QCD and precision calculations
3. Loop integrals, tools, NNLO

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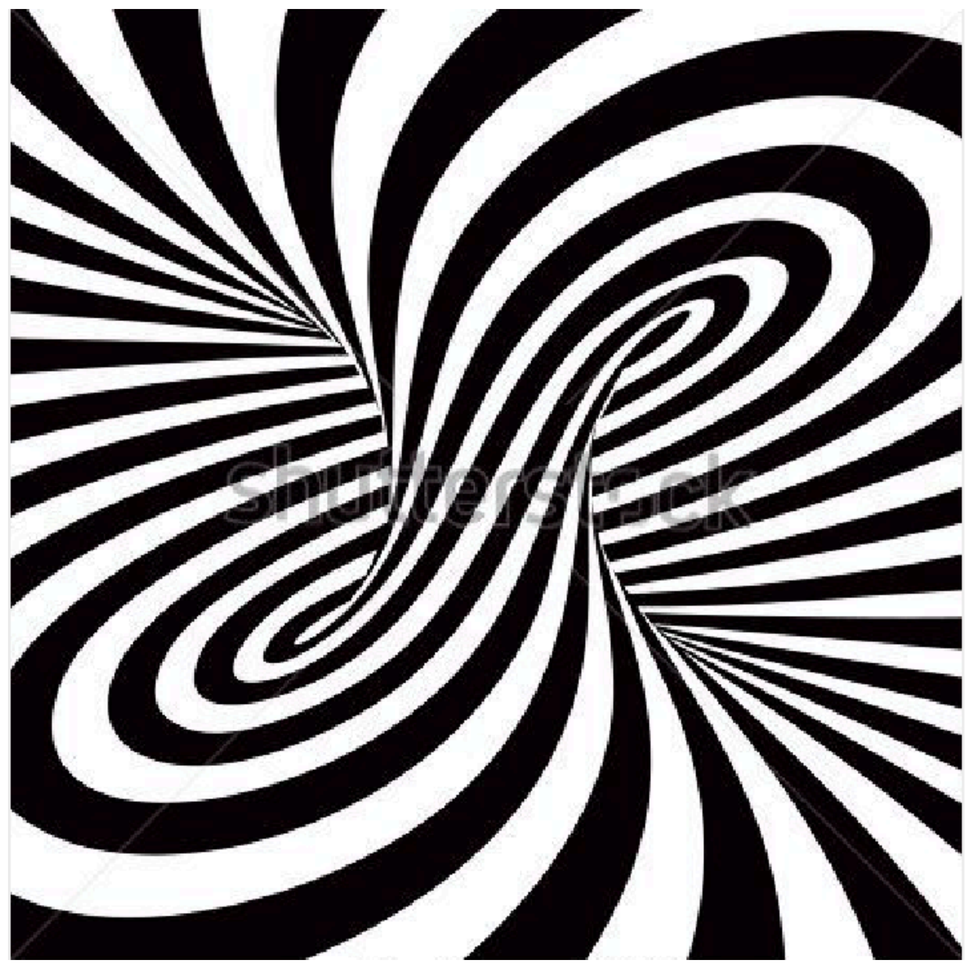

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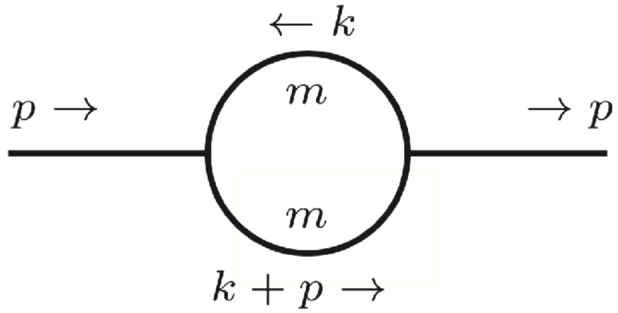
Contents

- one-loop integrals
- renormalisation
- running coupling
- automated NLO calculations
- integrand reduction
- outlook on NNLO calculations

one-loop integrals

let us calculate the one-loop bubble integral explicitly



general formula for Feynman parameters:

$$\underset{(\nu_i > 0)}{\frac{1}{d_1^{\nu_1} d_2^{\nu_2} \ldots d_N^{\nu_N}}} = \frac{\Gamma(\sum_{i=1}^N \nu_i)}{\prod_{i=1}^N \Gamma(\nu_i)} \int_0^\infty \prod_{i=1}^N dz_i \, z_i^{\nu_i - 1} \frac{\delta(1 - \sum_{j=1}^N z_j)}{[z_1 d_1 + z_2 d_2 + \ldots + z_N d_N]^{\sum_{i=1}^N \nu_i}}$$

$$I_2 = \int_{-\infty}^\infty d\kappa \, \frac{1}{[k^2 - m^2 + i\delta][(k+p)^2 - m^2 + i\delta]}$$

$$d_1 = k^2 - m^2 + i\delta$$

$$d_2 = (k+p)^2 - m^2 + i\delta$$

$$\kappa = d^D k / i\pi^{\frac{D}{2}}$$

$$= \Gamma(2) \int_0^\infty dz_1 dz_2 \int_{-\infty}^\infty d\kappa \, \frac{\delta(1 - z_1 - z_2)}{[z_1 \, (k^2 - m^2) + z_2 \, ((k+p)^2 - m^2) + i\delta]^2}$$

$$\nu_1 = \nu_2 = 1$$

$$= \Gamma(2) \int_0^1 dz_2 \int_{-\infty}^\infty d\kappa \, \frac{1}{[k^2 + 2\, k \cdot Q + A + i\delta]^2}$$

$$Q^\mu = z_2 \, p^\mu \ , \ A = z_2 \, p^2 - m^2$$

one-loop integrals

momentum integration: Gauss-integration in D dimensions

to complete the square, shift $l = k + Q$, rename $z_2 \to z$

$$I_2 = \Gamma(2) \int_0^1 dz \int_{-\infty}^{\infty} \frac{d^D l}{i\pi^{\frac{D}{2}}} \left[l^2 - R^2 + i\delta\right]^{-2}$$

$$R^2 = Q^2 - A = -p^2\, z\,(1-z) + m^2$$

now use general formula

$$\boxed{\int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{(l^2)^r}{[l^2 - R^2 + i\delta]^N} = (-1)^{N+r} \frac{\Gamma(r+\frac{D}{2})\Gamma(N-r-\frac{D}{2})}{\Gamma(\frac{D}{2})\Gamma(N)} \left[R^2 - i\delta\right]^{r-N+\frac{D}{2}}}$$

$$\Rightarrow \quad I_2 = \Gamma(2-\frac{D}{2}) \int_0^1 dz \left[-p^2\, z\,(1-z) + m^2 - i\delta\,\right]^{\frac{D}{2}-2}$$

for $D = 4 - 2\epsilon$ $\quad I_2 = \Gamma(\epsilon) \int_0^1 dz \left[-p^2\, z\,(1-z) + m^2 - i\delta\,\right]^{-\epsilon}$

UV pole

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$$

aside: D-dim. loop momentum integration

derivation of

$$\int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{1}{[l^2 - R^2 + i\delta]^N} = (-1)^N \frac{\Gamma(N - \frac{D}{2})}{\Gamma(N)} \left[R^2 - i\delta\right]^{-N+\frac{D}{2}}$$

in Minkowski space $\quad l^2 = l_0^2 - \vec{l}^{\,2}$

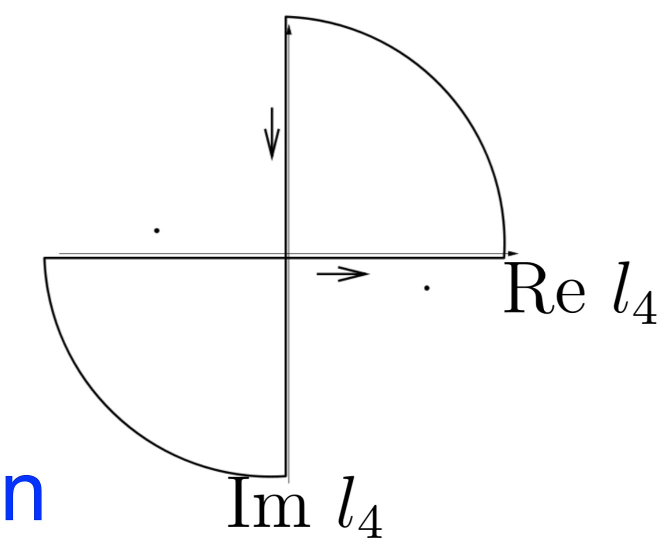
poles in complex energy plane: $\quad l_0^2 = R^2 + \vec{l}^2 - i\delta \Rightarrow l_0^\pm \simeq \pm\sqrt{R^2 + \vec{l}^2} \mp i\,\delta$

for integration move to Euclidean space where $\quad l_E^2 = \sum_{i=1}^{D} l_i^2$

this is achieved by the transformation $\quad l_0 \to i\, l_4$

$$\Rightarrow l^2 \to -l_E^2 = l_4^2 + \vec{l}^{\,2}$$

this implies rotating the integration contour
in the complex energy plane and is called Wick rotation



