4. Asymptotic Evaluation of Integrals

4.1 GENERAL CONSIDERATIONS

4.1a Transformation to a Canonical Form

Infinite integrals

Radiation and diffraction fields in open regions (with infinite cross section) are usually expressed by integral representations that cannot be evaluated in closed form. In many applications, however, the integrands contain a large parameter, Ω, in terms of which the integrals may be approximated. While such an evaluation can be treated for rather general functional dependences of the integrand on Ω, it will suffice within the present context to consider integrals of the following type:

$$I(\Omega) = \int_{\bar{P}} f(z) e^{\Omega q(z)}\, dz, \tag{1}$$

where f and q are analytic functions of the complex variable z along a path of integration $\bar{P}$ with endpoints at infinity, and where the large parameter Ω is assumed to be positive.†

Suppose that at the point z_s on $\bar{P}$, $\operatorname{Re} q(z)$ has a maximum value so that $\operatorname{Re} q(z) < \operatorname{Re} q(z_s)$ on the remainder of the path. Since Ω is very large, it follows that $A = |\exp[\Omega q(z)]|$ likewise has a maximum at z_s and decreases very rapidly away from z_s. It is then suggestive to approximate $I(\Omega)$ only by the path contribution from the vicinity of z_s, since the contribution from the remainder of the path will be exponentially small in comparison. If $f(z)$ is regular and slowly varying in the vicinity of z_s, it may be approximated there by $f(z_s)$

†If $\Omega = |\Omega| \exp(i \arg \Omega)$ is complex, the phase term is included in the definition of $q(z)$. Alternatively, it may be more convenient to obtain an asymptotic evaluation of $I(\Omega)$ for real values of Ω and then continue Ω analytically into a range of permitted complex values.

and taken outside the integrand in Eq. (1), thereby leaving in the integrand only the exponential. Approximate integration of the latter can be effected by expanding $q(z)$ in a power series about z_s and retaining only the first few terms; in a more accurate procedure described below, the integral is compared with a "canonical" one having similar properties and a simpler structure. This, in rough outline, constitutes the basis for an asymptotic approximation of $I(\Omega)$ for large values of Ω. In general, A will not have the above-described behavior along the given contour $\bar{P}$, but along some other path $\bar{P}_z$. One then attempts to deform $\bar{P}$ into $\bar{P}_z$, proper account being taken of any interfering singularities of $f(z)$ (such as poles or branch points) in the complex z plane.

It is shown in Sec. 4.1b that the point(s) z_s, where there occur maximum contributions to the integral, are "saddle" or "stationary points" of the function $q(z)$, defined by the vanishing of one or more of the derivatives of $q(z)$. The desired path $\bar{P}_z$ is one on which $\operatorname{Im} q(z)$ remains constant. Since only the vicinity of the various saddle points traversed by the path $\bar{P}_z$ is relevant, it is unnecessary to deal completely with the generally complicated function $q(z)$; instead, as mentioned above, a finite number of terms in the power-series expansion of $q(z)$ about z_s characterizes the asymptotic evaluation for large Ω. This feature may be exploited in a rigorous manner by transforming the given integral into a "canonical" form, wherein the function $q(z)$ is replaced by another function, a polynomial that describes in the simplest fashion the relevant saddle-point arrangement at z_s. The transformation will be phrased in terms of a new variable s and the polynomial $\tau(s)$:

$$\tau(s) = q(z), \tag{2}$$

the point z_s in the complex z plane being chosen to correspond to $s = 0$ in the complex s plane. Thus, Eq. (1) leads to the transformed integral

$$I(\Omega) = \int_{P'} G(s) e^{\Omega\tau(s)}\, ds \tag{3}$$

where

$$G(s) = f(z)\frac{dz}{ds} \qquad \text{and} \qquad \frac{dz}{ds} = \frac{\tau'(s)}{q'(z)}, \tag{3a}$$

with the prime denoting the derivative with respect to the argument. P' represents the mapping onto the s plane of the path $\bar{P}$ in the z plane.

By considerations analogous to those mentioned above, $I(\Omega)$ in the s plane is approximated most simply on a contour P, which passes in the vicinity of the origin and along which the magnitude of $\exp[\Omega\tau(s)]$ decreases rapidly away from $s = 0$. The desired contour P in the s plane, along which $\operatorname{Im}\tau(s) =$ constant and $\operatorname{Re}\tau(s) < \operatorname{Re}\tau(0)$ away from a region about $s = 0$, constitutes a mapping into the s plane of the path $\bar{P}_z$ in the z plane. Generally, P' will not be identical with P, and in any ensuing deformation of P' into P, the presence of singularities of the integrand in Eq. (3) may have to be taken into account. A typical case is shown in Fig. 4.1.1, where the desired path P and the given path P' are deformable one into the other at $|s| = \infty$. In view of the pole

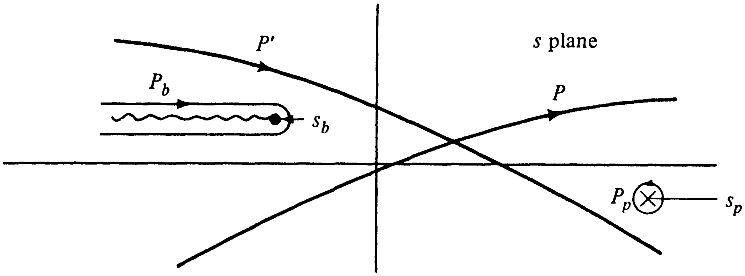


FIG. 4.1.1 Deformation of P' into P in the s plane.

singularity at s_p and the branch-point singularity at s_b, the integrals over the contours P' and P are related by Cauchy's theorem as follows:

$$\int_{P'} = \int_P + \int_{P_p} + \int_{P_b}, \tag{4}$$

with P_p and P_b denoting contours surrounding the pole and branch-cut singularities, respectively.

Assume now that the integral in Eq. (3) is to be evaluated over the desired contour P and that $G(s)$ is regular and slowly varying in the vicinity of $s = 0$. For large values of Ω the dominant contribution to the integral in Eq. (3) arises from the vicinity of the origin since the exponential term decreases rapidly away from the region $s \approx 0$. One may therefore write†

$$I(\Omega) = \int_P G(s) e^{\Omega\tau(s)}\, ds \sim G(0) \int_P e^{\Omega\tau(s)}\, ds, \qquad \Omega \to \infty, \tag{5}$$

where the symbol $\sim$ means "asymptotically represented by" and $G(0)$ is the value of the regular function $G(s)$ at $s = 0$. The last integral in Eq. (5) represents a "canonical" integral that provides a first-order approximation to the unknown integral $I(\Omega)$ of Eq. (1). Asymptotic approximations are discussed further in Sec. 4.2b.

The above remarks serve to highlight the motivation for a proper choice of the transformation (2). First, Re $\tau(s)$ along P should decrease most rapidly away from $s = 0$ so that the major contribution to the integral arises from the vicinity of the origin in the s plane, and Re $\tau(0)$ should be as small as possible; the latter condition facilitates the asymptotic evaluation. As shown in Sec. 4.1b, these requirements imply that the stationary or saddle points of $\tau(s)$ [the zeros of $\tau'(s)$] are situated near $s = 0$.‡ Viewed in the z plane, the path $\bar{P}_z$ (corresponding to P in the s plane) passes near one or more of the pertinent saddle points z_s of $q(z)$ [zeros of $q'(z)$], and Re $q(z)$ decreases along $\bar{P}_z$ away from the neighborhood of z_s. Thus, the transformation (2) should be chosen so that the

†This simple asymptotic formula is valid provided that Ω is the only relevant parameter. If $I(\Omega)$ depends on other parameters, uniform asymptotic approximations with respect to these are generally somewhat more involved (see Secs. 4.4–4.6).

‡If $\tau'(s)$ has an Mth-order zero at s_s, $\tau(s)$ is said to have a saddle point of order M at s_s.

neighborhood of the pertinent saddle points z_s in the z plane is mapped into the neighborhood of $s = 0$ in the s plane. Second, the mapping derivative dz/ds in Eq. (3a) must be finite near $s = 0$ in order to assure the assumed regularity of $G(s)$ near $s = 0$. Therefore, $\tau(s)$ must be selected so that at the points s_s the derivative $\tau'(s)$ possesses zeros of the same order as those of $q'(z)$ at z_s, where the points s_s in the s plane correspond to z_s in the z plane. The simplest $\tau(s)$ meeting these requirements will yield the simplest comparison integral on the right-hand side of Eq. (5). If $f(z)$ has singularities near z_s, $G(s)$ has singularities near $s = 0$; these must be isolated in their simplest form and require consideration of a new class of comparison integrals. A number of important canonical (comparison) integrals constructed in this manner may be expressed in terms of known functions. Included among these, and employed in subsequent discussion, are the gamma function, the error function or Fresnel integral, the Airy function, and the parabolic cylinder function.

We shall investigate, in detail, infinite integrals with the following configurations of saddle points and singularities:

1. $q'(z)$ has a simple (or multiple) zero at z_s and no other zero near z_s; $f(z)$ is regular near z_s (ordinary saddle-point procedure) (Secs. 4.2 and 4.3).

2. $q'(z)$ has simple zeros at $z_s = z_1$ and z_2, where z_1 lies arbitrarily near z_2; $f(z)$ is regular near z_1 and z_2 (double-saddle-point procedure) (Sec. 4.5a).

3. $q'(z)$ has equally spaced, collinear simple zeros at $z_s = z_1, z_2, z_3$ (i.e., $z_1 - z_2 = z_2 - z_3$) which may be arbitrarily near one another; $f(z)$ is regular near $z_{1,2,3}$ (triple-saddle-point procedure) (Sec. 4.5b).

4. $q'(z)$ has a simple zero at z_s and no other zero near z_s; $f(z)$ has a simple (or multiple) pole near z_s (saddle-point integration in the vicinity of a pole) (Secs. 4.4a and 4.4b).

5. $q'(z)$ has a simple zero at z_s and no other zeros near z_s; $f(z)$ has an algebraic branch-point singularity at or near z_s (saddle-point integration in the vicinity of a branch point) (Sec. 4.4c).

6. $q'(z)$ has a simple (or multiple) zero at z_s and no other zeros near z_s; $f(z)$ has an algebraic branch-point singularity at z_s. [By following the procedure in Sec. 4.3, this problem is reduced to the evaluation of integrals as in Eq. (4.3.5), except that n is not an integer. The asymptotic result therefore involves the gamma function and will not be discussed in detail.]

Emphasis is placed on uniform representations that reduce correctly to simpler ones under appropriate limiting conditions. For example, when $z_1 \not\approx z_2$, case 2 must reduce to the simple case 1; similarly, cases 4, 5, and 6 must reduce to case 1 when the singularity moves away from the saddle point. Corresponding integrals with finite endpoints are discussed separately below.

To illustrate the choice of the transformation in Eq. (2), and the principles set out above, we consider further the various cases.

Case 1. When $q'(z)$ has an Mth-order zero at z_s and no other zeros nearby, one may select the polynomial $\tau(s)$ as

$$q(z) = \tau(s) = q(z_s) - s^{M+1}, \tag{6}$$

whence $s = 0$ corresponds to $z = z_s$, and $dz/ds = \tau'(s)/q'(z)$ is finite at $s = 0$. The corresponding integral in Eq. (5) is expressible in terms of the gamma function (details are given in Sec. 4.2). The desired path P is one along which $s^{M+1} > 0$, or less stringently, $\operatorname{Re} s^{M+1} > 0$.

Case 2. when $q'(z)$ has two simple neighboring zeros $z_{1,2}$, one may select

$$q(z) = \tau(s) = a_0 + \sigma s - \frac{s^3}{3}, \tag{7}$$

where a_0 and σ are constants. Since $\tau'(s)$ has two simple zeros at $s_{1,2} = \pm\sqrt{\sigma}$, we let $s_1 = \sqrt{\sigma}$ correspond to z_1 and $s_2 = -\sqrt{\sigma}$ correspond to z_2 to assure the regularity of dz/ds at $s_{1,2}$. From the ensuing relations

$$q(z_1) = \tau(\sqrt{\sigma}) = a_0 + \tfrac{2}{3}\sigma^{3/2}, \tag{7a}$$

$$q(z_2) = \tau(-\sqrt{\sigma}) = a_0 - \tfrac{2}{3}\sigma^{3/2}, \tag{7b}$$

one obtains the following expressions for a_0 and σ in terms of $q(z_{1,2})$:

$$a_0 = \tau(0) = \tfrac{1}{2}\,[q(z_1) + q(z_2)], \tag{8a}$$

$$\tfrac{2}{3}\,\sigma^{3/2} = \tfrac{1}{2}\,[q(z_1) - q(z_2)]. \tag{8b}$$

The integral in Eq. (5) with $\tau(s)$ given in Eq. (7) is expressible in terms of the Airy function (details are given in Sec. 4.5a). It should be emphasized that the case of two neighboring simple zeros of $q'(z)$ must be treated separately from case 1 only when these zeros are arbitrarily close (i.e., when $\sigma \to 0$). If σ remains finite, the formal asymptotic evaluation of the integral in Eq. (1) as $\Omega \to \infty$ can be carried out in terms of the separate contributions from each relevant zero treated in isolation. Most of the problems encountered in practice belong to this class and can be treated by the techniques in category 1. Just how close the zeros of $\tau'(s)$ can approach each other before they must be considered jointly will be illustrated in the detailed discussion carried out in Sec. 4.5a. Analogous remarks apply to the remaining cases wherein saddle points are not isolated.

Case 3. It suffices in the case of three collinear, equally spaced zeros to select

$$q(z) = \tau(s) = a_0 - (a + s^2)^2, \tag{9}$$

since $\tau'(s)$ then has zeros at $s = 0, \pm i\sqrt{a}$ which correspond to z_2 and to $z_{1,3}$, respectively. Evidently,

$$a_0 = q(z_1) = q(z_3), \qquad a^2 = q(z_1) - q(z_2), \tag{9a}$$

and the integral resulting upon substitution of Eq. (9) into Eq. (5) may be expressed in terms of the parabolic cylinder function. Details of this evaluation are given in Sec. 4.5b.

Case 4. If $f(z)$ in Eq. (1) has a pole singularity near a simple zero z_s of $q'(z)$ [i.e., $G(s)$ in Eq. (3) has a pole near $s = 0$], procedure 1 can no longer be applied directly, since $G(s)$ is now rapidly varying near $s = 0$. A modified approach is sketched here for the case where $G(s)$ has a simple pole at $s = b$, with $b \to 0$. Suppose that $G(s)(s - b) \to a$ as $s \to b$, where a is a constant. Then one represents $G(s)$ as

$$G(s) = \frac{a}{s - b} + T(s), \tag{10}$$

so $T(s)$ is regular at both $s = b$ and $s = 0$, and is slowly varying near $s = 0$. The asymptotic approximation of $I(\Omega)$ in Eq. (3) as $\Omega \to \infty$ is now given by [see Eq. (5)]

$$I(\Omega) \sim T(0) \int_P e^{\Omega\tau(s)}\, ds + a \int_P \frac{e^{\Omega\tau(s)}}{s - b}\, ds, \tag{11}$$

with $\tau(s)$ given in Eq. (6) with $M = 1$. The first integral in Eq. (11) is identical with that encountered in case 1; the second integral containing the effect of the pole singularity can be evaluated in terms of the error function or Fresnel integral (see Sec. 4.4a).

Case 5. If $f(z)$ has a first-order branch-point singularity near a first-order saddle point z_s, the resulting canonical integral [see Eq. (4.4.28)] may be transformed into an integral as for case 3, so the parabolic cylinder function describes this case as well.

In the preceding discussion we have been concerned only with the first-order asymptotic evaluation [see Eq. (5)] for the integral $I(\Omega)$ in Eq. (1) as $\Omega \to \infty$. It is possible, however, to obtain complete asymptotic expansions for $I(\Omega)$ involving successively decreasing terms as $\Omega \to \infty$ so that higher-order approximations can also be calculated. Complete expansions are presented for several of the cases described above (for a general treatment, see Reference 1).

Integrals with finite endpoints

If the interval of integration in Eq. (1) ranges between the finite limits z_α and z_β, the asymptotic evaluation of the integral can be carried out analogously to that over the infinite path $\bar{P}$. The given finite path $\bar{P}_{\alpha\beta}$ with endpoints at z_α and z_β can be deformed to pass in the vicinity of one (or more) pertinent saddle point(s) z_s of $q(z)$. If $\operatorname{Re} q(z) < \operatorname{Re} q(z_s)$ along the portions of the path away from the saddle point(s), the dominant contribution to the integral as $\Omega \to \infty$ again arises from the vicinity of the point(s) z_s, and its asymptotic evaluation is identical with that for the infinite integral.

The finite integral

$$I_{\alpha\beta}(\Omega) = \int_{z_\alpha}^{z_\beta} f(z) e^{\Omega q(z)}\, dz \tag{12}$$

transforms into the s plane via Eq. (2) as

$$I_{\alpha\beta}(\Omega) = \int_{P_{\alpha\beta}} G(s) e^{\Omega\tau(s)}\, ds, \qquad \int_{P_{\alpha\beta}} = \int_{s_\alpha}^{s_\beta}, \tag{13}$$

where s_α and s_β correspond to z_α and z_β, respectively. The integral in Eq. (13) can be written as a sum of the following three integrals:

$$I_{\alpha\beta}(\Omega) = \left(\int_P + \int_{P_\alpha} - \int_{P_\beta}\right) G(s)e^{\Omega\tau(s)}\,ds = I(\Omega) + I_\alpha(\Omega) + I_\beta(\Omega), \tag{14}$$

where the paths P, P_α, and P_β are those shown in Fig. 4.1.2. P is the previously encountered infinite path along which $\tau(s)$ decreases most rapidly away from the vicinity of the pertinent saddle point(s) near the origin, and P_α and P_β are paths connecting the endpoints of P at $\pm\infty$ with the given endpoints s_α and

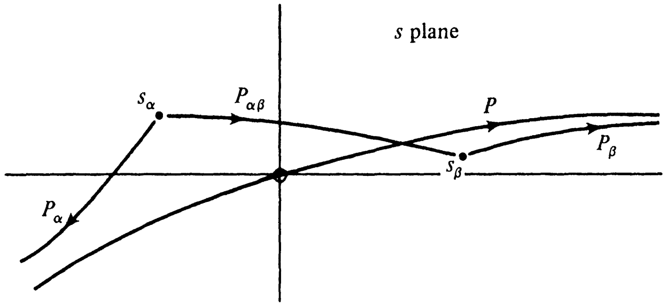


FIG. 4.1.2 Contours for evaluation of a finite integral.

s_β, respectively. It is assumed herein that $\operatorname{Re}\tau(s) < \operatorname{Re}\tau(0)$ along all path segments leading away from the origin $s = 0$, and that $\operatorname{Re}\tau(s) < \operatorname{Re}\tau(s_{\alpha,\beta})$ along the contours $P_{\alpha,\beta}$, respectively. It should also be emphasized that the presence of any singularities of $G(s)$ near $s = 0$, such as poles or branch points, must be taken into account in the deformation of $P_{\alpha\beta}$ into the three paths in Eq. (14) [see Eq. (4) and Fig. 4.1.1].

The asymptotic evaluation of the integral $I(\Omega)$ in Eq. (14) along the infinite path P proceeds as before. To estimate the contribution from the remaining integrals as $\Omega \to \infty$, we make use of the fact that $\tau'(s) \neq 0$ on P_α and P_β (i.e., there are no saddle points on the contours P_α and P_β). By an integration by parts one obtains

$$I_\alpha(\Omega) = \int_{P_\alpha} G(s)e^{\Omega\tau(s)}\,ds = \frac{1}{\Omega}\int_{P_\alpha} \frac{G(s)}{\tau'(s)}\frac{d}{ds}e^{\Omega\tau(s)}\,ds, \tag{15}$$

$$= -\frac{1}{\Omega}\frac{G(s_\alpha)e^{\Omega\tau(s_\alpha)}}{\tau'(s_\alpha)} - \frac{1}{\Omega}\int_{P_\alpha}\frac{d}{ds}\left[\frac{G(s)}{\tau'(s)}\right]e^{\Omega\tau(s)}\,ds, \tag{16}$$

where it has been recognized that the exponential term vanishes at infinity. The same procedure can be repeated for the integral in Eq. (16). In fact, the integration-by-parts technique is frequently an effective means for asymptotic evaluation of integrals whose integrands do not contain a saddle point on or near the contour of integration. One notes from Eq. (16) that the magnitude of $I_\alpha(\Omega)$ as $\Omega \to \infty$ is determined by the factor $(1/\Omega)e^{\Omega\tau(s_\alpha)}$. Similarly, the magnitude of the integral $I_\beta(\Omega)$ over the path P_β is determined by the factor

$(1/\Omega)e^{\Omega\tau(s_\beta)}$. On the other hand, the contribution to $I(\Omega)$ over the path P traversing a saddle point has an exponential dependence $e^{\Omega\tau(0)}$ [see Eqs. (5), (6), and (7)]. If $\operatorname{Re}\tau(s_{\alpha,\beta}) < \operatorname{Re}\tau(0)$, one notes that $|I_{\alpha,\beta}|/|I|$ is proportional to $e^{\Omega\operatorname{Re}[\tau(s_{\alpha,\beta})-\tau(0)]}$ (i.e., the ratio is exponentially small as $\Omega \to \infty$). Thus, just as for the infinite integral, the predominant contribution to the finite integral in Eq. (13) arises from the neighborhood of the saddle point(s) of $\tau(s)$, and we may write

$$I_{\alpha\beta}(\Omega) \sim I(\Omega) \qquad \text{as } \Omega \to \infty. \tag{17}$$

If the exponential term in the integrand of Eq. (13) has the same magnitude at the endpoint s_α as at the saddle point $s = 0$ [i.e., $\operatorname{Re}\tau(s_\alpha) = \operatorname{Re}\tau(0)$], the endpoint contribution in Eq. (14) can still be estimated as in Eq. (16) and its magnitude is proportional to $(1/\Omega)e^{\Omega\operatorname{Re}\tau(0)}$. In this instance one requires for the above comparison a better estimate of the asymptotic behavior of $I(\Omega)$. It is shown in Eq. (4.3.7) that for an Mth-order saddle point of $q(z)$ at $z = z_s$ [see Eq. (6)], $I(\Omega)$ behaves like $\Omega^{-1/(M+1)}e^{\Omega\tau(0)}$. Thus, the major contribution to $I_{\alpha\beta}(\Omega)$ as $\Omega \to \infty$ still arises from the vicinity of the saddle point(s), although the endpoint contribution is now smaller in magnitude only by an algebraic factor of the form $\Omega^{-\gamma}$, where γ is a positive number less than 1 whose exact value depends on the particular form of $\tau(s)$.

The applicability of the integration-by-parts technique fails when the endpoint approaches the saddle point, since $\tau'(s_\alpha) \to 0$ in Eq. (16). Under these circumstances, the saddle-point and endpoint contributions can no longer be treated separately and the asymptotic evaluation involves new canonical integrals. Two specific cases are considered: a single first-order saddle point (Sec. 4.6a), and two adjacent first-order saddle points (Sec. 4.6b) near an endpoint of the integration interval. The former leads to a canonical integral

$$\int_{s_\alpha}^{``\infty"} e^{-\Omega s^2}\, ds,$$

which is expressible in terms of the Fresnel integral, whereas the latter requires knowledge of the "incomplete" Airy function

$$\int_{s_\alpha}^{``\infty"} e^{\tau s - s^3/3}\, ds.$$

4.1b Saddle Points and Paths of Constant Level and Constant Phase[2-4]

Saddle points

The various classes of integrals of the type shown in Eq. (1) can be distinguished by the form of the argument function $q(z)$. A pictorial characterization is effected by a "level map" of the behavior of $q(z)$ over the complex z plane. The resulting picture of the $q(z)$ terrain with its mountains, valleys, and interconnecting passes provides an appropriate physical basis for distinguishing and selecting the integration paths $\bar{P}$ of Sec. 4.1a. Since the stationary points z_s of

$q(z)$ distinguish the passes, it is appropriate to investigate first the behavior of the analytic function $q(z)$ in the neighborhood of such a stationary point. Let us write

$$q(z) = u(x, y) + iv(x, y), \qquad z = x + iy, \tag{18}$$

where u, v, x, and y are real. A stationary point z_s of $q(z)$ is determined by

$$\frac{dq}{dz} = 0 \qquad \text{at } z = z_s; \tag{19}$$

in view of the analytic behavior of q about z_s, it is thereby implied that the functions u and v are also stationary at (x_s, y_s),

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \qquad \text{at } x = x_s, \quad y = y_s. \tag{20}$$

Although Eqs. (20) are satisfied, the surfaces $u(x, y) =$ constant and $v(x, y) =$ constant do not have absolute maximum or minimum values at (x_s, y_s). This is seen readily from the Cauchy–Riemann equations, satisfied by u and v,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{21}$$

from which it follows that for a first-order saddle point $(d^2q/dz_s^2 \neq 0)$,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}. \tag{22}$$

Thus, if at (x_s, y_s) the curvature of the surface $u(x, y) =$ constant, or $v(x, y) =$ constant, is positive along the x direction, it is negative along the (perpendicular) y direction. The stationary points z_s are therefore "saddle points," as noted in Fig. 4.1.3, and locate pass regions in the $q(z)$ terrain. Depending on the choice of path through z_s, the magnitude of $u(x, y)$ and therefore of $\exp |\Omega q(z)|$ may increase, decrease, or remain constant along the path.

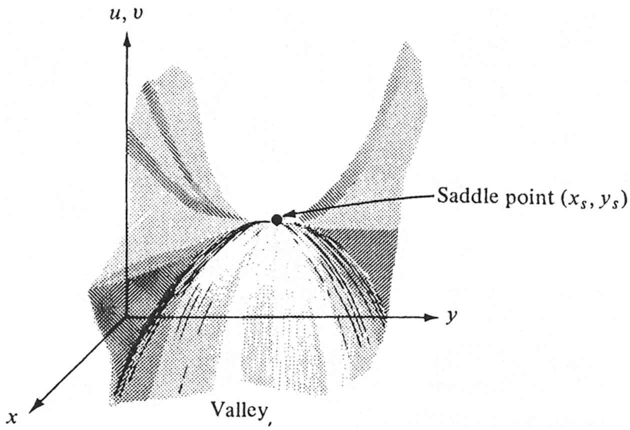


FIG. 4.1.3 Relief map of the functions u or v in the vicinity of a first-order saddle point.

Paths of constant level and constant phase

We now seek a criterion that determines the selection of the "steepest paths" through a saddle point (i.e., those paths along which the magnitude of $\exp[\Omega q(z)]$ changes most rapidly). In view of Eq. (18) these will be the paths along which $u(x, y)$ changes most rapidly. If, as shown in Fig. 4.1.4, ds denotes

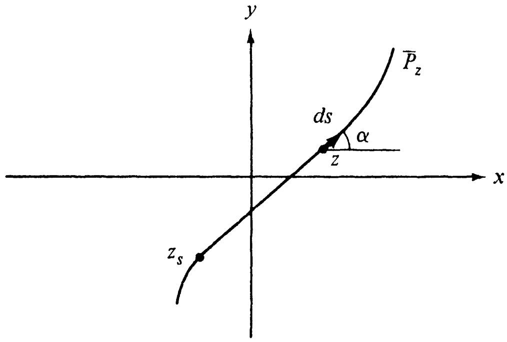


FIG. 4.1.4 Steepest-descent path in the z plane.

an element of length along a path of integration $\bar{P}_z$ through the saddle point z_s, the rate of change of u along $\bar{P}_z$ is given by

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha, \tag{23}$$

where α is the angle between the element ds and the x axis. The values of α for which du/ds is a maximum are defined by

$$\frac{\partial^2 u}{\partial\alpha\,\partial s} = 0 = -\frac{\partial u}{\partial x}\sin\alpha + \frac{\partial u}{\partial y}\cos\alpha, \tag{24a}$$

whence via the Cauchy–Riemann equations (21),

$$0 = -\frac{\partial v}{\partial y}\frac{dy}{ds} - \frac{\partial v}{\partial x}\frac{dx}{ds} = -\frac{dv}{ds}. \tag{24b}$$

Thus, $v =$ constant along the path on which u changes most rapidly, so the steepest path is a constant-phase path. Similarly, one can show from an examination of the maximum values of dv/ds that the magnitude of the exponential (i.e., the "level") is constant along the paths of most rapid phase variation. Although the path in Fig. 4.1.4 is shown passing through a saddle point, the above conclusions apply as well to constant-level and constant-phase paths through any given point z.

