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CALCULI OF COMPLEXITY: HOW PHENOMENA EMERGE FROM RULES

A REVIEW OF COMPLEXITY: A GUIDED TOUR BY MELANIE MITCHELL

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A complex system is one that generates emergent structures, usually by the repeated application of relatively simple rules. The structures are emergent in the sense that they are not specified in, and cannot be predicted from, the rules that produce them. Mitchell argues for a unified science of complexity but does not ultimately succeed in providing a coherent account. Her book is nevertheless useful as an introduction to some systems and algorithms that have been studied in complexity research and that may be applicable to natural phenomena, including the behavior of organisms. The book also contains interesting historical and biographical details about complexity research and some of the mathematicians and scientists who have contributed to it. A sample of the specific complex systems and algorithms Mitchell introduces is examined in this review, including iterated maps, cellular automata, genetic algorithms, and small world and random Boolean networks. The relevance of complexity theory for behavior analysis is also considered, including applications of simple rules, networks, cellular automata, and genetic algorithms.

Key words: complexity, emergence, genetic algorithms, cellular automata, networks, behavior dynamics

In *Complexity: A Guided Tour,* Melanie Mitchell seeks a unified science of complexity that can be applied to all complex systems. She acknowledges that this is a formidable task: "…neither a *single science of complexity* nor a single *complexity theory* exists yet (p.14, emphasis in the original)," but she wishes to develop such a science, and her purpose in writing this book was to describe the "struggle to define [the science's] central terms…the struggles to define such core concepts as *information, computation, order,* and *life* (p. 14, emphasis in the original)", and the "struggles to understand the many facets of complexity (p. 14)". In the last chapter of her book, Mitchell discusses the history of complexity research, including the cybernetics movement that was initiated by Norbert Wiener in the 1940s. Mitchell quotes William Aspray, an historian of science, who wrote that

> …Wiener's hope for a unified science of control and communication was not fulfilled. As one participant [in a conference]…explained, cybernetics had 'more extent than content'. It ranged over too disparate an array of subjects, and its theoretical apparatus was too meager and cumbersome to achieve the unification Wiener desired (p. 297).

The same can be said of complexity theory today. Mitchell's book does not come close to providing a coherent account of complexity, even from the information-theoretic perspective she favors. I suspect that most readers new to complexity, who appear to be Mitchell's intended audience, will come away from this book more bewildered than enlightened. This is unfortunate, because complexity research appears to have much to offer, both to science in general and to behavior analysis in particular.

An alternative to Mitchell's vision of a unified science of complexity is the view that complexity research is a kind of discrete, algorithmic mathematics that deals with a variety of often abstract topics that may or may not be applicable to natural phenomena. Most of these topics are interesting, and some have been applied to scientific problems with a degree of success. A sample of the topics Mitchell discusses will be considered in this review. First, however, it is

The title of this article was inspired by the title of an article by Crutchfield (1994) and by the title of a book by Holland (1995). I thank Nicholas Calvin and Andrei Popa for their helpful comments on an earlier version of this paper.

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probably a good idea to ask what complexity is—a question that Mitchell addresses in Chapter 1.

Complexity

The idea of complexity is simple. The essence of a complex system is that it exhibits emergence, which is

> …generally understood to be a process that leads to the appearance of structure not directly described by the defining constraints and instantaneous forces that control a system (Crutchfield, 1994, p. 12).

A simple example of emergent structure is the collective behavior of a flock of birds. The flock behaves as a unit with distinctive properties, even though each individual bird's behavior is likely governed only by local processes or rules, such as "avoid collisions" and "stay with the flock". Schools of fish and colonies of ants are other common examples of natural phenomena that exhibit emergent structure (e.g., NOVA, 2007). Complex systems sometimes are also said to be self-organizing, which means that they exhibit emergent structure, but emphasizes that the structure is not caused by an overall control process. Importantly, in complexity theory the emergent structure is considered to supervene on, that is, to be caused by, the local processes or rules.

Notice that one may talk abstractly about complexity, emergence, and self-organization, or one may talk more definitely about specific complex systems and specific emergent or self–organized structures. The former usage is common in the literature, and also in Mitchell's book, and it no doubt contributes to the bewilderment of readers who are new to complexity research. In Mitchell's book the bewilderment is compounded by a degree of terminological slippage. For example, she sometimes uses "complex" as a synonym for "complicated" even though these adjectives have distinct technical meanings. A system may be complicated, in the sense that it consists of many intricately constructed parts with detailed rules of operation, but if it does not generate emergent structure, then it is not complex.

