

3

3.1

3.2

3.3

x	q_0	q_1	
a	q_0	q_1	
b	q_1	\perp	
c	\perp	q_0	
aa	q_0	q_1	same as a
ab	q_1	\perp	same as b
ac	\perp	q_0	same as c
ba	q_1	\perp	same as b
bb	\perp	\perp	
bc	q_0	\perp	
ca	\perp	q_0	same as c
cb	\perp	q_1	
cc	\perp	\perp	same as bb
\vdots	\vdots	\vdots	

From the table we compute $S(M) = \{[a], [b], [c], [b^2], [bc], [cb]\}$. Suppose $S(M)$ is a group with $u, v \in \Sigma^*$, $\delta_u, \delta_v \in \langle \mathcal{F}(M) \rangle$ then there exists for an arbitrary but fixed δ_u another δ_v with $\delta_u \delta_v = id_Q$. Notice we can choose $q_1 \delta_b = \perp$ and we cannot find any δ_v with $\perp \delta_v = q_1$. Therefore $S(M)$ is not a group.

Now assume $S(M)$ is a monoid. Hence, there exists $u \in \Sigma^*$, $\delta_u \in \langle \mathcal{F}(M) \rangle$ with $\delta_u = id_Q$. Notice δ_a satisfies this condition and acts a neutral element. Thus, $S(M)$ is a monoid. \square

3.4

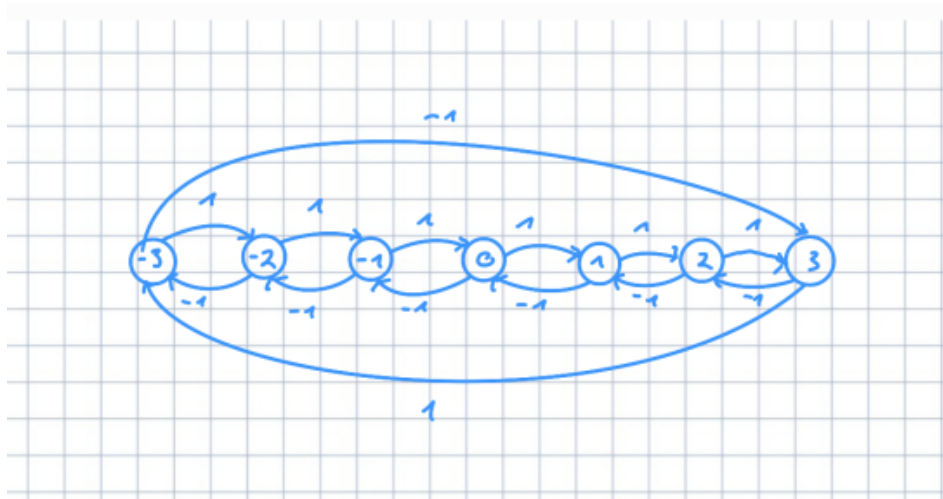
We construct the following state machine:

$$M = \{Q, \Sigma, \delta\}$$

$$Q = \{q_{-3}, q_{-2}, q_{-1}, q_0, q_1, q_2, q_3\}$$

$$\Sigma = \{-1, 1\}$$

$$\delta(q_i, j) = \begin{cases} q_{i+1}, & \text{for } j = 1 \wedge q_i \neq q_3 \\ q_{-3}, & \text{for } j = 1 \wedge q_i = q_3 \\ q_{i-1}, & \text{for } j = -1 \wedge q_i \neq q_{-3} \\ q_3, & \text{for } j = -1 \wedge q_i = q_{-3} \end{cases}$$



With the following operation table:

x	q_{-3}	q_{-2}	q_{-1}	q_0	q_1	q_2	q_3	
1-1	q_{-3}	q_{-2}	q_{-1}	q_0	q_1	q_2	q_3	
1	q_{-2}	q_{-1}	q_0	q_1	q_2	q_3	q_{-3}	shift left
11	q_{-1}	q_0	q_1	q_2	q_3	q_{-3}	q_{-2}	
111	q_0	q_1	q_2	q_3	q_{-3}	q_{-2}	q_{-1}	
-1	q_3	q_{-3}	q_{-2}	q_{-1}	q_0	q_1	q_2	shift right
-1-1	q_2	q_3	q_{-3}	q_{-2}	q_{-1}	q_0	q_1	
-1-1-1	q_1	q_2	q_3	q_{-3}	q_{-2}	q_{-1}	q_0	
1111	q_1	q_2	q_3	q_{-3}	q_{-2}	q_{-1}	q_0	same as -1-1-1
-1-1-1-1	q_0	q_1	q_2	q_3	q_{-3}	q_{-2}	q_{-1}	same as 111
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	

This gives us $S(M) = \{[-1-1-1], [-1-1], [-1], [1-1], [1], [11], [111]\}$. With the following isomorphism f we can map each equivalence class $[w]$ with $w \in \Sigma^*$ to an item of \mathbb{Z}_3 :

$$f([w]) = \begin{cases} (|\sum_{u \in w}(u)| + 3) \% 7 - 3 & \text{for } \sum_{u \in w}(u) \geq 0 \\ -(|\sum_{u \in w}(u)| + 3) \% 7 - 3 & \text{for } \sum_{u \in w}(u) < 0 \end{cases}$$

Here we sum up all right and left shift operations resulting in an all in all right or left shift (or no shift if 0). If we shift more than 3 in one direction, we switch back to shifting in the other direction, i.e. $\delta_{1111} = \delta_{-1-1-1}$.

$[w]$	\mathbb{Z}_3
$[-1-1-1]$	-3
$[-1-1]$	-2
$[-1]$	-1
$[1-1]$	0
$[1]$	1
$[11]$	2
$[111]$	3