Method of moments for thin wire antennas

1. Introduction

Computational electromagnetics (CEM) is defined as the discipline that uses the digital computer to obtain numerical results for electromagnetic problems.

The method of moments is an integral equation based numerical technique. Because it naturally incorporates the Sommerfeld radiation condition through the use of the appropriate Green's function, the discretization can be kept to a minimum when used to solve problems without finite boundaries (for instance the radiated field of an antenna or scattering applications). The main idea is to reduce the formal problem to a set of linear equations that can be solved by matrix inversion.

One of its advantages is that it can be used to find solutions for any kind of electromagnetic problems and guarantees convergence for a sufficiently dense discretization. It also fits for implementation on parallel computers since the elements of the impedance matrix (see next sections) can be computed independently.

However its principal disadvantage is that it results in full matrices whose treatment is a time-consuming process and requires excessive storage.

2. Method of moments (MoM) basics

The goal is to reduce a certain equation (satisfied in an infinite number of points) to finite problem, for instance to a solvable system of equations.

Let's consider the following operator equation:

$$O(\Psi) = g \tag{2.1}$$

where O is a ***linear*** operator, g represents a known function called ***excitation*** and Ψ is the unknown function to be determined.

The algorithm starts by approximating the unknown f as a series of ***basis*** (***expansion***) functions f_n:

$$\Psi \approx \sum_n I_n \cdot f_n \qquad (2.2)$$

I_n –unknown scalar coefficients.
Replacing Ψ in eq. (2.1) results:

$$O(\Psi) = O(\sum_n I_n \cdot f_n) = \sum_n I_n \cdot O(f_n) = g \qquad (2.3)$$

$$\sum_n I_n \cdot O(f_n) = g \qquad (2.4)$$

The inner product of two functions is defined as:

$$\langle u, v \rangle = \int_\Omega u \cdot v \cdot d\Omega \qquad (2.5)$$

with Ω being the domain of operator O.

Taking the inner product of both left and right sides in equation (2.4) with a set of ***weight* (*testing*)** functions W_m yields:

$$\langle W_m, O(\Psi) \rangle = \langle W_m, g \rangle \qquad (2.6)$$

$$\left\langle W_m, \sum_n I_n \cdot O(f_n) \right\rangle = \langle W_m, g \rangle \qquad (2.7)$$

The above procedure is called ***testing*** and its reason is to minimize the residual error $R = \Psi - \sum_n I_n \cdot f_n$ introduced by the expansion (2.2) and to enforce it in an average weighted sense.
Making the next matrix notations:

$$\mathbf{Z} = [Z_{mn}] = \langle W_m, O(f_n) \rangle = \iint_\Omega W_m \cdot O(f_n) \cdot d\Omega \qquad (2.8)$$

$$\mathbf{V} = [V_m] = \langle W_m, g \rangle = \iint_\Omega W_m \cdot g \cdot d\Omega \qquad (2.9)$$

$$\mathbf{I} = [I_n] = I_n \qquad (2.10)$$

equation (2.7) can be written in matrix form:

$$\mathbf{Z} \cdot \mathbf{I} = \mathbf{V} \qquad (2.11)$$

or:

$$\mathbf{I} = \mathbf{Z}^{-1} \cdot \mathbf{V} \tag{2.12}$$

Once the vector **I** (coefficients I_n) is found by the matrix inversion (2.12) the function Ψ can be approximated using relation (2.2).

The quantities Ψ, f, g can be electric currents, electric and/or magnetic fields etc. and the operator O is usually an integral or integro-differential one when the method is used to solve equations that govern the electromagnetic field.

A special attention must be given when choosing the basis and weight functions, they must be in agreement with the physics of the problem to be solved.

3. The electric field integral equation (EFIE)

The electric field integral equation for an arbitrary conducting body S is briefly presented as follows. On the surface of the body the tangential component of the electric field is zero:

$$\mathbf{n} \times (\mathbf{E}^i + \mathbf{E}^s) = 0 \text{ or}$$

$$\mathbf{n} \times \mathbf{E}^s = -\mathbf{n} \times \mathbf{E}^i \tag{3.1}$$

with **n** being the normal unit vector on the surface of the body pointing out of the surface, $\mathbf{E}^i$ is the incident electric field and $\mathbf{E}^s$ represents the scattered electric field.

