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Three research directions in non-uniform cellular automata

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ABSTRACT

The paper deals with recent developments about non-uniform cellular automata. After reviewing known results about structural stability we complete them by showing that also sensitivity to initial conditions is not structurally stable. The second part of the paper reports the complexity results about the main dynamical properties. Some proofs are shortened and clarified. The third part is completely new and starts the exploration of the fixed points set of non-uniform cellular automata.

1. Introduction and motivations

Last fifty years witness the growing interest of researchers in cellular automata (CA) both from the theoretical and applicative point of view. CA are indeed a formal model for complex systems [41,13,12,26]. They essentially consist of an infinite number of finite automata arranged on a regular lattice ($\mathbb{Z}$ in this paper). Each automaton takes a state chosen from a finite set. The state is updated at each time step according to a local rule on the basis of the state of the automaton itself and the ones of a fixed set of neighboring automata. All automata of the lattice apply the same local rule and have the same neighborhood pattern. All updates are synchronous. These few lines single out the three main characteristics of the model: locality, synchronicity and uniformity. Relaxing these properties originates variants of the model that have a great interest in their own, especially in practical applications. This paper surveys recent results about the dynamical behavior of non-uniform cellular automata (ν-CA), *i.e.*, those variants of CA in which each automaton can have a different local rule (and hence a possibly different neighborhood). In Section 3, the reader can immediately get convinced that ν-CA constitute a real stand-alone model. Indeed, many of the classical results concerning CA dynamics are disproved. For example, injective ν-CA are no longer necessarily surjective and expansive ν-CA do not need to be surjective.

Structural stability is one of the main motivations for the study of this new model. This notion has been introduced by Aleksandr Andronov and Lev Pontryagin in relation to the qualitative behavior of a dynamical system [4]. A property of a system is structurally stable if small perturbations of the system do not affect it. Therefore, the structural stability of a dynamical system is often interpreted as its robustness to failures.

In the context of cellular automata, there are several ways to model structural perturbations. The first possibility is to consider transient failures. In [30], Peter Gacs proposed a model in which each cell has some probability p of being updated and $1 - p$ of keeping its current state (*i.e.*, the identity local rule is applied). More recently, the case of transient failures

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is more viewed as a modification of the updating scheme for cellular automata. This turns into a model of asynchronous cellular automata. An ever-growing literature exists on this subject. The interested reader can begin with [43,27,17,18,16,28], for example.

Another possibility is to modify the topology of the lattice, *i.e.*, the links between cells can be rewired. As the neighborhood can vary, to be consistent with the definition of the local rule, this kind of perturbation is mainly used for totalistic rules, namely functions of the values in the cells wherever they are positioned [32].

This paper is more concerned with the case of permanent failures. Given a cellular automaton, the local rule is replaced in some positions by arbitrary local rules. The structural stability of several properties is investigated according to those perturbations. Section 4 shows that (among other things) neither sensitivity to the initial conditions nor almost equicontinuity are structurally stable properties for CA. Indeed, it seems that most of the dynamical properties are not structurally stable except for equicontinuity.

These results also point out that structural stability is influenced not only from the specific properties of the local rules perturbing the system but even from their relative distribution and cooperation. Section 5 tries then to characterize all the distributions of local rules inducing a given dynamical property. Indeed, if the radius and the set of states are fixed, the set of distinct local rules is finite. Therefore, a distribution of local rules is just a bi-infinite word over some finite alphabet and, seemingly, the set of distributions inducing a given dynamical property is nothing but a language of bi-infinite word. This simple remark leads us to characterize bi-infinite languages of distributions inducing ν-CA with some interesting dynamical property. The idea is that the complexity of the language quantifies, in a certain sense, the complexity of the dynamical property itself.

For example, we illustrate that distributions inducing surjective ν-CA form a sofic subshift while injective ν-CA are characterized by ζ-rational languages. Going more in deep along this direction seems difficult since one needs more properties on the intrinsic structure of the ν-CA to prove further results. For these reasons, we focused on linear non-uniform cellular automata, *i.e.*, ν-CA with an additive global rule (see [38,11] for the main results about additive CA). The additivity constraint allowed to show that for linear ν-CA also equicontinuity and sensitivity to the initial conditions are characterized by ζ-rational languages. Indeed, in the more general case, we know that none of those properties is decidable and hence the language cannot be ζ-rational.

The last part of the paper reports a recent research direction focusing on fixed points. These latter play a fundamental role in many modeling situations since they represent the viable/feasible ones (in systems biology, for example). Section 6 first writes down some more or less folklore results about the cardinality of the set of fixed points in CA. The proofs are essentially based on the well-known De Bruijn graphs. More or less the same results about cardinality can be restated also in the context of ν-CA. Moreover, we proved the following interesting characterization. Consider a set S of configurations which can represent a pointed ζ-rational language. Then, in the case of rν-CA (a subclass of ν-CA), the set of distributions having S as set of fixed points is a pointed ζ-rational language.

All the three research directions need further investigations and provide more questions than answers. Some of these questions are addressed in the last section.

2. Background

In this section, we briefly recall standard definitions about CA and discrete dynamical systems (see for instance [35,22,2,15,14,1,23,10] for introductory matter and recent results).

For all integers i and j with $i \le j$, let $[i, j]$ denote the set $\{i, i+1, \dots, j\}$. With the obvious meaning, we shall use the notations $]-\infty, i]$ and $[i, \infty[$.

Configurations and cellular automata Let A be a finite set (an *alphabet*). A *configuration* is a function from $\mathbb{Z}$ to A. The *configuration set* $A^{\mathbb{Z}}$ is usually equipped with the metric d defined as follows

$$\forall x, y \in A^{\mathbb{Z}}, \quad d(x, y) = 2^{-n}, \quad \text{where } n = \min\{i \in \mathbb{N}, x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}.$$

The set $A^{\mathbb{Z}}$ is a compact, totally disconnected and perfect topological space (*i.e.*, $A^{\mathbb{Z}}$ is a Cantor space). For all integers i and j with $i \le j$, and for all configurations $x \in A^{\mathbb{Z}}$, we denote by $x_{[i,j]}$ the word $x_i \cdots x_j \in A^{j-i+1}$, *i.e.*, the portion of x inside the interval $[i, j]$. Similarly, $u_{[i,j]} = u_i \cdots u_j$ is the factor of a word $u \in A^l$ inside $[i, j]$ (here, $i, j \in [0, l-1]$). For any word $u \in A^*$, $|u|$ denotes its length. A *cylinder* of block $u \in A^k$ and position $i \in \mathbb{Z}$ is the set $[u]_i = \{x \in A^{\mathbb{Z}} : x_{[i,i+k)} = u\}$. Cylinders are clopen sets w.r.t. the metric d and they form a basis for the topology induced by d. A configuration x is said to be *a-finite* for some $a \in A$ if there exists $k \in \mathbb{N}$ such that $x_i = a$ for all $i \in \mathbb{Z}$, $|i| > k$. In the sequel, the collection of the a-finite configurations for a certain a will be simply called set of finite configurations.

A (one-dimensional) *cellular automaton* (CA) is a structure $\langle A, r, f\rangle$, where A is the *alphabet*, $r \in \mathbb{N}$ is the *radius* and $f : A^{2r+1} \to A$ is the *local rule* of the automaton. The local rule f induces a *global rule* $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined as follows,

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad F(x)_i = f(x_{i-r}, \dots, x_{i+r}).$$

Recall that F is a uniformly continuous map w.r.t. the metric d.

A local rule f is extended to a function $f^*: A^* \to A^*$ which map any $u \in A^*$ to the word v such that $v = \epsilon$ (the empty word), if $|u| \leq 2r$; and v is the word of size $l - 2r$ defined by $v_i = f(u_{[i,i+2r]})$ for all $i \in [0, |v| - 1]$, otherwise. This definition still holds also for $u \in A^{\mathbb{N}}$. With an abuse of notation, we will still write f for f^*.

DTDS and dynamical properties A *discrete time dynamical system (DTDS)* is a pair (X, G) where X is a set equipped with a distance d and $G: X \to X$ is a map which is continuous on X with respect to the metric d. When $A^{\mathbb{Z}}$ is the configuration space equipped with the above introduced metric and F is the global rule of a CA, the pair $(A^{\mathbb{Z}}, F)$ is a DTDS. From now on, for the sake of simplicity, we identify a CA with the dynamical system induced by itself or even with its global rule F.

Given a DTDS (X, G), an element $x \in X$ is an *ultimately periodic point* if there exist $p, q \in \mathbb{N}$ such that $G^{p+q}(x) = G^q(x)$. If q can be chosen equal to 0, then x is a *periodic point*, *i.e.*, $G^p(x) = x$. The minimal p for which $G^p(x) = x$ holds is called *period* of x. A DTDS (X, G) is *surjective* (resp., *injective*) if G is surjective (resp., G is injective).

Recall that a DTDS (X, g) is *sensitive to the initial conditions* (or simply *sensitive*) if there exists a constant $\varepsilon > 0$ such that for any element $x \in X$ and any $\delta > 0$ there is a point $y \in X$ such that $d(y, x) < \delta$ and $d(G^n(y), G^n(x)) > \varepsilon$ for some $n \in \mathbb{N}$. A DTDS (X, G) is *positively expansive* if there exists a constant $\varepsilon > 0$ such that for any pair of distinct elements x, y we have $d(G^n(y), G^n(x)) \geq \varepsilon$ for some $n \in \mathbb{N}$. If X is a perfect set, positive expansivity implies sensitivity. Recall that a DTDS (X, G) is *(topologically) transitive* if for any pair of non-empty open sets $U, V \subseteq X$ there exists an integer $n \in \mathbb{N}$ such that $G^n(U) \cap V \neq \emptyset$.

Let (X, G) be a DTDS. An element $x \in X$ is an *equicontinuity point* for G if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in X$, $d(y, x) < \delta$ implies that $\forall n \in \mathbb{N}$, $d(G^n(y), G^n(x)) < \varepsilon$. For a CA F, the existence of an equicontinuity point is related to the existence of a special word, called *blocking word*. A word $u \in A^l$ is s-blocking ($s \in [1, l]$) for a CA F if there exists an offset $d \in [0, l - s]$ such that for any $x, y \in [u]_0$ and any $n \in \mathbb{N}$, $F^n(x)_{[d,d+s-1]} = F^n(y)_{[d,d+s-1]}$. A DTDS is said to be *equicontinuous* if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, $d(x, y) < \delta$ implies that $\forall n \in \mathbb{N}$, $d(G^n(x), G^n(y)) < \varepsilon$. If X is a compact set, a DTDS (X, G) is equicontinuous iff the set E of all its equicontinuity points is the whole X. A DTDS is said to be *almost equicontinuous* if E is residual (*i.e.*, E contains an intersection of dense open sets). In [36], Kůrka proved that a CA is almost equicontinuous if and only if it is non-sensitive if and only if it admits a r-blocking word.

Recall that two DTDS (X, G) and (X', G') are *topologically conjugated* if there exists a homeomorphism $\phi: X \to X'$ such that $G' \circ \phi = \phi \circ G$. In that case, (X, G) and (X', G') share some properties such as surjectivity, injectivity, transitivity. If ϕ is only an injective morphism, then (X, G) is called a subsystem of (X', G').

