Linear Algebra (Gilbert Strang) : 08. $Ax=b$ : row reduced form $R$.

□ Solvability Condition of $b$

↳ $Ax=b$ solvable iff) when $b$ is in $C(A)$.

iff) a combination of rows of $A$ gives zero row, then the same combinations of entries of $b$ must give 0.

□ To find complete solution to $Ax=b$

↳ ① $X_{particular}$ : 'How do you find one particular solution?'

set all free variables to zero. (but you must check if there is $0=0$ beforehand).
Then, solve $Ax=b$ for pivot variables.

② $X_{nullspace}$ :

$\therefore X = X_p + X_n$.

$$Ax_p = b. \quad Ax_n = 0 \quad A(x_p + x_n) = b.$$

□ m by n matrix A of rank r

↳ (know that the number of $r \le m$, $r \le n$)

Full column Rank means $\boxed{r=n}$ : all the columns have pivots in this case. no free variables.

→ unique sol. if exists.

$N(A) = \{\text{zero vector}\}$ solution to $Ax=b$ = $x = x_p$. unique solution if it exists. (0 or 1 solution).

Full row Rank means $\boxed{r=m}$ : Can solve $Ax=b$ for every $b$. Exists!

→ solution for every $b$.

Left with $\underbrace{n-r}_{=n-m}$ free variables.

ex) $A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix}$

Full Rank $r = m = n$ : ex) $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $R = I$

→ unique sol. (1 solution) exists for every $b$.

It is invertible. $N(A)$ is zero vector only.

□ Summary

| | | | | |
|---|---|---|---|---|
| ↳ | $r=m=n$ | $r=n<m$ | $r=m<n$ | $r<m,\ r<n$ |
| | $R=I$ | $R=\begin{bmatrix} I \\ 0 \end{bmatrix}$ | $R=\begin{bmatrix} I & F \end{bmatrix}$ | $R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ |
| to $Ax=b$ ⇒ | 1 solution | 0 or 1 solution | ∞ solutions | 0 or ∞ solutions |

when you don't get 0=0 for some bs