QCD and precision calculations
Lecture 1: basics

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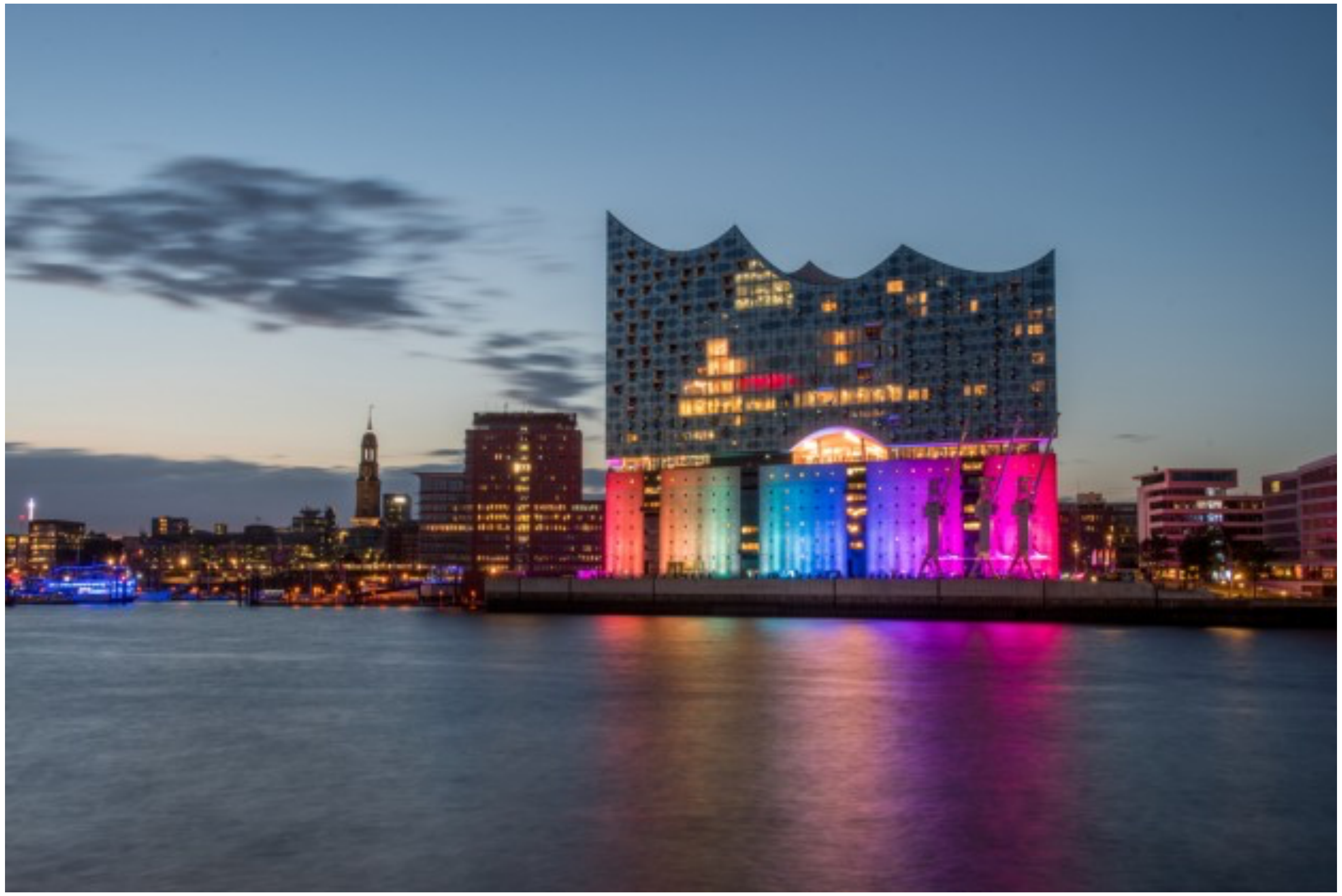


PREFIT School, March 2, 2020

MAX-PLANCK-GESELLSCHAFT

COST EUROPEAN COOPERATION IN SCIENCE & TECHNOLOGY

Max-Planck-Institut für Physik (Werner-Heisenberg-Institut)

Outline

1. Basics of QCD
    - QCD Lagrangian
    - Gauge invariance
    - Colour algebra
    - Feynman rules
2. Tree level amplitudes
    - Application of Feynman rules
    - Polarisation sums
3. NLO calculations

(details see lectures 2-4)

4. Beyond NLO

Some literature

- G. Dissertori, I. Knowles, M. Schmelling, *Quantum Chromodynamics: High energy experiments and theory* International Series of Monographs on Physics No. 115, Oxford University Press, Feb. 2003. Reprinted in 2005.
- T. Muta, *Foundations of QCD*, World Scientific (1997).
- R.K. Ellis, W.J. Stirling and B.R.Webber, *QCD and collider physics*, Cambridge University Press, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **8** (1996) 1.
- J. Wells and G. Altarelli, *Collider Physics within the Standard Model : A Primer*, Lect. Notes Phys. **937** (2017) 1. doi:10.1007/978-3-319-51920-3.
- M.E. Peskin, D.V. Schroeder, *Introduction to Quantum Field Theory*, Addison-Wesley 1995.
- J. Campbell, J. Huston and F. Krauss, *The Black Book of Quantum Chromodynamics: A Primer for the LHC Era* Oxford University Press, December 2017.
- V. A. Smirnov, *Analytic tools for Feynman integrals*, Springer Tracts Mod. Phys. **250** (2012) 1. doi:10.1007/978-3-642-34886-0.
- L. J. Dixon, *"Calculating scattering amplitudes efficiently"*, *Invited lectures presented at the Theoretical Advanced Study Institute in Elementary Particle Physics (TASI '95): QCD and Beyond, Boulder, CO, June 1995.* https://arxiv.org/abs/hep-ph/9601359.

QCD

QCD is a very rich field!

asymptotic freedom

confinement

strong CP-problem

QCD at finite temperature

spectroscopy

lattice gauge theory

quark-gluon-plasma

flavour puzzles

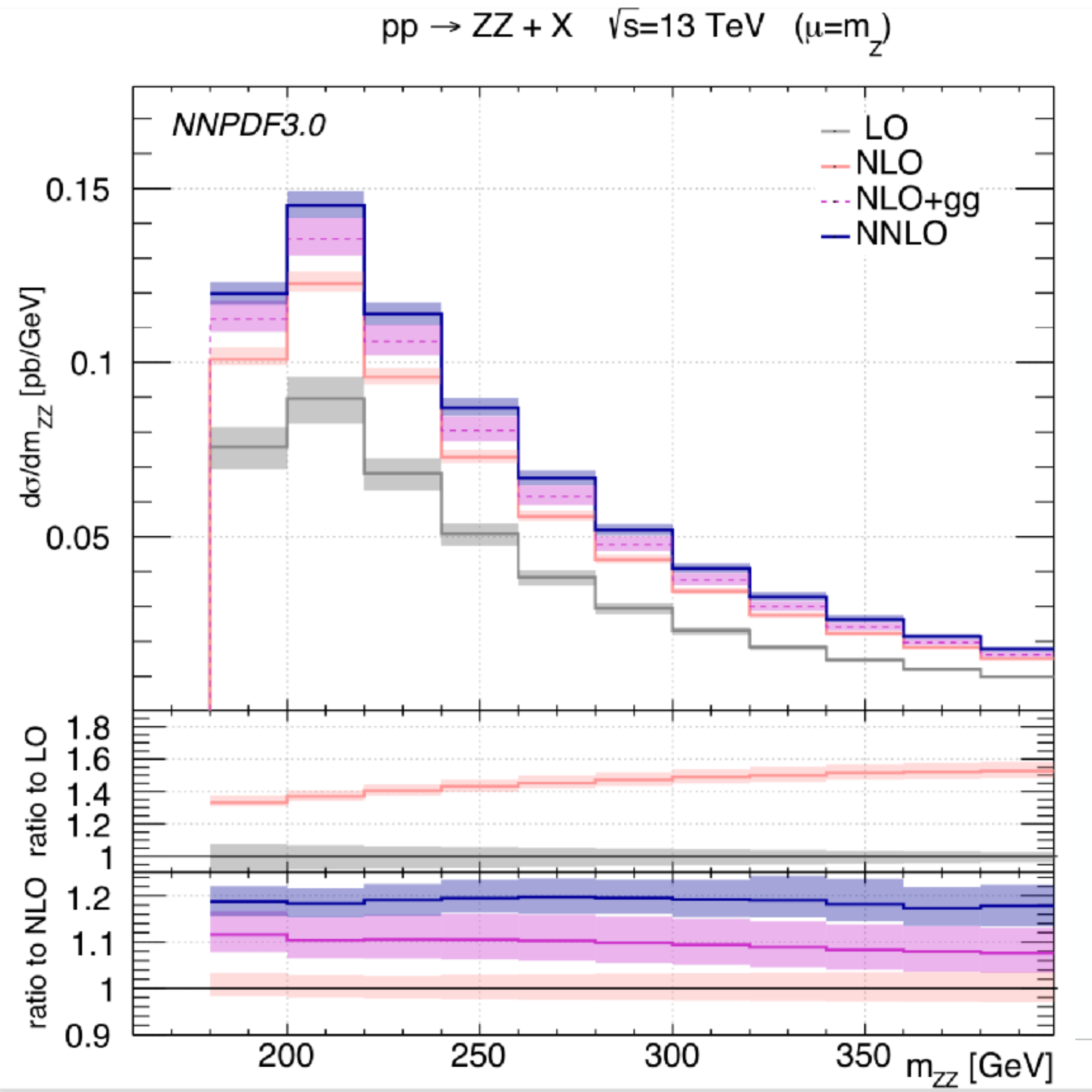
disclaimer:

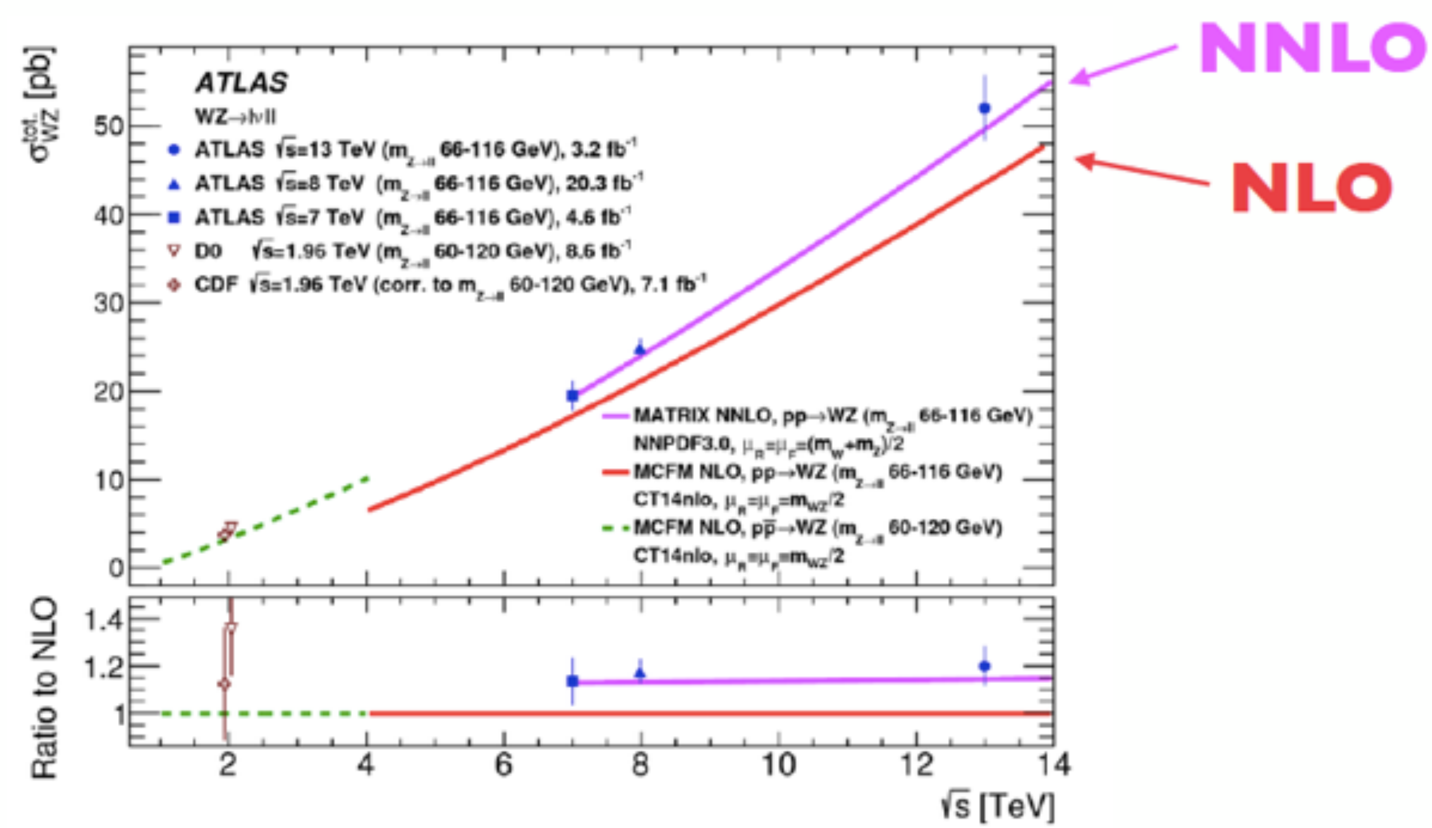
I am not doing justice to it by just considering perturbative QCD, and even that only at fixed order

