TVD and ENO Schemes for Multidimensional Steady and Unsteady Flows
A Comparative Analysis

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ABSTRACT: *This work deals with numerical solution of first order hyperbolic equation or system in 2D or 3D. In the first part, the solution of a 2D linear scalar equation is investigated. Using this model problem, the results of different numerical schemes are compared from the point of view of accuracy and parallel efficiency. The next part describes 2D and 3D steady numerical solutions of the transonic flow through a channel and a turbine or compressor cascade by several second order TVD methods on structured quadrilateral grids. The full TVD MacCormack and Causon's simplified version are used and an improved version is proposed. In the last part, the numerical simulation by ENO schemes of the unsteady interaction of a shock wave with a flow inhomogeneity described by Euler or Navier-Stokes equations is presented.*

KEY WORDS: *hyperbolic equations, Euler equations, ENO schemes, TVD schemes, transonic flow, shock-vortex interaction.*

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1. A 2D linear equation

The accuracy of different TVD and ENO schemes is investigated in the first section. The linear scalar equation:

$$u_t - yu_x + xu_y = 0 \tag{1}$$

equipped with the initial condition $u(x, y, 0) = u_0(x, y)$ is chosen as a model equation. The analytical solution can be calculated as:

$$u^{Ex}(x, y, t) = u_0(x \cos t + y \sin t, -x \sin t + y \cos t). \tag{2}$$

This problem is numerically solved on the regular Cartesian grid aligned with x and y coordinates. The numerical solution $u^n_{i,j}$ which approximates the exact one $u^n_{i,j} \approx u^{Ex}(i\Delta x, j\Delta y, n\Delta t)$ in appropriate sense i.e. mean value for finite volume methods (FVM) or point-wise value for finite difference methods (FDM) is computed with TVD and ENO schemes.

2.1 *Numerical methods*

The numerical solution is generally obtained by semi-discrete methods. It means that one solves the system of ordinary differential equations:

$$\frac{du_{i,j}(t)}{dt} = -L_h(u). \tag{3}$$

Here the operator L_h is so called space discretization and it approximates the divergence of the flux (in our case $L_h(u) \approx \partial_x(-yu) + \partial_y(xu)$. Different methods are used for the evaluation of $L_h(u)$. In general, the following conservative methods are implemented:

$$L_h(u) = \frac{1}{\Delta x}(f_{i+1/2,j} - f_{i-1/2,j}) + \frac{1}{\Delta y}(g_{i,j+1/2} - g_{i,j-1/2}) \tag{4}$$

Using different approximations of the numerical fluxes $f_{i+1/2,j}$, $g_{i,j+1/2}$ one can construct different spatial discretizations.

First order upwind (TVD) scheme: The well known first order upwind scheme can be obtained by the following formulas:

$$f_{i+1/2,j} = \begin{cases} f_{i,j} & \text{if } f'_{i+1/2,j} > 0 \\ f_{i+1,j} & \text{if } f'_{i+1/2,j} < 0 \end{cases} \tag{5}$$

Here $f_{i,j} = f(u_{i,j}) = -y_j u_{i,j}$ and $f'_{i+1/2,j} = -y_j$. The same formula is used for $g_{i,j+1/2}$ with $-y_j$ replaced by x_i.

Second order TVD scheme: For simplicity we suppose that f' is nonnegative, the case with $f' < 0$ is similar. The second order MUSCL scheme can be written as:

$$f_{i+1/2,j} = f_{i,j} + 0.5\text{minmod}(f_{i+1,j} - f_{i,j}, f_{i,j} - f_{i-1,j}). \tag{6}$$

Here minmod is defined by $\text{minmod}(a,b) = \text{sign}(a)\max(0, \min(|a|, \text{sign}(a)b))$. This scheme is second order accurate away from discontinuities and local extrema, but the accuracy reduces to first order near singularities.

