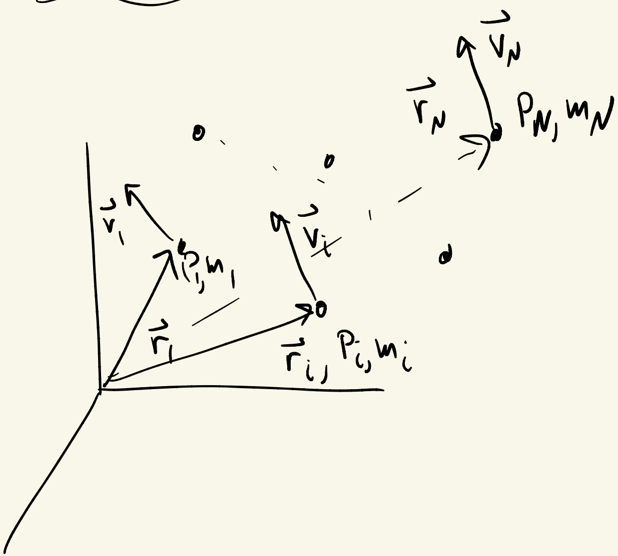
6062 Celestial Mechanics

The N-Body Problem:

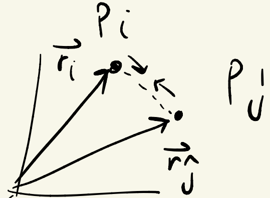
Consider N particles P_i placed in space with positions & velocities $\vec{r}_i$, $\vec{v}_i = \frac{d\vec{r}_i}{dt}$, relative to an inertial frame, each having a mass m_i; for $i = 1, 2, \ldots, N$.



And the bodies are mutually attracted by Newton's Law of Gravitation.

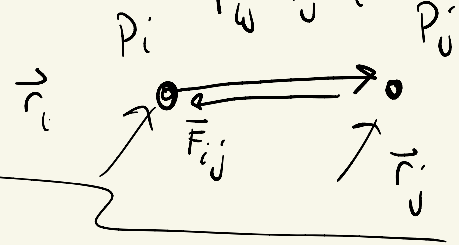
Mutual Gravitation Law

Consider P_i & P_j,



$$\vec{F}_{ij} = \frac{-G m_i m_j}{\underbrace{|\vec{r}_j - \vec{r}_i|}_{\vec{r}_{ij}}{}^3} \underbrace{(\vec{r}_j - \vec{r}_i)}_{\vec{r}_{ij}}$$

$$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$$



$$\boxed{\begin{aligned} \vec{F}_{ij} &= \frac{-G m_i m_j}{|\vec{r}_{ij}|^3} \vec{r}_{ij} \\ \vec{F}_{ji} &= \frac{-G m_j m_i}{|\vec{r}_{ji}|^3} \vec{r}_{ji} \end{aligned}}$$

$$|\vec{r}_{ij}| = |\vec{r}_{ji}|$$

$$\vec{r}_{ij} = -\vec{r}_{ji}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

$$|\vec{r}| = \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{x^2 + y^2 + z^2} = r$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}, \qquad \hat{r} \cdot \hat{r} = 1$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = [\tilde{\vec{a}}]_{3\times3} \vec{b}$$

$$[\tilde{\vec{a}}] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Newton's 2nd Law, $\vec{m a} = \vec{F}$

Eqn for P_i

$$\boxed{m_{\hat{i}} \ddot{\vec{r}}_i = \sum_{\substack{j=1 \\ j \neq \hat{i}}}^{N} \vec{F}_{j\hat{i}}} = \vec{F}_{\hat{i}} = \text{Total Force on } P_i, \quad \hat{i} = 1, 2, \dots, N$$

$\rightarrow$ EOM for the NBP

If we know the positions & velocities of all N bodies at a given time, t_0, $\vec{r}_i(t_0), \vec{v}_i(t_0)$, we have $3N$ well defined ODE & an initial state $\rightarrow$ We can solve this theoretical, well posed problem as long as $r_{ij} \neq 0$ at some future point.

Gravity is a conservative force & hence can be derived from a potential.

