CHAPTER 2

INTRODUCTION TO WAVES

2-1. The Wave Equation. A field that is a function of both time and space coordinates can be called a wave. We shall, however, be a bit more restrictive in our definition and use the term wave to denote a solution to a particular type of equation, called a wave equation. Electromagnetic fields obey wave equations, so the terms wave and field are synonymous for time-varying electromagnetism. In this chapter we shall consider a number of simple wave solutions to introduce and illustrate various a-c electromagnetic phenomena.

For the present, let us consider fields in regions which are source-free ($\mathbf{J}^i = \mathbf{M}^i = 0$), linear ($\hat{z}$ and $\hat{y}$ independent of $|E|$ and $|H|$), homogeneous ($\hat{z}$ and $\hat{y}$ independent of position), and isotropic ($\hat{z}$ and $\hat{y}$ are scalar). The complex field equations are then

$$\begin{aligned} \nabla \times \mathbf{E} &= -\hat{z}\mathbf{H} \\ \nabla \times \mathbf{H} &= \hat{y}\mathbf{E} \end{aligned} \qquad (2\text{-}1)$$

The curl of the first equation is

$$\nabla \times \nabla \times \mathbf{E} = -\hat{z}\nabla \times \mathbf{H}$$

which, upon substitution for $\nabla \times \mathbf{H}$ from the second equation, becomes

$$\nabla \times \nabla \times \mathbf{E} = -\hat{z}\hat{y}\mathbf{E}$$

The frequently encountered parameter

$$k = \sqrt{-\hat{z}\hat{y}} \qquad (2\text{-}2)$$

is called the *wave number* of the medium. In terms of k, the preceding equation becomes

$$\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = 0 \qquad (2\text{-}3)$$

which we shall call the *complex vector wave equation.* If we return to Eqs. (2-1), take the curl of the second equation, and substitute from the first equation, we obtain

$$\nabla \times \nabla \times \mathbf{H} - k^2\mathbf{H} = 0 \qquad (2\text{-}4)$$

Thus, $\mathbf{H}$ is a solution to the same complex wave equation as is $\mathbf{E}$.

The wave equation is often written in another form by defining an operation

$$\nabla^2\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

In rectangular components, this reduces to

$$\nabla^2\mathbf{A} = \mathbf{u}_x\nabla^2 A_x + \mathbf{u}_y\nabla^2 A_y + \mathbf{u}_z\nabla^2 A_z$$

where $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$ are the rectangular-coordinate unit vectors and ∇^2 is the Laplacian operator. It is implicit in the wave equations that

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0 \tag{2-5}$$

shown by taking the divergence of Eqs. (2-3) and (2-4). Using Eqs. (2-5) and the operation defined above, we can write Eqs. (2-3) and (2-4) as

$$\begin{aligned} \nabla^2\mathbf{E} + k^2\mathbf{E} &= 0 \\ \nabla^2\mathbf{H} + k^2\mathbf{H} &= 0 \end{aligned} \tag{2-6}$$

These we shall also call vector wave equations. They are not, however, so general as the previous forms, for they do not imply Eqs. (2-5). In other words, Eqs. (2-6) *and* Eqs. (2-5) are equivalent to Eqs. (2-3) and (2-4). Thus, the *rectangular components* of **E** and **H** satisfy the *complex scalar wave equation* or *Helmholtz equation*[1]

$$\nabla^2\psi + k^2\psi = 0 \tag{2-7}$$

We can construct electromagnetic fields by choosing solutions to Eq. (2-7) for E_x, E_y, and E_z or H_x, H_y, and H_z, such that Eqs. (2-5) are also satisfied.

To illustrate the wave behavior of electromagnetic fields, let us construct a simple solution. Take the medium to be a perfect dielectric, in which case $\hat{y} = j\omega\epsilon$, $\hat{z} = j\omega\mu$, and

$$k = \omega\sqrt{\epsilon\mu} \tag{2-8}$$

Also, take **E** to have only an x component independent of x and y. The first of Eqs. (2-6) then reduces to

$$\frac{d^2E_x}{dz^2} + k^2E_x = 0$$

which is the one-dimensional Helmholtz equation. Solutions to this are linear combinations of e^{jkz} and e^{-jkz}. In particular, let us consider a solution

$$E_x = E_0e^{-jkz} \tag{2-9}$$

This satisfies $\nabla \cdot \mathbf{E} = 0$ and is therefore a possible electromagnetic field.

[1] We shall use the symbol ψ to denote "wave functions," that is, solutions to Eq. (2-7). Do not confuse these ψ's with magnetic flux.

The associated magnetic field is found according to

$$j\omega\mu \mathbf{H} = -\nabla \times \mathbf{E} = \mathbf{u}_y jkE_x$$

which, using Eq. (2-8), can be written as

$$E_x = \sqrt{\frac{\mu}{\epsilon}}\, H_y \tag{2-10}$$

Ratios of components of **E** to components of **H** have the dimensions of impedance and are called *wave impedances*. The wave impedance associated with our present solution,

$$\eta = \frac{E_x}{H_y} = \sqrt{\frac{\mu}{\epsilon}} \tag{2-11}$$

is called the *intrinsic impedance* of the medium. In vacuum,

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 120\pi \approx 377 \text{ ohms} \tag{2-12}$$

We shall see later that the intrinsic impedance of a medium enters into wave transmission and reflection problems in the same manner as the characteristic impedance of transmission lines.

To interpret this solution, let E_0 be real and determine $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ according to Eq. (1-41). The instantaneous fields are found as

$$\begin{aligned} \mathcal{E}_x &= \sqrt{2}\, E_0 \cos(\omega t - kz) \\ \mathcal{H}_y &= \frac{\sqrt{2}}{\eta} E_0 \cos(\omega t - kz) \end{aligned} \tag{2-13}$$

This is called a *plane wave* because the phase (kz) of $\mathcal{E}$ and $\mathcal{H}$ is constant over a set of planes (defined by z = constant) called *equiphase surfaces*. It is called a *uniform* plane wave because the amplitudes (E_0 and E_0/η) of $\mathcal{E}$ and $\mathcal{H}$ are constant over the equiphase planes. $\mathcal{E}$ and $\mathcal{H}$ are said to be *in phase* because they have the same phase at any point. At some specific time, $\mathcal{E}$ and $\mathcal{H}$ are sinusoidal functions of z. The vector picture of Fig. 2-1 illustrates $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ along the z axis at $t = 0$. The direction of an arrow represents the direction of a vector, and the length of an arrow represents the magnitude of a vector. If we take a slightly later instant of time, the picture of Fig. 2-1 will be shifted in the $+z$ direction. We say that the wave is traveling in the $+z$ direction and call it a *traveling wave*. The term *polarization* is used to specify the behavior of $\boldsymbol{\mathcal{E}}$ lines. In this wave, the $\boldsymbol{\mathcal{E}}$ lines are always parallel to the x axis, and the wave is said to be *linearly polarized* in the x direction.

The velocity at which an equiphase surface travels is called the *phase* *velocity* of the wave. An equiphase plane $z = z_p$ is defined by

$$\omega t - kz_p = \text{constant}$$

that is, the argument of the cosine functions of Eq. (2-13) is constant. As t increases, the value of z_p must also increase to maintain this constancy, and the plane $z = z_p$ will move in the $+z$ direction. This is illustrated by Fig. 2-2, which is a plot of $\mathcal{E}$ for several instants of time. To obtain the phase velocity dz_p/dt, differentiate the above equation. This gives

$$\omega - k\frac{dz_p}{dt} = 0$$

The phase velocity of this wave is called the *intrinsic phase velocity* v_p of the dielectric and is, according to the above equation,

$$v_p = \frac{dz_p}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon\mu}} \tag{2-14}$$

In vacuum, this is the velocity of light: 3×10^8 meters per second.

The *wavelength* of a wave is defined as the distance in which the phase increases by 2π at any instant. This distance is shown on Fig. 2-2. The wavelength of the particular wave of Eqs. (2-13) is called the *intrinsic wavelength* λ of the medium. It is given by $k\lambda = 2\pi$, or

$$\lambda = \frac{2\pi}{k} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f} \tag{2-15}$$

where f is the frequency in cycles per second. The wavelength is often used as a measure of whether a distance is long or short. The range of wavelengths encountered in electromagnetic engineering is large. For example, the free-space wavelength of a 60-cycle wave is 5000 kilometers, whereas the free-space wavelength of a 1000-megacycle wave is only 30 centimeters. Thus, a distance of 1 kilometer is very short at 60 cycles,

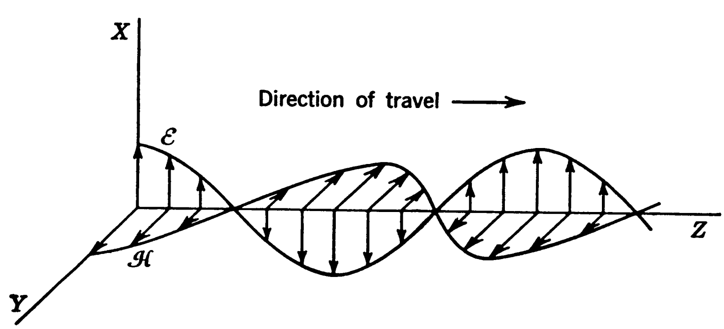


FIG. 2-1. A linearly polarized uniform plane traveling wave.

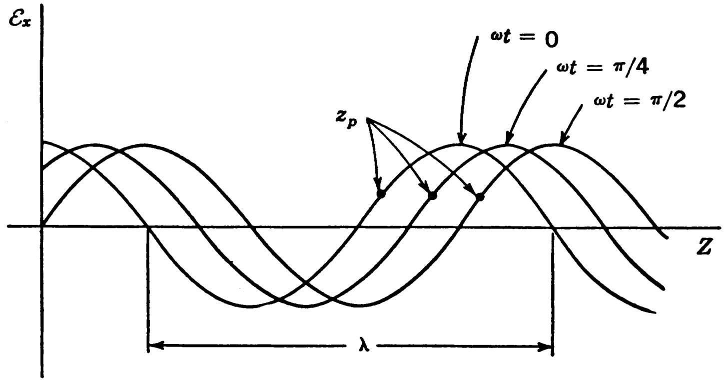


FIG. 2-2. $\mathcal{E}$ at several instants of time in a linearly polarized uniform plane traveling wave.

but very long at 1000 megacycles. The usual circuit theory is based on the assumption that distances are much shorter than a wavelength.

2.2. Waves in Perfect Dielectrics. In this section we shall consider the properties of uniform plane waves in perfect dielectrics, of which free space is the most common example. We have already given a special case of the uniform plane wave in the preceding section. To summarize,

$$E_x = E_0 e^{-jkz} \qquad H_y = \frac{E_0}{\eta} e^{-jkz}$$

where

$$k = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda} = \frac{\omega}{v_p} \tag{2-16}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

It is an x-polarized, $+z$ traveling wave. Because of the symmetry of the rectangular coordinate system, other uniform plane-wave solutions can be obtained by rotations of the coordinate axes, corresponding to cyclic interchanges of coordinate variables. We wish to restrict consideration to $+z$ and $-z$ traveling waves; so we shall consider only the transformations (x,y,z) to $(-y,x,z)$, to $(x,-y,-z)$, and to $(y,x,-z)$. This procedure, together with our original solution, gives us the four waves

$$\begin{aligned}
E_x^+ &= Ae^{-jkz} & H_y^+ &= \frac{A}{\eta} e^{-jkz} \\
E_y^+ &= Be^{-jkz} & H_x^+ &= \frac{-B}{\eta} e^{-jkz} \\
E_x^- &= Ce^{jkz} & H_y^- &= \frac{-C}{\eta} e^{jkz} \\
E_y^- &= De^{jkz} & H_x^- &= \frac{D}{\eta} e^{jkz}
\end{aligned} \tag{2-17}$$

where the previously used E_0 has been replaced by A, B, C, or D. The superscript + denotes a $+z$ traveling wave, and the superscript − denotes a $-z$ traveling wave. The most general uniform plane wave is a superposition of Eqs. (2-17).

We have already interpreted the first wave of Eqs. (2-17) in Sec. 2-1. This also constitutes an interpretation of the other three waves if the appropriate interchanges of coordinates are made. We have not yet mentioned power and energy considerations, so let us do so now. Given the traveling wave

$$E_x = E_0 e^{-jkz} \qquad H_y = \frac{E_0}{\eta} e^{-jkz}$$

we evaluate the various energy and power quantities as

$$\begin{aligned} w_e &= \frac{\epsilon}{2} \mathcal{E}^2 = \epsilon E_0^2 \cos^2 (\omega t - kz) \\ w_m &= \frac{\mu}{2} \mathcal{H}^2 = \epsilon E_0^2 \cos^2 (\omega t - kz) \\ \mathcal{S} &= \mathcal{E} \times \mathcal{H} = \mathbf{u}_z \frac{2}{\eta} E_0^2 \cos^2 (\omega t - kz) \\ \mathbf{S} &= \mathbf{E} \times \mathbf{H}^* = \mathbf{u}_z \frac{E_0^2}{\eta} \end{aligned} \tag{2-18}$$

Thus, the electric and magnetic energy densities are equal, half of the energy of the wave being electric and half magnetic. We can define a *velocity of propagation of energy* v_e as

$$v_e = \frac{\text{power flow density}}{\text{energy density}} = \frac{\mathcal{S}}{w_e + w_m} \tag{2-19}$$

For the uniform plane traveling wave, from Eqs. (2-18) and (2-19) we find

$$v_e = \frac{1}{\sqrt{\mu\epsilon}}$$

which is also the phase velocity [Eq. (2-14)]. These two velocities are not necessarily equal for other types of electromagnetic waves. In general, the phase velocity may be greater or less than the velocity of light, but the velocity of propagation of energy is never greater than the velocity of light.

Another property of waves can be illustrated by the *standing wave*

$$E_x = E_0 \sin kz \qquad H_y = j \frac{E_0}{\eta} \cos kz \tag{2-20}$$

obtained by combining the first and third waves of Eqs. (2-17) with

$A = -C = jE_0/2$. The corresponding instantaneous fields are

$$\mathcal{E}_x = \sqrt{2}\, E_0 \sin kz \cos \omega t \qquad \mathcal{H}_y = -\sqrt{2}\,\frac{E_0}{\eta} \cos kz \sin \omega t$$

Note that the phase is now independent of z, there being no traveling motion; hence the name *standing wave*. A picture of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ at some instant of time is shown in Fig. 2-3. The field oscillates in amplitude, with $\mathcal{E}$ reaching its peak value when $\mathcal{H}$ is zero, and vice versa. In other words, $\mathcal{E}$ and $\mathcal{H}$ are 90° out of phase. The planes of zero $\mathcal{E}$ and $\mathcal{H}$ are fixed in space, the zeros of $\mathcal{E}$ being displaced a quarter-wavelength from the zeros of $\mathcal{H}$. Successive zeros of $\mathcal{E}$ or of $\mathcal{H}$ are separated by a half-wavelength, as shown on Fig. 2-3. The wave is still a *plane* wave, for equiphase surfaces are planes. It is still a *uniform* wave, for its amplitude is constant over equiphase surfaces. It is still *linearly polarized*, for $\boldsymbol{\mathcal{E}}$ always points in the same direction (or opposite direction when $\mathcal{E}$ is negative).

The energy and power quantities associated with this wave are

$$\begin{aligned}
w_e &= \frac{\epsilon}{2}\,\mathcal{E}^2 = \epsilon E_0{}^2 \sin^2 kz \cos^2 \omega t \\
w_m &= \frac{\mu}{2}\,\mathcal{H}^2 = \epsilon E_0{}^2 \cos^2 kz \sin^2 \omega t \\
\boldsymbol{\mathcal{S}} &= \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}} = -\mathbf{u}_z \frac{E_0{}^2}{2\eta} \sin 2kz \sin 2\omega t \\
\mathbf{S} &= \mathbf{E} \times \mathbf{H}^* = -\mathbf{u}_z \frac{jE_0{}^2}{2\eta} \sin 2kz
\end{aligned} \tag{2-21}$$

The time-average Poynting vector $\bar{\mathcal{S}} = \text{Re}\,(\mathbf{S})$ is zero, showing no power flow on the average. The electric energy density is a maximum when the magnetic energy density is zero, and vice versa. A picture of energy

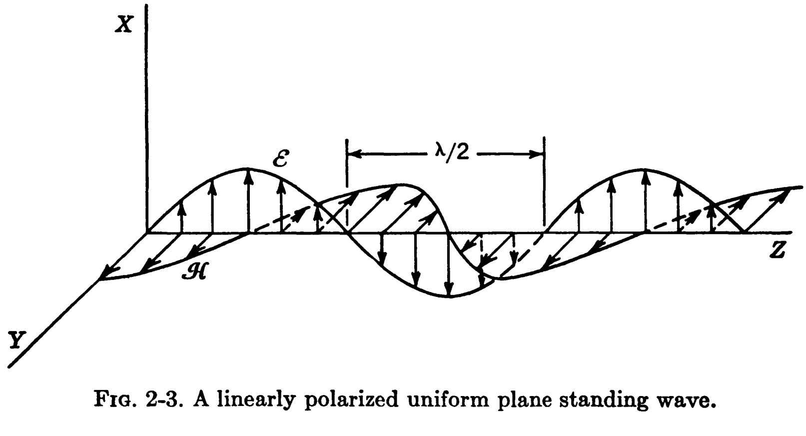


FIG. 2-3. A linearly polarized uniform plane standing wave.

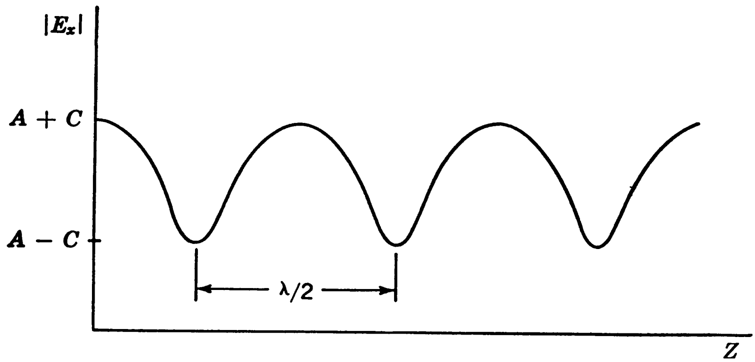


FIG. 2-4. Standing-wave pattern of two oppositely traveling waves of unequal amplitudes.

oscillating between the electric and magnetic forms can be used for this wave. Note that we have planes of zero electric intensity at $kz = n\pi$, n an integer. Thus, perfect electric conductors can be placed over one or more of these planes. If an electric conductor covers the plane $z = 0$, Eqs. (2-20) represent the solution to the problem of reflection of a uniform plane wave normally incident on this conductor. If two electric conductors cover the planes $kz = n_1\pi$ and $kz = n_2\pi$, Eqs. (2-20) represent the solution of a one-dimensional "resonator."