Languages. Recall that a *language* is any set $\mathcal{L} \subseteq A^*$ and a *finite automaton* is a tuple $\mathcal{A} = (Q, A, T, I, F)$, where Q is a finite set of states, A is the alphabet, $T \subseteq Q \times A \times Q$ is the set of *transitions*, and $I, F \subseteq Q$ are the sets of *initial* and *final* states, respectively. A *(finite) path* in $\mathcal{A}$ is a finite sequence of transitions $(q_i, a_i, q_{i+1})_{i \in [0,n]}$. The word $a_0 \cdots a_{n-1}$ is the label of the path. A path is *accepting* if $q_0 \in I$ and $q_n \in F$. The language $\mathcal{L}(\mathcal{A})$ of the automaton $\mathcal{A}$ is the set of the labels of all accepting paths in $\mathcal{A}$. A language $\mathcal{L}$ is *rational* if there exists a finite automaton $\mathcal{A}$ such that $\mathcal{L} = \mathcal{L}(\mathcal{A})$.

An infinite (ω-) language (resp., a bi-infinite (ζ-) language) is any subset of $A^{\mathbb{N}}$ (resp., $A^{\mathbb{Z}}$). Let $\mathcal{A} = (Q, A, T, I, F)$ be a finite automaton. An infinite (resp., bi-infinite) path in $\mathcal{A}$ is an infinite (resp., bi-infinite) sequence of transitions (q_i, a_i, q_{i+1}) for $i \in \mathbb{N}$ (resp., $i \in \mathbb{Z}$). The word a is the label of the path. An infinite path is *accepting* if $q_0 \in I$ and $\{i \in \mathbb{N} : q_i \in F\}$ is infinite. A bi-infinite path is *accepting* if the sets $\{i \in \mathbb{N} : q_{-i} \in I\}$ and $\{i \in \mathbb{N} : q_i \in F\}$ are infinite. The infinite (resp., bi-infinite) language $\mathcal{L}^{\omega}(\mathcal{A})$ (resp., $\mathcal{L}^{\zeta}(\mathcal{A})$) of the automaton $\mathcal{A}$ is the set of the labels of all accepting infinite (resp., bi-infinite) paths in $\mathcal{A}$. An infinite (resp., bi-infinite) language $\mathcal{L}$ is *ω-rational* (resp., *ζ-rational*) if there exists a finite automaton $\mathcal{A}$ such that $\mathcal{L} = \mathcal{L}^{\omega}(\mathcal{A})$ (resp., $\mathcal{L} = \mathcal{L}^{\zeta}(\mathcal{A})$).

For words u and v in $A^{\mathbb{N}}$, we denote by $[u, v]$ the bi-infinite word w such that $w_i = u_i$ and $w_{-i-1} = v_i$, for all $i \in \mathbb{N}$. For pairs U, V of ω-languages on the alphabet A, we denote by $[U, V]$ the set of bi-infinite words $\{[u, v] : u \in U, v \in V\}$. A bi-infinite language $\mathcal{L}$ is a *pointed ζ-rational language* if it is a finite union of languages $[U, V]$ where U and V are ω-rational languages. Classical ζ-rational languages are just closures of pointed ζ-rational languages under the shift operator, *i.e.*, the function $\sigma: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined as $\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \sigma(x)_i = x_{i+1}$.

A bi-infinite language X is a *subshift* if X is (topologically) closed and σ-invariant, *i.e.*, $\sigma(X) = X$. For any $\mathcal{F} \subseteq A^*$ let $X_{\mathcal{F}}$ be the bi-infinite language of all bi-infinite words x such that no word $u \in \mathcal{F}$ appears in x. A bi-infinite language X is a subshift if and only if $X = X_{\mathcal{F}}$ for some $\mathcal{F} \subseteq A^*$. The set $\mathcal{F}$ is a set of *forbidden words* for X. A subshift X is said to be a *subshift of finite type* (resp., *sofic*) iff $X = X_{\mathcal{F}}$ for some finite (resp., rational) $\mathcal{F}$.

For a more in deep introduction to the theory of formal languages, the reader can refer to [31] for rational languages, [6,37] for subshifts and [42] for ω-rational and ζ-rational languages.

3. Non-uniform cellular automata

This section gives the formal definition of ν-CA and the first basic properties. Moreover, main differences with classical CA are reported.

3.1. Definition and first properties

Definition 1 *(Non-uniform cellular automaton (ν-CA)).* A non-uniform cellular automaton (ν-CA) is a pair $(A, (\theta_i, r_i)_{i \in \mathbb{Z}})$ where A is a finite set called *alphabet* and the sequence $(\theta_i, r_i)_{i \in \mathbb{Z}}$, called *distribution of rules*, is such that $r_i \geq 0$ and $\theta_i : A^{2r_i+1} \to A$, for all integer i. The function θ_i is the *local rule* of *radius* r_i at index i in the distribution θ. The *global rule* induced by a ν-CA (or by the distribution of rules θ) is the function $H_\theta : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined as

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad H_\theta(x)_i = \theta_i(x_{[i-r_i, i+r_i]}).$$

Clearly, the global function of a ν-CA is uniformly continuous. From now on, for the sake of simplicity, we identify a ν-CA with its global rule H_θ or even with the dynamical system induced by itself. It is easy to see that ν-CA is characterized as the class of continuous functions on $A^{\mathbb{Z}}$. In other words, a function $H : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is the global rule of a ν-CA if and only if H is continuous.

The class of ν-CA is by far too wide to be dealt with. Therefore, we will focus on several subclasses of ν-CA endowed with some structure [9]. This allowed to study their behavior with more precision.

Definition 2 *(dν-CA, pν-CA, and rν-CA).* A ν-CA $H : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is

- a dν-CA if there exists a distribution θ, a local rule f and an integer $n \geq 0$ such that $H = H_\theta$ and $\forall k \in \mathbb{Z}, |k| > n \Rightarrow \theta_k = f$. The rule f is called the *default rule* and n is the *perturbation threshold* of H.
- a pν-CA if there exists a distribution θ and integers $n \geq 0$ and $p > 0$ such that $H = H_\theta$; and $\forall k > n, \theta_k = \theta_{k+p}$ and $\theta_{-k} = \theta_{-k-p}$. The integer n is called the *perturbation threshold* and p the *structural period* of H. If $p = 1$, θ_{n+1} and θ_{-n-1} are said to be the *right* and *left default rule*, respectively.
- an rν-CA if there exists a distribution θ and an integer $r \geq 0$ such that $H = H_\theta$ and for each θ_i is a local rule of radius r. In that case, the given ν-CA is said to have radius r.

Remark that the some notions above introduced (namely, default rule, perturbation threshold and radius) are not univocally defined for a given ν-CA (seen as a dynamical system) but only for a distribution inducing it. It is possible to entirely define them using minimality arguments, but, for practical purposes, it is more convenient to allow some flexibility. For instance an rν-CA of radius r can be viewed as an rν-CA of radius $r' > r$.

Example 1. Consider the ν-CA's H_0, H_1, H_2 and H_3 on the alphabet $A = \{0, 1\}$ defined as follows: $\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}$,

$$H_0(x)_i = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$H_1(x)_i = \begin{cases} 1 & \text{if } i \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$H_2(x)_i = \begin{cases} 1 & \text{if } |i| \text{ is prime}, \\ 0 & \text{otherwise}, \end{cases}$$

$$H_3(x)_i = x_{-i}.$$

Clearly, H_0 is a dν-CA but not a $\mathcal{CA}$, H_1 is a pν-CA but not a dν-CA, H_2 is an rν-CA but not a pν-CA, and H_3 is a ν-CA but not an rν-CA.

The following strict hierarchy holds among the classes of ν-CA introduced in Definition 2:

$$\mathcal{CA} \subsetneq d\nu\text{-}CA \subsetneq p\nu\text{-}CA \subsetneq r\nu\text{-}CA \subsetneq \nu\text{-}CA.$$

Proposition 1. *(See [20].) Any dν-CA is topologically conjugated to a dν-CA of radius 1 and perturbation threshold 0. Any pν-CA is topologically conjugated to a pν-CA of radius 1, perturbation threshold 0, and structural period 1. Any rν-CA is topologically conjugated to an rν-CA of radius 1.*

Proposition 2. *(See [20].) Any rν-CA is a sub-system of a CA.*

3.2. Differences with classical CA

We now illustrate some differences in dynamical behavior between CA and ν-$\mathcal{CA}$. The following properties which are really specific for CA are lost in the larger class of ν-$\mathcal{CA}$.

P1) *the set of ultimately periodic points is dense in* $A^{\mathbb{Z}}$,
P2) *surjectivity* $\Leftrightarrow$ *injectivity on finite configurations*,
P3) *surjectivity* $\Leftrightarrow$ *any configuration has a finite number of pre-images*,
P4) *expansivity* $\Rightarrow$ *transitivity*,

P5) *expansivity* $\Rightarrow$ *surjectivity,*
P6) *injectivity* $\Rightarrow$ *surjectivity.*

Some of the previous properties are not valid for the following ν-$\mathcal{CA}$.

Example 2. Consider the ν-CA of global rule H_4 on the alphabet $A = \{0, 1\}$ defined as follows

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad H_4(x)_i = \begin{cases} x_i & \text{if } i = 0 \\ x_{i-1} & \text{otherwise.} \end{cases}$$

Since $H_4^n([01]_0) \subseteq [0^{n+1}1]_0$ for every $n \in \mathbb{N}$, the cylinder $[01]_0$ contains no ultimately periodic configuration. Thus, P1 is not valid for H_4. Furthermore, any configuration $x \in A^{\mathbb{Z}}$ has 0 pre-images if $x_0 \neq x_1$. Otherwise, x admits as pre-images the 2 configurations y and z such that $\forall i \notin \{-1, 0\}$, $y_i = z_i = x_{i+1}$, $y_0 = z_0 = x_0$, $y_{-1} = 0$; $z_{-1} = 1$. Hence P3, and in particular the implication $\Leftarrow$, is not valid for H_4 (while the implication $\Rightarrow$ fails for the ν-CA of global rule H such that $\forall x \in A^{\mathbb{Z}}$, $\forall i \in \mathbb{Z}$, $H(x)_i = x_i$ if $i = 0$, $H(x)_i = x_{i+1}$, if $i > 0$, and $H(x)_i = x_{i-1}$, otherwise).

Example 3. Consider the ν-CA of global rule H_5 on the alphabet $A = \{0, 1\}$ defined as follows

$$H_5(x)_i = \begin{cases} x_i & \text{if } i = 0 \\ x_{i-1} & \text{if } i > 0 \\ x_{i+1} & \text{if } i < 0 \end{cases}$$

Since $H_5(A^{\mathbb{Z}}) = [000]_{-1} \cup [111]_{-1}$, H_5 is not surjective. Furthermore, any configuration $x \in [000]_{-1} \cup [111]_{-1}$ admits as unique pre-image the configuration y such that $y_0 = x_0$, $\forall i > 0$, $y_i = x_{i+1}$, and $\forall i < 0$, $y_i = x_{i-1}$. Thus, H_5 is injective and P6 is not valid for H_5.

