

6

6.1

1. An epimorphism $(\alpha, \beta) : M_1 \rightarrow M_2$ with complete machines implies $M_2 \leq M_1$:
We choose $\alpha : Q_1 \rightarrow Q_2$ as our $\eta : Q_1 \rightarrow Q_2$. We define $\xi : \Sigma_2 \rightarrow \Sigma_1$ as a function which picks an arbitrary element of $\xi_{a_2} = \{a_1 | \beta(a_1) = a_2, a_1 \in \Sigma_1\}$ for an arbitrary $a_2 \in \Sigma_2$ (1). There exists at least one a_1 for each a_2 because β is surjective (2). Since (α, β) is a state machine homomorphism and the machines are complete following holds for all $a_1 \in \Sigma_1$ and all $q_1 \in Q_1$ (Definition 3.37):

$$\alpha(q_1 \delta_{a_1}^1) = (\alpha(q_1)) \delta_{\beta(a_1)}^2$$

Now we show: $\eta(q_1) \delta_{a_2}^2 \subseteq \eta(q_1 \delta_{\xi(a_2)}^1)$ for all $q_1 \in Q_1$ and $a_2 \in \Sigma_2$.

$$\begin{aligned} \eta(q_1) \delta_{a_2}^2 &= \alpha(q_1) \delta_{a_2}^2 \\ &= \alpha(q_1) \delta_{\beta(a_1)}^2 \end{aligned} \tag{2}$$

$$\begin{aligned} &= \alpha(q_1 \delta_{a_1}^1) \\ &= \alpha(q_1 \delta_{\xi(a_2)}^1) \\ &= \eta(q_1 \delta_{\xi(a_2)}^1) \end{aligned} \tag{1}$$

2. An monomorphism $(\alpha, \beta) : M_1 \rightarrow M_2$ with complete machines implies $M_1 \leq M_2$:
We choose $\beta : \Sigma_1 \rightarrow \Sigma_2$ as our $\xi : \Sigma_1 \rightarrow \Sigma_2$. We define $\eta : Q_2 \rightarrow Q_1$ as the inverse function from $\alpha : Q_1 \rightarrow Q_2$ which is well defined (Lemma 2.20). For each $q_2 \in Q_2$ with $\alpha^{-1}(q_2) \neq \emptyset$ there exists exactly one $q_1 = \alpha^{-1}(q_2)$ because α is injective. We show: $\eta(q_2) \delta_{a_1}^1 \subseteq \eta(q_2 \delta_{\xi(a_1)}^2)$ for all $q_2 \in Q_2$ and $a_1 \in \Sigma_1$.

$$\begin{aligned} \eta(q_2) \delta_{a_1}^1 &= \alpha^{-1}(q_2) \delta_{a_1}^1 \\ &= \alpha^{-1}(q_2 \delta_{a_2}^2) \\ &= \alpha^{-1}(q_2 \delta_{\xi(a_1)}^2) \\ &= \eta(q_2 \delta_{\xi(a_1)}^2) \end{aligned}$$

6.2

For the covering relation to be a partial order it has to be *reflexive*, *antisymmetric* and *transitive*.

Proof:

Lets begin by showing that \leq is reflexive, for this let $M = (Q, \Sigma, \delta)$. We need to show that $M \leq M$ is true. For M to cover itself, there needs to exist a partial function $\eta : Q \rightarrow Q$ which is

1. surjective
2. $\eta(q')\delta_w = \eta(q'\delta'_w)$ for all $w \in \Sigma^*$ and all $q' \in \mathcal{D}(\eta)$
3. $\eta(q') = \emptyset$ if $q' \notin \mathcal{D}(\eta)$
4. $q\delta_w = \emptyset$ if $\delta(q, w)$ is undefined

We use id_Q for η , since, we do not need to change anything for M to cover it self. id_Q is surjective, because the identity function maps every element in Q onto the same element in Q , therefore, it hits all elements in Q . Since, $\delta_w = \delta'_w$ the second primitive holds. The third holds for the identity function, since, it can only map an element to itself and if q' is not in its domain (here Q itself) it can not map it onto anything thus $id_Q(q') = \emptyset$ if $q' \notin \mathcal{D}(\eta)$. And the fourth holds because M is complete.