A more general x-polarized field is one consisting of waves traveling in opposite directions with unequal amplitudes. This is a superposition of the first and third of Eqs. (2-17), or

$$\begin{aligned} E_x &= Ae^{-jkz} + Ce^{jkz} \\ H_y &= \frac{1}{\eta}(Ae^{-jkz} - Ce^{jkz}) \end{aligned} \tag{2-22}$$

If $A = 0$ or $C = 0$, we have a pure traveling wave, and if $|A| = |C|$, we have a pure standing wave. For $A \neq C$, let us take A and C real[1] and express the field in terms of an amplitude and phase. This gives

$$E_x = \sqrt{A^2 + C^2 + 2AC\cos 2kz}\, e^{-j\tan^{-1}\left(\frac{A-C}{A+C}\tan kz\right)} \tag{2-23}$$

The rms amplitude of E is

$$\sqrt{A^2 + C^2 + 2AC\cos 2kz}$$

which is called the *standing-wave pattern* of the field. This is illustrated by Fig. 2-4. The voltage output of a small probe (receiving antenna) connected to a detector would essentially follow this standing-wave pat-

[1] This is actually no restriction on the generality of our interpretation, for it corresponds to a judicious choice of z and t origins.

tern. For a pure traveling wave, the standing-wave pattern is a constant, and for a pure standing wave, it is of the form $|\cos kz|$, that is, a "rectified" sine wave. The ratio of the maximum of the standing-wave pattern to the minimum is called the *standing-wave ratio* (SWR). From Fig. 2-4, it is evident that

$$\text{SWR} = \frac{A + C}{A - C} \tag{2-24}$$

because the two traveling-wave components [Eqs. (2-22)] add in phase at some points and add 180° out of phase at other points. The distance between successive minima is $\lambda/2$. The standing-wave ratio of a pure traveling wave is unity, that of a pure standing wave is infinite. Plane traveling waves reflected by dielectric or imperfectly conducting boundaries will result in partial standing waves, with SWR's between one and infinity.

Let us now consider a traveling wave in which both E_x and E_y exist. This is a superposition of the first and second of Eqs. (2-17), that is,

$$\begin{aligned} \mathbf{E} &= (\mathbf{u}_x A + \mathbf{u}_y B) e^{-jkz} \\ \mathbf{H} &= (-\mathbf{u}_x B + \mathbf{u}_y A) \frac{1}{\eta} e^{-jkz} \end{aligned} \tag{2-25}$$

If $B = 0$, the wave is linearly polarized in the x direction. If $A = 0$, the wave is linearly polarized in the y direction. If A and B are both real (or complex with equal phases), we again have a linearly polarized wave, with the axis of polarization inclined at an angle $\tan^{-1} (B/A)$ with respect to the x axis. This is illustrated by Fig. 2-5*a*. If A and B are complex with different phase angles, $\mathcal{E}$ will no longer point in a single spatial direction. Letting $A = |A|e^{ja}$ and $B = |B|e^{jb}$, we have the instan-

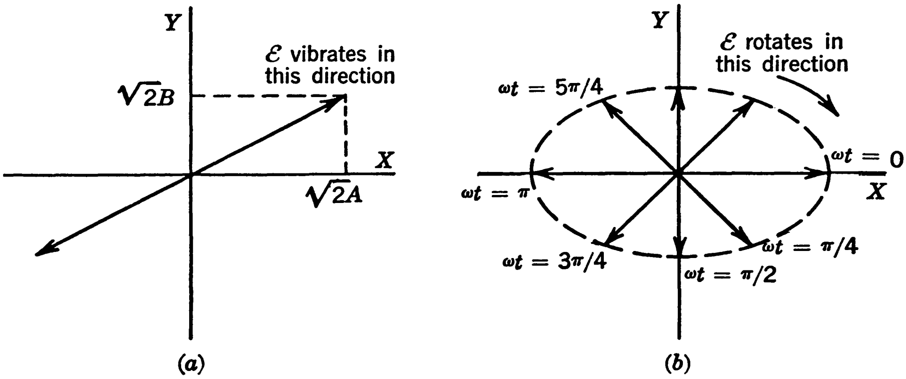


FIG. 2-5. Polarization of a uniform plane traveling wave. (*a*) Linear polarization; (*b*) elliptical polarization.

taneous electric intensity given by

$$\begin{aligned} \mathcal{E}_x &= \sqrt{2}\,|A| \cos(\omega t - kz + a) \\ \mathcal{E}_y &= \sqrt{2}\,|B| \cos(\omega t - kz + b) \end{aligned}$$

A vector picture of $\mathcal{E}$ for various instants of time changes in both amplitude and direction, going through this variation once each cycle. For example, let $|A| = 2\,|B|$, $a = 0$, and $b = \pi/2$. A plot of $\mathcal{E}$ for various values of t in the plane $z = 0$ is shown in Fig. 2-5*b*. The tip of the arrow in the vector picture traces out an ellipse, and the field is said to be *elliptically polarized.* Depending upon A and B, this ellipse can be of arbitrary orientation in the xy plane and of arbitrary axial ratio. Linear polarization can be considered as the special case of elliptic polarization for which the axial ratio is infinite.

If the axial ratio is unity, the tip of the arrow traces out a circle, and the field is said to be *circularly polarized.* The polarization is said to be *right-handed* if $\mathcal{E}$ rotates in the direction of the fingers of the right hand when the thumb points in the direction of propagation. The polarization is said to be *left-handed* if $\mathcal{E}$ rotates in the opposite direction. The specialization of Eq. (2-25) to right-handed circular polarization is obtained by setting $A = jB = E_0$, giving

$$\begin{aligned} \mathbf{E} &= (\mathbf{u}_x - j\mathbf{u}_y)E_0 e^{-jkz} \\ \mathbf{H} &= (\mathbf{u}_x - j\mathbf{u}_y)j\,\frac{E_0}{\eta}\,e^{-jkz} \end{aligned} \tag{2-26}$$

A vector picture of the type of Fig. 2-1 for this wave would show $\mathcal{E}$ and $\mathcal{H}$ in the form of two corkscrews, with $\mathcal{E}$ perpendicular to $\mathcal{H}$ at each point. As time increases, this picture would rotate giving a corkscrew type of motion in the z direction. The various energy and power quantities associated with this wave are

$$\begin{aligned} w_e &= \frac{\epsilon}{2}\,\mathcal{E}^2 = \epsilon E_0^2 \\ w_m &= \frac{\mu}{2}\,\mathcal{H}^2 = \epsilon E_0^2 \\ \mathcal{S} &= \mathcal{E} \times \mathcal{H} = \mathbf{u}_z \frac{2}{\eta} E_0^2 \\ \mathbf{S} &= \mathbf{E} \times \mathbf{H}^* = \mathbf{u}_z \frac{2}{\eta} E_0^2 \end{aligned} \tag{2-27}$$

Thus, there is no change in energy and power densities with time or space. Circular polarization gives a steady power flow, analogous to circuit-theory power transmission in a two-phase system.

As a final example, consider the circularly polarized standing-wave field specified by

$$\begin{aligned} \mathbf{E} &= (\mathbf{u}_x + j\mathbf{u}_y)E_0 \sin kz \\ \mathbf{H} &= (\mathbf{u}_x + j\mathbf{u}_y)\frac{E_0}{\eta} \cos kz \end{aligned} \tag{2-28}$$

This is the superposition of Eqs. (2-17) for which $A = -C = jE_0/2$, $D = -B = E_0/2$. The corresponding instantaneous fields are

$$\begin{aligned} \mathcal{E} &= (\mathbf{u}_x \cos \omega t - \mathbf{u}_y \sin \omega t)\sqrt{2}\, E_0 \sin kz \\ \mathcal{H} &= (\mathbf{u}_x \cos \omega t - \mathbf{u}_y \sin \omega t)\sqrt{2}\, \frac{E_0}{\eta} \cos kz \end{aligned}$$

Note that $\mathcal{E}$ and $\mathcal{H}$ are always *parallel* to each other. A vector picture of $\mathcal{E}$ and $\mathcal{H}$ at $t = 0$ is shown in Fig. 2-6. As time progresses, this picture rotates about the z axis, the amplitudes of $\mathcal{E}$ and $\mathcal{H}$ being independent of time. It is only the direction of $\mathcal{E}$ and $\mathcal{H}$ which changes with time. The amplitudes of $\mathcal{E}$ and $\mathcal{H}$ are, however, a function of z, giving a standing-wave pattern in the z direction. The energy and power densities associated with this wave are

$$\begin{aligned} w_e &= \frac{\epsilon}{2}\mathcal{E}^2 = \epsilon E_0^2 \sin^2 kz \\ w_m &= \frac{\mu}{2}\mathcal{H}^2 = \epsilon E_0^2 \cos^2 kz \\ \mathcal{S} &= \mathcal{E} \times \mathcal{H} = 0 \\ \mathbf{S} &= -\mathbf{u}_z \frac{j}{\eta} E_0^2 \sin 2kz \end{aligned} \tag{2-29}$$

It is interesting to note that the instantaneous energy and power densities are independent of time. This field can represent resonance between two perfectly conducting planes situated where E is zero. It thus seems that the picture of energy oscillating between the electric and magnetic forms

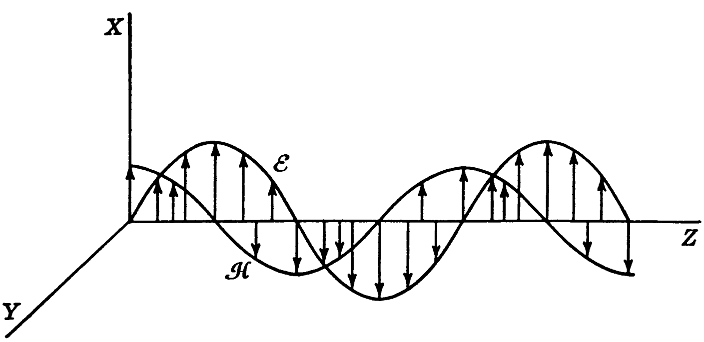


FIG. 2-6. A circularly polarized uniform plane standing wave.

is not generally valid for resonance. However, the circularly polarized standing wave is the sum of two linearly polarized waves which can exist independently of each other. We actually have two coincident resonances (called a *degenerate case*), and the picture of energy oscillating between electric and magnetic forms applies to each linearly polarized resonance.

2-3. Intrinsic Wave Constants. When the wave aspects of electromagnetism are emphasized, the wave number k and the intrinsic impedance η, given by

$$k = \sqrt{-\hat{z}\hat{y}} \qquad \eta = \sqrt{\frac{\hat{z}}{\hat{y}}} \tag{2-30}$$

play an important role. The second equation is a generalization of Eq. (2-11), obtained in the same manner as Eq. (2-11) when $\hat{z}$ and $\hat{y}$ are not specialized to the case of a perfect dielectric. We can solve Eqs. (2-30) for $\hat{z}$ and $\hat{y}$, obtaining

$$\hat{z} = jk\eta \qquad \hat{y} = \frac{jk}{\eta} \tag{2-31}$$

A knowledge of k and η is equivalent to a knowledge of $\hat{z}$ and $\hat{y}$, and hence specifies the characteristics of the medium.

The wave number is, in general, complex, and may be written as

$$k = k' - jk'' \tag{2-32}$$

where k' is the *intrinsic phase constant* and k'' is the *intrinsic attenuation constant.* We have already seen that when $k = k'$, it enters into the phase function of the wave. We shall see in the next section that k'' causes an exponential attenuation of the wave amplitude. The behavior of k can be illustrated by a complex diagram relating k to $\hat{z}$ and $\hat{y}$. This is shown in Fig. 2-7. In the expressions

$$\hat{y} = \sigma + \omega\epsilon'' + j\omega\epsilon'$$
$$\hat{z} = \omega\mu'' + j\omega\mu'$$

σ, ϵ'', and μ'' are always positive in source-free media, for they account for energy dissipation. The parameters ϵ' and μ' are usually positive but may be negative for certain types of atomic resonance. Thus, $\hat{z}$ and $\hat{y}$ usually lie in the first quadrant of the complex plane, as shown in Fig. 2-7. The product $-\hat{z}\hat{y}$ then usually lies in the bottom half of the complex

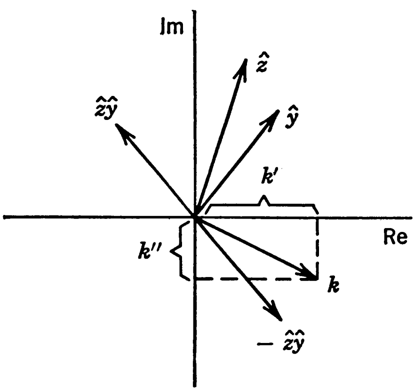


FIG. 2-7. Complex diagram relating k to $\hat{z}$ and $\hat{y}$.

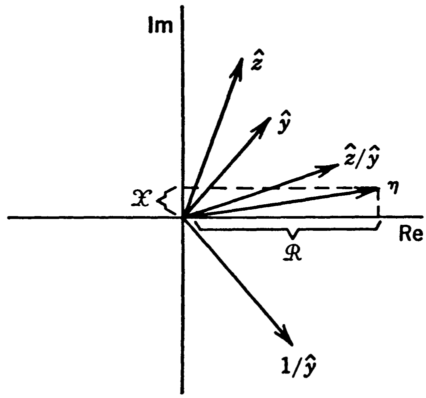


FIG. 2-8. Complex diagram relating η to $\hat{z}$ and $\hat{y}$.

plane. The principal square root, $k = \sqrt{-\hat{z}\hat{y}}$, lies in the fourth quadrant, showing that k' and k'' are usually positive. Even when ϵ' or μ' is negative, k'' is positive; it is only k' that could conceivably be negative. In lossless media, $\hat{y} = j\omega\epsilon$, $\hat{z} = j\omega\mu$, and k is real.

The intrinsic wave impedance can be considered in an analogous manner. Expressing η in rectangular components, we have

$$\eta = \mathcal{R} + j\mathcal{X} \tag{2-33}$$

where $\mathcal{R}$ is the *intrinsic wave resistance* and $\mathcal{X}$ is the *intrinsic wave reactance.* For a wave in a perfect dielectric, η is purely resistive and is therefore the ratio of the amplitude of $\mathcal{E}$ to $\mathcal{H}$. We shall see in Sec. 2-4 that $\mathcal{X}$ introduces a phase difference between $\mathcal{E}$ and $\mathcal{H}$. The complex diagram relating η to $\hat{y}$ and $\hat{z}$ in general is shown in Fig. 2-8. In source-free regions, σ, ϵ'', and μ'' are always positive, and ϵ' and μ' are usually positive. Thus $\hat{z}$ usually lies in the first quadrant and $1/\hat{y}$ in the fourth quadrant. The ratio $\hat{z}/\hat{y}$ therefore usually lies in the right half plane and η in the sector $\pm 45°$ with respect to the positive real axis. When ϵ' or μ' is negative, η may lie anywhere in the right half plane, but $\mathcal{R}$ is never negative. In lossless media, the wave impedance is real.

There are several special cases of particular interest to us. First, consider the case of no magnetic losses. From the first of Eqs. (2-31), we have

$$\eta = \frac{\hat{z}}{jk} = \frac{\hat{z}k^*}{jkk^*} = -\frac{jk^*\hat{z}}{|\hat{z}|\,|\hat{y}|}$$

the last equality following from Eqs. (2-30). Now for $\hat{z} = j\omega\mu = j|\hat{z}|$, we have

$$\eta = \frac{k^*}{|\hat{y}|} \qquad \text{no magnetic losses} \tag{2-34}$$

TABLE 2-1. WAVE NUMBER ($k = k' - jk''$) AND INTRINSIC IMPEDANCE ($\eta = \mathcal{R} + j\mathcal{X} = |\eta|e^{j\zeta}$)

	k'	k''	$\mathcal{R}$	$\mathcal{X}$
General	$\operatorname{Re}\sqrt{-\hat{z}\hat{y}}$	$-\operatorname{Im}\sqrt{-\hat{z}\hat{y}}$	$\operatorname{Re}\sqrt{\frac{\hat{z}}{\hat{y}}}$	$\operatorname{Im}\sqrt{\frac{\hat{z}}{\hat{y}}}$
No magnetic losses	$\operatorname{Im}\sqrt{j\omega\mu\hat{y}}$	$\operatorname{Re}\sqrt{j\omega\mu\hat{y}}$	$\frac{k'}{\lvert\hat{y}\rvert}$	$\frac{k''}{\lvert\hat{y}\rvert}$
Perfect dielectric	$\omega\sqrt{\mu\epsilon}$	0	$\sqrt{\frac{\mu}{\epsilon}}$	0
Good dielectric	$\omega\sqrt{\mu\epsilon'}$	$\frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}}$	$\sqrt{\frac{\mu}{\epsilon'}}$	$\frac{\epsilon''}{2\epsilon'}\sqrt{\frac{\mu}{\epsilon'}}$
Good conductor	$\sqrt{\frac{\omega\mu\sigma}{2}}$	$\sqrt{\frac{\omega\mu\sigma}{2}}$	$\sqrt{\frac{\omega\mu}{2\sigma}}$	$\sqrt{\frac{\omega\mu}{2\sigma}}$

Separation into real and imaginary parts is shown explicitly in row 2 of Table 2-1. A similar simplification can be made for the case of no electric losses. (See Prob. 2-13.) Three special cases of materials with no magnetic losses are (1) perfect dielectrics, (2) good dielectrics, and (3) good conductors. The perfect dielectric case is that for which

$$k = \omega\sqrt{\mu\epsilon} \qquad \eta = \sqrt{\frac{\mu}{\epsilon}}$$

This is summarized in row 3 of Table 2-1. A good dielectric is characterized by $\hat{z} = j\omega\mu$, $\hat{y} = \omega\epsilon'' + j\omega\epsilon'$, with $\epsilon' \gg \epsilon''$. In this case, we have

$$k = \omega\sqrt{\mu\epsilon'\left(1 - j\frac{\epsilon''}{\epsilon'}\right)} \approx \omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{2\epsilon'}\right)$$

$$\eta = \frac{k^*}{|\hat{y}|} \approx \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right)$$

which is summarized in row 4 of Table 2-1. Finally, a good conductor is characterized by $\hat{z} = j\omega\mu$, $\hat{y} = \sigma + j\omega\epsilon$, with $\sigma \gg \omega\epsilon$. In this case, we have

$$k = \sqrt{-j\omega\mu(\sigma + j\omega\epsilon)} \approx \sqrt{-j\omega\mu\sigma}$$

$$\eta = \frac{k^*}{|\hat{y}|} \approx \sqrt{\frac{j\omega\mu}{\sigma}}$$

The last row of Table 2-1 shows these parameters separated into real and imaginary parts.