Second order ENO scheme: This scheme can be obtained (for the case $f' > 0$) with:

$$f_{i+1/2,j} = f_{i,j} + 0.5 s_{i,j} \tag{7}$$

where

$$s_{i,j} = \begin{cases} f_{i+1,j} - f_{i,j} & \text{if } |f_{i+1,j} - f_{i,j}| < |f_{i,j} - f_{i-1,j}| \\ f_{i,j} - f_{i-1,j} & \text{if } |f_{i+1,j} - f_{i,j}| \geq |f_{i,j} - f_{i-1,j}|. \end{cases} \tag{8}$$

Third order ENO scheme: The finite difference ENO scheme of Shu and Osher can be written (for the case $f' > 0$) using the following formulas:

$$f^{-2}_{i,j} = \frac{1}{3} f_{i-2,j} - \frac{7}{6} f_{i-1,j} + \frac{11}{6} f_{i,j} \tag{9}$$

$$f^{-1}_{i,j} = -\frac{1}{6} f_{i-1,j} + \frac{5}{6} f_{i,j} + \frac{1}{3} f_{i+1,j} \tag{10}$$

$$f^{0}_{i,j} = \frac{1}{3} f_{i,j} + \frac{5}{6} f_{i+1,j} - \frac{1}{6} f_{i+2,j} \tag{11}$$

Then the ENO algorithm is applied for selecting the “smoothest” stencil.

1. Let $k = 0$.
2. If $|f_{i,j} - f_{i-1,j}| < |f_{i+1,j} - f_{i,j}|$ then $k = -1$.
3. If $|f_{i+k-1} - 2f_{i+k,j} + f_{i+k+1,j}| < |f_{i+k} - 2f_{i+k+1,j} + f_{i+k+2,j}|$ then $k = k-1$.

Then one takes

$$f_{i+1/2,j} = f^{k}_{i,j} \tag{12}$$

For details see [SO89].

Fifth order WENO scheme: This weighted ENO scheme of Jiang and Shu [JS95] is based on the previous one. The numerical flux is computed (for $f' > 0$) as a weighted average of f^{-2}, f^{-1} and f^{0} [9]:

$$f_{i+1/2,j} = \omega_0 f^{-2}_{i,j} + \omega_1 f^{-1}_{i,j} + \omega_2 f^{0}_{i,j} \tag{13}$$

Where $\omega_k = \alpha_k / (\sum_l \alpha_l)$ and

$$\alpha_0 = 0.1 / \left(\frac{13}{12} (f_{i-2,j} - 2f_{i-1,j} + f_{i,j})^2 + \frac{1}{4} (f_{i-2,j} - 4f_{i-1,j} + 3f_{i,j})^2 + \epsilon \right)^p$$

$$\alpha_1 = 0.6 / \left(\frac{13}{12} (f_{i-1,j} - 2f_{i,j} + f_{i+1,j})^2 + \frac{1}{4} (f_{i-1,j} - f_{i+1,j})^2 + \epsilon \right)^p$$

$$\alpha_2 = 0.3 / \left(\frac{13}{12} (f_{i,j} - 2f_{i+11,j} + f_{i+2,j})^2 + \frac{1}{4} (3f_{i,j} - 4f_{i+1,j} + f_{i+2,j})^2 + \epsilon \right)^p$$

with $p = 2$ and $\epsilon = 10^{-6}$. For more details see [JS95].

For the time integration of the system [3] the following methods were used: first order Euler method, second order TVD Runge-Kutta method, third order TVD Runge-Kutta method and fourth order non-TVD Runge-Kutta method. The above mentioned Runge-Kutta methods are more precisely described in [SO89] and [JS95].

2.2 *Numerical results*

As a test case we solved equation [1] on domain $\Omega = [-1, 1] \times [-1, 1]$ with the initial condition:

$$u_0(x, y) = \max\left(1 - \left((x - 0.5)^2 + y^2\right)/0.0225, 0\right) \tag{14}$$

and using Cartesian grid with $N \times N$ points. Then the numerical solution at the time $t = \pi$ is compared to the analytic one [2].

