\mu\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right) = \mathbf{v}\times\mathbf{c},$$

where $\mathbf{e}$ is a constant of integration.

Since $\mathbf{r}\cdot\mathbf{c} = 0$, it follows that $\mathbf{e}\cdot\mathbf{c} = 0$. Hence, if $c \neq 0$, the vectors $\mathbf{e}$ and $\mathbf{c}$ are perpendicular, so that $\mathbf{e}$ lies in the plane of motion. If $c = 0$, $\mathbf{r}/r = -\mathbf{e}$, so that $\mathbf{e}$ lies along the line of motion; in this case the length e of $\mathbf{e}$ is always 1.

We shall now find the interpretation of e when $c \neq 0$. Take the dot product of both sides of (4.2) with r. Then

$$\mu(\mathbf{e}\cdot\mathbf{r} + r) = \mathbf{r}\cdot\mathbf{v}\times\mathbf{c} = \mathbf{r}\times\mathbf{v}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{c}.$$

Consequently,

(4.3) $$\mathbf{e}\cdot\mathbf{r} + r = c^2/\mu.$$

There are two cases. If $e = 0$, then $r = c^2/\mu$, a constant. Therefore the motion is circular. Moreover, according to the formula $r^2v^2 = r^2\dot{r}^2 + c^2$ of Ex. 3.3, it follows that $rv = c$, $v = \mu/c$, so that the particle moves with constant speed. By the law of conservation of energy, $v^2/2 = \mu/r + h$. Therefore $h = -\mu^2/2c^2$, a negative number. Observe finally that $2T = U$.

Suppose now that $e \neq 0$. In the plane of motion indicated by Fig. 1, introduce the vector $\mathbf{e}$ as shown in Fig. 2. The fixed angle from the x-axis to $\mathbf{e}$ will be denoted by ω. If (r, θ) represents a position Q of the particle, the angle $\theta - \omega$ will be denoted by f. The same position of the particle can then be represented as (r, f) if $\mathbf{e}$ is used as the axis of coordinates. It follows that $e\cdot r = er\cos f$ and Eq. (4.3) becomes

$$r = \frac{c^2/\mu}{1 + e\cos f}. \tag{4.4}$$

Consider the dotted line L in Fig. 2 drawn at a distance $c^2/\mu e$ from O, perpendicular to e and on the side of O to which e is directed. Equation (4.4), which can also be written $r = e\left(\frac{c^2}{\mu e} - r\cos f\right)$, simply says that the

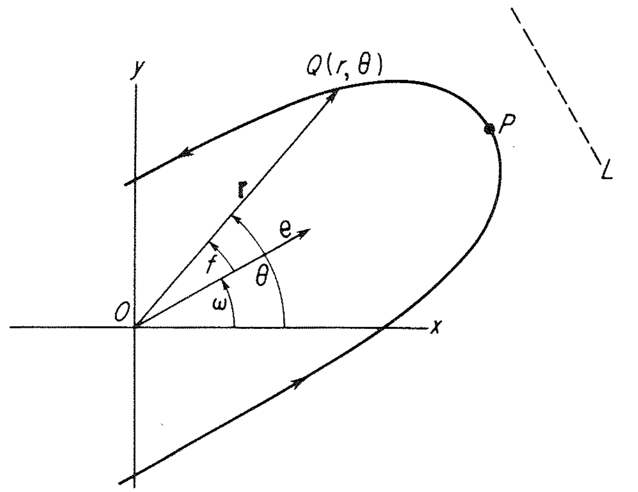

Figure 2

distance of the particle at Q from O is e times its distance from L. Consequently *the particle moves on a conic section of eccentricity e with one focus at O*. This is Kepler's *first law*.

As (4.4) shows, the value of r is smallest when $f = 0$, since $e > 0$. Therefore the vector $\mathbf{e}$ is of length equal to the eccentricity and points to the position P at which the particle is closest to the focus.

There is some traditional terminology used by the astronomers that the reader ought to know. The position P is called the *pericenter*, the angle f the *true anomaly*. Various names are given to the pericenter, according to the source of attraction at O. If the source is the sun, P is called *perihelion*; if the earth, *perigee*; if a star, *periastron*. In the study of the solar system, the x-axis of Fig. 1 is fixed by astronomical convention. In that case, ω is the *amplitude of pericenter*.

We return to the geometry. The word *orbit* will be used to describe the set of positions occupied by the particle without any indication of the time at which a particular position is occupied. From the theory of conics it follows that if $0 < e < 1$ the orbit falls on an ellipse; if $e = 1$, on a

parabola; and if $e > 1$, on a branch of hyperbola convex to the focus. Remember that in each case $c > 0$.

Since $r^2\dot{\theta} = c$ and $\dot{\theta} = \dot{f}$, it follows that $\dot{f} > 0$, so that the orbit is traced out in the direction of increasing f. This is indicated by the arrows on the curve in Fig. 2.

*EXERCISE 4.1. Show that if $0 < e < 1$ or $e > 1$ the semi-major axis of the corresponding conic has length a given by the formula

$$\mu a\,|e^2 - 1| = c^2.$$

EXERCISE 4.2. Use (4.2) to obtain the formula

$$\mu\mathbf{e} = \left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\mathbf{v}.$$

5. RELATIONS AMONG THE CONSTANTS

We pause at this point to remind the reader of some basic facts about differential equations. Let $f_i(z_1, \ldots, z_n)$, $i = 1, \ldots, n$ represent n functions with continuous first partial derivatives in some region of n-dimensional space, and let $(\zeta_1, \ldots, \zeta_n)$ be a particular point of this region. Then the system of differential equations

$$\dot{z}_i = f_i(z_1, \ldots, z_n), \qquad i = 1, \ldots, n \tag{5.1}$$

will have a unique solution $z_i(t)$ defined in a neighborhood of $t = 0$, such that $z_i(0) = \zeta_i$, $i = 1, \ldots, n$.

Now consider the basic Eqs. (1.1) with the additional assumption that f has a continuous derivative. This includes the special cases $f(r) = \mu r^{-p}$. Each of the two Eqs. (1.1) stands in place of three scalar equations, so that the pair constitutes a system of order six of the form (5.1). Specifically, let x, y, z denote the components of $\mathbf{r}$ in a rectangular coordinate system and let α, β, γ denote the components of $\mathbf{v}$. The equations become

$$\begin{aligned}
\dot{x} &= \alpha \\
\dot{y} &= \beta \\
\dot{z} &= \gamma \\
\dot{\alpha} &= -f(r)r^{-1}x \\
\dot{\beta} &= -f(r)r^{-1}y \\
\dot{\gamma} &= -f(r)r^{-1}z,
\end{aligned}$$

where $r^2 = x^2 + y^2 + z^2$. It follows that there is a unique solution satisfying six prescribed values of $x, y, z, \alpha, \beta, \gamma$ at $t = 0$. In vector form this says that the system (1.1) has a unique solution $\mathbf{r}(t)$, $\mathbf{v}(t)$ taking on prescribed values $\mathbf{r}_0$, $\mathbf{v}_0$ at time $t = 0$. These values can be prescribed arbitrarily.

In the special case $f(r) = \mu r^{-2}$, we have found that each of the quantities $\mathbf{c}$, $\mathbf{e}$, h remains constant during the motion and is therefore determined by its value at $t = 0$:

$$\mathbf{c} = \mathbf{r}_0 \times \mathbf{v}_0,$$
$$\mathbf{e} = \mu^{-1}(\mathbf{v}_0 \times \mathbf{c}) - \mu^{-1}r_0^{-1}\mathbf{r}_0,$$
$$h = \tfrac{1}{2}v_0^2 + \mu r_0^{-1}.$$

Since $\mathbf{c}$, $\mathbf{e}$, h constitute *seven* scalar quantities, it follows that there must be relations among them. We have already seen that there is a relation between $\mathbf{c}$ and $\mathbf{e}$, namely $\mathbf{c}\cdot\mathbf{e} = 0$. Therefore at most six of the seven quantities can be independent. Actually there is still another relation among the seven which reduces the number to *five*; it will be seen later that no further reduction is possible.

To obtain the new relation, square both sides of Eq. (4.2). Since v is perpendicular to c, we can replace $(\mathbf{v} \times \mathbf{c})^2$ by v^2c^2 to obtain

$$\mu^2\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right)^2 = v^2c^2$$

or

$$\mu^2\left(e^2 + \frac{2}{r}\mathbf{e}\cdot\mathbf{r} + 1\right) = v^2c^2.$$

Replace v^2 by $2h + (2\mu/r)$ and $\mathbf{e}\cdot\mathbf{r}$ by $(c^2/\mu) - r$, according to Eq. (4.3). Then

$$\mu^2(e^2 - 1) = 2hc^2. \tag{5.2}$$

Notice that this agrees with the earlier results that $e = 1$ if $c = 0$ and $h = -\mu^2/2c$ if $e = 0$.

Equation (5.2) has the following important consequences. If $c \neq 0$, then $e < 1$, $e = 1$ or $e > 1$ according to whether the energy h is negative, zero, or positive. If $h \neq 0$ and $c \neq 0$ and a is the semi-major axis of the conic (see Ex. 4.1), then

$$a = \tfrac{1}{2}\mu|h|^{-1}. \tag{5.3}$$

From this and the energy formula $\frac{1}{2}v^2 = (\mu/r) + h$, we obtain these basic formulas:

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right) \quad \text{if } h > 0;$$
$$v^2 = \frac{2\mu}{r} \quad \text{if } h = 0; \tag{5.4}$$
$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \quad \text{if } h < 0.$$

These formulas still hold if $c = 0$ provided we adopt (5.3) as the definition of a; we shall do so.

EXERCISE 5.1. What can you say about the orbit if $f(r) = -\mu r^{-2}$ rather than $f(r) = \mu r^{-2}$? This corresponds to a repulsive force rather than an attraction.

EXERCISE 5.2. Use (5.4) to prove that in the case of elliptical motion the speed of the particle at each position Q is the speed it would acquire in falling to Q from the circumference of a circle with center at O and radius equal to the major axis of the ellipse.

*EXERCISE 5.3. The area of an ellipse is $\pi a^2(1 - e^2)^{1/2}$. We already know that if $c \neq 0$ the particle sweeps out area at the rate $c/2$. Combine these facts to show that if $0 < e < 1$ the period p of a particle, that is, the time it takes to sweep out the area once, is given by the formula $p = (2\pi/\sqrt{\mu})a^{3/2}$. This is Kepler's *third law*.

*EXERCISE 5.4. Define the moment of inertia $2I$ by the formula $2I = mr^2$. Write $r^2 = (\mathbf{r}\cdot\mathbf{r})$ and prove that

$$\ddot{I} = 2T - U = T + h_1 = U + 2h_1.$$

In the case of circular motion I is constant so that $2T = U$, a result we already know from Sec. 4.