To examine the disposition of the constant-phase and constant-level paths in the neighborhood of a saddle point z_s, we expand $q(z)$ in a power series about the point z_s:

$$q(z) = q(z_s) + \frac{q''(z_s)}{2!}(z - z_s)^2 + \cdots, \tag{25}$$

where the double prime denotes the second derivative with respect to the argument. Then, for the case of a first-order saddle point,

$$e^{\Omega q(z)} \approx e^{\Omega q(z_s)} e^{(1/2)\Omega q''(z_s)(z-z_s)^2}, \tag{26a}$$

or

$$e^{\Omega q(z)} \approx e^{\Omega q(z_s)} e^{|(1/2)\Omega q''(z_s)(z-z_s)^2|(\cos 2\psi + i \sin 2\psi)}, \tag{26b}$$

where

$$\psi = \arg(z - z_s) + \tfrac{1}{2} \arg q''(z_s). \tag{26c}$$

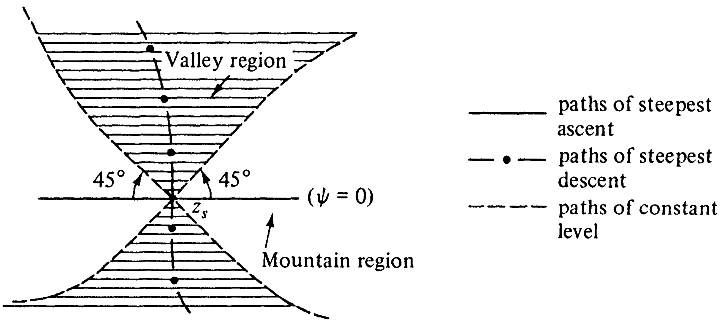
Since $\arg q''(z_s)$ is constant, ψ changes only with $\arg(z - z_s)$. One notes the following behavior of $\exp[\Omega q(z)]$ along the various paths $\psi =$ constant originating at z_s:

$\psi = \pm\pi/4, \pm 3\pi/4$ (*paths of constant level*): Phase of $e^{\Omega q(z)}$ varies most rapidly, $|e^{\Omega q(z)}| =$ constant.

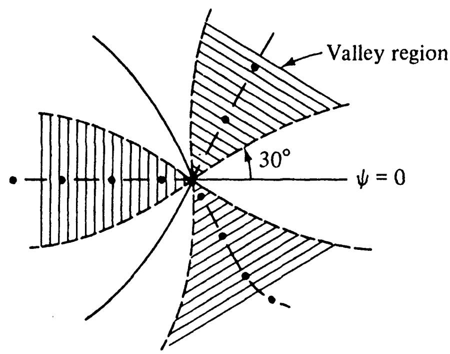
$\psi = 0, \pi$ (*paths of steepest ascent*): $|e^{\Omega q(z)}|$ increases most rapidly, phase of $e^{\Omega q(z)} =$ constant.

$\psi = \pm\pi/2$ (*paths of steepest descent*): $|e^{\Omega q(z)}|$ decreases most rapidly, phase of $e^{\Omega q(z)} =$ constant.

The above relations are illustrated in Fig. 4.1.5(a); the regions wherein the magnitude of $\exp[\Omega q(z)]$ increases or decreases characterize the mountain and valley regions, respectively.



(a) Saddle point of order 1



(b) Saddle point of order 2

FIG. 4.1.5 Behavior in the vicinity of a saddle point.

For a second-order saddle point, both $q'(z_s) = 0$ and $q''(z_s) = 0$. In this instance the expansion of $q(z)$ near the saddle point has the form

$$q(z) = q(z_s) + \frac{q^{(3)}(z_s)}{3!}(z - z_s)^3 + \cdots, \tag{27}$$

so

$$e^{\Omega q(z)} \approx e^{\Omega q(z_s)} \exp\left[\left|\frac{1}{3!}q^{(3)}(z_s)(z - z_s)^3\right|(\cos 3\psi + i \sin 3\psi)\right], \tag{27a}$$

where

$$\psi = \arg(z - z_s) + \tfrac{1}{3}\arg q^{(3)}(z_s). \tag{27b}$$

There exist now three mountain regions and three valley regions in the vicinity of the saddle point, as shown in Fig. 4.1.5(b). The angle between any two adjacent lines at the saddle point is 30°. A contour plot in the vicinity of an mth-order saddle point is constructed analogously and exhibits $(m + 1)$ mountain and valley regions.

If the given path of integration can be deformed into paths of steepest descent through one or more saddle points, then since the magnitude of the exponential term decreases most rapidly away from the saddle points, an approximate evaluation of the integral in Eq. (1) for large values of Ω can be phrased in terms of contributions only from the vicinity of the saddle points. On the other hand, if a constant-level path is chosen, an approximate evaluation for large Ω can also be carried out in terms of contributions only from the vicinity of the saddle points since the phase is stationary there and the exponential term oscillates so rapidly over the remainder of the path that the corresponding integral contributes negligibly to the overall result ("stationary-phase" procedure; see Sec. 4.2c). Since different paths of integration are involved in the two procedures, the asymptotic evaluation of the integral in Eq. (1) along a steepest-descent path need not yield the same result as that along a path of constant level. The two results will be identical only if the constant-level path can be continuously deformed into the steepest-descent path; this is possible if the two paths have the same termination (for example, at "infinity") and if $f(z)$ has no singularities in the region between the paths.

For rather general forms of the function $q(z)$, the determination of the complete steepest-descent paths [i.e., those paths along which $\operatorname{Im} q(z) =$ constant] may be quite complicated. In this instance, one may proceed less stringently by utilizing steepest-descent paths only in the immediate vicinity of the saddle points, where their progress is easily ascertained (see Fig. 4.1.5). The requirement on the remaining path segments L is merely that they proceed along a level lower than that at the saddle points. If a pertinent saddle point is at the level $q(z_s)$ and if $\operatorname{Re} q(z) \leq \operatorname{Re} q(z_1)$ along L, where $\operatorname{Re} q(z_1) < \operatorname{Re} q(z_s)$, then the relative error incurred in the asymptotic approximation by neglecting the contribution from L is $O\{\exp[\Omega q(z_1) - \Omega q(z_s)]\}$ (see Sec. 4.1a).† The difference

†The notation $g(\Omega) = O[h(\Omega)]$ as $\Omega \to \infty$ implies that $[g(\Omega)/h(\Omega)]$ remains bounded as $\Omega \to \infty$. The notation $g(\Omega) = o[h(\Omega)]$ as $\Omega \to \infty$ implies that $[g(\Omega)/h(\Omega)] \to 0$ as $\Omega \to \infty$.

between the asymptotic evaluation of the integral in the present case and the result obtained with the complete steepest-descent path SDP lies in the exponential error estimate, based here on the level at z_1, whereas for the complete SDP it involves the distance to the nearest singularity (see Sec. 4.2b). The results are therefore asymptotically equivalent if exponentially small terms are neglected. It must be emphasized that the retention of exponentially small contributions (e.g., from another saddle point, or from a singularity, situated at a lower level than the dominant saddle point) may be justified only if its magnitude exceeds that of the error inherent in the SDP approximation.

The simplified procedure described above has been employed in various problems discussed in this volume (see Figs. 7.5.9 and 8.3.4).

4.2 ISOLATED FIRST-ORDER SADDLE POINTS

4.2a First-order Approximation

If in the integrand of

$$I(\Omega) = \int_{\mathrm{SDP}} f(z) e^{\Omega q(z)}\, dz, \tag{1}$$

the function $f(z)$ has no singularities near an isolated first-order saddle point z_s of $q(z)$, where $q'(z_s) = 0$, $q''(z_s) \neq 0$, the asymptotic approximation of $I(\Omega)$ is given by [see Eq. (17) below for a complete asymptotic expansion]

$$I(\Omega) \sim \sqrt{\frac{-2\pi}{\Omega q''(z_s)}}\, f(z_s) e^{\Omega q(z_s)}, \qquad \Omega \to \infty. \tag{1a}$$

One must choose $\arg(\sqrt{\ }) = \varphi \equiv \arg(dz)_{z_s}$, with dz denoting an element at z_s along the steepest-descent path SDP [see Fig. 4.2.1 and Eq. (4.1.26)]. When

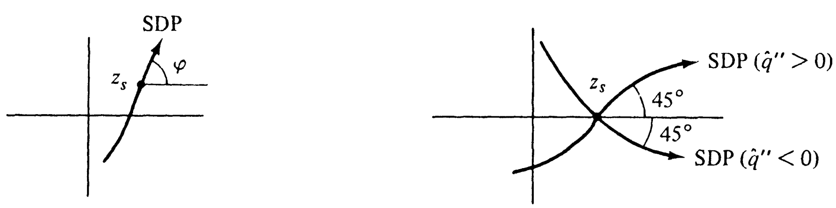


FIG. 4.2.1 Integration paths in the z plane.

$q(z) = i\hat{q}(z)$, with $\hat{q}$ denoting a real function of z, and z_s is real, Eq. (1a) may be written as [see Eq. (20a) below]

$$I(\Omega) = \int_{\mathrm{SDP}} f(z) e^{i\Omega \hat{q}(z)}\, dz \sim \sqrt{\frac{2\pi}{\Omega |\hat{q}''(z_s)|}}\, f(z_s) e^{i\Omega \hat{q}(z_s) \pm i\pi/4}, \qquad \hat{q}''(z_s) \gtrless 0, \tag{1b}$$

provided that Re (dz) increases along the SDP near z_s (see Fig. 4.2.1).

Analytical details

Since the isolated saddle point at z_s is of the first order, the pertinent change of variable from z to s is that given in Eq. (4.1.6), with $M = 1$:

$$q(z) = \tau(s) = q(z_s) - s^2. \tag{2}$$

The steepest-descent path P in the s plane, along which $\operatorname{Im} \tau(s) = $ constant, is clearly the real s axis. Thus, the integral in Eq. (4.1.5) becomes

$$I(\Omega) = e^{\Omega q(z_s)} \int_{-\infty}^{\infty} G(s) e^{-\Omega s^2}\, ds, \tag{3}$$

with $G(s)$ given via Eq. (4.1.3a) as

$$G(s) = f(z)\frac{dz}{ds}, \qquad \frac{dz}{ds} = \frac{-2s}{q'(z)}. \tag{3a}$$

Since $G(s)$ is assumed regular near $s = 0$, it can be expanded into a power series

$$G(s) = G(0) + G'(0)s + G''(0)\frac{s^2}{2!} + \cdots + G^{(n)}(0)\frac{s^n}{n!} + \cdots, \tag{4}$$

which converges uniformly inside a circle with finite radius r centered at $s = 0$, r being the distance to the nearest singularity of $G(s)$. Upon applying L'Hôpital's rule to the indeterminate form for dz/ds in Eq. (3a) when $s = 0$ (i.e., $z = z_s$), one evaluates the first coefficient of the expansion as

$$G(0) = f(z_s)\left(\frac{dz}{ds}\right)_{s=0}, \qquad \left(\frac{dz}{ds}\right)_{s=0} = \sqrt{\frac{-2}{q''(z_s)}}, \tag{5}$$

where $q''(z_s) \neq 0$ at the first-order saddle point. Since ds is positive along the path of integration and, in particular, at $s = 0$, $\arg(dz/ds)$ at $s = 0$ must be chosen equal to $\arg(dz)$ at z_s along the steepest descent path (see Fig. 4.2.1). This requirement specifies the square root function in Eq. (5). If $G(s)$ is approximated by $G(0)$ only (see Sec. 4.2b for a discussion of implications thereof), one obtains as the first-order asymptotic approximation for $I(\Omega)$ in Eq. (3):

$$I(\Omega) \sim G(0) e^{\Omega q(z_s)} \int_{-\infty}^{+\infty} e^{-\Omega s^2}\, ds \tag{6}$$

or, by Eq. (5), noting that the integral in Eq. (6) equals $\sqrt{\pi/\Omega}$:

$$I(\Omega) \sim \sqrt{\frac{-2\pi}{\Omega q''(z_s)}}\, f(z_s) e^{\Omega q(z_s)}, \qquad \Omega \to \infty. \tag{7}$$

These results continue to be applicable to integrals with finite endpoints if the conditions noted in Sec. 4.1a are satisfied.

Examples. To illustrate the use of Eq. (7), we treat two simple examples. First, consider the gamma function $\Gamma(\Omega + 1)$ defined by the integral

$$\Gamma(\Omega + 1) = \int_0^{\infty} e^{-x} x^{\Omega}\, dx = \Omega^{\Omega+1} \int_0^{\infty} e^{-\Omega z} z^{\Omega}\, dz = \Omega^{\Omega+1} \int_0^{\infty} e^{\Omega(\ln z - z)}\, dz, \tag{8}$$

whence $f(z) = 1$, $q(z) = \ln z - z$, $z_s = 1$, $q''(z_s) = -1$, and $\operatorname{Re} q(z) < -1$ for $z \neq 1$. Since the steepest-descent path in the z plane extends along the real z axis and the element of integration dz is positive along the path, one notes that $(dz/ds)_{s=0}$ is likewise positive. Thus, from Eq. (7),

$$\Gamma(\Omega + 1) \sim \Omega^{\Omega+1} \sqrt{\frac{2\pi}{\Omega}}\, e^{-\Omega}, \qquad \Omega \to \infty. \tag{9}$$

Since $q(0) < q(z_s)$, the endpoint contribution is negligible (see Sec. 4.1a).

Next, consider the modified Hankel function defined by the integral

$$K_0(\Omega) = \int_0^\infty e^{-\Omega \cosh z}\, dz, \qquad \Omega > 0. \tag{10}$$

Here, $f(z) = 1$, $q(z) = -\cosh z$, $z_s = 0$, $q''(z_s) = -1$. Again, Eq. (7) can be used directly except that we must include a factor $\frac{1}{2}$ since the stationary point is located at the endpoint $z = 0$ and hence the integral in Eq. (3) now runs only over a semiinfinite interval. Thus,

$$K_0(\Omega) \sim \sqrt{\frac{\pi}{2\Omega}}\, e^{-\Omega}, \qquad \Omega \to \infty. \tag{11}$$

4.2b Complete Asymptotic Expansion

A complete asymptotic expansion† for $I(\Omega)$ in Eq. (3) is obtained upon substituting for $G(s)$ the power-series expansion in Eq. (4) and integrating term by term:

$$I(\Omega) \sim e^{\Omega q(z_s)} \sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} I_n(\Omega). \tag{12}$$

The integral $I_n(\Omega)$ is evaluated in terms of the gamma function $\Gamma(z)$, defined in Eq. (8), as

$$I_n(\Omega) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} s^n e^{-\Omega s^2}\, ds = \frac{\Gamma[(1+n)/2]}{\Omega^{(1+n)/2}}, & n \text{ even}, \quad (13a) \\ 0, \quad n \text{ odd}, & (13b) \end{cases}$$

where Eq. (13b) is a consequence of the symmetrical integration interval and the fact that the integrand is an odd function of s. The values of $\Gamma(n + \frac{1}{2})$, $n = 0, 1, 2, \cdots$ are readily inferred from the recursion formula

$$\Gamma(z + 1) = z\Gamma(z), \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{14}$$

Alternatively, one may express $I_n(\Omega)$ as

---

†A function $I(\Omega)$ is said to have an asymptotic expansion

$$I(\Omega) \sim \sum_{n=0}^{\infty} a_n f_n(\Omega) \qquad \text{as } \Omega \to \infty$$

if, for any N and for $\arg \Omega$ in a given interval,

$$\frac{I(\Omega) - I_N(\Omega)}{f_N(\Omega)} \to 0 \qquad \text{as } \Omega \to \infty,$$

where $I_N(\Omega) = \sum_{n=0}^{N} a_n f_n(\Omega)$ [i.e., $f_{n+1}(\Omega)/f_n(\Omega) \to 0$ as $\Omega \to \infty$].[2-5]

$$I_n(\Omega) = (-1)^{n/2} \frac{d^{n/2}}{d\Omega^{n/2}} \int_{-\infty}^{\infty} e^{-\Omega s^2} \, ds \tag{15a}$$

$$= \left(-\frac{d}{d\Omega}\right)^{n/2} \sqrt{\frac{\pi}{\Omega}}, \qquad n \text{ even}. \tag{15b}$$

From Eqs. (15b) and (13a), one infers the recursion relation between I_n and I_{n+2}:

$$I_{n+2}(\Omega) = \left(-\frac{d}{d\Omega}\right) I_n(\Omega), \qquad n = 0, 2, 4, \ldots. \tag{16}$$

Thus, the complete asymptotic expansion for $I(\Omega)$ as $\Omega \to \infty$ is given by

$$I(\Omega) \sim \frac{e^{\Omega q(z_s)}}{\sqrt{\Omega}} \sum_{n=0}^{\infty} \frac{G^{(2n)}(0)}{(2n)!} \frac{\Gamma(n+\frac{1}{2})}{\Omega^n}, \tag{17}$$

or, alternatively,

$$I(\Omega) \sim e^{\Omega q(z_s)} \sum_{n=0}^{\infty} \frac{G^{(2n)}(0)}{(2n)!} \left(\sqrt{-\frac{d}{d\Omega}}\right)^{2n} \sqrt{\frac{\pi}{\Omega}}. \tag{18a}$$

Equation (18a) can be written in a convenient operator notation as

$$I(\Omega) \sim e^{\Omega q(z_s)} G_e\left(\sqrt{-\frac{d}{d\Omega}}\right) \sqrt{\frac{\pi}{\Omega}}, \tag{18b}$$

where the even function $G_e(x)$ is expressed in terms of a power series about $x = 0$ as follows:

$$G_e(x) = \sum_{n=0}^{\infty} \frac{G_e^{(2n)}(0)}{(2n)!} x^{2n}. \tag{18c}$$

That Eqs. (17) or (18a) indeed constitute the asymptotic expansion of $I(\Omega)$ as $\Omega \to \infty$ follows from the recognition that the ratio between successive terms of the series [i.e., between the $(N+1)$th and Nth terms], approaches zero as $\Omega \to \infty$, for any N (see footnote on p. 384)].

The term-by-term integration in Eq. (12) is not rigorously justifiable since the radius of convergence r of the power-series expansion is generally finite, so the series representation for $G(s)$ cannot be employed over the infinite range in s. The error incurred by this procedure may be estimated on dividing the integration interval into three parts. In the first, with $|s| = r_0 < r$, Eq. (4) is applicable while in the other two, with $-\infty < s < -r_0$ and $r_0 < s < \infty$, the function $G(s)$ is retained intact. The saddle-point contribution from the first interval is given by Eq. (12) or (17); the endpoint contribution is $O[\exp(-\Omega r_0^2)]$ [see Eq. (4.1.16)]. The remaining integrals from $|s| = r_0$ to ∞ are of the same exponential order of magnitude since the integrand decays along the entire path [$G(s)$ in Eq. (3) is dominated by $\exp(-\Omega s^2)$]. This exponentially small error, less significant than any of the algebraically small terms in the expansion of Eq. (17), assures the "asymptotic" validity of Eqs. (12) or (17) as $\Omega \to \infty$, provided that r_0 is finite [i.e., the saddle point is separated from singularities of $G(s)$].

One notes that the lowest-order approximation to $I(\Omega)$, which arises from the $n = 0$ term in Eqs. (17) or (18a), has been given in Eq. (7). For the evaluation of the higher-order terms, one requires a knowledge of the higher-order derivatives of $G(s) = f(z)dz/ds$ evaluated at $s = 0$. Formally, the derivatives $d^n z/ds^n$ can be obtained by successive differentiation of Eq. (3a) and evaluation of the resulting indeterminate forms at $s = 0$. An alternative procedure is to expand $q(z)$ in Eq. (2) in a power series about $z = z_s$ and thereby find s as a function of $z - z_s$. To obtain the required behavior of $z - z_s$ as a function of s, this power series must be inverted. One of the several possible techniques for the inversion of power series is presented in Appendix A. An explicit expression for $G^{(2)}(0)$ is given in Eq. (A6).

An important consideration in the use of asymptotic series is the error made on stopping the series after N terms. It may be shown[6] that if $\varphi(\zeta)$ is analytic on the real axis and, nearest the origin, possesses a singularity at $\zeta_1 = r \exp(i\alpha)$, $\alpha \neq 0$, where r is the modulus of ζ_1 and α its phase, then the integral

$$\hat{I} = \int_0^\infty e^{-\Omega\zeta} \zeta^\mu \varphi(\zeta)\, d\zeta \tag{19a}$$

has the exact representation

$$\hat{I} = \sum_{n=0}^{N-1} u_n + R_N, \qquad u_n = \frac{\varphi^{(n)}(0)\Gamma(n+\mu+1)}{\Omega^{n+\mu+1} n!}, \tag{19b}$$

where the u_n result from termwise integration of the power-series expansion (to N terms) of $\varphi(\zeta)$. The remainder R_N may be shown to be given approximately by $u_N(1 - e^{-i\alpha})^{-1}$, so the best approximation to $\hat{I}$ is obtained by stopping the series at the smallest term. The integrals considered in Eq. (3), or in Eq. (4.3.5a) for the more general case of a saddle point of order M, are readily transformed into $\hat{I}$ in Eq. (19a). In addition, there exist exponentially small error terms, as noted above.

Although it has been assumed throughout that Ω is real, it is evident that $I_n(\Omega)$ in Eq. (13a) can be continued analytically into the range $|\arg \Omega| < \pi/2$ since the integral converges in the extended range of Ω for which $\mathrm{Re}\,(\Omega s^2) > 0$. If Ω is complex, this process of analytic continuation is frequently more convenient than the inclusion of the factor $\exp(i \arg \Omega)$ in $q(z)$. Thus, in the range $|\arg \Omega| < \pi/2$, the asymptotic expansions in Eqs. (17) or (18a) are also valid for complex Ω, uniformly in $\arg \Omega$.

4.2c First-order, "Stationary-Phase" Evaluation of Finite Integrals

It was pointed out in Sec. 4.1a that the first-order asymptotic formula in Eq. (7), as well as the complete asymptotic expansion in Eq. (17), is valid for finite integrals, provided that $\mathrm{Re}\, q(z) < \mathrm{Re}\, q(z_s)$ on the portions of the path away from the saddle point z_s and that the integrand has no singularities near z_s. If the endpoint of a finite integral coincides with a saddle point z_s, then in the s plane one obtains integrals as in Eq. (13a) except that one of the limits of integration in zero. Thus, the contribution from such a saddle point is one half that given in Eq. (17). If $\operatorname{Re} q(z) = \operatorname{Re} q(z_s)$ along the contour, we are dealing with a constant-level path along which the phase of the exponential term $\exp[\Omega q(z)]$ varies most rapidly. As mentioned in Sec. 4.1b, a first-order asymptotic evaluation can then be carried out by a stationary-phase argument and yields the same result as the first-order steepest-descent evaluation, provided that $f(z)$ has no relevant singularities.

To highlight these remarks, let us introduce into the integral in Eq. (13a) (with $n = 0$) the change of variable $\bar{s} = s \exp(-i\pi/4)$. The resulting contour of integration in the $\bar{s}$ plane proceeds along a $-45°$ line which can be deformed into the real $\bar{s}$ axis. Thus, one obtains an integral as in Eq. (13a) except that Ω is replaced by $i\Omega$ so that the real axis in the $\bar{s}$ plane is a constant-level path. The desired result follows from Eq. (7) upon continuing the real variable Ω to imaginary values. In particular, if $q(x)$ is a real function of the real variable x, one obtains the "stationary-phase" formula [omitting the caret used in Eq. (1b)]

$$I(\Omega) = \int_{x_a}^{x_b} f(x)e^{i\Omega q(x)}\,dx \sim I_s(\Omega)U[(x_s - x_a)(x_b - x_s)] + I_e(\Omega) + O\left(\frac{1}{\Omega^{3/2}}\right), \qquad \Omega \to \infty, \tag{20}$$

where $\Omega > 0$, and $U(\alpha) = 1$ or 0 for $\alpha > 0$ and $\alpha < 0$, respectively. I_s is the lowest-order contribution from the stationary-phase point, namely,

$$I_s(\Omega) = \sqrt{\frac{2\pi}{\Omega|q''(x_s)|}}\,f(x_s)e^{i\Omega q(x_s)\pm i\pi/4}, \qquad q''(x_s) \gtrless 0, \tag{20a}$$

and $I_e(\Omega)$ is the lowest-order contribution from the endpoints [see Eq. (4.1.16)],

$$I_e(\Omega) = \frac{1}{i\Omega}\left[\frac{f(x_b)}{q'(x_b)}\,e^{i\Omega q(x_b)} - \frac{f(x_a)}{q'(x_a)}\,e^{i\Omega q(x_a)}\right]. \tag{20b}$$

The stationary point x_s [at which $q'(x_s) = 0$] is assumed to lie on the interval $x_a < x_s < x_b$; if the interval contains several stationary points, $I_s(\Omega)$ is a sum comprising terms representative of each x_s. When the saddle point coincides with either of the endpoints x_a or x_b, the corresponding endpoint contribution in Eq. (20b) is omitted and one takes $\frac{1}{2}$ times the stationary-point contribution in Eq. (20a). When the interval $x_a < x < x_b$ does not contain a stationary point, the Heaviside function vanishes and the integral $I(\Omega)$ is approximated by $I_e(\Omega)$ only. When an endpoint moves to infinity, the corresponding contribution is omitted. Equation (20) fails when the saddle point x_s approaches one of the endpoints; for example, when x_s moves continuously toward and across x_b, the $I_s(\Omega)$ term is discontinuous since the saddle point disappears from the integration interval, and the $I_e(\Omega)$ term diverges, since $q'(x_b) \to 0$ as $x_b \to x_s$. In this instance, one requires a more careful asymptotic evaluation that involves the error function or the Fresnel integral (see Sec. 4.6a).

Example

Use of Eq. (20) is illustrated by the asymptotic evaluation of the following integral representation for the Bessel function $J_n(\Omega)$:

$$J_n(\Omega) = \frac{e^{-in\pi/2}}{\pi} \int_0^{\pi} e^{i\Omega \cos x} \cos nx \, dx, \qquad n = 0, 1, 2, \cdots. \tag{21}$$

In this case, $f(x) = \cos nx$, $q(x) = \cos x$, $x_s = m\pi$, $m = 0, \pm 1, \pm 2, \cdots$. The interval of integration contains the two saddle points at $x_1 = 0$, $x_2 = \pi$, with $q''(x_1) = -1$, $q''(x_2) = 1$. Since the saddle points are situated at the endpoints of the interval of integration, a factor $\frac{1}{2}$ must be included in Eq. (20). Thus, as $\Omega \to \infty$,

$$J_n(\Omega) \sim \frac{e^{-in\pi/2}}{\pi} \frac{1}{2} \sqrt{\frac{2\pi}{\Omega}} [e^{i\Omega - i\pi/4} + e^{in\pi} e^{-i\Omega + i\pi/4}], \tag{22a}$$

which simplifies to

$$J_n(\Omega) \sim \sqrt{\frac{2}{\pi\Omega}} \cos\left[\Omega - \frac{n\pi}{2} - \frac{\pi}{4}\right], \qquad \Omega \to \infty. \tag{22b}$$

4.2d Steepest-Descent Evaluation of a Typical Diffraction Integral

Consider the integral

$$I_1(\Omega, \alpha, \beta) = \int_{\bar{P}} \frac{e^{i\Omega \cos(z-\alpha)}}{z - \beta} dz, \qquad 0 \leq \alpha < \frac{\pi}{2}, \tag{23}$$

taken over the path $\bar{P}$ shown in Fig. 4.2.2. This integral arises in several of the diffraction problems to be studied in the following chapters. Since

$$\operatorname{Im} \cos(z - \alpha) = -\sin(x - \alpha) \sinh y, \tag{24}$$

and Ω is assumed positive, the exponential term decays in the regions $y > 0$, $-\pi < x - \alpha < 0$, and $y < 0$, $0 < x - \alpha < \pi$, shown shaded in Fig. 4.2.2. The integrand contains a simple pole at $z = \beta$, where β is arbitrary. Upon

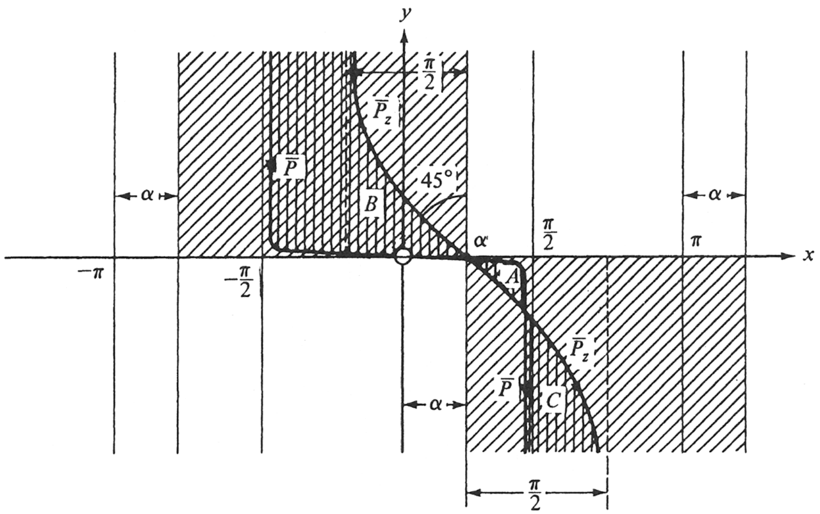


FIG. 4.2.2 Contours of integration in the z plane ($z = x + iy$).

comparison with Eq. (4.1.1) one notes that

$$q(z) = i \cos(z - \alpha), \qquad f(z) = \frac{1}{z - \beta}. \tag{25}$$

The pertinent saddle point z_s in the z plane is located at

$$\frac{d}{dz} q(z) = -i \sin(z - \alpha) = 0 \qquad \text{or at } z_s = \alpha. \tag{26a}$$

The steepest-descent path $\bar{P}_z$ through the saddle point is defined by

$$\operatorname{Im} q(z) = \operatorname{Im} q(z_s) = i. \tag{26b}$$

The change of variable to the s plane is then effected via Eq. (2) by

$$i \cos(z - \alpha) = i - s^2, \tag{27a}$$

or

$$s = \pm\sqrt{2}\, e^{i\pi/4} \sin\frac{z - \alpha}{2}. \tag{27b}$$

Although we could proceed immediately to the s plane via the transformation in Eq. (27b), we study first the nature of the steepest-descent path $\bar{P}_z$ in the z plane. Along $\bar{P}_z$, s is real, and the slope of $\bar{P}_z$ at $z = z_s = \alpha$ is inferred from Eq. (5) or Eq. (27b) to be

$$\left.\frac{dz}{ds}\right|_{s=0} = \pm\sqrt{2}\, e^{-i\pi/4}. \tag{28}$$

Thus, $\bar{P}_z$ makes an angle of $(-45°)$ with the x axis at $z = \alpha$. The direction of integration along $\bar{P}$ in Fig. 4.2.2 suggests a choice of the indicated direction along $\bar{P}_z$, so that $\arg(z - z_s) = -\pi/4$ along $\bar{P}_z$ near $z = z_s$, and the positive sign is chosen in Eqs. (28) and (27b). The complete steepest-descent path can be plotted readily from Eq. (26b) and requires that $\operatorname{Im}[i \cos(z - \alpha)] = i$ along $\bar{P}_z$, or

$$x - \alpha = \cos^{-1}(\operatorname{sech} y) \qquad \text{along } \bar{P}_z. \tag{29}$$

Equation (29) yields the x coordinate of any point on $\bar{P}_z$ for an assumed value of y. The resulting path is shown in Fig. 4.2.2 and is asymptotic to the lines $x = \alpha \pm \pi/2$.

