Chapter 2

Dyadics

Dyadics are linear functions of vectors. In real vector space they can be visualized through their operation on vectors, which for real vectors consists of turning and stretching the vector arrow. In complex vector space they correspondingly rotate and deform ellipses. Dyadic notation was introduced by GIBBS in the same pamphlet as the original vector algebra, in 1884, containing 30 pages of basic operations on dyadics. Double products of dyadics, which give the notation much of its power, were introduced by him in scientific journals (GIBBS 1886, 1891). Gibbs's work on dyadic algebra was compiled from his lectures by WILSON and printed a book *Vector analysis* containing 150 pages of dyadics (GIBBS and WILSON 1909). Of course, not all the formulas given by Gibbs were invented by Gibbs, quite a number of properties of linear vector functions were introduced earlier by Hamilton in his famous book on quaternions. In electromagnetics literature, dyadics and matrices are often used simultaneously. It is well recognized that the dyadic notation is best matched to the vector notation. Nevertheless, often the vector notation is suddenly changed to matrices, for example when inverse dyadics should be constructed, because the corresponding dyadic operations are unknown. The purpose of this section is to introduce the dyadic formalism, and subsequent chapters demonstrate some of its power. The contents of the present chapter are largely based on work given earlier by this author in report form (LINDELL 1968, 1973a, 1981).

2.1 Notation

2.1.1 Dyads and polyads

The dyadic product of two vectors $\mathbf{a}$, $\mathbf{b}$ (complex in general) is denoted without any multiplication sign by $\mathbf{ab}$ and the result is called a dyad. The order of dyadic multiplication is essential, $\mathbf{ab}$ is in general different from $\mathbf{ba}$.

A polyad is a string of vectors multiplying each other by dyadic products and denoted by $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3...\mathbf{a}_n$. For $n = 1$ we have a vector, $n = 2$ a dyad,

$n = 3$ a triad and, in general, an n-ad. Polyads of the same rank n generate a linear space, whose members are polynomials of n-ads. Thus, all polynomials of dyads, or dyadic polynomials, or in short dyadics, are of the form $\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + ... + \mathbf{a}_k\mathbf{b}_k$. Similarly, n-adic polynomials form a linear space of n-adics. A sum of two n-adics is an n-adic. Here we concentrate on the case $n = 2$, or dyadics, which are denoted by $\overline{\overline{A}}$, $\overline{\overline{B}}$, $\overline{\overline{C}}$, etc.

Dyadics and other polyadics arise in a natural manner in expressions of vector algebra, where a linear operator is separated from the quantity that is being operated upon. For example, projection of a vector $\mathbf{a}$ onto a line which has the direction of the unit vector $\mathbf{u}$ can be written as $\mathbf{u}(\mathbf{u}\cdot\mathbf{a})$. Here, the vectors $\mathbf{u}$ represent the operation on the vector $\mathbf{a}$. Separating these from each other by moving the brackets of vector notation, gives rise to the dyad $\mathbf{uu}$ in the expression $\mathbf{u}(\mathbf{u}\cdot\mathbf{a}) = (\mathbf{uu})\cdot\mathbf{a}$.

A dyad is bilinear in its vector multiplicants:

$$(\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2)\mathbf{b} = \alpha_1(\mathbf{a}_1\mathbf{b}) + \alpha_2(\mathbf{a}_2\mathbf{b}), \tag{2.1}$$

$$\mathbf{a}(\beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2) = \beta_1(\mathbf{a}\mathbf{b}_1) + \beta_2(\mathbf{a}\mathbf{b}_2). \tag{2.2}$$

This means that the same dyad or dyadic can be written in infinitely many different polynomial forms, just like a vector can be written as a sum of different vectors. Whether two forms in fact represent the same dyadic (the same element in dyadic space), can be asserted if one of them can be obtained from the other through these bilinear operations.

A dyadic $\overline{\overline{A}}$ can be multiplied by a vector $\mathbf{c}$ in many ways. Taking one dyad $\mathbf{ab}$ of the dyadic, the following multiplications are possible:

$$\mathbf{c}\cdot(\mathbf{ab}) = (\mathbf{c}\cdot\mathbf{a})\mathbf{b}, \tag{2.3}$$

$$\mathbf{c}\times(\mathbf{ab}) = (\mathbf{c}\times\mathbf{a})\mathbf{b}, \tag{2.4}$$

$$(\mathbf{ab})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}), \tag{2.5}$$

$$(\mathbf{ab})\times\mathbf{c} = \mathbf{a}(\mathbf{b}\times\mathbf{c}). \tag{2.6}$$

In dot multiplication of a dyad by a vector, the result is a vector, in cross multiplication, a dyad.

Likewise, double multiplications of a dyad $\mathbf{ab}$ by another dyad $\mathbf{cd}$ are defined as follows:

$$(\mathbf{ab}):(\mathbf{cd}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}), \tag{2.7}$$

$$(\mathbf{ab})\,{}^{\times}_{\times}(\mathbf{cd}) = (\mathbf{a}\times\mathbf{c})(\mathbf{b}\times\mathbf{d}), \tag{2.8}$$

$$(\mathbf{ab})\,{}^{\times}_{\cdot}(\mathbf{cd}) = (\mathbf{a}\times\mathbf{c})(\mathbf{b}\cdot\mathbf{d}), \tag{2.9}$$

$$(\mathbf{ab})\,{}^{\cdot}_{\times}(\mathbf{cd}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\times\mathbf{d}). \tag{2.10}$$

These expressions can be generalized to corresponding double products between dyadics, $\overline{\overline{A}} : \overline{\overline{B}}$, $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}$, $\overline{\overline{A}}{}^{\times}_{\cdot}\overline{\overline{B}}$ and $\overline{\overline{A}}{}^{\cdot}_{\times}\overline{\overline{B}}$, when dyads are replaced by dyadic polynomials and multiplication is made term by term. The double dot product produces a scalar, the double cross product, a dyadic, and the mixed products, a vector. These products, especially the double dot and double cross products, give more power to the dyadic notation. Their application requires, however, a knowledge of some identities, which are not in common use in the literature. These identities will be introduced later and they are also listed in Appendix A of this book.

The linear space of dyadics contains all polynomials of dyads as its elements. The representation of a dyadic by a dyadic polynomial is, however, not unique. Two polynomials correspond to the same dyadic if their difference can be reduced to the null dyadic by bilinear operations. Because of Gibbs' identity (1.42), any dyadic can be written as a sum of three dyads. In fact, taking three base vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ with their reciprocal base vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, any dyadic polynomial can be written as

$$\sum_{i=1}^{n} \mathbf{a}_i\mathbf{b}_i = \mathbf{a}\sum_{i=1}^{n}(\mathbf{a}'\cdot\mathbf{a}_i)\mathbf{b}_i + \mathbf{b}\sum_{i=1}^{n}(\mathbf{b}'\cdot\mathbf{a}_i)\mathbf{b}_i + \mathbf{c}\sum_{i=1}^{n}(\mathbf{c}'\cdot\mathbf{a}_i)\mathbf{b}_i. \tag{2.11}$$

This is of the trinomial form $\mathbf{ae} + \mathbf{bf} + \mathbf{cg}$, which is the most general form of dyadic in the three-dimensional vector space. If we can prove a theorem for the general dyadic trinomial, the theorem is valid for any dyadic. A sum sign $\sum$ without index limit values in this text denotes a sum from 1 to 3.

2.1.2 Symmetric and antisymmetric dyadics

The transpose operation for dyadics changes the order in all dyadic products:

$$\left(\sum \mathbf{a}_i\mathbf{b}_i\right)^T = \sum(\mathbf{a}_i\mathbf{b}_i)^T = \sum \mathbf{b}_i\mathbf{a}_i. \tag{2.12}$$

Because $(\overline{\overline{A}}{}^T)^T = \overline{\overline{A}}$, the eigenvalue problem $\overline{\overline{A}}{}^T = \lambda\overline{\overline{A}}$ has the eigenvalues $\lambda = \pm 1$ corresponding to symmetric $\overline{\overline{A}}_s$ and antisymmetric $\overline{\overline{A}}_a$ dyadics, which satisfy

$$\overline{\overline{A}}{}_s^T = \overline{\overline{A}}_s, \qquad \overline{\overline{A}}_a{}^T = -\overline{\overline{A}}_a. \tag{2.13}$$

Any dyadic can be uniquely decomposed into a symmetric and an antisymmetric part:

$$\overline{\overline{A}} = \frac{1}{2}(\overline{\overline{A}} + \overline{\overline{A}}{}^T) + \frac{1}{2}(\overline{\overline{A}} - \overline{\overline{A}}{}^T). \tag{2.14}$$

Every symmetric dyadic can be written as a polynomial of symmetric dyads:

$$\overline{\overline{A}}_s = \sum \mathbf{a}_i\mathbf{b}_i = \frac{1}{2}\sum(\mathbf{a}_i\mathbf{b}_i + \mathbf{b}_i\mathbf{a}_i) =$$

$$\frac{1}{2}\sum((\mathbf{a}_i + \mathbf{b}_i)(\mathbf{a}_i + \mathbf{b}_i) - \mathbf{a}_i\mathbf{a}_i - \mathbf{b}_i\mathbf{b}_i). \tag{2.15}$$

The number of terms in this polynomial is, however, in general higher than 3.

A single dyad cannot be antisymmetric, because from the condition $\mathbf{ab} = -\mathbf{ba}$, by multiplying by $\mathbf{a}^*\cdot$ and dividing by $\mathbf{a}\cdot\mathbf{a}^*$ we see that $\mathbf{b}$ must be of the form $\alpha\mathbf{a}$, whence $\mathbf{ab} = \alpha\mathbf{aa} = -\alpha\mathbf{aa} = 0$. Instead, any antisymmetric dyadic can be expressed in terms of two dyads in the form $\mathbf{ab} - \mathbf{ba}$. The vectors $\mathbf{a}$, $\mathbf{b}$ are not unique. This is seen from the following expansion with orthonormal unit vectors $\mathbf{u}_j$:

$$\overline{\overline{A}}_a = \sum_i \mathbf{a}_i\mathbf{b}_i = \frac{1}{2}\sum_i(\mathbf{a}_i\mathbf{b}_i - \mathbf{b}_i\mathbf{a}_i) =$$

$$\sum_i \frac{1}{2}(\mathbf{a}_i\sum_j(\mathbf{b}_i\cdot\mathbf{u}_j\mathbf{u}_j) - \mathbf{b}_i\sum_j(\mathbf{a}_i\cdot\mathbf{u}_j\mathbf{u}_j)) = \sum_j \mathbf{c}\times\mathbf{u}_j\mathbf{u}_j, \tag{2.16}$$

with

$$\mathbf{c} = -\frac{1}{2}\sum \mathbf{a}_i \times \mathbf{b}_i. \tag{2.17}$$

Thus, the general antisymmetric dyadic can be expressed in terms of a single vector $\mathbf{c}$. If we write $\mathbf{c} = -\mathbf{a}\times\mathbf{b}$, going (2.16) backwards we see that any antisymmetric dyadic can be expressed as $\mathbf{ab} - \mathbf{ba}$. The choice of orthonormal basis vectors does not affect this conclusion.

The linear space of dyadics is nine dimensional, in which the antisymmetric dyadics form a three-dimensional and the symmetric dyadics a six-dimensional subspace.

2.2 Dyadics as linear mappings

A dyadic serves as a linear mapping from a vector to another: $\mathbf{a} \rightarrow \mathbf{b} = \overline{\overline{D}}\cdot\mathbf{a}$. Conversely, any such linear mapping can be expressed in terms of a dyadic. This can be seen by expanding in terms of orthonormal basis vectors $\mathbf{u}_i$ and applying the property of linearity of the vector function $\mathbf{f}(\mathbf{a})$:

$$\mathbf{f}(\mathbf{a}) = \sum_i \mathbf{u}_i\mathbf{u}_i\cdot\mathbf{f}(\sum_j \mathbf{u}_j\mathbf{u}_j\cdot\mathbf{a}) = \Big(\sum_i\sum_j \mathbf{u}_i\cdot\mathbf{f}(\mathbf{u}_j)\mathbf{u}_i\mathbf{u}_j\Big)\cdot\mathbf{a}. \tag{2.18}$$

The quantity in brackets is of the dyadic form and it corresponds to the linear function $\mathbf{f}(\mathbf{a})$.

The unit dyadic $\overline{\overline{I}}$ corresponds to the identity mapping $\overline{\overline{I}} \cdot \mathbf{a} = \mathbf{a}$ for any vector $\mathbf{a}$. From Gibbs' identity (1.42) we see that for any base of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ with the reciprocal base $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, the unit dyadic can be written as

$$\overline{\overline{I}} = \mathbf{aa}' + \mathbf{bb}' + \mathbf{cc}'. \tag{2.19}$$

Taking an orthonormal base $\mathbf{u}_i$ with $\mathbf{u}_i' = \mathbf{u}_i$, the unit dyadic takes the form

$$\overline{\overline{I}} = \sum \mathbf{u}_i \mathbf{u}_i. \tag{2.20}$$

The unit dyadic is symmetric and satisfies $\overline{\overline{I}} \cdot \overline{\overline{D}} = \overline{\overline{D}} \cdot \overline{\overline{I}} = \overline{\overline{D}}$ for any dyadic $\overline{\overline{D}}$. This and (2.19) can be applied to demonstrate the relation between matrix and dyadic notations by writing

$$\overline{\overline{D}} = \overline{\overline{I}} \cdot \overline{\overline{D}} \cdot \overline{\overline{I}} = \sum_i \sum_j (\mathbf{a}_i' \cdot \overline{\overline{D}} \cdot \mathbf{a}_j) \mathbf{a}_i \mathbf{a}_j' = \sum_i \sum_j D_{ij} \mathbf{a}_i \mathbf{a}_j', \tag{2.21}$$

or any dyadic can be written in terms of nine scalars D_{ij}. These scalars can be conceived as matrix components of the dyadic with respect to the base $\{\mathbf{a}_i\}$. The matrix components of the unit dyadic $\overline{\overline{I}}$ are $\{\delta_{ij}\}$ in all bases.

From (2.16) it can be seen that the most general antisymmetric dyadic can be written as

$$\overline{\overline{A}}_a = \mathbf{ab} - \mathbf{ba} = (\mathbf{b} \times \mathbf{a}) \times \overline{\overline{I}} = \overline{\overline{I}} \times (\mathbf{b} \times \mathbf{a}), \tag{2.22}$$

as is seen if (2.20) is substituted in (2.22). Thus, dyadics of the form $\mathbf{c} \times \overline{\overline{I}} = \overline{\overline{I}} \times \mathbf{c}$ are antisymmetric. The vector $\mathbf{c}$ corresponding to the antisymmetric part of a general dyadic $\overline{\overline{D}}$ can be obtained through the following operation:

$$\mathbf{c}(\overline{\overline{D}}) = \frac{1}{2} \overline{\overline{I}} \overset{\times}{\cdot} \overline{\overline{D}}. \tag{2.23}$$

For an antisymmetric dyadic, (2.23) can be easily verified from (2.22). For a symmetric dyadic, $\mathbf{c}(\overline{\overline{D}}) = 0$ is also easily shown to be valid.

All dyadics can be classified in terms of their mapping properties.