EXERCISE 5.5. (*Hard.*) Use the preceding exercise to prove that if $c \neq 0$, $h > 0$, then $r/|t|$ approaches $\sqrt{2h}$ as $|t| \to \infty$. (The hypothesis $c \neq 0$ rules out the possibility of a collision with the origin in a finite time.)

6. ORBITS UNDER NON-NEWTONIAN ATTRACTION

The elegant method used in Sec. 4 to obtain orbits is essentially due to Laplace (who, however, did not have the vector concept available to him). It is applicable specifically to Newton's law of attraction. In the general case another method must be used. We know that if $c = 0$ the orbit is linear, so we shall assume that $c \neq 0$. Moreover, we assume that $f(r)$ has a continuous derivative.

Let us first dispose of the case of circular motion $r = r_0$. By the principle of conservation of energy, v is also a constant v_0 so the motion is uniform. The normal acceleration in the plane of motion is v_0^2/r_0 and this must be balanced by the attraction $f(r_0)$. Therefore, $v_0^2 = r_0 f(r_0)$. Since the velocity vector is perpendicular to the radius vector, it follows from $\mathbf{r} \times \mathbf{v} = \mathbf{c}$ that $rv = c$. Hence, $r_0v_0 = c$, so that $c^2 = r_0^3 f(r_0)$. On the other hand, according to Ex. 3.3, the law of conservation of energy can be written

$$r^2\dot{r}^2 + c^2 = 2r^2[f_1(r) + h]. \tag{6.1}$$

Since $\dot{r} = 0$, we conclude that $c^2 = 2r_0^2[f_1(r_0) + h]$. Therefore, circular motion implies the two relations

$$c^2 = r_0^3 f(r_0), \qquad c^2 = 2r_0^2[f_1(r_0) + h]. \tag{6.2}$$

Conversely, we shall show that if (6.2) holds for the value of r at some instant of time, say $t = 0$, then the particle moves uniformly in a circle of radius r_0. According to (6.1), the second of Eqs. (6.2) implies that $\dot{r}_0 = 0$.

We interrupt the argument at this point to obtain an important general formula. Starting with the equation $r^2 = \mathbf{r}\cdot\mathbf{r}$, we obtain $r\dot{r} = \mathbf{r}\cdot\mathbf{v}$ by differentiation. Another differentiation yields $r\ddot{r} + \dot{r}^2 = (\mathbf{r}\cdot\dot{\mathbf{v}}) + (\mathbf{v}\cdot\mathbf{v}) = (\mathbf{r}\cdot\dot{\mathbf{v}}) + v^2$. But (see Ex. 3.3) $v^2 = \dot{r}^2 + c^2r^{-2}$, so that $r\ddot{r} = (\mathbf{r}\cdot\dot{\mathbf{v}}) + c^2r^{-2}$. Since $\dot{\mathbf{v}} = -f(r)r^{-1}\mathbf{r}$, we have $(\mathbf{r}\cdot\dot{\mathbf{v}}) = -f(r)r^{-1}\mathbf{r}\cdot\mathbf{r} = -rf(r)$. Therefore $r\ddot{r} = -rf(r) + c^2r^2$, or, finally,

$$\ddot{r} - c^2r^{-3} = -f(r). \tag{6.3}$$

We resume the argument. According to the first of Eqs. (6.2), Eq. (6.3) has the constant solution $r = r_0$. Moreover, since the values of r and $\dot{r}$ at $t = 0$ are given, the uniqueness theorem described in Sec. 5 tells us that this is the only possible solution. This completes the case of circular motion.

In the general case it is customary to start with (6.3) and remove the dependence on time by substitution from $r^2\dot{\theta} = c$. Specifically, let $r = \rho^{-1}$. Then $\dot{r} = -\rho^{-2}\dot{\rho} = -\rho^{-2}\rho'\dot{\theta} = -\rho^{-2}\rho' cr^{-2} = -c\rho'$, where the prime (′) denotes differentiation with respect to θ. Hence $\ddot{r} = -c\rho''\dot{\theta} = -c^2\rho''\rho^2$. Equation (6.3) becomes

$$\rho'' + \rho = c^{-2}\rho^{-2}f\left(\frac{1}{\rho}\right). \tag{6.4}$$

In general, this cannot be solved for ρ in terms of θ in any recognizable form and we content ourselves with some special cases.

Suppose first that $f(r) = \mu r^{-2}$, the Newtonian case. Then $\rho'' + \rho = \mu/c^2$. It follows that ρ has the form $(\mu/c^2) + A\cos\theta + B\sin\theta$ and its reciprocal r has the form demanded by (4.4), since $f = \theta - \omega$.

Another easy case is $f(r) = \mu r^{-3}$. Then $\rho'' + \rho = \mu c^{-2}\rho$ or $\rho'' + (1 - \mu c^{-2})\rho = 0$. The solutions of this are well known.

EXERCISE 6.1. Classify the solutions in the case $f(r) = \mu r^{-3}$ according to the sign of $1 - \mu c^{-2}$. What if $1 - \mu c^{-2} = 0$?

EXERCISE 6.2. Show that for the direct first power law, $f(r) = \mu r$, the orbits are ellipses with center (*not* focus) at the origin.

EXERCISE 6.3. If we write Eq. (6.3) in the form $\ddot{r} - r\dot{\theta}^2 = -f(r)$, what is the physical meaning?

7. POSITION ON THE ORBIT: THE CASE $h = 0$

We return to the problem of motion under Newtonian attraction. It was shown in Sec. 5 that a knowledge of initial values $\mathbf{r}_0$, $\mathbf{v}_0$ determine the motion completely. In particular, these values give us $\mathbf{c}$ and $\mathbf{e}$, which by Secs. 2 and 4 determine the orbit. But there is still something missing: where is the particle located on its orbit at a prescribed time t_1?

It would be desirable to answer this question by giving the position $\mathbf{r}(t)$ as some explicit recognizable function of time. This is difficult to do directly. Instead, we adopt another procedure. We shall change from the original time t to a fictitious "time" u by a change of variable $t = t(u)$. If this change of variable is suitably chosen, it is easy to locate the particle for a prescribed value u_1 of u. In order to locate the particle at the real time t_1, it will be necessary to solve the equation $t_1 = t(u_1)$ for the corresponding value of u_1. With the choice of $t(u)$ made in this chapter, the variable u is called by the astronomers the *eccentric anomaly*.

We start with Eq. (6.1), remembering that in the case of Newtonian attraction the function $f_1(r)$ is μ/r. Then

$$(r\dot{r})^2 + c^2 = 2(\mu r + hr^2). \tag{7.1}$$

It will be supposed that u is chosen in such a fashion that $r\dot{u}$ is a constant k. Specifically, let

$$u = k\int_T^t \frac{d\tau}{r(\tau)}, \tag{7.2}$$

where k and T will be selected later. It is remarkable that the change of variable involves the still unknown function $r(t)$, but this will take care of itself. Since

$$\dot{r} = \frac{dr}{du}\dot{u} = \frac{dr}{du}kr^{-1},$$

Eq. (7.1) becomes

$$k^2(r')^2 + c^2 = 2(\mu r + hr^2), \tag{7.3}$$

where now the prime (′) denotes differentiation with respect to u.

The treatment of this equation depends on the sign of h. In this section we confine ourselves to the case $h = 0$. With the choice $k^2 = \mu$, Eq. (7.3) then reads

$$(r')^2 + \frac{c^2}{\mu} = 2r. \tag{7.4}$$

If we differentiate both sides, the result is $r'r'' = r'$. Therefore, since r' cannot vanish over an interval (or r would be a constant!), it follows that $r'' = 1$. Therefore r is a quadratic in u which we write $r = \frac{1}{2}(u - u_0)^2 + A$. Substitution into (7.4) shows that $A = c^2/2\mu$. Moreover, since u is unspecified within an arbitrary constant by (7.2), we may choose $u_0 = 0$. Then

$$r = \tfrac{1}{2}\left(u^2 + \frac{c^2}{\mu}\right).$$

According to (7.2), $du/dt = k/r$, or $r\,du = k\,dt$. Moreover $u = 0$ when $t = T$. Therefore

$$k\int_T^t dt = \int_0^u r\,du$$
$$= \tfrac{1}{2}\int_0^u \left(u^2 + \frac{c^2}{\mu}\right) du,$$

or, because $k^2 = \mu$,

$$\sqrt{\mu}\,(t - T) = \tfrac{1}{6}u^3 + \frac{c^2}{2\mu}\,u.$$

In summary,

$$\text{(7.5)}\qquad \begin{aligned} \sqrt{\mu}\,(t - T) &= \tfrac{1}{6}u^3 + \frac{c^2}{2\mu}\,u, \\ r &= \tfrac{1}{2}\left(u^2 + \frac{c^2}{\mu}\right). \end{aligned}$$

Observe that, by the first equation of the pair, t is a strictly increasing function of u. This means that this equation can be solved uniquely for u in terms of t. Call this solution $u(t)$. Then $r = \frac{1}{2}[(u(t))^2 + (c^2/\mu)]$. It is easily verified that this satisfies the differential equation (7.1) when $h = 0$.

For the interpretation of T, it is best to separate the case $c \neq 0$, and $c = 0$. If $c \neq 0$ and $h = 0$, then $e = 1$, and we obtain for the orbit the parabola

$$\text{(7.6)}\qquad r = \frac{c^2/\mu}{1 + \cos f}.$$

The smallest value of r is $c^2/2\mu$ and is achieved when $f = 0$. But this is the value of r when $u = 0$, or $t = T$. Therefore T is the time at which the particle is closest to the origin; it is called the time of *pericenter passage*. It can occur either before or after the initial time $t = 0$, but, since $\dot{f} > 0$, it can occur only once.

If $c = 0$, the equations read

$$\text{(7.7)}\qquad 6\sqrt{\mu}\,(t - T) = u^3; \qquad r = \tfrac{1}{2}u^2.$$

Therefore the time $t = T$ corresponds to collision with the origin. It must occur at some time. If $T > 0$, then it occurs after the initial time; the motion after the time T is no longer governed by the original equations, and we can talk about the motion only in the time interval $-\infty < t < T$. If $T < 0$, then the particle has been "emitted" from O at the time $t = T$ and we can speak of the motion only in the interval $T < t < \infty$.

To locate the position of the particle at time t, given $\mathbf{r}_0$ and $\mathbf{v}_0$, we proceed as follows. By the second of Eqs. (7.5), $\dot{r} = u\dot{u} = ukr^{-1}$. Therefore $r\dot{r} = (\mathbf{r}\cdot\mathbf{v}) = \sqrt{\mu}\,u$. Then the value u_0 at $t = 0$ is given by $\sqrt{\mu}\,u_0 = (\mathbf{r}_0\cdot\mathbf{v}_0)$. Now let $t = 0$, $u = u_0$ in the first of Eqs. (7.5). This determines T. In order

to find r for a given value of t we work backwards. Solve the first of Eqs. (7.5) for $u = u(t)$ and substitute into the second.