The transformation of the path $\bar{P}$ from the z plane to the s plane is accomplished via Eq. (27b), where it is recalled that the positive sign is chosen. The resulting contour P' is shown in Fig. 4.2.3 and has as its asymptotes in the lower half of the s plane the lines $\arg s = -\alpha/2$ and $\arg s = \pi + \alpha/2$. Corresponding regions in the z and s planes where the exponential term decays in magnitude are shown shaded with slanted lines. Since $ds/dz = 0$ at $z - \alpha = \pm\pi$, the transformation in Eq. (27b) gives rise to first-order branch-point singularities at $s = \pm s_b = \pm\sqrt{2}\exp(i\pi/4)$ in the s plane. These branch points and the associated choice of branch cuts are shown in Fig. 4.2.3. The steepest-descent path P in the s plane, corresponding to the path $\bar{P}_z$ in the z plane, extends along the real s axis.

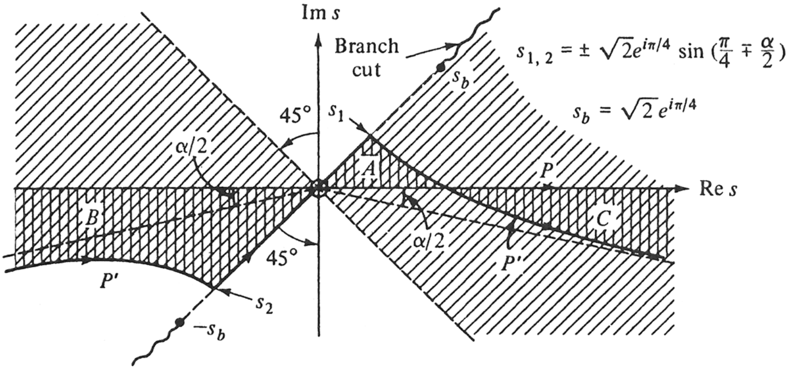


FIG. 4.2.3 Contours of integration in the s plane.

Since all paths considered begin and terminate in a shaded region of the z or s plane, it is evident that the contours $\bar{P}$ and P' can be deformed at infinity into the contours $\bar{P}_z$ and P, respectively. For deformation of the paths in the remainder of the z or s planes, attention must be given to the location of the pole singularity at $z = \beta$ in Eq. (23). If $z = \beta$ [or $s = s_\beta = \sqrt{2}\,e^{i\pi/4}\sin(\frac{1}{2}\beta - \frac{1}{2}\alpha)$] is situated in any of the vertically shaded regions A, B, C in Fig. 4.2.2 (or Fig. 4.2.3), the residue at the pole must be taken into account in the contour deformation. Thus,

$$I_1(\Omega, \alpha, \beta) = 2\pi i e^{i\Omega\cos(\beta-\alpha)}\epsilon(\beta) + e^{i\Omega}\int_{-\infty}^{\infty} G(s)e^{-\Omega s^2}\,ds, \tag{30}$$

where

$$\epsilon(\beta) = \begin{cases} +1, & \text{if } \beta \text{ (or } s_\beta\text{) lies in regions } B \text{ or } C, \\ -1, & \text{if } \beta \text{ (or } s_\beta\text{) lies in region } A, \\ 0, & \text{if } \beta \text{ (or } s_\beta\text{) lies outside regions } A, B, C. \end{cases} \tag{30a}$$

In view of the change of variable in Eqs. (27), $G(s)$ is given by

$$G(s) = \frac{1}{z-\beta}\frac{dz}{ds}, \qquad \frac{dz}{ds} = \frac{-2is}{\sin(z-\alpha)} = \frac{-2is}{\sqrt{1-\cos^2(z-\alpha)}}. \tag{30b}$$

The simple form of dz/ds permits a direct determination of the complete series expansion by the binomial theorem. One has

$$\begin{aligned} \frac{dz}{ds} &= \sqrt{2}\,e^{-i\pi/4}\left(1 - \frac{is^2}{2}\right)^{-1/2} \\ &= \sqrt{2}\,e^{-i\pi/4}\left(1 + \frac{is^2}{4} - \frac{3}{32}s^4 + \cdots\right), \end{aligned} \tag{30c}$$

which converges in the interior of a circle of radius $|s| = \sqrt{2}$ passing through the branch-point singularities. Thus, the range of convergence of the power-series expansion for $G(s)$ is $|s| < \sqrt{2}$ if $|s_\beta| > \sqrt{2}$ and is $|s| < |s_\beta|$ if $|s_\beta| < \sqrt{2}$ (i.e., within a circle of finite radius provided $|s_\beta| > 0$). The first two terms in the asymptotic expansion of $I_1(\Omega, \alpha, \beta)$ in Eq. (23) as $\Omega \to \infty$ are therefore given via Eqs. (30), (17), and (A2) by

$$I_1(\Omega, \alpha, \beta) \sim 2\pi i e^{i\Omega \cos(\beta-\alpha)} \epsilon(\beta) + \frac{e^{i(\Omega-\pi/4)}}{\alpha-\beta}\sqrt{\frac{2\pi}{\Omega}}\left\{1 - \frac{i}{4\Omega}\left[\frac{4}{(\alpha-\beta)^2} + \frac{1}{2}\right] + \cdots\right\}, \qquad \Omega \to \infty. \tag{31}$$

Concerning the residue contribution from the first term in Eq. (31), one notes that the magnitude of the exponential term behaves like

$$\exp\left[-\Omega \,|\sin(\beta_r - \alpha)\sinh \beta_i|\right],$$

where β_r and β_i are the real and imaginary parts of β, respectively. If $\beta_i \neq 0$ and $\beta_r \neq \alpha$, the pole contribution is exponentially small and can be neglected in comparison with the remaining terms. On the other hand, if $\beta_i \to 0$ or if $\beta_r \to \alpha$, the residue contribution may be the dominant one since its magnitude then remains constant as $\Omega \to \infty$. Care should be exercised in the explicit retention of an exponentially small residue contribution when β is complex, since the asymptotic expansion in Eq. (31) has itself an exponentially small error (see p. 385). If this error term decays more slowly than $\exp[i\Omega\cos(\beta - \alpha)]$, the residue term is not significant.

Although the asymptotic representation in Eq. (31) remains valid as $\Omega \to \infty$ for any $\beta \neq \alpha$, the proximity of the pole and the saddle point is seen to influence the accuracy of the approximation for large fixed values of Ω. If the pole approaches the saddle point $[(\alpha - \beta) \to 0]$, the radius of convergence of the series representation for $G(s)$ shrinks to zero. The representation in Eq. (31) then becomes inapplicable since examination of the second term in the expansion indicates that the error incurred by use of even the first term becomes large [see Eq. (19b)]. One notes in this connection that the quantity $\sqrt{\Omega}\,|\alpha - \beta|$ plays a special role in assessing whether the pole is near enough to the saddle point to invalidate Eq. (31). If for $\Omega \gg 1$, one also has $\sqrt{\Omega}\,|\alpha - \beta| \gg 1$, the pole can be considered to be far from the saddle point and Eq. (31) applies; on the other hand, if $\sqrt{\Omega}\,|\alpha - \beta| \lesssim 1$, with $\Omega \gg 1$, the terms in the asymptotic series are no longer small and the validity of the expansion is in question. The dependence of the asymptotic expansion of $I_1(\Omega, \alpha, \beta)$ on the magnitude of $\sqrt{\Omega}\,|\alpha - \beta|$ is studied in detail in Sec. 4.4a, where we explore the evaluation of integrals whose integrands contain a pole near a saddle point.

4.2e Integrands with Two Relevant Isolated Saddle Points: Asymptotic Expansion of the Airy Integral

Consider the Airy function $\mathrm{Ai}(\sigma)$ defined by the integral

$$\mathrm{Ai}(\sigma) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}z^3 + \sigma z\right) dz, \tag{32}$$

which can also be written as

$$\mathrm{Ai}(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(z^3/3+\sigma z)}\,dz, \tag{32a}$$

or, upon transforming z into $-iz$, as

$$\mathrm{Ai}(\sigma) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\sigma z - z^3/3}\,dz. \tag{32b}$$

The contour of integration in Eq. (32b) can be deformed away from the imaginary axis upon recognizing that the behavior of the integrand as $|z| \to \infty$ is determined by $\exp(-z^3/3)$. $\mathrm{Re}\,(z^3) > 0$ in the following sectors of the complex z plane: (1) $|\arg z| < \pi/6$; (2) $\pi/2 < \arg z < 5\pi/6$; (3) $-\pi/2 > \arg z > -5\pi/6$. Thus, by Cauchy's theorem, the integral in Eq. (32b) can be evaluated over any contour L_{32} in the complex z plane which, as shown in Fig. 4.2.4,

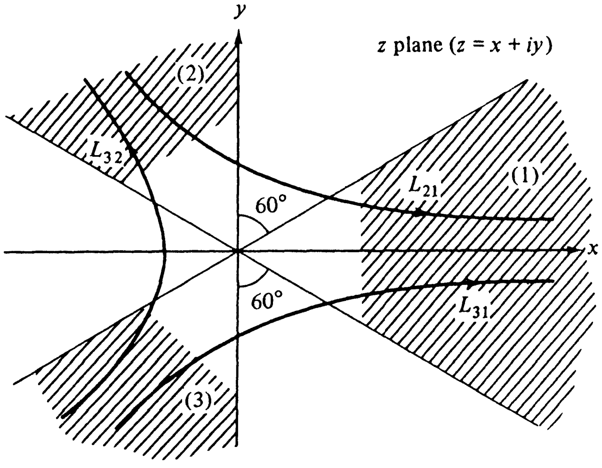


FIG. 4.2.4 Airy integral paths (Re $z^3 > 0$ in shaded regions).

begins at $|z| \to \infty$ in sector 3 and ends at $|z| \to \infty$ in sector 2. From Eq. (32b) one notes that $\mathrm{Ai}(\sigma)$ satisfies the differential equation

$$\left(\frac{d^2}{d\sigma^2} - \sigma\right)\mathrm{Ai}(\sigma) = -\frac{1}{2\pi i}\int_{L_{32}} e^{\sigma z - z^3/3}\, d\left(\sigma z - \frac{z^3}{3}\right) = 0, \tag{33}$$

where it has been recognized that the integrand is a perfect differential and vanishes at the endpoints of the path L_{32}.

A second independent solution of the differential equation (33) is defined as

$$\mathrm{Bi}(\sigma) = \frac{1}{2\pi}\int_{L_{21}+L_{31}} e^{\sigma z - z^3/3}\,dz \tag{34}$$

where the paths L_{21} and L_{31} are shown in Fig. 4.2.4.

The exponent in Eq. (32b) or (34) possesses stationary points of order 1 at

$$\sigma - z_s^2 = 0, \qquad \text{i.e., } z_s = \pm\sqrt{\sigma}. \tag{35}$$

If σ is large, the saddle points are widely separated and as $\sigma \to \infty$, one may obtain an asymptotic expansion of $\mathrm{Ai}(\sigma)$ and $\mathrm{Bi}(\sigma)$ by treating separately the contribution from each relevant saddle point.[4] To transform the exponent into the form $\Omega q(z)$, with $q(z)$ independent of Ω, we introduce the parameter

$$\Omega = \sigma^{3/2} \tag{36}$$

and change the variable in Eqs. (32b) and (34) from z to $\bar{z} = \Omega^{-1/3} z$ to obtain (assuming Ω and σ positive for the present)

$$\mathrm{Ai}(\sigma) = \frac{\Omega^{1/3}}{2\pi i} \int_{L_{32}} e^{\Omega(z - z^3/3)}\, dz, \tag{37a}$$

$$\mathrm{Bi}(\sigma) = \frac{\Omega^{1/3}}{2\pi} \int_{L_{21}+L_{31}} e^{\Omega(z - z^3/3)}\, dz. \tag{37b}$$

In Eqs. (37), the notation z instead of $\bar{z}$ has been retained for convenience. The saddle points of the exponent in the integrands are now located at

$$z_s = \pm 1. \tag{38}$$

The steepest-descent and steepest-ascent paths through the saddle points are determined by the constant-phase requirement: $\operatorname{Im} q(z) = \operatorname{Im} q(z_s)$. Since $q(z) = z - z^3/3$, where $z = x + iy$, one obtains for the equation of the steepest paths,

$$y(y^2 - 3x^2 + 3) = 0. \tag{39}$$

Thus, the steepest paths extend along the real z axis and along the hyperbola $3x^2 - y^2 = 3$. The steepest-descent paths through the saddle points at $z_s = \pm 1$ are shown in Fig. 4.2.5. It is evident from examination of $q(z_s)$ that the saddle point at $z_s = +1$ lies on a higher level than that at $z_s = -1$. Since the hyperbola in Fig. 4.2.5 is asymptotic to the lines $y = \pm\sqrt{3}\,x$ [i.e., $\arg z = \mp 2\pi/3$], one verifies from a comparison with Fig. 4.2.4 that the steepest-descent paths end in the shaded regions. Therefore, any of the contours in Fig. 4.2.4 can be deformed into appropriate steepest-descent paths through a saddle point.

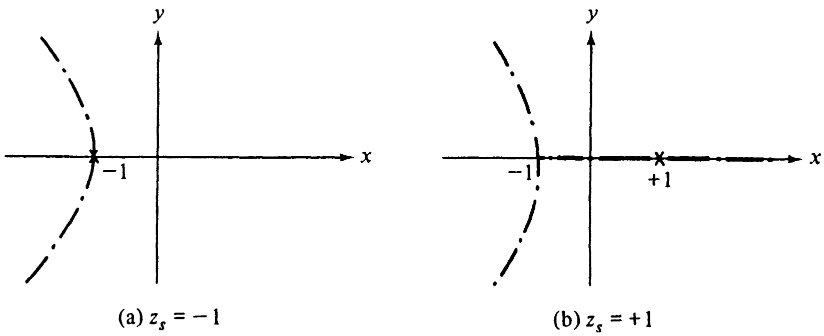


FIG. 4.2.5 Steepest-descent paths through saddle points $z_s = \pm 1$.

The formal asymptotic expansion of the integrals in Eqs. (37) can now be written down directly from Eq. (17). For $\mathrm{Ai}(\sigma)$, the pertinent saddle point is located at $z_s = -1$ and the steepest-descent paths are shown in Fig. 4.2.5(a). The transformation to the complex s plane is accomplished via

$$q(z) = z - \frac{z^3}{3} = -\frac{2}{3} - s^2, \tag{40}$$

where s is real and increases from $-\infty$ to $+\infty$ along the steepest-descent path. Thus,

$$\frac{dz}{ds} = \frac{-2s}{1 - z^2}, \tag{41a}$$

and, from Eq. (5),

$$\left.\frac{dz}{ds}\right|_{s=0} = \pm i. \tag{41b}$$

If it is assumed that $s = -\infty$ and $s = +\infty$ correspond, respectively, to the lower and upper endpoints of the path L_{32} in Fig. 4.2.4, then from the direction of integration along the path, one notes that $\arg(dz/ds) = +\pi/2$ at $z_s = -1$, so the plus sign is chosen in Eq. (41b). Thus, from Eqs. (18b), (36), and (37a), one obtains the formal asymptotic expansion

$$\operatorname{Ai}(\sigma) \sim \frac{\sqrt{\sigma}\, e^{-(2/3)\sigma^{3/2}}}{2\pi i} \left[G_e\left(\sqrt{-\frac{d}{d\Omega}}\right) \sqrt{\frac{\pi}{\Omega}} \right]_{\Omega=\sigma^{3/2}}, \qquad \sigma \to \infty, \tag{42}$$

whose first term is given in view of Eq. (41b) by

$$\operatorname{Ai}(\sigma) \sim \frac{1}{2\sqrt{\pi}\,\sigma^{1/4}} e^{-(2/3)\sigma^{3/2}}. \tag{42a}$$

It is relatively simple in the present case to obtain the complete expansion of $G(s) = dz/ds$ as a function of s and therefore the value of the general term $G^{(2n)}(0)$. The complete asymptotic expansion of $\operatorname{Ai}(\sigma)$ as in Eq. (42) is then found to be given explicitly by[4,5]

$$\operatorname{Ai}(\sigma) \sim \frac{1}{2\sqrt{\pi}\,\sigma^{1/4}} e^{-(2/3)\sigma^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma(3n + \frac{1}{2})}{\sqrt{\pi}\,(2n)!(-9\sigma^{3/2})^n}. \tag{43}$$

Although it has been assumed above that σ, and therefore Ω, are positive real, the results apply as well for complex Ω in the range $|\arg \Omega| < \pi/2$, as pointed out in Sec. 4.2b. In view of Eq. (36), the validity of Eqs. (42) and (43) can therefore be extended to complex values of σ lying in the sector $|\arg \sigma| < \pi/3$ [i.e., the expansion in Eq. (43) holds uniformly for $\arg \sigma$ in this range as $|\sigma| \to \infty$].

For the asymptotic evaluation of $\operatorname{Bi}(\sigma)$, defined in Eq. (37b), one deforms the contours L_{21} and L_{31} in Fig. 4.2.4 into the appropriate steepest paths through the saddle point at $z_s = +1$ in Fig. 4.2.5(b). For L_{21}, the path consists of the segment of the real z axis indicated in Fig. 4.2.5(b) and the upper branch of the hyperbola; for L_{31}, one employs the lower branch of the hyperbola in conjunction with the segment along the real axis. The transformation to the s plane for both L_{21} and L_{31} is now

$$q(z) = z - \frac{z^3}{3} = \frac{2}{3} - s^2, \qquad -\infty < s < \infty, \tag{44}$$

from which dz/ds is still given by Eq. (41a). However, from Eq. (5), $dz/ds = \pm 1$ at $s = 0$; from the direction of integration along L_{21} and L_{31} one notes that the positive sign must be chosen. Thus, as $\sigma \to \infty$, one obtains, to a first order,

$$\mathrm{Bi}(\sigma) \sim \frac{1}{\sqrt{\pi}\,\sigma^{1/4}} e^{(2/3)\sigma^{3/2}}, \tag{45}$$

or, as in Eq. (43), the complete asymptotic expansion[4,5]

$$\mathrm{Bi}(\sigma) \sim \frac{1}{\sqrt{\pi}\,\sigma^{1/4}} e^{(2/3)\sigma^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma(3n+\frac{1}{2})}{\sqrt{\pi}\,(2n)!} \frac{1}{(9\sigma^{3/2})^n}, \qquad \sigma \to \infty. \tag{45a}$$

As for $\mathrm{Ai}(\sigma)$, Eqs. (45) apply uniformly in $\arg\sigma$ for complex values of σ lying in the sector $|\arg\sigma| < \pi/3$.

To obtain the asymptotic behavior of $\mathrm{Ai}(\sigma)$ and $\mathrm{Bi}(\sigma)$ for large negative real values of σ, it is convenient to consider the functions $\mathrm{Ai}(-\sigma)$ and $\mathrm{Bi}(-\sigma)$, where $\sigma > 0$. These functions are defined by Eq. (32b) (with L_{32} as the contour of integration) and Eq. (34), and can be expressed analogously to Eqs. (37) as

$$\mathrm{Ai}(-\sigma) = \frac{\Omega^{1/3}}{2\pi i}\int_{L_{32}} e^{-\Omega(z+z^3/3)}\,dz, \qquad \Omega = \sigma^{3/2} > 0, \tag{46a}$$

$$\mathrm{Bi}(-\sigma) = \frac{\Omega^{1/3}}{2\pi}\int_{L_{21}+L_{31}} e^{-\Omega(z+z^3/3)}\,dz. \tag{46b}$$

The saddle points are now located at $z_s = \pm i$, so $|\exp \Omega q(z_s)| = 1$ at both saddle points, which thus have the same level. The steepest paths through the saddle points satisfy the equations $\mathrm{Im}\, q(z) = \mathrm{Im}\, q(z_s) = \pm 2/3$, or

$$y^3 - 3x^2y - 3y = \pm 2. \tag{47}$$

For large values of x and y, the cubic curves defined by Eq. (47) are asymptotic to the lines $y = 0$ and $y = \pm\sqrt{3}\,x$. Since the steepest-descent contours must end in the shaded regions in Fig. 4.2.4, one selects as the steepest-descent paths the curves shown in Fig. 4.2.6. Upon comparing Figs. 4.2.4 and 4.2.6, one notes that contours L_{21} and L_{31} can be deformed directly into the steepest-descent paths P_{21} and P_{31}, respectively, while the contour L_{32} must be deformed first along P_{31} and then along P_{21} so that it traverses both saddle points.

The transformation from the z to the s plane is given by

$$q(z) = -z - \frac{z^3}{3} = \mp\frac{2}{3}i - s^2, \qquad -\infty < s < \infty, \tag{48}$$

where the upper and lower signs apply for $z_s = \pm i$. Thus,

$$\frac{dz}{ds} = \frac{2s}{1+z^2}, \tag{49a}$$

and, from Eq. (5),

$$\left.\frac{dz}{ds}\right|_{s=0} = \begin{cases} e^{-i\pi/4} & \text{on } P_{21}, \\ e^{i\pi/4} & \text{on } P_{31}. \end{cases} \tag{49b}$$

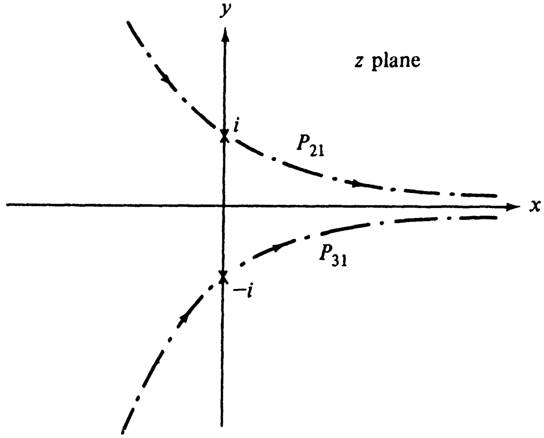


FIG. 4.2.6 Steepest-descent paths through the saddle points $z_s = \pm i$.

The asymptotic evaluation of the integrals in Eqs. (46) as $\Omega \to \infty$ can now be carried out directly from Eq. (17). To a first order one obtains

$$I_{21} = \int_{P_{21}} e^{-\Omega(z+z^3/3)}\,dz \sim e^{-i(2/3)\Omega - i\pi/4}\sqrt{\frac{\pi}{\Omega}}, \tag{50a}$$

$$I_{31} = \int_{P_{31}} e^{-\Omega(z+z^3/3)}\,dz \sim e^{i(2/3)\Omega + i\pi/4}\sqrt{\frac{\pi}{\Omega}}. \tag{50b}$$

Accordingly, the integral in Eq. (46a) is equal to $I_{31} - I_{21}$ while that in Eq. (46b) is equal to $I_{31} + I_{21}$; the positive sense on P_{21} and P_{31} in Fig. 4.2.6 is to be noted. Thus, one obtains the following first-order asymptotic representations as $\sigma \to \infty$ [see Eq. (36)]:

$$\mathrm{Ai}(-\sigma) \sim \frac{1}{\sqrt{\pi}\,\sigma^{1/4}} \sin\left(\frac{2}{3}\sigma^{3/2} + \frac{\pi}{4}\right), \tag{51a}$$

$$\mathrm{Bi}(-\sigma) \sim \frac{1}{\sqrt{\pi}\,\sigma^{1/4}} \cos\left(\frac{2}{3}\sigma^{3/2} + \frac{\pi}{4}\right). \tag{51b}$$

The complete asymptotic expansion of $\mathrm{Ai}(-\sigma)$ and $\mathrm{Bi}(-\sigma)$ as $\sigma \to \infty$ can be shown to be as follows[4,5]:

$$\mathrm{Ai}(-\sigma) = \frac{1}{\sqrt{\pi}\,\sigma^{1/4}} \left\{P(\sigma)\sin\left(\frac{2}{3}\sigma^{3/2} + \frac{\pi}{4}\right) - Q(\sigma)\cos\left(\frac{2}{3}\sigma^{3/2} + \frac{\pi}{4}\right)\right\}, \tag{52a}$$

$$\mathrm{Bi}(-\sigma) = \frac{1}{\sqrt{\pi}\,\sigma^{1/4}} \left\{P(\sigma)\cos\left(\frac{2}{3}\sigma^{3/2} + \frac{\pi}{4}\right) + Q(\sigma)\sin\left(\frac{2}{3}\sigma^{3/2} + \frac{\pi}{4}\right)\right\}, \tag{52b}$$

where

$$P(\sigma) \sim \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(6n + \frac{1}{2})}{(4n)!} \frac{(-1)^n}{(9\sigma^{3/2})^{2n}}, \tag{52c}$$

$$Q(\sigma) \sim \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(6n + \frac{7}{2})}{(4n + 2)!} \frac{(-1)^n}{(9\sigma^{3/2})^{2n+1}}. \tag{52d}$$

As previously, Eqs. (51) and (52) also apply if $|\sigma| \to \infty$ in the sector $|\arg \sigma| < \pi/3$.

When the argument of the Airy function is not negative real, the trigonometric functions may be approximated by the dominant exponential. It is then found that for $\pi/3 < \arg \sigma < \pi$, the dominant exponential in Eq. (51a) yields the same formula for $\mathrm{Ai}(\sigma)$ as Eq. (42a). Thus, it is suggestive to employ Eq. (42a) over the entire range $0 < \arg \sigma < \pi$ and then switch to Eq. (51a) when $\arg \sigma \geq \pi$. The validity of this argument may be established by a consideration of the Stokes phenomenon, a detailed discussion of which is contained in references 7 and 8. It may then be inferred that Eq. (42a) remains valid when $-2\pi/3 \leq \arg \sigma \leq 2\pi/3$, whereas Eq. (51a) is employed when $2\pi/3 \leq \arg \sigma \leq 4\pi/3$. The matching up of the two formulas has been performed at $\arg \sigma = 2\pi/3$, where the dominant exponential has its maximum value.

4.3 ISOLATED SADDLE POINTS OF HIGHER ORDER

If in the integrand of

$$I(\Omega) = \int_{z_s}^{``\infty"} f(z) e^{\Omega q(z)}\, dz, \tag{1}$$

the function $f(z)$ has no singularities near the isolated Mth-order saddle point z_s of $q(z)$, with $q^{(n)}(z_s) = 0, n = 1 \cdots M, q^{(M+1)}(z_s) \neq 0$, the asymptotic approximation of $I(\Omega)$ is given by

$$I(\Omega) \sim \left[\frac{-(M+1)!}{q^{(M+1)}(z_s)}\right]^{1/(M+1)} f(z_s) e^{\Omega q(z_s)} \frac{\Gamma[1/(M+1)]}{(M+1)\Omega^{1/(M+1)}}, \qquad \Omega \to \infty, \tag{2}$$

where the $(M+1)$th root in the first factor is chosen so that $\arg \{[\ ]^{1/(M+1)}\} = \varphi \equiv \arg (dz)_{z_s}$. dz denotes an element along the steepest-descent path, SDP, which begins at z_s and ends in an appropriate valley at "∞" (not necessarily along the real z axis) (Fig. 4.3.1). Since $M + 1$ valleys are accessible from an

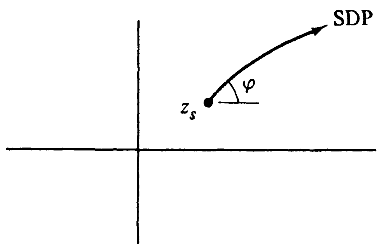


FIG. 4.3.1 Integration path in the z plane.

Mth-order saddle point, it is possible (for $M > 1$) to have non-collinear segments of the SDP leading to and from the saddle point, respectively. For this reason, each segment is treated separately and provides a contribution as in Eq. (1).

