Linearization of motorcycle magic formula equations

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Table of Contents


```
close("all"); clear; clc;
setmadsympath();
```

Slip ratio:

```
k = sym("kappa",'real');
```

Slip angle:

```
b = sym("beta",'real');
```

Camber:

```
g = sym("gamma",'real');
```

Normal force:

```
fz = sym("f_z",'real');
```

Nominal normal force:

```
fz0 = sym('f_z0');
```

Normal force ratio:

```
dfz = (fz - fz0)/fz0;
```

Longitudinal forces

Peak factor

```
pDx1 = sym('p_Dx1','real');
pDx2 = sym('p_Dx2','real');
Dx = (pDx1 + pDx2*dfz)*fz;
```

Curvature factor

```
pEx1 = sym('p_Ex1','real');
pEx2 = sym('p_Ex2','real');
pEx3 = sym('p_Ex3','real');
pEx4 = sym('p_Ex4','real');

Ex = (pEx1 + pEx2*dfz + pEx3*dfz^2)*(1 - pEx4*sign(k));
```

Shape and stiffness factors

```
pKx1 = sym('p_Kx1','real');
pKx2 = sym('p_Kx2','real');
pKx3 = sym('p_Kx3','real');
Kxk  = fz*(pKx1 + pKx2*dfz)*exp(pKx3*dfz);

Cx = sym('C_x','real');
Bx = Kxk/(Cx*Dx);
```

Pure slip

```
fx0 = Dx*sin(Cx*atan(Bx*k - Ex*(Bx*k - atan(Bx*k))));
```

Combined slip

```
rBx1 = sym('r_Bx1','real');
rBx2 = sym('r_Bx2','real');
Bxa = rBx1*cos(atan(rBx2*k));
Cxa = sym('C_x_alpha','real');
fx = cos(Cxa*atan(Bxa*b))*fx0;
```

Linear expressions

```
fx_lin = linearize(fx,[k,b],[0,0])
```

fx_lin =

$$\frac{f_z\,\kappa\,\mathrm{e}^{\frac{p_{\mathrm{Kx3}}\,(f_z-f_{\mathrm{z0}})}{f_{\mathrm{z0}}}}\,(f_{\mathrm{z0}}\,p_{\mathrm{Kx1}}+f_z\,p_{\mathrm{Kx2}}-f_{\mathrm{z0}}\,p_{\mathrm{Kx2}})}{f_{\mathrm{z0}}}$$

```
fx_lin0 = subs(fx_lin,fz,fz0)
```

fx_lin0 = $f_{\mathrm{z0}}\,\kappa\,p_{\mathrm{Kx1}}$

Verify against Pacejka's stated value linear equation:

```
disp(subs(Bx*Cx*Dx,fz,fz0)*k)
```

$f_{\mathrm{z0}}\,\kappa\,p_{\mathrm{Kx1}}$

Lateral forces

Slip

Peak factor

```
pDy1 = sym('p_Dy1','real');
pDy2 = sym('p_Dy2','real');
pDy3 = sym('p_Dy3','real');
Dy = fz*pDy1*exp(pDy2*dfz)/(1 + pDy3*g^2);
```

Curvature factor

```
pEy1 = sym('p_Ey1','real');
pEy2 = sym('p_Ey2','real');
pEy4 = sym('p_Ey4','real');
Ey = pEy1 + pEy2*g^2 + pEy4*g*sign(b);
```

Shape and stiffness factors

```
pKy1 = sym('p_Ky1','real');
pKy2 = sym('p_Ky2','real');
pKy3 = sym('p_Ky3','real');
pKy4 = sym('p_Ky4','real');
pKy5 = sym('p_Ky5','real');
ang = fz/((pKy3 + pKy4*g^2)*fz0);
Kya = pKy1*fz0*sin(pKy2*atan(ang))/(1 + pKy5*g^2);

Cy = sym('C_y','real');
By = Kya/(Cy*Dy);
```