- *Complete dyadics* $\overline{\overline{D}}$ define a linear mapping with an inverse, which is represented by an inverse dyadic $\overline{\overline{D}}^{-1}$. Thus, any vector $\mathbf{b}$ can be reached by mapping a suitable vector $\mathbf{a}$ by $\overline{\overline{D}} \cdot \mathbf{a} = \mathbf{b}$.

- *Planar dyadics* map all vectors in a two-dimensional subspace. If we take a base $\{\mathbf{a}_i\}$, the vectors $\{\overline{\overline{D}} \cdot \mathbf{a}_i\}$ do not form a base, because they are linearly dependent and satisfy $(\overline{\overline{D}} \cdot \mathbf{a}_1) \cdot (\overline{\overline{D}} \cdot \mathbf{a}_2) \times (\overline{\overline{D}} \cdot \mathbf{a}_3) = 0$. Writing the general $\overline{\overline{D}}$ in the trinomial form $\mathbf{ab} + \mathbf{cd} + \mathbf{ef}$, we can show that the vector triple $\mathbf{a}$, $\mathbf{c}$, $\mathbf{e}$ is linearly dependent and one of these vectors can be expressed in terms of other two. Thus, the most general *planar dyadic* can be written as a dyadic binomial $\mathbf{ab} + \mathbf{cd}$.

- *Strictly planar dyadics* are planar dyadics which cannot be written as a single dyad.

- *Linear dyadics* map all vectors in a one-dimensional subspace, i.e. parallel to a vector $\mathbf{c}$: $\overline{\overline{D}} \cdot \mathbf{a} = \alpha \mathbf{c}$. Thus, $\overline{\overline{D}}$ must obviously be of the form $\mathbf{cb}$. Linear dyadics can be written as a single dyad. Finally, we can distinguish between *strictly linear* dyadics and the *null dyadic*.

As examples, we note that the unit dyadic $\overline{\overline{I}}$ is complete, whereas an antisymmetric dyadic is either strictly planar or the null dyadic. The inverse dyadic of a given complete dyadic $\overline{\overline{D}} = \sum \mathbf{a}_i \mathbf{b}_i$ can be written quite straightforwardly in trinomial form. First, to be complete, the vector triples $\{\mathbf{a}_i\}$, $\{\mathbf{b}_i\}$ must be bases because from linear dependence of either base, a planar dyadic would result. Hence, there exist reciprocal bases $\{\mathbf{a}'_i\}$, $\{\mathbf{b}'_i\}$, with which we can write

$$\overline{\overline{D}}^{-1} = \sum \mathbf{b}'_i \mathbf{a}'_i. \tag{2.24}$$

That (2.24) satisfies $\overline{\overline{D}} \cdot \overline{\overline{D}}^{-1} = \overline{\overline{D}}^{-1} \cdot \overline{\overline{D}} = \overline{\overline{I}}$, can be easily verified.

2.3 Products of dyadics

Different products of dyadics play a role similar to dot and cross products of vectors, which introduce the operational power to the vector notation. The products of dyadics obey certain rules which are governed by certain identities summarized in Appendix A.

2.3.1 Dot-product algebra

The dot product between two dyadics has already been mentioned above and is defined in an obvious manner:

$$\overline{\overline{A}} \cdot \overline{\overline{C}} = (\sum \mathbf{a}_i \mathbf{b}_i) \cdot (\sum \mathbf{c}_j \mathbf{d}_j) = \sum \sum (\mathbf{b}_i \cdot \mathbf{c}_j) \mathbf{a}_i \mathbf{d}_j. \tag{2.25}$$

With this dot product, the dyadics form an algebra, where the unit dyadic, null dyadic and inverse dyadics are defined as above. This algebra corresponds to the matrix algebra, because the matrix (with respect to a given base) of $\overline{\overline{A}} \cdot \overline{\overline{B}}$ can be shown to equal the matrix product of the matrices of each dyadic. Thus properties known from matrix algebra are valid to dot-product algebra: associativity

$$\overline{\overline{A}} \cdot (\overline{\overline{B}} \cdot \overline{\overline{C}}) = (\overline{\overline{A}} \cdot \overline{\overline{B}}) \cdot \overline{\overline{C}}, \tag{2.26}$$

and (in general) non-commutativity, $\overline{\overline{A}} \cdot \overline{\overline{B}} \neq \overline{\overline{B}} \cdot \overline{\overline{A}}$. Further, we have

$$(\overline{\overline{A}} \cdot \overline{\overline{B}})^T = \overline{\overline{B}}^T \cdot \overline{\overline{A}}^T, \tag{2.27}$$

$$(\overline{\overline{A}} \cdot \overline{\overline{B}})^{-1} = \overline{\overline{B}}^{-1} \cdot \overline{\overline{A}}^{-1}. \tag{2.28}$$

Powers of dyadics, both positive and negative, are defined through the dot product (negative powers only for complete dyadics) in an obvious manner. For example, the antisymmetric dyadic $\overline{\overline{A}} = \mathbf{u} \times \overline{\overline{I}}$ with an NCP unit vector $\mathbf{u}$, satisfies for all $n > 0$.

$$\overline{\overline{A}}^{4n} = \overline{\overline{I}} - \mathbf{uu}, \tag{2.29}$$

$$\overline{\overline{A}}^{4n+1} = \overline{\overline{A}}, \tag{2.30}$$

$$\overline{\overline{A}}^{4n+2} = -\overline{\overline{I}} + \mathbf{uu}, \tag{2.31}$$

$$\overline{\overline{A}}^{4n+3} = -\overline{\overline{A}}. \tag{2.32}$$

Because for real $\mathbf{u}$, $\overline{\overline{A}}$ can be interpreted as a rotation by $\pi/2$ around $\mathbf{u}$, the powers of $\overline{\overline{A}}$ can be easily understood as multiples of that rotation.

Two dyadics do not commute in general in the dot product. It is easy to see that two antisymmetric dyadics only commute when one can be written as a multiple of the other. This is evident if we expand the dot product of two general antisymmetric dyadics:

$$(\mathbf{a} \times \overline{\overline{I}}) \cdot (\mathbf{b} \times \overline{\overline{I}}) = \mathbf{ba} - (\mathbf{a} \cdot \mathbf{b})\overline{\overline{I}}. \tag{2.33}$$

If this is required to be symmetric in $\mathbf{a}$ and $\mathbf{b}$, we should have $\mathbf{ab} = \mathbf{ba}$ or $(\mathbf{a} \times \mathbf{b}) \times \overline{\overline{I}} = 0$, which implies $\mathbf{a} \times \mathbf{b} = 0$ or $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

A dyadic commutes with an antisymmetric dyadic only if its symmetric and antisymmetric parts commute separately. In fact, writing $\overline{\overline{D}} = \overline{\overline{D}}_s + \mathbf{d} \times \overline{\overline{I}}$ in terms of its symmetric and antisymmetric parts, we can write

$$\overline{\overline{D}} \cdot (\mathbf{a} \times \overline{\overline{I}}) - (\mathbf{a} \times \overline{\overline{I}}) \cdot \overline{\overline{D}} = (\overline{\overline{D}}_s \times \mathbf{a}) + (\overline{\overline{D}}_s \times \mathbf{a})^T - (\mathbf{a} \times \mathbf{d}) \times \overline{\overline{I}}. \tag{2.34}$$

Equating the antisymmetric and symmetric parts of (2.34) to zero, shows us that the symmetric and antisymmetric parts of $\overline{\overline{D}}$ must commute with $\mathbf{a} \times \overline{\overline{I}}$ separately. Thus, the antisymmetric part of $\overline{\overline{D}}$ must be a multiple of $\mathbf{a}\times\overline{\overline{I}}$. The symmetric part of $\overline{\overline{D}}$ must be such that $\overline{\overline{D}}_s \times \mathbf{a}$ is antisymmetric, i.e. of the form $\mathbf{b} \times \overline{\overline{I}}$. Multiplying this by $\cdot\mathbf{a}$ gives zero, whence $\mathbf{b}$ must be a multiple of $\mathbf{a}$. It is easy to show that the symmetric dyadic must be of the form $\alpha\overline{\overline{I}} + \beta\mathbf{aa}$. Thus, the most general dyadic, which commutes with the antisymmetric dyadic $\mathbf{a} \times \overline{\overline{I}}$ is necessarily of the form

$$\overline{\overline{D}} = \alpha\overline{\overline{I}} + \beta\mathbf{aa} + \gamma\mathbf{a} \times \overline{\overline{I}}. \tag{2.35}$$

A dyadic of this special form is called *gyrotropic with axis* $\mathbf{a}$, which may also be a complex vector. From this, it is easy to show that if a dyadic commutes with its transpose, it must be either symmetric or gyrotropic.

2.3.2 Double-dot product

The double-dot product of two dyadics $\overline{\overline{A}} = \sum \mathbf{a}_i\mathbf{b}_i$, $\overline{\overline{B}} = \sum \mathbf{c}_j\mathbf{d}_j$ gives the scalar

$$\overline{\overline{A}} : \overline{\overline{B}} = \sum\sum(\mathbf{a}_i \cdot \mathbf{c}_j)(\mathbf{b}_i \cdot \mathbf{d}_j). \tag{2.36}$$

This is symmetric in both dyadics and satisfies

$$\overline{\overline{A}}^T : \overline{\overline{B}}^T = \overline{\overline{A}} : \overline{\overline{B}}, \tag{2.37}$$

$$(\mathbf{a} \times \overline{\overline{I}}) : (\mathbf{b} \times \overline{\overline{I}}) = 2\mathbf{a} \cdot \mathbf{b}, \tag{2.38}$$

$$\overline{\overline{A}} : \overline{\overline{I}} = \sum(\mathbf{a}_i \cdot \mathbf{b}_i) = \mathrm{tr}\overline{\overline{A}}. \tag{2.39}$$

The last operation gives a scalar which can be called the *trace* of $\overline{\overline{A}}$ because it gives the trace of the matrix of $\overline{\overline{A}}$ in any base $\{\mathbf{c}_i\}$. In fact, writing $\overline{\overline{A}} = \sum A_{ij}\mathbf{c}_i\mathbf{c}'_j$ gives us

$$\overline{\overline{A}} : \overline{\overline{I}} = (\sum\sum A_{ij}\mathbf{c}_i\mathbf{c}'_j) : (\sum \mathbf{c}'_k\mathbf{c}_k) = \sum A_{kk}. \tag{2.40}$$

As special cases we have $\overline{\overline{I}} : \overline{\overline{I}} = 3$ and $\overline{\overline{A}} : \overline{\overline{B}} = (\overline{\overline{A}} \cdot \overline{\overline{B}}^T) : \overline{\overline{I}} = (\overline{\overline{B}} \cdot \overline{\overline{A}}^T) : \overline{\overline{I}}$. A dyadic whose trace is zero is called trace free. Any dyadic can be written as a sum of a multiple of $\overline{\overline{I}}$ and a trace-free dyadic:

$$\overline{\overline{D}} = \frac{1}{3}(\overline{\overline{D}} : \overline{\overline{I}})\overline{\overline{I}} + (\overline{\overline{D}} - \frac{1}{3}(\overline{\overline{D}} : \overline{\overline{I}})\overline{\overline{I}}). \tag{2.41}$$

Antisymmetric dyadics are trace free. In fact, more generally, if $\overline{\overline{S}}$ is symmetric and $\overline{\overline{A}}$ antisymmetric, we have from (2.37)

$$\overline{\overline{A}} : \overline{\overline{S}} = \overline{\overline{A}}^T : \overline{\overline{S}}^T = -\overline{\overline{A}} : \overline{\overline{S}} = 0. \tag{2.42}$$

The scalar $\overline{\overline{D}} : \overline{\overline{D}}^*$ is a non-negative real number for any complex dyadic $\overline{\overline{D}}$. It is zero only for $\overline{\overline{D}} = 0$, which can be shown from the following:

$$\overline{\overline{D}} : \overline{\overline{D}}^* = (\overline{\overline{D}}^T \cdot \overline{\overline{D}}^*) : \overline{\overline{I}} = \sum \mathbf{u}_i \cdot \overline{\overline{D}}^T \cdot \overline{\overline{D}}^* \cdot \mathbf{u}_i = \sum |\overline{\overline{D}} \cdot \mathbf{u}_i|^2. \tag{2.43}$$

Here, $\{\mathbf{u}_i\}$ is a real orthonormal base. (2.43) is seen to give a non-negative number and vanish only if all $\overline{\overline{D}} \cdot \mathbf{u}_i$ vanish, whence $\overline{\overline{D}} = \sum \overline{\overline{D}} \cdot \mathbf{u}_i \mathbf{u}_i = 0$. We can define the norm of $\overline{\overline{D}}$ as

$$||\overline{\overline{D}}|| = \sqrt{\overline{\overline{D}} : \overline{\overline{D}}^*}. \tag{2.44}$$

2.3.3 Double-cross product

The double-cross product of two dyadics produces a third dyadic. Thus, it defines a double-cross algebra. Unlike the dot-product algebra, the double-cross algebra is commutative:

$$\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{B}} = \overline{\overline{B}} {}^{\times}_{\times} \overline{\overline{A}}, \tag{2.45}$$

and non-associative, because $\overline{\overline{A}} {}^{\times}_{\times} (\overline{\overline{B}} {}^{\times}_{\times} \overline{\overline{C}}) \neq (\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{B}}) {}^{\times}_{\times} \overline{\overline{C}}$ in general. The commutative property follows directly from the anticommutativity of the cross product: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, as is easy to see. It is also easy to show that there does not exist a unit element in this algebra.

A most useful formula for the expansion of dyadic expressions can be obtained from the following evaluation with dyads:

$$(\mathbf{ab}) {}^{\times}_{\times} [(\mathbf{cd}) {}^{\times}_{\times} (\mathbf{ef})] = [\mathbf{a} \times (\mathbf{c} \times \mathbf{e})][\mathbf{b} \times (\mathbf{d} \times \mathbf{f})] =$$

$$[\mathbf{ca} \cdot \mathbf{e} - \mathbf{a} \cdot \mathbf{ce}][\mathbf{db} \cdot \mathbf{f} - \mathbf{b} \cdot \mathbf{df}] =$$

$$(\mathbf{ab} : \mathbf{cd})\mathbf{ef} + (\mathbf{ab} : \mathbf{ef})\mathbf{cd} - \mathbf{ef} \cdot (\mathbf{ab})^T \cdot \mathbf{cd} - \mathbf{cd} \cdot (\mathbf{ab})^T \cdot \mathbf{ef}. \tag{2.46}$$

This expression is a trilinear identity for dyads. Thus, every dyad can be replaced by any dyadic polynomial because of linearity, whence (2.46) may be written for general dyadics:

$$\overline{\overline{A}} {}^{\times}_{\times} (\overline{\overline{B}} {}^{\times}_{\times} \overline{\overline{C}}) = (\overline{\overline{A}} : \overline{\overline{B}})\overline{\overline{C}} + (\overline{\overline{A}} : \overline{\overline{C}})\overline{\overline{B}} - \overline{\overline{B}} \cdot \overline{\overline{A}}^T \cdot \overline{\overline{C}} - \overline{\overline{C}} \cdot \overline{\overline{A}}^T \cdot \overline{\overline{B}}. \tag{2.47}$$

Use of this dyadic identity adds more power to the dyadic calculus. Its memorizing is aided by the fact that, because of the commutative property (2.45), $\overline{\overline{B}}$ and $\overline{\overline{C}}$ are symmetrical in (2.47).

