Chapter 1

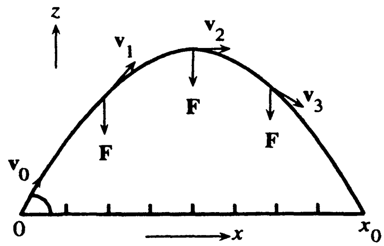
Vector and Dyadic Algebra

1-1 Representations of Vector Functions

A vector function has both magnitude and direction. The vector functions that we encounter in many physical problems are, in general, functions of space and time. In the first five chapters, we discuss only their characteristics as functions of spatial variables. Functions of space and time are covered in Chapter 6, dealing with a moving surface or a moving contour.

A vector function is denoted by $\mathbf{F}$. Geometrically, it is represented by a line with an arrow in a three-dimensional space. The length of the line corresponds to its magnitude, and the direction of the line represents the direction of the vector function. The convenience of using vectors to represent physical quantities is illustrated by a simple example shown in Fig. 1-1, which describes the motion of a mass particle in a frictionless air (vacuum) against a constant gravitational force. The particle is thrown into the space with an initial velocity $\mathbf{v}_0$, making an angle θ_0 with respect to the horizon. During its flight, the velocity function of the particle changes both its magnitude and direction, as shown by $\mathbf{v}_1$, $\mathbf{v}_2$, and so on, at subsequent locations. The gravitational force that acts on the particle is assumed to be constant, and it is represented by $\mathbf{F}$ in the figure. A constant vector function means that both the magnitude and the direction of the function are constant, being independent of the spatial variables, x and z in this case.

The rule of the addition of two vectors $\mathbf{a}$ and $\mathbf{b}$ is shown geometrically by Fig. 1-2a, b, or c. Algebraically, it is written in the same form as the addition of



$$z = x\left(1 - \frac{x}{x_0}\right)\tan\theta_0$$

$$x_0 = \frac{1}{g}v_0^2 \sin 2\theta_0$$

g = gravitational constant

Figure 1-1 Trajectory of a mass particle in a gravitational field showing the velocity **v** and the constant force vector **F** at different locations.

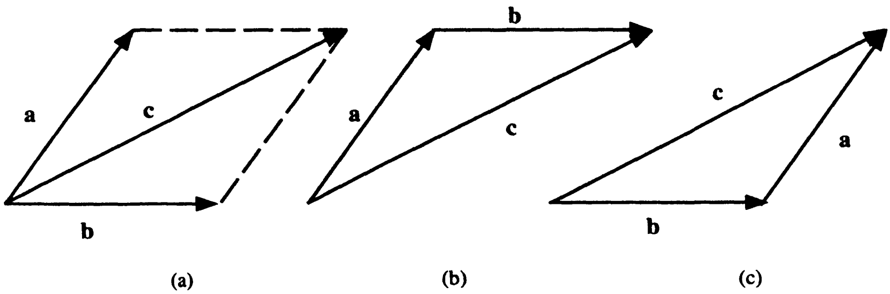


Figure 1-2 Addition of vectors, $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

two numbers of two scalar functions, that is,

$$\mathbf{c} = \mathbf{a} + \mathbf{b}. \tag{1.1}$$

The subtraction of vector **b** from vector **a** is written in the form

$$\mathbf{d} = \mathbf{a} - \mathbf{b}. \tag{1.2}$$

Now, $-\mathbf{b}$ is a vector that has the same magnitude as **b**, but of opposite direction; then (1.2) can be considered as the addition of **a** and ($-\mathbf{b}$). Geometrically, the meaning of (1.2) is shown in Fig. 1-3. The sum and the difference of two vectors obey the associative rule, that is,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \tag{1.3}$$

and

$$\mathbf{a} - \mathbf{b} = -\mathbf{b} + \mathbf{a}. \tag{1.4}$$

They can be generalized to any number of vectors.

The rule of the addition of vectors suggests that any vector can be considered as being made of basic components associated with a proper coordinate system. The

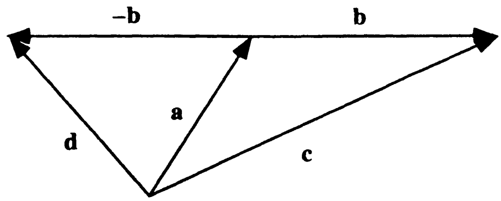


Figure 1-3 Subtraction of vectors, $\mathbf{a} - \mathbf{b} = \mathbf{d}$.

most convenient system to use is the Cartesian system or the rectangular coordinate system, or more specifically, a right-handed rectangular system in which, when x is turning to y, a right-handed screw advances to the z direction. The spatial variables in this system are commonly denoted by x, y, z. A vector that has a magnitude equal to unity and pointed in the positive x direction is called a *unit vector* in the x direction and is denoted by $\hat{x}$. Similarly, we have $\hat{y}$, $\hat{z}$. In such a system, a vector function $\mathbf{F}$ that, in general, is a function of position, can be written in the form

$$\mathbf{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}. \tag{1.5}$$

The three scalar functions F_x, F_y, F_z are called the *components* of $\mathbf{F}$ in the direction of $\hat{x}$, $\hat{y}$, and $\hat{z}$, respectively, while $F_x\hat{x}$, $F_y\hat{y}$, and $F_z\hat{z}$ are called the *vector components* of $\mathbf{F}$. The geometrical representation of $\mathbf{F}$ is shown in Fig. 1-4. It is seen that F_x, F_y, and F_z can be either positive or negative. In Fig. 1-4, F_x and F_z are positive, but F_y is negative.

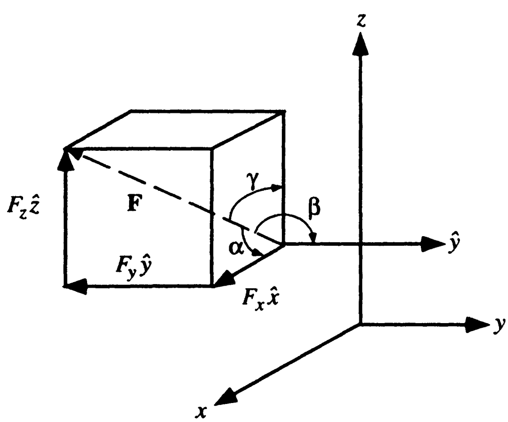


Figure 1-4 Components of a vector in a Cartesian system.

