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Self-organized criticality in non-conservative models

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Some recent results on non-conservative models of self-organized criticality are reviewed and discussed. A class of deterministic models introduced by Olami, Feder and Christensen exhibits $1/f$ noise with an exponent which depends on the degree of conservation. The "Game of Life", a cellular automaton mimicking a society of interacting organisms, also appears to evolve to a critical state, with avalanches obeying finite size scaling.

1. Introduction

"Self-organized criticality (SOC)" [1, 2] describes the tendency of dissipative systems to drive themselves to a critical state with a wide range of length and time scales. The idea provides a unifying concept for large scale behavior in systems with many degrees of freedom; it complements the concept of "chaos" wherein simple systems with a small number of degrees display quite complex behavior. The phenomenon is expected to be quite universal; indeed, it has been looked for in such diverse areas as geophysics, economics, condensed matter physics and astrophysics.

In the BTW model the dynamical variable is conserved. For some time there were some speculations [3] that any degree of non-conservation would necessarily lead to a finite correlation length. Also, randomness was believed to be essential. However, several classes of non-conservative SOC models have now been discovered. Here, some recent results on two such models will be discussed. The first is a deterministic "earthquake" model introduced by Olami, Feder and Christensen [4]. It exhibits $1/f$ noise with an exponent which depends on the degree of conservation [5]. The criticality is robust with respect to external white noise. The second is the "Game of Life", a cellular automaton showing complex local structures. The avalanches following single mutations obey power laws [6] with finite size scaling. For a discussion on another non-conserved model, a forest fire model first introduced as a toy model of turbulence [7], see the paper by Drossel and Schwabl in these proceedings [8].

2. SOC and $1/f$ noise

In the original paper on self-organized criticality it was suggested that the "$1/f$" noise originating from many sources in nature originates from a superposition of avalanches of all sizes occurring in the critical state [1, 2]. However, the spatio-temporal scaling in the self-organized critical state does not necessarily manifest itself in non-trivial exponents for the power spectrum. Jensen et al. [9] and later Kertész and Kiss [10] showed that the power spectrum or the BTW model was in fact $1/f^2$, i.e. the spectrum of a random walk.

The proper relationship between the distribution of avalanches and the power spectrum was worked out by Jensen et al. [9], assuming no interference between different avalanches. Introducing the weighted lifetime distribution.

$$\Lambda(t) = \sum s^2 P(S = s, T = t)$$

and assuming that $\Lambda(t)$ exhibits a scaling behavior

$$\Lambda(t) \approx t^{\mu} ,$$

they derived a power spectrum

$$S(f) = f^{-\phi} , \qquad \phi = 1 + \mu ,$$

for $0 < 1 + \mu < 2$. In order to get non-trivial exponents ϕ, the exponent μ of the weighted avalanche distribution must be between 1 and -1. In the BTW model the exponent was 2.1, 1.77, 1.69 and 1.56 in dimensions 2, 3, 4 and 5, respectively, which is outside this region.

Recently, Olami, Christensen and Feder [4] have introduced a class of deterministic models, related to block spring models of earthquakes, where the self-organized critical behavior is maintained for a conservation level down to 20%. The critical exponents depend on the level of conservation. In these models, a set of dynamical variables $F_{i,j}$ representing the local force is defined on a two-dimensional lattice. The values of F are increased uniformly at an essentially infinitely slow rate until somewhere the force exceeds a critical value. Then the values of F at the unstable site and its nearest neighbours (nn) are updated according to the simple toppling rule

$$F_{i,j} \to 0 , \qquad F_{\mathrm{nn}} \to F_{\mathrm{nn}} + \alpha F_{i,j} .$$

This initiates an avalanche which lasts for t time units (t parallel updatings) and involves a total of s topplings.

The difference between this model and the non-conservative BTW model, in which an F-independent amount of force was transferred to the neighbors, might seem insignificant, but nevertheless it has striking consequences. As F keeps increasing there will be more avalanches triggered by instability at other sites. During a long transient period which depends on the size of the system, the correlations grow, until saturating at a value limited by and scaling with the size of the system, indicative of the slow self-organizing process leading to the stationary critical state.