Analytical details

If $q(z)$ in Eq. (1) has one pertinent Mth-order saddle point at z_s [i.e., $q^{(n)}(z_s) = 0, n = 1, \ldots, M, q^{(M+1)}(z_s) \neq 0$], the appropriate transformation to the s plane is given by [see Eq. (4.1.6)]

$$q(z) = \tau(s) = q(z_s) - s^{M+1}. \tag{3}$$

The requirement Im $q(z)$ = constant (i.e., s^{M+1} positive along the steepest-descent paths) leads to the following possible contours in the s plane:

$$\arg s = 2\mu\pi/(M + 1), \qquad \mu = 0, 1, 2, \ldots, M. \tag{4}$$

Thus, the $M + 1$ steepest-descent paths originating at the saddle point z_s in the z plane [see Fig 4.1.5(b) for $M = 2$] map in the s plane into the $M + 1$ straight lines defined in Eq. (4). The latter originate at $s = 0$ and extend to $|s| = \infty$. Since s^{M+1} is positive along any of the paths defined in Eq. (4) we may consider without loss of generality the case $\mu = 0$, wherein the steepest-descent path maps into the positive real s axis. Then the integrals $I_n(\Omega)$ in Eq. (4.2.13) are given in view of Eq. (3) by

$$I_n(\Omega) = \int_0^\infty s^n e^{-\Omega s^{M+1}}\, ds \tag{5a}$$

$$= \Gamma\left(\frac{1 + n}{M + 1}\right) \frac{\Omega^{-(1+n)/(M+1)}}{M + 1}, \tag{5b}$$

where the integral in Eq. (5a) has been evaluated in terms of the gamma function defined in Eq. (4.2.8). The asymptotic nature of the resulting expansion in Eq. (4.2.17) as $\Omega \to \infty$ follows by the same considerations as those mentioned in Sec. 4.2b.

Upon expanding $q(z)$ in Eq. (3) in a power series about the point z_s, one obtains directly

$$\left.\frac{dz}{ds}\right|_{s=0} = \zeta\left[\frac{-(M + 1)!}{q^{(M+1)}(z_s)}\right]^{1/(M+1)}, \qquad \zeta = e^{i2\pi\mu/(M+1)}, \tag{6}$$

where ζ is the appropriate $(M + 1)$th root of unity. The choice of root is determined by the given steepest-descent path [see Eq. (4)], and the principal value of the $(M + 1)$th root is taken in the remaining term. From Eqs. (4.2.12) and (5), one obtains the following first-order approximation to the integral $\bar{I}(\Omega)$ (for $\mu = 0$):

$$\bar{I}(\Omega) \equiv \int_0^\infty G(s)e^{\Omega\tau(s)}\, ds \sim \left[\frac{-(M + 1)!}{q^{(M+1)}(z_s)}\right]^{1/(M+1)} f(z_s)e^{\Omega q(z_s)} \frac{\Gamma[1/(M + 1)]}{(M + 1)\Omega^{1/(M+1)}} \tag{7}$$

where it is recalled that $G(s) = f(z)(dz/ds)$. When $M = 1$, one recovers from Eq. (7) the previously derived result in Eq. (4.2.7), save for a factor of 2 which arises since the interval of integration in Eq. (4.2.6) extends from $s = -\infty$ to

$s = \infty$. If $G(s)$ has a branch-point singularity at $s = 0$, the integrand in Eq. (5a) involves instead of s^n the factor $s^{n+\beta}$, where $\beta > -1$ is non-integral. The resulting formula for $I_n(\Omega)$ is still given in terms of the gamma function.

4.4 FIRST-ORDER SADDLE POINT AND NEARBY SINGULARITIES

4.4a Simple Pole Singularity

If in the integrand of

$$I(\Omega) = \int_{SDP} f(z) e^{\Omega q(z)}\, dz, \tag{1}$$

the function $f(z)$ has a simple pole singularity at $z = z_0$ near the isolated first-order saddle point z_s of $q(z)$ [i.e., $q'(z_s) = 0$, $q''(z_s) \neq 0$], the asymptotic approximation of $I(\Omega)$, valid uniformly as $z_0 \to z_s$, is given by[9] [for a complete asymptotic expansion, see Eq. (16)]

$$I(\Omega) \sim e^{\Omega q(z_s)} \left\{ \pm i2a\sqrt{\pi}\, e^{-\Omega b^2} Q(\mp ib\sqrt{\Omega}) + \sqrt{\frac{\pi}{\Omega}}\, T(0) \right\},$$

$$\operatorname{Im} b \gtrless 0, \quad \Omega \to \infty, \tag{2}$$

where

$$a = \lim_{z \to z_0} [(z - z_0) f(z)], \qquad b = \sqrt{q(z_s) - q(z_0)}, \tag{2a}$$

$$T(0) = h f(z_s) + \frac{a}{b}, \qquad h = \sqrt{\frac{-2}{q''(z_s)}}, \tag{2b}$$

$$Q(y) = \int_y^\infty e^{-x^2}\, dx, \qquad Q(y) + Q(-y) = \sqrt{\pi}, \tag{2c}$$

The square root in Eq. (2b) is defined so that $\arg h = \varphi \equiv (\arg dz)_{z_s}$, where dz is an element along the steepest-descent path SDP, while $\arg b$ is defined so that $b \to (z_0 - z_s)/h$ as $z_0 \to z_s$. When $q(z) = i\hat{q}(z)$, with $\hat{q}$ denoting a real function of z, and z_s is real, $\arg h = \pm\pi/4$ when $\hat{q}''(z_s) \gtrless 0$, provided that Re (dz) increases along the SDP near z_s (see Fig. 4.2.1). The discontinuity in $I(\Omega)$ as Im b changes from positive to negative values is exactly equal to the residue of the integrand at the pole $z = z_0$ (see Fig. 4.4.1). In view of the last relation in Eq. (2c), the expression in Eq. (2) can be written either in terms of $Q(-ib\sqrt{\Omega})$ or $Q(+ib\sqrt{\Omega})$.

When Im $b = 0$, Eq. (2) is replaced by

$$I(\Omega) \sim e^{\Omega q(z_s)} \left\{ i2a\sqrt{\pi}\, e^{-\Omega b^2} Q(-ib\sqrt{\Omega}) - \pi i a e^{-\Omega b^2} + \sqrt{\frac{\pi}{\Omega}}\, T(0) \right\}. \tag{3}$$

In this instance, the pole z_0 lies *on* the SDP since in order to render b real, Im $q(z_0) =$ Im $q(z_s)$, Re $q(z_0) <$ Re $q(z_s)$. Equation 3 represents an asymptotic approximation of the principal value of the integral $I(\Omega)$ in Eq. (1). When the pole and the saddle point coincide, $b = 0$ and $Q(0) = \frac{1}{2}\sqrt{\pi}$, so

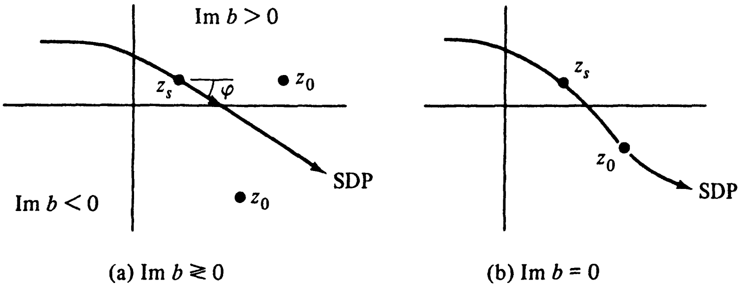


FIG. 4.4.1 Integration paths and pole locations in the z plane.

$$I(\Omega) \sim e^{\Omega q(z_s)} \sqrt{\frac{\pi}{\Omega}}\, T(0), \qquad z_0 = z_s, \tag{4a}$$

where $T(0)$ remains bounded as $z_0 \to z_s$ and is, in fact, given by

$$\begin{aligned} T(0) &= hf(z_s) + \frac{ah}{(z_0 - z_s)\{1 + [q^{(3)}(z_s)/3q''(z_s)](z_0 - z_s) + \cdots\}^{1/2}} \\ &= h\left[g(z_s) - a\frac{q^{(3)}(z_s)}{6q''(z_s)}\right], \qquad z_0 = z_s, \end{aligned} \tag{4b}$$

with $g(z) = f(z) - a(z - z_0)^{-1}$.

Analytical details

If $f(z)$ in the integrand of Eq. (1) has a simple pole singularity at $z = z_0$ near a first-order saddle point z_s, $G(s)$ in Eq. (4.1.3) will possess correspondingly a simple pole singularity at $s = b$ in the vicinity of $s = 0$. Suppose that $G(s)(s - b) \to a$ as $s \to b$; then $G(s)$ can be represented in the vicinity of $s = 0$ by

$$G(s) = \frac{a}{s - b} + T(s). \tag{5}$$

It will be convenient to employ the identity

$$\frac{a}{s - b} = \frac{as}{s^2 - b^2} + \frac{ab}{s^2 - b^2}, \tag{5a}$$

and to expand

$$T(s) = T(0) + T'(0)s + T''(0)\frac{s^2}{2!} + \cdots, \tag{5b}$$

which is regular at $s = b$ and has a radius of convergence uninfluenced by the presence of the pole.

Since the saddle point is of order 1, the transformation in Eq. (4.2.2) still applies and the integral in Eq. (1) can be written as (note that $s = [q(z_s) - q(z)]^{1/2}$, $h = (dz/ds)_{s=0}$, and $a = \lim_{z \to z_0} [(z - z_0)f(z)] = \lim_{s \to b} [(s - b)G(s)]$)

$$I(\Omega, b) = e^{\Omega q(z_s)} \int_{-\infty}^{\infty} G(s) e^{-\Omega s^2}\, ds, \tag{6}$$

whence the formal result in Eq. (4.2.18b) yields directly

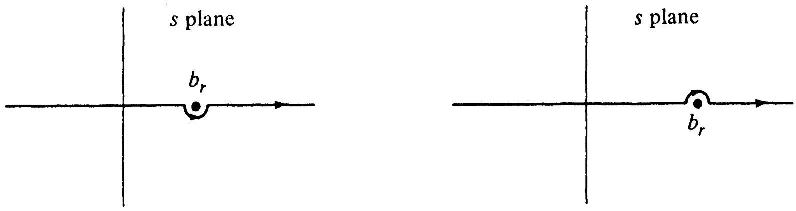
$$I(\Omega, b) \sim e^{\Omega q(z_s)} G_e\left(\sqrt{-\frac{d}{d\Omega}}\right)\sqrt{\frac{\pi}{\Omega}}, \tag{7}$$

with G_e obtained from Eqs. (5) as

$$G_e\left(\sqrt{-\frac{d}{d\Omega}}\right) = \frac{-ab}{(d/d\Omega) + b^2} + T_e\left(\sqrt{-\frac{d}{d\Omega}}\right). \tag{8}$$

The operation on $\sqrt{\pi/\Omega}$ implied by the last term in Eq. (8) is the same as in Eq. (4.2.18b) and leads to a formal asymptotic expansion as in Eq. (4.2.17).

It is to be noted that the integral $I(\Omega, b)$ in Eq. (6), with $G(s)$ given in Eq. (5), is defined only for $\operatorname{Im} b \neq 0$ and does not exist when b is real or zero. Viewed as a function of b, a study of the analytic properties of $I(\Omega, b)$ as $\operatorname{Im} b \to 0$ reveals that the integral is discontinuous across the real b axis. Suppose that b approaches the real b axis from above [i.e., $b \to b_r + i\bar{\delta}$, with $b_r, \bar{\delta}$ real and $\bar{\delta} > 0$]. Then the path of integration is indented at $s = b_r + i\bar{\delta}$ as shown in Fig. 4.4.2(a). Similarly, when $b \to b_r - i\bar{\delta}$, the appropriate path is



(a) Im b = 0+ (b) Im b = 0−

FIG. 4.4.2 Contours of integration.

that in Fig. 4.4.2(b). To exhibit the discontinuity in $I(\Omega, b)$ across the real b axis, one constructs the difference $I(\Omega, b_r + i\bar{\delta}) - I(\Omega, b_r - i\bar{\delta})$ and notes that the contributions to the integrals from the straight portions of the paths in Fig. 4.4.2 cancel; there then remains only a small circular contour enclosing the pole at $s = b_r$ in the positive sense. Since $T(s)$ in Eq. (5) is regular inside this circle, its contribution vanishes and one obtains from the residue at $s = b_r$†

$$I(\Omega, b_r + i\bar{\delta}) - I(\Omega, b_r - i\bar{\delta}) = 2\pi i a e^{-\Omega b_r^2} e^{\Omega q(z_s)}, \qquad \bar{\delta} \to 0. \tag{9}$$

The operation

$$A(\Omega, b) = \frac{-ab}{(d/d\Omega) + b^2}\sqrt{\frac{\pi}{\Omega}} \tag{10}$$

implied by the first term in Eq. (8) can be interpreted in terms of the ordinary first-order differential equation

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†To obtain Eq. (9), one also may use the generalized function[see Eq. (1.2.6c)]

$$\lim_{\bar{\delta} \to 0} \frac{1}{s - b \mp i\bar{\delta}} = P\frac{1}{s - b} \pm \pi i \delta(s - b),$$

where $\delta(x)$ is the delta function and P denotes the principal value.

$$\left(\frac{d}{d\Omega} + b^2\right) A(\Omega, b) = -ab\sqrt{\frac{\pi}{\Omega}}. \tag{10a}$$

To find a particular integral of Eq. (10), substitute

$$A(\Omega, b) = e^{-\Omega b^2} B(\Omega, b) \tag{11}$$

into Eq.(10a), whence

$$\frac{dB}{d\Omega} = -abe^{\Omega b^2}\sqrt{\frac{\pi}{\Omega}}. \tag{12}$$

Upon integrating Eq. (12) over Ω between the limits $\bar{\Omega}$ and ∞, one obtains

$$B(\bar{\Omega}, b) = ab\sqrt{\pi} \int_{\bar{\Omega}}^{\infty} e^{\Omega b^2} \Omega^{-1/2}\, d\Omega, \tag{13}$$

where it has been assumed for the moment that $b^2 < 0$,† so in view of Eq. (11) and for $A(\infty, b)$ finite, $B(\infty, b) = 0$.‡ A change of variable in Eq. (13) from Ω to $-x^2/b^2$, or $x = \mp ib\sqrt{\Omega}$, yields

$$B(\Omega, b) = 2a\sqrt{\pi}\, \frac{b}{\mp ib}\, Q[\mp ib\sqrt{\Omega}], \tag{14}$$

where $Q(y)$ is the "error-function complement"

$$Q(y) = \int_y^{\infty} e^{-x^2} dx. \tag{14a}$$

The ambiguity in sign introduced into Eq. (14) by the change of variable is resolved by the previously imposed requirement $B(\infty, b) = 0$ for $b^2 < 0$. Since the error-function integral in Eq. (14a) will vanish if the lower limit approaches infinity along the positive real axis, we require for $b^2 < 0$ a choice of sign such that

$$\mp ib > 0 \tag{14b}$$

(i.e., the minus sign when $b = i|b|$ and the plus sign when $b = -i|b|$).

The validity of Eq. (14) can be extended by analytic continuation to values other than $b^2 < 0$. By direct substitution of Eqs. (5) into the integral in Eq. (6) and comparison with Eqs. (8), (10), and (11), one notes that $B(\Omega, b)$ is also given in terms of the definite integral[10]

$$B(\Omega, b) = abe^{\Omega b^2} \int_{-\infty}^{\infty} \frac{1}{s^2 - b^2} e^{-\Omega s^2}\, ds = ae^{\Omega b^2} \int_{-\infty}^{+\infty} \frac{e^{-\Omega s^2}}{s - b}\, ds. \tag{15}$$

One verifies readily that $B(\Omega, b)$ as defined in Eq. (15) satisfies the differential equation (12). Since the expressions for B in Eqs. (14) and (15) represent

†More generally $\operatorname{Re} b^2 < 0$.

‡By an alternative procedure, if $b^2 < 0$,

$$\frac{1}{ab} B(\Omega, b) \equiv \int_{-\infty}^{\infty} \frac{e^{-\Omega(s^2 - b^2)}}{s^2 - b^2}\, ds = \int_{-\infty}^{\infty} ds \int_{\Omega}^{\infty} e^{-\xi(s^2 - b^2)}\, d\xi = \int_{\Omega}^{\infty} d\xi\, e^{\xi b^2} \int_{-\infty}^{\infty} e^{-\xi s^2}\, ds$$

$$= \sqrt{\pi} \int_{\Omega}^{\infty} \frac{e^{\xi b^2}}{\sqrt{\xi}}\, d\xi.$$

identical functions of b^2 when $b^2 < 0$, and since Eq. (15) clearly remains valid for all except positive and zero values of b^2, it follows that the expression in Eq. (14) can likewise be continued analytically for all values of b^2 except $b^2 \geq 0$. The analytic continuation from pure imaginary values of b ($b^2 < 0$) to complex values must be consistent with condition (14b). In view of the analytic properties of the error-function complement, this implies that the sign in Eq. (14) must be chosen according to the more general condition $\operatorname{Re}(\mp ib) > 0$ (i.e., the minus sign applies when $\operatorname{Im} b > 0$ and the plus sign when $\operatorname{Im} b < 0$).

From Eqs. (7), (10), and (14), we can now write down the asymptotic expansion of the integral in Eq. (6) for $\Omega \gg 1$ and for arbitrary values of b:

$$I(\Omega, b) \sim e^{\Omega q(z_s)}\left\{\pm\, i2a\sqrt{\pi}\, e^{-\Omega b^2} Q(\mp ib\sqrt{\Omega}) + T_e\left(\sqrt{-\frac{d}{d\Omega}}\right)\sqrt{\frac{\pi}{\Omega}}\right\}, \qquad \operatorname{Im} b \gtrless 0, \tag{16}$$

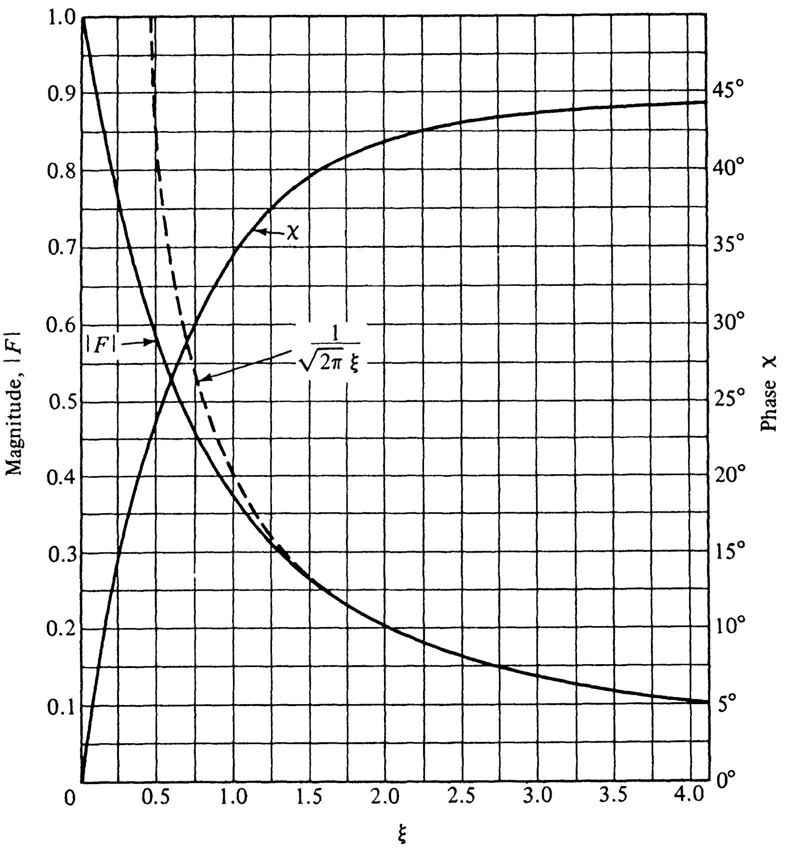


FIG. 4.4.3(a) Plot of $F = |F|e^{i\chi} = \dfrac{2}{\sqrt{\pi}} e^{-i2\xi^2} \displaystyle\int_{(1-i)\xi}^{\infty} e^{-y^2}\, dy$ $\left[\text{for large } \xi : F \sim \dfrac{e^{i\pi/4}}{\sqrt{2\pi}\xi} + O\left(\dfrac{1}{\xi^3}\right)\right]$.

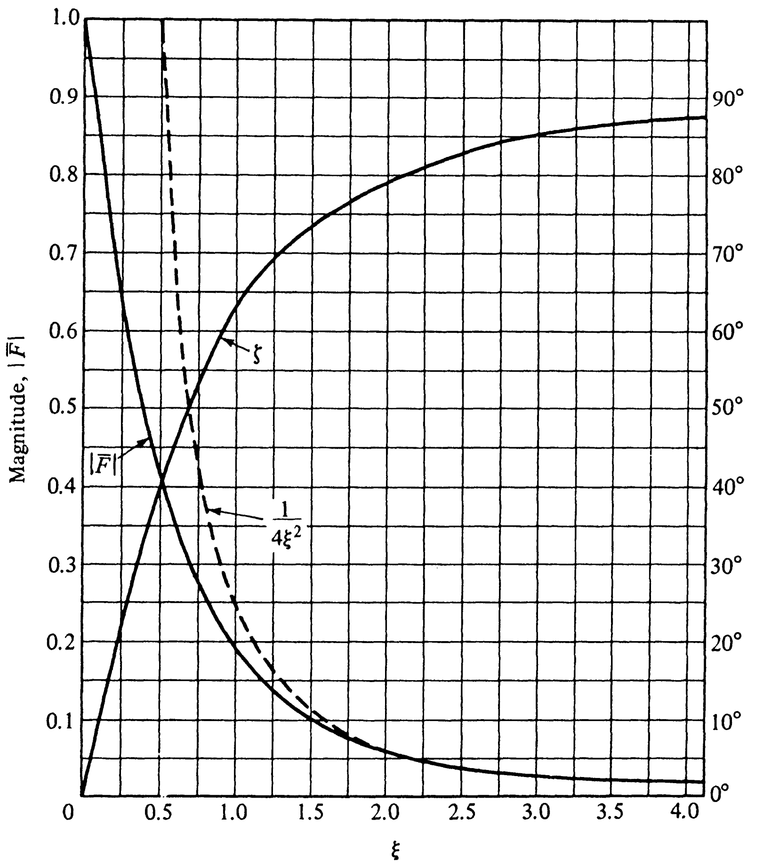


FIG. 4.4.3(b) Plot of $\bar{F} = |\bar{F}|e^{i\zeta} = 1 - 2\sqrt{2}\,\xi e^{-i(2\xi^2+\pi/4)} \int_{(1-i)\xi}^{\infty} e^{-y^2}\,dy$ $\left[\text{for large } \xi : \bar{F} \sim \frac{i}{4\xi^2} + O\left(\frac{1}{\xi^4}\right)\right]$.

the lowest-order term of which $[T_e \to T(0)]$ furnishes the result in Eq. (2). Thus, the asymptotic expansion of an integral whose integrand contains a simple pole near a saddle point has the same form as that for an integrand without a pole except for an additional term involving the error function Q. The function $e^{-\Omega b^2}Q(\mp ib\sqrt{\Omega})$ is tabulated for real and complex values of $b\sqrt{\Omega}$ (see Fig. 4.4.3)

It is of interest to verify from Eq. (16) the previously noted expression for the discontinuity in the value of $I(\Omega, b)$[Eq. (9)] when Im b changes from positive to negative values. As before, we define

$$b_{1,2} = b_r \pm i\bar{\delta}, \qquad b_r, \bar{\delta} \text{ real}, \quad \bar{\delta} \to +0. \tag{17}$$

Since $T_e(\sqrt{-d/d\Omega})\sqrt{\pi/\Omega}$ is continuous for all b, the jump $[I(\Omega, b_1) - I(\Omega, b_2)]$ in the value of $I(\Omega, b)$ is given by

$$I(\Omega, b)\Big|_{b=b_2}^{b=b_1} = 2ia\sqrt{\pi}\, e^{-\Omega b_r^2}[Q(-ib_r\sqrt{\Omega}) + Q(ib_r\sqrt{\Omega})]e^{\Omega q(z_s)}. \tag{18}$$

To treat the sum $[Q(i\alpha) + Q(-i\alpha)]$, we choose a path of integration for $Q(\pm i\alpha)$ in Eq. (14a) from $\pm i\alpha$ to 0 and then from 0 to ∞ along the real axis. Thus,

$$Q(i\alpha) + Q(-i\alpha) = \int_{-i\alpha}^{0} e^{-x^2}\,dx + \int_{i\alpha}^{0} e^{-x^2}\,dx + 2\int_{0}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}, \tag{18a}$$

since the first two terms in the first equality of Eq. (18a) cancel. Thus, Eq. (18) reduces to the previous result in Eq. (9).

It may be noted that the function

$$\hat{B}(\Omega, b) = i2a\sqrt{\pi}\, Q(-ib\sqrt{\Omega}) \tag{19a}$$

represents the integral in Eq. (15) for *all* complex values of b provided that the integral is taken as the definition of an *analytic function* $\hat{B}$ of b. This assertion may be verified by assuming initially that $\operatorname{Im} b > 0$, for which case the equivalence of Eqs. (15) and (19a) has been demonstrated. To extend the range of validity, we first deform the integration path in Eq. (15) into the lower half of the complex s plane. This deformation does not alter the value of the function $\hat{B}$, since no singularities are located between the two paths [it is convenient for the present argument to replace $ab/(s^2 - b^2)$ in the integrand by $a/(s - b)$]. The form of the function $\hat{B}$ defined along the new path remains unchanged even when the pole at $s = b$ moves across the real s axis as long as it does not intercept the deformed integration contour (evidently, the contour may be distorted sufficiently to accommodate in this manner an arbitrarily situated pole). Thus, the integral is represented by the error-function complement in Eq. (19a) for *arbitrary* b provided that the integration contour always passes *beneath* the pole. To obtain a representation involving an integration path along the real axis, one performs the path deformation in reverse and must now exhibit the residue from the pole at $s = b$. Thus, for $\operatorname{Im} b < 0$,

$$\begin{aligned}\hat{B}(\Omega, b) &= i2a\sqrt{\pi}\, Q(-ib\sqrt{\Omega}) = i2\pi a + a\int_{-\infty}^{\infty} \frac{e^{-\Omega s^2}}{s - b}\,ds \\ &= i2\pi a - i2a\sqrt{\pi}\, Q(ib\sqrt{\Omega}),\end{aligned} \tag{19b}$$

the last relation following from Eq. (14). Since $Q(x) + Q(-x) = \sqrt{\pi}$ [see Eq. (18a)], the two alternative forms in Eq. (19b) are equivalent and each may be used to represent the analytic function $\hat{B}(\Omega, b)$ for arbitrary values of b. These considerations may be utilized to simplify the formulas for the asymptotic representation of analytic functions defined by integrals containing pole singularities in the integrand [for example, Eq. (34) may be written in a simpler form].

When b is large enough so that $b\sqrt{\Omega}$ is likewise large, one may employ an asymptotic expansion for $Q(\mp ib\sqrt{\Omega})$ in Eq. (16). This expansion is obtained directly from the representation in Eq. (13) by repeated integrations by parts or by expansion of $(s^2 - b^2)^{-1}$ in Eq. (15) in powers of s^2:

$$B(\Omega, b) = \pm 2ia\sqrt{\pi}\, Q(\mp ib\sqrt{\Omega}) \sim -\frac{a}{b} e^{\Omega b^2} \sqrt{\frac{\pi}{\Omega}} \left[1 + \frac{1}{2b^2\Omega} + O\left(\frac{1}{b^4\Omega^2}\right)\right]. \tag{20}$$

The first-order asymptotic representation for $I(\Omega, b)$ in Eq. (16) is then given by

$$I(\Omega, b) \sim e^{\Omega q(z_s)} \sqrt{\frac{\pi}{\Omega}}\, G(0), \qquad |b|\sqrt{\Omega} \gg 1, \tag{21}$$

where

$$G(0) = -\frac{a}{b} + T(0). \tag{21a}$$

In this instance, the pole is situated "far" from the origin in the s plane and the expression in Eq. (21) is identical with that obtained in Eq. (4.2.7). Just how large $|b|\sqrt{\Omega}$ has to be, before Eq. (20) can be employed to retain a given accuracy, can be assessed by comparing Eq. (20) with the exact expression (14) whose values for a given b are found from numerical tables. A detailed comparison is made in Eqs. (38) et seq. for the special case $\arg(\pm b) = \pi/4$.

When $|b| \to 0$, $Q(\mp ib\sqrt{\Omega}) \to \frac{1}{2}\sqrt{\pi}$.

