Appendix C

Vector formulas

General formulas

$$\nabla(\alpha f(\mathbf{r})) = \alpha \nabla f(\mathbf{r})$$

$$\nabla[f(\mathbf{r})g(\mathbf{r})] = g(\mathbf{r})\nabla f(\mathbf{r}) + f(\mathbf{r})\nabla g(\mathbf{r})$$

$$\nabla \cdot [\alpha \mathbf{f}(\mathbf{r})] = \alpha \nabla \cdot \mathbf{f}(\mathbf{r})$$

$$\nabla \cdot [f(\mathbf{r})\mathbf{g}(\mathbf{r})] = [\nabla f(\mathbf{r})] \cdot \mathbf{g}(\mathbf{r}) + f(\mathbf{r})[\nabla \cdot \mathbf{g}(\mathbf{r})]$$

$$\nabla \times [\alpha \mathbf{f}(\mathbf{r})] = \alpha \nabla \times \mathbf{f}(\mathbf{r})$$

$$\nabla \times [f(\mathbf{r})\mathbf{g}(\mathbf{r})] = [\nabla f(\mathbf{r})] \times \mathbf{g}(\mathbf{r}) + f(\mathbf{r})[\nabla \times \mathbf{g}(\mathbf{r})]$$

$$\nabla \cdot [\mathbf{f}(\mathbf{r}) \times \mathbf{g}(\mathbf{r})] = [\nabla \times \mathbf{f}(\mathbf{r})] \cdot \mathbf{g}(\mathbf{r}) - \mathbf{f}(\mathbf{r}) \cdot [\nabla \times \mathbf{g}(\mathbf{r})]$$

$$\nabla \times [\mathbf{f} \times \mathbf{g}] = \mathbf{f}[\nabla \cdot \mathbf{g}] - \mathbf{g}[\nabla \cdot \mathbf{f}] + [\mathbf{g} \cdot \nabla]\mathbf{f} - [\mathbf{f} \cdot \nabla]\mathbf{g}$$

$$\nabla \times \nabla f(\mathbf{r}) = 0$$

$$\nabla \cdot [\nabla \times \mathbf{f}(\mathbf{r})] = 0$$

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla \cdot \nabla \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f}$$

$$\int_V \nabla \cdot \mathbf{f} dV = \oint_S \mathbf{f} \cdot d\mathbf{S} \quad \text{(Gauss)}$$

$$\int_S \nabla \times \mathbf{f} \cdot d\mathbf{S} = \oint_C \mathbf{f} \cdot d\ell \quad \text{(Stokes)}$$

Cartesian coordinates $x,\ y,\ z$

$$\nabla f = \mathbf{u}_x \frac{\partial}{\partial x} f + \mathbf{u}_y \frac{\partial}{\partial y} f + \mathbf{u}_z \frac{\partial}{\partial z} f$$

$$\nabla \cdot \mathbf{f} = \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z$$

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical coordinates $\rho,\ \varphi,\ z$

$$\nabla f = \mathbf{u}_\rho \frac{\partial}{\partial \rho} f + \mathbf{u}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} f + \mathbf{u}_z \frac{\partial}{\partial z} f$$

$$\nabla \cdot \mathbf{f} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho f_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \varphi} f_\varphi + \frac{\partial}{\partial z} f_z$$

$$\nabla \times \mathbf{f} = \frac{1}{\rho} \begin{vmatrix} \mathbf{u}_\rho & \rho \mathbf{u}_\varphi & \mathbf{u}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ f_\rho & \rho f_\varphi & f_z \end{vmatrix}$$

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \boldsymbol{\rho} = \nabla(\mathbf{r} - \mathbf{u}_z z) = \overline{\overline{I}} - \mathbf{u}_z \mathbf{u}_z = \overline{\overline{I}}_t, \qquad \nabla \cdot \boldsymbol{\rho} = 2, \qquad \nabla \times \boldsymbol{\rho} = 0$$

$$\rho = |\boldsymbol{\rho}| = |\mathbf{r} - \mathbf{u}_z z|, \qquad \nabla \rho = \mathbf{u}_\rho$$

$$\nabla \nabla \rho = \nabla \mathbf{u}_\rho = \frac{1}{\rho}(\overline{\overline{I}} - \mathbf{u}_z \mathbf{u}_z - \mathbf{u}_\rho \mathbf{u}_\rho) = \frac{1}{\rho} \mathbf{u}_z \mathbf{u}_z {}^{\times}_{\times} \mathbf{u}_\rho \mathbf{u}_\rho = \frac{1}{\rho} \mathbf{u}_\varphi \mathbf{u}_\varphi$$

$$\nabla^2 \rho = \nabla \cdot \mathbf{u}_\rho = \frac{1}{\rho}, \qquad \nabla \times \mathbf{u}_\rho = 0$$

