Appendix A

Transformation Between Unit Vectors

A-1 Cylindrical System

$$(v_1, v_2, v_3) = (r, \phi, z)$$
$$(h_1, h_2, h_3) = (1, r, 1)$$

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z$$

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{r}$	$\cos\phi$	$\sin\phi$	0
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0
$\hat{z}$	0	0	1

A-2 Spherical System

$$(v_1, v_2, v_3) = (R, \theta, \phi)$$
$$(h_1, h_2, h_3) = (1, R, R\sin\theta)$$

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta$$

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{R}$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
$\hat{\theta}$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0

A-3 Elliptical Cylinder

$$(v_1, v_2, v_3) = (\eta, \xi, z)$$

$$(h_1, h_2, h_3) = \left[c\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{1/2}, \; c\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{1/2}, \; 1\right]$$

$$x = c\eta\xi, \quad y = c\left[(1-\eta^2)(\xi^2-1)\right]^{1/2}, \quad z = z$$

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{\eta}$	$\dfrac{c\xi}{h_1}$	$\dfrac{-c\eta}{h_2}$	0
$\hat{\xi}$	$\dfrac{c\eta}{h_2}$	$\dfrac{c\xi}{h_1}$	0
$\hat{z}$	0	0	1

A-4 Parabolic Cylinder

$$(v_1, v_2, v_3) = (\eta, \xi, z)$$

$$(h_1, h_2, h_3) = \left[(\eta^2+\xi^2)^{1/2}, \; (\eta^2+\xi^2)^{1/2}, \; 1\right]$$

$$x = \frac{1}{2}(\eta^2-\xi^2), \quad y = \eta\xi, \quad z = z$$

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{\eta}$	$\dfrac{\eta}{h}$	$\dfrac{\xi}{h}$	0
$\hat{\xi}$	$\dfrac{-\xi}{h}$	$\dfrac{\eta}{h}$	0
$\hat{z}$	0	0	1

$$h = h_1 = h_2 = (\eta^2+\xi^2)^{1/2}$$

A-5 Prolate Spheroid

$$(v_1, v_2, v_3) = (\eta, \xi, \phi)$$

$$(h_1, h_2, h_3) = \left[c\left(\frac{\xi^2 - \eta^2}{1 - \eta^2}\right)^{1/2}, c\left(\frac{\xi^2 - \eta^2}{\xi^2 - 1}\right)^{1/2}, c\left(1 - \eta^2\right)^{1/2}\left(\xi^2 - 1\right)^{1/2}\right]$$

$$r = \left(x^2 + y^2\right)^{1/2} = c\left[\left(1 - \eta^2\right)\left(\xi^2 - 1\right)\right]^{1/2}, \qquad \phi = \phi, \qquad z = c\eta\xi$$

	$\hat{z}$	$\hat{r}$	$\hat{\phi}$
$\hat{\eta}$	$\frac{c\xi}{h_1}$	$\frac{-c\eta}{h_2}$	0
$\hat{\xi}$	$\frac{c\eta}{h_2}$	$\frac{c\xi}{h_1}$	0
$\hat{\phi}$	0	0	1

The unit vectors $\hat{r}$ and $\hat{\phi}$ can be expressed in terms of $\hat{x}$ and $\hat{y}$ covered in the cylindrical case; the same applies to the oblate spheroid.

A-6 Oblate Spheroid

$$(v_1, v_2, v_3) = (\eta, \xi, \phi)$$

$$(h_1, h_2, h_3) = \left[c\left(\frac{\xi^2 - \eta^2}{\xi^2 - 1}\right)^{1/2}, c\left(\frac{\xi^2 - \eta^2}{1 - \eta^2}\right)^{1/2}, c\xi\eta\right]$$

$$r = \left(x^2 + y^2\right)^{1/2} = c\xi\eta, \qquad z = c\left[\left(\xi^2 - 1\right)\left(1 - \eta^2\right)\right]^{1/2}$$

	$\hat{z}$	$\hat{r}$	$\hat{\phi}$
$\hat{\xi}$	$\frac{c\xi}{h_2}$	$\frac{c\eta}{h_1}$	0
$\hat{\eta}$	$\frac{-c\eta}{h_1}$	$\frac{c\xi}{h_2}$	0
$\hat{\phi}$	0	0	1

A-7 Bipolar Cylinders

$$(v_1, v_2, v_3) = (\eta, \xi, z)$$

$$(h_1, h_2, h_3) = \left(\frac{a}{\cosh\xi - \cos\eta}, \frac{a}{\cosh\xi - \cos\eta}, 1\right)$$

$$x = \frac{a \sinh\xi}{\cosh\xi - \cos\eta}, \qquad y = \frac{a \sin\eta}{\cosh\xi - \cos\eta}, \qquad z = z$$

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{\eta}$	$\frac{-h}{a}\sinh\xi\sin\eta$	$\frac{h}{a}(\cosh\xi\cos\eta - 1)$	0
$\hat{\xi}$	$\frac{-h}{a}(\cosh\xi\cos\eta - 1)$	$\frac{-h}{a}\sinh\xi\sin\eta$	0
$\hat{z}$	0	0	1

where

$$h = h_1 = h_2 = \frac{a}{\cosh\xi - \cos\eta}.$$

In all of these tables, the unit vectors are all arranged in the order of a right-handed system, that is, $\hat{x}_1 \times \hat{x}_2 = \hat{x}_3$.