Appendix A

Problems

The following set of problems covers a selection of topics discussed in Chapters 1 – 6. The symbols used in the problems are defined in the corresponding Chapters.

Complex vectors

1.1 Show through the corresponding time-harmonic vector that the complex vector $\mathbf{a} = \mathbf{a}_1 + j\mathbf{a}_2$ with real vectors $\mathbf{a}_1$, $\mathbf{a}_2$ satisfying $\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$, is in axial form, i.e., the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are on the two symmetry axes of the ellipse.

1.2 Show that the transformation $\mathbf{a} \to e^{j\phi}\mathbf{a}$, with real ϕ, does not change the ellipse of the complex vector but moves the phase vector $\mathbf{A}(0)$ on the ellipse.

1.3 Show that the length of the polarization vector $\mathbf{p}(\mathbf{a})$ of a complex vector $\mathbf{a}$ has the following geometrical properties

(a) $|\mathbf{p}(\mathbf{a})| = \sin\psi$, where ψ is any angle of the equilateral quadrangle whose diagonals are the axes of the ellipse of $\mathbf{a}$,

(b) $|\mathbf{p}(\mathbf{a})| = 2A/\pi|\mathbf{a}|^2$, where A is the area of the ellipse.

1.4 Study the polarization of the following complex vectors $\mathbf{a}$ in terms of the polarization vector $\mathbf{p}(\mathbf{a})$:

(a) $\mathbf{a} = \mathbf{u}_x \cos\alpha + j\mathbf{u}_y \sin\alpha$, ($\alpha$ a real number)

(b) $\mathbf{a} = \mathbf{b} + j\mathbf{u} \times \mathbf{b}$, ($\mathbf{b}$ a complex vector, $\mathbf{u}$ a real unit vector satisfying $\mathbf{u} \cdot \mathbf{b} = 0$)

(c) $\mathbf{a} = \mathbf{b} \times \mathbf{b}^*$, ($\mathbf{b}$ a complex vector)

1.5 Show that $\mathbf{p}(\mathbf{a} \times \mathbf{p}(\mathbf{a})) = \mathbf{p}(\mathbf{a})$ when $\mathbf{a}$ is not a linearly polarized vector.

1.6 Show that any complex vector $\mathbf{a}$ can be written as the projection of a circularly polarized vector $\mathbf{b}$ on the plane of $\mathbf{a}$. Find the possible expressions for $\mathbf{b}$.

1.7 Show that any complex vector $\mathbf{a}$ can be written as $\mathbf{a} = \mathbf{b} + \mathbf{c}$, where $\mathbf{b}$ and $\mathbf{c}$ are circularly polarized and $|\mathbf{b}| = |\mathbf{c}|$.

1.8 Find the most general complex vector $\mathbf{b}$ satisfying $\mathbf{p}(\mathbf{b}) = \mathbf{p}(\mathbf{a})$ when $\mathbf{a}$ is a given complex vector.

1.9 Show that if $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} \cdot \mathbf{b}^* = 0$, one of the vectors $\mathbf{a}$, $\mathbf{b}$ must be linearly polarized or zero.

1.10 Show that the reciprocal basis of the reciprocal basis equals the original basis of complex vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.

1.11 Determine the reciprocal basis of $\mathbf{a}_1 = \mathbf{a}$, $\mathbf{a}_2 = \mathbf{a}^*$, $\mathbf{a}_3 = \mathbf{a} \times \mathbf{a}^*$ when $\mathbf{a}$ is circularly polarized.

1.12 Show by expanding in the base $\mathbf{a}$, $\mathbf{a}^*$, $\mathbf{a} \times \mathbf{a}^*$ that the solutions vectors $\mathbf{b}_\pm$ to the equations $\mathbf{a} \times \mathbf{b}_\pm = \pm j \mathbf{b}_\pm$ are of the form

$$\mathbf{b}_\pm = \alpha_\pm [\mathbf{a} \times \mathbf{a}^* \mp j \mathbf{a} \times (\mathbf{a} \times \mathbf{a}^*)]$$

when $\mathbf{a}$ is not a circularly polarized vector. The coefficients $\alpha_\pm$ may be arbitrary.

1.13 Study the relation between the real and imaginary parts of the complex vector $\mathbf{k}$ when it satisfies $\mathbf{k} \cdot \mathbf{k} = k_o^2$ with real k_o.

