

# Efficient Aircraft Multidisciplinary Design Optimization and Sensitivity Analysis via Signomial Programming

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This paper proposes a new methodology for physics based aircraft multidisciplinary design optimization (MDO) and sensitivity analysis. The proposed architecture uses signomial programming (SP), a type of difference-of-convex optimization that is solved iteratively as a series of log-convex problems. A requirement of SP is that all constraints and objective functions must have explicit signomial formulas. The SP MDO architecture facilitates the low cost computation of optimal sensitivities through Lagrange duality. The specific example of commercial aircraft MDO is considered. Using SP, a small, medium, and large scale benchmark problem is solved 16, 39, and 26 times faster, respectively, than Transport Aircraft System Optimization (TASOPT), a comparable and widely used aircraft MDO tool. The SP solution times include computation of all optimal parameter and constraint sensitivities, a feature unique to the presented architecture. The reliability of SP is demonstrated by converging a commercial aircraft MDO problem for a number of different objective functions and evaluating both traditional and non-traditional aircraft configurations. While the presented example is commercial aircraft MDO, the SP MDO architecture is applicable to a range of engineering optimization problems.

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## Nomenclature

$AR_w$  = wing aspect ratio

$b_w$  = wing span

$D$  = drag

$\frac{L}{D}$  = aircraft lift to drag ratio

$F$  = thrust

$M_{\min}$  = minimum cruise Mach number

$t_{\text{total}}$  = total mission flight time

$V_{\text{ne}}$  = never exceed speed

$W_{\text{ailerons}}$  = aileron weight

$W_{\text{empty}}$  = aircraft empty weight

$W_{\text{engine}}$  = engine weight

$W_{\text{flaps}}$  = wing flap weight

$W_{f_{\text{total}}}$  = total fuel weight

$W_{\text{fuselage}}$  = fuselage weight

$W_{\text{lg}}$  = landing gear weight

$W_{\text{payload}}$  = payload weight

$W_{\text{slats}}$  = wing slat weight

$W_{\text{tail}}$  = horizontal and vertical tail weight

$W_{\text{skin}}$  = wing skin weight

$W_{\text{wing}}$  = wing weight

$W_{\text{wing box}}$  = wing box weight

$(\cdot)_i$  = flight segment  $i$  quantity

## I. Introduction

A key goal in conceptual engineering system design is to understand tradeoffs between various design parameters, system configurations, and mission objectives. Understanding the Pareto frontier early on in the design process requires system level optimization across a range of possible system configurations and missions. Performing this type of optimization has become increasingly difficult due to the multimodal nature and complexity of modern engineering systems. As noted by Martins and Lambe, a variety of multidisciplinary design optimization (MDO) architectures exist. However, there is still a need for new architectures that exhibit fast convergence for medium and large scale problems[1].

One technique for improving computational efficiency is to solve particular forms of optimization problems rather than general non-linear programs. Hoburg et al.[2] successfully formulate a basic aircraft design problem as a geometric program (GP), which is a type of convex optimization problem. GPs with thousands of design variables can be solved on a personal laptop in just a few seconds. Algorithms for solving GPs guarantee convergence to a global optimum, and use Lagrange duality to compute parameter and constraint sensitivities with minimal computational cost[3]. A caveat of GP is that all physical relationships must be expressed in a mathematical form compatible with GP. A number of physical relationships fit the required form, and GP has been used to perform engineering design analysis [4]. However, the restrictions on constraint and objective functions limit GP in its level of fidelity.

This work introduces signomial programming (SP) as a new methodology for medium fidelity physics based multidisciplinary design optimization (MDO) and sensitivity analysis. SPs, a specific type of difference-of-convex programs, are non-convex extensions of GPs. Restrictions on the form of SP constraints are less stringent than the restrictions on GP constraints. SPs have many of the advantages of GPs, such as their relative speed compared to general non-linear problems and low cost computation of optimal sensitivities. However, unlike GPs, SPs provide no guarantee of global optimality. A detailed description of geometric and signomial programming is provided in Section II.

The utility of this new architecture is demonstrated through the solution of a series of com-

mercial aircraft MDO problems. Prior work has developed physics based signomial programming compatible models for aircraft wings, fuselages, horizontal tails, vertical tails, landing gear, and turbofan engines[5, 6]. These models, along with a mission profile, have been combined to form a full system aircraft MDO tool of comparable fidelity to the widely used Transport Aircraft System Optimization (TASOPT) [7]. Section IV demonstrates that the presented methodology can model both different scales and configurations of aircraft by optimizing a 737, 777, and D8.2, which includes non-traditional features such as a double bubble fuselage and boundary layer ingestion[8]. A series of case studies are used to illustrate the advantages of signomial programming. In Section V A, the SP is shown to perform a single-mission aircraft optimization 16 times faster, a two-mission aircraft optimization 39 times faster, and a four-mission aircraft optimization problem 26 times faster than TASOPT. The SP solution times include computation of all optimal parameter and constraint sensitivities while the TASOPT solution times do not. Section V B presents a sample sensitivity analysis and validates the accuracy of sensitivities computed via Lagrange duality. Finally, the SP tool’s ability to reliably converge models across a range of objective functions is illustrated in Section V C. The code used to generate presented results is publicly available at <https://github.com/convexengineering/SPaircraft>.

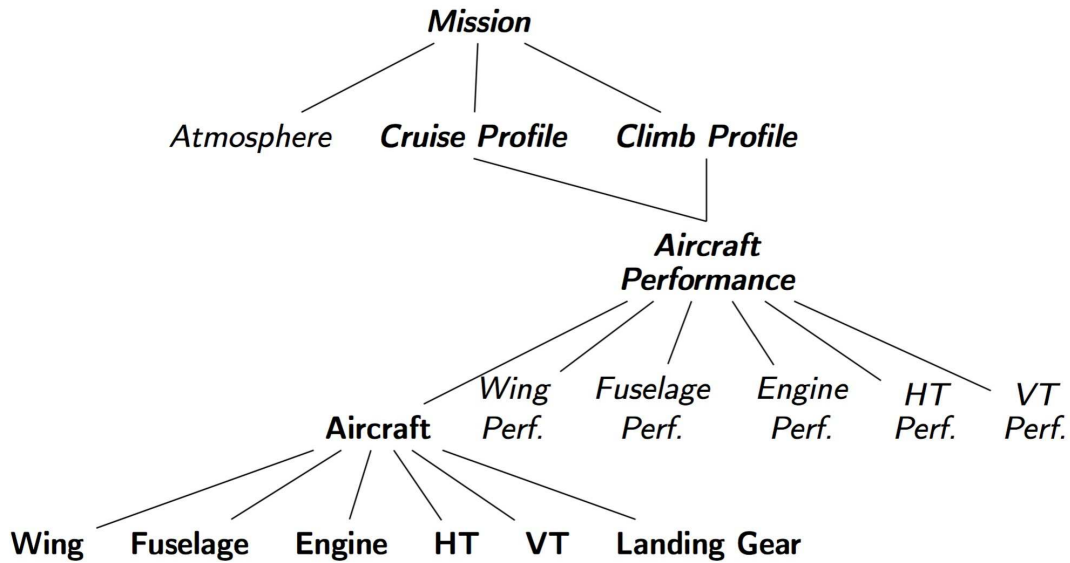
## II. Optimization Formulation

All presented models are SPs that were constructed with GPkit [9] and employ its built-in framework for multi-point optimization, which is described in Section IIA. A detailed description of GP and SP is provided in Sections IIB and IIC respectively. GPkit is an open source Python package developed at MIT that enables the fast and intuitive formulation of GPs and SPs. GPkit has a number of built-in heuristics for solving SPs via a series of GP approximations. This work employs the relaxed constants penalty function heuristic detailed in [5]. GPkit binds with open source and commercial primal-dual interior point solvers to solve the individual GPs. The presented SPs were solved with the commercially-available solver MOSEK [10]. Each individual GP is solved all-at-once. The presented methodology is stable enough that initial guesses are not required for any variables; at the beginning of a SP solve, all variables are set to one.

### A. Multi-Point Optimization Formulation

Each individual GP is a non-hierarchical collection of every constraint (or its local approximation) in the model. However, during model development, a hierarchy of models, such as that in Figure 1, helps ensure the appropriate constraining connections between subsystems are made. Many different modular decompositions of the full-system model are possible, but the combination of two design rules generally leads to a single obvious and highly reusable decomposition.

The first rule is a strict maintenance of hierarchy: models can reference only the variables of models that are at the same or lower level of hierarchy (depicted in Figure 1) than themselves. This rule is useful in the design of any component-decomposed system such as the SP aircraft model and helps determine when a constraint that seems like part of a low-level subsystem is best considered in a higher-level system.



**Fig. 1** Variable and constraint hierarchy. Models can access attributes of models lower than themselves in the hierarchy. Models that include sizing variables are bolded while models containing performance variables are italicized.

The second rule is a separation of "sizing" and "performance" variables into separate models. Sizing models contain all variables and constraints that do not change between operating points, such as component weights and dimensions. Performance models contain all constraints and variables

that change between operating points, such as air speeds, lift coefficients, and fuel quantities. Sizing models contain a pointer to their companion performance model. This model separation allows the modeler to specify a scalar performance model but create it in a ‘vectorized’ environment which extends each variable in the original scalar performance model across the vector of operating points. For example, the constraint thrust is greater than or equal to drag could be written in a scalar performance model as  $F \geq D$  and then be vectorized across  $N$  operating points. After vectorization, the original constraint  $F \geq D$  becomes the  $N$  unique constraints below.

$$\begin{aligned} F_1 &\geq D_1 \\ F_2 &\geq D_2 \\ &\vdots \\ F_N &\geq D_N \end{aligned}$$

Figure 2 provides a visual representation of sizing and performance models. The technique is not restricted to aerospace applications and can be used for any multi-point optimization. Vectorization allows the airplane design problem to be extended to a fleet design problem with a single line of code. Together, these two model development rules help specify simple submodels, making it easier for modelers to collaborate and use models written by others.

## B. Geometric Programming

Introduced in 1967 by Duffin et al. [11], a geometric program (GP) is a type of constrained optimization problem that becomes convex after a logarithmic change of variables. Modern interior point methods allow a typical sparse GP with tens of thousands of decision variables and constraints to be solved in minutes on a desktop computer [12]. These solvers do not require an initial guess, and guarantee convergence to a *global* optimum, assuming a feasible solution exists. If a feasible solution does not exist, the solver will return a certificate of infeasibility. These impressive properties are possible because a GP’s objective and constraints consist of only monomial and posynomial functions, which can be transformed into convex functions in log space.

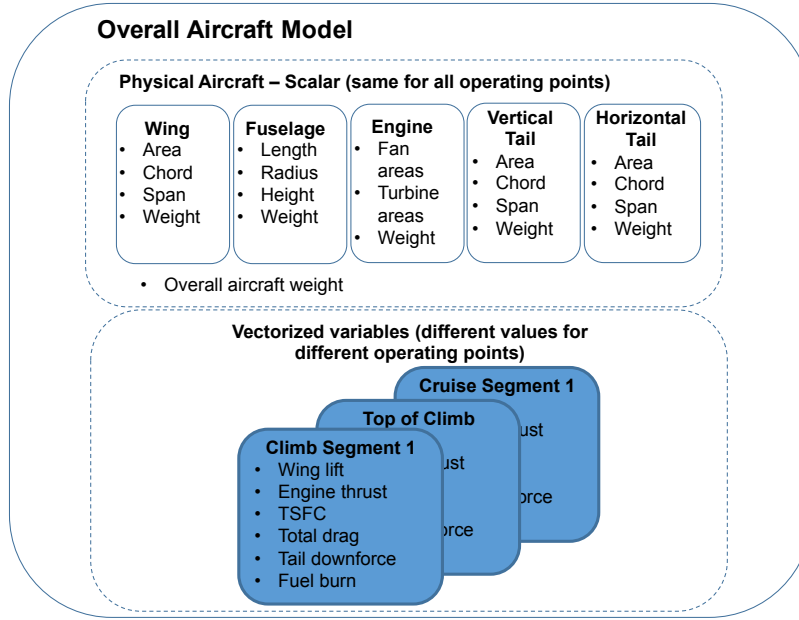


Fig. 2 Aircraft model architecture.

