Numeric, Non-Linear, Lifting Line Theory

These notes detail the iterative Prandtl lifting line theory method described by Anderson in *Fundamentals of Aerodynamics* Section 5.4. Prandtl lifting line theory enables the calculation of 3-dimensional lift distributions over a finite wing by utilizing the analytical concept of a distribution of horseshoe vortices to produce the downwash effect observed in real 3D flow that is induced by the wingtip vortices of the real wing. Prandtl's original method is a closed-form solution, which uses thin airfoil theory to model the sectional (properties at a given wingspan location) lift distribution along the wing.

Since Prandtl's time, more complex and accurate methods of modeling airfoil performance have been developed, such as vortex panel method, giving us the potential to more accurately model 3D wing performance characteristics, including viscous stall. Added complexity, however, means a closed-form solution like Prandtl's is no longer possible. Instead, an iterative method is adopted, where solutions start from an initial guess and then are gradually updated according to the equations of lifting line theory and Kutta-Joukowski theorem until a final, unchanging solution is converged. This method will now be detailed step-by-step in the following.

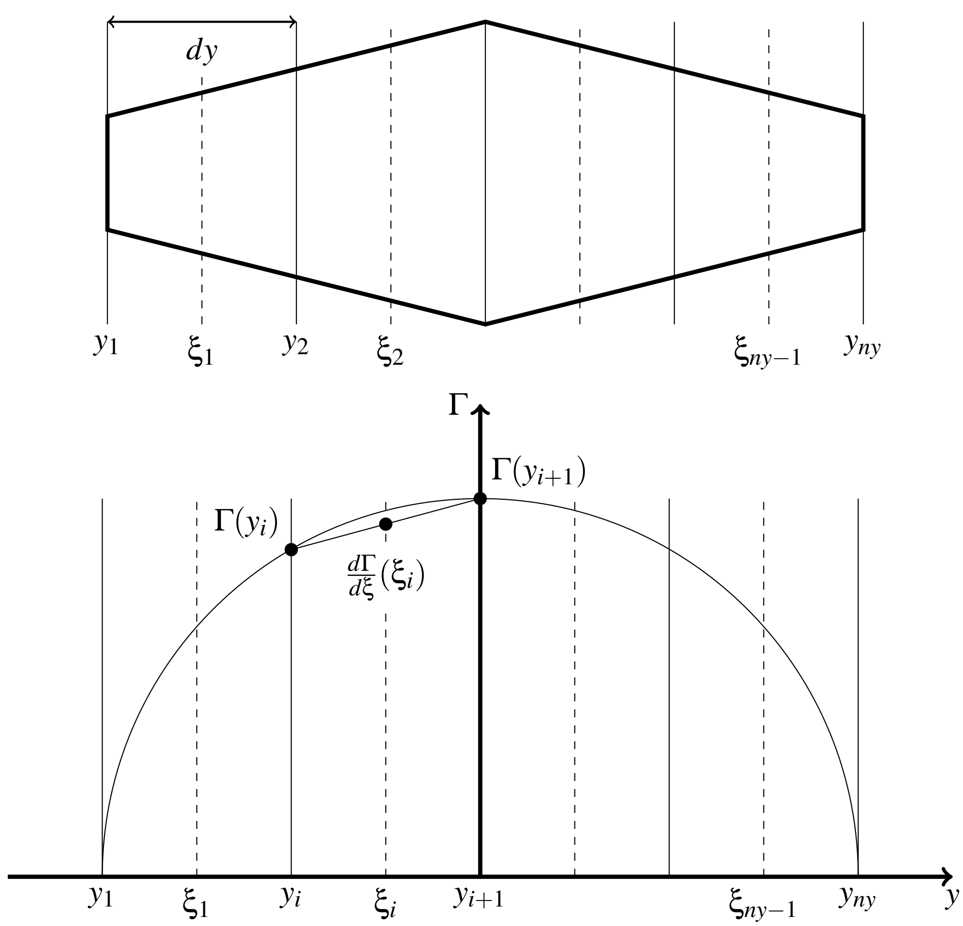

Fig. 1: Wing discretized into y and ξ (dummy variable) stations (Top). Wing circulation distribution and finite difference calculation of $\frac{d\Gamma}{d\xi}$ (Bottom)

