Chapter 7

Dyadic Analysis

7-1 Divergence and Curl of Dyadic Functions and Gradient of Vector Functions

In Chapter 1, the definition of dyadic functions in a rectangular system and the algebra of these functions were introduced. In this chapter, the calculus of dyadics, or dyadic analysis, will be developed.

The divergence of a dyadic function $\bar{\bar{F}}$, expressed in a rectangular system by (1.82) in Chapter 1, will be denoted by $\nabla \cdot \bar{\bar{F}}$, and is defined by

$$\nabla \cdot \bar{\bar{F}} = \sum_j (\nabla \cdot \mathbf{F}_j)\hat{x}_j = \sum_i \sum_j \frac{\partial F_{ij}}{\partial x_i}\hat{x}_j, \tag{7.1}$$

which is a vector function.

The curl of a dyadic function of the form $\bar{\bar{F}}$, denoted by $\nabla \times \bar{\bar{F}}$, is defined by

$$\nabla \times \bar{\bar{F}} = \sum_j (\nabla \times \mathbf{F}_j)\hat{x}_j = \sum_i \sum_j (\nabla F_{ij} \times \hat{x}_i)\hat{x}_j, \tag{7.2}$$

which is also a dyadic function. Here, we have used the vector identity

$$\nabla \times \mathbf{F}_j = \nabla \times \sum_i F_{ij}\hat{x}_i = \sum_i (\nabla F_{ij} \times \hat{x}_i) \tag{7.3}$$

to convert the single sum in (7.2) to a double sum.

Sometimes we need the gradient of a vector function in dyadic analysis, denoted by $\nabla\mathbf{F}$. In a rectangular coordinate system, its expression is given by (4.153), that is,

$$\nabla\mathbf{F} = \sum_j (\nabla\mathbf{F}_j)\hat{x}_j = \sum_{i,j} \frac{\partial F_{ij}}{\partial x_i}\hat{x}_i\hat{x}_j.$$

In OCS, it is defined by

$$\begin{aligned}\nabla\mathbf{F} &= \sum_i \frac{\hat{u}_i}{h_i}\frac{\partial\mathbf{F}}{\partial v_i} = \sum_i \frac{\hat{u}_i}{h_i}\frac{\partial}{\partial v_i}\sum_j F_j\hat{u}_j \\ &= \sum_{i,j}\frac{\hat{u}_i}{h_i}\left(\frac{\partial F_j}{\partial v_i}\hat{u}_j + F_j\frac{\partial\hat{u}_j}{\partial v_i}\right).\end{aligned} \tag{7.4}$$

The derivatives of the unit vector $\hat{u}_j$ in this equation can be expressed in terms of the other unit vectors $\hat{u}_i$, $\hat{u}_k$ with the aid of (2.59) and (2.61), which yields

$$\begin{aligned}\nabla\mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i}&\left[\left(\frac{\partial F_i}{\partial v_i} + \frac{F_j}{h_j}\frac{\partial h_i}{\partial v_j} + \frac{F_k}{h_k}\frac{\partial h_i}{\partial v_k}\right)\hat{u}_i\right. \\ &\left. + \left(\frac{\partial F_j}{\partial v_i} - \frac{F_i}{h_j}\frac{\partial h_i}{\partial v_j}\right)\hat{u}_j + \left(\frac{\partial F_k}{\partial v_i} - \frac{F_i}{h_k}\frac{\partial h_i}{\partial v_k}\right)\hat{u}_k\right]\end{aligned} \tag{7.5}$$

with $i, j, k = 1, 2, 3$ in cyclic order. Expression of $\nabla\mathbf{F}$ in GCS can be derived in a similar manner.

