5

5.1

We have $M = (Q, \Sigma, \delta)$ and $\pi = \{H_i\}_{i \in I}$ a admissible partion of Q. If M is complete then for all $i \in I$ and for all $a \in \Sigma$ there exists **exactly one** partion such that for $j \in I$ $H_i\delta_a \subseteq H_j$. By definition of π (Lemma 3.48) there exists **at least one** $j \in I$ with $H_i\delta_a \subseteq H_j$. Notice $H_i\delta_a \neq \emptyset$ because M is complete.
We show that there only exists **exactly one** $j \in I$. Suppose there exists $j, k \in I$ with $H_i\delta_a \subseteq H_{j,k}$ for all $a \in \Sigma$ and $j \neq k$. We choose an arbitrary $q \in H_i$ then the following must hold:

$$q\delta_a = q_j \in H_j$$
$$q\delta_a = q_k \in H_k$$

Notice $q_j \neq q_k$ because $H_j \cap H_k = \emptyset$. This is a contraction because $q\delta_a$ is not right unique anymore.

5.2

Let $\pi = \{H_i\}_{i \in I}$ be an admissible partition on the irreducable transformation semigroup (Q, S), than we know its equivalence classes are either singletons or the entire set Q. Now let's form the orbit for an arbitray $q \in Q$: $qS = \{q' | q' \in Q : qs = q'\}$.
Case 1:
In this case, the trivial partition on Q is Q itself. This means, that for all $q, q' \in Q$, we can find an action $s \in S$ such that $qs = q'$. Therefore, the cardinality of the orbit in this case is $|Q|$ itself.
Case 2:
In this case, the trivial partition on Q is the set of singleton classes of Q. Here, each state $q \in Q$, is only able to reach itself via some $s \in S$, therefore, the cardinality of the orbit in this case has to be 1. $\square$

5.3

1. Since, $Aut(M)$ is the set of all state machine automorphisms, this means that f is a bijective function on $Q \times Q$, therefore, it only permutates the states of M. As, Σ is mapped to Σ by the identity function, the transactions δ of M do not change. Given Lemma 2.89, we know that given a set Q, $(S_Q, \circ, id_Q)$ is a group. We have already established that $Aut(M)$ only permutates states of M, and as $Aut(M)$ includes 'all' state machine automorphisms, it also includes the identity function. This concludes the proof, that $Aut(M)$ is a group. □

2. Let $q_1, q_2 \in Q$ arbitrary but fixed with $q_1 = q_2\delta_w$ with $w \in \Sigma*$ and $q_1 \neq q_2$. Lets assume, that $f(q_1) = q_1$. Since, $Aut(M)$ is a state machine homomorphism with (f, id_Σ),
$$f(q\delta_w) \subseteq (f(q))\delta'_{id(w)} \tag{5.1}$$
holds true. For q_1, q_2 and w this results in:
$$f(q_1) = (f(q_2))\delta_w \tag{5.2}$$
$$q_1 = q'_2\delta_w | q_2 \in Q \tag{5.3}$$
For (5.3) to be true, q'_2 would have to equal q_2, since δ and Σ have not changed. This, however, would mean that if $f(q_1) = q_1 \rightarrow \forall q \in Q : f(q) = q$, since there exists a $w \in \Sigma*$ for all $q_1, q_2 \in Q$ with $q_1 = q_2\delta_w$, since, M is transitive.

3. Since, $\sim$ is an equivalence relation between q_1 and q_2, we can use it to impose a partition on Q which we will call π with $\pi = \{H_i\}_{i \in I}$. Now, we need to show that π is admissible. For this to hold true there needs to exists a $j \in I$ for all $i \in I$ and for all $a \in \Sigma$ such that $H_i\delta_a \subseteq H_j$. Since, (f, id_Σ) is an automorphism it is isomorphic and therefore f is bijective. This means that all equivalence classes of π are singletons. Since, all equivalence classes are singletons, there can never be the case that two elements of one equivalence class is mapped to two different equivalence classes. Therefore, π is admissible. □

5.4

joke's on you :)!