In addition to the representation by (1.5), it is sometimes desirable to express $\mathbf{F}$ in terms of its magnitude, denoted by $|\mathbf{F}|$, and its directional cosines, that is,

$$\mathbf{F} = |\mathbf{F}| \left(\cos\alpha\hat{x} + \cos\beta\hat{y} + \cos\gamma\hat{z}\right). \tag{1.6}$$

α, β, and γ are the angles **F** makes, respectively, with $\hat{x}$, $\hat{y}$, and $\hat{z}$, as shown in Fig. 1-4. It is obvious from the geometry of that figure that

$$|\mathbf{F}| = \left(F_x^2 + F_y^2 + F_z^2\right)^{1/2} \tag{1.7}$$

and

$$\cos\alpha = \frac{F_x}{|\mathbf{F}|}, \qquad \cos\beta = \frac{F_y}{|\mathbf{F}|}, \qquad \cos\gamma = \frac{F_z}{|\mathbf{F}|}. \tag{1.8}$$

Furthermore, we have the relation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{1.9}$$

In view of (1.9), only two of the directional cosine angles are independent. From the previous discussion, we observe that, in general, we need three parameters to specify a vector function. The three parameters could be F_x, F_y, and F_z or $|\mathbf{F}|$ and two of the directional cosine angles. Representations such as (1.5) and (1.6) can be extended to other orthogonal coordinate systems, which will be discussed in a later chapter.

1-2 Products and Identities

The scalar product of two vectors **a** and **b** is denoted by $\mathbf{a}\cdot\mathbf{b}$, and it is defined by

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta, \tag{1.10}$$

where θ is the angle between **a** and **b**, as shown in Fig. 1-5. Because of the notation used for such a product, sometimes it is called the *dot product*. By applying (1.10) to three orthogonal unit vectors $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$, one finds

$$\hat{u}_i\cdot\hat{u}_j = \left\{\begin{array}{ll} 1, & i = j \\ 0, & i \neq j \end{array}\right\}, \qquad i, j = 1, 2, 3. \tag{1.11}$$

The value of $\mathbf{a}\cdot\mathbf{b}$ can also be expressed in terms of the components of **a** and **b** in any orthogonal system. Let the system under consideration be the rectangular system, and let $\mathbf{c} = \mathbf{a} - \mathbf{b}$; then

$$|\mathbf{c}|^2 = |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta.$$

Hence

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta &= \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2}{2} \\ &= \frac{a_x^2 + a_y^2 + a_z^2 + b_x^2 + b_y^2 + b_z^2 - (a_x - b_x)^2 - \left(a_y - b_y\right)^2 - (a_z - b_z)^2}{2} \\ &= a_x b_x + a_y b_y + a_z b_z. \end{aligned} \tag{1.12}$$

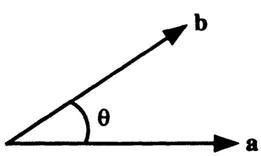


Figure 1-5 Scalar product of two vectors, $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$.

By equating (1.10) and (1.12), one finds

$$\begin{aligned}\cos\theta &= \frac{1}{|\mathbf{a}|\,|\mathbf{b}|}\left(a_x b_x + a_y b_y + a_z b_z\right) \\ &= \cos\alpha_a \cos\alpha_b + \cos\beta_a \cos\beta_b + \cos\gamma_a \cos\gamma_b,\end{aligned} \tag{1.13}$$

a relationship well known in analytical geometry. Equation (1.12) can be used to prove the validity of the distributive law for the scalar products, namely,

$$(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}. \tag{1.14}$$

According to (1.12), we have

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} &= (a_x+b_x)\,c_x + \left(a_y+b_y\right)c_y + (a_z+b_z)\,c_z \\ &= \left(a_x c_x + a_y c_y + a_z c_z\right) + \left(b_x c_x + b_y c_y + b_z c_z\right) \\ &= \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}.\end{aligned}$$

Once we have proved the distributive law for the scalar product, (1.12) can be verified by taking the sum of the scalar products of the individual terms of **a** and **b**.

The vector product of two vector functions **a** and **b**, denoted by $\mathbf{a}\times\mathbf{b}$, is defined by

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\hat{u}_c, \tag{1.15}$$

where θ denotes the angle between **a** and **b**, measured from **a** to **b**; $\hat{u}_c$ denotes a unit vector perpendicular to both **a** and **b** and is pointed to the advancing direction of a right-hand screw when we turn from **a** to **b**. Figure 1-6 shows the relative position of $\hat{u}_c$ with respect to **a** and **b**. Because of the notation used for the vector product, it is sometimes called the *cross product*, in contrast to the dot product or the scalar product. For three orthogonal unit vectors in a right-hand system, we have $\hat{u}_1\times\hat{u}_2 = \hat{u}_3$, $\hat{u}_2\times\hat{u}_3 = \hat{u}_1$, and $\hat{u}_3\times\hat{u}_1 = \hat{u}_2$. It is obvious that $\hat{u}_i\times\hat{u}_i = 0$, $i = 1, 2, 3$. From the definition of the vector product in (1.15), one finds

$$\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}. \tag{1.16}$$

The value of $\mathbf{a}\times\mathbf{b}$ as described by (1.15) can also be expressed in terms of the components of **a** and **b** in a rectangular coordinate system. If we let $\mathbf{a}\times\mathbf{b} = \mathbf{v} =$

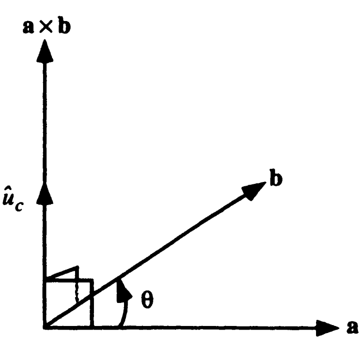


Figure 1-6 Vector product of two vectors, $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{u}_c$; $\hat{u}_c \perp \mathbf{a}$, $\hat{u}_c \perp \mathbf{b}$.

$v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$, which is perpendicular to both **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{v} = a_x v_x + a_y v_y + a_z v_z = 0, \tag{1.17}$$

$$\mathbf{b} \cdot \mathbf{v} = b_x v_x + b_y v_y + b_z v_z = 0. \tag{1.18}$$

Solving for v_x/v_z and v_y/v_z, from (1.17) and (1.18) we obtain

$$\frac{v_x}{v_z} = \frac{a_y b_z - a_z b_y}{a_x b_y - a_y b_x}, \qquad \frac{v_y}{v_z} = \frac{a_z b_x - a_x b_z}{a_x b_y - a_y b_x}.$$

Thus,

$$\frac{v_x}{a_y b_z - a_z b_y} = \frac{v_y}{a_z b_x - a_x b_z} = \frac{v_z}{a_x b_y - a_y b_x}.$$

Let the common ratio of these quantities be denoted by c, which can be determined by considering the case with $\mathbf{a} = \hat{x}$, $\mathbf{b} = \hat{y}$; then $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \hat{z}$; hence from the last ratio, we find $c = 1$ because $v_z = 1$ and $a_x = b_y = 1$, while $a_y = b_x = 0$. The three components of **v**, therefore, are given by

$$\left.\begin{aligned} v_x &= a_y b_z - a_z b_y \\ v_y &= a_z b_x - a_x b_z \\ v_z &= a_x b_y - a_y b_x \end{aligned}\right\}, \tag{1.19}$$

which can be assembled in a determinant form as

$$\mathbf{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{1.20}$$

We can use (1.20) to prove the distributive law of vector products, that is,

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}. \tag{1.21}$$

To prove (1.21), we find that the x component of $(\mathbf{a} + \mathbf{b}) \times \mathbf{c}$ according to (1.20) is equal to

$$\left(a_y + b_y\right) c_z - \left(a_z + b_z\right) c_y = \left(a_y c_z - a_z c_y\right) + \left(b_y c_z - b_z c_y\right). \tag{1.22}$$

The last two terms in (1.22) denote, respectively, the x component of $\mathbf{a} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$. The equality of the y and z components of (1.21) can be proved in a similar manner.