1 Discretize The Wing

All numeric solutions begin with a grid. For this method, the wing will be discretized into ny points where aerodynamic properties such as induced angle of attack α_i and section circulation $\Gamma(y)$ will be calculated (see Fig. 1). To avoid singularities in the numeric integration (detailed in the following sections), a dummy variable of integration ξ is introduced, with each ξ point set to be half-way between each y point and with the total number of ξ points $(ny-1)$ being one less than of y (ny). Pseudocode for creating the dummy variable is included below:

for i in range(0, ny-1):
  $\xi[i] = y[i] + \frac{dy}{2}$

where dy is the spacing between y points (and $dy = d\xi$).

2 Calculate α_{eff} (Prandtl Lifting Line Theory)

Once a circulation distribution has been set, the induced angle of attack α_i due to downwash at every section can be calculated using Prandtl lifting line theory (Eqn 1).

$$\alpha_i(y_n) = \frac{1}{4\pi V_\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{d\Gamma}{dy}}{y_n - y} dy \tag{1}$$

where y_n is the location at which α_i is being calculated.

Observing Eqn 1, it can be seen that a singularity will occur in the integral whenever $y = y_n$. This can be avoided by integrating against a dummy variable ξ, which is on the same axis as y, but occupies points offset from y such that y will never coincide with ξ (defined in the previous section). In a discretized system such as Fig 1, this means that the finite set of points that make up y and ξ will both be distributed along the y-axis (wingspan) but the points of each set will never overlap the other. Substituting ξ as the variable of integration in Eqn 1 yields Eqn 2 below:

$$\alpha_i(y_n) = \frac{1}{4\pi V_\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{d\Gamma}{d\xi}}{y_n - \xi} d\xi \tag{2}$$

The derivative in the numerator of the integral may be calculated using standard finite differencing methods. In the second part of Fig 1, we see that a first-order, forward difference of Γ in the y-direction will yield the approximation of the first derivative at the location halfway between the two y points, corresponding exactly to ξ. Thus, it can be shown that a forward difference in y is equivalent to a central difference in ξ, which is depicted in Eqn 3.

$$\left(\frac{d\Gamma}{d\xi}\right)_j = \frac{\Gamma(y_{j+1}) - \Gamma(y_j)}{\Delta\xi} \tag{3}$$

where $\Delta\xi$ is the distance between ξ points and is a constant.

Pseudocode in Python for calculating the induced angle of attack is as follows:

```
#evaluate α_i at all spanwise y locations
for n in range(0, ny):
    #only evaluate ξ inside the wingspan
    for j in range(0, ny-1):
        integral[j] = (dΓ/dξ[j]) / (y[n] − ξ[j])
    α_i[n] = 1/(4πV_∞) * np.trapz(integral, ξ)
```

After calculating induced angle of attack α_i, the effective angle of attack α_{eff} at each station may be calculated as the difference of the geometric angle of attack α_{geom} of the wing and α_i. Additionally, if the wing has any geometric twist t (downward angle of the tip section with respect to the root section, also known as washout), its influence on α_{eff} may be included as shown in Eqn 4.

$$\alpha_{eff}(y) = \alpha_{geom} - \alpha_i(y) - t(y) \tag{4}$$

3 Sectional Lift Distribution

Now that the effective angle of attack α_{eff} of each wing section is known, we can incorporate our 2D airfoil data into the calculation. Lift curve data ($C_l\ vs\ \alpha$) can be obtained via computational methods like XFOIL vortex panel method or through experimental methods like wind tunnel testing. **Be sure to obtain data for positive *and negative* α as well as data through and after stall to accurately model this behavior on the 3D wing.** With this data, a sectional lift coefficient $C_l(y)$ for each wing station y may be calculated by linear interpolation (i.e. numpy interp) of the lift curve data for each sectional effective angle of attack $\alpha_{eff}(y)$.

