7

7.1

We prove $M \wedge N \leq M \times N$. Notice for $M \wedge N$ if $\Sigma_m \neq \Sigma_n$ then $M \wedge N = \emptyset$ which is always a subset of any set. Therefore, we assume from now on $\Sigma_m = \Sigma_n$. Also checkout these definitions:

$$
\begin{aligned}
(q_m, q_n)(\delta_m \wedge \delta_n)_a =& (q_m \delta^m_a, q_n \delta^n_a) && (1)\\
(q_m, q_n)(\delta_m \times \delta_n)_{(a_m, a_n)} =& (q_m \delta^m_{a_m}, q_n \delta^n_{a_n}) && (2)
\end{aligned}
$$

We choose for the covering (η, ξ) $id_{(q_m, q_n)}$ as η and define $\xi : \Sigma_n \to \Sigma_m \times \Sigma_n; a_m \mapsto (a_m, a_m)$. which is well-defined due the assumption $\Sigma_n = \Sigma_m$.
Now we show $\eta((q_m, q_n))(\delta^m \wedge \delta^n)_{a_m} \subseteq \eta((q_m, q_n)(\delta^m \times \delta^n)_{\xi(a_m)})$:

$$
\begin{aligned}
\eta((q_m, q_n))(\delta^m \wedge \delta^n)_{a_m} =& (q_m, q_n)(\delta^m \wedge \delta^n)_{a_m} && \text{(def. } \eta\text{, 1)}\\
=& (q_m \delta^m_{a_m}, q_n \delta^n_{a_m}) && (2)\\
=& (q_m, q_n)(\delta_m \times \delta_n)_{(a_m, a_m)} && \text{(def. } \xi\text{)}\\
=& (q_m, q_n)(\delta_m \times \delta_n)_{\xi(a_m)} && \text{(def. } \eta\text{)}\\
=& \eta((q_m, q_n)(\delta^m \times \delta^n)_{\xi(a_m)})
\end{aligned}
$$

$\square$

7.2

7.3

We prove $P \leq N \omega R$ under the assumption $M \leq N$ and $P \leq M \omega R$. Let $\eta_1 : Q_n \to Q_m$ and $\eta_2 : Q_m \times Q_r \to Q_p$ be the η-functions of $M \leq N$ and $P \leq M \omega R$. Futhermore let $\xi_1 : \Sigma_p \to \Sigma_r$ be the ξ-function of $P \leq M \omega R$.
We define $\eta : Q_n \times Q_r \to Q_p; (q_n, q_r) \mapsto \eta_2(\eta_1(q_n), q_r)$. Moreover we choose ξ_1 as ξ.

Finally, we show $\eta((q_n, q_r))\delta^p_{a_p} \subseteq \eta((q_n, q_r)\delta^\omega_{\xi(a_p)})$.

$$\begin{aligned}
\eta((q_n, q_r))\delta^p_{a_p} &= \eta_2((\eta_1(q_n), q_r))\delta^p_{a_p} && \text{(def. } \eta, \eta_1) \\
&= \eta_2(q_m, q_r))\delta^p_{a_p} && \text{(def. } \eta_2) \\
&= q_p\delta^p_{a_p} && (P \leq M\omega R) \\
&\subseteq \eta_2((q_m, q_r)\delta^\omega_{\xi(a_p)}) && \text{(def. } \xi) \\
&= \eta_2((q_m, q_r)\delta^\omega_{a_r}) && \text{(def. } \delta^\omega) \\
&= \eta_2((q_m\delta^m_{\omega_{a_r}(q_r)}, q_r\delta^r_{a_r})) && \text{(def. } \omega) \\
&= \eta_2((q_m\delta^m_{a_m}, q_r\delta^r_{a_r})) && (M \leq N) \\
&= \eta_2((\eta_1(q_n\delta^n_{a_n}), q_r\delta^r_{a_r})) && \text{(def. } \eta) \\
&= \eta((q_n\delta^n_{a_n}, q_r\delta^r_{a_r})) && \text{(def. } \omega) \\
&= \eta((q_n\delta^n_{\omega_{a_r}(q_r)}, q_r\delta^r_{a_r})) && \text{(def. } \delta^\omega) \\
&= \eta((q_n, q_r)\delta^\omega_{a_r})) && \text{(def. } \xi) \\
&= \eta((q_n, q_r)\delta^\omega_{\xi(a_p)}))
\end{aligned}$$

This gets boring... □

7.4