It can be shown that the scattered electric field $\mathbf{E}^s$ can be expressed as

$$\mathbf{E}^s = -j\omega \cdot \mathbf{A} - \nabla\Phi \tag{3.2}$$

where **A** represents the magnetic vector potential and Φ is the electric scalar potential. Considering the current density **J** (charge density σ) on the surface of the body, the magnetic and electric potentials are given in terms of surface integrals:

$$\mathbf{A} = \mu \cdot \iint_S \mathbf{J}(\mathbf{r}') \cdot G_0(\mathbf{r}, \mathbf{r}') \cdot dS' \tag{3.3}$$

$$\Phi = \frac{1}{\varepsilon}\iint_S \sigma \cdot G_0(\mathbf{r},\mathbf{r}') \cdot dS' \tag{3.4}$$

$$G_0(\mathbf{r},\mathbf{r}') = \frac{e^{-jkR}}{4\pi R} \tag{3.5}$$

$$\sigma = -\frac{1}{j\omega}\nabla\mathbf{J} \tag{3.6}$$

G_0 is the free space Green's function and R the distance from the source point to the field point:

$$R = |\mathbf{r} - \mathbf{r}'| \tag{3.7}$$

Replacing equation (3.2) in (3.1) yields:

$$\mathbf{n} \times \mathbf{E}^i = \mathbf{n} \times (j\omega \cdot \mathbf{A} + \nabla\Phi) \tag{3.8}$$

Relation (3.8), applicable to perfectly conducting bodies of general geometry, represents the so-called electric field integral equation. Note that the use of free-space Green's function is possible in relations (2.3) and (2.4) because the body surface is replaced by the electric currents radiating in free-space.

A solution to this equation means to determine the electric current density **J**, of course the incident field $\mathbf{E}^i$ being known. Once the electric currents are known all the parameters of interest (for example input impedance, radiation pattern, gain etc. for an antenna system or the radar cross section for a scatterer) can be easily computed.

4. MoM solution of EFIE for thin wires

In this section the EFIE will be particularized for a wire antenna and solved with the help of the algorithm introduced in section 2.

For a thin wire antenna of total length L and wire diameter d (fig. 4.1) the next assumptions are made:

- $d \ll \lambda$
- $d \ll L$
- the current (confined on the surface of the wire) is constrained to flow in axial direction and is has no circumferential variation
- the current at the ends of the wire is zero

Due to these assumptions the potentials from (3.4) and (3.5) become:

$$\mathbf{A} = \mu \cdot \int_0^{2\pi} d\phi' \int_L \hat{\ell}' \cdot I(\ell') \cdot G_0(\mathbf{r},\mathbf{r}') \cdot d\ell = 2\pi\mu \cdot \int_L \hat{\ell}' \cdot I(\ell') \cdot G_0(\mathbf{r},\mathbf{r}') \cdot d\ell' \tag{4.1}$$

$$\Phi = -\frac{1}{j\omega\varepsilon} \int_0^{2\pi} d\phi' \int_L \frac{dI(\ell')}{d\ell'} \cdot G_0(\mathbf{r},\mathbf{r}') \cdot d\ell' = -\frac{2\pi}{j\omega\varepsilon} \cdot \int_L \frac{dI(\ell')}{d\ell'} \cdot G_0(\mathbf{r},\mathbf{r}') \cdot d\ell' \tag{4.2}$$

in which L is the wire contour, $\hat{\ell}$ is the unit vector along L and $\mathbf{J}$ has been replaced by the axial current $\hat{\ell} I$.

