Chapter 7

Exact image theory

Image sources can be defined as equivalent sources replacing physical structures such as regions of different media or boundaries with impedance conditions. As a classical example we have the mirror image due to a perfectly conducting plane surface. In this case, the image source located at its mirror image position replaces the conducting plane. It was seen in Chapter 4 that by applying the reflection transformation, the correct boundary conditions on the plane are ensured through the introduction of this kind of image source.

The image principle has also been applied in electrostatics to problems with charges in front of a dielectric half space. In this case, the image charge is in the mirror image position and its amplitude depends on the dielectric constants of the two media. In this case, the image source is determined through the property that the correct interface conditions for the electric field are satisfied. A similar principle is also valid in magnetostatics. When trying to extend these to time dependent electromagnetic problems, trouble arises because a simple image source at the mirror image location cannot satisfy the correct interface conditions for both the electric and the magnetic field. Thus, the image concept must be generalized in one way or another. In contrast to different existing approximate theories, the present exact image theory (EIT) was constructed in the 1980s.

7.1 General formulation for layered media

In the present section an introduction to the EIT theory will be given in a general form applicable to layered media with plane parallel interfaces. In the following sections, specific geometries are presented as special cases of the general formulation.

7.1.1 Fourier transformations

Let us consider the problem of piecewise homogeneous media, with all interfaces and boundaries parallel to the xy plane, in terms of Fourier transformation in the two dimensions of the plane. The transformation does not seem to be necessary, however, and it would be interesting to find a method of image construction without applying the Fourier transformation at all. As indicated in a paper by KELLER in 1981, some problems of this kind can be handled directly without the Fourier transformation, but the shortcut requires a certain theorem. In waiting for such an approach for the general problem, the Fourier method will be applied here to produce the results with some labour.

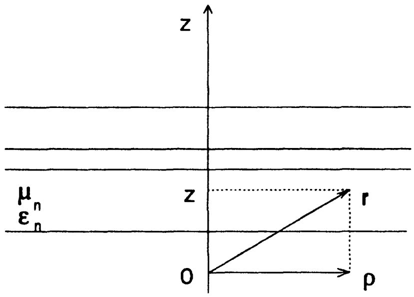

Fig. 7.1 Geometry of a multilayered medium with homogeneous layers of isotropic material.

Vector transmission-line equations

The fields in the layer n with parameters ϵ_n, μ_n and sources $\mathbf{J}_{mn}$, $\mathbf{J}_n$ satisfy the Maxwell equations

$$\nabla \times \mathbf{E}_n = -j\omega\mu_n\mathbf{H}_n - \mathbf{J}_{mn}, \tag{7.1}$$

$$\nabla \times \mathbf{H}_n = j\omega\epsilon_n\mathbf{E}_n + \mathbf{J}_n. \tag{7.2}$$

The two-dimensional Fourier transformation is defined as

$$f(\mathbf{r}) \to F(z, \mathbf{K}) = \int_{S_\rho} f(\mathbf{r})e^{j\mathbf{K}\cdot\boldsymbol{\rho}}dS_\rho, \tag{7.3}$$

with the inverse transformation

$$F(z, \mathbf{K}) \to f(\mathbf{r}) = \frac{1}{(2\pi)^2}\int_{S_K} F(z, \mathbf{K})e^{-j\mathbf{K}\cdot\boldsymbol{\rho}}dS_K, \tag{7.4}$$

where the integration domains S_ρ, S_K are the $\boldsymbol{\rho}$ and $\mathbf{K}$ planes, respectively.

Applying the transformation to (7.1), (7.2), we obtain the ordinary differential equations

$$\mathbf{u}_z \times \mathbf{E}_n' - j\mathbf{K} \times \mathbf{E}_n + j\omega\mu_n\mathbf{H}_n = -\mathbf{J}_{mn}, \tag{7.5}$$

$$\mathbf{u}_z \times \mathbf{H}_n' - j\mathbf{K} \times \mathbf{H}_n - j\omega\epsilon_n \mathbf{E}_n = \mathbf{J}_n, \tag{7.6}$$

where the prime denotes differentiation with respect to the coordinate z. To help in finding solutions, these equations are written as transmission-line equations for the vector field components transverse to the normal unit vector $\mathbf{u}_z$, denoted by $\mathbf{e}$, $\mathbf{h}$, by eliminating the normal components $\mathbf{u}_z \cdot \mathbf{E}$, $\mathbf{u}_z \cdot \mathbf{H}$. The resulting equations are

$$\mathbf{e}_n' + j\beta_n \overline{\overline{Z}}_n \cdot (-\mathbf{u}_z \times \mathbf{h}_n) = \mathbf{u}_z \times \mathbf{j}_{mn} + \frac{\eta_n}{k_n}\mathbf{K} j_{en}, \tag{7.7}$$

$$-\mathbf{u}_z \times \mathbf{h}_n' + j\beta_n \overline{\overline{Y}}_n \cdot \mathbf{e}_n = -\mathbf{j}_{en} - \frac{1}{\eta_n k_n}\mathbf{u}_z \times \mathbf{K} j_{mn}, \tag{7.8}$$

where the electric source has been denoted by $\mathbf{J}_n = \mathbf{j}_{en} + \mathbf{u}_z j_{en}$, and the magnetic source by $\mathbf{J}_{mn} = \mathbf{j}_{mn} + \mathbf{u}_z j_{mn}$ in components transverse and parallel to $\mathbf{u}_z$. Also, we denote

$$\beta_n = \sqrt{k_n^2 - K^2}, \qquad \eta_n = \sqrt{\frac{\mu_n}{\epsilon_n}}, \tag{7.9}$$

and the dyadics in (7.7), (7.8) have the form

$$\overline{\overline{Y}}_n = \frac{1}{k_n \beta_n \eta_n}(k_n^2 \overline{\overline{I}}_t - \mathbf{u}_z \mathbf{u}_z {}^{\times}_{\times} \mathbf{KK}), \qquad \overline{\overline{Z}}_n = \frac{\eta_n}{k_n \beta_n}(k_n^2 \overline{\overline{I}}_t - \mathbf{KK}), \tag{7.10}$$

β_n can be identified as the propagation factor of the Fourier plane wave component in $\mathbf{u}_z$ direction. The planar dyadics $\overline{\overline{Z}}_n$, $\overline{\overline{Y}}_n$ can easily be seen to satisfy the relation

$$\overline{\overline{Z}}_n \cdot \overline{\overline{Y}}_n = \frac{1}{k_n^2 \beta_n^2}(k_n^4 \overline{\overline{I}}_t - k_n^2 \mathbf{KK} - k_n^2 \mathbf{u}_z \mathbf{u}_z {}^{\times}_{\times} \mathbf{KK}) = \overline{\overline{I}}_t, \tag{7.11}$$

because of the identity

$$\frac{\mathbf{KK}}{K^2} + \frac{\mathbf{u}_z \mathbf{u}_z {}^{\times}_{\times} \mathbf{KK}}{K^2} = \overline{\overline{I}} - \mathbf{u}_z \mathbf{u}_z = \overline{\overline{I}}_t. \tag{7.12}$$

$\overline{\overline{Y}}_n$ is the two-dimensional inverse of $\overline{\overline{Z}}_n$ and conversely. Thus, we can write

$$\overline{\overline{Y}}_n = \frac{1}{\eta_n^2}\mathbf{u}_z \mathbf{u}_z {}^{\times}_{\times} \overline{\overline{Z}}_n = \overline{\overline{Z}}_n^{-1}, \quad \overline{\overline{Z}}_n = \eta_n^2 \mathbf{u}_z \mathbf{u}_z {}^{\times}_{\times} \overline{\overline{Y}}_n = \overline{\overline{Y}}_n^{-1}, \tag{7.13}$$

$$\mathrm{spm}\overline{\overline{Z}}_n = \frac{1}{\mathrm{spm}\overline{\overline{Y}}_n} = \eta_n^2, \tag{7.14}$$

in accord with the expression for the inverse of two-dimensional dyadics as discussed in Chapter 2.

The equations (7.7), (7.8) are obviously vector counterparts of transmission-line equations with the propagation factor β_n in the z direction. A comparison can be made with the scalar transmission-line equations

$$U'(z) + j\beta Z_c I(z) = u(z), \tag{7.15}$$

$$I'(z) + j\beta Y_c U(z) = i(z), \tag{7.16}$$

with characteristic impedance $Z_c = 1/Y_c$ and propagation factor β, as well as distributed generator voltage u and current i quantities.

This shows us that in the present case, $\mathbf{e}_n$ can be interpreted as a vector voltage and $-\mathbf{u}_z \times \mathbf{h}_n$ as a vector current. The characteristic impedance of the vector transmission line is a dyadic quantity $\overline{\overline{Z}}_n$.

Eliminating the vector $\mathbf{u}_z \times \mathbf{h}_n$, an equation for $\mathbf{e}_n$ can be derived from (7.7), (7.8):

$$\mathbf{e}_n'' + \beta_n^2 \mathbf{e}_n = \mathbf{s}_n, \tag{7.17}$$

with the distributed source function

$$\mathbf{s}_n = j\frac{\eta_n}{k_n}(k_n^2 \overline{\overline{I}}_t - \mathbf{KK}) \cdot \mathbf{j}_{en} + \frac{\eta_n}{k_n}\mathbf{K} j_{en}' + \mathbf{u}_z \times \mathbf{j}_{mn}' + j(\mathbf{u}_z \times \mathbf{K}) j_{mn}. \tag{7.18}$$

Likewise, we may derive

$$\mathbf{h}_n'' + \beta_n^2 \mathbf{h}_n = \mathbf{s}_{mn}, \tag{7.19}$$

with the distributed magnetic source function

$$\mathbf{s}_{mn} = -\mathbf{u}_z \times \mathbf{j}_{en}' - j(\mathbf{u}_z \times \mathbf{K}) j_{en} + \frac{j}{k_n \eta_n}(k_n^2 \overline{\overline{I}}_t - \mathbf{KK}) \cdot \mathbf{j}_{mn} + \frac{1}{k_n \eta_n}\mathbf{K} j_{mn}'. \tag{7.20}$$

Alternatively, these complicated-looking source expressions can be derived directly from the Helmholtz equation expressions, in homogeneous space outside the sources. For the electric field we can write

$$(\nabla^2 + k^2)\mathbf{E} = j\omega\mu\left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right) \cdot \mathbf{J} + \nabla \times \mathbf{J}_m. \tag{7.21}$$

In fact, Fourier transforming the transverse component of this equation gives us exactly the vector transmission-line equation (7.17) with the expression (7.18) for the source above. In the same manner, the magnetic source is obtained from the magnetic field expression. The sources $\mathbf{s}$ and $\mathbf{s}_m$ can be seen to correspond to each other through the duality transformation: $\mathbf{s}_d = j\eta \mathbf{s}_m$.

Reflection and transmission

Solutions of these equations in each region n can be written in terms of two waves propagating in opposite directions:

$$\mathbf{e}_n = \mathbf{a}_n^+ e^{-j\beta_n z} + \mathbf{a}_n^- e^{j\beta_n z}, \tag{7.22}$$

$$-\mathbf{u}_z \times \mathbf{h}_n = \mathbf{b}_n^+ e^{-j\beta_n z} + \mathbf{b}_n^- e^{j\beta_n z}. \tag{7.23}$$

For a wave propagating outside the sources in medium n there is a linear relation between the electric and magnetic field amplitudes. In fact, we can write from (7.7), (7.8) for sourceless regions

$$\mathbf{b}_n^+ = \overline{\overline{Y}}_n \cdot \mathbf{a}_n^+, \quad \mathbf{b}_n^- = -\overline{\overline{Y}}_n \cdot \mathbf{a}_n^-, \tag{7.24}$$

$$\mathbf{a}_n^+ = \overline{\overline{Z}}_n \cdot \mathbf{b}_n^+, \quad \mathbf{a}_n^- = -\overline{\overline{Z}}_n \cdot \mathbf{b}_n^-, \tag{7.25}$$

with $\overline{\overline{Y}}_n$, $\overline{\overline{Z}}_n$ defined in (7.10).

The amplitude vectors depend on the sources and the boundary conditions. At the interfaces, there is the continuity condition for the tangential fields. For example, at the boundary between media 1 and 2 at $z = d$ we can write

$$\mathbf{a}_1^+ e^{-j\beta_1 d} + \mathbf{a}_1^- e^{j\beta_1 d} = \mathbf{a}_2^+ e^{-j\beta_2 d} + \mathbf{a}_2^- e^{j\beta_2 d}, \tag{7.26}$$

$$\mathbf{b}_1^+ e^{-j\beta_1 d} + \mathbf{b}_1^- e^{j\beta_1 d} = \mathbf{b}_2^+ e^{-j\beta_2 d} + \mathbf{b}_2^- e^{j\beta_2 d}. \tag{7.27}$$

At the interface, the boundary condition gives rise to a linear relation for the four amplitude vectors $\mathbf{a}_1^+$, $\mathbf{a}_1^-$, $\mathbf{a}_2^+$, $\mathbf{a}_2^-$ of the form

$$\begin{pmatrix} \mathbf{a}_1^+ \\ \mathbf{a}_2^- \end{pmatrix} = \begin{pmatrix} \overline{\overline{R}}_{11} & \overline{\overline{T}}_{12} \\ \overline{\overline{T}}_{21} & \overline{\overline{R}}_{22} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_1^- \\ \mathbf{a}_2^+ \end{pmatrix}. \tag{7.28}$$

The reflection and transmission dyadics are easily obtained for $d = 0$ from the above relations in the form

$$\overline{\overline{R}}_{11} = -\overline{\overline{R}}_{22} = (\overline{\overline{Z}}_2 - \overline{\overline{Z}}_1) \cdot (\overline{\overline{Z}}_2 + \overline{\overline{Z}}_1)^{-1} =$$

$$(\overline{\overline{Y}}_1 + \overline{\overline{Y}}_2)^{-1} \cdot (\overline{\overline{Y}}_1 - \overline{\overline{Y}}_2), \tag{7.29}$$

$$\overline{\overline{T}}_{21} = \overline{\overline{I}}_t + \overline{\overline{R}}_{11} = 2\overline{\overline{Z}}_2 \cdot (\overline{\overline{Z}}_2 + \overline{\overline{Z}}_1)^{-1} = 2\overline{\overline{Y}}_1 \cdot (\overline{\overline{Y}}_1 + \overline{\overline{Y}}_2)^{-1}, \tag{7.30}$$

$$\overline{\overline{T}}_{12} = \overline{\overline{I}}_t + \overline{\overline{R}}_{22} = 2\overline{\overline{Z}}_1 \cdot (\overline{\overline{Z}}_2 + \overline{\overline{Z}}_1)^{-1} = 2\overline{\overline{Y}}_2 \cdot (\overline{\overline{Y}}_1 + \overline{\overline{Y}}_2)^{-1}. \tag{7.31}$$

If, instead of $z = 0$, the interface is at $z = d$, there are additional phase factors in the coefficient dyadics:

$$\overline{\overline{R}}_{11}(d) = \overline{\overline{R}}_{11}(0) e^{j\beta_1 2d}, \tag{7.32}$$

$$\overline{\overline{R}}_{22}(d) = \overline{\overline{R}}_{22}(0)e^{-j\beta_2 2d}, \tag{7.33}$$

$$\overline{\overline{T}}_{12}(d) = \overline{\overline{T}}_{12}(0)e^{-j(\beta_1-\beta_2)d}, \tag{7.34}$$

$$\overline{\overline{T}}_{21}(d) = \overline{\overline{T}}_{21}(0)e^{-j(\beta_1-\beta_2)d}. \tag{7.35}$$

The reflection dyadic expression (7.29) can be generalized to the case where, instead of the medium 2, in the half space $z < 0$ there is a stratified structure of some kind and in the half space $z > 0$, the homogeneous medium 1, if only we know the 'loading' admittance dyadic $\overline{\overline{Y}}_L$ due to the stratified structure, which is defined through the tangential fields at $z = 0$ by

$$-\mathbf{u}_z \times \mathbf{h} = -\overline{\overline{Y}}_L \cdot \mathbf{e}, \quad \text{or} \quad \mathbf{e} = -\overline{\overline{Z}}_L \cdot (-\mathbf{u}_z \times \mathbf{h}), \tag{7.36}$$

with $\overline{\overline{Z}}_L = \overline{\overline{Y}}_L^{-1}$. In fact, from reasoning similar to that above and $\mathbf{a}_1^+ = \overline{\overline{R}} \cdot \mathbf{a}_1^-$, we can write

$$\mathbf{a}_1^+ + \mathbf{a}_1^- = (\overline{\overline{R}} + \overline{\overline{I}}_t) \cdot \mathbf{a}_1^- = -\overline{\overline{Y}}_L^{-1} \cdot (\mathbf{b}_1^+ + \mathbf{b}_1^-) = -\overline{\overline{Y}}_L^{-1} \cdot \overline{\overline{Y}}_1 \cdot (\overline{\overline{R}} - \overline{\overline{I}}_t) \cdot \mathbf{a}_1^-, \tag{7.37}$$

from which $\overline{\overline{R}}$ can be solved in the form

$$\overline{\overline{R}} = (\overline{\overline{Y}}_1 + \overline{\overline{Y}}_L)^{-1} \cdot (\overline{\overline{Y}}_1 - \overline{\overline{Y}}_L) = (\overline{\overline{Z}}_L + \overline{\overline{Z}}_1)^{-1} \cdot (\overline{\overline{Z}}_L - \overline{\overline{Z}}_1). \tag{7.38}$$

This expression is the same as (7.29) above if we identify $\overline{\overline{Y}}_L$ with $\overline{\overline{Y}}_2$, but the last expression is only valid if the impedance dyadics $\overline{\overline{Z}}_1$ and $\overline{\overline{Z}}_L$ commute. In the more general case (for example, anisotropic loading impedance) the impedance expression reads

$$\overline{\overline{R}} = \overline{\overline{Z}}_1 \cdot (\overline{\overline{Z}}_L + \overline{\overline{Z}}_1)^{-1} \cdot (\overline{\overline{Z}}_L - \overline{\overline{Z}}_1) \cdot \overline{\overline{Z}}_1^{-1}. \tag{7.39}$$

By substituting the free-space admittance or impedance dyadics (7.10) for $\overline{\overline{Y}}_n$, $\overline{\overline{Z}}_n$ we have

$$\overline{\overline{Y}}_n = \frac{k_n}{\beta_n \eta_n} \frac{\mathbf{K}\mathbf{K}}{K^2} + \frac{\beta_n}{k_n \eta_n} \frac{\mathbf{u}_z \mathbf{u}_z {}^\times_\times \mathbf{K}\mathbf{K}}{K^2}, \tag{7.40}$$

$$\overline{\overline{Z}}_n = \frac{\beta_n \eta_n}{k_n} \frac{\mathbf{K}\mathbf{K}}{K^2} + \frac{k_n \eta_n}{\beta_n} \frac{\mathbf{u}_z \mathbf{u}_z {}^\times_\times \mathbf{K}\mathbf{K}}{K^2}. \tag{7.41}$$

Thus, the eigenvectors of these two-dimensional dyadics are $\mathbf{K}$, and $\mathbf{u}_z \times \mathbf{K}$, which correspond to TM and TE fields, respectively. (To check this, we may note that, outside of sources, from the divergence condition $\nabla \cdot \mathbf{E} = 0$, for a TE field in Fourier space we have $\mathbf{K} \cdot \mathbf{e} = 0$, implying that

$\mathbf{e}$ must be parallel to the vector $\mathbf{u}_z \times \mathbf{K}$. Analogously, for a TM field we have $\mathbf{u}_z \cdot \mathbf{h} = 0$ or $\mathbf{u}_z \times \mathbf{K} \cdot \mathbf{e} = 0$, whence $\mathbf{e}$ must be parallel to $\mathbf{K}$.)

The eigenvectors $\mathbf{K}$ and $\mathbf{u}_z \times \mathbf{K}$ are also shared by the two-dimensional reflection and transmission dyadics, provided the admittance and impedance dyadics of all media and impedance boundaries have the same eigenvectors. In this case, these symmetric dyadics can be written in terms of their eigenvalues as

$$\overline{\overline{R}} = R^{TM}\frac{\mathbf{KK}}{K^2} + R^{TE}\frac{\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times} \mathbf{KK}}{K^2}, \tag{7.42}$$

$$\overline{\overline{T}} = T^{TM}\frac{\mathbf{KK}}{K^2} + T^{TE}\frac{\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times} \mathbf{KK}}{K^2}. \tag{7.43}$$

For example, substituting (7.41) in (7.38), the reflection dyadic associated with an interface of two media takes on the form

$$\overline{\overline{R}}_{11} = \frac{\frac{\beta_2\eta_2}{k_2} - \frac{\beta_1\eta_1}{k_1}}{\frac{\beta_2\eta_2}{k_2} + \frac{\beta_1\eta_1}{k_1}}\frac{\mathbf{KK}}{K^2} + \frac{\frac{k_2\eta_2}{\beta_2} - \frac{k_1\eta_1}{\beta_1}}{\frac{k_2\eta_2}{\beta_2} + \frac{k_1\eta_1}{\beta_1}}\frac{\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times} \mathbf{KK}}{K^2}. \tag{7.44}$$

The problem can actually be handled in scalar form if we make the TE/TM decomposition to the original source, since the two polarizations do not couple to each other except if there are anisotropic media or surface impedances which possess other eigenvectors.

7.1.2 Image functions for reflection fields

To find the fields in physical space one must perform the inverse Fourier transformation. The result, often called the Sommerfeld integral, cannot be written in closed form because the reflection and transmission coefficients are too complicated functions of the Fourier parameter $\mathbf{K}$. There have been attempts to approximate the reflection coefficients by simpler functions for which the inverse transformation can be found, for example by PARHAMI*et al.* (1980). However, the accuracy of the result depends on the approximations and cannot be easily predicted.

The Sommerfeld integrals, despite their long history, are still a challenge to computers, since the integrands in most cases are highly oscillating. The idea behind EIT is to represent the reflection coefficient functions first as certain integral transformations of other functions which allow the inverse Fourier transformation from $\mathbf{K}$ space to $\boldsymbol{\rho}$ space to be performed exactly and what remains is the new transformation integral. This may just seem to be a substitution of one integral by another, but because the result can be interpreted as integration of an image source, it gains quasi-physical insight, which is helpful in setting up equations. Also, there is some numerical advantage in this approach. Thus, it becomes an interesting problem to find image sources corresponding to different geometries.

This method was possibly first sketched, although not labelled in terms of images, for the classical half-space problem by BOOKER and CLEMMOW in 1950. In fact, their result can be interpreted in terms of an exponentially diverging image. More recently, KUESTER and CHANG (1979) and MOHSEN (1982) applied a similar idea by making approximations for the reflection coefficient functions. The converging exact form of the image principle was introduced in 1983 by LINDELL and ALANEN.

The EIT method can be described for any layered structures, although it was first applied to the Sommerfeld problem of two homogeneous half-spaces. Its success depends on whether the reflection coefficient functions can be represented analytically in terms of suitable integral transformations. The basic image functions can then be obtained by comparing the resulting field expression to the free-space field expression.

Electric source problem

The EIT image functions are obtained by comparing actual field expressions with those arising from a line current source along the z axis. The free-space transverse field arising from a dipole source $\mathbf{J}(\mathbf{r}) = \mathbf{v}IL\delta(\mathbf{r} - \mathbf{u}_z h)$, where $\mathbf{v}$ is a unit vector not necessarily equal to $\mathbf{u}_z$, in physical space, obeys the law

$$\mathbf{e}(\mathbf{r}) = -jk\eta\left(\overline{\overline{I}}_t + \frac{1}{k^2}\nabla_t\nabla\right)\cdot G(\mathbf{r} - \mathbf{u}_z h)\mathbf{v}IL. \tag{7.45}$$

Making the Fourier transformation leaves us with the expression

$$\mathbf{e}(\mathbf{K}, z) = -jk\eta\overline{\overline{O}}_e \cdot \frac{e^{-j\beta|z-h|}}{2j\beta}\mathbf{v}IL, \tag{7.46}$$

where the (electric) operator $\overline{\overline{O}}_e$ is defined as

$$\overline{\overline{O}}_e = \overline{\overline{I}}_t - \frac{\mathbf{KK}}{k^2} - j\frac{\mathbf{K}\mathbf{u}_z}{k^2}\frac{d}{dz}. \tag{7.47}$$

The corresponding field from a line current $\mathbf{I}(z) = \mathbf{v}I(z)$, extending from $z = -z_o$ to $-\infty$, can be written as the integral

$$\mathbf{e}(\mathbf{r}) = \overline{\overline{I}}_t \cdot \left[-jk\eta\left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right)\cdot\int_{z_o}^{\infty} G(\mathbf{r} + \mathbf{u}_z z')\mathbf{I}(z')dz'\right] \tag{7.48}$$

and in Fourier space in the region $z > -z_o$ with no sources:

$$\mathbf{e}(\mathbf{K}, z) = -jk\eta\overline{\overline{O}}_e \cdot \int_{z_o}^{\infty} \frac{e^{-j\beta(z+z')}}{2j\beta}\mathbf{I}(z')dz'. \tag{7.49}$$

Let us now try to compare the free-space field expression (7.49) with the expression of the reflected field in the homogeneous half-space $z > 0$, due to a dipole source at height h: $\mathbf{J}(\mathbf{r}) = \mathbf{v}IL\delta(\boldsymbol{\rho})\delta(z-h)$, when the stratified structure exists for $z < 0$. The reflected field denotes the difference of the total field and the field from the dipole in free space (the incident field). The effect of the stratified half space is characterized, in Fourier space, through the reflection dyadic $\overline{\overline{R}}$ at $z = 0$. Another analysis for the transmission field for the region $z < 0$ will be subsequently made with the transmission dyadic $\overline{\overline{T}}$ replacing $\overline{\overline{R}}$.

The field incident from the dipole to the stratified structure, in the region $h > z > 0$, is

$$\mathbf{e}_i(\mathbf{K}, z) = -jk\eta\overline{\overline{O}}_e \cdot \frac{e^{-j\beta h}}{2j\beta}\mathbf{v}ILe^{j\beta z} = \mathbf{a}^- e^{j\beta z}, \tag{7.50}$$

giving rise to the reflected field

$$\mathbf{e}_r(\mathbf{K}, z) = \mathbf{a}^+ e^{-j\beta z} = \overline{\overline{R}}\cdot\mathbf{a}^- e^{-j\beta z} = -jk\eta\overline{\overline{R}}\cdot\overline{\overline{O}}_e\cdot\overline{\overline{C}}\cdot\frac{e^{-j\beta(z+h)}}{2j\beta}\mathbf{v}IL. \tag{7.51}$$

The mirror reflection dyadic

$$\overline{\overline{C}} = \overline{\overline{I}} - 2\mathbf{u}_z\mathbf{u}_z \tag{7.52}$$

arises because the differentiation d/dz in the operator $\overline{\overline{O}}_e$ now operates on $\exp(-j\beta z)$ and not on $\exp(j\beta z)$.