The method used above to obtain a dyadic identity from vector identities can be generalized by the following procedure.

1. A multilinear dyadic expression (linear in every dyadic) is written in terms of dyads, i.e. every dyadic is replaced by a dyad.
2. Vector identities are applied to change the expression into another form.
3. The result is grouped in such a way that the original dyads are formed.
4. The dyads are replaced by the original dyadics.

To demonstrate this procedure, let us expand the dyadic expression $(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}}):\overline{\overline{I}}$, which is linear in $\overline{\overline{A}}$ and $\overline{\overline{B}}$. Hence, we start by replacing them by $\mathbf{ab}$ and $\mathbf{cd}$, respectively, and applying the well-known vector identity $(\mathbf{ab}{}^\times_\times\mathbf{cd}):\overline{\overline{I}} = (\mathbf{a}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$. This can be grouped as $(\mathbf{ab}:\overline{\overline{I}})(\mathbf{cd}:\overline{\overline{I}}) - (\mathbf{ab}):(\mathbf{cd})^T$. Finally, going back to $\overline{\overline{A}}$ and $\overline{\overline{B}}$ leaves us with the dyadic identity

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}}):\overline{\overline{I}} = (\overline{\overline{A}}:\overline{\overline{I}})(\overline{\overline{B}}:\overline{\overline{I}}) - \overline{\overline{A}}:\overline{\overline{B}}^T, \tag{2.48}$$

or trace of $\overline{\overline{A}}{}^\times_\times\overline{\overline{B}}$ equals $\mathrm{tr}\overline{\overline{A}}\ \mathrm{tr}\overline{\overline{B}} - \mathrm{tr}(\overline{\overline{A}}\cdot\overline{\overline{B}})$.

New dyadic identities can also be obtained from old identities. As an example, let us write (2.48) in the form

$$[\overline{\overline{A}}{}^\times_\times\overline{\overline{I}} - (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}} + \overline{\overline{A}}^T]:\overline{\overline{B}} = 0. \tag{2.49}$$

To obtain this, we have applied the invariance in any permutation of the triple scalar product of dyadics, $\overline{\overline{A}}{}^\times_\times\overline{\overline{B}}:\overline{\overline{C}} = \overline{\overline{A}}{}^\times_\times\overline{\overline{C}}:\overline{\overline{B}} = \overline{\overline{B}}{}^\times_\times\overline{\overline{A}}:\overline{\overline{C}} = \cdots$. (Of course, the double-cross product must always be performed first.) Because (2.49) is valid for any dyadic $\overline{\overline{B}}$, the bracketed dyadic must be the null dyadic, and the following identity is obtained:

$$\overline{\overline{A}}{}^\times_\times\overline{\overline{I}} = (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}} - \overline{\overline{A}}^T. \tag{2.50}$$

That $\overline{\overline{D}}:\overline{\overline{B}} = 0$ for all $\overline{\overline{B}}$ implies $\overline{\overline{D}} = 0$, is easily seen by taking $\overline{\overline{B}} = \mathbf{u}_i\mathbf{u}_j$ from an orthonormal base $\{\mathbf{u}_i\}$, whence all matrix coefficients D_{ij} of $\overline{\overline{D}}$ can be shown to vanish. The identity (2.48) is obtained from (2.50) as a special case by operating it by $:\overline{\overline{B}}$.

An important identity for the double-cross product can be obtained by expanding the expression $(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}})^\times_\times(\overline{\overline{C}}{}^\times_\times\overline{\overline{D}})$ twice through (2.47) by considering one of the bracketed dyadics as a single dyadic, and equating the expressions. Setting $\overline{\overline{C}} = \overline{\overline{D}} = \overline{\overline{I}}$ we obtain

$$\overline{\overline{A}}{}^\times_\times\overline{\overline{B}} =$$

$$[(\overline{\overline{A}}:\overline{\overline{I}})(\overline{\overline{B}}:\overline{\overline{I}}) - \overline{\overline{A}}:\overline{\overline{B}}^T]\overline{\overline{I}} - (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{B}}^T - (\overline{\overline{B}}:\overline{\overline{I}})\overline{\overline{A}}^T + (\overline{\overline{A}}\cdot\overline{\overline{B}} + \overline{\overline{B}}\cdot\overline{\overline{A}})^T. \quad (2.51)$$

This identity could be also conceived as the definition of the double-cross product in terms of single-dot and double-dot products and the transpose operation. It is easily seen that (2.50) is a special case of (2.51). Also, $\overline{\overline{I}}{}^\times_\times\overline{\overline{I}} = 2\overline{\overline{I}}$ is obtained as a further special case. The operation $\overline{\overline{A}}{}^\times_\times\overline{\overline{I}}$ is in fact a mapping from dyadic to dyadic. Its properties can be examined through the following dyadic eigenproblem:

$$\overline{\overline{A}}{}^\times_\times\overline{\overline{I}} = \lambda\overline{\overline{A}}. \quad (2.52)$$

Taking the trace of (2.52) leaves us with $(2-\lambda)\overline{\overline{A}}:\overline{\overline{I}} = 0$, whence either $\lambda = 2$ or $\overline{\overline{A}}$ is trace free. Substituting (2.50) in (2.52) gives us the following different solutions:

- $\lambda = 2$ and $\overline{\overline{A}} = \alpha\overline{\overline{I}}$ where α is any scalar;
- $\lambda = 1$ and $\overline{\overline{A}}$ is antisymmetric;
- $\lambda = -1$ and $\overline{\overline{A}}$ is symmetric and trace free.

Any dyadic can be written uniquely as a sum of three components: a multiple of $\overline{\overline{I}}$, an antisymmetric dyadic and a trace-free symmetric dyadic,

$$\overline{\overline{A}} = \frac{1}{3}(\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}} + \frac{1}{2}(\overline{\overline{A}} - \overline{\overline{A}}^T) + [\frac{1}{2}(\overline{\overline{A}} + \overline{\overline{A}}^T) - \frac{1}{3}(\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}}], \quad (2.53)$$

respectively. It is a simple matter to check that the right-hand side of (2.53), each term multiplied by the corresponding eigenvalue λ, gives the same result as (2.50). There exists an inverse mapping to $\overline{\overline{I}}{}^\times_\times$ which, denoted by $\overline{\overline{L}}(\overline{\overline{A}})$ and satisfying $\overline{\overline{L}}(\overline{\overline{A}}{}^\times_\times\overline{\overline{I}}) = \overline{\overline{A}}$ for every $\overline{\overline{A}}$, can be written in terms of inverse eigenvalues, or in the simple form

$$\overline{\overline{L}}(\overline{\overline{A}}) = \frac{1}{2}(\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}} - \overline{\overline{A}}^T. \quad (2.54)$$

Finally, let us consider the double-cross mapping through an antisymmetric dyadic $\mathbf{d}\times\overline{\overline{I}}$. Every antisymmetric dyadic $\mathbf{a}\times\overline{\overline{I}}$ is mapped onto the planar symmetric dyadic

$$(\mathbf{a}\times\overline{\overline{I}})\,{}^{\times}_{\times}(\mathbf{d}\times\overline{\overline{I}}) = \mathbf{ad}+\mathbf{da}. \tag{2.55}$$

Every symmetric dyadic $\overline{\overline{S}}$ is mapped onto the antisymmetric dyadic:

$$\overline{\overline{S}}\,{}^{\times}_{\times}(\mathbf{d}\times\overline{\overline{I}}) = (\overline{\overline{S}}\cdot\mathbf{d})\times\overline{\overline{I}}. \tag{2.56}$$

The mapping of the general dyadic is obtained when it is decomposed into a symmetric and an antisymmetric part. There does not exist an inverse to the mapping ${}^{\times}_{\times}(\mathbf{d}\times\overline{\overline{I}})$, because all symmetric dyadics satisfying $\overline{\overline{S}}\cdot\mathbf{d}=0$ are mapped onto the null dyadic.

2.4 Invariants and inverses

In matrix algebra, invariants are such functions of a matrix that are independent of the basis in which the matrix is formed. Thus, they can be expressed as functions of the corresponding dyadic. The trace was defined in (2.39) as the scalar $\overline{\overline{A}}:\overline{\overline{I}}$, corresponding to the sum of diagonal terms in the matrix. The second invariant can be denoted, and expanded from (2.48), as

$$\mathrm{spm}\overline{\overline{A}} = \frac{1}{2}\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}:\overline{\overline{I}} = \frac{1}{2}[(\overline{\overline{A}}:\overline{\overline{I}})^2\overline{\overline{A}}:\overline{\overline{A}}^T], \tag{2.57}$$

and its counterpart in matrix algebra is the sum of principal minors. The third invariant corresponds to the determinant of the matrix and can be expressed as

$$\det\overline{\overline{A}} = \frac{1}{6}\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}:\overline{\overline{A}}. \tag{2.58}$$

Finally, the cross-product square of the dyadic is defined as

$$\overline{\overline{A}}^{(2)} = \frac{1}{2}\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}. \tag{2.59}$$

As an example, $\overline{\overline{I}}^{(2)}=\overline{\overline{I}}$ can be easily verified.

Writing the dyadic $\overline{\overline{A}}$ in the form $\sum\mathbf{a}_i\mathbf{b}_i$, from (2.59) we can find the following vector expression for the cross-product square:

$$\overline{\overline{A}}^{(2)} = (\mathbf{a}_1\times\mathbf{a}_2)(\mathbf{b}_1\times\mathbf{b}_2)+(\mathbf{a}_2\times\mathbf{a}_3)(\mathbf{b}_2\times\mathbf{b}_3)+(\mathbf{a}_3\times\mathbf{a}_1)(\mathbf{b}_3\times\mathbf{b}_1) =$$

$$\sum\mathbf{a}_i'\mathbf{b}_i'(\mathbf{a}_1\times\mathbf{a}_2\cdot\mathbf{a}_3)(\mathbf{b}_1\times\mathbf{b}_2\cdot\mathbf{b}_3), \tag{2.60}$$

where $\{\mathbf{a}'\}$ and $\{\mathbf{b}'\}$ are reciprocal to the bases $\{\mathbf{a}\}$, $\{\mathbf{b}\}$. If $\overline{\overline{A}}$ is not complete, the last expression in (2.60) is not meaningful, since the reciprocal bases do not both exist. Further, we can write for the determinant,

$$\mathrm{det}\overline{\overline{A}} = (\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3)(\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3). \tag{2.61}$$

Combining (2.24), (2.60) and (2.61) gives us the formula for the inverse of a dyadic:

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^{(2)T}}{\mathrm{det}\overline{\overline{A}}} = 3\frac{(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})^T}{\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}} : \overline{\overline{A}}}, \tag{2.62}$$

which implies

$$\frac{1}{2}(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})^T \cdot \overline{\overline{A}} = \frac{1}{2}\overline{\overline{A}} \cdot (\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})^T = \frac{1}{6}(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}} : \overline{\overline{A}})\overline{\overline{I}}. \tag{2.63}$$

Because $\mathrm{det}\overline{\overline{I}} = 1$, the inverse of $\overline{\overline{I}}$ is easily seen to be $\overline{\overline{I}}$, as expected. The inverse exists only if $\mathrm{det}\overline{\overline{A}} \neq 0$ is satisfied, which serves as a definition of the complete dyadic. In fact, the classification of dyadics can be written in the form

- complete dyadic if $\mathrm{det}\overline{\overline{A}} \neq 0$.
- strictly planar dyadic if $\mathrm{det}\overline{\overline{A}} = 0$ and $\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}} \neq 0$.
- linear dyadic if $\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}} = 0$.

There exists a relation between the cross-product square of a dyadic and its dot-product square. In fact, from (2.51) we can write

$$\overline{\overline{A}}^{(2)} = \overline{\overline{A}}^{2T} - (\mathrm{tr}\overline{\overline{A}})\overline{\overline{A}}^T + (\mathrm{spm}\overline{\overline{A}})\overline{\overline{I}}. \tag{2.64}$$

After transposing (2.64), dot-multiplying by $\overline{\overline{A}}$ and applying (2.62) we end up with a relation between powers of a dyadic $\overline{\overline{A}}$

$$\overline{\overline{A}}^3 - (\mathrm{tr}\overline{\overline{A}})\overline{\overline{A}}^2 + (\mathrm{spm}\overline{\overline{A}})\overline{\overline{A}} - (\mathrm{det}\overline{\overline{A}})\overline{\overline{I}} = 0, \tag{2.65}$$

which is called the Cayley–Hamilton equation, but actually it is an identity, since it is satisfied by all dyadics $\overline{\overline{A}}$.

As other properties of the cross-product square we can easily prove that

$$[\overline{\overline{A}}^{(2)}]^{(2)} = \frac{1}{8}(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})\,{}^{\times}_{\times}(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}) = (\mathrm{det}\overline{\overline{A}})\overline{\overline{A}}, \tag{2.66}$$

which tells us that if $\overline{\overline{A}}$ is planar, $\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}$ is a linear dyadic. In fact, a dyadic of the form $\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}$ cannot be strictly planar, it is either complete or linear.

To prove the property $\det(\overline{\overline{A}}^n) = (\det\overline{\overline{A}})^n$ known from matrix algebra, we need the identity

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}})\cdot(\overline{\overline{C}}{}^\times_\times\overline{\overline{D}}) = (\overline{\overline{A}}\cdot\overline{\overline{C}}){}^\times_\times(\overline{\overline{B}}\cdot\overline{\overline{D}}) + (\overline{\overline{A}}\cdot\overline{\overline{D}}){}^\times_\times(\overline{\overline{B}}\cdot\overline{\overline{C}}), \tag{2.67}$$

which can be derived with the method described above. The following is a special case:

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}})\cdot(\overline{\overline{B}}{}^\times_\times\overline{\overline{B}}) = 2(\overline{\overline{A}}\cdot\overline{\overline{B}}){}^\times_\times(\overline{\overline{A}}\cdot\overline{\overline{B}}), \tag{2.68}$$

which can also be written as

$$\overline{\overline{A}}^{(2)}\cdot\overline{\overline{B}}^{(2)} = (\overline{\overline{A}}\cdot\overline{\overline{B}})^{(2)}. \tag{2.69}$$

Applying this and (2.63) we have

$$\det(\overline{\overline{A}}\cdot\overline{\overline{B}}) = (\det\overline{\overline{A}})(\det\overline{\overline{B}}) \tag{2.70}$$

after some algebra. This in fact shows that the dot product of any number of dyadics has a determinant function which equals the product of determinant functions of individual dyadics. Thus, the product of dyadics is planar if any one of the dyadics is a planar dyadic.

2.5 Solving dyadic equations

In this section, some principles for and results of solving dyadic equations are considered. The presentation does not cover all the kinds of equations that may occur. By a dyadic equation we mean the form $\overline{\overline{A}} = \overline{\overline{B}}$, where $\overline{\overline{A}}$ and $\overline{\overline{B}}$ may contain unknown quantities. In the following cases, the equation $\overline{\overline{A}} = \overline{\overline{B}}$ reduces to $\overline{\overline{A}} = 0$ and $\overline{\overline{B}} = 0$, or is split up into two equations which are usually easier to solve than the original one.