When a dyadic function is formed by a scalar function f with an idemfactor $\bar{\bar{I}}$ in the form of

$$f\bar{\bar{I}} = \sum_i f\hat{x}_i\hat{x}_i, \tag{7.6}$$

the divergence of this dyadic function is then a vector function, and it is given by

$$\nabla(f\bar{\bar{I}}) = \sum_i \frac{\partial f}{\partial x_i}\hat{x}_i = \nabla f. \tag{7.7}$$

In the OCS, $\bar{\bar{I}}$ is defined by

$$\bar{\bar{I}} = \sum_i \hat{u}_i\hat{u}_i, \tag{7.8}$$

and the divergence of $f\bar{\bar{I}}$ is defined by

$$\begin{aligned}\nabla(f\bar{\bar{I}}) &= \sum_i \frac{\hat{u}_i}{h_i}\cdot\frac{\partial}{\partial v_i}\sum_j f\hat{u}_j\hat{u}_j \\ &= \sum_i \frac{\hat{u}_i}{h_i}\cdot\sum_j\left[\frac{\partial f}{\partial v_i}\hat{u}_j\hat{u}_j + f\frac{\partial\hat{u}_j}{\partial v_i}\hat{u}_j + f\hat{u}_j\frac{\partial\hat{u}_j}{\partial v_i}\right].\end{aligned} \tag{7.9}$$

In (7.9), for $j = i$,

$$\hat{u}_i \cdot \frac{\partial \hat{u}_i}{\partial v_i} = 0, \tag{7.10}$$

which can be proved by taking the derivative of

$$\hat{u}_i \cdot \hat{u}_i = 1$$

with respect to v_i, or any variable for that matter. With the aid of (2.59) and (2.61), the last two terms in (7.9) cancel each other; hence

$$\nabla(f\bar{\bar{I}}) = \sum_i \frac{1}{h_i}\frac{\partial f}{\partial v_i}\hat{u}_i = \nabla f. \tag{7.11}$$

Equation (7.11), therefore, is invariant to the coordinate system. We demonstrate here once more the invariance of $\boldsymbol{\nabla}$. By following the same approach, we find

$$\boldsymbol{\nabla}(f\bar{\bar{I}}) = \sum_j (\boldsymbol{\nabla} f\hat{x}_j)\hat{x}_j = \sum_j (\nabla f \times \hat{x}_j)\hat{x}_j = \nabla f \times \bar{\bar{I}}, \tag{7.12}$$

which is a dyadic. This identity is valid in any coordinate system.

To find the divergence of a dyadic in an OCS, one can transform all the functions defined in a rectangular system to the functions in a specified OCS. We let

$$\mathbf{F} = \sum_i F_i\hat{x}_i = \sum_j F'_j\hat{u}_j = \mathbf{F}', \tag{7.13}$$

where F_i and F'_j (with $i, j = 1, 2, 3$) denote, respectively, the components of the function $\mathbf{F}$ or $\mathbf{F}'$ in the two systems; then

$$F'_j = \sum_i F_i\hat{x}_i \cdot \hat{u}_j = \sum_i c_{ji}F_i, \tag{7.14}$$

where c_{ji} denotes the directional cosines between $\hat{x}_i$ and $\hat{u}_j$. These coefficients can be found by the method of gradient in Section 4-5. The inverse transform is

$$F_i = \sum_j c_{ji}F'_j. \tag{7.15}$$

Equation (7.15) also applies to the transformation of the unit vectors. For example,

$$\hat{x}_i = \sum_j c_{ji}\hat{u}_j. \tag{7.16}$$

By definition,

$$\begin{aligned}
\nabla \bar{\bar{F}} &= \sum_j (\nabla \mathbf{F}_j)\hat{x}_j = \sum_j (\nabla \mathbf{F}'_j)\hat{x}_j \\
&= \sum_j \sum_k \frac{1}{\Omega}\frac{\partial}{\partial v_k}\left(\frac{\Omega}{h_k}F'_{kj}\right)\sum_i c_{ij}\hat{u}_i \qquad (7.17)\\
&= \sum_{i,j,k} \frac{c_{ij}}{\Omega}\frac{\partial}{\partial v_k}\left(\frac{\Omega}{h_k}F'_{kj}\right)\hat{u}_i,
\end{aligned}$$

which is a vector function.