To obtain an image source giving the exact reflected field (7.51) as radiating from a line source in free space, in the form (7.49), the operator $\overline{\overline{O}}_e$ should appear in front of the expression. Applying the eigenvector expansion of the reflection dyadic $\overline{\overline{R}}$, with eigenvectors $\mathbf{K}$ and $\mathbf{u}_z \times \mathbf{K}$ corresponding to the respective TM and TE polarizations, we may write

$$\left(R^{TE}\frac{\mathbf{u}_z\mathbf{u}_z{}^{\times}_{\times}\mathbf{K}\mathbf{K}}{K^2} + R^{TM}\frac{\mathbf{K}\mathbf{K}}{K^2}\right)\cdot\left(\overline{\overline{I}}_t - \frac{\mathbf{K}\mathbf{K}}{k^2} - j\frac{\mathbf{K}\mathbf{u}_z}{k^2}\frac{d}{dz}\right) =$$

$$\left(\overline{\overline{I}}_t - \frac{\mathbf{K}\mathbf{K}}{k^2} - j\frac{\mathbf{K}\mathbf{u}_z}{k^2}\frac{d}{dz}\right)\cdot\left(R^{TE}\frac{\mathbf{u}_z\mathbf{u}_z{}^{\times}_{\times}\mathbf{K}\mathbf{K}}{K^2} + R^{TM}\left(\frac{\mathbf{K}\mathbf{K}}{K^2} + \mathbf{u}_z\mathbf{u}_z\right)\right), \tag{7.53}$$

or

$$\overline{\overline{R}} \cdot \overline{\overline{O}}_e = \overline{\overline{O}}_e \cdot \overline{\overline{R}}_e, \tag{7.54}$$

with the modified (electric) reflection dyadic defined as

$$\overline{\overline{R}}_e = R^{TE} \frac{\mathbf{u}_z \mathbf{u}_z \overset{\times}{\times} \mathbf{KK}}{K^2} + R^{TM} \left(\frac{\mathbf{KK}}{K^2} + \mathbf{u}_z \mathbf{u}_z \right). \tag{7.55}$$

This in fact implies that the operator $\overline{\overline{O}}_e$ commutes with the modified reflection dyadic $\overline{\overline{R}}_e$.

Now we are able to write (7.51) in a form resembling that of (7.49):

$$\mathbf{e}_r(\mathbf{K}, z) = -jk\eta \overline{\overline{O}}_e \cdot \frac{e^{-j\beta(z+h)}}{2j\beta} \overline{\overline{R}}_e \cdot \overline{\overline{C}} \cdot \mathbf{v} IL, \tag{7.56}$$

allowing a representation for the reflected field as arising from an image current source.

Dyadic image function

The EIT method is based on an integral representation of the modified reflection dyadic, defined by a function $H(\zeta)$:

$$\overline{\overline{R}}_e(\mathbf{K}) = \int_0^\infty \overline{\overline{f}}_e(\mathbf{K}, \zeta) e^{-j\beta H(\zeta)} d\zeta. \tag{7.57}$$

Here, ζ is an integration variable and $H(\zeta)$ is a function of ζ but not of the Fourier parameter $\mathbf{K}$. The dyadic function $\overline{\overline{f}}_e(\mathbf{K}, \zeta)$ is called the electric dyadic image function and its functional form depends on the stratified structure in question. Note that for the moment we consider $\mathbf{K}$ and β as independent variables so that $\mathbf{K}$ only appears in the vector form. Thus, $\overline{\overline{R}}_e$ is a function of both β and $\mathbf{K}$, and $\overline{\overline{f}}_e$ depends on $\mathbf{K}$ but not on β.

If a representation of the above kind cannot be found, it is sufficient to define a more general form

$$\overline{\overline{R}}_e(\mathbf{K}) = \sum_{n=1}^{\infty} \int_0^\infty \overline{\overline{f}}_{en}(\mathbf{K}, \zeta) e^{-j\beta H_n(\zeta)} d\zeta, \tag{7.58}$$

which gives us the possibility of expressing the structure in terms of a series of image sources.

Inserting (7.57) in (7.56), its form becomes similar to that of (7.49):

$$\mathbf{e}_r(\mathbf{K}, z) = -jk\eta \overline{\overline{O}}_e \cdot \int_0^\infty \frac{e^{-j\beta(z+h+H(\zeta))}}{2j\beta} \overline{\overline{f}}_e(\mathbf{K}, \zeta) \cdot \overline{\overline{C}} \cdot \mathbf{v} IL d\zeta. \tag{7.59}$$

Defining the following relation between the two integration variables z' and ζ:

$$z' = h + H(\zeta), \tag{7.60}$$

we can finally identify the image current function in the Fourier space in the form

$$\mathbf{I}(z') = \frac{1}{H'(\zeta)} \overline{\overline{f}}_e(\mathbf{K}, \zeta) \cdot \overline{\overline{C}} \cdot \mathbf{v} IL. \tag{7.61}$$

The inverse Fourier transformation to the transverse component of the reflection field can now be performed exactly, from which the total reflected electric field can be obtained in the form

$$\mathbf{E}_r(\mathbf{r}) = -jk\eta \left(\overline{\overline{I}} + \frac{1}{k^2} \nabla\nabla \right) \cdot \int_V \int_0^\infty G(\mathbf{r} - \mathbf{r}' + \mathbf{u}_z H(\zeta)) \mathbf{J}_i(\mathbf{r}', \zeta) dV' d\zeta. \tag{7.62}$$

Defining the mirror reflection dyadic $\overline{\overline{C}}$ by (7.52), the image source corresponding to the original dipole can obviously now be written in physical space as

$$\mathbf{J}_i(\mathbf{r}, \zeta) = \overline{\overline{f}}_e(j\nabla_t, \zeta) \cdot \overline{\overline{C}} \cdot \mathbf{v} IL\delta(z + h)\delta(\rho). \tag{7.63}$$

Thus, the functional form of the image source depends on the dyadic image function $\overline{\overline{f}}_e$, which again depends on the geometry of the problem. It is obvious from linearity that for an original volume source $\mathbf{J}(\mathbf{r})$ the image source has the general form

$$\mathbf{J}_i(\mathbf{r}, \zeta) = \overline{\overline{f}}_e(j\nabla_t, \zeta) \cdot \mathbf{J}_c(\mathbf{r}), \tag{7.64}$$

with the mirror image of the original source defined as

$$\mathbf{J}_c(\mathbf{r}) = \overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r}). \tag{7.65}$$

The dyadic image function is actually an operator operating on the mirror image of the primary source. The field from the image source must be calculated as a fourfold integral over the three space coordinates and the parameter ζ. For a point source, the image is a line source, which starts from the mirror image point at $z = -h$ and proceeds along a line parallel to the z axis:

$$z = -h - H(\zeta). \tag{7.66}$$

In many cases, $H(\zeta)$ may take complex values so that the image line lies in complex space. This does not, however, essentially complicate the field calculation process.

Scalar image functions

Since the reflection dyadic can be written in terms of its eigenvalues, the dyadic image function can also be written in terms of certain scalar functions. In fact, let us write the modified reflection dyadic (7.55) as

$$\overline{\overline{R}}_e = R^{TE}\frac{\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times}\mathbf{KK}}{K^2} + R^{TM}\frac{\mathbf{KK}}{K^2} + R^{TM}\mathbf{u}_z\mathbf{u}_z =$$

$$R^{TM}\overline{\overline{I}} - R_o\frac{1}{k^2}\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times}\mathbf{KK}, \tag{7.67}$$

with the third reflection coefficient

$$R_o = \frac{k^2}{K^2}(R^{TM} - R^{TE}). \tag{7.68}$$

For each scalar reflection coefficient function, corresponding scalar image functions $f^{TE}(\zeta)$, $f^{TM}(\zeta)$ and $f_o(\zeta)$ can be defined through

$$R^{TE}(\beta) = \int_0^\infty f^{TE}(\zeta)e^{-j\beta H(\zeta)}d\zeta, \tag{7.69}$$

$$R^{TM}(\beta) = \int_0^\infty f^{TM}(\zeta)e^{-j\beta H(\zeta)}d\zeta, \tag{7.70}$$

$$R_o(\beta) = \int_0^\infty f_o(\zeta)e^{-j\beta H(\zeta)}d\zeta. \tag{7.71}$$

In terms of these scalar functions, the dyadic image operator can be represented as

$$\overline{\overline{f}}_e(j\nabla_t, \zeta) = f^{TM}(\zeta)\overline{\overline{I}} + f_o(\zeta)\frac{1}{k^2}\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times}\nabla\nabla. \tag{7.72}$$

Thus, the image source can be explicitly written as

$$\mathbf{J}_i(\mathbf{r}, \zeta) = \left(f^{TM}(\zeta)\overline{\overline{I}} + \frac{1}{k^2}f_o(\zeta)\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times}\nabla\nabla\right)\cdot\mathbf{J}_c(\mathbf{r}). \tag{7.73}$$

This form is not unique, since the equivalent source is not unique. Other possible forms for the image source will be given subsequently.

Duality transformations

The image functions have certain relations through the simple duality transformation (Section 4.2.1), following from corresponding relations between the reflection coefficients. The duality transformation of the reflection dyadic can be obtained from a consideration of incident and reflected fields, related through

$$\mathbf{e}^+ = \overline{\overline{R}} \cdot \mathbf{e}^-. \tag{7.74}$$

The dual of this relation can be written

$$\mathbf{h}^+ = \overline{\overline{R}}_d \cdot \mathbf{h}^-, \quad \text{or} \quad -\mathbf{u}_z \times \mathbf{h}^+ = (\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times} \overline{\overline{R}}_d) \cdot (-\mathbf{u}_z \times \mathbf{h}^-). \tag{7.75}$$

Inserting $-\mathbf{u}_z \times \mathbf{h}^\pm = \pm\overline{\overline{Y}} \cdot \mathbf{e}^\pm$ leaves us with the relation

$$\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times} \overline{\overline{R}}_d = -\overline{\overline{Y}} \cdot \overline{\overline{R}} \cdot \overline{\overline{Y}}^{-1}. \tag{7.76}$$

This takes a simple form provided the eigenvectors of $\overline{\overline{Y}}$ and $\overline{\overline{R}}$ are the same, which is the case for example in isotropic problems, because the dyadics then commute. Thus, in this case, the relation reads

$$\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times} \overline{\overline{R}}_d = -\overline{\overline{R}} \quad \text{or} \quad \overline{\overline{R}}_d = -\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times} \overline{\overline{R}}. \tag{7.77}$$

This corresponds to the following duality relations between the scalar reflection coefficients:

$$R_d^{TE} = -R^{TM}, \qquad R_d^{TM} = -R^{TE}, \qquad R_{od} = \frac{k^2}{K^2}(R_d^{TM} - R_d^{TE}) = R_o. \tag{7.78}$$

These, finally, lead to duality relations between the image functions:

$$f_d^{TE}(\zeta) = -f^{TM}(\zeta), \qquad f_d^{TM}(\zeta) = -f^{TE}(\zeta), \qquad f_{od}(\zeta) = f_o(\zeta). \tag{7.79}$$

Note that the dual of a dual always gives the original quantity.

Thus, in dealing with the general isotropic medium, we need only derive, for example, the functions f^{TE}, f_o since f^{TM} can be obtained through the duality transformation. However, if the analysis is limited to, for example, dielectric media, this property cannot be applied.

Magnetic source problem

The duality transformation analysis can be applied for the construction of the image of a magnetic current source $\mathbf{J}_m(\mathbf{r})$. In fact, making the duality transformation for the image expression of the electric source (7.64) gives us

$$\mathbf{J}_{mi}(\mathbf{r}, \zeta) = \overline{\overline{f}}_m(j\nabla_t, \zeta) \cdot \mathbf{J}_{mc}(\mathbf{r}), \tag{7.80}$$

with

$$\overline{\overline{f}}_m(j\nabla_t, \zeta) = \overline{\overline{f}}_{ed}(j\nabla_t, \zeta) = -f^{TE}(\zeta)\overline{\overline{I}} + f_o(\zeta)\frac{1}{k^2}\mathbf{u}_z\mathbf{u}_z \substack{\times \\ \times} \nabla\nabla. \tag{7.81}$$

The total reflection field due to the magnetic image source can written similarly to that in (7.62):

$$\mathbf{E}_r(\mathbf{r}) = -\nabla \times \int_V \int_0^\infty G(\mathbf{r} - \mathbf{r}' + \mathbf{u}_z H(\zeta))\mathbf{J}_{mi}(\mathbf{r}', \zeta) dV' d\zeta. \tag{7.82}$$

Again, the image line corresponding to a magnetic point source lies on a line parallel to the z axis:

$$z = -h - H(\zeta), \tag{7.83}$$

extending from the mirror image point $z = -h$ to infinity, probably on the complex z plane.

Further properties of the image functions

The three image functions $f^{TE}(\zeta)$, $f^{TM}(\zeta)$ and $f_o(\zeta)$ depend on the reflection coefficient functions and, thus, on the geometry of the problem. This being the case, they must be determined for each problem separately. However, they share certain general properties in all problems.

Since there exists a relation between the three reflection coefficients R^{TE}, R^{TM} and R_o, we might expect a relation between the corresponding image functions, aside from that due to the duality properties. To find this, we write $K^2 = k^2 - \beta^2$, and apply the relation

$$K^2 R_o = k^2 R_o - \beta^2 R_o = k^2(R^{TM} - R^{TE}), \tag{7.84}$$

which leads to

$$\int_0^\infty k^2 f_o(\zeta) e^{-j\beta H(\zeta)} d\zeta - \int_0^\infty f_o(\zeta)\beta^2 e^{-j\beta H(\zeta)} d\zeta =$$

$$\int_0^\infty k^2 [f^{TM}(\zeta) - f^{TE}(\zeta)] e^{-j\beta H(\zeta)} d\zeta. \tag{7.85}$$

Since all the image functions are zero together with their derivatives at $\zeta \leq 0$, we can easily make partial integrations in the second integral term and discard the terms at the endpoint $\zeta = 0$. Also, we may anticipate functions vanishing at infinity, whence the corresponding endpoint terms are also discarded. Thus, equating the functions in the transformation integrands we finally have the resulting relation for the image function $f_o(\zeta)$ in the form of a differential equation, the prime denoting differentiation with respect to ζ:

$$\left(\frac{1}{H'(\zeta)}\left(\frac{f_o(\zeta)}{H'(\zeta)}\right)'\right)' + k^2 f_o(\zeta) = k^2[f^{TM}(\zeta) - f^{TE}(\zeta)]. \tag{7.86}$$

For the special case $H(\zeta) = \zeta$ this becomes simply

$$f_o''(\zeta) + k^2 f_o(\zeta) = k^2[f^{TM}(\zeta) - f^{TE}(\zeta)], \tag{7.87}$$

with the boundary conditions $f_o(0) = f_o(\infty) = 0$.

Another form of this condition can be simply written after inverse Fourier transformation as the operator equation, which is valid inside a field integral, where ∇_t operates on the current function:

$$-f_o(\zeta)\nabla_t^2 = k^2[f^{TM}(\zeta) - f^{TE}(\zeta)]. \tag{7.88}$$

This together with the following identity

$$\mathbf{u}_z\mathbf{u}_z {}^\times_\times \nabla\nabla + \nabla\nabla_t + (\nabla^2 + k^2)\mathbf{u}_z\mathbf{u}_z = \nabla_t^2\overline{\overline{I}} + \mathbf{u}_z\mathbf{u}_z \cdot (\nabla\nabla + k^2\overline{\overline{I}}), \tag{7.89}$$

which is easy to verify, gives us the possibility of transforming the dyadic image operator function $\overline{\overline{f}}_e$ given in (7.72) in the form

$$\overline{\overline{f}}_e(j\nabla_t, \zeta) = f^{TM}(\zeta)\overline{\overline{I}} + f_o(\zeta)\frac{1}{k^2}\mathbf{u}_z\mathbf{u}_z {}^\times_\times \nabla\nabla, \tag{7.90}$$

in a different form

$$\overline{\overline{f}}_e(j\nabla_t, \zeta) = f^{TE}(\zeta)\overline{\overline{I}} + f_o(\zeta)\mathbf{u}_z\mathbf{u}_z \cdot (\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla). \tag{7.91}$$

To arrive at this expression, one must note that the terms with $\nabla\nabla_t$ and $\nabla^2 + k^2$ in the image source function result in NR image components and can be omitted, as explained in Chapter 6. Further, we can easily develop other possible forms, for example:

$$\overline{\overline{f}}_e(j\nabla_t, \zeta) = f^{TE}(\zeta)\overline{\overline{I}}_t + \mathbf{u}_z\mathbf{u}_z \cdot \left(f^{TM}(\zeta)\overline{\overline{I}} + \frac{1}{k^2}f_o(\zeta)\nabla\nabla_t\right). \tag{7.92}$$

For the magnetic dyadic image operator we can correspondingly write from duality

$$\overline{\overline{f}}_m(j\nabla_t, \zeta) = -f^{TE}(\zeta)\overline{\overline{I}} + f_o(\zeta)\frac{1}{k^2}\mathbf{u}_z\mathbf{u}_z {}^\times_\times \nabla\nabla, \tag{7.93}$$

$$\overline{\overline{f}}_m(j\nabla_t,\zeta) = -f^{TM}(\zeta)\overline{\overline{I}} + f_o(\zeta)\mathbf{u}_z\mathbf{u}_z \cdot (\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla), \tag{7.94}$$

$$\overline{\overline{f}}_m(j\nabla_t,\zeta) = f^{TM}(\zeta)\overline{\overline{I}}_t - \mathbf{u}_z\mathbf{u}_z \cdot \left(f^{TE}(\zeta)\overline{\overline{I}} + \frac{1}{k^2}f_o(\zeta)\nabla\nabla_t\right). \tag{7.95}$$

With any of the above expressions for the dyadic image operators, the image sources can be compactly expressed as

$$J_i(\mathbf{r},\zeta) = \overline{\overline{f}}_e(j\nabla_t,\zeta)\cdot\mathbf{J}_c(\mathbf{r}), \tag{7.96}$$

$$J_{mi}(\mathbf{r},\zeta) = \overline{\overline{f}}_m(j\nabla_t,\zeta)\cdot\mathbf{J}_{mc}(\mathbf{r}). \tag{7.97}$$

The difference of the electric and magnetic dyadic image operators is seen from previous expressions to be

$$\overline{\overline{f}}_e(j\nabla_t,\zeta) - \overline{\overline{f}}_m(j\nabla_t,\zeta) = [f^{TE}(\zeta) + f^{TM}(\zeta)]\overline{\overline{I}}, \tag{7.98}$$

which shows that the anti-self-dual part of the $\overline{\overline{f}}_e$ operator is just a function of ζ times the unit dyadic. Correspondingly, we have

$$\overline{\overline{f}}_e(j\nabla_t,\zeta) + \overline{\overline{f}}_m(j\nabla_t,\zeta) = \frac{1}{k^2}f_o(\zeta)[2\mathbf{u}_z\mathbf{u}_z{}^{\times}_{\times}\nabla\nabla - \nabla_t^2\overline{\overline{I}}] =$$

$$\frac{2}{k^2}f_o(\zeta)\mathbf{u}_z\mathbf{u}_z{}^{\times}_{\times}\nabla\nabla + [f^{TM}(\zeta) - f^{TE}(\zeta)]\overline{\overline{I}}, \tag{7.99}$$

for twice the self-dual part of $\overline{\overline{f}}_e$.

7.1.3 Image functions for transmission fields

The EIT method can also be applied to problems of fields traversing a stratified layer. Let us assume that the transverse electric field (Fourier component) penetrating into medium 2 through a layer of any kind from medium 1 is

$$\mathbf{e}_T(\mathbf{K},z) = \mathbf{a}_2^- e^{j\beta_2 z} = \overline{\overline{T}}\cdot\mathbf{a}_1^- e^{j\beta_2 z}. \tag{7.100}$$

For simplicity in notation, the thickness of the layer in between the media 1 and 2 is taken as zero. This can be understood as a convention for the coordinates in the two media: medium 1 is defined for $z > 0$ and medium 2 for $z < 0$. All information on effects due to the layer on transmitted waves is included in the transmission dyadic $\overline{\overline{T}}$.

Let us follow a similar line of reasoning as in the derivation of the reflection image expressions. The transmitted field can be written in Fourier space as

$$\mathbf{e}_T(\mathbf{K},z) = \overline{\overline{T}}\cdot\left(-jk_1\eta_1\overline{\overline{O}}_{e1}\cdot\frac{e^{j\beta_2 z}e^{-j\beta_1 h}}{2j\beta_1}\mathbf{v}IL\right), \tag{7.101}$$

in analogy with (7.50), (7.51). The indices 1 and 2 denote parameters corresponding to media 1 and 2. The dyadic $\overline{\overline{O}}_{e1}$ in (7.47), when operating on the incident field $\mathbf{a}_1^- e^{j\beta_1 z}$, can be written as

$$\overline{\overline{O}}_{e1} = \overline{\overline{I}}_t - \frac{\mathbf{KK}}{k_1^2} + \beta_1 \frac{\mathbf{Ku}_z}{k_1^2}. \tag{7.102}$$

Since the transmission image source must be in homogeous space filled with material 2, we should be able to write (7.101) in the form resembling (7.56)

$$\mathbf{e}_T(\mathbf{K}, z) = -jk_2\eta_2 \overline{\overline{O}}_{e2} \cdot \frac{e^{j\beta_2 z}}{2j\beta_2} \overline{\overline{T}}_e e^{-j\beta_1 h} \cdot \mathbf{v}IL. \tag{7.103}$$

It is seen that the exponential term containing the 'wrong' β_1 coefficient must be treated otherwise than in the reflection problem, unless the media 1 and 2 are the same. The modified transmission dyadic turns out to have quite a complicated-looking appearance in the general case:

$$\overline{\overline{T}}_e = \frac{\epsilon_2}{\epsilon_1} T^{TM} \left(\frac{\beta_1}{\beta_2} \frac{\mathbf{KK}}{K^2} + \mathbf{u}_z \mathbf{u}_z \right) + \frac{\mu_1 \beta_2}{\mu_2 \beta_1} T^{TE} \frac{\mathbf{u}_z \mathbf{u}_z {}^{\times}_{\times} \mathbf{KK}}{K^2} =$$

$$T_u \mathbf{u}_z \mathbf{u}_z + T_I \overline{\overline{I}}_t + T_o \frac{\mathbf{KK}}{k_1^2}, \tag{7.104}$$

with

$$T_u = \frac{\epsilon_2}{\epsilon_1} T^{TM}, \tag{7.105}$$

$$T_I = \frac{\mu_1 \beta_2}{\mu_2 \beta_1} T^{TE}, \tag{7.106}$$

$$T_o = \left(\frac{\epsilon_2 \beta_1}{\epsilon_1 \beta_2} T^{TM} - \frac{\mu_1 \beta_2}{\mu_2 \beta_1} T^{TE} \right) \frac{k_1^2}{K^2}. \tag{7.107}$$

It can be argued that, in T_o, K^2 in the denominator is cancelled by the preceding bracketed factor for any layer. In fact, if the term in brackets is written as a Taylor expansion according to powers of K, it is clear that the series only contains even powers, because the terms are independent of the sign of K (actually they are independent of the direction of the vector $\mathbf{K}$). Thus, the series starts by $A + BK^2 + ...$ Because for $K = 0$ (normal plane-wave incidence), the TE and TM polarizations are both TEM, the transmission coefficients are actually the same, and from $\beta_1 = k_1$ and $\beta_2 = k_2$ the bracketed term can easily be shown to vanish. Thus, $A = 0$ and K^2 cancels out.

Transmission into a similar medium

We can now define the transmission dyadic image function $\overline{\overline{h}}_e$ analogously with (7.57). For the special case when the media 1 and 2 are the same, whence $k_1 = k_2 = k$, $\beta_1 = \beta_2 = \beta$ and $T_u = T^{TM}$, $T_I = T^{TE}$, we have

$$\overline{\overline{T}}_e = \int_0^\infty \overline{\overline{h}}_e(\mathbf{K}, \zeta) e^{-j\beta H(\zeta)} d\zeta, \tag{7.108}$$

with

$$\overline{\overline{h}}_e(\mathbf{K}, \zeta) = h^{TM}(\zeta)\overline{\overline{I}} - h_o(\zeta)\frac{1}{k^2}\mathbf{u}_z\mathbf{u}_z {}^\times_\times \mathbf{K}\mathbf{K}, \tag{7.109}$$

$$T^{TE}(\beta) = \int_0^\infty h^{TE}(\zeta) e^{-j\beta H(\zeta)} d\zeta, \tag{7.110}$$

$$T^{TM}(\beta) = \int_0^\infty h^{TM}(\zeta) e^{-j\beta H(\zeta)} d\zeta, \tag{7.111}$$

$$T_o(\beta) = \frac{k^2}{K^2}(T^{TM} - T^{TE}) = \int_0^\infty h_o(\zeta) e^{-j\beta H(\zeta)} d\zeta. \tag{7.112}$$

Transmission into a different medium

For the general case, the previous expressions do not work and, instead, we must define a dyadic image function based not on the transmission dyadic $\overline{\overline{T}}_e$ but on $\overline{\overline{T}}_e e^{-j\beta_1 h}$, which means that the image functions also become functions of the distance parameter h:

$$\overline{\overline{T}}_e e^{-j\beta_1 h} = \int_0^\infty \overline{\overline{h}}_e(\mathbf{K}, h, \zeta) e^{-j\beta_2 H(h,\zeta)} d\zeta. \tag{7.113}$$

Applying the expression (7.104) in the form

$$\overline{\overline{T}}_e = T_u \mathbf{u}_z\mathbf{u}_z + T_I \overline{\overline{I}}_t - \beta_2 T_o \frac{\mathbf{u}_z\mathbf{K}}{k_1^2} + (\beta_2\mathbf{u}_z + \mathbf{K}) T_o \frac{\mathbf{K}}{k_1^2}, \tag{7.114}$$

the dyadic image function can be expressed in terms of three scalar image functions, in the corresponding form

$$\overline{\overline{h}}_e(\mathbf{K}, h, \zeta) = h_u(h, \zeta)\mathbf{u}_z\mathbf{u}_z + h_I(h, \zeta)\overline{\overline{I}}_t -$$

$$h_0(h,\zeta)\frac{\mathbf{u}_z\mathbf{K}}{k_1^2} + (\beta_2\mathbf{u}_z + \mathbf{K})h_0(h,\zeta)\frac{\mathbf{K}}{\beta_2 k_1^2}. \tag{7.115}$$

The scalar transmission image functions are defined through

$$T_u e^{-j\beta_1 h} = \frac{\epsilon_2}{\epsilon_1}T^{TM}e^{-j\beta_1 h} = \int_0^\infty h_u(h,\zeta)e^{-j\beta_2 H(h,\zeta)}d\zeta, \tag{7.116}$$

$$T_I e^{-j\beta_1 h} = \frac{\mu_1\beta_2}{\mu_2\beta_1}T^{TE}e^{-j\beta_1 h} = \int_0^\infty h_I(h,\zeta)e^{-j\beta_2 H(h,\zeta)}d\zeta, \tag{7.117}$$

$$T_0 e^{-j\beta_1 h} = \beta_2 T_o e^{-j\beta_1 h} = \int_0^\infty h_0(h,\zeta)e^{-j\beta_2 H(h,\zeta)}d\zeta. \tag{7.118}$$

Other forms for the transmission image dyadic are also possible. The last term in (7.115) will be omitted, because a term starting with $-j\mathbf{K}-j\beta_2\mathbf{u}_z$, or in physical space with ∇, corresponds to an NR source. Thus, we can neglect the corresponding term in the transmission dyadic as well, and apply the expression

$$\overline{\overline{T}}_e = T_u\mathbf{u}_z\mathbf{u}_z + T_I\overline{\overline{I}}_t - \beta_2 T_o\frac{\mathbf{u}_z\mathbf{K}}{k_1^2}. \tag{7.119}$$

Also, note the difference between the coefficients T_o and $T_0 = \beta_2 T_o$.

