Aeroacoustics: Homework 3 (Due 25th March)

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1. Consider a uniform base flow consisting of velocity directed in the $+x-$direction of magnitude u_0; the base density and pressure are ρ_0 and p_0, respectively. The associated speed of sound for the base flow is c_0.

   (a) (5 points) Recall the energy equation as given in lecture notes:

   $$\rho_\dagger \left\{ \frac{\partial s_\dagger}{\partial t} + \underline{u}_\dagger \cdot (\nabla s_\dagger) \right\} = \frac{1}{T_\dagger} \left\{ \left(\nabla \underline{u}_\dagger\right) : \underline{\underline{\tau}}_\dagger - \nabla \cdot \underline{h}_\dagger \right\}.$$

   Here, we have changed the notation for the (full) flow quantities to have a † subscript, reserving the corresponding basic symbol for the fluctuation of the same flow quantity. For example, the (full) density $\rho_\dagger$ is decomposed into the base density ρ_0 plus the density fluctuation ρ; i.e., $\rho_\dagger = \rho_0 + \rho$. This reduces the subsequent notational clutter, since we use fluctuation quantities much more extensively than the full quantities.

   To do: specializing this to an inviscid flow without heat conduction (i.e., in the absence of molecular diffusion processes), **derive** the *linearized* energy equation in terms of pressure fluctuations as follows:

   $$\frac{\partial p}{\partial t} + u_0 \frac{\partial p}{\partial x} + \gamma p_0 \nabla \cdot \underline{u} = 0.$$

   In the above, p and $\underline{u}$ are the (weak) pressure fluctuations and velocity vector fluctuations, respectively; γ is the ratio of specific heats.

   (b) (5 points) The following non-dimensionalizations are used in the 2-D LEE: (quantities with superscript as $*$ represent dimensionless variables, whereas quantities without this superscript refer to the corresponding dimensional variables as before)

   $$\text{Lengths: } x^* = \frac{x}{\Delta x}, \quad y^* = \frac{y}{\Delta x}, \qquad \text{Velocities: } u^* = \frac{u}{c_0}, \quad v^* = \frac{v}{c_0}$$
   $$\text{Time: } t^* = \frac{t c_0}{\Delta x}, \qquad \text{Density: } \rho^* = \frac{\rho}{\rho_0}, \qquad \text{Pressure: } p^* = \frac{p}{\rho_0 c_0^2}.$$

   Then, starting from the dimensional linearized mass and momentum conservation equations derived in class and the energy equation derived in part (a), show that the non-dimensional 2-D LEE with base-flow Mach number M_x in x-direction, is

   $$\frac{\partial \underline{U}^*}{\partial t^*} + \frac{\partial \underline{E}^*}{\partial x^*} + \frac{\partial \underline{F}^*}{\partial y^*} = 0, \qquad \underline{U} := \begin{bmatrix} \rho^* \\ u^* \\ v^* \\ p^* \end{bmatrix}, \quad \underline{E} := \begin{bmatrix} M_x \rho^* + u^* \\ M_x u^* + p^* \\ M_x v^* \\ M_x p^* + u^* \end{bmatrix}, \quad \underline{F} := \begin{bmatrix} v^* \\ 0 \\ p^* \\ v^* \end{bmatrix}.$$

2. (25 points) Write a MATLAB/Python/C/C++ code to simulate the solution to the non-dimensional 2-D LEE of the first question. Make provision for the user to supply the initial condition separately (possibly through a function call). Use a 7-point DRP stencil for spatial discretization and a 4-level optimized time discretization stencil. Apply radiation and outflow boundary conditions assuming the base flow velocity is in the direction of increasing x. Create appropriate ghost grid on all sides to implement the boundary conditions. **Hint:** Note that, following the indicated non-dimensionalization, the non-dimensional grid spacing in the $x-$direction Δx^* is identically equal to unity.

3. (5 points) Exercise your 2-D LEE code with $M_x = 0.5$ and the following initial condition:

$$
\begin{aligned}
\rho^*(x^*, y^*, t=0) &= \varepsilon_a \exp\left[-\alpha_a \left\{(x^* - x_a^*)^2 + (y^* - y_a^*)^2\right\}\right] \\
&\quad + \varepsilon_e \exp\left[-\alpha_e \left\{(x^* - x_e^*)^2 + (y^* - y_e^*)^2\right\}\right], \\
u^*(x^*, y^*, t=0) &= \varepsilon_v \left(y^* - y_v^*\right) \exp\left[-\alpha_v \left\{(x^* - x_v^*)^2 + (y^* - y_v^*)^2\right\}\right], \\
v^*(x^*, y^*, t=0) &= -\varepsilon_v \left(x^* - x_v^*\right) \exp\left[-\alpha_v \left\{(x^* - x_v^*)^2 + (y^* - y_v^*)^2\right\}\right], \\
p^*(x^*, y^*, t=0) &= \varepsilon_a \exp\left[-\alpha_a \left\{(x^* - x_a^*)^2 + (y^* - y_a^*)^2\right\}\right].
\end{aligned}
$$

Evidently, Gaussian pulses are used with ε_η as the amplitude, α_η as the 'sharpness', and x_η^* & y_η^* as the center coordinates of the η component. The subscripts a, e and v stand for acoustic, entropic and vortical components of perturbation. Use the following values:

$$
\begin{aligned}
&\varepsilon_a = 0.01, \quad \varepsilon_e = 0.005, \quad \varepsilon_v = 0.007, \quad \alpha_a = \frac{\log 2}{9}, \quad \alpha_e = \frac{\log 2}{16}, \quad \alpha_v = \frac{\log 2}{25}, \\
&x_a^* = 0, \quad y_a^* = 10, \quad x_e^* = 67, \quad y_e^* = 5, \quad x_v^* = 67, \quad y_v^* = -6.
\end{aligned}
$$

Present contour plots of the solution (all 4 components) at times $t^* = 10$, 50, and 100.

Hint: In the absence of any justification to the contrary, you can set $\Delta y^* = \Delta x^*$ (i.e., a square grid). The non-dimensional size of the solution domain can be 200×200, centered at the origin. Also, you can use the non-dimensional time step of $\Delta t^* = 0.1$. You may check that this guarantees numerical stability and accuracy for the $M_x = 0.5$ of our problem.