Dyadics

2.1 Prove the following identity:

$$\mathbf{a} \times \overline{\overline{I}} = \overline{\overline{I}} \times \mathbf{a}$$

2.2 Prove the following identity:

$$(\mathbf{a} \times \overline{\overline{I}}) : (\mathbf{b} \times \overline{\overline{I}}) = 2 \mathbf{a} \cdot \mathbf{b}$$

2.3 Prove the following identity:

$$(\mathbf{a} \times \overline{\overline{A}}) : (\overline{\overline{B}} \times \mathbf{b}) = \mathbf{a} \cdot (\overline{\overline{A}} \,{}^\times_\times\, \overline{\overline{B}}) \cdot \mathbf{b}$$

2.4 Prove the following identity:

$$(\mathbf{a} \times \overline{\overline{I}})^{\times}_{\times}(\mathbf{b} \times \overline{\overline{I}}) = \mathbf{ab} + \mathbf{ba}$$

2.5 Prove the following identity:

$$(\overline{\overline{A}} \cdot \mathbf{a}) \times (\overline{\overline{A}} \cdot \mathbf{b}) = \overline{\overline{A}}^{(2)} \cdot (\mathbf{a} \times \mathbf{b})$$

2.6 Show that $\det(\overline{\overline{A}}^{\times}_{\times}\overline{\overline{A}}) = 8(\det\overline{\overline{A}})^2$. This implies that $\overline{\overline{A}}^{\times}_{\times}\overline{\overline{A}}$ is complete only when $\overline{\overline{A}}$ is complete.

2.7 Expand the inverse of the dyadic

$$\overline{\overline{A}} = \alpha\overline{\overline{I}} + \mathbf{a} \times \overline{\overline{I}}.$$

Check that $\overline{\overline{A}} \cdot \overline{\overline{A}}^{-1} = \overline{\overline{I}}$ is really satisfied.

2.8 Study the solutions α, $\overline{\overline{A}}$ of the following dyadic equation:

$$\overline{\overline{A}}^{\times}_{\times}\overline{\overline{A}} = \alpha\overline{\overline{A}},$$

when $\overline{\overline{A}}$ is restricted to be a symmetric dyadic.

2.9 Solve the following dyadic equation for the dyadic $\overline{\overline{X}}$:

$$(\alpha\overline{\overline{I}} + \mathbf{a} \times \overline{\overline{I}})^{\times}_{\times}\overline{\overline{X}} = \mathbf{a} \times \overline{\overline{I}}.$$

2.10 Defining the uniaxial dyadic as

$$\overline{\overline{D}} = \alpha\overline{\overline{I}}_t + \beta\mathbf{uu},$$

write its Cayley-Hamilton equation and find the eigenvalues and eigenvectors.

2.11 Defining the gyrotropic dyadic as

$$\overline{\overline{G}}(\beta, R, \theta) = \beta\mathbf{uu} + Re^{\overline{\overline{J}}\theta}, \qquad \overline{\overline{J}} = \mathbf{u} \times \overline{\overline{I}},$$

where the dyadic exponential function is understood as

$$e^{\overline{\overline{J}}\theta} = \overline{\overline{I}}_t + \overline{\overline{J}}\theta + \frac{1}{2!}\overline{\overline{J}}^2\theta^2 + \frac{1}{3!}\overline{\overline{J}}^3\theta^3 + \ldots$$

$$= \overline{\overline{I}}_t \cos\theta + \overline{\overline{J}}\sin\theta, \quad \overline{\overline{I}}_t = \overline{\overline{I}} - \mathbf{uu},$$

derive its eigenvalues and eigenvectors.

2.12 Show that the two conditions

$$\overline{\overline{R}}^{-1} = \overline{\overline{R}}^{T}, \qquad \det\overline{\overline{R}} = 1$$

for a real dyadic $\overline{\overline{R}}$ are sufficient to guarantee that if $\mathbf{b} = \overline{\overline{R}} \cdot \mathbf{a}$, we have

$$|\mathbf{b}|^2 = |\mathbf{a}|^2 \quad \text{and} \quad |\mathbf{b} \times \mathbf{b}^*|^2 = |\mathbf{a} \times \mathbf{a}^*|^2.$$

These mean that in the transformation $\mathbf{a} \to \mathbf{b} = \overline{\overline{R}} \cdot \mathbf{a}$ the magnitude and polarization of the complex vector $\mathbf{a}$ do not change. Thus, the transformation only moves the ellipse to another position and can be interpreted as a rotation operation.

