Appendix D

Dyadic identities

Definitions

$$\overline{\overline{A}}^2 = \overline{\overline{A}} \cdot \overline{\overline{A}}, \qquad \overline{\overline{A}}^{-2} = (\overline{\overline{A}}^{-1})^2 = (\overline{\overline{A}}^2)^{-1}$$

$$\overline{\overline{A}}^{(2)} = \frac{1}{2}\overline{\overline{A}}\,{}^{\times}_{\times}\,\overline{\overline{A}}, \qquad \overline{\overline{A}}^{(-2)} = (\overline{\overline{A}}^{-1})^{(2)} = (\overline{\overline{A}}^{(2)})^{-1}$$

$$\mathrm{tr}\overline{\overline{A}} = \overline{\overline{A}} : \overline{\overline{I}}$$

$$\mathrm{spm}\overline{\overline{A}} = \mathrm{tr}\overline{\overline{A}}^{(2)} = \frac{1}{2}\overline{\overline{A}}\,{}^{\times}_{\times}\,\overline{\overline{A}} : \overline{\overline{I}}$$

$$\det\overline{\overline{A}} = \frac{1}{6}\overline{\overline{A}}\,{}^{\times}_{\times}\,\overline{\overline{A}} : \overline{\overline{A}}$$

$$\det\overline{\overline{A}} \neq 0 \quad \leftrightarrow \quad \overline{\overline{A}} \text{ complete}$$

$$\det\overline{\overline{A}} = 0 \quad \leftrightarrow \quad \overline{\overline{A}} \text{ planar}$$

$$\overline{\overline{A}}^{(2)} = 0 \quad \leftrightarrow \quad \overline{\overline{A}} \text{ linear.}$$

Identities

$$\overline{\overline{A}}\,{}^{\times}_{\times}\,\overline{\overline{B}} = \overline{\overline{B}}\,{}^{\times}_{\times}\,\overline{\overline{A}} = \left[(\overline{\overline{A}} : \overline{\overline{I}})(\overline{\overline{B}} : \overline{\overline{I}}) - \overline{\overline{A}} : \overline{\overline{B}}^T\right]\overline{\overline{I}} -$$

$$(\overline{\overline{A}} : \overline{\overline{I}})\overline{\overline{B}}^T - (\overline{\overline{B}} : \overline{\overline{I}})\overline{\overline{A}}^T + [\overline{\overline{A}} \cdot \overline{\overline{B}} + \overline{\overline{B}} \cdot \overline{\overline{A}}]^T$$

$$\overline{\overline{A}}\,{}^{\times}_{\times}\,\overline{\overline{I}} = (\overline{\overline{A}} : \overline{\overline{I}})\overline{\overline{I}} - \overline{\overline{A}}^T$$

$$\overline{\overline{A}}\,{}^{\times}_{\times}\,(\mathbf{a} \times \overline{\overline{I}}) = \mathbf{a}(\overline{\overline{A}}\,{}^{\cdot}_{\times}\,\overline{\overline{I}}) + \overline{\overline{I}} \times (\mathbf{a} \cdot \overline{\overline{A}})$$

$$\overline{\overline{I}}\,{}^{\times}_{\times}\,\overline{\overline{I}} = 2\overline{\overline{I}}$$

$$(\mathbf{a}\times\overline{\overline{I}}){}^\times_\times\overline{\overline{I}} = \mathbf{a}\times\overline{\overline{I}}$$

$$\overline{\overline{S}}{}^\times_\times\overline{\overline{I}} = -\overline{\overline{S}} \quad (\overline{\overline{S}}\ \text{symmetric, trace free})$$

$$(\mathbf{a}\times\overline{\overline{I}}){}^\times_\times(\mathbf{b}\times\overline{\overline{I}}) = \mathbf{ab}+\mathbf{ba}$$

$$\overline{\overline{S}}{}^\times_\times(\mathbf{a}\times\overline{\overline{I}}) = (\overline{\overline{S}}\cdot\mathbf{a})\times\overline{\overline{I}} \quad (\overline{\overline{S}}\ \text{symmetric})$$

