3

3.1

3.2

3.3

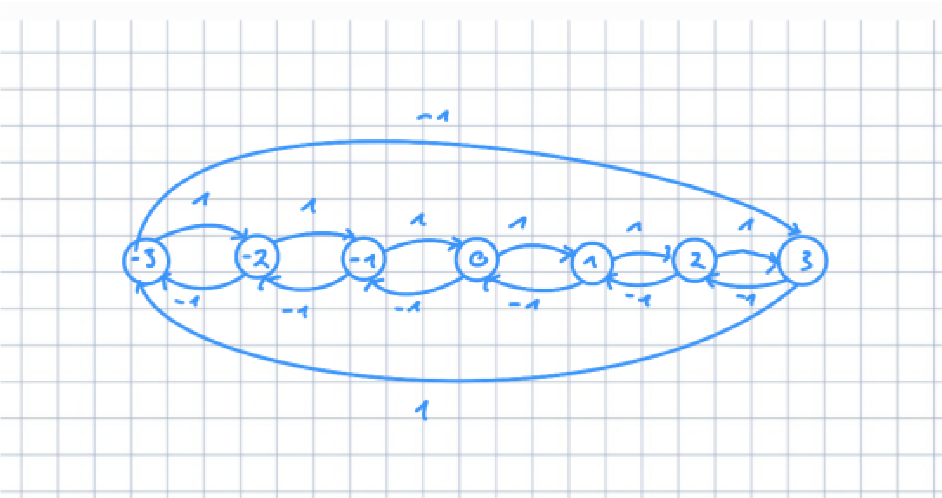
x	q_0	q_1	
a	q_0	q_1	
b	q_1	$\perp$	
c	$\perp$	q_0	
aa	q_0	q_1	same as a
ab	q_1	$\perp$	same as b
ac	$\perp$	q_0	same as c
ba	q_1	$\perp$	same as b
bb	$\perp$	$\perp$	
bc	q_0	$\perp$	
ca	$\perp$	q_0	same as c
cb	$\perp$	q_1	
cc	$\perp$	$\perp$	same as bb
$\vdots$	$\vdots$	$\vdots$	

From the table we compute $S(M) = \{[a], [b], [c], [b^2], [bc], [cb]\}$. Suppose $S(M)$ is a group with $u, v \in \Sigma^*, \delta_u, \delta_v \in \langle \mathcal{F}(M) \rangle$ then there exists for an arbitrary but fixed δ_u another δ_v with $\delta_u \delta_v = id_Q$. Notice we can choose $q_1 \delta_b = \perp$ and we cannot find any δ_v with $\perp \delta_v = q_1$. Therefore $S(M)$ is not a group.
Now assume $S(M)$ is a monoid. Hence, there exists $u \in \Sigma^*, \delta_u \in \langle \mathcal{F}(M) \rangle$ with $\delta_u = id_Q$. Notice δ_a satisfies this condition and acts a neutral element. Thus, $S(M)$ is a monoid. $\square$

3.4

We construct the following state machine:

$$
\begin{aligned}
M &= \{Q, \Sigma, \delta\} \\
Q &= \{q_{-3}, q_{-2}, q_{-1}, q_0, q_1, q_2, q_3\} \\
\Sigma &= \{-1, 1\} \\
\delta(q_i, j) &= \begin{cases} q_{i+1}, & \text{for } j = 1 \land q_i \neq q_3 \\ q_{-3}, & \text{for } j = 1 \land q_i = q_3 \\ q_{i-1}, & \text{for } j = -1 \land q_i \neq q_{-3} \\ q_3, & \text{for } j = -1 \land q_i = q_{-3} \end{cases}
\end{aligned}
$$



With the following operation table:

x	q_{-3}	q_{-2}	q_{-1}	q_0	q_1	q_2	q_3	
1-1	q_{-3}	q_{-2}	q_{-1}	q_0	q_1	q_2	q_3	
1	q_{-2}	q_{-1}	q_0	q_1	q_2	q_3	q_{-3}	shift left
11	q_{-1}	q_0	q_1	q_2	q_3	q_{-3}	q_{-2}	
111	q_0	q_1	q_2	q_3	q_{-3}	q_{-2}	q_{-1}	
-1	q_3	q_{-3}	q_{-2}	q_{-1}	q_0	q_1	q_2	shift right
-1-1	q_2	q_3	q_{-3}	q_{-2}	q_{-1}	q_0	q_1	
-1-1-1	q_1	q_2	q_3	q_{-3}	q_{-2}	q_{-1}	q_0	
1111	q_1	q_2	q_3	q_{-3}	q_{-2}	q_{-1}	q_0	same as -1-1-1
-1-1-1-1	q_0	q_1	q_2	q_3	q_{-3}	q_{-2}	q_{-1}	same as 111
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

This gives us $S(M) = \{[-1-1-1], [-1-1], [-1], [1-1], [1], [11], [111]\}$. With the following isomorphismus f we can map each equivalence class $[w]$ with $w \in \Sigma^*$ to an item of $\mathbb{Z}_3$:

$$f([w]) = \begin{cases} ((|\sum_{u\in w}(u)| + 3) \bmod 7) - 3 & \text{for } \sum_{u\in w}(u) \geq 0 \\ -((|\sum_{u\in w}(u)| + 3) \bmod 7) - 3 & \text{for } \sum_{u\in w}(u) < 0 \end{cases}$$

Here we sum up all right and left shift operations resulting in an all in all right or left shift (or no shift if 0). If we shift more than 3 in one direction, we switch back to shifting in the other direction, i.e. $\delta_{1111} = \delta_{-1-1-1}$.

$[w]$	$\mathbb{Z}_3$
$[-1-1-1]$	-3
$[-1-1]$	-2
$[-1]$	-1
$[1-1]$	0
$[1]$	1
$[11]$	2
$[111]$	3