5

5.1

We have $M = (Q, \Sigma, \delta)$ and $\pi = \{H_i\}_{i \in I}$ a admissible partion of Q. If M is complete then for all $i \in I$ and for all $a \in \Sigma$ there exists **exactly one** partion such that for $j \in I$ $H_i\delta_a \subseteq H_j$. By definition of π (Lemma 3.48) there exists **at least one** $j \in I$ with $H_i\delta_a \subseteq H_j$. Notice $H_i\delta_a \neq \emptyset$ because M is complete.
We show that there only exists **exactly one** $j \in I$. Suppose there exists $j, k \in I$ with $H_i\delta_a \subseteq H_{j,k}$ for all $a \in \Sigma$ and $j \neq k$. We choose an arbitrary $q \in H_i$ then the following must hold:

$$q\delta_a = q_j \in H_j$$
$$q\delta_a = q_k \in H_k$$

Notice $q_j \neq q_k$ because $H_j \cap H_k = \emptyset$. This is a contraction because $q\delta_a$ is not right unique anymore.

5.2

We proove for a transformation semigroup transformation (Q, S) which is irreducable that for all $q \in Q$ either $|qS| = 1$ or $qS = Q$. First we determine qS for both trivial partitions. Assume $|qS| = 1$ for any $q \in Q$. This means we find one arbitray but fixed $q' \in Q$ such that $qS = q'$. Moreover, $q = H_i$ and $q' = H_j$ with $i, j \in I$, it is the trival partition of singleton classes. Now assume $qS = Q$ for any $q \in Q$. This means for each $s \in S$ with $qs = q'$ we map to a different q' such that all $q' = Q$. Thus, qS is the trivial partition of Q itself.
Suppose $|qS| > 1 \wedge qS \neq Q$. Suppose we miss one $q' \in Q$ then qs would not build a trivial partion.
$\pi = [q]| \; \forall q \in Q$:

- cases for only one admissible partition and for $|Q|$ admissible partitions
- one equivalence class $\pi = H_1 \implies |qS| = 1$:
  - all states are related, $(q, q') \in R | \forall q, q' \in Q$

- $q \in H_1$ for all $q \in Q$ and therefore $Q = H_1 = [q]_R$
- $H_1 s = H_1$
- $qs = q'$ for all $q, q' \in H_1$
- suppose $q' \notin H_i$, thus $q \notin Q$ and therefore not included in the transformation semi group

- $|Q|$ equivalence classes $\implies qS = Q$:
  - $|Q|$ equivalence classes implies each state has own equivalence classes, $R = \{(q_i, q_i) | i \in [|Q|\}$
  - $qs = q$ for all $q \in Q$ and $s \in S$ with $q \in H_i, i \in [|Q|]$
  - suppose $qs = q'$ with $q \neq q'$ then $(q, q') \in R$ which contradicts first statement

5.3

1. Since, $Aut(M)$ is the set of all state machine automorphisms, this means that f is a bijective function on $Q \times Q$, therefore, it only permutates the states of M. As, Σ is mapped to Σ by the identity function, the transactions δ of M do not change. Given Lemma 2.89, we know that given a set Q, $(S_Q, \circ, id_Q)$ is a group. We have already established that $Aut(M)$ only permutates states of M, and as $Aut(M)$ includes 'all' state machine automorphisms, it also includes the identity function. This concludes the proof, that $Aut(M)$ is a group. $\square$

2. Assume, that there exists a $q \in Q$ such that $f(q) = q$.

5.4