However, as the focus will be on precision calculations, I will make a (highly biased) choice

Motivation

- at hadron colliders, QCD is everywhere
- without QCD corrections, (most of) the data are not well described





M.Wiesemann, Grazzini, Kallweit, Rathlev ‘17

GH, Jahn, Jones, Kerner, Pires ‘17

QCD corrections

perturbation theory in the strong coupling α_s

$$\sigma = \underbrace{\sigma^{\mathrm{LO}}}_{\text{leading order}} + \alpha_s \underbrace{\sigma^{\mathrm{NLO}}}_{\text{next-to-leading order}} + \alpha_s^2 \underbrace{\sigma^{\mathrm{NNLO}}}_{\text{next-to-next-to-leading order}} + \mathcal{O}(\alpha_s^3)$$

$\alpha_s(M_Z) \simeq 0.118 \;\Rightarrow$ NLO corrections typically $\mathcal{O}(10\%)$

NNLO corrections typically a few %

but there are prominent exceptions
(e.g. Higgs production, NLO corr. ~100%)

Why can we do such an expansion at all?

important concepts:
- asymptotic freedom
- factorisation

Factorisation

$$d\sigma_{pp\to X} = \sum_{i,j} \int_0^1 dx_1 dx_2\, f_{i/p_a}(x_1,\mu_f)\, f_{j/p_b}(x_2,\mu_f)\, d\hat{\sigma}_{ij\to X}(x_1,x_2,\alpha_s,\mu_r,\mu_f) \;+\; \mathcal{O}\left(\frac{\Lambda}{Q}\right)^p$$

parton distribution functions (PDFs)

partonic cross section

power-suppressed non-perturbative corrections

μ_f : factorisation scale (separate short- and long-distance dynamics)

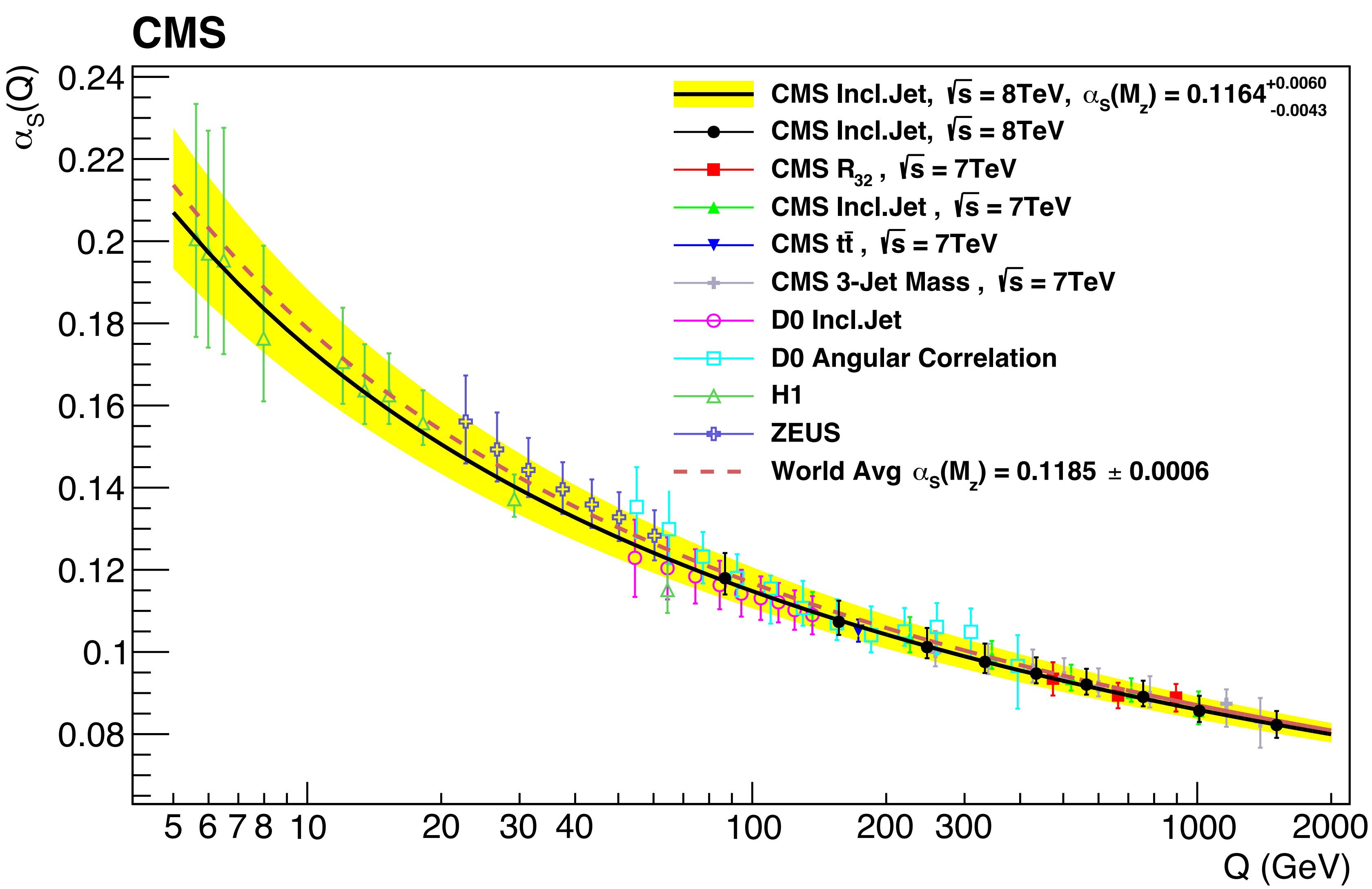
μ_r : renormalisation scale (UV divergences from loops, see later)

PDFs : from fits to data, only evolution with scale μ_f is calculable perturbatively

σ_{ij} : contains hard scattering matrix element, calculable in perturbation theory