Application of this expansion leads to a transmission image theory in which the form of the image function corresponding to a point source is dependent on the position of the source, which was not the case for the reflection image. In fact, writing the transverse component of the transmitted electric field in the Fourier space

$$\mathbf{e}_T(\mathbf{K},z) = -jk_2\eta_2\overline{\overline{O}}_{e2}\cdot\frac{e^{j\beta_2 z}}{2j\beta_2}\overline{\overline{T}}_e e^{-j\beta_1 h}\cdot\mathbf{v}IL =$$

$$-jk_2\eta_2\int_0^\infty \overline{\overline{O}}_{e2}\cdot\frac{e^{j\beta_2 z}}{2j\beta_2}\overline{\overline{h}}_e(\mathbf{K},h,\zeta)e^{-j\beta_2 H(h,\zeta)}\cdot\mathbf{v}ILd\zeta \tag{7.120}$$

we have for the total transmitted field in physical space corresponding to the original dipole $\mathbf{J}(\mathbf{r}') = \mathbf{v}IL\delta(\mathbf{r}'-\mathbf{u}_z h)$,

$$\mathbf{E}_T(\mathbf{r}) = -jk_2\eta_2\int_0^\infty\int_V \overline{\overline{G}}(D(\zeta))\cdot\mathbf{J}_i(\mathbf{r}',\zeta)dV'd\zeta, \tag{7.121}$$

with

$$D(\zeta) = \sqrt{\rho^2 + [z - H(z', \zeta)]^2}, \qquad (7.122)$$

$$\mathbf{J}_i(\mathbf{r}', \zeta) = \overline{\overline{h}}_e(j\nabla'_t, z', \zeta) \cdot \mathbf{v}IL\delta(\mathbf{r}', \zeta), \qquad (7.123)$$

which is dependent on the position $z' = h$ of the dipole. Note that this Green dyadic depends on the parameters of medium 2. More generally, for a three-dimensional source $\mathbf{J}(\mathbf{r}')$ we obtain the four-dimensional image

$$\mathbf{J}_i(\mathbf{r}', \zeta) = \overline{\overline{h}}_e(j\nabla'_t, z', \zeta) \cdot \mathbf{J}(\mathbf{r}'). \qquad (7.124)$$

7.1.4 Green functions

Instead of applying image functions each time when fields are being computed, one can define a dyadic Green function which takes into account the reflection or change in transmission due to the stratified geometry in question. This gives us an opportunity to apply solution routines designed for free-space problems by adding to the free-space dyadic Green function a new dyadic Green function.

Reflection problem

A new Green dyadic $\overline{\overline{K}}(\mathbf{r})$ corresponding to the reflection from the structure can be obtained by integrating out the parameter ζ in the field expressions. It can be seen that, to construct the dyadic image Green function, two scalar Green functions are required instead of only one as in the homogeneous-space problem. The reflection image Green dyadic is obtained from the reflection field expression:

$$\mathbf{E}_r(\mathbf{r}) = -jk\eta \int_V \int_0^\infty \overline{\overline{G}}(\mathbf{r} - \mathbf{r}' + \mathbf{u}_z H(\zeta)) \cdot \mathbf{J}_i(\mathbf{r}', \zeta) dV' d\zeta =$$

$$-jk\eta \int_V \left(\int_0^\infty \overline{\overline{G}}(\mathbf{r} - \mathbf{r}' + \mathbf{u}_z H(\zeta)) \cdot \overline{\overline{f}}_e(j\nabla'_t, \zeta) d\zeta \right) \cdot \mathbf{J}_c(\mathbf{r}') dV' =$$

$$-jk\eta \int_V \overline{\overline{K}}(\mathbf{r} - \mathbf{r}') \cdot \mathbf{J}_c(\mathbf{r}') dV', \qquad (7.125)$$

$$\overline{\overline{K}}(\mathbf{r}) = \int_0^\infty \overline{\overline{G}}(\mathbf{r} + \mathbf{u}_z H(\zeta)) \cdot \overline{\overline{f}}_e(j\nabla'_t, \zeta) d\zeta. \qquad (7.126)$$

The expression of the image current $\mathbf{J}_i$ contains functions of the ζ variable and the ∇ operator as separate factors, which makes it possible to integrate the ζ variable out and define the scalar Green functions

$$K^{TE}(\mathbf{r}) = \int_0^\infty G(\mathbf{r} + \mathbf{u}_z H(\zeta)) f^{TE}(\zeta) d\zeta, \tag{7.127}$$

$$K^{TM}(\mathbf{r}) = \int_0^\infty G(\mathbf{r} + \mathbf{u}_z H(\zeta)) f^{TM}(\zeta) d\zeta, \tag{7.128}$$

$$K_o(\mathbf{r}) = \int_0^\infty G(\mathbf{r} + \mathbf{u}_z H(\zeta)) f_o(\zeta) d\zeta. \tag{7.129}$$

The ∇' operators can be moved in front of the Green function through the following manipulations in the field integrals, recalling that the field point $\mathbf{r}$ never coincides with the image source point $\mathbf{r}'$:

$$\begin{gathered}
\int_V K_o(\mathbf{r} - \mathbf{r}')(\mathbf{u}_z \mathbf{u}_z \overset{\times}{\times} \nabla'\nabla') \cdot \mathbf{J}_c(\mathbf{r}') dV' = \\
\mathbf{u}_z \times \int_V \nabla' \big[K_o(\mathbf{u}_z \times \nabla') \cdot \mathbf{J}_c \big] dV' - \mathbf{u}_z \times \int_V (\nabla' K_o)(\mathbf{u}_z \cdot \nabla' \times \mathbf{J}_c) dV' = \\
\mathbf{u}_z \times \nabla \int_V K_o(\mathbf{u}_z \times \nabla' \cdot \mathbf{J}_c) dV' = (\mathbf{u}_z \mathbf{u}_z \overset{\times}{\times} \nabla\nabla) \cdot \int_V K_o(\mathbf{r} - \mathbf{r}') \mathbf{J}_c(\mathbf{r}') dV'.
\end{gathered} \tag{7.130}$$

Thus, the dyadic image Green function can be defined in terms of these scalar image Green functions as

$$\overline{\overline{K}}(\mathbf{r}) = \left(\overline{\overline{I}} + \frac{1}{k^2} \nabla\nabla \right) K^{TM}(\mathbf{r}) + \frac{1}{k^2} (\mathbf{u}_z \mathbf{u}_z \overset{\times}{\times} \nabla\nabla) K_o(\mathbf{r}), \tag{7.131}$$

or, taking another representation for the dyadic operator $\overline{\overline{f}}_e$,

$$\overline{\overline{K}}(\mathbf{r}) = \left(\overline{\overline{I}} + \frac{1}{k^2} \nabla\nabla \right) \cdot \left(K^{TE}(\mathbf{r}) \overline{\overline{I}} - \frac{1}{k^2} \nabla_t \nabla_t K_o(\mathbf{r}) \right). \tag{7.132}$$

The reflected field due to the image source can be expressed simply as

$$\mathbf{E}_r(\mathbf{r}) = -jk\eta \int_V \overline{\overline{K}}(\mathbf{r} - \mathbf{r}') \cdot \mathbf{J}_c(\mathbf{r}') dV'. \tag{7.133}$$

Note that the source here is the mirror image of the original source and not the original source. We could have included the mirror image operator c in the definition of the image Green dyadic. It is seen that only two image scalar Green functions are needed in the computation of the reflected fields, either of the functions K^{TM} and K^{TE} together with the function K_o.

From duality, there exist obvious relations between the image Green functions:

$$K_d^{TM}(\mathbf{r}) = -K^{TE}(\mathbf{r}), \tag{7.134}$$

$$K_d^{TE}(\mathbf{r}) = -K^{TM}(\mathbf{r}), \tag{7.135}$$

$$K_{od}(\mathbf{r}) = K_o(\mathbf{r}). \tag{7.136}$$

Thus, values for the Green functions K^{TE} and K^{TM} can be computed applying the same procedure with dual parameter values.

Asymptotic expressions for the reflection problem

Asymptotic expressions of the Green functions are of interest when computing fields far from the source region. Also, extracting the asymptotic term from the Green function leaves a function which is more easily approximated because it decays more quickly than the original Green function.

If the argument $\mathbf{r}$ becomes large, so that we may assume $|H(\zeta)| \ll |\mathbf{r}|$ for those ζ values for which the image function is not negligible, we may write the two-term Taylor expansion for the distance function as

$$D = \sqrt{(\mathbf{r} + \mathbf{u}_z H) \cdot (\mathbf{r} + \mathbf{u}_z H)} \approx r + \mathbf{u}_r \cdot \mathbf{u}_z H, \tag{7.137}$$

which inserted in the Green function gives us the basic approximation

$$G(\mathbf{r} + \mathbf{u}_z H(\zeta)) \approx G(\mathbf{r}) e^{-jk\mathbf{u}_r \cdot \mathbf{u}_z H(\zeta)}. \tag{7.138}$$

Thus, we have for the first-term approximation of any of the Green functions (7.127)–(7.129)

$$\int_0^\infty G(\mathbf{r} + \mathbf{u}_z H(\zeta)) f(\zeta) d\zeta \approx G(\mathbf{r}) \int_0^\infty e^{-jk\mathbf{u}_r \cdot \mathbf{u}_z H(\zeta)} f(\zeta) d\zeta. \tag{7.139}$$

The last integral expression can be interpreted in terms of the reflection coefficient function, applying the definitions (7.69)–(7.71). In fact, we can write

$$\int_0^\infty e^{-jk\mathbf{u}_r \cdot \mathbf{u}_z H(\zeta)} f(\zeta) d\zeta = R(\beta)\Big|_{\beta = k\cos\theta} = R(k\cos\theta), \tag{7.140}$$

denoting the angle of the vector $\mathbf{r}$ and the z axis by θ, whence we have the asymptotic expressions for the image Green functions

$$K^{TE}(\mathbf{r}) \approx R^{TE}(k\cos\theta)G(\mathbf{r}), \tag{7.141}$$

$$K^{TM}(\mathbf{r}) \approx R^{TM}(k\cos\theta)G(\mathbf{r}), \tag{7.142}$$

$$K_o(\mathbf{r}) \approx R_o(k\cos\theta)G(\mathbf{r}). \tag{7.143}$$

The corresponding approximation for the reflection dyadic image Green function

$$\overline{\overline{K}}(\mathbf{r}) \approx R^{TM}(k\cos\theta)\left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right)G(\mathbf{r})+$$

$$R_o(k\cos\theta)\frac{1}{k^2}(\mathbf{u}_z\mathbf{u}_z \overset{\times}{\times} \nabla\nabla)G(\mathbf{r}). \tag{7.144}$$

can be written, if we make the far field approximation $\nabla \to -jk\mathbf{u}_r$ and apply (7.68) (note that the Fourier parameter is now $K = k\sin\theta$), in the form

$$\overline{\overline{K}}(\mathbf{r}) \approx \left[R^{TE}(k\cos\theta)\mathbf{u}_\varphi\mathbf{u}_\varphi + R^{TM}(k\cos\theta)\mathbf{u}_\theta\mathbf{u}_\theta\right]G(\mathbf{r}). \tag{7.145}$$

This can be called the reflection-coefficient method, or RCM approximation, in which the reflection far field is obtained by replacing the exact image by the mirror image of the original source, and multiplying it by the dyadic $R^{TE}\mathbf{u}_\varphi\mathbf{u}_\varphi + R^{TM}\mathbf{u}_\theta\mathbf{u}_\theta$.

When computing the reflection field, the argument $\mathbf{r}$ of the Green function is replaced by the difference $\mathbf{r} - \mathbf{r}'$, where $\mathbf{r}'$ denotes a point of the mirror image source. The angles and distances are then measured from the point $\mathbf{r}'$ and not the origin.

Transmission problem

The transmission field can also be written in terms of a new transmission Green dyadic taking care of the geometry behind the transmission plane. In fact, writing the transmission field (7.121) in the form

$$\mathbf{E}_T(\mathbf{r}) = -jk_2\eta_2 \int_V \left(\int_0^\infty \overline{\overline{G}}(D(\zeta)) \cdot \overline{\overline{h}}_e(j\nabla_t', z', \zeta)d\zeta\right) \cdot \mathbf{J}(\mathbf{r}')dV' =$$

$$-jk_2\eta_2 \int_V \overline{\overline{K}}_T(\mathbf{r}-\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')dV', \tag{7.146}$$

we can define the transmission Green dyadic by

$$\overline{\overline{K}}_T(\mathbf{r}-\mathbf{r}') = \int_0^\infty \overline{\overline{G}}(D(\zeta)) \cdot \overline{\overline{h}}_e(j\nabla_t', z', \zeta) d\zeta, \tag{7.147}$$

$$D(\zeta) = \sqrt{[\mathbf{r} - \boldsymbol{\rho}' - \mathbf{u}_z H(z',\zeta)] \cdot [\mathbf{r} - \boldsymbol{\rho}' - \mathbf{u}_z H(z',\zeta)]}. \tag{7.148}$$

The asymptotic far field expression for the new Green dyadic can be written similarly with the reflection problem:

$$D(\zeta) \approx r - \mathbf{u}_r \cdot [\boldsymbol{\rho}' + \mathbf{u}_z H(z',\zeta)], \tag{7.149}$$

$$\overline{\overline{K}}_T(\mathbf{r}-\mathbf{r}') \approx \overline{\overline{G}}(r) e^{jk_2 \mathbf{u}_r \cdot \boldsymbol{\rho}'} \cdot \int_0^\infty \overline{\overline{h}}_e(j\nabla_t', z', \zeta) e^{jk_2 \mathbf{u}_r \cdot \mathbf{u}_z H(z',\zeta)} d\zeta. \tag{7.150}$$

The last integral can be interpreted in terms of the definition of the transmission dyadic, (7.113):

$$\overline{\overline{K}}_T(\mathbf{r}-\mathbf{r}') \approx \overline{\overline{G}}(r) e^{jk_2 \mathbf{u}_r \cdot \boldsymbol{\rho}'} \cdot \overline{\overline{T}}_e e^{-j\beta_1 z'}, \tag{7.151}$$

with the parameters in the transmission dyadic and exponent expressions given as

$$\beta_2 = -k_2 \mathbf{u}_r \cdot \mathbf{u}_z = k_2 \cos\theta_2, \tag{7.152}$$

$$\mathbf{K} = K\mathbf{u}_\rho = k_2(\overline{\overline{I}} - \mathbf{u}_z\mathbf{u}_z) \cdot \mathbf{u}_r, \qquad K = k_2 \sin\theta_2, \tag{7.153}$$

$$\beta_1 = \sqrt{k_1^2 - K^2} = k_1 \cos\theta_1 = \omega\sqrt{\epsilon_1\mu_1 - \epsilon_2\mu_2 \sin^2\theta_2}. \tag{7.154}$$

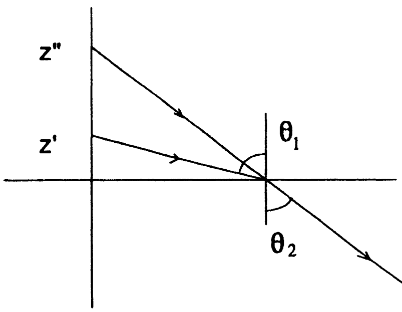

Fig. 7.2 The asymptotic far field approximation for the transmission image expression can be interpreted in terms of refracted rays starting from a source point at an apparent height z''.

The angles θ_1 and θ_2 are defined above. To obtain an interpretation, we can write

$$\overline{\overline{K}}_T(\mathbf{r}-\mathbf{r}') \approx \overline{\overline{G}}(D)\cdot\overline{\overline{T}}_e, \tag{7.155}$$

$$D \approx \sqrt{(\mathbf{r}-\boldsymbol{\rho}'-\mathbf{u}_z z'')\cdot(\mathbf{r}-\boldsymbol{\rho}'-\mathbf{u}_z z'')}, \tag{7.156}$$

$$z'' = \frac{\beta_1}{\beta_2}z' = \frac{k_1\cos\theta_1}{k_2\cos\theta_2}z' = \frac{\tan\theta_2}{\tan\theta_1}z'. \tag{7.157}$$

This expression can be interpreted geometrically as representing the point on the z axis from which a ray in the direction $\mathbf{u}_r$ is emanating, when the layers between the planes in the medium 1 and 2 are squeezed to a plane.

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7.2 Surface problems

As an application of the general theory, let us consider problems involving planar impedance surfaces and sheets. If the surface is impenetrable, it can be characterized by an impedance surface condition, as discussed in Chapter 3. For penetrable surfaces, the corresponding impedance sheet characterization is valid. As a last example in this section, the thin metallic grid problem is considered. In all these cases, the image functions can be quite easily written in explicit form involving only simple exponential functions.

7.2.1 Impedance surface

The planar impedance surface serves as the simplest example of a problem for which the image method can be applied. The idea has also been applied in the book by FELSEN and MARCUVITZ (1973, pp. 557–559).

Reflection coefficients

In the case of an isotropic planar impedance surface at $z = 0$ with surface impedance Z_s, defining the impedance dyadic $\overline{\overline{Z}}_s = Z_s \overline{\overline{I}}_t$, the dyadics $\overline{\overline{Z}}_s$ and $\overline{\overline{Z}}$ obviously commute and the reflection dyadic can be written from (7.38)

$$\overline{\overline{R}} = (\overline{\overline{Z}}_s + \overline{\overline{Z}})^{-1} \cdot (\overline{\overline{Z}}_s - \overline{\overline{Z}}) = R^{TE} \frac{\mathbf{u}_z \mathbf{u}_z {}^{\times}_{\times} \mathbf{KK}}{K^2} + R^{TM} \frac{\mathbf{KK}}{K^2}, \tag{7.158}$$

with the eigenvalues

$$R^{TE} = \frac{Z_s - \frac{\eta k}{\beta}}{Z_s + \frac{\eta k}{\beta}} = 1 - \frac{2\frac{\eta k}{Z_s}}{\beta + \frac{\eta k}{Z_s}}, \qquad R^{TM} = \frac{Z_s - \frac{\beta \eta}{k}}{Z_s + \frac{\beta \eta}{k}} = -1 + \frac{2\frac{Z_s k}{\eta}}{\beta + \frac{Z_s k}{\eta}}. \tag{7.159}$$

These coefficients obviously satisfy the duality condition (7.78) because the dual of Z_s/η is η/Z_s. As another check, both coefficients can be seen to tend to -1 as $Z_s \to 0$ and to $+1$ as $Z_s \to \infty$, corresponding respectively to perfect electric and magnetic conductor surfaces.

The third reflection coefficient takes the following forms:

$$R_o = \frac{k^2}{K^2}(R^{TM} - R^{TE}) = \frac{2k^2}{(\beta + \frac{Z_s k}{\eta})(\beta + \frac{\eta k}{Z_s})} =$$

$$-\frac{2k\eta Z_s}{Z_s^2 - \eta^2} \left(\frac{1}{\beta + \frac{Z_s k}{\eta}} - \frac{1}{\beta + \frac{\eta k}{Z_s}} \right), \tag{7.160}$$

and it is seen to vanish in both the perfect electric and magnetic conductor limiting cases.

Image functions

The image functions corresponding to the reflection coefficients can be found in a straightforward manner if the transformation is defined through the integral

$$R(\beta) = \int_0^\infty f(\zeta) e^{-j\beta\zeta} d\zeta. \tag{7.161}$$

This means that the function $H(t)$ of (7.57) in this case is simply defined as $H(\zeta) = \zeta$. The image functions can then be written as

$$f^{TE}(\zeta) = \delta_+(\zeta) - 2j\frac{\eta k}{Z_s} e^{-j\eta k\zeta/Z_s} U_+(\zeta), \tag{7.162}$$

$$f^{TM}(\zeta) = -\delta_+(\zeta) + 2j\frac{Z_s k}{\eta} e^{-jZ_s k\zeta/\eta} U_+(\zeta), \tag{7.163}$$

$$f_o(\zeta) = \frac{2jk\eta Z_s}{Z_s^2 - \eta^2}\left(e^{-jZ_s k\zeta/\eta} - e^{-j\eta k\zeta/Z_s}\right) U_+(\zeta). \tag{7.164}$$

The unit step function $U_+(\zeta)$ and the corresponding delta function $\delta_+(\zeta)$ are defined at the limits $U(\zeta - \Delta)$ and $\delta(\zeta - \Delta)$ as $\Delta \to 0$. Thus, all image functions vanish for $\zeta \le 0$. They can also be checked to satisfy the condition (7.87).

In terms of these functions, the reflected field can be expressed as arising from an image source, by applying one of the many forms for the dyadic image function $\overline{\overline{f}}_e$. For example, with (7.92) we have the following explicit expression for the image source corresponding to a primary source $\mathbf{J}(\mathbf{r})$:

$$\mathbf{J}_i(\mathbf{r}, \zeta) = \overline{\overline{f}}_e(j\nabla_t, \zeta) \cdot \mathbf{J}_c(\mathbf{r}) =$$

$$\left(f^{TM}(\zeta)\mathbf{u}_z\mathbf{u}_z + f^{TE}(\zeta)\overline{\overline{I}}_t + \mathbf{u}_z\frac{f_o'(\zeta)}{k^2}\nabla_t\right) \cdot \mathbf{J}_c(\mathbf{r}), \tag{7.165}$$

in which the image functions must be substituted from (7.162)–(7.164).

Convergence of image functions

To obtain a meaningful field integral, the integrand should converge. Because the Green function can be made converging through a proper choice of the branch of the distance function, it suffices that the image functions are non-divergent. It is seen immediately that for real parameter ζ values, for a lossless medium (real and positive k and η), the exponential functions are all non-diverging when Z_s is real. On the other hand, if Z_s has an imaginary part, some of the exponential functions turn out to be diverging.

In particular, for a lossless medium with real k and η, the condition for the TE image (7.162) to be non-divergent is obviously $\Im\{\zeta/Z_s\} = \Im\{\zeta Y_s\} \leq 0$, or the surface impedance should be inductive or pure resistive if ζ is real and positive. Correspondingly, for the TM image, from (7.163) we have the condition $\Im\{\zeta Z_s\} \leq 0$, or the surface impedance should be capacitive or pure resistive.

Thus, we see that unless the surface is purely resistive, one of the functions $f^{TE}(\zeta)$, $f^{TM}(\zeta)$ diverges for real and positive ζ and the $f_o(\zeta)$ function diverges. The image source diverging for real and positive ζ can be interpreted as giving rise to a surface-wave field.

The existence of a surface wave is associated with the poles of the reflection coefficient. Writing

$$R(\beta) = \frac{A}{\beta - \beta_p}, \tag{7.166}$$

a surface wave with z dependence as $\exp(-j\beta_p z)$ exists if the condition $\Im\{\beta_p\} < 0$ is valid. Because the pole corresponds to either $\beta_p^{TE} = -k\eta/Z_s$ or $\beta_p^{TM} = -kZ_s/\eta$, there exists a TE surface wave for $\Im\{Y_s\} > 0$, which defines a capacitive surface impedance. On the other hand, there exists a TM surface wave for $\Im\{Z_s\} > 0$, which defines an inductive surface impedance. These conditions are the same as those for diverging image functions for real and positive t. Exponentially diverging image functions corresponding to surface waves along the structure will also be encountered in other problems.

To obtain non-diverging image functions we may choose an integration path in complex ζ plane. It is sufficient to choose a path along the imaginary axis $\zeta = j\zeta'$, although this does not make the image function converge. However, since the distance function D is now complex, its branch can be chosen so that the Green function becomes exponentially decaying, whence the field integrand converges. However, for imaginary ζ, there is a singularity in the Green function when the source and field points are at the impedance plane, because the distance function D becomes zero for $\rho = \zeta'$. This singularity is clearly demonstrated by the following example.

Vertical electric dipole above an impedance plane

Let us consider a horizontal impedance plane and the field from a vertical electric dipole (VED) $\mathbf{J}(\mathbf{r}) = \mathbf{u}_z IL\delta(z-h)$. According to the EIT method, we can replace the impedance surface with the image source

$$\mathbf{J}_i(\mathbf{r},\zeta) = \overline{\overline{f}}_e(j\nabla_t,\zeta)\cdot[-\mathbf{u}_z IL\delta(\rho)\delta(z+h)] =$$
$$-\, f^{TM}(\zeta)\mathbf{u}_z IL\delta(\rho)\delta(z+h). \tag{7.167}$$

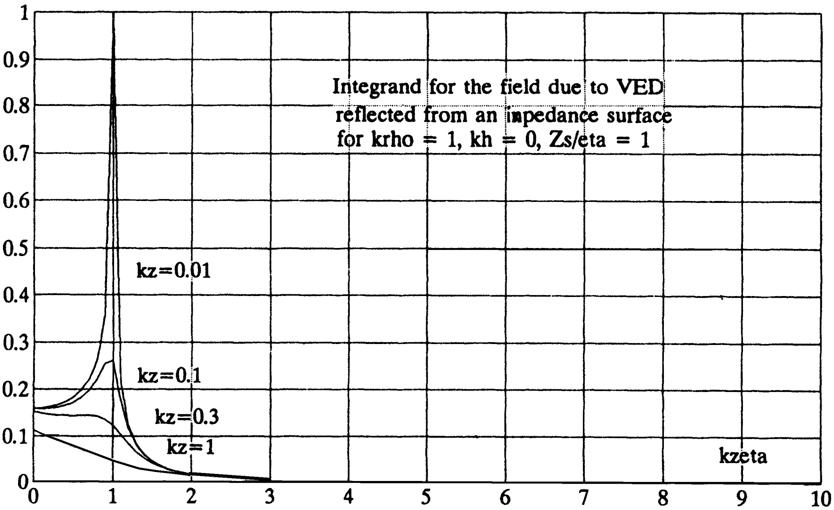

Fig. 7.3 Normalized integrand of the reflected field integral without the delta term, as a function of $k\zeta$ for different values of kz, at $h = 0$ and $\rho = 0$. A peak arises when $z + h \approx 0$, i.e. the source and field points are both close to the surface.