4.4b Multiple Pole Singularity

If $G(s)$ has a pole of order N at $s = b$, $G(s)$ is represented by the expression [see Eq. (5)]

$$G(s) = \frac{a_{-N}}{(s-b)^N} + \frac{a_{-N+1}}{(s-b)^{N-1}} + \cdots + \frac{a_{-1}}{s-b} + \bar{T}(s), \tag{22}$$

where $\bar{T}(s)$ is regular at $s = b$. To infer the asymptotic expansion of $I(\Omega, b)$ in this case, one must investigate integrals of the form

$$A_{-N}(\Omega, b) = \int_{-\infty}^{\infty} \frac{1}{(s-b)^N} e^{-\Omega s^2}\, ds. \tag{23}$$

For $N = 1$ (with $a_{-1} = 1$), the result has been obtained in Eq. (14)[with Eq. (11)] as

$$A_{-1}(\Omega, b) = \pm 2i\sqrt{\pi}\, e^{-\Omega b^2} Q(\mp ib\sqrt{\Omega}), \qquad \operatorname{Im} b \gtrless 0. \tag{24}$$

Since one notes from Eq. (23) that

$$A_{-N}(\Omega, b) = \frac{1}{N-1}\frac{d}{db} A_{-N+1}(\Omega, b), \qquad N = 2, 3, \ldots, \tag{25}$$

all A_{-N}, $N \geq 2$, can be inferred from A_{-1} in Eq. (24) by successive differentiation with respect to b. The integral in Eq. (23) is uniformly convergent for $\operatorname{Im} b \neq 0$, so the differentiation under the integral sign implied in Eq. (25) is permitted. Evaluation of the derivatives of Q is readily accomplished via the formula [see Eq. (14a)]

$$\frac{d}{dy} Q(y) = -e^{-y^2}, \tag{26}$$

so

$$A_{-2}(\Omega, b) = -2\sqrt{\Omega\pi} - 2b\Omega A_{-1}(\Omega, b), \tag{27a}$$

etc.

One notes that as $b \to 0$, the dependence on Ω is $O(\sqrt{\Omega})$ for A_{-2} and $O(1)$ for A_{-1}. Thus, a higher-order pole in the vicinity of the saddle point yields a larger value for the integral than a first-order pole. If b is large enough so that $|b|\sqrt{\Omega} \gg 1$, one can employ in Eq. (27a) the asymptotic expansion in Eq. (20). The result is found to be

$$A_{-2}(\Omega, b) \sim \sqrt{\frac{\pi}{\Omega}} \frac{1}{b^2}, \qquad \sqrt{\Omega}\,|b| \gg 1, \tag{27b}$$

which agrees with that obtained from Eq. (23) by a direct asymptotic evaluation.

4.4c Branch-Point Singularity

If $f(z)$ in Eq. (1) has an algebraic branch-point singularity z_b near a first-order saddle point z_s, then in the transformed integral of Eq. (4.2.3), G(s) will have a corresponding singularity at $s = b$ in the complex s plane. It is then required to evaluate integrals of the form

$$I(\Omega, b) = \int_{-\infty}^{\infty} s^n (s - b)^\alpha e^{-\Omega s^2}\, ds, \tag{28}$$

where $n = 0, 1, 2\ldots$ and $0 < \alpha < 1$. For $\alpha > 1$, factors of $(s - b)^m$, $m=$ integer, may be split off to keep α in the range $0 < \alpha < 1$; a dependence on $(s - b)^{-\beta}$, $\beta > 0$, may be accommodated by differentiating I with respect to b. By the change of variable

$$u = (s - b)^\alpha, \tag{29}$$

the integral in Eq. (28) is transformed into

$$I(\Omega, b) = \frac{1}{\alpha} \int (u^{1/\alpha} + b)^n u^{1/\alpha} e^{-\Omega (u^{1/\alpha} + b)^2}\, du, \tag{30}$$

with the endpoints of the integration path placed appropriately at infinity. Saddle points in the complex u plane are now located at $u = 0, (-b)^\alpha$. If $\alpha = 1/m$, $m = 2, 3, \ldots$, the $(m - 1)$th-order branch-point singularity in the s plane is seen to give rise to a cluster of m saddle points surrounding $u = 0$. For the special case $m = 2$, $I(\Omega, b)$ is expressible in terms of the parabolic cylinder function in Eq. (4.5.36). For a more general discussion, see Reference 11.

4.4d Uniform Asymptotic Evaluation of a Typical Diffraction Integral

We return now to the evaluation of the integral in Eq. (4.2.23) for the case when the pole at $z = \beta$ is situated near the saddle point $z_s = \alpha$. The representation of $I_1(\Omega, \alpha, \beta)$ as in Eq. (4.2.30) still applies. However, G(s) in Eq. (4.2.30b) should now be represented as in Eq. (5), with the pole contribution exhibited separately. To determine the behavior of $(z - \beta)^{-1}$ as a functions of s, we first expand s in Eq. (4.2.27b) (with the plus sign) in a power series about $z = \beta$:

$$s = \sqrt{2}\, e^{i\pi/4}\left[\left(\sin\frac{\beta-\alpha}{2}\right) + \tfrac{1}{2}\left(\cos\frac{\beta-\alpha}{2}\right)(z-\beta) - \tfrac{1}{8}\left(\sin\frac{\beta-\alpha}{2}\right)(z-\beta)^2 + \cdots\right], \tag{31}$$

and invert the series (see Appendix A) to obtain

$$z - \beta = \frac{\sqrt{2}}{e^{i\pi/4}\cos[(\beta-\alpha)/2]}(s-b) + \frac{\tan[(\beta-\alpha)/2]}{2i\cos^2[(\beta-\alpha)/2]}(s-b)^2 + \cdots, \tag{31a}$$

where

$$b = \sqrt{2}\, e^{i\pi/4}\sin\frac{\beta-\alpha}{2}. \tag{31b}$$

Thus,

$$\frac{1}{z-\beta} = \frac{e^{i\pi/4}\cos[(\beta-\alpha)/2]}{\sqrt{2}}\frac{1}{s-b} + (\text{terms finite at } s=b) \tag{32}$$

and a in Eq. (5) is given by [see also Eq. (2a)]

$$a = \lim_{s\to b}[G(s)(s-b)] = \frac{e^{i\pi/4}\cos[(\beta-\alpha)/2]}{\sqrt{2}}\left(\frac{dz}{ds}\right)_{s=b} = 1, \tag{33}$$

with the value of dz/ds at $s = b$ obtained from Eq. (31a). By Eq. (16) the asymptotic expansion for the integral in Eqs. (4.2.23) or (4.2.30) as $\Omega \to \infty$ can be written down directly as

$$I_1(\Omega, \alpha, \beta) \sim 2\pi i e^{i\Omega\cos(\beta-\alpha)}\epsilon(\beta) + e^{i\Omega}\left[\pm i2\sqrt{\pi}\, e^{-\Omega b^2}Q(\mp ib\sqrt{\Omega}) + T_e\left(\sqrt{-\frac{d}{d\Omega}}\right)\sqrt{\frac{\pi}{\Omega}}\right], \qquad \operatorname{Im} b \gtrless 0. \tag{34}$$

where $\epsilon(\beta)$ is defined in Eq. (4.2.30), b is given in Eq. (31b), and

$$T(s) = \frac{1}{z-\beta}\frac{dz}{ds} - \frac{1}{s-b}, \tag{34a}$$

$$T(0) = \frac{1}{\alpha-\beta}\sqrt{2}\, e^{-i\pi/4} + \frac{e^{-i\pi/4}}{\sqrt{2}\sin[(\beta-\alpha)/2]} \tag{34b}$$

It is of interest to note that Eq. (34) is a continuous function of b, although various terms therein are discontinuously represented. This is verified upon inspection of Fig. 4.2.3, Eq. (4.2.30), and Eq. (9). If the pole at $s = b$ crosses the real axis in Fig. 4.2.3, the term inside the brackets in Eq. (34) experiences a jump as in Eq. (9). However, the first term on the right-hand side of Eq. (34) also changes discontinuously under these conditions and compensates exactly for the first-mentioned discontinuity. For values of b such that $|b|\sqrt{\Omega} \gg 1$, Eq. (34) reduces via Eq. (20) to Eq. (4.2.31). For a more compact way of writing Eq. (34), see the remarks following Eqs. (19).

The special case where β is real is of particular importance in several of the diffraction problems to be treated subsequently. In this instance, $\sin[(\beta-\alpha)/2]$ in Eq. (31b) is real and, for $|\beta-\alpha|<\pi$,

$$\mp ib\sqrt{\Omega} = (1-i)\xi, \qquad \xi = \sqrt{\Omega}\left|\sin\frac{\beta-\alpha}{2}\right|, \qquad \operatorname{Im} b \gtrless 0. \tag{35}$$

Moreover, $\operatorname{Im} b > 0$ if $\beta-\alpha>0$, and $\operatorname{Im} b<0$ if $\beta-\alpha<0$. Thus, the following term in Eq. (34) can be rewritten for $\operatorname{Im} b \gtrless 0$ as

$$\pm i2\sqrt{\pi}\, e^{-\Omega b^2}Q(\mp ib\sqrt{\Omega}) = 2i\sqrt{\pi}\operatorname{sgn}(\beta-\alpha)e^{-2i\xi^2}Q[(1-i)\xi], \tag{36}$$

where

$$\operatorname{sgn}(\beta-\alpha) = \pm 1, \qquad \beta-\alpha \gtrless 0. \tag{36a}$$

The function

$$e^{i\pi/4}Q[(1-i)\xi] = e^{i\pi/4}\int_{(1-i)\xi}^{\infty} e^{-x^2}\,dx = e^{i\pi/4}\left[\frac{1}{2}\sqrt{\pi} - \int_0^{(1-i)\xi} e^{-x^2}\,dx\right] \tag{37a}$$

can be expressed via the change of variable

$$x = \frac{\sqrt{\pi}}{2}(1-i)t \tag{37b}$$

in terms of the well-tabulated Fresnel integrals $C(x)$ and $S(x)$ as

$$e^{i\pi/4}Q[(1-i)\xi] = \frac{1}{2}\sqrt{\pi}\, e^{i\pi/4} - \sqrt{\frac{\pi}{2}}\left[C\left(\frac{2\xi}{\sqrt{\pi}}\right) + iS\left(\frac{2\xi}{\sqrt{\pi}}\right)\right], \tag{37c}$$

where

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt, \qquad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt. \tag{37d}$$

To provide an estimate of how large ξ has to be before the asymptotic representation for $Q[(1-i)\xi]$ in Eq. (20) can be employed, we have plotted the function[12]

$$F(\xi) = \frac{2}{\sqrt{\pi}}e^{-i2\xi^2}Q[(1-i)\xi], \qquad \xi \geq 0. \tag{38}$$

For $\xi \to 0$, $F(\xi)\to 1$, while for $\xi \gg 1$ one has, from Eq. (20),

$$F(\xi) \sim \frac{e^{i\pi/4}}{\xi\sqrt{2\pi}}, \qquad \xi \gg 1 \tag{38a}$$

Upon comparing the plots of Eqs. (38) and (38a) shown in Fig. 4.4.3(a), one notes that the lowest-order asymptotic formula in Eq. (38a) holds with very good accuracy when $\xi \geq 3$. In terms of this estimate one finds from Eq. (35) that the "transition region," inside which the simple asymptotic representation in Eq. (4.2.31) fails, is given approximately by $|\beta-\alpha| \leq 6/\sqrt{\Omega}$. An analogous estimate can be found for the case of a double pole singularity, in which event one requires values for the derivative of $F(\xi)$[see Eq. (25)]. The function[13]

$$\bar{F}(\xi) = 1 - 2\sqrt{2}\,\xi e^{-i(2\xi^2+\pi/4)}Q[(1-i)\xi], \tag{39}$$

which occurs in this connection, as well as its asymptotic approximation

$$\bar{F}(\xi) = \frac{i}{4\xi^2}, \qquad \xi \gg 1, \tag{39a}$$

are plotted in Fig. 4.4.3(b).

4.5 NEARBY FIRST-ORDER SADDLE POINTS

4.5a Two Saddle Points

If in the integrand of

$$I(\Omega) = \int_{\mathrm{SDP}} f(z)e^{\Omega q(z)}\, dz, \tag{1}$$

the function $f(z)$ is regular near the two adjacent first-order saddle points $z_{1,2}$ of $q(z)$ [i.e., $q'(z_{1,2}) = 0$ and $q''(z_{1,2}) \neq 0$ unless $z_1 = z_2$], the asymptotic approximation of $I(\Omega)$, uniformly valid as $z_1 \to z_2$, is given by [see Eq. (23) for a complete asymptotic expansion]:

$$I(\Omega) \sim \frac{1}{2}[f(z_1)h_1 + f(z_2)h_2]\frac{e^{\Omega a_0}}{\Omega^{1/3}}C(\sigma\Omega^{2/3})$$
$$+ \frac{1}{2\sigma^{1/2}}[f(z_1)h_1 - f(z_2)h_2]\frac{e^{\Omega a_0}}{\Omega^{2/3}}C'(\sigma\Omega^{2/3}), \qquad \Omega \to \infty, \tag{2}$$

where

$$a_0 = \tfrac{1}{2}[q(z_1) + q(z_2)], \qquad \sigma^{1/2} = \{\tfrac{3}{4}[q(z_1) - q(z_2)]\}^{1/3}, \tag{2a}$$

$$h_{1,2} = \sqrt{\frac{\mp 2\sigma^{1/2}}{q''(z_{1,2})}}, \qquad C(\zeta) = \int_{\mathrm{SDP}} e^{\zeta t - t^3/3}\, dt, \qquad C'(\zeta) = \frac{dC(\zeta)}{d\zeta}. \tag{2b}$$

When $\sigma \to 0$ (i.e., $z_1 \to z_2$), one finds that $q''(z_{1,2}) \to 0$ at the resulting second-order saddle point. In this instance, $h_{1,2}$ takes on the limiting form

$$h_1 = h_2 = \left[\frac{-2}{q^{(3)}(z_s)}\right]^{1/3}, \qquad z_1 = z_2 = z_s. \tag{2c}$$

The multivalued expressions for $h_{1,2}$ and $\sigma^{1/2}$ may be defined by requiring that when $z_1 = z_2 = z_s$, $\arg h_{1,2} = \varphi \equiv \arg (dz)_{z_s}$, where dz is the integration element along the steepest-descent path SDP leading away from the second-order saddle point z_s (see Fig. 4.5.1). Arg $\sigma^{1/2}$ is then chosen to satisfy Eq. (2a) and to make the expression in Eq. (2b) compatible with Eq. (2c) when $z_1 \to z_2$. Alternatively, one may choose $\arg \sigma^{1/2}$ in a convenient manner and then deduce $\arg h_{1,2}$ by referring to the integration path. The integral in Eq. (2b) is identifiable in terms of the Airy integrals in Eqs. (4.2.32) and (4.2.34) for any specified allowable path P in Fig. 4.2.4 [see Eq. (14)]. When convenient, the integration variable t is defined to be real along the SDP leading away from z_s in Fig. 4.5.1(b); this implies that paths terminating in regions 2 or 3 of Fig. 4.2.4 are rotated into the real t axis.

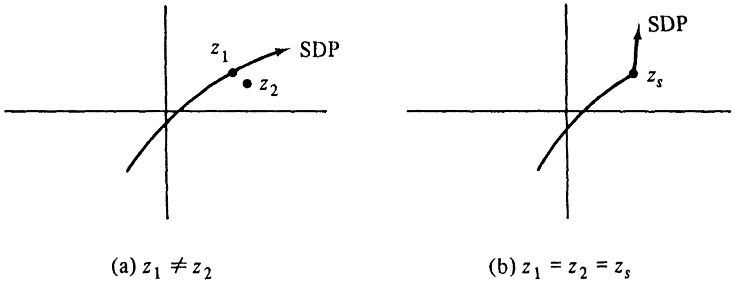


FIG. 4.5.1 Integration paths and saddle points.

When $z_1 = z_2$ ($\sigma = 0$), one employs Eq. (2c) and substitutes for $C(0)$ the following values into Eq. (2) [see Eq. (4.2.8)]:

$$C(0) = \begin{cases} i3^{-1/6}\Gamma(\frac{1}{3}), & \text{if } P = L_{32}, \\ e^{-i\pi/6}3^{-1/6}\Gamma(\frac{1}{3}), & \text{if } P = L_{21}, \\ e^{i\pi/6}3^{-1/6}\Gamma(\frac{1}{3}), & if\, P = L_{31}. \end{cases} \tag{3}$$

One verifies, on use of Eq. (3), that the leading term in Eq. (2) for $z_1 = z_2$ agrees with the second-order saddle-point result in Eq. (4.3.7) (with $M = 2$) provided that the path P for $C(0)$ is taken from $s = 0$ to $s = \infty$. In this case, the value for $C(0)$ as obtained from Eq. (4.2.8) is $3^{-2/3}\Gamma(\frac{1}{3})$. The expressions in Eq. (3) are given in terms of this result as follows:

$$\int_{L_{32}} e^{-s^3/3}\, ds = \int_{\infty e^{-i2\pi/3}}^{0} + \int_0^{\infty e^{i2\pi/3}} = (e^{i2\pi/3} - e^{-i2\pi/3}) \int_0^{\infty} = i3^{-1/6}\Gamma(\tfrac{1}{3}), \tag{4}$$

etc.

When $|\sigma|\Omega^{2/3} \gg 1$, one may employ the large-argument asymptotic approximation for the Airy-type function $C(\sigma\Omega^{2/3})$. To be specific, let it be assumed that the path SDP in Eq. (2b) is the same as L_{32} in Fig. 4.2.4. Then, from Eq. (14) and Eq. (4.2.42),

$$C(\sigma\Omega^{2/3}) = 2\pi i \operatorname{Ai}(\sigma\Omega^{2/3}) \sim \frac{i\sqrt{\pi}}{\sigma^{1/4}\Omega^{1/6}} e^{-(2/3)\Omega\sigma^{3/2}}, \tag{5a}$$

$$C'(\sigma\Omega^{2/3}) \sim -i\sqrt{\pi}\, \sigma^{1/4}\Omega^{1/6} e^{-(2/3)\Omega\sigma^{3/2}}, \tag{5b}$$

thereby reducing Eq. (2) to

$$I(\Omega) \sim \sqrt{\frac{-2\pi}{\Omega q''(z_2)}} f(z_2) e^{\Omega q(z_2)}, \tag{6}$$

the correct formula appropriate to an *isolated* first-order saddle point at z_2. Similar results are obtained when any of the other integration contours in Fig. 4.2.4 are involved.

Since $\Omega \to \infty$, the condition $|\sigma|\Omega^{2/3} \to \infty$, which defines widely separated saddle points, is satisfied when $|\sigma| \sim \Omega^{-(2/3)+\alpha}$, where α is positive but may be small. Thus, the uniform approximation in Eq. (2) need be employed only when

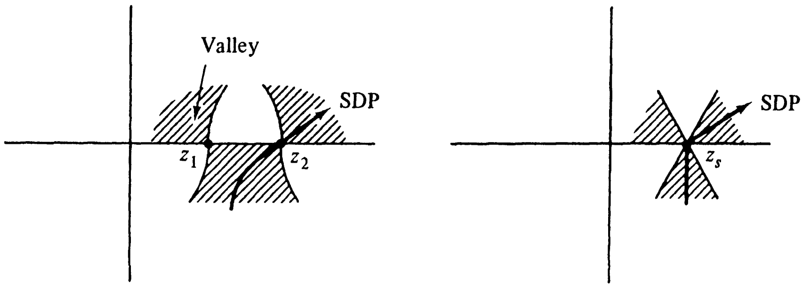
$\sigma = O(\Omega^{-2/3})$; for larger values of σ, each of the two saddle points may be treated independently of the other. In the range of small σ, the first term in Eq. (2) dominates and one may employ the simpler formula resulting from direct application of Eq. (4.1.5):

$$I(\Omega) \sim f(z_i)h_i \frac{e^{\Omega a_0}}{\Omega^{1/3}} C(\sigma\Omega^{2/3}), \qquad \sigma \text{ small}, \tag{7}$$

where the subscript i stands for either 1 or 2 since $f(z_i)h_i$ is slowly varying in the vicinity of $z_1 \approx z_2$. It must, however, be kept in mind that Eq. (7) is valid only when σ is small (although $\sigma\Omega^{2/3}$ may be large), whereas no such restriction is imposed on the more general formula in Eq. (2). The transition from Eq. (7) to Eq. (6) follows on use of Eqs. (5a) and (2b), subject to σ remaining small. Equation (6) may then be used for larger values of σ. Together, Eqs. (7) and (6), or some other relevant form for different integration paths, constitute non-uniform (though overlapping) approximations of $I(\Omega)$ with respect to the parameter σ, in constrast to the more complicated uniform approximation in Eq. (2) which applies for arbitrary σ.

An important special case of Eq. (2) pertains to $q(z) = i\hat{q}(z)$, where $\hat{q}(z)$ is a real function of z with real saddle points $z_{1,2}$. Let us assume that $\hat{q}''(z_1) < 0$, $\hat{q}''(z_2) > 0$, and that the integration path traverses the saddle point z_2 as shown in Fig. 4.5.2(a). It then follows from Fig. 4.5.2(b) that when $z_1 = z_2 = z_s$, $\varphi = \arg(dz)_{z_s} = \pi/6$, and, from Eq. (16c), $\hat{q}^{(3)}(z_s) > 0$. Thus, from Eq. (2c),

$$h_1 = h_2 = \left[\frac{2}{\hat{q}^{(3)}(z_s)}\right]^{1/3} e^{i\pi/6}, \text{ when } z_1 = z_2 = z_s, \tag{8}$$



(a) $z_1 \neq z_s; \hat{q}''(z_1) < 0, \hat{q}''(z_2) > 0$ (b) $z_1 = z_2 = z_s$

FIG. 4.5.2 Integration paths when $q(z) = i\hat{q}(z)$.

where the positive value of the cube root is taken. From Eq. (16b), $\arg \sigma^{3/2} = \pi/2$, whence $\arg \sigma^{1/2} = (\pi/6) + (2n\pi/3)$, $n = 0, 1, 2$; comparison with Eqs. (2b) and (8) requires the choice $n = 1$. Taking the integration variable t in the integral formula for $C(\zeta)$ in Eq. (2b) as positive along the path segment leading away from z_s in Fig. 4.5.2(b), one finds from Eq. (14) that $C(\zeta) = \pi[\mathrm{Bi}(\zeta) + i\,\mathrm{Ai}(\zeta)]$, where in view of the definition of $\arg \sigma^{1/2}$, $\zeta = \sigma\Omega^{2/3} = |\sigma|\Omega^{2/3} \times \exp(-i\pi/3) = -|\sigma|\Omega^{2/3} \exp(+i2\pi/3)$. Use of Eq. (B20c) then yields a formula wherein the arguments of Ai and Bi are negative real, so Eq. (2) becomes

$$I(\Omega) \sim \frac{\pi}{2}[f(z_1)h_1 + f(z_2)h_2]\frac{e^{\Omega a_0}}{\Omega^{1/3}}e^{-i\pi/6}[\mathrm{Ai}(-|\sigma|\Omega^{2/3}) + i\,\mathrm{Bi}(-|\sigma|\Omega^{2/3})]$$
$$+ \text{[analogous term with } C'], \tag{9}$$

where $q = i\hat{q}$ is to be employed in the definitions of a_0 and σ, and it is recalled that $\arg h_{1,2} = \pi/6$. For an extension of these results to the case where the saddle points move into the complex plane after coalescing at z_s, see Eqs. (5.8.59)–(5.8.61).

Analytical details

To transform the integral in Eq. (1) into the canonical form in Eq. (4.1.5), we use Eq. (4.1.7), as is appropriate when $q(z)$ has two adjacent saddle points at z_1 and z_2,

$$q(z) = \tau(s) = a_0 + \sigma s - \frac{s^3}{3}, \tag{10}$$

and obtain, as in Eq. (4.1.8),

$$a_0 = \tau(0) = \tfrac{1}{2}[q(z_1) + q(z_2)], \tag{11}$$

$$\tfrac{2}{3}\sigma^{3/2} = \tfrac{1}{2}[q(z_1) - q(z_2)]. \tag{12}$$

Special attention must be given to the choice of the proper branch of $(\sigma^{3/2})^{2/3}$ required for the evaluation of σ from Eq. (12). Upon substituting Eq. (10) into Eq. (4.1.5), one obtains the first-order asymptotic approximation for $I(\Omega)$ in Eq. (1) as $\Omega \to \infty$, valid for small values of σ:

$$I(\Omega) \sim \Omega^{-1/3}G(0)e^{a_0\Omega}C(\sigma\Omega^{2/3}), \tag{13}$$

with a_0 and σ defined as in Eqs. (11) and (12) and

$$G(0) = \left[f(z)\frac{dz}{ds}\right]_{s=0}, \tag{13a}$$

$$C(\zeta) = \int_P e^{\zeta t - t^3/3}\,dt. \tag{13b}$$

The integral in Eq. (13b) is readily identified in terms of the Airy integrals in Eqs. (4.2.37) for any specified allowable path P in Fig. 4.2.4. Since $\int_{L_{21}} - \int_{L_{31}} + \int_{L_{32}} = 0$, one notes that

$$C(\zeta) = \begin{cases} 2\pi i\,\mathrm{Ai}(\zeta), & \text{if } P = L_{32}, \\ \pi[\mathrm{Bi}(\zeta) - i\,\mathrm{Ai}(\zeta)], & \text{if } P = L_{21}, \\ \pi[\mathrm{Bi}(\zeta) + i\,\mathrm{Ai}(\zeta)], & \text{if } P = L_{31}. \end{cases} \tag{14}$$

Equation (13) is valid for small values of σ (i.e., for $z_1 \approx z_2$). Since $f(z)$ and dz/ds are assumed to be regular and slowly varying functions in the vinicity of $s = 0$, we may then write approximately

$$f(z)|_{s=0} \approx f(z_1) \approx f(z_2), \tag{15a}$$

and

$$\left(\frac{dz}{ds}\right)_{s=0} \approx \left(\frac{dz}{ds}\right)_{s=\sqrt{\sigma}} \approx \left(\frac{dz}{ds}\right)_{s=-\sqrt{\sigma}} \tag{15b}$$

From Eq. (10) one finds that

$$h_1 \equiv \left(\frac{dz}{ds}\right)_{s=\sqrt{\sigma}} = \sigma^{1/4}\sqrt{\frac{-2}{q''(z_1)}}, \qquad h_2 \equiv \left(\frac{dz}{ds}\right)_{s=-\sqrt{\sigma}} = \sigma^{1/4}\sqrt{\frac{2}{q''(z_2)}}, \tag{15c}$$

where the sign ambiguity introduced by the square roots is resolved on determining arg (dz/ds), as mentioned after Eq. (2c). This completes the derivation of the result in Eq. (7).

If the two first-order saddle points $z_{1,2}$ coincide, the point $z_1 = z_2 = z_s$ is a second-order saddle point, since both $q'(z_s)$ and $q''(z_s)$ vanish. The derivative $(dz/ds)_{s=0}$ must have a unique value, implying that $q''(z_2) \approx -q''(z_1)$ for $z_1 \approx z_2$ (i.e., $\sigma \approx 0$); it then follows that $q''(z_2) = q''(z_1) = 0$ if $z_1 = z_2$. Thus, in Eq. (10) for $\sigma = 0, \tau'(0) = \tau''(0) = 0$ and hence the origin in the s plane is likewise a second-order saddle point. To evaluate the resulting indeterminate form for $(dz/ds)_{s=0}$ in Eq. (15c), we observe first that, as $z_1 \to z_2$,

$$q^{(3)}(z_1) \approx q^{(3)}(z_2) \approx q^{(3)}(z_s), \qquad z_s = \frac{z_1 + z_2}{2}, \tag{16a}$$

where z_s is the second-order saddle point for which $q''(z_s) = 0$. This is verified from a power-series expansion of $q''(z_{1,2})$ about z_s after recalling that $q''(z_1) \approx -q''(z_2)$. Also, from Eq. (12),

$$\frac{4}{3}\sigma^{3/2} = q(z_1) - q(z_2) = q'(z_2)(z_1 - z_2) + \frac{q''(z_2)}{2}(z_1 - z_2)^2 + \frac{q^{(3)}(z_2)}{6}(z_1 - z_2)^3 + \cdots = -\frac{1}{12}q^{(3)}(z_s)(z_1 - z_2)^3 \tag{16b}$$

where we have utilized Eq. (16a), $q'(z_2) = 0$, and

$$q''(z_2) = q''(z_s) + q^{(3)}(z_s)(z_2 - z_s) + \cdots = -\tfrac{1}{2}q^{(3)}(z_s)(z_1 - z_2) + \cdots. \tag{16c}$$

Thus,

$$\sigma^{1/2} = \tfrac{1}{2}[-\tfrac{1}{2}q^{(3)}(z_s)]^{1/3}(z_1 - z_2) + \cdots, \tag{17}$$

and substitution of Eqs. (16c) and (17) into (15c) yields the result in Eq. (2c):

$$\left(\frac{dz}{ds}\right)_{s=0} = \left[\frac{-2}{q^{(3)}(z_s)}\right]^{1/3}, \qquad \text{when } z_1 = z_2 = z_s. \tag{18}$$

When $\sigma\Omega^{2/3} \gg 1$, one may employ Eq. (5a) to reduce Eq. (13), the integration path being taken along L_{32} in Fig. 4.2.4. Subject to Eqs. (15), the resulting form is that given in Eq. (6), so the large-argument asymptotic approximation of Eq. (13) connects onto the isolated saddle-point result. In all the considerations based on Eqs. (13) or (7), it has been necessary to keep σ small.

For larger σ, descriptive of isolated saddle points, one must deal not with Eq. (13) but with Eq. (6) and its counterpart for the saddle point at z_1. Thus, as noted earlier, the expression in Eq. (13) is non-uniform in the sense that it cannot accommodate all ranges of σ. A uniform expression valid for arbitrary values of σ is given in Eq. (2) and represents the first terms of a complete asymptotic expansion of the integral in Eq. (1).

