7

7.1

We prove $M \wedge N \leq M \times N$. Notice for $M \wedge N$ if $\Sigma_m \neq \Sigma_n$ then $M \wedge N = \emptyset$ which is always a subset of any set. Therefore, we assume from now on $\Sigma_m = \Sigma_n$. Also checkout these definitions:

$$(q_m, q_n)(\delta_m \wedge \delta_n)_a = (q_m \delta^m_a, q_n \delta^n_a) \tag{1}$$
$$(q_m, q_n)(\delta_m \times \delta_n)_{(a_m, a_n)} = (q_m \delta^m_{a_m}, q_n \delta^n_{a_n}) \tag{2}$$

We choose for the covering (η, ξ) $id_{(q_m, q_n)}$ as η and define $\xi : \Sigma_n \to \Sigma_m \times \Sigma_n; a_m \mapsto (a_m, a_m)$. which is well-defined due the assumption $\Sigma_n = \Sigma_m$.
Now we show $\eta((q_m, q_n))(\delta^m \wedge \delta^n)_{a_m} \subseteq \eta((q_m, q_n)(\delta^m \times \delta^n)_{\xi(a_m)})$:

$$\begin{aligned}
\eta((q_m, q_n))(\delta^m \wedge \delta^n)_{a_m} &= (q_m, q_n)(\delta^m \wedge \delta^n)_{a_m} && \text{(def. } \eta\text{, 1)} \\
&= (q_m \delta^m_{a_m}, q_n \delta^n_{a_m}) && (2) \\
&= (q_m, q_n)(\delta_m \times \delta_n)_{(a_m, a_m)} && \text{(def. } \xi) \\
&= (q_m, q_n)(\delta_m \times \delta_n)_{\xi(a_m)} && \text{(def. } \eta) \\
&= \eta((q_m, q_n)(\delta^m \times \delta^n)_{\xi(a_m)})
\end{aligned}$$

□

7.2

Let M_1, M_2, M_3 three state machines. The claim is that $(M_1 \times M_2) \times M_3 \simeq M_1 \times (M_2 \times M_3)$ holds true. Lets begin by looking at the definitions:

$$M_1 \times M_2 = (Q \times Q', \Sigma \times \Sigma', \delta \times \delta') \tag{7.1}$$

with

$$(q, q')(\delta \wedge \delta')_{(a,a')} = (q\delta_a, q'\delta'_{a'}) \mid q \in Q, q' \in Q', a \in \Sigma, a' \in \Sigma' \tag{7.2}$$

. Next we look at $(M_1 \times M_2) \times M_3$:

$$(M_1 \times M_2) \times M_3 = ((Q \times Q') \times Q'', (\Sigma \times \Sigma') \times \Sigma'', (\delta \times \delta') \times \delta'') \tag{7.3}$$

with

$$((q, q'), q'')((\delta \wedge \delta') \wedge \delta'')_{((a,a'),a'')} = ((q\delta_a, q'\delta'_{a'}), q''\delta''_{a''}) \tag{7.4}$$

. Likewise, $M_1 \times (M_2 \times M_3)$ looks like the following:

$$M_1 \times (M_2 \times M_3) = (Q \times (Q' \times Q''), \Sigma \times (\Sigma' \times \Sigma''), \delta \times (\delta' \times \delta'')) \tag{7.5}$$

with

$$(q, (q', q''))(\delta \wedge (\delta' \wedge \delta''))_{(a,(a',a''))} = (q\delta_a, (q'\delta'_{a'}, q''\delta''_{a''})) \tag{7.6}$$

To show that $(M_1 \times M_2) \times M_3 \simeq M_1 \times (M_2 \times M_3)$ holds true we need to find a bijection that translates $(M_1 \times M_2) \times M_3$ into $M_1 \times (M_2 \times M_3)$:
Define $\alpha : (Q \times Q') \times Q'' \to Q \times (Q' \times Q'') : ((q, q'), q'') \mapsto (q, (q', q'')$ and $\beta : (\Sigma \times \Sigma') \times \Sigma'' \to \Sigma \times (\Sigma' \times \Sigma'') : (a, a'), a'' \mapsto a, (a', a'')$ as bijective functions. Now we need to show that

$$\alpha((q, q'), q'')((\delta \wedge \delta') \wedge \delta'')_{((a,a'),a'')}) \subseteq (\alpha((q, q'), q''))(\delta \wedge (\delta' \wedge \delta'')_{\beta((a,a'),a'')} \tag{7.7}$$

Proof:

$$\begin{aligned} (q\delta_a, (q'\delta'_{a'}, q''\delta''_{a''})) &= \\ \alpha((q\delta_a, q'\delta'_{a'}), q''\delta''_{a''}) &= \\ \alpha((q, q'), q'')((\delta \wedge \delta') \wedge \delta'')_{((a,a'),a'')}) &\subseteq (q, (q', q''))(\delta \wedge (\delta' \wedge \delta'')_{a,(a',a'')} \\ &= (q\delta_a, (q'\delta'_{a'}, q''\delta''_{a''})) \end{aligned}$$

This conclude the proof for our claim $(M_1 \times M_2) \times M_3 \simeq M_1 \times (M_2 \times M_3)$. □

7.3

We prove $P \leq N\omega R$ under the assumption $M \leq N$ and $P \leq M\omega R$. Let $\eta_1 : Q_n \to Q_m$ and $\eta_2 : Q_m \times Q_r \to Q_p$ be the η-functions of $M \leq N$ and $P \leq M\omega R$. Furthermore, let $\xi_1 : \Sigma_p \to \Sigma_r$ be the ξ-function of $P \leq M\omega R$.

We define $\eta : Q_n \times Q_r \to Q_p; (q_n, q_r) \mapsto \eta_2(\eta_1(q_n), q_r)$. Moreover we choose ξ_1 as ξ. Finally, we show $\eta((q_n, q_r))\delta^p_{a_p} \subseteq \eta((q_n, q_r)\delta^\omega_{\xi(a_p)})$.

$$
\begin{aligned}
\eta((q_n, q_r))\delta^p_{a_p} =& \eta_2((\eta_1(q_n), q_r))\delta^p_{a_p} && \text{(def. } \eta, \eta_1) \\
=& \eta_2(q_m, q_r))\delta^p_{a_p} && \text{(def. } \eta_2) \\
=& q_p\delta^p_{a_p} && (P \leq M\omega R) \\
\subseteq& \eta_2((q_m, q_r)\delta^\omega_{\xi(a_p)}) && \text{(def. } \xi) \\
=& \eta_2((q_m, q_r)\delta^\omega_{a_r}) && \text{(def. } \delta^\omega) \\
=& \eta_2((q_m\delta^m_{\omega_{a_r}(q_r)}, q_r\delta^r_{a_r})) && \text{(def. } \omega) \\
=& \eta_2((q_m\delta^m_{a_m}, q_r\delta^r_{a_r})) && (M \leq N) \\
=& \eta_2((\eta_1(q_n\delta^n_{a_n}), q_r\delta^r_{a_r})) && \text{(def. } \eta) \\
=& \eta((q_n\delta^n_{a_n}, q_r\delta^r_{a_r})) && \text{(def. } \omega) \\
=& \eta((q_n\delta^n_{\omega_{a_r}(q_r)}, q_r\delta^r_{a_r})) && \text{(def. } \delta^\omega) \\
=& \eta((q_n, q_r)\delta^\omega_{a_r})) && \text{(def. } \xi) \\
=& \eta((q_n, q_r)\delta^\omega_{\xi(a_p)}))
\end{aligned}
$$

□

7.4

No time for that, buzy buying christmas presents.