$$(\overline{\overline{A}}\times\mathbf{a}){}^\times_\times(\overline{\overline{B}}\times\mathbf{a}) = (\overline{\overline{A}}{}^\times_\times\overline{\overline{B}})\cdot\mathbf{aa}$$

$$(\mathbf{a}\times\overline{\overline{A}}){}^\times_\times(\mathbf{a}\times\overline{\overline{B}}) = \mathbf{aa}\cdot(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}})$$

$$(\mathbf{a}\times\overline{\overline{I}}){}^\times_\times(\mathbf{a}\times\overline{\overline{I}}) = 2\mathbf{aa}$$

$$\overline{\overline{A}}{}^\times_\times(\overline{\overline{B}}{}^\times_\times\overline{\overline{C}}) = (\overline{\overline{A}}:\overline{\overline{C}})\overline{\overline{B}} + (\overline{\overline{A}}:\overline{\overline{B}})\overline{\overline{C}} - \overline{\overline{B}}\cdot\overline{\overline{A}}^T\cdot\overline{\overline{C}} - \overline{\overline{C}}\cdot\overline{\overline{A}}^T\cdot\overline{\overline{B}}$$

$$\overline{\overline{I}}{}^\times_\times(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}}) = (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{B}} + (\overline{\overline{B}}:\overline{\overline{I}})\overline{\overline{A}} - (\overline{\overline{A}}\cdot\overline{\overline{B}} + \overline{\overline{B}}\cdot\overline{\overline{A}})$$

$$\overline{\overline{I}}{}^\times_\times(\overline{\overline{I}}{}^\times_\times\overline{\overline{A}}) = \overline{\overline{A}} + (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}}$$

$$\overline{\overline{I}}{}^\times_\times(\overline{\overline{I}}{}^\times_\times\overline{\overline{I}}) = 4\overline{\overline{I}}$$

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}){}^\times_\times(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}) = 8(\overline{\overline{A}}^{(2)})^{(2)} = 8\det\overline{\overline{A}}\,\overline{\overline{A}}$$

$$(\overline{\overline{A}}^{(2)})^{(2)} = \overline{\overline{A}}\det\overline{\overline{A}}$$

$$\det(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}) = 8\det{}^2\overline{\overline{A}}$$

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}})\cdot(\overline{\overline{C}}{}^\times_\times\overline{\overline{D}}) = (\overline{\overline{A}}\cdot\overline{\overline{C}}){}^\times_\times(\overline{\overline{B}}\cdot\overline{\overline{D}}) + (\overline{\overline{A}}\cdot\overline{\overline{D}}){}^\times_\times(\overline{\overline{B}}\cdot\overline{\overline{C}})$$

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}})\cdot(\overline{\overline{B}}{}^\times_\times\overline{\overline{B}}) = 2(\overline{\overline{A}}\cdot\overline{\overline{B}}){}^\times_\times(\overline{\overline{A}}\cdot\overline{\overline{B}})$$

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}})^2 = (\overline{\overline{A}}^2){}^\times_\times(\overline{\overline{B}}^2) + (\overline{\overline{A}}\cdot\overline{\overline{B}}){}^\times_\times(\overline{\overline{A}}\cdot\overline{\overline{B}})$$

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}})^2 = 2(\overline{\overline{A}}^2){}^\times_\times(\overline{\overline{A}}^2)$$

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{I}})^2 = (\overline{\overline{A}}^2){}^\times_\times\overline{\overline{I}} + \overline{\overline{A}}{}^\times_\times\overline{\overline{A}}$$

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}})^T\cdot\overline{\overline{A}} = \overline{\overline{A}}\cdot(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}})^T = \frac{1}{3}(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}:\overline{\overline{A}})\overline{\overline{I}}$$

$$\overline{\overline{A}}^{(2)T}\cdot\overline{\overline{A}} = \overline{\overline{A}}\cdot\overline{\overline{A}}^{(2)T} = \det\overline{\overline{A}}\,\overline{\overline{I}}$$