Camber

Curvature factor

```
Eg = sym('E_gamma','real');
```

Shape and stiffness factors

```
pKy6 = sym('p_Ky6','real');
pKy7 = sym('p_Ky7','real');
Kyg = (pKy6 + pKy7*dfz)*fz;

Cg = sym('C_gamma','real');
Bg = Kyg/(Cg*Dy);
```

Pure slip

```
fy0b = Cy*atan(By*b - Ey*(By*b - atan(By*b)));
fy0g = Cg*atan(Bg*g - Eg*(Bg*g - atan(Bg*g)));
fy0 = Dy*sin(fy0b + fy0g);
```

Combined slip

```
rBy1 = sym('r_By1','real');
rBy2 = sym('r_By2','real');
rBy3 = sym('r_By3','real');
Byk = rBy1*cos(atan(rBy2*(b - rBy3)));
Cyk = sym('C_y_kappa','real');
fy = cos(Cyk*atan(Byk*k))*fy0;
```

Linear expressions

```
fy_lin = linearize(fy,[k,b,g],[0,0,0])
```

fy_lin =

$$\gamma \left(f_z p_{\mathrm{Ky6}} - f_z p_{\mathrm{Ky7}} + \frac{f_z^2 p_{\mathrm{Ky7}}}{f_{\mathrm{z0}}} \right) + \beta f_{\mathrm{z0}} p_{\mathrm{Ky1}} \sin\left(p_{\mathrm{Ky2}} \operatorname{atan}\left(\frac{f_z}{f_{\mathrm{z0}} p_{\mathrm{Ky3}}} \right) \right)$$

```
fy_lin0 = subs(fy_lin,fz,fz0)
```

```
fy_lin0 =
```

$$f_{z0}\,\gamma\,p_{Ky6} + \beta f_{z0}\,p_{Ky1}\,\sin\left(p_{Ky2}\,\mathrm{atan}\left(\frac{1}{p_{Ky3}}\right)\right)$$

Verify against Pacejka's stated value linear equation:

```
disp(subs(By*Cy*Dy,[fz,g],[fz0,0])*b + subs(Bg*Cg*Dy,[fz,g],[fz0,0])*g)
```

$$f_{z0}\,\gamma\,p_{Ky6} + \beta f_{z0}\,p_{Ky1}\,\sin\left(p_{Ky2}\,\mathrm{atan}\left(\frac{1}{p_{Ky3}}\right)\right)$$

Aligning moments

```
qHz3 = sym('q_Hz3','real');
qHz4 = sym('q_Hz4','real');
Shr = (qHz3 + qHz4*dfz)*g;
St = 1/sqrt(1 + b^2);
```

Twisting

Stiffness factor

```
qBz1 = sym('q_Bz1','real');
qBz2 = sym('q_Bz2','real');
qBz5 = sym('q_Bz5','real');
qBz6 = sym('q_Bz6','real');

ep = sym('epsilon');
absg = sqrt(g^2 + ep^2);
Bt = (qBz1 + qBz2*dfz)*(1 + qBz5*sqrt(g^2 + ep^2) + qBz6*g^2);
```

Peak factor

```
r0 = sym('R_0','real');
qDz1 = sym('q_Dz1','real');
qDz2 = sym('q_Dz2','real');
qDz3 = sym('q_Dz3','real');
qDz4 = sym('q_Dz4','real');
Dt = fz*(r0/fz0)*(qDz1 + qDz2*dfz)*(1 + qDz3*sqrt(g^2 + ep^2) + qDz4*g^2);
```

Shape factor

```
Ct = sym('C_t','real');
```

Curvature factor

```
qEz1 = sym('q_Ez1','real');
qEz2 = sym('q_Ez2','real');
qEz5 = sym('q_Ez5','real');
pi = sym('pi');
Et = (qEz1 + qEz2*dfz)*(1 + qEz5*g*(2/pi)*atan(Bt*Ct*b));
```