In addition to the scalar product and the vector product introduced before, there are two identities involving the triple products that are very useful in vector analysis. They are

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \tag{1.23}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}. \tag{1.24}$$

Identities described by (1.23) can be proved by writing $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ in a determinant form:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

According to the theory of determinants,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The last two determinants represent, respectively, $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ and $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$; hence we have the validity of (1.23). To prove (1.24), we observe that the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ lies in the plane containing $\mathbf{b}$ and $\mathbf{c}$, so we can treat $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ as being made of two components $\alpha\mathbf{b}$ and $\beta\mathbf{c}$, as shown in Fig. 1-7, that is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \alpha\mathbf{b} + \beta\mathbf{c}. \tag{1.25}$$

Because

$$\mathbf{a} \cdot [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})] = 0,$$

hence

$$\alpha\,(\mathbf{a} \cdot \mathbf{b}) + \beta\,(\mathbf{a} \cdot \mathbf{c}) = 0.$$

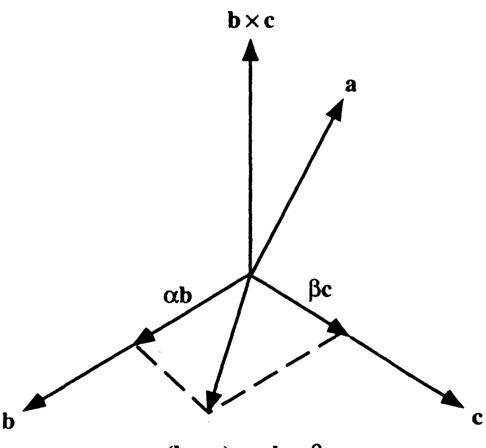


Figure 1-7 Orientation of various vectors in $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Equation (1.25), therefore, can be written in the form

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \alpha \left[\mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{c}} \mathbf{c} \right] = \alpha' \left[(\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} \right], \tag{1.26}$$

where α' is a constant to be determined. By considering the case $\mathbf{a} = \hat{y}$, $\mathbf{b} = \hat{x}$, $\mathbf{c} = \hat{y}$, we have

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \hat{x},$$
$$(\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} = \hat{x},$$
$$(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} = 0.$$

Hence $\alpha' = 1$. All other choices of **a**, **b**, and **c** yield the same answer. The validity of (1.23) and (1.24) is independent of the choice of the coordinate system in which these vectors are represented.

1-3 Orthogonal Transformation of Vector Functions

A vector function represented by (1.5) in a specified rectangular system can likewise be represented in another rectangular system. To discuss the relation between these two representations, we must first show the geometry of the two coordinate systems. The relative orientation of the axes of these two systems can be formed by three successive rotations originally due to Leonhard Euler (1707–1783).

Let the coordinates of the original system be denoted by (x, y, z). We first rotate the (x, y) axes by an angle ϕ_1 to form the (x_1, y_1) axes, keeping $z_1 = z$ as shown in Fig. 1-8; then the coordinates of a point (x, y, z) change to (x_1, y_1, z_1) with

$$x_1 = x\cos\phi_1 + y\sin\phi_1, \tag{1.27}$$

$$y_1 = -x\sin\phi_1 + y\cos\phi_1, \tag{1.28}$$

$$z_1 = z. \tag{1.29}$$

Now we turn the (y_1, z_1) axes by an angle ϕ_2 to form the (y_2, z_2) axes with $x_2 = x_1$; then,

$$y_2 = y_1\cos\phi_2 + z_1\sin\phi_2, \tag{1.30}$$

$$z_2 = -y_1\sin\phi_2 + z_1\cos\phi_2, \tag{1.31}$$

$$x_2 = x_1. \tag{1.32}$$

Finally, we rotate the (z_2, x_2) axes by an angle ϕ_3 to form the (z_3, x_3) axes with $y_3 = y_2$; then,

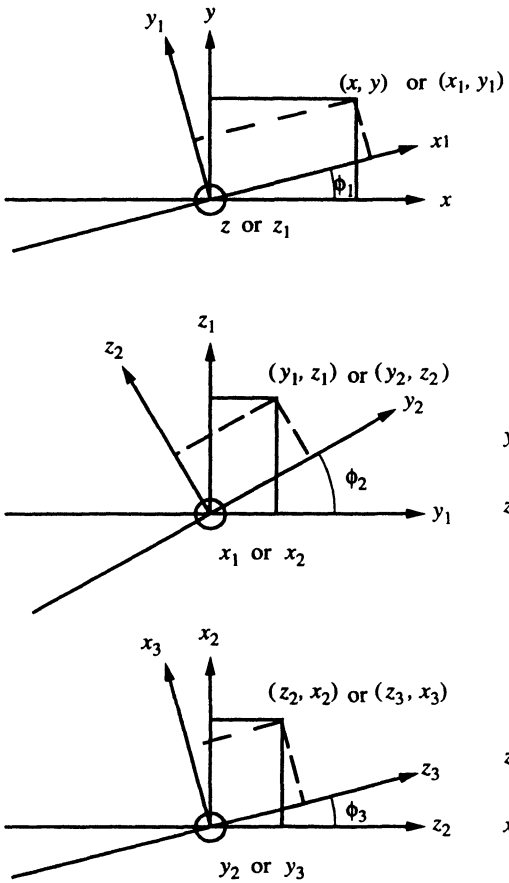
$$z_3 = z_2\cos\phi_3 + x_2\sin\phi_3, \tag{1.33}$$

$$x_3 = -z_2\sin\phi_3 + x_2\cos\phi_3, \tag{1.34}$$

$$y_3 = y_2. \tag{1.35}$$

By expressing (x_3, y_3, z_3) in terms of (x, y, z) and changing the letters (x, y, z) and (x_3, y_3, z_3), respectively, to the unprimed and primed indexed letters (x_1, x_2, x_3) and (x_1', x_2', x_3'), we obtain