Once the stationary state is reached, the distribution function $\Lambda(t)$ for various values of α can be measured. Indeed, it obeys a power law with μ depending on the dissipation [5]. The values of μ are in the range where non-trivial exponents for the power spectra are expected. For instance, for $\alpha = 0.15$, a value of $\mu = 0.61$ was found. Fig. 1 shows the power spectrum generated by randomly superimposing the avalanches for this value of α. The exponent $\phi = 1.56$ of the spectrum is consistent with the value 1.61 expected from the measured value of μ. The lower frequency cutoff scales with the size of the system while the upper frequency cutoff is a constant. Actually, it would be interesting to perform experiments on systems of varying size in order to check our assertion that the $1/f$ noise is a critical many-body effect.

Alternatively, the time sequence can be generated by a direct measurement of the activity in a slowly driven system. This yields the same exponent. It thus appears that the long term correlations ignored by the random superposition method do not affect the shorter time scales of interest here.

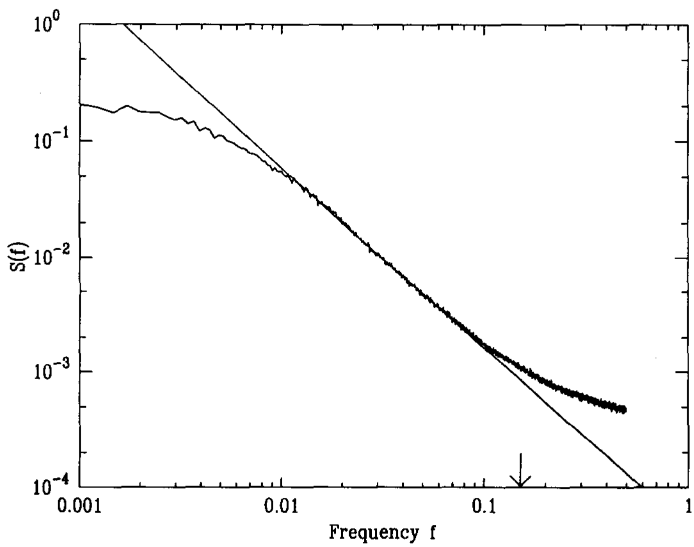

Fig. 1. Power spectrum of randomly superimposed avalanches for $\alpha = 0.15$. The arrow indicates the upper cutoff for the scaling region. The exponent ϕ of the power spectrum is $\phi = 1.56$.

3. Is Life a critical phenomenon?

A spectacular application of the SOC concept is to Conway's "Game of Life", a simple two-dimensional cellular automaton. The Game of Life may be thought of as a model of a simple society of living organisms. The organisms live on a two-dimensional square lattice of size $L \times L$. They can be either "alive" or "dead". Their fate depends on the states of the eight neighbors, at the up, down, left, right, and four corner positions, respectively. The states of all organisms are updated in parallel. A live organism remains alive as long as the number of live neighbors is not to small or too large. If the number of live neighbors is either larger than three or smaller than two it dies at the next time step. A new individual is born only at a dead site which has precisely three live neighbors.

If the system is initiated at a random configuration of live and dead individuals, it will come to rest after a while in a configuration of static clusters, and simple periodic states, "blinkers". The system is now perturbed by a mutation in which a single individual is added. This affects the number of live sites surrounding the neighbors and may cause an avalanche of extinction and creation events; after a number of parallel updatings the system will come to rest in a new static configuration. This procedure is repeated again and again. Eventually the system appears to organize itself into a statistically stationary state with avalanches of all sizes, i.e. the society has become globally connected.

The size s of an avalanche is measured as the total number of births and deaths following a single perturbation. Fig. 2 shows histograms for the distribution of avalanches for several sizes L of lattices, with closed boundary conditions. The linear behavior over three decades indicates a power-law distribution. The cutoff at large avalanches depends on the size of the system. This is seen by plotting the histograms for various L in terms of a rescaled coordinate s/L, rather than the size itself. The rescaling makes the curves for various L fall on a single curve; the system obeys finite size scaling. Only critical systems obey finite size scaling. The biggest catastrophic avalanches for the 256×256 system involved as many as 10 million events following a single perturbation.