Perhaps the clearest example of a complex system is a cellular automaton, which is a finite state machine developed by John von Neumann and Stanislaw Ulam in the 1940s. Mitchell discusses cellular automata in Chapter 10 of her book. An elementary cellular automaton consists of a row of squares, or cells, that exist in one of two states—for example, black (1) or white (0). An automaton is given an initial condition, such as a row of cells colored back or white at random, and then runs by generating new rows of cells below the initial row. Each cell in the current row either retains its state or changes state in the next row, depending on its state in the current row and the states of its left and right neighbors in the current row. For example, a white cell in the current row might change state if its left neighbor is black and right neighbor is white, but retain state if both neighbors are white. Because there are four neighbor conditions (both white, both black, left black right white, right white left black) for each of the two cell states, there are eight neighbor conditions for every elementary cellular automaton. These 8 conditions are listed in the first three columns of Table 1. The automaton's rules are completed by specifying the state of the target, or "self", cell in the next row (fourth column of Table 1) given the neighbor condition in the current row. Because each neighbor condition can be associated with two possible states of the target cell in the next row, there are 2^8 or 256 possible sets of rules, and hence 256 distinct elementary cellular automata.

Table 1

A set of cellular automaton rules.

Neighbor condition			
Left	Self	Right	Next Self
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

Figure 1 shows the output of a cellular automaton given an initial condition consisting of a row of 256 cells colored black or white (1 or 0) at random (top of the left panel). The rules listed in Table 1 were then applied for 885 time steps, that is, 885 new rows were generated, each directly below the previous

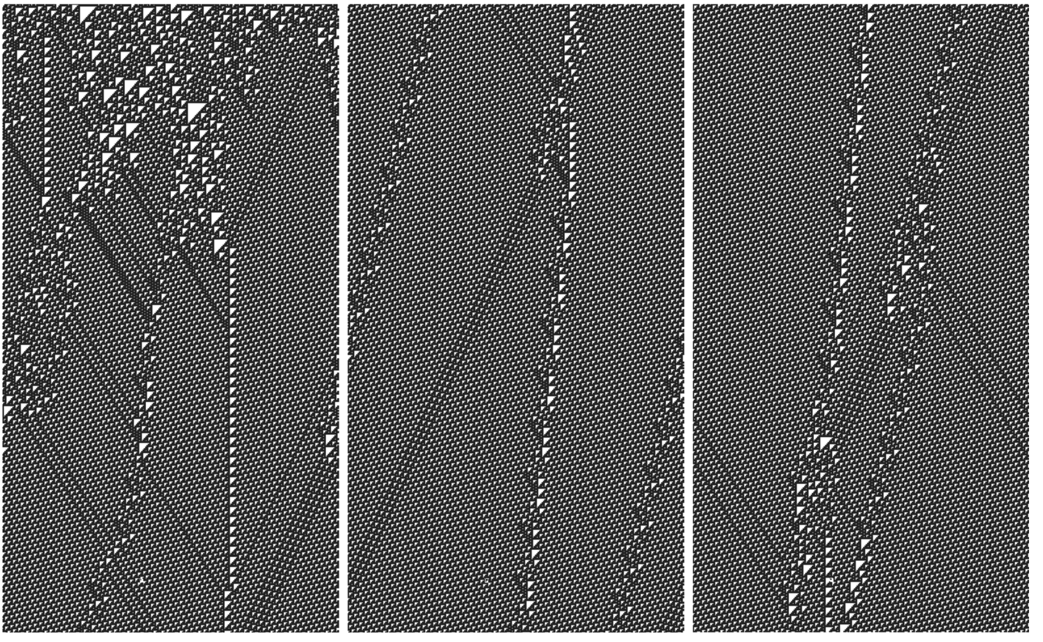

Fig. 1. Evolution in space (horizontal dimension) and time (vertical dimension) of the cellular automaton specified by the rules in Table 1. Each panel is 256 cells wide. The first and last cells in each row are neighbors. The automaton starts at the top of the left panel with an initial row of cells colored back or white at random, and then continues down the panel by applying the Table 1 rules. From the bottom of the left panel, the automaton continues to the top of the center panel and then on to the right panel. Reproduced from McDowell & Popa (2009, p. 346) by permission of Elsevier, B. V.

row. The rows continue from the bottom of the left panel to the top of the center panel, and then to the right panel. There is an initial burst of activity at the top of the left panel that eventually coalesces into well–defined structures. These structures move through space (horizontal dimension) and time (vertical dimension), and interact with each other, sometimes resulting in the structures' spatial dislocation or annihilation, and sometimes creating new structures. The automaton also generates a constant background material, evident as a pattern in the figure, through which the structures propagate.

