Chapter 4

Field transformations

There exist various transformations which can be applied to electromagnetic field problems to transform them into other electromagnetic problems. Thus, from known solutions of certain problems solutions of certain other problems can be obtained without going through the solving procedure. For example, the duality transformation changes an electromagnetic problem to a magnetoelectric problem and the affine transformation changes the metric of the space so that an isotropic space becomes anisotropic. Of course, only transformations which do not change the Maxwell equations are of interest, because otherwise the transformed fields would not be electromagnetic fields. A few such transformations are considered in this chapter.

4.1 Reversal transformations

The simplest examples of field transformations are changes of sign in polarity, time, space and frequency, which transform sources, fields and medium parameters so that the Maxwell equations remain invariant. Each of them will be discussed briefly below.

4.1.1 Polarity reversal

In this transformation all electromagnetic source and field quantities q are changed as $q \rightarrow q_C$, starting from the electric charge, which reverses its sign:

$$\varrho_C = -\varrho. \tag{4.1}$$

Assuming that the medium and boundary parameters do not change, from the linearity of the Maxwell equations it is easily seen that they do not change if all source and field quantities q change sign in this transformation:

$$q_C = -q. \tag{4.2}$$

The sign of the electric charge is actually a convention which could be changed, the present choice being due to Benjamin Franklin from the eighteenth century. If the medium and boundary quantities are the same, the fields for sources with change in sign are also changed in sign. Products of these quantities do not change because the two minus signs cancel. Thus power, energy and impedance remain unchanged in this transformation. For example, the force between two sources equals that of two similar sources of opposite sign.

4.1.2 Time reversal

In this transformation, the sign of time is reversed: $t_T = -t$. Let us assume that the electric charge does not change sign in this transformation: $\varrho_T(t) = \varrho(-t)$. It turns out that all other transformed quantities can be determined from the Maxwell equations. For example, we have from $\nabla \cdot \mathbf{J}_T = \partial \varrho_T / \partial t_T$ the rule $\mathbf{J}_T(t) = -\mathbf{J}(-t)$ and from $\nabla \cdot \mathbf{D}_T = \varrho_T$ the rule $\mathbf{D}_T(t) = \mathbf{D}(-t)$. Assuming $\overline{\overline{\epsilon}}_T = \overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}_T = \overline{\overline{\mu}}$, we see from medium equations that the other parameters must obey $\overline{\overline{\xi}}_T = -\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}_T = \overline{\overline{\zeta}}$. However, for dispersive media, the relation is not so simple.

Also, if assuming magnetic charges, they must obey the law $\varrho_{mT} = -\varrho_m$, $\mathbf{J}_{mT} = \mathbf{J}_m$.

Thus, all quantites fall into two groups: those which are invariant in the T transformation, ϱ, $\mathbf{J}_m$, $\mathbf{E}$, $\mathbf{D}$, $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$, and those which are anti-invariant, ϱ_m, $\mathbf{J}$, $\mathbf{H}$, $\mathbf{B}$, $\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}$.

Combining time reversal with charge reversal gives us a similar transformation in which the magnetic charge is invariant, $\varrho_{mCT} = \varrho_m$, and the electric charge anti-invariant, $\varrho_{CT} = -\varrho$.

Because reversal of time corresponds to changing the direction of motion of electric charges, it reverses the currents and magnetic fields but does not change the electric fields. The change of sign in magnetic sources can be understood if magnetic monopoles are omitted as non-physical and magnetic dipoles are replaced by electric current loops, which change sign in time reversal.

In the frequency domain, the T transformation corresponds to change of sign of the frequency as is obvious from the Fourier integral representation

$$q_T(\omega) = \int_{-\infty}^{\infty} q_T(t) e^{-j\omega t} dt = q(-\omega) = q^*(\omega). \tag{4.3}$$

Because the T transformation also implies conjugation, it is seen that the real parts of complex functions are even and the imaginary parts are odd functions of frequency.

4.1.3 Space inversion

In this transformation, space points $\mathbf{r}$ are reversed in the origin to points $\mathbf{r}_P = -\mathbf{r}$. Assuming again that the sign of the electric charge is not changed in the reversal, $\varrho_P(\mathbf{r}) = \varrho(-\mathbf{r})$, we can find from the Maxwell equations, again, that the quantities $\mathbf{H}$, $\mathbf{B}$, $\mathbf{J}_m$, $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$ do not change sign, wheras the quantities $\mathbf{J}$, $\mathbf{E}$, $\mathbf{D}$, ϱ_m, $\overline{\overline{\xi}}$, $\overline{\overline{\zeta}}$ transform like $\mathbf{J}_P(\mathbf{r}) = -\mathbf{J}(-\mathbf{r})$. Space inversion is a special case of the affine transformation discussed in more detail in a subsequent section.

4.1.4 Transformations of power and impedance

To see how power flow changes as a result of these transformations, let us study the Poynting vector $\mathbf{S}(t) = \mathbf{E} \times \mathbf{H}$. It is easy to see that we have

$$\mathbf{S}_C = \mathbf{S}, \tag{4.4}$$

or the power flow is not dependent on the sign of the charges nor the direction of the current. On the other hand we have

$$\mathbf{S}_T(t) = -\mathbf{S}(-t), \tag{4.5}$$

which means that the sign of the energy flow is reversed in the time reversal, as is most obvious. Finally, we can write

$$\mathbf{S}_P(\mathbf{r}) = -\mathbf{S}(-\mathbf{r}). \tag{4.6}$$

The frequency sign change has an effect on the complex Poynting vector:

$$\mathbf{S}_T(\omega) = -\mathbf{S}^*(-\omega). \tag{4.7}$$

Thus, the real power flow changes sign in this transformation, whereas the reactive imaginary part does not. Similarly, all impedance quantities change like

$$Z_T(\omega) = -Z^*(-\omega). \tag{4.8}$$

Again, the real part changes sign in time reversal. This can be understood so that power absorbing passive medium becomes active in time reversal, feeding power to the outside world, which is just like watching a movie backwards.

References

JACKSON, J.D. (1975). *Classical electrodynamics*, (2nd edn), pp. 245–51. Wiley, New York.

MITTER, H. (1990). *Elektrodynamik*, (2nd edn), pp. 209–15. BI Wissenschaftsverlag, Mannheim.

4.2 Duality transformations

The duality transformation makes use of the symmetry between the electric and magnetic quantities in the Maxwell equations. In its classical form, the electric quantities of an electromagnetic problem are changed to magnetic quantities and vice versa. In changing the fields and sources of the problem the parameters of media and boundaries are also changed. In many cases it is necessary that the medium remains unchanged, which makes the transformation dependent on the medium in question. However, in the form considered here, duality transformations can only be found for bi-isotropic media.

4.2.1 Simple duality

In the literature, a simple form of duality is often applied to transform an equation corresponding to a certain electromagnetic problem. The outcome is another equation corresponding to a dual problem. If the solution of the former is known, there is no need to solve the latter. Instead, by performing the duality transformation to the known solution, the unknown dual solution can be readily obtained. In the simple version, one just makes the substitution $\mathbf{E} \rightarrow \mathbf{H}$ and conversely $\mathbf{H} \rightarrow \mathbf{E}$, completed by the additional substitutions $\mathbf{B} \leftrightarrow -\mathbf{D}$, $\mathbf{J} \leftrightarrow -\mathbf{J}_m$ and $\overline{\overline{\epsilon}} \leftrightarrow -\overline{\overline{\mu}}$, $\overline{\overline{\xi}} \leftrightarrow -\overline{\overline{\zeta}}$. However, in this form the duality transformation changes the nature of the quantities so that their dimensions are changed. For example, the electric field is transformed into a magnetic field. Thus, it is not possible to add the original and transformed field quantities.

Let us consider duality transformations which transform quantities so that their nature does not change. Thus, the electric field is transformed into another electric field. Let us also limit the theory to the frequency domain.

4.2.2 Duality transformations for isotropic media

The original idea behind duality is to interchange electric and magnetic quantities so that one of the Maxwell equations is transformed to the other one and conversely. Because the equations can be multiplied by non-null constants without changing their validity, let us require that the corresponding quantities in the two Maxwell equations are transformed into each other through multiplying by constant factors α and β:

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} - \mathbf{J}_m \quad \rightarrow \quad \nabla \times \mathbf{H}_d = j\omega\mathbf{D}_d + \mathbf{J}_d, \tag{4.9}$$

$$\nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J} \quad \rightarrow \quad \nabla \times \mathbf{E}_d = -j\omega\mathbf{B}_d - \mathbf{J}_{md}. \tag{4.10}$$

The equation pair is invariant if we define

$$\mathbf{E}_d = \alpha \mathbf{H}, \quad \mathbf{B}_d = -\alpha \mathbf{D}, \quad \mathbf{J}_{md} = -\alpha \mathbf{J}, \tag{4.11}$$

$$\mathbf{H}_d = \beta \mathbf{E}, \quad \mathbf{D}_d = -\beta \mathbf{B}, \quad \mathbf{J}_d = -\beta \mathbf{J}_m. \tag{4.12}$$

The divergence equations transform to

$$\nabla \cdot \mathbf{D} = \varrho \quad \rightarrow \quad \nabla \cdot \mathbf{B}_d = \varrho_{md}, \tag{4.13}$$

$$\nabla \cdot \mathbf{B} = \varrho_m \quad \rightarrow \quad \nabla \cdot \mathbf{D}_d = \varrho_d, \tag{4.14}$$

if we define

$$\varrho_d = -\beta \varrho_m, \quad \varrho_{md} = -\alpha \varrho. \tag{4.15}$$

Further, from the condition that the medium equations transform to one another the parameters of the dual problem can be identified:

$$\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}_d = \begin{pmatrix} -\frac{\beta}{\alpha}\overline{\overline{\mu}} & -\overline{\overline{\zeta}} \\ -\overline{\overline{\xi}} & -\frac{\alpha}{\beta}\overline{\overline{\epsilon}} \end{pmatrix}. \tag{4.16}$$

Finally, from the impedance boundary conditions

$$\mathbf{n} \times \mathbf{H} = \overline{\overline{Y}}_s \cdot \mathbf{E}_t, \qquad \mathbf{E}_t = \overline{\overline{Z}}_s \cdot (\mathbf{n} \times \mathbf{H}) \tag{4.17}$$

with the unit normal vector $\mathbf{n}$ pointing out from the boundary, we have for the dual dyadics

$$\overline{\overline{Y}}_{sd} = -\frac{\beta}{\alpha}(\overline{\overline{Z}}_s {}^{\times}_{\times} \mathbf{nn}), \qquad \overline{\overline{Z}}_{sd} = -\frac{\alpha}{\beta}(\overline{\overline{Y}}_s {}^{\times}_{\times} \mathbf{nn}). \tag{4.18}$$

4.2.3 Left-hand and right-hand transformations

The definitions of the dual quantities above are valid for any values of the constants α, β. The simple duality transformation, applicable for the derivation of formulas, is obtained by taking $\alpha = \beta = 1$. The coefficients can also be determined by imposing certain conditions for the transformation. It appears natural to require the following properties.