There are now two cases. If $c = 0$, then this knowledge of r determines the position completely since the line $\mathbf{e}$ containing the motion is known. On the other hand, if $c \neq 0$, it follows from (7.6) that there are two possible values of f for each value of r. It is clear that we must take f positive if $t > T$, f negative $t < T$; alternatively, $f > 0$ if $u > 0$, $f < 0$ if $u < 0$. The coordinates (r, f) then locate the particle completely.

EXERCISE 7.1. There is a standard formula from algebra for solving the cubic in (7.5) for $u = u(t)$. Write out the solution explicitly.

EXERCISE 7.2. Excluding the cases of collision, show that if $h = 0$, then $r|t|^{-2/3} \to (\frac{9}{2}\mu)^{1/3}$ as $|t| \to \infty$. Compare this with the corresponding result in case $h > 0$. (See Ex. 5.5.)

EXERCISE 7.3. Show that $u = (c/\sqrt{\mu})\tan f/2$, thus relating the two anomalies in the case $c \neq 0$. Hint: equate r as given by (7.5) and by (7.6).

8. POSITION ON THE ORBIT: THE CASE $h \neq 0$

If $h \neq 0$, there are these possible motions: linear if $c = 0$, hyperbolic if $c \neq 0$, $h > 0$, and elliptic if $c \neq 0$, $h < 0$. We now turn to the problem of location on the orbit at a prescribed time t.

Once again we start with the Eqs. (7.3) with the independent variable u defined by (7.2). This time we choose $k^2 = 2|h|$, or according to (5.3), $k^2 = \mu/a$. On division by k^2, Eq. (7.3) becomes

$$(r')^2 + \frac{ac^2}{\mu} = 2ar + \sigma(h)r^2,$$

where $\sigma(h) = 1$ if $h > 0$, $\sigma(h) = -1$ if $h < 0$. Add $\sigma(h)a^2$ to both sides and use the fact that $c^2/\mu = a(e^2 - 1)\sigma(h)$, as in (5.2). We obtain

$$(r')^2 + a^2e^2\sigma(h) = \sigma(h)[a + \sigma(h)r]^2.$$

Now define a new function $\rho(u)$ by

$$\text{(8.1)}\qquad ea\rho = a + \sigma(h)r.$$

This converts the preceding equation for r' into

$$(\rho')^2 - \sigma(h)\rho^2 = -\sigma(h).$$

It is easily verified that if we rule out the "singular" solutions $\rho = \pm 1$ the equation is satisfied by $\rho = \cosh(u + k_1)$ if $h > 0$ and $\rho = \cos(u + k_2)$ if $h < 0$. According to (7.2), where the choice of T is not yet specified we are free to choose k_1 and k_2. Let them be zero. Then, by (8.1), we obtain $r = a(e\cosh u - 1)$ if h > 0 and $r = a(1 - e\cos u)$ if h < 0. According to (7.2), we have $k\,dt = r\,du$. Since $u = 0$ when $t = T$, we can integrate both sides to obtain $k(t - T) = \int_0^u r\,du$. Substituting for r each of the functions just obtained we get the parametric pairs

$$r = a(e \cosh u - 1)$$

(8.2) $\qquad$ *if* $h > 0$,

$$n(t - T) = e \sinh u - u$$

and

$$r = a(1 - e \cos u)$$

(8.3) $\qquad$ *if* $h < 0$.

$$n(t - T) = u - e \sin u$$

The coefficient n is defined by $n = k/a$ or

(8.4) $$n = \mu^{1/2} a^{-3/2},$$

and is called the *mean motion*. Observe that in the case of elliptic motion $n = 2\pi/p$, where p is the period (see Ex. 5.3), so that n is simply the frequency.

Observe that if $u = 0$, then $t = T$ and $r = a|e - 1|$. It follows from the equation of the orbit, namely

(8.5) $$r = \frac{a|e^2 - 1|}{1 + e \cos f},$$

that if $c \neq 0$, T is a time of pericenter passage. On the other hand, if $c = 0$, then $e = 1$, so that $r = 0$ and T is a time of collision with or emission from the origin.

From this point on it is well to separate the cases $h > 0$ and $h < 0$. This is done in Secs. 9 and 10.

EXERCISE 8.1. Show from the Eqs. (8.2) that if $h > 0$, then as $|t| \to \infty$ the ratio r/t approaches $2h$, provided that the value $r = 0$ is not reached at a finite value of t. This gives and alternative solution of Ex. 5.5.

EXERCISE 8.2. Show from the formula $r + \mathbf{e}\cdot\mathbf{r} = c^2/\mu$ that if $h > 0$, $c \neq 0$, the unit vector $\mathbf{r}/r$ approaches a limit vector $\mathbf{l}$ as $t \to \infty$ and that $\mathbf{e}\cdot\mathbf{l} = -1$. Then, according to the formula

$$\mu(\mathbf{c} \times \mathbf{e}) = c^2\mathbf{v} - \mu\frac{\mathbf{c} \times \mathbf{r}}{r},$$

easily derived from (4.2), the vector $\mathbf{v}$ also approaches a limit $\mathbf{V}$. What is the length of $\mathbf{V}$?

*EXERCISE 8.3. By matching each of Eqs. (8.2) and (8.3) with (8.5) pairwise, obtain these formulas connecting true and eccentric anomalies:

$$\tan\frac{f}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tanh\frac{u}{2}, \qquad h > 0$$

$$\tan\frac{f}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{u}{2}, \qquad h < 0.$$

*EXERCISE 8.4. Show that for each value of t each of the equations

$$n(t - T) = e \sinh u - u, \qquad e \geqq 1$$

$$n(t - T) = u - e \sin u, \qquad 0 < e \leqq 1$$

has a unique solution u. They are known as *Kepler's equations*.

9. POSITION ON THE ORBIT: THE CASE $h > 0$

We start with the Eqs. (8.2), which we reproduce here as

(9.1) $$r = a(e \cosh u - 1)$$

and

(9.2) $$n(t - T) = e \sinh u - u.$$

The first step is the determination of T from $\mathbf{r}_0$ and $\mathbf{v}_0$. Starting with the formulas

$$\mathbf{r}\cdot\mathbf{v} = r\dot{r} = rr'\dot{u} = rr'kr^{-1} = kr' = \sqrt{\mu a}\, e \sinh u,$$

we see that the value u_0 of u at $t = 0$ is given by $(\mathbf{r}_0\cdot\mathbf{v}_0) = \sqrt{\mu a}\, e \sinh u_0$. Now let $t = 0$ in (9.2) and we find that T is given by $-nT = e \sinh u_0 - u_0$. Remember that if $c = 0$, then time T corresponds to a collision or emission; hence (9.2) is valid only if $t < T$ in the first case and $t > T$ in the second.

Now to determine the location at a time t, we must solve (9.2) for u and then substitute into (9.1) to obtain the corresponding value of r. If $c = 0$ the motion is linear and the location is complete. If $c \neq 0$ there are two possible values of f which satisfy

$$r = \frac{a(e^2 - 1)}{1 + e \cos f}.$$

Clearly, we must choose $f > 0$ if $t > T$ and $f < 0$ if $t < T$.

The quantity $l = n(t - T)$ is known as the *mean anomaly*. If t is given, l is determined and the main problem in the preceding computation is the solution of $l = e \sinh u - u$ for u. A solution for the function $u = u(l)$ in some recognizable form is lacking, and the problem is usually treated as a numerical one. A simple procedure is this. For the given value of l, plot the line $y = l + u$ and the curve $y = e \sinh u$. Then their intersection yields a value u_0 which, because of the roughness of method, will generally be a first approximation to the answer.

Improved approximations can be obtained by Newton's method, as follows. Let $y = l + u - e \sinh u$. We seek the value of u for which y vanishes, starting with the approximation $u = u_0$. Draw the tangent to the curve at u_0 and find where this tangent hits the y-axis. This gives an improved value u_1 and the method can be repeated. Analytically, if u_n is the result of n successive uses of the method, then

$$u_{n+1} = u_n + \frac{l + u_n e \sinh u_n}{e \cosh u_n - 1}.^*$$

EXERCISE 9.1. Solve the equation

$$1.667 = 2 \sinh u - u$$

numerically.

10. POSITION ON THE ORBIT: THE CASE $h < 0$

The parametric equations in the case of negative energy read

$$r = a(1 - e \cos u), \tag{10.1}$$

and

$$l = u - e \sin u, \tag{10.2}$$

where l is the mean anomaly $n(t - T)$.

The quantity u has an important geometric meaning if $c \neq 0$. In fact, in most treatments of the subject, u is introduced by its geometric interpretation rather than as an analytical device. The motivation for following the procedure we have adopted is the fact that in the three-body problem to be discussed later an analogue of (7.2) has important significance, whereas the geometric meaning of u will be lost.

To describe the geometry, consider the ellipse of Fig. 3, which corresponds to an orbit. The center of attraction is O, P is the pericenter, and C is the center of the ellipse. The arrow indicates the direction of motion. Let Q be a position of the particle when the true anomaly is f. Project Q to that point S of the circle for which SQ is perpendicular to CP. Then the angle PCS is u. The proof follows from (10.1) and is left to the reader.

Observe that as Q moves around the ellipse, as indicated by the arrow, u and f each change by 2π every time Q goes through pericenter. As in the earlier cases, we must determine T. Since the particle goes through P periodically, T is not uniquely determined by $\mathbf{r}_0$, $\mathbf{v}_0$. We shall agree, however, to choose T as follows if $c \neq 0$. If at $t = 0$, $f_0 > 0$, that is, if the particle is on the upper half of the ellipse, then T is the first time before $t = 0$ that the particle went through P. On the other hand, if $f_0 < 0$, then

*For more about this subject consult P. Herget. *The Computation of Orbits*, privately printed, Cincinnati, Ohio, 1948.

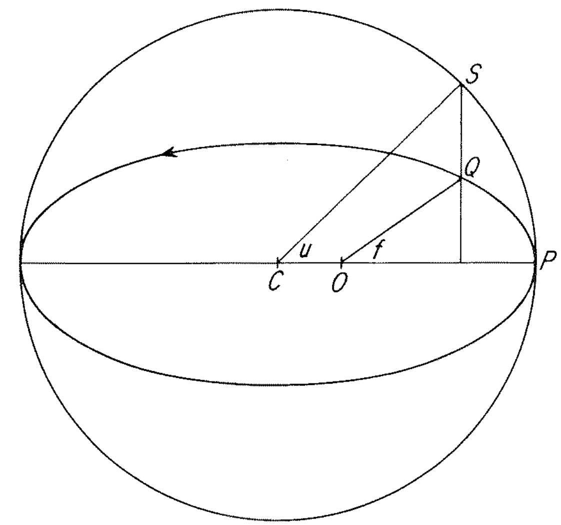

Figure 3

T is the first time after $t = 0$ that the particle will go through pericenter. Analytically the computation goes this way. Since

$$\begin{aligned} \mathbf{r} \cdot \mathbf{v} &= r\dot{r} = rr'\dot{u} = rr'kr^{-1} = \sqrt{\frac{\mu}{a}}\, r' \\ &= \sqrt{\mu a}\, e \sin u, \end{aligned} \tag{10.3}$$

it follows that u_0 must satisfy $\mathbf{r}_0 \cdot \mathbf{v}_0 = \sqrt{\mu a}\, e \sin u_0$. In addition, in the interval $-\pi < u \leq \pi$ there are, in general, exactly two values of u_0 which satisfy $r_0 = a(1 - e \cos u_0)$, each the negative of the other. But of these only one can satisfy the preceding relation involving $\mathbf{r}_0 \cdot \mathbf{v}_0$. Choose that one to be the value to be substituted into $-nT = u_0 - e \sin u_0$.