sum over parton types i, j

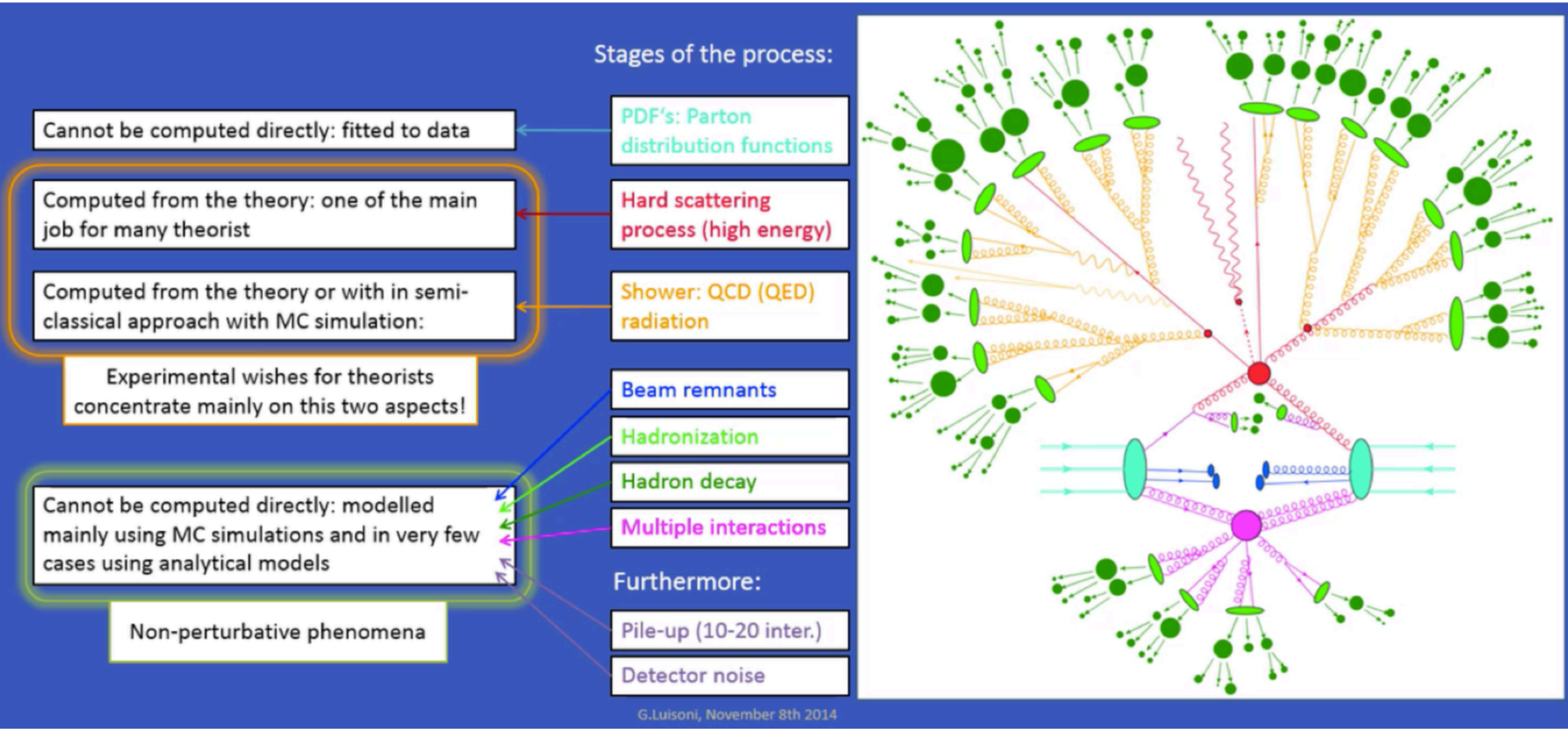
Asymptotic freedom



strong coupling becomes weaker as energy scale increases

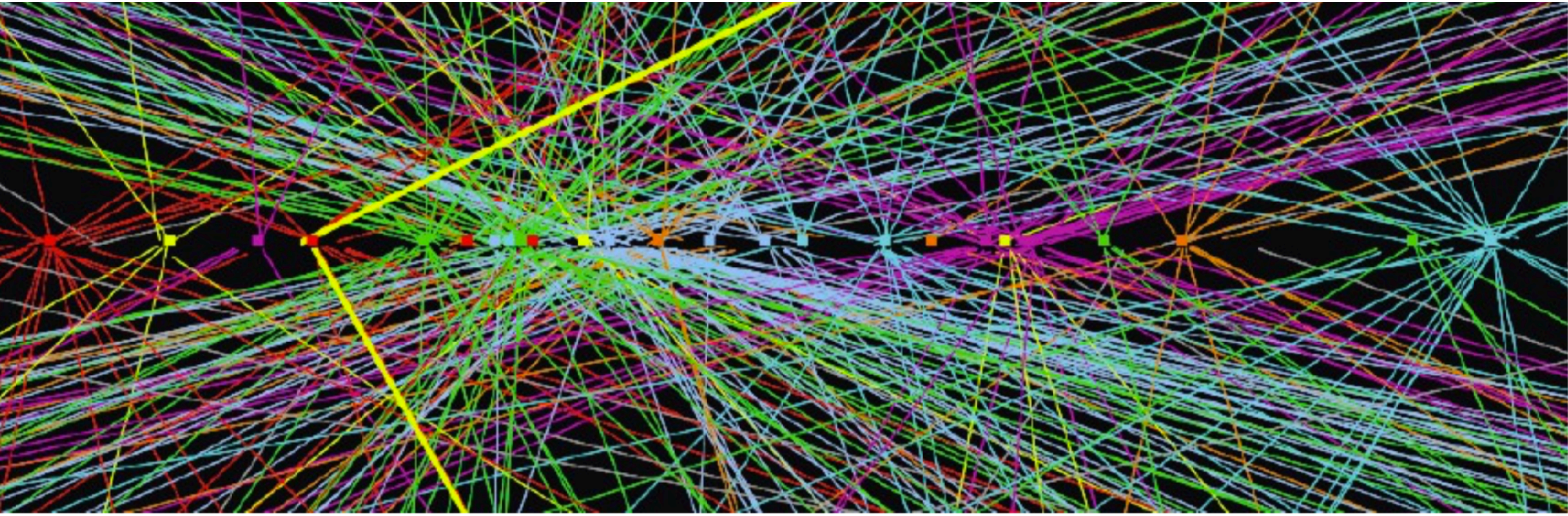
will be discussed in more detail later

LHC event



Stages of the process:
Cannot be computed directly: fitted to data
PDF's: Parton distribution functions
Computed from the theory: one of the main job for many theorist
Hard scattering process (high energy)
Computed from the theory or with in semi-classical approach with MC simulation:
Shower: QCD (QED) radiation
Experimental wishes for theorists concentrate mainly on this two aspects!
Beam remnants
Hadronization
Hadron decay
Multiple interactions
Cannot be computed directly: modelled mainly using MC simulations and in very few cases using analytical models
Furthermore:
Pile-up (10-20 inter.)
Detector noise
Non-perturbative phenomena
G.Luisoni, November 8th 2014

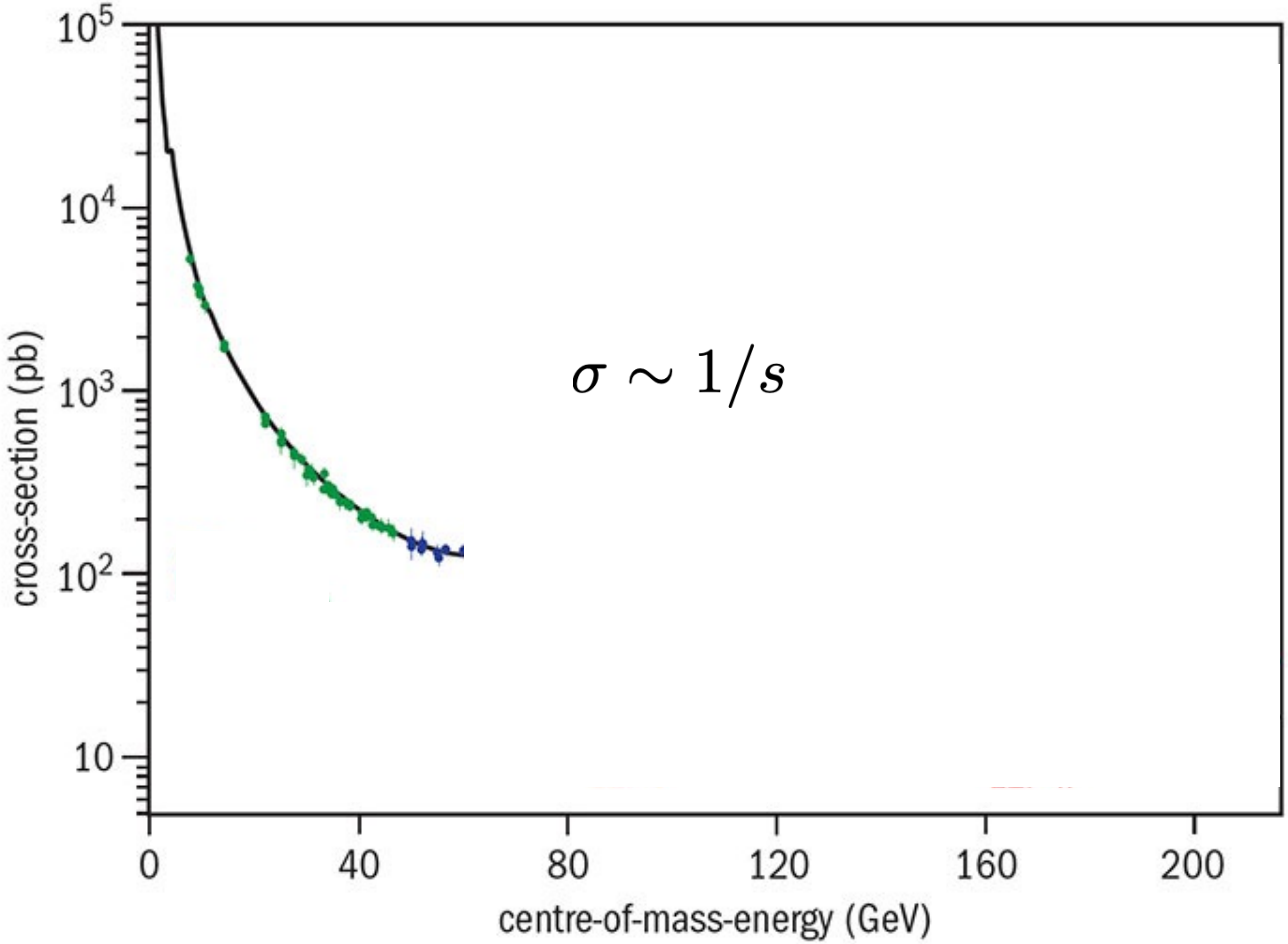
non-perturbative stuff



https://indico.cern.ch/event/505613/contributions/2230824/

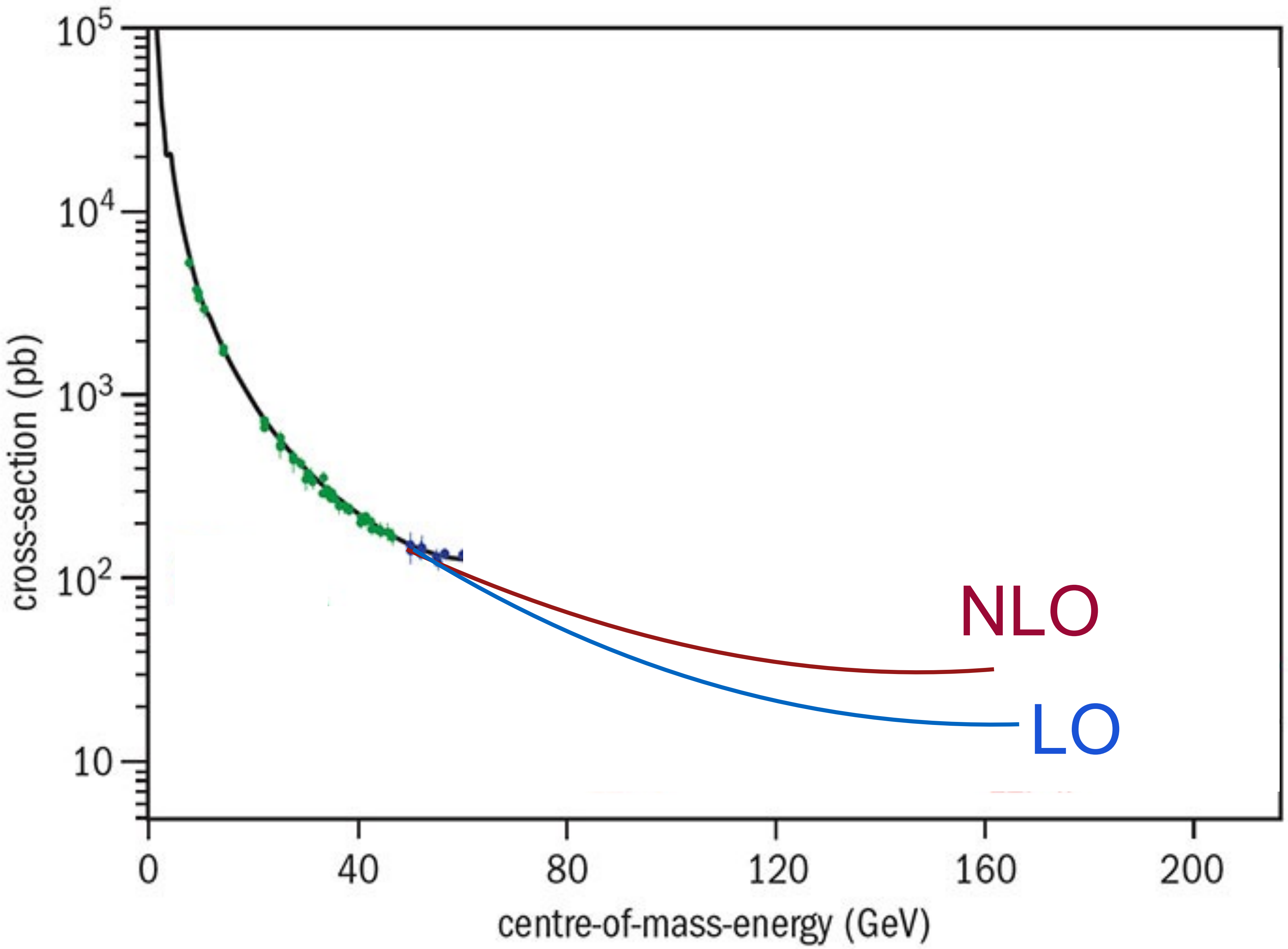
visualisation of pile-up in the ATLAS tracker (Run I)

QCD or New Physics?



a typical distribution, decreasing with energy

QCD or New Physics?