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j, \qquad i = 1, 2, 3, \tag{1.36}$$



$$x_1 = x \cos \phi_1 + y \sin \phi_1$$

$$y_1 = y \cos \phi_1 - x \sin \phi_1$$

$$y_2 = y_1 \cos \phi_2 - z_1 \sin \phi_2$$

$$z_2 = z_1 \cos \phi_2 - y_1 \sin \phi_2$$

$$z_3 = z_2 \cos \phi_3 + x_2 \sin \phi_3$$

$$x_3 = x_2 \cos \phi_3 - z_2 \sin \phi_3$$

Figure 1-8 Sequences of rotations of the axes of a rectangular coordinate system.

where

$$\begin{aligned}
a_{11} &= \cos\phi_1 \cos\phi_3 - \sin\phi_1 \sin\phi_2 \sin\phi_3, \\
a_{12} &= \sin\phi_1 \cos\phi_3 + \cos\phi_1 \sin\phi_2 \sin\phi_3, \\
a_{13} &= -\cos\phi_2 \sin\phi_3, \\
a_{21} &= -\sin\phi_1 \cos\phi_2, \\
a_{22} &= \cos\phi_1 \cos\phi_2, \\
a_{23} &= \sin\phi_2, \\
a_{31} &= \cos\phi_1 \sin\phi_3 + \sin\phi_1 \sin\phi_2 \cos\phi_3, \\
a_{32} &= \sin\phi_1 \sin\phi_3 - \cos\phi_1 \sin\phi_2 \cos\phi_3, \\
a_{33} &= \cos\phi_2 \cos\phi_3.
\end{aligned} \tag{1.37}$$

The coefficients a_{ij} correspond to the directional cosines between the x_i' and x_j axes, that is,

$$a_{ij} = \cos \beta_{ij}, \tag{1.38}$$

where β_{ij} denotes the angle between the two axes.

If we solve (x, y, z) in terms of (x_3, y_3, z_3) or x_j $(j = 1, 2, 3)$ in terms of x_i' $(i = 1, 2, 3)$ from (1.27) to (1.32), we obtain

$$x_j = \sum_{i=1}^{3} a_{ij} x_i', \qquad j = 1, 2, 3, \tag{1.39}$$

where a_{ij} denotes the same coefficients defined in (1.37). It should be observed that the summation indices in (1.36) and (1.39) are executed differently in these two equations. For example,

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \tag{1.40}$$

but

$$x_1 = a_{11}x_1' + a_{21}x_2' + a_{31}x_3', \tag{1.41}$$

and $a_{ij} \neq a_{ji}$ when $j \neq i$. Henceforth, whenever a summation sign is used, it is understood that the running index goes from 1 to 3 unless specified otherwise. A more efficient notation is to delete the summation sign in (1.36) and (1.39). When the summation index appears in two terms, we write (1.36) in the form

$$x_i' = a_{ij}x_j. \tag{1.42}$$

The single index (i) means $i = 1, 2, 3$ and the double index (j) represents a summation of the terms from $j = 1$ to $j = 3$. Such a notation, originally due to Einstein, can be applied to more than three variables. In this book we will use the summation sign in order to convey the meaning more vividly, particularly when several summation indices are involved in an equation. The summation index will be placed under the sign for one summation or several summations separated by a comma. The linear relations between the coordinates x_i' and x_j, as stated by (1.36) and (1.39), apply equally well to two sets of unit vectors $\hat{x}_i'$ and $\hat{x}_j$ and also to the components of a vector **A**, denoted respectively by A_i' and A_j in the two systems. This is evident from the processes by which the primed system is formed from the unprimed system. To recapitulate these relations, it is convenient to construct a 3×3 square matrix, as shown in Table 1-1. We identify i, the first subscript of a_{ij}, as the ordinal number of the rows, and j, the second subscript, as the ordinal number of the columns. The quantities involved in the transformation are listed at the side and the top.

The matrix can be used either horizontally or vertically, for example,

$$A_3' = a_{31}A_1 + a_{32}A_2 + a_{33}A_3,$$

Table 1-1: The Matrix of Transformation $[a_{ij}]$ for Quantities Defined in Two Orthogonal Rectangular Systems

$i\backslash j$	x_j or $\hat{x}_j$ or A_j		
x_i' or	a_{11}	a_{12}	a_{13}
$\hat{x}_i'$ or	a_{21}	a_{22}	a_{23}
A_i'	a_{31}	a_{32}	a_{33}

and

$$A_2 = a_{12}A_1' + a_{22}A_2' + a_{32}A_3',$$

which conform with (1.36) and (1.39) after x_i' and x_j are replaced by A_i' and A_j. For convenience, we will designate the a_{ij}'s as the directional coefficients and its matrix by $[a_{ij}]$. There are several important properties of the matrix that must be shown. In the first place, the determinant of $[a_{ij}]$, denoted by $|a_{ij}|$, is equal to unity for a right-hand system under consideration. In such a system, when one turns $\hat{x}_1$ to $\hat{x}_2$ using a right-hand screw, it advances to the $\hat{x}_3$ direction. To prove

$$|a_{ij}| = 1, \tag{1.43}$$

we consider a cubic made of $\hat{x}_i'$ with $i = 1, 2, 3$. Its volume is equal to unity, that is,

$$\hat{x}_1' \cdot (\hat{x}_2' \times \hat{x}_3') = 1. \tag{1.44}$$

The expression on the left side of (1.44) is given by

$$\sum_l a_{il}\hat{x}_l \cdot \left(\sum_m a_{jm}\hat{x}_m \times \sum_n a_{kn}\hat{x}_n\right) = |a_{ij}|, \tag{1.45}$$

where (l, m, n) and $(i, j, k) = (1, 2, 3)$ in cyclic order. The identity between (1.44) and (1.45) yields (1.43). A second identity relates the directional coefficients a_{ij} with the cofactors or the signed minors of $[a_{ij}]$. If we solve (x_1, x_2, x_3) in terms of (x_1', x_2', x_3') from (1.36) based on the theory of linear equations, we find

$$x_j = \frac{1}{|a_{ij}|}\sum_i A_{ij}x_i', \tag{1.46}$$

where A_{ij} denotes the cofactor or the signed minor of $[a_{ij}]$ obtained by eliminating the ith row and jth column. By comparing (1.46) with (1.39) and because $|a_{ij}| = 1$, we obtain

$$A_{ij} = a_{ij}. \tag{1.47}$$

An alternative derivation is to start with the relation

$$\hat{x}_j = \sum_i A_{ij}\hat{x}_i', \tag{1.48}$$

which is the same as (1.46) with $|a_{ij}| = 1$, and to replace x_j, x'_i with $\hat{x}_j$ and $\hat{x}_i$; then the scalar product of (1.48) with $\hat{x}'_i$ yields

$$a_{ij} = A_{ij}. \tag{1.49}$$

As an example, let $i = 1,\ j = 2$; then

$$a_{12} = -(a_{21}a_{33} - a_{23}a_{31}). \tag{1.50}$$

The validity of (1.49) can also be verified by using the expressions of a_{ij} defined in terms of the Eulerian angles listed in (1.37); that is,

$$\begin{aligned} -a_{21}a_{33} + a_{23}a_{31} &= \sin\phi_1 \cos\phi_2 \cos\phi_2 \cos\phi_3 \\ &\quad + \sin\phi_2(\cos\phi_1 \sin\phi_3 + \sin\phi_1 \sin\phi_2 \cos\phi_3) \\ &= \cos\phi_1 \sin\phi_2 \sin\phi_3 + \sin\phi_1 \cos\phi_3 \\ &= a_{12}. \end{aligned}$$

Equation (1.47) is a very useful identity in discussing the transformation of vector products.