D-dim. loop momentum integration

contour does not enclose poles

$$\Rightarrow \int_{-\infty}^{\infty} dl_0 f(l_0) = - \int_{i\infty}^{-i\infty} dl_0 f(l_0) = i \int_{-\infty}^{\infty} dl_4 f(l_4)$$

integral with *N* propagators:

$$I_N^D = \quad (-1)^N \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i \, \delta(1 - \sum_{l=1}^N z_l) \int_{-\infty}^\infty \frac{d^D l_E}{\pi^{\frac{D}{2}}} \quad \left[l_E^2 + R^2 - i\delta\right]^{-N}$$

go to polar coordinates: $\int_{-\infty}^\infty d^D l = \int_0^\infty dr \, r^{D-1} \int d\Omega_{D-1} \; , \; r = \sqrt{l_E^2} = \left(\sum_{i=1}^4 l_i^2 \right)^{\frac{1}{2}}$

$\int d\Omega_{D-1} = V(D) = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$ volume of a unit sphere in D dimensions

$$\boxed{I_N^D = \quad 2(-1)^N \frac{\Gamma(N)}{\Gamma(\frac{D}{2})} \int_0^\infty \prod_{i=1}^N dz_i \, \delta(1 - \sum_{l=1}^N z_l) \int_0^\infty dr \, r^{D-1} \, \frac{1}{\left[r^2 + R^2 - i\delta\right]^N}}$$

D-dim. loop momentum integration

$$I_N^D = \quad 2(-1)^N \frac{\Gamma(N)}{\Gamma(\frac{D}{2})} \int_0^\infty \prod_{i=1}^N dz_i \, \delta(1 - \sum_{l=1}^N z_l) \int_0^\infty dr \, r^{D-1} \frac{1}{[r^2 + R^2 - i\delta]^N}$$

substitute $r^2 = xR^2$

$$\int_0^\infty dr \, r^{D-1} \frac{1}{[r^2 + R^2 - i\delta]^N} = \frac{1}{2} [R^2 - i\delta]^{\frac{D}{2} - N} \int_0^\infty dx \, x^{D/2-1} \frac{1}{(x+1)^N}$$

and use

$$\int_0^\infty dz \, \frac{z^{a-1}}{(1+z)^{a+b}} = \int_0^1 dy \, y^{a-1} (1-y)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \equiv B(a,b)$$

Euler Beta-function

to arrive at $\displaystyle \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{1}{[l^2 - R^2 + i\delta]^N} = (-1)^N \frac{\Gamma(N - \frac{D}{2})}{\Gamma(N)} \left[R^2 - i\delta\right]^{-N + \frac{D}{2}}$

so $\displaystyle \boxed{I_N^D = (-1)^N \Gamma(N - \frac{D}{2}) \int_0^\infty \prod_{i=1}^N dz_i \, \delta(1 - \sum_{l=1}^N z_l) \left[R^2 - i\delta\right]^{\frac{D}{2} - N}}$

one-loop integrals

for a one-loop integral with N propagators

$$R^2 = -\frac{1}{2}\sum_{i,j=1}^{N} z_i\, z_j\, \mathcal{S}_{ij} \qquad\qquad \mathcal{S}_{ij} = (r_i - r_j)^2 - m_i^2 - m_j^2$$

keep $-i\delta$ for correct analytic continuation of the Feynman parameter integrals

bubble example: $I_2 = \Gamma(2)\int_0^1 dz \int_{-\infty}^{\infty} \frac{d^D l}{i\pi^{\frac{D}{2}}} \left[l^2 - R^2 + i\delta\right]^{-2}$ $\quad R^2 = -p^2\, z\,(1-z) + m^2$

after loop momentum integration: $I_2 = \Gamma(2-\frac{D}{2})\int_0^1 dz\, [-p^2\, z\,(1-z) + m^2 - i\delta\,]^{\frac{D}{2}-2}$

for $m^2 = 0:$ $\quad I_2 = \left(-p^2\right)^{\frac{D}{2}-2}\Gamma(2-D/2)\, B(D/2-1, D/2-1)$

closed solution for arbitrary *D*

$$= (-p^2)^{-\epsilon}\Gamma(\epsilon)\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \qquad (D = 4-2\epsilon)$$

one-loop integrals

include prefactors from Feynman rules

$$g^2\mu^{2\epsilon}\frac{i\pi^{\frac{D}{2}}}{(2\pi)^D}I_2 = (4\pi)^\epsilon\; i\,\frac{g^2}{(4\pi)^2}\Gamma(\epsilon)\left(-p^2/\mu^2\right)^{-\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

$$= i\,(4\pi)^\epsilon\;\frac{\alpha}{4\pi}\,r_\Gamma\left(-p^2/\mu^2\right)^{-\epsilon}\left(\frac{1}{\epsilon}+2+\mathcal{O}(\epsilon)\right)$$

$$\alpha=\frac{g^2}{4\pi}\;,\; r_\Gamma=\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\;,\;\epsilon\,\Gamma(\epsilon)=\Gamma(1+\epsilon)$$

note:

- UV poles are removed by renormalisation procedure
- how finite parts are dealt with defines a renormalisation scheme

$\overline{\text{MS}}$ subtraction scheme: combination $\Delta=\frac{1}{\epsilon}-\gamma_E+\ln(4\pi)$

always occurs together with a pole

$\Rightarrow$ subtract whole Δ in renormalisation procedure

one-loop integrals

- integrals containing no dimensionful scale like masses or external momenta are **zero** in dimensional regularisation

more precisely $\int_{-\infty}^{\infty} \frac{d^D k}{k^{2\rho}} = i\pi\, V(D)\, \delta(\rho - D/2)$

- tensor integrals: $I_N^{D,\,\mu_1\ldots\mu_r} = \int_{-\infty}^{\infty} \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{k^{\mu_1}\ldots k^{\mu_r}}{\prod_{i=1}^{N}(q_i^2 - m_i^2 + i\delta)}$

$q_i = k + r_i$

- direct calculation very similar to procedure outlined above
- can be algebraically reduced to scalar integrals
- various libraries exist to do this automatically
- nowadays in QCD at one loop mostly reduction at *integrand level* is used

(see later)

UV renormalisation basics

QCD is a renormalisable gauge theory

procedure:

write “bare” Lagrangian in terms of renormalised fields and counter terms

$$\mathcal{L}(A_0, q_0, \eta_0, m_0, g_0, \lambda_0) = \mathcal{L}(A, q, \eta, m, g\mu^\epsilon, \lambda) + \mathcal{L}_c(A, q, \eta, m, g\mu^\epsilon, \lambda)$$

$$A^\mu = Z_3^{-\frac{1}{2}} A_0^\mu, \ \lambda = Z_3^{-1} \lambda_0, \ q = Z_2^{-\frac{1}{2}} q_0, \ m = Z_m^{-1} m_0, \ \eta = \tilde{Z}_3^{-\frac{1}{2}} \eta_0$$

gluon field | gauge fixing parameter | quark field | mass | ghost field

important relations:

$$g_0 = g\mu^\epsilon \frac{Z_1}{Z_3^{\frac{3}{2}}} = g\mu^\epsilon \frac{\tilde{Z}_1}{\tilde{Z}_3 Z_3^{\frac{1}{2}}} = g\mu^\epsilon \frac{Z_1^F}{Z_2} = g\mu^\epsilon \frac{Z_4^{\frac{1}{2}}}{Z_3} = g\mu^\epsilon Z_g$$

$$\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_1^F}{Z_2} = \frac{Z_4}{Z_1}$$ Slavnov-Taylor identities

UV renormalisation basics

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$$\mathcal{L}(A_0, q_0, \eta_0, m_0, g_0, \lambda_0) = \mathcal{L}(A, q, \eta, m, g\mu^\epsilon, \lambda) + \mathcal{L}_c(A, q, \eta, m, g\mu^\epsilon, \lambda)$$

$$A^\mu = Z_3^{-\frac{1}{2}} A_0^\mu, \ \lambda = Z_3^{-1}\lambda_0, \ q = Z_2^{-\frac{1}{2}} q_0, \ m = Z_m^{-1} m_0, \ \eta = \tilde{Z}_3^{-\frac{1}{2}} \eta_0$$

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$$\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_1^F}{Z_2} = \frac{Z_4}{Z_1}$$

Slavnov-Taylor identities

(c.f. Ward Identities in QED)

$$Z_1^F = Z_2$$

quark-gluon-vertex quark-field

QCD beta-function

- the QCD coupling is not constant, it depends on the scale at which the interaction takes place

- at leading order $$\alpha_s(Q^2) = \frac{1}{b_0 \log\left(Q^2/\Lambda^2_{QCD}\right)}$$

$$b_0 = \frac{1}{4\pi}\left(\frac{11}{3}C_A - \frac{4}{3}T_R\,N_f\right)$$

Λ_{QCD} : scale where perturbative description breaks down

- to get an idea how this arises, consider the hadronic R-ratio

$$R(s) = \frac{\sigma(e^+e^- \to \text{ hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