The remarkable outcome shown in Figure 1 is complex because it contains emergent structures. The outcome supervenes on the rules listed in Table 1, but it cannot be predicted from them by any mathematical or computational shortcut. Instead, to obtain the outcome it is necessary to apply the rules, a property of some complex systems that is referred to as computational irreducibility (McDowell & Popa, 2009). Notice also that the cellular automaton that generated the outcome shown in the figure is very simple, inasmuch as it consists of only the eight rules listed in Table 1. Hence, just as a complicated system may not be complex, a complex system need not be complicated. Amazingly, Matthew Cook (2004) proved that given specific initial conditions, the automaton defined by the rules in Table 1 is capable of universal computation.

For the natural scientist, the promise of complexity is that the often confusing, detailed, and distracting world of phenomena that we observe in nature, which is analogous to the image shown in Figure 1, may supervene on simple rules. If so, then finding the rules is tantamount to obtaining a complete causal account of the phenomena.

Extent, and Specific Systems and Algorithms

The extent, in Aspray's sense (quoted earlier), of Mitchell's treatment of complexity is enormous. She considers chaotic dynamics, entropy and Maxwell's demon, Boltzmann's statistical mechanics, Shannon's information theory, Gödel's theorem, Turing machines and the decision and halting problems, evolution as understood by Darwin, Lamarck, and Stephen Jay Gould, genetics, fractals and the Koch curve, von Neumann's self-reproducing automata, genetic algorithms and evolutionary computation, cellular automata, the immune system, biological metabolism, analogy-making algorithms, computer simulations of the prisoner's dilemma, small–world and scale-free networks, power laws of scaling and fractals, and evolutionary developmental biology. She also discusses more general issues, such as the measurement of complexity. Regarding this issue, Mitchell considers a system's size, entropy, algorithmic information content, logical depth, thermodynamic depth, computational capacity, statistical complexity, fractal dimension, and degree of hierarchy as possible metrics, but finally concludes that complexity "probably can't be captured by a single measurement scale" (p. 111). This is an example of an unsuccessful (18–page) struggle to find a commonality among complex systems.

On the positive side, Mitchell includes many interesting details about the history of complexity research, as well as biographical details about some of the researchers who have contributed to it. For example, in Chapter 8 she discusses the remarkable John von Neumann and his sometimes controversial tenure at Princeton's Institute for Advanced Study (IAS). She quotes Freeman Dyson, who explained that

> The [IAS] School of Mathematics has a permanent establishment which is divided into three groups, one consisting of pure mathematicians, one consisting of theoretical physicists, and one consisting of Professor von Neumann (p. 126).

The historical and biographical details in Mitchell's book, which include numerous photographs, add a degree of interest and intimacy to her account.

Setting aside Mitchell's vision of a unified science of complexity and her struggles to assemble it, we can proceed to consider some of the interesting systems and algorithms she discusses. Four will be examined here; they are iterated maps (Chapter 2), cellular automata (Chapters 10 and 11), genetic algorithms (Chapter 9), and small world and random Boolean networks (Chapters 15, 16, and 18).

Iterated Maps

An iterated map is an equation with at least one so-called control parameter that generates a number from an initial value. This number is substituted back into the equation to generate a second number, which in turn is substituted back into the equation to generate a third number, and so on. Typically, the iterates of the equation are plotted as a time series and the plot is studied as a function of the control parameter and of the initial value used to generate the iterates.

