6

6.1

1. An epimorphism $(\alpha, \beta) : M_1 \to M_2$ with complete machines implies $M_2 \leq M_1$:
   We choose $\alpha : Q_1 \to Q_2$ as our $\eta : Q_1 \to Q_2$. We define $\xi : \Sigma_2 \to \Sigma_1$ as a function which picks an arbitrary element of $\xi_{a_2} = \{a_1 | \beta(a_1) = a_2, a_1 \in \Sigma_1\}$ for an arbitraty $a_2 \in \Sigma_2$ (1). There exists at least one a_1 for each a_2 because β is surjective (2). Since (α, β) is a state machine homomorphismus and the maschines are complete following holds for all $a_1 \in \Sigma_1$ and all $q_1 \in Q_1$ (Definition 3.37):

   $$\alpha(q_1 \delta^1_{a_1}) = (\alpha(q_1))\delta^2_{\beta(a_1)} \tag{3}$$

   Now we show: $\eta(q_1)\delta^2_{a_2} \subseteq \eta(q_1 \delta^1_{\xi(a_2)})$ for all $q_1 \in Q_1$ and $a_2 \in \Sigma_2$.

   $$\begin{aligned} \eta(q_1)\delta^2_{a_2} =& \alpha(q_1)\delta^2_{a_2} \\ =& \alpha(q_1)\delta^2_{\beta(a_1)} && (2) \\ =& \alpha(q_1 \delta^1_{a_1}) \\ =& \alpha(q_1 \delta^1_{\xi(a_2)}) && (1) \\ =& \eta(q_1 \delta^1_{\xi(a_2)}) \end{aligned}$$

2. A monomorphismus $(\alpha, \beta) : M_1 \to M_2$ with complete machines implies $M_1 \leq M_2$:
   We choose $\beta : \Sigma_1 \to \Sigma_2$ as our $\xi : \Sigma_1 \to \Sigma_2$ (4). We define $\eta : Q_2 \to Q_1$ as the inverse function from $\alpha : Q_1 \to Q_2$ which is well defined (Lemma 2.20) (5). For each $q_2 \in Q_2$ with $\alpha^{-1}(q_2) \neq \emptyset$ there exists exactly one $q_1 = \alpha^{-1}(q_2)$ because α is

injective. We show: $\eta(q_2)\delta^1_{a_1} \subseteq \eta(q_2\delta^2_{\xi(a_1)})$ for all $q_2 \in Q_2$ and $a_1 \in \Sigma_1$.

$$\begin{aligned}
\eta(q_2)\delta^1_{a_1} =& \alpha^{-1}(q_2)\delta^1_{a_1} && (5)\\
=& q_1\delta^1_{a_1} \\
=& \alpha^{-1}(\alpha(q_1\delta^1_{a_1})) && \text{wir sind so kluk, K. L. U. K.!} \quad (3)\\
=& \alpha^{-1}(\alpha(q_1)\delta^2_{\beta(a_1)}) \\
=& \alpha^{-1}(\alpha(q_1)\delta^2_{\beta(a_1)}) && (4)\\
=& \alpha^{-1}(\alpha(q_1)\delta^2_{\xi(a_1)}) \\
=& \alpha^{-1}(q_2\delta^2_{\xi(a_1)}) \\
=& \eta(q_2\delta^2_{\xi(a_1)})
\end{aligned}$$

6.2

For the covering relation to be a partial order it has to be *reflexive, antisymmetric* and *transitive.*

Proof:

Lets begin by showing that $\leq$ is *reflexive*, for this let $M = (Q, \Sigma, \delta)$. We need to show that $M \leq M$ is true. For M to cover itself, there needs to exist a partial function $\eta : Q \to Q$ which is

1. surjective
2. $\eta(q')\delta_w = \eta(q'\delta'_w)$ for all $w \in \Sigma^*$ and all $q' \in \mathcal{D}(\eta)$
3. $\eta(q') = \emptyset$ if $q' \notin \mathcal{D}(\eta)$
4. $q\delta_w = \emptyset$ if $\delta(q, w)$ is undefined

We use id_Q for η, since, we do not need to change anything for M to cover it self. id_Q is surjective, because the identity function maps every element in Q onto the same element in Q, therefore, it hits all elements in Q. Since, $\delta_w = \delta'_w$ the second primitive holds. The third holds for the identity function, since, it can only map an element to itself and if q' is not in its domain (here Q itself) it can not map it onto anything thus $id_Q(q') = \emptyset$ if $q' \notin \mathcal{D}(\eta)$. And the fourth holds because M is complete.

Next we show that $\leq$ is *antisymmetric*:

Let $M_1 = (Q, \Sigma, \delta), M_2 = (Q', \Sigma, , \delta')$ be two state machines with $M_1 \leq M_2 \wedge M_2 \leq M_1$. We need to show that for this $M_1 = M_2$. Since, $M_1 \leq M_2 \wedge M_2 \leq M_1$ we know there exists two partial functions $\eta : Q' \to Q \wedge \eta' : Q \to Q'$ which are surjective. Furthermore, we know that $\eta(q')\delta_w = \eta(q'\delta'_w)$ for all $w \in \Sigma^*$ and all $q' \in \mathcal{D}(\eta)$ and $\eta(q)\delta'_w = \eta(q\delta_w)$ for all $w \in \Sigma^*$ and all $q \in \mathcal{D}(\eta')$, therefore, the two machines have to be isometric.