Let $\boxed{\mathcal{U} = -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{G m_i m_j}{|\vec{r}_{ij}|} = -\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{G m_i m_j}{|\vec{r}_{ij}|}}$ Gravitational Potential of the NBP.

Then $\vec{F}_i = -\frac{\partial \mathcal{U}}{\partial \vec{r}_i} = -\nabla_{\vec{r}_i}\mathcal{U}$

Consider $-\frac{\partial}{\partial \vec{r}_i}\left(\frac{-G m_k m_j}{|\vec{r}_{kj}|}\right) = 0$ if $k, j \neq i$

$$-\frac{\partial}{\partial \vec{r}_i}\left(\frac{-G m_i m_j}{|\vec{r}_{ij}|}\right) = G m_i m_j \frac{\partial}{\partial \vec{r}_i}\left(\frac{1}{\sqrt{\vec{r}_{ij}\cdot\vec{r}_{ij}}}\right)$$

$$= G m_i m_j \left[-\frac{1}{2}\underbrace{\frac{1}{(\vec{r}_{ij}\cdot\vec{r}_{ij})^{3/2}}}_{r_{ij}^3}\frac{\partial}{\partial \vec{r}_i}(\vec{r}_{ij}\cdot\vec{r}_{ij})\right]$$

$$= G m_i m_j \left[-\frac{1}{2}\frac{1}{r_{ij}^3}\left\{\frac{\partial \vec{r}_{ij}}{\partial \vec{r}_i}\cdot\vec{r}_{ij} + \vec{r}_{ij}\cdot\frac{\partial \vec{r}_{ij}}{\partial \vec{r}_i}\right\}\right]$$

$$-\frac{d}{d\vec{r}_i}\left(\frac{-Gm_im_j}{|\vec{r}_{ij}|}\right) = -Gm_im_j\frac{1}{r_{ij}^3}\left(\frac{d\vec{r}_{ij}}{d\vec{r}_i}\cdot\vec{r}_{ij}\right) = \frac{Gm_im_j}{r_{ij}^3}\vec{r}_{ij} = -\frac{Gm_im_j}{r_{ij}^3}\vec{r}_{ji}$$

$$\frac{d}{d\vec{r}_i}\left(\vec{r}_{ij} = \vec{r}_j - \vec{r}_i\right) \qquad \frac{d\vec{r}_{ij}}{d\vec{r}_i} = \cancel{\frac{d\vec{r}_j}{d\vec{r}_i}}^{\,0} - \underbrace{\frac{d\vec{r}_i}{d\vec{r}_i}}_{\overline{\overline{U}}} = -\overline{\overline{U}}$$

Unity Dyad

Unity Dyad $\equiv$ Identity

$$\boxed{\vec{a}\cdot\overline{\overline{U}} = \vec{a} = \overline{\overline{U}}\cdot\vec{a}}$$

$$\overline{\overline{U}} = \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}$$

Thus

$$\boxed{\begin{aligned} m_i\ddot{\vec{r}}_i &= \frac{-\partial \mathcal{U}}{\partial \vec{r}_i}, \quad i = 1, 2, \ldots, N \\ \mathcal{U} &= -G\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{m_im_j}{r_{ij}} \end{aligned}}$$

☆

Given conditions $(\vec{r}, \vec{v})$ at a time t_0, can continue the solution to $t \to \pm\infty$ via numerical methods. So long as $r_{ij} \not\to 0$ for 2 or more bodies.

Can we simplify the system? Fundamental Integrals of motion:

There are 10 integrals of motion for the NBP.

- An Integral of motion is a function of positions, velocities & masses that is constant under the Equations of Motion (as the system evolves).

Let $\vec{X} \in \mathbb{R}^{6N}$, $\dot{\vec{X}} = \vec{F}(\vec{X})$

A function $h(\vec{X})$ is an integral of motion <u>if</u>

$$\boxed{\frac{d}{dt} h(\vec{X}) = \frac{\partial h}{\partial \vec{X}} \cdot \dot{\vec{X}} = \frac{\partial h}{\partial \vec{X}} \cdot \vec{F}(\vec{X}) = 0} \quad \forall \vec{X}$$

For every integral, we can reduce the dimension of our problem by $\underline{1}$.