- $\overline{\overline{A}} = \alpha\overline{\overline{C}}$ where $\overline{\overline{C}}$ is a complete dyadic, and $\overline{\overline{B}}$ is known to be planar. (Taking the determinant function of each side gives $\alpha = 0$.) If $\alpha \neq 0$, there is no solution at all.

- $\overline{\overline{A}} = \alpha\overline{\overline{I}}$ and $\overline{\overline{B}}$ is trace free. (Taking the trace operation of each side gives us $\alpha = 0$.)

- $\overline{\overline{A}}$ is symmetric and $\overline{\overline{B}}$ is antisymmetric.

- $\overline{\overline{A}}$ is antisymmetric and $\overline{\overline{B}}$ is linear dyadic. (Because an antisymmetric dyadic is either strictly planar or null dyadic, both are null dyadics.)

For example, we might encounter an equation of the type

$$\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}} = \mathbf{a} \times \overline{\overline{I}}. \tag{2.71}$$

Because $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}}$ is either complete, linear or null dyadic and the antisymmetric dyadic $\mathbf{a} \times \overline{\overline{I}}$ either strictly planar or null dyadic, we have $\mathbf{a} = 0$ and $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}} = 0$, or $\overline{\overline{A}}$ may be any linear dyadic.

2.5.1 Linear equations

Let us study the linear dyadic equation

$$\overline{\overline{A}} \cdot \overline{\overline{X}} = \overline{\overline{B}}. \tag{2.72}$$

If $\overline{\overline{A}}$ is complete, the solution $\overline{\overline{X}} = \overline{\overline{A}}^{-1} \cdot \overline{\overline{B}}$ is defined by (2.62). Problems arise when $\overline{\overline{A}}$ is planar. If in this case $\overline{\overline{B}}$ is complete, (2.72) does not possess a solution at all. Thus, for (2.72) to have a solution when $\overline{\overline{A}}$ is planar, $\overline{\overline{B}}$ must also be planar, which does not warrant a solution, however. If there is a solution, it is not unique unless we restrict it somehow. Let us study the problem (2.72) where $\overline{\overline{A}}$ is strictly planar, and try to solve for $\overline{\overline{X}}$.

Because $\overline{\overline{A}}$ is strictly planar, we have $\det\overline{\overline{A}} = 0$ and $\overline{\overline{A}}^{(2)} \neq 0$. In fact, the cross-product square of $\overline{\overline{A}}$ is a strictly linear dyadic and can be written in the form $\overline{\overline{A}}^{(2)} = \mathbf{ab} \neq 0$. From (2.63) we have $\overline{\overline{A}}^{(2)T} \cdot \overline{\overline{A}} = \overline{\overline{A}} \cdot \overline{\overline{A}}^{(2)T} = 0$, or $\mathbf{a} \cdot \overline{\overline{A}} = \overline{\overline{A}} \cdot \mathbf{b} = 0$. Thus, $\overline{\overline{A}}$ defines a class A of planar dyadics which are orthogonal to $\mathbf{a}$ from the left and to $\mathbf{b}$ from the right. This class can be used to define a unique solution to the equation (2.72).

From (2.72) it is seen that there does not exist a solution unless $\overline{\overline{B}}$ satisfies

$$\overline{\overline{A}}^{(2)T} \cdot \overline{\overline{B}} = 0, \tag{2.73}$$

or $\mathbf{a} \cdot \overline{\overline{B}} = 0$. $\overline{\overline{B}}$ need not belong to class A. Let us look for the solution of (2.72) in the form $\overline{\overline{X}} = \overline{\overline{D}} \cdot \overline{\overline{B}}$ and find a dyadic $\overline{\overline{D}}$, whose conjugate transpose is in class A. Such a dyadic is called the *planar inverse* or generalized inverse of $\overline{\overline{A}}$ and is simply denoted by $\overline{\overline{D}} = \overline{\overline{A}}^{-1}$. Because the inverse in the strict sense (2.62) does not exist for planar dyadics, there should not be any place for misinterpretation due to the notation. The planar inverse can be written in the form

$$\overline{\overline{A}}^{-1} = \frac{(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}}^{*(2)})^T}{\overline{\overline{A}}^{(2)} : \overline{\overline{A}}^{*(2)}}. \tag{2.74}$$

This is a solution whose conjugated transpose is in class A, as can be easily checked. To verify (2.74), we expand using (2.47) and (2.64):

$$\overline{\overline{A}}\cdot(\overline{\overline{A}}{}^{\times}_{\times}(\overline{\overline{A}}{}^{*}{}^{\times}_{\times}\overline{\overline{A}}{}^{*}))^T = 2[\mathrm{tr}(\overline{\overline{A}}\cdot\overline{\overline{A}}{}^{*T})(\overline{\overline{A}}\cdot\overline{\overline{A}}{}^{*T}) - (\overline{\overline{A}}\cdot\overline{\overline{A}}{}^{*T})^2] =$$
$$2[\mathrm{spm}(\overline{\overline{A}}\cdot\overline{\overline{A}}{}^{*T})\overline{\overline{I}} - (\overline{\overline{A}}\cdot\overline{\overline{A}}{}^{*T})^{(2)T}], \tag{2.75}$$

whence we can write

$$\overline{\overline{A}}\cdot\frac{(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}}{}^{*(2)})^T}{\mathrm{spm}(\overline{\overline{A}}{}^{*}\cdot\overline{\overline{A}}{}^{T})} = \overline{\overline{I}} - \frac{(\overline{\overline{A}}{}^{*}\cdot\overline{\overline{A}}{}^{T})^{(2)}}{\mathrm{spm}(\overline{\overline{A}}{}^{*}\cdot\overline{\overline{A}}{}^{T})} = \overline{\overline{I}}_p. \tag{2.76}$$

Here, the dyadic $\overline{\overline{I}}_p$ can be called the planar unit dyadic of class A.

Multiplying (2.76) by $\cdot\overline{\overline{B}}$ from the right and taking (2.69), (2.73) and the identity

$$\mathrm{spm}(\overline{\overline{A}}\cdot\overline{\overline{B}}) = \mathrm{tr}(\overline{\overline{A}}\cdot\overline{\overline{B}})^{(2)} = \mathrm{tr}(\overline{\overline{A}}{}^{(2)}\cdot\overline{\overline{B}}{}^{(2)}) = \overline{\overline{A}}{}^{(2)}:\overline{\overline{B}}{}^{(2)T} \tag{2.77}$$

into account, we see that $\overline{\overline{X}} = \overline{\overline{A}}{}^{-1}\cdot\overline{\overline{B}}$ is really a solution of (2.72). (2.74) is called the planar inverse of $\overline{\overline{A}}$. The denominator in (2.74) equals $\mathrm{spm}(\overline{\overline{A}}\cdot\overline{\overline{A}}{}^{*T})$ and is never zero for strictly planar dyadics $\overline{\overline{A}}$.

The most general solution to (2.72) can be written in the form

$$\overline{\overline{X}} = \overline{\overline{A}}{}^{-1}\cdot\overline{\overline{B}} + \overline{\overline{X}}_o, \tag{2.78}$$

where $\overline{\overline{X}}_o$ is any solution to the homogeneous equation

$$\overline{\overline{A}}\cdot\overline{\overline{X}}_o = 0. \tag{2.79}$$

Because $\overline{\overline{A}}$ is strictly planar, $\overline{\overline{X}}_o$ must be a linear dyadic, as is easily seen writing $\overline{\overline{A}} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2$. In fact, we obviously must have

$$\overline{\overline{X}}_o = \overline{\overline{A}}{}^{(2)T}\cdot\overline{\overline{C}}, \tag{2.80}$$

as the most general solution with an arbitrary dyadic $\overline{\overline{C}}$. The equation $\overline{\overline{X}}\cdot\overline{\overline{A}} = \overline{\overline{B}}$ can be solved in corresponding manner.

Let us now study the linear equation

$$\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{X}} = \overline{\overline{B}}, \tag{2.81}$$

where $\overline{\overline{A}}$ is a complete dyadic. Multiplying this by $(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}})^{\times}_{\times}$ and applying (2.47) we easily have for the solution

$$\overline{\overline{X}} = \overline{\overline{A}}{}^{-1T}{}^{\times}_{\times}\overline{\overline{B}} - \frac{\overline{\overline{A}}:\overline{\overline{B}}}{2\det\overline{\overline{A}}}\overline{\overline{A}}. \tag{2.82}$$

For example, if $\overline{\overline{A}} = \overline{\overline{I}}$, the expression (2.54) is obtained.

2.5.2 Quadratic equations

Let us study the quadratic equation

$$\overline{\overline{X}} {}^{\times}_{\times} \overline{\overline{X}} = \overline{\overline{A}}, \tag{2.83}$$

which is quite easy to solve if $\overline{\overline{A}}$ is a complete dyadic. Double-cross multiplying both sides with itself and applying (2.66) gives us the two solutions

$$\overline{\overline{X}} = \pm \frac{\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{A}}}{\sqrt{8 \det \overline{\overline{A}}}}, \tag{2.84}$$

which fails for planar dyadics satisfying $\det \overline{\overline{A}} = 0$.

If $\overline{\overline{A}}$ is strictly planar, $\overline{\overline{A}}^{(2)} \neq 0$, there is no solution at all, as was seen earlier. If $\overline{\overline{A}}$ is strictly linear, there exist infinitely many strictly planar solutions $\overline{\overline{X}}$. In fact, in this case $\overline{\overline{A}}$ defines a class of strictly planar dyadics satisfying $\overline{\overline{X}} \cdot \overline{\overline{A}}^T = \overline{\overline{A}}^T \cdot \overline{\overline{X}} = 0$. For $\overline{\overline{A}} = 0$, any linear dyadic is a solution.

Square roots of dyadics

The quadratic equation

$$\overline{\overline{X}} \cdot \overline{\overline{X}} = \overline{\overline{X}}^2 = \overline{\overline{A}} \tag{2.85}$$

is more difficult to solve than (2.83). Its solutions, *square roots* of $\overline{\overline{A}}$, may be infinite or zero in number, or something in between, depending on the dyadic $\overline{\overline{A}}$. For example, all dyadics of the form $\pm(\overline{\overline{I}} - 2\mathbf{uu})$ are square roots of the unit dyadic $\overline{\overline{I}}$ for all unit vectors $\mathbf{u}$.

A solution procedure can be based on the solution of (2.83) through the following identity, which can be derived by applying the identity for $\overline{\overline{A}} {}^{\times}_{\times} (\overline{\overline{B}} {}^{\times}_{\times} \overline{\overline{C}})$:

$$(\overline{\overline{X}} {}^{\times}_{\times} \overline{\overline{I}}) {}^{\times}_{\times} (\overline{\overline{X}} {}^{\times}_{\times} \overline{\overline{I}}) = 2(\mathrm{spm}\overline{\overline{X}})\overline{\overline{I}} + 2\overline{\overline{X}}^2. \tag{2.86}$$

The procedure is first to solve for the unknown scalar $\alpha = \mathrm{spm}\overline{\overline{X}}$ and then solve (2.86). An equation for α can be derived as follows. Denoting $\overline{\overline{D}}(\alpha) = \alpha\overline{\overline{I}} + \overline{\overline{X}}^2 = \alpha\overline{\overline{I}} + \overline{\overline{A}}$, we can write

$$(\overline{\overline{X}} {}^{\times}_{\times} \overline{\overline{I}})^{(2)} = \overline{\overline{D}}(\alpha), \tag{2.87}$$

$$[(\overline{\overline{X}} {}^{\times}_{\times} \overline{\overline{I}})^{(2)}]^{(2)} = [\det(\overline{\overline{X}} {}^{\times}_{\times} \overline{\overline{I}})]\overline{\overline{X}} {}^{\times}_{\times} \overline{\overline{I}} = \overline{\overline{D}}^{(2)}, \tag{2.88}$$

$$\det[(\overline{\overline{X}} {}^{\times}_{\times} \overline{\overline{I}})^{(2)}] = [\det(\overline{\overline{X}} {}^{\times}_{\times} \overline{\overline{I}})]^2 = \det\overline{\overline{D}}. \tag{2.89}$$

For $\overline{\overline{D}}(\alpha)$ complete, from (2.88) and (2.89) we can express the unknown dyadic $\overline{\overline{X}}{}^{\times}_{\times}\overline{\overline{I}}$ in terms of the unknown dyadic $\overline{\overline{D}}(\alpha)$:

$$\overline{\overline{X}}{}^{\times}_{\times}\overline{\overline{I}} = \frac{\overline{\overline{D}}^{(2)}}{\det(\overline{\overline{X}}{}^{\times}_{\times}\overline{\overline{I}})} = \pm\frac{\overline{\overline{D}}^{(2)}}{\sqrt{\det\overline{\overline{D}}}}, \tag{2.90}$$

whence by taking the spm() operation of this and applying the identity $(\mathrm{tr}\overline{\overline{X}})^2 = 2\mathrm{spm}\overline{\overline{X}} + \mathrm{tr}(\overline{\overline{X}}^2)$ we obtain the following quartic equation for the unknown scalar α:

$$\alpha^4 - 2\alpha^2\mathrm{spm}\overline{\overline{A}} - 8\alpha\,\det\overline{\overline{A}} + (\mathrm{spm}\overline{\overline{A}})^2 - 4\mathrm{tr}\overline{\overline{A}}\,\det\overline{\overline{A}} = 0. \tag{2.91}$$

Values of α substituted in (2.90) solved for $\overline{\overline{X}}$

$$\overline{\overline{X}} = \pm\frac{1}{2\sqrt{\det\overline{\overline{D}}}}\left((\mathrm{spm}\overline{\overline{D}})\overline{\overline{I}} - 2\overline{\overline{D}}^{(2)T}\right) \tag{2.92}$$

give us possible square root solutions corresponding to those cases for which the dyadic $\overline{\overline{D}}(\alpha)$ is complete. Written explicitly, the solution is

$$\overline{\overline{X}} = \overline{\overline{A}}^{1/2} = \pm\frac{1}{2\sqrt{\det(\overline{\overline{A}}+\alpha\overline{\overline{I}})}}\left((\mathrm{spm}(\overline{\overline{A}}+\alpha\overline{\overline{I}}))\overline{\overline{I}} - 2(\overline{\overline{A}}+\alpha\overline{\overline{I}})^{(2)T}\right). \tag{2.93}$$

If the completeness condition is not valid, there are either no or infinitely many solutions corresponding to that particular solution of (2.83). As an example of a dyadic with no square roots we may write

$$\overline{\overline{A}} = \mathbf{uv} + \mathbf{ww}, \tag{2.94}$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is an orthonormal set of unit vectors. In fact, this corresponds to $\det\overline{\overline{A}} = 0$, $\mathrm{spm}\overline{\overline{A}} = 0$, whence (2.91) only has the solution $\alpha = 0$. Thus, $\overline{\overline{D}}$ equals the strictly planar dyadic $\overline{\overline{A}}$, whence (2.88) does not possess a solution. A dyadic of this kind is called a shearer.