1. The duality transformation is an *involution*, i.e., the dual of a dual quantity equals the original quantity itself. This implies $\alpha\beta = 1$, as is seen by checking through all the equations.

2. A chosen isotropic medium is defined to be self-dual: its parameters ϵ_S, μ_S are not changed in the duality transformation. The medium chosen is usually the free space with $\overline{\overline{\epsilon}}_S = \epsilon_o \overline{\overline{I}}$, $\overline{\overline{\mu}}_S = \mu_o \overline{\overline{I}}$, $\overline{\overline{\xi}}_S = 0$,

$\overline{\overline{\zeta}}_S = 0$. The conditions $\overline{\overline{\epsilon}}_{Sd} = \epsilon_o \overline{\overline{I}}$, $\overline{\overline{\mu}}_{Sd} = \mu_o \overline{\overline{I}}$, $\overline{\overline{\xi}}_{Sd} = 0$, $\overline{\overline{\zeta}}_{Sd} = 0$ imply the condition $\alpha/\beta = -\mu_o/\epsilon_o = -\eta_o^2$. More generally, we could actually take any isotropic chiral medium with the parameters ϵ_S, μ_S, $\xi_S = -\zeta_S = -j\kappa_S\sqrt{\mu_o\epsilon_o}$ as the self-dual medium, because, as can be seen from (4.16), the dual of the chirality parameter coincides with that of the original medium: $\kappa_d = \kappa_S$.

From the two conditions $\alpha\beta = 1$ and $\alpha/\beta = -\eta_S^2$ we have two solutions for the parameters:

$$\alpha^l = j\eta_S, \qquad \beta^l = \frac{1}{j\eta_S}, \tag{4.19}$$

$$\alpha^r = -j\eta_S, \qquad \beta^r = -\frac{1}{j\eta_S}. \tag{4.20}$$

Two possible solutions means that, corresponding to a chosen self-dual medium, there in fact exist two duality transformations, neither of which is a more 'natural' one than the other. These two transformations will be denoted as follows: let the duality tranformation corresponding to the superscript l be called the *left-hand duality transformation*, and that corresponding to r the *right-hand duality transformation*. These labels refer to polarizations of two self-dual propagating fields, as will be explained subsequently.

From the previous knowledge we can write the following double duality transformation table for the fields and sources $f \to f^l$ and $f \to f^r$:

f^l	f	f^r	f^l	f	f^r
$j\eta_S\mathbf{H}$	$\mathbf{E}$	$-j\eta_S\mathbf{H}$	$\mathbf{E}/j\eta_S$	$\mathbf{H}$	$-\mathbf{E}/j\eta_S$
$-j\eta_S\mathbf{D}$	$\mathbf{B}$	$j\eta_S\mathbf{D}$	$-\mathbf{B}/j\eta_S$	$\mathbf{D}$	$\mathbf{B}/j\eta_S$
$-j\eta_S\mathbf{J}$	$\mathbf{J}_m$	$j\eta_S\mathbf{J}$	$-\mathbf{J}_m/j\eta_S$	$\mathbf{J}$	$\mathbf{J}_m/j\eta_S$
$-j\eta_S\varrho$	ϱ_m	$j\eta_S\varrho$	$\varrho_m/j\eta_S$	ϱ	$\varrho_m/j\eta_S$

and for the medium and impedance parameters, which transform in the same way in both transformations

f	$f^l = f^r$	f	$f^l = f^r$
$\overline{\overline{\epsilon}}$	$\overline{\overline{\mu}}/\eta_S^2$	$\overline{\overline{\mu}}$	$\eta_S^2\overline{\overline{\epsilon}}$
$\overline{\overline{\xi}}$	$-\overline{\overline{\zeta}}$	$\overline{\overline{\zeta}}$	$-\overline{\overline{\xi}}$
$\overline{\overline{Y}}_s$	$\eta_S^{-2}\overline{\overline{Z}}_s {}^\times_\times \mathbf{nn}$	$\overline{\overline{Z}}_s$	$\eta_S^2\overline{\overline{Y}}_s {}^\times_\times \mathbf{nn}$

It is seen clearly from the above table that the transformation equals its inverse whence only one half of the table is actually needed.

The two duality transformations possess the following properties.

- For a lossless medium (real η_S) the dual of $\mathbf{E}\times\mathbf{H}^*$ is $\mathbf{E}^*\times\mathbf{H}$, whence the real part of the Poynting vector is self dual. Thus, the field pattern representing the energy flow in a lossless medium is unchanged in both duality transformations.

- The PEC and PMC boundaries are dual to each other.

- The dual of a dielectric object in air with $\epsilon_S = \epsilon_o$, $\mu_S = \mu_o$, with the relative dielectric factor $\epsilon_r = A$, is a magnetic object with the relative permeability $\mu_r = A$, since the dual of $\overline{\overline{\epsilon}}/\epsilon_o$ in this case equals $\overline{\overline{\mu}}/\mu_o$ and conversely.

- The wave number of a plane wave, $k = \omega\sqrt{\mu\epsilon}$, in the dual of an isotropic medium equals the original propagation factor $k_d = k$.

- The wave impedance η of a propagating plane wave in the dual of an isotropic medium equals $\eta_d = \eta_S^2/\eta$.

- The dual of an electric dipole $\mathbf{J} = \mathbf{u}IL\delta(\mathbf{r})$ is the magnetic dipole $\mathbf{J}_m = -j\mathbf{u}\eta_S IL\delta(\mathbf{r})$.

4.2.4 Application of the duality transformations

Either of the two duality transformations can be applied for an electromagnetic problem in two ways because of the involutory property of the transformation:

1. attach the label d (representing either l or r superscript) to each quantity of the problem and then replace these quantities from the above definitions,

2. replace the original quantities in terms of transformed quantities from the definitions, substitute these in the problem and discard the labels d.

Both ways lead to the same result. It is, however, quite easy to get confused with the two possibilities and end up with a wrong result. This means that one has to be careful and pursue only one path. Of course, only one of the two possible duality transformations, either the left-hand or the right-hand transformation, must be systematically applied.

Dipole fields

Let us consider as an example of the method 1 above the problem of finding the far field expression of a magnetic dipole from that of an electric dipole.

The electric field of an electric dipole $\mathbf{J}(\mathbf{r}) = \mathbf{u}IL\delta(\mathbf{r})$ in the far field region in air is

$$\mathbf{E} = \mathbf{u}_\theta j\omega\mu_o \frac{e^{-jkr}}{4\pi r} IL\sin\theta. \tag{4.21}$$

Since the self-dual medium is air, we have $\eta_S = \eta_o$. Making the left-hand duality transformation (i.e. adding the superscript l and replacing transformed quantities from their definitions), for every quantity of this equation we obtain

$$j\eta_S\mathbf{H} = \mathbf{u}_\theta j\omega(\eta_o^2\epsilon_o)\frac{e^{-jkr}}{4\pi r}\left(\frac{-I_m}{j\eta_o}\right) L\sin\theta, \tag{4.22}$$

or

$$\mathbf{H} = \mathbf{u}_\theta j\omega\epsilon_o \frac{e^{-jkr}}{4\pi r} I_m L\sin\theta, \tag{4.23}$$

which is the expression for the magnetic field arising from a magnetic dipole in the far field region. The same result is obtained for the right-hand transformation and also for the simple duality transformation.

Scattering problem

As a second example, let us consider the duality transformation of a problem of electromagnetic plane wave scattering from a dielectric obstacle in air with $\epsilon_r = A$, $\mu_r = 1$. Let the self-dual medium be free space and the incident wave

$$\mathbf{E}_i(\mathbf{r}) = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}}, \qquad \mathbf{H}_i(\mathbf{r}) = \mathbf{H}_o e^{-j\mathbf{k}\cdot\mathbf{r}}. \tag{4.24}$$

Both duality transformations change the obstacle to a magnetic scatterer with $\mu_r = A$ and $\epsilon_r = 1$. Moreover, they also change the incident plane wave to

$$\mathbf{E}_i(\mathbf{r}) = \pm j\eta_o\mathbf{H}_o e^{-j\mathbf{k}\cdot\mathbf{r}}, \qquad \mathbf{H}_i(\mathbf{r}) = \pm\frac{\mathbf{E}_o}{j\eta_o} e^{-j\mathbf{k}\cdot\mathbf{r}}. \tag{4.25}$$

Also, the scattered field $\mathbf{E}_{sc}$, $\mathbf{H}_{sc}$ of the first problem can be transformed to give the scattered field of the second problem.

It must be noted, that in this way we obtain a solution to the dual problem involving the dual scatterer and the *dual incoming field*, which in general is different from the solution for the original incoming field, unless the incoming field is self dual. This can be applied as follows.

- If the incoming field is *left-hand self dual*, i.e. it coincides with its own left-hand transformation, the left-hand duality transformation will not change the incoming field and the dual scattering problem has the original incoming field.

- The same is valid if the incoming field is *right-hand self dual*, in which case we can apply the right-hand duality transformation.
- If the incoming field is none of these, we can write it as a sum of left-hand and right-hand self-dual fields, treat these separately as above and add the results. Thus, the scattered field for the dual scatterer and original incoming field can be obtained through two duality transformations, one for each self-dual parts of the problem.