Next we show that \leq is *antisymmetric*:

Let $M_1 = (Q, \Sigma, \delta)$, $M_2 = (Q', \Sigma, \delta')$ be two state machines with $M_1 \leq M_2 \wedge M_2 \leq M_1$. We need to show that for this $M_1 = M_2$. Since, $M_1 \leq M_2 \wedge M_2 \leq M_1$ we know there exists two partial functions $\eta : Q' \rightarrow Q \wedge \eta' : Q \rightarrow Q'$ which are surjective. Furthermore, we know that $\eta(q')\delta_w = \eta(q'\delta'_w)$ for all $w \in \Sigma^*$ and all $q' \in \mathcal{D}(\eta)$ and $\eta(q)\delta'_w = \eta(q\delta_w)$ for all $w \in \Sigma^*$ and all $q \in \mathcal{D}(\eta')$

6.3

6.4

We show $M \leq M'$ by using the following covering (η, ξ) :

$$\eta : Q' \hookrightarrow Q; q' \mapsto \begin{cases} q_0, & \text{for } q' = r_0, \\ q_1, & \text{for } q' = r_2, \end{cases}$$

$$\xi : \Sigma \rightarrow \Sigma'; u \mapsto \begin{cases} b', & \text{for } u = a, \\ a', & \text{for } u = b, \end{cases}$$

Moreover we prove η, ξ is well defined. First, ξ is a function because all $\text{dom}(\xi) = \Sigma$ and it is right unique. η is surjective as its range hits all states of Q . Also it is right unique and therefore a partial function. Finally, we show $\eta(q')\delta_w \subseteq \eta(q')\delta'_{\xi(w)}$ for all $q' \in Q'$ and $w \in \Sigma^*$. We build an operation table to get Σ^+ / \sim_M to get all relevant w .

x	q_0	q_1	
a	q_1	\perp	
b	q_0	q_1	
aa	\perp	\perp	
ab	q_1	\perp	same as a
bb	q_0	q_1	same as b
ba	q_1	\perp	same as a
aaa	\perp	\perp	same as aa
aab	\perp	\perp	same as aa
aba	\perp	\perp	same as aa
abb	q_1	\perp	same as a
bba	q_1	\perp	same as a
baa	q_1	\perp	same as aa
bab	q_1	\perp	same as a

We get $\Sigma^+ / \sim_M = \{[a], [b], [aa]\}$. Since $\eta(q') = \emptyset | q' \notin \mathcal{D}(\eta)$ we only observe for r_0, r_2 as states from M' . We check for all relevant $q' \in Q'$ and $\text{win}\Sigma$:

$$\begin{aligned}
\eta(r_0)\delta_a &= q_0\delta_a = q_1 \subseteq \eta(r_0\delta'_{\xi(a)}) = \eta(r_0\delta'_{b'}) = \eta(r_2) = q_1 \\
\eta(r_0)\delta_b &= q_0\delta_b = q_0 \subseteq \eta(r_0\delta'_{\xi(b)}) = \eta(r_0\delta'_{a'}) = \eta(r_0) = q_0 \\
\eta(r_0)\delta_{aa} &= q_0\delta_{aa} = \emptyset \subseteq \eta(r_0\delta'_{\xi(aa)}) = \eta(r_2\delta'_{b'b'}) = \eta(\emptyset) = \emptyset \\
\eta(r_2)\delta_a &= q_1\delta_a = \emptyset \subseteq \eta(r_2\delta'_{\xi(a)}) = \eta(r_2\delta'_{b'}) = \eta(r_3) = \emptyset \\
\eta(r_2)\delta_b &= q_1\delta_b = q_1 \subseteq \eta(r_2\delta'_{\xi(b)}) = \eta(r_2\delta'_{a'}) = \eta(r_2) = q_1 \\
\eta(r_2)\delta_{aa} &= q_1\delta_{aa} = \emptyset \subseteq \eta(r_2\delta'_{\xi(aa)}) = \eta(r_2\delta'_{b'b'}) = \eta(\emptyset) = \emptyset
\end{aligned}$$

For each combination $\eta(q')\delta_w \subseteq \eta(q'\delta'_{\xi(w)})$ holds which concludes the proof. \square