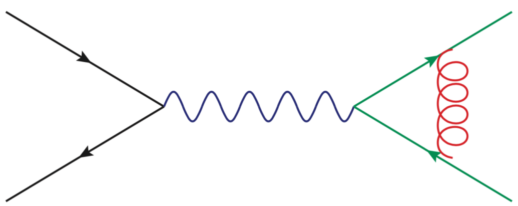
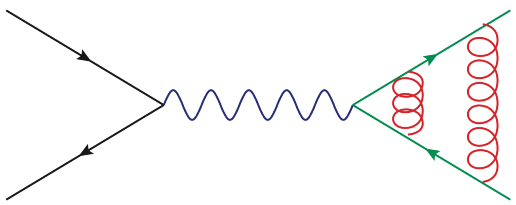
perturbative expansion: $R(s) = K_{QCD}(s)\,R_0\;,\;R_0 = N_c \sum_f Q_f^2$

$$K_{QCD}(s) = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \sum_{n\geq 2} C_n\left(\frac{s}{\mu^2}\right)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n$$

QCD beta-function

$$K_{QCD}(s) = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \sum_{n\geq 2} C_n\left(\frac{s}{\mu^2}\right)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n$$

(known up to n=4)

n=1  n=2  + …

explicit calculation: $$K_{QCD}(s) = 1 + \frac{\alpha_s^0}{\pi} + \left(\frac{\alpha_s^0}{\pi}\right)^2\left[c + b_0\pi\,\frac{1}{\epsilon}\left(\frac{s}{\mu_0^2}\right)^{-\epsilon}\right] + \mathcal{O}(\alpha_s^3)$$

redefine coupling: $$\alpha_s(\mu) = \alpha_s^0 + \alpha_s^2\, b_0\,\frac{1}{\epsilon}\left(\frac{\mu^2}{\mu_0^2}\right)^{-\epsilon}$$

expand consistently to order α_s^2

$$K_{QCD}^{\text{ren}}(\alpha_s(\mu),\mu^2/s) = 1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left[c + b_0\pi\,\log\frac{\mu^2}{s}\right] + \mathcal{O}(\alpha_s^3)$$

QCD beta-function

$$K^{\rm ren}_{QCD}(\alpha_s(\mu), \mu^2/s) = 1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left[c + b_0\pi \, \log\frac{\mu^2}{s}\right] + \mathcal{O}(\alpha_s^3)$$

poles disappeared but now depends on μ

physical quantity $R^{\rm ren} = R_0 \, K^{\rm ren}_{QCD}$ cannot depend on unphysical scale

$$\Rightarrow \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} R^{\rm ren}(\alpha_s(\mu), \mu^2/Q^2) = 0 = \left(\mu^2 \frac{\partial}{\partial\mu^2} + \mu^2 \frac{\partial\alpha_s}{\partial\mu^2}\frac{\partial}{\partial\alpha_s}\right) R^{\rm ren}(\alpha_s(\mu), \mu^2/Q^2)$$

renormalisation group equation

define $t = \ln\frac{Q^2}{\mu^2}$, $\beta(\alpha_s) = \mu^2 \, \partial\alpha_s/\partial\mu^2$

$$\Rightarrow \left(-\frac{\partial}{\partial t} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right) R = 0$$

implies $\dfrac{\partial\alpha_s(Q^2)}{\partial t} = \beta\left(\alpha_s(Q^2)\right)$ $\quad \beta(\alpha_s) = -b_0\alpha_s^2\left[1 + \sum_{n=1}^{\infty} b_n \, \alpha_s^n\right]$

running coupling

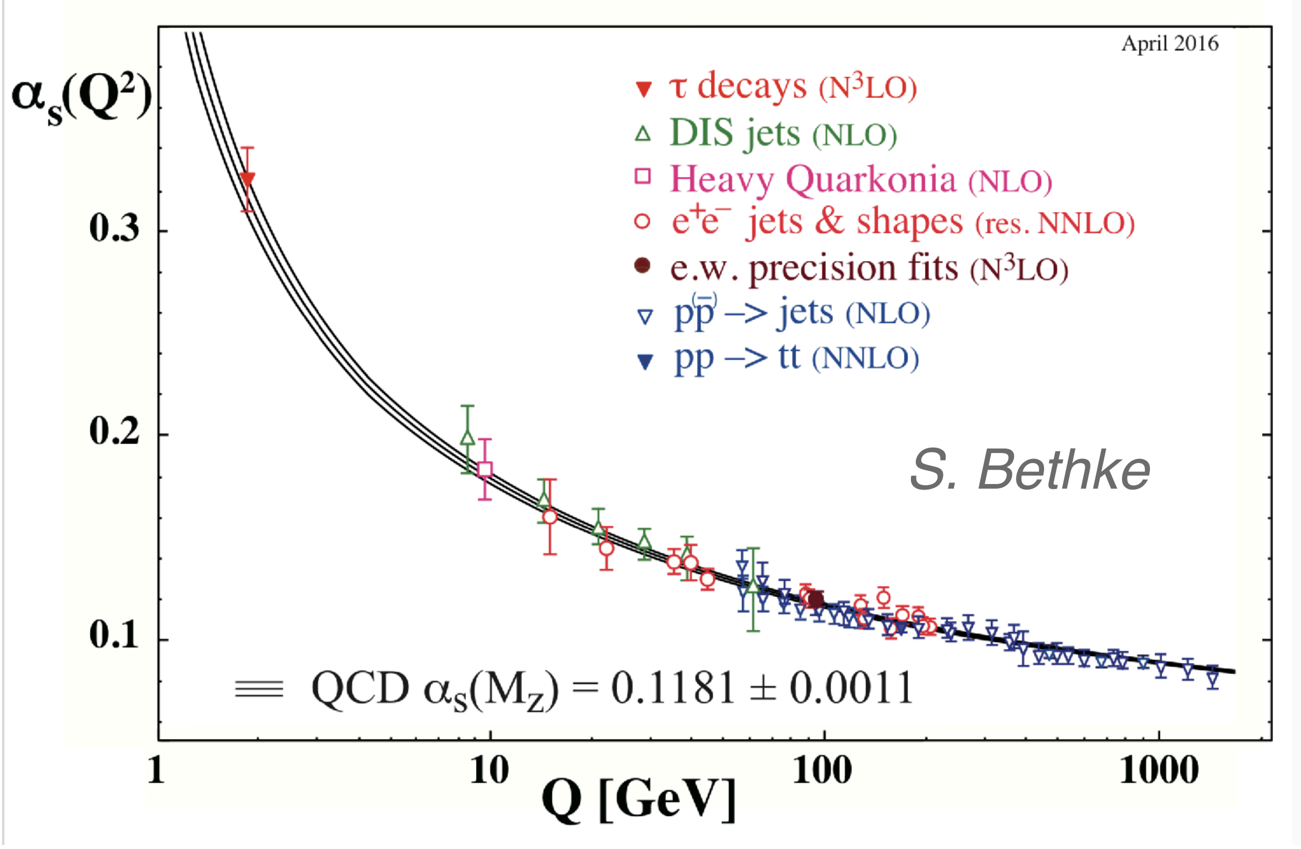
$$\frac{\partial \alpha_s(Q^2)}{\partial t} = \beta\left(\alpha_s(Q^2)\right)$$