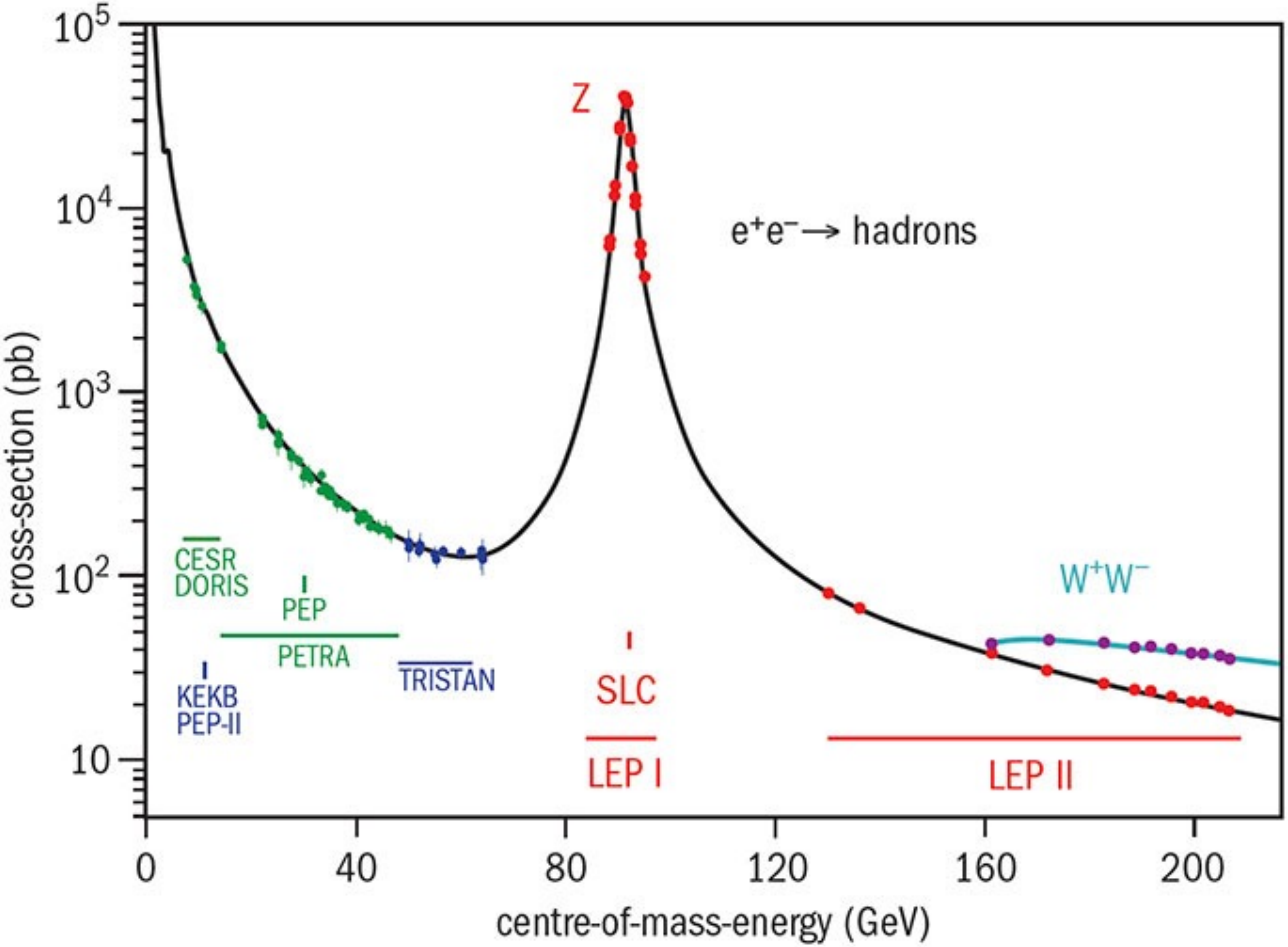


small discrepancy to data at max. available energy

due to lack of NNLO corrections?

or onset of New Physics?

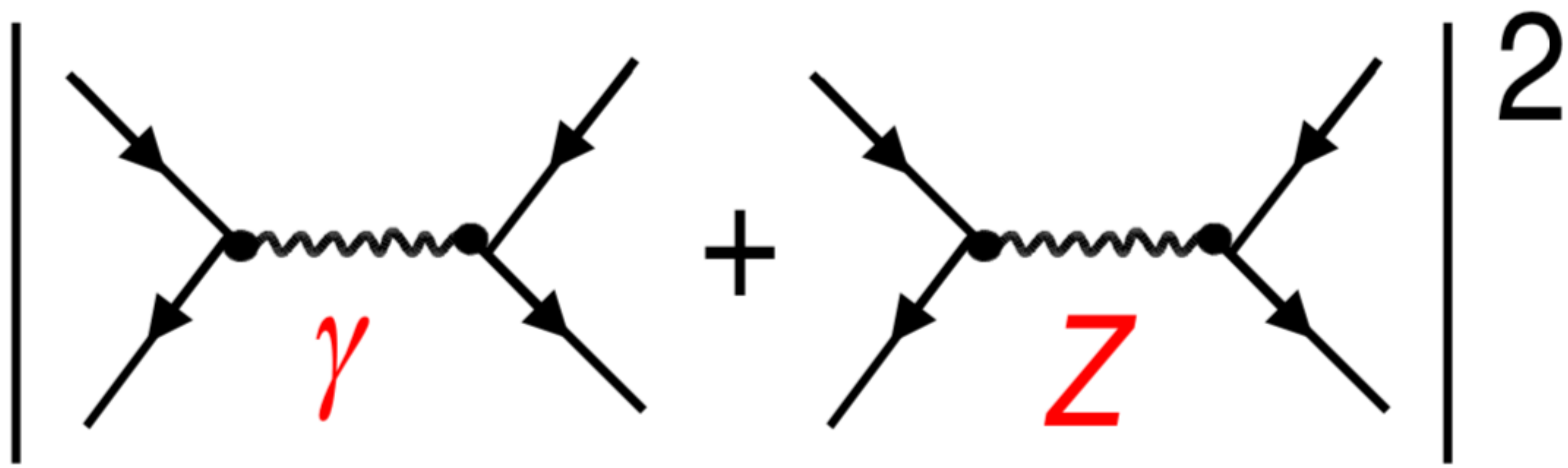
New Physics! (Z-boson 1983)



matrix element

image: P.Uwer

$$|M|^2 =$$



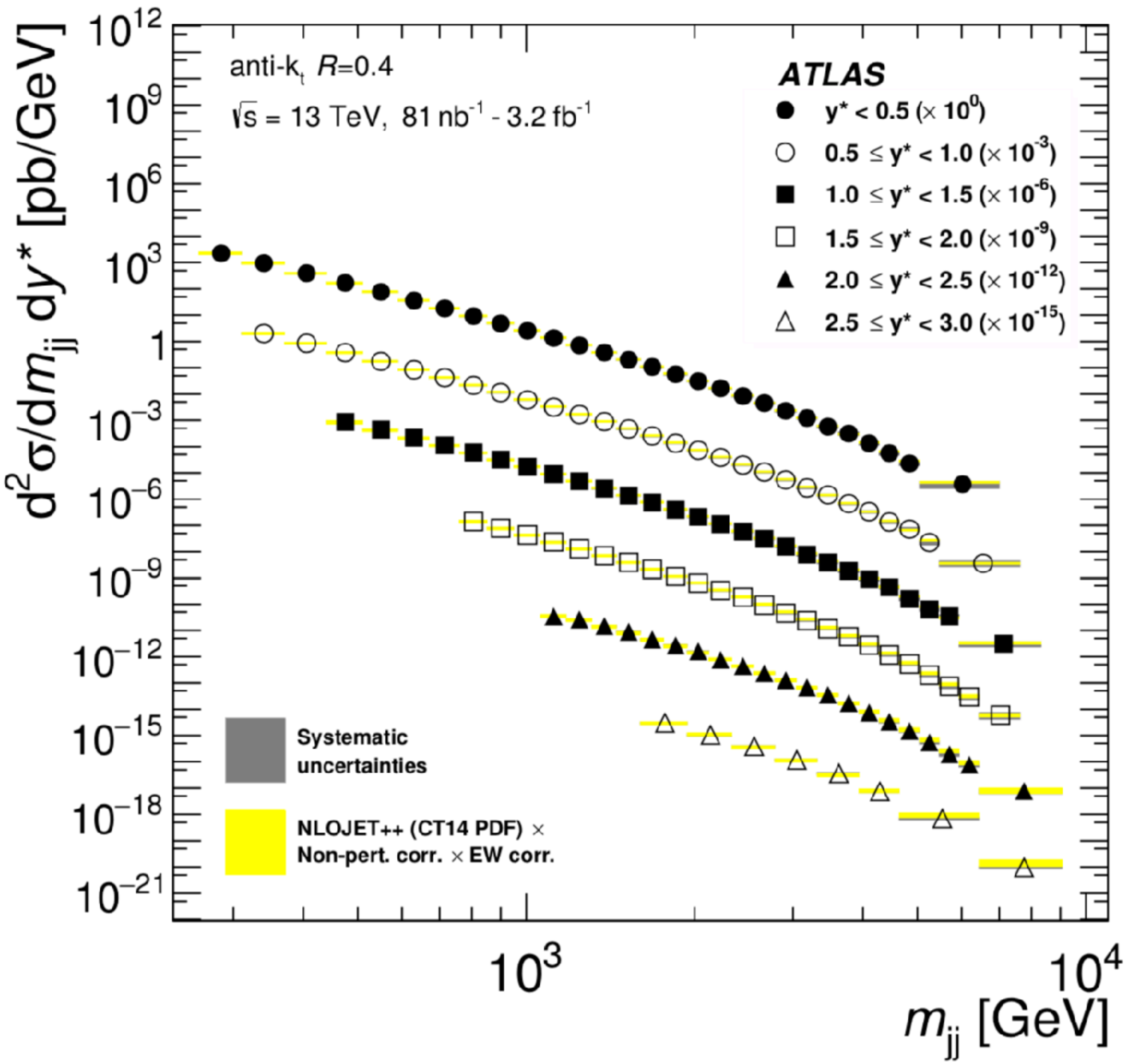
$$M \sim 1/q^2$$

$$M \sim 1/[\,(q^2 - M_Z^2) - iM_Z\Gamma_Z\,]$$

Z-boson resonance

note also: $1/(q^2 - M_Z^2) \to 1/M_Z^2$ for $q^2 \lesssim M_Z^2$ (see Effective Field Theory lecture)

Today



no sign of a resonance up to di-jet invariant masses of ~7 TeV

if New Physics scale is very large, only indirect effects visible $\Rightarrow$ need precision

QCD Lagrangian

strong interactions: $SU(N_c)$ gauge theory

$N_c = 3$ "colours" of quarks

non-Abelian structure of SU(3) related to gluon-self-interactions

purely gluonic part: Yang-Mills Lagrangian $\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$$

convention: doubly occurring indices are summed over

f^{abc} : structure constants of SU(3) (totally antisymmetric)

$[T^a, T^b] = i f^{abc} T^c$ $\quad T^a : N_c^2 - 1 \; (\widehat{=}\, 8)$ generators of SU(3)

8 gluons, in the adjoint representation of SU(3)

QCD Lagrangian

fermionic part: quark fields for flavour *f:* $q_f^i(x)$

$i = 1, 2, 3$: colour index

quarks are in the fundamental representation of SU(3)

$$\mathcal{L}_q(q_f\,,\,m_f) = \sum_{j,k=1}^{N_c} \bar{q}_f^j(x)\,(i\,\gamma_\mu D^\mu[A] - m_f)_{jk}\,q_f^k(x)$$

