4

4.1

We can find a state machine homomorphism (SMH) for M and M' which is neither monomorphic nor epimorphic and therefore not isomorphic. To prove this conjecture we will first show that there can not exist a epimorphic SMH. For a SMH f to be epimorphic it has to be surjective. This means that given $f = (\alpha, \beta)$:

$$\alpha : Q \to Q' \wedge \beta : \Sigma \to \Sigma' \tag{4.1}$$

the range of α and β have to be equal to Q' and Σ' respectively.
Let's suppose that α maps to Q' and β to Σ'. This means that we have to map the states of Q to the states of Q'. The state r_1 is isolated whilst the states r_2 and r_3 form a cyclic state machine with stern with stern length and cycle length of 1. Therefore, we would need to find a similar structure in Q to map the states to Q'. If we look for these structures in M we can find an isolated state q_4 when we disregard c. However, we can only find a cyclic state machine with stern if we either disregard a, c or b, c since the combination of b and c allows us to always move between the states of $q_\mathrm{i} \; | i \in 3$. Thus, we can not find a mapping of β to Σ', since, we need to omit either at least 2 out of 3 elements of Σ to be able to represent the structure of M, therefore, either α or β can not be surjective.
To prove that M and M' are not monomorphic we have to show there is no injective SMH. Since, M' has less states as well as a smaller alphabet and a homomorphism must be total, some states of M have to map to the same state in M' which contradicts injectivity.
Now we show that M and M' are morphic by providing (α, β).

$$\alpha : q \mapsto r_1 \;\forall q \in Q$$
$$\beta : u \mapsto a' \;\forall u \in \Sigma$$

Since, we map all states to the same state in M' and all letters to the same letter in Σ' it does not matter if we either first compute in M and then transform into M' or first transform into M' and then compute there. We always end up in r_1. This concludes that M and M' are morphic but not monomorphic, epimorphic or isomorphic. □

As stated above, M_1 can only be monomorphic to M_2 if $|Q_1| \leq |Q_2|$ and there exists a one to one mapping with the homomorphism criteria. Same applies to Σ_1 and Σ_2.
Moreover, M_1 can only be epimorphic to M_2 if $|Q_1| \geq |Q_2|$. Otherwise due right uniquness we can not map one state of M_1 to two or more states in M_2 and therefore can not hit the whole range of Q_2. This also applies to the alphabets.
For an isomorphismus both conditions of a monomorphismus and epimorphismus must apply. Thus, we have $|Q_1| = |Q_2|$ and $|\Sigma_1| = |\Sigma_2|$.

4.2

For Exercise 2 we need to prove that $TS(M') = \overline{\mathrm{TS(M)}}$ holds. The closure of a transformation semigroup $A = (Q, S)$ is defined as

$$\overline{\mathrm{A}} = (Q, \langle S \cup \overline{\mathrm{Q}} \rangle \tag{4.2}$$

. For $TS(M') = \overline{\mathrm{TS(M)}}$ to hold true, $TS(M')$ has to include all possible transformations from S and Q. Since, there exists a δ_{a} for all $a \in \Sigma$ as well as the identity function for each state of q with ${\delta'}_{\mathrm{a}}(q) = a$ for all $a \in Q$ there can not exist any other transformations in $TS(M')$ that are not included in the closure of $TS(M) = \overline{\mathrm{TS(M)}}$. Therefore, $TS(M')$ $= \overline{\mathrm{TS(M)}}$ holds true. □

4.3

Given $\mathbb{Z}_2$ we obtain the transformation semigroup $(\mathbb{Z}_2, \mathbb{Z}_2)$ with right multiplication as an action (Lemma 3.28). Let $a, b \in \mathbb{Z}_2$ and notice $\mathbb{Z}_2 = \{[0], [1]\}$. We compute $a + b$ by using the action and therefore adding its corresponding equivalence classes $[a] + [b]$. To obtain the equivalence classes we use the mudolo operator:

$$\begin{aligned} & [a] + [b] \\ = \; & a \bmod 2 + b \bmod 2 \\ = \; & a + b \bmod 2 \end{aligned}$$

By definition of the mudolo operator $a \bmod 2 + b \bmod 2 = a + b \bmod 2$ which is equal to $\delta(a, b)$. □

4.4

To prove that every group acts on itself we need show compatibility and faithfulness for a function $f : G \times G \to G; (g, g') \mapsto g'g$ with G being a group. Compatibility follows due to groups being associative. Let $g_1, g_2 \in G$. Let $gg_1 = gg_2(0)$ for all $g \in G$ suppose $g_1 \neq g_2$. To fulfill the equation g has to be the inverse element of g_1 and g_2. But because g is fixed it also has to be the same element. Furthermore the inverse element exists only for one other element in the group. Thus, g can not be the inverse of g_1 and g_2 at the same time. This contradicts $g_1 \neq g_2$. This concludes the proof.