4 Calculate New Circulation Distribution

Via the Kutta-Joukowski theorem, a new circulation distribution $\Gamma_{new}(y)$ can be calculated. Substituting the definition of coefficient of lift C_l for lift force L' in the KJ theorem results in Eqn 5.

$$\Gamma_{new}(y) = \frac{1}{2} V_\infty c(y) C_l(y) \tag{5}$$

From Eqn 5, it can be seen that Γ_{new} is a function of the wing's chord distribution as well as its lift distribution, making this method capable of simulating 3D effects over wings of *any* planform geometry.

5 Damp Iteration for Stability

Since this is an iterative method, it is beneficial to damp the convergence of the solution with Successive Under Relaxation (SUR), as shown in Eqn 6.

$$\Gamma_{new} = \Gamma_{old} + D\left(\Gamma_{new} - \Gamma_{old}\right) \tag{6}$$

where $D \approx 0.05$ is some damping factor used to control the rate of convergence. Stop iteration when solution is converged (usually 50-300 iterations). Convergence can be tested after each iteration but before SUR by calculating the residual as:

residual = max(abs($\Gamma_{new} - \Gamma old$))

A diagram depicting the iterative process can be found in Fig 2.

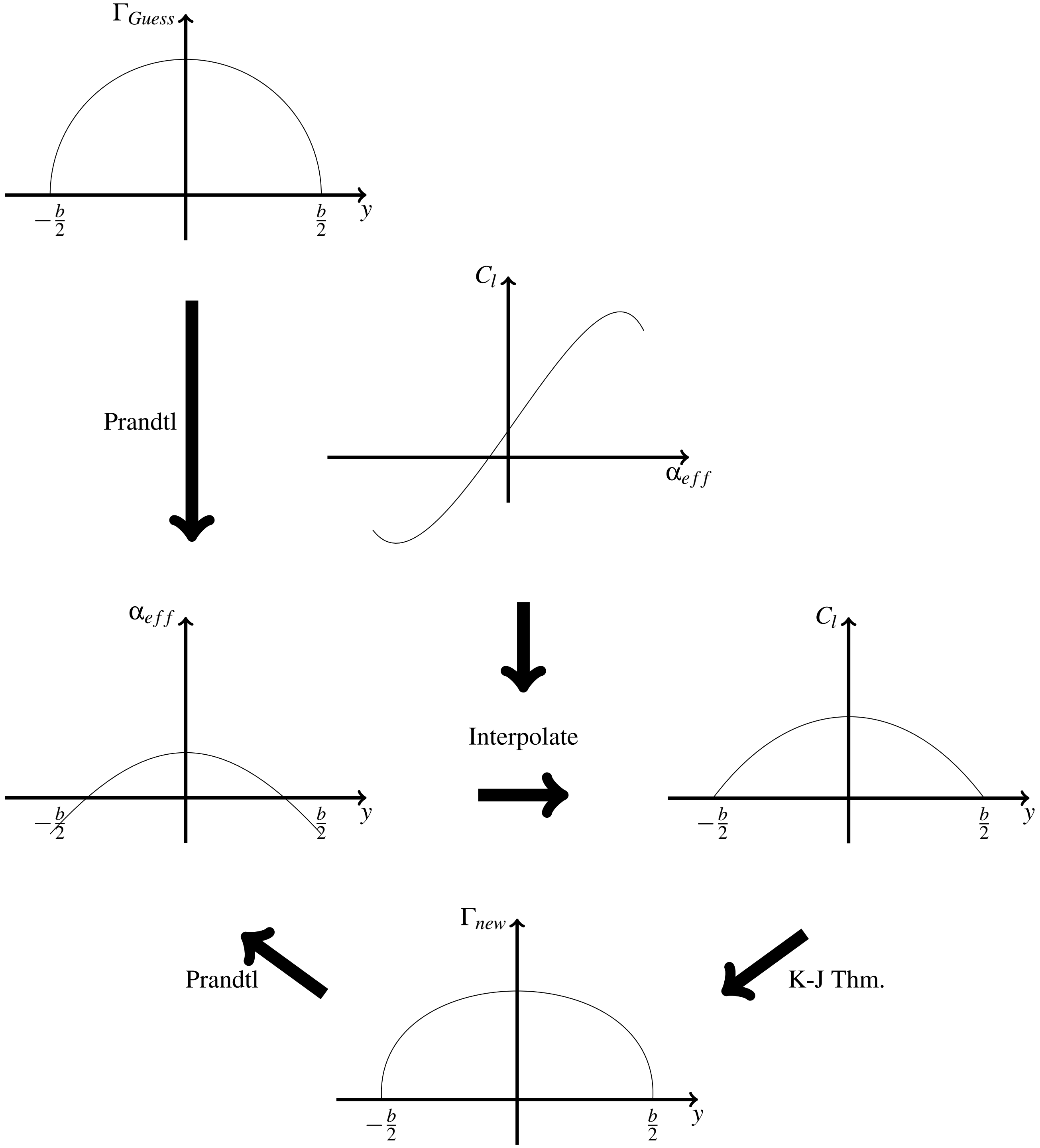

Fig. 2: Numeric lifting line iterative procedure: Guess circulation distribution, calculate α_{eff} through Prandtl lifting line theory, interpolate C_l distribution using XFOIL sectional data, calculate new circulation distribution via Kutta-Joukowski Theorem, and repeat until converged

6 3D Aerodynamic Analysis

Once a converged solution for circulation distribution $\Gamma(y)$ has been reached, aerodynamic forces may be calculated for the entire finite wing. Total circulation can be calculated by integrating the circulation distribution and lift can be found from total circulation via the Kutta-Joukowski theorem (Eqn 7).

