1

Fundamental Concepts

The material presented in this book is generally considered to be at the level of a graduate student or a practising engineer. Thus, the reader is assumed to have been exposed to electromagnetics either through a suitable graduate course or practical experience. In this chapter, fundamental electromagnetic concepts and theorems are presented along with the notation used throughout this text so that a common base is available to all readers.

Many good texts on general electromagnetics principles and techniques are currently in print. Some are considered classical treatises such as [1]–[2] while others are more recent vintage such as [3]–[4]. Although this chapter presents the minimal introductory material necessary for the study of the finite element method as applied to electromagnetics, the interested reader is encouraged to consult these references for a more complete treatment of electromagnetics.

Upon assumption of a harmonic field, the phasor or time-harmonic form of Maxwell's equations are presented along with complex material definitions which permit the incorporation of loss mechanisms. The natural boundary conditions are derived followed by fictitious, though useful, approximate resistive and impedance conditions. Electrostatic and magnetostatic formulations are discussed for use in later examples of the finite element method. Several useful electromagnetic concepts are presented including the Poynting, uniqueness, superposition, and duality theorems. Time-harmonic Maxwell's equations will be covered first.

1.1 TIME-HARMONIC MAXWELL'S EQUATIONS

Maxwell's equations were originally written as a set of coupled, time-dependent integral equations. However, of primary interest in this text is the study of harmonically varying fields (i.e., frequency domain) with an angular frequency of

$\omega = 2\pi f$ rad/sec since the finite element method for electromagnetics utilizes time-harmonic fields. The interested reader is referred to one of the excellent general electromagnetics texts cited in this chapter's introduction for a discussion of the time-dependent form of Maxwell's equations. For our purposes, we begin with the time-harmonic form of Maxwell's equations.

The time-harmonic electric field is related to the time-dependent electric field (assuming that $j = \sqrt{-1}$) by

$$\begin{aligned}\mathcal{E}(x, y, z; t) &= \mathrm{Re}[\mathbf{E}(x, y, z)e^{j\omega t}] \\ &= \hat{x}E_{x0}\cos(\omega t + \phi_x) + \hat{y}E_{y0}\cos(\omega t + \phi_y) + \hat{z}E_{z0}\cos(\omega t + \phi_z) \end{aligned} \tag{1.1}$$

where the complex vector

$$\mathbf{E}(x, y, z) = \hat{x}E_{x0}e^{j\phi_x} + \hat{y}E_{y0}e^{j\phi_y} + \hat{z}E_{z0}e^{j\phi_z} \tag{1.2}$$

is referred to as the field phasor, and similar representations can be employed for the other field quantities. Introducing these into the time-dependent Maxwell's equations [5], we obtain a simplified set

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E} \tag{1.3}$$

$$\nabla \times \mathbf{E} = -\mathbf{M} - j\omega\mu\mathbf{H} \tag{1.4}$$

$$\nabla \cdot (\mu\mathbf{H}) = \rho_m \tag{1.5}$$

$$\nabla \cdot (\epsilon\mathbf{E}) = \rho \tag{1.6}$$

where the corresponding vector field and current phasors are

$\mathbf{E}$ = electric field intensity in volts/meter (V/m)

$\mathbf{H}$ = magnetic field intensity in amperes/meter (A/m)

$\mathbf{J}$ = electric current density in amperes/meter^2 (A/m^2)

$\mathbf{M}$ = magnetic current density in volts/meter^2 (V/m^2)

and the two scalar charge phasors are

ρ = electric charge density in coulombs/meter^3 (C/m^3)

ρ_m = magnetic charge density in webers/meter^3 (Wb/m^3)

Both the magnetic current density ($\mathbf{M}$) and the magnetic charge density (ρ_m) are fictitious quantities introduced for convenience.

Implied in these time-harmonic equations are constitutive relations for an isotropic medium

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\epsilon_r\mathbf{E} \tag{1.7}$$

$$\mathbf{B} = \mu\mathbf{H} = \mu_0\mu_r\mathbf{H} \tag{1.8}$$

$$\mathbf{J} = \sigma\mathbf{E} \tag{1.9}$$

$$\mathbf{M} = \sigma_m\mathbf{H} \tag{1.10}$$

where two additional phasors

$\mathbf{D}$ = electric flux density in coulombs/meter^2 (C/m^2)

$\mathbf{B}$ = magnetic field density in webers/meter^2 (Wb/m^2)

are related to **E** and **H**, respectively. These constitutive equations are an important link between the original time-dependent form of Maxwell's equations and the time-harmonic form used in the finite element method. Similarly, the phasor forms of the continuity equations [5] are given by

$$\nabla \cdot \mathbf{J} + j\omega\rho = 0 \tag{1.11}$$

$$\nabla \cdot \mathbf{M} + j\omega\rho_m = 0 \tag{1.12}$$

In (1.3)–(1.12), the material constants are given by

ϵ_0 = free space permittivity = 8.854×10^{-12} farads/meter (F/m)

μ_0 = free space permeability = $4\pi \times 10^{-7}$ henrys/meter (H/m)

ϵ_r = medium's relative permittivity constant

μ_r = medium's relative permeability constant

σ = electric current conductivity in mhos/m ($\frac{1}{\Omega}$/m)

σ_m = magnetic current conductivity in ohms/m (Ω/m)

The first two of these (ϵ_0 and μ_0) are fundamental constants while the others describe the specific material. For example, ϵ_r is a measure of the material's electric storage capacity while σ is a measure of the material's ability to conduct electric currents or alternatively as an Ohmic loss mechanism. The relative permeability μ_r and magnetic conductivity σ_m are the magnetic field analogues to ϵ_r and σ, respectively. For the purposes of the finite element method, all four of these material quantities may vary spatially (inhomogeneous) and spectrally (dispersive).

The current densities **J** and **M** appearing in (1.3) and (1.4) do not include the presence of impressed sources. In general, **J** and **M** can be written as a sum of impressed (or excitation) and induced (or conduction) currents as

$$\mathbf{J} = \mathbf{J}_i + \mathbf{J}_c = \mathbf{J}_i + \sigma\mathbf{E} \tag{1.13}$$

$$\mathbf{M} = \mathbf{M}_i + \mathbf{M}_c = \mathbf{M}_i + \sigma_m\mathbf{H} \tag{1.14}$$

where the subscript "i" denotes impressed currents while the subscript "c" refers to conduction currents. When these are substituted into (1.3) and (1.4), the familiar form of Maxwell's equations are obtained

$$\nabla \times \mathbf{H} = \mathbf{J}_i + j\omega\epsilon_0\dot{\epsilon}_r\mathbf{E} \tag{1.15}$$

$$\nabla \times \mathbf{E} = -\mathbf{M}_i - j\omega\mu_0\dot{\mu}_r\mathbf{H} \tag{1.16}$$

where

$$\dot{\epsilon}_r = \epsilon_r - j\frac{\sigma}{\omega\epsilon_0} = \epsilon' - j\epsilon'' = \epsilon'(1 - j\tan\delta) \tag{1.17}$$

and

$$\dot{\mu}_r = \mu_r - j\frac{\sigma_m}{\omega\mu_0} = \mu' - j\mu'' = \mu'(1 - j\tan\delta_m) \tag{1.18}$$

represent equivalent relative complex permittivity and permeability constants. For notational convenience, the dot over the relative constitutive parameters will be omitted with the understanding that these still represent all possible material losses.

Any one of the representations given in (1.17) and (1.18) are likely to be found in the literature with the quantities

$$\tan\delta = \frac{\epsilon''}{\epsilon'} \tag{1.19}$$

$$\tan\delta_m = \frac{\mu''}{\mu'} \tag{1.20}$$

referred to as the material's electric and magnetic loss tangents, respectively.

To summarize, Maxwell's equations in phasor form for isotropic media are

$$\nabla \times \mathbf{H} = \mathbf{J}_i + j\omega\epsilon\mathbf{E} \tag{1.21}$$

$$\nabla \times \mathbf{E} = -\mathbf{M}_i - j\omega\mu\mathbf{H} \tag{1.22}$$

$$\nabla \cdot (\mu\mathbf{H}) = -(\nabla \cdot \mathbf{M}_i)/j\omega \tag{1.23}$$

$$\nabla \cdot (\epsilon\mathbf{E}) = -(\nabla \cdot \mathbf{J}_i)/j\omega \tag{1.24}$$

where the phasor form of (1.11) and (1.12) was employed to rewrite (1.5) and (1.6) as given above.

The corresponding integral representations of (1.21)–(1.24) are

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int\int_S (\mathbf{J}_i + j\omega\epsilon\mathbf{E}) \cdot d\mathbf{S} \tag{1.25}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int\int_S (\mathbf{M}_i + j\omega\mu\mathbf{H}) \cdot d\mathbf{S} \tag{1.26}$$

$$\oiint_{S_c} \mu\mathbf{H} \cdot d\mathbf{S} = -\int\int\int_V \frac{\nabla \cdot \mathbf{M}_i}{j\omega}\, dV = \int\int\int_V \rho_m\, dV \tag{1.27}$$

$$\oiint_{S_c} \epsilon\mathbf{E} \cdot d\mathbf{S} = -\int\int\int_V \frac{\nabla \cdot \mathbf{J}_i}{j\omega}\, dV = \int\int\int_V \rho\, dV \tag{1.28}$$

where C is the contour bounding the open surface S illustrated in Fig. 1.1 and $d\mathbf{S} = \hat{n}\, dS$. The circle through the single integral indicates integration over a closed contour, whereas the same symbol through the surface integral denotes integration over the closed surface S_c which encloses the corresponding volume V. The surface S associated with the integrals (1.25) and (1.26) is completely unrelated to S_c which encloses the volume V.

Expressions (1.21) and (1.22) imply six scalar equations for the solution of the six components associated with **E** and **H**. Thus, for time-harmonic fields, (1.21) and

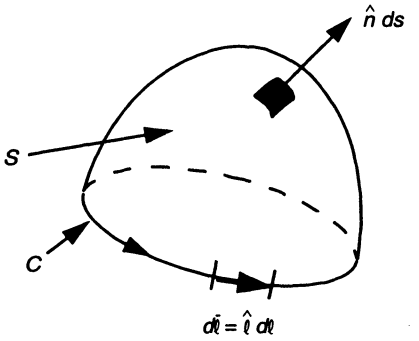


Figure 1.1 Illustration of the differential element ds and the contour C.

(1.22) or (1.25) and (1.26) are sufficient for a solution of the electric and magnetic fields. The divergence conditions (1.23) and (1.24), or their integral counterparts, (1.27) and (1.28), are superfluous. In fact, these two equations follow directly from the first two upon taking their divergence and observing that $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ [6] for any vector **A**. Equations (1.21)–(1.28) can be easily modified for anisotropic material as well. This requires that $\epsilon\mathbf{E}$ and $\mu\mathbf{H}$ be replaced by $\bar{\bar{\epsilon}} \cdot \mathbf{E}$ and $\bar{\bar{\mu}} \cdot \mathbf{H}$, respectively, where $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$ represent 3×3 tensors [7].

In this text, open scattering and radiation problems will be considered. Consequently, any valid and unique solution of the electric and/or magnetic fields must also satisfy the Sommerfeld radiation condition, which describes the field behavior at infinity

$$\lim_{r\to\infty} r\left[\nabla \times \begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} + jk_0\hat{r} \times \begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix}\right] = 0 \tag{1.29}$$

where k_0 is the free-space wavenumber ($k_0 = 2\pi/\lambda_0 = \omega\sqrt{\epsilon_0\mu_0}$) and λ_0 is the corresponding free space wavelength. This simply states that the field is outgoing and of the form e^{-jk_0r}/r as $r \to \infty$.