Moment

```
fy0g0 = subs(fy0,g,0);
mzt0 = -Dt*cos(Ct*atan(Bt*b - Et*(Bt*b - atan(Bt*b))))*St*fy0g0;
```

Rolling

Stiffness factor

```
qBz9 = sym('q_Bz9','real');
qBz10 = sym('q_Bz10','real');
Br = qBz9 + qBz10*By*Cy;
```

Peak factor

```
qDz8 = sym('q_Dz8','real');
qDz9 = sym('q_Dz9','real');
qDz10 = sym('q_Dz10','real');
qDz11 = sym('q_Dz11','real');
Dr = fz*r0*((qDz8 + qDz9*dfz)*g + (qDz10 + qDz11*dfz)*g*absg)*St;
```

Moment

```
mzr0 = Dr*cos(atan(Br*(b + Shr)));
```

Pure slip

```
mz0 = mzt0 + mzr0;
```

Combined slip

```
lr = sqrt(b^2 + (Kxk*k/subs(Kya,g,0))^2)*sign(b);
lt = sqrt((b + Shr)^2 + (Kxk*k/subs(Kya,g,0))^2)*sign(b + Shr);

fyg0 = cos(Cyk*atan(Byk*k))*fy0g0;
mzr = Dr*cos(atan(Br*lr));
mzt = -fyg0*Dt*cos(Ct*atan(Bt*lt - Et*(Bt*lt - atan(Bt*lt))))*St;

mz = mzr + mzt;
```

Linear expressions

```
mz_lin = linearize(mz,[k,b,g],[0,0,0])
```

mz_lin =

$$\frac{R_0 f_z \gamma \ (f_{z0} q_{Dz8} + f_z q_{Dz9} - f_{z0} q_{Dz9})}{f_{z0}} - \frac{R_0 \beta f_z p_{Ky1} \sin\left(p_{Ky2} \operatorname{atan}\left(\frac{f_z}{f_{z0} p_{Ky3}}\right)\right) \ (f_{z0} q_{Dz1} + f_z q_{Dz2} - f_{z0} q_{Dz2})}{f_{z0}}$$

```
mz_lin0 = subs(mz_lin,fz,fz0)
```

mz_lin0 =

$$R_0 f_{z0} \gamma \, q_{Dz8} - R_0 \beta f_{z0} p_{Ky1} q_{Dz1} \sin\left(p_{Ky2} \operatorname{atan}\left(\frac{1}{p_{Ky3}}\right)\right)$$

Verify against Pacejka's stated value linear equation:

```
Dt0 = fz*r0*(qDz1 + qDz2*dfz)/fz;
Kzg0 = fz*r0*(qDz8 + qDz9*dfz);
disp(subs(-Dt0*By*Cy*Dy,[g,fz],[0,fz0])*b + subs(Kzg0,[g,fz],[0,fz0])*g)
```

$$R_0 f_{z0} \gamma \, q_{Dz8} - R_0 \beta f_{z0} p_{Ky1} q_{Dz1} \sin\left(p_{Ky2} \operatorname{atan}\left(\frac{1}{p_{Ky3}}\right)\right)$$

```
function f_lin = linearize(f,q,q0)
    arguments
        f (:,1) sym;
        q (:,1) sym;
        q0 (:,1) double;
    end
    f0 = subs(f,q,q0);
    dq = sym('dq',[length(q),1],'real');
    q_perturbed = q0 + dq;
    f_perturbed = subs(f,q,q_perturbed);
    f_taylor = taylor(f_perturbed,dq,'Order',2);
    J_safe = jacobian(f_taylor,dq);
    J0 = simplify(expand(subs(J_safe,dq,zeros(size(dq)))));
    f_lin = f0 + J0*(q - q0);
    ep = sym('epsilon');
    if any(has(symvar(f_lin),ep))
        f_lin = subs(f_lin,ep,0);
    end
end
```