Define:

System Center of Mass $\qquad \vec{R}_c = \dfrac{1}{\sum_{i=1}^{N} m_i} \sum_{j=1}^{N} m_j \vec{r}_j$

" $\quad$ Linear Momentum $\qquad \vec{P} = \sum_{i=1}^{N} m_i \vec{v}_i$

" $\quad$ Angular Momentum $\qquad \vec{H} = \sum_{i=1}^{N} m_i \vec{r}_i \times \vec{v}_i$

System Energy $\qquad E = \frac{1}{2} \sum_{i=1}^{N} m_i v_i^2 + \mathcal{U}$

Thm: $\vec{P}$, $\vec{H}$ + E are Integrals of Motion of the NBP.

$\vec{R}_c(t_0)$ is also an Integral

Proof $\Rightarrow$ Direct Differentiation

$$\boxed{\dot{\vec{P}} = \sum_{j=1}^{N} m_j \dot{\vec{V}}_j = -\sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{G m_i m_j}{r_{ij}^3} \vec{r}_{ji} = \vec{0}}$$

$$m_j \dot{\vec{V}}_j = m_j \ddot{\vec{r}}_j = -\sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{G m_i m_j \vec{r}_{ji}}{r_{ij}^3}$$

For every $\vec{r}_{ji}$, $\exists$ $\vec{r}_{ij}$

$$\vec{P} = \text{const}$$

$$M = \sum_{i=1}^{N} m_i$$

$$M \dot{\vec{R}}_c = \sum_{j=1}^{N} m_j \vec{V}_j = \vec{P} \Rightarrow \text{Integrate to get}$$

$$\boxed{M \vec{R}_c = \underset{\sim\sim\sim\sim}{M \vec{R}_c(t_0)} + \vec{P}(t - t_0)} \Rightarrow$$

$$\frac{d\vec{H}}{dt} = \sum_{i=1}^{N} m_i \vec{r}_i \times \dot{\vec{V}}_i = -G \sum_{i=1}^{N} \vec{r}_i \times \left(\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ji} \right) = +G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{r_{ij}^3} \vec{r}_i \times \vec{r}_j = \vec{0}$$

$$\vec{H} = \text{Const.}$$

$$\frac{dE}{dt} = \sum_{i=1}^{N} m_i \vec{V}_i \cdot \dot{\vec{V}}_i + \sum_{i=1}^{N} \frac{\partial \mathcal{U}}{\partial \vec{r}_i} \cdot \underbrace{\dot{\vec{r}}_i}_{\vec{V}_i} = \sum_{i=1}^{N} \vec{V}_i \cdot \underbrace{\left[m_i \dot{\vec{V}}_i + \frac{\partial \mathcal{U}}{\partial \vec{r}_i} \right]}_{0} = 0$$

$$\boxed{E = \text{const.}}$$

Given initial conditions, can evaluate $\vec{P}$, $\vec{R}_c(t_0)$, $\vec{H}$, E

& they must be constant for all future time.

Simplification of the EOM.

1-Degree of Freedom = 2-dimension in position & velocity space

1 Body = 3 DOF, 6-Dim space

N Bodies = 3N DOF, 6N-Dimensions

- All motion can be made relative to the center of mass of the system ≡ "**Barycenter**"

Let $\vec{\rho}_i = \vec{r}_i - \vec{R}_c$, $\vec{\rho}_{ij} = \vec{r}_{ij} = \vec{\rho}_j - \vec{\rho}_i$

$$\ddot{\vec{\rho}}_i = \ddot{\vec{r}}_i \Rightarrow \star \boxed{m_i \ddot{\vec{\rho}}_i = -\frac{\partial U}{\partial \vec{\rho}_i}}$$

Don't care about $\vec{R}_c$

$i = 1, \dots, N-1$

$$\boxed{\sum_{i=1}^{N} m_i \vec{\rho}_i = \vec{0}}$$

Can eliminate 1 body from the system

$$\boxed{\vec{\rho}_N = \frac{-1}{m_N} \sum_{i=1}^{N-1} m_i \vec{\rho}_i}$$

- All motion can be made relative to a single body.