Because the axes of the two coordinate systems x'_i and x_j or the unit vectors $\hat{x}'_i$ and $\hat{x}_j$ are themselves orthogonal, then

$$\hat{x}'_i \cdot \hat{x}'_j = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{1.51}$$

and similarly,

$$\hat{x}_i \cdot \hat{x}_j = \delta_{ij}, \tag{1.52}$$

where δ_{ij} denotes the Kronecker δ function defined in (1.51). In terms of the unprimed unit vectors, (1.51) becomes

$$\sum_m a_{im}\hat{x}_m \cdot \sum_n a_{jn}\hat{x}_n = \delta_{ij}; \tag{1.53}$$

because

$$\hat{x}_m \cdot \hat{x}_n = \delta_{mn},$$

(1.53) reduces to

$$\sum_m a_{im}a_{jm} = \delta_{ij}, \tag{1.54}$$

and similarly, by expressing (1.52) in terms of $\hat{x}'_m$ and $\hat{x}'_n$, we obtain

$$\sum_m a_{mi}a_{mj} = \delta_{ij}. \tag{1.55}$$

Either (1.54) or (1.55) contains six identities. Looking at the rotational relations between the unprimed and the primed coordinates, we observe that the three Eulerian angles or parameters generate nine coefficients. Only three of them are therefore independent, provided they are not the triads of

$$\sum_j a_{ij}^2 = 1, \qquad i = 1, 2, 3, \tag{1.56}$$

or

$$\sum_i a_{ij}^2 = 1, \qquad j = 1, 2, 3. \tag{1.57}$$

The remaining six coefficients are therefore dependent coefficients that are related by (1.54) or (1.55) or a mixture of six relations from both of them. For example, if a_{11}, a_{12}, and a_{23} have been specified, then we can determine a_{13} from the equation

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1,$$

and subsequently the coefficient a_{33} from

$$a_{13}^2 + a_{23}^2 + a_{33}^2 = 1.$$

The remaining four coefficients a_{21}, a_{22}, a_{31}, a_{32} can be found from the equations

$$a_{11}\underline{a}_{21} + a_{12}\underline{a}_{22} + a_{13}a_{23} = 0,$$

and

$$a_{11}\underline{a}_{31} + a_{12}\underline{a}_{32} + a_{13}a_{33} = 0.$$

We underline the unknowns by placing a bar underneath these coefficients. They must also satisfy

$$a_{23}\underline{a}_{21} + a_{33}\underline{a}_{31} + a_{13}a_{11} = 0,$$

and

$$a_{23}\underline{a}_{22} + a_{33}\underline{a}_{32} + a_{13}a_{12} = 0.$$

For convenience, we summarize here the important formulas that have been derived:

$$x_i' = \sum_j a_{ij}x_j, \qquad i = 1, 2, 3, \tag{1.58}$$

$$x_j = \sum_i a_{ij}x_i', \qquad j = 1, 2, 3, \tag{1.59}$$

$$\hat{x}_i' = \sum_j a_{ij}\hat{x}_j, \qquad i = 1, 2, 3, \tag{1.60}$$

$$\hat{x}_j = \sum_i a_{ij}\hat{x}_i', \qquad j = 1, 2, 3, \tag{1.61}$$

$$A_i' = \sum_j a_{ij}A_j, \qquad i = 1, 2, 3, \tag{1.62}$$

$$A_j = \sum_i a_{ij}A_i', \qquad j = 1, 2, 3, \tag{1.63}$$

$$|a_{ij}| = 1, \tag{1.64}$$

$$a_{ij} = A_{ij}, \tag{1.65}$$

$$\sum_m a_{im}a_{jm} = \delta_{ij}, \tag{1.66}$$

$$\sum_m a_{mi}a_{mj} = \delta_{ij}. \tag{1.67}$$

In expressions (1.58) through (1.67), all the summation signs are understood to be executed from 1 to 3. The conglomerate puts these relations into a group.

Equations (1.62) and (1.63) are two important equations or requirements for the transformation of the components of a vector in two rectangular systems rotated with respect to each other. These relations also show that a vector function has an invariant form, namely,

$$\mathbf{A} = \sum_i A_i \hat{x}_i = \sum_j A'_j \hat{x}'_j \tag{1.68}$$

and

$$\mathbf{A} \cdot \mathbf{A} = \sum_i A_i^2 = \sum_j A'^2_j. \tag{1.69}$$

Equation (1.69) shows that the magnitude of a vector is an invariant scalar quantity, independent of the defining coordinate system. The speed of a car running 50 miles per hour is independent of its direction. However, its direction does depend on the reference system that is being used, namely, either $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ or $(\hat{x}'_1, \hat{x}'_2, \hat{x}'_3)$. The vector functions that transform according to (1.62) and (1.63) are called *polar vectors*, to be distinguished from another class of vectors that will be covered in the next section.

1-4 Transform of Vector Products

A vector product formed by two polar vectors $\mathbf{A}$ and $\mathbf{B}$ in the unprimed system and their corresponding expressions $\mathbf{A}'$ and $\mathbf{B}'$ in the primed system is

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{1.70}$$

or

$$\mathbf{C}' = \mathbf{A}' \times \mathbf{B}'. \tag{1.71}$$

According to the definition of a vector product, (1.19), its expression in a right-handed rectangular system is

$$C'_k = A'_i B'_j - A'_j B'_i \tag{1.72}$$

with $i, j, k = 1, 2, 3$ in cyclic order. Now,

$$A'_i = \sum_m a_{im} A_m, \tag{1.73}$$

$$B'_j = \sum_n a_{jn} B_n. \tag{1.74}$$

Hence

$$\begin{aligned} C'_k = A'_i B'_j - A'_j B'_i &= \sum_m \sum_n (a_{im} a_{jn} - a_{jm} a_{in}) A_m B_n \\ &= \sum_m \sum_n a_{im} a_{jn} (A_m B_n - A_n B_m). \end{aligned} \tag{1.75}$$