Square roots of the unit dyadic

As an example, let us consider the problem of finding all the square roots of the unit dyadic $\overline{\overline{A}} = \overline{\overline{I}}$. In this case, the quartic (2.91) can be written as

$$(\alpha - 3)(\alpha + 1)^3 = 0, \tag{2.95}$$

or the roots are $\alpha = 3$ and $\alpha = -1$. The equation (2.86) now reads

$$(\overline{\overline{X}}{}^{\times}_{\times}\overline{\overline{I}})^{(2)} = \alpha\overline{\overline{I}} + \overline{\overline{X}}^2 = (\alpha + 1)\overline{\overline{I}}, \tag{2.96}$$

whose right side equals $4\overline{\overline{I}}$ and zero for the respective two α values. The solution formula (2.93) can now be applied only for the case $\alpha = 3$ and it results in the obvious roots $\pm\overline{\overline{I}}$. The other roots are obtained as solutions of

$$(\overline{\overline{X}}{}^{\times}_{\times}\overline{\overline{I}})^{(2)} = 0, \tag{2.97}$$

which is valid if $\overline{\overline{X}}{}^{\times}_{\times}\overline{\overline{I}}$ is any linear dyadic. The solutions $\overline{\overline{X}}$ satisfying both this and $\overline{\overline{X}}^2 = \overline{\overline{I}}$ are of the general form

$$\overline{\overline{X}} = \pm\left(\overline{\overline{I}} - 2\frac{\mathbf{ab}}{\mathbf{a}\cdot\mathbf{b}}\right), \tag{2.98}$$

with $\mathbf{a}\cdot\mathbf{b} \neq 0$.

Square roots of the null dyadic

Solutions of

$$\overline{\overline{X}}^2 = 0 \tag{2.99}$$

must be linear dyadics, because, as is known from solutions of (2.72), at least one of two dyadics must be linear if their product is the null dyadic. Writing $\overline{\overline{X}} = \mathbf{ab}$ in (2.99), we see that any *trace-free linear dyadic*, i.e. a dyadic of the form $\mathbf{aa}\times\mathbf{c}$, is a solution of (2.99).

2.5.3 Shearers

Solutions of the cubic equation

$$\overline{\overline{X}}^3 = 0, \tag{2.100}$$

i.e. cubic roots of the null dyadic, are important because of their physical significance. If $\overline{\overline{X}}$ is a linear dyadic satisfying (2.100), obviously it also satisfies (2.99), or it is trace free. Thus, the other solutions must be strictly planar and can easily be shown to satisfy the three scalar equations $\mathrm{tr}\overline{\overline{X}} = 0$, $\mathrm{spm}\overline{\overline{X}} = 0$ and $\det\overline{\overline{X}} = 0$. If $\overline{\overline{X}}$ is written as a dyadic binomial, it is easily concluded that the most general solution of (2.100) is of the form

$$\overline{\overline{X}} = \mathbf{a}\times(\mathbf{bb}+\mathbf{ca}), \quad \text{or} \quad \overline{\overline{X}} = (\mathbf{ab}+\mathbf{cc})\times\mathbf{a}. \tag{2.101}$$

(2.101) also includes the solutions of (2.99) for $\mathbf{c} = 0$ or $\mathbf{b} = 0$.

Dyadics of the form (2.101) in general, including the trace-free linear dyadics, are called *trace-free shearers*, which are special cases of *shearers*, defined by the two equations

$$\det\overline{\overline{A}} = 0, \qquad \mathrm{spm}\overline{\overline{A}} = 0. \tag{2.102}$$

Note that this definition differs from that adopted by Gibbs, who calls dyadics of the form $\overline{\overline{A}} + \alpha\overline{\overline{I}}$ shearers. From the Cayley–Hamilton identity (2.65) we can write another condition equivalent to (2.102), for the definition of the shearer:

$$\overline{\overline{A}}^3 = (\mathrm{tr}\overline{\overline{A}})\overline{\overline{A}}^2. \tag{2.103}$$

It is easily seen that, in the trace-free case, (2.103) corresponds to (2.100).

The most general shearer can be written in the forms

$$\overline{\overline{A}} = \mathbf{a} \times (\mathbf{bc} + \mathbf{da}), \quad \text{or} \quad \overline{\overline{A}} = (\mathbf{ad} + \mathbf{cb}) \times \mathbf{a}. \tag{2.104}$$

It is not difficult to check that (2.102) and (2.103) are satisfied for any vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. As was seen just before, a strictly planar shearer does not have a square root.

2.6 The eigenvalue problem

Right and left eigenvalue problems with eigenvalues and eigenvectors α_i, $\mathbf{a}_i$ and, respectively, β_i, $\mathbf{b}_i$, are of the form

$$\overline{\overline{A}} \cdot \mathbf{a}_i = \alpha_i \mathbf{a}_i, \tag{2.105}$$

$$\mathbf{b}_i \cdot \overline{\overline{A}} = \beta_i \mathbf{b}_i. \tag{2.106}$$

Because the dyadic $\overline{\overline{A}} - \gamma\overline{\overline{I}}$ is planar when γ equals α_i or β_i, the eigenvalues satisfy the equation

$$-\det(\overline{\overline{A}} - \gamma\overline{\overline{I}}) = \gamma^3 - \gamma^2 \mathrm{tr}\overline{\overline{A}} + \gamma\, \mathrm{spm}\overline{\overline{A}} - \det\overline{\overline{A}} = 0. \tag{2.107}$$

Because both right and left eigenvalues satisfy the same problem, they have the same values which are denoted by α_i. There are either one, two or three different values for α_i. Because $\mathbf{b}_i \cdot \overline{\overline{A}} \cdot \mathbf{a}_j = (\alpha_i \mathbf{b}_i) \cdot \mathbf{a}_j = \mathbf{b}_i \cdot (\alpha_j \mathbf{a}_j)$, we see that if $\alpha_i \neq \alpha_j$, the left and right eigenvectors are orthogonal, i.e. they satisfy $\mathbf{b}_i \cdot \mathbf{a}_j = 0$.

Eigenvectors $\mathbf{b}_i$ and $\mathbf{a}_i$ corresponding to a solution α_i of (2.107) can be constructed using dyadic methods. The construction depends on the multitude of the particular eigenvalue, which depends on whether the dyadic $\overline{\overline{A}} - \alpha_i\overline{\overline{I}}$ is strictly planar, strictly linear or null. Let us consider these cases separately. For this we need the following identities:

$$\mathbf{a} \times (\overline{\overline{A}} {}^\times_\times \overline{\overline{B}}) = \overline{\overline{B}} \times (\mathbf{a} \cdot \overline{\overline{A}}) + \overline{\overline{A}} \times (\mathbf{a} \cdot \overline{\overline{B}}), \tag{2.108}$$

$$(\overline{\overline{A}} {}^\times_\times \overline{\overline{B}}) \times \mathbf{a} = (\overline{\overline{A}} \cdot \mathbf{a}) \times \overline{\overline{B}} + (\overline{\overline{B}} \cdot \mathbf{a}) \times \overline{\overline{A}}, \tag{2.109}$$

with the special cases

$$\mathbf{a} \times (\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{A}}) = 2\overline{\overline{A}} \times (\mathbf{a} \cdot \overline{\overline{A}}), \tag{2.110}$$

$$(\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{A}}) \times \mathbf{a} = 2(\overline{\overline{A}} \cdot \mathbf{a}) \times \overline{\overline{A}}. \tag{2.111}$$

These can be derived with the general method described in Section 2.3, for creating dyadic identities.

- *Strictly planar* $\overline{\overline{A}} - \alpha_i \overline{\overline{I}}$. Defining $\overline{\overline{B}}_i = (\overline{\overline{A}} - \alpha_i \overline{\overline{I}}) {}^{\times}_{\times} (\overline{\overline{A}} - \alpha_i \overline{\overline{I}}) \neq 0$, from (2.110), (2.111) we see that $\mathbf{b}_i \times \overline{\overline{B}}_i = \overline{\overline{B}}_i \times \mathbf{a}_i = 0$, whence there exists a scalar $\xi_i \neq 0$ such that $\overline{\overline{B}}_i = \xi_i \mathbf{b}_i \mathbf{a}_i$. Thus, from the knowledge of α_i the dyadic $\overline{\overline{B}}_i$ is known and the eigenvectors can be written in the form $\mathbf{a}_i = \mathbf{c} \cdot \overline{\overline{B}}_i$ and $\mathbf{b}_i = \overline{\overline{B}}_i \cdot \mathbf{c}$ with suitable vector $\mathbf{c}$. In this case α_i is a single root of (2.107).

- *Strictly linear* $\overline{\overline{A}} - \alpha_i \overline{\overline{I}}$. There exist non-null vectors $\mathbf{c}$, $\mathbf{d}$ such that $\overline{\overline{A}}$ is of the form $\alpha\overline{\overline{I}} + \mathbf{cd}$. This is a special type of dyadic, which is called *uniaxial.* (2.107) leaves us with the equation $(\gamma - \alpha)^2(\gamma - \alpha - \mathbf{c}\cdot\mathbf{d}) = 0$, which shows us that the eigenvalue $\alpha_i = \alpha$ is a double root of (2.107). Assuming $\mathbf{c} \cdot \mathbf{d} \neq 0$, the third root is a simple one, $\alpha_j = \alpha + \mathbf{c} \cdot \mathbf{d}$, for which the eigenvectors can be obtained through the expression above. The left and right eigenvectors corresponding to the double root are any vectors satisfying the conditions $\mathbf{b} \cdot \mathbf{c} = 0$ and $\mathbf{d} \cdot \mathbf{a} = 0$, respectively. For $\mathbf{c} \cdot \mathbf{d} = 0$, all three eigenvalues are the same, but there only exist two linearly independent eigenvectors, those just mentioned.

- *Null dyadic* $\overline{\overline{A}} - \alpha_i \overline{\overline{I}}$. In this case $\overline{\overline{A}} = \alpha\overline{\overline{I}}$, there is a triple eigenvalue α and any vector is an eigenvector. This happens if $\overline{\overline{A}}$ is a multiple of the unit dyadic $\overline{\overline{I}}$.

The previous classification was made in terms of the dyadic $\overline{\overline{A}} - \alpha_i \overline{\overline{I}}$, or multitude of a particular eigenvalue α_i. Let us now consider the number of different eigenvalues of a dyadic $\overline{\overline{A}}$, which can be 1, 2 or 3. Because $\mathrm{det}\overline{\overline{A}} = \alpha_1\alpha_2\alpha_3$, $\mathrm{spm}\overline{\overline{A}} = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$ and $\mathrm{tr}\overline{\overline{A}} = \alpha_1 + \alpha_2 + \alpha_3$, the Cayley–Hamilton equation (2.65) can be written as

$$(\overline{\overline{A}} - \alpha_1\overline{\overline{I}}) \cdot (\overline{\overline{A}} - \alpha_2\overline{\overline{I}}) \cdot (\overline{\overline{A}} - \alpha_3\overline{\overline{I}}) = 0. \tag{2.112}$$

The order of terms is immaterial here.

- *One eigenvalue* $\alpha_1 = \alpha_2 = \alpha_3$. Because $(\overline{\overline{A}} - \alpha_1 \overline{\overline{I}})^3 = 0$, from the solution of equation (2.100) we conclude that $\overline{\overline{A}} - \alpha_1 \overline{\overline{I}}$ is a trace-free shearer, whence the most general form for $\overline{\overline{A}}$ is $\alpha_1 \overline{\overline{I}} + (\mathbf{ab} + \mathbf{cc}) \times \mathbf{a}$. Taking the trace operation it is seen that $\alpha_1 = \mathrm{tr}\overline{\overline{A}}/3$. The number of eigenvectors is obviously that of the shearer term. Here we can separate the cases.

  - Strictly planar trace-free shearer with $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \neq 0$. It is easy to show that there exists only one eigenvector (left and right), which can be obtained from the expression $(\overline{\overline{A}} - \alpha_1 \overline{\overline{I}})^{\times}_{\times}(\overline{\overline{A}} - \alpha_1 \overline{\overline{I}}) = 2(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})(\mathbf{a} \times \mathbf{c})\mathbf{a}$. Thus, the left eigenvector is $\mathbf{a} \times \mathbf{c}$ and the right eigenvector $\mathbf{a}$. Only this type of dyadic has just one eigenvector.

  - Linear shearer, which can be written with $\mathbf{c} = 0$ in the above expression. Now $\overline{\overline{A}} - \alpha_1 \overline{\overline{I}}$ satisfies (2.99). There exist two eigenvectors, which are orthogonal to vectors $\mathbf{a}$ from the left and $\mathbf{b} \times \mathbf{a}$ from the right.

- *Two eigenvalues* $\alpha_1 \neq \alpha_2 = \alpha_3$. The dyadic $\overline{\overline{A}}$ satisfies an equation of the form (2.103): $\overline{\overline{B}}^2(\overline{\overline{B}} - \mathrm{tr}\overline{\overline{B}}\ \overline{\overline{I}}) = 0$ with $\overline{\overline{B}} = \overline{\overline{A}} - \alpha_2 \overline{\overline{I}}$, as is easily verified. Thus, the dyadic $\overline{\overline{B}}$ must be a general shearer and $\overline{\overline{A}}$ is of the form $\alpha_2 \overline{\overline{I}} + \mathbf{a} \times (\mathbf{bc} + \mathbf{da})$. The eigenvalue α_1 equals $\alpha_2 + \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$, whence the shearer here cannot be trace free in order that the two eigenvalues do not coincide. In this case there exist two linearly independent eigenvectors.

- *Three eigenvalues* $\alpha_1 \neq \alpha_2 \neq \alpha_3$. In this case there exist three linearly independent eigenvectors. In fact, assuming the eigenvector $\mathbf{a}_3$ to be a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, which are linearly independent, (2.105) will lead to the contradictory conditions $\alpha_3 = \alpha_1$, $\alpha_3 = \alpha_2$.

As a summary, the following table presents the different cases of dyadics with different numbers of eigenvalues (N in horizontal lines) and eigenvectors (M in vertical columns).

$M \vert N$	1	2	3
1	$\alpha\overline{\overline{I}} + (\mathbf{ab} + \mathbf{cc}) \times \mathbf{a}$	none	none
2	$\alpha\overline{\overline{I}} + \mathbf{aa} \times \mathbf{b}$	$\alpha\overline{\overline{I}} + (\mathbf{ab} + \mathbf{cd}) \times \mathbf{a}$	none
3	$\alpha\overline{\overline{I}}$	$\alpha\overline{\overline{I}} + \mathbf{ab}$	$\mathbf{ab} + \mathbf{cd} + \mathbf{ef}$

Because the left or right eigenvectors of a dyadic with three linearly independent eigenvectors form two bases $\{\mathbf{b}_i\}$, $\{\mathbf{a}_i\}$, the dyadic can be written in either base as

$$\overline{\overline{A}} = \overline{\overline{A}} \cdot \sum \mathbf{a}_i \mathbf{a}_i' = \sum \alpha_i \mathbf{a}_i \mathbf{a}_i' = \sum \alpha_i \mathbf{b}_i' \mathbf{b}_i. \tag{2.113}$$

From this we conclude that the left and right eigenvectors are in fact reciprocals of each other: $\mathbf{a}_i' = \mathbf{b}_i$, $\mathbf{b}_i' = \mathbf{a}_i$. If two dyadics commute, they have the same eigenvectors.

2.7 Hermitian and positive definite dyadics

Hermitian and positive definite dyadics are often encountered in electromagnetics. In fact, lossless medium parameter dyadics are hermitian or antihermitian depending on the definition. Also, from power considerations in a medium, positive definiteness of dyadics often follows.

