3

The Spectral Representation Method

3.1 INTRODUCTION

In this chapter, we continue our discussion of linear ordinary differential equations of second order. We begin with a discussion of eigenvalues and eigenfunctions. We follow with a description of the method of solution for self-adjoint SLP2 problems in terms of the eigenfunctions. We include a discussion of the determination of the eigenfunctions directly from the Green's function for the problem. This determination leads to a *spectral representation* of the delta function specific to a particular operator and its domain. We next consider problems on unbounded intervals (SLP3). We are able to expand our analysis to produce the appropriate spectral representations for many important unbounded interval problems. We conclude the chapter by emphasizing the connection between solutions by the Green's function method, and by the spectral representation method.

3.2 EIGENFUNCTIONS AND EIGENVALUES

A complex number μ is called an *eigenvalue* of the linear operator L if there exists a nonzero vector v in the domain of L such that

$$Lv = \mu v \tag{3.1}$$

The vector v is called an *eigenfunction* of the linear operator L. We remark that, although an eigenfunction is by definition nonzero, it can be associated with a zero eigenvalue.

So far, it is unclear whether or not there are any eigenfunctions and eigenvalues associated with a specific operator. However, if eigenvalues and eigenfunctions do exist, they have remarkable properties. We first show that if $u_1, u_2, \ldots, u_n$ are eigenfunctions corresponding to different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, associated with the operator L, then $\{u_k\}$ is a linearly independent sequence. Our proof is by induction. Let $n = 1$ and examine

$$\alpha_1 u_1 = 0$$

Since u_1 is an eigenfunction, $u_1 \neq 0$, and therefore $\alpha_1 = 0$. We now suppose that the linearly independent assertion is true for $n-1$ and examine

$$\sum_{k=1}^{n} \alpha_k u_k = 0$$

Operating on both sides with $L - \lambda_n$, we obtain

$$0 = (L - \lambda_n) \sum_{k=1}^{n} \alpha_k u_k = \sum_{k=1}^{n-1} \alpha_k (\lambda_k - \lambda_n) u_k$$

Since the $n - 1$ length sequence has been supposed independent,

$$\alpha_k (\lambda_k - \lambda_n) = 0, \qquad k = 1, 2, \ldots, n - 1$$

which implies that

$$\alpha_k = 0, \qquad k = 1, 2, \ldots, n - 1$$

The expression

$$\sum_{k=1}^{n} \alpha_k u_k = 0$$

then reduces to

$$\alpha_n u_n = 0$$

which implies that $\alpha_n = 0$. What we have shown is that if the $(n - 1)$-length sequence is independent, then the n-length sequence is independent. We have established the result for $n = 1$, and therefore it must be true for $n = 2$. By induction, it must therefore be true for arbitrary n. Further, since n is arbitrary, the countably infinite sequence $u_1, u_2, \ldots$ is linearly independent.

In the above proof of linear independence of the eigenfunctions, we assumed nothing about the operator L except linearity. If the linear operator L is self-adjoint, however, we may show that its eigenvalues are real. Indeed, let μ be an eigenvalue associated with the eigenfunction v. Then,

$$\mu\langle v, v\rangle = \langle \mu v, v\rangle = \langle Lv, v\rangle$$

But, since L is self-adjoint,

$$\mu\langle v, v\rangle = \langle v, Lv\rangle = \langle v, \mu v\rangle = \bar{\mu}\langle v, v\rangle$$

Therefore,

$$(\mu - \bar{\mu})\langle v, v\rangle = 0$$

Since $v \neq 0$, $\langle v, v\rangle > 0$ and we must have

$$\mu - \bar{\mu} = 0$$

which implies that $\mu \in \mathbf{R}$.

We next establish that eigenfunctions of a self-adjoint operator corresponding to different eigenvalues are orthogonal. Indeed, let

$$Lu_m = \lambda_m u_m$$

$$Lu_n = \lambda_n u_n$$

where $\lambda_m \neq \lambda_n$ Then,

$$\lambda_m\langle u_m, u_n\rangle = \langle \lambda_m u_m, u_n\rangle = \langle Lu_m, u_n\rangle$$

Since L is self-adjoint,

$$\lambda_m\langle u_m, u_n\rangle = \langle u_m, Lu_n\rangle = \langle u_m, \lambda_n u_n\rangle = \bar{\lambda}_n\langle u_m, u_n\rangle$$

But, we have established that λ_n is real. Therefore,

$$\lambda_m\langle u_m, u_n\rangle = \lambda_n\langle u_m, u_n\rangle$$

and

$$(\lambda_m - \lambda_n)\langle u_m, u_n\rangle = 0$$

Since $\lambda_m \neq \lambda_n$, we must have $\langle u_m, u_n\rangle = 0$.

We next state a central result for Hilbert space $\mathcal{L}_2(a, b)$. It can be shown that the eigenfunctions u_k of a self-adjoint operator form an orthogonal basis in $\mathcal{L}_2(a, b)$. Therefore, any $u \in \mathcal{L}_2(a, b)$ can be expanded:

$$u = \sum_{k=1}^{\infty} \alpha_k u_k$$

The equality is interpreted in the sense that

$$\lim_{n\to\infty} \|u - \sum_{k=1}^{n} \alpha_k u_k\| = 0$$

The proof of this property involves the theory of integral equations and is beyond the scope of this book. The interested reader is referred to the literature [1].

The fact that the eigenfunctions form an orthogonal basis allows us to solve the self-adjoint SLP2 problem in terms of the eigenfunctions u_n of the operator L. We begin by noting that the eigenfunctions can always be normalized, so that we shall assume that they are orthonormal. Therefore, if

$$u = \sum_{n} \alpha_n u_n \tag{3.2}$$

then by (1.58), the Fourier coefficients are given by

$$\alpha_n = \langle u, u_n \rangle \tag{3.3}$$

where the index n runs over all of the eigenfunctions. On the interval $x \in (a, b)$, consider the following SLP2 problem:

$$(L - \lambda)u = f \tag{3.4}$$

with associated boundary conditions

$$B_1(u) = 0 \tag{3.5}$$

$$B_2(u) = 0 \tag{3.6}$$

We assume that the operator L is self-adjoint. We associate with this SLP2 problem the following *eigenproblem*:

$$Lu_n = \lambda_n u_n \tag{3.7}$$

where $\lambda_n \in \mathbf{R}$. We assign to u_n the same boundary conditions as those we have assigned to u in (3.5) and (3.6), viz.

$$B_1(u_n) = 0 \tag{3.8}$$

$$B_2(u_n) = 0 \tag{3.9}$$

The subscript n is an integer indexing the sequences $\{u_n\}$ and $\{\lambda_n\}$. We form the following inner product relation:

$$\langle (L - \lambda)u, u_n \rangle = \langle u, (L - \bar{\lambda})u_n \rangle + J(u, u_n) \Big|_a^b \tag{3.10}$$

From the self-adjoint property, the conjunct J is zero. Substituting (3.4) and (3.7) into (3.10), we obtain

$$\langle f, u_n \rangle = \langle u, (\lambda_n - \bar{\lambda})u_n \rangle = (\lambda_n - \lambda)\langle u, u_n \rangle = (\lambda_n - \lambda)\alpha_n$$

Therefore,

$$\alpha_n = \frac{\langle f, u_n \rangle}{\lambda_n - \lambda} \tag{3.11}$$

Substituting (3.11) into (3.2) gives the solution in terms of the eigenvalues and eigenfunctions, viz.

$$u = \sum_n \frac{\langle f, u_n \rangle}{\lambda_n - \lambda} u_n \tag{3.12}$$

EXAMPLE 3.1 Using the eigenfunction–eigenvalue method, we wish to solve the following differential equation on $x \in (0, a)$:

$$(L - \lambda)u = f \tag{3.13}$$

where f and λ are complex and where

$$L = -\frac{d^2}{dx^2} \tag{3.14}$$

and

$$u'(0) = u'(a) = 0 \tag{3.15}$$

The problem is of class SLP2. The operator L with the given unmixed boundary conditions is self-adjoint. The associated eigenproblem is given by

$$-u_n'' = \lambda_n u_n \tag{3.16}$$

with

$$u_n'(0) = u_n'(a) = 0 \tag{3.17}$$

The orthonormal solutions to the eigenproblem are given by

$$u_n = \left(\frac{\epsilon_n}{a}\right)^{1/2} \cos\frac{n\pi x}{a}, \qquad n = 0, 1, \ldots \tag{3.18}$$

where

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 \tag{3.19}$$

and where ϵ_n is *Neumann's number*, given by

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \neq 0 \end{cases} \tag{3.20}$$

The solution in (3.18) can be easily verified by substitution into (3.16). We note that the factor $\sqrt{\epsilon_n/a}$ normalizes the eigenfunctions. Substitution of (3.18) and (3.19) in (3.12) yields

$$u(x) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{a} \frac{\int_0^a f(x') \cos \frac{n\pi x'}{a} dx'}{\left(\frac{n\pi}{a}\right)^2 - \lambda} \cos \frac{n\pi x}{a} \tag{3.21}$$

■

In Example 3.1, we have solved the differential equation in (3.13)–(3.15) by the eigenfunction–eigenvalue method. The method is also called the *spectral representation method.* The reader should compare the solution in Example 3.1 to the solution by Green's function methods, given in Example 2.13. It appears that the Green's function method might always be preferred since the spectral representation method contains a summation that must be performed before the mathematics can be reduced to a numerical answer. There are, however, many reasons why the spectral representation is important. First, consider (3.21) in a slightly different form, viz.

$$u(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{a} \tag{3.22}$$

where

$$A_n = \frac{\epsilon_n}{a\left[\left(\frac{n\pi}{a}\right)^2 - \lambda\right]} \int_0^a f(x') \cos(\frac{n\pi x'}{a}) dx' \tag{3.23}$$

We interpret (3.22) as a Fourier sum over the *natural modes* of the system with *modal coefficients* A_n. We note that it is the interaction between the forcing function $f(x)$ and the modes in (3.23) that determines the modal coefficients. Second, in dealing with multidimensional problems in later chapters, we shall encounter partial differential equations whose solutions are rarely expressible in terms of closed-form Green's functions. However, proper combination of the spectral representation and Green's function methods results in a powerful tool to solve many partial differential equations appearing in electromagnetic problems.