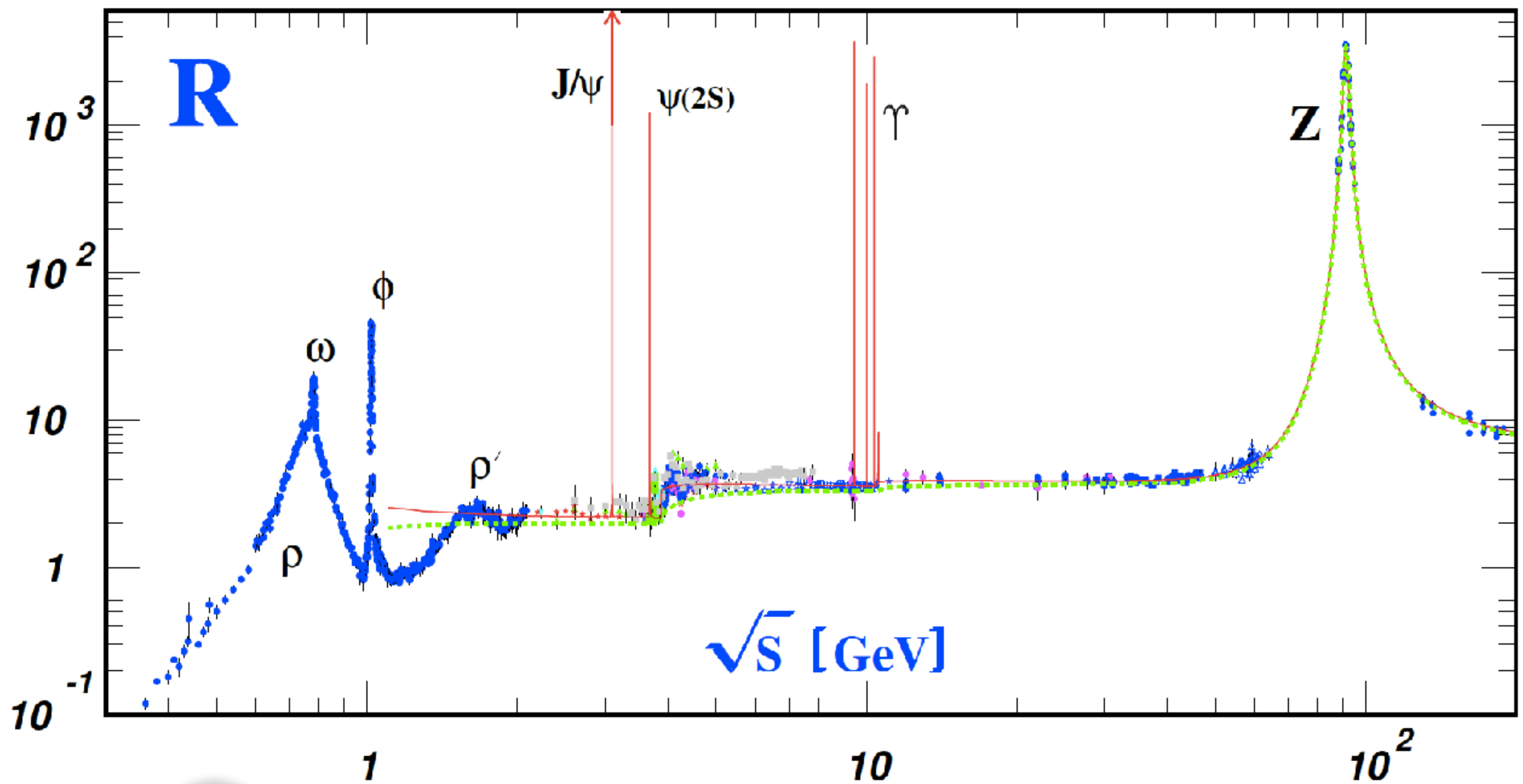
$(D^\mu[A])_{ij} = \delta_{ij}\partial^\mu + i\,g_s\,t^a_{ij}A^\mu_a$, define $\mathbf{A}^\mu = \sum_{a=1}^{8} t^a A^\mu_a$

$t^a_{ij} = \lambda^a_{ij}/2$ generators of SU(3) in fundamental representation

λ^a_{ij} : Gell-Mann matrices

Evidence for Colour

hadronic R-ratio: $$R(s) = \frac{\sigma(e^+e^- \to \text{ hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



$$R(s) = N_c \sum_{f=u,d,s,c,\ldots} e_f^2\,\theta(s - 4m_f^2) \qquad N_c = 3$$

$$R = 3\left((\frac{2}{3})^2 + (-\frac{1}{3})^2 + (-\frac{1}{3})^2 + \ldots \right)$$

u d s

threshold energy to produce quark-antiquark-pair of mass m_f

Gauge invariance

local gauge transformation with $U \in SU(3)$

$$q_i(x) \to q_i'(x) = U_{ij}(x) q_j(x), \; \bar{q}' = \bar{q}\, U^\dagger, \;\; \mathbf{A}(x) \to \mathbf{A}'(x) = ?$$

$$U_{ij}(x) = \exp\left\{ \frac{i}{2} \sum_{a=1}^{N_c^2-1} \lambda^a\, \theta^a(x) \right\}_{ij} = \delta_{ij} + \frac{i}{2} \sum_{a=1}^{N_c^2-1} \lambda^a_{ij}\, \theta^a(x) + \mathcal{O}(\theta^2)$$

remember

$$\mathcal{L}_q = \sum_{j,k=1}^{N_c} \bar{q}_f^j(x) \left(i\, \gamma_\mu D^\mu[\mathbf{A}] - m_f \right)_{jk} q_f^k(x)$$

$$D^\mu[\mathbf{A}] = \partial^\mu + i g_s \mathbf{A}^\mu$$ covariant derivative

for $\mathcal{L}_q$ to be invariant $D^\mu[\mathbf{A}] q(x) \stackrel{!}{\to} U \left(D^\mu[\mathbf{A}] q(x) \right)$

gauge invariance

$$(D^\mu[\mathbf{A}]q(x))' = (\partial_\mu - ig_s\mathbf{A}'_\mu(x))\,Uq(x) = U\left(\partial_\mu + U^\dagger(x)(\partial_\mu U(x)) - ig_s\mathbf{A}'_\mu(x)\right)q(x)$$

$$\stackrel{!}{=} U\left[D^\mu[\mathbf{A}]q(x)\right] = U[(\partial^\mu + ig_s\mathbf{A}^\mu)q(x)]$$

$$\Rightarrow U^\dagger(\partial_\mu U) - ig_s U^\dagger \mathbf{A}'_\mu U \stackrel{!}{=} -ig_s\mathbf{A}_\mu$$

$$\boxed{\Rightarrow \mathbf{A}'_\mu = U\,\mathbf{A}_\mu U^\dagger - \frac{i}{g_s}(\partial_\mu U)\,U^\dagger}$$

note: mass term ~ $\frac{1}{2}m\,\mathbf{A}_\mu(x)\mathbf{A}^\mu(x)$

would break gauge invariance

Colour Algebra

generators of SU(3): $[T^a, T^b] = i f^{abc} T^c$

fundamental representation: $t^a_{ij} = \lambda^a_{ij}/2 \quad i,j = 1,2,3; a = 1,\ldots,8$

λ^a_{ij} : Gell-Mann matrices (traceless, hermitian)

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Colour Algebra

adjoint representation:

dimension of the vector space on which it acts is equal to the dimension of the group itself

(dimension of the group = number of generators, $N^2 - 1$ for $SU(N)$)

matrices $(F^a)_{bc} = -i f^{abc}$ (8x8) - matrices

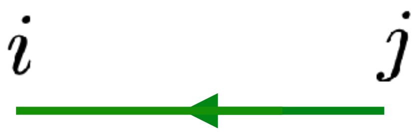
group invariants:

$$\sum_{j,a} t^a_{ij} t^a_{jk} = C_{\mathrm{F}}\, \delta_{ik} \,, \qquad \sum_{a,d} F^a_{bd} F^a_{dc} = C_{\mathrm{A}}\, \delta_{bc}$$

C_F, C_A : eigenvalues of Casimir operators in fundamental/adjoint representation

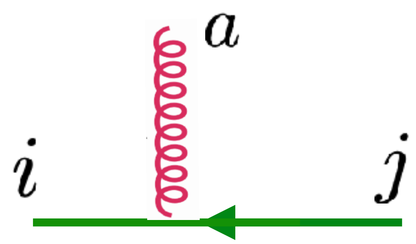
$$\boxed{C_F = T_R \frac{N_c^2 - 1}{N_c} \,,\; C_{\mathrm{A}} = 2\, T_R\, N_c}$$ usually $T_R = \frac{1}{2}$ (convention)

Colour Algebra pictorially

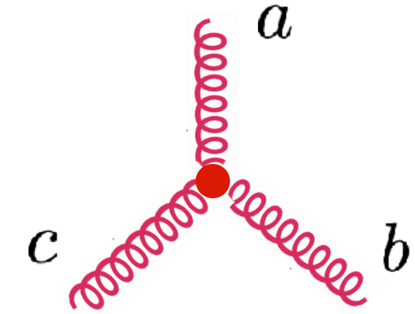


$\stackrel{\text{colour}}{=} \delta_{ij}$

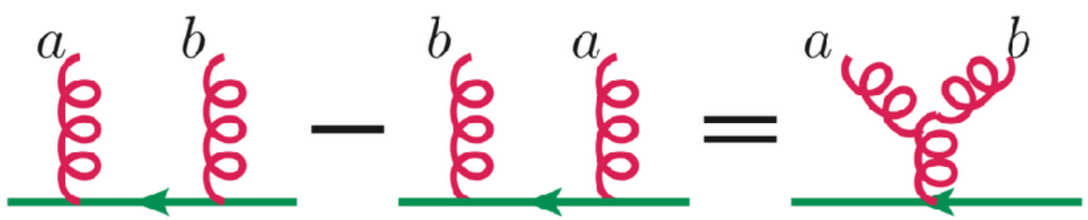
$\stackrel{\text{colour}}{=} \delta_{ab}$



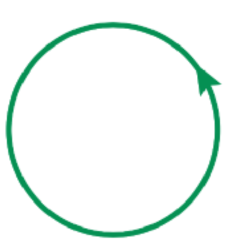
$\stackrel{\text{colour}}{=} t^a_{ij}$



$\stackrel{\text{colour}}{=} i f^{abc}$



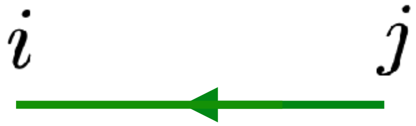
$$[t^a, t^b] = i f^{abc} t^c$$



$\stackrel{\text{colour}}{=} \delta_{ij}\delta^{ij} = N_c$

$\stackrel{\text{colour}}{=} \delta_{ab}\delta^{ab} = N_c^2 - 1$

Colour Algebra pictorially



$\stackrel{\text{colour}}{=} \delta_{ij}$

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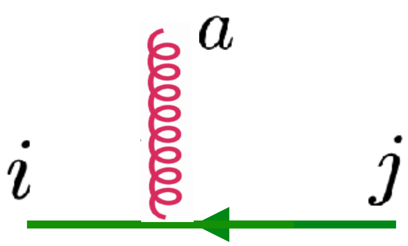
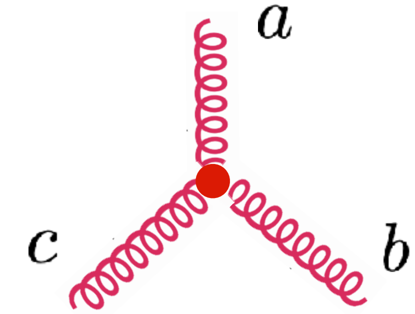
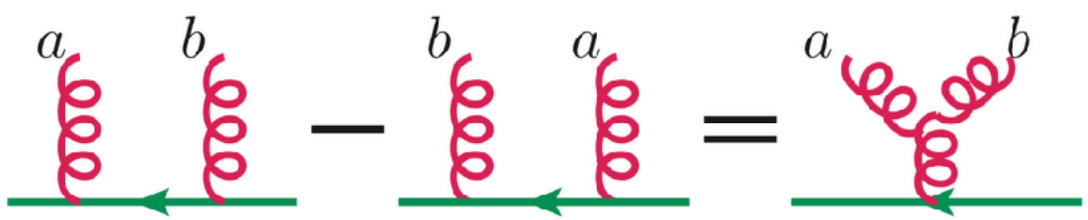


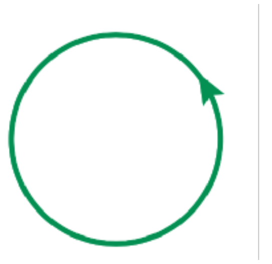
$\stackrel{\text{colour}}{=} t^a_{ij}$



$\stackrel{\text{colour}}{=} i f^{abc}$



$$[t^a, t^b] = i f^{abc} t^c$$



$\stackrel{\text{colour}}{=} \delta_{ij}\delta^{ij} = N_c$

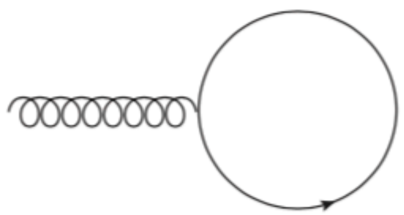
$\stackrel{\text{colour}}{=} \delta_{ab}\delta^{ab} = N_c^2 - 1$

more: see exercises

colour algebra exercise

calculate the colour factors for the following diagrams

(1)