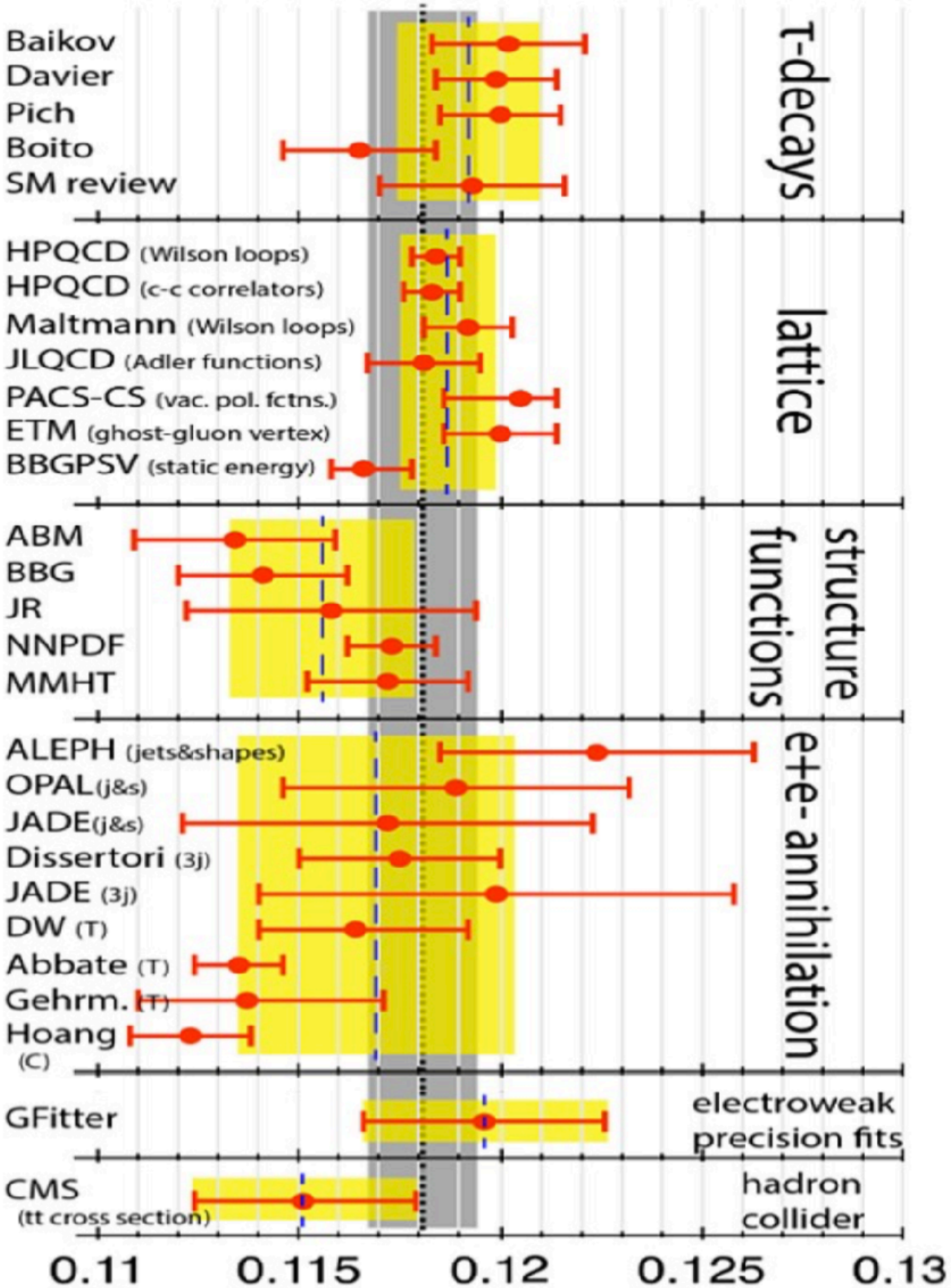
can be solved perturbatively with expansion $\beta(\alpha_s) = -b_0\alpha_s^2\left[1 + \sum_{n=1}^{\infty} b_n\,\alpha_s^n\right]$

$$\Rightarrow \quad \alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+b_0\,t\,\alpha_s(\mu^2)} \qquad t = \ln\frac{Q^2}{\mu^2}$$

$$\Rightarrow \quad \alpha_s(Q^2) \xrightarrow{Q^2\to\infty} \frac{1}{b_0 t} \xrightarrow{Q^2\to\infty} 0$$

asymptotic freedom





class averages:

$\alpha_s(M_Z) = 0.1192 \pm 0.0018$ (±1.5%)

$\alpha_s(M_Z) = 0.1184 \pm 0.0012$ (±1.0%)

$\alpha_s(M_Z) = 0.1156 \pm 0.0021$ (±1.8%)

$\alpha_s(M_Z) = 0.1169 \pm 0.0034$ (±2.9%)

$\alpha_s(M_Z) = 0.1196 \pm 0.0030$ (±2.5%)

$\alpha_s(M_Z) = 0.1151 \pm 0.0028$ (±2.5%)

unweighted χ^2 average: $\alpha_s(M_Z) = 0.1181 \pm 0.0011$ (±0.9%)

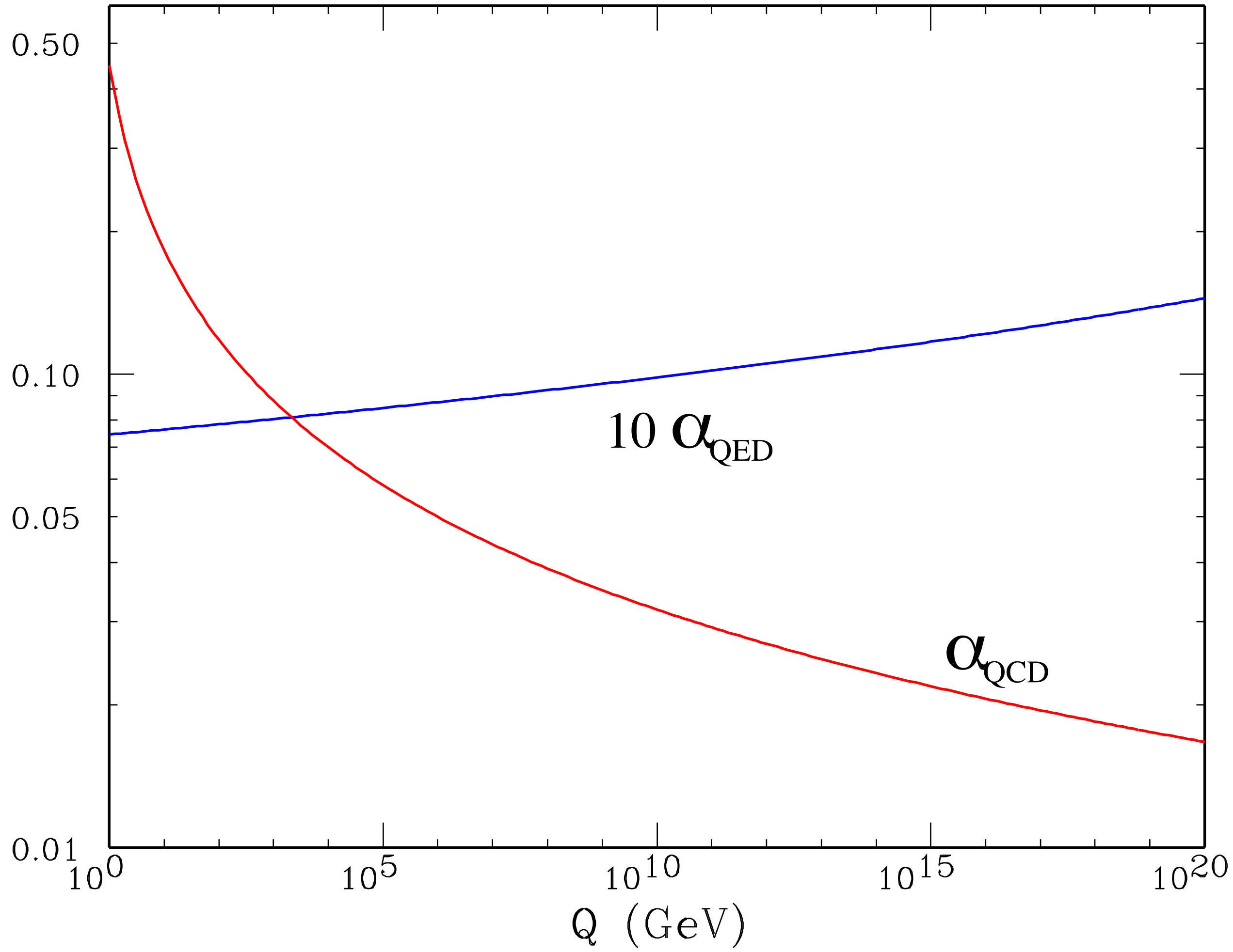
QCD beta-function

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+b_0\, t\, \alpha_s(\mu^2)} \quad \text{(LO)} \quad b_0 = \frac{1}{4\pi}\left(\frac{11}{3}C_A - \frac{4}{3}T_R N_f\right)$$

note that sign of b_0 is very important!

$$t = \ln\frac{Q^2}{\mu^2}$$

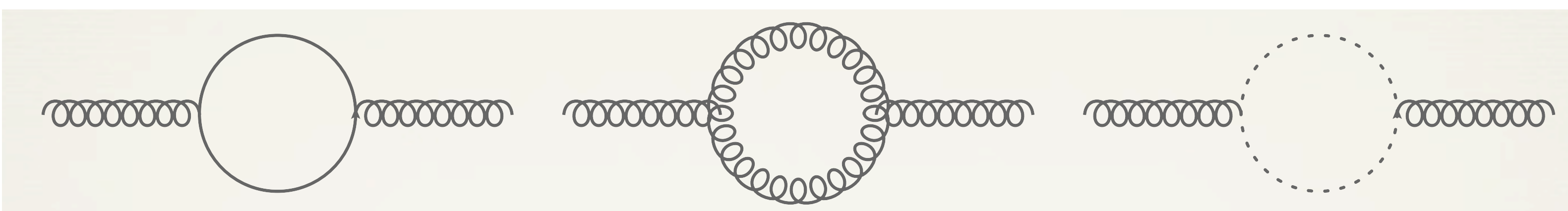
running of α_{QED} is the opposite way



running coupling

QCD: $$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\, b_0\, \ln\left(\frac{Q^2}{\mu^2}\right)}$$

coupling decreases with energy



(a) N_f

(b) N_c

(ghost loop)

$$b_0 = \frac{1}{12\pi}(11N_c - 2N_f)$$

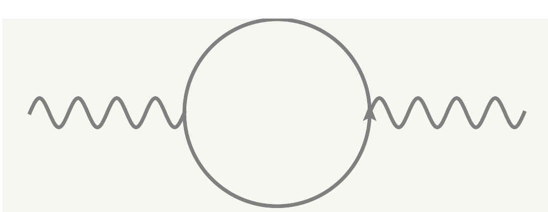
$b_0 > 0$ for

$$N_f < \frac{11}{2}N_c$$

QED: $$\alpha(Q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\, \ln\left(\frac{Q^2}{m_e^2}\right)}$$