It appears that the components $A_m B_n - A_n B_m$ in (1.75) do not transform like the components of a polar vector as in (1.62). However, if we inspect the terms in (1.75), for example, with $k = 1$, $i = 2$, $j = 3$, then

$$\begin{aligned} C_1' &= (a_{22}a_{33} - a_{23}a_{32})(A_2 B_3 - A_3 B_2) \\ &\quad + (a_{23}a_{31} - a_{21}a_{33})(A_3 B_1 - A_1 B_3) \\ &\quad + (a_{21}a_{32} - a_{22}a_{31})(A_1 B_2 - A_2 B_1). \end{aligned} \tag{1.76}$$

The terms involving the directional coefficients are recognized as the cofactors A_{11}, A_{12}, and A_{13} of $[a_{ij}]$ and, according to (1.65), they are equal to a_{11}, a_{12}, and a_{13}; hence

$$\begin{aligned} C_1' &= A_{11}C_1 + A_{12}C_2 + A_{13}C_3 \\ &= a_{11}C_1 + a_{12}C_2 + a_{13}C_3 \\ &= \sum_j a_{ij}C_j. \end{aligned} \tag{1.77}$$

Equation (1.77) obeys the same rule as the transformation of two polar vectors. Thus, in a three-dimensional Euclidean space, the vector product does transform like a polar vector even though its origin stems from the vector product of two polar vectors. From the physical point of view, the vector product is used to describe a quantity associated with rotation, such as the angular velocity of a rotating body, the moment of force, the vorticity in hydrodynamics, and the magnetic field in electrodynamics. For this reason such a vector was called a *skew vector* by J. Willard Gibbs (1837–1903), one of the founders of vector analysis. Nowadays, it is commonly called an *axial vector*. From now on, the word *vector* will be used to comprise both the polar and the axial vectors in a three-dimensional or Euclidean space. In a four-dimensional manifold as in the theory of relativity, the situation is different. In that case, we have to distinguish the polar vector, or the four-vector, from the axial vector, or the six-vector. This topic will be briefly discussed after the subjects of dyadic and tensor analysis are introduced. Even though the transformation rule of a polar vector applies to an axial vector, we must remember that we have defined an axial vector according to a right-hand rule. In a left-hand coordinate system, obtained by an inversion of the axes of a right-hand system, the components of a polar vector change their signs; then we must use a left-hand rotating rule to define a vector product to preserve the same rule of transformation between a polar vector and an axial vector. We would like to mention that in a left-hand system the determinant of the corresponding directional coefficients is equal to -1.

Before we close this section, we want to point out that as a result of the identical rule of transformation of the polar vectors and the axial vectors, the characteristics of the two triple products $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ can be ascertained. The scalar triple product is, indeed, an invariant scalar because

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{A}' \cdot (\mathbf{B}' \times \mathbf{C}'). \tag{1.78}$$

For the vector triple product, it behaves like a vector because by decomposing it into two terms using the vector identity (1.24),

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}, \tag{1.79}$$

we see that $\mathbf{A} \cdot \mathbf{C}$ and $\mathbf{A} \cdot \mathbf{B}$ are invariant scalars, and $\mathbf{B}$ and $\mathbf{C}$ are vectors; the sum at the right side of (1.79) is therefore again a vector. This synthesis may appear to be trivial but it does offer a better understanding of the nature of these quantities. The reader should practice constructing these identities when the vectors are defined in a left-hand coordinate system.

1-5 Definition of Dyadics and Tensors

A vector function $\mathbf{F}$ in a three-dimensional space defined in a rectangular system is represented by

$$\mathbf{F} = \sum_i F_i \hat{x}_i. \tag{1.80}$$

If we consider three independent vector functions denoted by

$$\mathbf{F}_j = \sum_i F_{ij} \hat{x}_i, \qquad j = 1, 2, 3, \tag{1.81}$$

then a dyadic function can be formed that will be denoted by $\bar{\bar{F}}$ and defined by

$$\bar{\bar{F}} = \sum_j \mathbf{F}_j \hat{x}_j. \tag{1.82}$$

The unit vector $\hat{x}_j$ is juxtaposed at the posterior position of $\mathbf{F}_j$. By substituting the expression of $\mathbf{F}_j$ into (1.82), we obtain

$$\bar{\bar{F}} = \sum_i \sum_j F_{ij} \hat{x}_i \hat{x}_j = \sum_{i,j} F_{ij} \hat{x}_i \hat{x}_j. \tag{1.83}$$

Equation (1.83) is the explicit expression of a dyadic function defined in a rectangular coordinate system. Sometimes the name *Cartesian dyadic* is used. A dyadic function, or simply a dyadic, therefore, consists of nine dyadic components; each component is made of a scalar component F_{ij} and a dyad in the form of a pair of unit vectors $\hat{x}_i \hat{x}_j$ placed in that order.

Because a dyadic is formed by three vector functions and three unit vectors, the transform of a dyadic from its representation in one rectangular system (the unprimed) to another rectangular system (the primed) can most conveniently be executed by applying (1.61) to (1.83); thus,

$$\begin{aligned} \bar{\bar{F}} &= \sum_{i,j} F_{ij} \sum_m a_{mi} \hat{x}'_m \sum_n a_{nj} \hat{x}'_n \\ &= \sum_{i,j,m,n} a_{mi} a_{nj} F_{ij} \hat{x}'_m \hat{x}'_n. \end{aligned} \tag{1.84}$$

If we denote

$$F'_{mn} = \sum_{i,j} a_{mi} a_{nj} F_{ij}, \tag{1.85}$$

then

$$\begin{aligned} \bar{\bar{F}} &= \sum_{m,n} F'_{mn} \hat{x}'_m \hat{x}'_n \\ &= \sum_{i,j} F_{ij} \hat{x}_i \hat{x}_j. \end{aligned} \tag{1.86}$$

Equation (1.85) describes the rule of transform of the scalar components F_{ij} of a dyadic in two rectangular systems. By starting with the expression of $\bar{\bar{F}}'$ in the primed system, we find

$$F_{ij} = \sum_{m,n} a_{mi} a_{nj} F'_{mn}. \tag{1.87}$$

When the 3×3 scalar components F_{ij} are arranged in matrix form, denoted by $[F_{ij}]$, it is designated as a tensor or, more precisely, as a tensor of rank 2 or a tensor of valance 2. The exact form of $[F_{ij}]$ is

$$[F_{ij}] = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}. \tag{1.88}$$

In tensor analysis, a vector is treated as a tensor of rank 1 and a scalar as a tensor of rank 0. In this book, tensor analysis is not one of our main topics. The subject has been covered by many excellent books such as Brand [1] and Borisenko and Tarapov [2]. However, many applications of tensor analysis can be treated equally well by dyadic analysis. In the previous section, we have already correlated a tensor of rank 2 in a Euclidean space with a dyadic. Tensor analysis is most useful in the theory of relativity, but one can formulate problems in the special theory of relativity using conventional vector analysis, as illustrated in Appendix E of this book.

