1

1.1

We show that $R^{-1} \subseteq Y \times X$ is a partial function if $R \subseteq X \times Y$ is injective by proofing R^{-1} satisfies the right uniqess condition

$$(y, x), (y, x') \in R^{-1} \Rightarrow x = x'.$$

Since R is injective the following holds true:

$$(x, y), (x', y') \in R | x \neq x' \Rightarrow y \neq y' \tag{1}$$

Thus

$$(y, x), (y, x') \in R^{-1} | (x, y), (x', y) \in R \Rightarrow x = x'$$

because $x \neq x'$ contradicts (1). $\square$

1.2

$\emptyset \subseteq \emptyset \times Y$: First we show that $\emptyset \subseteq \emptyset \times Y$ is a partial function. By definition

$$(x, y), (x', y') \in R \Rightarrow y = y'$$

the empty relation is right-unique because we have no elements in the relation. Further we show totalness with $dom(R) = \emptyset$. By Definition

$$dom(R) = \{a \in A | \exists a' \in A : (a, a') \in R\}$$

the domain of the empty relation is also empty due missing elements and therfore $dom(R) = \emptyset = \emptyset$.

$\emptyset \subseteq X \times \emptyset$: This relation is also a partial function because the relation is also empty.

But it is not total because the preimage is X and the domain is $\emptyset$. Therefore

$$dom(R) = \emptyset \neq X$$

$\square$

1.3

Let $(S, \cdot)$, $(T, *)$ be two semigroups and $f : S \to T$ be a partial semigroup homomorphismus, then we have $f(S) \leq_{sg} T$ with

$$f(s) * f(s') = f(s, s') \tag{1}$$

$f(S)$ is a subsemigroup of T iff $f(S)$ is closed under $*$:

$$\forall x, y \in f(S) : x * y \in f(S)$$

Since f is a semigroup homomorphismus (1) holds true, thus

$$\exists a, b \in S : x * y = f(a) * f(b) = f(a \cdot b) \tag{2}$$

and since $(S, \cdot)$ is a semigroup, it is closed under $\cdot$, hence

$$a \cdot b \in S$$

and with this (2) holds true and therefore the initial implication of Lemma 2.49. $\square$

1.4

To prove that $\sim_{\text{k}} \subseteq \ \Sigma^* \times \Sigma^*$ is a congruence relation for some given $k \in \mathcal{N}$ we first show that given two words $u, v \in \Sigma*$ which are $k - Simon - congruent$ they are also $(k-1) - Simon - congruent$ for $k \in \mathcal{N}$ with $ScatFact_{\text{k-1}}(u) = ScatFact_{\text{k-1}}(v)$.
We prove this claim using a prove by contradiction, therefore, we assume that we are given two words $u, v \in \Sigma^*$ which are $k - Simon - congruent$, however, not $(k-1) - Simon - congruent$. This means that the $ScatFact_{\text{k-1}}(u) \neq ScatFact_{\text{k-1}}(v)$. This means that there exists at least one object in WLOG. $ScatFact_{\text{k-1}}(u)$ which does not exist in $ScatFact_{\text{k-1}}(v)$, lets call this element w. Since, $ScatFact_{\text{k}}(u)$ contains all ScatterFactors

of u this means that there exists at least one $w' \in ScatFact_{\mathrm{k}}(u)$ which contains w (either $wx = w' \vee xw = w' \mid x \in u \wedge |x| = 1$). However, $ScatFact_{\mathrm{k}}(u) = ScatFact_{\mathrm{k}}(v)$ and therefore, $w' \in ScatFact_{\mathrm{k}}(v)$. Per definition of the $\mathrm{ScatFact}_{\mathrm{k}}$ set, $ScatFact_{\text{k-1}}(v)$ has to contain all ScatterFactors of v of length $k-1$, therefore also w since w' exists in v and by removing x from w' w is a ScatterFactor of v. This concludes the prove by contradiction. To now prove that $\sim_{\mathrm{k}}$ is a congruence relation for some k, we need to show that for $u, v, z \in \Sigma*, u \sim_{\mathrm{k}} v \implies uz \sim_{\mathrm{k}} vz$ to prove the right congruence. Since, v and u are k-simon-congruent we know that they have the same ScatterFactors for all $n \in k$. Since, we concatenate z to both words from the same side, all new ScatterFactors that can be formed, can be formed for both words, because all possible prefixes that can exist in u or v are the same. The prove for left congruence is symmetrical, which concludes the prove. $\square$