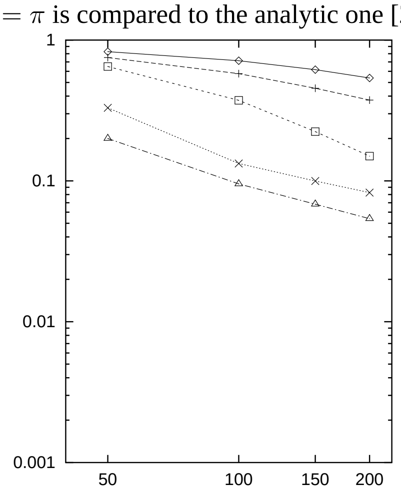

Fig. 1: *Errors in* $||.||_\infty$ *norm*

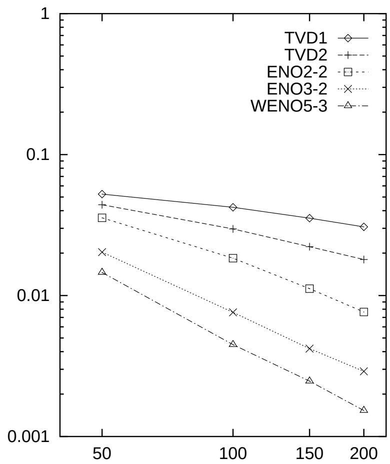

Fig. 2: *Errors in* $||.||_1$ *norm*

Figures 1 and 2 show the numerical error measured with the discrete norms $||.||_\infty$ and $||.||_1$ respectively in the spaces L^∞ and L^1 as a functions of grid size N. The results are plotted for (from the worst to the best) the first order TVD scheme, second order TVD, second order ENO with second order Runge-Kutta method, third order ENO combined with second order Runge-Kutta method and for fifth order WENO scheme with third order R-K method.

All the methods were written in HPF and compiled using ADAPTOR version 4.0 [Bra95]. Table 1 shows a parallel efficiency for the second order ENO scheme with different mesh sizes computed as $E = T_1/(N_P T_{N_P})$ where N_P is the number of processors and T_{N_P} is the elapsed time with N_P processors. The computations were performed on SGI Power Challenge with 6 R8000 processors and System V shared memory communication.

N_P	50×50	100×100	150×150	200×200
2	84.0%	95.6%	96.8%	97.9%
4	41.5%	74.4%	85.7%	88.9%

Table 1: *Parallel efficiency for different grid sizes on SGI Power Challenge*

2. Transonic flow through channels and cascades

Some results of the simulation of 2D and 3D transonic flows through channels and turbine or compressor cascades are presented in this section. The results are obtained by different second order TVD and high order ENO methods implemented on structured meshes for the discretisation of the system of Euler equations $\partial_t W + \partial_x F + \partial_y G = 0$:

1. Full TVD MacCormack scheme in the finite volume formulation [Yee87].
2. Causon's TVD MacCormack scheme in the finite volume formulation [Cau89]. For simplicity, only 1D version of this scheme is presented. It consists of two main steps: usual MacCormack predictor-corrector step and TVD damping.

$$\begin{aligned} W_i^{n+1/2} &= W_i^n - \lambda \left(F_i^n - F_{i-1}^n \right) \quad \lambda = \frac{\Delta t}{\Delta x} \\ \overline{W}_i^{n+1} &= \frac{1}{2} \left[W_i^n + W_i^{n+1/2} - \lambda \left(F_{i+1}^{n+1/2} - F_i^{n+1/2} \right) \right] \\ W_i^{n+1} &= \overline{W}_i^{n+1} + DW_i^n \end{aligned} \tag{15}$$