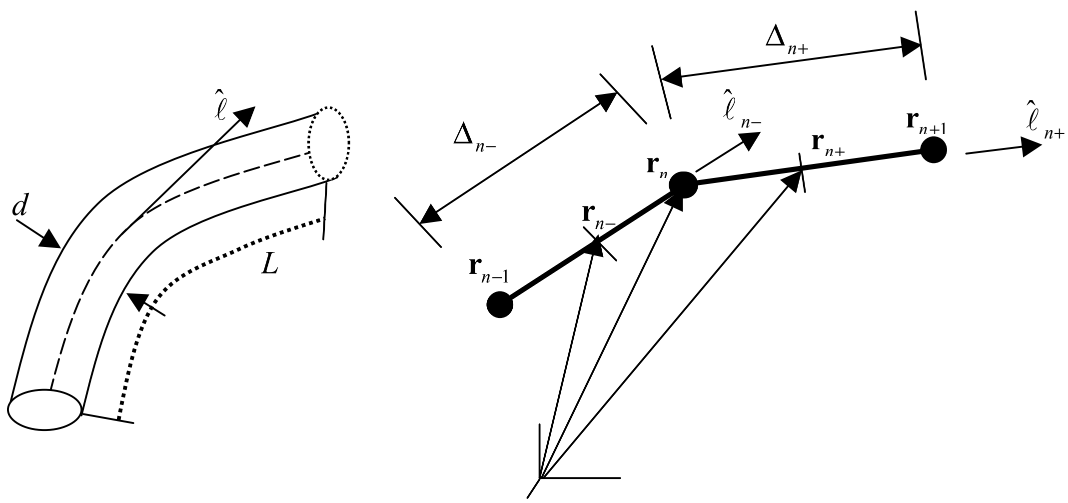

Fig 4.1 Wire geometry and segmentation

Equation (3.8) becomes:

$$j \cdot \frac{\eta}{k} \left\{ k^2 \cdot \int_L \hat{\ell}\hat{\ell}' \cdot I(\ell') \cdot G_0(\mathbf{r},\mathbf{r}') \cdot d\ell' + \frac{d}{d\ell} \int_L \frac{dI(\ell')}{d\ell'} \cdot G_0(\mathbf{r},\mathbf{r}') \cdot d\ell' \right\} = \hat{\ell} \cdot \mathbf{E}^i(\ell) \tag{4.3}$$

In order to solve the equation (4.3), first, one selects equidistant points on the wire axis (fig. 4.1). Two neighboring points form a linear segment. Each segment is defined by its length Δ, its middle point and its unit vector $\hat{\ell}$.

The basis functions will be triangular functions defined as follows:

$$T_n(\ell) = \begin{cases} \dfrac{\ell - \ell_{n-1}}{\Delta_{n-}}; \ell_{n-1} < \ell < \ell_n \\ \dfrac{\ell_{n+1} - \ell}{\Delta_{n+}}; \ell_n < \ell < \ell_{n+1} \\ 0; otherwise \end{cases} \tag{4.4}$$

with their first derivatives

$$\frac{dT_n(\ell)}{d\ell} = \begin{cases} \dfrac{1}{\Delta_{n-}}; \ell_{n-1} < \ell < \ell_n \\ -\dfrac{1}{\Delta_{n+}}; \ell_n < \ell < \ell_{n+1} \\ 0; otherwise \end{cases} \tag{4.5}$$

The weight (test) functions are:

$$B_m(\ell) = \begin{cases} 1; \ell_{m-} < \ell < \ell_{m+} \\ 0; otherwise \end{cases} \tag{4.6}$$

