Chap 3 插值和多项式估值

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拉格朗日插值多项式

- n阶拉格朗日多项式:$L_{n,k}(x)=\prod_{i=0}^{n}\frac{(x-x_i)}{(x_k-x_i)}(i\neq k)$, $P(x)=\sum_{k=0}^{n}f(x_k)L_{n,k}(x)$
- 误差$E=\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$,仅能用于计算最大误差,O(h^n)
- 通常用于阶数较小的情况,且常用于数值微分和积分
- 内维尔方法(可迭代)
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x_0	P_0				
x_1	P_1	$P_{0,1}$			
x_2	P_2	$P_{1,2}$	$P_{0,1,2}$		
x_3	P_3	$P_{2,3}$	$P_{1,2,3}$	$P_{0,1,2,3}$	
x_4	P_4	$P_{3,4}$	$P_{2,3,4}$	$P_{1,2,3,4}$	$P_{0,1,2,3,4}$

  - $P(x)=\frac{(x-x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x)-(x-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{(x_i-x_j)}$

差分多项式

- $f[x_0,x_1,\ldots,x_n]=\frac{f[x_1,x_2,\ldots,x_n]-f[x_0,x_1,\ldots,x_{n-1}]}{x_n-x_0}.$
- $$P_n(x)=f[x_0]+f[x_0,x_1](x-x_0)+a_2(x-x_0)(x-x_1)+\cdots+a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}).$$
- 误差O(h^n)
- 比拉氏多项式更适于计算,亦用于解决微分方程
- 缺陷:高阶项系数的微小误差将会带来巨大的波动.该点同样使用于拉格朗日多项式.

密切多项式(赫尔米特插值)

- 不仅保证了函数值相等,还保证了一阶导数乃至高阶导数值相等.
- $$H_{2n+1}(x)=\sum_{j=0}^{n}f(x_j)H_{n,j}(x)+\sum_{j=0}^{n}f'(x_j)\hat{H}_{n,j}(x),$$

  $$H_{n,j}(x)=[1-2(x-x_j)L'_{n,j}(x_j)]L^2_{n,j}(x)\quad\text{and}\quad\hat{H}_{n,j}(x)=(x-x_j)L^2_{n,j}(x).$$
- Moreover, if $f\in C^{2n+2}[a,b]$, then

  $$f(x)=H_{2n+1}(x)+\frac{(x-x_0)^2\ldots(x-x_n)^2}{(2n+2)!}f^{(2n+2)}(\xi(x)),$$

INPUT numbers $x_0, x_1, \ldots, x_n$; values $f(x_0), \ldots, f(x_n)$ and $f'(x_0), \ldots, f'(x_n)$.

OUTPUT the numbers $Q_{0,0}, Q_{1,1}, \ldots, Q_{2n+1,2n+1}$ where

$$\begin{aligned} H(x) = {} & Q_{0,0} + Q_{1,1}(x-x_0) + Q_{2,2}(x-x_0)^2 + Q_{3,3}(x-x_0)^2(x-x_1) \\ & + Q_{4,4}(x-x_0)^2(x-x_1)^2 + \cdots \\ & + Q_{2n+1,2n+1}(x-x_0)^2(x-x_1)^2 \cdots (x-x_{n-1})^2(x-x_n). \end{aligned}$$

Step 1 For $i = 0, 1, \ldots, n$ do Steps 2 and 3.

Step 2 Set $z_{2i} = x_i$;
$z_{2i+1} = x_i$;
$Q_{2i,0} = f(x_i)$;
$Q_{2i+1,0} = f(x_i)$;
$Q_{2i+1,1} = f'(x_i)$.

Step 3 If $i \neq 0$ then set

$$Q_{2i,1} = \frac{Q_{2i,0} - Q_{2i-1,0}}{z_{2i} - z_{2i-1}}.$$

Step 4 For $i = 2, 3, \ldots, 2n+1$
for $j = 2, 3, \ldots, i$ set $Q_{i,j} = \dfrac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}$.

Step 5 OUTPUT $(Q_{0,0}, Q_{1,1}, \ldots, Q_{2n+1,2n+1})$;
STOP

三次样条插值,本质就是分段三次赫尔米特插值(piecewise cubic Hermite interpolation)

- 自然边界:端点的二阶导均为零。唯一条件：f在[a,b]上有定义
- 压缩边界:端点的一阶导等于函数的一阶导。唯一条件：f在a，b可微