$$\nabla \varphi = \frac{1}{\rho} \mathbf{u}_\varphi, \qquad \nabla \mathbf{u}_\varphi = -\frac{1}{\rho} \mathbf{u}_\varphi \mathbf{u}_\rho, \qquad \nabla \cdot \mathbf{u}_\varphi = 0, \qquad \nabla \times \mathbf{u}_\varphi = \frac{1}{\rho} \mathbf{u}_z$$

$$g_2(\boldsymbol{\rho}) = -\frac{1}{2\pi} \ln(k|\boldsymbol{\rho}|), \qquad \nabla g_2(\boldsymbol{\rho}) = -\frac{\mathbf{u}_\rho}{2\pi \rho}$$

$$\nabla \nabla g_2(\boldsymbol{\rho}) = \mathrm{PV} \frac{1}{2\pi \rho^2} (\mathbf{u}_\rho \mathbf{u}_\rho - \mathbf{u}_\varphi \mathbf{u}_\varphi) - \frac{1}{2} \overline{\overline{I}}_t \delta(\boldsymbol{\rho})$$

$$\nabla^2 g_2(\boldsymbol{\rho}) = -\delta(\boldsymbol{\rho})$$

$$\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times} \nabla\nabla g_2(\boldsymbol{\rho}) = -\mathrm{PV}\frac{1}{2\pi\rho^2}(\mathbf{u}_\rho\mathbf{u}_\rho - \mathbf{u}_\varphi\mathbf{u}_\varphi) - \frac{1}{2}\overline{\overline{I}}_t\delta(\boldsymbol{\rho})$$

$$G_2(\boldsymbol{\rho}) = \frac{1}{4j}H_o^{(2)}(k|\boldsymbol{\rho}|), \qquad \nabla G_2(\boldsymbol{\rho}) = -\mathbf{u}_\rho\frac{k}{4j}H_1^{(2)}(k|\boldsymbol{\rho}|)$$

$$\nabla\nabla G_2(\boldsymbol{\rho}) = -\mathbf{u}_\rho\mathbf{u}_\rho k^2 G_2(k|\boldsymbol{\rho}|) + \mathrm{PV}(\mathbf{u}_\rho\mathbf{u}_\rho - \mathbf{u}_\varphi\mathbf{u}_\varphi)\frac{k}{4j\rho}H_1^{(2)}(k|\boldsymbol{\rho}|) - \frac{1}{2}\overline{\overline{I}}_t\delta(\boldsymbol{\rho})$$

$$\nabla^2 G_2(\boldsymbol{\rho}) = \overline{\overline{I}} : \nabla\nabla G_2(\boldsymbol{\rho}) = -k^2 G_2(\boldsymbol{\rho}) - \delta(\boldsymbol{\rho})$$

Spherical coordinates r, θ, φ

$$\nabla f = \mathbf{u}_r\frac{\partial}{\partial r}f + \mathbf{u}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}f + \mathbf{u}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}f$$

$$\nabla\cdot\mathbf{f} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}f_\varphi$$

$$\nabla\times\mathbf{f} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{u}_r & r\mathbf{u}_\theta & r\sin\theta\mathbf{u}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ f_r & rf_\theta & r\sin\theta f_\varphi \end{vmatrix}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

$$\nabla\mathbf{r} = \overline{\overline{I}}, \qquad \nabla\cdot\mathbf{r} = 3, \qquad \nabla\times\mathbf{r} = 0$$

$$\nabla r = \mathbf{u}_r, \qquad \nabla\mathbf{u}_r = \frac{1}{r}(\overline{\overline{I}} - \mathbf{u}_r\mathbf{u}_r), \qquad \nabla\cdot\mathbf{u}_r = \frac{2}{r}, \qquad \nabla\times\mathbf{u}_r = 0$$

$$\nabla(\mathbf{a}\times\mathbf{r}) = -\overline{\overline{I}}\times\mathbf{a} = -\mathbf{a}\times\overline{\overline{I}}$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{r}) = 0, \qquad \nabla\times(\mathbf{a}\times\mathbf{r}) = 2\mathbf{a}$$

$$G(\mathbf{r}) = \frac{e^{-jkr}}{4\pi r}, \qquad r = |\mathbf{r}|$$

$$\nabla G(\mathbf{r}) = -\frac{\mathbf{u}_r}{r}(1 + jkr)G(\mathbf{r})$$

$$\nabla\nabla G(\mathbf{r}) = -\mathbf{u}_r\mathbf{u}_r k^2 G(\mathbf{r}) - \mathrm{PV}\frac{1}{r^2}(1 + jkr)(\overline{\overline{I}} - 3\mathbf{u}_r\mathbf{u}_r)G(\mathbf{r}) - \frac{1}{3}\overline{\overline{I}}\delta(\mathbf{r})$$

$$\nabla^2 G(\mathbf{r}) = \overline{\overline{I}} : \nabla\nabla G(\mathbf{r}) = -k^2 G(\mathbf{r}) - \delta(\mathbf{r})$$