If $c = 0$, precisely the same argument will yield a value of T, but the geometric interpretation is altered. Since $\mathbf{r}_0 \cdot \mathbf{v}_0 = r_0 \dot{r}_0$, the choice makes $T > 0$ if $\dot{r}_0 < 0$ and $T < 0$ if $\dot{r}_0 > 0$.

From now on the procedure is the same as in the cases $h \geq 0$. The main problem is the solution of Kepler's equation (10.2). That can be accomplished numerically as in the case of positive energy, but a simplification should be observed. The equation is unchanged if we simultaneously add or subtract any multiple of 2π to both l and u. Therefore, when l is given, add or subtract a multiple of 2π to bring it into the range $-\pi \leq l \leq \pi$. Moreover, the equation is unchanged if l and u are simultaneously replaced by $-l$ and $-u$, respectively. This means that u is an odd function of l

and it is enough to solve the equation when $0 \leq l \leq \pi$. When $l = 0$, $u = 0$ and when $l = \pi$, $u = \pi$. Therefore, the problem is reduced to the range $0 < l < \pi$. It is clear from the graph of l against u (see Fig. 4) that the values of u also lie in the range $0 < u < \pi$.

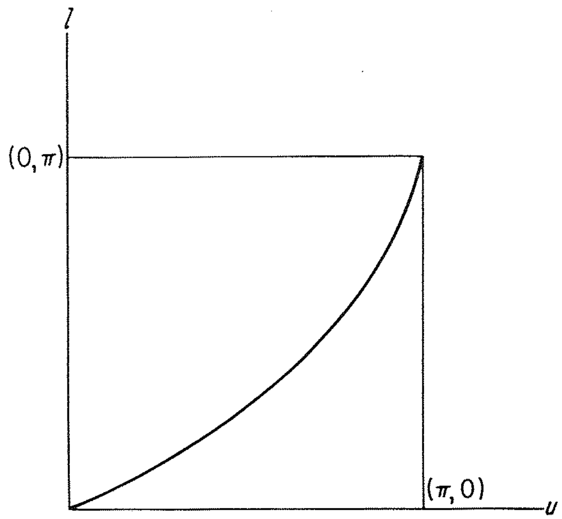

Figure 4

If the eccentricity is in the range $0 < e < 1$, there exist analytic solutions of the problem. We defer the discussion of Sec. 12.

*EXERCISE 10.1. Prove that if $0 < e < 1$, the function $u(l)$ defined by (10.2) has the property that $u(l) - l$ is periodic of period 2π in l, is odd, vanishes at $l = 0$, $l = \pi$ and has a continuous derivative. Therefore, it may be expanded in a uniformly convergent Fourier series

$$u(l) - l = \sum_{n=1}^{\infty} u_n \sin nl.$$

Prove that

$$u_n = \frac{2}{\pi n} \int_0^{\pi} \cos n(u - e \sin u)\, du.$$

*EXERCISE 10.2. Let Q_0, Q_1 be two positions on an elliptic orbit, and let u_0, u_1 be the corresponding eccentric anomalies. Assume $u_1 > u_0$. Prove that the distance Q_0Q_1 is $2a \sin \alpha \sin \beta$, where $\alpha = \frac{1}{2}(u_1 - u_0)$ and β is defined by $\cos \beta = e \cos \frac{1}{2}(u_1 + u_0)$, $0 < \beta < \pi$.

EXERCISE 10.3. Prove Lambert's theorem, which says that for an elliptic orbit the time occupied in moving from one position to another depends only on the sum of the distances from O of the two positions, and on the length of the chord joining the positions. (This will be proved in Sec. 11, but try it now, using Ex. 10.2.)

11. DETERMINATION OF THE PATH OF A PARTICLE

In the preceding theory we have solved the problem of the determination of the motion of a particle moving under the inverse square law $f(r) = \mu r^{-2}$ on the assumption that $\mathbf{r}_0$ and $\mathbf{v}_0$ are known at some time $t = 0$. In practice, $\mathbf{r}_0$ and $\mathbf{v}_0$ cannot be determined directly, so the problem arises of the determination of the motion when other types of data are given. We shall be content with one example, highly idealized for the sake of illustration. The realistic problems are treated definitively in Herget's book mentioned at the end of Sec. 9.

Suppose the center of attraction is the center of the earth, regarded as a point mass, and that the particle is an artificial satellite moving in elliptic motion. Its positions $\mathbf{r}_0$ and $\mathbf{r}_1$ are observed in succession at times τ units apart. It will be assumed that the angle g swept out by the radius vector $\mathbf{r}$ in moving from $\mathbf{r}_0$ to $\mathbf{r}_1$ is small enough so that the area caught between the chord joining the observed positions and the orbit itself does not contain O. It may, however, contain the "empty" focus F, that is, the focus which is not the center of attraction. This is illustrated in Fig. 5 by the shaded regions.

The plane of motion is determined by $\mathbf{r}_0$ and $\mathbf{r}_1$. Let $\mathbf{e}$ be the (unknown) eccentric axis and f the true anomaly measured from $\mathbf{e}$. Then the conic has the equation

$$r = \frac{a(1 - e^2)}{1 + e \cos f}.$$

Suppose now that a has been found by some means. We shall show how to find the remaining constants. Let f_0 be the true anomaly of the first position. Then $f_0 + g$ is the anomaly of the second. Hence, we have the relations

$$r_1 = \frac{a(1 - e^2)}{1 + e \cos (f_0 + g)},$$
$$r_0 = \frac{a(1 - e^2)}{1 + e \cos f_0}. \tag{11.1}$$

From these, the unknowns e and f_0 can be determined. This locates the eccentric axis, which is forward of $\mathbf{r}_0$ by the angle $-f_0$ if $f_0 < 0$, and back of it by f_0 if $f_0 > 0$. The orbit is now completely determined.

However, position on the orbit is not. For this we need to know $\mathbf{v}_0$, the velocity vector corresponding to $\mathbf{r}_0$. For then the problem becomes the

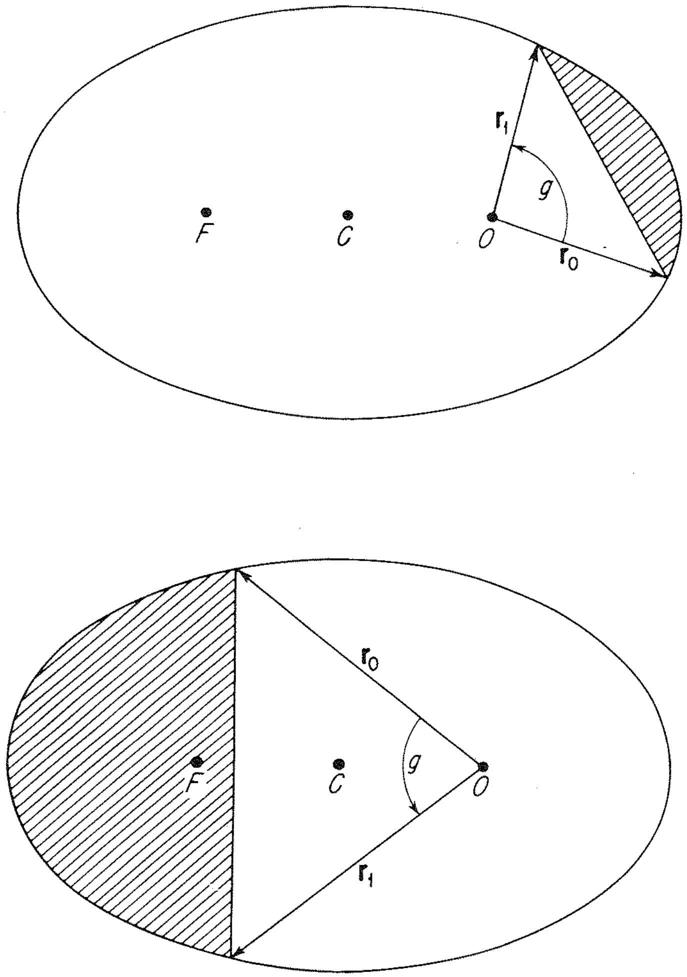

Figure 5

initial condition problem discussed in the earlier sections. Now the components of $\mathbf{v}_0$ are $\dot{r}_0$ in the direction $\mathbf{r}_0$, and c/r_0 perpendicular to it in the direction of motion (see Ex. 3.3). So all we need are the values of $\dot{r}_0$ and c. The latter can be found from $c^2 = \mu a(1 - e^2)$, the former from

$$\dot{r}_0^2 + \frac{c^2}{r_0^2} = \mu\left(\frac{2}{r_0} - \frac{1}{a}\right),$$

where $\dot{r}_0 > 0$ if $f_0 > 0$ and $\dot{r}_0 < 0$ if $f_0 < 0$.

There remains the determination of a whose value was assumed to be known in the preceding discussion. If the time τ between observations were

a period, then a could be found from Kepler's third law. But τ is less than a period and another method must be found. The key is Lambert's theorem anticipated in Ex. 10.3.