1-6 Classification of Dyadics

When the scalar components of a dyadic are symmetrical, such that

$$D_{ij} = D_{ji}, \tag{1.89}$$

it is called a *symmetric dyadic* and the corresponding tensor, a *symmetric tensor*. When the components are antisymmetric, such that

$$D_{ij} = -D_{ji}, \tag{1.90}$$

such a dyadic is called an *antisymmetric dyadic* and the corresponding tensor, the *antisymmetric tensor*. For an antisymmetric dyadic, (1.90) implies $D_{ii} = 0$. An antisymmetric tensor of this dyadic, therefore, has effectively only three distinct components, namely, D_{12}, D_{13}, and D_{23}. The other three components are $-D_{12}$, $-D_{13}$, and $-D_{23}$, which are not considered to be distinctly different.

Let us now introduce three terms with a single index, such that

$$D_1 = D_{23}, \qquad D_2 = D_{31}, \qquad D_3 = D_{12},$$

or

$$D_i = D_{jk}$$

with $i, j, k = 1, 2, 3$ in cyclic order; then the tensor of this antisymmetric dyadic has the form

$$[D_{ij}] = \begin{bmatrix} 0 & D_3 & -D_2 \\ -D_3 & 0 & D_1 \\ D_2 & -D_1 & 0 \end{bmatrix}. \tag{1.91}$$

The transform of these components to a primed system, according to (1.85), has the form

$$D'_{ij} = \sum_{m,n} a_{im} a_{jn} D_{mn}, \tag{1.92}$$

where we have interchanged the roles of the indices in (1.85). In terms of the single indexed components for D'_{ij},

$$D'_k = \sum_{m,n} a_{im} a_{jn} D_{mn}, \tag{1.93}$$

where (i, j, k) and $(l, m, n) = (1, 2, 3)$ in cyclic order. For example, the explicit expression of D'_1 is

$$\begin{aligned} D'_1 &= (a_{22}a_{33} - a_{23}a_{32})D_{23} \\ &\quad + (a_{23}a_{31} - a_{21}a_{33})D_{31} \\ &\quad + (a_{21}a_{32} - a_{22}a_{31})D_{12}. \end{aligned} \tag{1.94}$$

The coefficients attached to D_{mn} are recognized as three cofactors of $[a_{ij}]$; they are, respectively, A_{11}, A_{12}, and A_{13}, which are equal to a_{11}, a_{12}, and a_{13}. By changing D_{23}, D_{31}, and D_{12} to D_1, D_2, and D_3, we obtain

$$\begin{aligned} D'_1 &= A_{11}D_1 + A_{12}D_2 + A_{13}D_3 \\ &= a_{11}D_1 + a_{12}D_2 + a_{13}D_3 \\ &= \sum_j a_{ij}D_j. \end{aligned} \tag{1.95}$$

Equation (1.95) describes, precisely, the transform of a polar vector in the two-coordinate system. By tracing back the derivations, we see that an antisymmetric tensor is essentially an axial vector (defined in a right-hand system) and its components transform like a polar vector. The connection between an antisymmetric tensor, an axial vector, and a polar vector is well illustrated in this exercise. We now continue on to define some more quantities used in dyadic algebra.

When a symmetric dyadic is made of three dyads in the form

$$\bar{\bar{I}} = \hat{x}_1\hat{x}_1 + \hat{x}_2\hat{x}_2 + \hat{x}_3\hat{x}_3 = \sum_i \hat{x}_i\hat{x}_i, \tag{1.96}$$

it is called an *idemfactor*. Its significance and applications will be revealed shortly. When the positions of $\mathbf{F}_j$ and $\hat{x}_j$ are interchanged in (1.82), we form another dyadic that is called the *transpose* of $\bar{\bar{F}}$, and it is denoted by $[\bar{\bar{F}}]^{\mathrm{T}}$, that is,

$$[\bar{\bar{F}}]^{\mathrm{T}} = \sum_j \hat{x}_j\mathbf{F}_j = \sum_{j,i} F_{ij}\hat{x}_j\hat{x}_i = \sum_{i,j} F_{ji}\hat{x}_i\hat{x}_j. \tag{1.97}$$

The corresponding tensor will be denoted by

$$[F_{ij}]^{\mathrm{T}} = [F_{ji}] = \begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{bmatrix}. \tag{1.98}$$

We therefore transpose the columns in $[F_{ij}]$ to form the rows in $[F_{ij}]^{\mathrm{T}}$. It is obvious that the transpose of $[F_{ij}]^{\mathrm{T}}$ goes back to $[\bar{\bar{F}}]$.

1-7 Products Between Vectors and Dyadics

There are two scalar products between a vector $\mathbf{A}$ and a dyadic $\bar{\bar{D}}$. The anterior scalar product is defined by

$$\begin{aligned} \mathbf{B} = \mathbf{A} \cdot \bar{\bar{D}} &= \mathbf{A} \cdot \sum_j \mathbf{D}_j\hat{x}_j \\ &= \mathbf{A} \cdot \sum_{i,j} D_{ij}\hat{x}_i\hat{x}_j \\ &= \sum_{i,j} A_i D_{ij}\hat{x}_j, \end{aligned} \tag{1.99}$$

which is a vector; hence we use the notation $\mathbf{B}$. Following the rules of the transform of a vector and a dyadic, we find that the same vector becomes

$$\mathbf{B} = \sum_{m,n} A'_m D'_{mn}\hat{x}'_n = \mathbf{A}' \cdot \bar{\bar{D}}'. \tag{1.100}$$

Hence

$$\mathbf{B} = \mathbf{B}'. \tag{1.101}$$

That means the product, like the scalar product $\mathbf{A} \cdot \mathbf{B}$, is independent of the coordinate system in which it is defined, or it is form-invariant.

The posterior vector between $\mathbf{A}$ and $\bar{\bar{D}}$ is defined by

$$\mathbf{C} = \bar{\bar{D}} \cdot \mathbf{A} = \sum_{i,j} D_{ij}\hat{x}_i\hat{x}_j \cdot \mathbf{A} = \sum_{i,j} D_{ij}A_j\hat{x}_i. \tag{1.102}$$

The scalar components of (1.102) can be cast as the product between the square matrix or tensor $[D_{ij}]$ and a column matrix $[A_j]$, that is,

$$[C_i] = [D_{ij}][A_j]. \tag{1.103}$$

A typical term of (1.103) reads

$$C_1 = D_{11}A_1 + D_{12}A_2 + D_{13}A_3. \tag{1.104}$$

Linear relations like (1.103) occur often in solid mechanics, crystal optics, and electromagnetic theory. Equation (1.102) is a more complete representation of these relations because the unit vectors are also included in the equation. We then speak of a scalar product between a vector and a dyadic instead of a product between a tensor and a column matrix.

