BONDI ACCRETION (ISOTHERMAL CASE)

The equation of motion for steady-state, spherically-symmetric, non–self-gravitating, isothermal flow onto a point mass ($M_\star$) involves the co-moving acceleration of a fluid elememt at radius R,

$$\frac{Dv}{Dt} \;\equiv\; -\,v(R)\,\frac{dv}{dR} \;=\; \frac{G\,M_\star}{R^2} \;-\; \frac{a_{\rm O}^2}{\rho(R)}\frac{d\rho}{dR}\,, \tag{1}$$

where v is defined to be positive inwards, and the other variables have their standard identities.

We can use the steady-state assumption,

$$4\,\pi\,R^2\,\rho(R)\,v(R) \;=\; \dot{M}_\star\,, \tag{2}$$

to eliminate $\rho(R)$ in favour of $v(R)$, and obtain

$$\frac{dv}{dR} \;=\; \left\{\frac{2\,a_{\rm O}^2}{R} \;-\; \frac{G\,M_\star}{R^2}\right\}\left\{v(R) \;-\; \frac{a_{\rm O}^2}{v(R)}\right\}^{-1}\,. \tag{3}$$

Bondi (1952; *MNRAS* **112** 195) argues that the prefered solution is the one in which the flow is subsonic at large radii ($v \to 0$ and $\rho \to \rho_\infty$ as $R \to \infty$) and supersonic at small radii ($v \to [2GM_\star/R]^{1/2}$, i.e. freefall, and $\rho \to \infty$ as $R \to 0$). However, this solution must go through the sonic point (where $v(R) = a_{\rm O}$), and Eqn. 3 is evidently singular at this point. Therefore the solution can only have a finite slope dv/dR at the sonic point if this point occurs at radius

$$R_{\rm S} \;=\; \frac{G\,M_\star}{2\,a_{\rm O}^2}\,, \tag{4}$$

so that the numerator and denominator are both zero.

With this constraint, and given the boundary condition as $R \to \infty$, the accretion rate has to be

$$\dot{M}_\star \;=\; \frac{{\rm e}^{3/2}\,\pi\,G^2\,M_\star^2\,\rho_\infty}{a_{\rm O}^3}\,. \tag{5}$$

Care has to be exercised when integrating differential equations in the vicinity of a singular point. In this case the singular point is a saddle point, and therefore the following procedure is recommended: (i) linearise the equations in the vicinity of the singular point, to obtain an approximate analytic solution there; (ii) use this analytic solution to get off the sonic point and then integrate away from the singular point inwards (towards the centre) and outwards (towards the undisturbed background medium at infinity).

For this purpose we introduce dimensionless variables (i.e. a dimensionless radius and a dimensionless radial velocity)

$$x \;=\; \frac{R}{R_{\rm S}} \;=\; \frac{2\,a_{\rm O}^2\,R}{G\,M_\star}\,, \tag{6}$$

$$w \;=\; \frac{v(R)}{a_{\rm O}}\,. \tag{7}$$

Eqn. 3 then reduces to

$$\frac{dw}{dx} \;=\; \frac{2\;\{x^{-1}\,-\,x^{-2}\}}{\{w\,-\,w^{-1}\}}\,; \tag{8}$$

and the singular point is at $(x,w)=(1,1)$.

Next we substitute $x=1+\epsilon$, $w=1+A\epsilon$ in Eqn. 8, where A is the slope of the solution at $(x,w)=(1,1)$ (equivalently, $dv/dR=2Aa_{\rm O}^3/GM_\star$ at the sonic point) and introduce Taylor Series expansions for the terms x^{-1}, x^{-2}, w^{-1}. Eqn. 8 then reduces to

$$A \;=\; \frac{2\;\{[1-\epsilon+\mathcal{O}\,(\epsilon^2)]\,-\,[1-2\epsilon+\mathcal{O}\,(\epsilon^2)]\}}{\{[1+A\epsilon]\,-\,[1-A\epsilon+\mathcal{O}\,(\epsilon^2)]\}} \;=\; \frac{1}{A}\,. \tag{9}$$

It follows that $A=\pm 1$, and since we want the solution which goes from being subsonic at large radius to supersonic at small radius, we pick $A=-1$. We therefore put $x=1+\epsilon$ and $w=1-\epsilon$ with $\epsilon>1$ (<1) and integrate outwards (inwards).