N bodies $\hat{i} = 1, 2, \dots, N \Rightarrow \hat{j} = \underline{\underline{0}}, 1, 2, \dots, \underbrace{N-1}_{n}$

$N = n+1$

Let $\vec{r}_{\hat{i}} = \vec{\rho}_{\hat{i}} - \vec{\rho}_0 = \vec{\rho}_{0i}$;

$$\boxed{\sum_{i=1}^{n} m_i \vec{r}_i = -M \vec{\rho}_0}$$

$\vec{r}_{i\hat{j}} = \vec{\rho}_{ij} = \vec{r}_{ij}$

$$\boxed{\ddot{\vec{r}}_i = \ddot{\vec{\rho}}_{\hat{i}} - \ddot{\vec{\rho}}_0 = -\frac{1}{m_{\hat{i}}} \frac{\partial \mathcal{U}}{\partial \vec{\rho}_i} + \frac{1}{m_0} \frac{\partial \mathcal{U}}{\partial \vec{\rho}_0} = \frac{-G(m_0 + m_i)\vec{\rho}_{0i}}{\rho_{0i}^3} - G \sum_{\substack{j=1 \\ j \neq \hat{i}}}^{n} \left[\frac{m_{\hat{j}} \vec{\rho}_{ji}}{\rho_{ji}^3} - \frac{m_{\hat{j}} \vec{\rho}_{0j}}{\rho_{0j}^3} \right]}$$

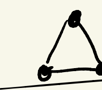
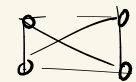
$$\boxed{\ddot{\vec{r}}_i = -\frac{\partial}{\partial \vec{r}_i} U_{0i} - \frac{\partial}{\partial \vec{r}_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} R_{ji} \qquad i = 1, 2, \ldots, n = N-1}$$

Relative EOM

Still Conserves $\underline{\underline{\vec{H}, E}}$.

$$U_{0i} = \frac{-G(m_0 + m_i)}{|\vec{r}_i|} \quad ; \quad R_{ji} = -\left[\underbrace{\frac{G m_j}{|\vec{r}_{ji}|}}_{\substack{\text{Direct} \\ \text{Attraction} \\ \text{of body } j}} - \underbrace{\frac{G m_j}{|\vec{r}_j|^3} \vec{r}_j \cdot \vec{r}_i}_{\substack{\text{Indirect} \\ \text{Attraction}}} \right]$$

All motion can be made relative to each other.

Bodies N	Total DoF	Reduced DoF	Relative DoF ($\vec{r}_{ij}$)	
1	3	0	0	Uniform Translation
2	6	3	3	2BP Integrable
3	9	6	9	
4	12	9	18	
5	15	12	30	$\ddot{\vec{r}}_{ij} = \vec{F}_{ij}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
N	$3N$	$3(N-1)$	$\frac{3}{2}N(N-1)$	

Can't write the EOM in relative form, but can express fundamental Quantities as such

Can Show That

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \frac{1}{2} M \dot{\vec{R}}_c \cdot \dot{\vec{R}}_c + \frac{1}{2M}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} m_i m_j \dot{\vec{r}}_{ij} \cdot \dot{\vec{r}}_{ij}$$

$$\vec{H} = \sum_{i=1}^{N} m_i \vec{r}_i \times \dot{\vec{r}}_i = M \vec{R}_c \times \dot{\vec{R}}_c + \frac{1}{M}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} m_i m_j \vec{r}_{ij} \times \dot{\vec{r}}_{ij}$$

$$\mathcal{U} = -G \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{m_i m_j}{|\vec{r}_{ij}|}$$