Another indication of criticality is the fact that the scaling depends on the type of boundary conditions, such as whether the boundaries are open or periodic. The lifetime t of an avalanche is defined as the total number of parallel updatings before the system comes to rest. The distribution of lifetimes obeys a similar power law distribution with finite size scaling. The temporal cutoff scales with L with an exponent less than one, i.e. the diffusion is faster than linear. For $L = 512$, the cutoff is at $t \approx 10\,000$. One might think of the

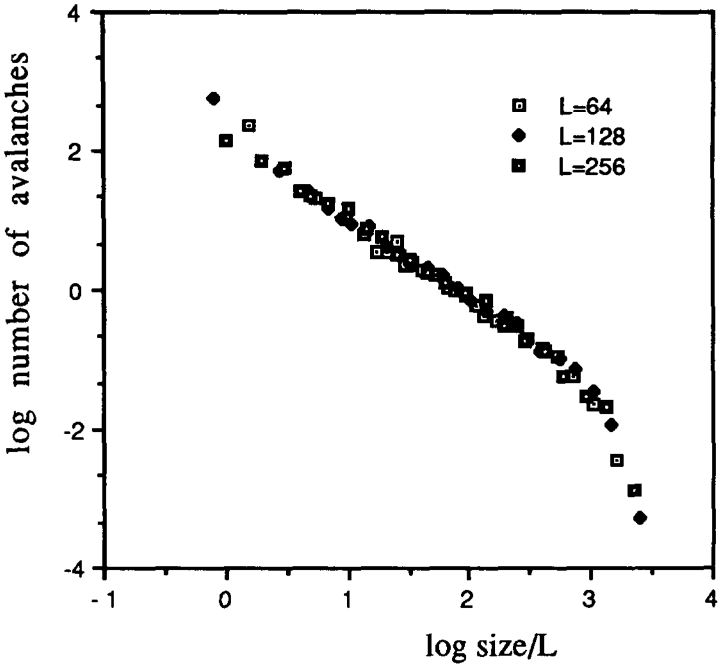

Fig. 2. Histogram of sizes of avalanches in the "Game of Life" for various linear dimensions L of the lattice. The linear behavior over 3 decades, with slope $b \approx 1.1$, indicates criticality. Finite-size scaling of the variables condenses all curves to a single one.

diffusion process as a non-Gaussian "Levy flight" with an appropriate algebraic distribution of jumps at each time step. For different boundary conditions, one would have to rescale the axis with different powers of L.

Our paper [6] has turned out to become somewhat controversial, with Bennett and Bourzutschky [11] claiming a size-independent cutoff for lattices larger than $L = 80$. They claim that there is a characteristic time scale of order $t = 50$ beyond which there is exponential decay but the finite scaling result shown above seems to refute this. The computational demands for establishing equilibrium for larger lattices are prohibitively large even for today's most powerful computers, so we cannot guarantee that there is not an intrinsic cutoff for $L > 500$. Moreover large scale computations by Creutz [12], and massive parallel computations by Herrmann [13] confirm our finding of criticality. The numerical work by Bennett and Bourzutschky thus appears seriously flawed.

It is quite interesting that John Conway, in his search for models exhibiting local complex structures, arrived at a model which at the same time is at criticality. Is this accidental, or does it suggest an intimate relation between global *criticality* and *local complexity* in self-organizing systems. Maybe complexity and criticality are synonymous concepts, with local complexity being a consequence of global self-organized integration.

Raup, a biologist, has argued that biological evolution is in fact intermittent rather than gradual [14]. Periods of stasis are interrupted by events where many species become extinct. The distribution of the magnitudes of those events seems to be power-law like, although the statistics are poor. Perhaps this can be taken as an indication that biology operates at a self-organized critical point, in which case no external cataclysmic force is necessary to bring about major disasters such as the extinction of the dinosaurs. Kauffmann and Johnson has studied models of evolving interacting species, and found evidence of self-organized criticality [15]. This idea is sometimes called "evolution to the edge of chaos", emphasizing that the critical state is between a disordered "chaotic" state and a frozen ordered state with small avalanches only. Moreover, Tom Ray [16] has performed a spectacular simulation of life evolving in the memory of a computer, and also found intermittent events with power law distribution. Maybe self-organized criticality constitutes the physical principle behind Lovelock's Gaia hypothesis [17], in which life on earth constitutes one single globally connected organism far out of equilibrium.

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