5

5.1

We have $M = (Q, \Sigma, \delta)$ and $\pi = \{H_i\}_{i \in I}$ a admissible partion of Q. If M is complete then for all $i \in I$ and for all $a \in \Sigma$ there exists **exactly one** partion such that for $j \in I$ $H_i\delta_a \subseteq H_j$. By definition of π (Lemma 3.48) there exists **at least one** $j \in I$ with $H_i\delta_a \subseteq H_j$. Notice $H_i\delta_a \neq \emptyset$ because M is complete.
We show that there only exists **exactly one** $j \in I$. Suppose there exists $j, k \in I$ with $H_i\delta_a \subseteq H_{j,k}$ for all $a \in \Sigma$ and $j \neq k$. We choose an arbitrary $q \in H_i$ then the following must hold:

$$q\delta_a = q_j \in H_j$$
$$q\delta_a = q_k \in H_k$$

Notice $q_j \neq q_k$ because $H_j \cap H_k = \emptyset$. This is a contraction because $q\delta_a$ is not right unique anymore.

5.2

We prove for a transformation semigroup (Q, S) which is irreducable that for all $q \in Q$ either $|qS| = 1$ or $qS = Q$. Assume $|qS| = 1$ for a given $q \in Q$. This means we find one arbitray but fixed $q' \in Q$ such that $qS = q'$. Thus, all states act the same and are equivalent, it is the trival partition of Q itself. Now assume $qS = Q$ for a given $q \in Q$. This means for each $s \in S$ with $qs = q'$ we map to a different q' such that $\bigcup q' = Q$. Thus, we find no relation between the state and get the trivial partition of singleton classes.
Finally suppose $|qS| > 1 \land qS \neq Q$ for all $q \in Q$. We can build the partition of Q by states reachable of q and not reachable by q. Let

$$\pi = \{qS, Q \setminus qS\} = \{H_1, H_2\}$$

. We show π is an admissible partition for all $i \in I$ and all $s \in S$ there exists $j \in I$ such that $H_is \subseteq H_js$.

5.3

1. Since, $Aut(M)$ is the set of all state machine automorphisms, this means that f is a bijective function on $Q \times Q$, therefore, it only permutates the states of M. As, Σ is mapped to Σ by the identity function, the transactions δ of M do not change. Given Lemma 2.89, we know that given a set Q, $(S_Q, \circ, id_Q)$ is a group. We have already established that $Aut(M)$ only permutates states of M, and as $Aut(M)$ includes 'all' state machine automorphisms, it also includes the identity function. This concludes the proof, that $Aut(M)$ is a group. □

2. Let $q_1, q_2 \in Q$ arbitrary but fixed with $q_1 = q_2\delta_w$ with $w \in \Sigma*$ and $q_1 \neq q_2$. Lets assume, that $f(q_1) = q_1$. Since, $Aut(M)$ is a state machine homomorphism with (f, id_Σ),
$$f(q\delta_w) \subseteq (f(q))\delta'_{id(w)} \tag{5.1}$$
holds true. For q_1, q_2 and w this results in:
$$f(q_1) = (f(q_2))\delta_w \tag{5.2}$$
$$q_1 = q'_2\delta_w | q_2 \in Q \tag{5.3}$$
For (5.3) to be true, q'_2 would have to equal q_2, since δ and Σ have not changed. This, however, would mean that if $f(q_1) = q_1 \rightarrow \forall q \in Q : f(q) = q$, since there exists a $w \in \Sigma*$ for all $q_1, q_2 \in Q$ with $q_1 = q_2\delta_w$, since, M is transitive.

3. Since, $\sim$ is an equivalence relation between q_1 and q_2, we can use it to impose a partition on Q which we will call π with $\pi = \{H_i\}_{i \in I}$. Now, we need to show that π is admissible. For this to hold true there needs to exists a $j \in I$ for all $i \in I$ and for all $a \in \Sigma$ such that $H_i\delta_a \subseteq H_j$. Since, (f, id_Σ) is an automorphism it is isomorphic and therefore f is bijective. This means that all equivalence classes of π are singletons. Since, all equivalence classes are singletons, there can never be the case that two elements of one equivalence class is mapped to two different equivalence classes. Therefore, π is admissible. □

5.4