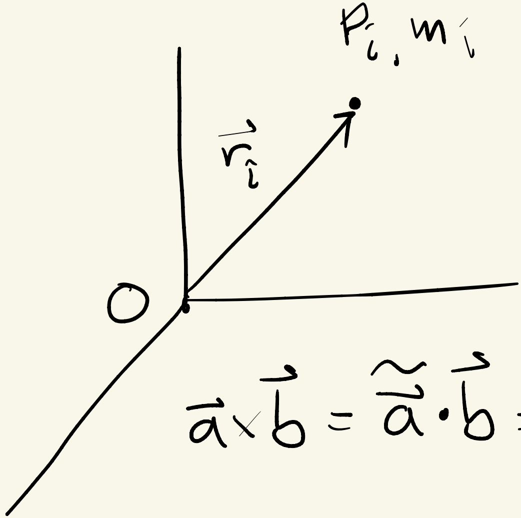
Given $\vec{r}_{ij} \in \mathbb{R}^{3\frac{N}{2}(N-1)}$, need $3(N-1)$

Others are a function of this reduced set, $\vec{r}_{12}, \vec{r}_{23}, \vec{r}_{34}$ 4BP

Ex: $\vec{r}_{14} = \vec{r}_{12} + \vec{r}_{23} + \vec{r}_{34}$, etc...

Additional Important Quantity

Inertia of the NBP



Inertia

$$\bar{\bar{I}}_i = -(\tilde{\vec r}_i \cdot \tilde{\vec r}_i)\, m_i$$

$\tilde{a}$ = cross product Operator

$$\vec a \times \vec b = \tilde{\vec a} \cdot \vec b = \vec a \cdot \tilde{\vec b} \; ; \quad \tilde{\vec a} = a_x(\hat z \hat y - \hat y \hat z) + a_y(\hat x \hat z - \hat z \hat x) + a_z(\hat y \hat x - \hat x \hat y)$$

$$\boxed{\vec a = a_x \hat x + a_y \hat y + a_z \hat z}$$

$\vec{a}, \vec{b},\quad \boxed{\tilde{a}\cdot\tilde{b} = \underbrace{\vec{b}\vec{a}}_{[\vec{b}][\vec{a}]^T} - (\vec{a}\cdot\vec{b})\,\bar{\bar{U}}}\;;$

$\bar{\bar{U}} \equiv$ unit dyad defined such that

$$\vec{a}\cdot\bar{\bar{U}} = \bar{\bar{U}}\cdot\vec{a} = \vec{a}$$

$$\bar{\bar{U}} = \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}$$

$$-\tilde{r}_i\cdot\tilde{r}_i = r_i^2\,\bar{\bar{U}} - \vec{r}_i\vec{r}_i$$

$$\bar{\bar{I}}_i = m_i\left[r_i^2\,\bar{\bar{U}} - \vec{r}_i\vec{r}_i\right] = m_i\begin{bmatrix} y_i^2+z_i^2 & -x_iy_i & -x_iz_i \\ -x_iy_i & x_i^2+z_i^2 & -y_iz_i \\ -z_iy_i & -y_iz_i & x_i^2+y_i^2 \end{bmatrix}$$

$$\boxed{\bar{\bar{I}} = \sum_{i=1}^{N} m_i\left(-\tilde{r}_i\cdot\tilde{r}_i\right) = -M\tilde{R}_c\cdot\tilde{R}_c + \frac{1}{M}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} m_i m_j\left(-\tilde{r}_{ij}\cdot\tilde{r}_{ij}\right)}$$

Note: $\mathrm{Tr}\left[-\tilde{a}\cdot\tilde{a}\right] = \mathrm{Tr}\left[a^2\,\bar{\bar{U}} - \vec{a}\vec{a}\right] = 3a^2 - a^2 = 2a^2$

$$\mathrm{Tr}[\overline{\overline{I}}] = \sum_{i=1}^{N} m_i \,\mathrm{Tr}[-\tilde{r}_i \cdot \tilde{r}_i] = \sum_{i=1}^{N} m_i (2 r_i^2) = 2 \underbrace{\sum_{i=1}^{N} m_i r_i^2}_{\text{Polar Moment of Inertia}}$$

$$\mathrm{Tr}[\overline{\overline{I}}] = 2 I_p$$

$$\boxed{I_p \equiv \text{Polar Moment of Inertia}}$$

$$\boxed{I_p = \frac{1}{2}\mathrm{Tr}[\overline{\overline{I}}] = M R_c^2 + \frac{1}{M}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} m_i m_j r_{ij}^2}$$

Moment of Inertia about an axis $\hat{a}$.

