4

4.1

We can find a state machine homomorphism (SMH) for M and M' which is monomorphic but not epimorphic or isomorphic. To prove this conjecture we will first show that there can not exist a epimorphic SMH. For a SMH f to be epimorphic it has to be surjective. This means that given $f = (\alpha, \beta)$:

$$\alpha : Q \to Q' \land \beta : \Sigma \to \Sigma' \tag{4.1}$$

the range of α and β have to be equal to Q' and Σ' respectively.
Lets assume that α maps to Q' and β to Σ'. This means that we have to map the states of Q to the states of Q'. The state r_1 is isolated whilst the states r_2 and r_3 form a cyclic state machine with stern with stern length and cycle length of 1. Therefore, we would need to find a similar structure in Q to map the states to Q'. If we look for these structures in M we can find an isolated state q_4 when we disregard c. However, we can only find a cyclic state machine with stern if we either disregard a, c or b, c since the combination of b and c allows us to always move between the states of $q_{\mathrm{i}} \mid i \in 3$. Thus, we can not find a mapping of β to Σ', since, we need to omit either at least 2 out of 3 elements of Σ to be able to represent the structure of M, therefore, either α or β can not be surjective.
To prove that M and M' are monomorphic we only have to provide a state machine homomorphism that is injective. For this we map a onto a' and q_4 onto r_1. α is injective because there only exists a single element in the relation, therefore, there can only exists a one-to-one mapping. For β this is the same case as only the tuple (q_4, r_1) exists in β.
Now we need to show that

$$\forall q \in Q \land \forall a \in \Sigma : \alpha(q\delta_{\mathrm{a}}) \subseteq (\alpha(q)){\delta'}_{\beta(a)} \tag{4.2}$$

Since, there only exist one element with one state we can apply the transition function for the element a on the state q_4 and then apply α. δ_{a} on q_4 results in q_4, and if we then apply α this results in r_1. If we first apply α on q_4 this results in r_1. And if we then apply ${\delta'}_{\beta(a)}$ on r_1 it stays in r_1 which proves that it is a state machine morphism.
This concludes that M and M' are monomorphic. $\square$

4.2

For Exercise 2 we need to prove that $TS(M') = \overline{\text{TS(M)}}$ holds. The closure of a transformation semigroup $A = (Q, S)$ is defined as

$$\overline{\text{A}} = (Q, \langle S \cup \overline{\text{Q}} \rangle \tag{4.3}$$

. For $TS(M') = \overline{\text{TS(M)}}$ to hold true, $TS(M')$ has to include all possible transformations from S and Q. Since, there exists a δ_{a} for all $a \in \Sigma$ as well as the identity function for each state of q with ${\delta'}_{\text{a}}(q) = a$ for all $a \in Q$ there can not exist any other transformations in $TS(M')$ that are not included in the closure of $TS(M) = \overline{\text{TS(M)}}$. Therefore, $TS(M')$ $= \overline{\text{TS(M)}}$ holds true. $\square$

4.3

We show that $\overline{TS(M)} = TS(M') = (Q, S')$. By definition

$$\overline{TS(M)} = (Q, \left\langle S \cup \overline{Q} \right\rangle)$$

S' must contain for all $q \in Q$ the corresponding $\overline{q}$. Notice $S' = \Sigma^+ / \sim_{M'}$. We assume that for each $q \in Q \subseteq \Sigma$ exists a congruence classes $[q] \in S'$.
Proof: Due to the definition of δ'_a for all $q_1, q_2 \in Q \subseteq \Sigma$ with $q_1 \neq q_2$ follows $\delta'_{q_1}(q) \neq \delta'_{q_2}(q)$ for all $q \in Q$. Therefore, for all $q \in Q$ exists $[q]$ and $[q] \in \Sigma^+ / \sim_{M'}$.
Since $\langle F(M') \rangle \simeq S(M')$ for all $q \in Q$ also $\delta'_q \in F(M')$. Finally $\delta'_q = \overline{q}$ because δ'_q maps all states to q and is therefore a constant mapping which proves that $\overline{Q} \subseteq S'$ and further $\overline{TS(M)} = (Q, \left\langle S \cup \overline{Q} \right\rangle) = TS(M') = (Q, \left\langle S' \cup \overline{Q} \right\rangle)$. $\square$

4.4

Given $\mathbb{Z}_2$ we obtain the transformation semigroup $(\mathbb{Z}_2, \mathbb{Z}_2)$ with right multiplication as an action (Lemma 3.28). Let $a, b \in \mathbb{Z}_2$ we compute $a + b$ by adding its corresponding equivalence classes $[a] + [b]$. To obtain the equivalence classes we compute the modulo:

$$a \mod 2 + b \mod 2$$

. By definition $a \mod 2 + b \mod 2 = a + b \mod 2$ which is equal to $\delta(a, b)$. $\square$

4.5

To prove that every group acts on itself we need show compatibility and faithfulness for a function $f : G \times G \to G; (g, g') \mapsto g'g$ with G being a group. Compatibility follows due to groups being associative. Let $g_1, g_2 \in G$. Let $gg_1 = gg_2(0)$ for all $g \in G$ assume $g_1 \neq g_2$. Set g to the neutral element ϵ which is allowed since (0) holds true for all $g \in G$. Hence, $\epsilon g_1 = \epsilon g_2 \iff g_1 = g_2$ which contradicts $g_1 \neq g_2$. This concludes the proof.