# Appendix C

# Integral Theorems

In this appendix, we use $\hat{n}$ to denote the normal unit vector to a surface that can be either open or closed. The two tangential vectors of an open surface will be denoted by $\hat{\ell}$ and $\hat{m}$; $\hat{\ell}$ is tangential to the edge of the contour and $\hat{m}$ is normal to the contour. Both are tangential to the surface. The triad forms an orthogonal relation $\hat{m} \times \hat{\ell} = \hat{n}$ and $d\mathbf{S} = \hat{n}\, ds,\ d\boldsymbol{\ell} = \hat{\ell}\, d\ell$.

1. Gauss theorem or divergence theorem:
$$\iiint \nabla \mathbf{F}\, dv = \oiint \hat{n} \cdot \mathbf{F}\, dS.$$
2. Curl theorem:
$$\iiint \nabla \mathbf{F}\, dV = \oiint \hat{n} \times \mathbf{F}\, dS.$$
3. Gradient theorem:
$$\iiint \nabla f\, dV = \oiint \hat{n} f\, dS.$$
4. Surface divergence theorem:
$$\iint \nabla_s \mathbf{F}\, dS = \oint \hat{m} \cdot \mathbf{F}\, d\ell.$$
5. Surface curl theorem:
$$\iint \nabla_s \mathbf{F}\, dS = \oint \hat{m} \times \mathbf{F}\, d\ell.$$

6. Surface gradient theorem:

$$\iint \nabla_s f \, dS = \oint \hat{m} f \, d\ell.$$

7. Cross-gradient theorem:

$$\iint \hat{n} \times \nabla f \, dS = \oint f \, d\ell.$$

8. Stokes's theorem:

$$\iint \hat{n} \cdot \boldsymbol{\nabla} \mathbf{F} \, dS = \oint \mathbf{F} \cdot d\ell.$$

9. Cross-$\boldsymbol{\nabla}$-cross theorem:

$$\iint (\hat{n} \times \boldsymbol{\nabla}) \times \mathbf{F} \, dS = \iint [\hat{n} \times \boldsymbol{\nabla} \mathbf{F} + \hat{n} \cdot \nabla \mathbf{F} - \hat{n} \boldsymbol{\nabla} \mathbf{F}] \, dS = -\oint \mathbf{F} \times d\ell.$$

10. First scalar Green's theorem:

$$\iiint [a \boldsymbol{\nabla} \nabla b + \nabla a \cdot \nabla b] \, dv = \oiint \hat{n} \cdot a \nabla b \, dS.$$

11. Second scalar Green's theorem:

$$\iiint (a \boldsymbol{\nabla} \nabla b - b \boldsymbol{\nabla} \nabla a) \, dV = \oiint \hat{n} \cdot (a \nabla b - b \nabla a) \, dS.$$

12. Second scalar surface Green's theorem:

$$\iint (a \boldsymbol{\nabla}_s \nabla_s b - b \boldsymbol{\nabla}_s \nabla_s a) \, dS = \oint \hat{m} \cdot (a \nabla_s b - b \nabla_s a) \, dS.$$

13. First scalar–vector Green's theorem:

Type 1: $$\iiint (f \boldsymbol{\nabla} \nabla \mathbf{F} + \nabla f \cdot \nabla \mathbf{F}) \, dV = \oiint \hat{n} \cdot f \nabla \mathbf{F} \, dS,$$

Type 2: $$\iiint (\mathbf{F} \boldsymbol{\nabla} \nabla f + \nabla f \cdot \nabla \mathbf{F}) \, dV = \oiint (\hat{n} \cdot \nabla f) \mathbf{F} \, dS.$$

14. Second scalar–vector Green's theorem:

$$\iiint (f \boldsymbol{\nabla} \nabla \mathbf{F} - \mathbf{F} \boldsymbol{\nabla} \nabla f) \, dV = \oiint \hat{n} \cdot [f \nabla \mathbf{F} - (\nabla f) \mathbf{F}] \, dS.$$

15. First vector Green's theorem:

$$\iiint [(\boldsymbol{\nabla} \mathbf{P}) \cdot (\boldsymbol{\nabla} \mathbf{Q}) - \mathbf{P} \cdot \boldsymbol{\nabla} \boldsymbol{\nabla} \mathbf{Q}] \, dV = \oiint \hat{n} \cdot (\mathbf{P} \times \boldsymbol{\nabla} \mathbf{Q}) \, dS.$$

16. Second vector Green's theorem:

$$\iiint (\mathbf{Q} \cdot \boldsymbol{\nabla} \boldsymbol{\nabla} \mathbf{P} - \mathbf{P} \cdot \boldsymbol{\nabla} \boldsymbol{\nabla} \mathbf{Q}) \, dV = \oiint \hat{n} \cdot (\mathbf{P} \times \boldsymbol{\nabla} \mathbf{Q} - \mathbf{Q} \times \boldsymbol{\nabla} \mathbf{P}) \, dS.$$

17. First vector–dyadic Green's theorem:

$$\iiint [(\nabla \mathbf{P}) \cdot \nabla \bar{\bar{Q}} - \mathbf{P} \cdot \nabla \nabla \bar{\bar{Q}}]\, dV = \oiint \hat{n} \cdot (\mathbf{P} \times \nabla \bar{\bar{Q}})\, dS.$$

18. Second vector–dyadic Green's theorem:

$$\iiint [(\nabla \nabla \mathbf{P}) \cdot \bar{\bar{Q}} - \mathbf{P} \cdot \nabla \nabla \bar{\bar{Q}}]\, dV = \oiint \hat{n} \cdot [\mathbf{P} \times \nabla \bar{\bar{Q}} + (\nabla \mathbf{P}) \times \bar{\bar{Q}}]\, dS.$$

19. First dyadic–dyadic Green's theorem:

$$\iiint \left\{ [\bar{\bar{Q}}]^{\mathrm{T}} \cdot \nabla \nabla \bar{\bar{P}} - [\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot \nabla \bar{\bar{P}} \right\} dV = \oiint [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\hat{n} \times \nabla \bar{\bar{P}})\, dS.$$

20. Second dyadic–dyadic Green's theorem:

$$\iiint \left\{ [\bar{\bar{Q}}]^{\mathrm{T}} \cdot \nabla \nabla \bar{\bar{P}} - [\nabla \nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot \bar{\bar{P}} \right\} dV$$

$$= \oiint \left\{ [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\hat{n} \times \nabla \bar{\bar{P}}) + [\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\hat{n} \times \bar{\bar{P}}) \right\} dS.$$

21. Helmholtz transport theorem:

$$\frac{d}{dt} \iint_{S(t)} \mathbf{F} \cdot d\mathbf{S} = \iint_{S(t)} \left[ \frac{\partial \mathbf{F}}{\partial t} + \mathbf{v} \nabla \mathbf{F} - \nabla (\mathbf{v} \times \mathbf{F}) \right] \cdot d\mathbf{S}.$$

22. Maxwell's theorem:

$$\frac{d}{dt} \oint_{L(t)} \mathbf{f} \cdot d\boldsymbol{\ell} = \int_{L(t)} \left( \frac{\partial \mathbf{f}}{\partial t} - \mathbf{v} \times \nabla \mathbf{f} \right) \cdot d\boldsymbol{\ell}.$$

23. Reynolds's transport theorem:

$$\frac{d}{dt} \iiint_{V(t)} \rho\, dV = \iiint_{V(t)} \left[ \frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{v}) \right] dV$$

$$= \iiint_{V(t)} \left( \frac{d\rho}{dt} + \rho \nabla \mathbf{v} \right) dV.$$