1.2 WAVE EQUATION

Ampère-Maxwell's Law (1.21) and Faraday's Law (1.22) are independent first-order vector equations, and as noted earlier, they lead to a unique solution subject to the specified boundary conditions. They may be combined together to yield a single second-order vector equation in terms of **E** or **H** known as the wave equation. The finite element method is used to numerically approximate the solution of the wave equation.

Specifically by taking the curl of (1.21) or (1.22) and making use of the other, the following vector wave equations are obtained:

$$\begin{aligned}\nabla \times \left\{\frac{\nabla \times \mathbf{E}}{\mu_r}\right\} - k_0^2\epsilon_r\mathbf{E} &= -jk_0Z_0\mathbf{J} - \nabla \times \left\{\frac{\mathbf{M}}{\mu_r}\right\}\\ \nabla \times \left\{\frac{\nabla \times \mathbf{H}}{\epsilon_r}\right\} - k_0^2\mu_r\mathbf{H} &= -jk_0Y_0\mathbf{M} + \nabla \times \left\{\frac{\mathbf{J}}{\epsilon_r}\right\}\end{aligned} \tag{1.30}$$

where the upper set of equations are for solution of the electric field while the lower set is for solution of the magnetic field. In (1.30), ϵ_r denotes the relative permittivity of the media and μ_r indicates the relative permeability of the media. For free space, these two quantities are both unity. Also, $Z_0 = 1/Y_0 = \sqrt{\mu_0/\epsilon_0}$ is the free space wave impedance. In materials other than free space, the wave impedance and wavenumber are given by $Z = 1/Y = \sqrt{\mu/\epsilon}$ and $k = \omega\sqrt{\epsilon\mu}$, respectively.

Utilizing the vector identity

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi\nabla \times \mathbf{A} \tag{1.31}$$

(1.30) can be written in a more convenient form

$$\left(\frac{1}{\mu_r}\right)\nabla \times \nabla \times \mathbf{E} - k_0^2\epsilon_r\mathbf{E} + \left[\nabla\left(\frac{1}{\mu_r}\right) \times \nabla \times \mathbf{E}\right] = -jk_0Z_0\mathbf{J} - \nabla \times \left\{\frac{\mathbf{M}}{\mu_r}\right\} \tag{1.32}$$

for the electric field and

$$\left(\frac{1}{\epsilon_r}\right)\nabla \times \nabla \times \mathbf{H} - k_0^2 \mu_r \mathbf{H} + \left[\nabla\left(\frac{1}{\epsilon_r}\right) \times \nabla \times \mathbf{H}\right] = -jk_0 Y_0 \mathbf{M} + \nabla \times \left\{\frac{\mathbf{J}}{\epsilon_r}\right\} \tag{1.33}$$

for the magnetic field. The significance of this form of the wave equation is that for homogeneous materials, the terms within the bracket are zero. Most implementations of the finite element method assume a homogeneous material within each finite element and hence this bracketed term can be set to zero for those cases.

Another important version of (1.30) for homogeneous media is obtained by utilizing the vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla\nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} \tag{1.34}$$

in (1.32) to get

$$-\nabla(\nabla \cdot \mathbf{E}) + \nabla^2 \mathbf{E} + k_0^2 \epsilon_r \mu_r \mathbf{E} = jk_0 Z_0 \mu_r \mathbf{J} + \nabla \times \mathbf{M} \tag{1.35}$$

In a source-free region, (1.35) simplifies to

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0 \tag{1.36}$$

These equations represent three vector field components each of which satisfies the Helmholtz or scalar wave equation

$$\nabla^2 \psi + k^2 \psi = 0 \tag{1.37}$$

where ψ denotes E_x, E_y, E_z. Similar partial differential equations can be formed for the magnetic field from (1.33).

1.3 ELECTROSTATICS AND MAGNETOSTATICS

Although for the majority of this text we are concerned with dynamic electromagnetics, some examples of static electromagnetics are included to illustrate basic finite element principles. Therefore, we present the basic equations of electrostatics and magnetostatics, namely Poisson's equation and the potential relations.

In this section, we present a basic review of electrostatic and magnetostatic expressions sufficient for this text. It is not a comprehensive review of either electro- or magnetostatics, and the interested reader is encouraged to study [8] or [9] for further information. We begin with electrostatics.

1.3.1 Electrostatics

The fundamental equations of electrostatics are forms of Faraday's equation (1.4) and Gauss' Law (1.6), namely

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 \\ \nabla \cdot (\epsilon \mathbf{E}) &= \rho_v \end{aligned} \tag{1.38}$$

The electric field can be expressed in terms of a gauge condition involving a scalar quantity, $\phi(\mathbf{r})$

$$\mathbf{E} = -\nabla\phi(\mathbf{r}) \tag{1.39}$$

With this field expression (1.39), (1.38) reduces to Poisson's equation (here given in terms of general scalar fields and sources since Poisson's equation is also used for magnetostatics)

$$\nabla \cdot [\epsilon \nabla \phi(\mathbf{r})] = f(\mathbf{r}) \tag{1.40}$$

For electrostatics, the scalar quantity is a voltage ($\phi(\mathbf{r}) = V(\mathbf{r})$) and the sources are volume charges ($f(\mathbf{r}) = -\rho_v(\mathbf{r})$), hence (1.40) reduces to

$$\nabla \cdot [\epsilon \nabla V(\mathbf{r})] = -\rho_v(\mathbf{r}) \tag{1.41}$$

Solution of (1.41), subject to the appropriate boundary conditions (Dirichlet, Neumann, and/or impedance conditions), is equivalent to solving the original Maxwell's equations.

Solution of (1.41) is often accomplished either in closed form or numerically. Closed form solutions are available for only a limited number of boundary conditions [9]. Hence, it is usually more practical to employ a numerical method such as the finite element or boundary element methods.

The potential attributed to a volume charge is given by the integral relation

$$V^i(\mathbf{r}) = \iiint_V \frac{\rho_v(\mathbf{r}')}{\epsilon} G^{3D}(\mathbf{r}, \mathbf{r}')\, dV' \tag{1.42}$$

where the three-dimensional static Green's function is given by

$$G^{3D}(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi R} \tag{1.43}$$

and volume charges are denoted by ρ_v. This representation is the solution of (1.41) in unbounded space and a pictorial relationship of the primed and unprimed parameters is shown in Fig. 1.2.

For two-dimensional situations (e.g., the sources of excitation run from $z = -\infty$ to $z = \infty$ and are invariant with respect to z), the following potential integral is appropriate

$$V^i(\mathbf{r}) = \iint_S \frac{\rho_s(\mathbf{r}')}{\epsilon} G^{2D}(\mathbf{r}, \mathbf{r}')\, dS' \tag{1.44}$$

where ρ_s denotes surface charges and G^{2D} is the two-dimensional static Green's function

$$G^{2D}(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \ln\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) \tag{1.45}$$

These integral relations are used to determine the potential, V^i, at some point in space due to an impressed charge distribution. They are used to derive integral equations for the total potential due to an impressed source subject to boundary conditions on surfaces within the domain. Such an integral equation (for surface problems) is given by

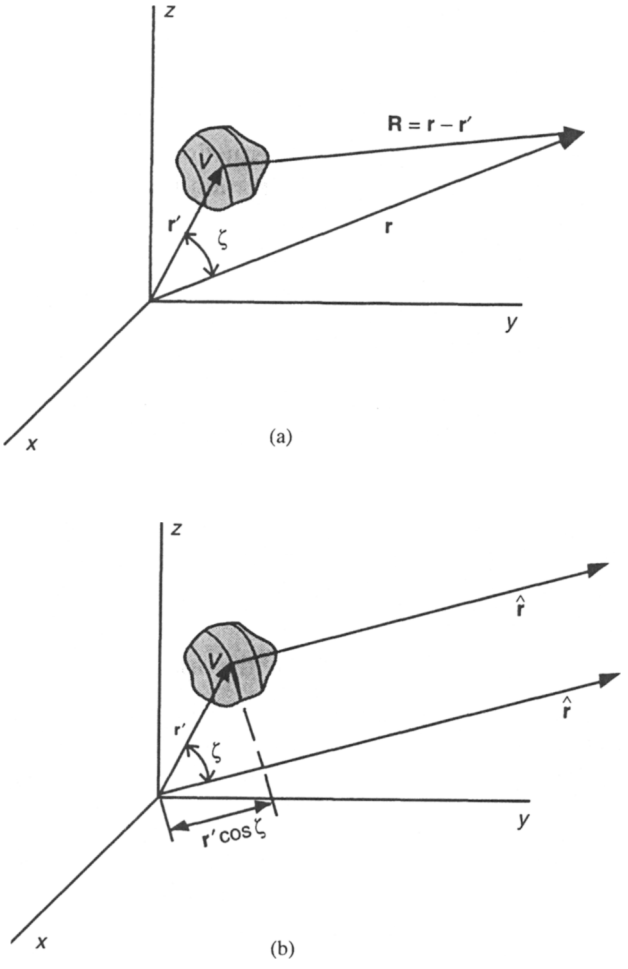


Figure 1.2 Illustration of the geometrical parameters associated with field representations; (a) near zone setup; (b) far zone setup.

$$V(\mathbf{r}) = V^i(\mathbf{r}) - \int\int_S \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial V(\mathbf{r}')}{\partial n'} - \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} V(\mathbf{r}') \right] dS' \tag{1.46}$$

where $\partial V(\mathbf{r})/\partial n = \hat{n} \cdot \nabla V(\mathbf{r})$, $\partial G(\mathbf{r}, \mathbf{r}')/\partial n' = \hat{n}' \cdot \nabla' G(\mathbf{r}, \mathbf{r}') = -\hat{n}' \cdot \nabla G(\mathbf{r}, \mathbf{r}')$ and $\hat{n}'$ denotes the outward normal to the integration surface S. Note that $\hat{n}' = n(\mathbf{r}')$ implies that the unit normal is a function of the integration variables, where $\hat{n} = \hat{n}(\mathbf{r})$ is a function of the observation variables.

For perfect electric conductors, (1.46) can be rewritten in terms of unknown surface charges. Specifically, by making use of (1.39), and relation $\rho_s/\epsilon = \hat{n} \cdot \mathbf{E} = -\hat{n} \cdot \nabla V(\mathbf{r})$, we obtain the usual expression

$$V(\mathbf{r}) = V^i(\mathbf{r}) + \int\int_S \left[G(\mathbf{r}, \mathbf{r}') \frac{\rho_s(\mathbf{r}')}{\epsilon} \right] dS' \tag{1.47}$$

where G represents either (1.43) or (1.45), as appropriate.

1.3.2 Magnetostatics

The solution of Maxwell's equations for a stationary magnetic field is similar to the procedure given above for electrostatics. In this case, Ampère's and Gauss' Magnetic Laws are

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \tag{1.48}$$

where the static field density is assumed to be related to the magnetic field intensity by the expression

$$\mathbf{B} = \mu \mathbf{H} \tag{1.49}$$

Note that in (1.48), we have not assumed a fictitious magnetic charge density. Rather, the fundamental sources of static magnetic fields are currents, **J**.