There is a variation on the procedure used to produce the result in (3.12). This variation results in a more direct approach to the solution of differential equations by the eigenfunction–eigenvalue method. In addition, the variation is very useful in the solution to the partial differential equations considered in later chapters. We proceed as follows. Suppose L is self-adjoint with associated orthonormal eigenfunctions u_n. Then,

$$u = \sum_n \alpha_n u_n \tag{3.24}$$

$$\alpha_n = \langle u, u_n \rangle \tag{3.25}$$

We say that (3.25) is a transformation of the function $u(x) \in \mathcal{L}_2(a, b)$ into coefficients α_n. Conversely, (3.24) is the inverse transformation of the coefficients α_n into the function $u(x)$. We represent this transformation relationship by $u(x) \Longleftrightarrow \alpha_n$. We now show that if L is self-adjoint and

$$u \Longleftrightarrow \alpha_n \tag{3.26}$$

then,

$$Lu \Longleftrightarrow \lambda_n \alpha_n \tag{3.27}$$

Indeed, forming the transformation defined in (3.25), we have

$$\langle Lu, u_n \rangle = \langle u, Lu_n \rangle = \lambda_n \langle u, u_n \rangle = \lambda_n \alpha_n \tag{3.28}$$

Having established the basic result in (3.27), we reconsider the original problem stated in (3.4)–(3.6), which we repeat here for convenience, viz.

$$(L - \lambda)u = f \tag{3.29}$$

with associated boundary conditions

$$B_1(u) = 0 \tag{3.30}$$

$$B_2(u) = 0 \tag{3.31}$$

where we assume that L is self-adjoint. If u_n are eigenfunctions and λ_n are eigenvalues of L, then

$$u \Longleftrightarrow \alpha_n \tag{3.32}$$

and

$$Lu \Longleftrightarrow \lambda_n \alpha_n \tag{3.33}$$

If we define

$$f(x) \Longleftrightarrow \beta_n \tag{3.34}$$

then (3.29) transforms into

$$(\lambda_n - \lambda)\alpha_n = \beta_n \tag{3.35}$$

Solving for α_n, we obtain

$$\alpha_n = \frac{\beta_n}{\lambda_n - \lambda} = \frac{\langle f, u_n \rangle}{\lambda_n - \lambda} \tag{3.36}$$

Substitution of this result into (3.24) yields (3.12).

3.3 SPECTRAL REPRESENTATIONS FOR SLP1 AND SLP2

In Example 3.1, we assumed that we had somehow obtained the eigenfunctions given by (3.18). In this section, we shall present a method of obtaining the eigenfunctions and eigenvalues of a self-adjoint operator directly from the Green's function for the problem. Since SLP1 is a special case contained in SLP2, we shall confine our attention to SLP2.

Note that the solution in (3.12) in terms of the eigenfunctions and eigenvalues is parametrically dependent on λ, viz.

$$u(x, \lambda) = -\sum_n \frac{\langle f, u_n \rangle}{\lambda - \lambda_n} u_n$$

Consider

$$\oint_{C_R} u(x, \lambda) d\lambda$$

where C_R is a circle of radius R centered at the origin in the complex λ-plane (Fig. 3-1). We have

$$\oint_{C_R} u(x, \lambda) d\lambda = -\sum_n \langle f, u_n \rangle u_n \oint_{C_R} \frac{d\lambda}{\lambda - \lambda_n}$$

where the sum is over those eigenvalues λ_n contained within the circle. The singularities of the integrand are simple poles with residue of unity at all $\lambda = \lambda_n$ within the contour. We note that since L is self-adjoint, the poles must lie on the real axis in the λ-plane. Taking the limit as $R \to \infty$, we enclose all of the singularities and obtain by the *Residue Theorem* [2]

$$\lim_{R \to \infty} \oint_{C_R} u(x, \lambda) d\lambda = -2\pi i \sum_n \langle f, u_n \rangle u_n \tag{3.37}$$

where the sum is now over all of the eigenfunctions. The summation is simply the Fourier expansion of the forcing function in terms of the eigenfunctions. Therefore, we find that

$$\frac{1}{2\pi i} \oint_C u(x, \lambda) d\lambda = -f(x) \tag{3.38}$$

where C is the contour at infinity obtained in the limiting operation in (3.37). There is an important special case to the result in (3.38). We note that the general forcing function $f(x)$ produces the response $u(x, \lambda)$.

Therefore, the specific forcing function $\delta(x-\xi)/w(x)$ must produce the Green's function $g(x, \xi, \lambda)$. We therefore obtain

$$\frac{1}{2\pi i}\oint_C g(x,\xi,\lambda)d\lambda = -\frac{\delta(x-\xi)}{w(x)} \tag{3.39}$$

The solution to the contour integral in (3.39) for a specific Green's function associated with a specific operator L and boundary conditions is called the *spectral representation of the delta function* [3] for the operator L. We shall demonstrate the utility of this result in an example.

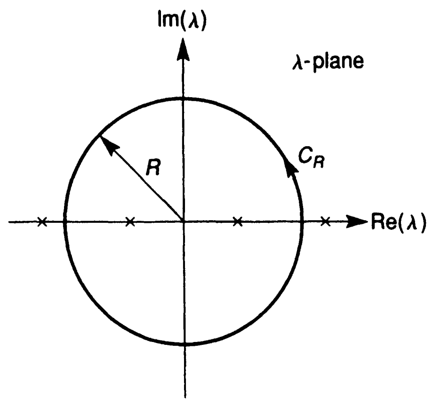

Fig. 3-1 Circular contour of radius R centered at the origin in the complex λ-plane. The $\times$ indicate possible simple pole locations at $\lambda = \lambda_n$.

EXAMPLE 3.2 Consider the following operator defined on $x \in (0, a)$:

$$L = -\frac{d^2}{dx^2} \tag{3.40}$$

with boundary conditions

$$u'(0) = u'(a) = 0 \tag{3.41}$$

The Green's function associated with L_λ with these boundary conditions has been previously derived in Example 2.13. We repeat it here for convenience, viz.

$$g(x,\xi,\lambda) = -\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}a}\begin{cases}\cos\sqrt{\lambda}x\cos\sqrt{\lambda}(a-\xi), & x<\xi\\ \cos\sqrt{\lambda}\xi\cos\sqrt{\lambda}(a-x), & x>\xi\end{cases} \tag{3.42}$$

First, consider the case $x < \xi$, so that

$$g(x,\xi,\lambda) = -\frac{\cos\sqrt{\lambda}x\cos\sqrt{\lambda}(a-\xi)}{\sqrt{\lambda}\sin\sqrt{\lambda}a} \tag{3.43}$$

Substitution into (3.39) gives

$$\delta(x-\xi) = \frac{1}{2\pi i}\oint_C \frac{\cos\sqrt{\lambda}x\cos\sqrt{\lambda}(a-\xi)}{\sqrt{\lambda}\sin\sqrt{\lambda}a}d\lambda \tag{3.44}$$

In order to solve this closed contour integral by the residue theorem, we first investigate the singularities of the integrand. Since $\sqrt{\lambda}$ is a multiple-valued function, we might expect that the integrand contains a branch cut with a branch point at $\lambda = 0$. We may show, however, that although $\sqrt{\lambda}$ has a branch cut, $g(x, \xi, \lambda)$ does not. Indeed, define

$$\lambda = |\lambda| e^{i\phi}, \qquad 2\pi > \phi > 0 \tag{3.45}$$

so that

$$\sqrt{\lambda} = |\lambda|^{1/2} e^{i\phi/2}, \quad \pi > \frac{\phi}{2} > 0 \tag{3.46}$$

This definition of λ results in a branch cut in $\sqrt{\lambda}$ along the positive-real axis in the λ-plane (Fig. 3-2). In fact,

$$\lim_{\phi\to 0} \sqrt{\lambda} = |\lambda|^{1/2}$$

$$\lim_{\phi\to 2\pi} \sqrt{\lambda} = -|\lambda|^{1/2}$$

Applying this result to (3.43), we find that

$$\lim_{\phi\to 0} g(x, \xi, \lambda) = -\frac{\cos|\lambda|^{1/2}x \cos|\lambda|^{1/2}(a-\xi)}{|\lambda|^{1/2}\sin|\lambda|^{1/2}a}$$

$$\lim_{\phi\to 2\pi} g(x, \xi, \lambda) = -\frac{\cos(-|\lambda|^{1/2}x)\cos[-|\lambda|^{1/2}(a-\xi)]}{-|\lambda|^{1/2}\sin(-|\lambda|^{1/2}a)}$$

Some minor algebraic manipulation shows that

$$\lim_{\phi\to 0} g(x, \xi, \lambda) = \lim_{\phi\to 2\pi} g(x, \xi, \lambda)$$

We conclude that, although there is a branch cut in $\sqrt{\lambda}$ along the positive-real axis, the Green's function is continuous there. We next consider the possible location and order of poles of $g(x, \xi, \lambda)$. Since the numerator of the Green's function in (3.43) is finite throughout the complex λ-plane, it is sufficient to search for poles caused by the denominator. Let

$$f(\lambda) = \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}a} = \frac{p(\lambda)}{q(\lambda)}$$

where

$$p(\lambda) = 1$$

$$q(\lambda) = \sqrt{\lambda}\sin\sqrt{\lambda}a$$

By a well-known theorem of complex analysis [4], $f(\lambda)$ has a simple pole at $\lambda = \lambda_n$ if $p(\lambda_n) \neq 0$, $q(\lambda_n) = 0$, and $q'(\lambda_n) \neq 0$. In addition, the residue at the simple pole location is given by

$$\operatorname{Res}\{f(\lambda); \lambda_n\} = \frac{p(\lambda_n)}{q'(\lambda_n)}$$

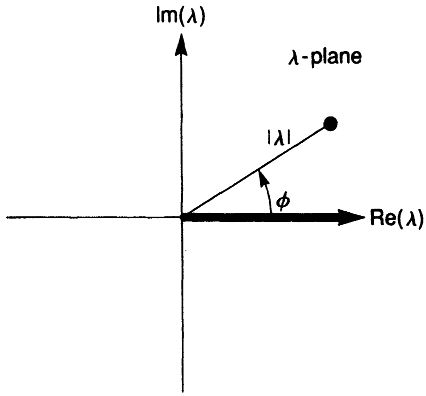

Fig. 3-2 Polar representation of λ in the complex λ-plane showing branch cut for $\sqrt{\lambda}$ along positive-real axis (thick line).

We note that $p(\lambda) \neq 0$ anywhere, and that $q(\lambda) = 0$ wherever $\sqrt{\lambda}a = 0, \pm\pi, \pm 2\pi, \ldots$. We conclude that $q(\lambda) = 0$ whenever

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \qquad n = 0, 1, 2, \ldots$$

Differentiating the denominator, we find that

$$q'(\lambda) = \frac{a}{2}\left(\cos\sqrt{\lambda}a + \frac{\sin\sqrt{\lambda}a}{\sqrt{\lambda}a}\right)$$

from which

$$q'(\lambda_n) = (-1)^n\left(\frac{a}{\epsilon_n}\right), \qquad n = 0, 1, \ldots$$

where ϵ_n is Neumann's number, defined in (3.20). We have shown that the singularities at λ_n are simple poles. For the residues, we have

$$\mathrm{Res}\{f(\lambda); \lambda_n\} = (-1)^n \frac{\epsilon_n}{a}$$

Returning to (3.44), we now have

$$\begin{aligned}\delta(x-\xi) &= \sum_{n=0}^{\infty} \cos\frac{n\pi x}{a} \cos\frac{n\pi}{a}(a-\xi)\mathrm{Res}\{f(\lambda); \lambda_n\} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\epsilon_n}{a} \cos\frac{n\pi x}{a} \cos\frac{n\pi}{a}(a-\xi)\end{aligned}$$

and finally,

$$\delta(x-\xi) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{a} \cos\frac{n\pi x}{a} \cos\frac{n\pi\xi}{a} \tag{3.47}$$

Equation (3.47) is the spectral representation of the delta function associated with the operator $L = -d^2/dx^2$ with boundary conditions $u'(0) = u'(a) = 0$. Recall that our result is for $x < \xi$. To produce the result for $x > \xi$, we interchange x and ξ in (3.47). Since this interchange produces no change in the result, (3.47) holds for all x and ξ on the interval $(0, a)$. The reader is cautioned that the series on the right side of (3.47) does not converge. It is, however, an extremely useful symbolic equality, as has been discussed in Section 2.2. Indeed, we may show that (3.47) is merely a disguised form of the Fourier cosine series. For any $s(x) \in \mathcal{L}_2(0, a)$, we have, from (2.9),

$$s(x) = \int_0^a \delta(x - \xi)s(\xi)d\xi \tag{3.48}$$

Substituting (3.47), we obtain, after an interchange of integration and summation,

$$s(x) = \sum_{n=0}^{\infty} \alpha_n \sqrt{\frac{\epsilon_n}{a}} \cos \frac{n\pi x}{a}$$

where

$$\alpha_n = \int_0^a s(\xi)\sqrt{\frac{\epsilon_n}{a}} \cos \frac{n\pi\xi}{a} d\xi$$

We note that we have produced the eigenfunctions and eigenvalues inferred in (3.18) and (3.19) directly from the Green's function associated with the operator L and its boundary conditions.