1/137

coupling grows with energy

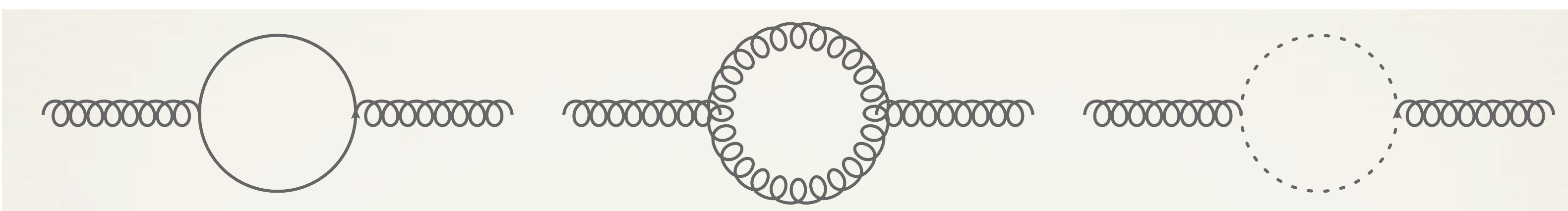


running coupling

QCD: $$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\, b_0 \, \ln\left(\frac{Q^2}{\mu^2}\right)}$$

coupling decreases with energy

$$b_0 = \frac{1}{12\pi}(11N_c - 2N_f)$$



(a) (b) (ghost loop)

N_f N_c

$b_0 > 0$ for

$$N_f < \frac{11}{2} N_c$$

only non-Abelian gauge theories can be asymptotically free (but don't have to)

QED: $$\alpha(Q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\, \ln\left(\frac{Q^2}{m_e^2}\right)}$$

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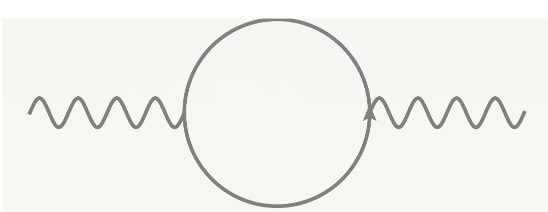


coupling grows with energy

QCD lambda parameter

It is useful to define a dimensionful parameter Λ (integration constant)

$$\ln\left(\frac{Q^2}{\Lambda^2}\right) = -\int_{\alpha_s(Q)}^{\infty} \frac{dx}{\beta(x)} = \int_{\alpha_s(Q)}^{\infty} \frac{dx}{b_0\, x^2\, (1 + b_1\, x + \ldots)}$$

Keeping only $b_0(\mathrm{LO}), b_1(\mathrm{NLO})$

$$\alpha_s(Q) = \frac{1}{b_0\ \ln\left(\frac{Q^2}{\Lambda^2}\right)} \quad \text{(LO)} \qquad \alpha_s(Q) = \frac{1}{b_0\ \ln\left(\frac{Q^2}{\Lambda^2}\right)}\left[1 - \frac{b_1\ \ln\ \ln\left(\frac{Q^2}{\Lambda^2}\right)}{b_0\ \ln\left(\frac{Q^2}{\Lambda^2}\right)}\right] \quad \text{(NLO)}$$

Note that Λ depends on the number of active flavours N_f

(and on renormalisation scheme)

Below scale Λ strong interactions become non-perturbative

$$\Lambda \simeq 200\,\mathrm{MeV}$$

Automated NLO calculations

- nowadays most NLO calculations can be performed by automated tools

  (see hands-on sessions about MadGraph5_aMC@NLO)

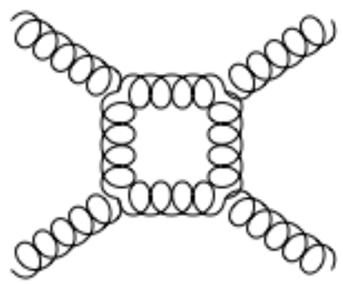
- this is sometimes called the “NLO revolution”

- today we are working on the “NNLO revolution” …

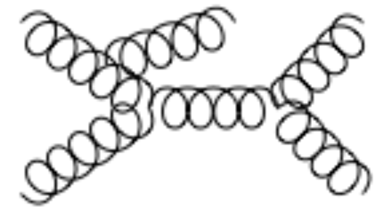
… let us first try to understand what caused the NLO revolution

NLO building blocks

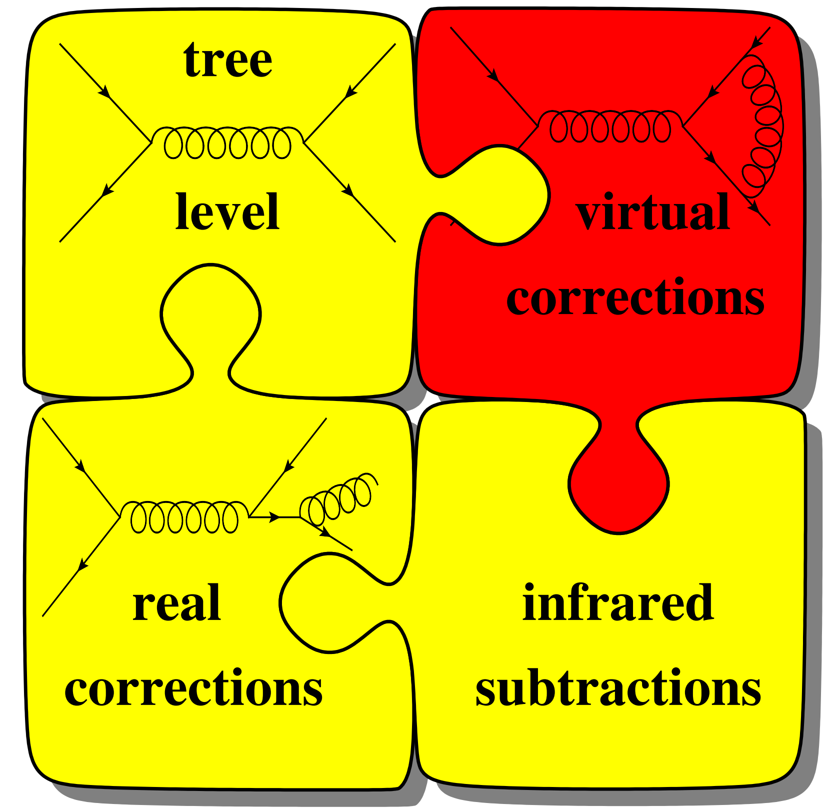
one loop (virtual) + extra real radiation + subtraction terms



virtual



real



individual contributions are **divergent**

requires the isolation of the singularities

➡ at differential level need a subtraction method for IR singularities of real radiation contributions

$$\sigma^{NLO} = \underbrace{\int_{m+1} \left[d\sigma^R - d\sigma^S \right]_{\epsilon=0}}_{\text{numerically}} + \underbrace{\int_m \left[\underbrace{d\sigma^V}_{\text{cancel poles}} + \underbrace{\int_s d\sigma^S}_{\text{analytically}} \right]_{\epsilon=0}}_{\text{numerically}}$$

NLO automation

Monte Carlo program

- tree amplitudes
- infrared subtractions
- phase space integration/ event generation
- parton shower (optional)

BLHA or custom made interface

One-loop provider

- virtual amplitude

- Powheg
- Sherpa
- Herwig7
- Geneva
- Vincia

- Recola
- OpenLoops
- MadLoop
- GoSam
- NJet
- FeynArts/FormCalc
- FeynCalc

all in one:

- MG5_aMC@NLO
- Helac-NLO

library of pre-computed processes:

- MCFM
- VBFNLO

NLO automation

Monte Carlo program
- tree amplitudes
- infrared subtractions
- phase space integration/ event generation
- parton shower (optional)

BLHA or custom made interface

One-loop provider
- virtual amplitude

- Powheg
- Sherpa
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see hands on sessions

all in one:
- MG5_aMC@NLO
- Helac-NLO

library of pre-computed processes:
- MCFM
- VBFNLO

What caused the NLO revolution?