We can define a magnetic vector potential in terms of these currents

$$\mathbf{A}(\mathbf{r}) = \mu \iiint_V \mathbf{J}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}')\, dV' \tag{1.50}$$

where the Green's function is the same three-dimensional function used for electrostatics (1.43) or the two-dimensional function (1.45). For the latter case, the integral must be reduced to a two-dimensional one over the domain of **J**. With the introduction of this vector potential, solution of (1.48) with (1.49) yields the expression

$$\mathbf{B} = \nabla \times \mathbf{A} = -\mu \iiint_V \mathbf{J}(\mathbf{r}') \times \nabla G(\mathbf{r}, \mathbf{r}')\, dV' \tag{1.51}$$

The derivation of (1.51) can be found in most introductory electromagnetic texts [2, 3, 5], and similar expressions are possible for two-dimensional currents where the integration is taken over a surface rather than a volume.

Integral equations may be formed for magnetostatics in a similar manner to electrostatics and the interested reader is referred to [9].

1.4 SURFACE EQUIVALENCE

Surface equivalent currents are very useful in the formulation and execution of a numerical solution of Maxwell's equations. Their introduction can be readily justified in the context of the surface equivalence principle, e.g., *two sources that produce the same field within a region are said to be equivalent within that region.*

The surface equivalence principle states that the field exterior (or interior) to a given (possibly fictitious) surface may be exactly represented by equivalent currents placed on that surface and allowed to radiate into the region external (or internal) to that surface. For the exterior case, these equivalent currents are given in terms of the total exterior (**E**, **H**) fields while the interior fields are assumed to be zero (this is Love's equivalence principle). The appropriate currents for representing the fields exterior to the surface are given by

$$\hat{n} \times \mathbf{H} = \mathbf{J}$$
$$\mathbf{E} \times \hat{n} = \mathbf{M} \tag{1.52}$$

For the interior fields, the negative of (1.52) are used. The radiated fields due to these equivalent currents are given by the integral expressions

$$\mathbf{E}(\mathbf{r}) = -\oiint_{S_c} \nabla \times \bar{\bar{G}}(R) \cdot \hat{n}' \times \mathbf{E}(\mathbf{r}')\, dS' + jk_0 Z_0 \oiint_{S_c} \bar{\bar{G}}(R) \cdot \hat{n}' \times \mathbf{H}(\mathbf{r}')\, dS' \tag{1.53}$$

$$\mathbf{H}(\mathbf{r}) = -\oiint_{S_c} \nabla \times \bar{\bar{G}}(R) \cdot \hat{n}' \times \mathbf{H}(\mathbf{r}')\, dS' - jk_0 Y_0 \oiint_{S_c} \bar{\bar{G}}(R) \cdot \hat{n}' \times \mathbf{E}(\mathbf{r}')\, dS' \tag{1.54}$$

where $R = |\mathbf{r} - \mathbf{r}'|$, $\mathbf{r}$ and $\mathbf{r}'$ denote the observation and integration point, respectively, and $\hat{n}'$ is the outward directed unit normal at the point $\mathbf{r}'$. The closed surface on which the equivalence theorem is applied is denoted by S_c. These geometrical quantities are illustrated in Figs. 1.2 and 1.3. In (1.53) and (1.54), a dyadic Green's function is required which at least satisfies the radiation condition (1.29). When $\mathbf{J}$ and $\mathbf{M}$ are radiating in free space, the dyadic Green's function is given in closed form by

$$\bar{\bar{G}}_0 = -\left(\bar{\bar{I}} + \frac{\nabla\nabla}{k_0^2}\right) G_0(R) \tag{1.55}$$

where $\bar{\bar{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$ is the unit dyad and the corresponding scalar Green's function is given by

$$G_0(R) = G_0(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk_0 R}}{4\pi R} \tag{1.56}$$

Also,

$$\nabla \times \bar{\bar{G}}_0(\mathbf{r}, \mathbf{r}') = -\nabla \times [\bar{\bar{I}} G_0(\mathbf{r}, \mathbf{r}')] = -\nabla G_0(\mathbf{r}, \mathbf{r}') \times \bar{\bar{I}} \tag{1.57}$$

implying $\nabla \times \bar{\bar{G}}_0(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}(\mathbf{r}') = -\nabla G_0(\mathbf{r}, \mathbf{r}') \times \mathbf{M}(\mathbf{r}')$.

When (1.55)–(1.57) are introduced into (1.53) and (1.54), and after the use of common vector and dyadic identities, we obtain the representations

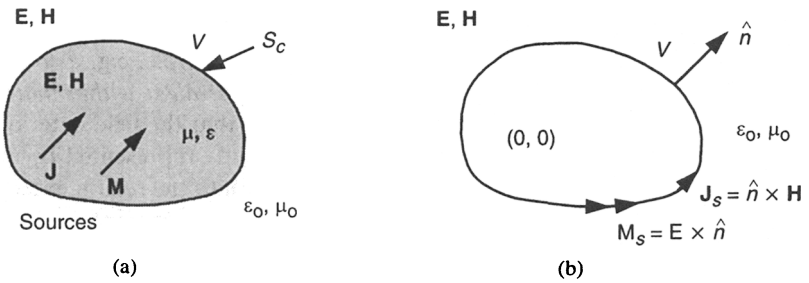


Figure 1.3 Illustration of the application of the surface equivalence principle.

$$\mathbf{E}(\mathbf{r}) = \oiint_{S_c} \left[\mathbf{M}(\mathbf{r}') \times \nabla G_0(\mathbf{r}, \mathbf{r}') - jk_0 Z_0 \mathbf{J}(\mathbf{r}') G_0(\mathbf{r}, \mathbf{r}') - j\frac{Z_0}{k_0} \mathbf{J}(\mathbf{r}') \cdot \nabla\nabla G_0(\mathbf{r}, \mathbf{r}') \right] dS' \tag{1.58}$$

$$\mathbf{H}(\mathbf{r}) = \oiint_{S_c} \left[-\mathbf{J}(\mathbf{r}') \times \nabla G_0(\mathbf{r}, \mathbf{r}') - j\frac{k_0}{Z_0} \mathbf{M}(\mathbf{r}') G_0(\mathbf{r}, \mathbf{r}') - j\frac{1}{k_0 Z_0} \mathbf{M}(\mathbf{r}') \cdot \nabla\nabla G_0(\mathbf{r}, \mathbf{r}') \right] dS' \tag{1.59}$$

More explicit expressions for $\mathbf{E}$ and $\mathbf{H}$ can be obtained by introducing the identities

$$\nabla G_0(\mathbf{r}, \mathbf{r}') = \frac{dG(R)}{dR} \nabla R = -\left(jk_0 + \frac{1}{R} \right) G_0(R) \hat{R}$$

$$\nabla\nabla G_0(\mathbf{r}, \mathbf{r}') = \hat{R}\hat{R} \left[\frac{1}{R^2} + \left(jk_0 + \frac{1}{R} \right)^2 \right] G_0(\mathbf{r}, \mathbf{r}') - (\bar{\bar{I}} - \hat{R}\hat{R}) \left(jk_0 + \frac{1}{R} \right) \frac{G_0(\mathbf{r}, \mathbf{r}')}{R}$$

in which $\hat{R} = (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$, as depicted in Fig. 1.2. Specifically, (1.58) becomes

$$\mathbf{E} = -jk_0 \oiint_{S_c} [\mathbf{M}(\mathbf{r}') \times \hat{R}] \left(1 + \frac{1}{jk_0 R} \right) G_0(\mathbf{r}, \mathbf{r}') \, dS' - jk_0 Z_0 \oiint_{S_c} \left\{ \left[1 - \frac{j}{k_0 R} - \frac{1}{(k_0 R)^2} \right] \mathbf{J}(\mathbf{r}') - \left[1 - \frac{3j}{k_0 R} - \frac{3}{(k_0 R)^2} \right] [\mathbf{J}(\mathbf{r}') \cdot \hat{R}] \hat{R} \right\} G_0(\mathbf{r}, \mathbf{r}') \, dS' \tag{1.60}$$

and a corresponding expression for $\mathbf{H}$ can be obtained by invoking duality ($\mathbf{E} \to \mathbf{H}$, $\mathbf{H} \to -\mathbf{E}$, $\mathbf{J} \to \mathbf{M}$, $\mathbf{M} \to -\mathbf{J}$, $\mu \to \epsilon$, $\epsilon \to \mu$, $Z_0 \to Y_0$, $Y_0 \to Z_0$).

We can rewrite (1.58) in a less singular form by noting the identities, $\nabla G = -\nabla' G$,

$$\nabla' \cdot [\mathbf{J}(\mathbf{r}') G_0(\mathbf{r}, \mathbf{r}')] = G_0(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{J}(\mathbf{r}') + \mathbf{J}(\mathbf{r}') \cdot \nabla' G_0(\mathbf{r}, \mathbf{r}') \tag{1.61}$$

$$\begin{aligned} \nabla[\nabla \cdot \mathbf{J}(\mathbf{r}') G_0(\mathbf{r}, \mathbf{r}')] &= \nabla \mathbf{J}(\mathbf{r}') \cdot \nabla G_0(\mathbf{r}, \mathbf{r}') + \mathbf{J}(\mathbf{r}') \cdot \nabla\nabla G_0(\mathbf{r}, \mathbf{r}') \\ &= \mathbf{J}(\mathbf{r}') \cdot \nabla\nabla G_0(\mathbf{r}, \mathbf{r}') \end{aligned} \tag{1.62}$$

to deduce that

$$\begin{aligned} \oiint_{S_c} \mathbf{J}(\mathbf{r}') \cdot \nabla\nabla G_0(\mathbf{r}, \mathbf{r}') \, dS' &= -\nabla \oiint_{S_c} \mathbf{J}(\mathbf{r}') \cdot \nabla' G_0(\mathbf{r}, \mathbf{r}') \, dS' \\ &= +\oiint_{S_c} \nabla' \cdot \mathbf{J}(\mathbf{r}') \nabla G_0(\mathbf{r}, \mathbf{r}') \, dS' \end{aligned}$$

Introducing these into (1.58) yields the expression

$$\mathbf{E} = \oiint_{S_c} \left\{ \mathbf{M}(\mathbf{r}') \times \nabla G_0(\mathbf{r}, \mathbf{r}') - jk_0 Z_0 \mathbf{J}(\mathbf{r}') G_0(\mathbf{r}, \mathbf{r}') - j \frac{Z_0}{k_0} [\nabla' \cdot \mathbf{J}(\mathbf{r}')] \nabla G_0(\mathbf{r}, \mathbf{r}') \right\} dS'$$

which is most commonly used for integral equation numerical solutions and is also valid for open surfaces since the normal components of **J** to the perimeter edges of the surface vanish. The corresponding **H** field expression is again obtained by duality.

For far zone computations ($r \to \infty$), the Green's function (1.56) can be simplified as

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk_0 r}}{4\pi r} e^{jk_0(\mathbf{r}' \cdot \hat{r})} \tag{1.63}$$

Using this in (1.58) and (1.59), carrying out the vector derivative operations, and retaining only the terms that decay[1] as $\mathcal{O}(1/r)$, we get

$$\mathbf{E}(\mathbf{r}) \approx jk_0 \frac{e^{-jk_0 r}}{4\pi r} \oiint_{S_c} [+\hat{r} \times \mathbf{M}(\mathbf{r}') + Z_0 \hat{r} \times (\hat{r} \times \mathbf{J}(\mathbf{r}'))] e^{jk_0(\mathbf{r}' \cdot \hat{r})} dS' \tag{1.64}$$

$$\mathbf{H}(\mathbf{r}) \approx jk_0 \frac{e^{-jk_0 r}}{4\pi r} \oiint_{S_c} [-\hat{r} \times \mathbf{J}(\mathbf{r}') + Y_0 \hat{r} \times (\hat{r} \times \mathbf{M}(\mathbf{r}'))] e^{jk_0(\mathbf{r}' \cdot \hat{r})} dS' \tag{1.65}$$

These are referred to as the far zone field expressions and are typically used for the evaluation of antenna radiated fields or for the calculation of the radar cross section (RCS) of a target. An acceptable criterion for using (1.64) and (1.65) is

$$r \geq \frac{2D^2}{\lambda_0} \tag{1.66}$$

where D is the largest antenna or target dimension. In this case, the phase error in the intervening approximations is maintained at less than $\frac{\pi}{8}$. A typical setup for computing the radiation from volume sources at points in the near and far zones is depicted in Fig. 1.2. Figure 1.4 also shows the spherical angles commonly used in electromagnetics and to be used throughout this text.