$$C_L = \frac{2}{V_\infty S} \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y) dy \tag{7}$$

Induced drag can then be found as a function of lift (Eqn 8).

$$C_{D,i} = \frac{2}{V_\infty S} \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y) \alpha_i(y) dy \tag{8}$$

Viscous drag can be estimated by computing the 2D sectional drag coefficient $C_{d,v}$ from XFOIL data based on the sectional value of α_{eff}. These coefficients can be converted to forces assuming that the 2D data was non-dimensionalized with the same parameters being used for the 3D lifting line analysis.

The sectional viscous drag forces are then integrated over the span to derive a total viscous drag force on the wing, which can be non-dimensionalized into a coefficient again (Eqn 9).

$$\begin{aligned} C_{d,v}(y) &= f(\alpha_{eff}(y)) \\ D_{v,2D}(y) &= C_{d,v}(y) \cdot q_\infty c(y) \\ D_{v,3D} &= \int_{-\frac{b}{2}}^{\frac{b}{2}} D_{v,2D}(y) dy \\ C_{D,v} &= \frac{D_{v,3D}}{q_\infty S} \end{aligned} \tag{9}$$

Finally, to obtain the *total* 3D drag coefficient, the two components of 3D drag coefficient $C_{D,i}$ and $C_{D,v}$ can be summed directly, assuming they have been non-dimensionalized by identical parameters (Eqn 10):

$$C_D = C_{D,i} + C_{D,v} \tag{10}$$

These equations can be utilized to calculate 3D lift curves and drag polars complete with wing stall characteristics. Span location of first stall can be determined for stability analysis. Wings of different planform shapes can be compared, and analysis can be performed for blended airfoil wings and wings with washout.