■

In the example spectral representation in (3.47), each term in the sum consists of the product of the orthonormal eigenfunction as a function of x with the same orthonormal eigenfunction as a function of ξ. We may show that this result can be generalized to all self-adjoint operators on SLP2. Indeed, if the forcing function in (3.12) is the delta function, we have

$$g(x, \xi, \lambda) = \sum_n \frac{\langle \frac{\delta(x'-\xi)}{w(x')}, u_n(x')\rangle_{x'}}{\lambda_n - \lambda} u_n(x)$$

where, as indicated, the inner product is with respect to x'. Performing the integration gives

$$g(x, \xi) = \sum_n \frac{u_n(x)\overline{u_n}(\xi)}{\lambda_n - \lambda} \tag{3.49}$$

This form of the Green's function is called the bilinear series form [5]. Substitution of (3.49) into (3.39) gives

$$\frac{\delta(x - \xi)}{w(x)} = -\frac{1}{2\pi i} \sum_n u_n(x)\overline{u_n}(\xi) \oint_C \frac{d\lambda}{\lambda_n - \lambda}$$

or

$$\frac{\delta(x-\xi)}{w(x)} = \sum_n u_n(x)\overline{u_n}(\xi) \tag{3.50}$$

Equation (3.50) gives the general spectral representation for self-adjoint operators on SLP2. The procedure for solving self-adjoint SLP2 problems by the spectral representation method can now be summarized as follows:

1. For a given self-adjoint operator L and given boundary conditions, solve the Green's function problem $L_\lambda g(x, \xi, \lambda) = \delta(x-\xi)/w(x)$.
2. Substitute the Green's function $g(x, \xi, \lambda)$ into (3.39) and solve for the spectral representation of the delta function. The resulting form should appear as in (3.50).
3. Substitute the normalized eigenfunctions and their eigenvalues, obtained in the spectral representation, into (3.12) to produce the solution $u(x)$.

In addition to Example 3.2, we have included in the problems several common examples to illustrate the method.

3.4 SPECTRAL REPRESENTATIONS FOR SLP3

The spectral representation of the delta function takes on a different character when the interval along the real axis becomes unbounded. Consider the problem in Example 3.2, defined on the interval $x \in (0, a)$. The Green's function $g(x, \xi, \lambda)$ was shown to have simple poles at $\lambda = (n\pi/a)^2$. Note that as a becomes larger, the poles become closer together. In the limit as $a \to \infty$, the poles become arbitrarily close. This behavior leads us to inquire into the form of the singularity along the positive-real axis in the λ-plane in this limiting case. We shall illustrate the obtaining of the spectral representation with an example.

EXAMPLE 3.3 Consider the following SLP3 differential equation:

$$-u'' - \lambda u = f, \qquad \lambda \in \mathbf{C} \tag{3.51}$$

$$u(0) = 0 \tag{3.52}$$

This problem is in the limit point case as $x \to \infty$. We therefore assign the limit condition

$$\lim_{x\to\infty} u(x) = 0 \tag{3.53}$$

The associated Green's function problem is

$$-\frac{d^2g}{dx^2} - \lambda g = \delta(x-\xi)$$
$$g(0,\xi) = 0$$
$$\lim_{x\to\infty} g(x,\xi) = 0$$

We have previously obtained this Green's function in (2.172) and repeat it here for convenience, viz.

$$g(x,\xi) = \frac{1}{\sqrt{\lambda}} \begin{cases} e^{-i\sqrt{\lambda}\xi} \sin\sqrt{\lambda}x, & x < \xi \\ e^{-i\sqrt{\lambda}x} \sin\sqrt{\lambda}\xi, & x > \xi \end{cases} \tag{3.54}$$

where

$$\mathrm{Im}\sqrt{\lambda} < 0 \tag{3.55}$$

To produce the restriction in (3.55), we define

$$\lambda = |\lambda|e^{i\phi}, \quad 2\pi < \phi < 4\pi \tag{3.56}$$

so that

$$\sqrt{\lambda} = |\lambda|^{1/2}e^{i\phi/2}, \quad \pi < \frac{\phi}{2} < 2\pi \tag{3.57}$$

The angular definition in (3.56) defines a *Riemann sheet* of the λ-plane. The angular restriction in (3.57) indicates that $\mathrm{Im}\sqrt{\lambda} < 0$ everywhere on this sheet. We shall refer to this sheet as the *proper Riemann sheet*. We note that once (3.55) has been invoked, any result with $\mathrm{Im}\sqrt{\lambda} > 0$ would violate the requirement for the proper Riemann sheet. The definition of λ in (3.56) results in a branch cut in $\sqrt{\lambda}$ along the positive-real axis in the λ-plane (Fig. 3-2). In this case,

$$\lim_{\phi\to 2\pi} \sqrt{\lambda} = -|\lambda|^{1/2}$$
$$\lim_{\phi\to 4\pi} \sqrt{\lambda} = |\lambda|^{1/2}$$

Unlike the situation in Example 3.2, however, the branch cut in $\sqrt{\lambda}$ produces a branch cut in $g(x,\xi,\lambda)$ along the positive-real axis. Indeed, for $x < \xi$,

$$\lim_{\phi\to 2\pi} g(x,\xi,\lambda) = \frac{e^{i|\lambda|^{1/2}\xi} \sin|\lambda|^{1/2}x}{|\lambda|^{1/2}}$$
$$\lim_{\phi\to 4\pi} g(x,\xi,\lambda) = \frac{e^{-i|\lambda|^{1/2}\xi} \sin|\lambda|^{1/2}x}{|\lambda|^{1/2}}$$

Since the exponential changes sign, $g(x,\xi,\lambda)$ is discontinuous across the positive-real axis. The result for $x > \xi$ is the same, except that x and ξ are interchanged.

We next produce the spectral representation by considering the closed contour in the complex λ-plane shown in Fig. 3-3. Since the contour excludes the branch cut and since $g(x, \xi, \lambda)$ has no other singularities, Cauchy's theorem gives

$$\oint_{C_R+C_1+C_\rho+C_2} g(x, \xi, \lambda)d\lambda = 0 \tag{3.58}$$

We examine the contributions from the various portions of the contour in the limit as $\rho \to 0$, $R \to \infty$, and $\Gamma \to 0$. Consider C_R. From (3.39), we have

$$\lim_{\Gamma\to 0}\lim_{R\to\infty}\int_{C_R} g(x, \xi, \lambda)d\lambda = -2\pi i\delta(x-\xi) \tag{3.59}$$

where we have assumed that, even in the presence of the branch cut, the integral around the circle of infinite radius centered at the origin produces the delta-function contribution. Consider C_ρ. Letting $\lambda = \rho\exp(i\phi)$, we have

$$\int_{C_\rho} g(x, \xi, \lambda)d\lambda = i\rho\int_{4\pi-\Gamma}^{2\pi+\Gamma} \frac{\sin(\rho^{1/2}e^{i\phi/2}x)}{\rho^{1/2}e^{i\phi/2}} e^{-i(\rho^{1/2}e^{i\phi/2}\xi)} e^{i\phi}d\phi$$

Since the integrand on the right side is bounded as $\rho \to 0$ and $\Gamma \to 0$, the integral is bounded and

$$\lim_{\Gamma\to 0}\lim_{\rho\to 0}\int_{C_\rho} g(x, \xi, \lambda)d\lambda = 0 \tag{3.60}$$

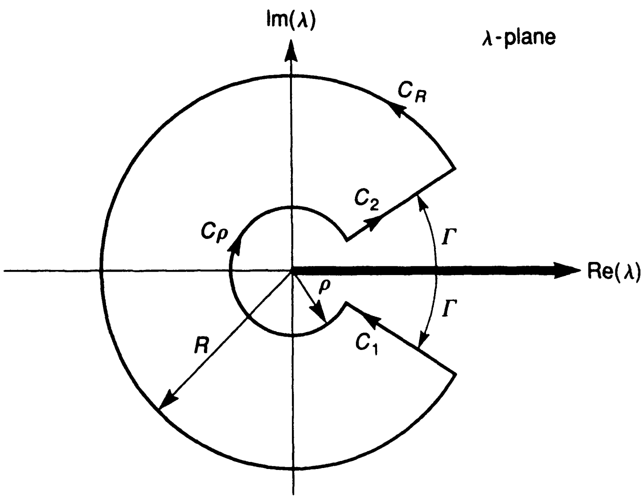

Fig. 3-3 Contour for evaluation of the spectral representation for Example 3.3.

Consider the integral along C_1 and C_2. On C_1, let

$$\lambda = re^{i(4\pi-\Gamma)}$$

On C_2, let

$$\lambda = re^{i(2\pi+\Gamma)}$$

We obtain

$$\int_{C_1+C_2} g(x,\xi,\lambda)d\lambda = \int_R^\rho \frac{\sin[r^{1/2}e^{i(2\pi-\Gamma/2)}x]}{r^{1/2}e^{i(2\pi-\Gamma/2)}} e^{-i[r^{1/2}e^{i(2\pi-\Gamma/2)}\xi]} e^{i(4\pi-\Gamma)}dr$$
$$+ \int_\rho^R \frac{\sin[r^{1/2}e^{i(\pi+\Gamma/2)}x]}{r^{1/2}e^{i(\pi+\Gamma/2)}} e^{-i[r^{1/2}e^{i(\pi+\Gamma/2)}\xi]} e^{i(2\pi+\Gamma)}dr \tag{3.61}$$

Taking the limits, we have

$$\lim_{\rho\to 0}\lim_{R\to\infty}\lim_{\Gamma\to 0}\int_{C_1+C_2} g(x,\xi,\lambda)d\lambda = \int_\infty^0 \frac{\sin r^{1/2}x}{r^{1/2}} e^{-ir^{1/2}\xi}dr$$
$$+ \int_0^\infty \frac{\sin r^{1/2}x}{r^{1/2}} e^{ir^{1/2}\xi}dr$$
$$= 2i\int_0^\infty \frac{\sin(r^{1/2}x)\sin(r^{1/2}\xi)}{r^{1/2}}dr \tag{3.62}$$

Combining the results in (3.58)–(3.62), we obtain

$$-2\pi i\delta(x-\xi) + 2i\int_0^\infty \frac{\sin(r^{1/2}x)\sin(r^{1/2}\xi)}{r^{1/2}}dr = 0 \tag{3.63}$$