A monomial is a function of the form

$$m(\mathbf{u}) = c \prod_{j=1}^n u_j^{a_j} \quad (1)$$

where  $a_j \in \mathbb{R}, c \in \mathbb{R}_{++}$  and  $u_j \in \mathbb{R}_{++}$ . An example of a monomial is the common expression for lift,  $\frac{1}{2}\rho V^2 C_L S$ . In this case,  $\mathbf{u} = (\rho, V, C_L, S)$ ,  $c = 1/2$ , and  $\mathbf{a} = (1, 2, 1, 1)$ .

A posynomial is a function of the form

$$p(\mathbf{u}) = \sum_{k=1}^K c_k \prod_{j=1}^n u_j^{a_{jk}} \quad (2)$$

where  $a_{jk} \in \mathbb{R}, c_k \in \mathbb{R}_{++}$  and  $u_j \in \mathbb{R}_{++}$ . A posynomial is a sum of monomials. Therefore, all monomials are one-term posynomials.

A GP minimizes a posynomial objective function subject to monomial equality and posynomial inequality constraints. A GP written in standard form is

$$\begin{aligned} & \text{minimize } p_0(\mathbf{u}) \\ & \text{subject to } p_i(\mathbf{u}) \leq 1, i = 1, \dots, n_p, \\ & m_i(\mathbf{u}) = 1, i = 1, \dots, n_m \end{aligned} \quad (3)$$

where  $p_i$  are posynomial functions,  $m_i$  are monomial functions, and  $\mathbf{u} \in \mathbb{R}_{++}^n$  are the decision variables. Once a problem has been formulated in the standard form (Equation 3), it can be solved efficiently.

There are a number of techniques for formulating engineering design problems as GPs [13]. With knowledge of how a problem pressures design variables, many posynomial equalities can be reformulated as posynomial inequalities that will be equivalent to the equality at the optimum. For example, in an aircraft design problem whose objective is to minimize fuel burn, the total aircraft weight may be given by  $W_{\text{total}} = W_{\text{tail}} + W_{\text{fuselage}} + W_{\text{engine}} + W_{\text{payload}}$ . The heavier the aircraft is, the more fuel will be burned. Thus, the objective and other constraints drive total weight to its minimum value and the non-GP compatible posynomial equality relationship can be relaxed into the GP compatible posynomial inequality  $W_{\text{total}} \geq W_{\text{tail}} + W_{\text{fuselage}} + W_{\text{engine}} + W_{\text{payload}}$ . At the optimum, the inequality constraint will be tight, meaning the left and right hand sides equal each other. The inequality constraint functions as an equality constraint. GPKit [9] has a framework to confirm the tightness of constraints relaxed in this way.

Other techniques to make constraints GP compatible include using variable transformations and Taylor approximations. Variable transformations are used in the tail cone model from [5] and the stagnation relations in Appendix B. Taylor approximations are used in the GP compatible Breguet range equation presented in [2] and the structural model in Appendix A. Additionally, empirical relations can be used to formulate constraints using the methods described in [14]. An example of GP-compatible fits can be found in [5] where XFOIL[15] data is used to generate a posynomial inequality constraint for TASOPT C-series airfoil parasitic drag coefficient as a function of wing thickness, lift coefficient, Reynolds number, and Mach number. The TASOPT C-series airfoils are representative of modern transonic airfoils[7]. A detailed description of how to take raw data and create a GP compatible fit can be found in [13].

### C. Signomial Programming

Because GPs require a special mathematical form, it is not always possible to formulate a design problem as a GP. This motivates the introduction of signomials. Signomials have the same form as



posynomials

$$s(\mathbf{u}) = \sum_{k=1}^K c_k \prod_{j=1}^n u_j^{a_{jk}} \quad (4)$$

but the coefficients,  $c_k \in \mathbb{R}$ , can now be any (including non-positive) real numbers.

A signomial program (SP) is a generalization of a GP where the inequality constraints can be composed of signomial constraints of the form  $s(u) \leq 0$ . The log transform of an SP is not a convex optimization problem, but is a difference-of-convex optimization problem that can be written in log-space as

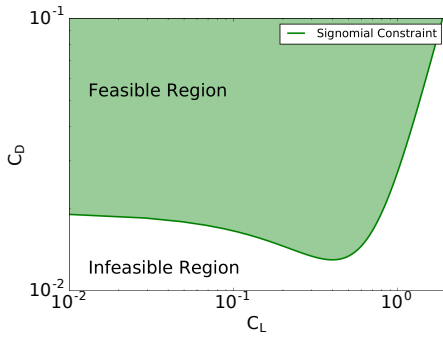
$$\begin{aligned} & \text{minimize } f_0(\mathbf{x}) \\ & \text{subject to } f_i(\mathbf{x}) - g_i(\mathbf{x}) \leq 0, i = 1, \dots, m \end{aligned} \quad (5)$$

where  $f_i$  and  $g_i$  are convex.

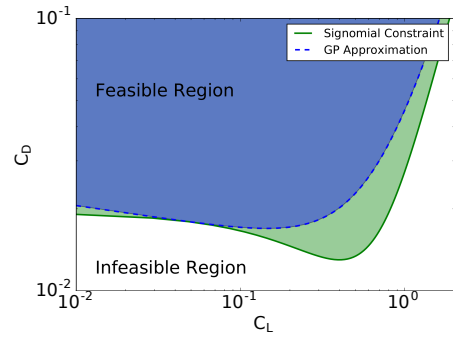
There are multiple algorithms that reliably solve signomial programs to *local* optima [16, 17]. A common solution heuristic, referred to as difference-of-convex programming or the convex-concave procedure, involves solving a sequence of GPs, where each GP is a local approximation to the SP, until convergence occurs. The introduction of even a single signomial constraint to any GP turns the GP into a SP, thus losing the guarantee of solution convergence to a global optimum. Despite the possibility of convergence to a local, not global, optimum, signomial programming is a powerful tool. The convex approximation,  $\hat{f}(x)$ , to the non-convex signomial in log-space,  $f(x) - g(x)$ , is constructed such that it always satisfies

$$\hat{f}(x) \geq f(x) - g(x) \quad \forall \quad x \quad (6)$$

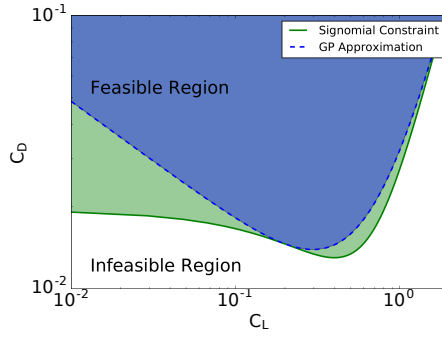
In other words, for each constraint, the feasible set of the convex approximation  $\hat{f}(x) \leq 0$  is a subset of the original SP's feasible set,  $f(x) - g(x) \leq 0$ . This means SP inequalities do not require a trust region, removing the need for trust region parameter tuning and making solving SPs substantially more reliable than solving general nonlinear programs. Figure 3, where a series of log-convex (GP compatible) constraints approximates a non-convex parabolic drag polar in log space, illustrates this property.



(a) Non-convex signomial inequality drag  
constraint



(b) Convex approximation about  $C_L = 0.05$ .



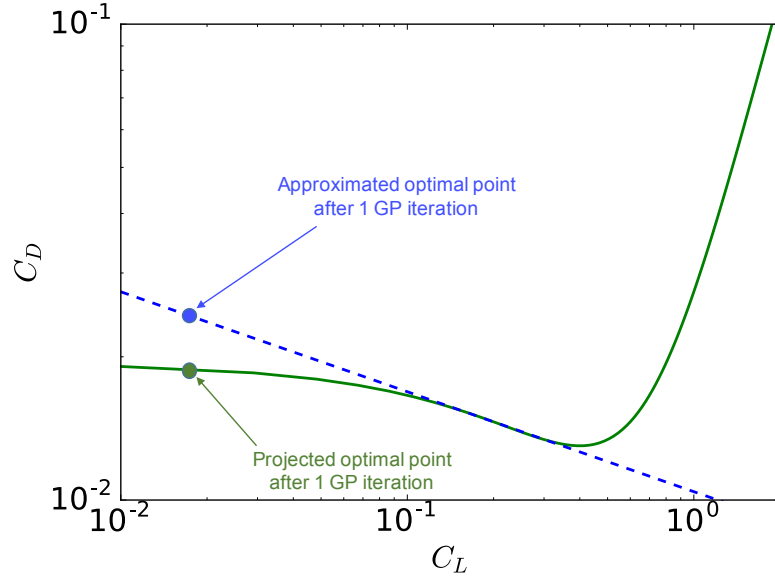
(c) Convex approximation about  $C_L = 0.20$ .

**Fig. 3 A signomial inequality constraint and GP approximations about two different points.**

Signomial equality constraints can be approximated with monomials using the methods described in [18]. As shown in Figure 4, the monomial approximation of a signomial equality constraint does not fully lie within the original constraint's feasible region. Consequently, signomial equalities are the least desirable type of constraint.

### III. Aircraft Model Overview

The full aircraft model is a set of coupled subsystem models. Individual sub-models are linked through shared variables. For example, the wing structural model depends on engine weight and the fuselage structural model depends on maximum tail aerodynamic loads. A full description of subsystem variable linking can be found in [5]. A qualitative description of all sub-models is provided in the following subsections.



**Fig. 4 The signomial equality constraint  $C_D = f(C_L)$  and its approximation.**

#### **A. Fuselage**

The fuselage model is adapted from the model in [5], and borrows heavily from TASOPT [7]. Modifications, described in Appendix A, were made to support double bubble fuselages in addition to traditional tube fuselages. Fuselage sizing constraints include pressure loads, y-axis and z-axis bending moments, and floor loads.

#### **B. Engine**

The SP model utilizes the full 1D core+fan flow path simulation turbofan engine model developed by York et al. [6]. An additional boundary layer ingestion model used for the SP D8.2 is described in Appendix B. The nacelle drag model in [7] is adopted with one modification: the nacelle skin friction coefficient is assumed to be that of a turbulent flat plate.

#### **C. Wing**

The wing model is taken directly from [5]. The model includes a physics based structural model, geometry based induced drag and lift curve slope estimation, drag fits to modern transonic airfoils, and a fuel tank volume model.

#### **D. Vertical and Horizontal Tails**

This work leverages the tail models in [5]. The vertical tail is sized for both takeoff engine out and a minimum required yaw acceleration rate at flare.

The horizontal tail is sized to provide a minimum static margin at the forward and aft CG locations (this work includes the CG model in [5]). The SP model can implement both pi-tails and conventional tails.

The vertical tail surfaces utilize the same structural model as the wing. To facilitate modeling pi-tails, the horizontal tail has a unique structural model described in Appendix C.