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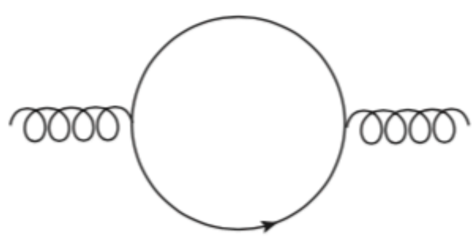

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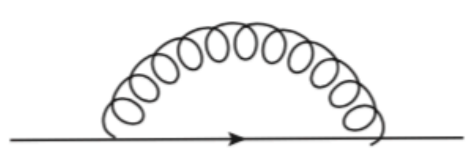
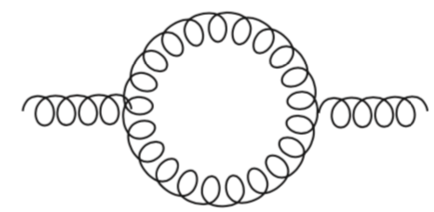
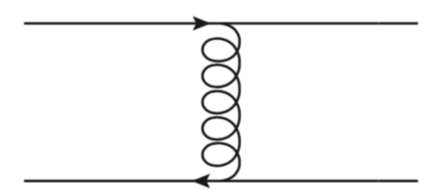


(4)



(5)



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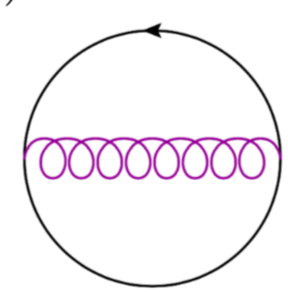
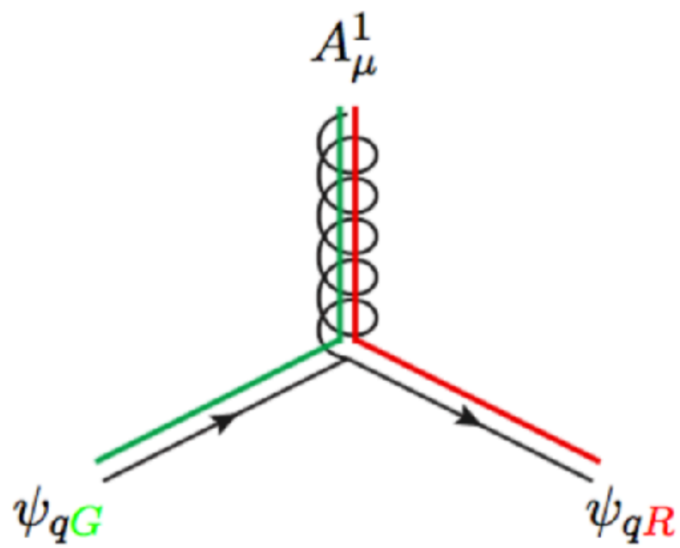


Colour Algebra

explicit example for a quark-gluon vertex $-ig_s\,\bar{\psi}_i\,\frac{\lambda^a_{ij}}{2}\psi_j\,A^a_\mu$

$i=1, j=2, a=1:$

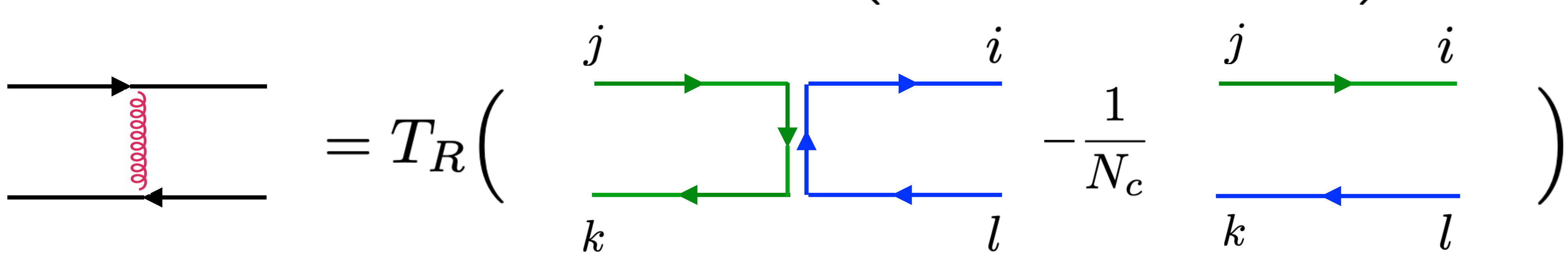


$$\begin{array}{ccccc} \propto & -\frac{i}{2}g_s & \bar{\psi}_{qR} & \lambda^1 & \psi_{qG} \\ = & -\frac{i}{2}g_s & \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{array}$$

figure: Peter Skands

gluon represented as double line of two colours

therefore relation $t^a_{ij}t^a_{kl} = T_R\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$ (second term because of traceless generators)



QCD Lagrangian

- the gluon propagator $\Delta^{ab}_{\mu\nu}(p) = \Delta_{\mu\nu}(p)\delta^{ab}$ is constructed from the inverse of the bilinear term in the gluon fields in the action

- in momentum space it should fulfil

$$i\,\Delta_{\mu\rho}(p)\,\left[p^2 g^{\rho\,\nu} - p^\rho p^\nu\right] = g^\nu_\mu \qquad (1)$$

however as $\left[p^2 g^{\rho\,\nu} - p^\rho p^\nu\right] p_\nu = 0$

Eq. (1) has zero modes, so [.] is not invertible

reason: $\mathcal{L}_q + \mathcal{L}_{\rm YM}$ contains physically equivalent configurations

$\Rightarrow$ need *gauge fixing:* add constraint on gluon fields with a Lagrange multiplier to the Lagrangian

QCD Lagrangian

covariant gauges: add condition $\partial_\mu A^\mu(x) = 0$

$$\Rightarrow \quad \mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2, \quad \lambda \in \mathbb{R}$$

$$i\left(p^2 g^{\mu\nu} - \left(1 - \frac{1}{\lambda}\right) p^\mu p^\nu\right)$$

$$\Delta_{\mu\nu}(p) = \frac{-i}{p^2 + i\,\varepsilon}\left[g_{\mu\nu} - (1-\lambda)\frac{p_\mu p_\nu}{p^2}\right]$$

$\lambda = 1$: Feynman gauge

$\lambda = 0$: Landau gauge

QCD Lagrangian

- however in covariant gauges unphysical (non-transverse) degrees of freedom also can propagate

- their effect is cancelled by *ghost fields* η^a

  coloured complex scalars obeying Fermi statistics

$$\mathcal{L}_{FP} = \eta_a^\dagger \, M^{ab} \, \eta_b$$

Faddeev-Popov matrix in Feynman gauge:

$$M^{ab}_{Feyn} = \delta^{ab} \, \partial_\mu \partial^\mu + g_s \, f^{abc} A^c_\mu \partial^\mu$$

$$\boxed{\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \mathcal{L}_q + \mathcal{L}_{GF} + \mathcal{L}_{FP}}$$

QCD Lagrangian

- unphysical degrees of freedom and the ghost fields can be avoided by choosing axial (physical) gauges

in axial gauges: n^μ vector with $p \cdot n \neq 0$

$$\mathcal{L}_{GF} = -\frac{1}{2\alpha}\left(n^\mu A_\mu\right)^2$$

gluon propagator:

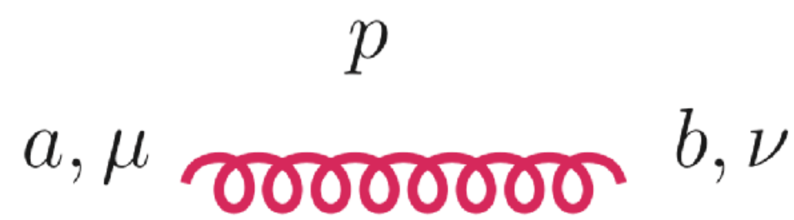
$$\Delta_{\mu\nu}(p,n) = \frac{-i}{p^2 + i\,\varepsilon}\left(g_{\mu\nu} - \frac{p_\mu n_\nu + n_\mu p_\nu}{p \cdot n} + \frac{n^2\, p_\mu p_\nu}{(p \cdot n)^2}\right)$$

light-cone gauge: $n^2 = 0$

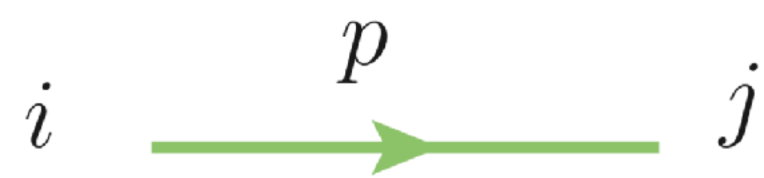
Feynman rules

Propagators:

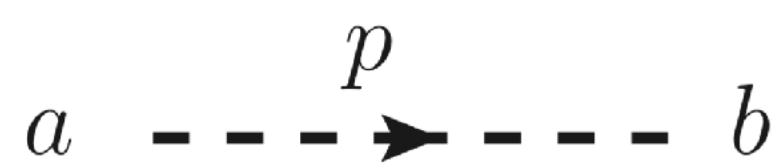
gluon $\Delta^{ab}_{\mu\nu}(p) = \delta^{ab}\Delta_{\mu\nu}(p)$



quark $\Delta^{ij}(p) = \delta^{ij}\dfrac{i(\not{p}+m)}{p^2 - m^2 + i\,\varepsilon}$



ghost $\Delta^{ab}(p) = \delta^{ab}\dfrac{i}{p^2 + i\,\varepsilon}$



$$\Delta_{\mu\nu}(p) = \frac{-i}{p^2 + i\,\varepsilon}\left[g_{\mu\nu} - (1-\lambda)\frac{p_\mu p_\nu}{p^2}\right]$$

covariant gauge

$$\Delta_{\mu\nu}(p,n) = \frac{-i}{p^2 + i\,\varepsilon}\left[g_{\mu\nu} - \frac{p_\mu n_\nu + n_\mu p_\nu}{p \cdot n}\right]$$

light-cone gauge

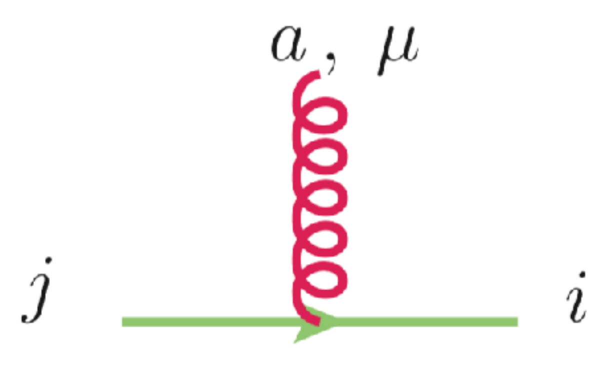
$$n^2 = 0$$

Feynman rules

Vertices:

