Jacobi Coordinates

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Given N bodies with masses m_i and positions $\mathbf{r}_i$, we wish to apply the Jacobi coordinate transformation to them. We define the Jacobi coordinates with the following definitions.

Let $M_0 = 0$ and $M_i = M_{i-1} + m_i$ for $i = 1, 2, \ldots, N$.

Define the center of mass of the bodies from $i = 1, 2, \ldots, j$ as

$$\mathbf{R}_j^C = \frac{1}{M_j} \sum_{i=1}^{j} m_i \mathbf{r}_i$$

and define $\mathbf{R}_0^C = \mathbf{0}$. Then the Jacobi coordinates are defined as following:

$$\mathbf{R}_i = \mathbf{r}_{i+1} - \mathbf{R}_i^C$$

or

$$\mathbf{r}_{i+1} = \mathbf{R}_i + \mathbf{R}_i^C$$

Note that the Jacobi position $\mathbf{R}_0 = \mathbf{r}_1$, but that this Jacobi term is ignorable and will not appear in the subsequent equations.

A few observations can be made. The Jacobi coordinates run from $i = 1, N-1$, and thus represent the reduction of the N body system by one body. While the term $\mathbf{R}_N$ is not defined, we note that $\mathbf{R}_N^C = \frac{1}{M_N} \sum_{i=1}^{N} m_i \mathbf{r}_i$ is defined and equals the center of mass of the system. We will see this explicitly in the later, derived equations.

Starting from this definition the following relationships between $\mathbf{R}_i$, $\mathbf{R}_i^C$ and $\mathbf{r}_i$ can be derived.

$$\mathbf{R}_{j+1}^C = \sum_{i=0}^{j} \frac{m_{i+1}}{M_{i+1}} \mathbf{R}_i$$

$$\mathbf{R}_j = \frac{M_{j+1}}{m_{j+1}} \left[\mathbf{R}_{j+1}^C - \mathbf{R}_j^C \right]$$

$$\mathbf{R}_j = \mathbf{r}_{j+1} - \frac{1}{M_j} \sum_{i=1}^{j} m_i \mathbf{r}_i$$

$$\mathbf{r}_{j+1} = \mathbf{R}_j + \sum_{i=0}^{j-1} \frac{m_{i+1}}{M_{i+1}} \mathbf{R}_i$$

These expressions can be used to derive the transformed form of the individual kinetic energy, moment of inertia and angular momentum terms. Specifically, it can be shown that:

$$\begin{aligned}
\frac{1}{2} m_{i+1} \mathbf{v}_{i+1} \cdot \mathbf{v}_{i+1} &= \frac{1}{2} \frac{M_i m_{i+1}}{M_{i+1}} \mathbf{V}_i \cdot \mathbf{V}_i + \frac{1}{2} M_{i+1} \mathbf{V}_{i+1}^C \cdot \mathbf{V}_{i+1}^C - \frac{1}{2} M_i \mathbf{V}_i^C \cdot \mathbf{V}_i^C \\
m_{i+1} \mathbf{r}_{i+1} \cdot \mathbf{r}_{i+1} &= \frac{M_i m_{i+1}}{M_{i+1}} \mathbf{R}_i \cdot \mathbf{R}_i + M_{i+1} \mathbf{R}_{i+1}^C \cdot \mathbf{R}_{i+1}^C - M_i \mathbf{R}_i^C \cdot \mathbf{R}_i^C \\
m_{i+1} \mathbf{r}_{i+1} \times \mathbf{v}_{i+1} &= \frac{M_i m_{i+1}}{M_{i+1}} \mathbf{R}_i \times \mathbf{V}_i + M_{i+1} \mathbf{R}_{i+1}^C \times \mathbf{V}_{i+1}^C - M_i \mathbf{R}_i^C \times \mathbf{V}_i^C \\
m_{i+1} \tilde{\mathbf{r}}_{i+1} \cdot \tilde{\mathbf{r}}_{i+1} &= \frac{M_i m_{i+1}}{M_{i+1}} \tilde{\mathbf{R}}_i \cdot \tilde{\mathbf{R}}_i + M_{i+1} \tilde{\mathbf{R}}_{i+1}^C \cdot \tilde{\mathbf{R}}_{i+1}^C - M_i \tilde{\mathbf{R}}_i^C \cdot \tilde{\mathbf{R}}_i^C
\end{aligned}$$