The image can be expressed in terms of the function f^{TM} alone, because the VED source produces a TM field. The reflected field can be written as

$$\mathbf{E}_r(\mathbf{r}) = -jk\eta\int_V\int_0^\infty \overline{\overline{G}}(\mathbf{r}-\mathbf{r}'+\mathbf{u}_z\zeta)\cdot\mathbf{J}_i(\mathbf{r}',\zeta)dV'd\zeta =$$

$$jk\eta\left(\overline{\overline{I}}+\frac{1}{k^2}\nabla\nabla\right)\cdot\mathbf{u}_z IL\, K^{TM}(\mathbf{r}+\mathbf{u}_z h), \tag{7.168}$$

when the Green function is defined as in (7.128) with $H(\zeta) = \zeta$:

$$K^{TM}(\mathbf{r}+\mathbf{u}_z h) = \int_0^\infty \frac{e^{-jkD(\zeta)}}{4\pi D(\zeta)} f^{TM}(\zeta)d\zeta =$$

$$- G(D(0)) + 2jk\frac{Z_s}{\eta}\int_0^\infty \frac{e^{-jkD(\zeta)}}{4\pi D(\zeta)} e^{-jZ_s k\zeta/\eta} d\zeta, \tag{7.169}$$

with the distance function defined by

$$D(\zeta) = \sqrt{\rho^2 + (z + h + \zeta)^2}. \tag{7.170}$$

To ensure the convergence of the integral in (7.169), the branch for the distance function is defined by $\Im\{D\} \leq 0$. It is obvious that for a capacitive surface impedance, the f^{TM} function makes the integrand converge exponentially. For an inductive surface impedance we write $\zeta = -j\zeta'$ and rewrite (7.169) as

$$K^{TM}(\mathbf{r} + \mathbf{u}_z h) = -G(D(0)) + \frac{2kZ_s}{\eta}\int_0^\infty \frac{e^{-jkD(-j\zeta')}}{4\pi D(-j\zeta')} e^{-Z_s k\zeta'/\eta} d\zeta'. \tag{7.171}$$

Again, the integrand converges exponentially because of the complex distance function D, even if Z_s is imaginary.

The singularity is clearly integrable, because at $z + h = 0$ it is of the type $1/\sqrt{\rho^2 - \zeta'^2}$. Actually, integration along a short distance Δ around the singularity point $\zeta' = \rho$ leaves us with the expression

$$\int_{\rho-\Delta/2}^{\rho+\Delta/2} \frac{2kZ_s}{\eta}\frac{e^{-jkD(-j\zeta')}}{4\pi D(-j\zeta')} e^{-Z_s k\zeta'/\eta} d\zeta' \approx (1-j)\sqrt{\frac{\Delta}{\rho}}\frac{e^{-Z_s k\rho/2\eta}}{4\pi}, \tag{7.172}$$

which shows us that its contribution varies roughly as $1/\sqrt{\rho}$ times an exponentially decaying function.

7.2.2 Impedance sheet

Reflection images

The previous isotropic impedance surface geometry can be transformed to impedance sheet geometry by adding a homogeneous half-space behind the surface. If the admittance in both homogeneous spaces is denoted by $\overline{\overline{Y}}$, and the surface admittance by $\overline{\overline{Y}}_s$, the reflection dyadic at the interface is, accordingly, of the form

$$\overline{\overline{R}} = [\overline{\overline{Y}} - (\overline{\overline{Y}}_s + \overline{\overline{Y}})] \cdot [\overline{\overline{Y}} + (\overline{\overline{Y}}_s + \overline{\overline{Y}})]^{-1} = -\overline{\overline{Y}}_s \cdot (2\overline{\overline{Y}} + \overline{\overline{Y}}_s)^{-1}. \tag{7.173}$$

For isotropic surface admittance $\overline{\overline{Y}}_s = Y_s \overline{\overline{I}}_t = Z_s^{-1}\overline{\overline{I}}_t$, we simply have

$$\overline{\overline{R}} = -Y_s(2\overline{\overline{Y}} + Y_s\overline{\overline{I}}_t)^{-1} = -(2Z_s\overline{\overline{Y}} + \overline{\overline{I}}_t)^{-1}. \tag{7.174}$$

When the admittance dyadic $\overline{\overline{Y}}$ is substituted in terms of its TE and TM eigenvalues (7.40), the reflection dyadic for the isotropic sheet can be written as

$$\overline{\overline{R}} = -\frac{Y_s}{Y_s + \frac{2\beta}{k\eta}} \frac{\mathbf{u}_z\mathbf{u}_z {}^\times_\times \mathbf{KK}}{K^2} - \frac{Y_s}{Y_s + \frac{2k}{\beta\eta}} \frac{\mathbf{KK}}{K^2}. \tag{7.175}$$

Thus, the reflection coefficients are

$$R^{TE} = -\frac{\frac{k\eta}{2Z_s}}{\beta + \frac{k\eta}{2Z_s}}, \tag{7.176}$$

$$R^{TM} = -\frac{\beta}{\beta + \frac{2kZ_s}{\eta}} = -1 + \frac{\frac{2kZ_s}{\eta}}{\beta + \frac{2kZ_s}{\eta}}, \tag{7.177}$$

$$R_o = \frac{k^2}{(\beta + \frac{2kZ_s}{\eta})(\beta + \frac{k\eta}{2Z_s})} = \frac{2k\eta Z_s}{\eta^2 - 4Z_s^2}\left(\frac{1}{\beta + \frac{2kZ_s}{\eta}} + \frac{1}{\beta + \frac{k\eta}{2Z_s}}\right). \tag{7.178}$$

Note that the duality check does not work here because the sheet is not the most general one (magnetic surface impedance is not included). The corresponding image functions are, again, combinations of exponential functions:

$$f^{TE}(\zeta) = -j\frac{k\eta}{2Z_s}e^{-j\eta k\zeta/2Z_s}\,U_+(\zeta), \tag{7.179}$$

$$f^{TM}(\zeta) = -\delta_+(\zeta) + j\frac{2Z_s k}{\eta}e^{-j2kZ_s\zeta/\eta}\,U_+(\zeta), \tag{7.180}$$

$$f_o(\zeta) = \frac{j2\eta Z_s}{\eta^2 - 4Z_s^2}\left(e^{-j2Z_s k\zeta/\eta} + e^{-j\eta k\zeta/2Z_s}\right)U_+(\zeta). \tag{7.181}$$

To check the results, let us consider the case $Z_s \to 0$, which is readily seen to lead to $f^{TM} = -\delta_+(\zeta)$ and vanishing of $f_o(\zeta)$. The function f^{TE} is in this case of the form $j\alpha \exp j\alpha\zeta$ with $\alpha \to \infty$, which inside the field integral can be seen to act like the function $-\delta_+(\zeta)$:

$$\int_0^\infty j\alpha e^{j\alpha\zeta}F(\zeta)d\zeta = j\int_0^\infty e^{jT}F(T/\alpha)dT \to jF(0)\int_0^\infty e^{jT}dT = -F(0). \tag{7.182}$$

This property is also seen from the corresponding reflection coeffient R^{TE}, which tends to the value 1 when $Z_s \to 0$.

Thus, the limit equals the perfectly conducting sheet case as expected. On the other hand, for $Z_s = \infty$, all image functions vanish, which again is in accord with the vanishing of the impedance sheet.

Transmission images

Transmission through the sheet is defined by the transmission dyadic

$$\overline{\overline{T}} = \overline{\overline{I}}_t + \overline{\overline{R}} = \overline{\overline{I}}_t - \overline{\overline{Y}}_s \cdot (\overline{\overline{Y}}_s + 2\overline{\overline{Y}})^{-1} = 2\overline{\overline{Y}} \cdot (\overline{\overline{Y}}_s + 2\overline{\overline{Y}})^{-1}, \tag{7.183}$$

with the eigenvalues

$$T^{TE} = 1 + R^{TE} = 1 - \frac{\frac{k\eta}{2Z_s}}{\beta + \frac{k\eta}{2Z_s}}, \tag{7.184}$$

$$T^{TM} = 1 + R^{TM} = \frac{\frac{2kZ_s}{\eta}}{\beta + \frac{2kZ_s}{\eta}}, \tag{7.185}$$

and the third transmission coefficient is simply

$$T_o = R_o = \frac{2k\eta Z_s}{\eta^2 - 4Z_s^2}\left(\frac{1}{\beta + \frac{2kZ_s}{\eta}} + \frac{1}{\beta + \frac{k\eta}{2Z_s}}\right). \tag{7.186}$$

The image functions can be defined through the transformations (7.110)–(7.112) with $H(\zeta) = \zeta$, or

$$T(\beta) = \int_0^\infty h(\zeta)e^{-j\beta\zeta}d\zeta, \tag{7.187}$$

whence we have

$$h^{TE}(\zeta) = \delta_+(\zeta) + f^{TE}(\zeta) = \delta_+(\zeta) - j\frac{\eta k}{2Z_s}e^{-j\eta k\zeta/2Z_s}U_+(\zeta), \tag{7.188}$$

$$h^{TM}(\zeta) = \delta_+(\zeta) + f^{TM}(\zeta) = 2j\frac{Z_s k}{\eta}e^{-j2Z_s k\zeta/\eta}U_+(\zeta), \tag{7.189}$$

$$h_o(\zeta) = f_o(\zeta). \tag{7.190}$$

Let us again check the limiting cases. For $Z_s = 0$ we have the perfect conductor sheet, whence h^{TM} and f_o functions are seen to vanish immediately and h^{TE} after a consideration similar to that above. For $Z_s = \infty$ both h functions reduce to $\delta_+(t)$ in accord with the vanishing of the sheet.

Convergence

Since the reflection image functions for the impedance sheet are similar to those of the impedance surface case, the same convergence considerations apply. In fact, a surface wave is again connected with the divergence of the corresponding function for real and positive ζ. In particular, because the pole expressions only differ by a factor of 2, the same conclusions concerning the appearance of a surface-wave for a certain complex Z_s apply as for the impedance surface.

Dielectric slab

Modelling a thin dielectric slab by an impedance sheet, we may write for the sheet impedance when the thickness Δ of the slab becomes small, according to (3.173):

$$Z_s = jX_s = \frac{-j\eta}{k\Delta(\epsilon_r - 1)}. \tag{7.191}$$

The TE image function (7.179) can be written in this case as

$$f^{TE}(\zeta) = \frac{d}{d\zeta}\exp\left(\frac{k^2\zeta\Delta}{2}(\epsilon_r - 1)\right). \tag{7.192}$$

This function diverges unless $\epsilon_r < 1$ or ζ is imaginary. It is well known that the basic surface wave (actually a waveguide mode) can exist in a dielectric slab for any thickness of the slab.

7.2.3 Wire grid

The problem of a planar wire grid with square cells and small mesh size can be treated by the theory of KONTOROVICH (1963) much in the same way as the impedance sheet, although the surface impedance dyadic is no longer a multiple of $\overline{\overline{I}}_t$. However, it turns out to have the previous eigenvectors, which means that the grid does not couple TE and TM waves, which makes the theory simple. On the other hand, if the cells were of more general anisotropic form, such coupling would occur and the theory should be modified accordingly.

The condition at the grid can be written for the transverse fields $\mathbf{e}$, $\mathbf{h}$ on each side as a double equation, which in Fourier space reads

$$\mathbf{e}_1 = \mathbf{e}_2 = j\eta\kappa\left(\overline{\overline{I}}_t - \frac{\mathbf{KK}}{2k^2}\right)\cdot[\mathbf{u}_z \times (\mathbf{h}_1 - \mathbf{h}_2)]. \tag{7.193}$$

Defining the grid impedance dyadic $\overline{\overline{Z}}_g$ through the sheet equation

$$\mathbf{e} = -\overline{\overline{Z}}_g \cdot [-\mathbf{u}_z \times (\mathbf{h}_1 - \mathbf{h}_2)], \tag{7.194}$$

we can write its expression from (3.200) after replacing the operator ∇_t by $-j\mathbf{K}$:

$$\overline{\overline{Z}}_g = j\eta\kappa\left(\overline{\overline{I}}_t - \frac{\mathbf{K}\mathbf{K}}{2k^2}\right) = Z_g^{TE}\frac{\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times}\mathbf{K}\mathbf{K}}{K^2} + Z_g^{TM}\frac{\mathbf{K}\mathbf{K}}{K^2}, \tag{7.195}$$

with

$$Z_g^{TE} = j\eta\kappa = (Y_g^{TE})^{-1}, \tag{7.196}$$

$$Z_g^{TM} = j\frac{\eta\kappa}{2k^2}(k^2 + \beta^2) = (Y_g^{TM})^{-1}. \tag{7.197}$$

$\overline{\overline{Y}}_g$, the two-dimensional inverse of $\overline{\overline{Z}}_g$, can now be directly substituted for $\overline{\overline{Y}}_s$ in the impedance sheet expression (7.173) to produce the reflection dyadic. The reflection coefficients are found after some algebraic manipulations to be of the form

$$R^{TE} = \frac{\frac{jk}{2\kappa}}{\beta - \frac{jk}{2\kappa}}, \tag{7.198}$$

$$R^{TM} = \frac{1}{\sqrt{1+4\kappa^2}}\left(\frac{\frac{jk}{s}}{\beta + \frac{jk}{s}} + \frac{jks}{\beta - jks}\right), \tag{7.199}$$

$$R_o = \frac{k^2}{K^2}(R^{TE} - R^{TM}) = \frac{-jk\kappa}{1+4\kappa^2}\left(\frac{2}{\beta - \frac{jk}{2\kappa}} - \frac{1}{\beta - jks} - \frac{1}{\beta + \frac{jk}{s}}\right), \tag{7.200}$$

with the definition

$$s = \frac{1}{2\kappa} + \sqrt{1 + \frac{1}{4\kappa^2}}. \tag{7.201}$$

If the TM reflection coefficient is compared with that obtained for the vertical electric dipole by LINDELL*et al.* (1986), a difference in sign is noted. This is due to the difference in the definition of the reflection coefficient, which in the reference was defined for the Hertzian potential and not for the electric field as in the present analysis.

As a check we see that for $\kappa \to 0$ or $s \to \infty$, corresponding to a very dense grid, we have $\overline{\overline{R}} \to -\overline{\overline{I}}_t$, which is the reflection dyadic of the perfectly conducting plane. Also, for the limiting case $\kappa \to \infty$ (vanishing grid), we have $s \to 1$ and all the reflection coefficients are seen to vanish.

Again, the transmission coefficients are in a simple relationship to the reflection coefficients:

$$T^{TE} = 1 + R^{TE}, \tag{7.202}$$

$$T^{TM} = 1 + R^{TM}, \tag{7.203}$$

which makes it possible to express the transmission image functions in terms of the reflection image functions as in the case of the impedance sheet discussed previously.

Image functions

The reflection image functions for the wire grid can be written down without problem, since the reflection coefficients are of such a simple form. In fact, applying the same integral transform as for the impedance sheet, we can write

$$f^{TE}(\zeta) = -\frac{k}{2\kappa} e^{-k\zeta/2\kappa}\, U_+(\zeta), \tag{7.204}$$

$$f^{TM}(\zeta) = -\frac{1}{\sqrt{1+4\kappa^2}} \frac{d}{d\zeta} \left(e^{k\zeta/s} - e^{-sk\zeta} \right) U_+(\zeta), \tag{7.205}$$

$$f_o(\zeta) = \frac{\kappa k}{1+4\kappa^2} \left(2e^{-k\zeta/2\kappa} - e^{-sk\zeta} - e^{k\zeta/s} \right) U_+(\zeta). \tag{7.206}$$

The transmission image functions have the corresponding form:

$$h^{TE}(\zeta) = \delta_+(\zeta) + f^{TE}(\zeta), \tag{7.207}$$

$$h^{TM}(\zeta) = \delta_+(\zeta) + f^{TM}(\zeta). \tag{7.208}$$

To check these different image expressions, let us again set $\kappa \to \infty$, or $s \to 1$ for the vanishing grid, whence all reflection image functions are seen to vanish. The case $\kappa \to 0$ or $s = 1/\kappa \to \infty$ for an infinitely dense grid or conducting plane, leads to $f^{TE}(\zeta) = f^{TM}(\zeta) = -\delta_+(\zeta)$, because the function $\alpha e^{-\alpha\zeta}$ is a delta sequence for $\alpha \to \infty$. The function f_o is seen to vanish in both of these limiting cases. Thus, the two cases give no image or negative mirror image corresponding to total transmission and total reflection, respectively, which checks with the physical pictures of vanishing grid and conducting plane.

When studying the convergence of the image functions, it is seen that, as for the impedance surface, for real k one of the exponential functions is bound to diverge unless ζ is taken to be imaginary. Obviously, this again corresponds to a TM surface wave (waveguide mode) attached to the grid.

Vertical electric dipole above a metallic grid

As an example of application of the image concept let us consider a horizontal metallic grid with a vertical electric dipole $\mathbf{J}(\mathbf{r}) = \mathbf{u}_z IL\delta(\mathbf{r} - \mathbf{u}_z h)$. The grid gives rise to the reflection image current, analogously to (7.167),

$$\mathbf{J}_i^r(\mathbf{r}, \zeta) = -\mathbf{u}_z f^{TM}(\zeta) IL\delta(\boldsymbol{\rho})\delta(z+h) =$$

$$\mathbf{u}_z \frac{IL}{\sqrt{1+4\kappa^2}} \frac{d}{d\zeta} \left(e^{k\zeta/s} - e^{-sk\zeta} \right) U_+(\zeta)\delta(\boldsymbol{\rho})\delta(z+h), \tag{7.209}$$

while the transmission image current can be written as

$$\mathbf{J}_i^t(\mathbf{r},\zeta) = -\mathbf{u}_z h^{TM}(\zeta) IL\delta(\boldsymbol{\rho})\delta(z-h) =$$

$$-\mathbf{u}_z IL\delta(\boldsymbol{\rho})\delta(z-h)\delta_+(\zeta) - \mathbf{u}_z ILf^{TM}(\zeta)\delta(\boldsymbol{\rho})\delta(z-h). \qquad (7.210)$$

As noted above, the TM image function diverges unless the integration is performed along the imaginary direction on the ζ plane. Let us write

$$\zeta = -j\zeta'. \qquad (7.211)$$

The reflected electric field expressed in terms of the reflection image, with the complex distance function defined as

$$D = \sqrt{(\rho-\rho')^2 + (z-z'-j\zeta')^2}, \quad \Im\{D\} \le 0, \qquad (7.212)$$

reads

$$\mathbf{E}_r(\mathbf{r}) = -k\eta \int_0^\infty \int_V \overline{\overline{G}}(D)\cdot \mathbf{J}_i^r(\mathbf{r}', -j\zeta')dV'd\zeta' =$$

$$k\eta IL\mathbf{u}_z\cdot \int_0^\infty \overline{\overline{G}}(\sqrt{\rho^2+(z+h-j\zeta')^2}) f^{TM}(-j\zeta')d\zeta'. \qquad (7.213)$$

The field can easily be computed numerically, because the integrand converges exponentially owing to the exponential decay of the Green function. Let us look at some asymptotic cases of the field. When the source and/or the field point are/is far from the surface, i.e. for $k(z+h) \gg 1$, the field can be approximated according to the RCM method given in Section 7.1.4. The reflection field can be approximately expressed as

$$\mathbf{E}_r(\mathbf{r}) \approx -jk\eta\overline{\overline{G}}(|\mathbf{r}+\mathbf{u}_z h|)\cdot \mathbf{u}_z ILR^{TM}(k\cos\theta), \qquad (7.214)$$

and interpreted as the field arising from a dipole at the mirror image point with the original amplitude multiplied by the TM reflection coefficient associated with the geometrical optics ray going from the original dipole to the field point with a reflection at the grid.

The field close to the surface with $\rho \gg z+h$ can be evaluated, applying the approximation

$$D \approx \rho + \frac{1}{2\rho}(z+h-j\zeta')^2, \qquad (7.215)$$

in the form

$$\int_0^\infty G(D) f^{TM}(-j\zeta')d\zeta' \approx$$

$$\frac{k}{4j\sqrt{1+4\kappa^2}}\sqrt{\frac{j}{2\pi k\rho}}\left(se^{w_1^2}\mathrm{erfc}(w_1)+\frac{1}{s}e^{w_2^2}\mathrm{erfc}(w_2)\right), \tag{7.216}$$

with erfc denoting the complementary error function and

$$w_1=\sqrt{\frac{j}{2k\rho}}k(z+h-j\rho s), \qquad w_2=\sqrt{\frac{j}{2k\rho}}k(z+h+j\frac{\rho}{s}). \tag{7.217}$$

This can be identified as a sum of two Sommerfeld surface-wave solutions with imaginary numerical distances $-jk\rho s^2/2$ and $jk\rho/2s^2$, as discussed in FELSEN and MARCUVITZ (1973, pp. 509-510). In the region $2\kappa^2 \ll k\rho \ll 2/\kappa^2$ with $k(z+h) \ll 1$ we have $|w_1| \gg 1$ and $|w_2| \ll 1$, whence (7.216) can be shown to reduce to

$$\int_0^\infty G(D)f^{TM}(-j\zeta')d\zeta' \approx \frac{\kappa k}{4j}\sqrt{\frac{j}{2\pi k\rho}}e^{-\kappa k(z+h)}e^{-j\kappa^2 k\rho/2}, \tag{7.218}$$

which is obviously of the form of a radial surface wave, decaying in the z direction and propagating in the radial direction along the grid.

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7.3 The Sommerfeld half-space problem

The simplest electromagnetic field problem involving sources and material media, beyond that of the homogeneous space, is the problem with a source in front of a planar interface of two homogeneous half spaces. This problem was first studied seriously by SOMMERFELD in 1909, which is why it is often called the Sommerfeld half-space problem.

Let us find the image functions associated with two half-spaces 1 ($z > 0$) and 2 ($z < 0$) with respective medium parameters denoted by $\epsilon_1\epsilon_o$, $\mu_1\mu_o$ and $\epsilon_2\epsilon$, $\mu_2\mu$. The primary source is assumed to lie in the half-space 1. Let us denote the parameter ratios by $\epsilon = \epsilon_2/\epsilon_1$, $\mu = \mu_2/\mu_1$.

For static fields, the image principle is well documented in elementary textbooks. For example, if a point charge Q is located at the point $\mathbf{r} = \mathbf{u}_z h$ in the air (ϵ_o, μ_o), the field reflected from the dielectric ground medium $(\epsilon\epsilon_o, \mu_o)$ can be shown to equal that arising from the reflection image Q_r located at the mirror image point $\mathbf{r}_i$, when the medium is all air:

$$Q_r = -\frac{\epsilon - 1}{\epsilon + 1}Q, \qquad \mathbf{r}_i = \overline{\overline{C}} \cdot \mathbf{u}_z h = -\mathbf{u}_z h. \tag{7.219}$$

Also, the field transmitted into the ground in the static case can be found from the transmission image charge Q_T,

$$Q_T = \frac{2\epsilon}{\epsilon + 1}Q, \qquad \mathbf{r}_i = \mathbf{u}_z h, \tag{7.220}$$

which appears at the location of the original charge when the medium is all ground. It turns out that the correct interface condition, continuity of the transversal total electric field, is exactly satisfied with these two image sources.

For the time-dependent problem the image solution is not that obvious. If the original static charge is set in harmonic motion, the idea of setting the static image in the corresponding harmonic motion does not work, since, at the interface, only the condition for the electric field and not for the magnetic field is guaranteed. Thus, possible image sources in this case must necessarily be more complicated, which gives a motivation for the EIT method.

7.3.1 Reflection coefficients

As was seen in the previous section, construction of the image sources requires three reflection coefficient functions and their integral transforms to be found.

In the case of an interface between two homogeneous and isotropic media with the parameters ϵ_1, μ_1, ϵ_2, μ_2, the reflection coefficient functions were already given in (7.44) based on the wave impedances of the TE and TM polarized fields:

$$Z^{TE} = \frac{k\eta}{\beta}, \qquad Z^{TM} = \frac{\beta\eta}{k}, \tag{7.221}$$

$$R^{TE}(\beta_1) = \frac{\frac{k_2\eta_2}{\beta_2} - \frac{k_1\eta_1}{\beta_1}}{\frac{k_2\eta_2}{\beta_2} + \frac{k_1\eta_1}{\beta_1}} =$$

$$\frac{\mu_2\beta_1 - \mu_1\beta_2}{\mu_2\beta_1 + \mu_1\beta_2} = \frac{\mu\beta_1 - \sqrt{\beta_1^2 + B^2}}{\mu\beta_1 + \sqrt{\beta_1^2 + B^2}} = R(\mu, q), \tag{7.222}$$

$$R^{TM}(\beta_1) = \frac{\frac{\beta_2\eta_2}{k_2} - \frac{\beta_1\eta_1}{k_1}}{\frac{\beta_2\eta_2}{k_2} + \frac{\beta_1\eta_1}{k_1}} =$$

$$-\frac{\epsilon_2\beta_1 - \epsilon_1\beta_2}{\epsilon_2\beta_1 + \epsilon_1\beta_2} = -\frac{\epsilon\beta_1 - \sqrt{\beta_1^2 + B^2}}{\epsilon\beta_1 + \sqrt{\beta_1^2 + B^2}} = -R(\epsilon, q). \tag{7.223}$$

Here we denote

$$\epsilon = \frac{\epsilon_2}{\epsilon_1}, \qquad \mu = \frac{\mu_2}{\mu_1}, \tag{7.224}$$

and

$$q = \frac{\beta_1}{B}, \qquad B = \sqrt{k_2^2 - k_1^2} = k_1\sqrt{\mu\epsilon - 1}, \tag{7.225}$$

where $k_i = \omega\sqrt{\epsilon_i\mu_i}$. The reflection function $R(\alpha, q)$ applied above is defined by

$$R(\alpha, q) = \frac{\alpha q - \sqrt{q^2 + 1}}{\alpha q + \sqrt{q^2 + 1}}. \tag{7.226}$$

The reflection coefficients obviously satisfy the duality condition (7.78). For the third reflection coefficient we can evaluate after some effort

$$R_o(\beta_1) = \frac{k_1^2}{K^2}(R^{TM} - R^{TE}) = \frac{2}{(\mu q + \sqrt{q^2 + 1})(\epsilon q + \sqrt{q^2 + 1})} =$$

$$-\frac{\epsilon^2 - 1}{\epsilon(\epsilon - \mu)}[R(\epsilon, q) - E] - \frac{\mu^2 - 1}{\mu(\mu - \epsilon)}[R(\mu, q) - M]. \tag{7.227}$$

with

$$E = \frac{\epsilon - 1}{\epsilon + 1}, \quad M = \frac{\mu - 1}{\mu + 1}. \tag{7.228}$$

The image functions can be found in a convenient series form if we expand the reflection coefficient functions in terms of a reflection parameter r defined by

$$r = R(1, q) = \frac{q - \sqrt{q^2 + 1}}{q + \sqrt{q^2 + 1}}. \tag{7.229}$$

In fact, the following expansion can be easily verified:

$$R(\alpha, q) = \frac{A + r}{1 + Ar}, \quad A = \frac{\alpha - 1}{\alpha + 1}. \tag{7.230}$$

Applying the Taylor expansion, we can write

$$R(\alpha,q) = (A+r)\sum_{n=0}^{\infty}(-Ar)^n = A - \frac{4\alpha}{\alpha^2-1}\sum_{n=1}^{\infty}(-Ar)^n. \tag{7.231}$$

Obviously, the series expansion converges best for small $|Ar|$ values.