2-4. Waves in Lossy Matter. The only difference between the wave equation, Eq. (2-7), for lossy media and loss-free media is that k is complex in lossy media and real in loss-free media. Thus, Eq. (2-9) is still a solution in lossy media. In terms of the real and imaginary parts of k, it is

$$E_x = E_0 e^{-jkz} = E_0 e^{-k''z} e^{-jk'z} \tag{2-35}$$

Also, **H** is still given by Eq. (2-10), except that η is now complex. Thus, the **H** associated with the **E** of Eq. (2-35) is

$$H_y = \frac{E_0}{\eta} e^{-jkz} = \frac{E_0}{|\eta|} e^{-j\zeta} e^{-k''z} e^{-jk'z} \tag{2-36}$$

where $\eta = |\eta| e^{j\zeta}$. The instantaneous fields corresponding to Eqs. (2-35) and (2-36) are

$$\begin{aligned} \mathcal{E}_x &= \sqrt{2}\, E_0 e^{-k''z} \cos(\omega t - k'z) \\ \mathcal{H}_y &= \sqrt{2}\, \frac{E_0}{|\eta|} e^{-k''z} \cos(\omega t - k'z - \zeta) \end{aligned} \tag{2-37}$$

Thus, in lossy matter, a traveling wave is attenuated in the direction of travel according to $e^{-k''z}$, and $\mathcal{H}$ is no longer in phase with $\mathcal{E}$. A sketch of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ versus z at some instant of time would be similar to Fig. 2-1 except that the amplitudes of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ would decrease exponentially with z, and $\boldsymbol{\mathcal{H}}$ would not be in phase with $\boldsymbol{\mathcal{E}}$ ($\boldsymbol{\mathcal{H}}$ usually lags $\boldsymbol{\mathcal{E}}$). A sketch of $\mathcal{E}_x$ versus z for several instants of time is shown in Fig. 2-9 for a case of fairly large attenuation. A sketch of $\mathcal{H}_y$ versus z would be similar in form.

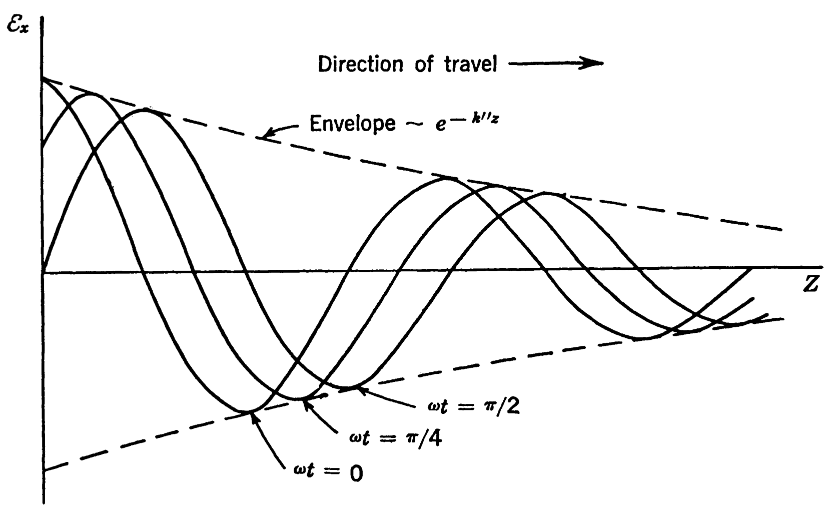


FIG. 2-9. $\mathcal{E}$ at several instants of time in a linearly polarized uniform plane traveling wave in dissipative matter.

The wave of Eq. (2-37) is still uniform, still plane, and still linearly polarized. So that our definitions of phase velocity and wavelength will be unchanged for lossy media, we should replace k and k' in the loss-free formulas, or

$$v_p = \frac{\omega}{k'} \qquad \lambda = \frac{2\pi}{k'} = \frac{v_p}{f} \tag{2-38}$$

Then v_p is still the velocity of a plane of constant phase, and λ is still the distance in which the phase increases by 2π.

Two cases of particular interest are (1) good dielectrics (low-loss), and (2) good conductors (high-loss). For the first case, we have (see Table 2-1)

$$\left.\begin{aligned} k' &= \omega\sqrt{\mu\epsilon'} \\ k'' &= \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} \\ |\eta| &= \sqrt{\frac{\mu}{\epsilon'}} \\ \zeta &= \tan^{-1}\frac{\epsilon''}{2\epsilon'} \end{aligned}\right\} \quad \text{in good dielectrics } (\epsilon'' \ll \epsilon') \tag{2-39}$$

Thus, the attenuation is very small, and $\mathcal{E}$ and $\mathcal{H}$ are nearly in phase. The wave is almost the same as in a loss-free dielectric. For example, in polystyrene (see Fig. 1-10), a 10-megacycle wave is attenuated only 0.5 per cent per kilometer, and the phase difference between $\mathcal{E}$ and $\mathcal{H}$ is only 0.003°. The intrinsic impedance of a dielectric is usually less than that of free space, since usually $\epsilon' > \epsilon_0$ and $\mu = \mu_0$. The intrinsic phase velocity and wavelength in a dielectric are also less than those of free space.

In the high-loss case (see Table 2-1), we have

$$\left.\begin{aligned} k' &= \sqrt{\frac{\omega\mu\sigma}{2}} \\ k'' &= \sqrt{\frac{\omega\mu\sigma}{2}} \\ |\eta| &= \sqrt{\frac{\omega\mu}{\sigma}} \\ \zeta &= \frac{\pi}{4} \end{aligned}\right\} \quad \text{in good conductors } (\sigma \gg \omega\epsilon) \tag{2-40}$$

Thus, the attenuation is very large, and $\mathcal{H}$ lags $\mathcal{E}$ by 45°. The intrinsic impedance of a good conductor is extremely small at radio frequencies, having a magnitude of 1.16×10^{-3} ohm for copper at 10 megacycles. The wavelength is also very small compared to the free-space wavelength. For example, at 10 megacycles the free-space wavelength is 30 meters, while in copper the wavelength is only 0.131 millimeter. The attenuation in a good conductor is very rapid. For the above-mentioned 10-megacycle wave in copper the attenuation is 99.81 per cent in 0.131 millimeter of travel. Thus, waves do not penetrate metals very deeply. A metal acts as a shield against electromagnetic waves.

A wave starting at the surface of a good conductor and propagating inward is very quickly damped to insignificant values. The field is localized in a thin surface layer, this phenomenon being known as *skin effect.* The distance in which a wave is attenuated to $1/e$ (36.8 per cent) of its initial value is called the *skin depth* or *depth of penetration* δ. This is defined by $k''\delta = 1$, or

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{k''} = \frac{\lambda_m}{2\pi} \tag{2-41}$$

where λ_m is the wavelength *in the metal.* The skin depth is very small for good conductors at radio frequencies, for λ_m is very small. For example, the depth of penetration into copper at 10 megacycles is only 0.021 millimeter. The density of power flow into the conductor, which must also be that dissipated within the conductor, is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}^* = \mathbf{u}_z |H_0|^2 \eta_m$$

where H_0 is the amplitude of $\mathbf{H}$ at the surface. The time-average power dissipation per unit area of surface cross section is the real part of the above power flow, or

$$\bar{\mathcal{P}}_d = |H_0|^2 \mathcal{R} \qquad \text{watts per square meter} \tag{2-42}$$

where $\mathcal{R} = \text{Re}\,(\eta_m)$ is the intrinsic resistance of the metal. $\mathcal{R}$ is also called the *surface resistance* and η_m the *surface impedance* of the metal. Eq. (2-42) is strictly true only when the wave propagates normally into the conductor. In the next section we shall see that this is usually so. In most problems Eq. (2-42) can be used to calculate power losses in conducting boundaries. (An important exception to this occurs at sharp points and corners extending outward from conductors.)

More general waves can be constructed by superposition of waves of the above type with various polarizations and directions of propagation. For waves uniform in the xy plane, the four basic waves, corresponding to Eqs. (2-17), are

$$\begin{aligned}
E_x^+ &= Ae^{-k''z}e^{-jk'z} & H_y^+ &= \frac{A}{\eta}\, e^{-k''z}e^{-jk'z} \\
E_y^+ &= Be^{-k''z}e^{-jk'z} & H_x^+ &= \frac{-B}{\eta}\, e^{-k''z}e^{-jk'z} \\
E_x^- &= Ce^{k''z}e^{jk'z} & H_y^- &= \frac{-C}{\eta}\, e^{k''z}e^{jk'z} \\
E_y^- &= De^{k''z}e^{jk'z} & H_x^- &= \frac{D}{\eta}\, e^{k''z}e^{jk'z}
\end{aligned} \tag{2-43}$$

The preceding discussion of this section applies to each of these waves if the appropriate interchange of coordinates is made.

A superposition of waves traveling in opposite directions, for example

$$
\begin{aligned}
E_x &= Ae^{-k''z}e^{-jk'z} + Ce^{k''z}e^{jk'z} \\
H_y &= \frac{1}{\eta}\left(Ae^{-k''z}e^{-jk'z} - Ce^{k''z}e^{jk'z}\right)
\end{aligned}
\tag{2-44}
$$

gives us standing-wave phenomena. However, it is no longer possible to have two "equal" waves traveling in opposite directions. One wave is attenuated in the $+z$ direction, the other in the $-z$ direction; hence they can be equal only at one plane. Suppose that the wave components are equal at $z = 0$, that is, $A = C$ in Eq. (2-44). There will then be standing waves in the vicinity of $z = 0$, which will die out in both the $+z$ and $-z$ directions. This is illustrated by Fig. 2-10 for a material having fairly large losses. Far in the $+z$ direction the $+z$ traveling wave has died out, leaving only the $-z$ traveling wave. Similarly, far in the $-z$ direction we have only the $+z$ traveling wave. The standing-wave ratio is now a function of z, being large in the vicinity of $z = 0$ and approaching unity as $|z|$ becomes large. For very small amounts of dissipation, say in a good dielectric, the attenuation of the wave is small, and standing-wave patterns are almost the same as for the dissipationless case.

Other superpositions of Eqs. (2-43) can be formed to give elliptically and circularly polarized waves. In a picture of a circularly polarized wave traveling in dissipative media, the "corkscrews" for $\mathcal{E}$ and $\mathcal{H}$ would be attenuated in the direction of propagation. Also, $\mathcal{E}$ would be somewhat out of phase with $\mathcal{H}$. A circularly polarized standing wave would be a localized phenomenon in dissipative media, just as a linearly polarized standing wave is localized.

2-5. Reflection of Waves. We saw in Sec. 1-14 that the tangential components of E and H must be continuous across a material boundary.

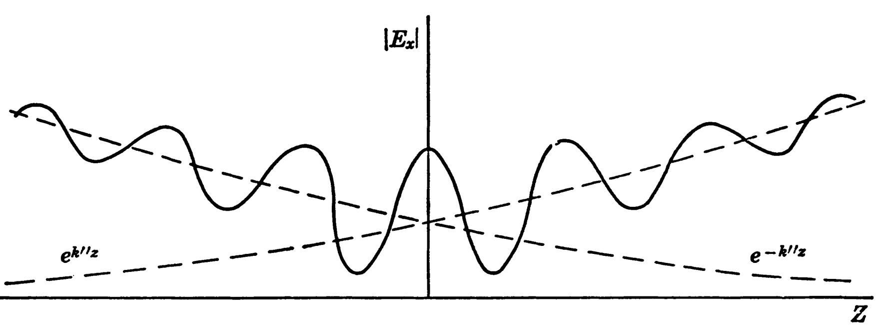


FIG. 2-10. Standing-wave pattern of two oppositely traveling waves in dissipative matter.

A ratio of a component of **E** to a component of **H** is called the wave impedance in the direction defined by the cross-product rule applied to the two components. Thus, continuity of tangential **E** and **H** requires that *wave impedances normal to a material boundary must be continuous.*

The simplest reflection problem is that of a uniform plane wave normally incident upon a plane boundary between two media. This is illustrated by Fig. 2-11. In region 1 the field will be the sum of an incident wave plus a reflected wave. The ratio of the reflected electric intensity to the incident electric intensity at the interface is defined to be the *reflection coefficient* Γ. Hence, for region 1

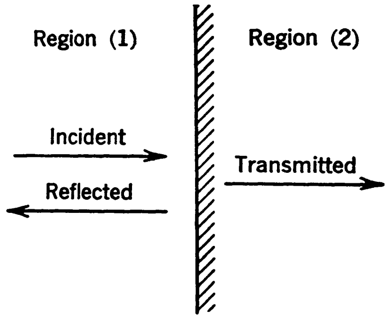


FIG. 2-11. Reflection at a plane dielectric interface, normal incidence.

$$E_x^{(1)} = E_0(e^{-jk_1z} + \Gamma e^{jk_1z})$$
$$H_y^{(1)} = \frac{E_0}{\eta_1}(e^{-jk_1z} - \Gamma e^{jk_1z})$$

In region 2 there will be a transmitted wave. The ratio of the transmitted electric intensity to the incident electric intensity at the interface is defined to be the transmission coefficient T. Hence, for region 2

$$E_x^{(2)} = E_0 T e^{-jk_2z}$$
$$H_y^{(2)} = \frac{E_0}{\eta_2} T e^{-jk_2z}$$

For continuity of wave impedance at the interface, we have

$$Z_z\Big|_{z=0} = \frac{E_x^{(1)}}{H_y^{(1)}}\Big|_{z=0} = \eta_1 \frac{1+\Gamma}{1-\Gamma} = \eta_2$$

where η_1 and η_2 are the intrinsic wave impedances of media 1 and 2. Solving for the reflection coefficient, we have

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \tag{2-45}$$

From the continuity of E_x at $z = 0$, we have the transmission coefficient given by

$$T = 1 + \Gamma = \frac{2\eta_2}{\eta_2 + \eta_1} \tag{2-46}$$

If region 1 is a perfect dielectric, the standing-wave ratio is

$$\text{SWR} = \frac{E_{\text{max}}^{(1)}}{E_{\text{min}}^{(1)}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \tag{2-47}$$

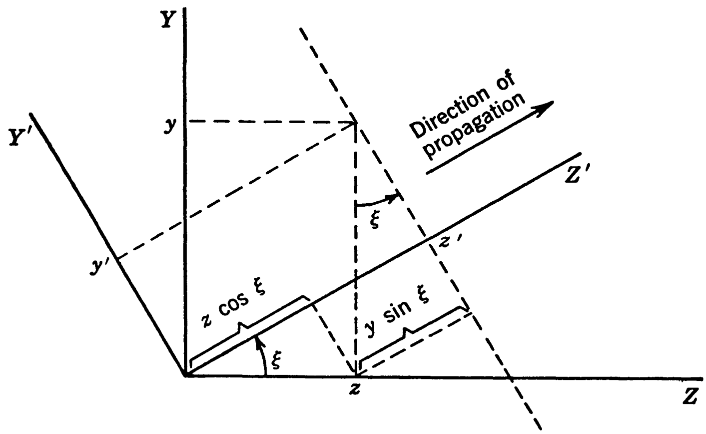


FIG. 2-12. A plane wave propagating at an angle ξ with respect to the x-z plane.

because the incident and reflected waves add in phase at some points and add 180° out of phase at other points. The density of power transmitted across the interface is

$$\mathcal{S}_{\text{trans}} = \operatorname{Re} \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{u}_z \Big|_{z=0} = \mathcal{S}_{\text{inc}}(1 - |\Gamma|^2) \tag{2-48}$$

where $\mathcal{S}_{\text{inc}} = E_0^2/\eta_1$ is the incident power density. The difference between the incident and transmitted power must be that reflected, or

$$\mathcal{S}_{\text{refl}} = \mathcal{S}_{\text{inc}}|\Gamma|^2 \tag{2-49}$$

We have used an x-polarized wave for the analysis, but the results are valid for arbitrary polarization, since the x axis may be in any direction tangential to the boundary. Those of us familiar with transmission-line theory should note the complete analogy between the above plane-wave problem and the transmission-line problem.

Another reflection problem of considerable interest is that of a plane wave incident at an angle upon a plane dielectric boundary. Before considering this problem, let us express the uniform plane wave in coordinates rotated with respect to the direction of propagation. Let Fig. 2-12 represent a plane wave propagating at an angle ξ with respect to the xz plane. An equiphase plane z' in terms of the unprimed coordinates is

$$z' = z \cos \xi + y \sin \xi$$

and the unit vector in the y' direction in terms of the unprimed coordinate unit vectors is

$$\mathbf{u}_{y'} = \mathbf{u}_y \cos \xi - \mathbf{u}_z \sin \xi$$

The expression for a uniform plane wave with **E** parallel to the $z = 0$ plane is the first of Eqs. (2-17) with all coordinates primed. Substituting from the above two equations, we have

$$\begin{aligned} E_x &= E_0 e^{-jk(y \sin \xi + z \cos \xi)} \\ \mathbf{H} &= (\mathbf{u}_y \cos \xi - \mathbf{u}_z \sin \xi) \frac{E_0}{\eta} e^{-jk(y \sin \xi + z \cos \xi)} \end{aligned} \tag{2-50}$$

The wave impedance in the z direction for this wave is

$$Z_z = \frac{E_x}{H_y} = \frac{\eta}{\cos \xi} \tag{2-51}$$

In a similar manner, from the second of Eqs. (2-17), the expression for a uniform plane wave with **H** parallel to the $z = 0$ plane is found to be

$$\begin{aligned} \mathbf{E} &= (\mathbf{u}_y \cos \xi - \mathbf{u}_z \sin \xi) E_0 e^{-jk(y \sin \xi + z \cos \xi)} \\ H_x &= -\frac{E_0}{\eta} e^{-jk(y \sin \xi + z \cos \xi)} \end{aligned} \tag{2-52}$$

The wave impedance in the z direction for this wave is

$$Z_z = -\frac{E_y}{H_x} = \eta \cos \xi \tag{2-53}$$

Thus, the z-directed wave impedance for **E** parallel to the $z = 0$ plane is always greater than the intrinsic impedance, and for **H** parallel to the $z = 0$ plane it is always less than the intrinsic impedance of the medium.