We let

$$k = r^{1/2}$$

so that

$$dk = \frac{dr}{2r^{1/2}}$$

We substitute into (3.63) and produce the following spectral representation:

$$\delta(x-\xi) = \frac{2}{\pi}\int_0^\infty \sin kx \sin k\xi dk \tag{3.64}$$

In a similar manner to that in Example 3.2, we now show that (3.64) is merely a disguised form of the Fourier sine transform. Indeed, for $f(x) \in \mathcal{L}_2(0,\infty)$, we have

$$f(x) = \int_0^\infty \delta(x-\xi)f(\xi)d\xi$$

Substituting (3.64), we obtain, after an interchange of integrations,

$$f(x) = \frac{2}{\pi} \int_0^\infty F(k) \sin kx dk \tag{3.65}$$

where

$$F(k) = \int_0^\infty f(\xi) \sin k\xi d\xi \tag{3.66}$$

We indicate the Fourier sine transform relationship symbolically by

$$f(x) \Longleftrightarrow F(k) \tag{3.67}$$

We now return to the solution to the differential equation considered in (3.51)–(3.53), repeated here for convenience. Consider $\mathcal{L}_2(0, \infty)$ with inner product

$$\langle u, v \rangle = \int_0^\infty u(x)v(x)dx \tag{3.68}$$

Consider

$$-u'' - \lambda u = f, \qquad \lambda \in \mathbf{C} \tag{3.69}$$

$$u(0) = 0 \tag{3.70}$$

$$\lim_{x \to \infty} u(x) = 0 \tag{3.71}$$

From Example 2.18 and the discussion following, this problem is self-adjoint. Using the results in (3.66) and (3.67), we expand $u(x)$ as follows:

$$u(x) = \frac{2}{\pi} \int_0^\infty U(k) \sin kx dk \tag{3.72}$$

where

$$U(k) = \langle u, \sin kx \rangle = \int_0^\infty u(\xi) \sin k\xi d\xi \tag{3.73}$$

Expression (3.72) is the equivalent to (3.24), except in this case we have an integral, rather than a sum. We note that $\sin kx$ plays the role that the eigenfunction u_n plays in (3.24). In (3.73), $U(k)$ is similar to the Fourier coefficient in (3.25), where $\sin kx$ plays the same role as the eigenfunction u_n in (3.25). To further investigate the similarity to the eigenfunction u_n, we note that

$$-\frac{d^2 \sin kx}{dx^2} = k^2 \sin kx \tag{3.74}$$

so that $\sin kx$ appears to be an eigenfunction of the self-adjoint operator $-d^2/dx^2$ with eigenvalue k^2. However, $\sin kx$ is not in $\mathcal{L}_2(0, \infty)$, and therefore cannot be an eigenfunction. We shall adopt the notation of Friedman [6] and call $\sin kx$ an *improper eigenfunction* with *improper eigenvalue* k^2. Fortunately, the procedure we adopted in (3.24)–(3.36) for solving differential equations by the eigenfunction–eigenvalue method can be extended to apply to improper eigenfunctions. We begin by showing that if

$$u \Longleftrightarrow U \tag{3.75}$$

then

$$-\frac{d^2u}{dx^2} \Longleftrightarrow k^2 U \tag{3.76}$$

Indeed, following (3.28), we form

$$\begin{aligned}\langle -\frac{d^2u}{dx^2}, \sin kx\rangle &= \langle u, -\frac{d^2 \sin kx}{dx^2}\rangle + J(u, \sin kx)\Big|_0^\infty \\ &= k^2U + \left(-\frac{du}{dx}\sin kx + ku\cos kx\right)\Bigg|_0^\infty \end{aligned} \tag{3.77}$$

Using the conditions in (3.70) and (3.71), we have

$$\langle -\frac{d^2u}{dx^2}, \sin kx\rangle = k^2U - \lim_{x\to\infty}\frac{du}{dx}\sin kx \tag{3.78}$$

We note that the improper eigenfunction $\sin kx$ does not vanish in the limit as $x \to \infty$. This behavior is contrary to what we found when dealing with eigenfunctions on finite intervals. Fortunately, in electromagnetic problems, when we have

$$\lim_{x\to\infty} u(x) = 0$$

then also

$$\lim_{x\to\infty}\frac{du(x)}{dx} = 0$$

For example, if u is a component of the electric field, then $\partial u/\partial x$ is a component of the magnetic field; if the E-field vanishes at infinity, then the H-field also vanishes. Therefore, in the usual cases in electromagnetics, we obtain

$$\langle -\frac{d^2u}{dx^2}, \sin kx\rangle = k^2U \tag{3.79}$$

which establishes (3.76). As a footnote, we remark that there are mathematical theorems that generalize this result to classes of functions possessing certain continuity and absolute integrability properties. The interested reader is referred to [7].

We now are able to solve the original differential equation in (3.69) using the spectral representation. Taking the Fourier sine transform of both sides of (3.69), we obtain

$$(k^2 - \lambda)U(k) = F(k) \tag{3.80}$$

Dividing both sides by $(k^2 - \lambda)$ and taking the inverse transform, we obtain

$$u(x) = \frac{2}{\pi} \int_0^\infty \frac{F(k)}{k^2 - \lambda} \sin kx dk \tag{3.81}$$

■

In the above example, we have indicated explicitly the various limiting operations involved in evaluating the integral of the Green's function around the closed contour indicated in Fig. 3-3. In subsequent discussions, we shall, whenever appropriate, simplify the contour (Fig. 3-4) such that the limiting operations have already taken place. The contour in Fig. 3-4 is to be interpreted as follows. The contour segments C_1 and C_2 are straight lines that are the result of limits as we approach the branch cut from below and above, respectively. By Cauchy's Theorem, if there are no singularities inside the contour, then the integral along C_R is the negative of the integral along $C_1 + C_2$. Therefore, we have

$$\int_{C_1+C_2} g(x, \xi, \lambda) d\lambda = -\int_{C_R} g(x, \xi, \lambda) d\lambda = \frac{2\pi i \delta(x - \xi)}{w(x)} \tag{3.82}$$

In (3.82), we have assumed that there is no contribution obtained from the integral along C_ρ (Fig. 3-3) in the limit as $\rho \to 0$. We use this abbreviated method in the following example.

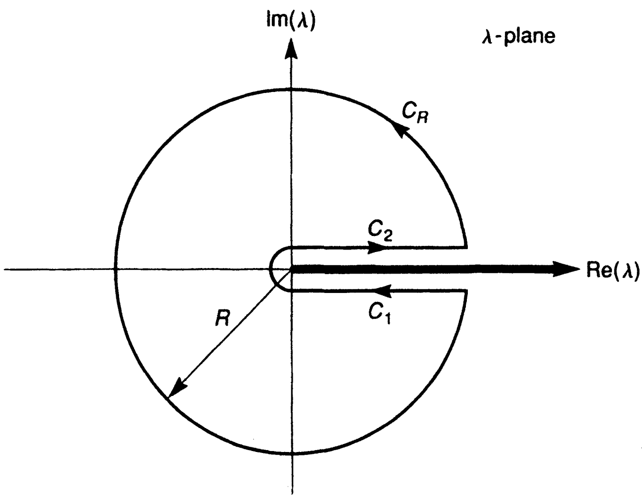

Fig. 3-4 Simplified contour for evaluation of the spectral representation of the delta function for SLP3 problems.

EXAMPLE 3.4 Consider Hilbert space $\mathcal{L}_2(-\infty, \infty)$ with inner product

$$\langle u, v\rangle = \int_{-\infty}^{\infty} u(x)\bar{v}(x)dx \tag{3.83}$$

We seek the spectral representation for the self-adjoint operator

$$L = -d^2/dx^2 \tag{3.84}$$

with limiting conditions

$$\lim_{x\to-\infty} u(x) = \lim_{x\to\infty} u(x) = 0 \tag{3.85}$$

In Example 2.20, we considered the following Green's function problem:

$$-\frac{d^2g}{dx^2} - \lambda g = \delta(x-\xi), \qquad \mathrm{Im}\sqrt{\lambda} < 0 \tag{3.86}$$

$$\lim_{x\to-\infty} g(x,\xi) = \lim_{x\to\infty} g(x,\xi) = 0 \tag{3.87}$$

The solution to this problem was given in (2.175), viz.

$$g(x,\xi) = \frac{e^{-i\sqrt{\lambda}|x-\xi|}}{2i\sqrt{\lambda}} \tag{3.88}$$

The singularities of $g(x,\xi)$ involve the branch cut associated with $\sqrt{\lambda}$. We define this branch cut using (3.56) and (3.57), and find for $x > \xi$

$$\lim_{\phi\to 2\pi} g(x,\xi,\lambda) = \frac{e^{i|\lambda|^{1/2}(x-\xi)}}{-2i|\lambda|^{1/2}} \tag{3.89}$$

$$\lim_{\phi\to 4\pi} g(x,\xi,\lambda) = \frac{e^{-i|\lambda|^{1/2}(x-\xi)}}{2i|\lambda|^{1/2}} \tag{3.90}$$

Therefore (Fig. 3-4), there is a branch cut in $g(x,\xi,\lambda)$ along the positive-real axis. Using (3.82), we obtain

$$2\pi i\delta(x-\xi) = \int_{\infty}^{0} \frac{e^{-i|\lambda|^{1/2}(x-\xi)}}{2i|\lambda|^{1/2}}d\lambda - \int_{0}^{\infty} \frac{e^{i|\lambda|^{1/2}(x-\xi)}}{2i|\lambda|^{1/2}}d\lambda \tag{3.91}$$

We let

$$k = |\lambda|^{1/2} \tag{3.92}$$

and find that

$$2\pi\delta(x-\xi) = -\int_{\infty}^{0} e^{-ik(x-\xi)}dk + \int_{0}^{\infty} e^{ik(x-\xi)}dk \tag{3.93}$$

Replacing k by $-k$ in the first integral gives the final result, viz.

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)}dk \tag{3.94}$$

We have obtained this result for the case $x > \xi$. However, this restriction can be removed. Indeed, to obtain the case $x < \xi$, we merely interchange x and ξ in (3.94). However,

$$\delta(x-\xi) = \delta(\xi - x)$$

which means that we can again reverse the interchange and reclaim the result in (3.94).