7.3.2 Reflection image functions

The reflection image sources are obtained in terms of a function of two variables, $f(\alpha,p)$, which can be identified through the following integral identity, found in any collection of Laplace transforms, for example, ABRAMOWITZ and STEGUN (1964).

$$\int_0^{\infty} 2n\frac{J_{2n}(p)}{p}e^{-qp}dp = [\sqrt{q^2+1}-q]^{2n} = (-r)^n, \tag{7.232}$$

where $J_{2n}(p)$ is the Bessel function of order $2n$.

In fact, if we write

$$R(\alpha,q) = \int_0^{\infty}\left[f(\alpha,p) + \frac{\alpha-1}{\alpha+1}\delta_+(p)\right]e^{-pq}dp, \tag{7.233}$$

the image function $f(\alpha,p)$ can be identified from the integral identity and the series expansion of the reflection coefficient in the form

$$f(\alpha,p) = -\frac{8\alpha}{\alpha^2-1}\sum_{n=1}^{\infty} n\left(\frac{\alpha-1}{\alpha+1}\right)^n \frac{J_{2n}(p)}{p}U_+(p). \tag{7.234}$$

The special case $\alpha = 1$ gives us

$$f(1,p) = -2\frac{J_2(p)}{p}U_+(p). \tag{7.235}$$

Values for the image function $f(\alpha,p)$ can be easily calculated from the above series, because Bessel functions with the same argument but different indices can be quickly computed with the aid of simple recursive formulas, as explained in ABRAMOWITZ and STEGUN (1964, pp.385–386).

There exists a differential equation for the image function $f(\alpha,p)$, from which it is easy to derive the Taylor expansion for the same function. In fact, it can be easily checked that the reflection coefficient function satisfies the algebraic equation

$$R(\alpha, q) - \frac{\alpha - 1}{\alpha + 1} = \alpha \frac{(q - \sqrt{q^2 + 1})^2 - \frac{\alpha-1}{\alpha+1}}{(\alpha^2 - 1)q^2 - 1} = -\alpha \frac{r + \frac{\alpha-1}{\alpha+1}}{(\alpha^2 - 1)q^2 - 1}, \quad (7.236)$$

which inserted in the integral identity (7.233) gives rise to the following differential equation for the image function:

$$(\alpha^2 - 1)\frac{d^2}{dp^2}f(\alpha, p) - f(\alpha, p) = -\alpha[f(1, p) + \frac{\alpha - 1}{\alpha + 1}\delta_+(p)]. \quad (7.237)$$

To define the solution uniquely, the following boundary conditions are needed:

$$f(\alpha, 0) = f(\alpha, \infty) = 0. \quad (7.238)$$

It is seen from the differential equation that the function $f(\alpha, p)$ is continuous at $p = 0_+$ but its first derivative (with respect to p) experiences a step discontinuity

$$f'(\alpha, 0_+) = -\frac{\alpha}{(\alpha + 1)^2}. \quad (7.239)$$

Taylor series expansion

Applying the differential equation (7.237), a Taylor series expansion can be written for the image function. The function $f(1, p)$ has the known expansion

$$f(1, p) = -2\frac{J_2(p)}{p}U_+(p) = -\sum_{n=0}^{\infty} \frac{(-1)^n (p/2)^{2n+1}}{n!(n + 2)!}U_+(p), \quad (7.240)$$

which can be substituted in the right-hand side of the differential equation (7.237). Obviously, like $f(1, p)$, $f(\alpha, p)$ is also an odd function of p and can be written as a power series with unknown coefficients:

$$f(\alpha, p) = \sum_{n=0}^{\infty} A_n \frac{p^{2n+1}}{(2n + 1)!}U_+(p). \quad (7.241)$$

The coefficients can be solved by equating the factors of equal powers of p on each side of the differential equation (7.237), which gives us a recursive formula. The second derivative gives rise to a delta discontinuity:

$$f''(\alpha, p) = A_0\delta_+(p) + \sum_{n=1}^{\infty} A_n \frac{p^{2n-1}}{(2n - 1)!}U_+(p). \quad (7.242)$$

Equating the coefficients of the delta functions gives us first

$$A_0 = -\frac{\alpha}{(\alpha+1)^2}. \tag{7.243}$$

Further coefficients are obtained from the recursive formula

$$A_{n+1} = \frac{A_n}{\alpha^2-1} + \frac{\alpha}{\alpha^2-1}\frac{(-1)^n(2n+1)!}{2^{2n+1}(n+2)!n!}, \tag{7.244}$$

whose solutions start with

$$A_1 = \frac{\alpha(\alpha+3)}{4(\alpha+1)^3}, \quad A_2 = -\frac{\alpha(\alpha^2+4\alpha+5)}{8(\alpha+1)^4},$$

$$A_3 = \frac{\alpha(5\alpha^3+25\alpha^2+47\alpha+35)}{64(\alpha+1)^5}. \tag{7.245}$$

Asymptotic expansion

To write an expansion for the image function $f(\alpha,p)$ valid for large p values, we must first take out the term corresponding to the pole of the reflection coefficient. In fact, writing (7.226) in the form

$$R(\alpha,q) = \frac{1}{\epsilon^2-1}\frac{(\epsilon q-\sqrt{q^2+1})^2}{q^2-\frac{1}{\epsilon^2-1}} \tag{7.246}$$

shows us that there are poles at

$$q_p = \pm\frac{1}{\sqrt{\epsilon^2-1}}, \tag{7.247}$$

which correspond to exponential image functions. With the convention $\Re\{\sqrt{\epsilon^2-1}\} > 0$ the pole $q_p = 1/\sqrt{\epsilon^2-1}$ leads to a converging image, while the other one can be shown to be cancelled by a zero of the numerator.

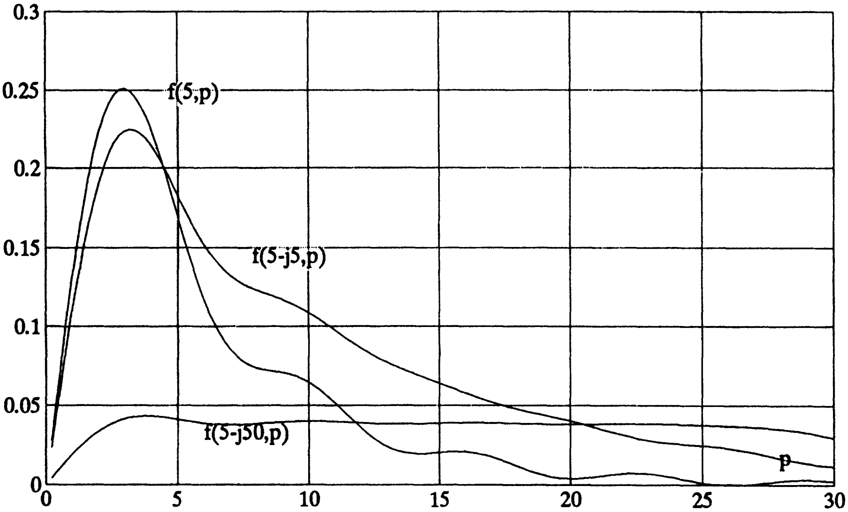

Fig. 7.4 Examples of the image function $f(\alpha, p)$ (absolute value) for different complex values of the parameter α.

Extracting the pole leaves us with an expression for the reflection coefficient function that has branch points at $q_b = \pm j$. The image function can be obtained by applying the Mellin inverse and Watson's lemma as was done by LINDELL and ALANEN (1984, part II) to produce an expression valid for large p values:

$$f(\alpha,p) \approx -\frac{2\alpha^2}{(\alpha^2-1)^{3/2}} e^{-p/\sqrt{\alpha^2-1}} + \frac{4}{\sqrt{2\pi}\alpha p^{3/2}} \sin(p+\frac{\pi}{4}) -$$

$$\frac{3(3\alpha^3-8)}{2\sqrt{2\pi}\alpha^3 p^{5/2}} \cos(p+\frac{\pi}{4}) - \frac{15(23\alpha^4-144\alpha^2+128)}{32\sqrt{2\pi}\alpha^5 p^{7/2}} \sin(p+\frac{\pi}{4}) +$$

$$\frac{105(91\alpha^6-472\alpha^4+896\alpha^2-512)}{256\sqrt{2\pi}\alpha^7 p^{9/2}} \cos(p+\frac{\pi}{4}) \ldots \tag{7.248}$$

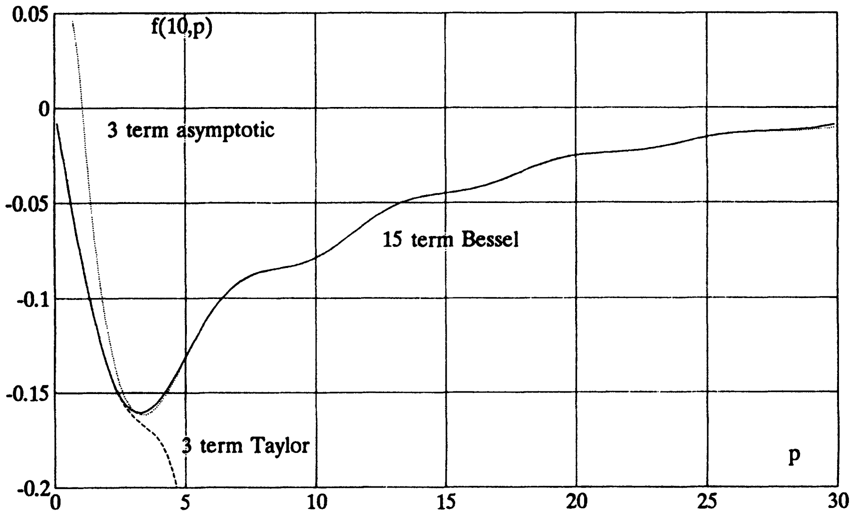

Fig. 7.5 Approximate values of the image function $f(\alpha, p)$ for $\alpha = 10$, as calculated from the 3 term Taylor expansion and 3 term asymptotic expansion, compared with the 15 term Bessel expansion, which may be considered as exact for $p < 30$.

In Fig. 7.4, values for the function $f(\alpha, p)$ are seen for different values of the parameter α. For real α the function is real, for complex α, only absolute values are given. It is seen that for large $|\alpha|$, the oscillations in the function are reduced and the exponential term in (7.248) is dominant.

TE and TM image functions

The relation between the image function $f(\alpha, p)$ and the functions $f^{TE}(\zeta)$, $f^{TM}(\zeta)$ defined in the Section 7.1 can be found if the integral expression (7.233) is compared with (7.69), (7.70). In this case we have

$$H(\zeta) = \zeta, \tag{7.249}$$

and the relation between the integration parameters p and ζ is

$$pq = \frac{\beta_1 p}{B} = j\beta_1\zeta, \tag{7.250}$$

or

$$p = jB\zeta = jk_1\zeta\sqrt{\mu\epsilon - 1}. \tag{7.251}$$

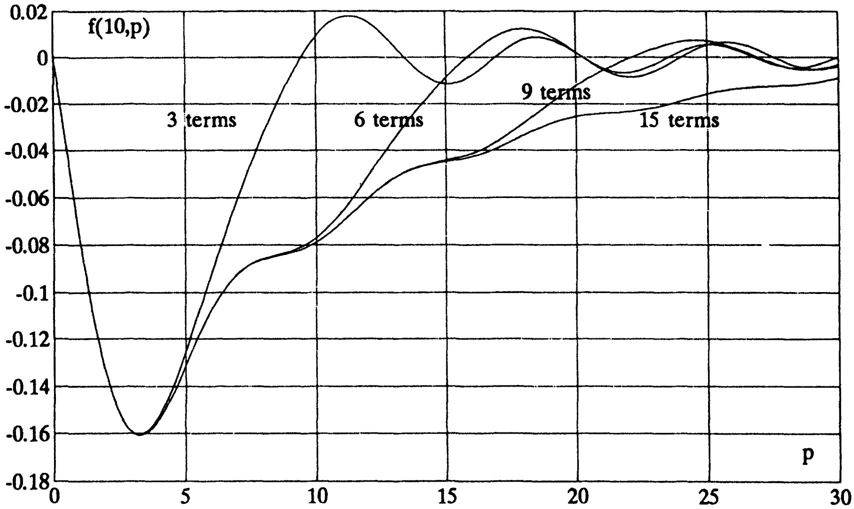

Fig. 7.6 Approximate values of the image function $f(\alpha, p)$ as calculated from the Bessel expansion with different numbers of terms. The accuracy of the Bessel series breaks down at approximately $p = 2N$, where N is the number of terms in the series.

Thus, we can write for the two image functions f^{TE} and f^{TM} the expressions

$$f^{TE}(\zeta) = jBf(\mu, jB\zeta) + M\delta_+(\zeta) =$$

$$-\frac{8\mu}{\mu^2-1}\sum_{n=1}^{\infty} n\left(\frac{\mu-1}{\mu+1}\right)^n \frac{J_{2n}(jB\zeta)}{\zeta} + \frac{\mu-1}{\mu+1}\delta_+(\zeta), \tag{7.252}$$

$$f^{TM}(\zeta) = -jBf(\epsilon, jB\zeta) - E\delta_+(\zeta) =$$

$$\frac{8\epsilon}{\epsilon^2-1}\sum_{n=1}^{\infty} n\left(\frac{\epsilon-1}{\epsilon+1}\right)^n \frac{J_{2n}(jB\zeta)}{\zeta} - \frac{\epsilon-1}{\epsilon+1}\delta_+(\zeta). \tag{7.253}$$

The third image function $f_o(\zeta)$ can be obtained directly through the definition of $R_o(\beta_1)$:

$$R_o(\beta_1) = -\frac{\epsilon^2-1}{\epsilon(\epsilon-\mu)}[R(\epsilon,q) - E] - \frac{\mu^2-1}{\mu(\mu-\epsilon)}[R(\mu,q) - M], \tag{7.254}$$

$$f_o(\zeta) = -\frac{\epsilon^2-1}{\epsilon(\epsilon-\mu)}jBf(\epsilon, jB\zeta) - \frac{\mu^2-1}{\mu(\mu-\epsilon)}jBf(\mu, jB\zeta). \tag{7.255}$$

It is readily seen that the $f_o(\zeta)$ function is invariant in the duality transformations, while $f(\epsilon, p)$ and $f(\mu, p)$ transform to one another.

It can also be verified, with some effort, that the differential equation (7.87) for $f_o(\zeta)$, valid for any layered structure, is really satisfied in the present Sommerfeld half-space problem case. To prove this, we must invoke the differential equation (7.237).

The dyadic image operator $\overline{\overline{f}}_e$, needed in the construction of the image source, can be written in any of the forms (7.90)–(7.92), for example, according to (7.92) as

$$\overline{\overline{f}}_e(j\nabla_t, \zeta) = f^{TM}(\zeta)\mathbf{u}_z\mathbf{u}_z + f^{TE}\overline{\overline{I}}_t + f_o(\zeta)\frac{1}{k^2}\mathbf{u}_z\mathbf{u}_z \cdot \nabla\nabla_t. \qquad (7.256)$$

Making a partial integration in the field integral, it is possible to transfer z differentiation $\mathbf{u}_z \cdot \nabla$ to ζ differentiation, whence it is possible to write instead of (7.256)

$$\overline{\overline{f}}_e(j\nabla_t, \zeta) = f^{TM}(\zeta)\mathbf{u}_z\mathbf{u}_z + f^{TE}\overline{\overline{I}}_t - f_o'(\zeta)\frac{1}{k^2}\mathbf{u}_z\nabla_t, \qquad (7.257)$$

which appears to be a very useful form, because it involves only one differentiation of the current source $\mathbf{J}$. The derivative of the function $f_o(\zeta)$ can be found analytically.

Convergence

Since the Bessel function $J_n(p)$ is known to converge for $|p| \to \infty$ only for real values of p, to obtain a converging image, the integration variable ζ should be chosen so that the parameter p in

$$\zeta = \frac{p}{jB} = \frac{p}{jk_1\sqrt{\mu\epsilon - 1}} \qquad (7.258)$$

is real. Thus it is wise to keep p as the integration parameter since $p = 0...\infty$ defines the image line. When B is not imaginary, ζ becomes complex, which means that the image line is defined on the complex ζ plane by

$$z = -h - \zeta = -h - \frac{p}{jB}. \qquad (7.259)$$

The image source and the reflection field

The dyadic image operator $\overline{\overline{f}}_e(j\nabla_t, \zeta)$ is now known and the expression for the image source in the Sommerfeld half-space problem can be expressed in terms of the image functions and the original source $\mathbf{J}(\mathbf{r})$ in the form

$$\mathbf{J}_i(\mathbf{r}, \zeta) = \overline{\overline{f}}_e(j\nabla_t, \zeta) \cdot \overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r}), \qquad (7.260)$$

whence the reflection field can be written as

$$\mathbf{E}_r(\mathbf{r}) = -jk_1\eta_1 \left(\overline{\overline{I}} + \frac{1}{k_1^2}\nabla\nabla\right)\cdot \int_V \int_0^\infty G(\mathbf{r}-\mathbf{r}'+\mathbf{u}_z\zeta)\mathbf{J}_i(\mathbf{r}',\zeta)dV'd\zeta. \quad (7.261)$$

This is the general solution for the reflected field and includes all special cases of sources, medium parameters and field points. For example, a VED source (vertical electric dipole above a horizontal interface) $\mathbf{J}(\mathbf{r}) = \mathbf{u}_z IL\delta(\mathbf{r}-\mathbf{u}_z h)$ gives rise to a vertical image source:

$$\mathbf{J}_i(\mathbf{r},\zeta) = -\mathbf{u}_z ILf^{TM}(\zeta)\delta(\mathbf{r}+\mathbf{u}_z h). \quad (7.262)$$

Because the field is everywhere transverse magnetic, only the function f^{TM} appears. On the other hand, a HED source (horizontal dipole) $\mathbf{J}(\mathbf{r}) = \mathbf{v}IL\delta(\mathbf{r}-\mathbf{u}_z h)$ with $\mathbf{v}\cdot\mathbf{u}_z = 0$ gives rise to both horizontal and vertical image currents:

$$\mathbf{J}_i(\mathbf{r},\zeta) = \mathbf{v}f^{TE}(\zeta)IL\delta(\boldsymbol{\rho})\delta(z+h) - \mathbf{u}_z f_o'(\zeta)\frac{IL}{k_1^2}[\mathbf{v}\cdot\nabla\delta(\boldsymbol{\rho})]\delta(z+h). \quad (7.263)$$

The vertical component is a double line current, like a transmission line, because of the term $\mathbf{v}\cdot\nabla\delta(\boldsymbol{\rho})$.

Limiting cases of the theory

Let us consider some limiting cases to test these results. For $\epsilon \to 1$ and $\mu \to 1$ we obviously have $E \to 0$, $M \to 0$ and $B \to 0$, whence $f^{TM}(\zeta) \to 0$, $f^{TE}(\zeta) \to 0$ and $f_o(\zeta) \to 0$. Thus, the image source vanishes, which is in accord with the vanishing interface.

Letting $\epsilon \to \infty$ and $\mu = 1$, we have $f^{TE}(\zeta) = jBf(1, jB\zeta)$, which for $B \to k_2 \to \infty$ acts like $\delta_+(\zeta)\int_0^\infty f(1,p)dp = -\delta_+(\zeta)$ in the field integral. Also, we have $f^{TM}(\zeta) \to -\delta_+(\zeta)$ because the continuous term in (7.253) vanishes. Further, we have $f_o(\zeta) \to 0$ for the same reason, whence the dyadic operator from (7.256) can be simply written as $\overline{\overline{f}}_e(j\nabla_t,\zeta) \to -\overline{\overline{I}}$ and the image source becomes $\mathbf{J}_i(\mathbf{r},\zeta) \to -\mathbf{J}_c(\mathbf{r}) = -\overline{\overline{C}}\cdot\mathbf{J}(\overline{\overline{C}}\cdot\mathbf{r})$. This is obviously the image in a PEC plane.

For large but finite ϵ, we can take the first term of the asymptotic expansion (7.248) to approximate

$$f^{TM} = -\delta_+(\zeta) - jBf(\epsilon, jB\zeta) \approx -\delta_+(\zeta) + \frac{2jB\epsilon^2}{(\epsilon^2-1)^{3/2}}e^{-jB\zeta/\sqrt{\epsilon^2-1}} \approx$$

$$-\delta_+(\zeta) + \frac{2jB}{\epsilon}e^{-jB\zeta/\epsilon}. \quad (7.264)$$

This is similar to the impedance-surface function (7.163), if we identify $Z_s = \sqrt{\mu_2/\epsilon_2}$, which is a small quantity. In the same way, f^{TE} and f_o functions can be checked for the same result. This shows us clearly, that at the limit of large ϵ the interface can be approximated by an inpedance surface.

Finally, the reflection-coefficient method (RCM) can be shown to arise as the far field limiting case of the EIT formalism. This was already done in general form for the image Green dyadic in Section 7.1 and is not repeated here.

Green functions

The different Green functions defined in Section 7.1.4 can be expressed in terms of just one scalar Green function $K(\alpha, \rho, z)$ for the Sommerfeld half-space problem. In fact, defining

$$K(\alpha, \rho, z) = \int_0^\infty G(D) f(\alpha, p) dp, \tag{7.265}$$

with

$$D = \sqrt{\rho^2 + (z + \frac{p}{jB})^2}, \tag{7.266}$$

we can write

$$K^{TM}(\rho, z) = -K(\epsilon, \rho, z) - EG(r), \tag{7.267}$$

$$K^{TE}(\rho, z) = K(\mu, \rho, z) + MG(r), \tag{7.268}$$

$$K_o(\rho, z) = -\frac{\epsilon^2 - 1}{\epsilon(\epsilon - \mu)} K(\epsilon, \rho, z) - \frac{\mu^2 - 1}{\mu(\mu - \epsilon)} K(\mu, \rho, z). \tag{7.269}$$

The Green function can be computed for different argument ρ, z values and stored for computer memory instead of working with the image functions.

For large arguments we can approximate as in Section 7.1.4, which results in the limiting case of the Green function

$$K(\alpha, \rho, z) \to G(r) \left(R(\alpha, \frac{z}{r\sqrt{\mu\epsilon - 1}}) + \frac{\alpha - 1}{\alpha + 1} \right). \tag{7.270}$$

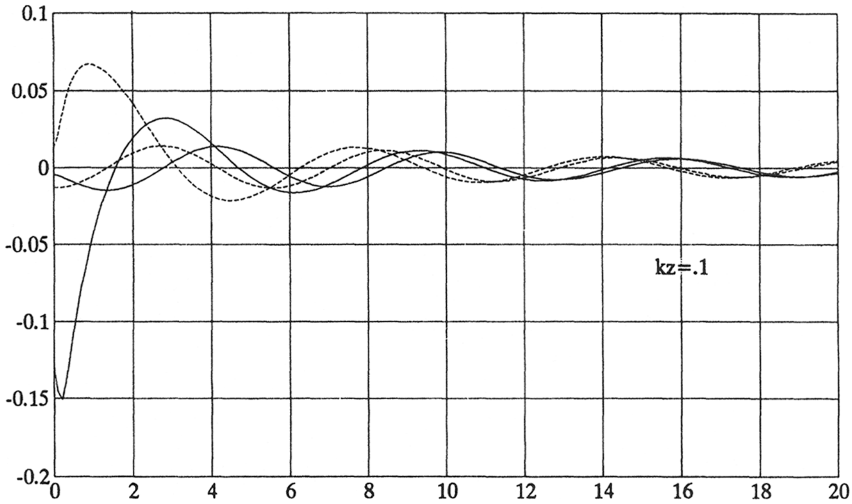

Fig. 7.7 Real (solid) and imaginary (dashed) parts of the Green function $K(\epsilon, \rho, z)$ for $\epsilon = 2$ and $kz = 0.1$, together with the asymptotic approximations (curves with larger amplitudes).