Now suppose that a uniform plane wave is incident at an angle $\xi = \theta_i$ upon a dielectric interface at $z = 0$, as shown in Fig. 2-13. Part of the wave will be reflected at an angle $\xi = \pi - \theta_r$, and part transmitted at an angle $\xi = \theta_t$. Each of these partial fields will be of the form of Eqs. (2-50) if **E** is parallel to the interface or of the form of Eqs. (2-52) if **H** is parallel to the interface. (Arbitrary polarization is a superposition of these two

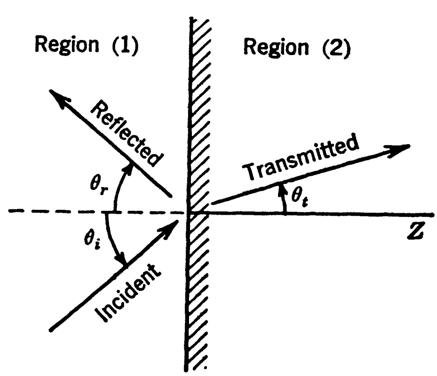


FIG. 2-13. Reflection at a plane dielectric interface, arbitrary angle of incidence.

cases.) For continuity of tangential **E** and **H** over the entire interface, the y variation of all three partial fields must be the same. This is so if

$$k_1 \sin \theta_i = k_1 \sin \theta_r = k_2 \sin \theta_t$$

From the first equality, we have

$$\theta_r = \theta_i \tag{2-54}$$

that is, *the angle of reflection is equal to the angle of incidence.* From the second equality, we have

$$\frac{\sin \theta_t}{\sin \theta_i} = \frac{k_1}{k_2} = \frac{v_2}{v_1} = \sqrt{\frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}} \tag{2-55}$$

where v is the phase velocity. Equation (2-55) is known as *Snell's law* of refraction. The direction of propagation of the transmitted wave is thus different from that of the incident wave unless $\epsilon_1\mu_1 = \epsilon_2\mu_2$. In practically all low-loss dielectrics, $\mu_1 = \mu_2 = \mu_0$. If medium 2 is free space and medium 1 is a nonmagnetic dielectric, the right-hand side of Eq. (2-55) becomes $\sqrt{\epsilon_1/\epsilon_0} = \sqrt{\epsilon_r}$, which is called the *index of refraction* of the dielectric.

The magnitudes of the reflected and transmitted fields depend upon the polarization. For **E** parallel to the interface, we have in region 1

$$E_x^{(1)} = A(e^{-jk_1z\cos\theta_i} + \Gamma e^{jk_1z\cos\theta_r})$$
$$H_y^{(1)} = \frac{A}{\eta_1}\cos\theta_i(e^{-jk_1z\cos\theta_i} - \Gamma e^{jk_1z\cos\theta_r})$$

where A includes the y dependence. Thus, the z-directed wave impedance in region 1 at the interface is

$$Z_z^{(1)} = \frac{E_x^{(1)}}{H_y^{(1)}} = \frac{\eta_1}{\cos\theta_i}\frac{1+\Gamma}{1-\Gamma}$$

This must be equal to the z-directed wave impedance in region 2 at the interface, which is Eq. (2-51) with $\xi = \theta_t$. Thus,

$$\Gamma = \frac{\eta_2 \sec\theta_t - \eta_1 \sec\theta_i}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_i} \tag{2-56}$$

Note that this is of the same form as the corresponding equation for normal incidence, Eq. (2-45). The intrinsic impedances are merely replaced by the z-directed wave impedances of single traveling waves. It should be apparent from the form of the equations that, for **H** parallel to the interface, the reflection coefficient is given by

$$\Gamma = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{2-57}$$

In both cases we have standing waves in the z direction, the standing-wave ratio being given by Eq. (2-47).

Two cases of special interest are (1) that of total transmission and (2) that of total reflection. The first case occurs when $\Gamma = 0$. For **E** parallel to the interface, we see from Eq. (2-56) that $\Gamma = 0$ when

$$\frac{\eta_2}{\cos\theta_t} = \frac{\eta_1}{\cos\theta_i}$$

Substituting for θ_t from Eq. (2-55) and for the η's from Eq. (2-11) we obtain

$$\sin\theta_i = \sqrt{\frac{\epsilon_2/\epsilon_1 - \mu_2/\mu_1}{\mu_1/\mu_2 - \mu_2/\mu_1}} \tag{2-58}$$

as the angle at which no reflection occurs. This does not always have a real solution for θ_i. In fact,

$$\sin\theta_i \xrightarrow[\mu_1 \to \mu_2]{} \infty$$

For nonmagnetic dielectrics ($\mu_1 = \mu_2 = \mu_0$) there is no angle of total transmission when **E** is parallel to the boundary. For the case of **H** parallel to the boundary, we find from Eq. (2-57) that $\Gamma = 0$ when

$$\sin\theta_i = \sqrt{\frac{\epsilon_2/\epsilon_1 - \mu_2/\mu_1}{\epsilon_2/\epsilon_1 - \epsilon_1/\epsilon_2}} \tag{2-59}$$

Again this does not always have a real solution for arbitrary μ and ϵ. But in the nonmagnetic case

$$\theta_i = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1 + \epsilon_2}} = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} \tag{2-60}$$

There is usually an angle of total transmission when **H** is parallel to the boundary. The angle specified by Eq. (2-60) is called the *polarizing angle* or *Brewster angle*. If an arbitrarily polarized wave is incident upon a nonmagnetic boundary at this angle, the reflected wave will be polarized with **E** parallel to the boundary.

The case of total reflection occurs when $|\Gamma| = 1$. We are considering lossless media; so the η's are real. It is apparent from Eqs. (2-56) and (2-57) that $|\Gamma| \neq 1$ for real values of θ_i and θ_t. However, when $\epsilon_1\mu_1 > \epsilon_2\mu_2$, Eq. (2-55) says that $\sin\theta_t$ can be greater than unity. What does this mean? Our initial assumption was that the transmitted wave was a uniform plane wave. But Eqs. (2-50) specify a solution to Maxwell's equations regardless of the value of $\sin\xi$. It can be real or complex. All that is changed is our interpretation of the field. To illustrate, suppose $\sin \xi > 1$ in Eqs. (2-50) and let

$$\begin{aligned} k \sin \xi &= \beta \\ k \cos \xi &= k\sqrt{1 - \sin^2 \xi} = \pm j\alpha \end{aligned} \tag{2-61}$$

If we choose the minus sign for α, Eqs. (2-50) become

$$\begin{aligned} E_x &= E_0 e^{-j\beta y} e^{-\alpha z} \\ \mathbf{H} &= -\left(\mathbf{u}_y \frac{j\alpha}{k} + \mathbf{u}_z \frac{\beta}{k}\right) \frac{E_0}{\eta} e^{-j\beta y} e^{-\alpha z} \end{aligned} \tag{2-62}$$

which is a field exponentially attenuated in the z direction. Note the 90° phase difference between E_x and H_y; so the wave impedance in the z direction is imaginary, and there is no power flow in the z direction. A similar interpretation applies to Eqs. (2-52) when $\sin \xi > 1$. Returning now to our reflection problem, from Eq. (2-55) it is evident that $\sin \theta_t$ is greater than unity when $\sin \theta_i > \sqrt{\epsilon_2\mu_2/\epsilon_1\mu_1}$. Thus, the point of transition from real values of θ_t (wave impedance real in region 2) to imaginary values of θ_t (wave impedance imaginary in region 2) is

$$\sin \theta_i = \sqrt{\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}} \tag{2-63}$$

The angle specified by Eq. (2-63) is called the *critical angle*. A wave incident upon the boundary at an angle equal to or greater than the critical angle will be totally reflected. Note that there is a real critical angle only if $\epsilon_1\mu_1 > \epsilon_2\mu_2$ or, in the nonmagnetic case, if $\epsilon_1 > \epsilon_2$. Thus, total reflection occurs only if the wave passes from a "dense" material into a "less dense" material. The reflection coefficient, Eq. (2-56) or Eq. (2-57), becomes of the form

$$\Gamma = \frac{R - jX}{R + jX}$$

when total reflection occurs. It is evident in this case that $|\Gamma|$ is unity. Remember that the field in region 2 is not zero when total reflection occurs. It is an exponentially decaying field, called a *reactive* field or an *evanescent* field. Optical prisms make use of the phenomenon of total reflection.

All the theory of this section can be applied to dissipative media if the η's and θ's are allowed to be complex. Of particular interest is the case of a plane wave incident upon a good conductor at an angle θ_i. When region 1 is a nonmagnetic dielectric and region 2 is a nonmagnetic conductor, Eq. (2-55) becomes

$$\frac{\sin \theta_t}{\sin \theta_i} = \frac{k_1}{k_2} \approx \sqrt{\frac{j\omega\epsilon}{\sigma}}$$

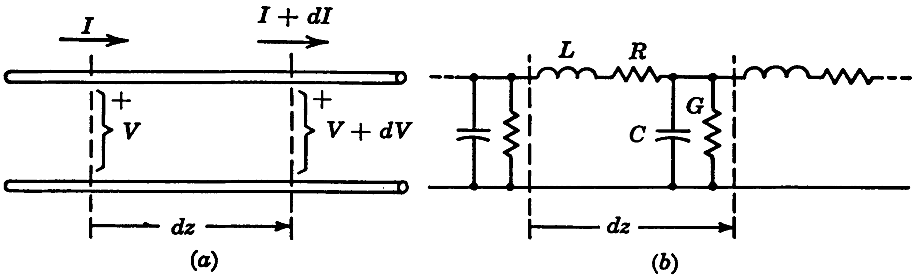


FIG. 2-14. A transmission line according to circuit concepts. (a) Physical line; (b) equivalent circuit.

This is an extremely small quantity for good conductors. For most practical purposes, the wave can be considered to propagate normally into the conductor regardless of the angle of incidence.

2-6. Transmission-line Concepts. Let us review the circuit concept of a transmission line and then show its relationship to the field concept. Let Fig. 2-14a represent a two-conductor transmission line. For each incremental length of line dz there is a series voltage drop dV and a shunt current dI. The circuit theory postulate is that the voltage drop is proportional to the line current I. Thus,

$$dV = -IZ\,dz$$

where Z is a series *impedance per unit length.* It is also postulated that the shunt current is proportional to the line voltage V. Thus,

$$dI = -VY\,dz$$

where Y is a *shunt admittance per unit length.* Dividing by dz, we have the *a-c transmission-line equations*

$$\frac{dV}{dz} = -IZ \qquad \frac{dI}{dz} = -VY \tag{2-64}$$

Implicit in this development are the assumptions that (1) no mutual impedance exists between incremental sections of line and (2) the shunt current dI flows in planes transverse to z. The transmission line is said to be *uniform* if Z and Y are independent of z.

Taking the derivative of the first of Eqs. (2-64) and substituting from the second, we obtain

$$\frac{d^2V}{dz^2} - ZYV = 0 \qquad \frac{d^2I}{dz^2} - ZYI = 0 \tag{2-65}$$

which are one-dimensional Helmholtz equations. The general solution

TABLE 2-2. COMPARISON OF TRANSMISSION-LINE WAVES TO UNIFORM PLANE WAVES

Transmission line	Uniform plane wave
$\frac{d^2V}{dz^2} - \gamma^2 V = 0$	$\frac{d^2E_x}{dz^2} + k^2E_x = 0$
$\frac{d^2I}{dz^2} - \gamma^2 I = 0$	$\frac{d^2H_y}{dz^2} + k^2H_y = 0$
$\gamma = \sqrt{ZY}$	$jk = \sqrt{\hat{z}\hat{y}}$
$V = V_0^+e^{-\gamma z} + V_0^-e^{\gamma z}$	$E_x = E_0^+e^{-jkz} + E_0^-e^{jkz}$
$I = I_0^+e^{-\gamma z} + I_0^-e^{\gamma z}$	$H_y = H_0^+e^{-jkz} + H_0^-e^{jkz}$
$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \sqrt{\frac{Z}{Y}}$	$\eta = \frac{E_0^+}{H_0^+} = -\frac{E_0^-}{H_0^-} = \sqrt{\frac{\hat{z}}{\hat{y}}}$
$P = VI^*$	$S_z = E_xH_y^*$

is a sum of a $+z$ traveling wave and a $-z$ traveling wave, with propagation constant

$$\gamma = \sqrt{ZY} \tag{2-66}$$

Choosing the $+z$ traveling wave

$$V^+ = V_0e^{-\gamma z} \qquad I^+ = I_0e^{-\gamma z}$$

we have from Eqs. (2-64) that

$$\frac{V^+}{I^+} = \frac{Z}{\gamma} = \frac{\gamma}{Y}$$

Substituting for γ from Eq. (2-66), we have

$$Z_0 = \frac{V^+}{I^+} = \sqrt{\frac{Z}{Y}} \tag{2-67}$$

which is called the *characteristic impedance* of the transmission line. The imaginary parts of Z and Y are usually positive, and it is common practice to write

$$Z = R + j\omega L \qquad Y = G + j\omega C \tag{2-68}$$

The equivalent circuit of the transmission line is then as shown in Fig. 2-14*b*. The reader has probably already noted the complete analogy between the linearly polarized plane wave and the transmission line. This analogy is summarized by Table 2-2.

In the circuit theory development, we assumed no mutual coupling between adjacent elements of the transmission line. From the field theory point of view, this is equivalent to assuming that no E_z or H_z exists. Such a wave is called *transverse electromagnetic,* abbreviated TEM. This is not the only wave possible on a transmission line, for Maxwell's equations show that infinitely many wave types can exist. Each possible wave is called a *mode,* and a TEM wave is called a *transmission-line mode.* All other waves, which must have an E_z or an H_z or both, are called *higher-order modes.* The higher-order modes are usually important only in the vicinity of the feed point, or in the vicinity of a discontinuity on the line. In this section we shall restrict consideration to transmission-line, or TEM, modes.

For the TEM mode to exist exactly, the conductors must be perfect, or else an E_z is required to support the z-directed current. Let us therefore specialize the problem to that of perfect conductors immersed in a homogeneous medium. We assume $E_z = H_z = 0$ and z dependence of the form $e^{-\gamma z}$. Expansion of the field equations, Eqs. (2-1), then gives

$$
\begin{aligned}
\gamma E_y &= -\hat{z} H_x & \qquad \gamma H_y &= \hat{y} E_x \\
\gamma E_x &= \hat{z} H_y & \qquad \gamma H_x &= -\hat{y} E_y \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= 0 & \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= 0
\end{aligned}
$$

It follows from these equations that

$$\gamma = jk \tag{2-69}$$

The propagation constant of any TEM wave is the intrinsic propagation constant of the medium. The proportionality of components of **E** to those of **H** expressed by the above equations can be written concisely as

$$\mathbf{E} = \eta \mathbf{H} \times \mathbf{u}_z \qquad \mathbf{H} = \frac{1}{\eta} \mathbf{u}_z \times \mathbf{E} \tag{2-70}$$

Thus, *the z-directed wave impedance of any TEM wave is the intrinsic wave impedance of the medium.* Finally, manipulation of the original six equations shows that each component of **E** and **H** satisfies the two-dimensional Laplace equation. We can summarize this by defining a *transverse Laplacian operator*

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2-71}$$

and writing $\qquad \nabla_t^2 \mathbf{E} = 0 \qquad \nabla_t^2 \mathbf{H} = 0$

The boundary conditions for the problem are

$$\left.\begin{aligned} E_t &= 0 \\ H_n &= 0 \end{aligned}\right\} \quad \text{at the conductors} \tag{2-72}$$

Thus, the boundary-value problem for **E** is the same as the electrostatic

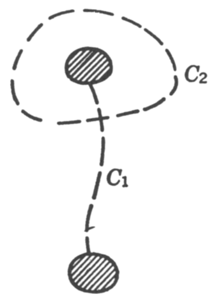


FIG. 2-15. Cross section of a transmission line.

problem having the same conducting boundaries. The boundary-value problem for **H** is the same as the magnetostatic problem having "anticonducting" (no H_n) boundaries. It is for this reason that "static" capacitances and inductances can be used for transmission lines even though the field is time-harmonic.

To show the relationship of the static L's and C's to the Z_0 of the transmission line, consider a cross section of the line as represented by Fig. 2-15. In the transmission-line problem, the line voltage and current are related to the fields by

$$V = \int_{C_1} \mathbf{E} \cdot d\mathbf{l} \qquad I = \int_{C_2} \mathbf{H} \cdot d\mathbf{l} \tag{2-73}$$

where C_1 and C_2 are as shown on Fig. 2-15. From the second of these and the second of Eqs. (2-70) we have

$$I = \frac{1}{\eta} \int_{C_2} \mathbf{u}_z \times \mathbf{E} \cdot d\mathbf{l} = \frac{1}{\eta} \int_{C_2} E_n \, dl$$

But in the corresponding electrostatic problem the capacitance is

$$C = \frac{q}{V} = \frac{\epsilon}{V} \int_{C_2} E_n \, dl$$

Thus, the characteristic impedance of the transmission line is related to the electrostatic capacitance per unit length by

$$Z_0 = \frac{V}{I} = \eta \frac{\epsilon}{C} \tag{2-74}$$

Similarly, from the first of Eqs. (2-73) and (2-70) we have

$$V = \eta \int_{C_1} \mathbf{H} \times \mathbf{u}_z \cdot d\mathbf{l} = \eta \int_{C_1} H_n \, dl$$

In the corresponding magnetostatic problem we have

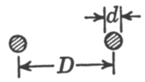
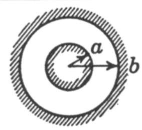
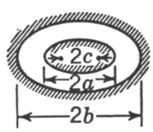
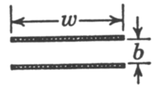
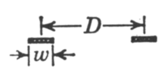
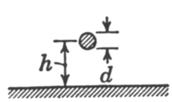
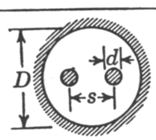
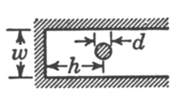
$$L = \frac{\psi}{I} = \frac{\mu}{I} \int_{C_1} H_n \, dl$$

Therefore, the characteristic impedance of the line is related to the magnetostatic inductance per unit length by

$$Z_0 = \frac{V}{I} = \eta \frac{L}{\mu} \tag{2-75}$$

Note also that L and C are related to each other through Eqs. (2-74) and (2-75). The electrostatic and magnetostatic problems have **E** and **H** everywhere orthogonal to each other and are called *conjugate problems.*

TABLE 2-3. CHARACTERISTIC IMPEDANCES OF SOME COMMON TRANSMISSION LINES

Line	Geometry	Characteristic impedance
Two wire		$Z_0 \approx \frac{\eta}{\pi} \log \frac{2D}{d} \qquad D \gg d$
Coaxial		$Z_0 = \frac{\eta}{2\pi} \log \frac{b}{a}$
Confocal elliptic		$Z_0 = \frac{\eta}{2\pi} \log \frac{b + \sqrt{b^2 - c^2}}{a + \sqrt{a^2 - c^2}}$
Parallel plate		$Z_0 \approx \eta \frac{b}{w} \qquad w \gg b$
Collinear plate		$Z_0 \approx \frac{\eta}{\pi} \log \frac{4D}{w} \qquad D \gg w$
Wire above ground plane		$Z_0 \approx \frac{\eta}{2\pi} \log \frac{4h}{d} \qquad h \gg d$
Shielded pair		$Z_0 \approx \frac{\eta}{\pi} \log \left(\frac{2s}{d} \frac{D^2 - s^2}{D^2 + s^2} \right) \qquad D \gg d, \; s \gg d$
Wire in trough		$Z_0 \approx \frac{\eta}{2\pi} \log \left(\frac{4w}{\pi d} \tanh \frac{\pi h}{w} \right) \qquad h \gg d, \; w \gg d$

Once the electrostatic C or the magnetostatic L is known, the Z_0 of the corresponding transmission line is given by Eq. (2-74) or Eq. (2-75). Table 2-3 lists the characteristic impedances of some common transmission lines.