Let u_0, u_1 be the eccentric anomalies at the two positions, where $-\pi < u_0 \leqq \pi$, $-\pi < u_1 \leqq \pi$. Then $r_1 = a(1 - e\cos u_1)$, $r_0 = a(1 - e\cos u_0)$ and

$$r_1 + r_0 = 2a[1 - e\cos\tfrac{1}{2}(u_1 - u_0)\cos\tfrac{1}{2}(u_1 + u_0)].$$

Therefore, using the notation of Ex. 10.2, $r_1 + r_0 = 2a(1 - \cos\alpha\cos\beta)$. Moreover, the distance ρ between the positions is given by $\rho = 2a\sin\alpha\sin\beta$. Therefore

$$r_1 + r_0 + \rho = 2a[1 - \cos(\alpha + \beta)] = 4a\sin^2\tfrac{1}{2}(\alpha + \beta),$$
$$r_1 + r_0 - \rho = 2a[1 - \cos(\alpha - \beta)] = 4a\sin^2\tfrac{1}{2}(\alpha - \beta).$$

Since $n(t - T) = u - e\sin u$ gives the eccentric anomaly at time t, it follows that the elapsed time τ between observations is given by

$$\begin{aligned} n\tau &= (u_1 - u_0) - e(\sin u_1 - \sin u_0) \\ &= (u_1 - u_0) - 2e\sin\tfrac{1}{2}(u_1 - u_0)\cos\tfrac{1}{2}(u_1 + u_0) \\ &= 2\alpha - 2\sin\alpha\cos\beta. \end{aligned}$$

Observe that ρ is known because r_1, r_0 and the angle g between the position vectors is known. In fact, by the cosine law $\rho^2 = r_1^2 - 2r_0r_1\cos g + r_0^2$. In summary, let $\epsilon = \alpha + \beta$, $\delta = \beta - \alpha$, and replace n by its value $\mu^{1/2}a^{-3/2}$. Then we have three equations

$$4a\sin^2\frac{\epsilon}{2} = r_1 + r_0 + \rho,$$

$$4a\sin^2\frac{\delta}{2} = r_1 + r_0 + \rho,$$

$$\mu^{1/2}\tau = a^{3/2}[\epsilon - \delta - (\sin\epsilon - \sin\delta)],$$

for the unknowns ϵ, δ, a. If ϵ and δ can be found from the first two, their values can be substituted into the third, giving one equation for the determination of a.

There is a difficulty here because the solutions for ϵ and δ are not unique. Since $-\pi < u_0 \leqq \pi$, $-\pi < u_1 \leqq \pi$ and $u_0 < u_1$, we know that $0 < \alpha \leqq \pi$. Also, $0 < \beta < \pi$ by its definition. Therefore $0 < \epsilon < 2\pi$. Similarly, $-\pi < \delta < \pi$. Hence, if (ϵ_1, δ_1) is the smallest pair of positive angles satisfying the equations for ϵ and δ, the remaining pairs are $(2\pi - \epsilon_1, \delta_1)$, $(\epsilon_1, -\delta_1)$ and $(2\pi - \epsilon_1, -\delta_1)$. It turns out that the last two cases are excluded by our assumption that the shaded areas of Fig. 5 do not contain O. This is discussed by H. C. Plummer.*

* *An Introductory Treatise on Dynamical Astronomy,* New York: Dover Publications, 1960, pp. 51–52.

He shows also that the proper choice of ϵ is ϵ_1 if the shaded area does not contain F, otherwise it is $2\pi - \epsilon_1$. Therefore the equation for a is

$$\mu^{1/2}\tau = a^{3/2}[\epsilon_1 - \delta_1 - (\sin \epsilon_1 - \sin \delta_1)]$$

in the first case, and

$$\mu^{1/2}\tau = a^{3/2}[2\pi - \epsilon_1 - \delta_1 + (\sin \epsilon_1 + \sin \delta_1)]$$

in the second.

It follows, therefore, that under the given conditions, two orbits satisfy the given data.

EXERCISE 11.1. Show how Eqs. (11.1) determine e and f_0.

12. EXPANSIONS IN ELLIPTIC MOTION

We have already seen in Ex. 10.1 that in case $0 < e < 1$ Kepler's equation

$$l = u - e \sin u \tag{12.1}$$

has a solution which permits expansion of $u(l) - l$ in a uniformly convergent sine series

$$u(l) - l = \sum_{n=1}^{\infty} u_n \sin nl. \tag{12.2}$$

According to the standard formula for the coefficients of a sine series,

$$u_n = \frac{2}{\pi}\int_0^\pi [u(l) - l] \sin nl\, dl$$

To evaluate the integral, write this as

$$u_n = -\frac{2}{\pi n}\int_0^\pi [u(l) - l]\, d \cos nl$$

and integrate by parts to obtain

$$\begin{aligned} u_n &= \frac{2}{\pi n}\int_0^\pi \cos nl\, d[u(l) - l] \\ &= \frac{2}{\pi n}\int_0^\pi \cos nl\, du(l) - \frac{2}{\pi n}\int_0^\pi \cos nl\, dl \\ &= \frac{2}{\pi n}\int_0^\pi \cos nl\, du(l). \end{aligned}$$

Now let $l = u - e \sin u$, according to (12.1). The limits of integration are unchanged, so that

$$u_n = \frac{2}{\pi n}\int_0^\pi \cos n(u - e \sin u)\, du.$$

The Bessel functions $J_n(x)$ are well-known in many parts of mathematics

and can be defined in a variety of equivalent ways. For our purpose this one is best:

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos (nu - x \sin u)\, du.$$

It follows that $u_n = (2/n)J_n(ne)$ and Eq. (12.2) takes the form

$$u = l + 2\sum_{n=1}^{\infty} n^{-1} J_n(ne) \sin nl,$$

so that by (12.1) once again,

$$e \sin u = 2\sum_{n=1}^{\infty} n^{-1} J_n(ne) \sin nl.$$

These expansions have many important consequences, including formulas for the position of the particle. A rigorous treatment is given by A. Wintner.*

Here we give only one formal consequence of the preceding theorem.

According to (12.1), $dl/du = 1 - e \cos u = r/a$. Therefore, if we differentiate the last series with respect to l we obtain

$$(e \cos u)\frac{a}{r} = 2\sum_{n=1}^{\infty} J_n(ne) \cos nl.$$

Since $e \cos u = 1 - (r/a)$,

$$\frac{a}{r} = 1 + 2\sum_{n=1}^{\infty} J_n(ne) \cos nl.$$

EXERCISE 12.1. Give a proof of the last formula starting with

$$(1 - e \cos u)^{-1} = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos nl,$$

where

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^\pi (1 - e \cos u)^{-1} \cos nl\, dl \\ &= \frac{2}{\pi}\int_0^\pi \cos nl\, du. \end{aligned}$$

13. ELEMENTS OF AN ORBIT

In the preceding treatment of the non-linear case $c \neq 0$, the coordinate system used is indeterminate in one respect. In the plane of motion perpendicular to $\mathbf{c}$ (see Figs. 1 and 2), a system of axes x, y is installed to form a right-handed coordinate system with respect to $\mathbf{c}$. Since $r^2\dot{\theta} = c$, the motion is in the direction of increasing θ. The orbit is completely determined by

* *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, 1947, pp. 204–22.

c, e and position on it by T, time of pericenter passage. Alternatively, we may say that *once the x-axis is in place* everything is determined by the quantities

$$e, \begin{Bmatrix} a \text{ if } e \neq 1, \\ c \text{ if } e = 1, \end{Bmatrix} \omega,\ T. \tag{13.1}$$

Now suppose, as is the case in practice, that a *prescribed* coordinate system X, Y, Z is given with its origin at O. The problem is now that of describing the motion in the prescribed system. Such a system is illustrated in Fig. 6, along with the position of the vector $\mathbf{c}$. What must be done is to

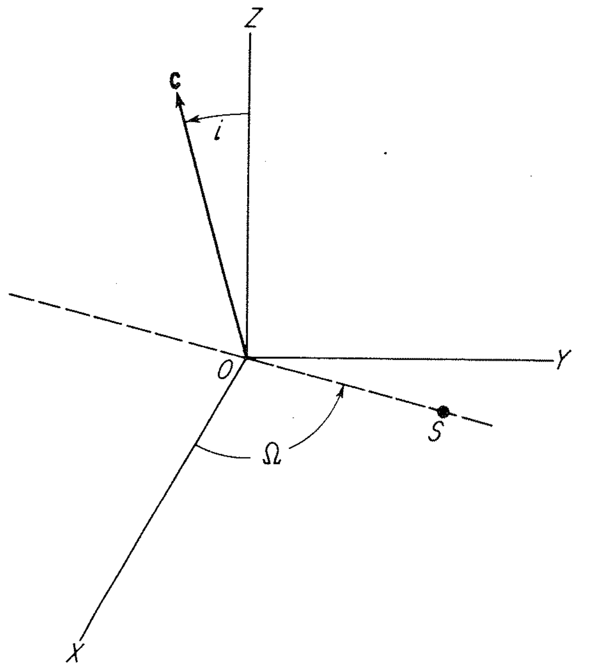

Figure 6

find a unique prescription for the x-axis. Then points in the x, y, $\mathbf{c}$ coordinate system can be described in the X, Y, Z system.

If $\mathbf{c}$ falls on the positive Z-axis, it is reasonable to choose the x-axis of Fig. 2 to fall along X; and if $\mathbf{c}$ falls on the negative Z-axis, it is reasonable to choose the x-axis to fall along Y in order to preserve the right-handed orientation.

Otherwise the plane of motion is determined by i, the angle from Z to $\mathbf{c}$, and by the line of intersection of that plane with the XY-plane. The

angle i is called the *angle of inclination*, or simply the inclination, and the line, shown dotted in Fig. 6, the *line of nodes*.

It is now customary to choose the x-axis in the plane of motion as follows. First exclude that rare case of non-elliptic motion in which the dotted line falls along the axis of the conic. Then the orbit will cut twice through the line of nodes, once on its way "up," the other on the way down. Let S be the point at which the particle cuts on its upward journey. S is called the *ascending* node, and OS is chosen as the positive x-axis. The angle XOS, measured counterclockwise as seen from the positive Z-axis, is called the *longitude of the ascending node*. The angles i and Ω accomplish the purpose of fixing the plane of motion. Therefore they, in conjunction with the numbers listed in (13.1), determine the motion completely. It is customary to use in place of ω the sum $\varpi = \Omega + \omega$, called the *longitude of pericenter*. Except for the rare cases just excluded, the orbit and position on it are then completely determined by the six numbers, called the *elements of the orbit*:

$$i,\ \Omega;\ e, \begin{Bmatrix} a \text{ if } e \neq 1, \\ c \text{ if } e = 1, \end{Bmatrix} \varpi;\ T. \tag{13.2}$$

The first two determine the plane of motion, the next three the orbit in the plane, the last the position of the mass particle on that orbit.

EXERCISE 13.1. Find formulas for changing the coordinates of the particle in its plane of motion to coordinates in the XYZ system.

14. THE TWO-BODY PROBLEM

Once the solution of the central force problem has been achieved, it is possible to solve what appears at first sight to be a more complicated problem: to describe the motion of a system of *two* mass particles moving according to their mutual gravitational attraction. This is known as the *two-body problem*, although the name *two-particle problem* would be a more accurate description.*

Let O represent a fixed point in the space of motion (see Fig. 7), let m_1, m_2 denote the masses of the two particles, $\mathbf{r}_1$, $\mathbf{r}_2$ their positions, and r the distance between them. Clearly, $r = |\mathbf{r}_2 - \mathbf{r}_1|$. According to Newton's law of universal gravitation, the force of attraction between the particles is $Gm_1m_2r^{-2}$, where G is a constant depending solely on the choice of units. The differential equations are then

$$\begin{aligned} m_1\ddot{\mathbf{r}}_1 &= \frac{Gm_1m_2}{r^2}\frac{\mathbf{r}_2 - \mathbf{r}_1}{r}, \\ m_2\ddot{\mathbf{r}}_2 &= \frac{Gm_2m_1}{r^2}\frac{\mathbf{r}_1 - \mathbf{r}_2}{r}, \end{aligned} \tag{14.1}$$

*The two-body problem for finite bodies is unsolved.