#### **E. Landing Gear**

The landing gear model is taken directly from [5]. The model includes aircraft geometry constraints as well as taxi and landing load cases.

#### **F. Mission Profile**

The mission profile is described in Appendix D. It includes climb and cruise segments, both of which can be discretized into an arbitrary number of sub-segments. For the purposes of this paper, three climb and two cruise segments were used. Climb performance is computed using an excess power formulation. For each cruise segment, the optimizer can either fly level or execute a cruise climb (cruise climb rate, cruise altitude, and lift coefficient are optimized).

#### **G. Atmosphere**

The atmosphere model is taken directly from [5].

### **IV. Example Solutions**

The SP MDO tool was used to optimize three different aircraft architectures: a single aisle airliner similar to a 737-800, a wide body airliner similar to a 777-300ER, and the D8.2[8]. Mission parameters are presented in Table 1. In all cases, the objective function was total fuel burn. Constant input parameters were selected to match TASOPT input parameters for the example files distributed with TASOPT version 2.16. SP model results are presented along side TASOPT results

**Table 1 737-800, D8.2 and 777-300ER model mission parameters.**

Quantity	737-800	D8.2	777-300ER
Range [nm]	3,000	3,000	6,000
Number of Passengers	180	180	450
Minimum Cruise Mach	0.80	0.72	0.84
Payload Weight [lbf]	38,716	38,716	103,541

**Table 2 Results for the SP and TASOPT 737-800 models.**

Quantity	SP	TASOPT
Takeoff Weight [lbf]	166,504	166,502
Required Fuel [lbf]	41,847	45,057
Empty Weight [lbf]	85,956	82,729.6
Wing Span [ft]	117.5	113.6

in Tables 2 through 4. Differences in the SP and TASOPT sub-system models, which manifest as solution differences, are discussed in [5], [6], and Appendices A-D. A few of the notable differences include the SP model's optimization of the engine on-design point versus the pre-definition of the on-design point in TASOPT, the SP model's use of a physics and geometry based landing gear model versus the fractional weights used in TASOPT, and the SP model's increased optimization of the flight profile relative to TASOPT.

The SP aircraft optimization tool was integrated with OpenVSP[19] to facilitate output visualization. Figures 5 through 7 show selected VSP output for the presented models. Figure 8 presents the SP mission profile overlaid with the TASOPT mission profile for each aircraft. The SP model optimizes cruise climb rates and cruise lift coefficients for each flight segment whereas TASOPT computes a cruise climb gradient to maintain a constant Mach number and lift coefficient. Objective function convergence plots are provided in Appendix E.

**Table 3 Results for the SP and TASOPT D8.2 models.**

Quantity	SP	TASOPT
Takeoff Weight [lbf]	143,421	134,758
Required Fuel [lbf]	27,529	26,959
Empty Weight [lbf]	77,129	69,084
Wing Span [ft]	140.0	140.0

**Table 4 Results for the SP and TASOPT 777-300ER models.**

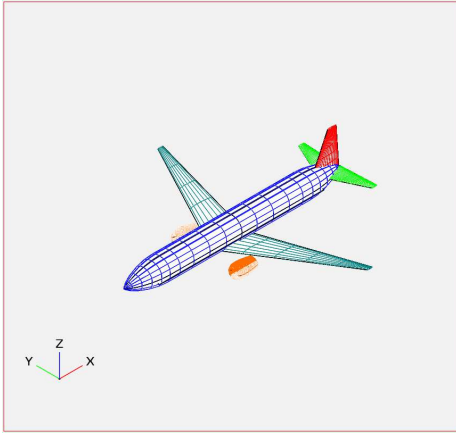
Quantity	SP	TASOPT
Takeoff Weight [lbf]	581,113	625,008
Required Fuel [lbf]	204,325	212,236
Empty Weight [lbf]	271,288	309,230
Wing Span [ft]	189.8	197.7

## V. Case Studies

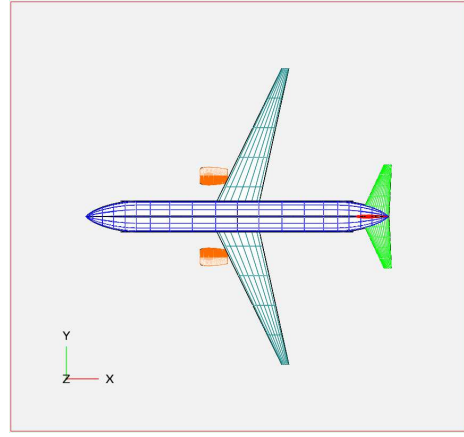
### A. Solution Time Comparison

Solution times for the SP model and TASOPT are presented in Table 5. It is important to note the SP solution time includes the computation of all optimal parameter and constraint sensitivities, which are discussed in the next section. TASOPT uses a traditional gradient based optimization method[7]. The two-mission optimization consists of optimizing a single aircraft to fly both a 3,000 nm and a 2,000 nm mission while the four-mission optimization consists of optimizing a single aircraft to fly a 3,000 nm, 2,500 nm, 2,000 nm, and 1,000 nm mission. In all cases, total fuel burn was the objective, and the payload of the 737 in Section IV was used. All models were solved on a laptop computer with a 2.5 GHz Intel Core i7 processor and 16GB of 1600 MHz DDR3 RAM.

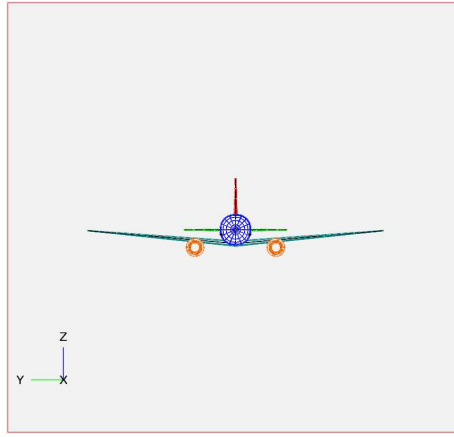
The SP model solves 16 times faster than TASOPT for the single mission case, 39 times faster than TASOPT for the two-mission case, and 26 times faster than TASOPT for the four-mission case. The SP model experiences a 6.4 times slow down when moving from the single to four-mission solve while TASOPT experiences a 10.2 times slow down. This suggests the SP formulation scales better to large problems than traditional gradient-based optimization formulations.



(a) 737 side view.



(b) 737 top view.



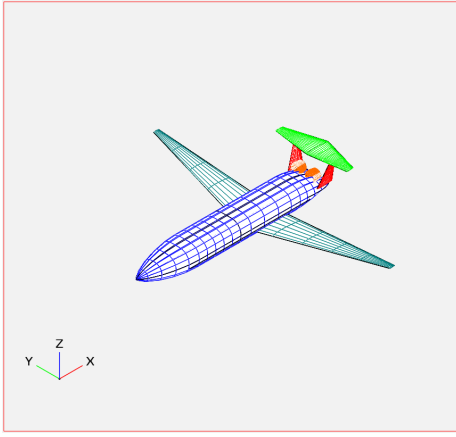
(c) 737 rear view.

**Fig. 5 SP 737 VSP outputs.**

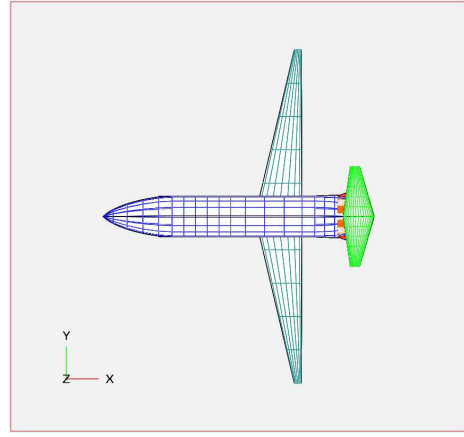
## B. Sensitivity Analysis

The SP model computes the sensitivity of each model parameter and constraint. Sensitivities are all local and computed about the optimum found in the last GP approximation of the SP. Equation 7 is the definition of parameter sensitivity while Equation 8 is the definition of constraint sensitivity[16]. Sensitivities represent the partial derivative of the computed optimum with respect to perturbations in constraints or model parameters.

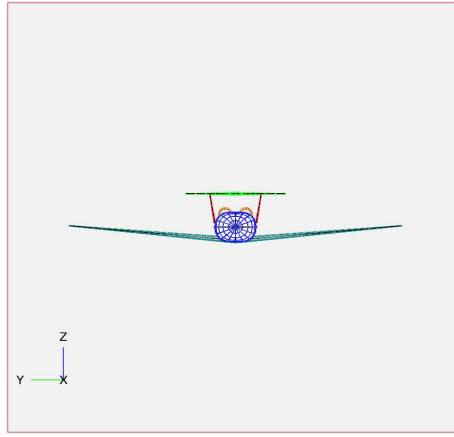
$$\text{Parameter Sensitivity} = \frac{\text{Objective-Function Percent Change}}{\text{Parameter Percent Change}} \quad (7)$$



(a) D8.2 side view.



(b) D8.2 top view.



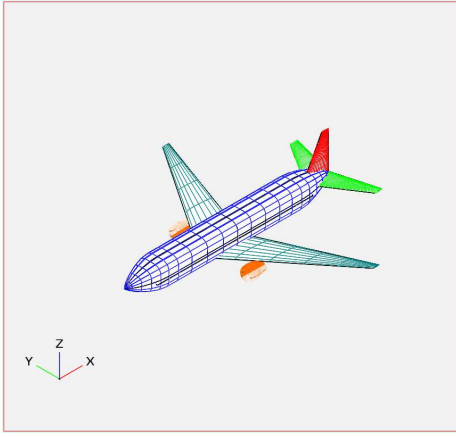
(c) D8.2 rear view.

**Fig. 6 SP D8.2 VSP output.**

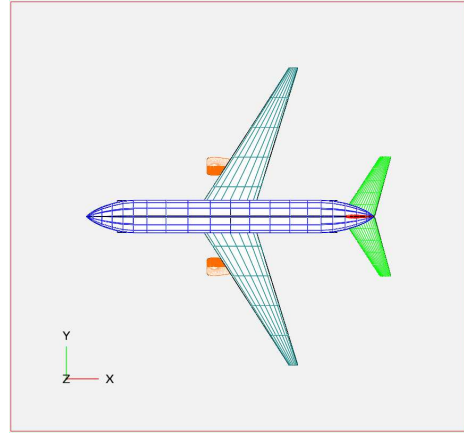
$$\text{Constraint Sensitivity} = \frac{\text{Objective-Function Percent Change}}{\text{Percent Change In Constraint Tightness}} \quad (8)$$

GPkit computes sensitivities via Lagrange duality using the methods in [3, 12]. Modern GP solvers that utilize primal-dual interior point methods, such as Mosek [10], determine the optimal primal and dual variables simultaneously (so long as both problems are feasible). Constraint sensitivities are simply the value of the optimal dual variables. Parameter sensitivities are equal to the dot product of the optimal dual variables and the parameter's exponents summed over all posynomial and signomial constraints containing the parameter. Thus, constraint sensitivities are determined for free and parameter sensitivities only require a sum of dot products. No finite differences or additional model evaluations are required.

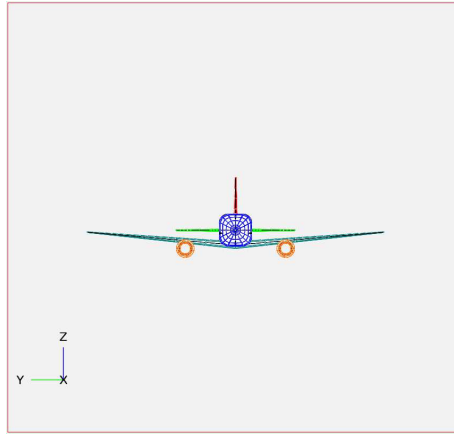




(a) 777 side view.