When the dielectric is lossy but the conductors still assumed perfect, all of our equations still apply. Z_0 (proportional to η) and γ ($= jk$)

become complex. The most important effect of this is that the wave is attenuated in the direction of travel. The attenuation constant in this case is the intrinsic attenuation constant of the dielectric (Table 2-1, column 2, row 4). When the conductors are imperfect, the field is no longer exactly TEM, and exact solutions are usually impractical. However, the waves will still be characterized by a propagation constant $\gamma = \alpha + j\beta$. Hence a $+z$-traveling wave will be of the form

$$V = V_0 e^{-(\alpha+j\beta)z} \qquad I = \frac{V}{Z_0}$$

and the power flow is given by

$$P_f = VI^* = \frac{|V_0|^2}{Z_0^*} e^{-2\alpha z} = P_0 e^{-2\alpha z}$$

or, in terms of time-average powers,

$$\bar{\mathcal{P}}_f = \text{Re}\,(P_f) = \text{Re}\,(P_0)e^{-2\alpha z}$$

The rate of decrease in $\bar{\mathcal{P}}_f$ versus z equals the time-average power dissipated per unit length $\bar{\mathcal{P}}_d$, or

$$\bar{\mathcal{P}}_d = -\frac{d\bar{\mathcal{P}}_f}{dz} = 2\alpha\bar{\mathcal{P}}_f$$

Thus, the attenuation constant is given by

$$\alpha = \frac{\bar{\mathcal{P}}_d}{2\bar{\mathcal{P}}_f} \tag{2-76}$$

While this equation is exact if $\bar{\mathcal{P}}_d$ and $\bar{\mathcal{P}}_f$ are determined exactly, its greatest use lies in approximating α by approximating $\bar{\mathcal{P}}_d$. For example, attenuation due to losses in imperfect conductors can be approximated by assuming that Eq. (2-42) holds at their surface. We shall carry out such a calculation for the rectangular waveguide in the next section.

2-7. Waveguide Concepts. The waves on a transmission line can be viewed as being guided by the conductors. This concept of wave guidance is quite general and applies to many configurations of matter. In general, systems which guide waves are called *waveguides*. Apart from transmission lines, the most commonly used waveguide is the *rectangular waveguide*, illustrated by Fig. 2-16. It is a hollow conducting tube

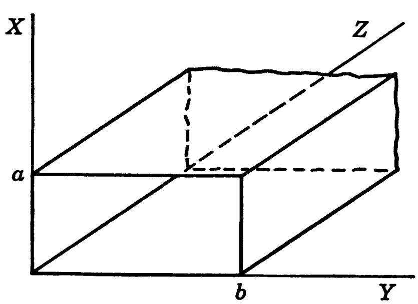


FIG. 2-16. The rectangular waveguide.

of rectangular cross section. Fields existing within this tube must be characterized by zero tangential components of **E** at the conducting walls.

Consider two uniform plane waves traveling at the angles ξ and $-\xi$ with respect to the xz plane (see Fig. 2-12). If the waves are x-polarized, we use Eq. (2-50) and write

$$\begin{aligned} E_x &= A(e^{-jky \sin \xi} - e^{jky \sin \xi})e^{-jkz \cos \xi} \\ &= -2jA \sin (ky \sin \xi)\, e^{-jkz \cos \xi} \end{aligned}$$

Let E_0 denote $(-2jA)$ and define

$$k_c = k \sin \xi \qquad \gamma = jk \cos \xi$$

In view of the trigonometric identity $\sin^2 \xi + \cos^2 \xi = 1$, the parameters γ and k_c are related by

$$\gamma^2 = k_c^2 - k^2 \tag{2-77}$$

The above field can now be written as

$$E_x = E_0 \sin (k_c y)\, e^{-\gamma z} \tag{2-78}$$

Let us see if this field can exist within the rectangular waveguide. There is only an E_x; so no component of **E** is tangential to the conductors $x = 0$ and $x = a$. Also, $E_x = 0$ at $y = 0$; so there is no tangential component of **E** at the wall $y = 0$. There remains the condition that $E_x = 0$ at $y = b$, which is satisfied if

$$k_c = \frac{n\pi}{b} \qquad n = 1, 2, 3, \ldots \tag{2-79}$$

These permissible values of k_c are called *eigenvalues*, or *characteristic values* of the problem.

Each choice of n in Eq. (2-79) determines a possible field, or *mode.* The modes in a waveguide are usually classified according to the existence of z components of the field. A mode having no E_z is said to be a *transverse electric* (TE) mode. One having no H_z is said to be a *transverse magnetic* (TM) mode. All the modes in the rectangular waveguide fall into one of these two classes. The modes represented by Eqs. (2-78) and (2-79) have no E_z and are therefore TE modes. The particular modes that we are considering are TE_{0n} modes, the subscript 0 denoting no variation with x, and the subscript n denoting the choice by Eq. (2-79). The complete system of modes will be considered in Sec. 4-3.

For k real (loss-free dielectric), the propagation constant γ can be expressed as

$$\gamma = \begin{cases} j\beta = j\sqrt{k^2 - \left(\dfrac{n\pi}{b}\right)^2} & k > \dfrac{n\pi}{b} \\ \alpha = \sqrt{\left(\dfrac{n\pi}{b}\right)^2 - k^2} & k < \dfrac{n\pi}{b} \end{cases} \tag{2-80}$$

where α and β are real. This follows from Eqs. (2-77) and (2-79). When $\gamma = j\beta$, we have wave propagation in the z direction, and the mode is called a *propagating mode.* When $\gamma = \alpha$, the field decays exponentially with z, and there is no wave propagation. In this case, the mode is called a *nonpropagating mode,* or an *evanescent mode.* The transition from one type of behavior to the other occurs at $\alpha = 0$ or $k = n\pi/b$. Letting $k = 2\pi f\sqrt{\epsilon\mu}$, we can solve for the transition frequency, obtaining

$$f_c = \frac{n}{2b\sqrt{\epsilon\mu}} \tag{2-81}$$

This is called the *cutoff frequency* of the TE_{0n} mode. The corresponding intrinsic wavelength

$$\lambda_c = \frac{2b}{n} \tag{2-82}$$

is called the *cutoff wavelength* of the TE_{0n} mode. At frequencies greater than f_c (wavelengths less than λ_c), the mode propagates. At frequencies less than f_c (wavelengths greater than λ_c), the mode is nonpropagating.

A knowledge of f_c or λ_c is equivalent to a knowledge of k_c; so they also are eigenvalues. In particular, from Eqs. (2-79), (2-81), and (2-82), it is evident that

$$k_c = \frac{2\pi}{\lambda_c} = 2\pi f_c\sqrt{\epsilon\mu} \tag{2-83}$$

Using the last equality and $k = 2\pi f\sqrt{\epsilon\mu}$ in Eq. (2-80), we can express γ as

$$\gamma = \begin{cases} j\beta = jk\sqrt{1 - \left(\dfrac{f_c}{f}\right)^2} & f > f_c \\ \alpha = k_c\sqrt{1 - \left(\dfrac{f}{f_c}\right)^2} & f < f_c \end{cases} \tag{2-84}$$

Thus, the phase constant β of a propagating mode is always less than the intrinsic phase constant k of the dielectric, approaching k as $f \to \infty$. The attenuation constant of a nonpropagating mode is always less than k_c, approaching k_c as $f \to 0$. When a mode propagates, the concepts of wavelength and phase velocity can be applied to the mode field as a whole. Thus, the *guide wavelength* λ_g is defined as the distance in which the phase of E increases by 2π, that is, $\beta\lambda_g = 2\pi$. Using β from Eq. (2-84), we have

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} \tag{2-85}$$

showing that the guide wavelength is always greater than the intrinsic wavelength of the dielectric. The *guide phase velocity* v_g is defined as the velocity at which a point of constant phase of $\mathcal{E}$ travels. Thus, in a manner analogous to that used to derive Eq. (2-14), we find

$$v_g = \frac{\omega}{\beta} = \frac{v_p}{\sqrt{1 - (f_c/f)^2}} \tag{2-86}$$

where v_p is the intrinsic phase velocity of the dielectric. The guide phase velocity is therefore greater than the intrinsic phase velocity.

Another important property of waveguide modes is the existence of a *characteristic wave impedance.* To show this, let us find **H** from the **E** of Eq. (2-78) according to $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$. The result is

$$\begin{aligned} E_x &= E_0 \sin(k_c y)\, e^{-\gamma z} \\ H_y &= \frac{\gamma}{j\omega\mu} E_0 \sin(k_c y)\, e^{-\gamma z} \\ H_z &= \frac{k_c}{j\omega\mu} E_0 \cos(k_c y)\, e^{-\gamma z} \end{aligned} \tag{2-87}$$

where E_x has been repeated for convenience. The wave impedance in the z direction is

$$Z_z = \frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma} \tag{2-88}$$

This is called the characteristic impedance of the mode and plays the same role in reflection problems as does the Z_0 of transmission lines. If we substitute into the above equation for γ from Eq. (2-84), we find

$$Z_0 = Z_z = \begin{cases} \dfrac{\eta}{\sqrt{1 - (f_c/f)^2}} & f > f_c \\ \dfrac{j\eta}{\sqrt{(f_c/f)^2 - 1}} & f < f_c \end{cases} \tag{2-89}$$

Thus, the characteristic impedance of a TE_{0n} propagating mode is always greater than the intrinsic impedance of the dielectric, approaching η as $f \to \infty$. The characteristic impedance of a nonpropagating mode is reactive and approaches zero as $f \to 0$.

All our discussion so far has dealt with waves traveling in the $+z$ direction. For each $+z$ traveling wave, a $-z$ traveling wave is possible, obtained by replacing γ by $-\gamma$ in Eqs. (2-87). The simultaneous existence of $+z$ and $-z$ traveling waves in the same mode gives rise to standing waves. The concepts of reflection coefficients, standing-wave ratios, etc., used in the case of uniform plane-wave reflection, also apply to waveguide problems.

The mode with the lowest cutoff frequency in a particular guide is called the *dominant mode.* The dominant mode in a rectangular waveguide, assuming $b > a$, is the TE_{01} mode. (This we have not shown, for

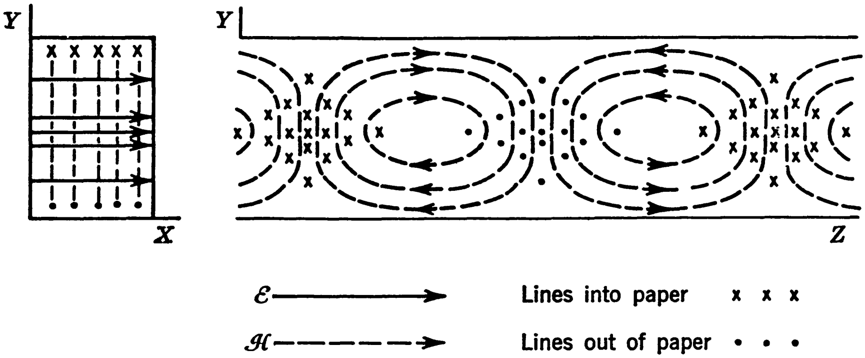


FIG. 2-17. Mode pattern for the TE_{01} waveguide mode.

we have not considered all modes.) From Eq. (2-82) with $n = 1$, we see that the cutoff wavelength of the TE_{01} mode is $\lambda_c = 2b$. Thus, wave propagation can take place in a rectangular waveguide only when its widest side is greater than a half-wavelength.[1] A sketch of the instantaneous field lines at some instant is called a *mode pattern.* The mode pattern of the TE_{01} mode in the propagating state is shown in Fig. 2-17. This figure is obtained by determining $\mathcal{E}$ and $\mathcal{H}$ from the **E** and **H** of Eqs. (2-87) and specializing the result to some instant of time. As time progresses, the mode pattern moves in the z direction.

It is admittedly confusing to learn that many modes exist on a given guiding system. It is not, however, so bad as it seems at first. If only one mode propagates in a waveguide, this will be the only mode of appreciable magnitude except near sources or discontinuities. The rectangular waveguide is usually operated so that only the TE_{01} mode propagates. This is therefore the only wave of significant amplitude along the guide except near sources and discontinuities.

Because of the importance of the TE_{01} mode, let us consider it in a little more detail. Table 2-4 specializes our preceding equations to this mode and includes some additional parameters which we shall now consider.

The power transmitted along the waveguide can be found by integrating the axial component of the Poynting vector over a guide cross section. This gives

$$P_f = \int_0^a \int_0^b E_x H_y^* \, dx \, dy = |E_0|^2 \frac{ab}{2Z_0^*}$$

which, above cutoff, is real and is therefore the time-average power transmitted. Below cutoff, the power is imaginary, indicating no time-average

[1] We are referring to the intrinsic wavelength of the dielectric filling the waveguide, which is usually free space.

TABLE 2-4. SUMMARY OF WAVEGUIDE PARAMETERS FOR THE DOMINANT MODE (TE_{01}) IN A RECTANGULAR WAVEGUIDE

Complex field	$E_x = E_0 \sin \frac{\pi y}{b} e^{-\gamma z}$ $H_y = \frac{E_0}{Z_0} \sin \frac{\pi y}{b} e^{-\gamma z}$ $H_z = \frac{E_0}{j\eta} \frac{f_c}{f} \cos \frac{\pi y}{b} e^{-\gamma z}$
Cutoff frequency	$f_c = \frac{1}{2b\sqrt{\epsilon\mu}}$
Cutoff wavelength	$\lambda_c = 2b$
Propagation constant	$\gamma = \begin{cases} j\beta = jk\sqrt{1 - (f_c/f)^2} & f > f_c \\ \alpha = \frac{2\pi}{\lambda_c}\sqrt{1 - (f/f_c)^2} & f < f_c \end{cases}$
Characteristic impedance	$Z_0 = \frac{j\omega\mu}{\gamma} = \begin{cases} \eta/\sqrt{1 - (f_c/f)^2} & f > f_c \\ j\eta/\sqrt{(f_c/f)^2 - 1} & f < f_c \end{cases}$
Guide wavelength	$\lambda_g = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}}$
Guide phase velocity	$v_g = \frac{v_p}{\sqrt{1 - (f_c/f)^2}}$
Power transmitted	$P = \frac{\lvert E_0\rvert^2 ab}{2Z_0}$
Attenuation due to lossy dielectric	$\alpha_d = \frac{\omega\epsilon''}{2} \eta \sqrt{1 - (f_c/f)^2}$
Attenuation due to imperfect conductor	$\alpha_c = \frac{\mathcal{R}}{a\eta\sqrt{1 - (f_c/f)^2}} \left[1 + \frac{2a}{b}\left(\frac{f_c}{f}\right)^2\right]$

power transmitted. (The preceding equation applies only at $z = 0$ below cutoff unless the factor $e^{-2\alpha z}$ is added.) It is also interesting to note that the time-average electric and magnetic energies per unit length of guide are equal above cutoff (see Prob. 2-32).

In contrast to the transmission-line mode, there is no unique voltage and current associated with a waveguide mode. However, the amplitude of a modal traveling wave (E_0 in Table 2-4) enters into waveguide reflection problems in the same manner as V in transmission-line problems.

To emphasize this correspondence, it is common to define a *mode voltage* V and a *mode current* I such that

$$Z_0 = \frac{V}{I} \qquad P = VI^* \tag{2-90}$$

From Table 2-4, it is evident that

$$V = E_0 \sqrt{\frac{ab}{2}}\, e^{-\gamma z} \qquad I = \frac{V}{Z_0} \tag{2-91}$$

satisfy this definition. Remember that we are dealing with only a $+z$ traveling wave. In the $-z$ traveling wave, $I = -V/Z_0$. When waves in both directions are present, the ratio V/I is a function of z. Other definitions of mode voltage, mode current, and characteristic impedance can be found in the literature. These alternative definitions will always be proportional to our definitions (see Prob. 2-34).

Our treatment has so far been confined to the ideal loss-free guide. When losses are present in the dielectric but not in the conductor, all our equations still apply, except that most parameters become complex. There is no longer a real cutoff frequency, for γ never goes to zero. Also, the characteristic impedance is complex at all frequencies. The behavior of $\gamma = \alpha + j\beta$ in the low-loss case is sketched in Fig. 2-18. The behavior of γ for the loss-free case is shown dashed. The most important effect of dissipation is the existence of an attenuation constant at all frequencies. In the low-loss case, we can continue to use the relationship

$$\gamma = \alpha + j\beta \approx jk\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

provided f is not too close to f_c. Letting $k = k' - jk''$ and referring to

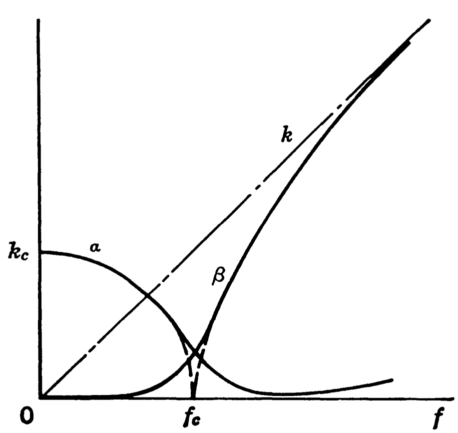


FIG. 2-18. Propagation constant for a lossy waveguide (loss-free case shown dashed).

Table 2-1, we find

$$\alpha_d \approx \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}}\sqrt{1-\left(\frac{f_c}{f}\right)^2} \tag{2-92}$$

This is the attenuation constant due to a lossy dielectric in the guide.

