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5.1

We have $M = (Q, \Sigma, \delta)$ and $\pi = \{H_i\}_{i \in I}$ a admissible partition of Q . If M is complete then for all $i \in I$ and for all $a \in \Sigma$ there exists **exactly one** partition such that for $j \in I$ $H_i \delta_a \subseteq H_j$. By definition of π (Lemma 3.48) there exists **at least one** $j \in I$ with $H_i \delta_a \subseteq H_j$. Notice $H_i \delta_a \neq \emptyset$ because M is complete.

We show that there only exists **exactly one** $j \in I$. Suppose there exists $j, k \in I$ with $H_i \delta_a \subseteq H_{j,k}$ for all $a \in \Sigma$ and $j \neq k$. We choose an arbitrary $q \in H_i$ then the following must hold:

$$q \delta_a = q_j \in H_j$$

$$q \delta_a = q_k \in H_k$$

Notice $q_j \neq q_k$ because $H_j \cap H_k = \emptyset$. This is a contraction because $q \delta_a$ is not right unique anymore.

5.2

We prove for a transformation semigroup transformation (Q, S) which is irreducible such that for all $q \in Q$ either $|qS| = 1$ or $qS = Q$. Assume $|qS| = 1$ for any $q \in Q$. This means we find one arbitrary but fixed $q' \in Q$ such that $qS = q'$. Moreover, $q = H_i$ and $q' = H_j$ with $i, j \in I$, it is the trivial partition of singleton classes. Now assume $qS = Q$ for any $q \in Q$. This means for each $s \in S$ with $qs = q'$ we map to a different q' such that $\bigcup q' = Q$. Thus, we get the trivial partition of Q .

Finally suppose $|qS| > 1 \wedge qS \neq Q$ for any $q \in Q$. Then qS maps into two or more partitions (up to $|Q| - 1$) which contradicts the condition of (Q, S) being irreducible.

5.3

1. Since, $\text{Aut}(M)$ is the set of all state machine automorphisms, this means that f is a bijective function on $Q \times Q$, therefore, it only permutes the states of M . As, Σ is mapped to Σ by the identity function, the transitions δ of M do not change.

Given Lemma 2.89, we know that given a set Q , (S_Q, \circ, id_Q) is a group. We have already established that $Aut(M)$ only permutes states of M , and as $Aut(M)$ includes 'all' state machine automorphisms, it also includes the identity function. This concludes the proof, that $Aut(M)$ is a group. \square

2. Assume, that there exists a $q \in Q$ such that $f(q) = q$.

5.4