To obtain such an expansion, we employ the canonical form in Eq. (4.1.5) with $\tau(s)$ given by Eq. (10). To achieve uniformity with respect to σ, one expands $G(s)$ not in a power series about $s = 0$, as in Eq. (4.2.4), but rather in series involving powers of the polynomial $\xi = \tau'(s) = \sigma - s^2$ and the coefficients $b_{n,m}$:[14]

$$G(s) = \sum_{n=0}^{\infty} b_{n,0}\xi^n + \sum_{n=0}^{\infty} b_{n,1}\xi^n s. \tag{19a}$$

This type of expansion permits an asymptotic-series representation for $I(\Omega)$,

$$I(\Omega) \sim \sum_{m=0}^{1}\sum_{n=0}^{\infty} b_{n,m} I_{n,m}(\Omega), \tag{19b}$$

that involves the integrals

$$I_{n,m}(\Omega) = e^{\Omega\tau(0)}\bar{I}_{n,m}(\Omega) \tag{19c}$$

$$\bar{I}_{n,m}(\Omega) = \int_P s^m \xi^n e^{\Omega(\sigma s - s^3/3)}\, ds, \qquad m = 0, 1. \tag{19d}$$

The lowest-order term ($m = n = 0$), $\bar{I}_{0,0}$, is identically the function $\Omega^{-1/3}C(\zeta)$, with $C(\zeta)$ defined in Eq. (13b). Upon introducing the change of variable $t = \Omega^{1/3}s$, one obtains

$$\bar{I}_{n,m}(\Omega) = \Omega^{-(2n+m+1)/3}\int_P t^m(\zeta - t^2)^n e^{\zeta t - t^3/3}\, dt, \qquad \zeta = \sigma\Omega^{2/3}. \tag{20}$$

From Eq. (13b) one finds

$$C^{(n)}(\zeta) \equiv \frac{d^n}{d\zeta^n}C(\zeta) = \int_P t^n e^{\zeta t - t^3/3}\, dt, \tag{21a}$$

and, from Eq. (4.2.33),

$$C''(\zeta) = \zeta C(\zeta). \tag{21b}$$

Expressions for the first few $C^{(n)}$ follow as

$$\begin{aligned} &C^{(3)} = C + \zeta C'; \quad C^{(4)} = \zeta^2 C + 2C'; \quad C^{(5)} = 4\zeta C + \zeta^2 C';\\ &C^{(6)} = (4 + \zeta^3)C + 6\zeta C'; \quad C^{(7)} = 9\zeta^2 C + (10 + \zeta^3)C';\\ &C^{(8)} = (28\zeta + \zeta^4)C + 12\zeta^2 C'. \end{aligned} \tag{21c}$$

It is not difficult to prove the recursion relations

$$\begin{aligned} &\bar{I}_{n,0} = -\frac{2n-1}{\Omega}\bar{I}_{n-2,1}, \qquad \bar{I}_{n,1} = -\frac{1}{\Omega}\bar{I}_{n-1,0} - \frac{2(n-1)}{\Omega}\bar{I}_{n-2,2},\\ &\bar{I}_{n,2} = \sigma\bar{I}_{n,0} - \bar{I}_{n+1,0}, \qquad \bar{I}_{n,1} = -\frac{1-2(n-1)}{\Omega}\bar{I}_{n-1,0} - \frac{2\sigma(n-1)}{\Omega}\bar{I}_{n-2,0}. \end{aligned} \tag{21d}$$

Upon expanding the factor $(\zeta - t^2)^n$ in Eq. (20) by the binomial theorem and employing Eqs. (21), one finds that

$$\bar{I}_{0,0} = \frac{1}{\Omega^{1/3}} C(\zeta), \quad \bar{I}_{1,0} = 0, \quad \bar{I}_{2,0} = \frac{2}{\Omega^{5/3}} C'(\zeta),$$
$$\bar{I}_{3,0} = \frac{-4}{\Omega^{7/3}} C(\zeta), \quad \bar{I}_{4,0} = \frac{12\zeta}{\Omega^{3}} C(\zeta) = \frac{12\sigma}{\Omega^{7/3}} C(\zeta), \tag{22a}$$

and

$$\bar{I}_{0,1} = \frac{1}{\Omega^{2/3}} C'(\zeta), \quad \bar{I}_{1,1} = \frac{-1}{\Omega^{4/3}} C(\zeta),$$
$$\bar{I}_{2,1} = \frac{2\zeta}{\Omega^{2}} C(\zeta) = \frac{2\sigma}{\Omega^{4/3}} C(\zeta), \quad \bar{I}_{3,1} = -\frac{10}{\Omega^{8/3}} C'(\zeta), \tag{22b}$$

where ζ is defined in Eq. (20). The largest contribution arises from the $\bar{I}_{0,0}$ term, which involves the factor $\Omega^{-1/3}$, while all other terms contain at least a factor $\Omega^{-2/3}$. For any given path P which begins and ends in a shaded region in Fig. 4.2.4, the integrals $C(\zeta)$ and $C'(\zeta)$ are readily identified in terms of the Airy functions in Eq. (14) and their derivatives, respectively.

In view of Eqs. (22) and the recursion relations in Eqs. (21), one notes that the asymptotic expansion in Eq. (19b) for the integral in Eq. (4.1.5) has the form

$$I(\Omega) \sim e^{\Omega\tau(0)} \left\{ \frac{b_{0,0}}{\Omega^{1/3}} C(\zeta) + \frac{b_{0,1}}{\Omega^{2/3}} C'(\zeta) + C(\zeta) \sum_{n=1}^{\infty} \left[\frac{c_n}{\Omega^{(6n+1)/3}} + \frac{d_n}{\Omega^{(6n-2)/3}} \right] \right.$$
$$\left. + C'(\zeta) \sum_{n=1}^{\infty} \left[\frac{e_n}{\Omega^{(6n+2)/3}} + \frac{f_n}{\Omega^{(6n-1)/3}} \right] \right\}, \qquad \Omega \to \infty, \tag{23}$$

where the coefficients c_n, d_n, e_n, and f_n are polynomials in σ. The asymptotic character of the expansion is verified on referring to the comments following Eq. (4.2.18c). Since from Eq. (19a), $G(s_i) = b_{0,0} + b_{0,1} s_i$, $i = 1, 2$, one finds that $b_{0,0} = \frac{1}{2}[G(\sqrt{\sigma}) + G(-\sqrt{\sigma})]$ represents the average of the values of $G(s)$ at the saddle points $s_{1,2} = \pm\sqrt{\sigma}$, whereas $b_{0,1} = 1/(2\sqrt{\sigma})[G(\sqrt{\sigma}) - G(-\sqrt{\sigma})]$. Since $\sigma = O(\Omega^{-2/3})$ in the range where the double-saddle-point procedure must be employed (i.e., where σ is very small), $b_{0,0}$ can be approximated by $G(0)$. The first term in the expansion in Eq. (23) thus yields the first-order result in Eqs. (7) or (13). By retaining the $b_{0,0}$ and $b_{0,1}$ terms, one obtains the uniform approximation given in Eq. (2).

Example: Asymptotic evaluation of Hankel function

Consider the Hankel function $H_\nu^{(1)}(\Omega)$ defined by the integral

$$H_\nu^{(1)}(\Omega) = \frac{1}{\pi} \int_P e^{\Omega q(z)}\, dz, \qquad (\nu, \Omega \text{ positive}), \tag{24}$$

$$q(z) = i\left[\cos z + \left(z - \frac{\pi}{2}\right) \sin\alpha\right], \qquad \sin\alpha = \frac{\nu}{\Omega}, \tag{24a}$$

where the path $\bar{P}$ is the same as in Fig. 4.2.2. The saddle points of $q(z)$ are located at

$$q'(z) = -i[\sin z - \sin\alpha] = 0, \tag{25}$$

whence the pertinent solutions are

$$z_1 = \alpha, \qquad z_2 = \pi - \alpha. \tag{26}$$

We assume that $\nu/\Omega \leq 1$, so $0 < \alpha \leq \pi/2$, and seek a first-order asymptotic evaluation of the integral in Eq. (24) as $\Omega \to \infty$ and $\alpha \to \pi/2$ (i.e., when the order and argument of the Hankel function are both large and almost equal).

From Eqs. (24),

$$q(z_{1,2}) = \pm i\left[\cos\alpha - \left(\frac{\pi}{2} - \alpha\right)\sin\alpha\right]. \tag{27}$$

Since $q(z_{1,2})$ is imaginary, $\sigma^{3/2}$ as defined in Eq. (12) is imaginary and can be satisfied by $\sigma < 0$. In the following, we choose

$$\sigma = e^{-i\pi}|\sigma| = e^{-i\pi}\eta, \qquad \eta > 0, \tag{28a}$$

and subsequently determine $\arg h_{1,2}$ in Eq. (2) by referring to the integration path. From Eq. (28a),

$$s_{1,2} = \pm\sqrt{\sigma} = \mp i\sqrt{\eta}, \qquad \sqrt{\eta} > 0. \tag{28b}$$

Then, from Eq. (12),

$$\frac{2}{3}\eta^{3/2} = \cos\alpha - \left(\frac{\pi}{2} - \alpha\right)\sin\alpha \geq 0, \tag{29a}$$

$$\tau(0) = 0, \tag{29b}$$

while, from Eq. (15c),

$$\left(\frac{dz}{ds}\right)_{s_1} = \left(\frac{dz}{ds}\right)_{s_2} = \pm i\eta^{1/4}\left(\frac{2}{\cos\alpha}\right)^{1/2}, \tag{29c}$$

with the sign ambiguity noted explicitly. Since $f(z) = 1$ in Eq. (24) and in view of Eq. (29c), Eqs. (15a) and (15b) are satisfied exactly in this case. It is implied thereby that the expressions for $I(\Omega)$ in Eqs. (2) and (7) are equivalent, with a consequent removal of the restriction on σ in Eq. (7). Near $\alpha = \pi/2$, one obtains from Eq. (29a) and the requirement that $\sqrt{\eta} \geq 0$:

$$\frac{2}{3}\eta^{3/2} \approx \frac{1}{3}\left(\frac{\pi}{2} - \alpha\right)^3, \qquad \text{i.e., } \eta^{1/2} \approx 2^{-1/3}\left(\frac{\pi}{2} - \alpha\right), \tag{30a}$$

whence, from Eq. (29c),

$$\left(\frac{dz}{ds}\right)_{s_1} = \left(\frac{dz}{ds}\right)_{s_2} = \pm i2^{1/3} \qquad \text{when } \eta = 0. \tag{30b}$$

To determine the contour of integration in the s plane, it suffices to consider the transformation in Eq. (10) as $\sigma \to 0$ (i.e., $\eta \to 0$) in order to establish the location of the endpoints of the transformed path. Thus, we examine

$$q(z) = i\left[\cos z - \left(\frac{\pi}{2} - z\right)\right] = -\frac{s^3}{3}, \qquad (\eta = 0), \tag{31}$$

which can be written near $z = \pi/2$ as

$$\frac{i}{6}\left(\frac{\pi}{2} - z\right)^3 \approx \frac{s^3}{3}. \tag{31a}$$

Upon taking the cube root of Eq. (31a), one obtains

$$s \approx 2^{-1/3}\left(\frac{\pi}{2} - z\right)e^{i\pi/6}e^{i2n\pi/3}, \qquad n = 0, 1, 2, \tag{32}$$

where the last factor expresses the three possible values of the cube root of unity. The branch to be chosen is that for which ds/dz evaluated at $z = \pi/2$ assumes one of the values in Eq. (30b). The proper choice is $n = 2$, so that

$$s \approx i2^{-1/3}\left(z - \frac{\pi}{2}\right), \qquad z \approx \frac{\pi}{2}. \tag{32a}$$

It is thereby implied that selection of the minus sign in Eqs. (30b) and (29c) resolves the sign ambiguity. One notes from Eq. (32a) that $\arg s = 0$ when $z = (\pi/2) + iz_i$, $z_i < 0$, and that $\arg s = -\pi/2$ when z is real and $z < \pi/2$. It then follows from Eq. (31) and a continuity argument that the entire line $\mathrm{Re}\, z = \pi/2$, $\mathrm{Im}\, z < 0$ maps into the positive real s axis, while the remainder of the path $\bar{P}$ terminates in the shaded region in the third quadrant of Fig. 4.2.4. Thus, the contour $\bar{P}$ is transformed in the s plane into a path L_{31}, as shown in Fig. 4.2.4. From the direction of integration along $\bar{P}$ in the neighborhood of $z = \pi/2$, one confirms the choice of sign noted above.

The first-order asymptotic representation as $\Omega \to \infty$ for $H_\nu^{(1)}(\Omega)$ in Eq. (24) can now be written down directly from Eqs. (2) or (7), (14), (27), and (29):

$$H_\nu^{(1)}(\Omega) \sim \left(\frac{2}{\cos\alpha}\right)^{1/2}\frac{\eta^{1/4}}{\Omega^{1/3}}[\mathrm{Ai}(-\eta\Omega^{2/3}) - i\,\mathrm{Bi}(-\eta\Omega^{2/3})], \tag{33}$$

where α and η are defined in Eqs. (24a) and (29a), respectively. When $\alpha = \pi/2$, use of Eqs. (3) and (30b) yields

$$H_\Omega^{(1)}(\Omega) \sim \left(\frac{2}{\Omega}\right)^{1/3}[\mathrm{Ai}(0) - i\,\mathrm{Bi}(0)] = \left(\frac{2}{\Omega}\right)^{1/3}\frac{3^{-1/6}}{\pi}\Gamma\left(\frac{1}{3}\right)e^{-i\pi/3}. \tag{34a}$$

When α is sufficiently different from $\pi/2$ to yield $\eta\Omega^{2/3}$ large, we may employ the asymptotic expressions for the Airy functions in Eqs. (4.2.51) and obtain via Eq. (29a) the Debye formula[15]

$$H_\nu^{(1)}(\Omega) \sim \left(\frac{2}{\pi\Omega\cos\alpha}\right)^{1/2}\exp\left\{i\Omega\left[\cos\alpha - \left(\frac{\pi}{2} - \alpha\right)\sin\alpha\right] - i\pi/4\right\}. \tag{34b}$$

Since from Eq. (30a),

$$\eta \approx 2^{-2/3}\cos^2\alpha = 2^{-2/3}\left[1 - \left(\frac{\nu}{\Omega}\right)^2\right] \approx 2^{1/3}\frac{\Omega - \nu}{\Omega}, \tag{35a}$$

the condition $\eta\Omega^{2/3} \gg 1$, required for the validity of Eq. (34b), can be phrased as

$$\Omega - \nu \gg \Omega^{1/3}. \tag{35b}$$

If $\Omega - \nu = O(\Omega^{1/3})$, one must employ Eq. (33).

4.5b Three Saddle Points

When $q(z)$ in Eq. (1) has three neighboring collinear, equidistant, saddle points $z_{1,2,3}$, a suitable comparison function $\tau(s)$ is given in Eq. (4.1.9). Substitution into Eq. (4.1.5) and comparison with the integral representation for the parabolic cylinder function[15]

$$D_{-\nu}(t) = \frac{2e^{t^2/4}}{\Gamma(\nu)} \int_0^\infty p^{2\nu-1} e^{-(1/2)(t+p^2)^2}\, dp, \qquad \operatorname{Re} \nu > 0, \tag{36}$$

then yields the lowest-order (non-uniform) asymptotic approximation as $\Omega \to \infty$ (see reference 1 for a uniform approximation)

$$\bar{I}(\Omega) \sim \frac{G(0)\sqrt{\pi}}{2(2\Omega)^{1/4}} e^{\Omega(a_0 - a^2/2)} D_{-1/2}(\sqrt{2\Omega}\, a), \qquad a \text{ small}, \tag{37}$$

where $G(0) = f(z_2)(dz/ds)|_{s=0}$ while a and a_0 are obtained from Eq. (4.1.9a). This result contains only the contribution from the portion of the integration path that begins at $s = 0$ and ends in a valley region at $s = \infty$ wherein the integral converges. For an evaluation of the mapping derivative

$$\frac{dz}{ds} = \frac{-4s(a + s^2)}{q'(z)}, \qquad a = \sqrt{q(z_1) - q(z_2)}, \tag{38a}$$

at $s = 0$, L'Hôpital's rule may be applied and furnishes alternative expressions that remain valid when $z_1 \to z_2$ (i.e., $a \to 0$) and verify the slowly varying character of dz/ds near $s = 0$,

$$\left.\frac{dz}{ds}\right|_{s=0} = \sqrt{\frac{-4a}{q''(z_2)}} \approx \sqrt{\frac{8a}{q''(z_1)}} \approx i\left[\frac{24}{-q^{(4)}(z_2)}\right]^{1/4}; \tag{38b}$$

the proper definition of the radicals [i.e., arg (dz/ds)] remains to be carried out from a study of the transformation of the original integration path. The approximate equality of the various formulas in Eq. (38b) follows from the series expansions of $q(z)$ and its derivatives in the vicinity of the interior saddle-point location $z = z_2$; since the saddle points are assumed to be collinear and equidistant, $q(z - z_2) \cong q(z_2 - z)$ when $z \approx z_2$:

$$q(z) = q(z_2) + q''(z_2)\frac{(z - z_2)^2}{2} + q^{(4)}(z_2)\frac{(z - z_2)^4}{4!} + \cdots \tag{39a}$$

$$q'(z) = q''(z_2)(z - z_2) + q^{(4)}(z_2)\frac{(z - z_2)^3}{6} + \cdots \tag{39b}$$

$$q''(z) = q''(z_2) + q^{(4)}(z_2)\frac{(z - z_2)^2}{2} + \cdots \tag{39c}$$

Using $q'(z_1) = 0$, one finds from Eq. (39b) by letting $z = z_1$,

$$q''(z_2) \approx -\frac{q^{(4)}(z_2)(z_1 - z_2)^2}{6}, \tag{40a}$$

and from this result combined with Eq. (39c),

$$q''(z_1) \approx \tfrac{1}{3} q^{(4)}(z_2)(z_1 - z_2)^2 \approx -2q''(z_2). \tag{40b}$$

When $|\sqrt{2\Omega}\, a| \gg 1$, the parabolic cylinder function in Eq. (37) may be replaced by its large-argument approximation in Eqs. (7.5.66), and the resulting formula may be reduced to the conventional one for isolated saddle points.† Since $\Omega \gg 1$, $a = [q(z_1) - q(z_2)]^{1/2}$ may be small and still yield $|\sqrt{2\Omega}\, a| \gg 1$, thereby requiring the above analysis to be carried out only for $a = O(\Omega^{-1/2})$. The transition to the conventional result occurs when z_1, z_2, and z_3 are still approximately equal, under which circumstances the slowly varying function $f(z)$ may be approximated by $f(z_1) \approx f(z_2) \approx f(z_3)$.

In the limit $z_1 = z_2 = z_3$, the formula

$$D_{-\nu}(0) = \frac{\sqrt{\pi}}{2^{\nu/2}\Gamma[(1+\nu)/2]}, \qquad \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}, \tag{41}$$

may be employed to reduce Eq. (37) to the correct asymptotic expression for $\bar{I}(\Omega)$ when the integrand has a third-order saddle point at z_2 [see Eq. (4.3.7)]:

$$\bar{I}(\Omega) \sim i\left[\frac{24}{-q^{(4)}(z_2)}\right]^{1/4} \frac{\Gamma(\frac{1}{4})}{4\Omega^{1/4}} f(z_2) e^{\Omega q(z_2)}. \tag{42}$$

Again, the proper choice of the radical remains to be clarified in any given situation.

Implicit in the preceding analysis is the assumption that $G(0) \neq 0$. If this condition is not satisfied and the first non-vanishing term in the power-series expansion of $G(s)$ about $s = 0$ is $G^{(n)}(0)s^n/n!$, where n is a positive integer, then one has, instead of Eq. (4.1.5),

$$\bar{I}(\Omega) \sim \frac{G^{(n)}(0)}{n!} \int_0^\infty s^n e^{\Omega\tau(s)}\, ds, \tag{43}$$

with $\tau(s)$ given in Eq. (4.1.9). From a comparison with Eq. (36), one derives the result

$$\bar{I}(\Omega) \sim \frac{G^{(n)}(0)\Gamma[(n+1)/2]}{2(2\Omega)^{(n+1)/4} n!} e^{\Omega(a_0 - a^2/2)} D_{-(n+1)/2}(\sqrt{2\Omega}\, a), \tag{44}$$

which contains Eq. (37) as the special case $n = 0$. Applications of these formulas may be found in Secs. 7.5d and 7.5e.

If the three saddle points of $q(z)$ are not equidistant and collinear, the canonical integral involves the more general exponent

$$q(z) = \tau(s) = \hat{a}_0 + a_2 s^2 + a_3 s^3 - s^4, \tag{45a}$$

whence $\tau'(s)$ has zeros at $s = 0$ and at the arbitrary points $s_{1,2}$. Alternatively, by not requiring that one of the zeros of $\tau'(s)$ is located at the origin, one may employ

†When a is positive, the integration path passes only through the saddle point at $s = 0$, while for negative real a another saddle point at $s = \pm i\sqrt{a}$ also lies on the path segment extending from $s = 0$ to $s = \infty$.

$$q(z) = \tau(s) = \hat{a}_0 + a_1 s + a_2 s^2 - s^4 \tag{45b}$$

The resulting canonical integral has been studied in Reference 16, and a summary of results and certain numerical data have been presented in Reference 17.

4.6 SADDLE POINTS NEAR AN ENDPOINT

4.6a Single Saddle Point

If in the integrand of

$$I(\Omega) = \int_{z_a}^{``\infty"} f(z) e^{\Omega q(z)}\, dz \tag{1}$$

the function $q(z)$ has a first-order saddle point at z_s [i.e.$q'(z_s) = 0$, $q''(z_s) \neq 0$] and the function $f(z)$ is regular in the vicinity of z_a and z_s, the asymptotic approximation of $I(\Omega)$, valid uniformly as $z_s \to z_a$, is given by

$$I(\Omega) \sim e^{\Omega q(z_s)} \left\{ \frac{f(z_s) h_s}{\sqrt{\Omega}} Q(\sqrt{\Omega}\, s_a) + \frac{e^{-\Omega s_a^2}}{2\Omega} \frac{[f(z_a) h_a - f(z_s) h_s]}{s_a} \right\}, \qquad \Omega \to \infty, \tag{2}$$

where

$$s_a = \sqrt{q(z_s) - q(z_a)}, \quad Q(y) = \int_y^\infty e^{-x^2}\, dx, \; h_a = \frac{-2s_a}{q'(z_a)},$$

$$h_s = \sqrt{\frac{-2}{q''(z_s)}} = h_a \bigg|_{z_a = z_s}. \tag{2a}$$

$\arg h_s \equiv \varphi = \arg (dz)_{z_s}$, where dz is an element along the steepest-descent path SDP through z_s, and $\arg s_a$ is defined so that $h_a \to h_s$ as $s_a \to 0$.

When $z_a = z_s$, one has $Q(0) = \frac{1}{2}\sqrt{\pi}$, whence Eq. (2) reduces to the result for coincident saddle point and endpoint [see Eq. (4.2.7) with inclusion of the factor $\frac{1}{2}$ to account for the semiinfinite integration interval]:

$$I(\Omega) \sim \frac{1}{2}\sqrt{\frac{-2\pi}{\Omega q''(z_s)}} f(z_s) e^{\Omega q(z_s)} \left[1 + O\left(\frac{1}{\sqrt{\Omega}}\right) \right], \qquad z_s = z_a; \tag{3}$$

for $\sqrt{\Omega}\,|s_a| \gg 1$, use of the asymptotic formula [Eq. (4.4.20)],

$$Q(\sqrt{\Omega}\, s_a) \sim \frac{e^{-\Omega s_a^2}}{2 s_a \sqrt{\Omega}} + \sqrt{\pi}\, U(-\operatorname{Re} s_a) \tag{4}$$

yields the result for isolated saddle point and endpoint,

$$I(\Omega) \sim \sqrt{\frac{-2\pi}{\Omega q''(z_s)}} f(z_s) e^{\Omega q(z_s)} U(-\operatorname{Re} s_a) - \frac{f(z_a) e^{\Omega q(z_a)}}{\Omega q'(z_a)}, \tag{5}$$

where $U(\alpha)$ equals unity or zero for $\alpha > 0$ and $\alpha < 0$ respectively.

An important special case arises when $q(z) = i\hat{q}(z)$, where $\hat{q}$ is a real function of z so that $\hat{q}(z_{s,a})$ is real when $z_{s,a}$ is real. In this instance, for real $z_{s,a}$,

$$I(\Omega) = \int_{z_a}^{\infty} f(z) e^{i\Omega\hat{q}(z)}\, dz$$
$$\sim e^{i\Omega\hat{q}(z_s)}\left\{\frac{f(z_s)h_s}{\sqrt{\Omega}} Q(\hat{s}_a\sqrt{\Omega}\, e^{\mp i\pi/4}) + \frac{e^{\pm i(\Omega\hat{s}_a^2+\pi/4)}}{2\Omega\hat{s}_a}[f(z_a)h_a - f(z_s)h_s]\right\},$$
$$\hat{q}''(z_s) \gtrless 0 \qquad (6)$$

where $\hat{s}_a = \pm|\hat{q}(z_s) - \hat{q}(z_a)|^{1/2}$ for $(z_a - z_s) \gtrless 0$, and

$$h_s = \sqrt{\frac{2}{|\hat{q}''(z_s)|}}\, e^{\pm i\pi/4}, \quad h_a = \frac{2|\hat{s}_a|e^{\pm i\pi/4}}{|\hat{q}'(z_a)|}, \qquad \text{for } \hat{q}''(z_s) \gtrless 0. \qquad (6a)$$

The reduction for $|\hat{s}_a|\sqrt{\Omega} \gg 1$ follows from Eq. (4) or (5),

$$I(\Omega) \sim \sqrt{\frac{2\pi}{\Omega|\hat{q}''(z_s)|}} f(z_s) e^{i\Omega\hat{q}(z_s)\pm i\pi/4} U(z_s - z_a) - \frac{f(z_a)e^{i\Omega\hat{q}(z_a)}}{i\Omega\hat{q}'(z_a)}, \qquad \hat{q}''(z_s) \gtrless 0, \qquad (6b)$$

and agrees with the result in Eq. (4.2.20).

Analytical details

Since $q(z)$ is assumed to have a first-order saddle point, the relevant transformation to the canonical form is $q(z) = \tau(s) = q(z_s) - s^2$, $s^2 > 0$ [see Eq. (4.1.6)] Thus, if s_a corresponds to z_a,

$$I(\Omega) = e^{\Omega q(z_s)} \int_{s_a}^{\infty} G(s) e^{-\Omega s^2}\, ds, \qquad G(s) = f(z)\frac{dz}{ds} = f(z)\frac{-2s}{q'(z)}, \qquad (7)$$

with s_a defined in Eq. (2a). Upon writing the integral in Eq. (7) as

$$\bar{I}(\Omega) = G(0)\int_{s_a}^{\infty} e^{-\Omega s^2}\, ds - \frac{1}{2\Omega}\int_{s_a}^{\infty} \frac{[G(s) - G(0)]}{s} \frac{d}{ds} e^{-\Omega s^2}\, ds \qquad (7a)$$

and, integrating the second term by parts, one finds

$$\bar{I}(\Omega) = \frac{G(0)}{\sqrt{\Omega}} Q(\sqrt{\Omega}\, s_a) + \frac{e^{-\Omega s_a^2}}{2\Omega s_a}[G(s_a) - G(0)]$$
$$+ \frac{1}{2\Omega}\int_{s_a}^{\infty} e^{-\Omega s^2} \frac{d}{ds}\left[\frac{G(s) - G(0)}{s}\right] ds, \qquad (7b)$$

which furnishes Eq. (2) via the definitions $h_s = (dz/ds)_{s=0}$ and $h_a = (dz/ds)_{s_a}$. The last term in Eq. (7b), omitted in Eq. (2),provides higher-order corrections; an asymptotic expansion of $I(\Omega)$ may be constructed by repeated decomposition of the remainder integrals as in Eq. (7a), with subsequent integration by parts.[18] When $s_a \to 0$, $z_a \to z_s$, one has $q'(z_a) \approx q''(z_s)(z_a - z_s)$, $s_a \approx \sqrt{-q''(z_s)/2} \times (z_a - z_s)$, so $h_a \to h_s$.

For the special case $q(z) = i\hat{q}(z)$, with $\hat{q}$ a real function of z, one has, as in Eq. (4.2.20),

$$h_s = \sqrt{\frac{2}{|\hat{q}''(z_s)|}}\, e^{\pm i\pi/4}, \qquad \text{for } \hat{q}''(z_s) \gtrless 0. \qquad (8a)$$

Since $\hat{q}'(z_s) = 0$, it follows that $\operatorname{sgn} \hat{q}'(z_a) = \pm \operatorname{sgn}(z_a - z_s)$ for $\hat{q}''(z_s) \gtrless 0$. Thus, from Eq. (2a) with $s_a = \hat{s}_a \exp(\mp i\pi/4)$ when $\hat{q}''(z_s) \gtrless 0$,

$$h_a = \frac{\pm i2s_a \operatorname{sgn}(z_a - z_s)}{|\hat{q}'(z_a)|} = \frac{2\hat{s}_a \operatorname{sgn}(z_a - z_s) e^{\pm i\pi/4}}{|\hat{q}'(z_a)|}, \qquad \text{for } \hat{q}''(z_s) \gtrless 0, \tag{8b}$$

where the definition of $\hat{s}_a$ in Eq. (6) ensures that $\arg h_a = \arg h_s$. Use of these relations in Eq. (2) furnishes the formula in Eq. (6).