Even more important is the attenuation due to imperfectly conducting guide walls. Our solution is no longer exact in this case, because the boundary conditions are changed. The tangential component of **E** is now not quite zero at the conductor. However, for good conductors, the tangential component of **E** is very small, and the field is only slightly changed, or "perturbed," from the loss-free solution. The loss-free solution is used to approximate **H** at the conductor, and Eq. (2-42) is used to approximate the power dissipated in the conductor. Such a procedure is called a *perturbational method* (see Chap. 7). The power per unit length dissipated in the wall $y = 0$ is

$$\begin{aligned}\bar{\mathcal{P}}_d\Big|_{y=0} &= \mathcal{R}\int_0^a |H_z|^2\,dx = \mathcal{R}|E_0|^2\left(\frac{f_c}{\eta f}\right)^2\int_0^a dx \\ &= \mathcal{R}|E_0|^2 a\left(\frac{f_c}{\eta f}\right)^2\end{aligned}$$

and an equal amount is dissipated in the wall $y = b$. The power per unit length dissipated in the wall $x = 0$ is

$$\begin{aligned}\bar{\mathcal{P}}_d\Big|_{x=0} &= \mathcal{R}\int_0^b (|H_y|^2 + |H_z|^2)\,dy \\ &= \mathcal{R}|E_0|^2\int_0^b\left[\frac{\sin^2(\pi y/b)}{Z_0^2} + \left(\frac{f_c}{\eta f}\right)^2\cos^2\frac{\pi y}{b}\right]dy \\ &= \mathcal{R}|E_0|^2\left[\frac{b}{2Z_0^2} + \left(\frac{f_c}{\eta f}\right)^2\frac{b}{2}\right]\end{aligned}$$

and an equal amount is dissipated in the wall $x = a$. The total power dissipated per unit length is the sum of that for the four walls, or

$$\bar{\mathcal{P}}_d = \mathcal{R}|E_0|^2\left[\frac{b}{Z_0^2} + \left(\frac{f_c}{\eta f}\right)^2(2a+b)\right]$$

Equation (2-76) is valid for any traveling wave; so using the above $\bar{\mathcal{P}}_d$, and $\bar{\mathcal{P}}_f = P$ of Table 2-4, we have

$$\begin{aligned}\alpha_c &= \frac{\mathcal{R}Z_0}{ab}\left[\frac{b}{Z_0^2} + \left(\frac{f_c}{\eta f}\right)^2(2a+b)\right] \\ &= \frac{\mathcal{R}}{a\eta\sqrt{1-(f_c/f)^2}}\left[1 + \frac{2a}{b}\left(\frac{f_c}{f}\right)^2\right]\end{aligned} \tag{2-93}$$

This is the attenuation constant due to conductor losses. When both

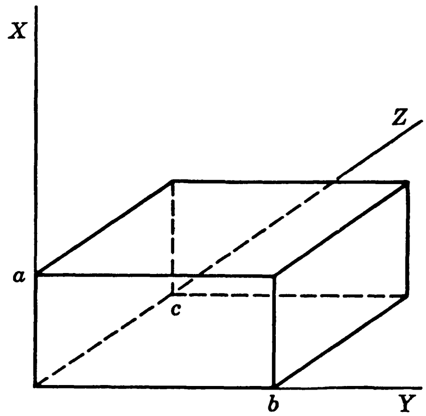


FIG. 2-19. The rectangular cavity.

dielectric losses and conductor losses need to be considered, the total attenuation constant is

$$\alpha = \alpha_d + \alpha_c \tag{2-94}$$

for by Eq. (2-76) we merely add the two losses.

2-8. Resonator Concepts. In Sec. 2-2 we noted a similarity between standing waves and circuit theory resonance. In the loss-free case, electromagnetic fields can exist within a source-free region enclosed by a perfect conductor. These fields can exist only at specific frequencies, called *resonant frequencies*. When losses are present, a source must exist to sustain oscillations. The input impedance seen by the source behaves, in the vicinity of a resonant frequency, like the impedance of an *LC* circuit. Resonators can therefore be used for the same purposes at high frequencies as *LC* resonators are used at lower frequencies.

To illustrate resonator concepts, consider the "rectangular cavity" of Fig. 2-19. This consists of a conductor enclosing a dielectric, both of which we will assume to be perfect at present. We desire to find solutions to the field equations having zero tangential components of **E** over the entire boundary. The TE_{01} waveguide mode already satisfies this condition over four of the walls. We recall that standing waves have planes of zero field, which suggests trying the standing-wave TE_{01} field. For E_x to be zero at $z = 0$, we choose

$$\begin{aligned} E_x = E_x^+ + E_x^- &= A \sin \frac{\pi y}{b} (e^{-j\beta z} - e^{j\beta z}) \\ &= E_0 \sin \left(\frac{\pi y}{b}\right) \sin \beta z \end{aligned}$$

For E_x to be zero at $z = c$, we choose $\beta c = \pi$, which, according to Table 2-4, is

$$\pi = ck \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = c2\pi f \sqrt{\epsilon\mu} \sqrt{1 - \frac{1}{(2b \sqrt{\epsilon\mu}\, f)^2}}$$

Solving for the resonant frequency $f = f_r$, we have

$$f_r = \frac{1}{2bc} \sqrt{\frac{b^2 + c^2}{\epsilon\mu}} \tag{2-95}$$

When a is the smallest cavity dimension, this is the resonant frequency of the dominant mode, called the TE_{011} mode. The additional subscript 1 indicates that we have chosen the first zero of $\sin \beta z$. The higher zeros give higher-order modes, that is, modes with higher resonant frequencies. Setting $\beta = \pi/c$ in the above expression for E_x and determining $\mathbf{H}$ from the Maxwell equations, we have for the TE_{011} mode

$$\begin{aligned} E_x &= E_0 \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \\ H_y &= \frac{jbE_0}{\eta \sqrt{b^2 + c^2}} \sin \frac{\pi y}{b} \cos \frac{\pi z}{c} \\ H_z &= -\frac{jcE_0}{\eta \sqrt{b^2 + c^2}} \cos \frac{\pi y}{b} \sin \frac{\pi z}{c} \end{aligned} \tag{2-96}$$

Note that E and H are 90° out of phase; so $\mathcal{E}$ is maximum when $\mathcal{H}$ is minimum and vice versa. A sketch of the instantaneous field lines at some time when both $\mathcal{E}$ and $\mathcal{H}$ exist is given in Fig. 2-20. Also of interest is the energy stored within the cavity. From the conservation of complex power, Eq. (1-68), we know that $\overline{\mathcal{W}}_m = \overline{\mathcal{W}}_e$. Thus, the time-average electric and magnetic energies are

$$\overline{\mathcal{W}}_m = \overline{\mathcal{W}}_e = \frac{\epsilon}{2} \iiint\limits_{\text{cavity}} |E|^2 \, d\tau = \frac{\epsilon}{8} |E_0|^2 abc \tag{2-97}$$

We also know from conservation of energy, Eq. (1-39), that the total energy within the resonator is independent of time. If we choose a time for which $\mathcal{H}$ is zero, $\mathcal{W}_m$ will be zero, and $\mathcal{W}_e$ will be maximum and twice its average value. Therefore,

$$\mathcal{W} = 2\overline{\mathcal{W}}_e = \frac{\epsilon}{4} |E_0|^2 abc \tag{2-98}$$

is the total energy stored within the cavity.

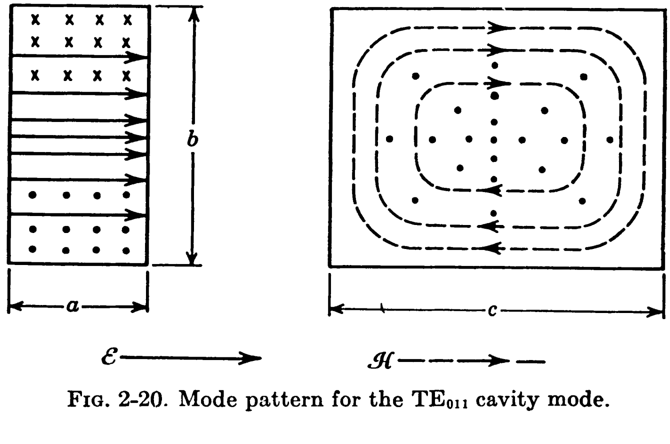


FIG. 2-20. Mode pattern for the TE_{011} cavity mode.

When the resonator has losses, we define its quality factor as

$$Q = \frac{\omega \times \text{energy stored}}{\text{average power dissipated}} = \frac{\omega \mathcal{W}}{\bar{\mathcal{P}}_d} \tag{2-99}$$

by analogy to the Q of an LC circuit. If the losses are dielectric losses, we have

$$Q_d = \frac{\omega\epsilon' \iiint |E|^2 \, d\tau}{\omega\epsilon'' \iiint |E|^2 \, d\tau} = \frac{\epsilon'}{\epsilon''} \tag{2-100}$$

so the Q of the resonator is that of the dielectric, Eq. (1-79). This is valid for any mode in a cavity of arbitrary shape. Usually more important in determining the Q is the loss due to imperfect conductors. This is determined to the same approximation as we used for waveguide attenuation. We assume H at walls to be that of the loss-free mode and calculate $\bar{\mathcal{P}}_d$ by Eq. (2-42). To summarize,

$$\bar{\mathcal{P}}_d = \mathcal{R} \oiint_{\substack{\text{cavity}\\\text{walls}}} |H|^2 \, ds = \frac{\mathcal{R}|E_0|^2}{2\eta^2(b^2 + c^2)} [bc(b^2 + c^2) + 2a(b^3 + c^3)]$$

Substituting this, Eq. (2-98), and Eq. (2-95) into Eq. (2-99), we have

$$Q_c = \frac{\pi\eta}{2\mathcal{R}} \frac{a(b^2 + c^2)^{3/2}}{bc(b^2 + c^2) + 2a(b^3 + c^3)} \tag{2-101}$$

From the symmetry of Q_c in b and c, it is evident that $b = c$ for maximum Q. For a "square-base" cavity ($b = c$), we have

$$f_r = \frac{1}{b\sqrt{2\epsilon\mu}} \qquad Q_c = \frac{1.11\eta}{\mathcal{R}(1 + b/2a)} \tag{2-102}$$

The Q also increases as a increases, but if $a > b$ we no longer have the dominant mode. As an example of the Q's obtainable, consider a cubic cavity constructed of copper. In this case we have

$$Q_c = 1.07 \times 10^9/\sqrt{f} \tag{2-103}$$

which, at microwave frequencies, gives Q's of several thousand. This idealized Q will, however, be lowered in practice by the introduction of a feed system, by imperfections in the construction, and by corrosion of the metal. When both conductor losses and dielectric losses are considered, the Q of the cavity becomes

$$\frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c} \tag{2-104}$$

which is evident from Eq. (2-99).

2-9. Radiation. We shall now show that a source in unbounded space is characterized by a radiation of energy. Consider the field equations

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \qquad \nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J} \tag{2-105}$$

where $\mathbf{J}$ is the source, or impressed, current. These equations apply explicitly to a perfect dielectric, but the extension to lossy media is effected by replacing $j\omega\mu$ by $\hat{z}$ and $j\omega\epsilon$ by $\hat{y}$. In homogeneous media, the divergence of the first equation is

$$\nabla \cdot \mathbf{H} = 0$$

Any divergenceless vector is the curl of some other vector; so

$$\mathbf{H} = \nabla \times \mathbf{A} \tag{2-106}$$

where $\mathbf{A}$ is called a *magnetic vector potential.*[1] Substituting Eq. (2-106) into the first of Eqs. (2-105), we have

$$\nabla \times (\mathbf{E} + j\omega\mu\mathbf{A}) = 0$$

Any curl-free vector is the gradient of some scalar. Hence,

$$\mathbf{E} + j\omega\mu\mathbf{A} = -\nabla\Phi \tag{2-107}$$

where Φ is an *electric scalar potential.* To obtain the equation for $\mathbf{A}$, substitute Eqs. (2-106) and (2-107) into the second of Eqs. (2-105). This gives

$$\nabla \times \nabla \times \mathbf{A} - k^2\mathbf{A} = \mathbf{J} - j\omega\epsilon\nabla\Phi \tag{2-108}$$

which, by a vector identity, becomes

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} - k^2\mathbf{A} = \mathbf{J} - j\omega\epsilon\nabla\Phi$$

Only $\nabla \times \mathbf{A}$ was specified by Eq. (2-106). We are still free to choose $\nabla \cdot \mathbf{A}$. If we let

$$\nabla \cdot \mathbf{A} = -j\omega\epsilon\Phi \tag{2-109}$$

the equation for $\mathbf{A}$ simplifies to

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mathbf{J} \tag{2-110}$$

This is the Helmholtz equation, or complex wave equation. Solutions to Eq. (2-110) are called *wave potentials.* In terms of the magnetic wave potential, we have

$$\begin{aligned} \mathbf{E} &= -j\omega\mu\mathbf{A} + \frac{1}{j\omega\epsilon}\nabla(\nabla \cdot \mathbf{A}) \\ \mathbf{H} &= \nabla \times \mathbf{A} \end{aligned} \tag{2-111}$$

[1] In general electromagnetic theory it is more common to let **A** be the vector potential of **B**. In homogeneous media the two potentials are in the ratio μ, a constant.

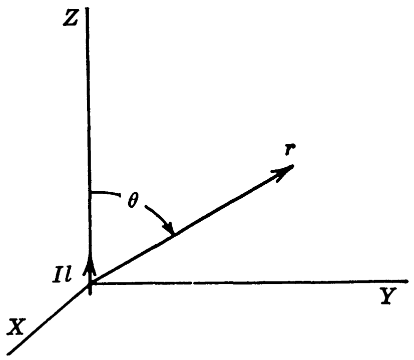


FIG. 2-21. A z-directed current element at the coordinate origin.

obtained from Eqs. (2-106), (2-107), and (2-109). The principal advantages of using **A** instead of **E** or **H** are (1) rectangular components of **A** have corresponding rectangular components of **J** as their sources and (2) **A** need not be divergenceless.

Let us first determine **A** for a current I extending over an incremental length l, forming a *current element* or *electric dipole* of moment Il. Take this current element to be z-directed and situated at the coordinate origin, as shown in Fig. 2-21. The current is z-directed; so we take **A** to have only a z component, satisfying

$$\nabla^2 A_z + k^2 A_z = 0$$

everywhere except at the origin. The scalar quantity A_z has a point source Il and should therefore be spherically symmetric. Thus, let $A_z = A_z(r)$, and the above equation reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dA_z}{dr}\right) + k^2 A_z = 0$$

This has the two independent solutions

$$\frac{1}{r}e^{-jkr} \qquad \frac{1}{r}e^{jkr}$$

the first of which represents an outward-traveling wave, and the second an inward-traveling wave. (In dissipative media, $k = k' - jk''$, and the first solution vanishes as $r \to \infty$, and the second solution becomes infinite.) We therefore choose the first solution, and take

$$A_z = \frac{C}{r}e^{-jkr}$$

where C is a constant.[1] As $k \to 0$, Eq. (2-110) reduces to Poisson's equation, for which the solution is

$$A_z = \frac{Il}{4\pi r}$$

[1] To be precise, C might be a function of k, but the solution must also reduce to the static field as $r \to 0$. Hence, C is not a function of k.

Our constant C must therefore be

$$C = \frac{Il}{4\pi}$$

and hence

$$A_z = \frac{Il}{4\pi r} e^{-jkr} \tag{2-112}$$

is the desired solution for the current element of Fig. 2-21. The outward-traveling wave represented by Eq. (2-112) is called a *spherical wave,* since surfaces of constant phase are spheres.

The electromagnetic field of the current element is obtained by substituting Eq. (2-112) into Eqs. (2-111). The result is

$$\begin{aligned} E_r &= \frac{Il}{2\pi} e^{-jkr} \left(\frac{\eta}{r^2} + \frac{1}{j\omega\epsilon r^3} \right) \cos\theta \\ E_\theta &= \frac{Il}{4\pi} e^{-jkr} \left(\frac{j\omega\mu}{r} + \frac{\eta}{r^2} + \frac{1}{j\omega\epsilon r^3} \right) \sin\theta \\ H_\phi &= \frac{Il}{4\pi} e^{-jkr} \left(\frac{jk}{r} + \frac{1}{r^2} \right) \sin\theta \end{aligned} \tag{2-113}$$

Very close to the current element, the $\mathbf{E}$ reduces to that of a static charge dipole, the $\mathbf{H}$ reduces to that of a constant current element, and the field is said to be *quasi-static.* Far from the current element, Eqs. (2-113) reduce to

$$\left.\begin{aligned} E_\theta &= \eta \frac{jIl}{2\lambda r} e^{-jkr} \sin\theta \\ H_\phi &= \frac{jIl}{2\lambda r} e^{-jkr} \sin\theta \end{aligned}\right\} \qquad r \gg \lambda \tag{2-114}$$

which is called the *radiation field.* At intermediate values of r the field is called the *induction field.* The outward-directed complex power over a sphere of radius r is

$$\begin{aligned} P_f &= \oiint \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{s} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta\, E_\theta H_\phi^* \\ &= \eta \frac{2\pi}{3} \left| \frac{Il}{\lambda} \right|^2 \left[1 - \frac{j}{(kr)^3} \right] \end{aligned} \tag{2-115}$$

The time-average power radiated is the real part of P_f, or

$$\bar{\mathcal{P}}_f = \eta \frac{2\pi}{3} \left| \frac{Il}{\lambda} \right|^2 \tag{2-116}$$

This is independent of r and can be most simply obtained from the radiation field, Eq. (2-114). The reactive power, which is negative, indicates that there is an excess of electric energy over magnetic energy in the near field.

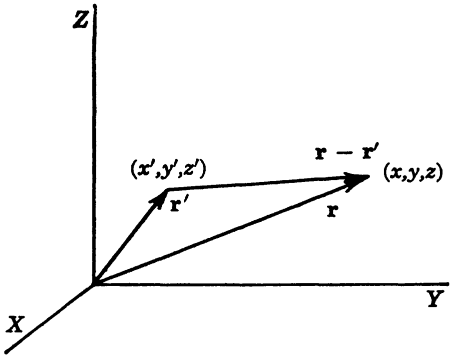


FIG. 2-22. Radius vector notation.

To obtain the field of an arbitrary distribution of electric currents, we need only superimpose the solutions for each element, for the equations are linear. A superposition of vector potentials is usually the most convenient one. For this purpose, we shall use the radius vector notation illustrated by Fig. 2-22. The "field coordinates" are specified by

$$\mathbf{r} = \mathbf{u}_x x + \mathbf{u}_y y + \mathbf{u}_z z$$

and the "source coordinates" by

$$\mathbf{r}' = \mathbf{u}_x x' + \mathbf{u}_y y' + \mathbf{u}_z z'$$

In Eq. (2-112), r is the distance from the source to the field point. For Il not at the coordinate origin, r should be replaced by

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

Note the direction of the vector potential is that of the current; so Eq. (2-112) can be generalized to a current element of arbitrary orientation by replacing Il by $I\mathbf{l}$ and A_z by $\mathbf{A}$. Thus, the vector potential from current element of arbitrary location and orientation is

$$\mathbf{A} = \frac{I\mathbf{l}\, e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

To emphasize that $\mathbf{A}$ is evaluated at the field point (x,y,z) and $I\mathbf{l}$ is situated at the source point (x',y',z'), we shall use the notation $\mathbf{A}(\mathbf{r})$ and $I\mathbf{l}(\mathbf{r}')$. The above equation then becomes

$$\mathbf{A}(\mathbf{r}) = \frac{I\mathbf{l}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{2-117}$$

Finally, for a current distribution $\mathbf{J}$, the current element contained in a volume element $d\tau$ is $\mathbf{J}\, d\tau$, and a superposition over all such elements is

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}\, d\tau' \tag{2-118}$$

The prime on $d\tau'$ emphasizes that the integration is over the source coordinates. Equation (2-118) is called the *magnetic vector potential integral.* It is intended to include the cases of surface currents and filamentary currents by implication. We therefore have a formal solution for any problem characterized by electric currents in an unbounded homogeneous medium. The medium may be dissipative if k is considered to be complex.