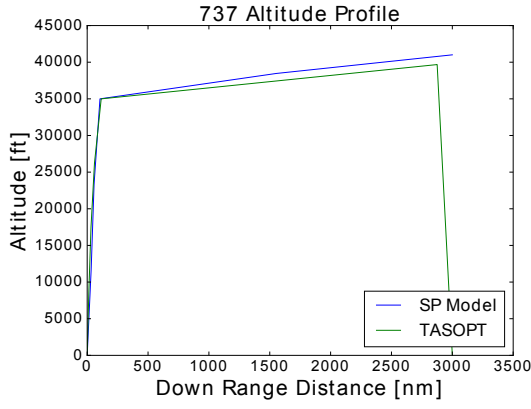
(b) 777 top view.



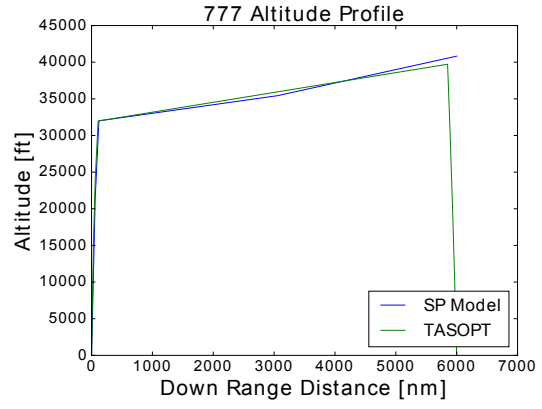
(c) 777 rear view.

**Fig. 7 SP 777 VSP output.**

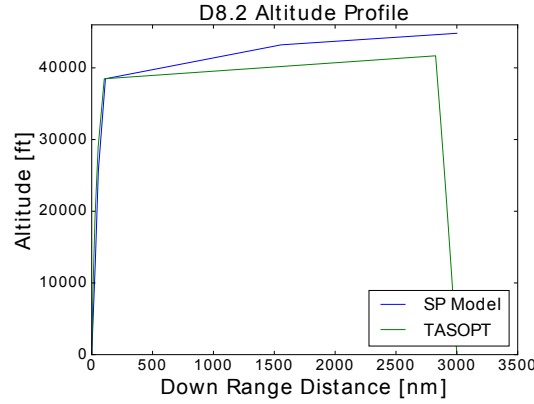
Sensitivities are useful in engineering design for two reasons. The first is to determine which areas of a physical design should be improved. For example, if the sensitivity to the engine burner pressure ratio (a fixed parameter) is large in magnitude, it is advantageous to focus effort on increasing the engine burner pressure ratio in order to improve the objective. Sensitivities are also a useful guide for model development. If the sensitivity to a fixed parameter is high, then it is important to either know the value of that parameter with a high degree of certainty or replace the parameter with a more detailed model. However, if the sensitivity of a fixed parameter is low, variations in the value of the parameter are unlikely to have a large effect on the model's solution. Constraint sensitivities can be used to gain intuition about the sensitivity of the objective function to design variables in specific constraints. Consider a constraint such as  $W_{\text{wing}} \geq W_{\text{wingbox}} + W_{\text{skin}} + W_{\text{flaps}} + W_{\text{slats}} + W_{\text{ailerons}}$



(a) 737 mission profile.



(b) 777 mission profile.



(c) D8.2 mission profile. The SP D8.2 initially has a higher cruise climb rate than TASOPT.

**Fig. 8 SP and TASOPT mission profiles for the 737, 777, and D8.2 models presented in Section IV.**

where wing box weight, skin weight, etc. are set by independent sub-models. If the sensitivity to this constraint is high, then the model is sensitive to wing weight and it may be advantageous to decrease wing weight. However, if the sensitivity to the constraint is low, the model is not sensitive to wing weight and the benefit of decreasing wing weight will be low. Table 6 presents selected sensitivity information for the optimal 737 model presented in Section IV. Table 7 presents the same parameter sensitivities for the D8.2 model discussed in Section IV.

The accuracy of GPKIT computed sensitivities is demonstrated by comparing them to sensitivities found through finite differencing. The 737 model was optimized for minimum fuel burn twice, once with a range of 3,000 nm and once with a range of 2,995 nm. The 3,000 nm fuel burn was 41,847 pounds while the 2,995 nm fuel burn was 41,757 pounds. The sensitivity to mission range

**Table 5 Comparison of SP and TASOPT solution times for different 737-800 models. The SP model experiences a 6.4 times slow down when moving from the single to four-mission solve while TASOPT experiences a 10.2 times slow down.**

Model	SP Solve Time	TASOPT Solve Time	Number of Variables in SP Model
Pure Analysis	N/A	< 1 sec	N/A
Single Point Optimization	7.66 sec	2 min 4 sec	1,902
Two Mission Optimization	20.1 sec	12 min 58 sec	3,133
Four Mission Optimization	49.3 sec	21 min 5 sec	5,593

**Table 6 Selected sensitivities for the optimal 737 presented in Section IV**

Parameter	Sensitivity
Avg. Passenger Weight (incl. payload)	0.84
Wing Max Allowed Tensile Stress	-0.30
Range	1.3
$V_{ne}$	0.30
Reserve Fuel Fraction	0.26
$M_{min}$	0.46
Max Fuselage Skin Stress	-0.035
Engine Burner Efficiency	-1.3
Engine Burner pressure ratio	-0.51

**Table 7 Selected sensitivities for the optimal D8.2 presented in Section IV.**

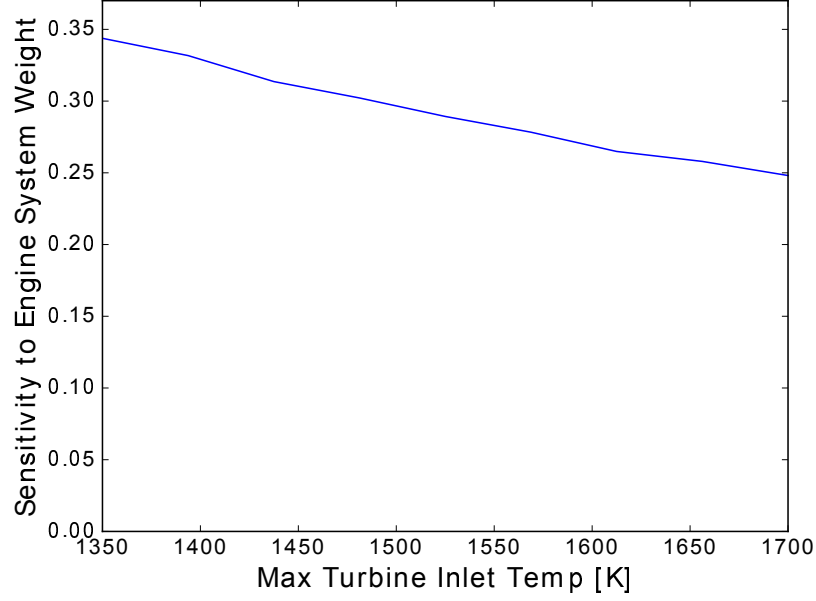
Parameter	Sensitivity
Avg. Passenger Weight (incl. payload)	0.72
Wing Max Allowed Tensile Stress	-0.26
Range	1.1
$V_{ne}$	0.28
Reserve Fuel Fraction	0.21
$M_{min}$	0.21
Max Fuselage Fuselage Skin Stress	-0.04
Engine Burner Efficiency	-1.2
Engine Burner pressure ratio	-0.39

was computed by dividing the percent change in fuel burn by the percent change in range. This finite difference method yields a sensitivity of 1.290. By solving the SP model, GPkit determines the sensitivity to range to be 1.266 for the 3,000 nm range and 1.267 for the 2,995 nm range. The difference in the sensitivity values is partially attributable to solver convergence criteria. The iterative SP solution procedure is terminated when the relative change in the value of the objective for successive GP solves falls beneath a preset tolerance. For the 3,000 nm mission, decreasing this tolerance from 1e-2 to 1e-6 decreases optimal fuel burn by 0.7% and increases the sensitivity to range to 1.274. As the convergence tolerance is moved closer to zero, the GPkit computed sensitivity will converge to the exact local derivative, similar to how finite difference sensitivities converge as the difference shrinks.

It is interesting to analyze how sensitivities change as parameters vary. An example is presented in Figure 9. As the max allowed turbine inlet temperature increases, the engine’s power density increases and weight decreases. As engine weight becomes a smaller proportion of total aircraft weight, further reductions in engine weight have decreasing returns with respect to overall system performance. Hence, the sensitivity to engine system weight decreases as max allowed turbine inlet temperature increases.

### C. Model Robustness Across Objective Functions

SPs are bags of constraints that are solved all at once via a convex-concave procedure [16, 17]. In the SP model, there are no parameter tuning or weight convergence loops and the naive initial guess of one is used for all variables. These factors, along with the mathematically favorable structure of SPs, allow the model solve reliably and efficiently across a variety of objective functions. Table 8 presents key design variables obtained when solving the optimal 737 model for a variety of objective functions. All results are normalized by the result obtained when the objective function is total fuel burn. For example, the value of total fuel burn ( $W_{f_{total}}$ ) when the objective is engine weight is listed as 1.4. This means the aircraft produced when optimizing for engine weight burns 1.4 times more fuel than the aircraft produced when optimizing for total fuel burn. Table 8 does not present an exhaustive set of objectives for which the model converges. The SP model can be solved for



**Fig. 9** Sensitivity to engine system weight versus max allowed turbine inlet temperature ( $T_{t4.1}$ ).

**Table 8** Key design variables for a 737 class aircraft optimized for a variety of objective functions.

Objective	$W_{f_{total}}$	$W_{empty}$	$b_w$	$AR_w$	$W_{engine}$	$t_{total}$	Initial Cruise $\frac{L}{D}$	$W_{lg}$
$W_{f_{total}}$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$W_{empty}$	1.52	0.72	0.72	0.56	0.70	0.99	0.85	0.97
$b_w$	2.2	1.0	0.60	0.29	1.4	0.97	0.58	1.58
$AR_w$	4.2	1.61	0.71	0.23	2.4	0.96	0.53	2.4
$W_{engine}$	1.4	0.78	0.92	0.94	0.52	1.0	1.1	0.98
$t_{total}$	3.5	1.3	1.0	0.60	1.7	0.89	0.84	2.2
Initial Cruise $\frac{D}{L}$	1.5	0.98	1.0	0.94	0.61	1.0	1.2	1.2
$W_{lg}$	1.4	0.87	0.93	0.74	0.87	0.99	0.99	0.67

any weighted sum of objective functions in Table 8 and supports net present value models. This capability enables aircraft performance to be assessed from the perspective of multiple stakeholders, such as operators and manufacturers.

## VI. Conclusion

This paper has proposed performing physics based MDO and sensitivity analysis via signomial programming. Through a series of aircraft MDO case studies, benefits of the signomial programming architecture are presented. SP is used to solve a single-mission, two-mission, and four-mission commercial aircraft MDO problem and to perform a sensitivity analysis 16, 39, and 26 times faster, respectively than TASOPT, a comparable existing tool, that performs aircraft optimization with no sensitivity analysis. The ability to compute accurate optimal parameter and constraint sensitivities with Lagrange duality is demonstrated in an example sensitivity analysis. Finally, the reliability of SP is illustrated by the convergence of a single aircraft MDO problem for eight unique objectives. The presented MDO architecture can be applied to any optimization problem where constraints can be either written or approximated in an explicit signomial form. Continued research into optimization via geometric and signomial programming will likely unearth additional unique capabilities and advantages of these methods.