7.3.3 Antenna above the ground

As an example of applying the reflection image theory, let us consider a thin horizontal $\lambda/2$ dipole antenna above the ground with the purpose of finding the effect of the ground on its impedance. The general problem is quite involved and could be treated through an integral equation for the surface current of the antenna, in which the effect of the ground is taken into account in terms of a second Green dyadic. Let us, however, simplify the problem by assuming a known sinusoidal current distribution for the dipole, which is a fair assumption if the dipole is thin enough and not very close to the ground. Thus, the original source is

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}_x I_o \cos(k_1 x)\delta(y)\delta(z-h), \qquad |x| \leq \frac{\lambda}{4} \tag{7.271}$$

and zero elsewhere. Knowing the total field $\mathbf{E}(\mathbf{r})$ at the dipole, the impedance can be computed from the stationary functional

$$Z = -\frac{1}{I_o^2}\int_V \mathbf{E}\cdot\mathbf{J}dV = -\frac{1}{I_o}\int_{-\lambda/4}^{\lambda/4} \mathbf{E}(x)\cdot\mathbf{u}_x \cos(k_1 x)dx. \tag{7.272}$$

Writing the total field in terms of the incident and reflected fields $\mathbf{E} = \mathbf{E}_i + \mathbf{E}_r$, the impedance functional can be written as $Z = Z_i + Z_r$, where Z_i is the impedance of the dipole in free space and Z_r represents the effect of the ground. To compute Z_r, $\mathbf{E}$ can be replaced by the expression (7.261) for $\mathbf{E}_r$ in (7.272) in terms of the image source applying (7.257)

$$\mathbf{J}_i(\mathbf{r},\zeta) = \overline{\overline{f}}_e(j\nabla_t,\zeta)\cdot\mathbf{J}_c(\mathbf{r}) =$$

$$[f^{TE}(\zeta)\mathbf{u}_x I_o\cos(k_1x) + f_o'(\zeta)\mathbf{u}_z\frac{I_o}{k_1}\sin(k_1x)]\delta(y)\delta(z), \tag{7.273}$$

valid for $|x| < \lambda/4$. (7.272) can thus be written as

$$Z_r = jk_1\eta_1\int_{-\lambda/4}^{\lambda/4}\int_{-\lambda/4}^{\lambda/4}\int_0^\infty G(D)\Big[f^{TE}(\zeta)\cos(k_1x')\cos(k_1x)+$$

$$\frac{f_o'(\zeta)}{k_1}\sin(k_1x')\cos(k_1x)\Big]d\zeta dx dx' =$$

$$jk_1\eta_1\int_{-\lambda/4}^{\lambda/4}\int_{-\lambda/4}^{\lambda/4}\int_0^\infty G(D)\Big[f^{TE}(\zeta)\cos(k_1x)\cos(k_1x')+$$

$$[\frac{1}{\epsilon}f^{TM}(\zeta) + \frac{\epsilon-1}{\epsilon}\delta_+(\zeta)]\sin(k_1x)\sin(k_1x')\Big]d\zeta dx dx'. \tag{7.274}$$

For the last expression we have made partial integrations and applied (7.87) for the expansion of $f_o''(\zeta)$ to obtain

$$\frac{1}{k_1^2}f_o''(\zeta) = -f_o(\zeta) + f^{TM}(\zeta) - f^{TE}(\zeta) =$$

$$-\frac{1}{\epsilon}f^{TM}(\zeta) - \frac{\epsilon-1}{\epsilon}\delta_+(\zeta) - f^{TE}(\zeta). \tag{7.275}$$

At this point it is wise to change the integration variables from x, x' to $\xi = x + x'$ and $\tau = x - x'$. The distance D does not depend on the parameter ξ at all:

$$D = \sqrt{(x-x')^2 + (2h+\zeta)^2} = \sqrt{\tau^2 + (2h+\zeta)^2}. \tag{7.276}$$

Writing $2\cos(k_1x)\cos(k_1x') = \cos(k_1\xi) + \cos(k_1\tau)$ and $2\sin(k_1x)\sin(k_1x') = -\cos(k_1\xi) + \cos(k_1\tau)$, the parameter ξ can be integrated out in (7.274). Denoting $\theta = k_1\tau$ we have

$$Z_r = j\frac{\eta_1}{k_1}\int_0^\infty\int_0^\pi G(D)\Big[f^{TE}(\sin\theta + (\pi-\theta)\cos\theta)+$$

$$[\frac{1}{\epsilon}f^{TM} + \frac{\epsilon-1}{\epsilon}\delta_+](-\sin\theta + (\pi-\theta)\cos\theta)\Big] d\theta d\zeta. \tag{7.277}$$

Values for this integral are quite easily computed for different values of complex ϵ and height h. The ζ integration can also be performed separately and the final result expressed in terms of the image Green function $K(\epsilon,\theta)$ as defined in (7.265) and in LINDELL*et al.* (1985):

$$Z_r = j\frac{2\eta_1}{k_1}\int_0^\pi \Big[K(1,\theta)(\sin\theta + (\pi-\theta)\cos\theta) +$$

$$[\frac{1}{\epsilon}K(\epsilon,\theta) + \frac{\epsilon-1}{\epsilon}G(D_o(\theta))](-\sin\theta + (\pi-\theta)\cos\theta)\Big] d\theta, \tag{7.278}$$

with

$$K(\epsilon,\theta) = \int_0^\infty G(D)f(\epsilon,p)dp, \tag{7.279}$$

$$k_1 D = \sqrt{\theta^2 + (2k_1h + k_1\zeta)^2},$$

$$k_1 D_o = k_1|\mathbf{u}_x(x-x') + \mathbf{u}_z 2h| = \sqrt{\theta^2 + (2k_1h)^2}. \tag{7.280}$$

For large k_1h values, (7.278) can be shown to give the limiting value

$$Z_r \to \frac{-j\eta_1}{2\pi k_1 h}\frac{\sqrt{\epsilon}-1}{\sqrt{\epsilon}+1}e^{-j2k_1h}, \tag{7.281}$$

which can also be directly obtained by applying the reflection-coefficient method.

7.3.4 Transmission coefficients

Considering the problem of wave transmission through a planar interface between two isotropic media 1 and 2, the transmission coefficients can be written in the form

$$T^{TM} = 1 + R^{TM} = \frac{2\beta_2}{\epsilon\beta_1 + \beta_2}, \tag{7.282}$$

$$T^{TE} = 1 + R^{TE} = \frac{2\mu\beta_1}{\mu\beta_1 + \beta_2}, \tag{7.283}$$

$$T_o = \frac{k_1^2}{K^2}(T^{TM} - T^{TE}) = \frac{k_1^2}{K^2}(R^{TM} - R^{TE}) = R_o. \tag{7.284}$$

The transmission image source can be written in terms of the dyadic transmission image function $\overline{\overline{h}}_e$ (7.115), which can be found if the transmission dyadic is first written in the form

$$\overline{\overline{T}}_e = \frac{\epsilon_2}{\epsilon_1}T^{TM}\mathbf{u}_z\mathbf{u}_z + \frac{\mu_1\beta_2}{\mu_2\beta_1}T^{TE}\frac{\mathbf{u}_z\mathbf{u}_z{}^\times_\times\mathbf{KK}}{K^2} + \frac{\epsilon_2\beta_1}{\epsilon_1\beta_2}T^{TM}\frac{\mathbf{KK}}{K^2}, \quad (7.285)$$

and then in not exactly the same but the equivalent form in the sense that it produces the same transmitted field:

$$\overline{\overline{T}}_e = T_u\mathbf{u}_z\mathbf{u}_z + T_I\overline{\overline{I}}_t - T_0\frac{\mathbf{u}_z\mathbf{K}}{k_1^2}, \quad (7.286)$$

where, denoting $q = \beta_1/B$ and applying the definition (7.226), we have

$$T_u = \epsilon T^{TM} = \frac{2\epsilon\beta_2}{\epsilon\beta_1+\beta_2} = \epsilon[1-R(\epsilon,q)], \quad (7.287)$$

$$T_I = \frac{\beta_2}{\mu\beta_1}T^{TE} = \frac{2\beta_2}{\mu\beta_1+\beta_2} = 1-R(\mu,q), \quad (7.288)$$

$$T_0 = \beta_2\left(\frac{\epsilon_2\beta_1}{\epsilon_1\beta_2}T^{TM} - \frac{\mu_1\beta_2}{\mu_2\beta_1}T^{TE}\right)\frac{k_1^2}{K^2} = \frac{2\beta_2 B^2}{(\mu\beta_1+\beta_2)(\epsilon\beta_1+\beta_2)} =$$

$$\beta_2 T_o = \frac{B^2}{(\mu-\epsilon)\beta_1}[R(\mu,q)-R(\epsilon,q)]. \quad (7.289)$$

Note, again, the difference between T_o and T_0. Only the latter is needed here. The expressions (7.287)–(7.289) will be expanded in terms of powers of the reflection parameter r as given above, to find the transmission image functions needed for the construction of the transmission image source.

7.3.5 Transmission image functions

The transmission image source corresponding to a volume source $\mathbf{J}(\mathbf{r}')$ was shown in Section 7.1 to be of the form

$$\mathbf{J}_i(\mathbf{r}',\zeta) = \overline{\overline{h}}_e(j\nabla_t, z', \zeta)\cdot\mathbf{J}(\mathbf{r}'), \quad (7.290)$$

giving the total transmission field in the form

$$\mathbf{E}_T(\mathbf{r}) = -jk_2\eta_2\int_0^\infty\int_V \overline{\overline{G}}(D)\cdot\mathbf{J}_i(\mathbf{r}',\zeta)dV'd\zeta, \quad (7.291)$$

$$D = \sqrt{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2 + (z-H(z',\zeta))^2}. \quad (7.292)$$

The dyadic transmission image function $\overline{\overline{h}}_e(\mathbf{K}, h, \zeta)$ and the function $H(z, \zeta)$ are defined through the integral expression

$$\overline{\overline{T}}_e e^{-j\beta_1 h} = \int_0^\infty \overline{\overline{h}}_e(\mathbf{K}, h, \zeta) e^{-j\beta_2 H(h,\zeta)} d\zeta, \tag{7.293}$$

with the definitions (7.115)–(7.118), written again for convenience,

$$\overline{\overline{h}}_e(\mathbf{K}, h, \zeta) = h_u(h, \zeta)\mathbf{u}_z\mathbf{u}_z + h_I(h, \zeta)\overline{\overline{I}}_t - h_0(h, \zeta)\frac{1}{k_1^2}\mathbf{u}_z\mathbf{K}, \tag{7.294}$$

$$T_u e^{-j\beta_1 h} = \int_0^\infty h_u(h, \zeta) e^{-j\beta_2 H(h,\zeta)} d\zeta, \tag{7.295}$$

$$T_I e^{-j\beta_1 h} = \int_0^\infty h_I(h, \zeta) e^{-j\beta_2 H(h,\zeta)} d\zeta, \tag{7.296}$$

$$T_0 e^{-j\beta_1 h} = \int_0^\infty h_0(h, \zeta) e^{-j\beta_2 H(h,\zeta)} d\zeta. \tag{7.297}$$

Because of the exponential factor $\exp(-j\beta_1 h)$, an integral identity of a more general type than that applied for the reflection problem must be found. As such we adapt an identity which can be found, for example, in the reference GRADSHTEYN and RYZHIK (1980, equation 6.646), and through a change of variables can be written in the form

$$r^n \frac{e^{-j\beta_1 h}}{j\beta_1} = \int_0^\infty \left(\frac{H-h}{H+h}\right)^n J_{2n}(B\zeta) e^{-j\beta_2 H} H' d\zeta, \tag{7.298}$$

where ζ is the integration parameter, $r = (\beta_1 - \beta_2)/(\beta_1 + \beta_2)$ the reflection parameter and H denotes the function $H(h, \zeta)$:

$$H(h, \zeta) = \sqrt{\zeta^2 + h^2}, \qquad H' = \frac{d}{d\zeta} H(h, \zeta) = \frac{\zeta}{H(h, \zeta)}. \tag{7.299}$$

Let us now define a transmission image function $F(\alpha, h, \zeta)$ through the integral expression

$$[1 + R(\alpha, q)]\frac{e^{-j\beta_1 h}}{\beta_1} = \frac{2\alpha}{\alpha\beta_1 + \beta_2} e^{-j\beta_1 h} = \int_0^\infty jF(\alpha, h, \zeta) e^{-j\beta_2 H} H' d\zeta. \tag{7.300}$$

From the expansion (7.231) applied in (7.298) and (7.300), the transmission image function can be written explicitly in the form

$$F(\alpha,h,\zeta)=\frac{2\alpha}{\alpha+1}J_0(B\zeta)-$$

$$\frac{4\alpha}{\alpha^2-1}\sum_{n=1}^{\infty}\left(-\frac{\alpha-1}{\alpha+1}\right)^n\left(\frac{H-h}{H+h}\right)^n J_{2n}(B\zeta). \tag{7.301}$$

By making partial integration in (7.300), we can write

$$\frac{2\alpha\beta_2}{\alpha\beta_1+\beta_2}e^{-j\beta_1 h}=\alpha[1-R(\alpha,q)]e^{-j\beta_1 h}=$$

$$\int_0^\infty\left(\frac{2\alpha}{\alpha+1}\delta_+(\zeta)+F'(\alpha,h,\zeta)\right)e^{-j\beta_2 H}d\zeta. \tag{7.302}$$

In these expressions, the prime denotes differentiation with respect to the parameter ζ. (7.302) can be applied directly to find the image functions corresponding to the transmission coefficients T_u and T_I defined by (7.287), (7.288):

$$h_u(h,\zeta)=\frac{2\epsilon}{\epsilon+1}\delta_+(\zeta)+F'(\epsilon,h,\zeta), \tag{7.303}$$

$$h_I(h,\zeta)=\frac{2}{\mu+1}\delta_+(\zeta)+\frac{1}{\mu}F'(\mu,h,\zeta). \tag{7.304}$$

The third coefficient T_0 defined by (7.289) corresponds to the image function h_0 which can be written with the aid of (7.300) as follows:

$$h_0(h,\zeta)=jH'(h,\zeta)\frac{B^2}{\mu-\epsilon}[F(\mu,h,\zeta)-F(\epsilon,h,\zeta)]. \tag{7.305}$$

Thus, substituting in (7.294), the transmission image source for the general current source $\mathbf{J}(\mathbf{r})$ is finally obtained in the form

$$\mathbf{J}_i(\mathbf{r},\zeta)=\overline{\overline{h}}_e(j\nabla_t,z,\zeta)\cdot\mathbf{J}(\mathbf{r})=\left(\frac{2\epsilon}{\epsilon+1}\delta_+(\zeta)+F'(\epsilon,z,\zeta)\right)\mathbf{u}_zJ_z+$$

$$\left(\frac{2}{\mu+1}\delta_+(\zeta)+\frac{1}{\mu}F'(\mu,z,\zeta)\right)\overline{\overline{I}}_t\mathbf{J}_t+$$

$$H'(z,\zeta)\frac{\mu\epsilon-1}{\mu-\epsilon}[F(\mu,z,\zeta)-F(\epsilon,z,\zeta)]\mathbf{u}_z(\nabla_t\cdot\mathbf{J}). \tag{7.306}$$

The transmission field can now be computed from the expression

$$\mathbf{E}_T(\mathbf{r}) = -jk_2\eta_2 \int_0^\infty \int_V \overline{\overline{G}}(D) \cdot \mathbf{J}_i(\mathbf{r}', \zeta) dV' d\zeta, \tag{7.307}$$

$$D = \sqrt{(\rho - \rho')^2 + (z - H(z', \zeta))^2}. \tag{7.308}$$

It is noted that, as for the reflection case, a VED source gives rise to a vertical ($\mathbf{u}_z$ directed) image source while a horizontal source brings about both horizontal and vertical image components, unless the horizontal source is solenoidal ($\nabla \cdot \mathbf{J} = 0$), in which case the vertical component vanishes.

Corresponding expressions for a magnetic source can be directly written applying the duality transformation.

Asymptotic expression

Similarly to the reflection field case, we can write a simple asymptotic far field expression for the transmission field starting from the approximation

$$D \approx r + \mathbf{u}_r \cdot [\boldsymbol{\rho}' + \mathbf{u}_z H(z', \zeta)] =$$

$$r + r' \sin\theta \sin\theta' \cos(\varphi - \varphi') + H(z', \zeta) \cos\theta. \tag{7.309}$$

Here, we have applied spherical coordinates to the source and field points.

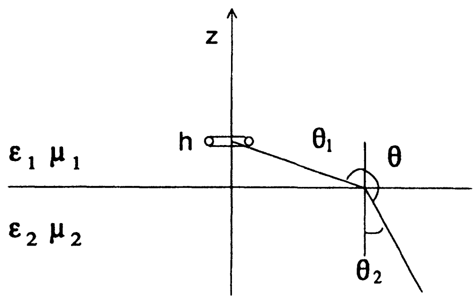

Fig. 7.8 Geometry of far field radiation through an interface.

Inserting (7.309) into (7.307), with the far field approximation for the Green dyadic, leads us to

$$\mathbf{E}_T(\mathbf{r}) \approx -jk_2\eta_2 \overline{\overline{G}}(r) \cdot \int_0^\infty \int_V e^{jk_2 \mathbf{u}_r \cdot [\boldsymbol{\rho}' + \mathbf{u}_z H(z', \zeta)]} \mathbf{J}_i(\mathbf{r}', \zeta) dV' d\zeta, \tag{7.310}$$

in which (7.290) can be substituted. The parameters of the Green dyadic are those of the medium 2. Comparison with (7.293) gives us the field in terms of the transmission dyadic:

$$\mathbf{E}_T(\mathbf{r}) \approx -jk_2\eta_2\overline{\overline{G}}(r)\cdot\int_V e^{jk_2\mathbf{u}_r\cdot\boldsymbol{\rho}'}\overline{\overline{T}}_e(\mathbf{K})e^{-j\beta_1 z'}\cdot\mathbf{J}(\mathbf{r}')dV', \tag{7.311}$$

with

$$\beta_2 = -k_2\mathbf{u}_r\cdot\mathbf{u}_z = k_2\cos\theta_2, \tag{7.312}$$

$$\mathbf{K} = k_2(\overline{\overline{I}} - \mathbf{u}_z\mathbf{u}_z)\cdot\mathbf{u}_r = \mathbf{u}_\rho k_2\sin\theta_2, \tag{7.313}$$

$$\beta_1 = \sqrt{k_1^2 - K^2} = k_1\sqrt{1-\epsilon\mu\sin^2\theta_2} = k_1\cos\theta_1, \tag{7.314}$$

$$\theta_2 = \pi - \theta, \qquad k_1\sin\theta_1 = k_2\sin\theta_2. \tag{7.315}$$

The expression (7.311) is the field arising from an image source whose amplitude is the original one multiplied by the transmission dyadic, which depends on the angle θ_2 of the field point.

An interpretation is obtained by considering the transmission from a point source $\mathbf{J}(\mathbf{r}) = \mathbf{v}IL\delta(\rho)\delta(z-h)$, for which we can write the expression (7.311) in the form

$$\mathbf{E}_T(\mathbf{r}) \approx -jk_2\eta_2\overline{\overline{G}}(r)e^{-j\beta_1 h}\cdot\overline{\overline{T}}_e(\mathbf{K})\cdot\mathbf{v}IL. \tag{7.316}$$

Writing

$$\overline{\overline{G}}(r)e^{-j\beta_1 h} \approx \overline{\overline{G}}(D), \tag{7.317}$$

$$D = \sqrt{(\mathbf{r}-\mathbf{u}_z h')\cdot(\mathbf{r}-\mathbf{u}_z h')}, \qquad h' = \frac{\beta_1}{\beta_1}h = \frac{\tan\theta_2}{\tan\theta_1}h, \tag{7.318}$$

we see that, actually, the position of the apparent image source can be interpreted geometrically as the point on the z axis from where the ray appears to be emanating.

It has been demonstrated by this author (LINDELL, 1988) that the original source may also be in complex space and yet the image theory is valid. In fact, representing a Gaussian beam by a point source in complex space, the well-known Goos–Hänchen shift at the interface reflection can be seen to arise together with the correct transmission beam.

7.3.6 Radiation from a loop antenna into the ground

As an example of the transmission image theory, let us consider a horizontal loop antenna in the air above non-magnetic ground with $\mu = 1$. Assuming the loop perimeter $2\pi b$ to be equal to one wavelength in the air, or $k_1 b = 1$, and the wire thin enough, the current can be approximated by the sinusoidal function

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}_\varphi I_o \cos\varphi \delta(\rho - b)\delta(z - h). \tag{7.319}$$

The exact transmission image current from (7.306) is then

$$\begin{aligned} \mathbf{J}_i(\mathbf{r}, \zeta) = [\delta_+(\zeta) + F'(1, z, \zeta)]\mathbf{J}(\mathbf{r}) - \\ H'(z, \zeta)[F(1, z, \zeta) - F(\epsilon, z, \zeta)]\mathbf{u}_z[\nabla \cdot \mathbf{J}(\mathbf{r})] = \\ I_o \delta(z - h)\delta(\rho - b)\Bigg[[\delta_+(\zeta) + F'(1, z, \zeta)]\mathbf{u}_\varphi \cos\varphi - \\ \frac{H'}{b}[F(1, z, \zeta) - F(\epsilon, z, \zeta)]\mathbf{u}_z \sin\varphi\Bigg]. \end{aligned} \tag{7.320}$$

When considering radiation into the ground, the far field approximation can be obtained from (7.311) with the same symbols

$$\mathbf{E}_T(\mathbf{r}) \approx -jk_2\eta_2 \overline{\overline{G}}(r) \cdot \int_S e^{jk_2 \mathbf{u}_r \cdot \boldsymbol{\rho}'} \overline{\overline{T}}_e(\mathbf{K}) e^{-j\beta_1 h} \cdot \mathbf{J}(\boldsymbol{\rho}') dS'. \tag{7.321}$$

This gives rise to the following radiation field expression:

$$\mathbf{E}_T(\mathbf{r}) \approx -jk_2 b \eta_2 I_o \frac{e^{-j(k_2 r + \beta_1 h)}}{4\pi r} \mathbf{A}, \tag{7.322}$$

with

$$\mathbf{A} = (\overline{\overline{I}} - \mathbf{u}_r \mathbf{u}_r) \cdot \overline{\overline{T}}_e \cdot \int_0^{2\pi} e^{jKb\cos(\varphi' - \varphi)} \mathbf{u}'_\varphi \cos\varphi' d\varphi', \tag{7.323}$$

$$\mathbf{K} = \mathbf{u}_\rho K, \qquad K = k_2 \sin\theta_2. \tag{7.324}$$

The integral can be evaluated after some algebra in the form

$$\int_0^{2\pi} e^{jKb\cos(\varphi' - \varphi)} \mathbf{u}'_\varphi \cos\varphi' d\varphi' = \mathbf{u}_\varphi 2\pi \cos\varphi J_1'(Kb) + \mathbf{u}_\rho 2\pi \sin\varphi \frac{J_1(Kb)}{Kb} \tag{7.325}$$

by invoking the integral identity

$$\int_0^{2\pi} e^{jz\cos\varphi'}\cos n\varphi' d\varphi' = 2j^n\pi J_n(z). \tag{7.326}$$

Thus, we can evaluate, writing the transmission dyadic $\overline{\overline{T}}_e$ from (7.285), with $\beta_2 = k_2\cos\theta_2$, $\beta_1 = k_1\sqrt{1-\epsilon\sin^2\theta_2}$:

$$\mathbf{A} = 2\pi(\overline{\overline{I}} - \mathbf{u}_r\mathbf{u}_r)\left(\epsilon T^{TM}\mathbf{u}_z\mathbf{u}_z + \frac{\beta_2}{\beta_1}T^{TE}\mathbf{u}_\varphi\mathbf{u}_\varphi + \frac{\epsilon\beta_1}{\beta_2}T^{TM}\mathbf{u}_\rho\mathbf{u}_\rho\right)$$

$$\cdot\left(\mathbf{u}_\varphi\cos\varphi J_1'(Kb) + \mathbf{u}_\rho\sin\varphi\frac{J_1(Kb)}{Kb}\right) =$$

$$2\pi\left(\mathbf{u}_\varphi\frac{2\beta_2}{\beta_1+\beta_2}\cos\varphi J_1'(Kb) + \mathbf{u}_\theta\frac{2\epsilon\beta_1\beta_2}{k_2(\epsilon\beta_1+\beta_2)}\frac{J_1(Kb)}{Kb}\right). \tag{7.327}$$

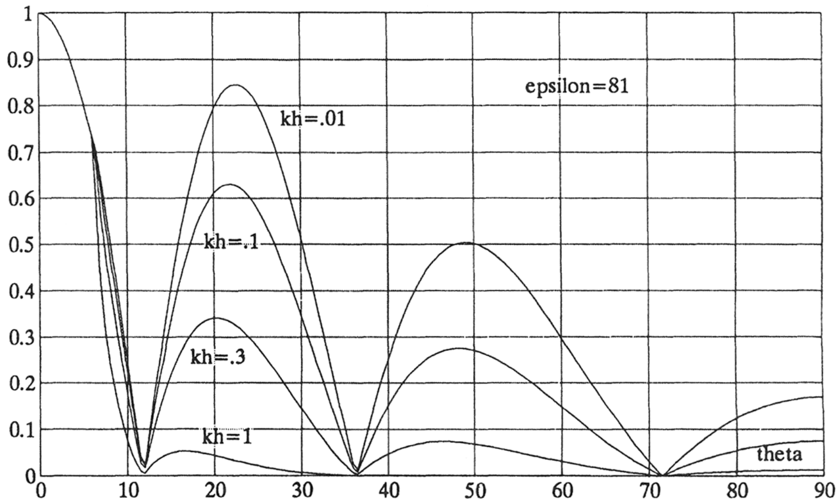

Fig. 7.9 Radiation pattern for a one-wavelength loop in air above the dielectric ground with $\epsilon = 81$, at different values of the height h.

The radiation pattern is obtained by normalizing the transmission field by the field amplitude at $\theta_2 = 0$:

$$\mathbf{E}_T(-\mathbf{u}_z r) \approx -jk_2\eta_2 I_o\frac{e^{-j(k_2r+k_1h)}}{2r}\frac{\sqrt{\epsilon}}{1+\sqrt{\epsilon}}\mathbf{u}_y, \tag{7.328}$$

whence we have the radiation pattern defined by the vector

$$\mathbf{F}(\theta_2, \varphi) = e^{-j(\beta_1 - k_1)h} \left(\frac{\sqrt{\epsilon} + 1}{\sqrt{\epsilon}} \right) \left(\mathbf{u}_\varphi \frac{2\beta_2}{\beta_1 + \beta_2} \cos\varphi J_1'(Kb) + \mathbf{u}_\theta \frac{2\epsilon\beta_1\beta_2}{k_2(\epsilon\beta_1 + \beta_2)} \frac{J_1(Kb)}{Kb} \right). \tag{7.329}$$

The radiation pattern in the xz plane, with $\varphi = 0$ is shown in Figs. 7.9 and 7.10 for some values of $k_1 h$ and ϵ. It is seen that the larger the permittivity ϵ, the narrower the transmitted radiation. The nulls in the radiation pattern correspond to nulls of the function $J_1'(\sqrt{\epsilon}\theta)$.