2-10. Antenna Concepts. A device whose primary purpose is to radiate or receive electromagnetic energy is called an antenna. To illustrate antenna concepts, we shall consider the *linear antenna* of Fig. 2-23. It consists of a straight wire carrying a current $I(z)$. When it is energized at the center, it is called a *dipole antenna.* The magnetic vector potential, Eq. (2-118), for this particular problem is

$$A_z = \frac{1}{4\pi}\int_{-L/2}^{L/2} \frac{I(z')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,dz' \qquad (2\text{-}119)$$

where

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + z'^2 - 2rz'\cos\theta} \qquad (2\text{-}120)$$

The radiation field (r large) is of primary interest, in which case

$$|\mathbf{r}-\mathbf{r}'| \approx r - z'\cos\theta \qquad r \gg z' \qquad (2\text{-}121)$$

and

$$A_z \approx \frac{e^{-jkr}}{4\pi r}\int_{-L/2}^{L/2} I(z')e^{jkz'\cos\theta}\,dz' \qquad r \gg L \qquad (2\text{-}122)$$

Note that the second term of Eq. (2-121) must be retained in the "phase term" $e^{-jk|\mathbf{r}-\mathbf{r}'|}$, but not in the "amplitude term" $|\mathbf{r}-\mathbf{r}'|^{-1}$. To obtain the field components, substitute Eq. (2-122) into Eqs. (2-111) and retain only the $1/r$ terms. This gives

$$\left.\begin{aligned} E_\theta &= j\omega\mu \sin\theta\, A_z \\ H_\phi &= \frac{1}{\eta} E_\theta \end{aligned}\right\} \qquad r \text{ large} \qquad (2\text{-}123)$$

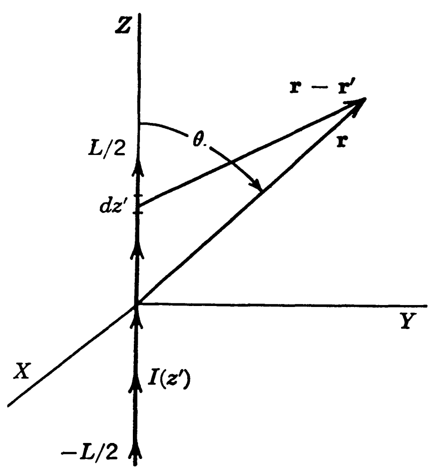


FIG. 2-23. The linear antenna.

This result is equivalent to superimposing Eqs. (2-114) for all elements of current.

To evaluate the radiation field, we must know the current on the antenna. An exact determination of the current requires the solution to a boundary-value problem. Fortunately, the radiation field is relatively insensitive to minor changes in current distribution, and much useful information can be obtained from an approximate current distribution. We have already seen that on transmission lines the current is a harmonic function of kz. This is also true for the principal mode on a single thin wire. The current on the dipole antenna must be zero at the ends of the wire, symmetrical in z, and continuous at the source ($z = 0$). Thus, we choose

$$I(z) = I_m \sin \left[k \left(\frac{L}{2} - |z| \right) \right] \tag{2-124}$$

The vector potential in the radiation zone can now be evaluated as

$$\begin{aligned} A_z &= \frac{I_m e^{-jkr}}{4\pi r} \int_{-L/2}^{L/2} \sin \left[k \left(\frac{L}{2} - |z'| \right) \right] e^{jkz' \cos \theta} \, dz' \\ &= \frac{I_m e^{-jkr}}{4\pi r} \, \frac{2 \left[\cos \left(k \frac{L}{2} \cos \theta \right) - \cos \left(k \frac{L}{2} \right) \right]}{k \sin^2 \theta} \end{aligned}$$

From Eq. (2-123), the radiation field is

$$E_\theta = \frac{j\eta I_m e^{-jkr}}{2\pi r} \left[\frac{\cos \left(k \frac{L}{2} \cos \theta \right) - \cos \left(k \frac{L}{2} \right)}{\sin \theta} \right] \tag{2-125}$$

with $H_\phi = E_\theta / \eta$. Note that the radiation field is linearly polarized, for there is only an E_θ. The density of power radiated is the r component of the Poynting vector

$$S_r = E_\theta H_\phi^* = \frac{\eta |I_m|^2}{(2\pi r)^2} \left[\frac{\cos \left(k \frac{L}{2} \cos \theta \right) - \cos \left(k \frac{L}{2} \right)}{\sin \theta} \right]^2 \tag{2-126}$$

The total power radiated is obtained by integrating S_r over a large sphere, or

$$\begin{aligned} \bar{\mathcal{P}}_f &= \int_0^{2\pi} \int_0^{\pi} S_r \, r^2 \sin \theta \, d\theta \, d\phi \\ &= \frac{\eta |I_m|^2}{2\pi} \int_0^{\pi} \frac{\left[\cos \left(k \frac{L}{2} \cos \theta \right) - \cos \left(k \frac{L}{2} \right) \right]^2}{\sin \theta} \, d\theta \end{aligned} \tag{2-127}$$

The radiation resistance R_r of an antenna is defined as

$$R_r = \frac{\bar{\mathcal{P}}_f}{|I|^2} \tag{2-128}$$

where I is some arbitrary reference current. For the dipole antenna, the reference current is usually picked as I_m. Hence,

$$R_r = \frac{\eta}{2\pi} \int_0^{\pi} \frac{\left[\cos \left(k \frac{L}{2} \cos \theta \right) - \cos \left(k \frac{L}{2} \right) \right]^2}{\sin \theta} \, d\theta \tag{2-129}$$

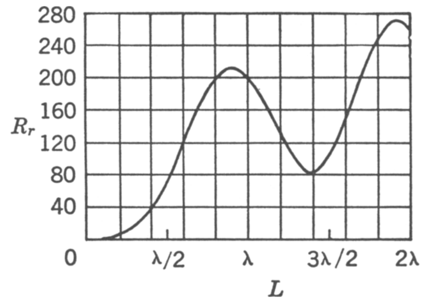


Fig. 2-24. Radiation resistance of the dipole antenna.

This integral can be evaluated in terms of tabulated functions (see Prob. 2-44). A graph of R_r versus L is given in Fig. 2-24.

The *radiation field pattern* of an antenna is a plot of $|E|$ at constant r in the radiation zone. For a dipole antenna, the radiation field pattern is essentially the bracketed term of Eq. (2-125). This is shown in Fig. 2-25 for kL small (short dipole), $kL = \pi$ (half-wavelength dipole), and $kL = 2\pi$ (full-wavelength dipole). The *radiation power pattern,* defined as a plot of $|S_r|$ at constant r, is an alternative method of showing radiation characteristics. When the radiation field is linearly polarized, as it is for the dipole antenna, the power pattern is the square of the field pattern. The *gain g* of an antenna in a given direction is defined as the ratio of the power required from an omnidirectional antenna to the power

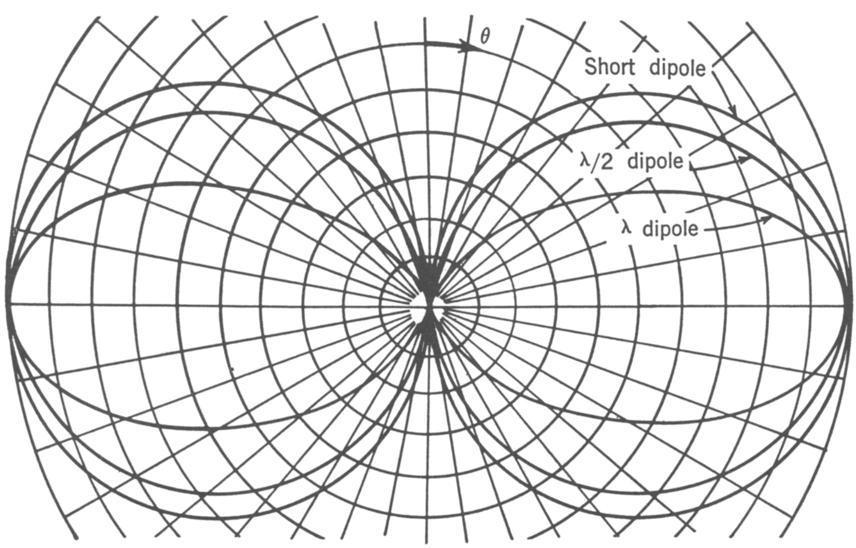


Fig. 2-25. Radiation field patterns for the dipole antenna.

required from the actual antenna, assuming equal power densities in the given direction. Thus,

$$g(\theta) = \frac{4\pi r^2 S_r(\theta)}{\bar{\mathcal{P}}_f} \tag{2-130}$$

For $L \leq \lambda$, the maximum gain of a dipole antenna occurs at $\theta = \pi/2$. From Eqs. (2-126) and (2-128), we have

$$g\left(\frac{\pi}{2}\right) = \frac{\eta|I_m|^2\left(1 - \cos\frac{kL}{2}\right)^2}{\pi\bar{\mathcal{P}}_f} = \frac{\eta\left(1 - \cos\frac{kL}{2}\right)^2}{\pi R_r} \tag{2-131}$$

In the limit $kL \to 0$, we have $g(\pi/2) = 1.5$; so the maximum gain of a short dipole is 1.5. For a half-wave dipole, we can use Fig. 2.24 and calculate a maximum gain of 1.64. Similarly, for a full-wave dipole, the maximum gain is 2.41.

The *input impedance* of an antenna is the impedance seen by the source, that is, the ratio of the complex terminal voltage to the complex terminal current. A knowledge of the reactive power, which cannot be obtained from radiation zone fields, is needed to evaluate the input reactance. The input resistance accounts for the radiated power (and dissipated power if losses are present). We define the input resistance of a loss-free antenna as

$$R_i = \frac{\bar{\mathcal{P}}_f}{|I_i|^2} \tag{2-132}$$

where $\bar{\mathcal{P}}_f$ is the power radiated and I_i is the input current. If losses are present, a "loss resistance" must be added to Eq. (2-132) to obtain the input resistance. For the dipole antenna,

$$I_i = I_m \sin\frac{kL}{2}$$

and the input resistance is

$$R_i = \frac{R_r}{\sin^2[k(L/2)]} \tag{2-133}$$

In the limit as kL is made small, we find

$$R_i = \frac{\eta(kL)^2}{24\pi} \qquad L \ll \lambda \tag{2-134}$$

The short dipole therefore has a very small input resistance. For example, if $L = \lambda/10$, the input resistance is about 2 ohms. For the half-wavelength dipole, we use Fig. 2-24 and Eq. (2-133) and find

$$R_i = R_r = 73.1 \text{ ohms} \qquad L = \frac{\lambda}{2} \tag{2-135}$$

For the full-wavelength dipole, Eq. (2-133) shows $R_i = \infty$. This incorrect result is due to our initial choice of current, which has a null at the source. The input resistance of the full-wavelength dipole is actually large, but not infinite, and depends markedly on the wire diameter (see Fig. 7-13).

2-11. On Waves in General. A complex function of coordinates representing an instantaneous function according to Eq. (1-40) is called a *wave function.* A wave function ψ, which may be either a scalar field or the component of a vector field, may be expressed as

$$\psi = A(x,y,z)e^{j\Phi(x,y,z)} \tag{2-136}$$

where A and Φ are real. The corresponding instantaneous function is

$$\sqrt{2}\, A(x,y,z) \cos\,[\omega t + \Phi(x,y,z)] \tag{2-137}$$

The *magnitude* A of the complex function is the rms amplitude of the instantaneous function. The *phase* Φ of the complex function is the initial phase of the instantaneous function. Surfaces over which the phase is constant (instantaneous function vibrates in phase) are called *equiphase surfaces.* These are defined by

$$\Phi(x,y,z) = \text{constant} \tag{2-138}$$

Waves are called *plane, cylindrical,* or *spherical* according as their equiphase surfaces are planes, cylinders, or spheres. Waves are called *uniform* when the amplitude A is constant over the equiphase surfaces. Perpendiculars to the equiphase surfaces are called *wave normals.* These are, of course, in the direction of $\nabla\Phi$ and are the curves along which the phase changes most rapidly.

The rate at which the phase decreases in some direction is called the *phase constant* in that direction. (The term phase constant is used even though it is not, in general, a constant.) For example, the phase constants in the cartesian coordinate directions are

$$\beta_x = -\frac{\partial \Phi}{\partial x} \qquad \beta_y = -\frac{\partial \Phi}{\partial y} \qquad \beta_z = -\frac{\partial \Phi}{\partial z} \tag{2-139}$$

These may be considered as components of a *vector phase constant* defined by

$$\boldsymbol{\beta} = -\nabla\Phi \tag{2-140}$$

The maximum phase constant is therefore along the wave normal and is of magnitude $|\nabla\Phi|$.

The *instantaneous phase* of a wave is the argument of the cosine function of Eq. (2-137). A *surface of constant phase* is defined as

$$\omega t + \Phi(x,y,z) = \text{constant} \tag{2-141}$$

that is, the instantaneous phase is constant. At any instant, the surfaces of constant phase coincide with the equiphase surfaces. As time increases, Φ must decrease to maintain the constancy of Eq. (2-141), and the surfaces of constant phase move in space. For any increment $d\mathbf{s}$ the change in Φ is

$$\nabla\Phi \cdot d\mathbf{s} = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$$

To keep the instantaneous phase constant for an incremental increase in time, we must have

$$\omega\, dt + \nabla\Phi \cdot d\mathbf{s} = 0$$

That is, the total differential of Eq. (2-141) must vanish. The *phase velocity* of a wave in a given direction is defined as the velocity of surfaces of constant phase in that direction. For example, the phase velocities along cartesian coordinates are

$$\begin{aligned} v_x &= -\frac{\omega}{\partial\Phi/\partial x} = \frac{\omega}{\beta_x} \\ v_y &= -\frac{\omega}{\partial\Phi/\partial y} = \frac{\omega}{\beta_y} \\ v_z &= -\frac{\omega}{\partial\Phi/\partial z} = \frac{\omega}{\beta_z} \end{aligned} \tag{2-142}$$

The phase velocity along a wave normal ($d\mathbf{s}$ in the direction of $-\nabla\Phi$) is

$$v_p = -\frac{\omega}{|\nabla\Phi|} = \frac{\omega}{\beta} \tag{2-143}$$

which is the *smallest* phase velocity for the wave. Phase velocity is *not* a vector quantity.

We can also express the wave function, Eq. (2-136), as

$$\psi = e^{\Theta(x,y,z)} \tag{2-144}$$

where Θ is a complex function whose imaginary part is the phase Φ. A *vector propagation constant* can be defined in terms of the rate of change of Θ as

$$\boldsymbol{\gamma} = -\nabla\Theta = \boldsymbol{\alpha} + j\boldsymbol{\beta} \tag{2-145}$$

where $\boldsymbol{\beta}$ is the phase constant of Eq. (2-140) and $\boldsymbol{\alpha}$ is the *vector attenuation constant.* The components of $\boldsymbol{\alpha}$ are the logarithmic rates of change of the magnitude of ψ in the various directions.

In the electromagnetic field, ratios of components of $\mathbf{E}$ to components of $\mathbf{H}$ are called *wave impedances.* The direction of a wave impedance is defined according to the right-hand "cross-product" rule of component $\mathbf{E}$

rotated into component **H**. For example,

$$\frac{E_x}{H_y} = Z_{xy}^{+} = Z_z \tag{2-146}$$

is a wave impedance in the $+z$ direction, while

$$-\frac{E_x}{H_y} = Z_{xy}^{-} = Z_{-z} \tag{2-147}$$

is a wave impedance in the $-z$ direction. The wave impedance in the $+z$ direction involving E_y and H_x is

$$\frac{-E_y}{H_x} = Z_{yx}^{+} = -Z_{yx}^{-} \tag{2-148}$$

The Poynting vector can be expressed in terms of wave impedances. For example, the z component is

$$\begin{aligned} S_z = (\mathbf{E} \times \mathbf{H}^*)_z &= E_x H_y^* - E_y H_x^* \\ &= Z_{xy}^{+}|H_y|^2 + Z_{yx}^{+}|H_x|^2 \end{aligned} \tag{2-149}$$

The concept of wave impedance is most useful when the wave impedances are constant over equiphase surfaces.

Let us illustrate the various concepts by specializing them to the uniform plane wave. Consider the x-polarized z-traveling wave in lossy matter,

$$\begin{aligned} E_x &= E_0 e^{-k''z} e^{-jk'z} \\ H_y &= \frac{E_0}{\eta} e^{-k''z} e^{-jk'z} \end{aligned}$$

The amplitude of E_x is $E_0 e^{-k''z}$ and its phase is $-k'z$. Equiphase surfaces are defined by $-k'z$ = constant, or, since k' is constant, by z = constant. These are planes; so the wave is a plane wave. The amplitude of E_x is constant over each equiphase surface; so the wave is uniform. The wave normals all point in the z direction. The cartesian components of the phase constant are $\beta_x = \beta_y = 0$, $\beta_z = k'$; so the vector phase constant is $\boldsymbol{\beta} = \mathbf{u}_z k'$. The phase velocity in the direction of the wave normals is $v_p = \omega/k'$. The cartesian components of the attenuation constant are $\alpha_x = \alpha_y = 0$, $\alpha_z = k''$; so the vector attenuation constant is $\boldsymbol{\alpha} = \mathbf{u}_z k''$. The vector propagation constant is

$$\boldsymbol{\gamma} = \boldsymbol{\alpha} + j\boldsymbol{\beta} = \mathbf{u}_z(k'' + jk') = \mathbf{u}_z jk$$

The wave impedance in the z direction is $Z_z = Z_{xy}^{+} = E_x/H_y = \eta$. Note that the various parameters specialized to the uniform plane traveling wave are all intrinsic parameters. This is, by definition, the meaning of the word "intrinsic."

PROBLEMS

2-1. Show that $E_z = E_0 e^{-jkz}$ satisfies Eq. (2-6) but not Eq. (2-5). Show that it does not satisfy Eq. (2-3). This is *not* a possible electromagnetic field.

2-2. Derive the "wave equations" for inhomogeneous media

$$\nabla \times (\hat{z}^{-1}\nabla \times \mathbf{E}) + \hat{y}\mathbf{E} = 0$$
$$\nabla \times (\hat{y}^{-1}\nabla \times \mathbf{H}) + \hat{z}\mathbf{H} = 0$$

Are these valid for nonisotropic media? Do Eqs. (2-5) hold for inhomogeneous media?