## Appendix

### A. Fuselage Modifications

Modifications to the fuselage model in [5] were made to support double bubble fuselages in addition to conventional fuselages.

#### 1. Fuselage Terminology

$A_{\text{db}}$  = web cross-sectional area

$A_{\text{fuse}}$  = fuselage cross-sectional area

$I_{\text{h\_shell}}$  = shell horizontal bending inertia

$I_{\text{v\_shell}}$  = shell vertical bending inertia

$M_r$  = root moment per vertical tail root chord

$R_{\text{fuse}}$  = fuselage radius

$S_{\text{bulk}}$  = bulkhead surface area

$S_{\text{nose}}$  = nose surface area

$V_{\text{cone}}$  = cone skin volume

$V_{\text{db}}$  = web volume

$W_{\text{insul}}$  = insulation material weight

$W''_{\text{insul}}$  = weight/area density of insulation material

$W_{\text{shell}}$  = shell weight

$W_{\text{skin}}$  = skin weight

$W_{\text{web}}$  = web weight

$\Delta P_{\text{over}}$  = cabin overpressure

$\Delta R_{\text{fuse}}$  = fuselage extension height

$\lambda_{\text{cone}}$  = tailcone radius taper ratio

$\rho_{\text{skin}}$  = skin density

$\sigma_{\text{skin}}$  = max allowable skin stress

$\tau_{\text{cone}}$  = shear stress in tail cone

$\theta_{\text{db}}$  = double bubble fuselage joining angle

$c_{\text{root}_{\text{vt}}}$  = vertical tail root chord

$f_{\text{fadd}}$  = fractional added weight of local reinforcements

$f_{\text{frame}}$  = fractional frame weight

$f_{\text{string}}$  = fractional stringer weight

$h_{\text{db}}$  = web half-height

$h_{\text{fuse}}$  = fuselage height

$l_{\text{cone}}$  = cone length

$l_{\text{shell}}$  = shell length

$t_{\text{db}}$  = web thickness

$t_{\text{shell}}$  = shell thickness

$t_{\text{skin}}$  = skin thickness

$w_{\text{db}}$  = DB added half-width

$w_{\text{fuse}}$  = fuselage half-width

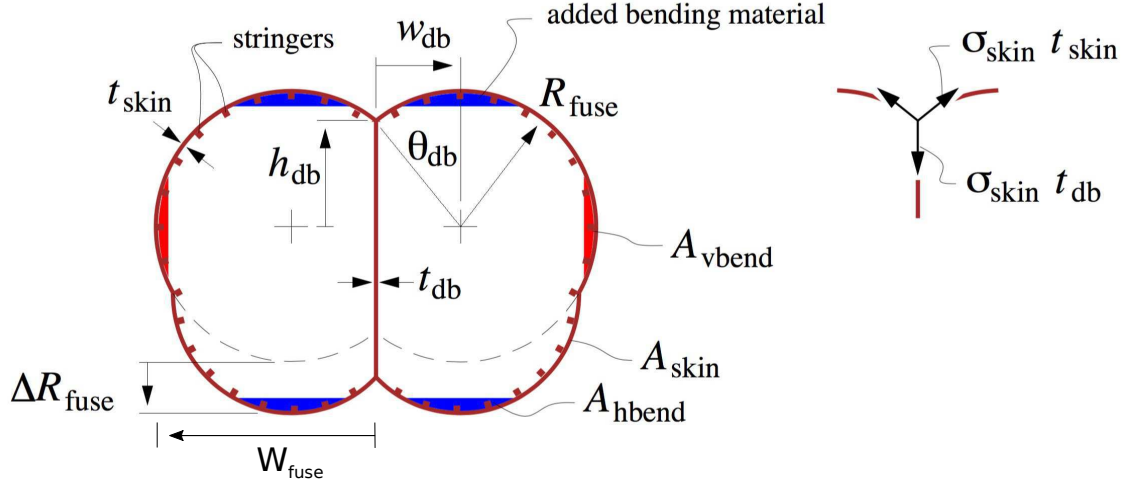


Fig. 10 Internal double bubble fuselage dimensions[7].

Figure 10 presents a cross sectional view of the double bubble fuselage. The added half-floor width due to the double bubble structure,  $w_{db}$ , is approximated with a first order Taylor expansion of the sine function.

$$\theta_{db} = \frac{w_{db}}{R_{fuse}} \quad (9)$$

A central tension web is added to account for the pressure forces in the fuselage center section. The web thickness depends on the internal pressure and the added floor half-width.

$$t_{db} = 2 \frac{\Delta P_{over} w_{db}}{\sigma_{skin}} \quad (10)$$

The half-height of the web is lower bounded with a second order Taylor expansion of the cosine function.

$$-0.5 R_{fuse} \theta_{db}^2 + R_{fuse} \leq h_{db} \quad (11)$$

The half-width of the fuselage is incremented by the half-width of the central fuselage section.



$$R_{\text{fuse}} + w_{\text{db}} \geq w_{\text{fuse}} \quad (12)$$

A fuselage extension height,  $\Delta R_{\text{fuse}}$ , augments the fuselage height and is constrained with a signomial equality constraint.

$$h_{\text{fuse}} = 0.5\Delta R_{\text{fuse}} + R_{\text{fuse}} \quad (13)$$

$\Delta R_{\text{fuse}}$  contributes to the shear web cross-sectional area and total material volume.

$$A_{\text{db}} \geq 2h_{\text{db}}t_{\text{db}} + \Delta R_{\text{fuse}}t_{\text{db}} \quad (14)$$

$$V_{\text{db}} = A_{\text{db}}l_{\text{shell}} \quad (15)$$

Web weight  $W_{\text{db}}$  is included in the total shell weight.

$$W_{\text{db}} = V_{\text{db}}\rho_{\text{skin}}g \quad (16)$$

$$W_{\text{shell}} \geq W_{\text{db}} + W_{\text{skin}} + W_{\text{skin}}f_{\text{add}} + W_{\text{skin}}f_{\text{frame}} + W_{\text{skin}}f_{\text{string}} \quad (17)$$

The skin cross sectional area, skin and bulkhead surface areas, and the tail cone volume are modified due to changing external geometry.

$$A_{\text{skin}} \geq 2\Delta R_{\text{fuse}}t_{\text{skin}} + 4R_{\text{fuse}}\theta_{\text{db}}t_{\text{skin}} + 2\pi R_{\text{fuse}}t_{\text{skin}} \quad (18)$$

$$S_{\text{nose}} \geq 4R_{\text{fuse}}^2\theta_{\text{db}} + 2\pi R_{\text{fuse}}^2 \quad (19)$$

$$S_{\text{bulk}} \geq 4R_{\text{fuse}}^2\theta_{\text{db}} + 2\pi R_{\text{fuse}}^2 \quad (20)$$

$$V_{\text{cone}} \geq \frac{M_r c_{\text{rootvt}}}{(1 + \lambda_{\text{cone}})\tau_{\text{cone}}} \frac{\pi + 2\theta_{\text{db}}}{\pi + 4\theta_{\text{db}}} \frac{l_{\text{cone}}}{R_{\text{fuse}}} \quad (21)$$

The cross-sectional area of the fuselage, used for the calculation of cabin volume, is lower bounded as follows.

$$A_{\text{fuse}} \geq -R_{\text{fuse}}^2 \theta_{\text{db}}^3 + 2R_{\text{fuse}} \Delta R_{\text{fuse}} + \pi R_{\text{fuse}}^2 + 4R_{\text{fuse}}^2 \theta_{\text{db}} \quad (22)$$

The insulation weight constraint is incremented due to the increased surface area of the fuselage.

$$W_{\text{insul}} \geq W_{\text{insul}}''(1.1\pi + 2\theta_{\text{db}})R_{\text{fuse}}l_{\text{shell}} + 0.55(S_{\text{nose}} + S_{\text{bulk}}) \quad (23)$$

The bending model, shown in Figure 11 and defined in [5], is modified due to the double bubble geometry and shear web, which provides additional bending reinforcement. These differences are captured in the bending area moments of inertia  $I_{\text{hshell}}$  and  $I_{\text{vshell}}$ .

$$I_{\text{hshell}} \leq \left[ (\pi + 4\theta_{\text{db}})R_{\text{fuse}}^2 + 8 \left( 1 - \frac{\theta_{\text{db}}^2}{2} \right) \left( \frac{\Delta R_{\text{fuse}}}{2} \right) R_{\text{fuse}} + \right. \quad (24)$$

$$\left. (2\pi + 4\theta_{\text{db}}) \left( \frac{\Delta R_{\text{fuse}}}{2} \right)^2 \right] R_{\text{fuse}} t_{\text{shell}} + \frac{2}{3} \left[ h_{\text{db}} + \frac{\Delta R_{\text{fuse}}}{2} \right]^3 t_{\text{db}}$$

$$I_{\text{vshell}} \leq \left[ \pi R_{\text{fuse}}^2 + 8w_{\text{db}} R_{\text{fuse}} + (2\pi + 4\theta_{\text{db}})w_{\text{db}}^2 \right] R_{\text{fuse}} t_{\text{shell}} \quad (25)$$

With the aforementioned modifications to the constraints from [5], the SP aircraft model can optimize both conventional tube and double bubble fuselages, with the fuselage joint angle parameter  $\theta_{\text{db}}$  adjusting the geometry.

## B. Boundary Layer Ingestion

A boundary layer ingestion (BLI) model is required to model the D8.2. The D8.2 engine configuration is illustrated in Figure 12. As noted by Hall et al.[20], BLI on the D8.2 results in a reduction in required propulsor mechanical power of 9 percent. Three percent of the power savings comes from reduced jet dissipation while the remainder comes from a roughly three percent increase in propulsive efficiency and decreased airframe dissipation.

### 1. BLI Terminology

$D$  = drag

$F$  = engine thrust

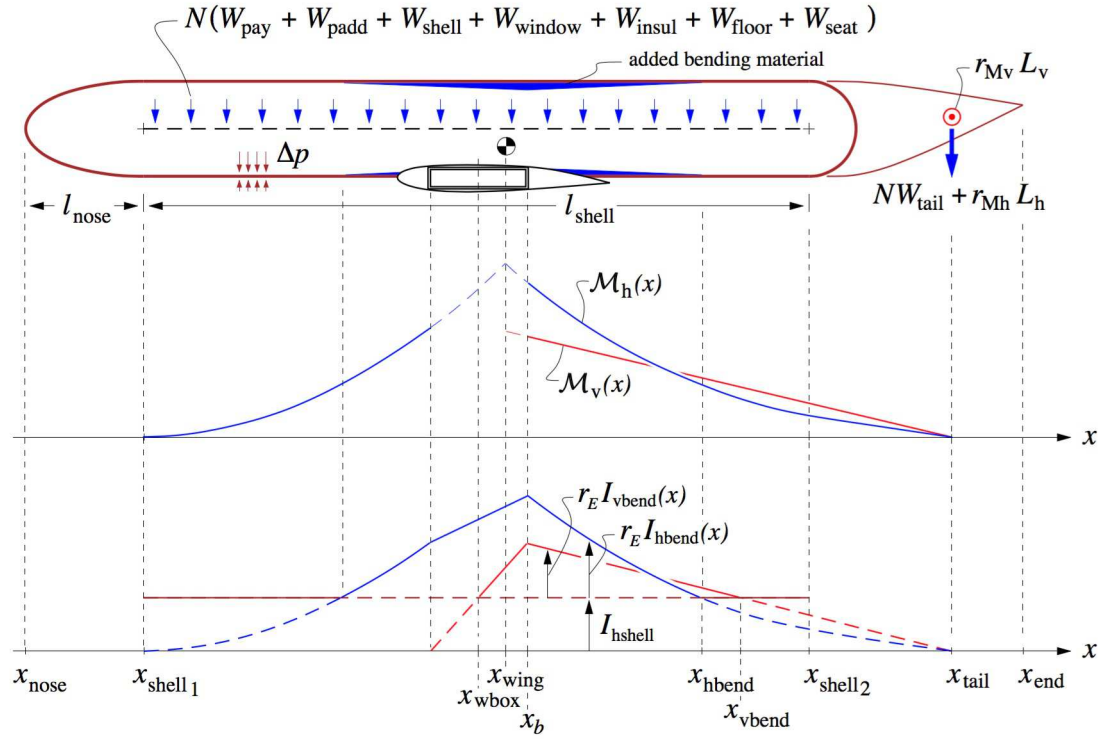


Fig. 11 TASOPT fuselage bending models[7]. The top graph shows the bending load distribution on the fuselage, whereas the bottom graph shows the area moment of inertia distribution.