$$I_a = \hat{a} \cdot \overline{\overline{I}} \cdot \hat{a}$$

Note $\hat{a} \cdot (-\tilde{b} \cdot \tilde{b}) \cdot \hat{a}$

$$= \hat{a} \cdot \left[b^2 \overline{\overline{U}} - \vec{b}\vec{b} \right] \cdot \hat{a}$$

$$= b^2 - (\vec{b} \cdot \vec{a})^2 =$$

$$I_a = M\left[R_c^2 - (\hat{a}\cdot\vec{R}_c)^2\right] + \frac{1}{M}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} m_i m_j\left[r_{ij}^2 - (\vec{r}_{ij}\cdot\hat{a})^2\right]$$

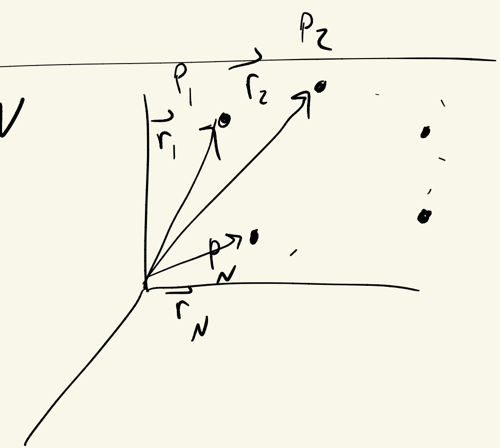
$$= \sum_{i=1}^{N} m_i\left[r_i^2 - (\vec{r}_i\cdot\hat{a})^2\right],$$ Note $\hat{a} \perp$ all $\vec{r}_{ij}$,

special cases

$$\boxed{I_a = I_p}$$

Jacobi Coordinates

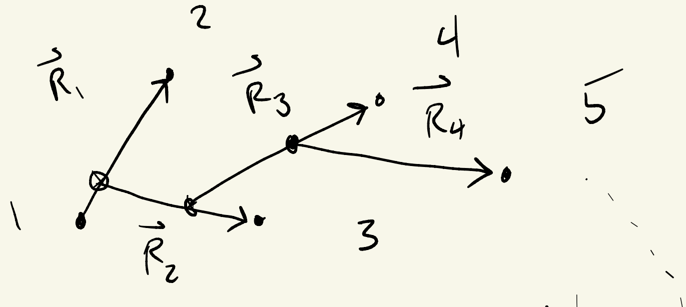
$i = 1, 2, \dots, N$



$$\vec{R}_1 = \vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$\vec{R}_2 = \vec{r}_3 - \frac{1}{m_1+m_2}\left[m_1\vec{r}_1 + m_2\vec{r}_2\right]$$

$$= \frac{1}{m_1+m_2}\left[m_1(\vec{r}_3 - \vec{r}_1) + m_2(\vec{r}_3 - \vec{r}_2)\right]$$



$$\vec{R}_{N-1} = \vec{r}_N - \frac{1}{\sum_{i=1}^{N-1} m_i} \sum_{i=1}^{N-1} m_i \vec{r}_i = \frac{1}{\sum_{i=1}^{N-1} m_i} \sum_{i=1}^{N-1} m_i \vec{r}_{iN}$$

$$\boxed{\vec{R}_N = \vec{R}_c}$$

Reversion of Jacobi Coords

$$\vec{r}_{12} = \vec{R}_1$$

$$\vec{r}_{23} = \vec{R}_2 - \frac{m_1}{m_1+m_2}\vec{R}_1 \qquad \vec{r}_{13} = \vec{r}_{12} + \vec{r}_{23} = \vec{R}_2 + \frac{m_2}{m_1+m_2}\vec{R}_1$$

$$\vec{r}_{34} = \vec{R}_3 - \frac{(m_1+m_2)}{(m_1+m_2+m_3)}\vec{R}_2$$

$$\vec{r}_{14} = \vec{r}_{12} + \vec{r}_{23} + \vec{r}_{34}$$

$$\vec{r}_{24} = \vec{r}_{23} + \vec{r}_{34}$$

$$\vec{r}_{k,k+1} = \vec{R}_k - \frac{\sum_{i=1}^{k-1} m_i}{\sum_{i=1}^{k} m_i} \vec{R}_{k-1}$$