2.7.1 Hermitian dyadics

By definition, the hermitian dyadic satisfies

$$\overline{\overline{A}}^T = \overline{\overline{A}}^*, \tag{2.114}$$

whereas the antihermitian dyadic is defined by

$$\overline{\overline{A}}^T = -\overline{\overline{A}}^*. \tag{2.115}$$

Any dyadic can be written as a sum of a hermitian and an antihermitian dyadic in the form

$$\overline{\overline{A}} = \frac{1}{2}(\overline{\overline{A}} + \overline{\overline{A}}^{T*}) + \frac{1}{2}(\overline{\overline{A}} - \overline{\overline{A}}^{T*}). \tag{2.116}$$

Any hermitian dyadic can be written in the form $\sum \pm \mathbf{c}\mathbf{c}^*$ and antihermitian, in the form $\sum \pm j\mathbf{c}\mathbf{c}^*$. Conversely, these kinds of dyadics are always hermitian and antihermitian, respectively.

From (2.114), (2.115) it follows that the symmetric part of a hermitian dyadic is real and the antisymmetric part imaginary, whence the most general hermitian dyadic $\overline{\overline{H}}$ can be written in the form

$$\overline{\overline{H}} = \overline{\overline{S}} + j\mathbf{h} \times \overline{\overline{I}}, \tag{2.117}$$

with real and symmetric $\overline{\overline{S}}$ and real $\mathbf{h}$. Any antihermitian dyadic can be written as $j\overline{\overline{H}}$, where $\overline{\overline{H}}$ is a hermitian dyadic.

Hermitian dyadics form a subspace in the linear space of dyadics. The dot product of two hermitian dyadics is not necessarily hermitian, but the double-cross product is, as is seen from

$$(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}})^T = \overline{\overline{A}}{}^T{}^{\times}_{\times}\overline{\overline{B}}{}^T = \overline{\overline{A}}{}^*{}^{\times}_{\times}\overline{\overline{B}}{}^* = (\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}})^*, \tag{2.118}$$

where $\overline{\overline{A}}$ and $\overline{\overline{B}}$ are hermitian. Also, the double-dot product of two hermitian dyadics is a real number, as is easy to see from the sum expression. Thus, the scalars $\mathrm{tr}\overline{\overline{A}}$, $\mathrm{spm}\overline{\overline{A}}$, $\det\overline{\overline{A}}$ are real if $\overline{\overline{A}}$ is hermitian. This implies that the inverse of a hermitian dyadic is hermitian.

The following theorem is very useful when deriving identities for hermitian dyadics:

$$\overline{\overline{A}} : \mathbf{a}\mathbf{a}^* = 0 \quad \text{for all } \mathbf{a}, \ \Rightarrow \ \overline{\overline{A}} = 0. \tag{2.119}$$

This can be proved by setting first $\mathbf{a} = \mathbf{b} + \mathbf{c}$ and then $\mathbf{a} = \mathbf{b} + j\mathbf{c}$, whence the condition $\overline{\overline{A}} : \mathbf{bc} = 0$ for all vectors $\mathbf{b}$, $\mathbf{c}$ will result from (2.119), making the matrix components of $\overline{\overline{A}}$ vanish. For comparison, $\overline{\overline{A}} : \mathbf{aa} = 0$ for all $\mathbf{a}$ implies $\overline{\overline{A}}$ antisymmetric, as is easy to prove. From (2.119) we can show that if $\overline{\overline{A}} : \mathbf{aa}^*$ is real for all vectors $\mathbf{a}$, $\overline{\overline{A}}$ is hermitian. In fact, this implies $\overline{\overline{A}} : \mathbf{aa}^* - \overline{\overline{A}}{}^* : \mathbf{a}^*\mathbf{a} = 0$ or $(\overline{\overline{A}} - \overline{\overline{A}}{}^{*T}) : \mathbf{aa}^* = 0$, whence $\overline{\overline{A}}$ is hermitian.

A hermitian dyadic always has three eigenvectors no matter how many eigenvalues it has, as can be shown. The right and left eigenvectors corresponding to the same eigenvalues are complex conjugates of each other, because from $\overline{\overline{A}}\cdot\mathbf{a} = \alpha\mathbf{a}$ we have $\overline{\overline{A}}{}^*\cdot\mathbf{a}^* = \mathbf{a}^*\cdot\overline{\overline{A}} = \alpha^*\mathbf{a}^*$. But eigenvalues are real and eigenvectors conjugate orthogonal, because $(\alpha_i - \alpha_j^*)\mathbf{a}_i\cdot\mathbf{a}_j^* = 0$, whence $\alpha_i - \alpha_i^* = 0$ and $\mathbf{a}_i\cdot\mathbf{a}_j^* = 0$ for $\alpha_i \neq \alpha_j$. Thus, the general hermitian dyadic can be written in terms of its eigenvalues and eigenvectors as

$$\overline{\overline{A}} = \sum \alpha_i \frac{\mathbf{a}_i\mathbf{a}_i^*}{\mathbf{a}_i\cdot\mathbf{a}_i^*}. \tag{2.120}$$

2.7.2 Positive definite dyadics

By definition, a dyadic $\overline{\overline{D}}$ is positive definite (PD), if it satisfies

$$\overline{\overline{D}} : \mathbf{aa}^* > 0, \quad \text{for all} \quad \mathbf{a} \neq 0. \tag{2.121}$$

A PD dyadic is always hermitian, as is evident. Other properties, whose proofs are partly omitted, follow.

- PD dyadics are complete. If $\overline{\overline{D}}$ were planar, there would exist a vector $\mathbf{a}$ such that $\overline{\overline{D}}\cdot\mathbf{a} = 0$, in contradiction with (2.121). Thus, the inverse of a PD dyadic always exists.

- PD dyadics possess positive eigenvalues. This is seen by dot multiplying the expansion (2.120) by $\mathbf{a}_j^*\mathbf{a}_j/\mathbf{a}_j\cdot\mathbf{a}_j^*$, and the result is α_j, which must be greater than 0 because of (2.121).

- if $\overline{\overline{A}}$ is PD, its symmetric part is PD.

- $\overline{\overline{A}}$ is PD exactly when its invariants $\mathrm{tr}\overline{\overline{A}}$, $\mathrm{spm}\overline{\overline{A}}$, $\det\overline{\overline{A}}$ are real and positive.

- $\overline{\overline{A}}$ and $\overline{\overline{B}}$ PD implies $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}$ PD.

- A dyadic of the form $\overline{\overline{A}}\cdot\overline{\overline{A}}^{*T}$ is positive semidefinite and PD if $\overline{\overline{A}}$ is complete.

2.8 Special dyadics

In this section we consider some special classes of dyadics appearing in practical electromagnetic problems. Rotation and reflection dyadics emerge in symmetries of various structures whereas uniaxial and gyrotropic dyadics are encountered when electromagnetic fields in special materials are analysed. Parameters of some media like the sea ice can be approximated in terms of a uniaxial dyadic, while others like magnetized ferrite or magnetoplasma may exhibit properties which can be analysed using gyrotropic dyadics.

2.8.1 Rotation dyadics

In real vector space, the rotation of a vector by an angle θ in the right-hand direction around the axis defined by the unit vector $\mathbf{u}$ can be written in terms of the following dyadic:

$$\overline{\overline{R}}(\mathbf{u},\theta) = \mathbf{uu} + \sin\theta(\mathbf{u}\times\overline{\overline{I}}) + \cos\theta(\overline{\overline{I}} - \mathbf{uu}), \qquad (2.122)$$

It can also be written in the form $e^{\mathbf{u}\times\overline{\overline{I}}\theta}$, as is seen if the dyadic exponential function is written as a Taylor series. The rotation dyadic obeys the properties

$$\overline{\overline{R}}(-\mathbf{u},\theta) = \overline{\overline{R}}(\mathbf{u},-\theta) = \overline{\overline{R}}^T(\mathbf{u},\theta) = \overline{\overline{R}}^{-1}(\mathbf{u},\theta), \qquad (2.123)$$

$$\overline{\overline{R}}(\mathbf{u},\theta_1)\cdot\overline{\overline{R}}(\mathbf{u},\theta_2)=\overline{\overline{R}}(\mathbf{u},\theta_1+\theta_2), \tag{2.124}$$

$$\overline{\overline{R}}(\mathbf{u},\theta)\overset{\times}{\times}\overline{\overline{R}}(\mathbf{u},\theta)=2\overline{\overline{R}}(\mathbf{u},\theta), \tag{2.125}$$

$$\det\overline{\overline{R}}(\mathbf{u},\theta)=\frac{1}{3}\overline{\overline{R}}(\mathbf{u},\theta):\overline{\overline{R}}(\mathbf{u},\theta)=1. \tag{2.126}$$

It is not difficult to show that the properties $\det\overline{\overline{R}}=1$ and $\overline{\overline{R}}^T=\overline{\overline{R}}^{-1}$ uniquely define the form (2.122) of the dyadic $\overline{\overline{R}}(\mathbf{u},\theta)$, so that they could be given as the definition. Also, (2.125) with $\overline{\overline{R}}\neq 0$ would do. In general, θ and/or $\mathbf{u}$ may be complex, which means that the geometrical interpretation is lost. The resulting dyadic is also called the rotation dyadic in the complex case.

The dot product of two rotation dyadics with arbitrary axes and angles is another rotation dyadic. This is seen from

$$(\overline{\overline{R}}_1\cdot\overline{\overline{R}}_2)\overset{\times}{\times}(\overline{\overline{R}}_1\cdot\overline{\overline{R}}_2)=\frac{1}{2}(\overline{\overline{R}}_1\overset{\times}{\times}\overline{\overline{R}}_1)\cdot(\overline{\overline{R}}_2\overset{\times}{\times}\overline{\overline{R}}_2)=2(\overline{\overline{R}}_1\cdot\overline{\overline{R}}_2), \tag{2.127}$$

where use has been made of the identity (2.67). It is not very easy to find the axis and angle of the resulting rotation dyadic. This can be done perhaps most easily by using a representation in terms of a special gyrotropic dyadic. The gyrotropic dyadic was defined in (2.35) and it is the most general non-symmetric dyadic which commutes with its own transpose. The special gyrotropic dyadic of interest here is of the type

$$\overline{\overline{G}}(\mathbf{q})=\overline{\overline{I}}+\mathbf{q}\times\overline{\overline{I}}, \tag{2.128}$$

and the rotation dyadic can be written as

$$\overline{\overline{R}}=\overline{\overline{G}}\cdot(\overline{\overline{G}}^T)^{-1}=(\overline{\overline{I}}+\mathbf{q}\times\overline{\overline{I}})\cdot(\overline{\overline{I}}-\mathbf{q}\times\overline{\overline{I}})^{-1}. \tag{2.129}$$

In fact, because

$$(\overline{\overline{I}}-\mathbf{q}\times\overline{\overline{I}})^{-1}=\frac{1}{1+q^2}(\overline{\overline{I}}+\mathbf{q}\mathbf{q}+\mathbf{q}\times\overline{\overline{I}}), \tag{2.130}$$

(2.129) can be seen to be of the form (2.122) if we write $\mathbf{q}=\mathbf{u}\tan(\theta/2)$, or $\mathbf{u}=\mathbf{q}/\sqrt{\mathbf{q}\cdot\mathbf{q}}$ and $\theta=2\tan^{-1}\sqrt{\mathbf{q}\cdot\mathbf{q}}$. Thus, $\mathbf{q}$ is not a CP vector. For real $\mathbf{q}$, its length q determines the angle of rotation. $q=0$ corresponds to $\theta=0$, $q=\infty$ to $\theta=\pi$ and $q=1$ to $\theta=\pi/2$.

It is straightforward, although a bit tedious, to prove the following identity between the dot product of two rotation dyadics $\overline{\overline{R}}_1$, $\overline{\overline{R}}_2$ and the corresponding $\mathbf{q}$ vectors:

$$\mathbf{q}(\overline{\overline{R}}_1\cdot\overline{\overline{R}}_2)=\frac{\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_2\times\mathbf{q}_1}{1-\mathbf{q}_1\cdot\mathbf{q}_2}. \tag{2.131}$$

This equation shows us that two rotations do not commute in general, since $\overline{\overline{R}}_1 \cdot \overline{\overline{R}}_2$ and $\overline{\overline{R}}_2 \cdot \overline{\overline{R}}_1$ lead to the same $\mathbf{q}$ vector only if $\mathbf{q}_1$ and $\mathbf{q}_2$ are parallel vectors, i.e. the two rotation dyadics have the same axes.

2.8.2 Reflection dyadics

The symmetric dyadic of the form

$$\overline{\overline{C}}(\mathbf{u}) = \overline{\overline{I}} - 2\mathbf{u}\mathbf{u} \tag{2.132}$$

with a unit vector $\mathbf{u}$ is called the reflection dyadic because, when $\mathbf{u}$ is real, mapping the position vector $\mathbf{r}$ through $\overline{\overline{C}} \cdot \mathbf{r} = \mathbf{r} - 2\mathbf{u}(\mathbf{u} \cdot \mathbf{r})$ obviously performs a reflection in the plane $\mathbf{u} \cdot \mathbf{r} = 0$. The reflection dyadic can be also presented as a negative rotation by an angle π around $\mathbf{u}$ as the axis

$$\overline{\overline{C}}(\mathbf{u}) = -\overline{\overline{R}}(\mathbf{u}, \pi), \tag{2.133}$$

as is seen from the definition of the rotation dyadic (2.122). In fact, the unit dyadic $\overline{\overline{I}}$ and the negative of the reflection dyadic are the only rotation dyadics that are symmetric.

The reflection dyadic satisfies

$$\overline{\overline{C}}^2(\mathbf{u}) = \overline{\overline{I}}, \quad \text{or} \quad \overline{\overline{C}}^{-1} = \overline{\overline{C}}, \tag{2.134}$$

$$\overline{\overline{C}}(\mathbf{u}) {\times \atop \times} \overline{\overline{C}}(\mathbf{u}) = -2\overline{\overline{C}}(\mathbf{u}), \tag{2.135}$$

$$\mathrm{tr}\overline{\overline{C}} = 1, \quad \mathrm{spm}\overline{\overline{C}} = -1, \quad \det\overline{\overline{C}} = -1, \tag{2.136}$$

The most general square root of the unit dyadic is not the reflection dyadic, but a dyadic of the form $\pm(\overline{\overline{I}} - 2\mathbf{ab})$ with either $\mathbf{a} \cdot \mathbf{b} = 1$ or $\mathbf{ab} = 0$.

It is easy to see that both rotation and reflection dyadics preserve the inner product of two vectors. In fact, because they both satisfy $\overline{\overline{A}}^T \cdot \overline{\overline{A}} = \overline{\overline{I}}$, we have

$$(\overline{\overline{A}} \cdot \mathbf{a}) \cdot (\overline{\overline{A}} \cdot \mathbf{b}) = \mathbf{a} \cdot (\overline{\overline{A}}^T \cdot \overline{\overline{A}}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}, \tag{2.137}$$

for any vectors $\mathbf{a}$, $\mathbf{b}$. The cross product is transformed differently through rotation

$$(\overline{\overline{R}} \cdot \mathbf{a}) \times (\overline{\overline{R}} \cdot \mathbf{b}) = \frac{1}{2}(\overline{\overline{R}} {\times \atop \times} \overline{\overline{R}}) \cdot (\mathbf{a} \times \mathbf{b}) = \overline{\overline{R}} \cdot (\mathbf{a} \times \mathbf{b}), \tag{2.138}$$

than through reflection

$$(\overline{\overline{C}} \cdot \mathbf{a}) \times (\overline{\overline{C}} \cdot \mathbf{b}) = -\overline{\overline{C}} \cdot (\mathbf{a} \times \mathbf{b}). \tag{2.139}$$

This equation shows us that a reflection transformed electromagnetic field is not an electromagnetic field, because if the electric and magnetic fields are transformed through reflection, the Poynting vector is not. The converse is, however, true for the rotation transformation.