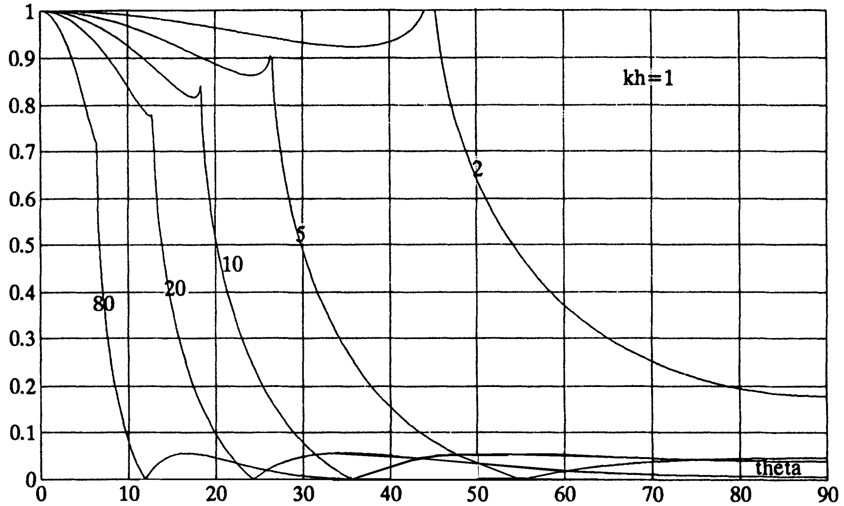

Fig. 7.10 Same as in Fig. 7.9, but the height h is kept constant, $k_1 h = 1$, and ϵ is varied from 2 to 80.

7.3.7 Scattering from an object in front of an interface

To demonstrate the application of the EIT in formulating a problem let us consider a scatterer in air in front of a half-space of isotropic medium. The incident field $\mathbf{E}_i(\mathbf{r})$ at the scatterer is the field from the original source plus its image due to the half-space without the effect of the scatterer and it is assumed to be known from the previous analysis. For a distant source, the incident field is approximately a plane wave plus its reflection from the interface.

For simplicity, let us assume that the scatterer is a dielectric object located at position $\mathbf{r} = \mathbf{u}_z h$ and sufficiently small so that its scattered field can be approximated by the field arising from a dipole with the dipole moment

$$\mathbf{p} = \overline{\overline{\alpha}} \cdot \mathbf{E}_t, \tag{7.330}$$

where $\overline{\overline{\alpha}}$, a dyadic, is the tensor dielectric polarizability. Explicit expressions for $\overline{\overline{\alpha}}$ are available for isotropic and anisotropic spheres and ellipsoids in the literature. $\mathbf{E}_t$ is the total field at the location of the scattering object: the sum of the incident field E_i and the reflected field due to the polarized dielectric object itself. The last component is unknown and EIT can be applied to its computation.

The goal is to solve for the unknown dipole moment vector $\mathbf{p}$. The corresponding current density vector can be written as

$$\mathbf{J}(\mathbf{r}) = j\omega\mathbf{p}\delta(\mathbf{r} - \mathbf{u}_z h). \tag{7.331}$$

The image of this dipole due to the interface can be written as

$$\mathbf{J}_i(\mathbf{r}, \zeta) = \left(f^{TM}(\zeta)\overline{\overline{I}} + \frac{1}{k_1^2} f_o(\zeta)\mathbf{u}_z\mathbf{u}_z \overset{\times}{\times} \nabla\nabla \right) \cdot \mathbf{J}_c(\mathbf{r}). \tag{7.332}$$

The mirror image of the current dipole is

$$\mathbf{J}_c(\mathbf{r}) = \overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r}) = (\overline{\overline{I}} - 2\mathbf{u}_z\mathbf{u}_z) \cdot j\omega\mathbf{p}\delta(\mathbf{r} + \mathbf{u}_z h). \tag{7.333}$$

The total electric field at the object, needed for the determination of the dipole moment, can be written as

$$\mathbf{E}_t(\mathbf{r}) = \mathbf{E}_i(\mathbf{r}) + \mathbf{E}_s(\mathbf{r}), \tag{7.334}$$

where the field $\mathbf{E}_s$ from the image of the dipole is

$$\mathbf{E}_s(\mathbf{r}) = -j\omega\mu_o \int\limits_V \int\limits_C \overline{\overline{G}}(\mathbf{r} - \mathbf{r}' + \mathbf{u}_z\zeta) \cdot \mathbf{J}_i(\mathbf{r}', z) dV' d\zeta, \tag{7.335}$$

where V denotes the integration volume of the mirror image and C is a line in the complex ζ plane ranging from the origin to infinity in such a way that the image function integrals converge, i.e. along a path such that

$$\arg\{\zeta\} = -\pi/2 - \arg\{\sqrt{\epsilon\mu - 1}\}. \tag{7.336}$$

Applying the reflection Green dyadic:

$$\overline{\overline{K}}(\mathbf{r}) = \left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla \right) K^{TM}(\mathbf{r}) + \frac{1}{k^2}(\mathbf{u}_z\mathbf{u}_z \overset{\times}{\times} \nabla\nabla) K_o(\mathbf{r}), \tag{7.337}$$

the field from the image source can now be expressed more concisely as

$$\mathbf{E}_s(\mathbf{r}) = -j\omega\mu_o \int_V \overline{\overline{K}}(\mathbf{r} - \mathbf{r}') \cdot \mathbf{J}_c(\mathbf{r}')dV'. \tag{7.338}$$

Inserting the proper mirror image current function from (7.333) we have

$$\mathbf{E}_s(\mathbf{r}) = \omega^2 \mu_o \overline{\overline{K}}(\mathbf{r} + \mathbf{u}_z h) \cdot (\overline{\overline{I}} - 2\mathbf{u}_z\mathbf{u}_z) \cdot \mathbf{p}, \tag{7.339}$$

which contains the unknown dipole moment vector $\mathbf{p}$, for which a final equation can now be formed. In fact, combining (7.330), (7.334) and (7.338) for the point $\mathbf{r} = \mathbf{u}_z h$, we arrive at an algebraic equation for the moment vector:

$$\mathbf{p} = \overline{\overline{\alpha}} \cdot \mathbf{E}_t(\mathbf{u}_z h) = \overline{\overline{\alpha}} \cdot \mathbf{E}_i(\mathbf{u}_z h) + \omega^2 \mu_o \overline{\overline{\alpha}} \cdot \overline{\overline{K}}(2\mathbf{u}_z h) \cdot \overline{\overline{C}} \cdot \mathbf{p}. \tag{7.340}$$

This has the solution

$$\mathbf{p} = [\overline{\overline{I}} - \omega^2 \mu_o \overline{\overline{\alpha}} \cdot \overline{\overline{K}}(2\mathbf{u}_z h) \cdot \overline{\overline{C}}]^{-1} \cdot \overline{\overline{\alpha}} \cdot \mathbf{E}^i(\mathbf{u}_z h). \tag{7.341}$$

Since all the terms on the right-hand side are known, it is possible to compute the moment vector for any exciting field $\mathbf{E}_i$. The only limitation for the applicability of this result is the dipole approximation of the scatterer, which may be poor if the object is located too close to the interface even if it is valid in free space. However, for small objects the range of validity probably extends to distances from the interface comparable with the object size. Computations have been made by LINDELL*et al.* (1991).

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7.4 Microstrip geometry

As another geometry let us consider the substrate of the microstrip consisting of a dielectric slab backed by a PEC plane, forming the basis for microstrip circuits and antennas. The microstrip was introduced as a lightweight, inexpensive and easily fabricable circuit and antenna structure in the early 1950s and its significance has never decreased. Despite its simplicity, the structure has been a challenge to scientists in electrical engineering for decades, because a simple method exact enough for the computation has not emerged.

The microstrip can be analysed with the EIT method after finding the proper image functions. To find the fields in the air above the structure, the dielectric slab with the ground plane is replaced by the image of the original source. Thus, for example, integral equations arising in microstrip problems can be formulated with these image sources. Defining Green functions for the structure gives an added advantage of reducing the number of integrations in field calculations. However, the Green functions must be stored in the computer in numerical form.

7.4.1 Reflection coefficients and image functions

Consider the typical microstrip geometry: planar interface at $z = 0$ between a homogeneous half space (air) $z > 0$ with the permittivity ϵ_o and a dielectric layer of thickness d and permittivity $\epsilon\epsilon_o$ above a perfectly conducting plane at $z = -d$. Permeability is assumed to be μ_o in the whole space.

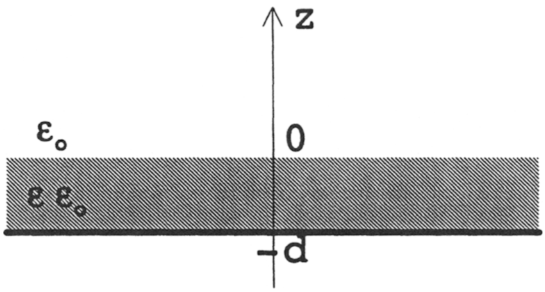

Fig. 7.11 The microstrip geometry: dielectric slab with ground plane.

Reflection coefficients

The reflection coefficients at the dielectric interface $z = 0$ of the microstrip structure can be derived from the transmission-line analogy, because the impedance dyadic at the interface, in Fourier space, is obviously

$$\overline{\overline{Z}}_L = \overline{\overline{Z}}_2 j \tan(\beta_2 d), \tag{7.342}$$

with $\overline{\overline{Z}}_2$ denoting the characteristic impedance dyadic of the dielectric medium as given in (7.10). (7.38) gives us the reflection dyadic with the TE and TM eigenvalues

$$R^{TE} = \frac{j\beta_1 \tan\beta_2 d - \beta_2}{j\beta_1 \tan\beta_2 d + \beta_2} = \frac{r^{TE} - e^{-2j\beta_2 d}}{1 - r^{TE} e^{-2j\beta_2 d}}, \tag{7.343}$$

$$R^{TM} = \frac{j\beta_2 \tan\beta_2 d - \epsilon\beta_1}{j\beta_2 \tan\beta_2 d + \epsilon\beta_1} = \frac{r^{TM} - e^{-2j\beta_2 d}}{1 - r^{TM} e^{-2j\beta_2 d}}. \tag{7.344}$$

Here we denote the partial reflection coefficients corresponding to the interface reflection by r^{TE}, r^{TM} and define

$$r^{TE} = \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} = r, \qquad r^{TM} = \frac{\beta_2 - \epsilon\beta_1}{\beta_2 + \epsilon\beta_1}. \tag{7.345}$$

Note that r^{TE} equals the reflection parameter r of Section 7.3. The third reflection coefficient R_o can also be easily derived:

$$R_o = \frac{k_1^2}{K^2}(R^{TM} - R^{TE}) = \frac{2j(\epsilon - 1)k_1^2 \tan\beta_2 d}{(j\beta_1 \tan\beta_2 d + \beta_2)(j\beta_2 \tan\beta_2 d + \epsilon\beta_1)}$$

$$= \frac{1 - e^{-4j\beta_2 d}}{(1 - r^{TE} e^{-2j\beta_2 d})(1 - r^{TM} e^{-2j\beta_2 d})} r_o, \tag{7.346}$$

with

$$r_o = \frac{k_1^2}{K^2}(r^{TM} - r^{TE}) = \frac{\epsilon + 1}{\epsilon}(r^{TM} + E). \tag{7.347}$$

Here, again we denote $E = (\epsilon - 1)/(\epsilon + 1)$. r_o given here equals what was denoted by R_o in the Sommerfeld problem (7.57) with $\mu = 1$. The expressions given here can also be quite straightforwardly generalized for the case $\mu \neq 1$.

Let us only consider the problem where the original source lies at the interface like a current in a microstrip circuit. To obtain the corresponding image functions, we should find the integral representations for the reflection coefficient functions of β_1 in the form

$$R(\beta_1) = \int_0^\infty f(\zeta)e^{-j\beta_1\zeta}d\zeta, \tag{7.348}$$

i.e., with $H(\zeta) = \zeta$ in (7.57).

Transverse electric image function

To find the function $f^{TE}(\zeta)$, the integral identity (7.298) must be written in another form:

$$\frac{r^m}{j\beta_2}e^{-j\beta_2 h} = \int_0^\infty \left(\frac{\zeta - h}{\zeta + h}\right)^m I_{2m}(B\sqrt{\zeta^2 - h^2})U_+(\zeta - h)e^{-j\beta_1\zeta}d\zeta. \tag{7.349}$$

Here, $I_n(x)$ denotes the modified Bessel function.

In forming the image function expressions the function $F_n(h, \zeta)$ to be defined below appears helpful. It has no connection with the function $F(\alpha, h, \zeta)$ of the previous section. The definition is

$$F_0(h, \zeta) = \frac{Bh}{\sqrt{\zeta^2 - h^2}} I_1(B\sqrt{\zeta^2 - h^2})U_+(\zeta - h), \tag{7.350}$$

$$F_m(h, \zeta) = -\frac{B}{2\sqrt{\zeta^2 - h^2}}\Big[(\zeta - h)I_{2m+1}(B\sqrt{\zeta^2 - h^2}) -$$

$$(\zeta + h)I_{2m-1}(B\sqrt{\zeta^2 - h^2})\Big]\left(\frac{\zeta - h}{\zeta + h}\right)^m U_+(\zeta - h), \quad m > 0. \tag{7.351}$$

The identity (7.349) can be written in terms of this function as

$$r^m e^{-j\beta_2 h} = \int_0^\infty \left[F_m(h, \zeta) + \delta_{m0}\delta_+(\zeta - h)\right] e^{-j\beta_1\zeta}d\zeta, \tag{7.352}$$

where δ_{m0} is the Kronecker symbol. This can be applied in finding the image function $f^{TE}(\zeta)$. First, we expand the total TE reflection coefficient as a Taylor series

$$R^{TE} = r + \sum_{n=1}^{\infty} \left[r^{n+1} - r^{n-1}\right] e^{-j\beta_2(2nd)}, \tag{7.353}$$

which can be interpreted as a sum of partial waves reflecting between the interfaces in the microstrip. Applying now the integral identity (7.352), we may write the image function as an expression involving the newly defined function $F_n(h,\zeta)$:

$$f^{TE}(\zeta) = -\delta_+(\zeta - 2d) + F_1(0,\zeta) + \sum_{n=1}^{\infty}[F_{n+1}(2nd,\zeta) - F_{n-1}(2nd,\zeta)]. \tag{7.354}$$

The result can also be expressed as a Taylor series by expanding the modified Bessel functions as

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}, \tag{7.355}$$

whence the function $F_n(h,\zeta)$ can be written for $n > 0$ as

$$F_n(h,\zeta) = -\frac{B}{2}\sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{y^{2n+k+1}}{(2n+k+1)!} - \frac{y^{2n+k-1}}{(2n+k-1)!}\right] U_+(\zeta - h), \tag{7.356}$$

with

$$x = \frac{B(\zeta+h)}{2}, \quad y = \frac{B(\zeta-h)}{2}. \tag{7.357}$$

The representation of the image function $f^{TE}(\zeta)$ as a series of modified Bessel functions I_n, individually divergent, may seem non-physical, but the sum function turns out to converge. However, such an expression may pose problems to the computer, because numbers of increasingly large values are being added and subtracted. It is wise to start summing from the large values of indices, i.e. from the smallest values of the terms.

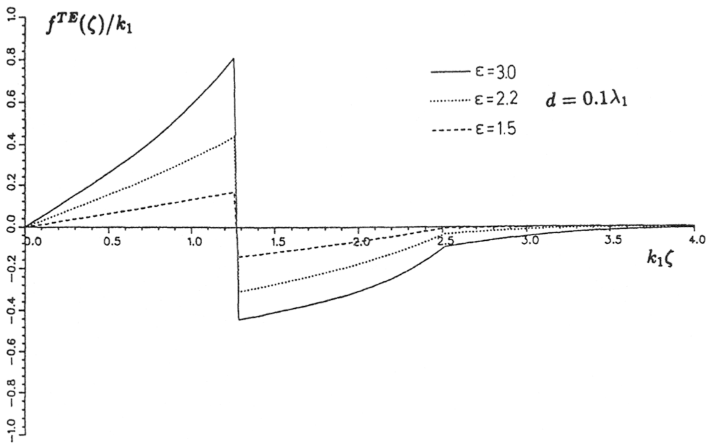

Fig. 7.12 Normalized TE image function for the microstrip for different relative permittivities ϵ of the substrate.

Transverse magnetic image function

When finding the TM image function the duality transform for the TE function does not work since we assumed the special case $\mu = 1$. Thus, f^{TM} must be evaluated from the start. Writing the reflection coefficient (7.344) as a Taylor series

$$R^{TM} = r^{TM} + \sum_{n=0}^{\infty} \left[r^{TM(n+1)} - r^{TM(n-1)} \right] e^{-j\beta_2(2nd)}, \tag{7.358}$$

we may apply the identity (7.352) if the reflection coefficient r^{TM} is expressed in terms of $r^{TE} = r$:

$$r^{TM} = -\frac{E + r}{1 + Er}. \tag{7.359}$$

Now we need the expansions

$$(1 + x)^n = \sum_{i=0}^{n} \frac{n!}{(n-i)!i!} x^i, \tag{7.360}$$

$$(1 - x)^{-n} = \sum_{k=0}^{\infty} \frac{(n-1+k)!}{(n-1)!k!} x^k, \quad n > 0, \tag{7.361}$$

which both converge for $|x| < 1$. Thus, we may write

$$\left(r^{TM}\right)^n = \sum_{k=0}^{\infty} C_k^n(E) r^k, \tag{7.362}$$

where the coefficient functions are polynomials defined by

$$C_k^n(x) = \sum_{i=0}^{n|k} \frac{(-1)^i n(n+k-i-1)!}{(n-i)!i!(k-i)!}(-x)^{n+k-2i}, \quad n > 0, \tag{7.363}$$

$$C_k^0(x) = \delta_{0k}. \tag{7.364}$$

The symbol $n|k$ here denotes 'smaller of the numbers n, k'. The coefficient function $C_k^n(x)$ can also be defined through the expansion

$$\tanh^n(\theta - \phi) = \sum_{k=0}^{\infty} C_k^n(\tanh\theta)\tanh^k\phi, \tag{7.365}$$

and it obeys the following rules

$$kC_k^n(x) = nC_n^k(x), \qquad n, k > 0, \tag{7.366}$$

$$C_k^1(x) = (-x)^{k+1} - (-x)^{k-1}, \qquad k > 0, \tag{7.367}$$

$$C_0^n(x) = (-x)^n, \tag{7.368}$$

$$C_1^n(x) = n[(-x)^{n+1} - (-x)^{n-1}]. \tag{7.369}$$

With this function we can write the reflection coefficient function in the following form

$$R^{TM} = \sum_{k=0}^{\infty} C_k^1(E) r^k + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \left[C_k^{n+1}(E) - C_k^{n-1}(E)\right] r^k e^{-j\beta_2(2nd)}. \tag{7.370}$$

Now applying the integral identity (7.352) we finally have an expression for the TM image function:

$$f^{TM}(\zeta) = C_o^1(E)\delta_+(\zeta) + \sum_{n=0}^{\infty}\left[C_o^{n+1}(E) - C_o^{n-1}(E)\right]\delta_+(\zeta - 2nd) +$$

$$\sum_{k=1}^{\infty} C_k^1(E) F_k(0,\zeta) + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[C_k^{n+1}(E) - C_k^{n-1}(E)\right] F_k(2nd,\zeta). \tag{7.371}$$

The image function f_o

The image function $f_o(\zeta)$ needed in the EIT method can be constructed much in the same way but with somewhat more complicated expressions. Writing again for the reflection coefficient R_o a Taylor series expression

$$R_o = \frac{\epsilon+1}{\epsilon}\left[1 - e^{4j\beta_2 d}\right]\left(r^{TM} + E\right)\times$$

$$\sum_{m=0}^{\infty}\left(r^{TE}e^{-2j\beta_2 d}\right)^m \sum_{n=0}^{\infty}\left(r^{TM}e^{-2j\beta_2 d}\right)^n =$$

$$\frac{\epsilon+1}{\epsilon}\sum_{m=0}^{\infty}\left[e^{-2jm\beta_2 d} - e^{-2j(m+2)\beta_2 d}\right]\times$$

$$\sum_{i=0}^{m}\left(r^{TE}\right)^{m-i}\left[E\left(r^{TM}\right)^i + \left(r^{TM}\right)^{i+1}\right], \tag{7.372}$$

we may again apply the identity if the reflection coefficient r^{TM} is expressed in terms of the parameter r. The resulting image function is

$$f_o(\zeta) = \frac{\epsilon+1}{\epsilon}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\sum_{i=0}^{m} D_k^i(E)\times$$

$$\left[F_{m+k-i}(2md,\zeta) - F_{m+k-i}(2(m+2)d,\zeta)\right], \tag{7.373}$$

with

$$D_k^i(E) = EC_k^i(E) + C_k^{i+1}(E), \quad k>0, \quad D_0^n(E) = 0. \tag{7.374}$$

These formulas are not too difficult to use in practice although the modified Bessel functions $I_n(x)$ are diverging, because the sum functions can be seen to converge, except for simple exponentially diverging parts, which can be extracted and handled otherwise. In Fig. 13, examples of the image functions are given for some parameter values.

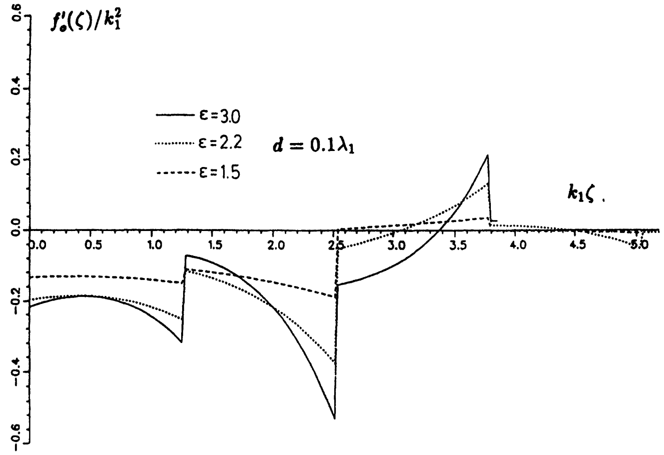

Fig. 7.13 Normalized derivative of the image function f_o for the microstrip for different relative permittivities ϵ of the substrate.

There is the connection (7.87) between the three image functions:

$$f_o''(\zeta) = k_1^2 \left(f^{TM}(\zeta) - f^{TE}(\zeta) - f_o(\zeta)\right), \tag{7.375}$$

which will save us from numerical differentiation of the f_o function.

The $f^{TM}(\zeta)$ and $f_o(\zeta)$ functions do not, however, converge as such. The lack of convergence is due to certain poles of the reflection coefficient, which correspond to propagating waveguide modes of the grounded slab waveguide, a situation similar to that of the impedance surface. Their contribution can, however, be extracted and made to converge by reorientating their location on the complex plane.

7.4.2 Fields at the interface

The image dyadic operator can be written in the many ways discussed in Section 7.1, for example, as

$$\overline{\overline{f}}_e(j\nabla_t, \zeta) = f^{TE}(\zeta)\overline{\overline{I}} + f_o(\zeta)\mathbf{u}_z\mathbf{u}_z \cdot (\overline{\overline{I}} + \frac{1}{k_1^2}\nabla\nabla), \tag{7.376}$$

which is perhaps the most preferable because it contains the simple function f^{TE}. The reflected fields in the half-space $z > 0$ due to the image sources can be written as

$$\mathbf{E}_r(\mathbf{r}) = -j\omega\mu_o \int_V \int_0^\infty \overline{\overline{G}}(D) \cdot \mathbf{J}_i(\mathbf{r}', \zeta) dV' d\zeta, \tag{7.377}$$

with

$$D = D(\mathbf{r}, \mathbf{r}', \zeta) = \sqrt{(\rho - \rho')^2 + (z - z' - \zeta)^2}, \qquad \Im\{D\} \leq 0. \tag{7.378}$$

Surface image sources

Restricting the original current source on the planar interface $z = 0$, $\mathbf{J}(\mathbf{r}) = \mathbf{J}_s(\rho)\delta(z)$, the transversal component of the electric field due to the image source represents the reaction of the grounded slab to the original field and is of importance for microstrip circuit analysis. This field added to the field due to the original source in free space gives the total transversal field, which is required for formulation of the integral equation of the original source problem. The formulation is made simpler by explicitly expressing the surface charge density

$$\varrho_s = -\frac{\nabla \cdot \mathbf{J}_s}{j\omega} \tag{7.379}$$

in the image source and field expressions (7.64), (7.377)

$$\mathbf{J}_i(\mathbf{r}, \zeta) = f^{TE}(\zeta)\mathbf{J}_s(\rho)\delta(z) - \mathbf{u}_z \frac{j\omega}{k_1^2} f_o(\zeta)\varrho_s(\rho)\delta'(z), \tag{7.380}$$

$$\varrho_i(\mathbf{r}, \zeta) = f^{TE}(\zeta)\varrho_s(\rho)\delta(z) + \frac{1}{k_1^2} f_o(\zeta)\varrho_s(\rho)\delta''(z), \tag{7.381}$$

$$\mathbf{E}_r(\mathbf{r}) = -j\omega\mu_o \int_V \int_0^\infty G(D)\mathbf{J}_i(\mathbf{r}', \zeta)dV'd\zeta +$$

$$\frac{1}{\epsilon_o} \int_V \int_0^\infty [\nabla' G(D)]\varrho_i(\mathbf{r}', \zeta)dV'd\zeta. \tag{7.382}$$

Here, the integration volume V is any volume containing the original source surface S. Note that the double differentiation within the Green dyadic of (7.377) is reduced through partial integration in (7.382) so that only one differentiation of $G(D)$ is present. This is necessary in reducing the singularity for the practical computation of the integrals. When taking the integration volume V large enough so that the original sources are zero on its boundary, we may perform further partial integrations with respect to z and ζ variables so that the differentiations can be transferred from the delta functions to the functions of ζ.