Note that use of (1.55) requires both equivalent currents (1.52). Alternatively, when **J** and **M** radiate in the presence of certain bodies such as an infinite metallic plane, a cylinder or a half-plane, a dyadic Green's function can be chosen to satisfy the boundary conditions appropriate for that surface, hence avoiding the need for currents to be placed on that surface. For example, the Dirichlet condition

$$\hat{n} \times \overline{\overline{G}}_1 = 0 \qquad \text{on S} \tag{1.67}$$

can be imposed for relating the electric or magnetic fields in the exterior region to the magnetic or electric currents, respectively. The radiated field expressions, (1.53) and (1.54), now simplify to

[1]The notation $\mathcal{O}(1/r)$ is read as "order of $1/r$."

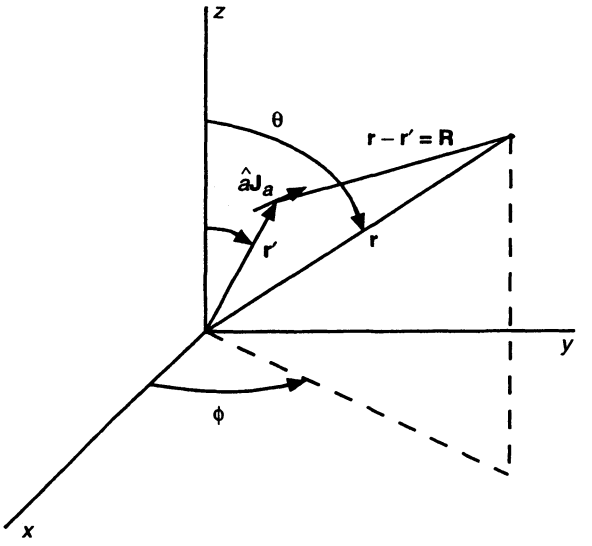


Figure 1.4 Illustration of an infinitesimal source $\mathbf{J} = \hat{a}J_a\, d\ell$ and the associated coordinates and angles.

$$\mathbf{E}(\mathbf{r}) = -\oiint_{S_c} \nabla \times \overline{\overline{G}}_1(R) \cdot \hat{n}' \times \mathbf{E}(\mathbf{r}')\, dS' \tag{1.68}$$

$$\mathbf{H}(\mathbf{r}) = -\oiint_{S_c} \nabla \times \overline{\overline{G}}_1(R) \cdot \hat{n}' \times \mathbf{H}(\mathbf{r}')\, dS' \tag{1.69}$$

implying that only a single current is required for the representation of each field quantity. One use of this Green's function is in the calculation of the scattering by a perfect magnetic conductor (a fictitious material) through the use of equivalent electric currents.

If the Neumann boundary condition is imposed

$$\hat{n} \times \nabla \times \overline{\overline{G}}_2 = 0 \qquad \text{on S} \tag{1.70}$$

the resulting field integral expressions are again given in terms of only one current

$$\mathbf{E}(\mathbf{r}) = +jk_0 Z_0 \oiint_{S_c} \overline{\overline{G}}_2(R) \cdot \hat{n}' \times \mathbf{H}(\mathbf{r}')\, dS' \tag{1.71}$$

$$\mathbf{H}(\mathbf{r}) = -jk_0 Y_0 \oiint_{S_c} \overline{\overline{G}}_2(R) \cdot \hat{n}' \times \mathbf{E}(\mathbf{r}')\, dS' \tag{1.72}$$

This Green's function ($\overline{\overline{G}}_2$) is useful for calculating the scattering by a perfect electric conductor using electric currents. Also, this Green's function will be used to relate the electric field quantities of an interior finite element formulation to the magnetic field of the bounding surface.

The above dyadic Green's functions, $\overline{\overline{G}}_1$ and $\overline{\overline{G}}_2$, are commonly termed the first and the second kind dyadic Green's functions, respectively. A good discussion of dyadic Green's functions used in electromagnetics is given in [10].

The above expressions are for three-dimensional fields. In the case of two-dimensional fields (e.g., one dimension, such as z, is invariant), similar expressions are used. These are scalar and typically written in terms of TM_z and TE_z polariz-

ations. For TM_z ($H_z = E_x = E_y = 0$), the electric field on or outside a contour C due to fields on that surface is given by

$$E_z = \oint_C E_z(\mathbf{r}')\left[\frac{\partial}{\partial n'} G^{2D}(\mathbf{r}, \mathbf{r}')\right] dl' - jk_0 Z_0 \oint_C H_t(\mathbf{r}')[G^{2D}(\mathbf{r}, \mathbf{r}')]\, dl' \tag{1.73}$$

since $\partial E_z/\partial n = +jk_0 Z_0 H_t$. The corresponding expression for TE_z ($E_z = H_x = H_y = 0$) polarization is

$$H_z = \oint_C H_z(\mathbf{r}')\left[\frac{\partial}{\partial n'} G^{2D}(\mathbf{r}, \mathbf{r}')\right] dl' + jk_0 Y_0 \oint_C E_t(\mathbf{r}')[G^{2D}(\mathbf{r}, \mathbf{r}')]\, dl' \tag{1.74}$$

In (1.73) and (1.74), the subscript "t" denotes the tangential component of the field along the unit vector $\hat{t} = \hat{\ell}$, where $\hat{n} \times \hat{t} = \hat{z}$. Also, the two-dimensional Green's function is given by

$$G^{2D}(\mathbf{r}, \mathbf{r}') = -\frac{j}{4} H_0^{(2)}(k_0|\mathbf{r} - \mathbf{r}'|) \tag{1.75}$$

In (1.75), $H_0^{(2)}(\cdot)$ denotes the zeroth-order Hankel function of the second kind. We observe that (1.73) and (1.74) can be reduced from (1.46) by replacing the potential V with E_z or H_z and making use of the equivalent current relations (1.52) and Maxwell's equations to relate $\mathbf{E}$ and $\mathbf{H}$. For the special cases of the TM_z and TE_z polarizations, the first two of Maxwell's equations imply

$$\mathbf{H} = -\frac{jY_0}{k_0}\, \hat{z} \times \nabla E_z, \qquad \mathbf{E} = \frac{jZ_0}{k_0}\, \hat{z} \times \nabla H_z \tag{1.76}$$

All of the expressions presented in this section are given for currents radiating in free space. Fields within a homogeneous media can be determined by replacing k_0 with k and Z_0 with Z in all of these expressions. Throughout this text, k will denote the wavenumber in the homogeneous media ($k = \sqrt{\epsilon_r \mu_r}\, k_0$) and Z is the intrinsic impedance of that material ($Z = \sqrt{\mu/\epsilon}$).

1.5 NATURAL BOUNDARY CONDITIONS

Maxwell's equations cannot be solved without the specification of the required boundary conditions at material interfaces. The pertinent boundary conditions can be derived directly from the integral form of Maxwell's equations. Specifically, (1.25) is applied to the contour illustrated in Fig. 1.5(a) with S being the area enclosed by C. Assuming $\Delta\ell$ is small, $\Delta h \to 0$ and $\Delta\ell \gg \Delta h$, (1.25) gives

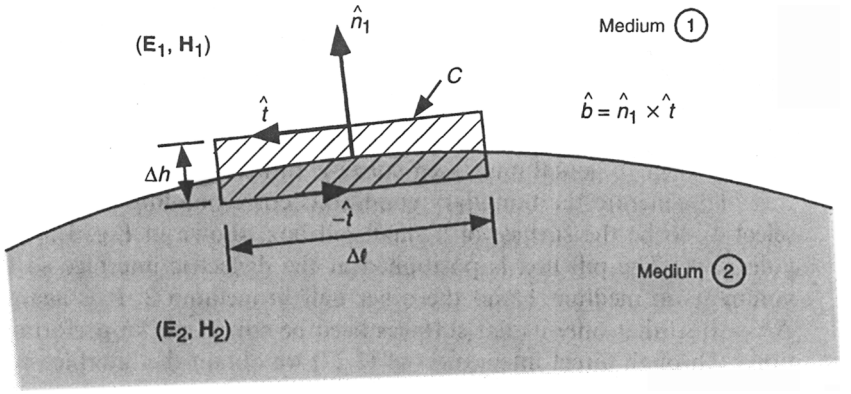
$$(\mathbf{H}_1 - \mathbf{H}_2) \cdot \hat{t} = [\mathbf{J}_i \cdot (\hat{n}_1 \times \hat{t})]\, \Delta h \tag{1.77}$$

and in deriving this we set

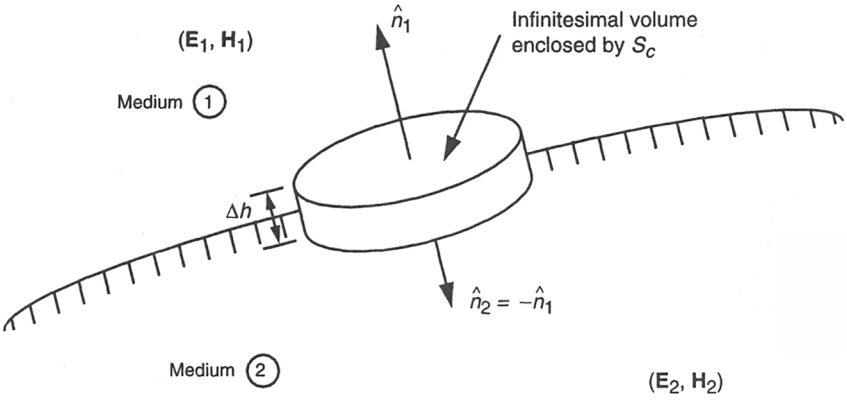
$$\lim_{\Delta h \to 0} \tfrac{1}{2} \Delta h[\epsilon_1 \mathbf{E}_1 + \epsilon_2 \mathbf{E}_2] \cdot \hat{n}_1 = 0$$

which is valid provided $\epsilon\mathbf{E}$ is finite at the interface. When (1.26) is applied to the same contour in Fig. 1.5(a) we find that

$$(\mathbf{E}_1 - \mathbf{E}_2) \cdot \hat{t} = -[\mathbf{M}_i \cdot (\hat{n}_1 \times \hat{t})]\, \Delta h \tag{1.78}$$



(a)



(b)

Figure 1.5 Geometries for deriving the boundary conditions (a) for tangential components, and (b) for normal components.

where we have again set

$$\lim_{\Delta h \to 0} \tfrac{1}{2} \Delta h[\mu_1 \mathbf{H}_1 + \mu_2 \mathbf{H}_2] = 0$$

implying that $\mu \mathbf{H}$ is finite at the dielectric interface.