4.6b Two First-Order Saddle Points

In the integrand of

$$I(\Omega) = \int_{z_a}^{``\infty"} f(z) e^{\Omega q(z)}\, dz = \int_{s_a}^{``\infty"} G(s) e^{\Omega \tau(s)}\, ds, \tag{9}$$

the function $q(z)$ has two first-order saddle points at $z_{1,2}$ [i.e., $q'(z_{1,2}) = 0$ and $q''(z_{1,2}) \neq 0$ unless $z_1 = z_2$]. If the function $f(z)$ is regular in the vicinity of z_1, z_2, and z_a, the asymptotic approximation of $I(\Omega)$, valid uniformly as $z_1 \to z_2 \to z_a$, is given by[19]

$$\begin{aligned} I(\Omega) \sim{}& [G(s_1) + G(s_2)] \frac{e^{\Omega a_0}}{2\Omega^{1/3}} C_a(s_1^2 \Omega^{2/3}) + [G(s_1) - G(s_2)] \frac{e^{\Omega a_0}}{2\Omega^{2/3} s_1} C_a'(s_1^2 \Omega^{2/3}) \\ &- \left[G(s_a) - \left(1 + \frac{s_a}{s_1}\right) \frac{G(s_1)}{2} - \left(1 - \frac{s_a}{s_1}\right) \frac{G(s_2)}{2} \right] \frac{e^{\Omega \tau(s_a)}}{\Omega(s_1^2 - s_a^2)}, \qquad \Omega \to \infty, \end{aligned} \tag{10}$$

where the variable s is related to z by the transformation

$$\begin{gathered} q(z) = \tau(s) = a_0 + \sigma s - \frac{s^3}{3}, \quad q(z_i) = \tau(s_i), \qquad i = 1, 2, a, \\ a_0 = \tfrac{1}{2}[q(z_1) + q(z_2)], \qquad \sigma^{1/2} = \{\tfrac{3}{4}[q(z_1) - q(z_2)]\}^{1/3} \equiv s_1 = -s_2, \end{gathered} \tag{10a}$$

and

$$\begin{gathered} G(s) = f(z)\frac{dz}{ds}, \quad C_a(\zeta) = \int_{\Omega^{1/3} s_a}^{``\infty"} e^{\zeta t - t^3/3}\, dt, \quad C_a'(\zeta) = \frac{dC_a(\zeta)}{d\zeta}, \\ \left(\frac{dz}{ds}\right)_{s_{1,2}} = \sqrt{\frac{\mp 2s_1}{q''(z_{1,2})}}, \qquad \left(\frac{dz}{ds}\right)_{s_a} = \frac{s_1^2 - s_a^2}{q'(z_a)}. \end{gathered} \tag{10b}$$

In order not to complicate the notation, the results in this section are given in terms of the variable s rather than the original variable z. The multivaluedness in the definition of σ and $(dz/ds)_{s_i}$ is resolved as in Sec. 4.5. The function $C_a(\zeta)$ may be termed an "incomplete" Airy function, in comparison with the Airy function $C(\zeta)$ in Eq. (4.5.14) for which both endpoints of the integration path lie at infinity; the location of the upper limit "∞" in the complex t plane is determined by comparison with the original path in the complex z plane as in Sec. 4.5 [for functions related to $C_0(\zeta)$ see Reference 20].

A number of special cases may be deduced from Eq. (10), which applies for arbitrary values of $z_{1,2,a}$ (or $s_{1,2,a}$). When $z_1 = z_2 = z_s$ (corresponding to $s_{1,2} =$

0), the two first-order saddle points coalesce into a single second-order saddle point. In this instance, one derives the following uniform asymptotic approximation when a second-order saddle point is situated near an endpoint of the integration interval:

$$I(\Omega) \sim G(0)\frac{e^{\Omega q(z_s)}}{\Omega^{1/3}}C_a(0) + G'(0)\frac{e^{\Omega q(z_s)}}{\Omega^{2/3}}C_a'(0)$$
$$+ \frac{e^{\Omega[q(z_s)-s_a^3/3]}}{\Omega s_a^2}[G(s_a) - G(0) - s_a G'(0)], \tag{11}$$

with $(dz/ds)_0 = [-2/q^{(3)}(z_s)]^{1/3}$ as in Eq. (4.5.18). When $z_1 = z_2 = z_a$ (corresponding to $s_a = s_1 = s_2 = 0$), the second-order saddle point coincides with the endpoint and Eq. (11) furnishes the expression

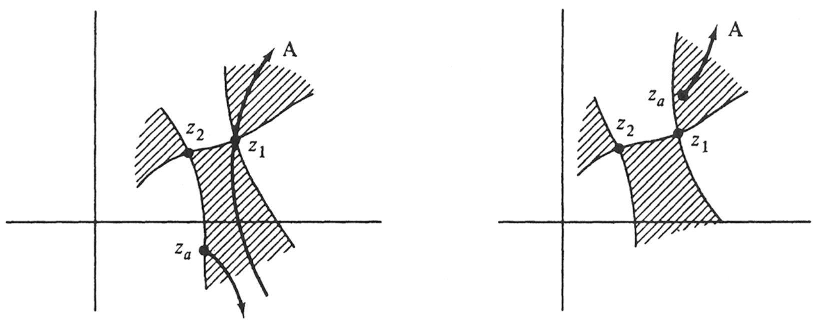
$$I(\Omega) \sim \frac{G(0)e^{\Omega q(z_s)}}{\Omega^{1/3}}\int_0^{\text{``}\infty\text{''}} e^{-t^3/3}\,dt + O\left(\frac{1}{\Omega^{2/3}}\right) \tag{12}$$

which agrees with the result in Eq. (4.3.7) since the integral can be expressed in terms of the gamma function of order $\frac{1}{3}$ [see Eq. (4.3.5)].

When neither z_1 nor z_2 is near z_a (i.e., $s_{1,2} \not\approx s_a$), the endpoint contribution to the incomplete Airy function may be evaluated separately from the saddle-point contribution, so by an integration by parts,

$$C_a(s_1^2\Omega^{2/3}) \sim \frac{\exp[\Omega\tau(s_a) - \Omega a_0]}{\Omega^{2/3}(s_a^2 - s_1^2)} + UC(\sigma\Omega^{2/3}), \tag{13}$$

where $C(\zeta)$ is the complete Airy function defined in Eq. (4.5.13b), descriptive of two adjacent saddle points situated far from other critical points in the integrand. U is a discontinuous factor which equals unity when the original integration path traverses the saddle-point region but equals zero when the original integration path proceeds from z_a to infinity in a valley without passing through the saddle-point region. Examples are shown in Fig. 4.6.1, where the integration path is assumed to end in the valley designated by A. When z_a is situated as in Fig. 4.6.1(a), it is necessary to pass through the saddle point at



(a) Endpoint plus saddle point contribution (b) Endpoint contribution only

FIG. 4.6.1 Integration paths in the complex z plane.

z_1, whence $U = 1$ in Eq. (13), whereas $U = 0$ for the case depicted in Fig. 4.6.1(b). The analogous formula for C'_a differs from Eq. (13) in that C is replaced by C' and the endpoint contribution is multiplied by $\Omega^{1/3}s_a$. Thus, Eq. (10) reduces for $s_{1,2} \not\approx s_a$ to

$$I(\Omega) \sim -G(s_a)\frac{e^{\Omega\tau(s_a)}}{\Omega(s_1^2 - s_a^2)} + \left[\frac{G(s_1) + G(s_2)}{2\Omega^{1/3}} e^{\Omega a_0} C(s_1^2\Omega^{2/3}) + \frac{G(s_1) - G(s_2)}{2\Omega^{2/3}s_1} e^{\Omega a_0} C'(s_1^2\Omega^{2/3})\right] U. \tag{14}$$

This formula accounts correctly for an isolated endpoint and for the contribution from two arbitrarily closely spaced saddle points (see Eqs. (5), (10b), and (4.5.2); it is noted that the first term in Eq. (14) can be written as $-f(z_a)[\Omega q'(z_a)]^{-1}$ $\exp[\Omega\tau(s_a)]$). When s_1 and s_2 are not adjacent, the last two terms in Eq. (14) may be simplified as in Eqs. (4.5.5) to furnish the isolated first-order saddle-point contributions. When s_1 and s_2 coalesce into a second-order saddle point at $s = 0$, the last two terms in Eq. (14) may be expressed as in Eqs. (4.5.2), with (4.5.2c) and (4.5.3).

A different reduction occurs when the two saddle points are not near one another ($z_1 \not\approx z_2$, or $s_{1,2} \not\approx 0$), but one of the saddle points is near the endpoint ($z_1 \approx z_a$, or $s_1 \approx s_a$). In this instance, it is found that [see Eqs. (26) and (27)]

$$C_a(s_1^2\Omega^{2/3}) - U_2\Omega^{1/3}\sqrt{\frac{\pi}{\Omega s_2}}\, e^{\Omega[\tau(s_2)-a_0]} \sim \frac{1}{\Omega^{1/3}s_1} C'_a(s_1^2\Omega^{2/3}) - \frac{e^{\Omega[\tau(s_a)-a_0]}}{\Omega^{2/3}s_1(s_1+s_a)}$$
$$\sim \Omega^{1/3}e^{\Omega[\tau(s_1)-a_0]}\left[\frac{Q(\alpha\sqrt{\Omega})}{\sqrt{\Omega s_1}} - \frac{e^{-\Omega\alpha^2}}{2\Omega\alpha}\left(\frac{2\alpha}{s_1^2 - s_a^2} + \frac{1}{\sqrt{s_1}}\right)\right], \qquad s_1 \not\approx 0, \tag{15}$$

where

$$\alpha = \sqrt{\tau(s_1) - \tau(s_a)}, \qquad Q(y) = \int_y^\infty e^{-x^2}\,dx, \tag{15a}$$

and U_2 equals 1 or 0, depending on whether or not the integration path traverses the isolated saddle point at s_2. Substitution of these formulas into Eq. (10) yields the reduced expression

$$I(\Omega) \sim \frac{G(s_1)e^{\Omega\tau(s_1)}Q(\alpha\sqrt{\Omega})}{\sqrt{\Omega s_1}} - \frac{e^{\Omega\tau(s_a)}}{\Omega}\left[\frac{G(s_1)}{2\alpha\sqrt{s_1}} + \frac{G(s_a)}{s_1^2 - s_a^2}\right] + U_2 G(s_2)e^{\Omega\tau(s_2)}\sqrt{\frac{\pi}{\Omega s_2}}, \tag{16}$$

which is the correct result for a first-order saddle point near an endpoint of the integration interval [see Eq. (2)], the last term providing the contribution from the isolated saddle point at s_2 when required. When $\alpha \to 0$, the saddle point at s_1 coincides with the endpoint at s_a, and Eq. (16) leads correctly to the expression [Note: $Q(0) = \sqrt{\pi}/2$]

$$I(\Omega) \sim \tfrac{1}{2}G(s_1)e^{\Omega\tau(s_1)}\sqrt{\frac{\pi}{\Omega s_1}}\left[1 + O\left(\frac{1}{\Omega}\right)\right] + U_2 G(s_2)e^{\Omega\tau(s_2)}\sqrt{\frac{\pi}{\Omega s_2}}. \tag{17}$$

When $\alpha\sqrt{\Omega} \gg 1$, the saddle point at s_1 and the endpoint at s_a are "widely separated" and permit use of the asymptotic formula [see Eqs. (4.4.18a) and (4.4.20)]

$$Q(\alpha\sqrt{\Omega}) \sim \frac{e^{-\Omega\alpha^2}}{2\alpha\sqrt{\Omega}} + \sqrt{\pi}\,U(-\operatorname{Re}\alpha); \tag{18}$$

hence Eq. (16) reduces to the conventional result where the two saddle points are separated from one another and from the endpoint:

$$I(\Omega) \sim U_1 G(s_1)e^{\Omega\tau(s_1)}\sqrt{\frac{\pi}{\Omega s_1}} - G(s_a)\frac{e^{\Omega\tau(s_a)}}{\Omega(s_1^2 - s_a^2)} + U_2 G(s_2)^{\Omega\tau(s_2)}\sqrt{\frac{\pi}{\Omega s_2}}. \tag{19}$$

$U_{1,2}$ equals unity when the integration path traverses the saddle points s_1 and s_2, respectively, but vanishes otherwise. In terms of functions defined in the original z plane of Eq. (9), Eq. (19) may be written equivalently as

$$I(\Omega) \sim U_1 f(z_1)e^{\Omega q(z_1)}\sqrt{\frac{-2\pi}{\Omega q''(z_1)}} - f(z_a)\frac{e^{\Omega q(z_a)}}{\Omega q'(z_a)} + U_2 f(z_2)e^{\Omega q(z_2)}\sqrt{\frac{-2\pi}{\Omega q''(z_2)}}. \tag{20}$$

Analytical details

Since $q(z)$ in Eq. (9) is assumed to have two neighboring first-order saddle points, the relevant transformation to the canonical form in the complex s plane is given as in Eq. (4.5.10), thereby leading to Eq. (10a) and to the second of Eqs. (9). The transformed integral in the variable s may now be written as

$$I(\Omega) = b_{0,0}\int_{s_a}^{\text{``}\infty\text{''}} e^{\Omega\tau(s)}\,ds + b_{0,1}\int_{s_a}^{\text{``}\infty\text{''}} s e^{\Omega\tau(s)}\,ds + \frac{1}{\Omega}\int_{s_a}^{\text{``}\infty\text{''}} \frac{G(s) - b_{0,0} - sb_{0,1}}{s_1^2 - s^2}\frac{d}{ds}e^{\Omega\tau(s)}\,ds, \tag{21}$$

where $b_{0,0}$ and $b_{0,1}$ are the first two coefficients in the uniform asymptotic expansion of the infinite double-saddle-point integral [see Eq. (4.5.23)]:

$$b_{0,0} = \tfrac{1}{2}[G(s_1) + G(s_2)], \qquad b_{0,1} = \frac{1}{2s_1}[G(s_1) - G(s_2)], \qquad s_2 = -s_1. \tag{21a}$$

Since $[G(s) - b_{0,0} - sb_{0,1}](s_1^2 - s^2)^{-1}$ remains bounded at $s = \pm s_1$, integration by parts on the third integral I_3 is allowed, with the result

$$I_3 = -\frac{1}{\Omega}\frac{G(s_a) - b_{0,0} - s_a b_{0,1}}{s_1^2 - s_a^2}e^{\Omega\tau(s_a)} - \frac{1}{\Omega}\int_{s_a}^{\text{``}\infty\text{''}}\frac{d}{ds}\left[\frac{G(s) - b_{0,0} - sb_{0,1}}{s_1^2 - s^2}\right]e^{\Omega\tau(s)}\,ds. \tag{22}$$

By regarding the function (d/ds) [ ] in the integrand of Eq. (22) as a new function $G(s)$, one may apply the decomposition in Eq. (21) successively to generate terms with increasing inverse powers of Ω. To a lowest order in Ω, the integral in Eq. (22) is omitted and Eq. (21) thus furnishes the result in Eq. (10), after changes of variable from s to t and the introduction of the parameter ζ

as in Eq. (4.5.2). From $dz/ds = \tau'(s)/q'(z) = (s_1^2 - s^2)/q'(z)$, one obtains the various forms of $(dz/ds)_{s_i}$ specified in Eq. (10b). As mentioned earlier, due regard must be given to the definition of the radicals appearing in these expressions.

The formulas for $I(\Omega)$ accommodating various special cases are derived from the general result in Eq. (10) by substitution of appropriate reductions of the canonical integrals $C_a(\zeta)$ and $C_a'(\zeta)$. The derivation of Eqs. (11) and (12) pertaining to $s_{1,2} = 0$ and $s_{1,2,a} = 0$, respectively, is evident. When $s_{1,2} \not\approx s_a$, the endpoint and saddle points are isolated and may be treated separately. As in Eq. (4.1.15), the endpoint contribution follows from integration by parts,

$$\frac{e^{\Omega a_0} C_a(s_1^2 \Omega^{2/3})}{\Omega^{1/3}} = \int_{s_a}^{``\infty"} e^{\Omega\tau(s)}\, ds = \frac{1}{\Omega} \int_{s_a}^{``\infty"} \frac{ds}{s_1^2 - s^2} \frac{d}{ds} e^{\Omega\tau(s)} \sim -\frac{1}{\Omega} \frac{e^{\Omega\tau(s_a)}}{s_1^2 - s_a^2}, \tag{23a}$$

which leads to the first term in Eq. (13). As noted in the discussion following Eq. (13), it may be necessary to add to this contribution the saddle-point result if the integration path traverses the vicinity of $s_{1,2}$. Similarly,

$$\frac{e^{\Omega a_0} C_a'(s_1^2 \Omega^{2/3})}{\Omega^{2/3}} = \int_{s_a}^{``\infty"} s e^{\Omega\tau(s)}\, ds \sim -\frac{1}{\Omega} \frac{s_a e^{\Omega\tau(s_a)}}{s_1^2 - s_a^2}, \tag{23b}$$

plus the saddle-point contribution, when required.

When the two saddle points are widely spaced ($s_{1,2} \not\approx 0$), but one saddle point is near the endpoint ($s \approx s_a$), the incomplete Airy integral may be reduced to an incomplete gamma function (Fresnel integral). Since $s_1 \not\approx s_2$, only first-order saddle points occur and one may transform the function $\tau(s)$ into the canonical function $\hat{\tau}(\mu)$ to describe the contribution from the first-order saddle point at s_1:

$$\tau(s) = a_0 + s_1^2 s - \frac{s^3}{3} = \tau(s_1) - \mu^2, \qquad \tau(s_1) = a_0 + \tfrac{2}{3} s_1^3, \tag{24}$$

whence $\mu = 0$ corresponds to $s = s_1$. If $\mu = \alpha$ is taken to correspond to $s = s_a$, then

$$\hat{I} = \int_{s_a}^{``\infty"} e^{\Omega\tau(s)}\, ds = e^{\Omega\tau(s_1)} \int_{\alpha}^{``\infty"} e^{-\Omega\mu^2} \left(\frac{ds}{d\mu}\right) d\mu, \qquad \frac{ds}{d\mu} = -\frac{2\mu}{s_1^2 - s^2}. \tag{25}$$

The uniform asymptotic approximation of the μ integral, valid as the endpoint $\mu = \alpha$ approaches the saddle point $\mu = 0$, is given in Eq. (2) and yields

$$\hat{I} \sim e^{\Omega\tau(s_1)} \left\{ \left(\frac{ds}{d\mu}\right)_0 \frac{Q(\sqrt{\Omega}\,\alpha)}{\sqrt{\Omega}} + \frac{e^{-\Omega\alpha^2}}{2\Omega\alpha} \left[\left(\frac{ds}{d\mu}\right)_\alpha - \left(\frac{ds}{d\mu}\right)_0 \right] \right\}, \tag{26}$$

where $\alpha = [\tau(s_1) - \tau(s_a)]^{1/2}$ from Eq. (24), and $Q(y)$ is the Fresnel integral defined in Eq. (15a). The mapping derivative $(ds/d\mu)_\alpha$ is calculated by direct substitution into Eq. (25), whereas from an evaluation of the resulting indeterminate form, $(ds/d\mu)_0 = s_1^{-1/2}$. Recalling the definitions of C_a and $\hat{I}$ in Eqs. (23a) and (25) and adding the conventional contribution from the isolated saddle point at s_2 when relevant, one is led from Eq. (26) to Eq. (15). In the analogous treatment via Eq. (25) of

$$\hat{I}' = \int_{s_a}^{``\infty"} s e^{\Omega\tau(s)}\,ds = \frac{d\hat{I}}{\Omega d(s_1^2)} \sim s_1\hat{I} - \frac{e^{\Omega[\tau(s_1)-\alpha^2]}}{\Omega}\left(\frac{ds}{d\mu}\right)_\alpha \frac{d\alpha}{d(s_1^2)}, \tag{27}$$

one obtains via the definition of C_a' in Eq. (23b) the result in Eq. (15) since in view of $\alpha^2 = \tau(s_1) - \tau(s_a)$,

$$\frac{d\alpha}{d(s_1^2)} = \frac{1}{2\alpha}\frac{d}{d(s_1^2)}[\tau(s_1) - \tau(s_a)] = \frac{s_1 - s_a}{2\alpha}. \tag{27a}$$

In the last portion of Eq. (27), a term involving the derivative $d^2s/d(s_1^2)d\mu$ in the integrand has been omitted since it may be shown to represent a higher-order contribution.

4.7 MULTIPLE INTEGRALS

If in the integrand of the real multiple integral

$$I_n(\Omega) = \int \cdots \int_{-\infty}^{\infty} f(\mathbf{x})e^{-\Omega q(\mathbf{x})}\,dx_1 \cdots dx_n, \qquad \Omega > 0, \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, the function q has a simple stationary point $\mathbf{x}_s = (x_{1s}, \ldots, x_{ns})$ defined by

$$\frac{\partial q}{\partial x_i} = 0 \text{ at } x_{is}, \qquad i = 1, \cdots, n, \tag{2}$$

and the function f is regular near $\mathbf{x}_s$, then the asymptotic approximation of $I_n(\Omega)$ for large Ω is given by

$$I_n(\Omega) \sim f(\mathbf{x}_s)e^{-\Omega q(\mathbf{x}_s)}\left(\frac{2\pi}{\Omega}\right)^{n/2}\frac{1}{|\det(\partial^2 q/\partial x_{is}\partial x_{js})|^{1/2}}. \tag{3}$$

One encounters frequently a modified form $\hat{I}_n(\Omega)$ of Eq. (1). If in the integrand of the multiple integral

$$\hat{I}_n(\Omega) = \int \cdots \int_{-\infty}^{\infty} f(\mathbf{x})e^{i\Omega q(\mathbf{x})}\,dx_1 \cdots dx_n, \qquad \Omega > 0, \tag{4}$$

the real function q has a simple, real stationary point $\mathbf{x}_s$ defined as in Eq. (2), and if f is regular near $\mathbf{x}_s$, then the asymptotic approximation of $\hat{I}_n(\Omega)$ for large Ω is given by

$$\hat{I}_n(\Omega) \sim f(\mathbf{x}_s)e^{i\Omega q(\mathbf{x}_s)}\left(\frac{2\pi}{\Omega}\right)^{n/2}\frac{e^{i(\pi/4)\sigma}}{|\det(\partial^2 q/\partial x_{is}\partial x_{js})|^{1/2}}, \tag{5}$$

where

$$\sigma = \sum_{i=1}^{n} \operatorname{sgn} d_i, \tag{5a}$$

and d_i are the eigenvalues of the matrix comprising the elements $\partial^2 q/\partial x_{is}\partial x_{js}$, $i, j = 1, \ldots, n$. Equation (5) is a generalization of the stationary-phase formula in Eq. (4.2.20a) for single integrals. If several stationary points exist, each contributes as in Eqs. (3) or (5).

Analytical details

The stationary points $\mathbf{x}_s$ of a function q of n variables $(x_1, \ldots, x_n)$ are defined as in Eq. (2) and, for large Ω, the principal contribution to the integrals arises from the vicinity of $\mathbf{x}_s$ [possible contributions from relevant singularities of $f(\mathbf{x})$ are not included in the present discussion]. Near $\mathbf{x}_s$ one may write $f(\mathbf{x}) \sim f(\mathbf{x}_s)$ and expand

$$q(\mathbf{x}) \sim q(\mathbf{x}_s) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i - x_{is})(x_j - x_{js}), \qquad a_{ij} \equiv \frac{\partial^2 q}{\partial x_{is}\partial x_{js}}, \tag{6}$$

with linear terms absent in view of Eq. (2). On substituting into Eqs. (1) or (4) and translating the origin to $\mathbf{x}_s$, one obtains

$$\bar{J}_n(\Omega) \sim f(\mathbf{x}_s)e^{\alpha\Omega q(\mathbf{x}_s)}\int_{-\infty}^{\infty}\cdots\int \exp\left(\frac{\alpha}{2}\Omega\sum_{i,j=1}^{n} a_{ij}x_i x_j\right) dx_1 \cdots dx_n, \tag{7}$$

where $\bar{J}_n \equiv I_n$ when $\alpha = -1$ and $\bar{J}_n \equiv \hat{I}_n$ when $\alpha = i$. To evaluate the multiple integral in Eq. (7), we introduce a transformation such that

$$\sum_{i,j=1}^{n} a_{ij}x_i x_j = \sum_{i=1}^{n} d_i y_i^2, \qquad x_i = \sum_{j=1}^{n} r_{ij}y_j, \tag{8a}$$

or, in matrix notation, with A denoting the $n \times n$ matrix comprising the elements a_{ij},

$$\tilde{\mathbf{x}}A\mathbf{x} = \tilde{\mathbf{y}}D\mathbf{y}, \qquad \mathbf{x} = R\mathbf{y}, \tag{8b}$$

where $\sim$ indicates the transposed quantity and D is a diagonal matrix. Since the diagonalization of A involves a coordinate rotation, R is an orthogonal matrix with $\det R = 1$. It follows from Eq. (8b) that

$$D = \tilde{R}AR, \qquad \det D = (\det R)^2 \det A = \det A, \tag{9a}$$

and that the Jacobian of the transformation from $\mathbf{x}$ to $\mathbf{y}$ is

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)} = \det R = 1. \tag{9b}$$

Thus, on changing variables to $\mathbf{y}$, one has, for the integral of Eq. (7),

$$\int_{-\infty}^{\infty}\cdots\int \exp\left(\frac{\alpha}{2}\Omega\sum_{i=1}^{n} d_i y_i^2\right)dy_1 \cdots dy_n = \prod_{i=1}^{n}\int_{-\infty}^{\infty} \exp\left(\frac{\alpha}{2}\Omega d_i y_i^2\right)dy_i. \tag{10}$$

When $\alpha = -1$ and $d_i > 0$, the single integral in Eq. (10) has the value $(2\pi)^{1/2}(\Omega d_i)^{-1/2}$; when $\alpha = i$, its value is $(2\pi)^{1/2}(\Omega|d_i|)^{-1/2} \exp[i(\pi/4) \operatorname{sgn} d_i]$ [see Eq. (4.2.20)]. Since $\prod_{i=1}^{n} d_i = \det D = \det A$, the results in Eqs. (3) and (5) are established. For higher-order terms in the asymptotic expansion, see Reference 21.

4.8 INTEGRATION AROUND A BRANCH POINT

Integrands of diffraction integrals often exhibit branch-point singularities that contribute to the asymptotic solution if they are crossed during the deformation of the original integration path into the steepest-descent path [see

Fig. 4.1.1 and Eq. (4.1.4)]. It is then necessary to calculate the contribution from the branch-cut integral, a generic form of which is given by

$$I_b = \int_{\bar{P}_b} f(z) e^{\Omega q(z)}\, dz, \tag{1}$$

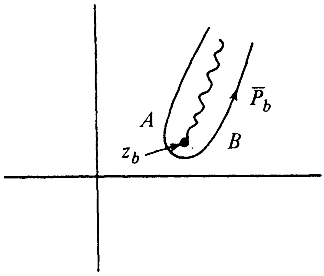


FIG. 4.8.1 First-order branch point singularity in the complex z plane.

where $\bar{P}_b$ is a contour encircling the branch cut in the positive sense (Fig. 4.8.1). In typical problems, $f(z)$ behaves like

$$f(z) \cong a + b\sqrt{z - z_b} \quad \text{near } z_b, \qquad \text{i.e., } b = 2[\sqrt{z - z_b}\, f'(z)]_{z_b}, \tag{2}$$

where a and b are constants; $q(z)$ is regular in the vicinity of z_b, and $\exp[\Omega q(z)]$ decays along $\bar{P}_b$. For large Ω, I_b has the following asymptotic approximation:

$$I_b \sim \frac{2\sqrt{\pi}}{[\Omega|q'(z_b)|]^{3/2}} [\sqrt{z - z_b}\, f'(z)]_{z_b} \exp\{\Omega q(z_b) - i\tfrac{3}{2}\arg[-q'(z_b)]\}. \tag{3}$$

Analytical details

The asymptotic evaluation of I_b in Eq. (1) may be performed in a manner analogous to that employed for a saddle-point integration if the following change of variable to s is introduced:

$$s^2 = q(z_b) - q(z), \tag{4}$$

with the integration path near z_b adjusted so that $s^2 > 0$ along $\bar{P}_b$. (Since $\exp[\Omega q(z)]$ is assumed to decay along $\bar{P}_b$, one has $\operatorname{Re} s^2 > 0$; the condition $s^2 > 0$ requires a legitimate path distortion which facilitates the analysis and defines essentially the path of steepest descent away from the branch point). Thus, for $z \approx z_b$ or $s \approx 0$,

$$s = \sqrt{-q'(z_b)}\,\sqrt{z - z_b}\,[1 + O(z - z_b)], \tag{5a}$$

or, upon inversion,

$$\sqrt{z - z_b} = \frac{s}{\sqrt{-q'(z_b)}}[1 + O(s^2)], \tag{5b}$$

and

$$\frac{dz}{ds} = \frac{2s}{-q'(z_b)} + O(s^3), \tag{6a}$$

$$f(z) = a + \frac{bs}{\sqrt{-q'(z_b)}} + O(s^2). \tag{6b}$$

For definiteness, we choose $s < 0$ on portion A and $s > 0$ on portion B of $\bar{P}_b$ in Fig. 4.8.1, implying that

$$\begin{aligned} \arg(z - z_b) &= -\arg[-q'(z_b)] && \text{on } B, \\ \arg(z - z_b) &= -\arg[-q'(z_b)] - 2\pi && \text{on } A. \end{aligned} \tag{7}$$

Upon substitution of Eq. (4) into Eq. (1) one obtains

$$I_b = e^{\Omega q(z_b)} \int_{-``\infty"}^{``\infty"} \left[f(z) \frac{dz}{ds} \right] e^{-\Omega s^2} \, ds, \tag{8}$$

which may be evaluated asymptotically by power series expansion and termwise integration of the bracketed quantity. The first term in the expansion involves the odd power s^1, which integrates out to zero. The first non-vanishing contribution derives from the s^2 term and yields

$$I_b \sim \frac{b\sqrt{\pi}}{[-\Omega q'(z_b)]^{3/2}} e^{\Omega q(z_b)}, \tag{9}$$

from which Eq. (3) follows. Higher-order terms in the asymptotic series may be generated by carrying further the power-series expansion of $[f(z)dz/ds]$.