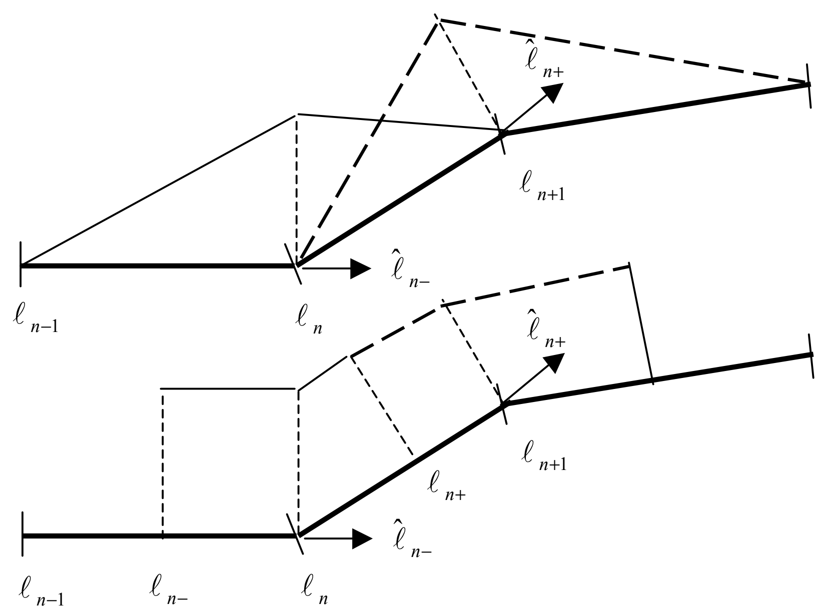

Fig. 4.2 Triangular basis (top) and pulse weight (bottom) functions

Note that the basis functions as well as the test functions are defined over two adjacent segments (the "-" and the "+" segment). So for a two-end wire, if there are *N+2* points, there will be *N+1* segments and *N* basis and test functions and implicitly *N* unknowns.

Following the procedure from section 2 the current *I* is expanded as a linear combination of weighted triangle basis functions:

$$I = \sum_{n=1}^{N} I_n \cdot T_n(\ell) \tag{4.7}$$

Replacing the approximation of *I* in (4.3) and taking into account the linearity of the integral operator, we have:

$$j \cdot \frac{\eta}{k}\left\{ k^2 \cdot \sum_{n=1}^{N} I_n \int_{\ell_{n-1}}^{\ell_{n+1}} \hat{\ell}\hat{\ell}'_n \cdot T_n(\ell') \cdot G_0(\mathbf{r},\mathbf{r'}) \cdot d\ell' + \sum_{n=1}^{N} I_n \frac{d}{d\ell} \int_{\ell_{n-1}}^{\ell_{n+1}} \frac{dT_n(\ell')}{d\ell'} \cdot G_0(\mathbf{r},\mathbf{r'}) \cdot d\ell' \right\} = \hat{\ell} \cdot \mathbf{E}^i(\ell)$$

(4.8)

Testing (4.8) with the above weight functions results:

$$j \cdot \frac{\eta}{k}\left\{ k^2 \cdot \sum_{n=1}^{N} I_n \int_{\ell_{m-}}^{\ell_{m+}} B_m(\ell) \int_{\ell_{n-1}}^{\ell_{n+1}} \hat{\ell}_m \hat{\ell}'_n \cdot T_n(\ell') \cdot G_0(\mathbf{r},\mathbf{r'}) \cdot d\ell' d\ell + \sum_{n=1}^{N} I_n \int_{\ell_{m-}}^{\ell_{m+}} B_m(\ell) \frac{d}{d\ell} \int_{\ell_{n-1}}^{\ell_{n+1}} \frac{dT_n(\ell')}{d\ell'} \cdot G_0(\mathbf{r},\mathbf{r'}) \cdot d\ell' d\ell \right\} = \int_{\ell_{m-}}^{\ell_{m+}} B_m(\ell) \cdot \hat{\ell}_m \cdot \mathbf{E}^i(\ell) \cdot d\ell \tag{4.9}$$

Analog to section 2, the next matrix notation are made:

$$\mathbf{Z} = [Z_{mn}] = j\frac{\eta}{k}\left\{k^2 \cdot \int_{\ell_{m-}}^{\ell_{m+}}\int_{\ell_{n-1}}^{\ell_{n+1}} B_m(\ell)\cdot\hat{\ell}_m\hat{\ell}'_n \cdot T_n(\ell')\cdot G_0(\mathbf{r},\mathbf{r}')\cdot d\ell' d\ell + \right.$$
$$\left. \int_{\ell_{m-}}^{\ell_{m+}} B_m(\ell)\frac{d}{d\ell}\int_{\ell_{n-1}}^{\ell_{n+1}} \frac{dT_n(\ell')}{d\ell'}\cdot G_0(\mathbf{r},\mathbf{r}')\cdot d\ell' d\ell \right\} \tag{4.10}$$

$$\mathbf{I} = [I_n] = I_n \tag{4.11}$$

$$\mathbf{V} = [V_m] = \int_{\ell_{m-}}^{\ell_{m+}} B_m(\ell)\cdot\hat{\ell}_m \cdot \mathbf{E}^i(\ell)\cdot d\ell \tag{4.12}$$

and equation (4.9) becomes:

$$\mathbf{Z}\cdot\mathbf{I} = \mathbf{V} \tag{4.13}$$

with the solution:

$$\mathbf{I} = \mathbf{Z}^{-1}\cdot\mathbf{V} \tag{4.14}$$

The relation

$$\frac{df(\ell)}{d\ell} \approx \frac{1}{\Delta\ell}\left[f\left(\ell+\frac{\Delta\ell}{2}\right) - f\left(\ell-\frac{\Delta\ell}{2}\right)\right] \tag{4.15}$$

is used to approximate the ℓ - related derivative in (4.10) and

$$\int_{\ell_{m-1}}^{\ell_{m+1}} f(\ell)\cdot\hat{\ell}I(\ell)\cdot B_m(\ell)\cdot d\ell = \frac{1}{2}\cdot f(\ell_m)\cdot\left[\Delta_{m-}\hat{\ell}_{m-} + \Delta_{m+}\hat{\ell}_{m+}\right] \tag{4.16}$$

is used to approximate test integral of the first term in (4.10).

Considering these approximations, as well as the definitions of the basis functions (rel. (4.4)), their first derivatives (rel. (4.5)) and the weight functions (rel. (4.6)), a more detailed expression for the impedance matrix **Z** can be written:

$$Z_{mn} = -j\frac{\eta}{k}\left\{\frac{k^2}{2}\left(\Delta_{m-}\hat{\ell}_{m-}\hat{\ell}_{n-} + \Delta_{m+}\hat{\ell}_{m+}\hat{\ell}_{n-}\right)\int_{\ell_{n-1}}^{\ell_n} T_n(\ell')\cdot G_0(R_m)\cdot d\ell + \right.$$

$$\frac{k^2}{2}\left(\Delta_{m-}\hat{\ell}_{m-}\hat{\ell}_{n+} + \Delta_{m+}\hat{\ell}_{m+}\hat{\ell}_{n+}\right)\int_{\ell_n}^{\ell_{n+1}} T_n(\ell)\cdot G_0(R_m)\cdot d\ell + \tag{4.17}$$

$$\left.\frac{1}{\Delta_{n-}}\int_{\ell_{n-1}}^{\ell_n}\left[G_0(R_{m+}) - G_0(R_{m-})\right]d\ell' + \frac{1}{\Delta_{n+}}\int_{\ell_n}^{\ell_{n+1}}\left[G_0(R_{m-}) - G_0(R_{m+})\right]d\ell'\right\}$$

with :

$$R_m = |\mathbf{r}_m - \mathbf{r}'|,\ R_{m+} = |\mathbf{r}_{m+} - \mathbf{r}'|,\text{ and } R_{m-} = |\mathbf{r}_{m-} - \mathbf{r}'|. \tag{4.18}$$

In order to determine the incident field $\mathbf{E}^i$, the delta gap generator model is used for a transmitting antenna (fig. 4.3).

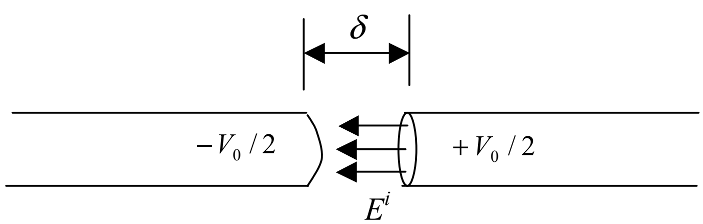

Fig. 4.3 Delta gap source model

For this model, the resulting incident electric field is:

$$\mathbf{E}^i = \hat{\ell}\cdot\frac{V_0}{\delta} \tag{4.19}$$

After the matrix inversion (4.14) the expansion coefficients I_n are found and the current distribution is easily computed using the approximation (4.7). Once the current distribution is known, the calculation of the interest parameters of the antenna is straightforward.

5. Homework

- Derive the expressions of the impedance matrix (relation (4.10)) and the excitation vector (relation (4.12)) for a z-directed straight wire antenna.
- Write the expressions of the input impedance, radiated far field and gain for an arbitrary shape wire antenna as well as for a straight wire antenna.
- Find out the expression of the excitation vector **V** for a receiving antenna (the incident electric field is a plane wave).