- automation of subtraction methods for IR divergent real radiation

  Frixione, Kunszt, Signer '95; Catani, Seymour '96 → MadDipole, AutoDipole, FxFx, …

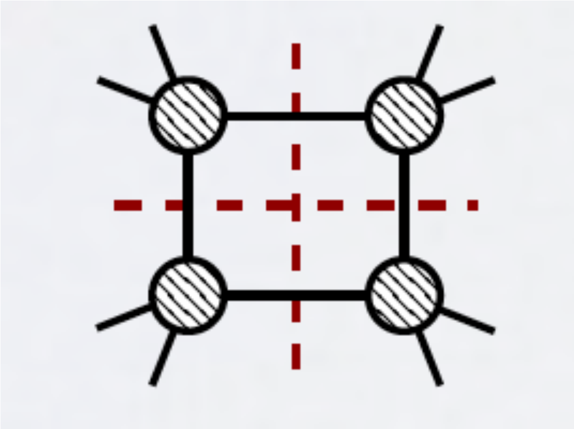
- unitarity-inspired methods for virtual corrections

  gauge dependent off-shell states introduce "spurious" terms

  → try to use on-shell quantities as building blocks

- construct N-point one-loop amplitudes from tree amplitudes

  Bern, Dixon, Dunbar, Kosower '94



- use complex momenta in generalised cuts

  Britto, Cachazo, Feng '04

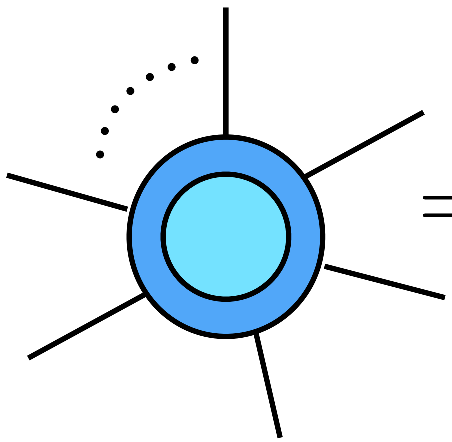
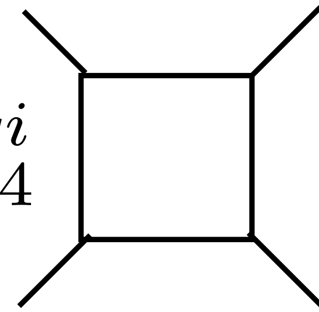
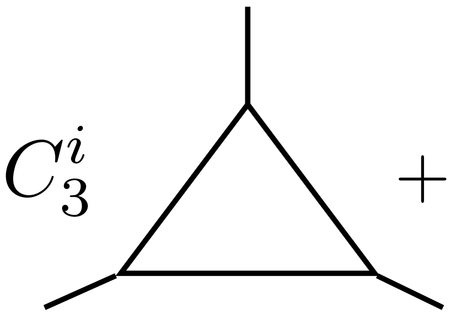
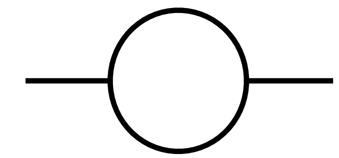
- numerical reduction *at integrand level*

  Ossola, Papadopoulos, Pittau '06

- D-dimensional unitarity

  Anastasiou, Britto, Feng, Kunszt, Mastrolia '06;
  Forde '07; Giele, Kunszt, Melnikov '08

one-loop n-point amplitudes

 $= \sum_i C_4^i$  $+ \sum_i C_3^i$  $+ \sum_i C_2^i$  $+ \mathcal{R}$

rational part

C_n^i can be obtained by numerical reduction at integrand level

"master integrals": boxes, triangles, bubbles, tadpoles

most complicated functions are dilogarithms

very different at two loops (and beyond)!

master integrals not a priori known

reduction at integrand level

integrand of a one-loop N-point amplitude:

$$A(q) = \frac{N(q)}{D_0 D_1 \dots D_{N-1}}$$

$q:$ loop momentum

$D_i = (q + p_i)^2 - m_i^2$

numerator in 4 dimensions can be written as

$$\begin{aligned}\hat{N}(\hat{q}) = &\sum_{i_0<i_1<i_2<i_3}^{N-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(\hat{q}; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{N-1} \hat{D}_i + \sum_{i_0<i_1<i_2}^{N-1} \left[c(i_0 i_1 i_2) + \tilde{c}(\hat{q}; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{N-1} \hat{D}_i \\ &+ \sum_{i_0<i_1}^{N-1} \left[b(i_0 i_1) + \tilde{b}(\hat{q}; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{N-1} \hat{D}_i + \sum_{i_0}^{N-1} \left[a(i_0) + \tilde{a}(\hat{q}; i_0)\right] \prod_{i \neq i_0}^{N-1} \hat{D}_i + \tilde{P}(\hat{q}) \prod_{i}^{N-1} \hat{D}_i \,.\end{aligned}$$

$\tilde{d}, \tilde{c}, \dots, \tilde{P}$ terms vanish upon integration

coefficients *d,c,b,a*, can be extracted by sampling $\hat{N}(\hat{q})$ a sufficient number of times and inverting the system

reduction at integrand level

example box coefficient: $d(0123) = \frac{1}{2}\left[\frac{N(\hat{q}_0^+)}{\prod_{i=4}^{N-1} D_i(\hat{q}_0^+)} + \frac{N(\hat{q}_0^-)}{\prod_{i=4}^{N-1} D_i(\hat{q}_0^-)}\right]$

$\hat{q}_0^{\pm}$ are the two solutions that solve the linear system

if this is done entirely in 4 dimensions the rational part $\mathcal{R}$ will be missed

possible way out: promote momenta to D-dimensions

$$q^\mu = q^\mu_{(4)} + n^\mu_\epsilon (q \cdot n_\epsilon), \quad q^2 = q^2_{(4)} + (n_\epsilon \cdot q)^2$$

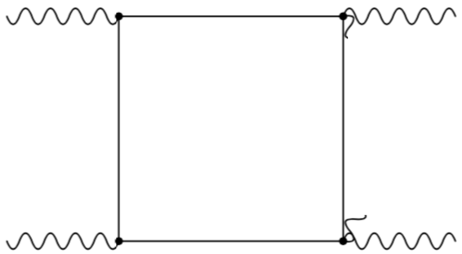
n_ϵ unit vector in (D-4) - dimensional space

rational terms have to do with $\mathcal{O}(\epsilon)$ in the numerator hitting $\frac{1}{\epsilon_{\mathrm{UV}}}$

however a UV finite amplitude also can have rational terms!

example 4-photon amplitude $\mathcal{A}^{++++} = 8\alpha^2$

purely rational



Tools for one-loop integrals

tensor reduction and scalar integrals:

- LoopTools — T. Hahn
- OneLoop — A. van Hameren
- golem95 — Binoth, Guillet, GH et al
- Collier — Denner, Dittmaier, Hofer
- Package-X — H. Patel

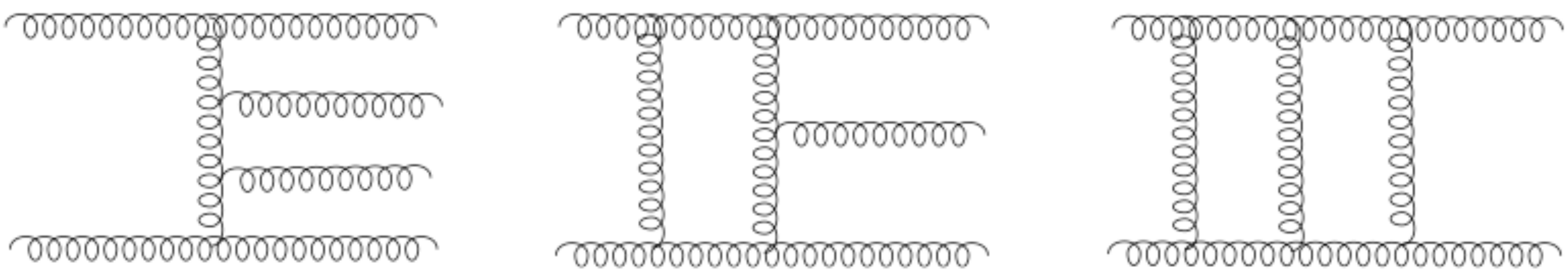
reduction at integrand level:

- CutTools — Ossola, Papadopoulos, Pittau
- Samurai — Mastrolia, Ossola, Reiter, Tramontano
- Ninja — T. Peraro

scalar integrals only:

- QCDloop — Carrazza, Ellis, Zanderighi

NNLO corrections: building blocks



double real | 1-loop virtual $\otimes$ single real | 2-loop virtual

$\downarrow$ implicit IR poles (PS integration)

$\downarrow$ explicit and implicit poles

$\downarrow$ explicit poles $1/\epsilon^{2L}$

bottlenecks: IR subtraction

$\uparrow$ harder with more massless particles (intricate IR singularity structure)

two-loop integrals

$\uparrow$ harder with more massive/off-shell particles (more scales $\rightarrow$ more complicated analytic structure)

NNLO

- enormous progress in the last few years
- caused mainly by development of IR subtraction schemes and improved techniques for loop integrals
- automation possible to some extent

  $\longrightarrow$ collections of processes in same framework available

- NNLOJet Durham/Uni Zurich++
- MCFM v8,9 Campbell, Ellis, Giele, Neumann, Williams et al.
- MATRIX Grazzini, Kallweit, Wiesemann et al.