Example 4. Consider the ν-CA of global rule H_6 on the alphabet $A = \{0, 1\}$ defined as follows

$$H_6(x)_i = \begin{cases} 0 & \text{if } i = 0 \\ x_{i-1} + x_{i+1} \pmod 2 & \text{otherwise} \end{cases}$$

Clearly, H_6 is not surjective, and hence not transitive. Let x, y be two distinct finite configurations with $x_i \neq y_i$, for some integer i that we can assume to be positive and maximal. Since $x_{i+2} = y_{i+2}$, it follows that $x_i \oplus x_{i+2} = H_6(x)_{i+1} \neq H_6(y)_{i+1} = y_i \oplus y_{i+2}$, and so H_6 is injective on the finite configurations. Hence, P2, and in particular the implication $\Leftarrow$, is not valid for H_6 (while the implication $\Rightarrow$ fails for the ν-CA of global rule H such that $\forall x \in A^{\mathbb{Z}}$, $\forall i \in \mathbb{Z}$, $H(x)_i = x_{2i}$).

Furthermore, if x and y are two distinct configuration with $d(x, y) < \frac{1}{2}$, then $d(H_6(x), H_6(y)) = 2d(x, y)$. So, there exists a natural $n > 0$ such that $d(H_6^n(x), H_6^n(y)) = \frac{1}{2}$. Thus, H_6 is positively expansive with expansivity constant $\frac{1}{2}$ and both P4 and P5 are not valid for H_6.

4. Structural stability and permanent failures

This section deals with the problem of the structural stability for cellular automata. Formally, a model of perturbation of a CA F with local rule f is any dν-CA with default rule f. A property is structurally stable for F if it is true for any model of perturbation of F (and then for F itself).

4.1. Surjectivity and injectivity

We consider first the structural stability of surjectivity and injectivity. These properties are of a great interest in modeling real-world phenomena. Indeed, reasonable models in physics or biology have to be reversible if the modeled phenomenon is [34]. We will see that neither surjectivity nor injectivity is structurally stable, but both properties have interesting relations in the models of perturbations.

Proposition 3. *Let F be any CA with local rule f of radius r and alphabet A with* $\mathrm{Card}(A) \geq 2$. *There exist always a non-surjective and non-injective model of perturbation of F.*

Proof. Let a and b be two distinct letters of A and let g be the constant local rule equal to a. Consider the distribution $\theta \in [g^{2r+1}]_{-r}$ with $\theta_i = f$ for all $i \notin [-r, r]$. The dν-CA H_θ is a model of perturbation of F which is not surjective since $H_\theta(A^{\mathbb{Z}}) \subseteq [a^{2r+1}]_{-r}$. It is not even injective since $H_\theta(x) = H_\theta(y)$ for all distinct configurations x and y with $x_i = y_i$ for all $i \neq 0$. $\square$

There exist necessary conditions on a CA to assure surjectivity and injectivity for its models of perturbation. Moreover, any injective model of perturbation is also surjective. As every dν-CA is the model of perturbation of some CA, injectivity and reversibility are equivalent for dν-CA.

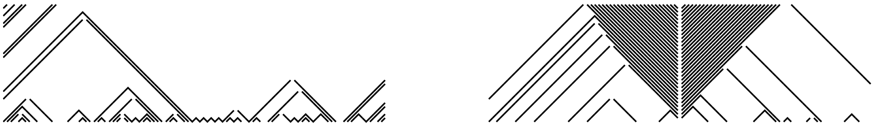

Fig. 1. Example of dynamics of F (on the left) and H (on the right). Time goes upward. A symbol 1 is represented by a / and 2 by \.

Proposition 4. *(See [20].) Let F be a CA and H a model of perturbation of F. It holds that*

1. *if H is surjective, then F is surjective,*
2. *if H is injective, then F is bijective and H is surjective.*

Proposition 4 shows that any injective dν-CA is in fact bijective. We already know that it is not true for pν-CA as shown in Example 3. Note that item 2 of Proposition 4 is not a consequence of some "Garden of Eden"-like theorem for dν-CA (the theorem for classical CA states that surjectivity is equivalent to injectivity on finite configurations). Indeed, the dν-CA from Example 4 is not surjective, and so not injective, but is injective on finite configurations.

4.2. Dynamical properties

We now study the structural stability of sensitivity, almost equicontinuity and equicontinuity. These properties are strictly related to blocking words [36,7]. A blocking word in a configuration separates the CA dynamics of the configuration to its left from the dynamics to its right and makes them independent.

In order to study the structural stability of sensitivity, consider the following example.

Example 5. Let $A = \{0, 1, 2\}$ and define the local rule $f : A^3 \to A$ as follows

$$\forall x, y, z \in A, \quad f(x, y, z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } (y = 1 \text{ or } y, z \neq 2) \\ 2 & \text{if } z = 2 \text{ and } (y = 2 \text{ or } x, y \neq 1) \\ 0 & \text{otherwise.} \end{cases}$$

The behavior of the CA F with local rule f can be described by two kinds of signals (1 and 2) that propagate towards opposite directions and annihilate whenever they meet (see Fig. 1).

Claim. *F is sensitive.*

Proof. Choose arbitrarily $n \in \mathbb{N}$ and $u \in A^{2n+1}$. Let $x, y \in [u]_{-n}$ be such that $x_i = 0$ and $y_i = 1$ for every $i \notin [-n, n]$. By induction, we obtain that

$$\forall k \in \mathbb{N}, \forall i \in \mathbb{Z}, \quad \begin{array}{l} i < -n + k \quad \Rightarrow \quad F^k(x)_i \neq 1 \\ i > n - k \quad \Rightarrow \quad F^k(x)_i \neq 2 \\ i < -n \quad \Rightarrow \quad F^k(y)_i = 1 \\ i > n - k \quad \Rightarrow \quad F^k(y)_i \neq 2 \end{array}$$

By a combination of the first, resp., last, two statements, we get that $\forall k > n$, $F^k(x)_0 = 0$, resp., $\forall k \in \mathbb{N}$, $\forall i \in \mathbb{Z}$, $F^{2n+1+k}(y)_i \neq 2$. The latter gives $i < -n + k \Rightarrow F^{2n+1+k}(y)_i = 1$, and then $F^{3n+2+k}(y)_0 = 1$.

Since any configuration $z \in [u]_{-n}$ is such that either $d(F^{3n+2}(z), F^{3n+2}(x)) = 1$ or $d(F^{3n+2}(z), F^{3n+2}(y)) = 1$, we conclude that F is sensitive with sensitivity constant 1. □

Let H be the model of perturbation of F defined as

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad H(x)_i = \begin{cases} 2 & \text{if } i = -1 \\ 1 & \text{if } i = 1 \\ f(x_{[i-1,i+1]}) & \text{otherwise.} \end{cases}$$

Roughly speaking, H acts as F but has two generators of signals (see Fig. 1).

Claim. *H is almost equicontinuous.*

Proof. For any $n \in \mathbb{N}$, consider the set $T_n = \bigcup_{k>n} [0^{2k-2}]_{-3k+2} \cap [0^{2k-2}]_{k+1}$. Every T_n is open and dense, so the set $T = \bigcap_{n \in \mathbb{N}} T_n$ is residual. We are going to prove that any element in T is an equicontinuity point for H. This allows to concludes that H is almost equicontinuous.

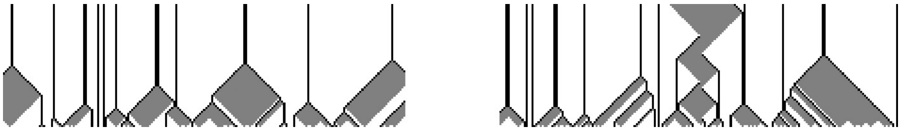

Fig. 2. Example of dynamics of F (on the left) and H (on the right). Cells in state 0, resp. 1, resp., 2, are white, resp., grey, resp. black. Time goes upward.

Since positive and negative cells do not interact each other under the application H and the dynamics is symmetric with respect to the origin of the lattice, we only consider the action of H on $A^{\mathbb{N}}$.

Choose arbitrarily $x \in T$. For any $n > 0$, $x \in T_n$ and so there exists $k > n$ such that $x_i = 0$ for all $i \in [k+1, 3k-2]$. The action of H assures that $H^t(x)_0 = 0$ for all $t > 1$, $H^{k-1}(x)_1 = 1$, $H^{k-1}(x)_j \neq 2$ for all $j \in [2, 2k-1]$, and $H^{2k-2}(x)_j = 1$ for all $j \in [1, k]$. As a consequence, one obtains by induction over all integers $i \geq 2k-2$ that $H^i(x)_k \neq 2$ and $H^i(x)_j = 1$ for every $j \in [1, k-1]$. In particular, for all $i \geq 2k-2$ it holds that $H^i(x)_j = 1$ for every $j \in [1, n]$, and by symmetry, $H^i(x)_{-j} = 2$ for every $j \in [1, n]$. Summarizing, for any $n > 0$ there exists $k > n$ such that $H^i(x)_{[-n,n]}$ does not depend on x for all $i \geq k$.

Thus, for any $n > 0$ there exists $m = n + k$ such that if y is any configuration with $y_{[-m,m]} = x_{[-m,m]}$, then $y \in T_n$, and so $H^i(x)_{[-n,n]} = H^i(y)_{[-n,n]}$ for all $i \in \mathbb{N}$. Hence, x is an equicontinuity point for H. $\square$

The following result immediately follows from the two claims proved in the Example 5.

Theorem 5. *Sensitivity to the initial conditions is not structurally stable.*

We now illustrate an almost equicontinuous CA admitting a sensitive model of perturbation. This CA has already been presented in [20]. We now propose an alternative, more simple and short proof of the existence of a sensitive model of perturbation.

Example 6. Let $A = \{0, 1, 2\}$. Define the local rule $f : A^3 \to A$ as $\forall x, y \in A$:

$$f(x,0,y) = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$f(x,1,y) = \begin{cases} 2 & \text{if } x = 2 \text{ or } y = 2 \\ 1 & \text{otherwise} \end{cases}$$
$$f(x,2,y) = \begin{cases} 0 & \text{if } x = 1 \text{ or } y = 1 \\ 2 & \text{otherwise.} \end{cases}$$

The CA F of local rule f has the following behavior: 0 is a neutral state; a cell in state 1, resp., 2, spreads its own state to each neighbor in state 0, resp., 1; state 2 is annihilated by the state 1 in some neighbor cell.

Since every word 20^k2 ($k > 0$) is k-blocking, the following fact is immediate.

Claim. *F is almost equicontinuous.*

Let H be the model of perturbation of F defined as

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad H(x)_i = \begin{cases} 1 & \text{if } i = 0 \\ f(x_{[i-1,i+1]}) & \text{otherwise.} \end{cases}$$

Roughly speaking, H acts as F but has one generator of 1 (see Fig. 2).

Since positive and negative cells do not interact each other under the application of H, we only consider the action of H on $A^{\mathbb{N}}$.