4.2.5 Self-dual problems

Quantities that are invariant in either of the two duality transformations are called self-dual quantities. The two self-dual concepts will be distinguished by the subscript + corresponding to the r and - corresponding to the l transformation. Medium parameters and impedances are not affected by the handedness of the transformation.

Obviously the sum of any quantity and its proper dual is a self-dual quantity and their difference is an antiself-dual quantity. It is easy to see that a quantity self dual with respect to the l transformation is antiself dual with respect to the r transformation and conversely. Any quantity can be written as a sum of a self-dual part and an antiself-dual part, or, what is the same thing, as a sum of two self-dual parts

$$q = q_+ + q_-, \qquad q_+ = \frac{1}{2}(q + q^r), \qquad q_- = \frac{1}{2}(q + q^l). \tag{4.26}$$

Self-dual fields

The self-dual electromagnetic fields can be written as

$$\mathbf{E}_\pm = \frac{1}{2}(\mathbf{E} \mp j\eta_S \mathbf{H}), \quad \mathbf{H}_\pm = \frac{1}{2}(\mathbf{H} \mp \frac{1}{j\eta_S}\mathbf{E}), \tag{4.27}$$

and they are seen to satisfy

$$\mathbf{E}_\pm = \mp j\eta_S \mathbf{H}_\pm. \tag{4.28}$$

Further, we can write

$$\mathbf{B}_\pm = \frac{1}{2}(\mathbf{B} \pm j\eta_S \mathbf{D}), \quad \mathbf{D}_\pm = \frac{1}{2}(\mathbf{D} \pm \frac{1}{j\eta_S}\mathbf{B}) = \pm\frac{1}{j\eta_S}\mathbf{B}_\pm. \tag{4.29}$$

As an example of self-dual fields (also called wave fields) let us consider a self-dual plane wave propagating in an isotropic chiral medium in the direction of the unit vector $\mathbf{u}$. Because the fields of a plane wave satisfy

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_o e^{-jk\mathbf{u}\cdot\mathbf{r}}, \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}_o e^{-jk\mathbf{u}\cdot\mathbf{r}}, \tag{4.30}$$

with

$$\mathbf{H}_o = \frac{1}{\eta_S}\mathbf{u} \times \mathbf{E}_o, \qquad \mathbf{E}_o = -\eta_S \mathbf{u} \times \mathbf{H}_o, \tag{4.31}$$

the self-dual fields are of the form

$$\mathbf{E}_\pm = \frac{1}{2}(\mathbf{E} \mp j\eta_S \mathbf{H}) = \frac{1}{2}(\mathbf{E} \mp j\mathbf{u} \times \mathbf{E}). \tag{4.32}$$

The two self-dual fields corresponding to the two duality transforms are obviously circularly polarized, because they satisfy $\mathbf{E}_\pm \cdot \mathbf{E}_\pm = 0$. We can form the polarization vectors for the two fields and obtain

$$\mathbf{p}(\mathbf{E}_\pm) = \pm\mathbf{u}. \tag{4.33}$$

The upper sign corresponds to a right-hand polarized and the lower sign a left-hand polarized CP wave when looking in the direction of propagation $\mathbf{u}$, thus justifying the labels 'left-hand duality transformation' and 'right-hand duality transformation' with respect to which these two respective polarizations are self dual.

Self-dual sources and media

Self-dual sources can be written similarly as

$$\mathbf{J}_\pm = \frac{1}{2}(\mathbf{J} \pm \frac{1}{j\eta_S}\mathbf{J}_m), \quad \mathbf{J}_{m\pm} = \frac{1}{2}(\mathbf{J}_m \pm j\eta_S \mathbf{J}) = \pm j\eta_S \mathbf{J}_\pm, \tag{4.34}$$

$$\varrho_\pm = \frac{1}{2}(\varrho \pm \frac{1}{j\eta_S}\varrho_m), \quad \varrho_{m\pm} = \frac{1}{2}(\varrho_m \pm j\eta_S \varrho) = \pm j\eta_S \varrho_\pm. \tag{4.35}$$

A single electric or magnetic current cannot be a self-dual source, instead, there must be a suitable combination of both of them. The self-dual source combination is obtained from any electric current source $\mathbf{J}$, by adding the magnetic current source $\mathbf{J}_{m\pm} = \pm j\eta_S \mathbf{J}$.

For example, a dipole current $\mathbf{J} = \mathbf{u}IL\delta(\mathbf{r})$, requires the magnetic dipole current $\mathbf{J}_m = \pm \mathbf{u}j\eta_S IL\delta(\mathbf{r})$ for a self-dual source. The magnetic dipole must be parallel to the electric dipole and have $\pm 90^o$ phase shift and amplitude $\eta_S I$. For fields outside the sources, the magnetic dipole can be replaced by an equivalent electric current loop perpendicular to the electric dipole.

The isotropic chiral medium assumed self dual with respect to the two duality transformations was denoted by the parameters ϵ_S, μ_S, $\xi_S = -\zeta_S = -j\kappa\sqrt{\mu_o\epsilon_o}$. Since only the impedance $\eta_S = \sqrt{\mu_S/\epsilon_S}$ is needed in defining the two duality transformations, all isotropic chiral media with the same impedance appear self dual with respect to these transformations.

Self-dual boundaries

The self-dual boundary impedance dyadic $\overline{\overline{Z}}_s$ must satisfy the equation

$$\overline{\overline{Z}}_{sd} = \eta_S^2(\overline{\overline{Y}}_s {}^\times_\times \mathbf{nn}) = \eta_S^2 \frac{\overline{\overline{Z}}_s^T}{\mathrm{spm}\overline{\overline{Z}}_s} = \overline{\overline{Z}}_s. \tag{4.36}$$

Taking the spm operation of this, we have $(\mathrm{spm}\overline{\overline{Z}}_s)^2 = \eta_S^4$, which has two solutions $\mathrm{spm}\overline{\overline{Z}}_s = \pm\eta_S^2$. Substituting in (4.36), we have the condition $\overline{\overline{Z}}_s^T = \pm\overline{\overline{Z}}_s$, whence the impedance dyadic must be either symmetric or antisymmetric to be self dual. Let us study these cases separately.

The antisymmetric self-dual two-dimensional dyadic is obviously of the form $\overline{\overline{Z}}_s = Z_s\mathbf{n}\times\overline{\overline{I}}$, where $\mathbf{n}$ is the unit vector normal to the surface. Equating $\mathrm{spm}\overline{\overline{Z}}_s$ with $-\eta_S^2$ gives us finally two antisymmetric self-dual surface impedance solutions

$$\overline{\overline{Z}}_s = \pm j\eta_S\mathbf{n}\times\overline{\overline{I}}. \tag{4.37}$$

Studying the most general symmetric boundary impedance dyadic

$$\overline{\overline{Z}}_s = Z_1\mathbf{u}_1\mathbf{u}_1 + Z_2\mathbf{u}_2\mathbf{u}_2, \tag{4.38}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the two orthonormal eigenvectors of $\overline{\overline{Z}}_s$ perpendicular to $\mathbf{n}$, from $\mathrm{spm}\overline{\overline{Z}}_s = \eta_S^2$ we have $Z_1Z_2 = \eta_S^2$ for the self-duality condition. As one limiting case we have the isotropic impedance $\overline{\overline{Z}}_s = \eta_S\overline{\overline{I}}_t$, like the Silver–Müller radiation condition at the spherical boundary in infinity. As another limit we have the anisotropic surface with $Z_1 \to 0$, $Z_2 \to \infty$, which approximates a dense tuned corrugated surface, i.e. a conducting surface with grooves parallel to $\mathbf{u}_1$ and a quarter wavelength of depth.

4.2.6 Self-dual field decomposition

The decomposition of electromagnetic fields in self-dual constituents $\mathbf{E}_+$ and $\mathbf{E}_-$ in (4.27), also called *wave fields*, leads to a useful way of treating electromagnetic problems in isotropic chiral media, instead of the conventional analysis based on electric and magnetic fields.

In fact, substituting fields and sources in terms of self-dual components, for *homogeneous* isotropic chiral media, the Maxwell equations are seen to split into two non-coupled sets, each self-dual system of fields and sources acting in an isotropic non-chiral medium of its own. The governing equations can be written together with double subscripts as follows:

$$\nabla\times\mathbf{E}_\pm = -j\omega\mu_\pm\mathbf{H}_\pm - \mathbf{J}_{m\pm} \tag{4.39}$$

$$\nabla \times \mathbf{H}_\pm = j\omega\epsilon_\pm \mathbf{E}_\pm + \mathbf{J}_\pm, \tag{4.40}$$

where the effective non-chiral parameters of the two media are denoted by

$$\epsilon_\pm = \epsilon(1 \pm \frac{\kappa}{n}), \qquad \mu_\pm = \mu(1 \pm \frac{\kappa}{n}), \qquad n = \sqrt{\epsilon_r \mu_r}. \tag{4.41}$$

It must be emphasized, however, that this is valid only for homogeneous media, in inhomogeneous media the two self-dual fields couple to each other.

It is also seen that in an isotropic chiral medium the wave impedances of the two self-dual fields are the same and independent of the chirality parameter κ

$$\eta_\pm = \sqrt{\frac{\mu_\pm}{\epsilon_\pm}} = \sqrt{\frac{\mu}{\epsilon}} = \eta, \tag{4.42}$$

whereas the wave numbers of plane waves in the two media are different:

$$k_\pm = \omega\sqrt{\mu_\pm \epsilon_\pm} = k(1 \pm \frac{\kappa}{n}) = k_o(n \pm \kappa) = k_o n_\pm. \tag{4.43}$$

The total electric and magnetic fields can be obtained any time from the wave fields as follows:

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-, \qquad \mathbf{H} = \frac{j}{\eta}(\mathbf{E}_+ - \mathbf{E}_-). \tag{4.44}$$

Power relations

Consider the real part of the Poynting vector in terms of the self-dual wave fields for a homogeneous *lossless* chiral medium:

$$\frac{1}{2}\Re\{\mathbf{E} \times \mathbf{H}^*\} = \Re\left\{\frac{\mathbf{E}_+ \times \mathbf{E}_+^*}{2j\eta}\right\} - \Re\left\{\frac{\mathbf{E}_- \times \mathbf{E}_-^*}{2j\eta}\right\}. \tag{4.45}$$

It is seen that the power is propagated independently in the two self-dual fields because there are no cross terms present.