At large x the solution should asymptote towards

$$w \;\longrightarrow\; \frac{{\rm e}^{3/2}}{x^2}\,,\qquad\quad x\gg 1\,; \tag{10}$$

and at small x towards

$$w \;\longrightarrow\; \frac{2}{x^{1/2}}\,,\qquad\quad x\ll 1\,. \tag{11}$$

We also need to determine a dimensionless density and a dimensionless mass,

$$y \;=\; \frac{\rho(R)}{\rho_\infty} \;=\; \frac{{\rm e}^{3/2}}{x^2 w}\,, \tag{12}$$

$$z \;=\; \frac{M(R)}{4\,\pi\,R_{\rm S}^3\,\rho_\infty}\,, \tag{13}$$

where $M(R)$ is the mass of inflowing gas interior to radius R, i.e. it does not include the central point mass, $M_\star$. $z(x)$ can only be obtained by integrating a second equation,

$$\frac{dz}{dx} \;=\; \frac{\mathrm{e}^{3/2}}{w} \tag{14}$$

and then applying the boundary condition $z(0) = 0$. $w(x)$ is needed for stretching a uniform-density sphere to produce the initial particle distribution.

Eqns. 8 and 14 have been integrated outwards to $x = 10^3$, and inwards to $x = 10^{-7}$. The results $(n, \log_{10}[x(n)], \log_{10}[z(n)], \log_{10}[y(n)], \log_{10}[w(n)])$ are tabulated in BONDI.DAT at uniform logarithmic intervals in x ($\Delta \log_{10}[x] = 0.01$), i.e. 1000 tabulated values, where $n = 100 \log_{10}(x(n)/10^{-7})$.

Note that when this setup is simulated, (a) the central sink should not increase in mass (even though it assimilates SPH particles); and (b) the inflowing gas should not be self-gravitating (the gravitational field is entirely due to the constant central mass). These assumptions are acceptable provided the central mass satisfies

$$M_\star \;\ll\; \frac{6\, a_{\mathrm{O}}^6}{\pi\, \mathrm{e}^{3/2}\, G^3\, \rho_\infty}\,, \tag{15}$$

i.e. – even though this may sound paradoxical – the central mass is much *less* than the Jeans mass in the background medium.

Suppose that you have picked the isothermal sound speed, a_{O}, the background density, ρ_∞, and the central point mass (satisfying Eqn. 15). You also know the sonic radius, from Eqn. 4. The only choice remaining is x_{B}, i.e. the dimensionless radius at the edge of the computational domain. The outer radius of the (spherical) initial cloud is at

$$R_{\mathrm{B}} = x_{\mathrm{B}}\, R_{\mathrm{S}}\,. \tag{16}$$

Therefore, if you choose x_{B} too large, all the computational effort will go into reproducing the subsonic flow region outside the sonic point. Conversely, if you choose x_{B} too small, all the effort will go into reproducing the supersonic flow region interior to the sonic point.

The total mass of inflowing gas is

$$M_{\mathrm{O}} \;=\; \frac{\pi\, G^3\, M_\star^3\, \rho_\infty\, z_{\mathrm{B}}}{2\, a_{\mathrm{O}}^6}\,. \tag{17}$$

The stretching transformation which converts a uniform-density, unit-radius sphere into a Bondi flow is

$$r \;\longrightarrow\; R_{\mathrm{S}}\, x\,, \tag{18}$$

where x is the solution of the equation

$$z(x) \;=\; r^3\, z_{\mathrm{B}}\,. \tag{19}$$

The local density and inward radial velocity are then $\rho = \rho_\infty y(x)$ and $v = a_{\mathrm{O}} w(x)$.