Lemma 6. *For any $s \in \mathbb{N}$ and any $u \in A^s$, let $x \in [u]_0$ be such that $x_i = 2$ for all $i \geq s$. Then, for every $n \in \mathbb{N}$, there exists $k \geq n$ such that $H^k(x)_1 = 2$.*

Proof. Choose any integer $n > 0$. Since $H^n(x)_0 = 1$ and $H^n(x)$ is 2-finite,

$$\alpha_n = \max\{i \in \mathbb{N} : H^n(x)_{[0,i]} = 1^{i+1}\},$$
$$\beta_n = \min\{i \in \mathbb{N} : H^n(x)_i = 2\}, \quad \text{and}$$
$$\gamma_n = \min\{i \in \mathbb{N} : H^n(x)_i = 2 \text{ and } \forall j > i, H^n(x)_j \neq 1\}$$

are well-defined integers with $0 \leq \alpha_n < \beta_n \leq \gamma_n$. We prove that there exists $k \geq n$ such that $\beta_k = 1$ and this concludes the proof. There are two cases:

Table 1
Possible values for v and $f(v)$.

v	0100	0102	0110	0111	1210	1211	2000	2002	2021	2210	2211
$f(v)$	11	11	11	11	02	02	00	00	00	02	02

1. $\alpha_n + 1 = \beta_n$. Then, for all $i \in [0, \beta_n - 1]$, it holds that $\alpha_{n+i} + 1 = \beta_{n+i} = \beta_n - i$, and so $\beta_k = 1$ with $k = n + \beta_n - 1$.
2. $H^n(x)_{\alpha_n+1} = 0$. Then, $\alpha_{n+1} \geq \alpha_n + 1$ and $\gamma_{n+1} \leq \gamma_n$. So, $0 < \gamma_{n+1} - \alpha_{n+1} < \gamma_n - \alpha_n$ and $\alpha_m + 1 = \beta_m$ for some $m > n$, *i.e.*, we fall in case 1. $\square$

Lemma 7. *For any $s \in \mathbb{N}$ and $u \in A^s$, the sequence $(u^{(n)})_{n\in\mathbb{N}} \in (A^s)^{\mathbb{N}}$*

$$\begin{cases} u^{(n+1)} = f(1u^{(n)}0) & \forall n \in \mathbb{N} \\ u^{(0)} = u \end{cases}$$

is such that $u^{(k)}_{s-1} \neq 2$ for some $k \in \mathbb{N}$.

Proof. For the sake of argument assume that $u^{(n)}_{s-1} = 2$ for all $n \in \mathbb{N}$. Then,

$$\alpha_n = \min\bigl\{i \in [0, s-1] : u^{(n)}_i = 2 \text{ and } \forall j \in [i+1, s-1], u^{(n)}_j \neq 1\bigr\},$$

$$\beta_n = \min\bigl\{i \in [0, s-1] : \forall j \in [i, s-1], u^{(n)}_j \neq 1\bigr\} \quad \text{and}$$

$$\gamma_n = \mathrm{Card}\bigl\{i \in [0, s-1] : u^{(n)}_i = 2\bigr\},$$

are well-defined naturals for every $n \in \mathbb{N}$.

As done in the proof of Lemma 6, one can show that there exists $k \geq 0$ such that $\alpha_k = 0$. So, without loss of generality we can assume that $u^{(0)} \in X$ where $X = \{1\}^*\{0, 2\}^*$. Since $f(X) \subseteq X$, we get that $u^{(n)} \in X$ for all $n \in \mathbb{N}$.

We are now going to prove that $\forall n \in \mathbb{N}$, $\delta_{n+1} \prec \delta_n$ where $\delta_n = (\gamma_n, \alpha_n, \alpha_n - \beta_n)$ and $\prec$ is the dominance order. Since $\prec$ is well-founded, we obtain a contradiction. Let $n \in \mathbb{N}$. We have 4 cases.

1. $\alpha_n = \beta_n = 0$. There exists $v \in \{0, 2\}^*$ such that $u^{(n)} = 2v$. Then, $u^{(n+1)} = 0v$ and so $\gamma_{n+1} = \gamma_n - 1$.
2. $\alpha_n = \beta_n > 0$. There exists $v \in \{0, 2\}^*$ such that $u^{(n)} = 1^{\beta_n}2v$. Then, $u^{(n+1)} = 1^{\beta_n-1}20v$. So, $\gamma_{n+1} = \gamma_n$ and $\alpha_{n+1} < \alpha_n$.
3. $\alpha_n > \beta_n > 0$. There exists $v \in \{0, 2\}^*$ such that $u^{(n)} = 1^{\beta_n}0^{\alpha_n-\beta_n}2v$. Then, $u^{(n+1)} = 1^{\beta_n+1}0^{\alpha_n-\beta_n-1}2v$. So, $\gamma_{n+1} = \gamma_n$, $\alpha_{n+1} = \alpha_n$, and $\beta_{n+1} = \beta_n + 1$.
4. $\alpha_n > \beta_n = 0$. There exists $v \in \{0, 2\}^*$ such that $u^{(n)} = 0^{\alpha_n-\beta_n}2v$. Then, $u^{(n+1)} = 10^{\alpha_n-\beta_n-1}2v$. So, $\gamma_{n+1} = \gamma_n$, $\alpha_{n+1} = \alpha_n$, and $\beta_{n+1} = \beta_n + 1$. $\square$

Lemma 8. *For any $s \in \mathbb{N}$ and any $u \in A^s$, let $x \in [u]_0$ be such that $x_i = 0$ for all $i \geq s$. Then, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, $H^k(x)_1 \neq 2$.*

Proof. Take $S = \{00, 02, 10, 11, 21\}$ and $\mathcal{C} = \{y \in A^{\mathbb{N}} : \forall i \in \mathbb{N}, y_{[i,i+1]} \in S\}$. For all $n \in \mathbb{N}$, $H^n(x)$ is 0-finite, and so $\alpha_n = \min\{i \in \mathbb{N}, H^n(x)_{[i,\infty[} \in \mathcal{C}\}$ is a well-defined natural. We now prove that $\forall n \in \mathbb{N}$, $\alpha_{n+1} \leq \alpha_n$ and $\lim \alpha_n = l = 0$.

Since $f(\mathcal{C}) \subseteq \mathcal{C}$, $\forall n \in \mathbb{N}, \alpha_{n+1} \leq \alpha_n + 1$. To conclude that $\alpha_{n+1} \leq \alpha_n$ it is enough to show that $H^{n+1}(x)_{[\alpha_n,\alpha_n+1]} \in S$ for every n. There are two cases:

1. $\alpha_n = 0$. Then, $H^n(x)_0 = 1$ and $H^n(x)_{[0,1]}, H^n(x)_{[1,2]} \in S$. So, $H^n(x)_{[0,2]} \in \{100, 102, 110, 111\}$ and $H^{n+1}(x)_1 = 1$. Finally, $H^{n+1}(x)_{[0,1]} = 11 \in S$.
2. $\alpha_n > 0$. Then $H^n(x)_{[\alpha_n-1,\alpha_n]} \notin S$ and $H^n(x)_{[i,i+1]} \in S$, $i \in \{\alpha_n, \alpha_n + 1\}$. If $v = H^n(x)_{[\alpha_n-1,\alpha_n+2]}$, then $f(v) = H^{n+1}(x)_{[\alpha_n,\alpha_n+1]} \in S$ (see Table 1).

Let $n_0 \in \mathbb{N}$ be such that, $\forall n \geq n_0$, $\alpha_n = l$. By contradiction, assume that $l > 0$. So, we have that

$$H^n(x)_{[l-1,l+2]} \in \{0100, 0102, 0110, 0111, 1210, 1211, 2000, 2002, 2021, 2210, 2211\}$$

for every $n \geq n_0$. If $H^n(x)_{[l-1,l+2]}$ were one among the first four patterns, we would have $\alpha_{n+1} < \alpha_n = l$. Thus, for all $n > n_0$, $H^n(x)_l = 0$ and then $H^n(x)_{[l-1,l]} = 20$ (see Table 1). Define now $u^{(n)} = H^{n+n_0+1}(x)_{[1,l-1]}$, for all $n \in \mathbb{N}$. Since for all $n \in \mathbb{N}$, $u^{(n+1)} = f(1u^{(n)}0)$, by Lemma 7 there exists $n_1 \in \mathbb{N}$ such that $u^{(n_1)}_{l-2} \neq 2$, *i.e.*, $H^{n_0+n_1+1}(x)_{l-1} \neq 2$, which is a contradiction.

Therefore, $l = 0$ and since $H^n(x)_{[0,1]} \in S$ and $H^n(x)_0 = 1$ for every $n > n_0$, it follows that $H^n(x)_1 \neq 2$ for all integer $n > n_0$. $\square$

Claim. *H is sensitive to the initial conditions.*

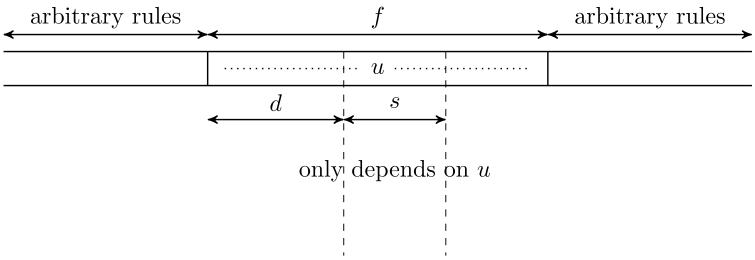

Fig. 3. A strongly blocking word. Time goes downward.

Proof. Choose arbitrarily $k \in \mathbb{N}$, $u \in A^{2k+1}$ and $z \in [u]_{-k}$. Let $x, y \in [u]_{-k}$ be such that $x_i = 2$ and $y_i = 0$ for all $i \in \mathbb{Z}$ with $|i| > k$. By Lemma 8, there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $H^n(y)_1 \neq 2$. By Lemma 6, there exists $n_1 \geq n_0$ such that $H^{n_1}(x)_1 = 2$. Finally, either $d(H^{n_1}(z), H^{n_1}(x)) = \frac{1}{2}$ or $d(H^{n_1}(z), H^{n_1}(y)) = \frac{1}{2}$ and hence H is sensitive. $\square$

Theorem 9. *(See [20].) Almost equicontinuity is not structurally stable.*

Theorem 9 says that blocking words of cellular automata are not necessarily blocking in their models of perturbations. It is possible to define a stronger notion of blocking words which forces this behavior.

Definition 3. (See [20].) Let $l \in \mathbb{N}$ with $l > 0$ and $s \in [1, l]$. A word $u \in A^l$ is *strongly s-blocking* for a CA F of local rule f and alphabet A if there exists an *offset* $d \in [0, l - s]$ such that for any rule distribution θ with $\theta_i = f$ for every integer $i \in [0, l - 1]$, it holds that

$$\forall x, y \in [u]_0, \forall n \in \mathbb{N}, \quad H^n_\theta(x)_{[d,d+s-1]} = H^n_\theta(y)_{[d,d+s-1]}.$$

Roughly speaking, a word u is blocking as soon as appears in a configuration at positions where the rule f is applied (see Fig. 3). It actually separates the dynamics on its left from the one on its right if both it is strongly s-blocking and every θ_i has radius at most s. It is obvious that any strongly s-blocking word is also s-blocking.