Applying the real polarization vector concept from Chapter 1, of a complex vector $\mathbf{a}$, $\mathbf{p}(\mathbf{a}) = -j(\mathbf{a} \times \mathbf{a}^*)/(\mathbf{a} \cdot \mathbf{a}^*)$, we can write

$$\frac{1}{2}\Re\{\mathbf{E} \times \mathbf{H}^*\} = W_+ v_+ \mathbf{p}(\mathbf{E}_+) - W_- v_- \mathbf{p}(\mathbf{E}_-). \tag{4.46}$$

Here, the quantities

$$W_\pm = \frac{\epsilon_\pm}{2}\mathbf{E}_\pm \cdot \mathbf{E}_\pm^* = \frac{\mu_\pm}{2}\mathbf{H}_\pm \cdot \mathbf{H}_\pm^*, \qquad v_\pm = \frac{1}{\sqrt{\mu_\pm \epsilon_\pm}} \tag{4.47}$$

can be identified as energy densities and phase velocities of the two wave fields in their respective isotropic nonchiral media. The two real polarization vectors $\mathbf{p}(\mathbf{E}_\pm)$ point to the positive (right-hand) directions of the two wave-field vectors and their lengths, which give the ratio of energy velocity to the phase velocity, vary from 0 for LP to 1 for CP fields.

Thus, we can state that the power in each of the self-dual fields propagates in a direction which is perpendicular to the plane of polarization and in the right-hand direction of the '+' field and the left-hand direction of the '−' field. This further justifies the basis for labeling the two duality transformations as 'right hand' and 'left hand'. It is seen that self-dual LP fields are standing waves because the power does not propagate at all. The energy velocities of the two wave fields are different from the phase velocities except for CP waves.

Circularly polarized self-dual fields

We can make further conclusions by splitting each of the wave fields into two CP components with labels R, L to be defined shortly. Let us write

$$\mathbf{E}_\pm = E_\pm^R \mathbf{u}_\pm^R + E_\pm^L \mathbf{u}_\pm^L, \tag{4.48}$$

with two pairs of CP unit vectors $\mathbf{u}_\pm^R$, $\mathbf{u}_\pm^L$. The vectors are pairwise coplanar, $\mathbf{u}_+^R$, $\mathbf{u}_+^L$ with $\mathbf{E}_+$ and $\mathbf{u}_-^R$, $\mathbf{u}_-^L$ with $\mathbf{E}_-$ and they satisfy

$$(\mathbf{u}_\pm^R)^* = \mathbf{u}_\pm^L, \qquad (\mathbf{u}_\pm^L)^* = \mathbf{u}_\pm^R, \tag{4.49}$$

$$\mathbf{u}_\pm^R \cdot \mathbf{u}_\pm^R = \mathbf{u}_\pm^L \cdot \mathbf{u}_\pm^L = 0, \quad \mathbf{u}_\pm^R \cdot \mathbf{u}_\pm^L = |\mathbf{u}_\pm^R|^2 = |\mathbf{u}_\pm^L|^2 = 1. \tag{4.50}$$

The polarization vectors of the CP vectors are unit vectors and parallel or antiparallel to those of the corresponding $\mathbf{E}_\pm$ vectors. They can be seen to obey the relations

$$\mathbf{p}(\mathbf{u}_\pm^R) = -\mathbf{p}(\mathbf{u}_\pm^L) = -j\mathbf{u}_\pm^R \times \mathbf{u}_\pm^L, \tag{4.51}$$

$$\mathbf{p}(\mathbf{u}_\pm^R) \times \mathbf{u}_\pm^R = j\mathbf{u}_\pm^R, \quad \mathbf{p}(\mathbf{u}_\pm^R) \times \mathbf{u}_\pm^L = -j\mathbf{u}_\pm^L. \tag{4.52}$$

Thus, the total power flow expression can be written in terms of CP components of the two self-dual fields as

$$\frac{1}{2}\Re\{\mathbf{E}\times\mathbf{H}^*\} = W_+v_+\frac{|E_+^R|^2}{|\mathbf{E}_+|^2}\mathbf{p}(\mathbf{u}_+^R) + W_+v_+\frac{|E_+^L|^2}{|\mathbf{E}_+|^2}\mathbf{p}(\mathbf{u}_+^L)$$
$$- W_-v_-\frac{|E_-^R|^2}{|\mathbf{E}_-|^2}\mathbf{p}(\mathbf{u}_-^R) - W_-v_-\frac{|E_-^L|^2}{|\mathbf{E}_-|^2}\mathbf{p}(\mathbf{u}_-^L). \tag{4.53}$$

This shows us that the field can be written in terms of four components, two self-dual fields each with two coplanar CP components, which do not couple energy between one another in a homogeneous medium. From the signs of the different terms it is seen that the + waves are right handed and the − waves left handed with respect to their individual directions of propagation.

However, since $\mathbf{u}_\pm^R$ point oppositely to $\mathbf{u}_\pm^L$, the polarizations of the total + and − fields depend on the amplitude ratios of the partial waves. Defining the labels so that $|E_+^R| \geq |E_+^L|$, the direction of the energy flow of the + wave points along $\mathbf{p}(\mathbf{u}_+^R)$ wave whereas for $|E_-^L| \geq |E_-^R|$ the energy of the − wave flows in the direction of $\mathbf{p}(\mathbf{u}_-^L)$. (This makes sense only if the self-dual fields are not LP fields, which do not propagate at all.) Thus, it is seen that the power in the electromagnetic field in a homogeneous isotropic chiral medium propagates in two self-dual fields each with two uncoupled CP components so that the + wave contains two oppositely propagating right-hand waves and the − wave two oppositely propagating left-hand waves. The velocities of the two + waves are different from those of the two − waves if the chirality parameter is non-zero. If both self-dual fields carry energy in the same direction, it is a consequence that the components E_+^R and E_-^L are two forward waves and the components E_+^L and E_-^R two backward waves with different velocities of propagation.

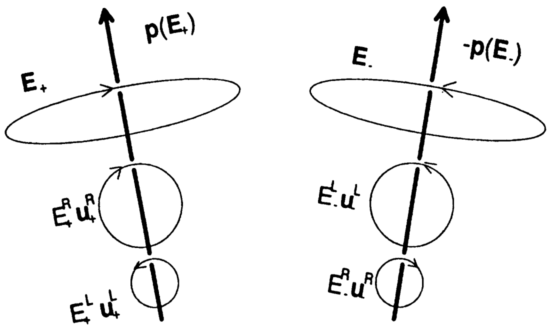

Fig. 4.1 Electromagnetic field in an isotropic chiral medium can be decomposed into its self-dual parts each with two oppositely polarized CP components.

4.2.7 Duality transformations for bi-isotropic media

A non-reciprocal bi-isotropic medium cannot serve as the self-dual medium in the previous duality transformation. However, a more general form for the duality transformation can be obtained by writing (LINDELL and

VIITANEN, 1992)

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_d = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{4.54}$$

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}_d = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}, \tag{4.55}$$

$$\begin{pmatrix} \mathbf{J} \\ \mathbf{J}_m \end{pmatrix}_d = \begin{pmatrix} I & J \\ K & L \end{pmatrix} \begin{pmatrix} \mathbf{J} \\ \mathbf{J}_m \end{pmatrix}. \tag{4.56}$$

Inserting these in the Maxwell equations for dual quantities

$$\nabla \times \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_d = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[j\omega \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}_d + \begin{pmatrix} \mathbf{J} \\ \mathbf{J}_m \end{pmatrix}_d \right], \tag{4.57}$$

we should obtain the original Maxwell equations, which gives rise to equations for the coefficients $A,...,L$:

$$H = L = A, \quad G = K = -B, \quad F = J = -C, \quad E = I = D. \tag{4.58}$$

For the transformation to be an involution: $()_{dd} = ()$, we must have

$$A^2 + BC = D^2 + BC = 1, \quad C(A + D) = B(A + D) = 0. \tag{4.59}$$

One possible solution is $C = B = 0$, which is not, however, very interesting because it reproduces the original problem. The second solution satisfies

$$A = -D = \pm\sqrt{1 - BC}. \tag{4.60}$$

The coefficients B and C can be determined if we choose one particular bi-isotropic medium with the parameters ϵ_S, μ_S, $\xi_S = (\chi_S - j\kappa_S)\sqrt{\mu_o\epsilon_o}$ and $\zeta_S = (\chi_S + j\kappa_S)\sqrt{\mu_o\epsilon_o}$ as the self-dual medium. Solving B and C we arrive, again, at two duality transformations

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_d = \frac{\pm j}{\cos\theta} \begin{pmatrix} \sin\theta & \eta_S \\ -1/\eta_S & -\sin\theta \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{4.61}$$

$$\sin\theta = \chi_S \frac{\sqrt{\mu_o\epsilon_o}}{\mu_S\epsilon_S} = \frac{\chi_S}{n_S}. \tag{4.62}$$

These expressions generalize the previous duality transformations in the special case of a reciprocal chiral medium, as is seen if $\chi_S = 0$, implying $\theta = 0$, is substituted in (4.61):

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_d = \begin{pmatrix} 0 & \mp j/\eta_S \\ \pm j\eta_S & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \tag{4.63}$$

Again, the upper sign can be called the left-hand transformation and the lower sign the right-hand transformation in analogy with the previous convention. The self-dual fields are thus

$$\mathbf{E}_{\pm} = \frac{1}{2\cos\theta}(e^{\mp j\theta}\mathbf{E} \mp j\eta_S\mathbf{H}), \tag{4.64}$$

which are seen to be generalizations of the self-dual fields for reciprocal bi-isotropic media.