The best known iterated map is the logistic,

$$x_{t+1} = rx_t(1 - x_t),$$

in which r is the control parameter. Given a fixed value of r, an initial x_t generates an iterate, x_{t+1}, which is used to generate the next iterate, x_{t+2}, and so on. For $r < 3$, the map converges on a single value after several iterations, at which point all future iterates are the same. This final value, which is called an attractor of the map (or system), does not depend on the initial value of x_t. As r increases slightly beyond 3, the map reaches a final state that consists of a strict alternation between two numbers. At this point the map is said to have bifurcated into a period–two attractor. Period-two attractors persist as r is increased further until, at a value of about 3.4, the map bifurcates again. It becomes a periodic alternation among four numbers, that is, a period-four attractor. As the control parameter is increased still further, the map bifurcates again, becoming a period-eight attractor, and again, becoming a period–sixteen attractor, and so on. Throughout this range, a given value of r produces the same attractor regardless of the initial value of x_t. As r is increased beyond about 3.56, the bifurcations become infinite and the attractor no longer oscillates periodically among a finite set of values. Instead it moves unpredictably among an infinite set of essentially random values, a condition that is referred to as chaos. Furthermore, in the chaotic regime, even very small changes in the initial value of x_t produce different sets of iterates and ultimately, after many iterations, can cause large divergences in the attractors associated with the different initial values. Hence, this simple deterministic system produces periodic and then unpredictable outputs as its control parameter increases. Emergent structure in the output of the logistic map is evident in its period-doubling cascade, which is shown in Figure 2. In this so-called bifurcation diagram, all points in an attractor are plotted as a function of the map's control parameter; the diagram shows the period–doubling cascade to chaos. It turns out that any map, $x_{t+1} = f(x_t)$, that is unimodal shows a similar period–doubling cascade. Moreover, as Mitchell explains in Chapter 2, Feigenbaum found that the rate of period doubling for all such maps is the same, which is another remarkable emergent property of these complex systems.

Iterated maps are interesting because they show, among other things, that apparently random behavior can be produced by simple deterministic systems. Hence, it is possible that apparent randomness observed in natural phenomena is generated by deterministic rules, rather than being truly stochastic. According to Mitchell, "physical dynamical systems, including fluid flow, electronic circuits, lasers, and chemical reactions" (p. 36), along with "computer models of weather…, electrical power systems, the heart, solar variability, and many other systems" (p. 37) have shown period–doubling cascades characterized by Feigenbaum's constant.

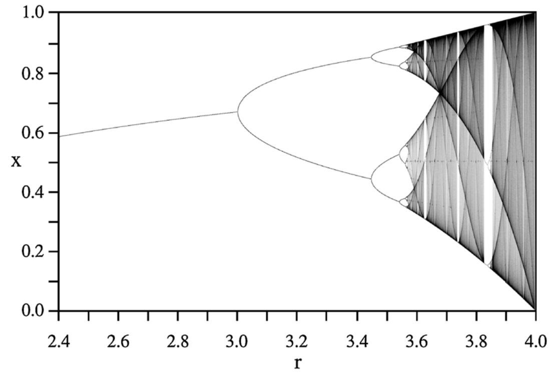

Fig. 2. Bifurcation diagram for the logistic map. All iterates, x, are plotted as a function of the map's control parameter, r. Structure in addition to period doubling, for example, boundary and iterate density curvature, is evident in the diagram. Notice also that at various values of r beyond about 3.56, the attractor falls out of the chaotic regime and returns to periodicity. For example, there is a period-3 attractor at $r \cong 3.83$. These returns to periodicity are followed by period-doubling cascades back to chaos. (Image in the public domain, retrieved September 4, 2012 from http://en.wikipedia.org/wiki/File:LogisticMap_BifurcationDiagram.png.)

Cellular Automata

These finite state machines were described earlier. They have been studied extensively, especially by Wolfram (2002), and also by Mitchell (e.g., Mitchell, Crutchfield, & Das, 1997). For example, Wolfram, in his massive 1280-page tome (reviewed by McDowell & Popa, 2009), considers cellular automata with more than two states, and that depend on more than just the left and right neighbors. He also considers cellular automata with continuous rather than discrete states, and with states that are continuous functions of the states of various combinations of neighbors. Beyond the 256 elementary forms, the field of cellular automata expands rapidly. Consider, for example, adding just one finite state to a cellular automaton such that a cell can exist in one of three, instead of just two, states, for example, white (0), black (1), and gray (2). For such an automaton the number of neighbor conditions that can determine a cell's next state expands from 8 (listed in Table 1) to 27, and so the number of possible sets of rules expands from 2^8 to 3^{27}, which works out to 7,625,597,484,987 distinct cellular automata. To learn the outputs of these 7+ trillion automata, each must be run individually, starting either from a random initial condition or from one of a very large number of periodic initial conditions (e.g., strictly alternating white, black, and gray cells). More variations, and therefore possible outputs, are obtained by using a cell's four orthogonal neighbors (called the von Neumann neighborhood) to determine its subsequent state, or else all eight of its neighbors (called the Moore neighborhood). In Chapter 10, Mitchell briefly discusses a 29–state cellular automaton invented by John von Neumann to demonstrate the logic of a self–replicating machine. In Chapter 11 she discusses Crutchfield's (1994; Crutchfield & Hanson, 1993) very interesting methods of filtering and stochastically describing cellular automata, which reveal additional structures in their outputs.