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^{(2)T}}{\det\overline{\overline{A}}} = \frac{3(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}})^T}{\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}:\overline{\overline{A}}} \quad (\overline{\overline{A}} \text{ complete})$$

$$\overline{\overline{A}}^{-1} = \frac{(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}^{(2)*})^T}{\overline{\overline{A}}^{(2)}:\overline{\overline{A}}^{(2)*}} \quad (\text{planar inverse})$$

$$\overline{\overline{A}}^{-1}\cdot\overline{\overline{A}} = \overline{\overline{I}} - \frac{\overline{\overline{A}}^{(2)*}\cdot\overline{\overline{A}}^{(2)T}}{\overline{\overline{A}}^{(2)}:\overline{\overline{A}}^{(2)*}} \quad (\overline{\overline{A}} \text{ planar})$$

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^T{}^\times_\times\mathbf{uu}}{\mathrm{spm}\overline{\overline{A}}} \quad (\text{two}-\text{dimensional inverse})$$

$$\overline{\overline{A}}^{-1}\cdot\overline{\overline{A}} = \overline{\overline{I}}_t = \overline{\overline{I}} - \mathbf{uu} \quad (\overline{\overline{A}} \text{ two}-\text{dimensional})$$

$$\overline{\overline{A}}:\overline{\overline{B}} = \overline{\overline{B}}:\overline{\overline{A}} = (\overline{\overline{A}}\cdot\overline{\overline{B}}^T):\overline{\overline{I}}$$

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}}):\overline{\overline{C}} = \overline{\overline{A}}:(\overline{\overline{B}}{}^\times_\times\overline{\overline{C}}) \quad (\text{with all permutations})$$

$$\mathrm{spm}\overline{\overline{A}} = \mathrm{tr}\overline{\overline{A}}^{(2)} = \frac{1}{2}[(\overline{\overline{A}}:\overline{\overline{I}})^2 - \overline{\overline{A}}:\overline{\overline{A}}^T]$$

$$\mathrm{spm}(\overline{\overline{A}}\cdot\overline{\overline{B}}) = \mathrm{tr}(\overline{\overline{A}}^{(2)}\cdot\overline{\overline{B}}^{(2)}) = \overline{\overline{A}}^{(2)}:\overline{\overline{B}}^{(2)T}$$

$$\det\overline{\overline{A}} = \frac{1}{3}\overline{\overline{A}}^3:\overline{\overline{I}} - \frac{1}{2}(\overline{\overline{A}}^2:I)(\overline{\overline{A}}:\overline{\overline{I}}) + \frac{1}{6}(\overline{\overline{A}}:\overline{\overline{I}})^3$$

$$\det(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}) = 8\det(\overline{\overline{A}}^{(2)}) = 8(\det\overline{\overline{A}})^2$$

$$\det(\overline{\overline{A}}\cdot\overline{\overline{B}}) = \det\overline{\overline{A}}\,\det\overline{\overline{B}}$$

$$\det(\overline{\overline{A}}\cdot\overline{\overline{B}} + \alpha\overline{\overline{I}}) = \det(\overline{\overline{B}}\cdot\overline{\overline{A}} + \alpha\overline{\overline{I}})$$

$$\mathbf{a}\times(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}}) = \overline{\overline{B}}\times(\mathbf{a}\cdot\overline{\overline{A}}) + \overline{\overline{A}}\times(\mathbf{a}\cdot\overline{\overline{B}})$$

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}})\times\mathbf{a} = (\overline{\overline{A}}\cdot\mathbf{a})\times\overline{\overline{B}} + (\overline{\overline{B}}\cdot\mathbf{a})\times\overline{\overline{A}}$$

$$(\overline{\overline{A}}\cdot\mathbf{a})\times(\overline{\overline{A}}\cdot\mathbf{b}) = \frac{1}{2}(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}})\cdot(\mathbf{a}\times\mathbf{b}) = \overline{\overline{A}}^{(2)}\cdot(\mathbf{a}\times\mathbf{b})$$

$$(\mathbf{a}\cdot\overline{\overline{A}})\times(\mathbf{b}\cdot\overline{\overline{A}}) = \frac{1}{2}(\mathbf{a}\times\mathbf{b})\cdot(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}) = (\mathbf{a}\times\mathbf{b})\cdot\overline{\overline{A}}^{(2)}$$