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4.3 Affine transformations

An affine transformation changes the metric of the space. An example of such a transformation is one that squeezes the space so that all distances in one direction are changed in the same proportion. A sphere is then transformed to an ellipsoid and a cube to a parallelepiped. Also, making the space the mirror image of itself in a plane is an affine transformation. The affine transformations considered here are defined by a constant dyadic $\overline{\overline{A}}$, which moves every space point $\mathbf{r}$ to another point according to the law

$$\mathbf{r} \rightarrow \mathbf{r}_a = \overline{\overline{A}} \cdot \mathbf{r}. \tag{4.65}$$

The question remains how to transform sources, fields, media and boundary conditions so that the Maxwell equations remain valid in the transformed space. Knowing this, it is possible to obtain the solution for a transformed problem from that of the original problem by just transforming the solution.

4.3.1 Transformation of fields and sources

Let us assume that in conjuction with the affine transformation of the space, the fields are transformed as

$$\mathbf{E}(\mathbf{r}) \to \mathbf{E}_a(\mathbf{r}) = \overline{\overline{B}} \cdot \mathbf{E}(\mathbf{r}_a) = \overline{\overline{B}} \cdot \mathbf{E}(\overline{\overline{A}} \cdot \mathbf{r}), \tag{4.66}$$

$$\mathbf{H}(\mathbf{r}) \to \mathbf{H}_a(\mathbf{r}) = \overline{\overline{C}} \cdot \mathbf{H}(\mathbf{r}_a) = \overline{\overline{C}} \cdot \mathbf{H}(\overline{\overline{A}} \cdot \mathbf{r}). \tag{4.67}$$

The dyadics $\overline{\overline{B}}$ and $\overline{\overline{C}}$ should be chosen to depend on $\overline{\overline{A}}$ so that the transformed fields satisfy the Maxwell equations. In fact, after the affine transformation the equations in the frequency domain should read as

$$\nabla \times \mathbf{E}_a(\mathbf{r}) = -j\omega \mathbf{B}_a(\mathbf{r}) - \mathbf{J}_{ma}(\mathbf{r}), \tag{4.68}$$

$$\nabla \times \mathbf{H}_a(\mathbf{r}) = j\omega \mathbf{D}_a(\mathbf{r}) + \mathbf{J}_a(\mathbf{r}). \tag{4.69}$$

To be sure of this, we apply the knowledge that the original fields satisfy the Maxwell equations in the $\mathbf{r}_a$ space.

The expression of the gradient in the transformed space, ∇_a, can be obtained from the definition

$$\nabla_a \mathbf{r}_a = \nabla \mathbf{r} = \overline{\overline{I}}. \tag{4.70}$$

Replacing $\mathbf{r}_a$ by $\overline{\overline{A}} \cdot \mathbf{r} = \mathbf{r} \cdot \overline{\overline{A}}^T$, we have

$$\nabla_a \mathbf{r} = \overline{\overline{A}}^{-1T} \quad \Rightarrow \quad \overline{\overline{A}}^T \cdot \nabla_a \mathbf{r} = \overline{\overline{I}}, \tag{4.71}$$

whence

$$\nabla_a = \overline{\overline{A}}^{-1T} \cdot \nabla, \qquad \text{or} \qquad \nabla = \overline{\overline{A}}^T \cdot \nabla_a. \tag{4.72}$$

The curls of a field vector in the original and transformed spaces are related through the dyadic identity

$$\frac{1}{2}(\overline{\overline{K}} {}^\times_\times \overline{\overline{K}}) \cdot (\mathbf{a} \times \mathbf{b}) = \overline{\overline{K}}^{(2)} \cdot (\mathbf{a} \times \mathbf{b}) = (\overline{\overline{K}} \cdot \mathbf{a}) \times (\overline{\overline{K}} \cdot \mathbf{b}), \tag{4.73}$$

which allows us to write

$$\nabla \times \mathbf{E}_a(\mathbf{r}) = (\overline{\overline{A}}^T \cdot \nabla_a) \times (\overline{\overline{B}} \cdot \mathbf{E}(\mathbf{r}_a)) = \overline{\overline{A}}^{T(2)} \cdot [\nabla_a \times (\overline{\overline{A}}^{-1T} \cdot \overline{\overline{B}} \cdot \mathbf{E}(\mathbf{r}_a))]$$

$$= (\det\overline{\overline{A}})\overline{\overline{A}}^{-1} \cdot [\nabla_a \times (\overline{\overline{A}}^{-1T} \cdot \overline{\overline{B}} \cdot \mathbf{E}(\mathbf{r}_a))]. \tag{4.74}$$

Thus, the equations (4.68), (4.69) can be written as

$$\nabla_a \times \left[\overline{\overline{A}}^{-1T} \cdot \overline{\overline{B}} \cdot \mathbf{E}(\mathbf{r}_a)\right] = -\frac{j\omega\overline{\overline{A}}}{\det\overline{\overline{A}}} \cdot \mathbf{B}_a(\mathbf{r}) - \frac{\overline{\overline{A}}}{\det\overline{\overline{A}}} \cdot \mathbf{J}_{ma}(\mathbf{r}), \tag{4.75}$$

$$\nabla_a \times \left[\overline{\overline{A}}^{-1T} \cdot \overline{\overline{C}} \cdot \mathbf{H}(\mathbf{r}_a)\right] = \frac{j\omega\overline{\overline{A}}}{\det\overline{\overline{A}}} \cdot \mathbf{D}_a(\mathbf{r}) + \frac{\overline{\overline{A}}}{\det\overline{\overline{A}}} \cdot \mathbf{J}_a(\mathbf{r}). \tag{4.76}$$

When these equations are compared with the Maxwell equations in the transformed space,

$$\nabla_a \times \mathbf{E}(\mathbf{r}_a) = -j\omega\mathbf{B}(\mathbf{r}_a) - \mathbf{J}_m(\mathbf{r}_a), \tag{4.77}$$

$$\nabla_a \times \mathbf{H}(\mathbf{r}_a) = j\omega\mathbf{D}(\mathbf{r}_a) + \mathbf{J}(\mathbf{r}_a), \tag{4.78}$$

by identifying the corresponding terms, the following relation between the dyadics $\overline{\overline{B}}$ and $\overline{\overline{C}}$ and the original dyadic $\overline{\overline{A}}$ are obtained:

$$\overline{\overline{B}} = \alpha\overline{\overline{A}}^T, \quad \overline{\overline{C}} = \beta\overline{\overline{A}}^T, \tag{4.79}$$

where α and β may be any scalar constants.

Thus, we arrive at the following affine transformation for the fields and sources:

$$\mathbf{E}_a(\mathbf{r}) = \alpha\overline{\overline{A}}^T \cdot \mathbf{E}(\overline{\overline{A}} \cdot \mathbf{r}), \tag{4.80}$$

$$\mathbf{H}_a(\mathbf{r}) = \beta\overline{\overline{A}}^T \cdot \mathbf{H}(\overline{\overline{A}} \cdot \mathbf{r}), \tag{4.81}$$

$$\mathbf{B}_a(\mathbf{r}) = \alpha(\det\overline{\overline{A}})\overline{\overline{A}}^{-1} \cdot \mathbf{B}(\overline{\overline{A}} \cdot \mathbf{r}), \tag{4.82}$$

$$\mathbf{D}_a(\mathbf{r}) = \beta(\det\overline{\overline{A}})\overline{\overline{A}}^{-1} \cdot \mathbf{D}(\overline{\overline{A}} \cdot \mathbf{r}), \tag{4.83}$$

$$\mathbf{J}_{ma}(\mathbf{r}) = \alpha(\det\overline{\overline{A}})\overline{\overline{A}}^{-1} \cdot \mathbf{J}_m(\overline{\overline{A}} \cdot \mathbf{r}), \tag{4.84}$$

$$\mathbf{J}_a(\mathbf{r}) = \beta(\det\overline{\overline{A}})\overline{\overline{A}}^{-1} \cdot \mathbf{J}(\overline{\overline{A}} \cdot \mathbf{r}), \tag{4.85}$$

which depends on the dyadic $\overline{\overline{A}}$ and two scalars α, β.

The coefficients α, β control the amplitudes of the transformed fields. Let us restrict their values by requiring that the affine transformation defined by the inverse dyadic $\overline{\overline{A}}^{-1}$ give the inverse transformation, i.e. return the quantities transformed through $\overline{\overline{A}}$ back to the original ones. It is readily seen that this property requires the condition for the parameters

$$\alpha^2 = \beta^2 = 1 \tag{4.86}$$

to be valid.

Several important transformations fall under the general case of the affine transformation. For example, rotation around an axis by an angle, space reversion at a point and reflection at a plane are special cases of the affine transformation.

4.3.2 Transformation of media

The affine transformation also changes the medium parameters and their formulas are obtained by comparison of terms. If the constitutive equations for the general bianisotropic medium are written after the affine transformation as

$$\mathbf{D}_a = \overline{\overline{\epsilon}}_a \cdot \mathbf{E}_a + \overline{\overline{\xi}}_a \cdot \mathbf{H}_a, \tag{4.87}$$

$$\mathbf{B}_a = \overline{\overline{\zeta}}_a \cdot \mathbf{E}_a + \overline{\overline{\mu}}_a \cdot \mathbf{H}_a, \tag{4.88}$$

and the transformed fields are substituted from the above equations, we are able to identify the transformed medium dyadics in terms of the original medium dyadics in the form

$$\overline{\overline{\epsilon}}_a = \gamma(\det\overline{\overline{A}})\overline{\overline{A}}^{-1} \cdot \overline{\overline{\epsilon}} \cdot \overline{\overline{A}}^{-1T}, \tag{4.89}$$

$$\overline{\overline{\xi}}_a = (\det\overline{\overline{A}})\overline{\overline{A}}^{-1} \cdot \overline{\overline{\xi}} \cdot \overline{\overline{A}}^{-1T}, \tag{4.90}$$

$$\overline{\overline{\zeta}}_a = (\det\overline{\overline{A}})\overline{\overline{A}}^{-1} \cdot \overline{\overline{\zeta}} \cdot \overline{\overline{A}}^{-1T}, \tag{4.91}$$

$$\overline{\overline{\mu}}_a = \gamma(\det\overline{\overline{A}})\overline{\overline{A}}^{-1} \cdot \overline{\overline{\mu}} \cdot \overline{\overline{A}}^{-1T}, \tag{4.92}$$

where the parameter $\gamma = \alpha\beta$ has either the value of 1 or -1.

