2

2.1

Since the group is finite, we can generate the powers of an element $s \in S$ infinetly. However there a only a finite amount of elements in S. Thus, there exists an integer x such that $s^x = s$. If $x = 2$ we are done, else we multiply both sides with s^{x-2}.

$$\begin{aligned} s^2 \cdot s^{x-2} &= s \cdot s^{x-2} \\ (s^x \cdot s^{-1}) \cdot (s^x \cdot s^{-1}) &= s^{x-1} \\ s^{x-1} \cdot s^{x-1} &= s^{x-1} \\ (s^{x-1})^2 &= s^{x-1} \end{aligned}$$

Then we substitute $s^{(x-1)}$ with y and the result is $y^2 = y'$. □

2.2

First we proof $g, g' \in G, g \sim_H g'$ if $g = hg'$ is an equivalence relation by showing it is reflexive, simmetric and transitive.
Reflexivity: For all $g \in G$ we have to prove that $g = hg'$, since H is a Subgroup of G, it contains the neutral Element of G therefore, we can always choose the neutral element for h with $g = hg'$. Thus $g = g'$ and $g \sim_H g$.
Symmetry: Assume $g \sim_H g'$ with $g = hg'$. Then we can add in inverse element h^{-1}, which is also in H because it is a group, resulting in $h^{-1}g' = g$. Hence, $g' \sim_H g$ is also true.
Transitivity: Assume $g \sim_H g'$ and $g' \sim_H g''$ with $g'' \in G$ and $h_1, h_2 \in H : g = h_1 g'$, $g' = h_2 g''$. We can substitute g' resulting in $g = h_1 h_2 g''$. Since H is closed $h_1 h_2$ is also in H. Hence, $g \sim_H g''$ is also true.
Since we proofed that $\sim_H$ is a equivalence relation, G/H is partition induced by $\sim_H$. □

2.3

We proof if G is a permutation group on Q then G acts on Q. Therefore we show that G fulfills the two conditions

$$\forall q \in Q, g_1, g_2 \in G : q(g_1 g_2) = (qg_1)g_2 \tag{1}$$

$$g_1, g_2 \in G : \text{if } qg_1 = qg_2 \text{ for all } q \in Q \text{ then } g_1 = g_2 \tag{2}$$

Hence, G is a group by Lemma 2.89 it is associative and satisfies (1). Also permutations are bijective which satisfies (2). □

2.4

First we prove that f is a semigroup homomorphism. For this the following hast to hold true:

$$f(s) * f(s') = f(s \cdot s') \tag{2.1}$$

given $w, w' \in \Sigma^* with |w| = n, |w'| = m : n, m \in \mathbb{N}_0$

$$f(w) + f(w') = n + m = f(ww') \tag{2.2}$$

To now prove that f is a monoid homomorphism we show that f(e) = e' with $e \in \Sigma^* \wedge e' \in \mathbb{N}_0$ with e and e' being the respective neutral elements. THe neutral element of Σ^* is ϵ the empty word which has a length of 0, which is the neutral element of the $\mathbb{N}_0$, therefore, the mapping of f is correct.

For a given word $w \in \Sigma^*$ its equivalence class is $[|w|]$ with all other words with the same length. Since the words of Σ^* can be arbitrary long, the order of the quotion set is infinte. □