   The TVD correction DW_i^n uses a simplified numerical viscosity term is defined by:

$$\begin{aligned} DW_i^n &= \left(G_i^+ + G_{i+1}^- \right) \Delta W_{i+1/2} - \left(G_{i-1}^+ + G_i^- \right) \Delta W_{i-1/2} \\ G_i^\pm &= \frac{1}{2} C(\nu) \left[1 - \Phi(r^\pm) \right] \quad \Delta W_{i+1/2} = W_{i+1} - W_i \\ C(\nu) &= \begin{cases} \nu(1-\nu) & \nu \le \epsilon \\ \epsilon(1-\epsilon) & \nu > \epsilon, \ \epsilon \in]0, 1/2] \end{cases} \\ \nu &= \frac{\Delta t}{\Delta x} \rho_A = \frac{\Delta t}{\Delta x} (|u| + a), \ a^2 = \kappa \frac{p}{\rho} \\ r_i^\pm &= (\Delta W_{i-1/2}, \Delta W_{i+1/2}) / ||\Delta W_{i \pm 1/2}||^2 \\ \Phi(r) &= \operatorname{minmod}(2r, 1). \end{aligned} \tag{16}$$

3. Modified Causon's TVD MacCormack scheme. Numerical experiments show that the original Causon's scheme is very dissipative. A simple modification can improve the accuracy. Jameson suggested in [Jam95] to use the minimal (in absolute value) eigenvalue instead of the maximal one. Thus, for the Euler's

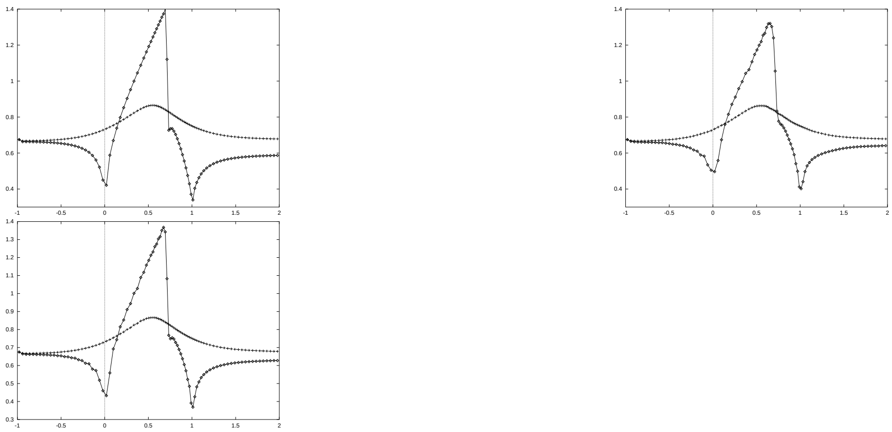

Figure 3: *Mach number distribution in Ron-Ho-Ni channel; full TVD MC scheme, Causon's TVD MC scheme and modified Causon's scheme (from left to right).*

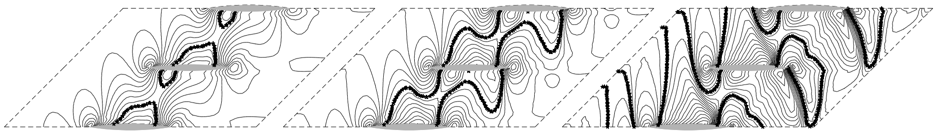

Figure 4: *Mach number distribution for 3D DCA 8% cascade.*

equations one choses $\nu = \frac{\Delta t}{\Delta x} \min\left(|u-a|, |u|, |u+a|\right)$ in [16] instead of $\nu = \frac{\Delta t}{\Delta x}(|u|+a)$ or better the so called entropy corrected absolute value of this ν.

First, a transonic flow through a 2D Ron-Ho-Ni channel is solved with inlet Mach number $M_1 = 0.675$. Figure 3 shows the distribution of the Mach number along the walls computed by the full TVD MacCormack scheme, Causon's TVD MacCormack scheme and modified Causon's scheme.

