6

6.1

6.2

An epimorphism $(\alpha, \beta) : M_1 \to M_2$ with complete machines implies $M_2 \leq M_1$:
We choose $\alpha : Q_1 \to Q_2$ as our $\eta : Q_1 \to Q_2$. We define $\xi : \Sigma_2 \to \Sigma_1$ as a function which picks an arbitrary element of $\xi_{a_2} = \{a_1 | \beta(a_1) = a_2, a_1 \in \Sigma_1\}$ for an arbitraty $a_2 \in \Sigma_2$ (1). There exists at least one a_1 for each a_2 because β is surjective (2). Since (α, β) is a state machine homomorphismus and the maschines are complete following holds for all $a_1 \in \Sigma_1$ and all $q_1 \in Q_1$:

$$\alpha(q_1\delta^1_{a_1}) = (\alpha(q_1))\delta^2_{\beta(a_1)}$$

Now we show: $\eta(q_1)\delta^2_{a_2} \subseteq \eta(q_1\delta^1_{\xi(a_2)})$ for all $q_1 \in Q_1$ and $a_2 \in \Sigma_2$.

$$\begin{aligned}
\eta(q_1)\delta^2_{a_2} =& \alpha(q_1)\delta^2_{a_2} \\
=& \alpha(q_1)\delta^2_{\beta(a_1)} && (2) \\
=& \alpha(q_1\delta^1_{a_1}) \\
=& \alpha(q_1\delta^1_{\xi(a_2)}) && (1) \\
=& \eta(q_1\delta^1_{\xi(a_2)})
\end{aligned}$$

6.3

6.4

We show $M \leq M'$ by using the following covering (η, ξ):

$$\eta : Q' \hookrightarrow Q; q' \mapsto \begin{cases} q_0, & \text{for } q' = r_0, \\ q_1, & \text{for } q' = r_2, \end{cases}$$

$$\xi : \Sigma \to \Sigma'; u \mapsto \begin{cases} b', & \text{for } u = a, \\ a', & \text{for } u = b, \end{cases}$$

Moreover we prove η, ξ is well defined. First, ξ is a function because all $dom(\xi) = \Sigma$ and it is right unique. η is surjective as its range hits all states of Q. Also it is right unique and therefore a partial function. Finally, we show $\eta(q')\delta_w \subseteq \eta(q'\delta'_{\xi(w)})$ for all $q' \in Q'$ and $w \in \Sigma^*$. We build an operation table to get $\Sigma^+/\sim_M$ to get all relevant w.

x	q_0	q_1	
a	q_1	$\perp$	
b	q_0	q_1	
aa	$\perp$	$\perp$	
ab	q_1	$\perp$	same as a
bb	q_0	q_1	same as b
ba	q_1	$\perp$	same as a
aaa	$\perp$	$\perp$	same as aa
aab	$\perp$	$\perp$	same as aa
aba	$\perp$	$\perp$	same as aa
abb	q_1	$\perp$	same as a
bba	q_1	$\perp$	same as a
baa	q_1	$\perp$	same as aa
bab	q_1	$\perp$	same as a

We get $\Sigma^+/\sim_M = \{[a], [b], [aa]\}$. Since $\eta(q') = \emptyset | q' \notin \mathcal{D}(\eta)$ we only observe for r_0, r_2 as states from M'. We check for all relevant $q' \in Q'$ and $win\Sigma$:

$$
\begin{aligned}
\eta(r_0)\delta_a = q_0\delta_a = q_1 &\subseteq \eta(r_0\delta'_{\xi(a)}) = \eta(r_0\delta'_{b'}) = \eta(r_2) = q_1 \\
\eta(r_0)\delta_b = q_0\delta_b = q_0 &\subseteq \eta(r_0\delta'_{\xi(b)}) = \eta(r_0\delta'_{a'}) = \eta(r_0) = q_0 \\
\eta(r_0)\delta_{aa} = q_0\delta_{aa} = \emptyset &\subseteq \eta(r_0\delta'_{\xi(aa)}) = \eta(r_2\delta'_{b'b'}) = \eta(\emptyset) = \emptyset \\
\eta(r_2)\delta_a = q_1\delta_a = \emptyset &\subseteq \eta(r_2\delta'_{\xi(a)}) = \eta(r_2\delta'_{b'}) = \eta(r_3) = \emptyset \\
\eta(r_2)\delta_b = q_1\delta_b = q_1 &\subseteq \eta(r_2\delta'_{\xi(b)}) = \eta(r_2\delta'_{a'}) = \eta(r_2) = q_1 \\
\eta(r_2)\delta_{aa} = q_1\delta_{aa} = \emptyset &\subseteq \eta(r_2\delta'_{\xi(aa)}) = \eta(r_2\delta'_{b'b'}) = \eta(\emptyset) = \emptyset
\end{aligned}
$$

For each combination $\eta(q')\delta_w \subseteq \eta(q'\delta'_{\xi(w)})$ holds which concludes the proof. $\square$