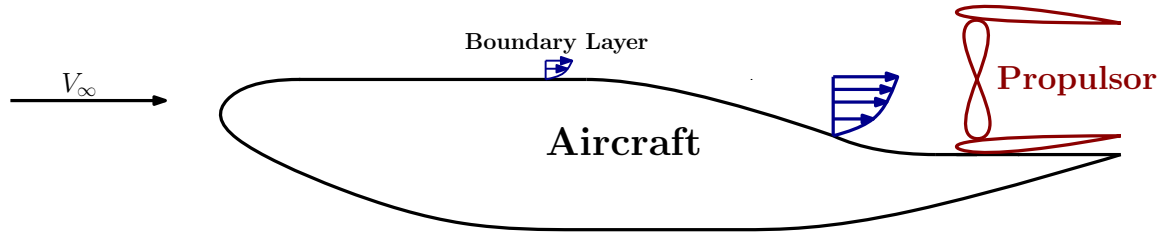


Fig. 12 Cartoon illustrating boundary layer growth on a BLI equipped aircraft similar to the D8.2.

$f_{\text{BLI}}$  = boundary layer ingestion fraction

$f_{\text{BLIP}}$  = BLI induced engine inlet stagnation pressure loss factor

$f_{\text{BLIV}}$  = BLI induced engine inlet velocity loss factor

$f_{\text{wake}}$  = wake dissipation fraction

$M_{\text{min}}$  = minimum cruise mach number

$P$  = pressure

$\Phi$  = dissipation rate

$\rho$  = density

$u$  = engine working fluid velocity

$(\cdot)_{\dots 0}$  = free stream quantity

$(\cdot)_{\dots 6}$  = core exhaust quantity

$(\cdot)_{\dots 8}$  = fan exhaust quantity

$(\cdot)_{\dots \text{atm}}$  = ambient atmospheric quantity

$(\cdot)_{\dots t}$  = stagnation quantity

## 2. Fuselage Dissipation Model

The reduction in jet dissipation is modeled with a drag reduction factor,  $\delta$ . Following Hall's[20] analysis, it is assumed the propulsor ingests 40 percent of the fuselage boundary layer ( $f_{\text{BLI}} = 0.4$ ). It is further assumed one third of total dissipation ( $\Phi$ ) is surface dissipation ( $\Phi_{\text{surf}}$ ). The wake dissipation fraction,  $f_{\text{wake}}$ , is defined by Equation 26 and assumed equal to 0.08. After noting  $\Phi = DV_{\infty}$ , this analysis yields Equations 27 and 28.

$$f_{\text{wake}} = \frac{\Phi_{\text{wake}}}{\Phi_{\text{wake}} + \Phi_{\text{surf}}} \quad (26)$$

$$\delta = f_{\text{BLI}} * 0.33 * 0.08 \quad (27)$$

$$D_{\text{total}} = \delta(D_{\text{induced}} + D_{\text{airframe}}) \quad (28)$$

## 3. Engine Boundary Layer Ingestion

BLI engines ingest air with lower average velocity, and in turn lower stagnation pressure, than free stream air. Three constraints from [6] were modified to account for BLI. Engine inlet stagnation pressure was reduced by the factor  $f_{\text{BLIP}}$ . Note  $f_{\text{BLIP}}$  represents the average drop in stagnation pressure across the entire inlet. Following [6],  $Z_0$  replaces the non-GP compatible expression  $1 + \frac{\gamma-1}{2}(M_0)^2$  in stagnation relations.

$$P_{t_0} = f_{\text{BLIP}} P_{\text{atm}} Z_0^{3.5} \quad (29)$$

Thrust is equal to the working fluid's rate of momentum change. The factor  $f_{\text{BLIV}}$  was introduced to fan and core thrust constraints to account for the decrease in average free stream velocity. Again,  $f_{\text{BLIV}}$  is the average velocity drop across the entire fan.

$$\frac{F_8}{\alpha \dot{m}_{\text{core}}} + f_{\text{BLIV}} u_0 \leq u_8 \quad (30)$$

$$\frac{F_6}{f_o \dot{m}_{\text{core}}} + f_{\text{BLIV}} u_0 \leq u_6 \quad (31)$$

Determining  $f_{\text{BLIP}}$  and  $f_{\text{BLIV}}$  can be difficult. As of now, there are no GP or SP compatible boundary layer models so either  $f_{\text{BLIP}}$  or  $f_{\text{BLIV}}$  must be estimated. Using the experimental results presented by Hall et al.[21]  $f_{\text{BLIV}}$  was estimated to be 0.0727.  $f_{\text{BLIP}}$  was then determined using Equation 32.

$$f_{\text{BLIP}} = \frac{P_{\text{atm}} + \rho_{\text{atm}}(f_{\text{BLIV}} M_{\text{min}} a)^2}{P_{\text{atm}} + \rho_{\text{atm}}(M_{\text{min}} a)^2} \quad (32)$$

Finally, it is important to note BLI fan distortion effects will decrease fan efficiency to approximately 90%[22].

### C. Horizontal Tail Structural Model Modifications

An update to the structural model in [5] was required to accurately model the bending and shear loads on horizontal pi-tails. This section derives and presents a new set of constraints, which are compatible with both conventional tail and pi-tail architectures.

#### 1. Assumptions

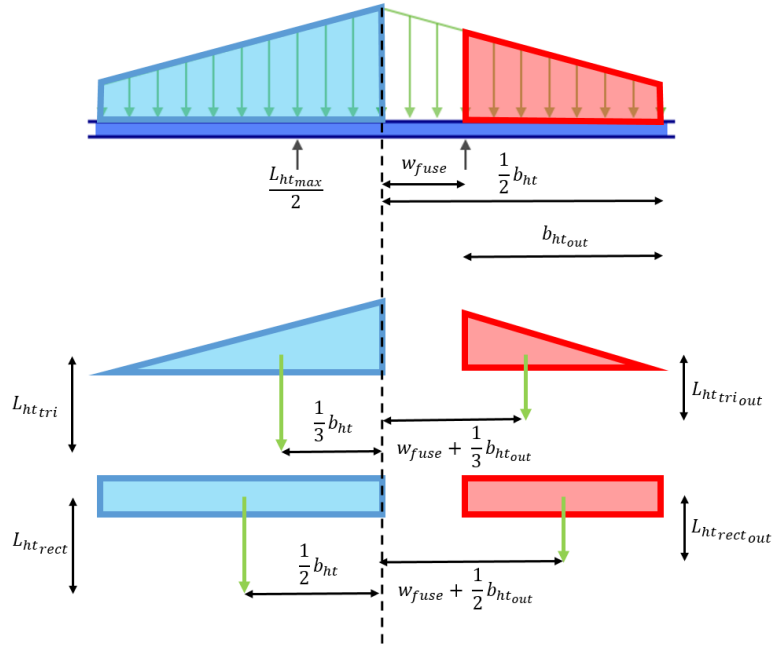
1. The lift per unit span is proportional to local chord.
2. The horizontal tail has a constant taper ratio.
3. The horizontal and vertical tail joint is a fuselage width away from the centerline of the aircraft.

4. The horizontal and vertical tail interface is a pin joint. Therefore, the joint does not exert a moment on the horizontal tail.
5. The shear and moment distributions on the horizontal tail are linearized.

The pin-joint assumption ensures the vertical tail structural constraints do not need to be modified for the pi-tail configuration.

## 2. Sample Free Body Diagram and Load Distributions

With the aforementioned assumptions, the free body diagram diagram of the pi-tail is shown at the top of Figure 13.



**Fig. 13** Free body diagram of the forces on the horizontal tail. The distributed lift force, which is assumed to be proportional to local chord, is partitioned into triangular and rectangular components.

Shear and moment diagrams are presented in Figures 14 and 15 respectively. The diagrams include both the distributed lift loads (green arrows in Figure 13) and the point loads of imposed on the pin joints by the vertical tails.

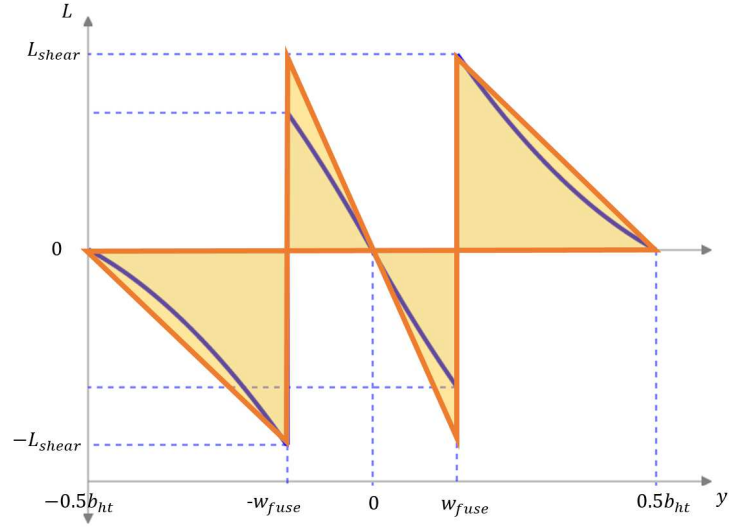


Fig. 14 Shear diagram of the pi-tail. The curved lines show the actual loading, while the straight lines show the conservative assumed load distribution.

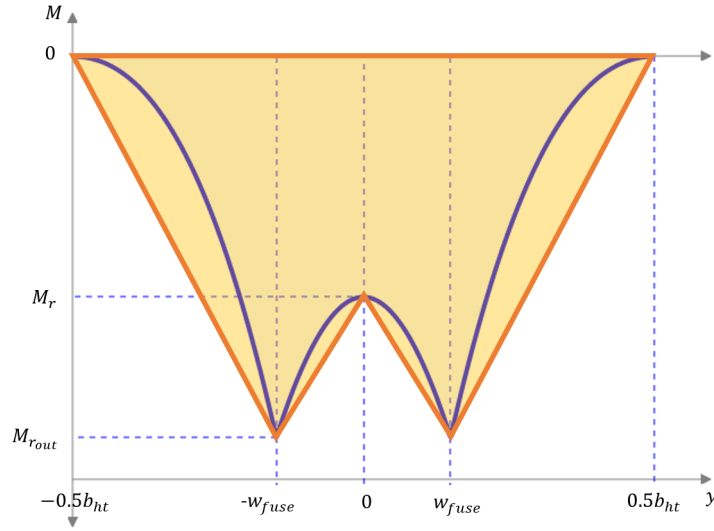


Fig. 15 Moment diagram of the pi-tail. The curved lines show the actual loading, while the straight lines show the conservative assumed load distribution.