2.8.3 Uniaxial dyadics

By definition, a uniaxial dyadic is of the general form

$$\overline{\overline{D}} = \alpha\overline{\overline{I}} + \mathbf{ab}. \tag{2.140}$$

A condition for $\overline{\overline{D}}$ to be uniaxial is obviously that there exist a scalar α such that $\overline{\overline{D}} - \alpha\overline{\overline{I}}$ is a linear dyadic, or

$$(\overline{\overline{D}} - \alpha\overline{\overline{I}})^{\times}_{\times}(\overline{\overline{D}} - \alpha\overline{\overline{I}}) = 0. \tag{2.141}$$

It is easier to define conditions not for $\overline{\overline{D}}$ itself but for its trace-free part $\overline{\overline{C}} = \overline{\overline{D}} - \frac{\mathrm{tr}\overline{\overline{D}}}{3}\overline{\overline{I}}$. $\overline{\overline{D}}$ is uniaxial exactly when $\overline{\overline{C}}$ is uniaxial, or of the form $\overline{\overline{C}} = \beta\overline{\overline{I}} + \mathbf{ab}$, with $\beta = -\mathbf{a}\cdot\mathbf{b}/3$, whence it satisfies

$$(\overline{\overline{C}} - \beta\overline{\overline{I}})^{\times}_{\times}(\overline{\overline{C}} - \beta\overline{\overline{I}}) = 0. \tag{2.142}$$

Taking the trace operation leaves us an equation for β:

$$\beta^2 = -\frac{\mathrm{spm}\overline{\overline{C}}}{3} = \frac{\overline{\overline{C}} : \overline{\overline{C}}^T}{6}. \tag{2.143}$$

Either root of (2.143) inserted in (2.142) would result in a dyadic equation which is satisfied for any trace-free uniaxial dyadic $\overline{\overline{C}}$. Applying (2.50) and (2.51) gives us the two equations

$$\overline{\overline{C}}^2 \pm j\sqrt{\frac{\mathrm{spm}\overline{\overline{C}}}{3}}\,\overline{\overline{C}} + 2\frac{\mathrm{spm}\overline{\overline{C}}}{3}\overline{\overline{I}} = 0. \tag{2.144}$$

Conversely, if a trace-free dyadic $\overline{\overline{C}}$ satisfies either of (2.144), or what is equivalent, $(\overline{\overline{C}} \pm 2j\sqrt{\frac{\mathrm{spm}\overline{\overline{C}}}{3}}\,\overline{\overline{I}})\cdot(\overline{\overline{C}} \mp j\sqrt{\frac{\mathrm{spm}\overline{\overline{C}}}{3}}\,\overline{\overline{I}}) = 0$, it is uniaxial, because either of the dyadic factors must be a linear dyadic.

The left-hand axis vector $\mathbf{a}$ is the right-hand eigenvector of a uniaxial dyadic and conversely. The corresponding eigenvalue is $\alpha + \mathbf{a}\cdot\mathbf{b}$.

If the vectors $\mathbf{a}$ and $\mathbf{b}$ are fixed, all uniaxial dyadics of the form

$$\overline{\overline{D}}(\alpha|\beta) = \alpha\overline{\overline{I}} + \beta\mathbf{ab} \tag{2.145}$$

form a set of co-uniaxial dyadics defined by the vector pair $(\mathbf{a}, \mathbf{b})$. These dyadics form a linear space, in which all sums and dot products of co-uniaxial dyadics are co-uniaxial. In fact, we can easily show that

$$\overline{\overline{D}}(\alpha_1|\beta_1) + \overline{\overline{D}}(\alpha_2|\beta_2) = \overline{\overline{D}}(\alpha_1 + \alpha_2|\beta_1 + \beta_2), \tag{2.146}$$

$$\overline{\overline{D}}(\alpha_1|\beta_1) \cdot \overline{\overline{D}}(\alpha_2|\beta_2) = \overline{\overline{D}}(\alpha_1\alpha_2|\alpha_1\beta_2 + \alpha_2\beta_1 + \beta_1\beta_2\mathbf{a}\cdot\mathbf{b}). \tag{2.147}$$

From the symmetry of (2.147) we see that the dot product of two co-uniaxial dyadics commutes. Further we have

$$[\overline{\overline{D}}(\alpha_1|\beta_1){}^\times_\times\overline{\overline{D}}(\alpha_2|\beta_2)]^T =$$

$$\overline{\overline{D}}(2\alpha_1\alpha_2 + (\alpha_1\beta_2 + \alpha_2\beta_1)\mathbf{a}\cdot\mathbf{b}| - \alpha_1\beta_2\alpha_2\beta_1). \tag{2.148}$$

Thus, the inverse of a uniaxial dyadic is co-uniaxial:

$$\overline{\overline{D}}^{-1}(\alpha|\beta) = \overline{\overline{D}}(\frac{1}{\alpha}|\frac{-\beta}{\alpha(\alpha+\beta\mathbf{a}\cdot\mathbf{b})}), \tag{2.149}$$

and it exists if the determinant

$$\det\overline{\overline{D}}(\alpha|\beta) = \alpha^2(\alpha + \beta\mathbf{a}\cdot\mathbf{b}) \tag{2.150}$$

is non-zero. Thus, any power of a uniaxial dyadic is co-uniaxial:

$$\overline{\overline{D}}^n(\alpha|\beta) = \overline{\overline{D}}(\alpha^n|\frac{(\alpha+\beta\mathbf{a}\cdot\mathbf{b})^n - \alpha^n}{\mathbf{a}\cdot\mathbf{b}}). \tag{2.151}$$

For $n = 0$, the resulting unit dyadic is obviously co-uniaxial. For $n < 0$, (2.151) is valid only if (2.150) is non-zero. (2.151) is also valid for non-integer values of n, whence it defines co-uniaxial roots of uniaxial dyadics. Of course, there exist other roots as well. For $\mathbf{a}\cdot\mathbf{b} \to 0$, the right-hand side of (2.151) approaches $\overline{\overline{D}}(\alpha^n|n\beta\alpha^{n-1})$

2.8.4 Gyrotropic dyadics

The gyrotropic dyadic $\overline{\overline{G}}$ was defined in Section 2.3.1 as any dyadic commuting with an antisymmetric dyadic $\mathbf{g}\times\overline{\overline{I}}$:

$$\overline{\overline{G}}\times\mathbf{g} = \mathbf{g}\times\overline{\overline{G}}. \tag{2.152}$$

The vector $\mathbf{g}$ is called the axis of $\overline{\overline{G}}$ and all gyrotropic dyadics with the same axis vector are called *coaxially gyrotropic* (CG). In the following $\mathbf{g}$ is assumed NCP and normalized so that $\mathbf{g}\cdot\mathbf{g} = 1$. The most general gyrotropic dyadic has the form (2.35) and is written in the coaxial form

$$\overline{\overline{G}}(\alpha|\beta|\gamma) = \alpha\overline{\overline{I}} + \beta\mathbf{g}\mathbf{g} + \gamma\mathbf{g}\times\overline{\overline{I}}. \tag{2.153}$$

Coaxially gyrotropic dyadics form a linear space with the properties

$$\overline{\overline{G}}(\alpha_1|\beta_1|\gamma_1) + \overline{\overline{G}}(\alpha_2|\beta_2|\gamma_2) = \overline{\overline{G}}(\alpha_1+\alpha_2|\beta_1+\beta_2|\gamma_1+\gamma_2), \tag{2.154}$$

$$\overline{\overline{G}}(\alpha_1|\beta_1|\gamma_1)\cdot\overline{\overline{G}}(\alpha_2|\beta_2|\gamma_2) =$$

$$\overline{\overline{G}}(\alpha_1\alpha_2\gamma_1\gamma_2|\beta_1\beta_2+\gamma_1\gamma_2+\alpha_1\beta_2+\beta_1\alpha_2|\alpha_1\gamma_2+\gamma_1\alpha_2). \tag{2.155}$$

Because of the symmetry in the indices in (2.155), two CG dyadics always commute in the dot product. The square is a special case of (2.155):

$$\overline{\overline{G}}^2(\alpha|\beta|\gamma) = \overline{\overline{G}}(\alpha^2\gamma^2|\beta^2+\gamma^2+2\alpha\beta|2\alpha\gamma) = \overline{\overline{G}}(\rho|\sigma|\tau). \tag{2.156}$$

The CG square-root of a gyrotropic dyadic can be obtained from (2.156) if the parameters α, β, γ are solved in terms of ρ, σ, τ from the equations

$$\rho = \alpha^2\gamma^2, \tag{2.157}$$

$$\sigma = \beta^2+\gamma^2+2\alpha\beta, \tag{2.158}$$

$$\tau = 2\alpha\gamma. \tag{2.159}$$

The solutions for α can be written

$$\alpha = \frac{1}{2}(\sqrt{\rho\pm j\tau}+\sqrt{\rho\mp j\tau}\,), \tag{2.160}$$

and the other parameters are obtained from (2.157)–(2.159).

The double-cross product of two CG dyadics is the CG dyadic

$$\overline{\overline{G}}(\alpha_1|\beta_1|\gamma_1)\overset{\times}{\times}\overline{\overline{G}}(\alpha_2|\beta_2|\gamma_2) = \overline{\overline{G}}((\alpha_1+\beta_1)\alpha_2+(\alpha_2+\beta_2)\alpha_1|$$

$$2\gamma_1\gamma_2-\alpha_1\beta_2-\beta_1\alpha_2|(\alpha_1+\beta_1)\gamma_2+\gamma_1(\alpha_2+\beta_2)). \tag{2.161}$$

This has the special case

$$[\overline{\overline{G}}(\alpha|\beta|\gamma)]^{(2)} = \overline{\overline{G}}(\alpha(\alpha+\beta)|\gamma^2-\alpha\beta|\gamma(\alpha+\beta)). \tag{2.162}$$

The inverse dyadic of a gyrotropic dyadic is CG and of the form

$$\overline{\overline{G}}^{-1}(\alpha|\beta|\gamma) = \overline{\overline{G}}(\frac{\alpha}{\alpha^2+\gamma^2}|\frac{\gamma^2-\alpha\beta}{(\alpha+\beta)(\alpha^2+\gamma^2)}|\frac{-\gamma}{\alpha^2+\gamma^2}). \tag{2.163}$$

This obviously exists for a non-zero determinant of the gyrotropic dyadic:

$$\det\overline{\overline{G}}(\alpha|\beta|\gamma) = (\alpha+\beta)(\alpha^2+\gamma^2). \tag{2.164}$$

The eigenvalues of the gyrotropic dyadic $\overline{\overline{G}}(\alpha|\beta|\gamma)$ of (2.153) are thus $\alpha+\beta$ and $\alpha\pm j\gamma$. The former corresponds to the left and right eigenvector $\mathbf{g}$. The left and right eigenvectors $\mathbf{b}_\pm$, $\mathbf{a}_\pm$ corresponding to the eigenvalues

$\alpha \pm j\gamma$ can be constructed with the aid of conclusions following (2.111), provided $\gamma \neq 0$ and $\beta \mp j\gamma \neq 0$:

$$\mathbf{b}_\pm \mathbf{a}_\pm = \overline{\overline{G}}(\mp j\gamma|\beta|\gamma) {}^\times_\times \overline{\overline{G}}(\mp j\gamma|\beta|\gamma) = 2\gamma(\beta \mp j\gamma)\overline{\overline{G}}(\mp j| \pm j|1) =$$

$$\mp 2j\gamma(\beta \mp j\gamma)(\overline{\overline{I}} - \mathbf{gg} \pm j\mathbf{g} \times \overline{\overline{I}}). \tag{2.165}$$

The dyadic $\overline{\overline{I}} - \mathbf{gg} \pm j\mathbf{g} \times \overline{\overline{I}}$ is really a linear dyadic, because it is of the form $\overline{\overline{G}}(1| - 1| \pm j)$, which inserted in (2.161) gives zero. Explicit expressions for the eigenvectors can be written in terms of any unit vector $\mathbf{v}$ satisfying $\mathbf{v} \cdot \mathbf{v} = 1$ and $\mathbf{v} \cdot \mathbf{g} = 0$, because if the dyadic $\mathbf{vv} {}^\times_\times \mathbf{gg}$ is evaluated through (2.51), we can write

$$\overline{\overline{I}} - \mathbf{gg} \pm j\mathbf{g} \times \overline{\overline{I}} = \mathbf{vv} + \mathbf{gg} {}^\times_\times \mathbf{vv} \pm j(\mathbf{vv} \times \mathbf{g} - \mathbf{v} \times \mathbf{gv}) =$$

$$(\mathbf{v} \mp j\mathbf{v} \times \mathbf{g})(\mathbf{v} \pm j\mathbf{v} \times \mathbf{g}). \tag{2.166}$$

Thus, the eigenvectors corresponding to the eigenvalues $\alpha \pm j\gamma$ are, respectively, $\mathbf{a} = \mathbf{v} \pm j\mathbf{v} \times \mathbf{g}$ and $\mathbf{b} = \mathbf{v} \mp j\mathbf{v} \times \mathbf{g}$, which are easily seen to be circularly polarized. If the gyrotropic dyadic is hermitian, the vector $\mathbf{g}$ must be real and the scalars α, β real and γ imaginary, which makes the eigenvalues real.

This is all easily seen if we write the general gyrotropic dyadic (2.153) in the form

$$\alpha\overline{\overline{I}} + \beta\mathbf{gg} + \gamma\mathbf{g} \times \overline{\overline{I}} = (\alpha + \beta)\mathbf{gg} +$$

$$\frac{\alpha + j\gamma}{2}(\mathbf{v} + j\mathbf{v} \times \mathbf{g})(\mathbf{v} - j\mathbf{v} \times \mathbf{g}) + \frac{\alpha - j\gamma}{2}(\mathbf{v} - j\mathbf{v} \times \mathbf{g})(\mathbf{v} + j\mathbf{v} \times \mathbf{g}). \tag{2.167}$$

Expanding the right-hand side, it is seen that the unit vector $\mathbf{v}$ really has no effect if it satisfies $\mathbf{v} \cdot \mathbf{g} = 0$.

2.9 Two-dimensional dyadics

By definition, a planar dyadic $\overline{\overline{A}}$ is two dimensional, if there is a vector $\mathbf{a} \neq 0$ satisfying

$$\overline{\overline{A}} \cdot \mathbf{a} = \mathbf{a} \cdot \overline{\overline{A}} = 0. \tag{2.168}$$

We restrict $\mathbf{a}$ to be NCP, whence there exists the unit vector $\mathbf{u} = \mathbf{a}/\sqrt{\mathbf{a} \cdot \mathbf{a}}$.