APPENDIX 4A. HIGHER-ORDER DERIVATIVES OF $G(s) = f(z)dz/ds$

Use of the asymptotic expansion in Eq. (4.2.12) requires knowledge of the derivatives of

$$G(s) = f(z)\varphi(s), \quad \varphi(s) = \frac{dz}{ds}, \tag{A1}$$

evaluated at $s = 0$ ($z = z_s$). By direct differentiation of Eq. (A1) one finds

$$G(0) = f(z_s)\varphi(0), \tag{A2a}$$

$$G'(0) = f'(z_s)[\varphi(0)]^2 + f(z_s)\varphi'(0), \tag{A2b}$$

$$G^{(2)}(0) = f^{(2)}(z_s)[\varphi(0)]^3 + 3f'(z_s)\varphi(0)\varphi'(0) + f(z_s)\varphi^{(2)}(0), \tag{A2c}$$

etc.

To obtain the value of the derivatives of $\varphi(s)$ at $s = 0$, we expand the function $q(z) = \tau(s)$ about the saddle point z_s for which $q'(z_s) = 0$ and invert the resulting series. The procedure is illustrated for the case of a first-order saddle point at z_s, in which case the relation $q(z) = q(z_s) - s^2$ applies. Thus, if $\zeta = z - z_s$,

$$-s^2 = q^{(2)}(z_s)\frac{\zeta^2}{2!} + q^{(3)}(z_s)\frac{\zeta^3}{3!} + q^{(4)}(z_s)\frac{\zeta^4}{4!} + \cdots. \tag{A3}$$

Equation (A3) has an inverse solution which is regular and vanishes at $s = 0$,†

†For an alternative procedure utilizing Cauchy's theorem, see Reference 22.

so in view of the definition of $\varphi(s)$ in Eq. (A1), we may employ the power-series expansion

$$\zeta(s) = \varphi(0)s + \varphi'(0)\frac{s^2}{2!} + \varphi^{(2)}(0)\frac{s^3}{3!} + \varphi^{(3)}(0)\frac{s^4}{4!} + \cdots, \tag{A4}$$

whence

$$\zeta^2 = [\varphi(0)]^2 s^2 + \varphi(0)\varphi'(0)s^3 + \left\{\left[\frac{\varphi'(0)}{2!}\right]^2 + \frac{\varphi^{(2)}(0)}{3}\varphi(0)\right\}s^4 + \cdots, \tag{A5a}$$

$$\zeta^3 = [\varphi(0)]^3 s^3 + \{\tfrac{3}{2}[\varphi(0)]^2\varphi'(0)\}\, s^4 + \cdots, \tag{A5b}$$

$$\zeta^4 = [\varphi(0)]^4 s^4 + \cdots, \tag{A5c}$$

etc. Upon substituting Eqs. (A5) into Eq. (A3) and equating coefficients of like powers of s on both sides of the resulting equation, one obtains

Coefficient of s^2

$$-1 = \tfrac{1}{2}[\varphi(0)]^2 q^{(2)}(z_s) \qquad \text{or} \qquad \varphi(0) = \pm\sqrt{\frac{-2}{q^{(2)}(z_s)}}. \tag{A6a}$$

Coefficient of s^3

$$0 = \frac{q^{(2)}(z_s)}{2!}\varphi(0)\varphi'(0) + \frac{q^{(3)}(z_s)}{3!}[\varphi(0)]^3,$$

or

$$\varphi'(0) = -\frac{1}{3}\frac{q^{(3)}(z_s)}{q^{(2)}(z_s)}[\varphi(0)]^2 = \frac{2}{3}\frac{q^{(3)}(z_s)}{[q^{(2)}(z_s)]^2}. \tag{A6b}$$

Coefficient of s^4

$$0 = \frac{q^{(2)}(z_s)}{2!}\left\{\left[\frac{\varphi'(0)}{2}\right]^2 + \frac{\varphi^{(2)}(0)\varphi(0)}{3}\right\} + \frac{q^{(3)}(z_s)}{3!}\frac{3}{2}[\varphi(0)]^2\varphi'(0) + \frac{q^{(4)}(z_s)}{4!}[\varphi(0)]^4,$$

or

$$\varphi^{(2)}(0) = \frac{-3}{\varphi(0)}\left\{\left[\frac{\varphi'(0)}{2}\right]^2 + \frac{2}{q^{(2)}(z_s)}\left[\frac{q^{(4)}(z_s)}{4!}[\varphi(0)]^4 + \frac{q^{(3)}(z_s)}{4}[\varphi(0)]^2\varphi'(0)\right]\right\}, \tag{A6c}$$

etc.

APPENDIX 4B. PROPERTIES OF THE AIRY FUNCTIONS

The Airy functions $\mathrm{Ai}(\sigma)$ and $\mathrm{Bi}(\sigma)$ may be defined either by the differential equation (4.2.33) (with appropriate boundary conditions), or by the integral representations in Eqs. (4.2.32) and (4.2.34). Section 4.2e contains a discussion of the properties of these functions when the argument σ is large. Other properties are discussed in this appendix. For tabulated results, see Reference 23.

To obtain a convergent series expansion useful for evaluation of $\mathrm{Ai}(\sigma)$ for small σ, one may substitute the power series for the exponential function $\exp(\sigma z)$ in the integrand of Eq. (4.2.32b) to find

$$\mathrm{Ai}(\sigma) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\sigma^n}{n!}a_n, \qquad a_n = \int_{L_{32}} z^n e^{-z^3/3}\,dz, \tag{B1}$$

after a permissible interchange of summation and integration. The coefficient a_n is defined by the convergent integral over the contour L_{32} in Fig. 4.2.4, which can be chosen along the straight lines $\arg z = 4\pi/3$ and $\arg z = 2\pi/3$. Since

$$\int_{\infty e^{i4\pi/3}}^{0} z^n e^{-z^3/3}\,dz = e^{i2(n+1)\pi/3}\int_{\infty e^{i2\pi/3}}^{0} z^n e^{-z^3/3}\,dz, \tag{B2a}$$

one may write

$$a_n = [1 - e^{i2(n+1)\pi/3}]\int_0^{\infty e^{i2\pi/3}} z^n e^{-z^3/3}\,dz, \tag{B2b}$$

or, upon changing variables to $\eta = z\exp(-i2\pi/3)$ and employing Eq. (4.2.8),

$$a_n = 2i(-1)^n 3^{(n-2)/3}\,\Gamma\left(\frac{n+1}{3}\right)\sin\left[\frac{(n+1)\pi}{3}\right]. \tag{B3}$$

Thus,

$$\mathrm{Ai}(\sigma) = \frac{1}{3^{2/3}\pi}\sum_{n=0}^{\infty}\frac{\sin[(n+1)\pi/3]\,\Gamma[(n+1)/3]3^{n/3}}{n!}(-\sigma)^n, \tag{B4}$$

and, upon use of the formula

$$\sin\left[(n+1)\frac{\pi}{3}\right]\Gamma\left(\frac{n+1}{3}\right) = \frac{\pi}{\Gamma[1-(n+1)/3]} = \frac{\pi}{\Gamma[(2-n)/3]}, \tag{B5}$$

one has, alternatively,

$$\mathrm{Ai}(\sigma) = \frac{1}{3^{2/3}}\sum_{n=0}^{\infty}\frac{3^{n/3}(-\sigma)^n}{n!\,\Gamma[(2-n)/3]}. \tag{B6}$$

Since $n! = \Gamma(n+1)$ and [see Eq. (4.2.9)]

$$\Gamma(\nu+\alpha) \sim \sqrt{\frac{2\pi}{\nu}}\left(\frac{\nu}{e}\right)^{\nu}\nu^{\alpha}, \qquad |\nu|\to\infty, \quad |\arg\nu| < \pi, \tag{B7}$$

the power-series expansions (B4) or (B6) are convergent for all σ. By a similar procedure, one derives

$$\mathrm{Bi}(\sigma) = \frac{1}{3^{1/6}}\sum_{n=0}^{\infty}\frac{3^{n/3}\sigma^n}{n!\,|\Gamma[(2-n)/3]|}. \tag{B8}$$

Because $\Gamma(-n) = \infty$, $n = 0, 1, 2, \ldots,$ the series in Eqs. (B6) and (B8) separate into two parts—one with powers σ^{3n} and the other with σ^{3n+1}. Repeated use of $\Gamma(y+1) = y\Gamma(y)$ leads to the alternative formulations

$$\mathrm{Ai}(\sigma) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})}W_1(\sigma) - \frac{1}{3^{1/3}\Gamma(\frac{1}{3})}W_2(\sigma), \tag{B9a}$$

$$\mathrm{Bi}(\sigma) = \frac{1}{3^{1/6}\Gamma(\frac{2}{3})}W_1(\sigma) + \frac{3^{1/6}}{\Gamma(\frac{1}{3})}W_2(\sigma), \tag{B9b}$$

where

$$W_1(\sigma) = \sum_{n=0}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3n+1)}{(3n+1)!} \sigma^{3n},$$
$$W_2(\sigma) = \sum_{n=0}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n+2)}{(3n+2)!} \sigma^{3n+1}. \tag{B9c}$$

The functions $W_1(-\sigma)$ and $W_2(-\sigma)$ can be expressed in terms of Bessel functions of order $\frac{1}{3}$. We begin with the series

$$J_{1/3}(y) = \left(\frac{y}{2}\right)^{1/3} \sum_{n=0}^{\infty} \left(\frac{iy}{2}\right)^{2n} \frac{1}{n!\Gamma(\frac{1}{3}+n+1)}. \tag{B10}$$

and use for $n!\Gamma(\frac{1}{3}+n+1)$ the product formula for the gamma function,

$$\Gamma(3x) = \frac{1}{2\pi\sqrt{3}} 3^{3x}\Gamma(x)\Gamma\left(x+\tfrac{1}{3}\right)\Gamma\left(x+\tfrac{2}{3}\right), \qquad x = n+1. \tag{B11}$$

In view of Eq. (B5) with $n = 0$, one obtains

$$\begin{aligned} J_{1/3}(\tfrac{2}{3}\sigma^{3/2}) &= \frac{\sqrt{\sigma}}{2\pi 3^{1/3}\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(n+1+\frac{2}{3})3^{n+3}}{\Gamma(3n+3)} \sigma^{3n} \\ &= -\frac{3^{2/3}}{\Gamma(\frac{1}{3})} \frac{1}{\sqrt{\sigma}} W_2(-\sigma), \end{aligned} \tag{B12a}$$

and, in a similar manner,

$$J_{-1/3}(\tfrac{2}{3}\sigma^{3/2}) = \frac{3^{1/3}}{\Gamma(\frac{2}{3})} \frac{1}{\sqrt{\sigma}} W_1(-\sigma). \tag{B12b}$$

Thus, Eqs. (B9) may be written as

$$\mathrm{Ai}(-\sigma) = \frac{\sqrt{\sigma}}{3}[J_{-1/3}(\tfrac{2}{3}\sigma^{3/2}) + J_{1/3}(\tfrac{2}{3}\sigma^{3/2})], \tag{B13a}$$

$$\mathrm{Bi}(-\sigma) = \sqrt{\frac{\sigma}{3}}[J_{-1/3}(\tfrac{2}{3}\sigma^{3/2}) - J_{1/3}(\tfrac{2}{3}\sigma^{3/2})]. \tag{B13b}$$

The combination $\mathrm{Ai}(-\sigma) \pm i\,\mathrm{Bi}(-\sigma)$ can likewise be expressed simply in terms of Hankel functions of order $\frac{1}{3}$, or more conveniently in terms of an Airy function with shifted argument. Via the formula

$$H_\nu^{(1,2)}(y) = \frac{J_{-\nu}(y) - e^{\mp i\nu\pi}J_\nu(y)}{\pm i \sin \nu\pi}, \tag{B14}$$

one obtains from Eqs. (B13), after a simple calculation [see also Eq. (6. A32)],

$$A_{2,1}(-\sigma) \equiv \mathrm{Ai}(-\sigma) \pm i\,\mathrm{Bi}(-\sigma) = \sqrt{\frac{\sigma}{3}} e^{\mp i\pi/6} H_{1/3}^{(2,1)}(\tfrac{2}{3}\sigma^{3/2}) \tag{B15a}$$

or, from Eqs. (B18)–(B20),

$$A_{2,1}(-\sigma) = -2e^{\pm i2\pi/3}\mathrm{Ai}(-\sigma e^{\pm i2\pi/3}). \tag{B15b}$$

The Wronskian of the Airy functions $\mathrm{Ai}(\sigma)$ and $\mathrm{Bi}(\sigma)$ can be calculated from the asymptotic formulas in Eqs. (4.2.42) and (4.2.45) as $\sigma \to \infty$, or from the power-series expansions (B4) and (B8) when $\sigma \to 0$. Either procedure is valid since the Wronskian is a constant, and the calculation by one method serves as a check on the result derived by the other. From the expressions for $\sigma \to \infty$, one finds directly,

$$\mathrm{Ai}(\sigma)\mathrm{Bi}'(\sigma) - \mathrm{Bi}(\sigma)\mathrm{Ai}'(\sigma) = \frac{1}{\pi}, \tag{B16}$$

while the calculation at $\sigma \to 0$ likewise yields $2\,[3^{1/2}\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})] = 1/\pi$, via Eq. (B5).

The two linearly independent solutions $\mathrm{Ai}(\sigma)$ and $\mathrm{Bi}(\sigma)$ are convenient because both are real when σ is real. However, it is evident from Fig. 4.2.4 that the contour integral taken only over path L_{21} or L_{31} appears simpler than the one for $\mathrm{Bi}(\sigma)$ in Eq. (4.2.34), and that the resulting functions are likewise linearly independent with respect to $\mathrm{Ai}(\sigma)$. If we define

$$\int_{L_{21}} e^{\sigma z - z^3/3}\, dz \equiv -\pi i A_2(\sigma), \tag{B17a}$$

$$\int_{L_{31}} e^{\sigma z - z^3/3}\, dz \equiv \pi i A_1(\sigma), \tag{B17b}$$

then, from Cauchy's theorem,

$$2\mathrm{Ai}(\sigma) = A_1(\sigma) + A_2(\sigma), \tag{B18a}$$

and, from Eq. (4.2.34),

$$2\mathrm{Bi}(\sigma) = i[A_1(\sigma) - A_2(\sigma)]. \tag{B18b}$$

Conversely, $A_{1,2}$ is expressed in terms of Ai and Bi as

$$A_{1,2}(\sigma) = \mathrm{Ai}(\sigma) \mp i\,\mathrm{Bi}(\sigma). \tag{B18c}$$

Figure 4.2.4 also shows that A_1, A_2, and Ai can be related to one another by changing the phase of the integration variable by $\pm\,(2\pi/3)$. Thus, if $\xi = z\exp(-i2\pi/3)$,

$$\int_{L_{32}} e^{\sigma z - z^3/3}\, dz = e^{i2\pi/3} \int_{L_{21}} e^{\sigma e^{i2\pi/3}\xi - \xi^3/3}\, d\xi, \tag{B19}$$

whence

$$\mathrm{Ai}(\sigma) = -\frac{e^{i2\pi/3}}{2} A_2(\sigma e^{i2\pi/3}). \tag{B20a}$$

Similarly,

$$\mathrm{Ai}(\sigma) = -\frac{e^{-i2\pi/3}}{2} A_1(\sigma e^{-i2\pi/3}), \tag{B20b}$$

and

$$A_1(\sigma) = e^{-i2\pi/3} A_2(\sigma e^{-i2\pi/3}). \tag{B20c}$$

Also, from Eqs. (B16) and (B18c),

$$A_{1,2}(\sigma)\mathrm{Ai}'(\sigma) - \mathrm{Ai}(\sigma)A'_{1,2}(\sigma) = \pm\,\frac{i}{\pi}, \tag{B21a}$$

whence

$$\mathrm{Ai}(\sigma)\frac{d}{d\sigma}\mathrm{Ai}(\sigma e^{\pm i2\pi/3}) - \mathrm{Ai}'(\sigma)\mathrm{Ai}(\sigma e^{\pm i2\pi/3}) = \frac{e^{\mp i\pi/6}}{2\pi}. \tag{B21b}$$

PROBLEMS

1. For large Ω, perform an asymptotic evaluation of the integral

$$I(\Omega) = \int_{\bar{P}_s} f(z)e^{\Omega q(z)}\,dz, \tag{1}$$

where $\bar{P}_s$ is the steepest-descent path through the first-order saddle point z_s of $q(z)$ (i.e., $q'(z_s) = 0$), by writing

$$e^{\Omega q(z)} = \exp\left[\Omega q(z_s) + \frac{\Omega q''(z_s)(z - z_s)^2}{2!}\right] M(\Omega, z) \tag{2}$$

with

$$M(\Omega, z) = \exp\left[\Omega \sum_{n=3}^{\infty} \frac{q^{(n)}(z_s)(z - z_s)^n}{n!}\right]. \tag{3}$$

Expand the regular functions $f(z)$ and $M(\Omega, z)$ in power series about $z = z_s$ and perform termwise integration to obtain the asymptotic expansion for $I(\Omega)$. Compare the results with those obtained by the procedure of Secs. 4.2a and 4.2b.

2. Through use of continued integration by parts, obtain the expansion for the exponential integral,

$$E_1(\Omega) = \int_{\Omega}^{\infty} \frac{e^{-y}}{y}\,dy \sim \frac{e^{-\Omega}}{\Omega} \sum_{n=0}^{\infty} \frac{n!}{\Omega^n}, \tag{4}$$

in inverse powers of Ω. Show that this series diverges in the ordinary sense but is a valid asymptotic expansion as $\Omega \to \infty$.

3. The following contour integral representation† for the parabolic cylinder function is valid for arbitrary values of ν:

$$D_{-\nu}(z) = -\frac{\Gamma(1-\nu)}{2\pi i} e^{-z^2/4} \int_L e^{-z\xi - \xi^2/2}(-\xi)^{\nu-1}\,d\xi, \qquad |\arg(-\xi)| \le \pi, \tag{5}$$

where the contour L encircles in the counterclockwise sense a branch cut extending from $\xi = 0$ to $\xi = \infty$ along the positive real ξ axis.

(a) Show that this expression may be reduced to the one in Eq. (4.5.36) when $\mathrm{Re}\,\nu > 0$. Introduce into Eq. (5) the new variable $\zeta = \sqrt{\xi}$ to show that for $(2\nu - 1)$ equal to a positive even integer or zero,

$$D_{-\nu}(z) = -\frac{1}{\Gamma(\nu)} e^{-z^2/4} I_{13}, \tag{6}$$

where

$$I_{13} = \int_{L_{13}} \zeta^{2\nu-1} e^{-z\zeta^2 - \zeta^4/2}\,d\zeta = 2I_{10} = 2I_{03}, \tag{7}$$

†E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, 1952, pp. 348-349.

and L_{ij} is a contour that begins at infinity in sector i, and ends at infinity in sector j, of the complex ζ plane as shown in Fig. P4.1. When i or j is equal to zero, the corresponding endpoint of the path is at the origin $\zeta = 0$.

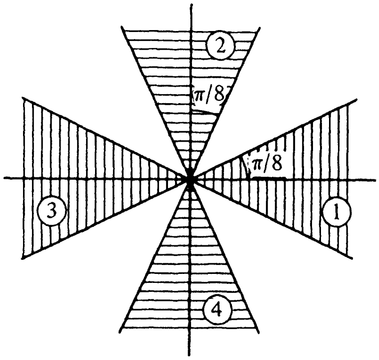


FIG. P4-1 Regions in the complex ζ plane.

Show from the integral representations that the parabolic cylinder functions satisfy the differential equation

$$\left[\frac{d^2}{dz^2} + \left(\frac{1}{2} - \nu - \frac{z^2}{4}\right)\right] D_{-\nu}(z) = 0. \tag{8}$$

(b) Assume that $(2\nu - 1)$ = positive even integer, i.e., $\nu = m + \frac{1}{2}$, where m is a positive integer or zero. Observe that since $I_{13} \exp(z^2/4)$ in Eq. (6) satisfies the differential equation (8), it follows that $I_{24} \exp(z^2/4)$ also satisfies the differential equation and furnishes a function linearly independent of $D_{-\nu}(z)$ (since the contour L_{24} cannot be deformed into the contour L_{13}, different solutions are involved). Show that

$$I_{02} = I_{40} = \frac{e^{i\nu\pi}}{2} \Gamma(\nu) e^{z^2/4} D_{-\nu}(-z). \tag{9}$$

Note that if $D_{-\nu}(z)$ satisfies Eq. (8), then so does $D_{\nu-1}(\pm iz)$, and that since $D_{-\nu}(\pm z)$ are two linearly independent solutions of the differential equation, it must be possible to employ them to represent $D_{\nu-1}(\pm iz)$. The relation may be shown to be

$$D_{-\nu}(z) - e^{\pm \nu\pi i} D_{-\nu}(-z) = \frac{\sqrt{2\pi}}{\Gamma(\nu)} e^{\mp i(1-\nu)\pi/2} D_{\nu-1}(\pm iz). \tag{10}$$

Show that

$$I_{12} = i\sqrt{\frac{\pi}{2}}\, e^{i\nu\pi/2} e^{z^2/4} D_{\nu-1}(iz), \tag{11a}$$

$$I_{14} = i\sqrt{\frac{\pi}{2}}\, e^{i\nu\pi/2} e^{z^2/4} D_{\nu-1}(iz) - \Gamma(\nu) e^{i\nu\pi} e^{z^2/4} D_{-\nu}(-z). \tag{11b}$$

4. (a) Explain why, for an asymptotic evaluation of I_{01} (and generally of I_{ij}) in Problem 3 for large values of z, it is convenient to transform Eq. (7) into

$$I_{01} = \Omega^{\nu/2} e^{i\nu\alpha} \int_0^{\infty e^{-i\alpha/2}} \mu^{2\nu-1} e^{-\Omega e^{i2\alpha}[\mu^2 + (\mu^4/2)]}\, d\mu, \tag{12}$$

where $\Omega = |z|^2, \alpha = \arg z, \mu = \zeta z^{-1/2}$. The integrand has saddle points at $\mu_s = 0, \pm i$; show that the direction of the steepest-descent paths at the saddle points is as follows:

$$\arg(d\mu)|_{\mu_s=0} = \begin{cases} -\alpha \\ -\alpha \pm \pi \end{cases}, \quad \arg(d\mu)|_{\mu_s=i} = -\alpha \pm \frac{\pi}{2},$$

$$\arg(d\mu)|_{\mu_s=-i} = -\alpha \pm \frac{\pi}{2}. \tag{13}$$

(b) Show that for $0 < \alpha < \pi/2$, the integration path can be deformed into the steepest-descent path without passing the saddle point at $\mu_s = -i$ ($\mu_s = +i$ is irrelevant for the range of α considered); that for $\pi/2 \leq \alpha < 3\pi/4$, both saddle points are traversed but that the dominant contribution arises from $\mu_s = 0$; that the saddle points $\mu_s = 0$ and $\mu_s = -i$ contribute equally for $\alpha = 3\pi/4$; and that the contribution from $\mu_s = -i$ is dominant when $3\pi/4 < \alpha < 5\pi/4$. Carry out the analogous considerations for $\alpha < 0$.

(c) From the information in (b) and the consequent asymptotic evaluation of the integral I_{01} for large $|z|$, show that

$$D_{-\nu}(z) \sim e^{-z^2/4} z^{-\nu} \qquad -\pi/2 < \arg z < \pi/2, \tag{14a}$$

$$D_{-\nu}(z) \sim e^{-z^2/4} z^{-\nu} - \frac{\sqrt{2\pi}\, e^{-i\nu\pi}}{\Gamma(\nu)} e^{z^2/4} z^{\nu-1}, \qquad \pi/2 \leq \arg z < 5\pi/4. \tag{14b}$$

(Note: Since $\nu = m + \frac{1}{2}, m = 0, 1, 2, \ldots$, the asymptotic result for $D_{-\nu}(z) \exp(-z^2/4)$ may be inferred from that for $m = 0$ by repeated differentiation with respect to z.) Show that formula (14b) applies also when $-\pi/2 \geq \arg z > -5\pi/4$ provided that $\exp(-i\nu\pi)$ is replaced by $\exp(i\nu\pi)$. (Equations (14a) and (14b) may actually be shown to hold for arbitrary ν satisfying the inequality $|\nu| \ll |z|$.)

(d) Deduce asymptotic expressions for the ranges $\pi/2 \leq \arg z < 5\pi/4$ and $-\pi/2 \geq \arg z > -5\pi/4$ from the one in Eq. (14a) through use of Eq. (10).

5. The associated Legendre function $P_\nu^{-\mu}[\cos(t/\nu)]$ may be defined by the integral†

$$P_\nu^{-\mu}\left(\cos\frac{t}{\nu}\right) = \frac{1}{\Gamma(\nu + \mu + 1)} \int_0^\infty e^{-z\cos(t/\nu)} z^\nu J_\mu\left(z \sin\frac{t}{\nu}\right) dz, \tag{15}$$

valid for $0 < (t/\nu) < \pi/2$, $\mathrm{Re}(\mu + \nu + 1) > 0$. Evaluate the integral asymptotically as $\nu \to \infty$ to show that

$$\lim_{\nu\to\infty} \nu^\mu P_\nu^{-\mu}\left(\cos\frac{t}{\nu}\right) = J_\mu(t). \tag{16}$$

$$\left(\text{Note: } \int_0^\infty e^{-z} z^\nu dz = \Gamma(\nu + 1), \qquad \mathrm{Re}(\nu + 1) > 0.\right)$$

6. Using the integral representation for the Legendre function†

$$P_n(\cos\theta) = \frac{1}{2\pi} \int_0^{2\pi} (\cos\theta + i \sin\theta \cos w)^n\, dw, \tag{17}$$

derive the asymptotic approximation for large n:

†W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics*, Chelsea Publishing Co., New York, 1954, pp. 67-68.

$$P_n(\cos\theta) \sim \sqrt{\frac{2}{\pi n \sin\theta}} \cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right], \qquad \sin\theta \not\approx 0. \tag{18}$$

7. The free-space Green's function has the following integral representation [cf. Eq. (5.4.7a)]:

$$G_f = \frac{e^{ikr}}{4\pi r} = \frac{i}{8\pi}\int_{\infty e^{i\pi}}^{\infty} \xi H_0^{(1)}(\xi\rho)\frac{e^{i\sqrt{k^2-\xi^2}|z|}}{\sqrt{k^2-\xi^2}}\,d\xi, \tag{19}$$

where $r = (\rho^2 + z^2)^{1/2}$, and the integration path passes above the branch points at $\xi = -k, 0$ and below the branch point at $\xi = k$. Assuming ρ large, use the asymptotic expansion for $H_0^{(1)}(\xi\rho)$ given in Eq. (6.4.8a), and then perform the asymptotic expansion of the integral in Eq. (19) by the methods of Secs. 4.2a and 4.2b. Show that the higher-order terms in the expansion vanish so that the leading term furnishes the exact result.

8. Using the integral representation [cf. Eq. (5.4.36c)],

$$H_0^{(1)}(k\rho) = \frac{1}{\pi}\int_{\bar{P}} e^{ik\rho\cos w}\,dw, \tag{20}$$

where $\bar{P}$ is the integration contour in Fig. 5.3.6(b), derive the asymptotic expansion in Eq. (6.4.8a) by the method of Sec. 4.2b.

9. Perform the operations implied in Eq. (4.2.42) to obtain the complete asymptotic expansion in Eq. (4.2.43).

10. Utilize the double integral representation in Eq. (5.4.12b),

$$G_f = \frac{i}{8\pi^2}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, \frac{\exp\,[i(\xi x + \eta y) + i\sqrt{k^2-\xi^2-\eta^2}|z|]}{\sqrt{k^2-\xi^2-\eta^2}} \tag{21}$$

to derive the asymptotic approximation for large $r = \sqrt{x^2+y^2+z^2}$.

(a) Perform the asymptotic evaluation by using the results of Sec. 4.7.
(b) Perform first the asymptotic evaluation of the η integral via the procedure of Sec. 4.2a, and then perform the evaluation of the ξ integral. Compare results of the two procedures.

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