The conditions (1.77) and (1.78) can be rewritten in vector form and more compactly by introducing the definitions

$$\mathbf{J}_{is} = \mathbf{J}_i \Delta h \tag{1.79}$$

$$\mathbf{M}_{is} = \mathbf{M}_i \Delta h \tag{1.80}$$

giving the conditions

$$\hat{n}_1 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_{is} \tag{1.81}$$

$$\hat{n}_1 \times (\mathbf{E}_1 - \mathbf{E}_2) = -\mathbf{M}_{is} \tag{1.82}$$

The quantities $\mathbf{J}_{is}$ and $\mathbf{M}_{is}$ are referred to as the impressed electric and magnetic surface current densities in A/m and V/m, respectively, at the interface. Note that if $\mathbf{E}_2$ and $\mathbf{H}_2$ are zero, these conditions are identical to (1.52) except that in this case $\mathbf{J}_{is}$ and $\mathbf{M}_{is}$ refer to actual impressed currents rather than equivalent currents.

To generate the boundary conditions corresponding to (1.27) and (1.28), we select S_c to be the surface of a small pill box, shown in Fig. 1.5(b), enclosing the volume V. The pill box is positioned at the dielectric interface so that half of its volume is in medium 1 and the other half in medium 2. It is again assumed that $\Delta h \to 0$ so that only its flat surfaces need be considered in performing the integrations. Through direct integration of (1.27) we obtain the interface conditions

$$\hat{n}_1 \cdot (\mu_1 \mathbf{H}_1 - \mu_2 \mathbf{H}_2) = \rho_{ms} \tag{1.83}$$

$$\hat{n}_1 \cdot (\epsilon_1 \mathbf{E}_1 - \epsilon_2 \mathbf{E}_2) = \rho_s \tag{1.84}$$

where ρ_s denotes the unbounded electric surface charge density in C/m^2 at the interface and ρ_{ms} is the corresponding fictitious surface magnetic charge density in Wb/m^2.

The boundary/interface conditions (1.81)–(1.84), although derived for time-harmonic fields, are applicable for instantaneous fields as well. In the time-harmonic case, only (1.81) and (1.82) are required in conjunction with (1.23) and (1.24) for a unique solution of the fields.

If we ignore the fictitious magnetic currents and charges appearing in (1.81)–(1.84), the boundary conditions are

$$\hat{n}_1 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_{is} \tag{1.85}$$

$$\hat{n}_1 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{1.86}$$

$$\hat{n}_1 \cdot (\mu_1 \mathbf{H}_1 - \mu_2 \mathbf{H}_2) = 0 \tag{1.87}$$

$$\hat{n}_1 \cdot (\epsilon_1 \mathbf{E}_1 - \epsilon_2 \mathbf{E}_2) = \rho_s \tag{1.88}$$

The first two of these state that the tangential electric fields are continuous across the interface whereas the tangential magnetic fields are discontinuous at the same location by an amount equal to the impressed electric current. Unless a source (i.e., free charge) is actually placed at the interface, $\mathbf{J}_{is}$ is also zero and in that case, the tangential magnetic fields will be continuous across the media as well.

When medium 2 is a perfect electric conductor then $\mathbf{E}_2 = \mathbf{H}_2 = 0$. In addition, M_{is} and ρ_{ms} vanish and (1.81)–(1.84) reduce to

$$\hat{n}_1 \times \mathbf{H}_1 = \mathbf{J}_{is} \tag{1.89}$$

$$\hat{n}_1 \times \mathbf{E}_1 = 0 \tag{1.90}$$

$$\hat{n}_1 \cdot (\mu_1 \mathbf{H}_1) = 0 \tag{1.91}$$

$$\hat{n}_1 \cdot (\epsilon_1 \mathbf{E}_1) = \rho_s \tag{1.92}$$

The first two of these now imply that the tangential electric field vanishes on the surface of the perfect electric conductor whereas the tangential magnetic field is equal to the impressed electric surface current on the conductor.

1.6 APPROXIMATE BOUNDARY CONDITIONS

In the previous section, the boundary conditions which must be imposed at the interface of different dielectrics were presented. Sometimes, it is difficult to utilize these conditions since excessive computational cost is required or the resulting formulation is numerically unstable such as the case of a thin dielectric sheet. In many cases, much simpler approximate boundary conditions that account for the presence of an inhomogeneous medium, coated metallic surface, or a thin dielectric layer can be employed to simulate the actual surface. Below we discuss two types of such approximate conditions: impedance boundary and sheet transition conditions. The interested reader is directed to [11] for a general treatment of approximate conditions.

1.6.1 Impedance Boundary Conditions

The most widely applied approximate conditions are referred to as the Standard Impedance Boundary Conditions (SIBC) or Leontovich Boundary conditions. It is derived by considering a plane wave impinging upon a material half-space. Consider a material-air interface which corresponds to the $y = 0$ plane. The SIBC takes the form

$$E_z = -\eta Z_0 H_x, \qquad E_x = \eta Z_0 H_z \tag{1.93}$$

where the free space impedance is given by $Z_0 = \sqrt{\mu_0/\epsilon_0}$ and the normalized material characteristic impedance is η. An important concept to understand is that these conditions are applied slightly above the interface (assuming a plane wave originating in the upper half-space) at $y = 0^+$. Combining these two conditions, the vector form of the SIBC is given by

$$\hat{n} \times (\hat{n} \times \mathbf{E}) = -\eta Z_0 \hat{n} \times \mathbf{H} \tag{1.94}$$

where the outward directed unit normal, $\hat{n}$, is shown in Fig. 1.6. This form of the SIBC is not restricted to a particular interface [as is the case with (1.93)] and is commonly applied to convex surfaces such as a sphere, cylinder, etc.

All of the quantities used in (1.94) are familiar and well defined except for the normalized impedance, η. One means of deriving this quantity is to demand that the reflected field attributed to (1.94) is identical to that due to the natural boundary conditions. Then,

$$\eta = \sqrt{\frac{\mu_r}{\epsilon_r}} \tag{1.95}$$

This is exact for an infinite planar interface while it is approximate for a curved boundary provided that

$$|\mathrm{Im}\,(\sqrt{\epsilon_r \mu_r})| k_0 r_i \gg 1 \tag{1.96}$$

where $\mathrm{Im}\,(\cdot)$ denotes the imaginary part of the complex argument and the principle radii of curvature, r_i, is associated with the surface at a point. This condition assures that the material is sufficiently lossy so that the fields which penetrate into the material does not re-emerge at some other point.

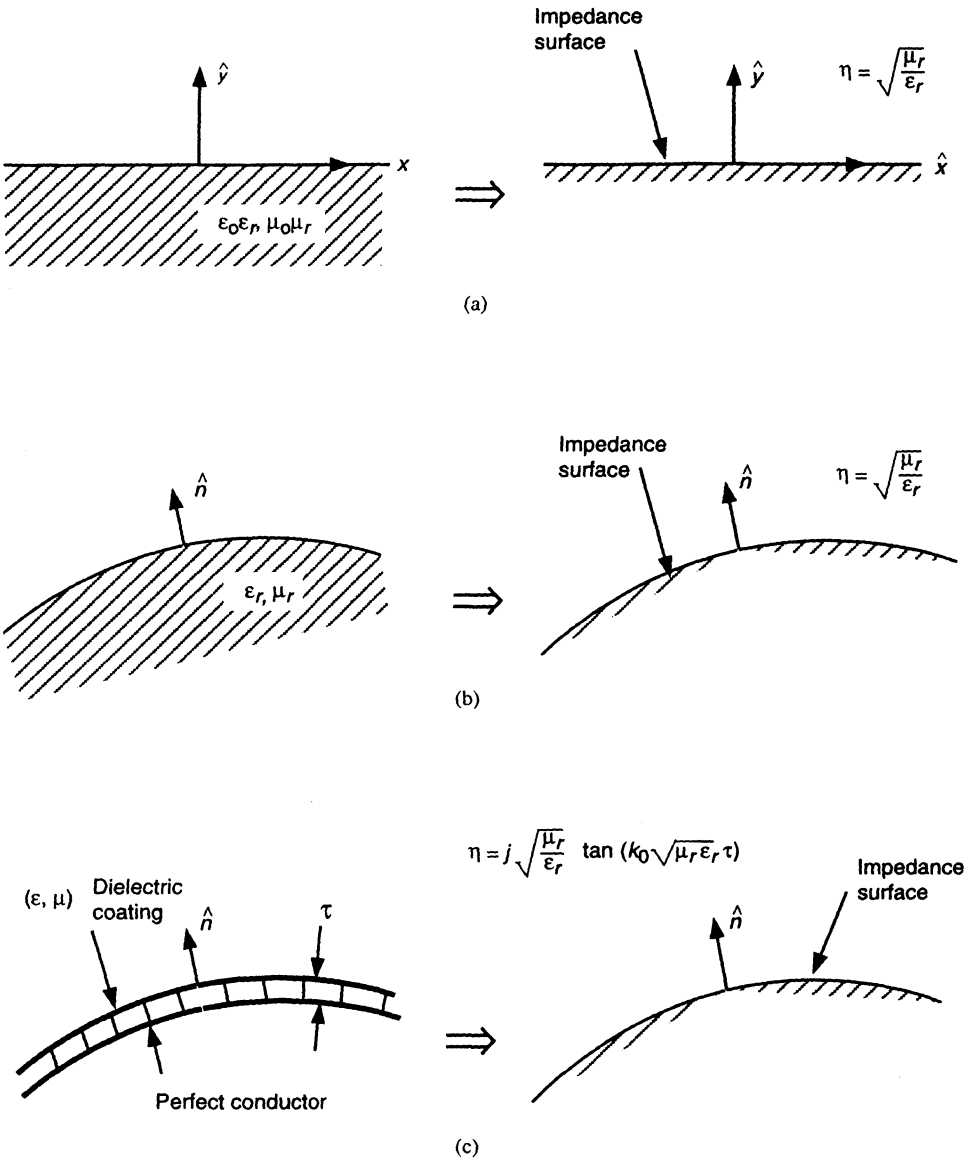


Figure 1.6 Simulation of dielectric boundaries and coatings with SIBCs.

For a coated conductor, the choice of η typically is found by considering a shorted transmission line model with length corresponding to the coating thickness, τ:

$$\eta = j\sqrt{\frac{\mu_r}{\epsilon_r}} \tan(k_0\sqrt{\epsilon_r\mu_r}\,\tau) \tag{1.97}$$

However, this condition is derived at normal incidence and deteriorates at oblique angles with increasing inaccuracy for thicker coatings.

The SIBCs can be applied for modeling surfaces whose material properties vary slowly in the transverse plane. For a planar interface, the coating can have a varying composition in the normal dimension, and Rytov [12] found the following impedance

$$\eta = \sqrt{\frac{\mu_r}{\epsilon_r}}\left\{1 + \frac{1}{j2k_0N}\frac{\partial}{\partial y}\ln(Z_0N) + \mathcal{O}(N^{-2})\right\} \tag{1.98}$$

is useful where $N = \sqrt{\mu_r\epsilon_r}$ is the index of refraction and the normal derivative is applied at the surface.

More accurate approximate conditions can be developed by incorporating higher order derivatives in their constructions. These are referred to as *Generalized Impedance Boundary Conditions* (*GIBCs*), and these are discussed in [11].

1.6.2 Sheet Transition Conditions

A thin dielectric layer may be replaced with an equivalent sheet model to simplify the analysis. Consider a thin dielectric layer with thickness τ, as shown in Fig. 1.7. This layer has conductivity σ and will support a volume current density

$$\mathbf{J} = \sigma\mathbf{E} \tag{1.99}$$

where $\mathbf{E}$ is the electric field within the layer. For $\tau \ll \lambda$, this volume current may be replaced with an equivalent sheet current (having units of A/m)

$$\mathbf{J}_s = \tau\mathbf{J} \tag{1.100}$$

and from (1.99)

$$\mathbf{E} = \frac{\mathbf{J}_s}{\tau\sigma} = Z_0R_e\mathbf{J}_s \tag{1.101}$$

This condition is referred to as a *resistive* sheet transition condition which only supports a single surface current, $\mathbf{J}_s$. The parameter R_e is the normalized *resistivity* of the sheet and is measured in Ohms per square.