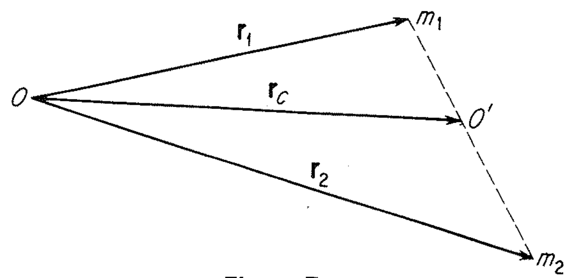

Figure 7

and it is assumed that initial values of $\mathbf{r}_1$, $\mathbf{r}_2$, $\dot{\mathbf{r}}_1$, $\dot{\mathbf{r}}_2$ are given.

It is possible to reduce the problem to the central force problem by the following procedure, called the *reduction to relative coordinates*. Divide the first of Eqs. (14.1) by m_1, the second by m_2 and subtract the first from the second. If $\mathbf{r}$ denotes $\mathbf{r}_2 - \mathbf{r}_1$ we find that

$$\ddot{\mathbf{r}} = -\mu r^{-3}\mathbf{r}, \qquad \mu = G(m_1 + m_2). \tag{14.2}$$

Clearly, initial values of $\mathbf{r}$ and $\mathbf{v}$ are known from the corresponding values for the original system (14.1). But (14.2) is precisely *the central force problem with a special choice of* μ, and all the preceding theory is applicable. Once $\mathbf{r}$ is determined, so is the right-hand side of each Eq. (14.1), from which both $\mathbf{r}_1$ and $\mathbf{r}_2$ can be obtained. In summary, *each* particle moves as if it were a unit mass attracted to a fixed center located at the other mass, with $\mu = G(m_1 + m_2)$. The orbit of each, as seen from the other, is called a *relative orbit*. Equation (14.2) is unchanged if $\mathbf{r}$ is replaced by $-\mathbf{r}$. Therefore, the relative orbits are geometrically identical.

Another procedure, called *the reduction to barycentric coordinates*, is also important. First add Eqs. (14.1) together as they stand. Then $m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0$. This has an important interpretation. Let

$$\mathbf{r}_c = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

denote the position vector of the center of mass O' of the two particles. Clearly it lies on the line joining them (see Fig. 7). Then $\ddot{\mathbf{r}}_c = 0$. It follows that

$$\mathbf{r}_c = \mathbf{a}t + \mathbf{b}, \tag{14.3}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors determined by the initial conditions. This gives the principle of *conservation of linear momentum*: the center of mass moves in a straight line with uniform velocity. The system (14.1) is of order twelve (two vector equations make six scalar equations, and each is of the second order). The vectors $\mathbf{a}$ and $\mathbf{b}$ provide six constants of the motion, which leaves six more to be accounted for.

To discover the other six, we move the origin of coordinates to the center of mass. This means that in (14.1) we replace $\mathbf{r}_1$ by $\mathbf{r}_1 - \mathbf{r}_c$ and $\mathbf{r}_2$ by $\mathbf{r}_2 - \mathbf{r}_c$. Since $\ddot{\mathbf{r}}_c = 0$, the Eqs. (14.1) remain unaltered by the change, and we may suppose from this point on that the origin is *fixed* at O', the center of mass. O' itself moves according to (14.3) and we are now studying the motion of m_1 and m_2 relative to O', which we now rename O. $\mathbf{r}_1$ and $\mathbf{r}_2$ are positions relative to the center of mass.

We now proceed in this way. Let r_1 and r_2 denote the lengths of $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. Then

$$r = r_1 + r_2, \qquad m_1 r_1 = m_2 r_2, \qquad m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0. \tag{14.4}$$

This enables us to rewrite (14.1) as a pair of equations which are formally independent of one another, namely:

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= -(Gm_2^3 M^{-2}) r_1^{-3}\mathbf{r}_1, \\ \ddot{\mathbf{r}}_2 &= -(Gm_1^3 M^{-2}) r_2^{-3}\mathbf{r}_2. \end{aligned} \tag{14.5}$$

Actually *one* of these suffices since $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$. Since each is of the form (1.1), with a special value of μ, we have accounted for six more constants, namely the elements of either orbit relative to the center of mass.

The conclusion is that the *center of mass moves uniformly and each of the particles moves with respect to that center of mass as if a fictitious force of attraction were located there with* $\mu = Gm_2^3 M^{-2}$ *for the first mass,* $\mu = Gm_1^3 M^{-2}$ *for the second.*

In what follows, we suppose the origin fixed at the center of mass. The *potential energy* of the system is defined to be $-U^*$, where

$$U^* = Gm_1 m_2 r^{-1}, \tag{14.6}$$

and the *kinetic energy* T^* is defined to be

$$T^* = \tfrac{1}{2}(m_1 v_1^2 + m_2 v_2^2), \tag{14.7}$$

where $\mathbf{v}_1 = \dot{\mathbf{r}}_1$ and $\mathbf{v}_2 = \dot{\mathbf{r}}_2$. Now let us examine each of the Eqs. (14.5) as if it corresponds to a central force problem. According to (3.1), each corresponds to a constant total "energy" defined, respectively, by

$$h_1 = \tfrac{1}{2} m_1 v_1^2 - Gm_1 m_2^3 M^{-2} r_1^{-1} \equiv T_1 - U_1$$

and

$$h_2 = \tfrac{1}{2} m_2 v_2^2 - Gm_2 m_1^3 M^{-2} r_2^{-1} \equiv T_2 - U_2.$$

Using (14.4), we can conclude that

$$T^* = T_1 + T_2, \qquad U^* = U_1 + U_2$$

Moreover,

$$\frac{h_1}{h_2} = \frac{U_1}{U_2} = \frac{T_1}{T_2} = \frac{m_2}{m_1}.$$

Therefore the various energies (kinetic, potential, total) are split between the masses m_1 and m_2 in the ratio m_2/m_1.

EXERCISE 14.1. The shape of an orbit in the central force problem is determined by the sign of h. Prove from this that in the two-body problem the orbit of each mass, relative to the center of mass, is the same kind of conic for each, although the eccentricities may differ.

*EXERCISE 14.2. Starting with Eq. (14.2) for the relative motion of two particles, study the behavior of r at an instant of collision. Notice that (7.1) applies with $c = 0$, $\mu = G(m_1 + m_2)$, so that $r\dot{r}^2 = 2(\mu + hr)$. Since $r \to 0$ at a collision we have $r\dot{r}^2 \to 2\mu$ when $t \to t_1$, the time of collision. This is independent of the sign of h. Conclude that $|\dot{r}|r^{1/2} \to \sqrt{2\mu}$ and hence that $r|t - t_1|^{-2/3} \to (9/2\,\mu)^{1/3}$ as $t \to t_1$.

15. THE SOLAR SYSTEM

The real solar system is very complicated. Mainly for the purpose of illustrating the preceding theory, we describe a simplified solar system. It consists of ten particles, one of which, the *sun*, carries most of the total mass. The other nine are *planets*. Since most of the mass is in the sun, it will be supposed that each of the planets moves independently of the others and is acted on only by the sun. The result is that we have nine independent two-body systems each consisting of the sun and one planet. Motion will be discussed relative to the sun, in accordance with the first part of Sec. 14. Then each planet is governed by Eq. (14.2), with $\mu = G(m_s + m_p)$, m_s being the mass of the sun and m_p that of the planet. Consistent with this, each planet moves in an ellipse with the sun at one focus. Let n_p and a_p denote the mean motion and the semi-major axis, respectively. Then, according to Kepler's third law (8.8) $n_p^2 a_p^3 = G(m_s + m_p)$. It follows that for two distinct planets p and q we have the law

$$\frac{n_p^2 a_p^3}{n_q^2 a_q^3} = \frac{1 + m_p/m_s}{1 + m_q/m_s}. \tag{15.1}$$

Since m_s is very large compared to m_p and m_q, the ratio on the right-hand side is very close to 1. Therefore, $n_p^2 a_p^3$ is almost (but not quite) the same for each of the planets. This is the original form of Kepler's third law.

To describe the actual orbits of the planet, it is customary to list the elements relative to the following coordinate system. (See Fig. 8, which is a special example of Fig. 6.) The origin is taken as the sun, the plane of the earth's orbit is the XY-plane. This orbit is known as the *ecliptic*, the XY-plane as the *plane of the ecliptic*. The X-axis is directed towards a point among the stars known as the *vernal equinox*. A precise definition can be found in the textbooks on astronomy. All that matters for our purpose is that it is to be regarded as fixed. Each orbit is then defined by its elements

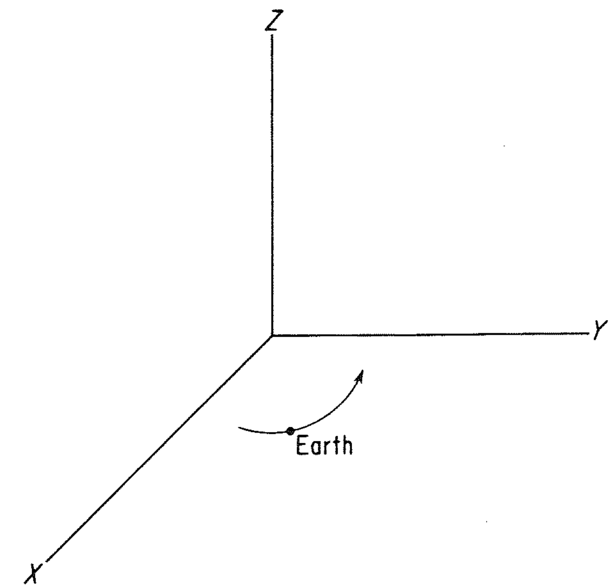

Figure 8

i, Ω, which give the plane of motion; a, e, ϖ, which describe the conic in that plane; and position on the orbit can be found from T, the date of perihelion passage.

We append a table of the elements of the nine major planets. In addition, we include the period p (measured in days) and the mass M (relative to the earth, which is taken to be of mass 1). Distance is measured in astronomical units, where one unit is the length of the semi-major axis of the earth. Time of perihelion passage T is the first date of this event after December 31,1899. Angles are given in degrees.

16. DISTURBED MOTION

We return to the problem of central attraction according to the inverse square law. The governing differential equation is $\ddot{\mathbf{r}} = -\mu r^{-3}\mathbf{r}$. Suppose that in addition to the central force, the moving particle is subjected to an additional force. This may be due to the attraction of some other body, to air resistance, or any other cause. The equation becomes

$$\dot{\mathbf{v}} = \ddot{\mathbf{r}} = -\mu r^{-3}\mathbf{r} + \mathbf{F}. \tag{16.1}$$

We shall call the motion subject to the extra force *disturbed*, and the motion with $\mathbf{F} = 0$ *undisturbed*.