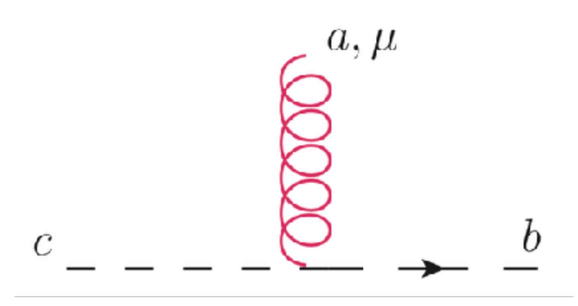
quark-gluon

$$\Gamma^{\mu,\,a}_{gq\bar{q}} = -i\,g_s\,(t^a)_{ij}\gamma^\mu$$



ghost-gluon

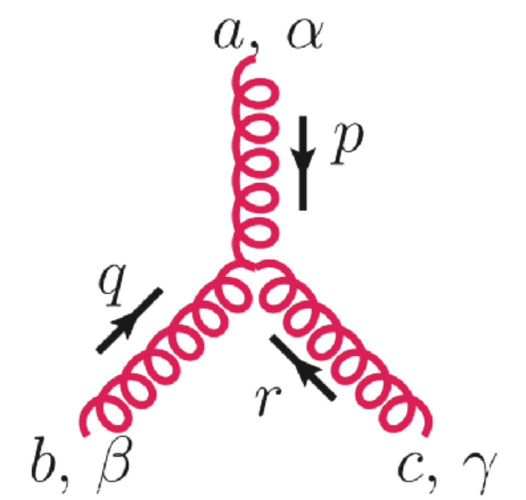
$$\Gamma^{\mu,\,a}_{g\eta\bar{\eta}} = -i\,g_s\,(F^a)_{bc}\,p^\mu$$
$$= -g_s\,f^{abc}\,p^\mu$$



Feynman rules

3-gluon

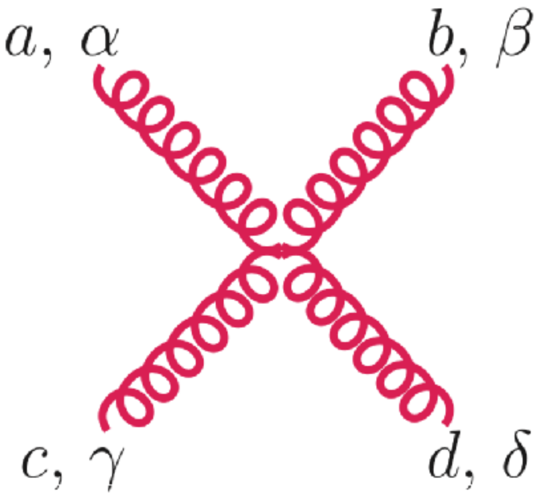
$$\Gamma^{abc}_{\alpha\beta\gamma}(p,q,r) = -i\,g_s\,(F^a)_{bc}\,V_{\alpha\beta\gamma}(p,q,r)$$



$$V_{\alpha\beta\gamma}(p,q,r) = (p-q)_\gamma g_{\alpha\beta} + (q-r)_\alpha g_{\beta\gamma} + (r-p)_\beta g_{\alpha\gamma}$$

4-gluon

$$\Gamma^{abcd}_{\alpha\beta\gamma\delta} = -i\,g_s^2\left[\begin{array}{l} +f^{xac}\,f^{xbd}\,(g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\beta\gamma}) \\ +f^{xad}\,f^{xcb}\,(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\beta}g_{\gamma\delta}) \\ +f^{xab}\,f^{xdc}\,(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}) \end{array}\right]$$



Feynman rules

spinors:

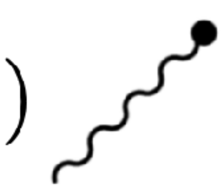
incoming fermion $u(p,s)$ 

incoming anti-fermion $\bar{v}(p,s)$ 

outgoing fermion $\bar{u}(p,s)$ 

outgoing anti-fermion $v(p,s)$ 

polarisation vectors:

incoming vector boson $\varepsilon_\mu(k,\lambda)$ 

outgoing vector boson $\varepsilon^*_\mu(k,\lambda)$ 

Feynman rules

further rules:

- momentum conservation at each vertex
- factor (-1) for each closed fermion loop
- factor (-1) for switching identical external fermions
- integrate over loop momenta with $\int \frac{d^4 l}{(2\pi)^4}$

spinor relations

useful identities:

$$(\not{p} - m)\, u(p,s) = 0 \qquad \bar{u}(p,s)(\not{p} - m) = 0$$
$$(\not{p} + m)\, v(p,s) = 0 \qquad \bar{v}(p,s)(\not{p} + m) = 0$$

(Dirac equation)

$$\bar{v}(p,r)u(p,s) = 0 \qquad \bar{u}(p,r)u(p,s) = 2m\,\delta_{rs}$$
$$\bar{u}(p,r)v(p,s) = 0 \qquad \bar{v}(p,r)v(p,s) = -2m\,\delta_{rs}$$

(orthogonality)

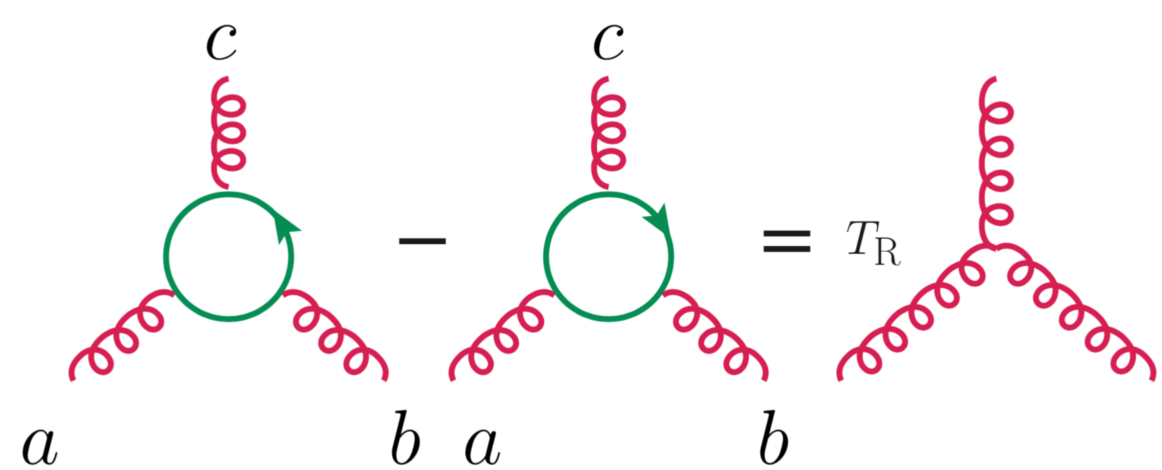
$$\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m$$
$$\sum_s v(p,s)\bar{v}(p,s) = \not{p} - m$$

(completeness)

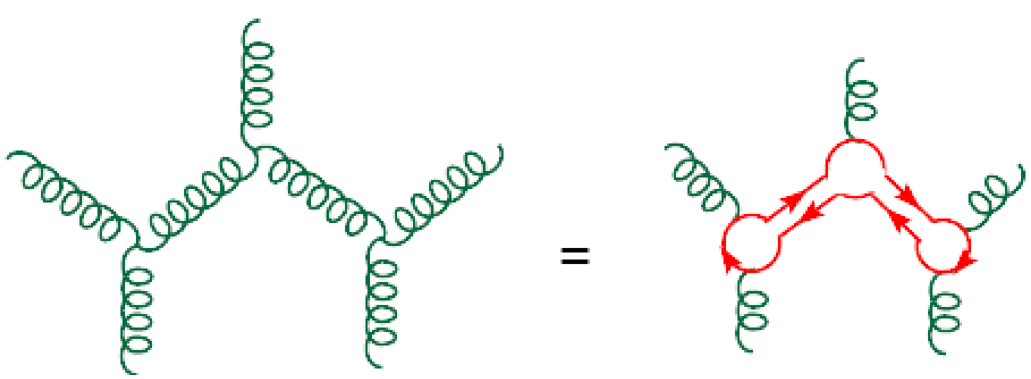
colour decomposition

using the “double line notation”, we can express gluon amplitudes entirely in terms of generators t^a_{ij}

based on



$$\mathrm{Trace}(t^a t^b t^c) - \mathrm{Trace}(t^c t^b t^a) = i\, T_R f^{abc}$$



+ *permutations*

$$\mathrm{Tr}(t^{a_1} t^{a_2} \ldots t^{a_n})$$ + all non-cyclic permutations (with corresponding signs)

similarly $q\bar{q}gggg\ldots \rightarrow \mathrm{Tr}(t^{a_1} t^{a_2} \ldots t^{a_n})_{ij}$ + permutations

colour decomposition

- therefore the colour part of the amplitude can be completely separated from the kinematic part

$$\mathcal{M}_n^{\rm tree}(\{p_i,a_i,h_i\}) = g^{n-2} Tr(t^{A_1}t^{A_2}\cdots t^{A_n})\, M_n^{\rm tree}(1^{h_1},2^{h_2}\ldots n^{h_n}) + \text{all non-cyclic permutations}$$

momenta, colour, helicities

colour ordered partial amplitude, colour factors stripped off

useful property: as $M_n^{\rm tree}(1^{h_1},2^{h_2}\ldots n^{h_n})$ comes from diagrams with cyclic ordering of external legs, it only can have infrared singularities in adjacent invariants

$$s_{i,i+1} = (p_i+p_{i+1})^2$$

- colour decomposition at loop level is straightforward, but then single and double trace structures are generated

$$\begin{aligned}
&\mathcal{A}_n^{1-\rm loop}(\{k_i,\lambda_i,a_i\}) \\
&= g^n \Bigg[\sum_{\sigma\in S_n/Z_n} N_c\, \mathrm{Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}\right)\, A_{n;1}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n})) \\
&\quad + \sum_{c=2}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_n/S_{n;c}} \mathrm{Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(c-1)}}\right)\, \mathrm{Tr}\left(T^{a_{\sigma(c)}}\cdots T^{a_{\sigma(n)}}\right) \\
&\qquad \times A_{n;c}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n})) \Bigg].
\end{aligned}$$

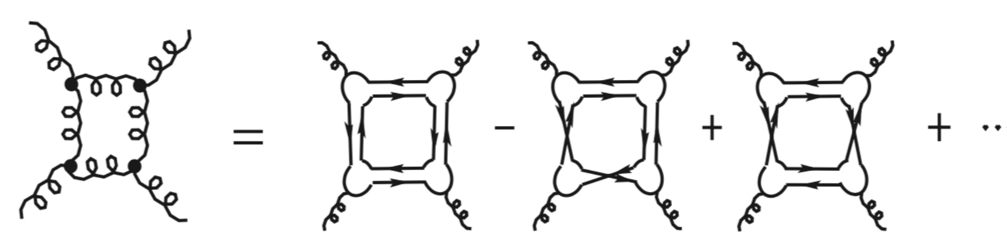
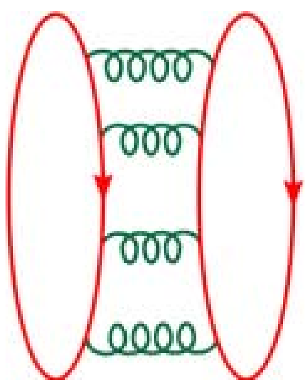


figure: Lance Dixon

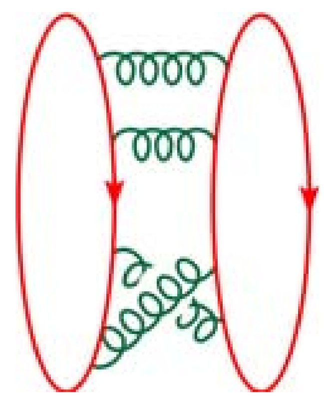
colour expansion

$$d\sigma^{\text{tree}}(\{p_i, a_i, h_i\}) \sim \sum_{a_i} \sum_{h_i} \left| M_n^{\text{tree}}(\{p_i, a_i, h_i\}) \right|^2$$

insert colour ordered amplitude and perform the colour sum:



$$= N_c^n$$



$$= N_c^n \times \frac{1}{N_c^2}$$

$$d\sigma^{\text{tree}}(\{p_i, a_i, h_i\}) \sim N_c^n \sum_{\sigma \in S_n/Z_n} \sum_{h_i} \left| M_n^{\text{tree}}(\sigma(1^{h_1}), \sigma(2^{h_2}) \ldots \sigma(n^{h_n})) \right|^2 + \mathcal{O}(N_c^{n-2})$$