2.13 The gyrotropic dyadic can be defined by

$$\overline{\overline{G}}(\beta, R, \theta) = \beta\mathbf{uu} + Re^{\overline{\overline{J}}\theta}.$$

Determine its square root dyadic satisfying the condition

$$[\overline{\overline{G}}(\beta, R, \theta)]^{1/2} \cdot [\overline{\overline{G}}(\beta, R, \theta)]^{1/2} = \overline{\overline{G}}(\beta, R, \theta).$$

2.14 Show the following properties of the dyadic $\overline{\overline{A}}$:

(a) $\overline{\overline{A}} : \mathbf{aa} = 0$ for all vectors $\mathbf{a}$ implies $\overline{\overline{A}}$ antisymmetric.

(b) $\overline{\overline{A}} : (\mathbf{ab} - \mathbf{ba}) = 0$ for all vectors $\mathbf{a}$, $\mathbf{b}$ implies $\overline{\overline{A}}$ symmetric.

(c) $\overline{\overline{A}} : \mathbf{aa}^* = 0$ for all vectors $\mathbf{a}$ implies $\overline{\overline{A}} = 0$.

Field equations

3.1 Derive the Helmholtz dyadic operator $\overline{\overline{H}}_e(\nabla)$ for the bi-isotropic medium and show that it can be factorized in the form

$$\overline{\overline{H}}_e(\nabla) = \overline{\overline{H}}_1(\nabla) \cdot \overline{\overline{H}}_2(\nabla).$$

What are the operators $\overline{\overline{H}}_1(\nabla)$ and $\overline{\overline{H}}_2(\nabla)$?

3.2 An electromagnetic shield is comprised of three layers of media: two dielectric layers of permittivity ϵ_1 and thickness t_1 and, in the middle, a third magnetic layer with permeability μ_2 and thickness t_2. Determine the relation between these parameters so that there would be no reflection of a normally incident plane wave from the shield. The thicknesses are assumed to be very small and $\epsilon_1 t_1$, $\mu_2 t_2$ finite. Hint: consider the input impedance of the equivalent network.

3.3 In some frequency regions the bi-anisotropic medium can be approximated by a lossless and nondispersive medium whose medium parameter dyadics are independent of frequency. Consider the expression of the energy density

$$W = \frac{1}{4}(\mathbf{E}\ \mathbf{H}) \cdot \mathsf{M} \cdot \begin{pmatrix} \mathbf{E}^* \\ \mathbf{H}^* \end{pmatrix}, \quad \mathsf{M} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}.$$

and require that the energy density be positive ($W > 0$) for all possible fields **E**, **H**. Show that this leads to the condition that the following four dyadics connected to the material six-dyadic M must be positive definite:

$$\overline{\overline{\epsilon}}, \quad \overline{\overline{\mu}}, \quad \overline{\overline{\epsilon}} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}}, \quad \overline{\overline{\mu}} - \overline{\overline{\zeta}} \cdot \overline{\overline{\epsilon}}^{-1} \cdot \overline{\overline{\xi}}.$$

3.4 What is the condition for the parameters corresponding to that above if we require a sharper condition $W \geq W_o$, where

$$W_o = \frac{1}{4}(\epsilon_o|\mathbf{E}|^2 + \mu_o|\mathbf{H}|^2)$$

is the energy density in vacuum?

3.5 Consider the special case of a bi-isotropic medium of the previous problems 3.3 and 3.4. Derive the conditions of losslessness for the scalar medium parameters ϵ, μ, κ, χ without applying the result of the problem 3.3.

3.6 Find the conditions of losslessness for the impedance parameters Z_1 and Z_2 of a bi-isotropic impedance surface with the impedance dyadic

$$\overline{\overline{Z}}_s = Z_1\overline{\overline{I}} + Z_2\mathbf{n} \times \overline{\overline{I}}$$

when the boundary consition is

$$\mathbf{n} \times \mathbf{E} = -\overline{\overline{Z}}_s \cdot \mathbf{H}.$$

Field transformations

4.1 Study the special duality transformation

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_d = \mathsf{T}(\alpha) \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

defined by the transformation matrix

$$\mathsf{T}(\alpha) = \frac{1}{\sqrt{1-\sin 2\alpha}} \begin{pmatrix} 1 & \sqrt{2}\sin\alpha\eta_o \\ \frac{\sqrt{2}}{\eta_o}\cos\alpha & 1 \end{pmatrix}.$$

(a) Show that $\det\mathsf{T}(\alpha) = 1$.