For mixed dyadics made of two independent vector functions of the form

$$\bar{\bar{F}} = \mathbf{M}(\mathbf{R})\,\mathbf{N}(\mathbf{R}'), \tag{7.18}$$

where $\mathbf{R}$ and $\mathbf{R}'$ represent two independent position vectors in some coordinate system, the divergence and the curl of $\bar{\bar{F}}$ with respect to the unprimed coordinates are defined as

$$\nabla\,\bar{\bar{F}} = [\nabla\mathbf{M}(\mathbf{R})]\,\mathbf{N}(\mathbf{R}'), \tag{7.19}$$

$$\boldsymbol{\nabla}\,\bar{\bar{F}} = [\boldsymbol{\nabla}\,\mathbf{M}(\mathbf{R})]\,\mathbf{N}(\mathbf{R}'). \tag{7.20}$$

The divergence and the curl of $\bar{\bar{F}}$ with respect to the primed coordinates are not defined, but

$$\nabla'[\bar{\bar{F}}]^{\mathrm{T}} = [\nabla'\mathbf{N}(\mathbf{R}')]\,\mathbf{M}(\mathbf{R}) \tag{7.21}$$

and

$$\boldsymbol{\nabla}'[\bar{\bar{F}}]^{\mathrm{T}} = [\boldsymbol{\nabla}'\mathbf{N}(\mathbf{R}')]\,\mathbf{M}(\mathbf{R}). \tag{7.22}$$

These functions are found in the application of dyadic analysis to electromagnetic theory [3].

A vector–dyadic identity to be quoted in the next chapter is derived here to show its origin. We have previously derived an identity, (4.158), showing that when a dyadic is formed by two vectors in the form of $\mathbf{ab}$,

$$\nabla(\mathbf{ab}) = (\nabla\mathbf{a})\mathbf{b} + \mathbf{a}\cdot\nabla\mathbf{b}. \tag{7.23}$$

Similarly,

$$\nabla(\mathbf{ba}) = (\nabla\mathbf{b})\mathbf{a} + \mathbf{b}\cdot\nabla\mathbf{a}. \tag{7.24}$$

The dyadic $\mathbf{ba}$ is the transpose of $\mathbf{ab}$. By taking the difference between the last two equations, we obtain

$$\nabla(\mathbf{ba} - \mathbf{ab}) = (\nabla\mathbf{b})\mathbf{a} - (\nabla\mathbf{a})\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{b}. \tag{7.25}$$

The right side of (7.25), according to (4.159), is equal to $\boldsymbol{\nabla}(\mathbf{a}\times\mathbf{b})$; hence

$$\nabla(\mathbf{ba} - \mathbf{ab}) = \boldsymbol{\nabla}(\mathbf{a}\times\mathbf{b}). \tag{7.26}$$

This identity was listed in Appendix B of reference [20] without derivation.

7-2 Dyadic Integral Theorems

There are several integral theorems in dyadic analysis that can be derived by changing the vector functions in the vector Green's theorems to dyadic functions.

1. **First vector–dyadic Green's theorem**

The first vector theorem stated by (4.211) will be written in the following form:

$$\iiint_V [(\nabla \mathbf{P}) \cdot (\nabla \mathbf{Q}) - \mathbf{P} \cdot \nabla \nabla \mathbf{Q}]\, dV = \oint_S \mathbf{n} \cdot (\mathbf{P} \times \nabla \mathbf{Q})\, dS. \tag{7.27}$$

We have purposely placed the function $\mathbf{Q}$ in the posterior position in (7.27), a practice that is used to change a vector to a dyadic. Consider now three distinct $\mathbf{Q}_j$ with $j = 1, 2, 3$ so that we have three identities of the same form as (7.27). By juxtaposing a unit vector $\hat{x}_j$ at the posterior position of each of the three equations and summing them, we obtain

$$\iiint_V [(\nabla \mathbf{P}) \cdot (\nabla \bar{\bar{Q}}) - \mathbf{P} \cdot \nabla \nabla \bar{\bar{Q}}]\, dV = \oint_S \mathbf{n} \cdot (\mathbf{P} \times \nabla \bar{\bar{Q}})\, dS, \tag{7.28}$$

where, by definition,

$$\nabla \bar{\bar{Q}} = \sum_j (\nabla \mathbf{Q}_j)\hat{x}_j \tag{7.29}$$

and

$$\nabla \nabla \bar{\bar{Q}} = \sum_j (\nabla \nabla \mathbf{Q}_j)\hat{x}_j. \tag{7.30}$$

