1

1.1

We show that $R^{-1} \subseteq Y \times X$ is a partial function if $R \subseteq X \times Y$ is injective by proofing R^{-1} satisfies the right uniquess condition

$$(y, x), (y, x') \in R^{-1} \Rightarrow x = x'.$$

Since R is injective the following holds true:

$$(x, y), (x', y') \in R | x \neq x' \Rightarrow y \neq y' \tag{1}$$

Thus

$$(y, x), (y, x') \in R^{-1} | (x, y), (x', y) \in R \Rightarrow x = x'$$

because $x \neq x'$ contradicts (1). $\square$

1.2

$\emptyset \subseteq \emptyset \times Y$: First we show that $\emptyset \subseteq \emptyset \times Y$ is a partial function. By definition

$$(x, y), (x', y') \in R \Rightarrow y = y'$$

the empty relation is right-unique because we have no elements in the relation. Further we show totalness with $dom(R) = \emptyset$. By Definition

$$dom(R) = \{a \in A | \exists a' \in A : (a, a') \in R\}$$

the domain of the empty relation is also empty due missing elements and therfore $dom(R) = \emptyset = \emptyset$.

$\emptyset \subseteq X \times \emptyset$: This relation is also a partial function because the relation is also empty.

But it is not total because the preimage is X and the domain is $\emptyset$. Therefore

$$dom(R) = \emptyset \neq X$$

$\square$

1.3

Let $(S, \cdot)$, $(T, *)$ be two semigroups and $f : S \to T$ be a partial semigroup homomorphismus, then we have $f(S) \leq_{sg} T$ with

$$f(s) * f(s') = f(s, s') \tag{1}$$

$f(S)$ is a subsemigroup of T iff $f(S)$ is closed under $*$:

$$\forall x, y \in f(S) : x * y \in f(S)$$

Since f is a semigroup homomorphismus (1) holds true, thus

$$\exists a, b \in S : x * y = f(a) * f(b) = f(a \cdot b) \tag{2}$$

and since $(S, \cdot)$ is a semigroup, it is closed under $\cdot$, hence

$$a \cdot b \in S$$

and with this (2) holds true and therefore the initial implication of Lemma 2.49. $\square$

1.4