### 3. Horizontal Tail Terminology

$I_{\text{cap}}$  = non-dimensional spar cap area moment of inertia

$L_{\text{ht}}$  = horizontal tail downforce

$L_{\text{ht}_{\text{max}}}$  = maximum horizontal tail downforce

$L_{\text{ht}_{\text{rect}}}$  = rectangular horizontal tail load

$L_{\text{htrectout}}$  = rectangular horizontal tail load outboard

$L_{\text{httri}}$  = triangular horizontal tail load

$L_{\text{httriout}}$  = triangular horizontal tail load outboard

$L_{\text{shear}}$  = maximum shear load at pin-joint

$M_{\text{r}}$  = moment per chord at horizontal tail root

$M_{\text{rout}}$  = moment per chord at pin-joint

$N_{\text{lift}}$  = horizontal tail loading multiplier

$S_{\text{ht}}$  = horizontal tail area

$W_{\text{cap}}$  = weight of spar caps

$W_{\text{struct}}$  = horizontal tail wingbox weight

$W_{\text{web}}$  = weight of shear web

$\lambda_{\text{ht}}$  = horizontal tail taper ratio

$\nu$  = dummy variable =  $(t^2 + t + 1)/(t + 1)^2$

$\pi_{\text{M-fac}}$  = pi-tail bending structural factor

$\rho_{\text{cap}}$  = density of spar cap material

$\rho_{\text{web}}$  = density of shear web material

$\sigma_{\text{max,shear}}$  = allowable shear stress

$\sigma_{\text{max}}$  = allowable tensile stress

$\tau_{\text{ht}}$  = horizontal tail thickness/chord ratio

$b_{\text{ht}}$  = horizontal tail span

$b_{\text{htout}}$  = horizontal tail outboard half-span

$c_{\text{attach}}$  = horizontal tail chord at the pin-joint

$c_{\text{rootht}}$  = horizontal tail root chord

$c_{\text{tipht}}$  = horizontal tail tip chord

$g$  = gravitational acceleration

$q_{\text{ht}}$  = substituted variable =  $1 + \text{taper}$

$r_h$  = fractional wing thickness at spar web

$t_{\text{cap}}$  = non-dim. spar cap thickness



$t_{\text{web}}$  = non-dim. shear web thickness

$w$  = wingbox width-to-chord ratio

$w_{\text{fuse}}$  = fuselage half-width

#### 4. Load Derivation

$L_{\text{htrect}}$  is defined to be half the lift generated by the ‘rectangular’ section of the wing (the rectangle in the left half of Figure 13).

$$L_{\text{htrect}} \geq \frac{L_{\text{htmax}} c_{\text{tipht}} b_{\text{ht}}}{2S_{\text{ht}}} \quad (33)$$

Similarly,  $L_{\text{httri}}$  is defined to be half the lift generated by the ‘triangular’ section of the wing (the triangle in the left half of Figure 13).

$$L_{\text{httri}} \geq \frac{L_{\text{htmax}} (1 - \lambda_{\text{ht}}) c_{\text{rootht}} b_{\text{ht}}}{4S_{\text{ht}}} \quad (34)$$

After defining the horizontal tail half-span outboard of the pin joint ( $b_{\text{htout}}$ ), the outboard components of the lift loads can be computed with respect to  $L_{\text{htrect}}$  and  $L_{\text{httri}}$ . The outboard loads are shown in the right half of Figure 13.

$$b_{\text{htout}} \geq 0.5b_{\text{ht}} - w_{\text{fuse}} \quad (35)$$

$$L_{\text{httriout}} \geq L_{\text{httri}} \frac{b_{\text{htout}}}{(0.5b_{\text{ht}})^2} \quad (36)$$

$$L_{\text{htrectout}} \geq L_{\text{htrect}} \frac{b_{\text{htout}}}{0.5b_{\text{ht}}} \quad (37)$$

The horizontal-vertical tail pin joint is assumed to be exactly at  $w_{\text{fuse}}$ . This is a conservative estimate. In most pi-tail configurations the vertical tails are canted outwards. The local chord at the pin joint is constrained with the following monomial equality.

$$c_{\text{attach}} = \frac{b_{\text{ht}} \lambda_{\text{ht}} c_{\text{rootht}}}{2w_{\text{fuse}}} \quad (38)$$

The maximum moment at the joint is determined by summing the bending moment contributions from loads outboard of the joint.

$$M_{\text{rout}} c_{\text{attach}} \geq L_{\text{ht}_{\text{rect}_{\text{out}}}} \frac{1}{2} b_{\text{ht}_{\text{out}}} + L_{\text{ht}_{\text{tri}_{\text{out}}}} \frac{1}{3} b_{\text{ht}_{\text{out}}}, \quad (39)$$

The maximum shear at the joint is the sum of the outboard shear loads. The maximum root moment is the sum of the bending loads from lift and the pin-joint load.

$$L_{\text{shear}} \geq L_{\text{ht}_{\text{rect}_{\text{out}}}} + L_{\text{ht}_{\text{tri}_{\text{out}}}} \quad (40)$$

$$M_{\text{r}} c_{\text{root}_{\text{ht}}} \geq L_{\text{ht}_{\text{rect}}} \frac{1}{4} b_{\text{ht}} + L_{\text{ht}_{\text{tri}}} \frac{1}{6} b_{\text{ht}} - \frac{1}{2} L_{\text{ht}_{\text{max}}} w_{\text{fuse}} \quad (41)$$

Finally, the wingtip moment is set equal to zero with a signomial equality constraint.

$$\frac{b_{\text{ht}}}{4} L_{\text{ht}_{\text{rect}}} + \frac{b_{\text{ht}}}{3} L_{\text{ht}_{\text{tri}}} = b_{\text{ht}_{\text{out}}} \frac{L_{\text{ht}_{\text{max}}}}{2} \quad (42)$$

### 5. Structural Sizing

Equations from [2] for wing structural sizing were adapted using a linearization of the moment and shear load distributions from Appendix C2. The constraints can be applied to both conventional and pi-tails.

$$0.92 w \tau_{\text{ht}} t_{\text{cap}}^2 + I_{\text{cap}} \leq \frac{0.92^2}{2} w \tau_{\text{ht}}^2 t_{\text{cap}} \quad (43)$$

$$8 \geq N_{\text{lift}} M_{\text{rout}} (AR_{\text{ht}}) q_{\text{ht}}^2 \frac{\tau_{\text{ht}}}{S_{\text{ht}} I_{\text{cap}} \sigma_{\text{max}}} \quad (44)$$

$$12 \geq \frac{2 L_{\text{shear}} N_{\text{lift}} q^2}{\tau_{\text{ht}} S t_{\text{web}} \sigma_{\text{max-shear}}} \quad (45)$$

The changes to the model in [2] are:

- In the shear constraint replacing  $L_{\text{ht}_{\text{max}}}$  with  $2L_{\text{shear}}$ . This is done because the shear loads for the pi-tail are different than the maximum lift loads for the conventional tail.
- Replacing  $M_{\text{r}}$  with  $M_{\text{rout}}$ , the moment per unit chord at the pin joint. For a pi-tail, maximum bending loads occur at the pin joint.

The linearization of the shear and bending load distributions simplifies the derivation of the structural web and cap weights. Shear web sizing relies on the assumption that the maximum shear ( $L_{\text{shear}}$ ) occurs at the pin-joint and the weight of the shear web of the pi-tail under  $L_{\text{shear}}$  is equal to the shear web weight of a conventional tail subjected to the the same maximum shear load at its root. This is a conservative approximation, the load distribution implied by this assumption (shown in yellow in Figure 14) has a larger internal area than the actual load distribution. Intuitively, the  $L_{\text{shear}}$  for a pi-tail is strictly smaller than the  $L_{\text{shear}}$  a conventional tail of the same size and loading. The pi-tail more efficient in shear.

The cap weight of the pi-tail is determined by scaling the cap weight of a conventional tail with the same geometry as the pi-tail and a root moment of  $M_{\text{rout}} c_{\text{attach}}$ . The scaling factor,  $\pi_{\text{M-fac}}$ , is the ratio of the total shaded bending moment area in Figure 15 to the sum of the outboard shaded areas multiplied by the ratio of the outboard half-span to the total half-span.

$$\pi_{\text{M-fac}} \geq \left[ \frac{\frac{1}{2}(M_{\text{rout}} c_{\text{attach}} + M_{\text{r}} c_{\text{rootht}}) w_{\text{fuse}}}{\frac{1}{2} M_{\text{rout}} c_{\text{attach}} b_{\text{htout}}} + 1.0 \right] \frac{b_{\text{htout}}}{0.5 b_{\text{ht}}}, \quad (46)$$

Given the calculated loads and structural factors, the bending material and shear web weight can be calculated.

$$W_{\text{cap}} \geq \frac{\pi_{\text{M-fac}} 8 \rho_{\text{cap}} g w t_{\text{cap}} S_{\text{ht}}^{1.5} \nu}{3 A R_{\text{ht}}^{0.5}} \quad (47)$$

$$W_{\text{web}} \geq \frac{8 \rho_{\text{web}} g r_h \tau_{\text{ht}} t_{\text{web}} S_{\text{ht}}^{1.5} \nu}{3 A R_{\text{ht}}^{0.5}} \quad (48)$$

$$W_{\text{struct}} \geq W_{\text{web}} + W_{\text{cap}} \quad (49)$$

The value for  $t_{\text{cap}}$  is notional in the derivation above. Rather than being the spar cap thickness of a pi-tail, it is the spar cap thickness required for a conventional tail of the same geometry and a root moment ( $M_{\text{rout}} c_{\text{attach}}$ ) as a pi-tail. With a similar reasoning as for the shear loads,  $\pi_{\text{M-fac}} t_{\text{cap}}$  for a pi-tail is strictly smaller than the  $t_{\text{cap}}$  for a conventional tail of the same geometry and loading, making the pi-tail more efficient in bending than a traditional tail.

## D. Mission Profile

The mission profile includes weight, drag, and altitude build up constraints as well as a series of aircraft performance constraints. The mission profile can be discretized into an arbitrary number of climb and cruise segments. The profile allows for the possibility of a cruise climb. The descent portion of the flight was neglected due to the small percent of mission time and fuel burn it encompasses. Neglecting descent results in a slight over-estimation of total mission fuel burn.