The electric field was assumed to be tangential to the sheet in the above derivation. A more general expression for the resistive sheet condition is given by

$$\hat{n} \times (\hat{n} \times \mathbf{E}) = -Z_0R_e\mathbf{J}_s \tag{1.102}$$

Furthermore, it is desirable to utilize fields just outside or inside the sheet surface. Since the tangential electric field is continuous across the sheet

$$\begin{aligned} \hat{n} \times [\hat{n} \times (\mathbf{E}^+ + \mathbf{E}^-)] &= -2Z_0R_e\mathbf{J}_s \\ \hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) &= 0 \end{aligned} \tag{1.103}$$

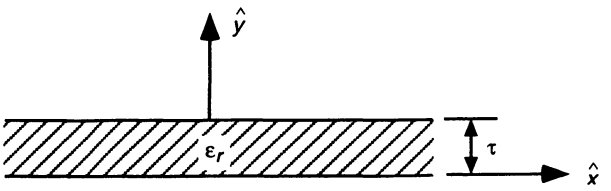


Figure 1.7 Simulation of thin dielectric layer with a sheet condition.

The superscripts ± denote the fields just above and below the sheet, and the introduction of the second equation in (1.103) is necessary to maintain equivalency with (1.102). The natural boundary conditions may be used to rewrite (1.103) as

$$\begin{aligned} \hat{n} \times [\hat{n} \times (\mathbf{E}^+ + \mathbf{E}^-)] &= -2Z_0 R_e \hat{n} \times (\mathbf{H}^+ - \mathbf{H}^-) \\ \hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) &= 0 \end{aligned} \tag{1.104}$$

As long as the loss in the layer is sufficient to assure that no multiple field penetrations will occur, these resistive transition conditions may be used for curved layers.

The dual to the resistive sheet condition is the *conductive* sheet condition which supports only a surface *magnetic* current. It is given by

$$\begin{aligned} \hat{n} \times [\hat{n} \times (\mathbf{H}^+ + \mathbf{H}^-)] &= 2Y_0 R_m \hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) \\ \hat{n} \times (\mathbf{H}^+ - \mathbf{H}^-) &= 0 \end{aligned} \tag{1.105}$$

The normalized *conductivity* of this sheet is denoted by R_m with units Mhos per square. This condition is required for the simulation of materials which have nontrivial permeability. Also, a special combination of coincident resistive and conductive sheets with respective resistivity and conductivity

$$R_e = \frac{\eta}{2}, \qquad R_m = \frac{1}{2\eta} \tag{1.106}$$

yields the same result as an impedance sheet with impedance η and (1.106) implies that $4R_eR_m = 1$. This set of sheets are useful in simplifying the analysis of a planar impedance sheet since coplanar resistive and conductive sheets are uncoupled.

The resistivity of a dielectric layer can be determined by considering the equivalent polarization current

$$\mathbf{J} = jk_0 Y_0(\epsilon_r - 1)\mathbf{E} \tag{1.107}$$

and (1.100). It follows that the tangential components of the field are given by

$$E_t = Z_0 R_e J_{st} \tag{1.108}$$

with

$$R_e = \frac{-j}{k_0 \tau(\epsilon_r - 1)} \tag{1.109}$$

A dual conductive sheet is given by

$$R_m = \frac{-j}{k_0 \tau(\mu_r - 1)} \tag{1.110}$$

More accurate representations can be formed using *Generalized Sheet Transition Conditions* (*GSTCs*) [11] which incorporate higher order derivatives in their construction.

1.7 POYNTING'S THEOREM

The quantity ("*" indicates complex conjugation)

$$\mathbf{S} = \tfrac{1}{2}\mathbf{E} \times \mathbf{H}^* \tag{1.111}$$

is known as the complex Poynting vector and has units of Watts/m^2. It represents the complex power density of the wave, and it is therefore important to understand the source and nature of this power. To do so, we refer to (1.21) and (1.22), where by dotting each equation with $\mathbf{E}$ and $\mathbf{H}^*$, we have

$$\mathbf{E}\cdot\nabla\times\mathbf{H}^* = \mathbf{J}_i^*\cdot\mathbf{E} - j\omega\epsilon^*\mathbf{E}^*\cdot\mathbf{E} = \mathbf{J}_i^*\cdot\mathbf{E} - j\omega\epsilon^*|\mathbf{E}|^2 \tag{1.112}$$

$$\mathbf{H}^*\cdot\nabla\times\mathbf{E} = -\mathbf{M}_i\cdot\mathbf{H}^* - j\omega\mu\mathbf{H}\cdot\mathbf{H}^* = -\mathbf{M}^i\cdot\mathbf{H}^* - j\omega\mu|\mathbf{H}|^2 \tag{1.113}$$

From the vector identity [6]

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) = \mathbf{H}^*\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H}^* \tag{1.114}$$

we then obtain

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) = j\omega\epsilon^*|\mathbf{E}|^2 - j\omega\mu|\mathbf{H}|^2 - \mathbf{J}_i^*\cdot\mathbf{E} - \mathbf{M}_i\cdot\mathbf{H}^* \tag{1.115}$$

which is an identity valid everywhere in space. Integrating both sides of this over a volume V containing all sources, and invoking the divergence theorem yields

$$\frac{1}{2}\oiint_{S_c}(\mathbf{E}\times\mathbf{H}^*)\cdot d\mathbf{s} = \frac{1}{2}\iiint_V [j\omega\epsilon^*|\mathbf{E}|^2 - j\omega\mu|\mathbf{H}|^2 - \mathbf{J}_i^*\cdot\mathbf{E} - \mathbf{M}_i\cdot\mathbf{H}^*]\,dv \tag{1.116}$$

which is commonly referred to as Poynting's theorem. Since S_c is closed, based on energy conservation one deduces that the right hand side of (1.116) must represent the sum of the power stored or radiated, i.e., escaping, out of the volume V. Each term of the volume integral of (1.116) is associated with a specific type of power but before proceeding with their identification, it is instructive that ϵ^* be first replaced by $\epsilon_0\epsilon_r + j\frac{\sigma}{w}$. Equation (1.116) can then be rewritten as

$$\frac{1}{2}\,\mathrm{Re}\oiint_{S_c}(\mathbf{E}\times\mathbf{H}^*)\cdot d\mathbf{s} = P_{ei} + P_{mi} - P_d \tag{1.117}$$

$$\frac{1}{2}\,\mathrm{Im}\oiint_{S_c}(\mathbf{E}\times\mathbf{H}^*)\cdot d\mathbf{s} = 2\omega[W_e - W_m] - \frac{1}{2}\,\mathrm{Im}\iiint_V [\mathbf{J}_i^*\cdot\mathbf{E} + \mathbf{M}_i\cdot\mathbf{H}^*]\,dv \tag{1.118}$$

where

$$P_{ei} = -\frac{1}{2}\iiint_V \mathrm{Re}(\mathbf{J}_i^*\cdot\mathbf{E})\,dv = \text{averaging outgoing power due to the impressed current } \mathbf{J} \tag{1.119}$$

$$P_{mi} = -\frac{1}{2}\iiint_V \mathrm{Re}(\mathbf{M}_i\cdot\mathbf{H}^*)\,dv = \text{average outgoing power due to the impressed current } \mathbf{M}_i \tag{1.120}$$

$$P_d = \frac{1}{2}\iiint_V \sigma|\mathbf{E}|^2\,dv = \text{average power dissipated in } V \tag{1.121}$$

$$W_e = \frac{1}{4}\iiint_V \epsilon_0\epsilon_r|\mathbf{E}|^2\,dv = \text{average electric energy stored in } V \tag{1.122}$$

$$W_m = \frac{1}{4}\iiint_V \mu_0\mu_r|\mathbf{H}|^2\,dv = \text{average magnetic energy stored in } V \tag{1.123}$$

The time-averaged power delivered to the electromagnetic field outside V is clearly the sum of P_{ei} and P_{mi}, whereas P_d is that dissipated in V due to conductor losses. Thus, we may consider

$$P_{\text{av}} = \frac{1}{2}\,\text{Re}\oiint_{S_c} (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} \tag{1.124}$$

to be the average or radiated power outside V if σ is zero in V. Expression (1.118) gives the reactive power, i.e., that which is stored within V and is not allowed to escape outside the boundary of S_c.

1.8 UNIQUENESS THEOREM

Whenever one pursues a solution to a set of equations it is important to know a priori whether this solution is unique and if not, what are the required conditions for a unique solution. This is important because depending on the application, different analytical or numerical methods will likely be used for the solution of Maxwell's equations. Given that Maxwell's equations (subject to the appropriate boundary conditions) yield a unique solution, one is then comforted to know that any convenient method of analysis will yield the correct solution to the problem.

The most common form of the uniqueness theorem is: *In a region V completely occupied with dissipative media, a harmonic field* $(\mathbf{E}, \mathbf{H})$ *is uniquely determined by the impressed currents in that region plus the tangential components of the electric or magnetic fields on the closed surface S_c bounding V.* This theorem may be proved by assuming for the moment that two solutions exist, denoted by $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$. Both fields must, of course, satisfy Maxwell's equations (1.21) and (1.22) with the same impressed currents $(\mathbf{J}_i, \mathbf{M}_i)$. We have

$$\begin{aligned} \nabla \times \mathbf{H}_1 &= \mathbf{J}_i + j\omega\epsilon\mathbf{E}_1, & \nabla \times \mathbf{H}_2 &= \mathbf{J}_i + j\omega\epsilon\mathbf{E}_2 \\ \nabla \times \mathbf{E}_1 &= -\mathbf{M}_i - j\omega\mu\mathbf{H}_1, & \nabla \times \mathbf{E}_2 &= -\mathbf{M}_i - j\omega\mu\mathbf{H}_2 \end{aligned} \tag{1.125}$$

and when these are subtracted we obtain

$$\nabla \times \mathbf{H}' = j\omega\epsilon\mathbf{E}' \tag{1.126}$$

$$\nabla \times \mathbf{E}' = -j\omega\mu\mathbf{H}' \tag{1.127}$$

where $\mathbf{E}' = \mathbf{E}_1 - \mathbf{E}_2$ and $\mathbf{H}' = \mathbf{H}_1 - \mathbf{H}_2$. To prove the theorem it is then necessary to show that $(\mathbf{E}', \mathbf{H}')$ are zero or equivalently, if no sources are enclosed by a volume V, the fields in that volume are zero for a given set of tangential electric and magnetic fields on S_c.

As a corollary to the uniqueness theorem, it can be shown that *if a harmonic field has a zero tangential electric or magnetic field on a surface enclosing a source-free region V occupied by dissipative media, the field vanishes everywhere within V.*

The usual proof of the uniqueness theorem can be found in many electromagnetics texts (see, for example, [5]).