Next, the transonic flow through the 2D DCA 8% cascade of Institute of Thermomechanics of Czech Republic is computed for the inlet Mach number $M_1 = 1.08$. All the methods are also examined on more industrial cases such as transonic flows through a 2D turbine cascade of Škoda Plzeň with inlet Mach number $M_1 = 0.4$ and outlet $M_2 = 1.2$ and compressor cascade of DFVLR Köln with $M_1 = 1.03$ and $p_2/p_1 = 1.53$.

The Causon's TVD scheme and the improved Causon's scheme are extended for the 3D case. Figure 4 shows the Mach number distribution in three different cross-sections $z = const.$ for the case of 3D transonic flows through the DCA 8% cascade. Our results show a good agreement both with the results obtained by Kozel, Nhac and Vavřincová with the classical MacCormack scheme using an artificial viscosity of Jameson's type [KNV89], and with the measurements carried out at the Institute of

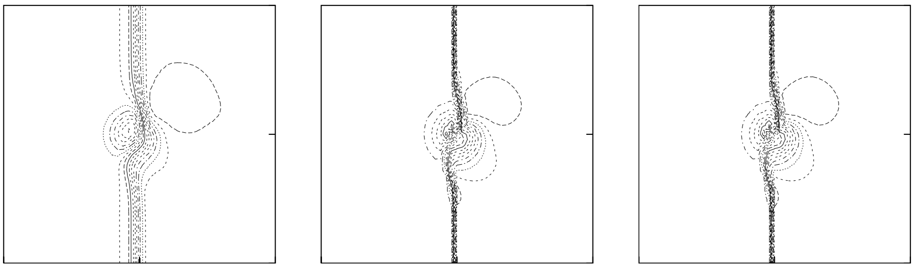

Figure 5: *Inviscid shock-vortex interaction, pressure distribution, 5-order WENO scheme with different flux-splitting methods.*

Thermomechanics of the Czech Academy of Sciences.

3. The shock-vortex interaction

The numerical solution of the interaction of a steady shock with a flow inhomogeneity as vortex or temperature perturbation is described in this section. The results are obtained by third order ENO schemes of Shu and Osher and the new WENO approach of Jiang and Shu with a third order TVD Runge-Kutta method for the 2D and 3D Euler's and 2D Navier-Stokes equations. Different flux-splitting methods are investigated:

1. Lax-Friedrichs method with $F^{\pm}(W) = \frac{1}{2}\left(F(W) \pm \max(|u| + a)W\right)$ where $\max$ is taken over the whole domain,
2. local Lax-Friedrichs method with $F^{\pm}(W) = \frac{1}{2}\left(F(W) \pm (|u| + a)W\right)$
3. Roe upwind splitting.

Different ENO approaches for systems were also tested: the characteristic reconstruction and the reconstruction in conservative variables. Although the characteristic reconstruction with Roe upwinding was more accurate, we decided to use much more simple conservative variable reconstruction with the Lax-Friedrichs splitting combined with a multi-level mesh refinement algorithm on Cartesian grids [BC89].

Figure 5 shows the pressure isolines obtained for the inviscid shock-vortex interaction by the fifth order WENO scheme with (from left to right) Lax-Friedrichs, local Lax-Friedrichs and Roe decompositions. One can observe that the shock wave is smeared for the case of Lax-Friedrichs splitting. Next, the combination of WENO scheme for inviscid terms and second order central difference scheme for viscous terms was used for computation for the case of compressible Navier-Stokes equation. The production of the vorticity will be investigated for the case of viscous interaction of the shock ($M_1 = 1.2$) with a temperature perturbation at $Re = 50$. The same case was solved by Lee, Moin and Lele by modified Padé schemes on a nonuniform mesh in [LML92].

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