The above formulas (4.89) – (4.92) show that an isotropic medium with parameters ϵ, μ is transformed to an anisotropic medium in the general affine transformation:

$$\overline{\overline{\epsilon}}_a = \gamma\epsilon(\det\overline{\overline{A}})(\overline{\overline{A}}^T \cdot \overline{\overline{A}})^{-1}, \tag{4.93}$$

$$\overline{\overline{\mu}}_a = \gamma\mu(\det\overline{\overline{A}})(\overline{\overline{A}}^T \cdot \overline{\overline{A}})^{-1}. \tag{4.94}$$

From these expressions it is seen that the transformed $\overline{\overline{\epsilon}}_a$ and $\overline{\overline{\mu}}_a$ dyadics are multiples of the same symmetric dyadic and they satisfy the relation

$$\overline{\overline{\epsilon}}_a \cdot \overline{\overline{\mu}}_a^{-1} = \overline{\overline{\mu}}_a^{-1} \cdot \overline{\overline{\epsilon}}_a = \frac{\epsilon}{\mu}\overline{\overline{I}}, \quad \text{or} \quad \frac{\overline{\overline{\epsilon}}_a}{\epsilon} = \frac{\overline{\overline{\mu}}_a}{\mu}. \tag{4.95}$$

It is alluring to search for an affine transformation that would transform a given anisotropic medium into an isotropic medium. However, from the previous equations we see that this is only possible for a limited class of anisotropic media. In fact, solving for $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$ from (4.89), (4.92) with $\overline{\overline{\epsilon}}_a = \epsilon_a \overline{\overline{I}}$, $\overline{\overline{\mu}}_a = \mu_a \overline{\overline{I}}$ shows us that the original medium dyadics $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ must satisfy an equation of the form

$$\overline{\overline{\epsilon}} \cdot \overline{\overline{\mu}}^{-1} = \frac{\epsilon_a}{\mu_a} \overline{\overline{I}}, \tag{4.96}$$

or $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ must be multiples of the same dyadic. Also, because $\overline{\overline{A}} \cdot \overline{\overline{A}}^T$ is a symmetric dyadic, both $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ must be multiples of the same symmetric complete dyadic $\overline{\overline{S}}$. Only in this case it is possible to find a dyadic $\overline{\overline{A}}$ defining the affine transformation needed to make the anisotropic medium isotropic. It is easy to see that the dyadic $\overline{\overline{A}}$ can then be taken as symmetric and a multiple of the dyadic $\overline{\overline{\epsilon}}^{1/2}$. In fact, we can write

$$\overline{\overline{A}} = \epsilon_a \sqrt{\det \overline{\overline{\epsilon}}}\, \overline{\overline{\epsilon}}^{1/2} = \mu_a \sqrt{\det \overline{\overline{\mu}}}\, \overline{\overline{\mu}}^{1/2}. \tag{4.97}$$

Thus, only a very special type of anisotropic medium can actually be transformed into an isotropic medium and the corresponding class can be called *affinely isotropic*. This can be generalized to a relation for bianisotropic media which can be transformed to bi-isotropic media by adding a similar requirement to the skew parameter dyadics, i.e. that they be multiples of the same symmetric dyadic as $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$. Such a medium might be called *affinely bi-isotropic.*

Finally, we might be interested to study what kind of anisotropic medium can be transformed to a symmetric uniaxial anisotropic medium with medium dyadics of the form $\overline{\overline{\epsilon}}_a = \epsilon_1 \overline{\overline{I}} + \epsilon_2 \mathbf{uu}$ and $\overline{\overline{\mu}}_a = \mu_1 \overline{\overline{I}} + \mu_2 \mathbf{uu}$. From the above expressions it can be shown, after some algebra, that the original medium dyadics must be of the general symmetric form

$$\overline{\overline{\epsilon}} = \epsilon_1' \overline{\overline{S}} + \epsilon_2' \mathbf{vv}, \qquad \overline{\overline{\mu}} = \mu_1' \overline{\overline{S}} + \mu_2' \mathbf{vv}, \tag{4.98}$$

where $\overline{\overline{S}}$ is a symmetric complete dyadic and $\mathbf{v}$ is a certain unit vector. In this case, the required transformation dyadic $\overline{\overline{A}}$ can be taken as a multiple of $\overline{\overline{S}}^{1/2}$ and the unit vector $\mathbf{u}$ will be a multiple of $\overline{\overline{S}}^{1/2} \cdot \mathbf{v}$. A medium of this kind of can be called *affinely uniaxial.*

4.3.3 Involutory affine transformations

Finally, let us consider affine transformations that are involutions like the duality transformation, which means that the original electromagnetic field is obtained through two consecutive transformations, or, the equivalent, the affine transformation equals its inverse. To find the condition, let us consider the double transformation

$$\mathbf{E}_{aa}(\mathbf{r}) = \alpha^2(\overline{\overline{A}}^T)^2 \cdot \mathbf{E}(\overline{\overline{A}}^2 \cdot \mathbf{r}) = \mathbf{E}(\mathbf{r}), \tag{4.99}$$

from which we have the condition

$$\overline{\overline{A}}^2 = \overline{\overline{I}}. \tag{4.100}$$

The same condition is obtained from all transformation formulas. Of course, the condition $\alpha^2 = \beta^2 = 1$ must also be valid.

Thus, for an involutory affine transformation the coefficients α and β can only have values $+1$ or -1 and the dyadic $\overline{\overline{A}}$ must be a square root of the unit dyadic, whence $\det\overline{\overline{A}} = \pm 1$. The trivial solutions $\overline{\overline{A}} = \pm\overline{\overline{I}}$ are two special cases of the most general square root of the unit dyadic, which is of the uniaxial form

$$\overline{\overline{A}} = \pm(\overline{\overline{I}} - \mathbf{ab}) \quad \text{with} \quad \mathbf{a} \cdot \mathbf{b} = 2. \tag{4.101}$$

4.4 Reflection transformations

As an important special case of the involutory affine transformation we consider the most general symmetric square root of the unit dyadic, which can be written in the form

$$\overline{\overline{C}} = \overline{\overline{I}} - 2\mathbf{uu}, \tag{4.102}$$

where $\mathbf{u}$ is a real unit vector. This actually defines a reflection transformation, which maps any vector $\mathbf{r}$ to the vector $\mathbf{r} - 2\mathbf{u}(\mathbf{u} \cdot \mathbf{r})$, i.e. its mirror image with respect to the plane with $\mathbf{r} \cdot \mathbf{u} = 0$. The uniaxial reflection dyadic $\overline{\overline{C}}$ satisfies the basic conditions

$$\overline{\overline{C}}^2 = \overline{\overline{I}}, \qquad \overline{\overline{C}}^T = \overline{\overline{C}}^{-1} = \overline{\overline{C}}, \qquad \det\overline{\overline{C}} = -1. \tag{4.103}$$

4.4.1 Invariance of media

The reflection transformation is often applied in conjunction with an image principle, in which original and transformed fields are combined. This makes sense only if the medium does not change in the reflection transformation, a property which is not valid for arbitrary media. Let us study what kind of media are invariant in reflection. Requiring $\overline{\overline{\epsilon}}_a = \overline{\overline{\epsilon}}$ for $\overline{\overline{A}} = \overline{\overline{C}}$ in (4.89), we obtain the condition

$$\overline{\overline{\epsilon}} = -\gamma\overline{\overline{C}} \cdot \overline{\overline{\epsilon}} \cdot \overline{\overline{C}}. \tag{4.104}$$

Taking the determinant function of both sides and assuming that $\det\overline{\overline{\epsilon}} \neq 0$ (otherwise the medium could not be polarized in a certain direction), we arrive at $\gamma = -1$. Thus, for an involutory transformation we have two possibilities: either $\alpha = -1$ and $\beta = 1$ or $\alpha = 1$ and $\beta = -1$.

Requiring the invariance of medium parameters in reflection transformation gives us four dyadic equations for the bianisotropic parameters:

$$\overline{\overline{C}} \cdot \overline{\overline{\epsilon}} = \overline{\overline{\epsilon}} \cdot \overline{\overline{C}}, \qquad \overline{\overline{C}} \cdot \overline{\overline{\xi}} = -\overline{\overline{\xi}} \cdot \overline{\overline{C}}, \tag{4.105}$$

$$\overline{\overline{C}} \cdot \overline{\overline{\zeta}} = -\overline{\overline{\zeta}} \cdot \overline{\overline{C}}, \qquad \overline{\overline{C}} \cdot \overline{\overline{\mu}} = \overline{\overline{\mu}} \cdot \overline{\overline{C}}. \tag{4.106}$$

Thus, $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ can be any dyadics which commute, and $\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}$ any dyadics that anticommute, with the reflection dyadic $\overline{\overline{C}}$. Writing each dyadic in terms of components parallel and transverse to $\mathbf{u}$ we can see that the medium parameter dyadics must be of the form

$$\overline{\overline{\epsilon}} = \epsilon_{uu}\mathbf{u}\mathbf{u} + \overline{\overline{\epsilon}}_t, \qquad \overline{\overline{\xi}} = \mathbf{u}\mathbf{a}_t + \mathbf{b}_t\mathbf{u}, \tag{4.107}$$

$$\overline{\overline{\zeta}} = \mathbf{u}\mathbf{c}_t + \mathbf{d}_t\mathbf{u}, \qquad \overline{\overline{\mu}} = \mu_{uu}\mathbf{u}\mathbf{u} + \overline{\overline{\mu}}_t, \tag{4.108}$$

where the dyadics $\overline{\overline{\epsilon}}_t$, $\overline{\overline{\mu}}_t$ and the vectors $\mathbf{a}_t$,...,$\mathbf{d}_t$ are transverse to $\mathbf{u}$.