At last we show *transitivity*: Let $M_1 = (Q_1, \Sigma, \delta), M_2 = (Q_2, \Sigma, \delta_2), M_3 = (Q_3, \Sigma, \delta_3)$ be state machines. Let $M_1 \leq M_2 \leq M_3$ with $\eta_1 : Q_2 \to Q_1 \wedge \eta_2 : Q_3 \to Q_2$ such that

$$\eta_1(q_2)\delta^1_w = \eta_1(q_2\delta^2_w) \mid \forall w \in \Sigma^*, \forall q_2 \in Q_2 \tag{6.1}$$

$$\eta_2(q_3)\delta^2_w = \eta_2(q_3\delta^3_w) \mid \forall w \in \Sigma^*, \forall q_3 \in Q_3 \tag{6.2}$$

The claim is that if this holds, $M_1 \leq M_3$ also holds. For $M_1 \leq M_3$ we need to find an $\eta_3 : Q_3 \to Q_1$. Our claim is that the concatenation of η_1 and η_2 will hold for η_3

$$\begin{aligned}
\eta_3(q_3)\delta^1_w =& \eta_1(\eta_2(q_3))\delta^1_w \\
=& \eta_1(\eta_2(q_3)\delta^2_w) \\
=& \eta_1(\eta_2(q_3\delta^3_w)) \\
=& \eta_3(q_3\delta^3_w)
\end{aligned}$$

This concludes the proof. $\square$

6.3

6.4

We show $M \leq M'$ by using the following covering (η, ξ):

$$\eta : Q' \hookrightarrow Q; q' \mapsto \begin{cases} q_0, & \text{for } q' = r_0, \\ q_1, & \text{for } q' = r_2, \end{cases}$$

$$\xi : \Sigma \to \Sigma'; u \mapsto \begin{cases} b', & \text{for } u = a, \\ a', & \text{for } u = b, \end{cases}$$

Moreover we prove η, ξ is well defined. First, ξ is a function because all $dom(\xi) = \Sigma$ and it is right unique. η is surjective as its range hits all states of Q. Also it is right unique and therefore a partial function. Finally, we show $\eta(q')\delta_w \subseteq \eta(q'\delta'_{\xi(w)})$ for all $q' \in Q'$ and $w \in \Sigma^*$. We build an operation table to get $\Sigma^+/\sim_M$ to get all relevant w.

x	q_0	q_1	
a	q_1	$\perp$	
b	q_0	q_1	
aa	$\perp$	$\perp$	
ab	q_1	$\perp$	same as a
bb	q_0	q_1	same as b
ba	q_1	$\perp$	same as a
aaa	$\perp$	$\perp$	same as aa
aab	$\perp$	$\perp$	same as aa
aba	$\perp$	$\perp$	same as aa
abb	q_1	$\perp$	same as a
bba	q_1	$\perp$	same as a
baa	q_1	$\perp$	same as aa
bab	q_1	$\perp$	same as a

We get $\Sigma^+/\sim_M = \{[a], [b], [aa]\}$. Since $\eta(q') = \emptyset | q' \notin \mathcal{D}(\eta)$ we only observe for r_0, r_2 as states from M'. We check for all relevant $q' \in Q'$ and $win\Sigma$:

$$\begin{aligned}
\eta(r_0)\delta_a = q_0\delta_a = q_1 \subseteq& \eta(r_0\delta'_{\xi(a)}) = \eta(r_0\delta'_{b'}) = \eta(r_2) = q_1 \\
\eta(r_0)\delta_b = q_0\delta_b = q_0 \subseteq& \eta(r_0\delta'_{\xi(b)}) = \eta(r_0\delta'_{a'}) = \eta(r_0) = q_0 \\
\eta(r_0)\delta_{aa} = q_0\delta_{aa} = \emptyset \subseteq& \eta(r_0\delta'_{\xi(aa)}) = \eta(r_2\delta'_{b'b'}) = \eta(\emptyset) = \emptyset \\
\eta(r_2)\delta_a = q_1\delta_a = \emptyset \subseteq& \eta(r_2\delta'_{\xi(a)}) = \eta(r_2\delta'_{b'}) = \eta(r_3) = \emptyset \\
\eta(r_2)\delta_b = q_1\delta_b = q_1 \subseteq& \eta(r_2\delta'_{\xi(b)}) = \eta(r_2\delta'_{a'}) = \eta(r_2) = q_1 \\
\eta(r_2)\delta_{aa} = q_1\delta_{aa} = \emptyset \subseteq& \eta(r_2\delta'_{\xi(aa)}) = \eta(r_2\delta'_{b'b'}) = \eta(\emptyset) = \emptyset
\end{aligned}$$

For each combination $\eta(q')\delta_w \subseteq \eta(q'\delta'_{\xi(w)})$ holds which concludes the proof. □