In this case, the equivalence $\partial/\partial z = \partial/\partial\zeta$ can be applied when operating on the source function of both ζ and $\mathbf{r}$ inside the field integral (7.382), allowing us to replace (7.380) and (7.381) by the following equivalent surface image sources

$$\mathbf{J}_{si}(\boldsymbol{\rho},\zeta) = f^{TE}(\zeta)\mathbf{J}_s(\boldsymbol{\rho}) - \mathbf{u}_z \frac{j\omega}{k_1^2} f_o'(\zeta)\varrho_s(\boldsymbol{\rho}), \tag{7.383}$$

$$\varrho_{si}(\boldsymbol{\rho},\zeta) = \left[f^{TE}(\zeta) + \frac{1}{k_1^2} f_o''(\zeta) \right] \varrho_s(\boldsymbol{\rho}). \tag{7.384}$$

The transverse electric field

When calculating only the transverse component $\mathbf{e}$ of the electric field $\mathbf{E}$, the $\mathbf{u}_z$ directed terms in the expressions of $\mathbf{J}_{si}$ can be omitted. The z integration in (7.382) can now be easily performed because of the delta functions and the transverse reflection field at the interface $z = 0$ reads

$$\mathbf{e}_r(\boldsymbol{\rho}) = -j\omega\mu_o \int_S \int_0^\infty G(D)\mathbf{J}_{si}(\boldsymbol{\rho}',\zeta)dS'd\zeta +$$

$$\frac{1}{\epsilon_o} \int_S \int_0^\infty [\nabla' G(D)]\varrho_i(\boldsymbol{\rho}',\zeta)dS'd\zeta, \tag{7.385}$$

with

$$D = \sqrt{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2 + \zeta^2}. \tag{7.386}$$

For the double differentiation of the $f_o(\zeta)$ function needed in the above expressions, an analytic form is desirable, to save us from numerical differentiation in the actual calculation

$$f_o''(\zeta) = \frac{B^2}{4} \left[\frac{\epsilon+1}{\epsilon} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \sum_{i=0}^{m} D_k^i \times \right.$$

$$[Q_{m+k-i}(2md,\zeta) - Q_{m+k-i}(2(m+2)d,\zeta)] -$$

$$\left. \frac{B^2}{\epsilon+1} \left(\delta_+(\zeta) - E\delta_+(\zeta-2d) - \frac{4\epsilon}{(\epsilon-1)^2} \sum_{m=2}^{\infty} (-E)^m \delta_+(\zeta - 2md) \right) \right], \tag{7.387}$$

with a new function defined as

$$Q_n(x,\zeta) = F_{n+1}(x,\zeta) + 2F_n(x,\zeta) + F_{n-1}(x,\zeta). \tag{7.388}$$

Denoting $D_o = |\rho - \rho'|$, the transverse total field in the same plane as the surface currents can also be written by defining two-dimensional Green functions:

$$\mathbf{e}(\rho) = -j\omega\mu_o \int_S [G(D_o) + K^{TE}(D_o)]\mathbf{J}_s(\rho')dS' +$$

$$\frac{1}{\epsilon_o}\int_S \nabla'[G(D_o) + L(D_o)]\varrho_s(\rho')dS'. \tag{7.389}$$

It is seen that the field can be expressed in terms of the free-space Green function G for the direct field and two new Green functions K^{TE} and L for the reflected field. These Green functions can be calculated in terms of the image functions as follows:

$$K^{TE}(D_o) = \int_0^\infty G(D)f^{TE}(\zeta)d\zeta = -G(D_1) +$$

$$\int_0^\infty G(D)\left(F_1(0,\zeta) + \sum_{n=1}^{\infty}[F_{n+1}(2nd,\zeta) - F_{n-1}(2nd,\zeta)]\right)d\zeta, \tag{7.390}$$

$$L(D_o) = \int_0^\infty G(D)\left[f^{TE}(\zeta) + \frac{1}{k_1^2}f_o''(\zeta)\right]d\zeta = K^{TE}(D_o) - EG(D_o) +$$

$$E^2G(D_1) + \frac{4\epsilon}{\epsilon^2-1}\sum_{m=2}^{\infty}(-E)^m G(D_m) + \frac{\epsilon^2-1}{4\epsilon}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\sum_{i=0}^{m} D_k^i \times$$

$$\int_0^\infty G(D)[Q_{m+k-i}(2md,\zeta) - Q_{m+k-i}(2(m+2)d,\zeta)]d\zeta, \tag{7.391}$$

with

$$D_m = \sqrt{(\rho-\rho')^2 + (2md)^2} = \sqrt{D_o^2 + (2md)^2}. \tag{7.392}$$

Note that the definition of the Green function L can be expressed as $K^{TM} - K_o$ in earlier notation.

In solving integral equations for the surface current problem it is not necessary to deal with the image sources at all if we directly compute the Green functions defined above and store them in the computer memory for further use.

7.4.3 The guided modes

As shown by LINDELL*et al.* (1987), the image function $f_o(\zeta)$ does not converge as ζ approaches infinity owing to certain poles of the reflection coefficient. To obtain convergence, it is necessary to extract a finite number of terms, responsible for the loss of convergence, and treat them separately. Each of these terms corresponds to a propagating mode guided by the grounded slab. For a sufficiently small thickness d and/or if ϵ is close to 1, there is only one such mode. This case is of particular interest to microstrip antenna design, because the guided mode increases coupling between the antenna elements, which is not generally what is wanted.

If only one guided mode propagates, we can write the image function $f_o(\zeta)$ in two parts

$$f_o(\zeta) = f_{oc}(\zeta) + f_{od}(\zeta), \tag{7.393}$$

with f_{oc} converging and f_{od} exponentially diverging

$$f_{od}(\zeta) = jAe^{\alpha\zeta}. \tag{7.394}$$

Here, α is the basic solution of the modal equation:

$$\tan(\sqrt{B^2 - \alpha^2}d) = \frac{\epsilon\alpha}{\sqrt{B^2 - \alpha^2}}. \tag{7.395}$$

The amplitude coefficient A can be obtained from the residue of the reflection coefficient R_o:

$$A = \frac{j2\epsilon k_1^2 \alpha d^2[(\epsilon - 1)(k_1 d)^2 - (\alpha d)^2]}{(\epsilon - 1)[(k_1 d)^2 + (\alpha d)^2][(\epsilon + \alpha d)(k_1 d)^2 + (\epsilon + 1)(\alpha d)^3]}. \tag{7.396}$$

Also, the corresponding Green function $L(D_o)$ can be calculated in two parts:

$$L(D_o) = L_c(D_o) + L_d(D_o), \tag{7.397}$$

with

$$L_d(D_o) = \frac{1}{k_1^2}\int_0^\infty G(D) f''_{od}(\zeta) d\zeta = j\frac{\alpha^2 A}{k_1^2}\int_0^\infty G(D) e^{\alpha\zeta} d\zeta. \tag{7.398}$$

Because this integral does not converge on the negative z axis, we must change the integration path in an imaginary direction by writing $\zeta = -j\zeta'$:

$$L_d(D_o) = \frac{\alpha^2 A}{k_1^2}\int_0^\infty G(D') e^{-j\alpha\zeta'} d\zeta', \tag{7.399}$$

$$D' = \sqrt{(\rho - \rho')^2 - \zeta'^2}. \tag{7.400}$$

Although the exponential term now oscillates and does not converge, the integral converges because the argument of the Green function exponent becomes imaginary when ζ' passes the value $|\rho - \rho'|$. Thus, the determination of the Green function $L(D_d)$ is defined in two converging integrations.

7.4.4 Properties of the Green functions

Singularities

Finally, the nature of the singularity of the new Green functions in the transversal field integrals can be questioned. Applying the Green dyadic expression, the double-nabla term is seen to be too singular to give the field within the surface current source itself, even with the definition of principal value integral, which is applied in volume current integrals by YAGHJIAN (1980). When one of the nablas is transferred by partial integration in front of the current function, the principal value can be defined. In fact, we may write for the limit of the field point $\mathbf{r}$ approaching the surface S from the normal direction $\mathbf{n}$ according to VAN BLADEL (1964)

$$\lim_{\mathbf{r}\to S} \int_S [\nabla' G(D)] f(\mathbf{r}') dS' = PV \int_S [\nabla' G(D)] f(\mathbf{r}') dS' + \frac{1}{2}\mathbf{n} f(\mathbf{r}). \tag{7.401}$$

Thus, the normal component of the field is discontinuous through the surface current. Considering only the transverse field component, we see that the principal value integral equals the limit when the field point approaches the surface. Thus, we may write for a point ρ on the surface S in the transversal field expression (7.385)

$$\mathbf{e}(\rho) = -j\omega\mu_o PV \int_S \int_0^\infty G(D)\mathbf{J}_{si}(\rho', \zeta) dS' d\zeta +$$

$$\frac{1}{\epsilon_o} PV \int_S \int_0^\infty [\nabla' G(D)] \varrho_i(\rho', \zeta) dS' d\zeta. \tag{7.402}$$

For the Sommerfeld problem with a planar interface between two homogeneous half spaces and the present microstrip problem, the same singularity properties can be seen to apply. Thus, the free-space Green function G in (7.402) must be replaced by the correct Green functions, taking into account the properties of the half space $z < 0$.

Computation of Green functions

For numerical computation of the Green functions, it is wise first to extract their asymptotic far field limit expressions, leaving simpler difference functions for raw computation. The asymptotic forms of $K^{TE}(\rho)$, $L(\rho)$ are obtained from in the limit $\rho \to \infty$ by approximating the distance function

$$D(\zeta) = \sqrt{\rho^2 + \zeta^2} \approx \rho \tag{7.403}$$

whence we may write

$$K^{TE}(\rho) = \int_0^\infty G(D) f^{TE}(\zeta) d\zeta \approx G(\rho) \int_0^\infty f^{TE}(\zeta) d\zeta. \tag{7.404}$$

From the definition of the reflection coefficients (7.348) we have

$$\int_0^\infty f^{TE}(\zeta) d\zeta = R^{TE}(\beta_1)\Big|_{\beta_1 = 0} = -1. \tag{7.405}$$

Since $G(\rho)$ is singular at $\rho = 0$ whereas $K^{TE}(\rho)$ is not, we can write

$$K^{TE}(\rho) = -G(D_1) + \Delta K(\rho), \tag{7.406}$$

because

$$D_1 = \sqrt{\rho^2 + (2d)^2} \tag{7.407}$$

approaches the same limit ρ as D. The function ΔK vanishes quickly and thus takes little storage space.

Likewise, we can write for the second Green function $L(\rho)$

$$L(\rho) \approx K^{TE}(\rho) + \frac{1}{k_1^2} G(\rho) \int_0^\infty f_o''(\zeta) d\zeta. \tag{7.408}$$

Applying (7.375), implying

$$f_o'' = k_1^2 (f^{TM} - f^{TE} - f_o), \tag{7.409}$$

gives us

$$L(\rho) \approx G(\rho) \int_0^\infty [f^{TM}(\zeta) - f_o(\zeta)] d\zeta = G(\rho) \Big[R^{TM} - R_o \Big]_{\beta_1 = 0} = -G(\rho). \tag{7.410}$$

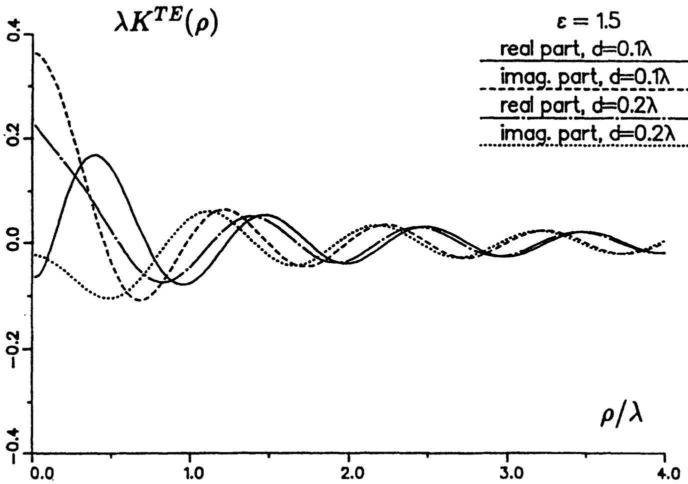

Fig. 7.14 Microstrip Green function $K^{TE}(\rho)$ for two values of the slab thickness d normalized by the free-space wave length.

The second Green function can now be written

$$L(\rho) = -G(D_1) + \Delta L(\rho), \tag{7.411}$$

with rapidly vanishing term $\Delta L(\rho)$.

The waveguide mode term must be taken care of separately. For large ρ values we can write the limiting expression

$$L_d(\rho) \approx \frac{A\alpha^2}{k_1^2} G(\rho) \int_0^\infty e^{-j\alpha\zeta'} d\zeta' = \frac{A\alpha}{k_1^2} G(\rho). \tag{7.412}$$

In Fig. 7.14, examples of computed Green functions are given for some parameter ϵ and d values. It is seen that the Green functions are unsensitive to the thickness d of the substrate at large distances ρ except for the waveguide term.

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7.5 Anisotropic half space

To extend the image theory to problems with anisotropic media, let us consider the simple case of two homogeneous half-spaces of which one (1) is isotropic and the other one (2) uniaxially anisotropic with the axis normal to the planar interface. The dielectric dyadic of the medium 2 can thus be written as

$$\overline{\overline{\epsilon}}_2 = \epsilon_{2z}\mathbf{u}_z\mathbf{u}_z + \epsilon_{2t}\overline{\overline{I}}_t. \tag{7.413}$$

In this special anisotropic case, the TE and TM fields do not couple to each other, which simplifies the analysis. As a practical problem involving this type of medium the sea ice may be mentioned, since it contains almost vertically elongated pockets of brine, which makes the medium approximately macroscopically anisotropic.

The theory given in previous sections must now be modified since the parameter ϵ_2 is not a scalar. Let us very briefly consider the transmission-line analogy in the present uniaxially anisotropic case. From the Maxwell equations, the following Fourier-transformed vector equations for the transverse field components in medium 2 can be straightforwardly derived:

$$\mathbf{e}_2'(\mathbf{K}, z) = \overline{\overline{A}}_2 \cdot (-\mathbf{u}_z \times \mathbf{h}_2) + \mathbf{u}_z \times \mathbf{j}_{2m} + \frac{1}{\omega\epsilon_{2z}}\mathbf{K}j_{2e}, \tag{7.414}$$

$$-\,\mathbf{u}_z \times \mathbf{h}_2'(\mathbf{K}, z) = \overline{\overline{B}}_2 \cdot \mathbf{e}_2 - \mathbf{j}_{2e} - \frac{1}{\omega\mu_2}(\mathbf{u}_z \times \mathbf{K})j_{2m}. \tag{7.415}$$

Here, the dyadics $\overline{\overline{A}}_2$ and $\overline{\overline{B}}_2$ are defined by

$$\overline{\overline{A}}_2 = -j\omega\mu_2\overline{\overline{I}}_t + \frac{j}{\omega\epsilon_{2z}}\mathbf{KK} = -j\overline{\overline{\beta}}_2 \cdot \overline{\overline{Z}}_2, \tag{7.416}$$

$$\overline{\overline{B}}_2 = -j\omega\overline{\overline{\epsilon}}_{2t} + \frac{j}{\omega\mu_2}\mathbf{u}_z\mathbf{u}_z{}^{\times}_{\times}\mathbf{KK} = -j\overline{\overline{\beta}}_2 \cdot \overline{\overline{Y}}_2. \tag{7.417}$$

All these dyadics obviously have the same eigenvectors $\mathbf{K}$, $\mathbf{u}_z \times \mathbf{K}$ as before, whence they commute in the dot product. Defining $\overline{\overline{Y}}_2$ to be the two-dimensional inverse of $\overline{\overline{Z}}_2$, we can derive from the transmission-line equations the second-order equation for the transverse electric field:

$$\mathbf{e}_2'' + \overline{\overline{\beta}}_2^2 \cdot \mathbf{e}_2 = -\overline{\overline{A}}_2 \cdot \mathbf{j}_{2e} + \frac{1}{\omega\epsilon_{2z}}\mathbf{K} j_{2e} + \mathbf{u}_z \times \mathbf{j}_{2m} + j(\mathbf{u}_z \times \mathbf{K}) j_{2m}, \quad (7.418)$$

with the propagation dyadic $\overline{\overline{\beta}}_2$ satisfying

$$\overline{\overline{\beta}}_2^2 = -\overline{\overline{A}}_2 \cdot \overline{\overline{B}}_2 = -\omega^2 \mu_2 \epsilon_{2t} \overline{\overline{I}}_t - \frac{\epsilon_{2t}}{\epsilon_{2z}} \mathbf{K}\mathbf{K} - \mathbf{u}_z\mathbf{u}_z {}^\times_\times \mathbf{K}\mathbf{K} =$$

$$(\beta_2^{TM})^2 \frac{\mathbf{K}\mathbf{K}}{K^2} + (\beta_2^{TE})^2 \frac{\mathbf{u}_z\mathbf{u}_z {}^\times_\times \mathbf{K}\mathbf{K}}{K^2}. \quad (7.419)$$

The eigenvalues correspond to the two different propagation factors of the two eigenpolarizations TM and TE. They have the explicit expressions

$$\beta_2^{TM} = \sqrt{\frac{\epsilon_{2t}}{\epsilon_{2z}}}\sqrt{k_{2z}^2 - K^2} = \sqrt{k_{2t}^2 - \frac{\epsilon_{2t}}{\epsilon_{2z}} K^2}, \qquad \beta_2^{TE} = \sqrt{k_{2t}^2 - K^2}, \quad (7.420)$$

with the two wave numbers defined as

$$k_{2z} = \omega\sqrt{\mu_2 \epsilon_{2z}}, \qquad k_{2t} = \omega\sqrt{\mu_2 \epsilon_{2t}}. \quad (7.421)$$

The TE case appears simpler, because it corresponds to an isotropic medium with the parameters ϵ_{2t}, μ. For the isotropic limiting case $\epsilon_{2t} \to \epsilon_{2z} \to \epsilon_2$, these expressions obviously reduce to the earlier isotropic results.

Finally, the impedance dyadic can be solved from

$$\overline{\overline{Z}}_2^2 = \overline{\overline{A}}_2 \cdot \overline{\overline{B}}_2^{-1} = \frac{\omega^2 \mu_2^2}{\omega^2 \mu \epsilon_{2t} - K^2} \frac{\mathbf{u}_z\mathbf{u}_z {}^\times_\times \mathbf{K}\mathbf{K}}{K^2} + \frac{\omega^2 \mu_2 \epsilon_{2z} - K^2}{\omega^2 \epsilon_{2z}\epsilon_{2t}} \frac{\mathbf{K}\mathbf{K}}{K^2} =$$

$$(Z_2^{TM})^2 \frac{\mathbf{K}\mathbf{K}}{K^2} + (Z_2^{TE})^2 \frac{\mathbf{u}_z\mathbf{u}_z {}^\times_\times \mathbf{K}\mathbf{K}}{K^2} \quad (7.422)$$

in terms of its eigenvalues

$$Z_2^{TE} = \frac{\omega\mu_2}{\beta_2^{TE}}, \qquad Z_2^{TM} = \frac{\beta_2^{TM}}{\omega\epsilon_{2t}}. \quad (7.423)$$

The reflection image

Let us now consider reflection from the planar interface of an isotropic medium with scalar parameters ϵ_1, μ and a uniaxial anisotropic medium with the dyadic parameters $\overline{\overline{\epsilon}}_2 = \overline{\overline{\epsilon}}\epsilon_1$, $\mu\overline{\overline{I}}$. The relative permittivity dyadic is denoted by $\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}_2/\epsilon_1 = \epsilon_z \mathbf{u}_z \mathbf{u}_z + \epsilon_t \bar{\bar{I}}_t$. If the source is in medium 1, the reflection dyadic can be written in terms of its TE and TM eigenvalues as

$$R^{TE} = \frac{Z_2^{TE} - Z_1^{TE}}{Z_2^{TE} + Z_1^{TE}} = \frac{\beta_1 - \beta_2^{TE}}{\beta_1 + \beta_2^{TE}}, \tag{7.424}$$

$$R^{TM} = \frac{Z_2^{TM} - Z_1^{TM}}{Z_2^{TM} + Z_1^{TM}} = \frac{\beta_2^{TM} - \epsilon_t \beta_1}{\beta_2^{TM} + \epsilon_t \beta_1}. \tag{7.425}$$

The reflection image functions can now be identified by comparison with the isotropic problem of Section 7.3. Defining the integral transforms by

$$R(\beta_1) = \int_0^\infty f(\zeta) e^{-j\beta_1 \zeta} d\zeta, \tag{7.426}$$

we are able to find the image functions corresponding to the reflection coefficients.

The TE case is straightforward since the anisotropic medium acts just like an isotropic medium with $\epsilon_2 = \epsilon_{2t}$, which gives us the possibility of writing the image function in terms of the $f(\alpha, p)$ function defined in (7.234):

$$f^{TE}(\zeta) = -\frac{2J_2(jB_t\zeta)}{\zeta} = jB_t f(1, jB_t\zeta), \qquad B_t = \sqrt{k_{2t}^2 - k_1^2}. \tag{7.427}$$

The path of integration goes in the complex ζ plane from 0 to ∞ so that the parameter p_t defined by

$$p_t = jB_t\zeta \tag{7.428}$$

is real and positive for $f^{TE}(\zeta)$ to converge.

Also, writing the TM reflection coefficient in the form

$$R^{TM} = -\frac{\sqrt{\epsilon_z}\sqrt{\epsilon_t}\beta_1 - \sqrt{\beta_1^2 + B_z^2}}{\sqrt{\epsilon_z}\sqrt{\epsilon_t}\beta_1 - \sqrt{\beta_1^2 + B_z^2}}, \qquad B_z = \sqrt{k_{2z}^2 - k_1^2}, \tag{7.429}$$

the corresponding image function can again be written from analogy with isotropic media as

$$f^{TM}(\zeta) = -\frac{\sqrt{\epsilon_z \epsilon_t} - 1}{\sqrt{\epsilon_z \epsilon_t} + 1}\delta_+(\zeta) - jB_z f(\sqrt{\epsilon_z \epsilon_t}, jB_z\zeta). \tag{7.430}$$

The image corresponds to that of an isotropic medium with effective parameters $\epsilon_e = \sqrt{\epsilon_z \epsilon_t}\epsilon_1$, $\mu_e = \sqrt{\epsilon_z/\epsilon_t}\mu_1$. The path of integration must now follow another line such that the parameter

$$p_z = jB_z\zeta \tag{7.431}$$

is real and positive. Thus, there is a difference compared with the isotropic medium case in that the images of TE and TM sources may take different paths in complex space. However, for lossless media both lines coincide.

To find the image for the general source, the third reflection coefficient R_o should be expanded in a form from which the image function can be identified:

$$R_o = \frac{k_1^2}{K^2}(R^{TM} - R^{TE}) = \frac{2k_1^2}{K^2}\frac{\beta_2^{TE}\beta_2^{TM} - \epsilon_t\beta_1^2}{(\beta_2^{TM} + \epsilon_t\beta_1)(\beta_2^{TE} + \beta_1)}. \tag{7.432}$$

After a few hours of algebra, the following form for R_o can be found:

$$R_o = \sqrt{\frac{1}{\epsilon_t}}\left(\frac{\sqrt{\epsilon_t} - 1}{\sqrt{\epsilon_t} + 1} + \frac{\beta_2^{TE} - \sqrt{\epsilon_t}\beta_1}{\beta_2^{TE} + \sqrt{\epsilon_t}\beta_1}\right) + \frac{\epsilon_z\sqrt{\epsilon_t} - 1}{\epsilon_z(\sqrt{\epsilon_t} + 1)} +$$

$$\frac{\sqrt{\epsilon_t}(1 - \epsilon_z)}{\epsilon_z(\epsilon_t - 1)}\frac{\beta_2^{TM} - \sqrt{\epsilon_t}\beta_1}{\beta_2^{TM} + \sqrt{\epsilon_t}\beta_1} + \frac{\epsilon_z\epsilon_t - 1}{\epsilon_z(\epsilon_t - 1)}\frac{\beta_2^{TM} - \epsilon_t\beta_1}{\beta_2^{TM} + \epsilon_t\beta_1}. \tag{7.433}$$

This expression requires some checks to be credible. For the limiting case $\epsilon_t \to \epsilon_z = \epsilon$ the medium becomes an isotropic medium. In this case the expression (7.433) can be shown to reduce to

$$R_o \to \frac{\epsilon + 1}{\epsilon}\left(\frac{\epsilon - 1}{\epsilon + 1} - \frac{\epsilon\beta_1 - \beta_2}{\epsilon_2\beta_1 + \beta_2}\right) = \frac{\epsilon + 1}{\epsilon}\left(E - R(\epsilon, q)\right), \tag{7.434}$$

which coincides with the corresponding expression of the isotropic Sommerfeld problem (7.255) given in Section 7.3.

As another check we may take the limit $\beta_1 \to 0$, whence we can show that both (7.432) and (7.433) tend to the same value $R_o \to 2$.

Finally, for $K \to \infty$, both (7.432) and (7.433) can be seen to give zero. The limit zero means that in the corresponding image function $f_o(\zeta)$ there are no delta singularities.

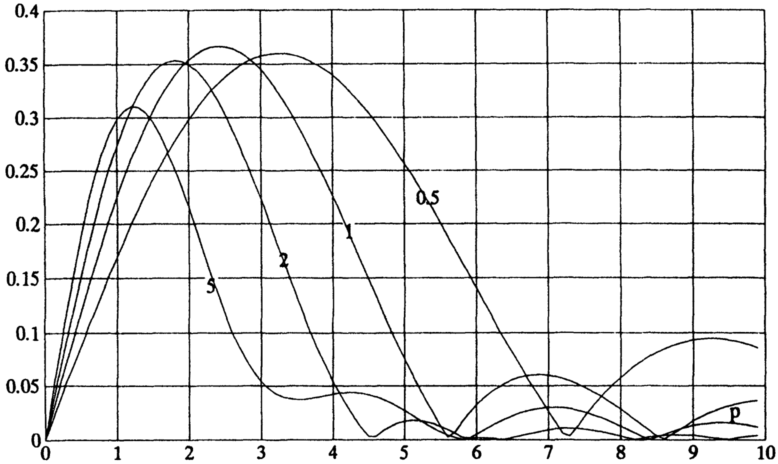

Fig. 7.15 Examples of the normalized image function $|f^{TM}/B_z|$ for $\epsilon_{2t} = 2\epsilon_1$, $\mu_2 = \mu_1$ and different values of $\epsilon_z = \epsilon_{2z}/\epsilon_1$. The argument is $p = jk_1\zeta$. The delta function part is not shown.

The image function $f_o(\zeta)$ associated with the reflection coefficient R_o can now be written without problem, because the terms in (7.433) are of the same form as (7.429):

$$f_o(\zeta) = \frac{8}{\epsilon_t - 1}\left\{ \sum_{n=1}^{\infty} \left(\frac{\sqrt{\epsilon_t} - 1}{\sqrt{\epsilon_t} + 1} \right)^n \frac{nJ_{2n}(jB_t\zeta)}{\zeta} + \right.$$

$$\left. \frac{\sqrt{\epsilon_t}}{\sqrt{\epsilon_z}} \sum_{n=1}^{\infty} \left[\left(\frac{\sqrt{\epsilon_z}\sqrt{\epsilon_t} - 1}{\sqrt{\epsilon_z}\sqrt{\epsilon_t} + 1} \right)^n - \left(\frac{\sqrt{\epsilon_z} - 1}{\sqrt{\epsilon_z} + 1} \right)^n \right] \frac{nJ_{2n}(jB_z\zeta)}{\zeta} \right\} U_+(\zeta) \quad (7.435)$$

References

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