Equation (7.28) is designated as the first vector–dyadic Green's theorem of Type A because it involves a vector function $\mathbf{P}$ and a dyadic function $\bar{\bar{Q}}$. By interchanging $\mathbf{P}$ with $\mathbf{Q}$ in (7.27) and then raising the level of $\mathbf{Q}$ to a dyadic, we obtain

$$\iiint_V [(\nabla \mathbf{P}) \cdot (\nabla \bar{\bar{Q}}) - (\nabla \nabla \mathbf{P}) \cdot \bar{\bar{Q}}]\, dV = -\oint_S \mathbf{n} \cdot [(\nabla \mathbf{P}) \times \bar{\bar{Q}}]\, dS. \tag{7.31}$$

Equation (7.31) is designated as the first vector–dyadic Green's theorem of Type B. Except for the term $(\nabla \mathbf{P}) \cdot (\nabla \bar{\bar{Q}})$, which is common to (7.28) and (7.31), the rest are different.

2. **Second vector–dyadic Green's theorem**

By subtracting (7.28) from (7.31), we obtain the second vector–dyadic Green's theorem:

$$\iiint_V [\mathbf{P} \cdot \nabla \nabla \bar{\bar{Q}} - (\nabla \nabla \mathbf{P}) \cdot \bar{\bar{Q}}]\, dV = -\oint_S \mathbf{n} \cdot [(\mathbf{P} \times \nabla \bar{\bar{Q}}) + (\nabla \mathbf{P}) \times \bar{\bar{Q}}]\, dS. \tag{7.32}$$

This theorem is probably the most useful formula in the application of dyadic analysis to electromagnetic theory [3].

3. **First dyadic–dyadic Green's theorem**

Equation (7.28) can be elevated to a higher level by moving **P** and ∇ **P** into the posterior position and transposing the dyadic terms into the anterior position so that

$$\iiint_V \left\{[\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\nabla \mathbf{P}) - [\nabla \nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot \mathbf{P}\right\} dV = \oint_S [\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\mathbf{n} \times \mathbf{P})\, dS. \tag{7.33}$$

The vector function **P** can now be elevated to a dyadic level that yields the first dyadic–dyadic Green's theorem:

$$\iiint_V \left\{[\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\nabla \bar{\bar{P}}) - [\nabla \nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot \bar{\bar{P}}\right\} dV = \oint_S [\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\mathbf{n} \times \bar{\bar{P}})\, dS. \tag{7.34}$$

By doing the same thing with (7.31), we obtain

$$\iiint_V \left\{[\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\nabla \bar{\bar{P}}) - [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\nabla \nabla \bar{\bar{P}})\right\} dV = -\oint_S [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\mathbf{n} \times \nabla \bar{\bar{P}})\, dS. \tag{7.35}$$

4. **Second dyadic–dyadic Green's theorem**

By taking the difference between (7.34) and (7.35), we obtain the second dyadic–dyadic Green's theorem:

$$\iiint_V \left\{[\nabla \nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot \bar{\bar{P}} - [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\nabla \nabla \bar{\bar{P}})\right\} dV$$
$$= -\oint_S \left\{[\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\mathbf{n} \times \bar{\bar{P}}) + [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\mathbf{n} \times \nabla \bar{\bar{P}})\right\} dS. \tag{7.36}$$

The two dyadic–dyadic Green's theorems involve two dyadics; hence we have the name. They can be used to prove the symmetrical property of the electric and magnetic dyadic Green's functions [3]. We have now assembled all of the important formulas in dyadic analysis, with the hope that they will be useful in digesting technical articles involving dyadic analysis, particularly in its application to electromagnetic theory.