Clearly, the study of cellular automata is a very large field. Wolfram (2002), for one, also sees it as an important field inasmuch as he believes that the universe and everything we observe in it are likely caused by the operation of a single, simple, as yet to be discovered cellular automaton, or similar machine. Considerably less ambitiously, McDowell & Popa (2009) have used cellular automata to model the behavior of biological organisms, an application that will be discussed later.

Genetic Algorithms

There is no doubt that organic evolution is complex. Mitchell puts it this way: "…the result of evolution by natural selection is the appearance of 'design' but with no designer. The appearance of design comes from chance, natural selection, and long periods of time" (p. 79). The design is the emergent structure that supervenes on the evolutionary processes. "Macroscale phenomena, such as the origin of new species, can be explained by the microscopic process of gene variation and natural selection" (p. 83).

Mitchell gives an interesting brief history of evolutionary theory in Chapter 5, beginning with pre–Darwinian ideas, and then considering the ideas and theories of Lamarck, Darwin, and Mendel, including disputes between the Darwinians and the Mendelians about, for example, the role of mutation, and whether variation is discrete or continuous. She also discusses the important mathematical and statistical contributions of Sir Ronald Fisher, J. B. S. Haldane, and Sewall Wright, whose development of population genetics brought the Darwinians and Mendelians together in what is usually called the modern synthesis. Mitchell adds an interesting biographical note by discussing the bitter dispute between Fisher and Wright about the relative importance of natural selection and random genetic drift in producing evolutionary change, a dispute that led them to publish acrimonious articles opposing each other's point of view, and finally brought an end to their association. Mitchell completes her history of evolutionary theory by discussing some challenges to the modern synthesis, such as those advanced by Stephen Jay Gould and Niles Eldredge, but resisted by Ernst Mayr and Richard Dawkins.

The evolutionary process as a complex system is brought into computational form by the genetic algorithm. John Holland is usually credited with inventing the genetic algorithm, although ideas about evolutionary computing preceded him (Fogel, 1998). Genetic algorithms most often entail populations of possible solutions to a problem, where each solution consists of a number of components, and the quality or fitness of each solution can be calculated in some way. The objective in most applications of genetic algorithms is to find the components that produce the best, or fittest, solution to a problem that is specified by the researcher. Of course one could exhaustively examine all possible solutions, but this would quickly become cumbersome and impractical as the number of solution components, and the number of components required to construct each solution, increased. A more reasonable approach might be to call in an expert who has knowledge of the problem and knowledge of possible solution components; this expert might be able to construct a reasonable solution. A third approach is to attempt to evolve the best solution using a genetic algorithm.

To implement a genetic algorithm, an initial population of solutions is created, perhaps by assembling solution components at random, and then the fitness of each solution is calculated. A subset of solutions is selected on the basis of their fitness, with fitter solutions being more likely to be selected than less fit solutions. Pairs of solutions from this relatively fit subset are then recombined to produce new solutions that constitute the next generation. Recombination entails swapping components among relatively fit solutions. In addition to recombination, a small number of additional components may be swapped among solutions at random in a process analogous to mutation. The cycle then repeats. The fitness of each solution in the new population is calculated, a subset of solutions is selected on the basis of fitness, the selected solutions are recombined, a small amount of random component swapping occurs, and so on, until an acceptably fit solution emerges. It turns out that the fitness of an emergent solution is sometimes difficult to explain logically, and it may be one that the expert would not have considered.

The specific methods of representing individual solutions, defining fitness, selecting solutions on the basis of fitness, recombining relatively fit solutions, and mutating solutions, have varied widely. Genetic algorithms have been applied successfully to specific engineering problems, such as the design of nozzles and antennae, and also to problems in other areas of complexity. For example, Mitchell has studied how genetic algorithms can be used to evolve rules for cellular automata that carry out desired computations. Some of this work is described in Chapter 11, where Mitchell also explains how the evolved finite state machines may be said to process information. This discussion is interesting (and revealing, most notably on p. 168) because it shows how an information-processing interpretation adds a layer of storytelling to the causal processes at work, which are the rules that produce the outputs. Mitchell favors adding the information-processing story, but it follows from her discussion that it is *de trop*.