1.9 SUPERPOSITION THEOREM

The superposition theorem states that for a linear medium, the total field intensity due to two or more sources is equal to the sum of the field intensities attributed to each individual source radiating independent of the others. In particular, let us consider two electric sources $\mathbf{J}_1$ and $\mathbf{J}_2$. On the basis of the superposition theorem, to find the total field caused by the simultaneous presence of both sources, we can consider the field due to each individual source in isolation. The fields $(\mathbf{E}_1, \mathbf{H}_1)$ due to $\mathbf{J}_1$ satisfy the equations

$$\nabla \times \mathbf{H}_1 = \mathbf{J}_1 + j\omega\epsilon\mathbf{E}_1 \tag{1.128}$$

$$\nabla \times \mathbf{E}_1 = -j\omega\mu\mathbf{H}_1 \tag{1.129}$$

and the fields corresponding to $\mathbf{J}_2$ satisfy

$$\nabla \times \mathbf{H}_2 = \mathbf{J}_2 + j\omega\epsilon\mathbf{E}_2 \tag{1.130}$$

$$\nabla \times \mathbf{E}_2 = -j\omega\mu\mathbf{H}_2 \tag{1.131}$$

By adding these two sets of equations, it is clear that the total field due to both sources combined is given by

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \qquad \mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 \tag{1.132}$$

where $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$ are obtained by solving separately (1.128)–(1.129) and (1.130)–(1.131), respectively.

1.10 DUALITY THEOREM

The duality theorem relates to the interchangeability of the electric and magnetic fields, currents, charges, or material properties. We observe from (1.3) and (1.4) that the first can be obtained from the second via the interchanges

$$\begin{aligned} \mathbf{M} &\rightarrow -\mathbf{J} \\ \mathbf{E} &\rightarrow \mathbf{H} \\ \mathbf{H} &\rightarrow -\mathbf{E} \\ \mu &\rightarrow \epsilon \end{aligned} \tag{1.133}$$

Similarly, (1.4) can be obtained from (1.3) via the interchanges

$$\begin{aligned} \mathbf{J} &\rightarrow \mathbf{M} \\ \mathbf{E} &\rightarrow \mathbf{H} \\ \mathbf{H} &\rightarrow -\mathbf{E} \\ \epsilon &\rightarrow \mu \end{aligned} \tag{1.134}$$

The duality theorem can reduce formulation and computational effort when one is able to invoke it for a particular application.

1.11 NUMERICAL TECHNIQUES

For numerical solutions, all governing equations can be written in operator form as

$$\mathcal{L}u - f = 0 \tag{1.135}$$

subject to appropriate boundary or transition conditions

$$B(u) = 0 \tag{1.136}$$

within the domain (Ω) and on its boundary ($S_c = \partial\Omega$). In these, the operator $\mathcal{L}$ is based on one of the following: an integral representation of the fields such as (1.52)–(1.53), on the vector wave equation (1.30), or the Helmholtz equation (1.36) for scalar fields. It is understood that u must be replaced by a vector field $\mathbf{u}$ when dealing with the vector wave equations (1.30) or (1.35). The forcing function f is a known excitation function while u or $\mathbf{u}$ is the unknown quantity. Throughout this text, u or $\mathbf{u}$ will denote a field or current density.

Unfortunately, very few analytical solutions for (1.135) are available in electromagnetics. One such solution, the fields due to a magnetic dipole in the presence of an infinite metallic plane or cylinder, will be used in Chapter 7 to form the appropriate dyadic Green's function for those geometries. However, most useful electromagnetic scattering and radiation problems cannot be solved using analytical methods. Rather, an approximate numerical solution is sought which in some way closely resembles the exact solution. Two methods of formulating such an approximate solution are: the Ritz method and the method of weighted residuals.

1.11.1 The Ritz Method

The Ritz or Rayleigh–Ritz method[2] [13, pp. 74–78], [14, pp. 13–63] seeks a stationary point of a variational functional. For operators which are *self-adjoint* and *positive-definite* (see later subsection for definitions), the stationary point of the following functional

$$F(\tilde{u}) = \tfrac{1}{2}\langle \mathcal{L}\tilde{u}, \tilde{u}\rangle - \langle f, \tilde{u}\rangle \tag{1.137}$$

is an approximate solution of (1.135). In (1.137), the *inner product* over the domain Ω (volume, surface, or contour) of the two functions is defined as

$$\langle a, b\rangle = \int_{\Omega} ab\, d\Omega \tag{1.138}$$

or for vector functions

$$\langle \mathbf{a}, \mathbf{b}\rangle = \int_{\Omega} \mathbf{a}\cdot\mathbf{b}\, d\Omega \tag{1.39}$$

The choice of this inner product extends the validity of the variational expressions to vectorial fields. When the operator $\mathcal{L}\tilde{u}$ and f in (1.137) are chosen as

[2]The method was originally introduced by Rayleigh in 1877 and was extended by Ritz in 1909.

$$\mathcal{L}\mathbf{u} = \nabla \times \left\{ \frac{\nabla \times \mathbf{u}}{\mu_r} \right\} - k_0^2 \epsilon_r \mathbf{u} \tag{1.140}$$

$$\mathbf{f} = -jk_0 Z_0 \mathbf{J} - \nabla \times \left(\frac{\mathbf{M}}{\mu_r} \right) \tag{1.141}$$

it can be shown that setting the first variation of $F(\mathbf{u})$ to zero is equivalent to satisfying the vector wave equation (1.30) over the computational domain Ω. Similarly, when

$$\mathcal{L}u = \nabla^2 u + k_0^2 u \tag{1.142}$$

setting the first variation of $F(u)$ to zero is equivalent to satisfying the inhomogeneous Helmholtz wave equation

$$\nabla^2 u + k_0^2 u = f \tag{1.143}$$

To show that setting the first variation of $F(u)$ to zero is equivalent to satisfying the Helmholtz equation (1.143), we begin by rewriting the functional (we assume a two-dimensional domain)

$$\begin{aligned} F(u) &= \tfrac{1}{2} \langle \nabla^2 u + k_0^2 u, u \rangle - \langle f, u \rangle \\ &= \frac{1}{2} \iint_\Omega [\nabla^2 u + k_0^2 u] u \, d\Omega - \iint_\Omega f u \, d\Omega \end{aligned} \tag{1.144}$$

as

$$F(u) = \frac{1}{2} \iint_\Omega (-\nabla u \cdot \nabla u + k_0^2 u^2) \, d\Omega + \frac{1}{2} \oint_C u \frac{\partial u}{\partial n} \, dl - \iint_\Omega f u \, d\Omega \tag{1.145}$$

in which $\nabla u \cdot \hat{n} = \partial u / \partial n$, C denotes the contour enclosing the region Ω (see Fig. 1.8) and $\hat{n}$ is the unit normal vector to C. Note that in deriving (1.145) we used the identity

$$\psi(\nabla \cdot \nabla u) = -\nabla u \cdot \nabla \psi + \nabla \cdot (\psi \nabla u) \tag{1.146}$$

and the divergence theorem

$$\iint_\Omega \nabla \cdot (\nabla u) \, d\Omega = \oint_C \nabla u \cdot \hat{n} \, ds \tag{1.147}$$

Next we proceed to evaluate the first variation of $F(u)$ given by

$$\delta F = F(u + \Delta u) - F(u) \tag{1.148}$$

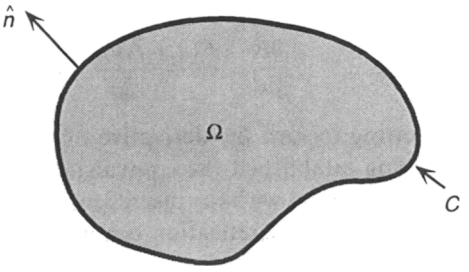


Figure 1.8 Illustration of the region Ω and the enclosing contour C.

where $\Delta \to 0$ is a scalar quantity. The evaluation of δF involves the quantities

$$(u + \Delta u)^2 = u^2 + 2(\Delta u)u + (\Delta u)^2 \tag{1.149}$$

$$[\nabla u + \nabla(\Delta u)] \cdot [\nabla u + \nabla(\Delta u)] = \nabla u \cdot \nabla u + 2\nabla(\Delta u) \cdot \nabla u + \nabla(\Delta u) \cdot \nabla(\Delta u) \tag{1.150}$$

$$(u + \Delta u) \frac{\partial}{\partial n} (u + \Delta u) = u\frac{\partial u}{\partial n} + u\frac{\partial(\Delta u)}{\partial n} + \Delta u\frac{\partial u}{\partial n} + \Delta u\frac{\partial(\Delta u)}{\partial n} \tag{1.151}$$

These can be simplified by neglecting the last term of each expansion which is of order Δ^2. Doing so yields the approximation

$$\begin{aligned} F(u + \Delta u) \approx{}& \frac{1}{2}\iint_\Omega [-\nabla u \cdot \nabla u + k_0^2 u^2]\, d\Omega - \iint_\Omega fu\, d\Omega \\ &+ \frac{1}{2}\oint u \frac{\partial u}{\partial n}\, dl \\ &+ \Delta \iint_\Omega [-u\nabla u \cdot \nabla u + k_0^2 u^2]\, d\Omega \\ &+ \frac{\Delta}{2}\oint_C \left[u \frac{\partial u}{\partial n} + u \frac{\partial u}{\partial n}\right] dl - \Delta \iint_\Omega fu\, d\Omega \end{aligned} \tag{1.152}$$

When this expression is compared with that for $F(u)$ in (1.145), we have

$$\begin{aligned} F(u + \Delta u) \approx{}& F(u) + \Delta \iint_\Omega u[\nabla \cdot \nabla u + k_0^2 u - f]\, d\Omega \\ &- \Delta \oint_C u \frac{\partial u}{\partial n} dl + \frac{\Delta}{2}\oint_C \left[u \frac{\partial u}{\partial n} + u \frac{\partial u}{\partial n}\right] dl \end{aligned} \tag{1.153}$$

where we also used the divergence theorem (1.147) and the identity (1.146) to obtain the second and third terms. Clearly, the last two terms in (1.153) cancel each other leading to

$$\begin{aligned} \delta F &= F(u + \Delta u) - F(u) \\ &= \Delta \iint_\Omega u[\nabla^2 u + k_0^2 u - f]\, d\Omega \end{aligned} \tag{1.154}$$

Thus, setting $\delta F = 0$ implies that $\nabla^2 u + k_0^2 u = f$ provided Δu is nonzero. That is, from (1.154) we conclude that the extremization of F obtained by setting

$$\delta F = 0 \quad \text{or} \quad \frac{\partial F}{\partial u} = 0$$

is equivalent to enforcing the Helmholtz wave equation over the domain Ω. In practice, the condition $\delta F = 0$ is enforced by setting

$$\frac{\partial F}{\partial u} = \frac{F(u + \Delta u) - F(u)}{\Delta u}\bigg|_{\Delta \to 0} = \frac{\delta F}{\Delta u}\bigg|_{\Delta \to 0} = 0 \tag{1.155}$$

i.e., by setting to zero the derivative of the functional with respect to u.