(b) Show that $\mathsf{T}^{-1}(\alpha) = \mathsf{T}(\alpha+\pi)$.

(c) Derive the transformation rules for the medium parameters and show that the parameter $\overline{\overline{\xi}} - \overline{\overline{\zeta}}$ transforms to itself for all α.

(d) Show that a reciprocal isotropic medium with parameters ϵ and μ is in general transformed to a nonreciprocal bi-isotropic medium with $\xi_d, \zeta_d \neq 0$.

(e) Which α transforms $\overline{\overline{\mu}}$ to itself? What are the other transformed parameters?

(f) Study whether a given nonreciprocal bi-isotropic medium can always be transformed to a reciprocal isotropic medium with $\chi_d = (\xi_d + \zeta_d)/2\sqrt{\mu_o\epsilon_o} = 0$. Find the dependence of the angle α on the parameters μ, ϵ and χ in this case.

4.2 Applying a suitable affine transformation, solve the basic electrostatic problem in an anisotropic dielectric: point charge Q at the origin $\mathbf{r} = 0$ in a medium with the permittivity dyadic $\overline{\overline{\epsilon}}_r\epsilon_o$. The dyadic $\overline{\overline{\epsilon}}_r$ is assumed symmetric, real and positive definite.

In particular, solve the scalar potential $\phi(\mathbf{r})$ satisfying the Poisson equation

$$\nabla\cdot[\overline{\overline{\epsilon}}_r\cdot\nabla\phi(\mathbf{r})] = -\frac{Q}{\epsilon_o}\delta(\mathbf{r})$$

together with the electric field $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ and the flux density vector $\mathbf{D}(\mathbf{r}) = \overline{\overline{\epsilon}}\cdot\mathbf{E}(\mathbf{r})$.

The following identity may be of some help:

$$\delta(\overline{\overline{A}}\cdot\mathbf{r}) = \frac{\delta(\mathbf{r})}{|\det\overline{\overline{A}}|},$$

where $\overline{\overline{A}}$ is a real dyadic.

4.3 Find the image charge for the previous problem when the original charge lies at the point $\mathbf{r} = \mathbf{u}_z h$, $h > 0$, and the previous anisotropic medium is bounded by a perfectly conducting plane at $z = 0$.

Electromagnetic field solutions

5.1 Find the two-dimensional Green dyadic for the bi-isotropic medium. In particular, find the solution for

$$\overline{\overline{H}}(\nabla) \cdot \overline{\overline{G}}(\rho) = -\delta(\rho)\overline{\overline{I}},$$

with the Helmholtz operator defined by

$$\overline{\overline{H}}(\nabla) = -(\nabla \times \overline{\overline{I}} - j\omega\xi\overline{\overline{I}}) \cdot (\nabla \times \overline{\overline{I}} + j\omega\zeta\overline{\overline{I}}) + k^2\overline{\overline{I}}.$$

Apply the symbols

$$\xi = (\chi - j\kappa)\sqrt{\mu_o\epsilon_o}, \quad \zeta = (\chi + j\kappa)\sqrt{\mu_o\epsilon_o}$$

and

$$k_\pm = k(\cos\vartheta \pm \kappa_r), \quad \text{where} \quad \chi = n\sin\vartheta, \quad \kappa = n\kappa_r, \quad n = \sqrt{\mu_r\epsilon_r}.$$

Note that the dyadics are not two-dimensional, only the Green dyadic does not depend on z.

5.2 Find the delta singularity of the Green dyadic in the bi-isotropic medium by comparing its expression to that of the isotropic medium. Note that delta singularities arise from the scalar Green functions obeying $1/r$ law in the double differentiations.

5.3 Find the field in real space from a dipole in complex space. Assume a dipole with the current density

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}_y IL\delta(x)\delta(y)\delta(z - ja)$$

where $a \gg \lambda$ is real. Find the field (amplitude and polarization) close to the z axis for large $|z|$ values.