Because genetic algorithms have typically been used to solve specific problems, their conceptual connection to organic evolution is often muted. In a typical application, fitness is determined by the nature of the problem that requires solution. But in organic evolution, fitness is determined by the environment, often capriciously. For problem-solving applications, a final solution emerges that fits the niche, so to speak, defined by the fitness criterion that the problem imposes. But in organic evolution, the fitness landscape changes over time and adaptation occurs continuously; in other words, there is no final solution. Hence in addition to solving specific problems, genetic algorithms can be used to model the evolutionary process itself. This use of genetic algorithms is not common in the literature, but it seems to be a natural way to model the behavior dynamics of biological organisms based on the idea of selection by consequences. Such an application will be considered later.

Small World and Random Boolean Networks

Networks consist of nodes and their connections. In a fully connected network, all nodes are connected to all other nodes. Networks can be characterized by their average path length, which is the average number of links along the shortest paths connecting all possible pairs of nodes. Fully connected networks tend to have large average path lengths. Networks are also characterized by the number of connections to each node, which is referred to as the degree of the node. In a fully connected network, all nodes have the same degree, but it is possible to construct networks with nodes of different degrees. Consider, for example, a network that consists of a few high-degree nodes, sometimes referred to as hubs, and many low–degree nodes. These networks often contain clusters of hubs having many connections to lower degree nodes, with the hubs themselves interconnected in some way. The many paths within clusters tend to be relatively short, whereas the fewer connections between clusters tend to be relatively long. Networks with these properties have much smaller, often dramatically smaller, average path lengths than fully connected networks with the same number of nodes. They are called small world networks. In a small world network, local communication can occur within a cluster, clusters can communicate with each other, and in general, communication throughout the network is faster than in a fully connected network with the same number of nodes.

In Chapter 15, Mitchell describes the pioneering work of Duncan Watts and Steven Strogatz on small world networks. These researchers also showed that some real social networks have small world properties, as does the electrical power grid in the western United States, and the brain of the nematode, *C. elegans*. It is also possible that vertebrate brains have small world properties, given what is known about the functioning of specific anatomically or functionally defined brain structures (clusters), and theories of whole brain functioning that entail longer–distance communication among such structures (e.g., Bassett & Bullmore, 2006; Edelman, 1987; Tononi & Edelman, 1998). Networks may or may not produce complex outcomes, depending on their rules of operation. For example, the electrical power grid, although no doubt very complicated, is probably not complex. Artificial neural networks, on the other hand, can produce complex outcomes. These networks, which are often fully connected, work by the repeated stimulation of receptor units. The stimulation propagates through the network and causes output units to fire, which may generate patterns with emergent structures.

In Chapter 18 Mitchell discusses the very interesting random Boolean networks invented and originally studied by Stuart Kauffman. Random Boolean networks are networks that behave somewhat like cellular automata (each node can be either on or off—hence Boolean) and that generate outputs somewhat like iterated maps (producing single, periodic, or chaotic attractors). In a random Boolean network, nodes are connected to each other at random and the state of each node is updated at discrete time steps by rules based on the states of the nodes to which it is connected. These networks differ from cellular automata in that the neighbor nodes affecting a given node's state are chosen at random (but once chosen are fixed), as opposed to neighbor cells in a cellular automaton being adjacent, and each node has its own rule, as opposed to all cells in a cellular automaton having the same rules. Like cellular automata, random Boolean networks produce complex outcomes from perfect order (viz., their rules). This is in contrast to organic evolution and genetic algorithms, which produce complex outcomes from random variation and selection. In the former systems, complexity emerges from order, whereas in the latter it emerges from randomness. This has led Kauffman, by analogy with random Boolean networks, and Wolfram (2002), by analogy with cellular automata, to speculate that the complex outcome of organic evolution may not be due to natural selection from random variants at all, or at least not entirely. Instead it might be due at least in part to the nature of the organic material initially produced by evolution, which itself generates complexity from perfect order by functioning as an analogue of either a cellular automaton or a random Boolean network. Needless to say, these views are controversial.