From the above the following can be seen.

- A bi-isotropic medium is invariant in the reflection transformation if and only if it is isotropic, i.e. if $\xi = \zeta = 0$. In fact, a chiral medium changes handedness in reflection and thus cannot be invariant.

- A bianisotropic medium is invariant only if $\mathbf{u}$ is one of the eigenvectors of the medium dyadics $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$ from both left and right and the other eigenvectors are transverse to $\mathbf{u}$. This is the case, for example, for a uniaxial symmetric medium with an optical axis normal to the reflection plane. Moreover, the dyadics $\overline{\overline{\xi}}$, $\overline{\overline{\zeta}}$ must be planar and of a special form to be invariant in reflection.

4.4.2 Electric and magnetic reflections

It was seen that there exist two reflection transformations of electromagnetic fields and sources. Let us call the transformation defined by $\alpha = 1$ and $\beta = -1$ *the electric reflection*, because the electric fields and sources are transformed to their mirror images, whereas the magnetic fields and sources are transformed to their negative mirror images:

$$\mathbf{E}_c(\mathbf{r}) = \overline{\overline{C}} \cdot \mathbf{E}(\overline{\overline{C}} \cdot \mathbf{r}), \qquad \mathbf{H}_c(\mathbf{r}) = -\overline{\overline{C}} \cdot \mathbf{H}(\overline{\overline{C}} \cdot \mathbf{r}), \tag{4.109}$$

$$\mathbf{D}_c(\mathbf{r}) = \overline{\overline{C}}\cdot\mathbf{D}(\overline{\overline{C}}\cdot\mathbf{r}), \qquad \mathbf{B}_c(\mathbf{r}) = -\overline{\overline{C}}\cdot\mathbf{B}(\overline{\overline{C}}\cdot\mathbf{r}), \tag{4.110}$$

$$\mathbf{J}_c(\mathbf{r}) = \overline{\overline{C}}\cdot\mathbf{J}(\overline{\overline{C}}\cdot\mathbf{r}), \qquad \mathbf{J}_{mc}(\mathbf{r}) = -\overline{\overline{C}}\cdot\mathbf{J}_m(\overline{\overline{C}}\cdot\mathbf{r}). \tag{4.111}$$

The second possible reflection transformation with $\alpha = -1$, $\beta = 1$ is called *the magnetic reflection* and defined by

$$\mathbf{E}_c(\mathbf{r}) = -\overline{\overline{C}}\cdot\mathbf{E}(\overline{\overline{C}}\cdot\mathbf{r}), \qquad \mathbf{H}_c(\mathbf{r}) = \overline{\overline{C}}\cdot\mathbf{H}(\overline{\overline{C}}\cdot\mathbf{r}), \tag{4.112}$$

$$\mathbf{D}_c(\mathbf{r}) = -\overline{\overline{C}}\cdot\mathbf{D}(\overline{\overline{C}}\cdot\mathbf{r}), \qquad \mathbf{B}_c(\mathbf{r}) = \overline{\overline{C}}\cdot\mathbf{B}(\overline{\overline{C}}\cdot\mathbf{r}), \tag{4.113}$$

$$\mathbf{J}_c(\mathbf{r}) = -\overline{\overline{C}}\cdot\mathbf{J}(\overline{\overline{C}}\cdot\mathbf{r}), \qquad \mathbf{J}_{mc}(\mathbf{r}) = \overline{\overline{C}}\cdot\mathbf{J}_m(\overline{\overline{C}}\cdot\mathbf{r}). \tag{4.114}$$

It is important to note that the naive reflection transformation where all fields and sources are transformed either through $\overline{\overline{C}}$ or $-\overline{\overline{C}}$ alone, does not satisfy the Maxwell equations in a medium which is invariant in the reflection transformation. Without different signs in electric and magnetic field vectors, the Poynting vector $\mathbf{S} = (1/2)\mathbf{E}\times\mathbf{H}^*$ would not transform to its mirror image but to the negative of its mirror image.

4.4.3 The mirror image principle

Any electromagnetic field and the corresponding source quantity q can be written as a sum of two self-reflecting parts, each invariant in one of the two reflection transformations:

$$q = q^e + q^m. \tag{4.115}$$

Here, the superscript e denotes the quantity invariant in the electric reflection (electrically symmetric) and m, in the magnetic reflection (magnetically symmetric). For example, the electric and magnetic fields can be written as

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^m, \qquad \mathbf{H} = \mathbf{H}^e + \mathbf{H}^m, \tag{4.116}$$

with

$$\mathbf{E}^e(\mathbf{r}) = \frac{1}{2}[\mathbf{E}(\mathbf{r}) + \overline{\overline{C}}\cdot\mathbf{E}(\overline{\overline{C}}\cdot\mathbf{r})], \qquad \mathbf{E}^m(\mathbf{r}) = \frac{1}{2}[\mathbf{E}(\mathbf{r}) - \overline{\overline{C}}\cdot\mathbf{E}(\overline{\overline{C}}\cdot\mathbf{r})], \tag{4.117}$$

$$\mathbf{H}^e(\mathbf{r}) = \frac{1}{2}[\mathbf{H}(\mathbf{r}) - \overline{\overline{C}}\cdot\mathbf{H}(\overline{\overline{C}}\cdot\mathbf{r})], \qquad \mathbf{H}^m(\mathbf{r}) = \frac{1}{2}[\mathbf{H}(\mathbf{r}) + \overline{\overline{C}}\cdot\mathbf{H}(\overline{\overline{C}}\cdot\mathbf{r})]. \tag{4.118}$$

Denoting $\mathbf{r} = \mathbf{u}(\mathbf{u}\cdot\mathbf{r}) + \boldsymbol{\rho}$, on the plane $\mathbf{u}\cdot\mathbf{r} = 0$ we have $\overline{\overline{C}}\cdot\mathbf{r} = \overline{\overline{C}}\cdot\boldsymbol{\rho} = \boldsymbol{\rho} = \mathbf{r}$, whence the arguments of the field vectors and their reflection images are the same. Thus, we can write

$$\mathbf{E}^e(\boldsymbol{\rho}) = \frac{1}{2}[\overline{\overline{I}} + \overline{\overline{C}}]\cdot\mathbf{E}(\boldsymbol{\rho}) = \overline{\overline{I}}_t\cdot\mathbf{E}(\boldsymbol{\rho}), \tag{4.119}$$

$$\mathbf{E}^m(\boldsymbol{\rho}) = \frac{1}{2}[\overline{\overline{I}} - \overline{\overline{C}}] \cdot \mathbf{E}(\boldsymbol{\rho}) = \mathbf{uu} \cdot \mathbf{E}(\boldsymbol{\rho}), \tag{4.120}$$

$$\mathbf{H}^e(\boldsymbol{\rho}) = \frac{1}{2}[\overline{\overline{I}} - \overline{\overline{C}}] \cdot \mathbf{E}(\boldsymbol{\rho}) = \mathbf{uu} \cdot \mathbf{H}(\boldsymbol{\rho}), \tag{4.121}$$

$$\mathbf{H}^m(\boldsymbol{\rho}) = \frac{1}{2}[\overline{\overline{I}} + \overline{\overline{C}}] \cdot \mathbf{H}(\boldsymbol{\rho}) = \overline{\overline{I}}_t \cdot \mathbf{H}(\boldsymbol{\rho}). \tag{4.122}$$

From the conditions

$$\mathbf{u} \cdot \mathbf{E}^e = 0, \qquad \mathbf{u} \times \mathbf{H}^e = 0, \tag{4.123}$$

we see that electrically symmetric fields satisfy the PMC boundary conditions on the plane of symmetry. Correspondingly, from

$$\mathbf{u} \times \mathbf{E}^m = 0, \qquad \mathbf{u} \cdot \mathbf{H}^m = 0, \tag{4.124}$$

we see that magnetically symmetric fields satisfy the PEC boundary conditions.

Electromagnetic fields can be decomposed into electrically and magnetically symmetric parts by decomposing their sources into symmetric parts

$$\mathbf{J}^e(\mathbf{r}) = \frac{1}{2}[\mathbf{J}(\mathbf{r}) + \overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r})], \tag{4.125}$$

$$\mathbf{J}_m^e(\mathbf{r}) = \frac{1}{2}[\mathbf{J}_m(\mathbf{r}) - \overline{\overline{C}} \cdot \mathbf{J}_m(\overline{\overline{C}} \cdot \mathbf{r})], \tag{4.126}$$

$$\mathbf{J}^m(\mathbf{r}) = \frac{1}{2}[\mathbf{J}(\mathbf{r}) - \overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r})], \tag{4.127}$$

$$\mathbf{J}_m^m(\mathbf{r}) = \frac{1}{2}[\mathbf{J}_m(\mathbf{r}) + \overline{\overline{C}} \cdot \mathbf{J}_m(\overline{\overline{C}} \cdot \mathbf{r})]. \tag{4.128}$$

Because the symmetric fields satisfy either PMC or PEC boundary conditions at the reflection plane $\mathbf{u} \cdot \mathbf{r} = 0$, the problem 'original source and plane boundary' can be replaced by the problem 'original and reflection-transformed sources with no plane boundary'. This is called the mirror image principle because the transformed source is either the mirror image or the negative of the mirror image of the original source.

The image sources for the PEC plane are obtained through magnetic reflection:

$$\mathbf{J}_i(\mathbf{r}) = -\overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r}), \qquad \mathbf{J}_{mi}(\mathbf{r}) = \overline{\overline{C}} \cdot \mathbf{J}_m(\overline{\overline{C}} \cdot \mathbf{r}), \tag{4.129}$$

and for the PMC plane through electric reflection

$$\mathbf{J}_i(\mathbf{r}) = \overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r}), \qquad \mathbf{J}_{mi}(\mathbf{r}) = -\overline{\overline{C}} \cdot \mathbf{J}_m(\overline{\overline{C}} \cdot \mathbf{r}). \tag{4.130}$$

The advantage of the image principle is in replacing a boundary value problem by a source problem, which in most cases is easier to handle. The present image principle is, however, only associated with PEC or PMC planes and cannot be simply extended to impedance surfaces or interface problems. A method for solving problems of this kind in terms of a more complicated image principle will be discussed in Chapter 7.

