1

1.1

We show that $R^{-1} \subseteq Y \times X$ is a partial function if $R \subseteq X \times Y$ is injective by proofing R^{-1} satisfies the right uniqess condition

$$(y, x), (y, x') \in R^{-1} \Rightarrow x = x'.$$

Since R is injective the following holds true:

$$(x, y), (x', y') \in R | x \neq x' \Rightarrow y \neq y' \tag{1}$$

Thus

$$(y, x), (y, x') \in R^{-1} | (x, y), (x', y) \in R \Rightarrow x = x'$$

because $x \neq x'$ contradicts (1). $\square$

1.2

$\emptyset \subseteq \emptyset \times Y$: First we show that $\emptyset \subseteq \emptyset \times Y$ is a partial function. By definition

$$(x, y), (x', y') \in R \Rightarrow y = y'$$

the empty relation is right-unique because we have no elements in the relation. Further we show totalness with $dom(R) = \emptyset$. By Definition

$$dom(R) = \{a \in A | \exists a' \in A : (a, a') \in R\}$$

the domain of the empty relation is also empty due missing elements and therfore $dom(R) = \emptyset = \emptyset$.

$\emptyset \subseteq X \times \emptyset$: This relation is also a partial function because the relation is also empty.

But it is not total because the preimage is X and the domain is $\emptyset$. Therefore

$$dom(R) = \emptyset \neq X$$

$\square$

1.3

Let $(S, \cdot)$, $(T, *)$ be two semigroups and $f : S \to T$ be a partial semigroup homomorphismus, then we have $f(S) \leq_{sg} T$ with

$$f(s) * f(s') = f(s, s') \tag{1}$$

$f(S)$ is a subsemigroup of T iff $f(S)$ is closed under $*$:

$$\forall x, y \in f(S) : x * y \in f(S)$$

Since f is a semigroup homomorphismus (1) holds true, thus

$$\exists a, b \in S : x * y = f(a) * f(b) = f(a \cdot b) \tag{2}$$

and since $(S, \cdot)$ is a semigroup, it is closed under $\cdot$, hence

$$a \cdot b \in S$$

and with this (2) holds true and therefore the initial implication of Lemma 2.49. $\square$

1.4

To prove that $\sim_{\text{k}} \subseteq \ \Sigma^* \times \Sigma^*$ is a congruence relation for some given $k \in \mathcal{N}$ we first show that given two words $u, v \in \Sigma*$ which are $k - Simon - congruent$ they are also $(k-1) - Simon - congruent$ for $k \in \mathcal{N}$ with $ScatFact_{\text{k-1}}(u) = ScatFact_{\text{k-1}}(v)$.
We prove this claim using a prove by contradiction, therefore, we assume that we are given two words $u, v \in \Sigma^*$ which are $k - Simon - congruent$, however, not $(k-1) - Simon - congruent$. This means that the $ScatFact_{\text{k-1}}(u) \neq ScatFact_{\text{k-1}}(v)$. This means that there exists at least one object in WLOG. $ScatFact_{\text{k-1}}(u)$ which does not exist in $ScatFact_{\text{k-1}}(v)$, lets call this element w. Since, $ScatFact_{\text{k}}(u)$ contains all ScatterFactors

of u this means that there exists at least one $w' \in ScatFact_{\mathrm{k}}(u)$ which contains w (either $wx = w' \vee xw = w' \mid x \in u \wedge |x| = 1$). However, $ScatFact_{\mathrm{k}}(u) = ScatFact_{\mathrm{k}}(v)$ and therefore, $w' \in ScatFact_{\mathrm{k}}(v)$. Per definition of the $\mathrm{ScatFact_k}$ set, $ScatFact_{\mathrm{k-1}}(v)$ has to contain all ScatterFactors of v of length $k-1$, therefore also w since w' exists in v and by removing x from w' w is a ScatterFactor of v. This concludes the prove by contradiction. To now prove that $\sim_{\mathrm{k}}$ is a congruence relation for some k, we need to show that for $u, v, z \in \Sigma*, u \sim_{\mathrm{k}} v \implies uz \sim_{\mathrm{k}} vz$ to prove the right congruence. Since, v and u are k-simon-congruent we know that they have the same ScatterFactors for all $n \in k$. Since, we concatenate z to both words from the same side, all new ScatterFactors that can be formed, can be formed for both words, because all possible prefixes that can exist in u or v are the same. The prove for left congruence is symmetrical, which concludes the prove. $\square$

2

2.1

We can find a state machine homomorphism (SMH) for M and M' which is monomorphic but not epimorphic or isomorphic. To prove this conjecture we will first show that there can not exist a epimorphic SMH. For a SMH f to be epimorphic it has to be surjective. This means that given $f = (\alpha, \beta)$:

$$\alpha : Q \to Q' \wedge \beta : \Sigma \to \Sigma' \tag{2.1}$$

the range of α and β have to be equal to Q' and Σ' respectively.
Lets assume that α maps to Q' and β to Σ'. This means that we have to map the states of Q to the states of Q'. The state r_1 is isolated whilst the states r_2 and r_3 form a cyclic state machine with stern with stern length and cycle length of 1. Therefore, we would need to find a similar structure in Q to map the states to Q'. If we look for these structures in M we can find an isolated state q_4 when we disregard c. However, we can only find a cyclic state machine with stern if we either disregard a, c or b, c since the combination of b and c allows us to always move between the states of $q_\mathrm{i} \mid i \in 3$. Thus, we can not find a mapping of β to Σ', since, we need to omit either at least 2 out of 3 elements of Σ to be able to represent the structure of M, therefore, either α or β can not be surjective.
To prove that M and M' are monomorphic we only have to provide a state machine homomorphism that is injective. For this we map a onto a' and q_4 onto r_1. α is injective because there only exists a single element in the relation, therefore, there can only exists a one-to-one mapping. For β this is the same case as only the tuple (q_4, r_1) exists in β. Now we need to show that

$$\forall q \in Q \wedge \forall a \in \Sigma : \alpha(q\delta_\mathrm{a}) \subseteq (\alpha(q)){\delta'}_{\beta(a)} \tag{2.2}$$

Since, there only exist one element with one state we can apply the transition function for the element a on the state q_4 and then apply α. δ_a on q_4 results in q_4, and if we then apply α this results in r_1. If we first apply α on q_4 this results in r_1. And if we then apply ${\delta'}_{\beta(a)}$ on r_1 it stays in r_1 which proves that it is a state machine morphism.
This concludes that M and M' are monomorphic. $\square$