### 1. Mission Profile Terminology

$a$  = speed of sound

$D$  = total aircraft drag

$D_{\text{components}}$  = drag on aircraft subsystems

$D_{\text{induced}}$  = induced drag

$\Delta h$  = altitude change

$F$  = engine thrust

$f_{\text{fuel}_{\text{res}}}$  = reserve fuel fraction

$h_{\text{cruise,min}}$  = minimum cruise altitude

$L$  = sum of wing and fuselage lift

$L_{\text{ht}}$  = horizontal tail down force

$\frac{L}{D}$  = Aircraft lift to drag ratio

$M$  = mach number

$M_{\text{min}}$  = minimum cruise mach number

$N_{\text{eng}}$  = aircraft's number of engines

$\theta$  = climb angle

$h$  = altitude

$P_{\text{excess}}$  = excess power

$R$  = downrange distance covered

$R_{\text{req}}$  = total required range

$\text{RC}$  = rate of climb

$t$  = flight segment duration

$t_{\text{climb,max}}$  = max allowed time to climb

TSFC = thrust specific fuel consumption

$V$  = aircraft speed

$W$  = takeoff weight

$W_{\text{buoy}}$  = buoyancy force

$W_{\text{avg}}$  = average flight segment aircraft weight

$W_{\text{dry}}$  = aircraft dry weight

$W_{\text{end}}$  = aircraft flight segment end weight

$W_{\text{engine}}$  = engine weight

$W_{\text{fuel}}$  = flight segment fuel weight burned

$W_{\text{fuse}}$  = fuselage weight

$W_{\text{ht}}$  = horizontal tail weight

$W_{\text{lg}}$  = landing gear weight

$W_{\text{misc}}$  = miscellaneous system weight

$W_{\text{payload}}$  = payload weight

$W_{f_{\text{primary}}}$  = total fuel weight less reserves

$W_{\text{fuel}_{\text{total}}}$  = total fuel weight

$W_{\text{start}}$  = aircraft flight segment start weight

$W_{\text{vt}}$  = vertical tail weight

$W_{\text{wing}}$  = wing weight

$(\cdot)_{0..i..N}$  = flight segment  $i$  quantity

## 2. *Weight and Drag Build Ups*

Downward optimization pressure on weight and drag allows basic posynomial weight and drag build ups to be used.

$$D_i \geq \sum D_{\text{components}_i} + D_{\text{induced}_i} \quad (50)$$

$$(51)$$

$W_{\text{avg}_i}$  is the geometric mean of a segments start and end weight. Average weight is used instead of either the segment start or end weight. This increases model accuracy and stability.

$$W_{\text{dry}} \geq W_{\text{wing}} + W_{\text{fuse}} + W_{\text{vt}} + W_{\text{ht}} + W_{\text{lg}} + W_{\text{eng}} + W_{\text{misc}} \quad (52)$$

$$\sum_{i=1}^N W_{\text{fuel}_i} \leq W_{f_{\text{primary}}} \quad (53)$$

$$W \geq W_{\text{dry}} + W_{\text{payload}} + f_{\text{fuel}_{\text{res}}} W_{f_{\text{primary}}} \quad (54)$$

$$W_{\text{start}_i} \geq W_{\text{end}_i} + \sum_{n=1}^i W_{\text{fuel}_n} \quad (55)$$

$$W_{\text{start}_0} = W \quad (56)$$

$$W_{\text{end}_N} \geq W_{\text{dry}} + W_{\text{payload}} + f_{\text{fuel}_{\text{res}}} W_{f_{\text{primary}}} \quad (57)$$

$$W_{\text{start}_{i+1}} = W_{\text{end}_i} \quad (58)$$

$$W_{\text{avg}_i} \geq \sqrt{W_{\text{start}_i} W_{\text{end}_i}} + W_{\text{buoy}_i} \quad (59)$$

$$(60)$$

### 3. General Performance Constraints

The sum of segment ranges is constrained to be greater than or equal to the required range.

$$\sum_{i=1}^N R_i \geq R_{\text{req}} \quad (61)$$

$$(62)$$

Segment fuel burn is a function of TSFC, thrust, and segment flight time.

$$W_{\text{fuel}_i} = N_{\text{eng}} \text{TSFC}_i t_i F_i \quad (63)$$

Altitude change during each segment is a function of climb rate and total segment time. Equation 65 uses a small angle approximation to compute the downrange distance covered during each segment.

$$\Delta h_i = t_i \text{RC}_i \quad (64)$$

$$t_i V_i = \text{Range}_i \quad (65)$$

Standard lift to drag and Mach number definitions are used.

$$M_i = \frac{V_i}{a_i} \quad (66)$$

$$\left(\frac{L}{D}\right)_i = \frac{W_{avg_i}}{D_i} \quad (67)$$

#### 4. *Climb Performance Constraints*

Climb rates are computed with an excess power formulation[23]. During the first climb segment, the climb rate is constrained to be greater than 2,500 ft/min. For all remaining climb segments, the climb rate is constrained to be greater than 500 ft/min. The climb angle,  $\theta$ , is set using a small angle approximation.

$$P_{\text{excess}} + V_i D_i \leq V_i N_{\text{eng}} F_i \quad (68)$$

$$\text{RC}_i = \frac{P_{\text{excess}}}{W_{\text{avg}_i}} \quad (69)$$

$$\theta_i V_i = \text{RC}_i \quad (70)$$

There can be either an upward or downward pressure on  $h$ . Thus, a signomial equality constraint must be used to constrain altitude.

$$h_i = h_{i-1} + \Delta h_i \quad (71)$$

In the above formulation,  $h_i$  is equivalent to segment end altitude.

$$h_0 = \Delta h_0 \quad (72)$$

Climb segments are constrained to have equal altitude changes and the final climb segment altitude is constrained to be greater than a user specified minimum cruise altitude. If no minimum cruise altitude is specified,  $h_{\text{cruise,min}}$  is optimized.

$$\Delta h_{i+1} = \Delta h_i \quad (73)$$

$$h_{N_{\text{climb}}} \geq h_{\text{cruise,min}} \quad (74)$$

Time to climb is constrained to be less than a user specified maximum value. If no maximum value is specified,  $t_{\text{climb,max}}$  is optimized.

$$\sum_0^{N_{\text{climb}}} t_i \leq t_{\text{climb,max}} \quad (75)$$

Finally, the climb gradient at top of climb is constrained to be greater than 0.015 radians.

$$\theta_{N_{\text{climb}}} \geq 0.015 \quad (76)$$

##### 5. Cruise Performance Constraints

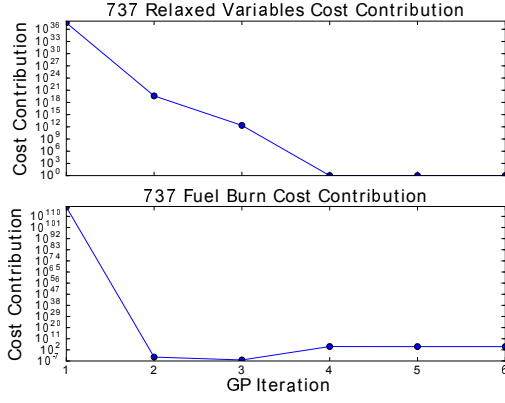
Cruise range segments are constrained to be equal length. Cruise Mach number is constrained to be greater than a user specified minimum. If no minimum is specified,  $M_{\text{min}}$  is optimized.

$$R_{i+1} = R_i \quad (77)$$

$$M_i \geq M_{\text{min}} \quad (78)$$

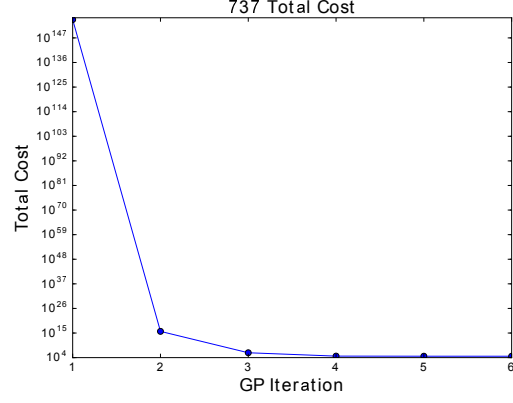
The cruise climb angle is assumed to be small. The sum of wing and fuselage lift is set equal to weight plus horizontal tail down force. Thrust must overcome both drag and the portion of aircraft





(a) 737 fuel and relaxed variable cost versus

iteration.



(b) 737 total cost versus iteration.

**Fig. 16 Cost evolution during solution of the 737 model.**

weight acting in the direction of thrust. These constraints are a conservative approximation of flight physics.

$$L_i \geq W_{\text{avg}_i} + L_{\text{ht}_i} \quad (79)$$

$$N_{\text{eng}} F_i \geq D_i + W_{\text{avg}_i} \theta_i \quad (80)$$

In cruise, there is a downward pressure on segment end altitude, removing the need for a signomial equality.

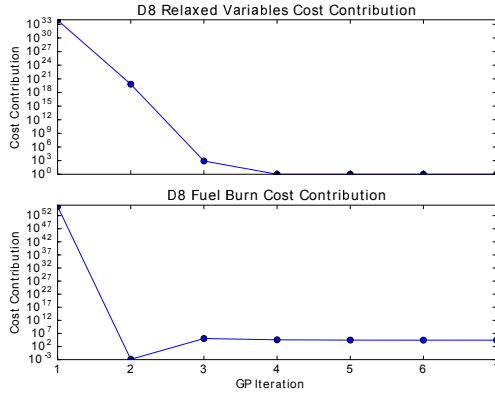
$$h_i \geq h_{i-1} + \Delta h_i \quad (81)$$

### E. Model Convergence Plots

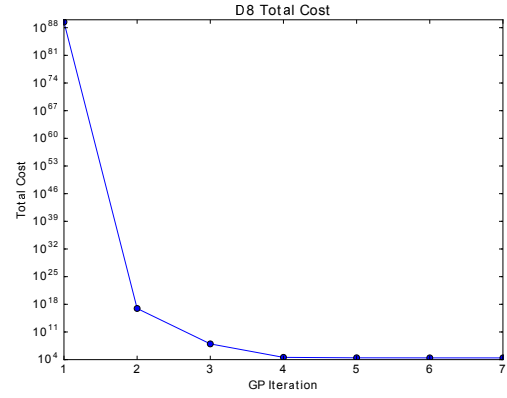
All models were solved using the relaxed constants penalty function solution heuristic from [5]. Consequently, the total cost can be decomposed into a contribution from relaxed variables and aircraft fuel burn. Figures 16 to 18 present the optimization history for each model. As expected, the cost contribution of relaxed variables is one before the final iteration.

### Acknowledgments

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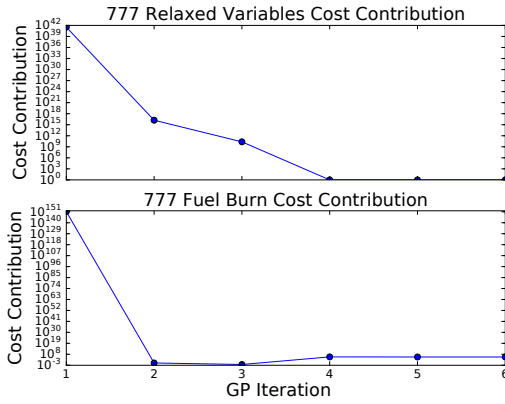


(a) D8.2 fuel and relaxed variable cost versus iteration.

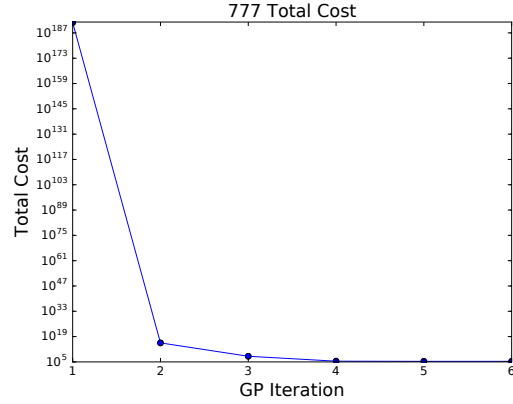


(b) D8.2 total cost versus iteration.

**Fig. 17 Cost evolution during solution of the D8.2 model.**



(a) 777 fuel and relaxed variable cost versus iteration.



(b) 777 total cost versus iteration.

**Fig. 18 Cost evolution during solution of the 777 model.**

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