4.4.4 Images in parallel planes

The reflection transformation above was defined with respect to a plane passing through the origin. A more general reflection transformation can be written in the form

$$\mathbf{r} \to \mathbf{r}_c = \overline{\overline{C}} \cdot \mathbf{r} + 2\mathbf{u}\mathbf{u} \cdot \mathbf{r}_o, \tag{4.131}$$

which defines reflection with respect to a plane with the normal unit vector $\mathbf{u}$ and passing through the point $\mathbf{r}_o$. The image sources corresponding to a PMC plane are now

$$\mathbf{J}_i(\mathbf{r}) = \overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r} + 2\mathbf{u}\mathbf{u} \cdot \mathbf{r}_o), \tag{4.132}$$

$$\mathbf{J}_{mi}(\mathbf{r}) = -\overline{\overline{C}} \cdot \mathbf{J}_m(\overline{\overline{C}} \cdot \mathbf{r} + 2\mathbf{u}\mathbf{u} \cdot \mathbf{r}_o), \tag{4.133}$$

and corresponding to a PEC plane,

$$\mathbf{J}_i(\mathbf{r}) = -\overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r} + 2\mathbf{u}\mathbf{u} \cdot \mathbf{r}_o), \tag{4.134}$$

$$\mathbf{J}_{mi}(\mathbf{r}) = \overline{\overline{C}} \cdot \mathbf{J}_m(\overline{\overline{C}} \cdot \mathbf{r} + 2\mathbf{u}\mathbf{u} \cdot \mathbf{r}_o). \tag{4.135}$$

These expressions make it possible to extend the image theory to a region bounded by two parallel PEC or PMC planes. As a simple example take the case of two PEC planes 1 and 2 going through the points $\mathbf{r}_1 = -\mathbf{u}r_1$ and $\mathbf{r}_2 = \mathbf{u}r_2$ with the distance $d = r_1 + r_2$. If the source $\mathbf{J}(\mathbf{r})$ lies between the planes, we can substitute the plane 1 by an unknown image source $\mathbf{J}_1(\mathbf{r})$ and the plane 2 by another unknown image source $\mathbf{J}_2(\mathbf{r})$. The original plus the image sources must be magnetically symmetric with respect to each of the two planes:

$$\mathbf{J}_1(\mathbf{r}) = -\overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r} + 2\mathbf{r}_1) - \overline{\overline{C}} \cdot \mathbf{J}_2(\overline{\overline{C}} \cdot \mathbf{r} + 2\mathbf{r}_1), \tag{4.136}$$

$$\mathbf{J}_2(\mathbf{r}) = -\overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r} + 2\mathbf{r}_2) - \overline{\overline{C}} \cdot \mathbf{J}_1(\overline{\overline{C}} \cdot \mathbf{r} + 2\mathbf{r}_2). \tag{4.137}$$

From these equations we can obtain, by elimination, equations for a single unknown source, either $\mathbf{J}_1$ or $\mathbf{J}_2$:

$$\mathbf{J}_1(\mathbf{r}) - \mathbf{J}_1(\mathbf{r} - 2\mathbf{u}d) = -\overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r} - 2\mathbf{u}d) + \mathbf{J}(\mathbf{r} - 2\mathbf{u}d), \tag{4.138}$$

$$\mathbf{J}_2(\mathbf{r}) - \mathbf{J}_2(\mathbf{r} + 2\mathbf{u}d) = -\overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r} + 2\mathbf{u}d) + \mathbf{J}(\mathbf{r} + 2\mathbf{u}d). \tag{4.139}$$

The difference equations for the unknown image currents (4.138), (4.139) can be solved to give

$$\mathbf{J}_1(\mathbf{r}) = -\overline{\overline{C}} \cdot \sum_{n=0}^{\infty} \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r} - 2n\mathbf{u}d + 2\mathbf{r}_1) + \sum_{n=1}^{\infty} \mathbf{J}(\mathbf{r} - 2n\mathbf{u}d), \tag{4.140}$$

$$\mathbf{J}_2(\mathbf{r}) = -\overline{\overline{C}} \cdot \sum_{n=0}^{\infty} \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r} + 2n\mathbf{u}d + 2\mathbf{r}_2) + \sum_{n=1}^{\infty} \mathbf{J}(\mathbf{r} + 2n\mathbf{u}d). \tag{4.141}$$

Each of these sources consists of two sets of periodic sources extending to infinity. The theory also works backwards: an infinite periodic structure can be replaced by a finite structure bounded by PEC or PMC planes.

In addition to sources, boundaries and obstacles also transform by reflection with respect to a PEC or PMC plane. To see this, we can temporarily replace the obstacle by its equivalent source, for example, a dielectric obstacle by its polarization current $\mathbf{J}_p = j\omega(\epsilon - \epsilon_o)\mathbf{E}$. Since the polarization current transforms like the electric field, the transformed electric field induces exactly the transformed polarization current at the mirror image position of the dielectric sphere.

4.4.5 Babinet's principle

A disadvantage of the duality transformation is that a PEC boundary is always transformed to a PMC boundary, which is unphysical. However, if the boundary is planar, the duality transformation can be applied in conjunction with the image principle, so that PMC boundaries can be avoided. This combination leads to a method called Babinet's principle, which relates two problems involving complementary PEC boundary planes. Two planes containing PEC and open regions (holes) are called complementary if the holes in one of the planes correspond to PEC regions in the other and conversely.

It is possible to obtain a solution for the diffraction field due to a metallic planar structure, a disk for example, if the solution is known for the corresponding complementary structure, such as a hole in the metallic plane. However, the sources of the two problems must be dual to each other. If self-dual, they are the same sources in both problems.

Let the original source be $\mathbf{J}(\mathbf{r}) = \mathbf{f}(\mathbf{r})$ in the half space $\mathbf{u}\cdot\mathbf{r} > 0$ bounded by the plane S defined by $\mathbf{u} \cdot \mathbf{r} = 0$. Let $S = S_1 + S_2$ consist of a PEC (metal) part S_1 and open part S_2. If the original source is written as a sum of two combined sources $\mathbf{J}_A(\mathbf{r})$, $\mathbf{J}_B(\mathbf{r})$ defined by

$$\mathbf{J}_A(\mathbf{r}) = \frac{1}{2}[\mathbf{f}(\mathbf{r}) + \overline{\overline{C}} \cdot \mathbf{f}(\overline{\overline{C}} \cdot \mathbf{r})], \tag{4.142}$$

$$\mathbf{J}_B(\mathbf{r}) = \frac{1}{2}[\mathbf{f}(\mathbf{r}) - \overline{\overline{C}} \cdot \mathbf{f}(\overline{\overline{C}} \cdot \mathbf{r})], \tag{4.143}$$

the problem can be split into two problems. The source $\mathbf{J}_B$ is magnetically symmetric and the part S_2 of the boundary plane S can be covered with PEC because the fields satisfy the correct boundary conditions. Thus, B corresponds to a problem of reflection from an intact metallic plane S. In the A problem, the S_2 part of the interface can be covered by PMC, while S_1 is still a PEC surface, or it is a problem with an inhomogeneous plane boundary.

The original problem of whole space and an interface plane has thus been transformed into two problems with plane boundaries. The complementary boundary problem with S_1 empty and PEC on S_2 can be approached through a similar method backwards after making the duality transform to problem A with the inhomogeneous boundary.

Let us start from a problem with the complementary boundary and the dual source:

$$\mathbf{J}_m(\mathbf{r}) = -j\eta\mathbf{J}_d(\mathbf{r}) = -j\eta\mathbf{f}(\mathbf{r}). \tag{4.144}$$

Likewise, this source can be split into two parts

$$\mathbf{J}_{mC}(\mathbf{r}) = -\frac{j\eta}{2}[\mathbf{f}(\mathbf{r}) + \overline{\overline{C}} \cdot \mathbf{f}(\overline{\overline{C}} \cdot \mathbf{r})], \tag{4.145}$$

$$\mathbf{J}_{mD}(\mathbf{r}) = -\frac{j\eta}{2}[\mathbf{f}(\mathbf{r}) - \overline{\overline{C}} \cdot \mathbf{f}(\overline{\overline{C}} \cdot \mathbf{r})], \tag{4.146}$$

of which the first one (C) corresponds to PEC boundary conditions on the whole plane, whence the plane can be either completed or removed without changing the fields. The second D source gives rise to PMC conditions on the S_1 part of the surface. This problem is thus dual to the A problem with both dual sources and dual boundary conditions. Hence, the fields are also dual:

$$(\mathbf{E}_D,\ \mathbf{H}_D) = (\mathbf{E}_A,\ \mathbf{H}_A)_d. \tag{4.147}$$

On the other hand, the fields B and C are easily calculated because of the planar PEC conditions. Thus, the original problem 1 and the problem 2 with complementary boundary and dual source are related through

$$(\mathbf{E}_2,\ \mathbf{H}_2) = (\mathbf{E}_D,\ \mathbf{H}_D) + (\mathbf{E}_C,\ \mathbf{H}_C) = (\mathbf{E}_A,\ \mathbf{H}_A)_d + (\mathbf{E}_C,\ \mathbf{H}_C)$$
$$= (\mathbf{E}_1,\ \mathbf{H}_1)_d - (\mathbf{E}_B,\ \mathbf{H}_B)_d + (\mathbf{E}_C,\ \mathbf{H}_C). \tag{4.148}$$

This means that there is a simple relation between the diffraction patterns of the two complementary problems, because the problems B and D involve no diffraction. However, the sources must be dual in the complementary problems, which is a limitation. For example, the diffraction from a hole in a metallic plane for a horizontally polarized plane wave has the same pattern as the diffraction from a complementary metallic disk for a vertically polarized plane wave. If the original source is a self-dual or antiself-dual source, this is no problem, because the dual source is then either the original source or the negative of it. The sign of the self-dual source depends of course on the chosen duality transformation. For an incoming plane wave the self-dual field is circularly polarized.

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