Suppose that the particle is moving subject to the disturbing force

Table of Elements, 1900

	ι	Ω	a	e	ϖ	T	P	m
Mercury	7°.00	47°.14	.387	.206	75°.90	Mar. 3, 1900	87.97	.053
Venus	3°.59	75°.78	.723	.007	130°.15	Apr. 1, 1900	224.7	.815
Earth	0°.00	0°.00	1.000	.017	101°.22	Jan. 1, 1900	365.26	1.000
Mars	1°.85	48°.78	1.524	.093	334°.22	Mar. 18, 1900	686.98	.107
Jupiter	1°.31	99°.44	5.203	.048	12°.72	June 1, 1904	4,332.6	318.00
Saturn	2°.5	112°.79	9.546	.056	91°.09	Feb. 20, 1915	10,759.	95.22
Uranus	0°.77	73°.48	19.20	.047	169°.05	May 20, 1966	30,687.	14.55
Neptune	1°.78	130°.68	30.09	.009	43°.83	Sept. 15, 2042	60,184.	17.23
Pluto	17°.14	108°.95	39.5	.247	222°.8	Aug. 5, 1989	90,700.	.9(?)

which at some instant t is suddenly wiped out. Let $\mathbf{r}(t)$, $\mathbf{v}(t)$ represent the position and velocity at that instant. From then on the particle will move according to the theory described earlier in the chapter. In particular, we can define the vectors $\mathbf{c}$, $\mathbf{e}$ and the time of pericenter passage T just as before, regarding $\mathbf{r}(t)$ and $\mathbf{v}(t)$ as the initial data. But $\mathbf{c}$ and $\mathbf{e}$ are dependent on the instant t at which $\mathbf{F}$ is wiped out. They are, therefore, functions of t.

At each instant t during the disturbed motion we can look at the particle in two ways: it is moving on its real orbit, or it is about to move on its undisturbed orbit, called the *osculating* orbit. With this as the clue, we are going to study the real orbit by finding how the undisturbed orbit changes with time. In other words, we shall see how $\mathbf{c}$, $\mathbf{e}$ and T change with time. Since at each instant of time these quantities determine the elements of the undisturbed orbit, this will enable us to find how the elements of the undisturbed orbit change with time.

We shall start with the definition $\mathbf{c} = \mathbf{r} \times \mathbf{v}$, where $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity on the disturbed orbit, so that $\mathbf{c}$ depends on t. Then $\dot{\mathbf{c}} = \mathbf{r} \times \dot{\mathbf{v}}$, or by (16.1),

$$\dot{\mathbf{c}} = \mathbf{r} \times (-\mu r^{-3}\mathbf{r} + \mathbf{F}) = \mathbf{r} \times \mathbf{F}, \tag{16.2}$$

since $\mathbf{r} \times \mathbf{r} = 0$.

We define the vector $\mathbf{e}$ by the equation

$$\mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right) = \mathbf{v} \times \mathbf{c}. \tag{16.3}$$

Since $\mathbf{e}$ is a function of time we can conclude that

$$\mu\left(\frac{d}{dt}\frac{\mathbf{r}}{r} + \dot{\mathbf{e}}\right) = \dot{\mathbf{v}} \times \mathbf{c} + \mathbf{v} \times \dot{\mathbf{c}}.$$

Now replace $\dot{\mathbf{v}}$ according to (16.1), $\dot{\mathbf{c}}$ according to (16.2) and $(d/dt)(\mathbf{r}/r)$ according to (2.3). Then

$$\mu\dot{\mathbf{e}} = \mathbf{F} \times \mathbf{c} + \mathbf{v} \times (\mathbf{r} \times \mathbf{F}). \tag{16.4}$$

Let t be an instant of time at which $c \neq 0$ and $e \neq 0$ and let f be the angle from $\mathbf{e}$ to $\mathbf{r}$. Then f is the true anomaly of the particle regarded as being on its undisturbed orbit at that instant.

We introduce a coordinate system at the instant t. Its origin is O and the axes are $\mathbf{c}$, $\mathbf{r}$ and $\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}$ is defined by $\boldsymbol{\alpha} = \mathbf{c} \times \mathbf{r}$. (See Fig. 9.) Clearly

$$\boldsymbol{\alpha} = \mathbf{c} \times \mathbf{r}, \qquad r^2\mathbf{c} = \mathbf{r} \times \boldsymbol{\alpha}, \qquad c^2\mathbf{r} = \boldsymbol{\alpha} \times \mathbf{c}. \tag{16.5}$$

The vector $\mathbf{v}$ lies in the plane perpendicular to $\mathbf{c}$, so that

$$\mathbf{v} = A\mathbf{r} + B\boldsymbol{\alpha}. \tag{16.6}$$

We proceed to compute A and B. We have $\mathbf{c} = \mathbf{r} \times \mathbf{v} = \mathbf{r} \times (A\mathbf{r} + B\boldsymbol{\alpha}) = A(\mathbf{r} \times \mathbf{r}) + B(\mathbf{r} \times \boldsymbol{\alpha})$, so that, by (16.5), $\mathbf{c} = B r^2 \mathbf{c}$ or $B = 1/r^2$. Also by

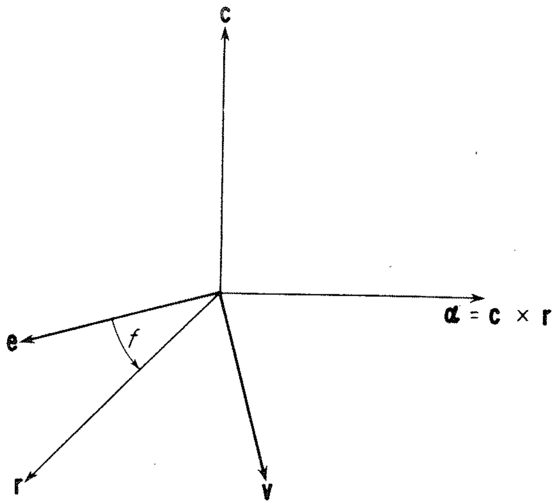

Figure 9

(16.6), $\mathbf{r}\cdot\mathbf{v} = A\mathbf{r}\cdot\mathbf{r} = Ar^2$, since $\boldsymbol{\alpha}\cdot\mathbf{r} = 0$. To finish the calculation of A we note that $\mathbf{e}\cdot\boldsymbol{\alpha} = e\alpha\cos(f + 90°)$. Also $\alpha = cr$. Therefore $\mathbf{e}\cdot\boldsymbol{\alpha} = -ecr\sin f$. Since $\mathbf{e}\cdot\mathbf{r} = er\cos f$, it follows on taking the dot product of both sides of (16.6) with $\mathbf{e}$ that $\mathbf{e}\cdot\mathbf{v} = Aer\cos f - Becr\sin f$. But according to (16.3), $\mathbf{r}\cdot\mathbf{v} + r(\mathbf{e}\cdot\mathbf{v}) = 0$. Therefore,

$$\begin{aligned} Ar^2 &= \mathbf{r}\cdot\mathbf{v} = -r(\mathbf{e}\cdot\mathbf{v}) \\ &= -Aer^2\cos f + Becr^2\sin f. \end{aligned}$$

But $Br^2 = 1$. It follows that $Ar^2(1 + e\cos f) = ec\sin f$. Since, at the instant t, $r = (c^2/\mu)(1 + e\cos f)^{-1}$, we get $A = \mu er^{-1}c^{-1}\sin f$.

Substitute from (16.6) into (16.4) to get rid of $\mathbf{v}$. Using the fact that $Br^2 = 1$ and expanding the triple products, we find that

(16.7) $$\mu\dot{\mathbf{e}} = \mathbf{F}\times\mathbf{c} - Ar^2\mathbf{F} + [A(\mathbf{F}\cdot\mathbf{r}) + r^{-2}(\mathbf{F}\cdot\boldsymbol{\alpha})]\mathbf{r}.$$

We interrupt with an exercise.

*EXERCISE 16.1. Write $\mathbf{F}$ in terms of its components F_c, F_r, F_α in the direction of the coordinate axes, that is, $\mathbf{F} = F_c c^{-1}\mathbf{c} + F_r r^{-1}\mathbf{r} + F_\alpha \alpha^{-1}\boldsymbol{\alpha}$. Show that the basic equations (16.2) and (16.7) become, respectively,

(16.8) $$\dot{\mathbf{c}} = rc^{-1}F_\alpha\mathbf{c} - c^{-1}F_c\boldsymbol{\alpha}$$

and

(16.9) $$\mu\dot{\mathbf{e}} = 2cr^{-1}F_\alpha\mathbf{r} - (r^{-1}F_r + Arc^{-1}F_\alpha)\boldsymbol{\alpha} - Ar^2c^{-1}F_c\mathbf{c},$$

where, as before, $A = \mu er^{-1}c^{-1}\sin f$.

Dot multiply both sides of (16.8) by $\mathbf{c}$ to obtain

(16.10) $$\dot{c} = rF_\alpha.$$

17. DISTURBED MOTION: VARIATION OF THE ELEMENTS

Now let X, Y, Z be a coordinate system, as described in Sec. 13. We wish to determine how the disturbed motion looks in this coordinate system. At each instant of time we shall regard the particle as being on its undisturbed orbit with the associated constants i, Ω, ω e, c, T and ask how these vary with the time as the particle moves through its successive undisturbed orbits.

We already know from (16.10) that

(17.1) $$\dot{c} = rF_\alpha.$$

Now let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote unit vectors in the X, Y, Z directions (see Fig. 10)

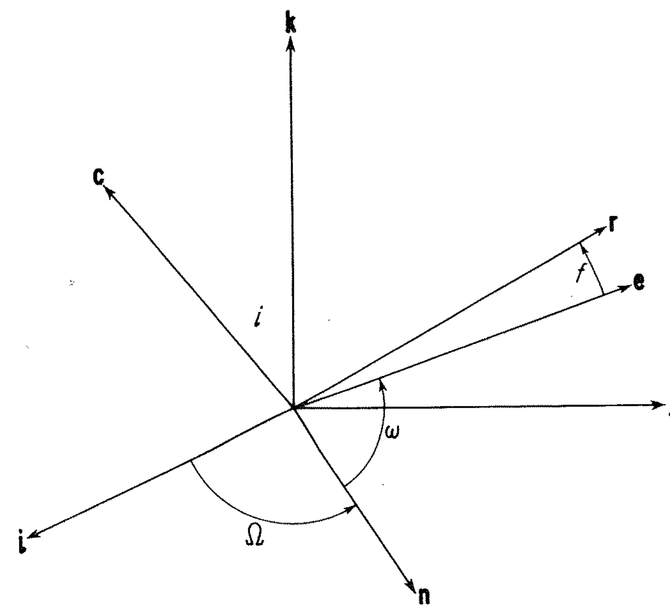

Figure 10

and let the line of nodes be directed along $\mathbf{n}$, where $\mathbf{n} = \mathbf{k}\times\mathbf{c}$. Clearly, $n = kc\sin i = c\sin i$. Also, because $\boldsymbol{\alpha} = \mathbf{c}\times\mathbf{r}$ we know that $\mathbf{k}\cdot\boldsymbol{\alpha} = \mathbf{k}\cdot(\mathbf{c}\times\mathbf{r}) = \mathbf{k}\times\mathbf{c}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{r} = nr\cos(\omega + f) = cr\sin i\cos(\omega + f)$.