Proposition 10. *(See [20].) Let F be a CA of radius r. If F admits a strongly r-blocking word, then all models of perturbation of F are almost equicontinuous. Furthermore, if all words of a certain size are strongly r-blocking, then all models of perturbation of F are equicontinuous.*

More precisely, the equicontinuity behavior for models of perturbation of an equicontinuous CA is expressed by the following

Theorem 11. *(See [20].) Let F be a CA of radius r with alphabet A. The following statements are equivalent:*

1. *F is equicontinuous.*
2. *There exists $k > 0$ such that all words in A^k are strongly r-blocking.*
3. *Every model of perturbation of F is ultimately periodic.*

The following is a direct consequence of Theorem 11.

Theorem 12. *(See [20].) Equicontinuity is structurally stable.*

Remark that the existence for a CA of a strongly blocking word is not a necessary condition in order to admit equicontinuous models of perturbation as illustrated in the following example.

Example 7. Let F be the CA of radius 1 on the alphabet $A = \{0, 1\}$ and with local rule f defined as $\forall a, b, c \in A$, $f(a, b, c) = \max\{a, b, c\}$. The CA F is not equicontinuous but admits equicontinuous models of perturbation as, for instance, the dν-CA induced by the distribution θ defined as $\theta_i = f$, if $i \neq 0$, $\theta_i = g$, otherwise, where g is the constant rule giving 1.

5. Characterization and decidability of properties by distributions of rules

Given a (finite) set $\mathcal{R}$ of allowed local rules, the distributions over $\mathcal{R}$ related to interesting ν-CA properties are characterized. Indeed, distributions are viewed as infinite words on the alphabet $\mathcal{R}$ and, hence, a formal language point of view can be adopted. Every property is associated with the languages of distributions inducing ν-CA exhibiting that property. The study of those languages provides machines which recognize them and allow to prove decidability results.

This is of a great interest when modeling systems based on local interactions by ν-CA. Indeed, once local interactions have been represented by local rules, it is possible to know how to assign them to the different lattice positions in order to assure some desired properties to the model. Conversely, by the knowledge of the distribution of a ν-CA it is possible to verify if the ν-CA exhibits a certain property.

Fix an alphabet A and a finite set of local rule $\mathcal{R}$ on the alphabet A. Without loss of generality all rules in $\mathcal{R}$ can be assumed to have the same radius r. The set of all distributions on $\mathcal{R}$ is denoted by Θ. In this case, Θ is viewed as the set $\mathcal{R}^{\zeta}$ of all bi-infinite words on $\mathcal{R}$ and we call language any set of distributions. The purpose of this section is to characterize the language $\mathcal{L} \subseteq \Theta$ such that $\theta \in \mathcal{L}$ if and only if H_θ has a given property. The considered properties are number conservation, surjectivity, injectivity, sensitivity to the initial conditions and equicontinuity.

As $\mathcal{R}$ contains only rules or radius r, we can define finite distributions and the mapping they induce. A *finite distribution* on $\mathcal{R}$ is any word in $\mathcal{R}^*$. Every finite distribution ψ of length n induces a mapping $h_\psi : A^{n+2r} \to A^n$ defined by

$$\forall u \in A^{n+2r}, \forall i \in [0, n-1], \quad h_\psi(u)_i = \psi_i(u_{[i,i+2r]}).$$

Finite distributions are related to whole distributions by the fact that

$$\forall \theta \in \mathcal{R}^{\mathbb{Z}}, \forall x \in A^{\mathbb{Z}}, \forall i, j \in \mathbb{Z}, i \le j, \quad H_\theta(x)_{[i,j]} = h_{\theta_{[i,j]}}(x_{[i-r,j+r]}).$$

5.1. Number conservation

In physics, most of transformations are conservative: a certain quantity remains invariant along time (conservation laws of mass and energy, conservation of the number of atoms in chemical reactions...). Those systems often consist of a huge number of particles in mutual interaction and the power of those interactions is neglectable for sufficiently far particles. Therefore, cellular automata, and their non-uniform generalization, are particularly suitable to model such systems and the question of the translation of conservation laws naturally occurs. The case of uniform CA has been treated in a number of papers, see for instance [8,24]. The results for ν-CA have been generalized in [21].

Throughout this section we consider the alphabet $A = \{0, 1, \ldots, s-1\}$ for some integer $s \ge 2$. The results presented here can be generalized to more complex alphabets (see [29]) but we only consider the case of numerical alphabets. Denote by $\underline{0}$ the empty configuration, *i.e.* the configuration such that $\underline{0}_i = 0$ for all integer i. In the sequel, finite configurations refer to 0-finite configurations.

For all configurations $x \in A^{\mathbb{Z}}$, define the *partial charge* of x between the indexes $-n$ and n as $\mu_n(x) = \sum_{i=-n}^{n} x_i$ and the *global charge* of x as $\mu(x) = \lim_{n\to\infty} \mu_n(x)$. Clearly $\mu(x) = \infty$ if and only if x is not a finite configuration.

There exist three usual ways to define number conserving CA: the number conservation either on the finite configurations, or on the periodic configurations, or on all the configurations. As to ν-CA, the second way has no meaning as it implicitly assumes an uniform application of a unique local rule, while the other ones are similar to the CA case.

Definition 4. A ν-CA H is *number conserving on finite configurations* (FNC) if for all finite configurations x, $\mu(x) = \mu(H(x))$, while it is said to be *number conserving* (NC) if both the following conditions hold

1. $H(\underline{0}) = \underline{0}$
2. $\forall x \in A^{\mathbb{Z}} \setminus \{\underline{0}\}$, $\lim_{n\to\infty} \frac{\mu_n(H(x))}{\mu_n(x)} = 1$.

It is known that all the definitions of number conservation are equivalent for the uniform case (see [24]). It is clear that any number conserving ν-CA is FNC. The converse holds for rν-CA.

Proposition 13. *(See [21].) Any rν-CA is FNC if and only if it is NC.*

In fact, for rν-CA condition 1 from Definition 4 is implied by condition 2. However, there exist ν-CA for which the condition 2 holds while condition 1 does not. In other words, there are ν-CA which are FNC but not NC.

Theorem 14. *(See [21].) The language $\mathcal{L}$ of distributions on $\mathcal{R}$ inducing a NC ν-CA is a subshift of finite type.*

A possible set of forbidden blocks for the SFT from Theorem 14 is

$$\mathcal{F} = \left\{ \psi \in \mathcal{R}^{2r+1} : \exists u \in A^{2r+1}, \psi_{2r}(u) \ne u_0 + \sum_{i=0}^{2r-1} \psi_{i+1}\big(0^{2r-i} u_{[1,i+1]}\big) - \psi_i\big(0^{2r-i} u_{[0,i]}\big) \right\}.$$

Corollary 15. *Number conservation is decidable for pν-CA.*

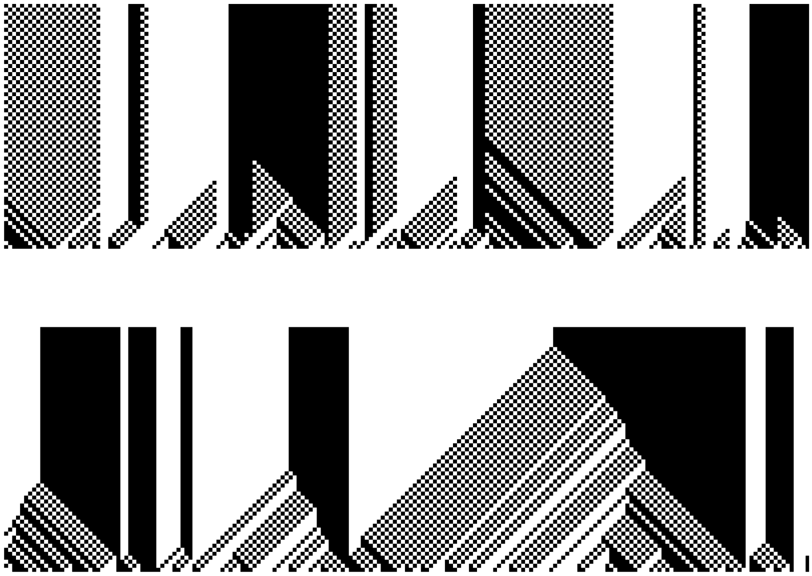

Fig. 4. Example of dynamics for automata with rules in $\{f, g, h\}$. The automaton at the top is number conserving, while the one at the bottom is not. Time goes upward.

Proof. The sets of patterns of a fixed size inside the distribution inducing a pν-CA is finite. Hence, the thesis is true. □

Example 8. Let $\mathcal{R} = \{f, g, h\}$ where f, g, h are the elementary rules 136, 184 and 252, respectively. The rule g is also known as the traffic rule in relation with its use for traffic flow modeling [41]. According to the SFT and the set $\mathcal{F}$ from Theorem 14, after a reduction we get $\{ff, gf, hh, hg\}$ as the set of minimal forbidden patterns for the SFT. Since gg is not a forbidden pattern while ff and hh are, the CA defined by g is NC while the CAs defined by f and g are not. However, there exist suitable distributions $\theta \in \Theta$ on $\mathcal{R}$ inducing number conserving rν-CA (see Fig. 4).

5.2. Surjectivity and injectivity

In classical CA settings, surjectivity and injectivity are fundamental properties which are strongly linked. Indeed, a CA is surjective if and only if it is injective on finite configurations [39,40] and then injectivity is equivalent to reversibility. As to dν-CA, this last result is still true as Proposition 4 states. However, surjectivity is no longer equivalent to injectivity on finite configurations (see Example 4). We are now interested in the case of rν-CA.

It is well-known that both properties are decidable for one-dimensional CA and undecidable in higher dimensions [3,33]. To deal with rν-CA, De Bruijn graphs and product graphs as presented in [45] have been generalized. This has allowed to extend decidability results to rν-CA.

Definition 5. Let $\mathcal{R}$ be a finite set of rules of radius r. The *De Bruijn graph of* $\mathcal{R}$ is the labeled multi-edge graph $\mathcal{G} = (V, E)$, where $V = A^{2r}$ and edges in E are all the pairs (aw, wb) with label $(f, f(awb))$, obtained varying $a, b \in A$, $w \in A^{2r-1}$, and $f \in \mathcal{R}$. The *language* $\mathcal{L}_\mathcal{G}$ of $\mathcal{G}$ is the set of all labels (ψ, u) of paths in $\mathcal{G}$.

If $\mathcal{G}$ is the De Bruijn graph of a finite set $\mathcal{R}$ of rules of radius r, then $\mathcal{L}_\mathcal{G} = \{(\psi, u) \in (\mathcal{R} \times A)^* : h_\psi^{-1}(u) \neq \emptyset\}$ and $\mathcal{L}_\mathcal{G}$ is a recognizable language.