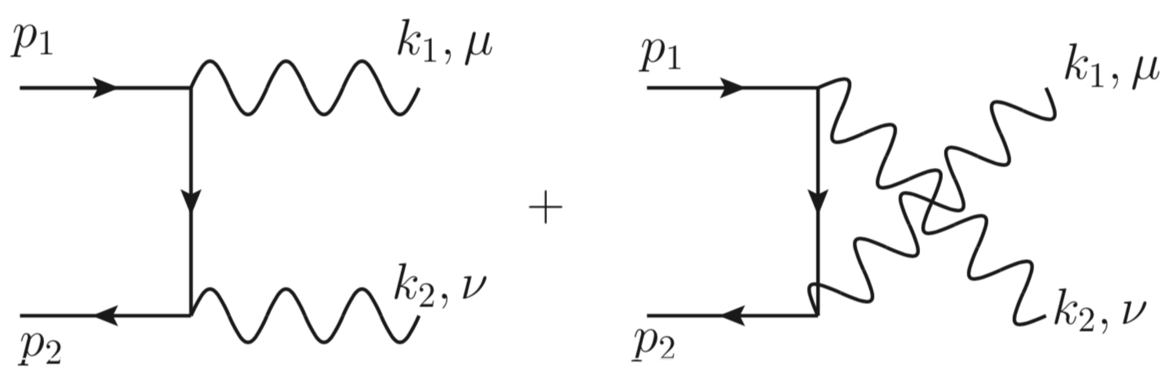
Non-planar topologies are subleading in colour

Note: parton showers usually do not take subleading colour into account

Tree level amplitudes

the non-Abelian structure of QCD leads to important differences compared to QED (physical polarisations, beta-function, …)

consider first a simple QED process: $e^+e^- \to \gamma\gamma$



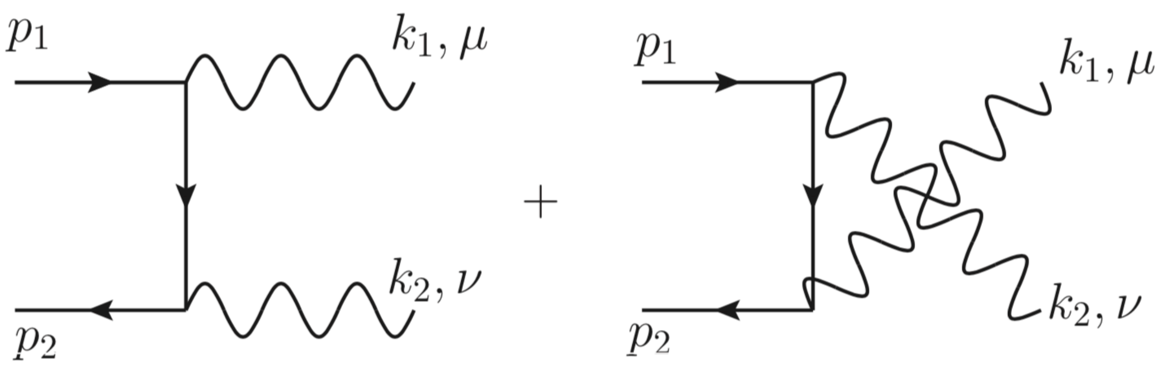
$$\mathcal{M} = -i\, e^2\, \epsilon_1^\mu(k_1)\epsilon_2^\nu(k_2)\, M_{\mu\nu} \quad , \qquad M_{\mu\nu} = M_{\mu\nu}^{(1)} + M_{\mu\nu}^{(2)}$$

$$M_{\mu\nu}^{(1)} = \bar{v}(p_2)\,\gamma_\nu \frac{\not{p}_1 - \not{k}_1}{(p_1 - k_1)^2}\gamma_\mu\, u(p_1)$$

$$M_{\mu\nu}^{(2)} = \bar{v}(p_2)\,\gamma_\mu \frac{\not{p}_1 - \not{k}_2}{(p_1 - k_2)^2}\gamma_\nu\, u(p_1)$$

Tree level amplitudes

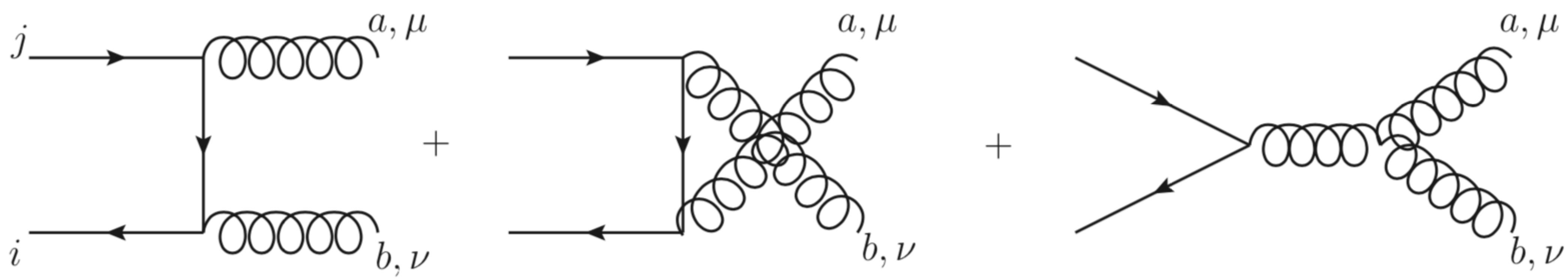
$$e^+e^- \to \gamma\gamma$$



$$\mathcal{M} = -i\, e^2\, \epsilon_1^\mu(k_1)\epsilon_2^\nu(k_2)\, M_{\mu\nu}$$

consider the current $J_\mu \equiv \epsilon_2^\nu M_{\mu\nu}$ charge conservation implies $k_1^\mu J_\mu = 0$

QCD analogue: $e^+e^- \to gg$



$$\mathcal{M} = -i\, g_s^2\, \epsilon_1^\mu(k_1)\epsilon_2^\nu(k_2)\, M_{\mu\nu}^{\rm QCD}$$

$$M_{\mu\nu}^{\rm QCD} = (t^b t^a)_{ij}\, M_{\mu\nu}^{(1)} + (t^a t^b)_{ij}\, M_{\mu\nu}^{(2)} + M_{\mu\nu}^{(3)}$$

Tree level amplitudes

use $(t^b t^a)_{ij} = (t^a t^b)_{ij} - i\, f^{abc} t^c_{ij}$

$$M^{\rm QCD}_{\mu\nu} = (t^a t^b)_{ij} \left[M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu} \right] - i\, f^{abc} t^c_{ij}\, M^{(1)}_{\mu\nu} + M^{(3)}_{\mu\nu}$$

term in square brackets is the same as in QED, so

$$k_i^\mu \left[M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu} \right] = 0$$

for the remaining terms we find

$$k_1^\mu M^{(1)}_{\mu\nu} = -\bar{v}(p_2)\, \gamma_\nu\, u(p_1)$$

$$k_1^\mu M^{(3)}_{\mu\nu} = \underbrace{i\, f^{abc} t^c_{ij} \bar{v}(p_2)\, \gamma_\nu\, u(p_1)}_{\text{cancels with contribution from } M^{(1)}_{\mu\nu}} - i\, f^{abc} t^c_{ij}\, \bar{v}(p_2) \;\; \not{k}_1\, u(p_1)\, \frac{k_{2,\nu}}{2 k_1 \cdot k_2}$$

vanishes only when contracted with the polarisation vector of a physical gluon, i.e. if $\epsilon^\nu(k_2) \cdot k_2 = 0$

Polarisation sums

QCD:

$$k_1^\mu \epsilon^\nu(k_2) M_{\mu\nu} \sim \epsilon(k_2)\cdot k_2 \quad \Rightarrow$$ vanishes only for physical gluons

QED:

$$k_1^{\mu_1} \ldots k_n^{\mu_n} \mathcal{M}_{\mu_1 \ldots \mu_n} = 0$$ regardless whether $\epsilon(k_j)\cdot k_j = 0$ or not

for cross sections we need $|\mathcal{M}|^2$

$$|\mathcal{M}|^2 \sim \sum_{\text{phys. pol}} \epsilon_{\mu_1}(k_1)\epsilon_{\nu_1}(k_2)\mathcal{M}^{\mu_1\nu_1}\epsilon^\star_{\mu_2}(k_1)\epsilon^\star_{\nu_2}(k_2)\left(\mathcal{M}^{\mu_2\nu_2}\right)^\star$$

let us consider just k_1 (the second boson is treated analogously)

Polarisation sums

In QED, we can make the replacement

$$\sum_{\text{phys. pol}} \epsilon_{\mu_1}(k_1)\epsilon^{\star}_{\mu_2}(k_1) \to -g_{\mu_1\mu_2}$$

In QCD, this will in general lead to the wrong result. Why?

sum over *physical* polarisations:

$$\sum_{i=L,R} \epsilon_i^{\mu}(k)\epsilon_i^{\nu,\star}(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}$$

$$\epsilon_{L,R} = (0,1,\pm i,0)/\sqrt{2} \qquad k = (k^0,0,0,k^0) \qquad n = (k^0,0,0,-k^0)$$

define $\mathcal{M}_\mu = \epsilon^\nu(k_2)M_{\mu\nu}$, in QED $k^\mu\mathcal{M}_\mu = 0 \Rightarrow$ second part of pol. sum will not contribute

Polarisation sums

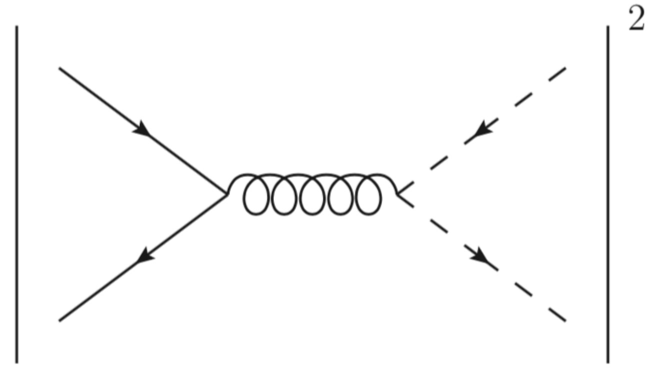
in QCD: $k_1^\mu \mathcal{M}_\mu \sim \epsilon(k_2) \cdot k_2$

therefore, if $\epsilon(k_2) \cdot k_2 \neq 0$ we can **not** just use $-g^{\mu\nu}$ for the polarisation sum

it can be shown that

$$S_{\text{unphys.}} \equiv \sum_{\text{unphysical pol.}} |\epsilon_\mu(k_1)\epsilon_\nu(k_2)\mathcal{M}^{\mu\nu}|^2 = \left| i\, g_s^2 f^{abc} t^c \bar{v}(p_2)\, \frac{\not{k}_1}{(k_1+k_2)^2}\, u(p_1) \right|^2$$

calculating this ghost contribution



results in $-S_{\text{unphys.}}$

ghost degrees of freedom cancel the unphysical gluon polarisations!

true in polarisation sums as well as virtual loop diagrams

$\Rightarrow$ when calculation in a non-physical gauge (e.g. Feynman gauge), need also ghost loops in virtual corrections

Summary

- Hadron Collider experiments need QCD corrections
- Description as SU(3) local gauge theory has important consequences
- We got familiar with the basics of the colour algebra and the QCD Feynman rules
- Application to tree level example $e^+e^- \to gg$

We saw how to sum over gluon polarisations when squaring an amplitude

next: NLO!

Exercise

calculate $d\sigma/d\Omega$ for electron-muon scattering

$$e^-(p_1) + \mu^-(p_2) \to e^-(p_3) + \mu^-(p_4)$$