2-3. Show that for any lossless nonmagnetic dielectric

$$k = k_0\sqrt{\epsilon_r} \qquad \eta = \frac{\eta_0}{\sqrt{\epsilon_r}}$$
$$\lambda = \frac{\lambda_0}{\sqrt{\epsilon_r}} \qquad v_p = \frac{c}{\sqrt{\epsilon_r}}$$

where ϵ_r is the dielectric constant and k_0, η_0, λ_0, and c are the intrinsic parameters of vacuum.

2-4. Show that the quantities of Eqs. (2-18) satisfy Eq. (1-35). Repeat for Eqs. (2-21), (2-27), and (2-29).

2-5. For the field of Eqs. (2-20), show that the velocity of propagation of energy as defined by Eq. (2-19) is

$$v_e = \frac{1}{\sqrt{\epsilon\mu}}\,\frac{\sin 2kz \sin 2\omega t}{1 - \cos 2kz \cos 2\omega t} \le \frac{1}{\sqrt{\epsilon\mu}}$$

2-6. For the field of Eqs. (2-22), show that the phase velocity is

$$v_p = \frac{1}{\sqrt{\epsilon\mu}}\left(\frac{A+C}{A-C}\cos^2 kz + \frac{A-C}{A+C}\sin^2 kz\right)$$

2-7. For the field of Eqs. (2-28), show that the z-directed wave impedances are

$$Z_{xy}{}^{+} = \frac{E_x}{H_y} = -j\eta \tan kz$$
$$Z_{yx}{}^{+} = \frac{-E_y}{H_x} = -j\eta \tan kz$$

Would you expect $Z_{xy}{}^{+} = Z_{yx}{}^{+}$ to be true for all a-c fields?

2-8. Given a uniform plane wave traveling in the $+z$ direction, show that the wave is circularly polarized if

$$\frac{E_x}{E_y} = \pm j$$

being right-handed if the ratio is $+j$ and left-handed if the ratio is $-j$.

2-9. Show that the uniform plane traveling wave of Eq. (2-25) can be expressed as the sum of a right-hand circularly polarized wave and a left-hand circularly polarized wave.

2-10. Show that the uniform plane traveling wave of Eq. (2-25) can be expressed as

$$\mathbf{E} = (\mathbf{E}_1 + j\mathbf{E}_2)e^{-jkz}$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are real vectors lying in the xy plane. Relate $\mathbf{E}_1$ and $\mathbf{E}_2$ to A and B.

2-11. Show that the tip of the arrow representing $\mathcal{E}$ for an arbitrary complex $\mathbf{E}$ traces out an ellipse in space. [*Hint:* let $\mathbf{E} = \text{Re}\,(\mathbf{E}) + j\,\text{Im}\,(\mathbf{E})$ and use the results of Prob. 2-10.]

2-12. For the frequencies 10, 100, and 1000 megacycles, determine $k = k' - jk''$ and $\eta = \mathcal{R} + j\mathcal{X}$ for (*a*) polystyrene, Fig. 1-10, (*b*) Plexiglas, Fig. 1-11, (*c*) Ferramic A, Fig. 1-12, $\epsilon_r = 10$, and (*d*) copper, $\sigma = 5.8 \times 10^7$.

2-13. Show that when all losses are of the magnetic type ($\sigma = \epsilon'' = 0$),

$$\eta = \frac{k}{|y|} = \frac{k'}{\omega\epsilon} - j\frac{k''}{\omega\epsilon}$$

2-14. Show that for nonmagnetic dielectrics

$$\left.\begin{aligned} k' &\approx \omega\sqrt{\mu\epsilon'}\left(1 + \frac{1}{8Q^2}\right) \\ k'' &\approx \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}}\left(1 - \frac{1}{8Q^2}\right) \\ \mathcal{R} &\approx \sqrt{\frac{\mu}{\epsilon'}}\left(1 - \frac{3}{8Q^2}\right) \\ \mathcal{X} &\approx \frac{\epsilon''}{2\epsilon'}\sqrt{\frac{\mu}{\epsilon'}}\left(1 - \frac{5}{8Q^2}\right) \end{aligned}\right\} \quad Q \gg 1$$

where Q is defined by Eq. (1-79).

2-15. Show that for nonmagnetic conductors

$$\left.\begin{aligned} k' &\approx \sqrt{\frac{\omega\mu\sigma}{2}}\left(1 + \frac{Q}{2}\right) \\ k'' &\approx \sqrt{\frac{\omega\mu\sigma}{2}}\left(1 - \frac{Q}{2}\right) \\ \mathcal{R} &\approx \sqrt{\frac{\omega\mu}{2\sigma}}\left(1 + \frac{Q}{2}\right) \\ \mathcal{X} &\approx \sqrt{\frac{\omega\mu}{2\sigma}}\left(1 - \frac{Q}{2}\right) \end{aligned}\right\} \quad Q \ll 1$$

where Q is defined by Eq. (1-79).

2-16. Show that for metals

$$\eta = \mathcal{R}(1 + j) \qquad k = \frac{1}{\delta}(1 - j) \qquad \mathcal{R} = \frac{I}{\sigma\delta}$$

where $\mathcal{R}$ is the surface resistance, δ is the skin depth, and σ is the conductivity.

2-17. Derive the following formulas

$$\begin{aligned} \mathcal{R}\ (\text{silver}) &= 2.52 \times 10^{-7}\sqrt{f} \\ \mathcal{R}\ (\text{copper}) &= 2.61 \times 10^{-7}\sqrt{f} \\ \mathcal{R}\ (\text{gold}) &= 3.12 \times 10^{-7}\sqrt{f} \\ \mathcal{R}\ (\text{aluminum}) &= 3.26 \times 10^{-7}\sqrt{f} \\ \mathcal{R}\ (\text{brass}) &= 5.01 \times 10^{-7}\sqrt{f} \end{aligned}$$

where f is the frequency in cycles per second.

2-18. Find the power per square meter dissipated in a copper sheet if the rms magnetic intensity at its surface is 1 ampere per meter at (*a*) 60 cycles, (*b*) 1 megacycle, (*c*) 1000 megacycles.

2-19. Make a sketch similar to Fig. 2-6 for a circularly polarized standing wave in dissipative media. Give a verbal description of $\mathcal{E}$ and $\mathcal{H}$.

2-20. Given a uniform plane wave normally incident upon a plane air-to-dielectric interface, show that the standing-wave ratio is

$$\text{SWR} = \sqrt{\epsilon_r} = \text{index of refraction}$$

where ϵ_r is the dielectric constant of the dielectric (assumed nonmagnetic and loss-free).

2-21. Take the index of refraction of water to be 9, and calculate the percentage of power reflected and transmitted when a plane wave is normally incident on a calm lake.

2-22. Calculate the two polarizing angles (internal and external) and the critical angle for a plane interface between air and (*a*) water, $\epsilon_r = 81$, (*b*) high-density glass, $\epsilon_r = 9$, and (*c*) polystyrene, $\epsilon_r = 2.56$.

2-23. Suppose a uniform plane wave in a dielectric just grazes a plane dielectric-to-air interface. Calculate the attenuation constant in the air [α as defined by Eq. (2-61)] for the three cases of Prob. 2-22. Calculate the distance from the boundary in which the field is attenuated to $1/e$ (36.8 per cent) of its value at the boundary. What is the value of α at the critical angle?

2-24. From Eqs. (2-66) and (2-68), show that when $R \ll \omega L$ and $G \ll \omega C$

$$\alpha \approx \frac{R}{2\sqrt{L/C}} + \frac{G\sqrt{L/C}}{2}$$
$$\beta \approx \omega\sqrt{LC}$$

where $\gamma = \alpha + j\beta$.

2-25. Show that G and C of a transmission line are related by

$$G = \frac{\omega\epsilon''}{\epsilon'} C = \frac{\omega\epsilon''\eta}{Z_0}$$

when the dielectric is homogeneous. Show that R of a transmission line is approximately equal to the d-c resistance per unit length of hollow conductors having thickness δ (skin depth) provided H is approximately constant over each conductor and the radius of curvature of the conductors is large compared to δ.

2-26. Using results of Prob. 2-25, show that for the two-wire line of Table 2-3

$$R \approx \frac{2\mathcal{R}}{\pi d} \qquad \begin{matrix} d \gg \delta \\ D \gg d \end{matrix}$$

and that for the coaxial line

$$R \approx \frac{\mathcal{R}}{2\pi}\frac{a+b}{ab} \qquad a \gg \delta$$

and that for the parallel-plate line

$$R \approx \frac{2\mathcal{R}}{w} \qquad w \gg b$$

2-27. Verify Eqs. (2-70).

2-28. Consider a parallel-plate waveguide formed by conductors covering the planes $y = 0$ and $y = b$. Show that the field

$$E_x = E_0 \sin\frac{n\pi y}{b} e^{-\gamma z} \qquad n = 1, 2, 3, \ldots$$

defines a set of TE_n modes and the field

$$H_x = H_0 \cos \frac{n\pi y}{b} e^{-\gamma z} \qquad n = 0, 1, 2, \ldots$$

defines a set of TM_n modes, where

$$\gamma = \sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}$$

in both cases. Show that the cutoff frequencies of the TE_n and TM_n modes are

$$f_c = \frac{n}{2b\sqrt{\epsilon\mu}}$$

Show that Eqs. (2-83) to (2-86) apply to the parallel-plate waveguide modes.

2-29. Show that the power transmitted per unit width (x direction) of the parallel-plate waveguide of Prob. 2-28 is

$$P = \frac{b|E_0|^2}{2\eta} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

for the TE_n modes, and

$$P = \frac{b|H_0|^2\eta}{2} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

for the TM_n modes ($n \neq 0$).

2-30. For the parallel-plate waveguide of Prob. 2-28, show that the attenuation due to conductor losses is

$$\alpha_c = \frac{2\mathcal{R}(f_c/f)^2}{b\eta\sqrt{1 - (f_c/f)^2}}$$

for the TE_n modes, and

$$\alpha_c = \frac{2\mathcal{R}}{b\eta\sqrt{1 - (f_c/f)^2}}$$

for the TM_n modes ($n \neq 0$).

2-31. Show that the TM_0 mode of the parallel-plate waveguide as defined in Prob. 2-28 is actually a TEM mode. Show that for this mode the attenuation due to conductor losses is

$$\alpha_c = \frac{\mathcal{R}}{b\eta}$$

Compare this with α obtained by using the results of Probs. 2-26 and 2-24.

2-32. For the TE_{01} rectangular waveguide mode, show that the time-average electric and magnetic energies per unit length are

$$\bar{\mathcal{W}}_e = \bar{\mathcal{W}}_m = \frac{\epsilon_0}{4} |E_0|^2 ab$$

Can this equality of $\bar{\mathcal{W}}_e$ and $\bar{\mathcal{W}}_m$ be predicted from Eq. (1-62)?

2-33. Show that the time-average velocity of propagation of energy down a rectangular waveguide is

$$\bar{v}_e = \frac{\bar{\mathcal{S}}_z}{\bar{\mathcal{W}}} = \frac{1}{\sqrt{\epsilon\mu}} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

for the TE_{01} mode.

2-34. For the TE_{01} rectangular waveguide mode, define a voltage V as $\int \mathbf{E} \cdot d\mathbf{l}$ across the center of the guide and a current I as the total z-directed current in the guide wall $x = 0$. Show that these are

$$V = aE_0 e^{-j\beta z} \qquad I = \frac{2bE_0}{\pi Z_0} e^{-j\beta z}$$

Show that $P \neq VI^*$. Why? Define a characteristic impedance $Z_{VI} = V/I$ and show that it is proportional to Z_0 of Table 2-4.

2-35. Let a rectangular waveguide have a discontinuity in dielectric at $z = 0$, that is, ϵ_1, μ_1 for $z < 0$ and ϵ_2, μ_2 for $z > 0$. Show that the reflection and transmission coefficients for a TE_{01} wave incident from $z < 0$ are

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \qquad T = \frac{2Z_{02}}{Z_{02} + Z_{01}}$$

where Z_{01} and Z_{02} are the characteristic impedances $z < 0$ and $z > 0$, respectively. These results are valid for any waveguide mode.

2-36. Show that there is no reflected wave for the TE_{01} mode in Prob. 2-35 when

$$\frac{f}{f_{c1}} = \sqrt{\frac{\epsilon_1(\mu_1{}^2 - \mu_2{}^2)}{\mu_2(\mu_1\epsilon_2 - \mu_2\epsilon_1)}}$$

where f_{c1} is the cutoff frequency $z < 0$. Note that we cannot have a reflectionless interface when both dielectrics are nonmagnetic. This result is valid for any TE mode.

2-37. Take a parallel-plate waveguide with ϵ_1, μ_1 for $z < 0$ and ϵ_2, μ_2 for $z > 0$. Show that there is no reflected wave for a TM mode incident from $z < 0$ when

$$\frac{f}{f_{c1}} = \sqrt{\frac{\mu_1(\epsilon_1{}^2 - \epsilon_2{}^2)}{\epsilon_2(\epsilon_1\mu_2 - \epsilon_2\mu_1)}}$$

For nonmagnetic dielectrics, this reduces to

$$\frac{f}{f_{c1}} = \sqrt{\frac{\epsilon_1 + \epsilon_2}{\epsilon_2}}$$

Compare this to Eq. (2-60). These results are valid for any TM mode.

2-38. Design a square-base cavity with height one-half the width of the base to resonate at 1000 megacycles (a) when it is air-filled and (b) when it is polystyrene-filled. Calculate the Q in each case.

2-39. For the rectangular cavity of Fig. 2-19, define a voltage V as that between mid-points of the top and bottom walls and a current I as the total x-directed current in the side walls. Show that

$$V = E_0 a \qquad I = \frac{j4E_0}{\pi\eta} \sqrt{b^2 + c^2}$$

Define a mode conductance G as $G = \bar{\mathcal{P}}_d/|V|^2$ and show that

$$G = \frac{\mathcal{R}[bc(b^2 + c^2) + 2a(b^3 + c^3)]}{2\eta^2 a^2 (b^2 + c^2)}$$

Define a mode resistance R as $R = \bar{\mathcal{P}}_d/|I|^2$ and show that

$$R = \frac{\pi^2 \mathcal{R}[bc(b^2 + c^2) + 2a(b^3 + c^3)]}{32(b^2 + c^2)^2}$$

2-40. Derive Eqs. (2-123).

2-41. Consider the small loop of constant current I as shown in Fig. 2-26. Show that the magnetic vector potential is

$$A_\phi = A_y \bigg|_{\phi=0} = \frac{Ia}{4\pi} \int_0^{2\pi} f \cos \phi' \, d\phi'$$

where
$$f = \frac{\exp\left(-jk\sqrt{r^2 + a^2 - 2ra \sin\theta \cos\phi'}\right)}{\sqrt{r^2 + a^2 - 2ra \sin\theta \cos\phi'}}$$

Expand f in a Maclaurin series about $a = 0$ and show that

$$A_\phi \xrightarrow[a \to 0]{} \frac{I\pi a^2}{4\pi} e^{-jkr} \left(\frac{jk}{r} + \frac{1}{r^2}\right) \sin\theta$$

The quantity $I\pi a^2 = IS$ is called the magnetic moment of the loop.

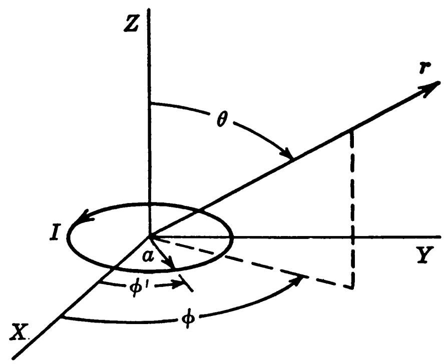


FIG. 2-26. A circular loop of current.

2-42. Show that the field of the small current loop of Prob. 2-41 is

$$H_r = \frac{IS}{2\pi} e^{-jkr} \left(\frac{jk}{r^2} + \frac{1}{r^3}\right) \cos\theta$$
$$H_\theta = \frac{IS}{4\pi} e^{-jkr} \left(-\frac{k^2}{r} + \frac{jk}{r^2} + \frac{1}{r^3}\right) \sin\theta$$
$$E_\phi = \frac{\eta IS}{4\pi} e^{-jkr} \left(\frac{k^2}{r} - \frac{jk}{r^2}\right) \sin\theta$$

Show that the radiation resistance of the small loop referred to I is

$$R_r = \eta \frac{2\pi}{3} \left(\frac{kS}{\lambda}\right)^2$$

2-43. Consider the current element of Fig. 2-21 and the current loop of Fig. 2-26 to exist simultaneously. Show that the radiation field is everywhere circularly polarized if

$$Il = kIS$$

2-44. In terms of the tabulated functions

$$\text{Si}(x) = \int_0^x \frac{\sin x}{x} dx \qquad \text{Ci}(x) = -\int_x^\infty \frac{\cos x}{x} dx$$

show that Eq. (2-129) can be expressed as

$$R_r = \frac{\eta}{2\pi}\left[C + \log kL - \mathrm{Ci}\, kL + \sin kL \left(\tfrac{1}{2}\mathrm{Si}\, 2kL - \mathrm{Si}\, kL\right) + \tfrac{1}{2}\cos kL \left(C + \log k\frac{L}{2} + \mathrm{Ci}\, 2kL - 2\mathrm{Ci}\, kL\right)\right]$$

where $C = 0.5772 \cdots$ is Euler's constant.

2-45. If the linear antenna of Fig. 2-23 is an integral number of half-wavelengths long, the current will assume the form

$$I(z) = I_m \sin k\left(z + \frac{L}{2}\right)$$

regardless of the position of the feed as long as it is not near a current null. Such an antenna is said to be of *resonant length.* Show that the radiation field of the antenna is

$$E_\theta = \frac{j\eta I_m}{2\pi r} e^{-jkr} \frac{\cos\left(\frac{n\pi}{2}\cos\theta\right)}{\sin\theta} \qquad n \text{ odd}$$

$$E_\theta = \frac{\eta I_m}{2\pi r} e^{-jkr} \frac{\sin\left(\frac{n\pi}{2}\cos\theta\right)}{\sin\theta} \qquad n \text{ even}$$

where $n = 2L/\lambda$ is an integer.

2-46. For an antenna of resonant length (Prob. 2-45), show that the radiation resistance referred to I_m is

$$R_r = \frac{\eta}{4\pi}\left[C + \log 2n\pi - \mathrm{Ci}(2n\pi)\right]$$

where $n = 2L/\lambda$, $C = 0.5772$, and Ci is as defined in Prob. 2-44. Show that the input resistance for a loss-free antenna with feed point at $z = a\lambda$ is

$$R_i = \frac{R_r}{\sin 2\pi(a + n/4)}$$

Specialize this result to $L = \lambda/2$, $a = 0$ (the half-wave dipole) and show that $R_i = 73$ ohms.