We next use the spectral representation in (3.94) to produce the *Fourier transform*. We write

$$u(x) = \int_{-\infty}^{\infty} u(\xi)\delta(x-\xi)d\xi \tag{3.95}$$

Substitution of (3.94) followed by a change in the order of integration yields

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k)e^{ikx}dk \tag{3.96}$$

where

$$U(k) = \int_{-\infty}^{\infty} u(x)e^{-ikx}dx = \langle u, e^{ikx}\rangle \tag{3.97}$$

In (3.96) and (3.97), we identify $\exp(ikx)$ as an improper eigenfunction with improper eigenvalue k^2. We indicate the Fourier transform relationship symbolically by

$$u(x) \Longleftrightarrow U(k) \tag{3.98}$$

We may use the Fourier transform to solve the following differential equation by the spectral representation method:

$$-u'' - \lambda u = f, \qquad \lambda \in \mathbf{C} \tag{3.99}$$

$$\lim_{x\to-\infty} u(x) = 0 \tag{3.100}$$

$$\lim_{x\to\infty} u(x) = 0 \tag{3.101}$$

We begin by showing that if

$$u(x) \Longleftrightarrow U(k)$$

then

$$-u''(x) \Longleftrightarrow k^2U(k) \tag{3.102}$$

Indeed,

$$\begin{aligned}\langle -u'', e^{ikx}\rangle &= \langle u, -\frac{d^2e^{ikx}}{dx^2}\rangle + J(u, e^{ikx})\Big|_{-\infty}^{\infty} \\ &= k^2U(k) + \left(-u'e^{-ikx} - ikue^{-ikx}\right)\Big|_{-\infty}^{\infty}\end{aligned} \tag{3.103}$$

Using the conditions in (3.100) and (3.101), we have

$$\langle -u'', e^{ikx}\rangle = k^2U(k) - \left(u'e^{-ikx}\right)\Big|_{-\infty}^{\infty}$$

From the discussion associated with (3.78), we know that in the usual cases in electromagnetics, (3.100) and (3.101) imply that

$$\lim_{x\to\pm\infty} u'(x) = 0$$

and therefore,

$$\langle -u'', e^{ikx}\rangle = k^2U(k) \tag{3.104}$$

which proves (3.102). We now take the Fourier transform of both sides of (3.99) and obtain

$$(k^2 - \lambda)U(k) = F(k)$$

Dividing both sides by $(k^2 - \lambda)$ and taking the inverse Fourier transform, we have

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(k)}{k^2 - \lambda} e^{ikx}\, dk \tag{3.105}$$

We note that the result in (3.102) could also be obtained by twice differentiating (3.96), *provided* that we can interchange differentiation and integration on the right side. Our method of proof provides a justification of this interchange in this case.

■

We next provide two examples leading to solutions involving Bessel functions. These examples will be useful in problems in cylindrical coordinates to be considered in later chapters.

EXAMPLE 3.5 Consider the following differential equation on $x \in (0, \infty)$:

$$(L - \lambda)u = f \tag{3.106}$$

where

$$L = -\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{d}{dx}\right)\right] \tag{3.107}$$

From Example 2.21, this problem is in the limit circle case as $x \to 0$ and the limit point case as $x \to \infty$. Furthermore, the operator L is self-adjoint. We invoke

$$\lim_{x\to\infty} u(x) = 0 \tag{3.108}$$

$$\lim_{x\to 0} u(x) \text{ finite} \tag{3.109}$$

We seek the spectral representation of the operator in (3.107) with the limiting conditions given in (3.108) and (3.109). The Green's function associated with this operator has been obtained previously in (2.184) and is repeated here for convenience, as follows:

$$g(x,\xi) = \frac{\pi}{2i} \begin{cases} H_0^{(2)}(\sqrt{\lambda}\xi)J_0(\sqrt{\lambda}x), & x < \xi \\ H_0^{(2)}(\sqrt{\lambda}x)J_0(\sqrt{\lambda}\xi), & x > \xi \end{cases} \tag{3.110}$$

where

$$\mathrm{Im}\sqrt{\lambda} < 0 \tag{3.111}$$

To assure the condition in (3.111), we restrict λ as follows:

$$\lambda = |\lambda|e^{i\phi}, \qquad -2\pi < \phi < 0 \tag{3.112}$$

so that

$$\sqrt{\lambda} = |\lambda|^{1/2}e^{i\phi/2}, \qquad -\pi < \frac{\phi}{2} < 0 \tag{3.113}$$

We may show that this definition produces a branch cut in $g(x,\xi,\lambda)$ along the positive-real axis in the λ-plane. Consider the case $x > \xi$. Approaching the positive-real axis from above, we have

$$\lim_{\phi\to-2\pi} H_0^{(2)}(\sqrt{\lambda}x)J_0(\sqrt{\lambda}\xi) = H_0^{(2)}(e^{-i\pi}|\lambda|^{1/2}x)J_0(e^{-i\pi}|\lambda|^{1/2}\xi) \tag{3.114}$$

But, using a well-known Bessel function identity [8], we have

$$J_0(e^{-i\pi}|\lambda|^{1/2}\xi) = J_0(|\lambda|^{1/2}\xi) \tag{3.115}$$

and using a well-known Hankel function identity [9], we have

$$H_0^{(2)}(e^{-i\pi}|\lambda|^{1/2}x) = -H_0^{(1)}(|\lambda|^{1/2}x) \tag{3.116}$$

so that

$$\lim_{\phi\to-2\pi} H_0^{(2)}(\sqrt{\lambda}x)J_0(\sqrt{\lambda}\xi) = -H_0^{(1)}(|\lambda|^{1/2}x)J_0(|\lambda|^{1/2}\xi) \tag{3.117}$$

On the other hand, approaching the positive-real axis from below, we have

$$\lim_{\phi\to 0} H_0^{(2)}(\sqrt{\lambda}x)J_0(\sqrt{\lambda}\xi) = H_0^{(2)}(|\lambda|^{1/2}x)J_0(|\lambda|^{1/2}\xi) \tag{3.118}$$

We note that (3.117) and (3.118) indicate a jump in the Green's function as we cross the positive real axis. To produce the spectral representation, we again consider the contour in Fig. 3-3 and write

$$\oint_{C_R+C_1+C_\rho+C_2} g(x,\xi,\lambda)d\lambda = 0 \tag{3.119}$$

In a manner similar to Example 3.3, we may show that the contribution along C_ρ vanishes as $\Gamma \to 0$ and $\rho \to 0$. We leave this for the reader to verify. Along $C_1 + C_2$, we have

$$\begin{aligned}\lim_{\rho\to 0}\lim_{R\to\infty}\lim_{\Gamma\to 0}\int_{C_1+C_2} g(x,\xi,\lambda)d\lambda &= \frac{\pi}{2i}\left[\int_{\infty}^{0} H_0^{(2)}(|\lambda|^{1/2}x)J_0(|\lambda|^{1/2}\xi)d\lambda\right.\\ &\quad \left. - \int_0^\infty H_0^{(1)}(|\lambda|^{1/2}x)J_0(|\lambda|^{1/2}\xi)d\lambda\right]\\ &= -\frac{\pi}{i}\int_0^\infty J_0(|\lambda|^{1/2}x)J_0(|\lambda|^{1/2}\xi)d\lambda\end{aligned} \tag{3.120}$$

Along C_R, we have

$$\lim_{\Gamma\to 0}\lim_{R\to\infty}\int_{C_R} g(x,\xi,\lambda)d\lambda = -2\pi i\frac{\delta(x-\xi)}{x} \tag{3.121}$$

Taking the appropriate limits in (3.119) and substituting (3.120) and (3.121), we obtain

$$\frac{\delta(x-\xi)}{x} = \frac{1}{2}\int_0^\infty J_0(|\lambda|^{1/2}x)J_0(|\lambda|^{1/2}\xi)d\lambda \tag{3.122}$$

Letting $k = |\lambda|^{1/2}$, we find that

$$\frac{\delta(x-\xi)}{x} = \int_0^\infty J_0(kx)J_0(k\xi)kdk \tag{3.123}$$

which is the required spectral representation. Although this representation has been obtained with the restriction $x > \xi$, the restriction can now be removed in the same manner as in Example 3.3 because of the symmetry of the Green's function.

The representation in (3.123) leads to the *Fourier–Bessel Transform* of order zero. Indeed, consider a Hilbert space $\mathcal{L}_2(0,\infty)$ with inner product

$$\langle s,t\rangle = \int_0^\infty s(x)t(x)xdx \tag{3.124}$$

For any $s(x) \in \mathcal{L}_2(0,\infty)$, we have

$$s(x) = \int_0^\infty s(\xi)\frac{\delta(x-\xi)}{x}\xi d\xi \tag{3.125}$$

Substituting (3.123) into (3.125), we produce the Fourier–Bessel transform pair

$$S(k) = \int_0^\infty s(x)J_0(kx)xdx \tag{3.126}$$

$$s(x) = \int_0^\infty S(k) J_0(kx) k dk \tag{3.127}$$

Symbolically, we write

$$s(x) \Longleftrightarrow S(k)$$

A very useful relation is obtained by noting that for $u(x) \in \mathcal{L}_2(0, \infty)$,

$$\begin{aligned}
\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{du}{dx}\right)\right] &= \int_0^\infty U(k)\left\{\frac{1}{x}\frac{d}{dx}\left[x\frac{dJ_0(kx)}{dx}\right]\right\} kdk \\
&= \int_0^\infty [-k^2 U(k)] J_0(kx) kdk
\end{aligned} \tag{3.128}$$

Therefore,

$$\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{du}{dx}\right)\right] \Longleftrightarrow -k^2 U(k) \tag{3.129}$$

We note that the result in (3.128) depends on the interchange of differentiation and integration. This operation can be justified by using the procedure followed in (3.76)–(3.79) and in (3.102)–(3.104). The details are left for the problems.

■

EXAMPLE 3.6 We wish to find the spectral representation of the operator

$$L = -x\left[\frac{d}{dx}\left(x\frac{d}{dx}\right)\right] - (kx)^2, \qquad \mathrm{Im}(k) < 0 \tag{3.130}$$

on $\mathcal{L}_2(0, \infty)$. By examining (2.22), we identify $p(x) = x$ and $w(x) = 1/x$. The Green's function differential equation associated with L_λ is given by

$$-x\left[\frac{d}{dx}\left(x\frac{dg}{dx}\right)\right] - (kx)^2 g - \lambda g = x\delta(x - \xi) \tag{3.131}$$

where we have identified $\delta(x - \xi)/w(x) = x\delta(x - \xi)$. We investigate limit point and limit circle conditions as $x \to 0$ and as $x \to \infty$ by examining solutions to the homogeneous equation

$$-x\left[\frac{d}{dx}\left(x\frac{du}{dx}\right)\right] - (kx)^2 u - \lambda u = 0$$

For $\lambda = 0$, two independent solutions are given by

$$u_1 = H_0^{(2)}(kx)$$

and

$$u_2 = H_0^{(1)}(kx)$$

Let ξ be an arbitrary interior point on the interval $x \in (0, \infty)$. As $x \to 0$, both u_1 and u_2 are logarithmically singular. The singularity is weak enough, however, that they are both in $\mathcal{L}_2(0, \xi)$. We therefore have the limit circle case as $x \to 0$. The solution u_2 diverges exponentially as $x \to \infty$, and thus is not in $\mathcal{L}_2(\xi, \infty)$. We therefore have the limit point case as $x \to \infty$. We invoke the limit conditions

$$\lim_{x\to\infty} u(x) = 0 \tag{3.132}$$

$$\lim_{x\to 0} u(x) \text{ finite} \tag{3.133}$$

The limit conditions associated with the Green's function are

$$\lim_{x\to\infty} g(x, \xi) = 0 \tag{3.134}$$

$$\lim_{x\to 0} g(x, \xi) \text{ finite} \tag{3.135}$$

Define a parameter ν by

$$\nu = e^{i\pi/2}\sqrt{\lambda} = i\sqrt{\lambda} \tag{3.136}$$

Then, for $x \neq \xi$, we have

$$x\left[\frac{d}{dx}\left(x\frac{dg}{dx}\right)\right] + \left[(kx)^2 - \nu^2\right] g = 0 \tag{3.137}$$