real radiation IR subtraction schemes

- antenna subtraction analytically integrated subtraction terms

  [Gehrmann-DeRidder, Gehrmann, Glover '05]

- qt "subtraction" slicing, (colourless final states)

  [Catani, Grazzini '07]

- N-jettiness slicing

  [Gaunt, Stahlhofen, Tackmann, Walsh '15]

  [Boughezal, Focke, Liu, Petriello '15]

- sector-improved residue subtraction numerically integrated subtraction terms

  [Czakon, Heymes, Mitov '10; Czakon, Heymes '14] [Boughezal et al. '11]

- nested subtraction [Caola, Melnikov, Röntsch '17, '19-]

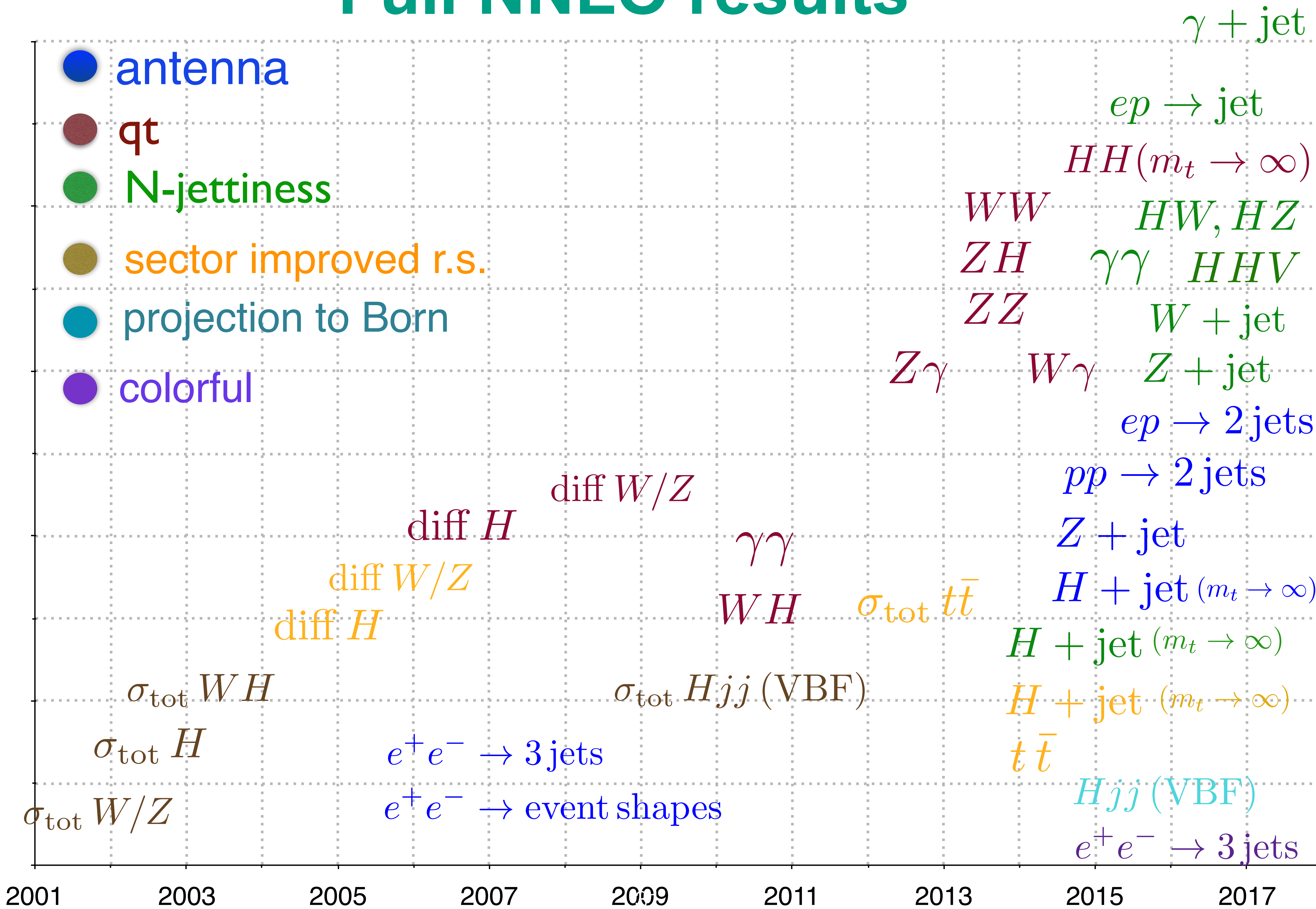
- projection to Born / structure function approach only special kinematics

  [Goa, Li, Zhu '12] [Brucherseifer, Caola, Melnikov '14] [Cacciari, Dreyer, Karlberg, Salam, Zanderighi '15]

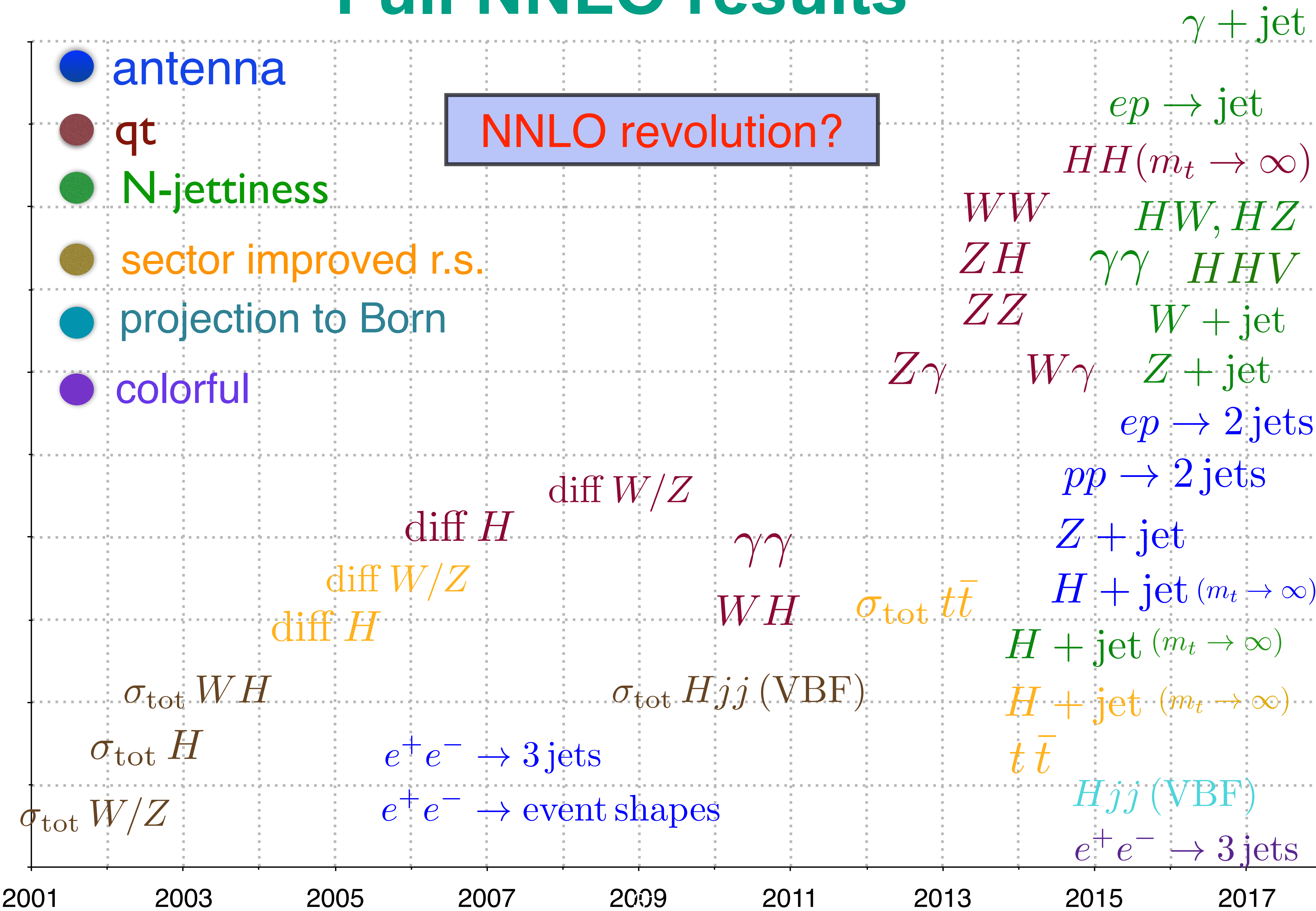
- colorful only final state colour so far [Del Duca, Somogyi, Trocsanyi et al '05-]

- local analytic sector subtraction [Magnea, Maina, Pellicioli, Signorile, Torielli, Uccurati '19-]