We start again with $\mathbf{c}\cdot\mathbf{k} = ck\cos i = c\cos i$, so that, according to (17.1),

$$\dot{\mathbf{c}}\cdot\mathbf{k} = rF_\alpha \cos i - c \sin i \frac{di}{dt}.$$

From (16.8) and our computation of $\mathbf{k}\cdot\boldsymbol{\alpha}$, we get

$$\dot{\mathbf{c}}\cdot\mathbf{k} = rc^{-1}F_\alpha c \cos i - c^{-1}F_c cr \sin i \cos(\omega + f).$$

From the last two equations it follows that

$$(17.2) \qquad \frac{di}{dt} = rc^{-1}F_c \cos(\omega + f).$$

We turn to the computation of $\dot{e}$. According to Fig. 9, we know that $\boldsymbol{\alpha}\cdot\mathbf{e} = \alpha e \cos(f + 90^\circ) = -rce \sin f$. Now dot multiply both sides of (16.9) by $\mathbf{e}$ to obtain

$$\begin{aligned}\mu e\dot{e} &= 2cr^{-1}F_\alpha\Big(\frac{c^2}{\mu} - r\Big) - (r^{-1}F_r + Arc^{-1}F_\alpha)(-rce \sin f).\\ &= ceF_r \sin f + ce(1 + e\cos f)^{-1}F_\alpha(e + 2\cos f + e\cos^2 f),\end{aligned}$$

or

$$\mu c^{-1}\dot{e} = F_r \sin f + F_\alpha(e + 2\cos f + e\cos^2 f)(1 + e\cos f)^{-1}.$$

Now $-\mathbf{j}\cdot\mathbf{c} = \mathbf{i}\times\mathbf{k}\cdot\mathbf{c} = \mathbf{i}\cdot\mathbf{k}\times\mathbf{c} = \mathbf{i}\cdot\mathbf{n}$ so that

$$(17.3) \qquad -\mathbf{j}\cdot\mathbf{c} = c \sin i \cos\Omega.$$

Also, as the reader may demonstrate,

$$(17.4) \qquad \boldsymbol{\alpha}\cdot\mathbf{j} = rc[-\sin(\omega + f)\sin\Omega + \cos(\omega + f)\cos\Omega\cos i].$$

Therefore, if we take the dot product of both sides of (16.8) with $\mathbf{j}$, the result is

$$\begin{aligned}\mathbf{j}\cdot\dot{\mathbf{c}} = &-rF_\alpha \sin i \cos\Omega\\ &+rF_c \sin(\omega + f)\sin\Omega\\ &-rF_c\cos(\omega + f)\cos\Omega\cos i,\end{aligned}$$

which, according to (17.1) and (17.2), may also be written

$$\begin{aligned}-\mathbf{j}\cdot\dot{\mathbf{c}} = \dot{c}\sin i\cos\Omega &- rF_c\sin(\omega + f)\sin\Omega\\ &+ c\cos i\cos\Omega\frac{di}{dt}.\end{aligned}$$

A direct differentiation of (17.3) shows agreement with the last equation, provided that

$$c\dot{\Omega}\sin i = rF_c \sin(\omega + f).$$

The computation of $\dot{\omega}$ starts with the observation that $\mathbf{n}\times\mathbf{e} = (\mathbf{k}\times\mathbf{c})\times\mathbf{e} = (\mathbf{k}\cdot\mathbf{e})\mathbf{c}$, so that $ne\sin\omega = (\mathbf{k}\cdot\mathbf{e})c$, or

$$(17.5) \qquad \mathbf{k}\cdot\mathbf{e} = e\sin i\sin\omega.$$

In addition, we have the formula

$$(17.6) \qquad \mathbf{k}\cdot\mathbf{r} = r\sin i\,\sin(\omega + f),$$

easily obtained by substituting $c^{-2}(\boldsymbol{\alpha}\times\mathbf{c})$ for $\mathbf{r}$, and the formula

$$(17.7) \qquad (\mathbf{k}\cdot\boldsymbol{\alpha}) = cr\sin i\cos(\omega + f),$$

derived at the beginning of the section. The remaining steps are these. Differentiate (17.5) and replace $\dot{\mathbf{e}}$, $\dot{e}$, di/dt by their equivalents obtained in this section and the preceding one. This yields an equation for $\dot{\omega}$ in terms of $(\mathbf{k}\cdot\dot{\mathbf{e}})$. Now dot multiply both sides of (16.9) with $\mathbf{k}$, substituting from (17.6) and (17.7). This gives us an evaluation of $(\mathbf{k}\cdot\dot{\mathbf{e}})$ which, on comparison with the preceding one, yields a formula for $\dot{\omega}$ given below.

There still remains the determination of $\dot{T}$. This we leave to the next section. In summary, we have found these formulas:

$$(17.8) \qquad \begin{aligned}&\dot{c} = rF_\alpha,\\ &\mu c^{-1}\dot{e} = F_r\sin f + F_\alpha(e + 2\cos f + e\cos^2 f)(1 + e\cos f)^{-1},\\ &\frac{di}{dt} = rc^{-1}F_c\cos(\omega + f),\\ &c\dot{\Omega}\sin i = rF_c\sin(\omega + f),\\ &\dot{\omega} = -c\mu^{-1}e^{-1}(\cos f)F_r - rc^{-1}\cot i\sin(\omega + f)F_c\\ &\qquad + (\mu ec)^{-1}(c^2 + r\mu)(\sin f)F_\alpha.\end{aligned}$$

EXERCISE 17.1. Prove (17.4) by consulting Fig. 10. Recall that $\boldsymbol{\alpha} = \mathbf{c}\times\mathbf{r}$.

EXERCISE 17.2. Give a detailed proof of (17.6).

EXERCISE 17.3. Verify the formula for $\dot{\omega}$.

18. DISTURBED MOTION: GEOMETRIC EFFECTS

To complete the calculation summarized by (17.8) we now suppose that the undisturbed motion is elliptical. In that case, $0 < e < 1$ and $c^2 = \mu a(1 - e^2)$. Since $\dot{c}$ and $\dot{e}$ have already been found, it is easy to calculate $\dot{a}$ from this last equation. The result is

$$(18.1) \qquad \dot{a} = 2a^2ec^{-1}(\sin f)F_r + 2a^2c\mu^{-1}r^{-1}F_\alpha.$$

Since $n = \mu^{1/2}a^{-3/2}$, we know that $\dot{n} = -\frac{3}{2}na^{-1}\dot{a}$.

Finally, we determine $\dot{T}$. At the instant t, let a, n, e be the customary quantities associated with location on an elliptic orbit. Then

$$\begin{aligned}r &= a(1 - e\cos u),\\ n(t - T) &= u - e\sin u.\end{aligned}$$

We know that $r\dot{r} = \sqrt{\mu a}\,e\sin u$, by (10.3). If we use this fact, then differentiation of the first equation of the pair yields

$$r^{-1}\sqrt{\mu a}\, e \sin u = \dot{a}(1 - e\cos u) + a(e\dot{u}\sin u - \dot{e}\cos u).$$

The second equation of the pair gives

$$\dot{n}(t - T) + n(1 - \dot{T}) = (1 - e\cos u)\dot{u} - \dot{e}\sin u.$$

If we (i) eliminate $\dot{u}$ between the last equations; (ii) replace $\dot{n}$ by $-\frac{3}{2}na^{-1}\dot{a}$, $1 - e\cos u$ by ra^{-1}, $\sin u$ by $r(1-e^2)^{-1/2}a^{-1}\sin f$; (iii) solve for $\dot{T}$, the result is

$$(18.2) \qquad \dot{T}\mu e \sin f = a^{-1}[rc - \tfrac{3}{2}\mu e(t-T)\sin f]\dot{a} - ac(\cos f)\dot{e}.$$

It is important to observe which of the elements is affected by which of the components F_r, F_c, F_α. The results are tabulated below.

F_r	*affects*	$\dot{e}$, $\dot{\omega}$, $\dot{a}$, $\dot{T}$,
F_c	*affects*	$\frac{di}{dt}$, $\dot{\Omega}$, $\dot{\omega}$,
F_α	*affects*	$\dot{c}$, $\dot{e}$, $\dot{\omega}$, $\dot{a}$, $\dot{T}$.

The major applications of the formulas (17.8) and (18.2) will come in our later study of perturbation theory. Here we shall be content to illustrate their use with a simple example. Suppose a mass moving in an elliptic orbit, $0 < e < 1$, encounters a region of resistance, due, say, to atmosphere. The force will sometimes be of the form $\mathbf{F} = -q\mathbf{v}$, where q is positive, although not necessarily a constant. What is the effect on the elements of the orbit? To solve the problem, observe that, according to (16.6),

$$\mathbf{F} = -qA\mathbf{r} - qB\boldsymbol{\alpha}.$$

Therefore, $F_r = -qAr$, $F_\alpha = -qB\alpha = -qBrc$. Using the computed values of A and B, we find that $F_r = -q\mu ec^{-1}\sin f$, $F_\alpha = -qr^{-1}c$. Clearly, $F_c = 0$. Substituting into (17.8) and (18.1), we get for the geometric elements of the orbit

$$\mu\dot{e} = -2q\mu(e + \cos f),$$

$$\frac{di}{dt} = 0,$$

$$\dot{\Omega}\sin i = 0,$$

$$\dot{\omega} = -2qe^{-1}\sin f,$$

$$\dot{a} = -2qa^2c^{-2}(1 + 2e\cos f + e^2).$$

The following conclusions are immediate. The eccentricity e increases if $e + \cos f < 0$ and increases if $e + \cos f > 0$. (These correspond, respectively, to the left and right half of the ellipse.) The inclination is unchanged. The longitude of nodes Ω is unchanged, provided $i \neq 0$. (If $i = 0$, the angle Ω is, of course, undefined.) The amplitude of pericenter ω decreases in the upper half of the ellipse and increases in the lower half. The major axis always decreases.

EXERCISE 18.1. Verify that the formulas (17.8) and (18.1) are dimensionally correct. Use L for r and a, L^3T^{-2} for μ (why?), L^2T^{-1} for c, LT^{-2} for components of force, while e and angles are dimensionless.

EXERCISE 18.2. Find analogous formulas for the variation of the elements when the force $\mathbf{F}$ is decomposed in the directions $\mathbf{c}$, $\mathbf{v}$, $\mathbf{c} \times \mathbf{v}$.