We identify (3.137) as Bessel's equation of order ν and argument kx [10]. The Green's function can be composed of linear combinations of various Bessel functions as follows:

$$g = \begin{cases} AJ_\nu(kx) + CJ_{-\nu}(kx), & x < \xi \\ BH_\nu^{(2)}(kx) + DH_\nu^{(1)}(kx), & x > \xi \end{cases} \tag{3.138}$$

where J_ν, $J_{-\nu}$, $H_\nu^{(2)}$, and $H_\nu^{(1)}$ are linearly independent solutions to Bessel's equation of order ν [10]. For $\text{Im}(k) < 0$, $H_\nu^{(1)}$ diverges as $x \to \infty$ [11]. Therefore, $D = 0$. We may set $C = 0$ by using the following argument. For $x \to 0$, we have

$$J_\nu(kx) \longrightarrow \frac{(kx)^\nu}{2^\nu \nu!}$$

$$J_{-\nu}(kx) \longrightarrow \frac{2^\nu (kx)^{-\nu}}{(-\nu)!}$$

Therefore, if we choose $\text{Re}(\nu) > 0$, $J_{-\nu}$ diverges as $x \to 0$ and we must set $C = 0$. The Green's function can now be written as follows:

$$g = \begin{cases} AJ_\nu(kx), & x < \xi \\ BH_\nu^{(2)}(kx), & x > \xi \end{cases} \tag{3.139}$$

The evaluation of the coefficients A and B proceeds in a manner similar to that in Example 2.21. Invoking the continuity and jump conditions at $x = \xi$, we find that

$$A = \frac{\pi}{2i} H_\nu^{(2)}(k\xi)$$

$$B = \frac{\pi}{2i} J_\nu(k\xi)$$

Substitution into (3.139) gives

$$g = \frac{\pi}{2i} \begin{cases} H_\nu^{(2)}(k\xi) J_\nu(kx), & x < \xi \\ H_\nu^{(2)}(kx) J_\nu(k\xi), & x > \xi \end{cases} \tag{3.140}$$

where

$$\text{Im}(k) < 0 \tag{3.141}$$

$$\text{Re}(\nu) > 0 \tag{3.142}$$

The last step in the determination of the Green's function involves making the transformation from ν to λ in accordance with (3.136). If we define

$$\lambda = |\lambda| e^{i\phi}, \qquad 0 > \phi > -2\pi \tag{3.143}$$

then

$$\sqrt{\lambda} = |\lambda|^{1/2} e^{i\phi/2}, \qquad 0 > \frac{\phi}{2} > -\pi \tag{3.144}$$

This result implies that

$$\text{Im}\sqrt{\lambda} < 0 \tag{3.145}$$

Substituting (3.144) into (3.136) gives

$$\nu = |\lambda|^{1/2} e^{i(\phi+\pi)/2}, \qquad \frac{\pi}{2} > \frac{\phi + \pi}{2} > -\frac{\pi}{2} \tag{3.146}$$

The angular range in (3.146) is consistent with the restriction on ν in (3.142). We therefore have

$$g(x, \xi, \lambda) = \frac{\pi}{2i} \begin{cases} H_{i\sqrt{\lambda}}^{(2)}(k\xi) J_{i\sqrt{\lambda}}(kx), & x < \xi \\ H_{i\sqrt{\lambda}}^{(2)}(kx) J_{i\sqrt{\lambda}}(k\xi), & x > \xi \end{cases} \tag{3.147}$$

where the branch cut in $\sqrt{\lambda}$ lies along the positive-real axis and is explicitly determined by (3.143).

Our next step is to determine the spectral representation of $x\delta(x - \xi)$ by integrating over the Green's function with respect to λ in a similar manner to that performed in Example 3.5. We first consider the case $x < \xi$. We find that the branch cut in $\sqrt{\lambda}$, defined in (3.143), produces a branch cut in $g(x, \xi, \lambda)$ along the positive-real axis. Indeed,

$$\lim_{\phi \to -2\pi} H^{(2)}_{i\sqrt{\lambda}}(k\xi) J_{i\sqrt{\lambda}}(kx)\, d = H^{(2)}_{-i|\lambda|^{1/2}}(k\xi) J_{-i|\lambda|^{1/2}}(kx)$$

$$\lim_{\phi \to 0} H^{(2)}_{i\sqrt{\lambda}}(k\xi) J_{i\sqrt{\lambda}}(kx) = H^{(2)}_{i|\lambda|^{1/2}}(k\xi) J_{i|\lambda|^{1/2}}(kx)$$

Since J_ν and $J_{-\nu}$ are linearly independent for any $\nu \in \mathbf{C}$, we conclude that there is a jump in the Green's function across the positive-real axis, resulting in a branch cut. The appropriate contour is the one shown in Fig. 3-4. Substituting (3.147) into (3.82), we find for $x < \xi$

$$2\pi i x \delta(x - \xi)$$
$$= \frac{\pi}{2i}\left[\int_\infty^0 H^{(2)}_{i|\lambda|^{1/2}}(k\xi) J_{i|\lambda|^{1/2}}(kx) d\lambda + \int_0^\infty H^{(2)}_{-i|\lambda|^{1/2}}(k\xi) J_{-i|\lambda|^{1/2}}(kx) d\lambda\right] \tag{3.148}$$

But [12],

$$H^{(2)}_{-i|\lambda|^{1/2}} = e^{-i\pi(i|\lambda|^{1/2})} H^{(2)}_{i|\lambda|^{1/2}} \tag{3.149}$$

Substituting (3.149) into (3.148) and combining integrals gives

$$-4x\delta(x - \xi) = \int_0^\infty e^{-i\pi(i|\lambda|^{1/2})} H^{(2)}_{i|\lambda|^{1/2}}(k\xi)$$
$$\cdot \left[J_{-i|\lambda|^{1/2}}(kx) - e^{i\pi(i|\lambda|^{1/2})} J_{i|\lambda|^{1/2}}(kx)\right] d\lambda \tag{3.150}$$

But for any $\nu \in \mathbf{C}$ [13],

$$J_{-\nu}(z) - e^{i\pi\nu} J_\nu(z) = \frac{H^{(2)}_\nu(z)}{i \csc(\nu\pi)} \tag{3.151}$$

Substitution into (3.150) gives

$$4x\delta(x - \xi) = i \int_0^\infty e^{-i\pi(i|\lambda|^{1/2})} \sin(i\pi|\lambda|^{1/2}) H^{(2)}_{i|\lambda|^{1/2}}(k\xi) H^{(2)}_{i|\lambda|^{1/2}}(kx) d\lambda \tag{3.152}$$

Let

$$\beta = i|\lambda|^{1/2} \tag{3.153}$$

Then,

$$x\delta(x - \xi) = \frac{1}{4}\int_0^{i\infty} \left(e^{-i2\pi\beta} - 1\right) H^{(2)}_\beta(kx) H^{(2)}_\beta(k\xi) \beta d\beta \tag{3.154}$$

We note that this result is not altered by interchanging x and ξ. Therefore, our restriction $x < \xi$ can be removed. Equation (3.154) gives the spectral representation of the delta function for the operator defined in (3.130)–(3.133).

The representation in (3.154) leads to the *Kantorovich–Lebedev Transform* [14],[15]. Indeed, consider a Hilbert space $\mathcal{L}_2(0, \infty)$ with inner product

$$\langle s, t\rangle = \int_0^\infty s(x)t(x)\frac{dx}{x} \tag{3.155}$$

For any $s(x) \in \mathcal{L}_2(0, \infty)$, we have

$$s(x) = \int_0^\infty s(\xi)x\delta(x-\xi)\frac{d\xi}{\xi} \tag{3.156}$$

Substituting (3.154) into (3.156), we produce the Kantorovich–Lebedev transform pair

$$F(\beta) = \int_0^\infty f(x)H_\beta^{(2)}(kx)\frac{dx}{x} \tag{3.157}$$

$$f(x) = \frac{1}{4}\int_0^{i\infty} \left(e^{-i2\pi\beta} - 1\right) F(\beta)H_\beta^{(2)}(kx)\beta d\beta \tag{3.158}$$

Alternately, we can manipulate (3.158) to produce

$$f(x) = \frac{1}{4}\int_{i\infty}^{-i\infty} F(\beta)H_\beta^{(2)}(kx)\beta d\beta \tag{3.159}$$

The details of producing (3.159) from (3.158) are left for Problem 3.6. We indicate the Kantorovich–Lebedev transform relationship by

$$f(x) \Longleftrightarrow F(\beta) \tag{3.160}$$

If we apply the operator L in (3.130) to both sides of (3.159), we produce the useful relationship

$$\left\{-x\left[\frac{d}{dx}\left(x\frac{d}{dx}\right)\right] - (kx)^2\right\} f(x) \Longleftrightarrow -\beta^2 F(\beta) \tag{3.161}$$

The interchange of differentiation and integration used to produce (3.161) can be justified in the same manner as in the procedure in (3.76)–(3.79) and in (3.102)–(3.104). The details are left for the problems. The Kantorovich–Lebedev transform is useful in solving certain electromagnetic problems in cylindrical coordinates, as we shall discover in the next chapter.

■

In the mathematical literature, the spectral contribution resulting from pole contributions, such as in (3.47), is called the *discrete spectrum*, whereas the contribution from the branch cut, such as in (3.64), is called the *continuous spectrum*. We next inquire if it is possible to have both a continuous and discrete spectrum associated with an operator. The example we choose involves an operator that is not self-adjoint. The theory of nonself-adjoint operators is both difficult and incomplete. However, in the simple example to follow, we are able to obtain the spectral representation in a straightforward manner.

EXAMPLE 3.7 We consider the spectral representation of the operator $L = -d^2/dx^2$, with boundary and limiting conditions

$$\lim_{x\to\infty} u(x) = 0 \tag{3.162}$$

$$u'(0) = \alpha u(0), \qquad \mathrm{Re}(\alpha) < 0 \tag{3.163}$$

Since α is complex, the operator L is nonself-adjoint. The associated Green's function problem is given by

$$-\frac{d^2g}{dx^2} - \lambda g = \delta(x-\xi)$$

$$\frac{dg(0,\xi)}{dx} = \alpha g(0,\xi)$$

$$\lim_{x\to\infty} g(x,\xi) = 0$$

We have previously obtained this Green's function in Example 2.23. We repeat the result given in (2.216) for convenience, viz.

$$g(x,\xi) = \frac{1}{i\sqrt{\lambda}+\alpha}\begin{cases} e^{-i\sqrt{\lambda}x}(\cos\sqrt{\lambda}\xi + \dfrac{\alpha}{\sqrt{\lambda}}\sin\sqrt{\lambda}\xi), & x > \xi \\ e^{-i\sqrt{\lambda}\xi}(\cos\sqrt{\lambda}x + \dfrac{\alpha}{\sqrt{\lambda}}\sin\sqrt{\lambda}x), & x < \xi \end{cases} \tag{3.164}$$

where

$$\mathrm{Im}\sqrt{\lambda} < 0 \tag{3.165}$$

The restriction in (3.165) can again be assured by defining $\sqrt{\lambda}$ as in (3.56) and (3.57) so that, once again (Fig. 3-2),

$$\lim_{\phi\to 2\pi}\sqrt{\lambda} = -|\lambda|^{1/2}$$

$$\lim_{\phi\to 4\pi}\sqrt{\lambda} = |\lambda|^{1/2}$$

The branch cut in $\sqrt{\lambda}$ along the positive-real axis results in a branch cut in the same location in $g(x,\xi,\lambda)$. Indeed, for $x < \xi$, we obtain in (3.164)

$$\lim_{\phi\to 2\pi} g(x,\xi,\lambda) = \frac{e^{i|\lambda|^{1/2}\xi}}{\alpha - i|\lambda|^{1/2}}(\cos|\lambda|^{1/2}x + \frac{\alpha}{|\lambda|^{1/2}}\sin|\lambda|^{1/2}x)$$

$$\lim_{\phi\to 4\pi} g(x,\xi,\lambda) = \frac{e^{-i|\lambda|^{1/2}\xi}}{\alpha + i|\lambda|^{1/2}}(\cos|\lambda|^{1/2}x + \frac{\alpha}{|\lambda|^{1/2}}\sin|\lambda|^{1/2}x)$$

The result for $x > \xi$ is obtained by interchanging x and ξ. In addition to the branch cut on the positive-real axis, $g(x, \xi, \lambda)$ has an isolated singularity at the location λ_0, given by solving

$$i\sqrt{\lambda} + \alpha = 0$$

with the result

$$\lambda_0 = -\alpha^2$$

We now show that this singularity is on the proper Riemann sheet. Indeed, we have $\sqrt{\lambda_0} = i\alpha$. From this relationship, we easily find that the relation $\mathrm{Im}\sqrt{\lambda} < 0$ implies that $\mathrm{Re}(\alpha) < 0$, as assumed in the problem statement.