2.9.1 Eigendyadics

The dyadic eigenvalue problem

$$\mathbf{uu}{}^{\times}_{\times}\overline{\overline{A}} = \lambda\overline{\overline{A}} \tag{2.169}$$

is essential in the classification of two-dimensional dyadics. The solutions can be found by operating (2.169) by $\mathbf{uu}{}^{\times}_{\times}$ and applying (2.47) to be $\lambda = 1$ and $\lambda = -1$. The eigendyadics are obviously two dimensional and they span the whole two-dimensional subspace, because any of its members can be expressed as

$$\overline{\overline{A}} = \frac{1}{2}(\overline{\overline{A}} + \overline{\overline{A}}{}^{\times}_{\times}\mathbf{uu}) + \frac{1}{2}(\overline{\overline{A}} - \overline{\overline{A}}{}^{\times}_{\times}\mathbf{uu}). \tag{2.170}$$

It can easily be seen that the first term corresponds to the eigenvalue $\lambda = 1$ and the second one to $\lambda = -1$. Applying (2.51) for $\overline{\overline{A}}{}^{\times}_{\times}\mathbf{uu}$, we can further write (2.170) in the form

$$\overline{\overline{A}} = \frac{1}{2}[(\mathrm{tr}\overline{\overline{A}})\overline{\overline{I}}_t + \overline{\overline{A}} - \overline{\overline{A}}^T] + \frac{1}{2}[\overline{\overline{A}} + \overline{\overline{A}}^T - (\mathrm{tr}\overline{\overline{A}})\overline{\overline{I}}_t], \tag{2.171}$$

with the two-dimensional unit dyadic

$$\overline{\overline{I}}_t = \overline{\overline{I}} - \mathbf{uu}. \tag{2.172}$$

Denoting the dyadic corresponding to 90^o rotation around $\mathbf{u}$ by

$$\overline{\overline{J}} = \mathbf{u} \times \overline{\overline{I}}, \tag{2.173}$$

it is easy to show that the most general antisymmetric two-dimensional dyadic $\overline{\overline{A}} - \overline{\overline{A}}^T$ must be a multiple of $\overline{\overline{J}}$. From the form of the first term in (2.171) it is seen that the most general eigendyadic of (2.169) corresponding to $\lambda = 1$ is a multiple of the two-dimensional (complex) rotation dyadic

$$\overline{\overline{R}}(\mathbf{u}, \theta) = \cos\theta\overline{\overline{I}}_t + \sin\theta\overline{\overline{J}}. \tag{2.174}$$

The eigendyadics corresponding to $\lambda = -1$ are symmetric and trace free, as can be seen from the second term of (2.171) and $\mathrm{tr}\overline{\overline{I}}_t = 2$. They can be interpreted as multiples of (complex) reflection dyadics. Denoting basic reflection dyadics by $\overline{\overline{K}}$ and $\overline{\overline{L}}$, the most general two-dimensional dyadic can thus be written as

$$\overline{\overline{A}} = \alpha\overline{\overline{I}}_t + \beta\overline{\overline{J}} + \gamma\overline{\overline{K}} + \delta\overline{\overline{L}}. \tag{2.175}$$

There is no obvious way to define the dyadics $\overline{\overline{K}}$ and $\overline{\overline{L}}$. Any multiple of a reflection is necessarily of the form $\gamma\overline{\overline{K}} + \delta\overline{\overline{L}}$ and there can be shown to exist a two-dimensional vector $\mathbf{p}$ such that this dyadic is of the form $\mathbf{pp} - \mathbf{uu}{}^\times_\times\mathbf{pp}$, or another two-dimensional vector $\mathbf{q}$ such that the same dyadic is of the form $\mathbf{qq}\times\mathbf{u} + \mathbf{u}\times\mathbf{qq}$. If the vector denoted by $\mathbf{u}$ in the reflection dyadic (2.132) is replaced by $\mathbf{p}/\sqrt{\mathbf{p}\cdot\mathbf{p}}$, its two-dimensional part is most easily seen to be a multiple of the above form.

2.9.2 Base dyadics

Taking an orthonormal base of vectors $\mathbf{u},\ \mathbf{v},\ \mathbf{w}$ satisfying $\mathbf{u}\times\mathbf{v} = \mathbf{w}$, with the two-dimensional sub-base $\mathbf{v},\ \mathbf{w}$, the dyadics can be defined as

$$\overline{\overline{I}}_t = \mathbf{vv} + \mathbf{ww}, \tag{2.176}$$

$$\overline{\overline{J}} = \mathbf{wv} - \mathbf{vw}, \tag{2.177}$$

$$\overline{\overline{K}} = \mathbf{vv} - \mathbf{ww}, \tag{2.178}$$

$$\overline{\overline{L}} = \mathbf{vw} + \mathbf{wv}. \tag{2.179}$$

It must be remembered that $\overline{\overline{I}}_t$ and $\overline{\overline{J}}$ are independent and $\overline{\overline{K}}$, $\overline{\overline{L}}$ dependent on the chosen base. Denoting the base dyadics by $\overline{\overline{B}}_i$, they can be shown to satisfy the following orthogonality conditions:

$$\overline{\overline{B}}_i : \overline{\overline{B}}_j = 2\delta_{ij}, \tag{2.180}$$

whence we may write for any two-dimensional dyadic $\overline{\overline{A}}$ the expansion

$$\overline{\overline{A}} = \frac{1}{2}[(\overline{\overline{A}}:\overline{\overline{I}}_t)\overline{\overline{I}}_t + (\overline{\overline{A}}:\overline{\overline{J}})\overline{\overline{J}} + (\overline{\overline{A}}:\overline{\overline{K}})\overline{\overline{K}} + (\overline{\overline{A}}:\overline{\overline{L}})\overline{\overline{L}}]. \tag{2.181}$$

Because for any two two-dimensional dyadics we have

$$\overline{\overline{A}}{}^\times_\times\overline{\overline{B}} = (\overline{\overline{A}}{}^\times_\times\overline{\overline{B}} : \mathbf{uu})\mathbf{uu} = (\overline{\overline{A}}{}^\times_\times\mathbf{uu}) : \overline{\overline{B}}\mathbf{uu}, \tag{2.182}$$

and with the eigenproblem (2.169) solutions

$$\mathbf{uu}{}^\times_\times\overline{\overline{I}}_t = \overline{\overline{I}}_t,\quad \mathbf{uu}{}^\times_\times\overline{\overline{J}} = \overline{\overline{J}},\quad \mathbf{uu}{}^\times_\times\overline{\overline{K}} = -\overline{\overline{K}},\quad \mathbf{uu}{}^\times_\times\overline{\overline{L}} = -\overline{\overline{L}}, \tag{2.183}$$

the double-cross product of different base dyadics can be obtained from the double-dot product expression (2.180). Thus, $\overline{\overline{B}}_i{}^\times_\times\overline{\overline{B}}_j = 0$ if $i \neq j$ and

$$\overline{\overline{I}}_t{}^\times_\times\overline{\overline{I}}_t = 2\mathbf{uu},\quad \overline{\overline{J}}{}^\times_\times\overline{\overline{J}} = 2\mathbf{uu},\quad \overline{\overline{K}}{}^\times_\times\overline{\overline{K}} = -2\mathbf{uu},\quad \overline{\overline{L}}{}^\times_\times\overline{\overline{L}} = -2\mathbf{uu}. \tag{2.184}$$

The dot product $\overline{\overline{B}}_i\cdot\overline{\overline{B}}_j$ of different base dyadics $\overline{\overline{B}}_i$ (in the vertical columns) and $\overline{\overline{B}}_j$ (in the horizontal rows) can be obtained from the following multiplication table:

$i\ ,\ j$	$\overline{\overline{I}}_t$	$\overline{\overline{J}}$	$\overline{\overline{K}}$	$\overline{\overline{L}}$
$\overline{\overline{I}}_t$	$\overline{\overline{I}}_t$	$\overline{\overline{J}}$	$\overline{\overline{K}}$	$\overline{\overline{L}}$
$\overline{\overline{J}}$	$\overline{\overline{J}}$	$-\overline{\overline{I}}_t$	$\overline{\overline{L}}$	$-\overline{\overline{K}}$
$\overline{\overline{K}}$	$\overline{\overline{K}}$	$-\overline{\overline{L}}$	$\overline{\overline{I}}_t$	$-\overline{\overline{J}}$
$\overline{\overline{L}}$	$\overline{\overline{L}}$	$\overline{\overline{K}}$	$\overline{\overline{J}}$	$\overline{\overline{I}}_t$

2.9.3 The inverse dyadic

Further, from (2.51) we can write for any two-dimensional dyadic

$$\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}} = 2\overline{\overline{A}}^{T2} - 2(\mathrm{tr}\overline{\overline{A}})\overline{\overline{A}}^{T} + 2(\mathrm{spm}\overline{\overline{A}})\overline{\overline{I}} = 2(\mathrm{spm}\overline{\overline{A}})\mathbf{uu}, \qquad (2.185)$$

which is the two-dimensional counterpart of the Cayley–Hamilton identity (2.65):

$$\overline{\overline{A}}^2 - (\mathrm{tr}\overline{\overline{A}})\overline{\overline{A}} + (\mathrm{spm}\overline{\overline{A}})\overline{\overline{I}}_t = 0. \qquad (2.186)$$

The function $\mathrm{spm}\overline{\overline{A}}$ serves as a two-dimensional determinant function, because $\det\overline{\overline{A}} = 0$ for planar dyadics such as these. The two-dimensional inverse of the dyadic $\overline{\overline{A}}$ can be directly written from (2.186) by writing it in the form $\overline{\overline{A}}\cdot[\overline{\overline{A}} - (\mathrm{tr}\overline{\overline{A}})\overline{\overline{I}}_t] = [\overline{\overline{A}} - (\mathrm{tr}\overline{\overline{A}})\overline{\overline{I}}_t]\cdot\overline{\overline{A}} = -(\mathrm{spm}\overline{\overline{A}})\overline{\overline{I}}_t$, as

$$\overline{\overline{A}}^{-1} = \frac{(\mathrm{tr}\overline{\overline{A}})\overline{\overline{I}}_t - \overline{\overline{A}}}{\mathrm{spm}\overline{\overline{A}}} = \frac{\overline{\overline{A}}^{T}{}^{\times}_{\times}\mathbf{uu}}{\mathrm{spm}\overline{\overline{A}}}. \qquad (2.187)$$

This is a generalization of the planar inverse (2.74), because $\overline{\overline{A}}^{(2)} = (\mathrm{spm}\overline{\overline{A}})\mathbf{uu}$. The two-dimensional inverse can be written in terms of the expansion (2.175) as

$$\overline{\overline{A}}^{-1} = \frac{\alpha\overline{\overline{I}}_t - \beta\overline{\overline{J}} - \gamma\overline{\overline{K}} - \delta\overline{\overline{L}}}{\alpha^2 + \beta^2 - \gamma^2 - \delta^2}. \qquad (2.188)$$

2.9.4 Dyadic square roots

Finally, let us find the two-dimensional square root of a two-dimensional dyadic, i.e. the solution to the equation $\overline{\overline{X}}\cdot\overline{\overline{X}} = \overline{\overline{A}}$. The problem is much easier than the general three-dimensional case. From (2.67) we have $(\overline{\overline{X}}{}^{\times}_{\times}\overline{\overline{X}})^2 = 2\overline{\overline{X}}^2{}^{\times}_{\times}\overline{\overline{X}}^2$ and, hence, $\mathrm{spm}\overline{\overline{A}} = (\mathrm{spm}\overline{\overline{X}})^2$. From (2.186) we have

$$\overline{\overline{A}} = \overline{\overline{X}}^2 = (\mathrm{tr}\overline{\overline{X}})\overline{\overline{X}} - (\mathrm{spm}\overline{\overline{X}})\overline{\overline{I}}_t = (\mathrm{tr}\overline{\overline{X}})\overline{\overline{X}} \mp \sqrt{\mathrm{spm}\overline{\overline{A}}}\,\overline{\overline{I}}_t. \qquad (2.189)$$

Taking the trace operation allows us to solve for $\mathrm{tr}\overline{\overline{X}}$ with the result

$$\mathrm{tr}\overline{\overline{X}} = \sqrt{\mathrm{tr}\overline{\overline{A}} \pm 2\sqrt{\mathrm{spm}\overline{\overline{A}}}}. \tag{2.190}$$

This actually stands for two pairs of values with two different signs attached to the first square root independent of the $\pm$ sign. Thus, (2.189) can be solved to give two pairs of solutions without showing the first double sign:

$$\overline{\overline{A}}^{1/2} = \frac{\overline{\overline{A}} \pm \sqrt{\mathrm{spm}\overline{\overline{A}}}\,\overline{\overline{I}}_t}{\sqrt{\mathrm{tr}\overline{\overline{A}} \pm 2\sqrt{\mathrm{spm}\overline{\overline{A}}}}}. \tag{2.191}$$

Here, the two $\pm$ signs correspond to each other.

The solution fails if the denominator vanishes. This corresponds to special solutions satisfying $\mathrm{tr}\overline{\overline{X}} = 0$, which from (2.189) are seen to be valid only for dyadics of the form $\overline{\overline{A}} = \alpha\overline{\overline{I}}_t$. Square roots in this case are obtained through the following expressions:

$$\overline{\overline{X}}{}^{\times}_{\times}\overline{\overline{X}} = (\overline{\overline{X}}{}^{\times}_{\times}\overline{\overline{X}} : \mathbf{uu})\mathbf{uu} = 2(\mathrm{spm}\overline{\overline{X}})\mathbf{uu} =$$

$$[(\mathrm{tr}\overline{\overline{X}})^2 - \mathrm{tr}(\overline{\overline{X}}^2)]\mathbf{uu} = -2\alpha\mathbf{uu}, \tag{2.192}$$

$$(\overline{\overline{X}} \pm \sqrt{\alpha}\overline{\overline{I}}_t){}^{\times}_{\times}(\overline{\overline{X}} \pm \sqrt{\alpha}\overline{\overline{I}}_t) = -2\alpha\mathbf{uu} \pm 2\sqrt{A}(\mathrm{tr}\overline{\overline{X}})\mathbf{uu} + 2\alpha\mathbf{uu} = 0. \tag{2.193}$$

Thus, the dyadic $\overline{\overline{X}} \pm \sqrt{\alpha}\overline{\overline{I}}_t$ in this case must be linear and we can write the corresponding solutions in terms of any two two-dimensional vectors $\mathbf{a}$, $\mathbf{b}$ satisfying $\mathbf{a}\cdot\mathbf{b} \neq 0$

$$\overline{\overline{X}} = \pm\sqrt{\alpha}(\overline{\overline{I}}_t - 2\frac{\mathbf{ab}}{\mathbf{a}\cdot\mathbf{b}}). \tag{2.194}$$

Obviously, this solution satisfies both $\overline{\overline{X}}^2 = \alpha\overline{\overline{I}}_t$ and $\mathrm{tr}\overline{\overline{X}} = 0$. It also includes as special cases multiples of the reflection dyadics $\overline{\overline{X}} = \mp\sqrt{\alpha}\overline{\overline{K}}$ and $= \mp\sqrt{\alpha}\overline{\overline{L}}$ if we choose $\mathbf{a} = \mathbf{b} = \mathbf{w}$ and $\mathbf{a} = \mathbf{b} = \mathbf{v} - \mathbf{w}$, respectively, where $\mathbf{v}$, $\mathbf{w}$ are orthonormal two-dimensional vectors.

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