Consider now surjectivity. By compactness, a distribution $\theta \in \Theta$ induces a surjective rν-CA H_θ if and only if h_ψ is surjective for all factor ψ of θ. Then, it follows that the language $\mathcal{L}$ of distributions inducing surjective rν-CA is a subshift. The fact that the set $L = \{\psi \in \mathcal{R}^* : \exists u \in A^*, h_\psi^{-1}(u) = \emptyset\}$ consists of the finite distributions on $\mathcal{R}$ inducing non-surjective functions and that L is the projection of $\mathcal{L}_\mathcal{G}^c$ on its first component are the ideas underlying the following result.

Theorem 16. *(See [21].) The language $\mathcal{L}$ of distributions on $\mathcal{R}$ inducing a surjective ν-CA is a sofic subshift.*

Similarly to CA, De Bruijn graphs allow to represent a configuration as a sequence of vertexes and its image obtained by some distribution as a sequence of edges. Clearly, the coupling of $\mathcal{G}$ with itself allows to individuate two configurations and the corresponding images obtained by the same distribution. In addition, the main idea of the product graph is that the two configurations have the same image.

Definition 6. Let $\mathcal{R}$ be a finite set of rules of radius r and $\mathcal{G} = (V, E)$ be the De Bruijn graph of $\mathcal{R}$. The *product graph* $\mathcal{P}$ of $\mathcal{R}$ is the labeled graph $(V \times V, W)$ where $((u, u'), (v, v')) \in W$ with label $f \in \mathcal{R}$ if and only if there exists $a \in A$ such that (u, v) and (u', v') belong to E both with the same label (f, a).

So, any path $((u_i, v_i), \theta_i)_{i\in\mathbb{Z}}$ in $\mathcal{P}$ defines two configurations x and y such that for all integer i, x_i and y_i are the $(r+1)$-th letter of u_i and v_i, respectively. Moreover, $H_\theta(x) = H_\theta(y)$. This allows to deal with injectivity.

Then, a distribution $\theta \in \Theta$ will induce a non-injective ν-CA if and only if θ is the label of some path such that $x \neq y$ if and only if θ is the label of some path reaching a vertex (u, v) with $u \neq v$. The set of all those labels defines a ζ-rational language and, as ζ-rational languages are closed under complementation, its complement is also ζ-rational. All the previous facts are the ideas underlying the following result.

Theorem 17. *(See [21].) The language $\mathcal{L}$ of distributions on $\mathcal{R}$ inducing injective ν-CA is a ζ-rational language.*

We now propose a proof for the next result from the language point of view.

Theorem 18. *(See [20].) Surjectivity and injectivity are decidable for pν-CA.*

Proof. Let H be a pν-CA defined by a distribution θ. By Proposition 1, we can assume without loss of generality that both the perturbation threshold and the structural period of H are equal to 1. Then, there exist local rules f, g and h such that $\theta_i = f$ for $i < 0$, $\theta_0 = g$ and $\theta_i = h$ for $i > 0$. It is possible to assimilate the distribution θ to the ζ-rational language $f^{-\omega}gh^{\omega}$. By Theorems 16 and 17, the languages of distributions inducing surjective and injective ν-CA on the set of rules $\{f, g, h\}$ are ζ-rational (sofic subshifts are a special case of ζ-rational languages). As the inclusion of ζ-rational languages is decidable, we can check if H is surjective or injective. $\square$

5.3. Equicontinuity and sensitivity of linear ν-CA

Equicontinuity and sensitivity are important properties of dynamical systems. It is known than none of them is decidable for CA [25]. Therefore, if we are interesting in the language of distributions from Θ inducing one of those properties, we know that this language is not ζ-rational. However, in the case of linear CA it is possible to decide both properties [38]. In this section, we extend the notion of linearity to ν-CA and we show that distributions in Θ inducing equicontinuous linear ν-CA are fully characterized by the existence of an infinite number of walls, which are a kind of mono-directional blocking words. If $\mathcal{R}$ only contains rules of radius 1, we can go further and prove that all those distributions forms a ζ-rational language.

From now on, we assume the alphabet A is endowed with two binary operations $+ : A \times A \to A$ and $\cdot : A \times A \to A$ such that $(A, +, \cdot)$ is a unitary finite ring. The main examples of such a ring is $(\frac{\mathbb{Z}}{m\mathbb{Z}}, +, \cdot)$ of integers modulo m for $m \geq 2$ with the usual addition and multiplication. Of course, $(A^n, +, \cdot)$ and $(A^{\mathbb{Z}}, +, \cdot)$ are also commutative rings where sum and product are defined component-wise and they are denoted by the same symbols.

Definition 7. A local rule f on A with radius r is *linear* if there exists $\lambda \in A^{2r+1}$ such that $\forall u \in A^{2r+1}$, $f(u) = \sum_{i\in[0,2r]} \lambda_i \cdot u_i$. A ν-CA is *linear* if there exists a distribution of linear local rules which defines it.

In fact, the notion of linearity for ν-CA matches with the usual notion of linearity from linear algebra: a ν-CA H is linear if and only if it is an endomorphism of the module $(A^{\mathbb{Z}}, +, \cdot)$, *i.e.*, $\forall a \in A, \forall x, y \in A^{\mathbb{Z}}, H(x + a \cdot y) = H(x) + a \cdot H(y)$.

It is known that any almost equicontinuous linear cellular automaton is actually equicontinuous. This property is still true in the non-uniform case.

Proposition 19. *(See [21].) A linear ν-CA is either sensitive or equicontinuous.*

We now assume that $\mathcal{R}$ is a set of linear local rules of radius r and we focus on the distributions inducing either sensitive or equicontinuous linear rν-CA. The following notion is essential to characterize equicontinuous rν-CA.

Definition 8. A *right-wall* is a finite distribution $\psi \in \mathcal{R}^*$ of length $n \geq r$ such that, for all $v \in A^r$, the sequence $u_\psi(v) : \mathbb{N} \to A^n$ defined by

$$
\begin{aligned}
&u_\psi(v)_0 = 0^n\\
&u_\psi(v)_1 = h_\psi\left(0^r u_\psi(v)_0 v\right)\\
&u_\psi(v)_{k+1} = h_\psi\left(0^r u_\psi(v)_k 0^r\right) \quad \text{for every integer } k > 1
\end{aligned}
$$

verifies $\forall k \in \mathbb{N}$, $(u_\psi(v)_k)_{[0,r-1]} = 0^r$. *Left-walls* are defined symmetrically.

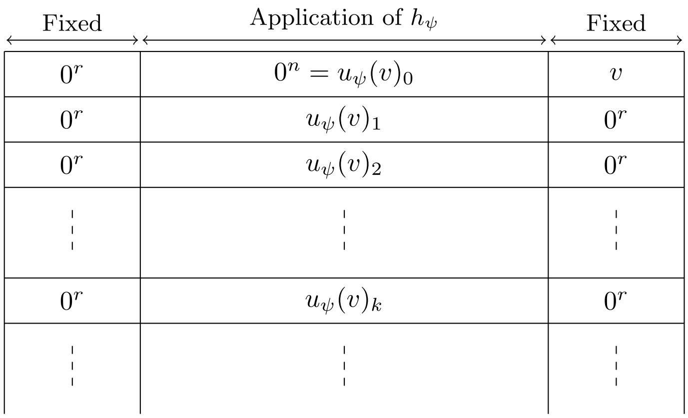

Fig. 5. The sequence $u_\psi(v)$. Time goes downward.

Roughly speaking, a right (resp., left) wall is a finite distribution that, when it is a factor of a distribution, blocks any signal coming from the right (resp., left) in the induced automaton. As to a right-wall, the sequence $u_\psi(v)$ is a partial dynamic induced by ψ on an empty initial configuration with a signal v coming from the right (see Fig. 5). After the first step, there is no longer any signal coming from the right neither from the left. Indeed, the fact that ψ is a right-wall prevents the signal from interacting with the leftmost part of the configuration.

Next two propositions illustrate two properties of walls: they are extendable and may entirely constrain the dynamic induced by a full distribution.

Proposition 20. *(See [21].) If ψ is a right-wall and ψ', ψ'' are any two finite distributions on $\mathcal{R}$, then $\psi'\psi\psi''$ is a right-wall.*

Proposition 21. *(See [21].) Let $\theta \in \Theta$, $x \in A^{\mathbb{Z}}$, m and $n \in \mathbb{Z}$ such that $m > n$ and $x_{[n+1,m]} = 0^{m-n}$. Let ψ denote $\theta_{[n+1,m]}$ and, for every $i \in \mathbb{N}$, α_i denote $H^i_\theta(x)_{[m+1,m+r]}$. If ψ is a right-wall, then, for all $k \in \mathbb{N}$,*

$$\left(\forall i \in [0,k),\, H^i_\theta(x)_{[n-r+1,n]} = 0^r\right) \quad\Rightarrow\quad H^k_\theta(x)_{[n+1,m]} = \sum_{j=0}^{k} u_\psi(\alpha_{k-j})_j.$$

Proposition 21 says that if a right-wall ψ appears as factor somewhere in a distribution θ, assuming that no signal comes from the left of ψ, the dynamics of H_θ in that place is just the superposition of the propagation of the signals coming from the right. As ψ is a right-wall, the propagation never reaches the left of the wall. Summarizing, a right-wall makes independent the dynamics to its left from the one to its right. Therefore, the dynamics between a left-wall and a right-wall is completely independent from the rest of the dynamics and so walls are used to characterize equicontinuity for rν-CA.

Theorem 22. *(See [21].) Let $\theta \in \Theta$, H_θ is equicontinuous if and only if both the two following conditions hold:*

1. *For all $n \in \mathbb{N}$, there exists $m \geq n$ such that $\theta_{[n+1,m]}$ is a right-wall.*
2. *For all $n \in \mathbb{N}$, there exists $m \geq n$ such that $\theta_{[-m,-n-1]}$ is a left-wall.*

Note that it is effectively decidable to know if a finite distribution ψ of length n is a right or left-wall. Indeed, for every $v \in A^r$, the sequence $u_\psi(v)$ is ultimately periodic and at most $\mathrm{Card}(A)^n$ different terms can appear. Then, the condition to be a wall can be checked on all these sequences by means of a Turing machine. That shows that, in some sense, the language $\mathcal{L}$ of distributions on $\mathcal{R}$ inducing equicontinuous linear rν-CA is recursive. More formally, let $\overleftarrow{\cdot}$ be the operator which inverts the indices of a word in $A^{\mathbb{N}}$ to produce its mirror in $A^{-\mathbb{N}}$ and consider the sets $\mathcal{L}^+ = \{x \in A^{\mathbb{N}} : \exists y \in \mathcal{L}, x = y_{[0,+\infty[}\}$ and $\mathcal{L}^- = \{x \in A^{\mathbb{N}} : \exists y \in \mathcal{L}, \overleftarrow{x} = y_{]-\infty,0]}\}$. We have that $\mathcal{L} = \{\overleftarrow{y}x : x \in \mathcal{L}^+, y \in \mathcal{L}^-\}$, and $\mathcal{L}^+$ and $\mathcal{L}^-$ are recursive according to [44]. However, in the general case it is not known if a smaller class of ζ-languages contains all the languages $\mathcal{L}$ induced by a finite set of rules $\mathcal{R}$. In the restricted case where $\mathcal{R}$ only contains rules of radius 1, we are going to see that the language $\mathcal{L}$ is ζ-rational.