These, when summed over all component bodies, leads to the total values

$$\begin{aligned}
T &= \frac{1}{2} \sum_{i=1}^{N-1} \frac{M_i m_{i+1}}{M_{i+1}} \mathbf{V}_i \cdot \mathbf{V}_i + \frac{1}{2} M_N \mathbf{V}_N^C \cdot \mathbf{V}_N^C \\
I_P &= \sum_{i=1}^{N-1} \frac{M_i m_{i+1}}{M_{i+1}} \mathbf{R}_i \cdot \mathbf{R}_i + M_N \mathbf{R}_N^C \cdot \mathbf{R}_N^C \\
\mathbf{H} &= \sum_{i=1}^{N-1} \frac{M_i m_{i+1}}{M_{i+1}} \mathbf{R}_i \times \mathbf{V}_i + M_N \mathbf{R}_N^C \times \mathbf{V}_N^C
\end{aligned}$$

Of special interest is the Lagrangian function for the N body problem, which we use to derive the equations of motion in the Jacobi frame, using the coordinates and velocities $\mathbf{R}_i$ and $\mathbf{V}_i$ respectively. We see that the partials of the Kinetic Energy with respect to velocity are then trivial. The partials of the gravitational potential with respect to the Jacobi coordinates is more complicated and involved.

First, using the above definitions we find:

$$\frac{\partial \mathbf{r}_i}{\partial \mathbf{R}_j} = \begin{cases} \mathbf{0} & j \geq i \\ \mathbf{U} & j = i-1 \\ \frac{m_{j+1}}{M_{j+1}}\mathbf{U} & j \leq i-2 \end{cases}$$

where $\mathbf{U}$ is the identity dyad.

Next, for the gravitational potential, we need to consider every possible combination $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$. Rewriting this vector difference in terms of Jacobi coordinates we find the result

$$\mathbf{r}_{ij} = \mathbf{R}_{j-1} - \mathbf{R}_{i-1} + \mathbf{R}^C_{j-1} - \mathbf{R}^C_{i-1}$$

where we assume that $j > i$. We note that in general $\mathbf{R}^C_{j-1} = \sum_{k=0}^{j-2} \frac{m_{k+1}}{M_{k+1}}\mathbf{R}_k$, and thus $\mathbf{R}^C_{j-1} - \mathbf{R}^C_{i-1} = \sum_{k=i-1}^{j-2} \frac{m_{k+1}}{M_{k+1}}\mathbf{R}_k$. We note that since $j > i$, we can have $j - 2 \geq i - 1$, or $j \geq i + 1$, meaning that the summation limits may be equal if j and i are subsequent bodies. Thus the relative vectors can be expressed as

$$\mathbf{r}_{ij} = \mathbf{R}_{j-1} - \mathbf{R}_{i-1} + \sum_{k=i-1}^{j-2} \frac{m_{k+1}}{M_{k+1}}\mathbf{R}_k$$

Then the partial derivatives can be directly found if we consider the partial of $\mathbf{r}_{ij}$ with respect to an arbitrary Jacobi position vector $\mathbf{R}_l$.

$$\frac{\partial \mathbf{r}_{ij}}{\partial \mathbf{R}_l} = \begin{cases} \mathbf{0} & l > j-1 \\ \mathbf{U} & l = j-1 \\ \frac{m_{l+1}}{M_{l+1}}\mathbf{U} & j-2 \geq l > i-1 \\ -\frac{M_{i-1}}{M_i}\mathbf{U} & l = i-1 \\ \mathbf{0} & l < i-1 \end{cases}$$

Note that if $j = i + 1$ that the term $j - 2 \geq l > i - 1$ is not defined and is skipped.

We can also verify this using the direct partial of $\mathbf{r}_i$ with respect to the Jacobi position coordinate.