Given $\vec{R}_i$, $\dot{\vec{R}}_i$, define $M_{\hat{i}} = \sum_{j=1}^{\hat{i}} m_j$; $\vec{R}_N \equiv$ system COM

$$T = \frac{1}{2} \sum_{i=1}^{N-1} \frac{M_{i-1} m_i}{M_{\hat{i}}} \dot{\vec{R}}_i \cdot \dot{\vec{R}}_i + \frac{1}{2} M_N \dot{\vec{R}}_N \cdot \dot{\vec{R}}_N$$

$$\vec{H} = \sum_{i=1}^{N-1} \frac{M_{i-1} m_i}{M_{\hat{i}}} \vec{R}_i \times \dot{\vec{R}}_i + M_N \vec{R}_N \times \dot{\vec{R}}_N$$

$\mathcal{U}(\vec{r}_{ij} ; i,j = 1, \dots, N \; i \neq j) \Rightarrow \vec{r}_{ij}$ are a function of $\vec{R}_k$

$$\bar{\bar{I}} = \sum_{i=1}^{N-1} \frac{M_{i-1} m_i}{M_{\hat{i}}} \left[R_i^2 \bar{\bar{U}} - \vec{R}_i \vec{R}_i \right] + M_N \left[R_N^2 \bar{\bar{U}} - \vec{R}_N \vec{R}_N \right]$$

$$I_p = \frac{1}{2} Tr[\overset{=}{I}] = \sum_{i=1}^{N-1} \frac{M_{i-1} m_i}{M_i} R_i^2 + \cdots \underline{CoM}$$

Lagrange Equations derivation of the EOM.

Lagrangian $L = T - \mathcal{U}$

Describe system with $\underline{\underline{n}}$ dof with coords q_i + velocities $\dot{q}_i = \frac{d}{dt} q_i$

$n = 3N$

Write $T(q_i, \dot{q}_i \,;\, i=1,2,\cdots,n)$ $\Rightarrow$ $L(q_i, \dot{q}_i \,;\, i=1,\cdots,n)$

$\mathcal{U}(q_i \,;\, i=1,2,\cdots,n)$

$L = T - \mathcal{U}$

EOM of the system are:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \;;\; j=1,2,\cdots,n} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_j}\right) = \frac{\partial L}{\partial \vec{r}_j}$$

Let $T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{r}_i^2$; $\mathcal{U}(\vec{r}_i)$

$$\frac{\partial T}{\partial \dot{\vec{r}}_j} = m_j \dot{\vec{r}}_j \;;\; \frac{d}{dt} m_j \dot{\vec{r}}_j = m_j \ddot{\vec{r}}_j \Rightarrow \boxed{m_j \ddot{\vec{r}}_j = -\frac{\partial \mathcal{U}}{\partial \vec{r}_j}}$$

$$\frac{\partial L}{\partial \vec{r}_j} = -\frac{\partial \mathcal{U}}{\partial \vec{r}_j} = \vec{F}_j$$

Consider Jacobi Coords (set $\vec{R}_N \equiv \vec{0}$)

$$L = T - \mathcal{U} \qquad T(\dot{\vec{R}}_j \, ; \, i = 1, \ldots, N-1)$$

$$\mathcal{U}(\vec{r}_{ij}(\vec{R}_k))$$

$$\frac{\partial L}{\partial \dot{\vec{R}}_j} = \frac{\partial T}{\partial \dot{\vec{R}}_j} = \frac{M_{j-1} m_j}{M_j} \dot{\vec{R}}_j \; ; \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{R}}_j}\right) = \frac{M_{j-1} m_j}{M_j} \ddot{\vec{R}}_j$$

$$\mathcal{U} = -\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{G m_i m_j}{|\vec{r}_{ij}|} \; ; \; \frac{\partial \mathcal{U}}{\partial \vec{R}_k} = -G \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij} \cdot \underbrace{\frac{\partial \vec{r}_{ij}}{\partial \vec{R}_k}}$$