Then, in the remaining part of this section, we assume that $\mathcal{R}$ is a finite set of linear rules of radius 1. In this case, for any rule $f \in \mathcal{R}$ there exist coefficients $\lambda^-_f = f(1,0,0)$, $\tilde{\lambda}_f = f(0,1,0)$ and $\lambda^+_f = f(0,0,1)$ in A such that for all $a,b,c \in A$, $f(a,b,c) = \lambda^-_f \cdot a + \tilde{\lambda}_f \cdot b + \lambda^+_f \cdot c$. In this context, walls are characterized as follows.

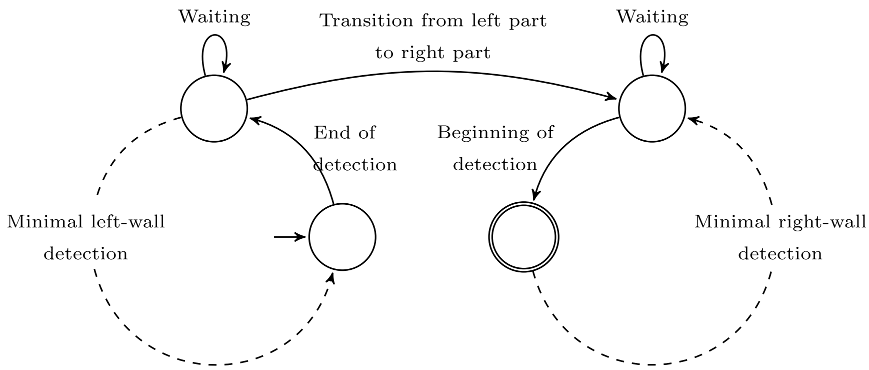

Fig. 6. Conceptual view of the automaton $\mathcal{A}$.

Proposition 23. *(See [21].) A finite distribution $\psi \in \mathcal{R}^n$ is a right-wall (resp., left-wall) if and only if $\prod_{i=0}^{n-1} \lambda^+_{\psi_i} = 0$ (resp., $\prod_{i=0}^{n-1} \lambda^-_{\psi_i} = 0$).*

Proposition 23 allows to design a finite automaton which detects right and left-wall on its paths. Let $\mathcal{A} = (\Sigma, Q, T, I, F)$ be the finite automaton such that the alphabet Σ is $\mathcal{R}$, the set of states Q is $\{-,+\} \times A$, $I = \{(-,0)\}$ and $F = \{(+,0)\}$ are the set of initial and final states, respectively, and the set T of transitions is as follows

1. $((-,a), f, (-, \lambda^-_f \cdot a))$, $\forall a \in A \setminus \{0\}, \forall f \in \mathcal{R}$ (minimal left-wall detection).
2. $((-,0), f, (-,1))$, $\forall f \in \mathcal{R}$ (end of detection).
3. $((-,1), f, (-,1))$, $\forall f \in \mathcal{R}$ (waiting).
4. $((-,1), f, (+,1))$, $\forall f \in \mathcal{R}$ (transition from left part to right part).
5. $((+,1), f, (+,1))$, $\forall f \in \mathcal{R}$ (waiting).
6. $((+,1), f, (+,0))$, $\forall f \in \mathcal{R}$ (beginning of detection).
7. $((+, \lambda^+_f \cdot a), f, (+,a))$, $\forall a \in A \setminus \{0\}, \forall f \in \mathcal{R}$ (minimal right-wall detection).

The automaton $\mathcal{A}$ consists of two symmetric parts for the detection of the left and right-walls (see Fig. 6) plus a transition from left to right. We just explain the left part: the automaton tries to detect a left-wall of minimal length and starting from $(-,1)$ it enters the states $(-, \prod \lambda^-_f)$ for those rules f read since the beginning of the detection. A left-wall is actually detected when it enters the state $(-,0)$. A distribution θ is recognized by this automaton if and only if there exists some path having this distribution as label and such that the states $(-,0)$ and $(+,0)$ are visited infinitely many times, *i.e.*, if and only if θ contains an infinite number of both left-walls and right-walls at negatives and positives indexes, respectively. Theorem 22, Proposition 19, Proposition 23, and the aforementioned automaton $\mathcal{A}$ lead to the following result.

Proposition 24. *(See [21].) Both the languages of distributions on $\mathcal{R}$ inducing linear equicontinuous and sensitive $r\nu$-CA are ζ-rational.*

In [21], an improvement has been proposed to reduce the number of states of $\mathcal{A}$. We consider the equivalence relation $\sim$ on A defined by $a \sim b$ if and only if there exists an invertible element $c \in A$ such that $a = b \cdot c$. We denote by $\tilde{A}$ the set of all equivalence classes. This relation is compatible with the multiplication of A, *i.e.*, $x \sim x'$ and $y \sim y'$ imply $x \cdot x' \sim y \cdot y'$. Then, we can extend the multiplication on A to $\tilde{A}$ and we can merge in $\mathcal{A}$ all states $(-,a)$ (resp., $(+,a)$) for the elements $a \in A$ which are in the same equivalence class. Finally, the automaton $\mathcal{A}$ can be constructed with $2 \times \mathrm{Card}(\tilde{A})$ states instead of $2 \times \mathrm{Card}(A)$ states.

One can prove that $\mathrm{Card}(\tilde{A}) = d(m)$ where number $d(m)$ is the number of divisors of m. Since $d(m) = o(m^c)$ for any $c > 0$ (see [5]), the improvement is effective. However, remark that if A is any boolean ring, the only invertible element is the multiplicative neutral one and then $\mathrm{Card}(\tilde{A}) = \mathrm{Card}(A)$, that is to say that there are no improvements at all.

6. A first characterization of fixed points

Let $F : X \to X$, we denote by $\mathrm{Fix}(F)$ the set of fixed points of F, *i.e.*, $\mathrm{Fix}(F) := \{x \in X : F(x) = x\}$. For a set S of such functions, $\mathrm{Fix}(S)$ denote the set of common fixed points of functions in S, *i.e.*, $\mathrm{Fix}(S) = \bigcap_{F \in S} \mathrm{Fix}(F)$.

In this section, we propose a study of fixed points of ν-CA using De Bruijn graphs.

6.1. Case of a CA

Let F be a cellular automaton of local rule f and radius $r > 0$. Let $\mathcal{G}_F$ be the De Bruijn graph of F, *i.e.*, the graph (V, E) where $V = A^{2r}$ and the set E is $\{(au, f(aub), ub) : a, b \in A, u \in A^{2r-1}\}$. This graph is the same as the De Bruijn graph of the set of local rules $\{f\}$ (see Section 5.2) without the second component on edges.

The fixed points graph of the CA F is the subgraph $\widetilde{\mathcal{G}_F}$ of $\mathcal{G}_F$ in which the edges are the pairs (au, ub), $a, b \in A$, $u \in A^{2r-1}$, such that $u_{r-1} = f(aub)$. It is clear that $x \in \mathrm{Fix}(F)$ if and only if x is the label of a path in $\widetilde{\mathcal{G}_F}$. In other words, $\mathrm{Fix}(F)$ is the SFT $X_{\mathcal{F}}$ where $\mathcal{F} = \{u \in A^{2r+1} : f(u) \neq u_r\}$.

Proposition 25. $\mathrm{Fix}(F)$ *has cardinality*

1. $2^{\aleph_0}$ *(of the continuum), if* $\widetilde{\mathcal{G}_F}$ *admits a strongly connected component containing two distinct cycles;*
2. $\aleph_0$, *if all the strongly connected components of* $\widetilde{\mathcal{G}_F}$ *are cycles and there exists a path between two of them.*
3. *In all the other cases,* $\mathrm{Fix}(F)$ *has finite cardinality.*

Proof.

1. It is clear that $\mathrm{Fix}(F)$ contains the subset consisting of all the bi-infinite juxtapositions of the blocks u and v possibly separated by w and w', where u and v are the labels of the two cycles and w and w' are the labels of the paths connecting them. Clearly, such a subset has cardinality $2^{\aleph_0}$.
2. In this case $\mathrm{Fix}(F)$ is the finite union of countable subsets each of which contains either a finite number of configurations or all the configurations $u^{-\omega}wv^{\omega}$, where u and v are the labels of each pair of cycles connected by some path π and w is label of π. In this second case, the position of w allows to encode any integer.
3. In this case, the strongly connected components are disconnected cycle. Every cycle encodes a finite number of spatially periodic configuration (periodic configurations for the shift) and those configurations are the only fixed points. □

6.2. Case of a pν-CA

Let H be a pν-CA. Without loss of generality, we can assume that H has radius, perturbation threshold and structural period all equal to 1 and we denote by f, g and h its left default rule, its right default rule, and the rule applied at index 0, respectively.

We define the De Bruijn graph of H to be the graph $\mathcal{G}_H = (V, E)$ where $V = A^2 \times \{f, g\}$ and E is the set

$$\Big\{\big((ab, \alpha), \alpha(abc), (bc, \alpha)\big) : a, b, c \in A, \alpha \in \{f, g\}\Big\} \cup \Big\{\big((ab, f), h(abc), (bc, g)\big) : a, b, c \in A\Big\}.$$

The fixed points graph of H is the subgraph $\widetilde{\mathcal{G}_H} = (\tilde{V}, \tilde{E})$ of $\mathcal{G}_H$ where $\tilde{V}$ is the set of vertices in V accessible from $A^2 \times \{f\}$ and co-accessible from $A^2 \times \{g\}$, and where $\tilde{E} = (\tilde{V} \times A \times \tilde{V}) \cap \{((ab, \alpha), b, (bc, \beta)) : a, b, c \in A, \alpha, \beta \in \{f, g\}\}$. It is clear that $x \in \mathrm{Fix}(H)$ if and only if x is the label of a path in $\widetilde{\mathcal{G}_H}$ which uses an edge $((ab, f), b, (bc, g))$ for some a, b and c in A at index 0. This proves that $\mathrm{Fix}(H)$ is a closed subset of $A^{\mathbb{Z}}$.

Proposition 26. $\mathrm{Fix}(H)$ *has cardinality*

1. $2^{\aleph_0}$ *(of the continuum), if* $\widetilde{\mathcal{G}_H}$ *admits a strongly connected component containing two distinct cycles;*
2. $\aleph_0$, *if all the strongly connected components of* $\widetilde{\mathcal{G}_H}$ *are cycles and there exists a path between two of them, both included in* $A^2 \times \{f\}$ *or in* $A^2 \times \{g\}$.
3. *In all the other cases,* $\mathrm{Fix}(H)$ *has finite cardinality.*

Proof. The proof is similar to the one of Proposition 25 except for the second case. When all the strongly connected component are cycles but the only paths between them use an edge of the kind $((ab, f), b, (bc, g))$ for some $a, b, c \in A$, this edge must appear at position 0 and therefore it is not possible to encode an integer. □

Corollary 27. *Let F and G be the CA of local rules f and g, respectively. If* $\mathrm{Fix}(F)$ *and* $\mathrm{Fix}(G)$ *are finite (resp., countable), then* $\mathrm{Fix}(H)$ *is finite (resp., countable).*

6.3. rν-CA

We now consider the fixed point set $\mathrm{Fix}(\{H_\theta : \theta \in \Theta\})$ for a set Θ of distributions from a given family $\mathcal{R}$ of rules with radius r. With a little abuse of notation, we will denote it by $\mathrm{Fix}(\Theta)$.

