9

9.1

We prove that if $M \leq N_1\omega_1 N_2...N_{n-1}\omega_{n-1}N_n$ then $TS(M) \leq TS(N_1) \wr ... \wr TS(N_n)$. By 4.10

$$TS(M) \leq TS(N_1\omega_1 N_2...N_{n-1}\omega_{n-1}N_n)$$

holds true. By 6.22 the cascade product is covered by the wreath product which results in

$$TS(N_1\omega_1 N_2...N_{n-1}\omega_{n-1}N_n) \leq TS(N_1 \wr ... \wr N_n)$$

. This concludes the proof.

9.2

We show $A \leq A/\pi \times A/\tau$. Notice, with $S', S'' \subseteq S$:

$$\begin{aligned} A &= (Q, S) \\ A/\pi \times A/\tau &= (\pi, S') \times (\tau, S'') \to (\pi \times \tau, S' \times S''); \\ &(q', s') \times (q'', s'') \mapsto (q's', q''s'') \end{aligned}$$

We define $\eta : (\pi \times \tau) \to Q; (g, h) \mapsto g \cap h$ which is surjective partial and results into one singleton of Q or is the emptyset due to the orthogonal property.

$$\pi \cap \tau = id_Q$$

Finally, we show $\eta((g, h))s \subseteq \eta((g, h)(s', s''))$ with $g \in \pi, h \in \tau, s' \in S_{\sim_\pi}, s'' \in S_{\sim_\tau}$.

$$\begin{aligned} \eta((g, h))s &= qs \\ &= q' \qquad (1) \\ &\subseteq \eta((g', h')) \qquad (2) \\ &= \eta((gs', hs'')) \\ &= \eta((g, h)(s', s'')) \end{aligned}$$

(1) By definition of η there exists for each $q \in Q$ one block of each partition that maps

to q or is empty which is still valid.
(2) By definition of admissible partitions for any $g' \in H$ there exists an $s' \in S/\sim$ and a partition $g \in H$ with $g' = gs'$.

9.3

Let $M = (Q, \Sigma, \delta)$ be a reset machine with at least two states. The claim is that then for all $q_1, q_2 \in Q$, the partition $\pi = \{\{q_1, q_2\}, Q \backslash \{q_1, q_2\}\}$ is admissible and orthogonal. Using Lemma 7.13 we can skip the proof that π is an admissible partition, and we only need to show that it is orthogonal. To show this we need to find another admissible partition τ of M s.t. $\pi \cap \tau = id_Q$.
Lets look at the characteristics that τ needs to have in order for the intersection to yield only id_Q. For this, there can not exist q_1 and q_2 can not be together in one block of the partition, since its intersection would yield q_1 and q_2 as a result. Therefore, we take the partition WLOG. $\tau = \{\{q_1, q_3\}, \{q_2\}, \{q_i\}_{i \in (|Q|-3) \backslash \{1,2,3\}}\}$, if we intersect τ with π, we will only get singleton blocks as a result because:

$$
\begin{aligned}
\{q_1, q_2\} \cap \{q_1, q_3\} &= \{q_1\} \\
\{q_1, q_2\} \cap \{q_2\} &= \{q_2\} \\
\{Q \backslash \{q_1, q_2\}\} \cap \{q_1, q_3\} &= \{q_3\} \\
\{q_1, q_2\} \cap \{q_i\}_{i \in (|Q|-3) \backslash \{1,2,3\}} &= \{\} \\
\{q_i\}_{i \in (|Q|-3) \backslash \{1,2,3\}} \cap \{Q \backslash \{q_1, q_2\}\} &= \{q_i\}_{i \in (|Q|-3) \backslash \{1,2,3\}}
\end{aligned}
$$

$$
\bigcup \pi \cap \tau = Q
$$

And since M is a reset machine, all partitions of it are admissible and therefore, also τ. This concludes the proof. $\square$

9.4

We use the partition $\pi = \{\{q_0, q_2, q_4\}, \{q_1, q_3\}\}$ and $\tau = \{\{q_0, q_1\}, \{q_2, q_3\}, \{q_4\}\}$. First we show that both π and τ are admissible partitions and after that we show that they are orthogonal.

$$
\begin{aligned}
\{q_0, q_2, q_4\}\delta_a &= \{q_0, q_2, q_4\} \\
\{q_0, q_2, q_4\}\delta_b &= \{q_1, q_3\} \\
\{q_1, q_3\}\delta_a &= \{q_1, q_3\} \\
\{q_1, q_3\}\delta_b &= \{q_1, q_3\}
\end{aligned}
$$

$$
\begin{aligned}
\{q_0, q_1\}\delta_a &= \{q_2, q_3\} \\
\{q_0, q_1\}\delta_b &= \{q_0, q_1\} \\
\{q_2, q_3\}\delta_a &= \{q_4\} \\
\{q_2, q_3\}\delta_b &= \{q_0, q_1\} \\
\{q_4\}\delta_a &= \{q_2, q_3\} \\
\{q_4\}\delta_b &= \{q_2, q_3\}
\end{aligned}
$$

Next we show that $\pi \cap \tau = id_Q$:

$$
\begin{aligned}
\{q_0, q_2, q_4\} \cap \{q_0, q_1\} &= \{q_0\} \\
\{q_0, q_2, q_4\} \cap \{q_2, q_3\} &= \{q_2\} \\
\{q_0, q_2, q_4\} \cap \{q_4\} &= \{q_4\} \\
\{q_1, q_3\} \cap \{q_0, q_1\} &= \{q_1\} \\
\{q_1, q_3\} \cap \{q_2, q_3\} &= \{q_3\} \\
\{q_1, q_3\} \cap \{q_4\} &= \{\}
\end{aligned}
$$

This concludes the proof. $\square$