Complexity in Behavior Analysis

Complex systems and algorithms are interesting in their own right, just as are topics in pure mathematics. But to the natural scientist they no doubt are most interesting if they can be applied to natural phenomena. Mitchell explains how complexity theory has been applied to problems in various natural science disciplines. It turns out that the fundamental idea of complexity, as well some specific complex systems and algorithms, have also been applied in behavior analysis. For example, Shimp (1966) found that in some situations, conformance of behavior to Herrnstein's (1961) matching law (McDowell, in press) emerges from the simple rule that each response occurs on the concurrent alternative with the higher momentary probability of reinforcement. This theory of momentary maximizing entails the fundamental idea of complexity, which is that the repeated application of rules may generate an emergent outcome, in this case, behavior that conforms to the matching equation. Catania's (2005) theory of the reflex reserve is another example of a theory that entails the fundamental idea of complexity. He showed that when certain rules were used to increment the reflex reserve following reinforcement and to decrement it following nonreinforcement, the behavior generated by the reserve on single schedules resembled the behavior of live organisms in various ways. In their present forms, however, Shimp's and Catania's theories appear to have limitations (Berg & McDowell, 2011; Heyman, 1979; Nevin, 1979).

In an application of a specific complex system, Donahoe and his colleagues (Donahoe, Burgos, & Palmer, 1993; Donahoe & Palmer, 1994) showed that fully connected artificial neural networks following relatively simple weighting and operating rules were able to reproduce some of the basic phenomena of classical conditioning, including acquisition, extinction, faster reacquisition of a conditional response following extinction, and blocking. Calvin (2012) found that these networks could also produce some phenomena of instrumental conditioning, although with limitations.

In another application of a specific complex system, McDowell & Popa (2009) mapped the output of a cellular automaton with a "next self" rule (column 4 in Table 1) of 00011001 onto behavior by considering single columns of output cells to represent target (black) and nontarget (white) behavior. This cellular automaton has the interesting property of calculating the quantity, $(n + n \bmod 2)/2$, given an initial condition of n adjacent black cells. The result of the calculation is displayed as that quantity of diagonal shooters traveling across the automaton's space–time diagram. Some cumulative records produced by McDowell & Popa's (2009) mapping of this automaton's output are shown in Figure 3. The mapping produced relatively constant response rates when the automaton was given initial conditions consisting of cells colored black with specific probabilities (given in each panel of Figure 3). In other words, the cumulative records were similar to those produced by live organisms working on variable–interval schedules. Moreover, higher probabilities, and hence proportions, of black cells in the initial condition produced higher response rates (as shown in the figure), and McDowell & Popa (2009) found that a plot of the mapped response rate *versus* the initial black cell probability conformed to

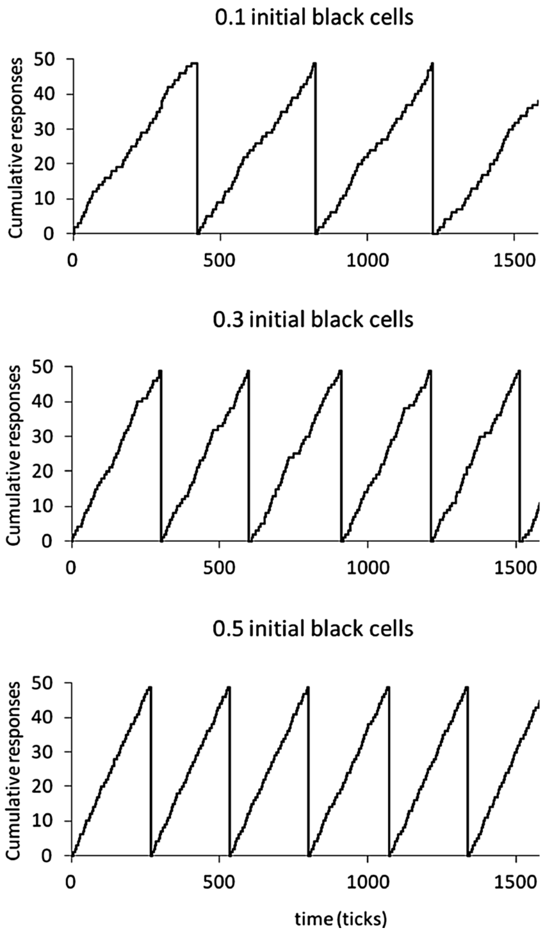

Fig. 3. Cumulative records produced by columnar mappings from an elementary cellular automaton with a "next self" rule of 00011001. In the top, middle, and bottom panels the automaton's initial condition consisted of 10%, 30%, or 50% of the cells in the initial row colored black at random. Notice the fine grain irregularities in the records, including pauses and periods of response rate acceleration and deceleration. Adapted from McDowell & Popa (2009, p. 349) by permission of Elsevier, B. V.