It is clear that $\mathrm{Fix}(\Theta)$ is a closed set. For any set Θ of rule distributions and any integer $i \in \mathbb{Z}$, denote by $\mathrm{App}(\Theta, i) := \{f \in \mathcal{R} : \exists \theta \in \Theta, \theta_i = f\}$ the set of rules appearing in position i in some distribution from Θ. It immediately follows that

Lemma 28. *For any $x \in A^{\mathbb{Z}}$ it holds that $x \in \mathrm{Fix}(\Theta)$ if and only if $\forall i \in \mathbb{Z}, \forall f \in \mathrm{App}(i)$, $f(x_{[i-r,i+r]}) = x_i$.*

Proposition 29. *If the sequences $\mathrm{App}(\Theta, i)_{i\in\mathbb{N}}$ and $\mathrm{App}(\Theta, -i)_{i\in\mathbb{N}}$ are ultimately periodic then $\mathrm{Fix}(\Theta)$ is a pointed ζ-rational language.*

Proof. Let p and q be the period and pre-period of $\mathrm{App}(\Theta, i)_{i\in\mathbb{N}}$. Consider the graph (V, E) where $V = A^{2r} \times \{0, \dots, p + q - 1\}$ and E contains the labeled edges $((au, i), u_{r-1}, (ub, i+1))$ and $((au, p+q-1), u_{r-1}, (ub, q))$ for all $a, b \in A$, $u \in A^{2r-1}$, and $i \in [0, p + q - 1)$ such that $\forall f \in \mathrm{App}(i)$, $f(aub) = u_{r-1}$. For each $u \in A^{2r}$, denote by U_u the set of words that are the labels of all paths with initial vertex $(u, 0)$. In a similar way, the language V_u is obtained starting from $\mathrm{App}(\Theta, -i - 1)_{i\in\mathbb{N}}$. Since $\mathrm{Fix}(\Theta) = \bigcup_{u\in A^{2r}}[V_u, U_u]$, we conclude that $\mathrm{Fix}(\Theta)$ is a pointed ζ-rational language. □

Proposition 30. *If Θ is a pointed ζ-rational language, then both the sequences $\mathrm{App}(\Theta, i)_{i\in\mathbb{N}}$ and $\mathrm{App}(\Theta, -i)_{i\in\mathbb{N}}$ are ultimately periodic.*

Proof. Let $\mathcal{L}$ be any ω-language. We show that if $\mathcal{L}$ is ω-rational then $\mathrm{App}(\mathcal{L}, i)_{i\in\mathbb{N}}$ is ultimately periodic. Let $A^k(\mathcal{L}) = \bigcup_{u\in A^k} u^{-1}\mathcal{L}$. Since $\mathcal{L}$ is ω-rational, $\mathcal{L}$ admits a finite number of residual languages and there exists $k < k'$ such that $A^k(\mathcal{L}) = A^{k'}(\mathcal{L})$. Since $\mathrm{App}(\mathcal{L}, j) = \mathrm{App}(A^i(\mathcal{L}), j - i)$ for all integers i, j with $i \le j$, it follows that

$$\forall i \in \mathbb{N}, \quad \mathrm{App}\big(\mathcal{L}, k' + i\big) = \mathrm{App}\big(A^{k'}(\mathcal{L}), i\big) = \mathrm{App}\big(A^k(\mathcal{L}), i\big) = \mathrm{App}(\mathcal{L}, k + i).$$

Hence, $\mathrm{App}(\mathcal{L}, i)_{i\in\mathbb{N}}$ is ultimately periodic with pre-period k and period $k' - k$. By a similar argument, one can prove that $\mathrm{App}(\mathcal{L}, -i)_{i\in\mathbb{N}}$ is ultimately periodic. The fact that Θ is a pointed ζ-rational language concludes the proof. □

Let $\mathcal{R}$ be a family of rules with radius r and $\mathcal{C} \subseteq A^{\mathbb{Z}}$. We now want to study the set $\Theta(\mathcal{C}) := \{\theta \in \mathcal{R}^{\mathbb{Z}} : \mathcal{C} \subseteq \mathrm{Fix}(H_\theta)\}$.

Proposition 31. *$\Theta(\mathcal{C})$ is closed and $\Theta(\mathcal{C}) = \Theta(\mathrm{Adh}(\mathcal{C}))$.*

Proof. It is clear that $\Theta(\mathcal{C})$ is closed. Since $\mathcal{C} \subseteq \mathrm{Fix}(\Theta(\mathcal{C}))$, it holds that $\mathrm{Adh}(\mathcal{C}) \subseteq \mathrm{Adh}(\mathrm{Fix}(\Theta(\mathcal{C}))) = \mathrm{Fix}(\Theta(\mathcal{C}))$, and hence $\Theta(\mathcal{C}) \subseteq \Theta(\mathrm{Adh}(\mathcal{C}))$. The converse inclusion is immediate. □

Proposition 32. *Let $\mathcal{R}$ be a family of rules with radius r and $\mathcal{C} \subseteq A^{\mathbb{Z}}$. If $\mathcal{C}$ is a pointed ζ-rational language, then $\Theta(\mathcal{C})$ is too.*

Proof. With an abuse of notation, for any $x \in A^{\mathbb{N}}$ and any $\theta \in \mathcal{R}^{\mathbb{N}}$, in this proof we means by $H_\theta(x)$ the element from $A^{\mathbb{N}}$ such that $H_\theta(x)_i = \theta_i(x_{[i,i+2r]})$ for all $i \in \mathbb{N}$.

First of all, we are going to prove that for any ω-rational language $\mathcal{L}$ and any $u \in A^{2r}$,

$$\mathcal{L}_u := \big\{\theta \in \mathcal{R}^{\mathbb{N}} : \forall x \in \mathcal{L} \cap u_{[r,2r-1]}A^\omega, H_\theta(u_{[0,r-1]}x) = x\big\}$$

is an ω-rational language. Consider an arbitrary ω-rational language $\mathcal{L}$ and a word $u \in A^{2r}$ and let $\mathcal{A} = (Q, A, T, i, F)$ be an automaton recognizing $\mathcal{L}$. Build the automaton $\mathcal{A}' = (Q', \mathcal{R}', T', (i, u), F')$ where $Q' = Q \times A^{2r}$, $\mathcal{R}' = \mathcal{R} \cup \{\mathrm{id}\}$, $F' = F \times A^{2r}$, and

$$\begin{aligned} T' = \big\{&\big((p, av), f, (q, vb)\big) : p, q \in Q, f \in \mathcal{R}, a, b \in A, v \in A^{2r-1}, \\ &\big(p, f(avb), q\big) \in T \text{ and } f(avb) = v_{r-1}\big\} \end{aligned}$$

The automaton $\mathcal{A}'$ endows $\mathcal{A}$ with the structure of the De Bruijn graph of $\mathcal{R}$ in such a way that accepting paths in $\mathcal{A}$ and accepting paths in $\mathcal{A}'$ with label id^ω are in a one-to-one correspondence. As it is possible to rebuild the label of a path in $\mathcal{A}$ using the state sequence of the corresponding path in $\mathcal{A}'$, there is no loss of information. Delete now all nodes and edges which do not belong to some accepting path. For the sake of simplicity, we use the same notation $\mathcal{A}'$ for the simplified graph.

Consider now the automaton $\mathcal{A}'' = (Q' \cup \{\perp\}, \mathcal{R}, T'', (i, u), \{\perp\})$ where

$$\begin{aligned} T'' = &\big(T' \cap \big(Q' \times \mathcal{R} \times Q'\big)\big) \\ &\cup \big\{\big((p, av), f, \perp\big) : p \in Q, f \in \mathcal{R}, a \in A, v \in A^{2r-1}, \\ &\quad \exists q \in Q, \exists b \in A, \big((p, av), \mathrm{id}, (q, vb)\big) \in T' \text{ and } \big((p, av), f, (q, vb)\big) \notin T'\big\} \\ &\cup \big\{(\perp, f, \perp) : f \in \mathcal{R}\big\} \end{aligned}$$

The intersection $T' \cap (Q' \times \mathcal{R} \times Q')$ allows to eliminate from $\mathcal{A}'$ the edges originated by the addition of the identity to $\mathcal{R}$ (if any). Furthermore, transitions leading to the accepting state are added. Their labels are those rules which do not keep unchanged a letter of some element of $\mathcal{L} \cap u_{[r,2r-1]}A^{\omega}$ as it should be with the identity rule. Then, $\mathcal{L}_u = \mathcal{L}(\mathcal{A}'')^c$ and so $\mathcal{L}_u$ is ω-rational.

By hypothesis, $\mathcal{C} = \bigcup_{i=1}^{n} [U^{(i)}, V^{(i)}]$ for some ω-rational languages $U^{(1)}, \ldots, U^{(n)}$, and $V^{(1)}, \ldots, V^{(i)}$. Finally, it holds that $\Theta(\mathcal{C}) = \bigcup_{i=1}^{n} \bigcup_{u \in A^{2r}} [U^{(i)}_{\tilde{u}}, V^{(i)}_{u}]$ where $\tilde{u}$ is the mirror of u and this concludes the proof. $\square$

7. Conclusions

The paper deals with three main research directions in the study of ν-CA dynamics. The first one simply aims at understanding if and how classical results of CA theory translate into the new setting. Although most of the results are no-go theorems, they are interesting since they can guide the search for further results more properly related to the ν-CA.

An important issue in this context is structural stability, namely the study of properties of the dynamics which are robust to small perturbations in the distribution of local rules. A question arises naturally here. Is there any interesting robust property other than equicontinuity? What does it determine robustness?

The second research direction tries to classify the complexity of dynamical properties by means of the complexity of languages of distributions of local rules. We have seen that there is a gap between the complexity of surjectivity (sofic subshift language) and injectivity (ζ-rational language). All the dynamical properties that we have been able to characterize fall in ζ-rational languages. It would be therefore interesting to refine results in order to understand if these properties fit know proper subsets of ζ-rational languages. Of course, as a preliminary step, perhaps one should refine the class structure of ζ-rational languages, similarly to what has been done for ω-rational language in [19].

The third direction aims at completing the understanding of basic set properties of ν-CA as the set of fixed points. It would be nice to understand how the properties of the local rules used to build a perturbation in a CA influence its set of fixed points. Moreover, classical and not fully understood CA questions also apply here when focusing on periodic points. Is it possible to characterize the Artin–Mazur zeta function (*i.e.*, the function that associates with each integer n the cardinality of the set of periodic points of period n)? Of course, finding analytic expressions for the zeta function is pretty ambitious but finding them at least in some simple cases might reveal important information about the whole class of ν-CA.

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