We now show that this singularity is a simple pole. For $x < \xi$, we write

$$g(x, \xi, \lambda) = f_1(\lambda) f_2(x, \xi, \lambda) \tag{3.166}$$

where

$$f_1(\lambda) = \frac{1}{i\sqrt{\lambda} + \alpha} \tag{3.167}$$

$$f_2(x, \xi, \lambda) = e^{-i\sqrt{\lambda}\xi}(\cos\sqrt{\lambda}x + \frac{\alpha}{\sqrt{\lambda}}\sin\sqrt{\lambda}x) \tag{3.168}$$

We note that $f_2(x, \xi, \lambda)$ is regular at λ_0. Consider

$$f_1(\lambda) = \frac{p(\lambda)}{q(\lambda)}$$

where $p(\lambda) = 1$ and

$$q(\lambda) = i\sqrt{\lambda} + \alpha$$

We have

$$q(-\alpha^2) = 0$$

$$q'(-\alpha^2) = \frac{1}{2\alpha}$$

We conclude that $f_1(\lambda)$ has a simple pole at λ_0 with residue

$$\mathrm{Res}\{f_1(\lambda); \lambda_0\} = \frac{p(\lambda_0)}{q'(\lambda_0)} = 2\alpha$$

To obtain the spectral representation, we integrate the Green's function $g(x, \xi, \lambda)$ around the closed contour shown in Fig. 3-5. For $x > \xi$, we have

$$\begin{aligned}
\oint g(x, \xi, \lambda) d\lambda &= 2\pi i \mathrm{Res}\{g(x, \xi, \lambda); \lambda_0\} \\
&= 2\pi i e^{\alpha x}[\cos(i\alpha\xi) - i\sin(i\alpha\xi)]\mathrm{Res}\{f_1(\lambda); \lambda_0\} \\
&= 2\pi i (2\alpha) e^{\alpha(x+\xi)}
\end{aligned} \tag{3.169}$$

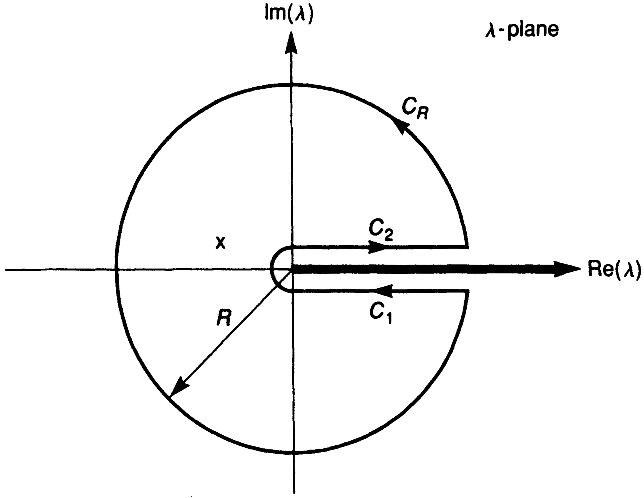

Fig. 3-5 Contour for evaluation of the spectral representation for Example 3.4. Contour includes the simple pole (×) at λ_0.

The integral around the closed contour (Fig. 3-5) consists of the integral around the circle of radius R plus the integrals along either side of the branch cut. Therefore, in the limit as $R \to \infty$, we obtain

$$-\delta(x-\xi) + \frac{1}{2\pi i} \lim_{R\to\infty} \int_{C_1+C_2} g(x,\xi,\lambda)d\lambda = 2\alpha e^{\alpha(x+\xi)} \tag{3.170}$$

Evaluating the integrals along C_1 and C_2 in a similar manner to the process in Example 3.3, we obtain

$$\lim_{R\to\infty} \int_{C_1+C_2} = 2i \int_0^\infty (\cos|\lambda|^{1/2}x + \frac{\alpha}{|\lambda|^{1/2}} \sin|\lambda|^{1/2}x)$$
$$\cdot (\cos|\lambda|^{1/2}\xi + \frac{\alpha}{|\lambda|^{1/2}} \sin|\lambda|^{1/2}\xi) \frac{|\lambda|^{1/2}d\lambda}{\alpha^2+|\lambda|} \tag{3.171}$$

We let $k = |\lambda|^{1/2}$ and obtain for the spectral representation

$$\delta(x-\xi) = -2\alpha e^{\alpha(x+\xi)} + \frac{2}{\pi}\int_0^\infty (\cos kx + \frac{\alpha}{k}\sin kx)(\cos k\xi + \frac{\alpha}{k}\sin k\xi)\frac{k^2dk}{\alpha^2+k^2} \tag{3.172}$$

The first term on the right side gives the discrete spectral contribution, while the second term gives the continuous spectral contribution. Although the delta function representation has been obtained with the restriction $x > \xi$, we note in (3.164) that the case $x < \xi$ can be obtained by interchanging x and ξ. Since such an interchange leaves the result in (3.172) unaltered, the restriction can be removed.

In Problem 5.6 given at the end of Chapter 5, the spectral representation in (3.172) will be used to characterize a source over a flat surface characterized by a surface impedance. We shall discover that we may associate the first term with a surface wave bound to the surface and the second term with radiation carried away from the surface.

We remark that we assumed $\mathrm{Re}(\alpha) < 0$ in the problem statement. In addition, we found that $\mathrm{Im}\sqrt{\lambda} < 0$ implies $\mathrm{Re}(\alpha) < 0$, as required. If the problem had stated that $\mathrm{Re}(\alpha) \geq 0$, the pole at $\lambda_0 = -\alpha^2$ would have been on the improper Riemann sheet and the discrete spectral term in (3.172) would be missing.

In a similar manner to that in Examples 3.2 and 3.3, we now cast (3.172) in a form that we shall call the *impedance transform*. For $u(x) \in \mathcal{L}_2(0, \infty)$, we have

$$u(x) = \int_0^\infty \delta(x - \xi)u(\xi)d\xi \tag{3.173}$$

Substituting (3.172) and interchanging the order of integration, we obtain

$$u(x) = -2\alpha e^{\alpha x}U_0 + \frac{2}{\pi}\int_0^\infty U(k)(\cos kx + \frac{\alpha}{k}\sin kx)\frac{k^2dk}{k^2+\alpha^2} \tag{3.174}$$

where

$$U_0 = \int_0^\infty u(x)e^{\alpha x}dx \tag{3.175}$$

$$U(k) = \int_0^\infty u(x)(\cos kx + \frac{\alpha}{k}\sin kx)dx \tag{3.176}$$

Equations (3.175) and (3.176) comprise the impedance transform, yielding the spectral coefficients U_0 and $U(k)$. Equation (3.174) is the inverse impedance transform. We call (3.175) the *zeroth-order impedance transform*, while (3.176) is the *kth-order impedance transform*. In (3.174), we identify

$$e^{\alpha x}$$

as an eigenfunction of the operator $-d^2/dx^2$ with boundary conditions given in (3.162) and (3.163). In addition,

$$\cos kx + \frac{\alpha}{k}\sin kx$$

is an improper eigenfunction of the same operator with the same boundary conditions. We define an inner product for the space by

$$\langle u, v\rangle = \int_0^\infty u(x)\overline{v}(x)dx$$

With this definition, we may write (3.175) and (3.176) as follows:

$$U_0 = \langle u, e^{\overline{\alpha}x}\rangle \tag{3.177}$$

$$U(k) = \langle u, \cos kx + \frac{\overline{\alpha}}{k} \sin kx \rangle \tag{3.178}$$

We identify

$$e^{\overline{\alpha}x}$$

as an adjoint eigenfunction of the operator $-d^2/dx^2$ with adjoint boundary conditions given by

$$v'(0) = \overline{\alpha} v(0)$$
$$\lim_{x\to\infty} v(x) = 0 \tag{3.179}$$

In addition,

$$\cos kx + \frac{\overline{\alpha}}{k} \sin kx$$

is an improper adjoint eigenfunction of the same operator with the adjoint boundary condition given in (3.179).

We next use the impedance transform to solve the following SLP3 problem:

$$-u'' - \lambda u = f, \qquad \lambda \in \mathbf{C} \tag{3.180}$$

with boundary condition

$$u'(0) = \alpha u(0), \qquad \mathrm{Re}(\alpha) < 0 \tag{3.181}$$

This problem is in the limit point case as $x \to \infty$. We therefore invoke the limiting condition

$$\lim_{x\to\infty} u(x) = 0 \tag{3.182}$$

In order to solve the differential equation in (3.180) by the use of the impedance transform, we require the zeroth-order and kth-order impedance transform of $-d^2u/dx^2$. For the zeroth order, we have

$$\langle -u'', e^{\overline{\alpha}x} \rangle = \langle u, -\frac{d^2}{dx^2} e^{\overline{\alpha}x} \rangle + \left(-u' e^{\alpha x} + u \frac{d}{dx} e^{\alpha x} \rangle \right) \Bigg|_0^\infty = -\alpha^2 \langle u, e^{\overline{\alpha}x} \rangle = -\alpha^2 U_0 \tag{3.183}$$

where we have used (3.181), (3.182), and

$$\lim_{x\to\infty} e^{\alpha x} = 0$$

Symbolically, we indicate this zeroth-order transform by

$$-u'' \stackrel{0}{\Longrightarrow} -\alpha^2 U_0 \tag{3.184}$$

Similarly, for the kth-order transform, we have

$$
\begin{aligned}
\langle -u'', \cos kx + \frac{\overline{\alpha}}{k}\sin kx\rangle &= \langle u, -\frac{d^2}{dx^2}(\cos kx + \frac{\overline{\alpha}}{k}\sin kx)\rangle \\
&\quad + \left[-u'(\cos kx + \frac{\alpha}{k}\sin kx) + u\frac{d}{dx}(\cos kx + \frac{\alpha}{k}\sin kx) \right]\Bigg|_0^\infty \\
&= k^2 \langle u, \cos kx + \frac{\overline{\alpha}}{k}\sin kx\rangle = k^2 U(k)
\end{aligned} \tag{3.185}
$$

where we have used (3.181) and (3.182). In addition, we have used the fact that, in the usual cases in electromagnetics,

$$\lim_{x\to\infty} u(x) = 0$$

implies that

$$\lim_{x\to\infty} u'(x) = 0$$

Symbolically, we indicate this transform by

$$-u'' \overset{k}{\Longrightarrow} k^2 U(k) \tag{3.186}$$

We now apply the zeroth-order transform to (3.180) to give

$$-(\alpha^2 + \lambda)U_0 = F_0$$

where F_0 is the zeroth-order transform of $f(x)$. Rearranging, we have

$$U_0 = -\frac{F_0}{\alpha^2 + \lambda} \tag{3.187}$$

Similarly, for the kth-order transform, we find that

$$(k^2 - \lambda)U(k) = F(k)$$

where $F(k)$ is the kth-order transform of $f(x)$. Rearranging, we have

$$U(k) = \frac{F(k)}{k^2 - \lambda} \tag{3.188}$$

Substituting these results into (3.174) gives the final result, viz.

$$u(x) = \frac{2\alpha e^{\alpha x} F_0}{\alpha^2 + \lambda} + \frac{2}{\pi}\int_0^\infty F(k)\frac{(\cos kx + \frac{\alpha}{k}\sin kx)}{k^2 - \lambda}\frac{k^2 dk}{k^2 + \alpha^2} \tag{3.189}$$

■

We have now concluded our presentation of the spectral representation method. This method and the Green's function method together comprise a powerful tool for the solution of many of the partial differential equations found in electromagnetic radiation, scattering, and diffraction. In subsequent chapters, we shall study electromagnetic source representations, and then develop the solution methods for a large class of electromagnetic problems. We shall conclude this chapter with a brief discussion of the connection between the Green's function method and the spectral representation method.

