# 4

## 4.1

We can find a state machine homomorphism (SMH) for $M$ and $M'$ which is monomorphic but not epimorphic or isomorphic. To prove this conjecture we will first show that there can not exist a epimorphic SMH. For a SMH $f$ to be epimorphic it has to be surjective. This means that given $f = (\alpha, \beta)$:

$$\alpha : Q \rightarrow Q' \wedge \beta : \Sigma \rightarrow \Sigma' \tag{4.1}$$

the range of $\alpha$ and $\beta$ have to be equal to $Q'$ and $\Sigma'$ respectively.
Lets assume that $\alpha$ maps to $Q'$ and $\beta$ to $\Sigma'$. This means that we have to map the states of $Q$ to the states of $Q'$. The state $r_1$ is isolated whilst the states $r_2$ and $r_3$ form a cyclic state machine with stern with stern length and cycle length of 1. Therefore, we would need to find a similar structure in $Q$ to map the states to $Q'$. If we look for these structures in $M$ we can find an isolated state $q_4$ when we disregard $c$. However, we can only find a cyclic state machine with stern if we either disregard $a, c$ or $b, c$ since the combination of $b$ and $c$ allows us to always move between the states of $q_\mathrm{i} \; | i \in 3$. Thus, we can not find a mapping of $\beta$ to $\Sigma'$, since, we need to omit either at least 2 out of 3 elements of $\Sigma$ to be able to represent the structure of $M$, therefore, either $\alpha$ or $\beta$ can not be surjective.
To prove that $M$ and $M'$ are monomorphic we only have to provide a state machine homomorphism that is injective. For this we map $a$ onto $a'$ and $q_4$ onto $r_1$. $\alpha$ is injective because there only exists a single element in the relation, therefore, there can only exists a one-to-one mapping. For $\beta$ this is the same case as only the tuple $(q_4, r_1)$ exists in $\beta$. Now we need to show that

$$\forall q \in Q \wedge \forall a \in \Sigma : \alpha(q\delta_\mathrm{a}) \subseteq (\alpha(q)){\delta'}_{\beta(a)} \tag{4.2}$$

Since, there only exist one element with one state we can apply the transition function for the element $a$ on the state $q_4$ and then apply $\alpha$. $\delta_\mathrm{a}$ on $q_4$ results in $q_4$, and if we then apply $\alpha$ this results in $r_1$. If we first apply $\alpha$ on $q_4$ this results in $r_1$. And if we then apply ${\delta'}_{\beta(a)}$ on $r_1$ it stays in $r_1$ which proves that it is a state machine morphism.
This concludes that $M$ and $M'$ are monomorphic. $\square$

## 4.2

For Exercise 2 we need to prove that $TS(M') = \overline{\mathrm{TS(M)}}$ holds. The closure of a transformation semigroup $A = (Q, S)$ is defined as

$$\overline{\mathrm{A}} = (Q, \langle S \cup \overline{\mathrm{Q}}\rangle \tag{4.3}$$

. For $TS(M') = \overline{\mathrm{TS(M)}}$ to hold true, $TS(M')$ has to include all possible transformations from $S$ and $Q$. Since, there exists a $\delta_{\mathrm{a}}$ for all $a \in \Sigma$ as well as the identity function for each state of q with $\delta'_{\mathrm{a}}(q) = a$ for all $a \in Q$ there can not exist any other transformations in $TS(M')$ that are not included in the closure of $TS(M) = \overline{\mathrm{TS(M)}}$. Therefore, $TS(M')$ $= \overline{\mathrm{TS(M)}}$ holds true. □

## 4.3

We show that $\overline{TS(M)} = TS(M') = (Q, S')$. By definition

$$\overline{TS(M)} = (Q, \left\langle S \cup \overline{Q} \right\rangle)$$

$S'$ must contain for all $q \in Q$ the corresponding $\overline{q}$. Notice $S' = \Sigma^{+} / \sim_{M'}$. We assume that for each $q \in Q \subseteq \Sigma$ exists a congruence classes $[q] \in S'$.
Proof: Due to the definition of $\delta'_a$ for all $q_1, q_2 \in Q \subseteq \Sigma$ with $q_1 \neq q_2$ follows $\delta'_{q_1}(q) \neq \delta'_{q_2}(q)$ for all $q \in Q$. Therefore, for all $q \in Q$ exists $[q]$ and $[q] \in \Sigma^{+} / \sim_{M'}$.
Since $\langle F(M') \rangle \simeq S(M')$ for all $q \in Q$ also $\delta'_q \in F(M')$. Finally $\delta'_q = \overline{q}$ because $\delta'_q$ maps all states to $q$ and is therefore a constant mapping which proves that $\overline{Q} \subseteq S'$ and further $\overline{TS(M)} = (Q, \left\langle S \cup \overline{Q} \right\rangle) = TS(M') = (Q, \left\langle S' \cup \overline{Q} \right\rangle)$. □

## 4.4

## 4.5

To prove that every group acts on itself we need show compatibility and faithfulness for a function $f : G \times G \to G; (g, g') \mapsto g'g$ with G being a group. Compatibility follows due to groups being associative. Let $g_1, g_2 \in G$. Let $gg_1 = gg_2(0)$ for all $g \in G$ assume $g_1 \neq g_2$. Set $g$ to the neutral element $\epsilon$ which is allowed since (0) holds true for all $g \in G$. Hence, $\epsilon g_1 = \epsilon g_2 \iff g_1 = g_2$ which contradicts $g_1 \neq g_2$. This concludes the proof.