Herrnstein's (1970) hyperbola. However, additional unpublished research has shown that this plot in fact deviates systematically from a hyperbolic form. Furthermore, in an exhaustive study of all 256 elementary cellular automata using this particular mapping and set of initial conditions, none produced the hyperbola as an emergent property. It may be that other mappings or initial conditions, or other approaches, such as implementing a three–state cellular automaton (McDowell & Popa, 2009), would produce better results. In related work, Popa (personal communication, September, 2011) has developed a novel class of cellular automata by adding levels of mutation (random state switching) to the outputs of otherwise normally functioning machines. These automata have not been studied thoroughly, but they generate unusual emergent structures, and mappings to their outputs may produce better results.

Considerably more successful than any of these forays into complex systems has been an evolutionary theory of behavior dynamics that implements the idea of selection by consequences (McDowell, 2004). The theory was developed independently of the genetic algorithm literature and includes features not found in that literature, but it can be seen as a kind of genetic algorithm *ex post facto*. The theory implements rules of selection, recombination, and mutation that operate on a population of potential behaviors represented by integers (phenotypes) and their corresponding binary expressions, or bit strings (genotypes). Reinforcement causes parent behaviors to be selected on the basis of their fitness, which is determined by the behaviors' phenotypic similarity to the just-reinforced behavior. Parent behavior bit strings recombine to produce child behaviors that populate the next generation, and then a small amount of mutation is added to the new population by flipping random bits in the population's bit strings. At each tick of time, a behavior is emitted at random from the population, and the resulting stream of behavior can be recorded and analyzed just as if it were the behavior of a live organism.

This dynamic evolutionary theory produces behavior that conforms to every empirically valid equation of matching theory (McDowell, in press) in both single (McDowell, 2004; McDowell & Caron, 2007) and concurrent schedules (McDowell, Caron, Kulubekova, & Berg, 2008; McDowell & Popa, 2010), and yields emergent parameter estimates, including power–function matching exponents, that fall in ranges similar to those produced by live organisms. The theory also generates behavior that conforms to the bivariate matching equation (Baum & Rachlin, 1969) in environments that vary both the rate and magnitude of reinforcement, again with ranges of exponents on the independent variable ratios that are similar to those produced by live organisms (McDowell, Popa, & Calvin, 2012). In addition to these emergent large-scale structures (viz., the correctly parameterized equations of matching theory), many detailed properties of behavior generated by the evolutionary theory are consistent with live–organism behavior. These include changeover patterns (McDowell et al., 2008; McDowell et al., 2012) and responsivity to the changeover delay (Popa & McDowell, 2010) on concurrent schedules, forms of interresponse time distributions on single schedules (Kulubekova & McDowell, 2008), and patterns of preference and specific sequences of behavior on concurrent schedules where the reinforcement rate ratio changes frequently and unpredictably within sessions (Kulubekova, 2012). A possible neural mechanism for the evolutionary theory has also been discussed (McDowell, 2010), and the theory has been used in translational research in an attempt to understand the verbal behavior of antisocial boys as a function of their level of social deviance (McDowell & Caron, 2010).

Conclusion

Natural scientists who are familiar with the vast field of complexity research may see useful connections between the natural phenomena that interest them and the sometimes abstract and esoteric systems and algorithms that are studied by complexity researchers. The principal benefit of Mitchell's book is that it provides an introduction to this field. However, her information-theoretic bias and her effort to assemble a unified science of complexity tend to distract from the details of the systems she considers. The details are important for determining whether specific systems and algorithms might be applicable to natural phenomena. It also may be worth emphasizing that Mitchell's treatments are introductory; many of the topics she discusses have extensive literatures that the interested theoretician would benefit from exploring.

Theoretical physicist, Stephen Hawking, is widely quoted (e.g., Sengupta, 2006) as having said that the present century is the century of complexity. Whether this assessment extends to the science of behavior remains to be seen, but at this point in the century it seems that applications of complexity to behavior are worth pursuing further.

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