3.5 GREEN'S FUNCTIONS AND SPECTRAL REPRESENTATIONS

There is an important connection between the Green's function method and the spectral representation method. Indeed, consider the result in Example 3.3, given in (3.81), viz.

$$u(x) = \frac{2}{\pi}\int_0^\infty \frac{F(k)}{k^2-\lambda}\sin kx\,dk \tag{3.190}$$

where

$$F(k) = \int_0^\infty f(\xi)\sin k\xi\,d\xi \tag{3.191}$$

If we substitute (3.191) into (3.190) and interchange the order of integration, we obtain

$$u(x) = \int_0^\infty f(\xi)\left[\frac{2}{\pi}\int_0^\infty \frac{\sin kx\sin k\xi}{k^2-\lambda}dk\right]d\xi \tag{3.192}$$

We identify the term in square brackets as the Green's function

$$g(x,\xi) = \frac{2}{\pi}\int_0^\infty \frac{\sin kx\sin k\xi}{k^2-\lambda}dk \tag{3.193}$$

We compare this result with the Green's function obtained in Example 2.18, given in (2.172), viz.

$$g(x,\xi) = \frac{1}{\sqrt{\lambda}}\begin{cases} e^{-i\sqrt{\lambda}\xi}\sin\sqrt{\lambda}x, & x<\xi \\ e^{-i\sqrt{\lambda}x}\sin\sqrt{\lambda}\xi, & x>\xi \end{cases} \tag{3.194}$$

Although (3.193) and (3.194) appear very different, they are different representations of the same Green's function. Indeed, comparing (3.193) and (3.194), we must have

$$\int_0^\infty \frac{\sin kx\sin k\xi}{k^2-\lambda}dk = \frac{\pi}{2\sqrt{\lambda}}\begin{cases} e^{-i\sqrt{\lambda}\xi}\sin\sqrt{\lambda}x, & x<\xi \\ e^{-i\sqrt{\lambda}x}\sin\sqrt{\lambda}\xi, & x>\xi \end{cases} \tag{3.195}$$

as we could easily verify by the calculus of residues. Since (3.193) requires an integration and (3.194) does not, it would appear that the spectral representation is not as useful in practice. Its utility, however, becomes clear as soon as we begin considering partial rather than ordinary differential equations in subsequent chapters.

PROBLEMS

3.1. For the operator $L = -d^2/dx^2$ and boundary conditions $u(0) = u(a) = 0$, begin with the Green's function for L_λ and show that the spectral representation of the delta function is given by

$$\delta(x-\xi) = \frac{2}{a}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{a}\sin\frac{n\pi\xi}{a}$$

Use this spectral representation to obtain the solution to the differential equation

$$L_\lambda u = f$$

with the operator L and the boundary conditions given above.

3.2. For the operator $L = -d^2/dx^2$ and boundary conditions $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)$, the Green's function for L_λ was found in Problem 2.18. The result is repeated here for reference, viz.

$$g(x,\xi) = -\frac{1}{2\sqrt{\lambda}\sin\sqrt{\lambda}\pi}\begin{cases} \cos\sqrt{\lambda}(\xi - x - \pi), & x < \xi \\ \cos\sqrt{\lambda}(x - \xi - \pi), & x > \xi \end{cases}$$

Beginning with this Green's function, show that the spectral representation of the delta function is given by

$$\delta(x-\xi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}(\cos nx\cos n\xi + \sin nx \sin n\xi)$$

By using Euler's identity, show that an alternate representation is given by

$$\delta(x-\xi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(x-\xi)}$$

Show that this alternate representation leads to the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n\sqrt{\frac{1}{2\pi}}e^{inx}$$

$$a_n = \int_0^{2\pi} f(x)\sqrt{\frac{1}{2\pi}}e^{-inx}\,dx$$

3.3. For the operator $L = -d^2/dx^2$ with boundary condition $u'(0) = 0$ and limiting condition

$$\lim_{x\to\infty} u(x) = 0$$

begin with the Green's function for L_λ and show that the spectral representation of the delta function is given by

$$\delta(x-\xi) = \frac{2}{\pi}\int_0^\infty \cos kx \cos k\xi \, dk$$

Use this spectral representation to obtain the solution to the differential equation

$$L_\lambda u = f$$

with the operator L and the boundary conditions given above.

3.4. In Example 3.5, it was shown that

$$\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{du}{dx}\right)\right] = \int_0^\infty U(k)\left\{\frac{1}{x}\frac{d}{dx}\left[x\frac{dJ_0(kx)}{dx}\right]\right\} kdk$$
$$= \int_0^\infty [-k^2U(k)]J_0(kx)kdk$$

and therefore,

$$\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{du}{dx}\right)\right] \Longleftrightarrow -k^2U(k)$$

This result involves the interchange of differential operator and integration. Justify this result by following the procedure used in (3.76)–(3.79) and in (3.102)–(3.104).

3.5. Consider the following Green's function problem associated with Bessel's equation of order ν:

$$(L-\lambda)g = \frac{\delta(x-\xi)}{x}$$

where

$$L = -\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{d}{dx}\right)\right] + \frac{\nu^2}{x^2}$$

with limiting condition

$$\lim_{x\to\infty} g(x,\xi) = 0$$

where λ is complex. By invoking the condition that g must be finite as $x \to 0$, show that

$$g(x,\xi,\lambda) = \frac{\pi}{2i}\begin{cases} H_\nu^{(2)}(\sqrt{\lambda}\xi)J_\nu(\sqrt{\lambda}x), & x < \xi \\ H_\nu^{(2)}(\sqrt{\lambda}x)J_\nu(\sqrt{\lambda}\xi), & x > \xi \end{cases}$$

Note: This result can be obtained directly from Example 3.6 by identifying λ above with k^2 in (3.131) in Example 3.6. By integrating the Green's function around the closed contour in the complex λ-plane, as described in (3.119), show that the spectral representation of the delta function for the operator L and boundary conditions given above is given by

$$\frac{\delta(x-\xi)}{x} = \int_0^\infty J_\nu(kx)J_\nu(k\xi)kdk$$

Show that this representation leads to the *Fourier–Bessel Transform* of order ν, given by the pair

$$S(k) = \int_0^\infty s(x)J_\nu(kx)xdx$$
$$s(x) = \int_0^\infty S(k)J_\nu(kx)kdk$$

Produce the following Fourier–Bessel transform pair:

$$\left\{-\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{d}{dx}\right)\right] + \frac{\nu^2}{x^2}\right\} s(x) \Longleftrightarrow k^2 S(k)$$

3.6. In the development of the Kantorovich–Lebedev transform, carefully complete the steps necessary to produce (3.159) from (3.158).

3.7. Justify the result in (3.161) by following the procedure used in (3.76)–(3.79) and (3.102)–(3.104).

3.8. In the development of the Kantorovich–Lebedev transform in Example 3.6, we considered the operator

$$L = -x\left[\frac{d}{dx}\left(x\frac{d}{dx}\right)\right] - (kx)^2, \qquad \mathrm{Im}(k) < 0$$

where

$$\lim_{x\to\infty} u(x) = 0$$
$$\lim_{x\to 0} u(x) \text{ finite}$$

In [14], Stakgold considers the Kantorovich–Lebedev problem for a slightly different operator, viz.

$$L_s = -x\left[\frac{d}{dx}\left(x\frac{d}{dx}\right)\right] + \mu x^2, \qquad \mu > 0$$

His resulting spectral representation is given by

$$x\delta(x-\xi) = \frac{2}{\pi^2}\int_0^\infty \sinh \pi\beta K_{i\beta}(\sqrt{\mu}x)K_{i\beta}(\sqrt{\mu}\xi)\beta d\beta \tag{3.196}$$

where $K_\lambda(y)$ is the modified Bessel function of the second kind, λth order [16]. By making the appropriate transformation between k and μ, show that the spectral representation in (3.154) transforms to (3.196).

REFERENCES

[1] Stakgold, I. (1979), *Green's Functions and Boundary Value Problems*. New York: Wiley Interscience, 414–415.

[2] Carrier, G.F., M. Krook, and C.E. Pearson (1966), *Functions of a Complex Variable*. New York: McGraw-Hill, 56–59.

[3] Friedman, B. (1956), *Principles and Techniques of Applied Mathematics*. New York: Wiley, 213–222.

[4] Churchill, R.V. (1960), *Complex Variables and Applications*. New York: McGraw-Hill, 159–161.

[5] Arfken, G. (1985), *Mathematical Methods for Physicists*, 3rd Edition. New York: Academic, 529–530.

[6] op.cit. Friedman, 234–237.

[7] Sneddon, I.N. (1972), *The Use of Integral Transforms*. New York: McGraw-Hill, 46–49.

[8] Abramowitz, M., and I.A. Stegun (Eds.) (1964), *Handbook of Mathematical Functions*, National Bureau of Standard, Applied Mathematics Series, 55, Superintendent of Documents, U.S. Government Printing Office, Washington, DC 20402, 361, eq. 9.1.35.

[9] Ibid., eq. 9.1.39.

[10] Ibid., 358, sect. 9.1.1.

[11] Harrington, R.F. (1961, reissued 1987), *Time-Harmonic Electromagnetic Fields*. New York: McGraw-Hill, 461–462.

[12] op.cit. Abramowitz and Stegun, eq. 9.1.6.

[13] Ibid., eq. 9.1.4.

[14] Stakgold, I. (1967), *Boundary Value Problems of Mathematical Physics*. New York: Macmillan, 317.

[15] Felsen, L.B., and N. Marcuvitz (1973), *Radiation and Scattering of Waves*. Englewood-Cliffs, NJ: Prentice-Hall, 323–326.

[16] op.cit. Harrington, 363.