2

2.1

Since the group is finite, we can generate the powers of an element $s \in S$ infinetly. However there a only a finite amount of elements in S. Thus, there exists an integer x such that $s^x = s$. If $x = z$ we are done, else we multiply both sides with s^{x-2}.

$$\begin{aligned} s^2 \cdot s^{x-2} &= s \cdot s^{x-2} \\ (s^x \cdot s^{-1}) \cdot (s^x \cdot s^{-1}) &= s^{x-1} \\ s^{x-1} \cdot s^{x-1} &= s^{x-1} \\ (s^{x-1})^2 &= s^{x-1} \end{aligned}$$

$\square$

2.2

First we proof $g, g' \in G, g \sim_H g'$ if $g = hg'$ is an equivalence relation by showing it is reflexive, simmetric and transitive.
Reflexivity: Assume $g \sim_H g'$ then we always can choose the neutral element as h with $hg' = g'$. Thus $g = g'$ and $g \sim_H g$.
Symmetry: Assume $g \sim_H g'$ with $g = hg'$. Then we can add in inverse element h^{-1}, which is also in H because it is a group, resulting in $h^{-1}g' = g$. Hence, $g' \sim_H g$ is also true.
Transitivity: Assume $g \sim_H g'$ and $g' \sim_H g''$ with $g'' \in G$ and $h_1, h_2 \in H : g = h_1 g'$, $g' = h_2 g''$. We can substitute g' resulting in $g = h_1 h_2 g''$. Since H is closed $h_1 h_2$ is also in H. Hence, $g \sim_H g''$ is also true.
Since we proofed that $\sim_H$ is a equivalence relation, G/H is partition induced by $\sim_H$. $\square$

2.3

We proof if G is a permutation group on Q then G acts on Q. Therefore we show that G fullfills the two conditions

$$\forall q \in Q, g_1, g_2 \in G : q(g_1 g_2) = (q g_1) g_2 \tag{1}$$

$$g_1, g_2 \in G : \text{if } q g_1 = q g_2 \text{ for all } q \in Q \text{ then } g_1 = g_2 \tag{2}$$

Hence, G is a group by Lemma 2.89 it is associative and satifies (1). Also permutations are bijective which satifies (2). □

2.4

f is a monoid morphismus as you can first can combine two words of $\sigma*$ and count the length or count first the length and then add them up. For a given word $w \in \sigma*$ its equivalence class is $[|w|]$ with all other words with the same length. Since the words of $\sigma*$ can be arbitrary long, the order of the quotion set is infinte. □