By transforming A_j, D_{ij}, and $\hat{x}_i$ into the primed functions, we find

$$\mathbf{C} = \mathbf{C}' = \sum_{m,n} A'_n D'_{mn}\hat{x}'_m. \tag{1.105}$$

The anterior scalar product between $\mathbf{A}$ and $[\bar{\bar{D}}]^{\mathrm{T}}$, denoted by $\mathbf{T}$ for the time being, is given by

$$\mathbf{T} = \mathbf{A} \cdot [\bar{\bar{D}}]^{\mathrm{T}} = \mathbf{A} \cdot \sum_{i,j} D_{ji}\hat{x}_i\hat{x}_j \tag{1.106}$$

$$= \sum_{i,j} A_i D_{ji}\hat{x}_j = \sum_{i,j} A_j D_{ij}\hat{x}_i, \tag{1.107}$$

which is equal to $\mathbf{C}$ given by (1.102); hence we have a very useful identity:

$$\mathbf{A} \cdot [\bar{\bar{D}}]^{\mathrm{T}} = \bar{\bar{D}} \cdot \mathbf{A}. \tag{1.108}$$

Similarly, one finds

$$[\bar{\bar{D}}]^{\mathrm{T}} \cdot \mathbf{A} = \mathbf{A} \cdot \bar{\bar{D}}. \tag{1.109}$$

For a symmetric dyadic, denoted by $\bar{\bar{D}}_{\mathrm{s}}$,

$$[\bar{\bar{D}}_{\mathrm{s}}]^{\mathrm{T}} = \bar{\bar{D}}_{\mathrm{s}}. \tag{1.110}$$

Hence

$$\mathbf{A} \cdot \bar{\bar{D}}_{\mathrm{s}} = \bar{\bar{D}}_{\mathrm{s}} \cdot \mathbf{A}. \tag{1.111}$$

When $\bar{\bar{D}}_s$ is the idemfactor defined by (1.96), we have

$$\mathbf{A} \cdot \bar{\bar{I}} = \bar{\bar{I}} \cdot \mathbf{A} = \mathbf{A}. \tag{1.112}$$

The tensor of $\bar{\bar{I}}$ can be called a *unit tensor*, with the three diagonal terms equal to unity and the rest are null.

There are two vector products between $\mathbf{A}$ and $\bar{\bar{D}}$. These products are both dyadics. The anterior vector product is defined by

$$\begin{aligned}\bar{\bar{B}} &= \mathbf{A} \times \bar{\bar{D}} \\ &= \mathbf{A} \times \sum_{i,j} D_{ij}\hat{x}_i\hat{x}_j \\ &= \sum_{i,j,k} D_{ij}A_k(\hat{x}_k \times \hat{x}_i)\hat{x}_j,\end{aligned} \tag{1.113}$$

where $i, j, k = 1, 2, 3$ in cyclic order. The posterior vector product is defined by

$$\bar{\bar{C}} = \bar{\bar{D}} \times \mathbf{A} = \sum_{i,j,k} D_{ij}A_k\hat{x}_i(\hat{x}_j \times \hat{x}_k). \tag{1.114}$$

One important triple product involving three vectors is given by (1.23):

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}). \tag{1.115}$$

In dyadic analysis, we need a similar product, with one of the vectors changed to a dyadic. We can obtain such an identity by first changing (1.115) into the form

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}. \tag{1.116}$$

Now we let

$$\mathbf{C} = \bar{\bar{C}} \cdot \mathbf{F},$$

where $\mathbf{F}$ is an arbitrary vector function and $\bar{\bar{C}}$ is a dyadic. Then

$$\mathbf{A} \cdot (\mathbf{B} \times \bar{\bar{C}}) \cdot \mathbf{F} = -\mathbf{B} \cdot (\mathbf{A} \times \bar{\bar{C}}) \cdot \mathbf{F} = (\mathbf{A} \times \mathbf{B}) \cdot \bar{\bar{C}} \cdot \mathbf{F}. \tag{1.117}$$

Because this identity is valid for any arbitrary $\mathbf{F}$, we obtain

$$\mathbf{A} \cdot (\mathbf{B} \times \bar{\bar{C}}) = -\mathbf{B} \cdot (\mathbf{A} \times \bar{\bar{C}}) = (\mathbf{A} \times \mathbf{B}) \cdot \bar{\bar{C}}. \tag{1.118}$$

An alternative method of deriving (1.118) is to consider three sets of identities like (1.116) with three distinct $\mathbf{C}_j$, with $j = 1, 2, 3$. Then, by juxtaposing a unit vector $\hat{x}_j$ at the posterior position of each of these sets and summing the resultant equations, we again obtain (1.118) with

$$\bar{\bar{C}} = \sum_j \mathbf{C}_j\hat{x}_j.$$

Other dyadic identities can be derived in a similar manner. Many of them will be given in Chapter 7, which deals with dyadic analysis.

Finally, we want to introduce another class of dyadics in the form of

$$\begin{aligned}\bar{\bar{S}} &= \mathbf{MN} \\ &= \sum_i M_i \hat{x}_i \sum_j N_j \hat{x}_j \\ &= \sum_{i,j} M_i N_j \hat{x}_i \hat{x}_j;\end{aligned} \tag{1.119}$$

then,

$$S_{ij} = M_i N_j. \tag{1.120}$$

Because there are three components of **M** and three components of **N**, all together six functions, and they have generated nine dyadic components of $\bar{\bar{L}}$, three of the relations in (1.120) must be dependent. For example, we can write

$$\begin{aligned}S_{23} = M_2 N_3 &= (M_2 N_2) N_3 / N_2 \\ &= (M_2 N_2)(M_1 N_3)/(M_1 N_2) \\ &= S_{22} S_{13} / S_{12}.\end{aligned} \tag{1.121}$$

When the vectors in (1.120) are defined in two different coordinate systems unrelated to each other, we have a mixed dyadic. Let

$$\mathbf{M} = \mathbf{M}(x_1, x_2, x_3)$$

and

$$\mathbf{N}'' = \mathbf{N}''(x_1'', x_2'', x_3''),$$

where we use a double primed system to avoid a conflict of notation with (x_1', x_2', x_3'), which have been used to denote a system rotated with respect to (x_1, x_2, x_3). Here x_j'' with $j = 1, 2, 3$ are independent of x_i with $i = 1, 2, 3$. The mixed dyadic then has the form

$$\begin{aligned}\bar{\bar{T}} = \mathbf{MN}'' &= \sum_i M_i \hat{x}_i \sum_j N_j'' \hat{x}_j'' \\ &= \sum_{i,j} M_i N_j'' \hat{x}_i \hat{x}_j''.\end{aligned} \tag{1.122}$$

We can form an anterior scalar product of $\bar{\bar{T}}$ with a vector function **A** defined in the x_i system, but the posterior scalar product between **A** and $\bar{\bar{T}}$ is undefined or meaningless. Mixed dyadics can be defined in any two unrelated coordinate systems not necessarily rectangular, such as two spherical systems. These dyadics are frequently used in electromagnetic theory [3]. Many commonly used coordinate systems are introduced in the following chapter.