5.4 Show that any plane wave propagating in the general uniaxial anisotropic medium with the parameter dyadics

$$\overline{\overline{\epsilon}} = \epsilon_t\overline{\overline{I}}_t + \epsilon_v\mathbf{v}\mathbf{v}, \quad \overline{\overline{\mu}} = \mu_t\overline{\overline{I}}_t + \mu_v\mathbf{v}\mathbf{v},$$

is either TE or TM to $\mathbf{v}$, i.e., either satisfies $\mathbf{v} \cdot \mathbf{E} = 0$ or $\mathbf{v} \cdot \mathbf{H} = 0$, unless the material satisfies the condition $\mu_t\epsilon_v = \epsilon_t\mu_v$. Explain the special behavior occurring at this special material condition.

5.5 Study the plane-wave propagation in a bi-anisotropic medium with the parameter dyadics defined as

$$\overline{\overline{\epsilon}} = \epsilon\overline{\overline{I}}, \quad \overline{\overline{\mu}} = \mu\overline{\overline{I}}, \quad \overline{\overline{\xi}} = -\overline{\overline{\zeta}} = -j\overline{\overline{\kappa}}_r\sqrt{\mu\epsilon},$$

with

$$\overline{\overline{\kappa}}_r = \kappa_r(\overline{\overline{I}}_t - \mathbf{u}_z\mathbf{u}_z).$$

This kind of a medium can be fabricated by taking similar right-handed and left-handed helices and mixing them in a base medium so that N left-handed helices are parallel to the z axis and $2N$ right handed helices are isotropically orthogonal to the z axis.

In particular, find the wave-number surfaces of the two plane waves. Study the optical axis directions in which the wave numbers are the same. What happens when the parameter κ approaches the value $n = \sqrt{\mu_r\epsilon_r}$? ($n$ is the refraction factor of a plane wave in isotropic medium with $\overline{\overline{\kappa}}_r = 0$.) Also determine the eigenpolarizations for propagation along the z axis. Because of axial symmetry, write $\mathbf{k} = \mathbf{u}kN(\theta)$, with $k = \omega\sqrt{\mu\epsilon}$, $\mathbf{u}$ is a unit vector which makes the angle θ with the z direction. $N(\theta)$ is the refraction factor of the plane wave propagating at the angle θ in the present chiral medium.

Source equivalence

6.1 Find the equivalent electric source corresponding to a magnetic dipole, $\mathbf{J}_m = \mathbf{u}_z I_m L\delta(\mathbf{r})$, in an isotropic chiral medium with parameters ϵ, μ and $\kappa = \kappa_r k/k_o$.

6.2 Show that a radial source of the form $\mathbf{J}(\mathbf{r}) = \mathbf{u}_r f(r)$ does not radiate outside the support of the function $f(r)$.

6.3 Find the equivalent magnetic source of an electric surface-current source

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}_\varphi J_s\delta(\rho - a)U(h^2 - z^2).$$

6.4 Find the equivalent magnetic volume current of the coaxial current

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}_z I(\frac{\delta(\rho - a)}{2\pi a} - \frac{\delta(\rho - b)}{2\pi b}).$$

Assume that the equivalent magnetic current is in the volume between the surface currents and that it is of the form $\mathbf{J}_m(\mathbf{r}) = \mathbf{u}_\varphi J_m(\rho)$.

6.5 Show that the approximation of a current source $\mathbf{J}(\mathbf{r})$ by a dipole of moment $\mathbf{P}$ at a position $\mathbf{r} = \mathbf{a}$ can also be done by minimizing the error function $\mathbf{R}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) - \mathbf{P}\delta(\mathbf{r} - \mathbf{a})$ in the following sense: require $\int \mathbf{R}(\mathbf{r})dV = 0$ and minimize the norm of $\int \mathbf{r}\mathbf{R}(\mathbf{r})dV$.

6.6 Formulate the Huygens principle to electrostatic problems with charges and magnetic currents as sources. The space is assumed homogeneous and isotropic. Start from the equations

$$\nabla \times \mathbf{E} = \mathbf{J}_m, \qquad \nabla \cdot \mathbf{D} = \varrho, \qquad \mathbf{D} = \epsilon \mathbf{E}.$$

Write the expression for the electric field in a volume V surrounded by the surface S with sources truncated in V and Huygens sources on S. Study the possibility to replace the magnetic Huygens current by equivalent electric charge on the surface S.