Full NNLO results



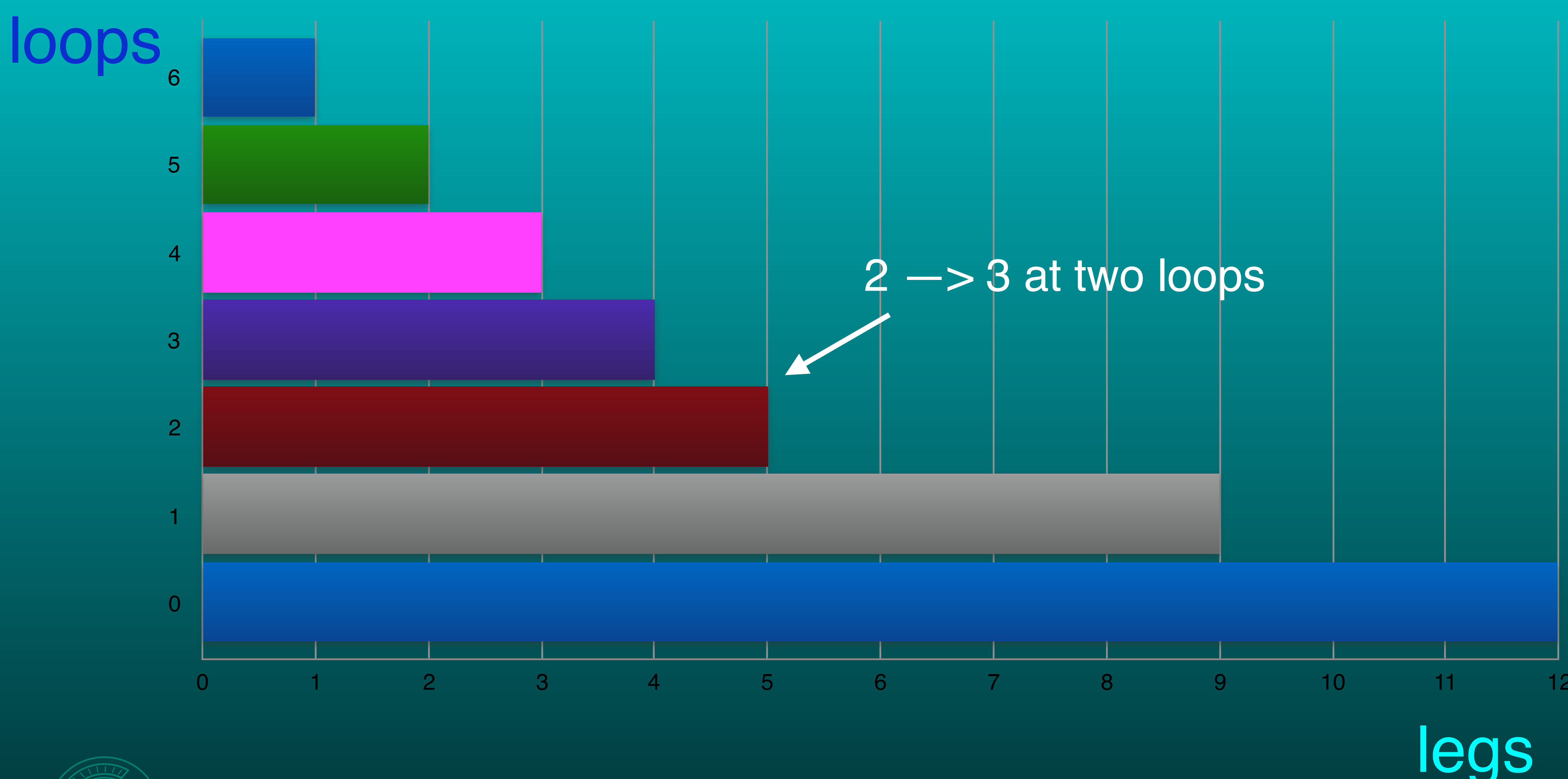
Full NNLO results



measure of complexity

#loops + #legs + **#scales (masses, off-shellness)**

complexity does not scale linearly!



(refers to physical results, not individual integrals) https://loopedia.mpp.mpg.de

Multiple Upload (expert)

Loopedia

Ex.: Edge list [(1,2),(2,3),(2,3),(3,4)] or 1 2 2 3 2 3 3 4 — Nickel index e11|e|

Enter your graph by its edge list (adjacency list) or Nickel index

or browse:

Loops = any Legs = any Scales = any

Fulltext must contain: must not contain:

Search Reset

If you wish to add a new integral to the database, start by searching for its graph first.

The Loopedia Team is C. Bogner, S. Borowka, T. Hahn, G. Heinrich, S. Jones, M. Kerner, A. von Manteuffel, M. Michel, E. Panzer, V. Papara.
Software version of 08 Sep 2017 18:53 UTC. In case of technical difficulties with this site please contact Thomas Hahn or Viktor Papara.
This Web site uses the GraphState library [arXiv:1409.8227] for all graph-theoretical operations
and the neato component of Graphviz for drawing graphs.

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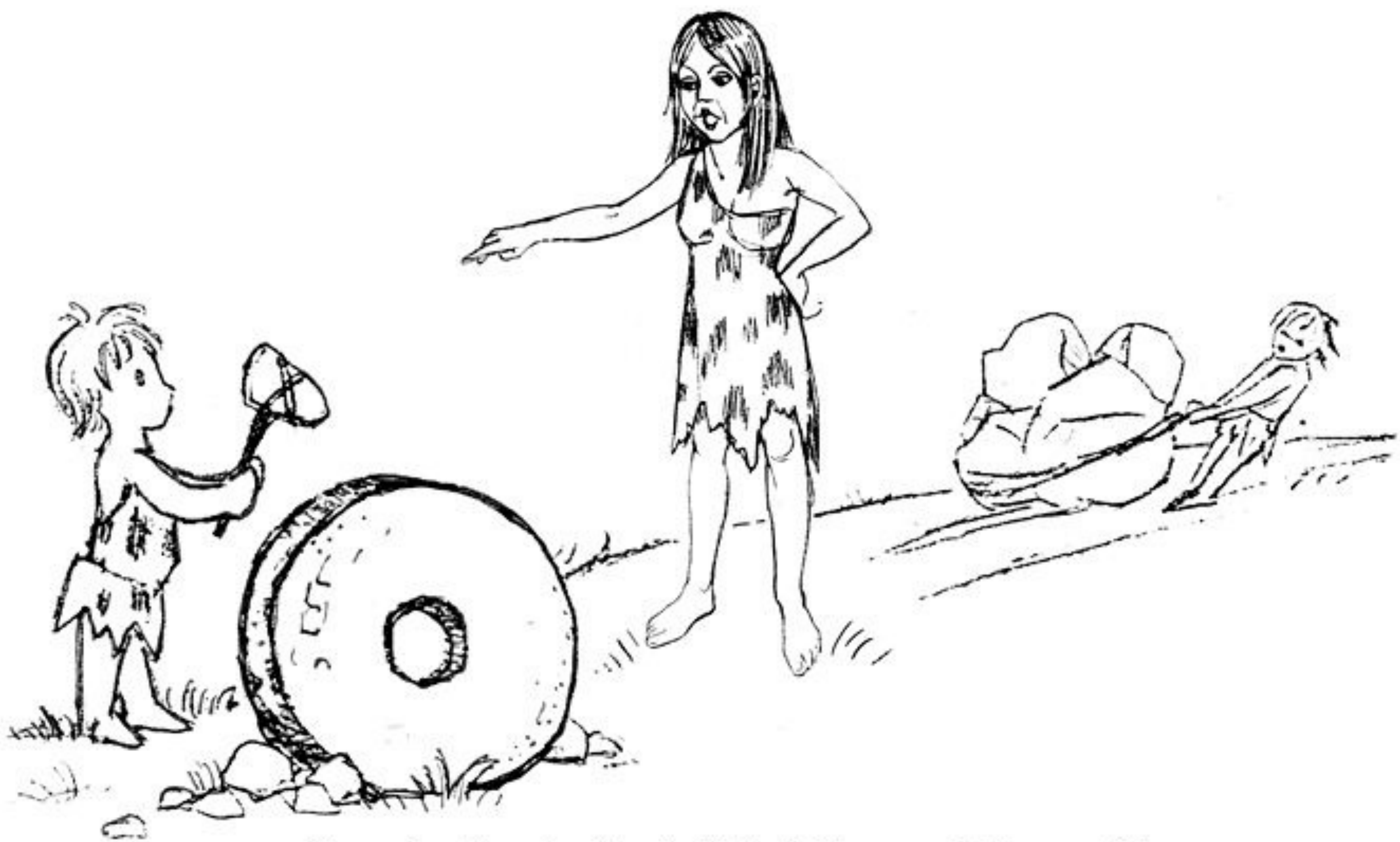
C. Bogner, S. Borowka, T. Hahn, GH, S. Jones, M. Kerner, A. von Manteuffel, M. Michel, E. Panzer, V. Papara, **arXiv:1709.01266**

open database for loop integrals

further plan: check all submissions numerically (SecDec)



All you've done is chisel all day! Do something useful, like helping your brother drag those rocks up the hill.