Having established the equivalence between $\delta F = 0$ and the Helmholtz equation we can proceed with the discretization of $F(u)$ and δF to obtain a discrete system of equations. The discretization begins with the trial function, $\tilde{u}$, expanded in terms of N basis functions

$$\tilde{u} = \sum_{j=1}^{N} u_j w_j = \{u\}^T \{w\} \tag{1.156}$$

where w_j are the basis functions and u_j are the unknown expansion coefficients. In (1.156), column data vectors are denoted with $\{\cdot\}$ while row data vectors involve a transposition $\{\cdot\}^T$. Substituting (1.156) into (1.137), the functional becomes

$$F(\tilde{u}) = \tfrac{1}{2}\{u\}^T \left[\iint_\Omega \{w\} \mathcal{L}\{w\}^T \, d\Omega \right] \{u\} - \{u\}^T \iint_\Omega \{w\} f \, d\Omega \tag{1.157}$$

where we used the innerproduct definition

$$\langle \{u\}, \{v\} \rangle = \{u\}^T \{v\}$$

for discrete data vectors. This functional is extremized by allowing all partial derivatives with respect to the coefficients, $\{u\}$, to vanish

$$\frac{\partial}{\partial u_i} F(\tilde{u}) = \frac{1}{2} \left[\iint_\Omega w_i \mathcal{L}\{w\}^T \, d\Omega \right] \{u\} + \tfrac{1}{2}\{u\}^T \iint_\Omega \{w\} \mathcal{L} w_i \, d\Omega - \iint_\Omega w_i f \, d\Omega = 0 \tag{1.158}$$

A single equation is obtained by differentiating with respect to each u_i. For $i = 1, 2, \ldots, N$ we obtain N equations which can be written as a matrix system

$$[A]\{u\} = \{b\} \tag{1.159}$$

The elements of the matrix $[A]$ and excitation vector $\{b\}$ are given by

$$\begin{aligned} A_{ij} &= \iint_\Omega w_i \mathcal{L} w_j \, d\Omega \\ b_i &= \iint_\Omega w_i f \, d\Omega \end{aligned} \tag{1.160}$$

A word of caution: electromagnetics differs from other branches of engineering in that no physical significance can be attached to the stationary point of the functional (1.137). In mechanical systems, for example, minimizing this functional represents minimization of the total potential energy of the system. However, since electromagnetics involves complex quantities, such a statement may not be asserted.

1.11.2 Functionals for Anisotropic Media

In three dimensions with anisotropic media [15, 16] the appropriate operator is of the form

$$\mathcal{L}(\mathbf{u}) = \begin{cases} \nabla \times (\bar{\bar{\mu}}_r^{-1} \cdot \nabla \times \mathbf{u}) - k_0^2 \bar{\bar{\epsilon}}_r \cdot \mathbf{u}, & \mathbf{u} = \mathbf{E} \\ \nabla \times (\bar{\bar{\epsilon}}_r^{-1} \cdot \nabla \times \mathbf{u}) - k_0^2 \bar{\bar{\mu}}_r \cdot \mathbf{u}, & \mathbf{u} = \mathbf{H} \end{cases} \tag{1.161}$$

with the corresponding source function given by

$$\mathbf{f} = \begin{cases} -j\omega\mu_0 \mathbf{J} - \nabla \times (\bar{\bar{\mu}}_r^{-1} \cdot \mathbf{M}), & \mathbf{u} = \mathbf{E} \\ -j\omega\epsilon_0 \mathbf{M} + \nabla \times (\bar{\bar{\epsilon}}_r^{-1} \cdot \mathbf{J}), & \mathbf{u} = \mathbf{H} \end{cases} \tag{1.162}$$

The associated functional to be extremized is then of the form (for $\mathbf{u} = \mathbf{H}$)

$$F(\mathbf{H}) = \frac{1}{2}\int_{\Omega} [\nabla \times \mathbf{H} \cdot \bar{\bar{\epsilon}}_r^{-1} \cdot \nabla \times \mathbf{H} - k_0^2 \mathbf{H} \cdot \bar{\bar{\mu}}_r \cdot \mathbf{H}]\, dV - \int_{\Omega} \mathbf{H} \cdot \mathbf{f}\, dV \tag{1.163}$$

where

$$\bar{\bar{\epsilon}} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}, \qquad \bar{\bar{\mu}}_r = \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix} \tag{1.164}$$

are the permittivity and permeability tensors of the media and Ω represents a volume. In general, for arbitrary anisotropy, this functional will lead to an asymmetric (non-Hermitian) system. One way to obtain a symmetric system is to use the functional

$$F(\mathbf{u}) = \langle \mathcal{L}\mathbf{u}, \mathbf{u}_a \rangle - \langle \mathbf{u}, \mathbf{f}_a \rangle - \langle \mathbf{u}_a, \mathbf{f} \rangle \tag{1.165}$$

where $\mathbf{u}_a$ and $\mathbf{f}_a$ satisfy the partial differential equation

$$\mathcal{L}_a \mathbf{u}_a = f_a \tag{1.166}$$

in which $\mathcal{L}_a$ is the adjoint operator to $\mathcal{L}$. That is,

$$\langle \mathcal{L}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathcal{L}_a \mathbf{u} \rangle \tag{1.167}$$

1.11.3 Method of Weighted Residuals

The method of weighted residuals [17], [18] begins with the residual

$$\mathcal{R} = \mathcal{L}\tilde{u} - f \tag{1.168}$$

and seeks a solution for $\tilde{u} = u$ by satisfying the condition $\mathcal{R} = 0$ within the domain Ω. In general, such a solution cannot be achieved at all points in Ω. Instead, it is more practical to find a solution which satisfies the residual condition in some average or weighted sense over N subdomains of Ω, viz.

$$\int_{\Omega} [t_i \mathcal{L}\{w\}^T \{u\} - t_i f]\, d\Omega = 0, \qquad i = 1, 2, 3, \ldots, N \tag{1.169}$$

In general any testing function, t_i (also referred to as trial or weighting functions), may be used; however, since these functions modify the enforcement of the boundary conditions throughout the domain, the choice of testing functions affects the quality of the solution (1.169). One popular testing procedure is called *collocation* or *point matching*. In this, the testing or weighting function is a Dirac delta function, $t_i = \delta(x - x_i)$, which implies enforcement of the boundary conditions only at discrete points (e.g., x_i, $i = 1, 2, 3, \ldots, N$). Another popular choice is termed *Galerkin's procedure*. When employing the Galerkin's testing procedure, the testing function is identical to the expansion function used in (1.156), e.g., $t_i = w_i$ and the weighted residual equation is given by

$$\left[\int_{\Omega} w_i \mathcal{L} w_j\, d\Omega\right] \{u\} = \int_{\Omega} w_i f\, d\Omega \tag{1.170}$$

which is identical to the Ritz procedure given above. Thus, Galerkin's method leads to the same linear system (1.159) as the Ritz method.

As a generalization, when $F(u)$ is chosen as

$$F(u) = \tfrac{1}{2}\langle \mathcal{L}u, u^* \rangle - \langle f, u^* \rangle \tag{1.171}$$

where the "*" indicates complex conjugation, the extremization of $F(u)$ leads to a linear system that is identical to that obtained from the weighted residual method with $t_i = w_i^*$.

1.11.4 Vector and Matrix Norms in Linear Space

A norm is a real valued function that provides a measure of the size or "length" of a multicomponent mathematical quantity such as a vector or a matrix. It is used in numerical analysis to provide a measure of how well a given vector approximates the exact solution. For matrices, norms provide a single value to quantify the "size" of the matrix $[A]$. They are often used in evaluating the numerical system's condition, which in turn affects the stability of the solution. That is, of interest is how a small change in the excitation or right hand side of the matrix system (1.159) affects the solution data vector.

1.11.4.1 Vector Norms

EUCLIDEAN NORM. The most popular form for a given discrete data vector $\{u\} = \{u_1, u_2, u_3, \ldots, u_N\}^T$ is the *Euclidean* norm. It is defined by[3]

$$\|\mathbf{u}\|_2 = \sqrt{\sum_{i=1}^{N}(u_i)^2} = \langle \{u\}, \{u\} \rangle = \{u\}^T\{u\} \tag{1.172}$$

for real valued $\{u\}$ and by

$$\|\mathbf{u}\|_2 = \sqrt{\sum_{i=1}^{N}|u_i|^2} = \langle \{u\}, \{u\} \rangle = \{u\}^T\{u^*\} \tag{1.173}$$

for complex vector $\{u\}$. Here $|u_i|$ implies the absolute value of the quantity. Throughout the book, the notation $\|u\|$ will imply the Euclidean norm of a vector or data column unless otherwise noted.

INFINITY NORM. The infinity norm of a data vector is defined by

$$\|\mathbf{u}\|_\infty = \text{max of } |u_i| \qquad 1 \le i \le N \tag{1.174}$$

This norm is also referred to as the *uniform vector* norm or *maximum magnitude* norm.

HÖLDER NORM. The *Hölder* or *p-norm* is a generalization of the Euclidean norm and is defined as

$$\|\mathbf{u}_p\| = \left\{ \sum_{i=1}^{N} |u_i|^p \right\}^{1/p} \tag{1.175}$$

where $|u_i|^p$ denotes the pth power of the quantity $|u_i|$.

[3]Here **u** is a simpler notation for $\{u\}$.

1.11.4.2 Matrix Norms

FROBENIUS NORM. The *Frobenius* matrix norm is a generalization of the Euclidean vector norm. For a square matrix $[A]$, it is given by

$$\|\mathcal{A}\|_F = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}|A_{ij}|^2} \tag{1.176}$$

where A_{ij} denotes the (i,j) entry of the $[A]$ matrix.[4] Note that (1.176) can be generalized to nonsquare matrices and to the p-matrix norms.

MATRIX INFINITY NORM. The *infinity* or *uniform* matrix norm is defined by

$$\|\mathcal{A}\|_\infty = \max_{1\le j\le N} \text{ of } \left(\sum_{i=1}^{N}|A_{ij}|\right) \tag{1.177}$$

This specific norm is also referred to as the *row-sum* norm. Similarly the *column-sum* matrix norm is given by

$$\|\mathcal{A}\|_\infty = \max_{1\le i\le N} \text{ of } \left(\sum_{j=1}^{N}|A_{ij}|\right) \tag{1.178}$$

The infinity norm is referred to as the *natural* norm of $[A]$. It can be shown that

$$\|\mathcal{A}\|_\infty = \max_{\|u\|=1}(\|[A]\{u\}\|) = \max_{1\le i\le N} \text{ of } \left(\sum_{j=1}^{N}|A_{ij}|\right) \tag{1.179}$$

For Hermitian matrices (1.178) is identical to (1.177).

MATRIX CONDITION NUMBER. The condition of a matrix is related to the natural norm of $[A]$ as

$$\text{Cond}(\mathcal{A}) = \|\mathcal{A}\|\|\mathcal{A}^{-1}\| \ge \frac{|\lambda_{\max}|}{|\lambda_{\min}|} \tag{1.180}$$

where $\|\mathcal{A}^{-1}\|$ refers to the natural norm of the inverse of $[A]$ (the Frobenius norm is not a natural norm). Here $\lambda_{\max}$ and $\lambda_{\min}$ denote the maximum and minimum eigenvalues of the matrix, respectively. Since $|\lambda_{\max}|$ is a lower bound for the natural norm of $[A]$, the ratio $|\lambda_{\max}|/|\lambda_{\min}|$ gives a conservative estimate for the condition of $[A]$.

As an example of the importance of $\text{Cond}(\mathcal{A})$, let us assume that due to truncation or arithmetic errors, $[A]$ is instead approximated by $[A_\Delta]$. A computer with t decimal digits of accuracy gives

$$\frac{\|\mathcal{A}-\mathcal{A}_\Delta\|}{\|\mathcal{A}\|} = 10^{-t}$$

[4]Here we use the notation $\|\mathcal{A}\| = \|[A]\|$, i.e., the calligraphic capital letter implies a matrix. Such a notation will be used later in Chapter 9.

which is a measure of the normalized error in approximating $[A]$. If the condition of the matrix is $c = \text{Cond}(\mathcal{A})$, the corresponding (normalized) error in the computed solution $\{u\}$ will then be [19, 20]

$$\frac{\|\mathbf{u} - \mathbf{u}_\Delta\|}{\|\mathbf{u}\|} = 10^{-s}$$

where

$$s \geq t - \log_{10} c \tag{1.181}$$

That is, s is always smaller than t, implying a larger error for the final solution vector. More specifically, an error in the seventh decimal place for the norm of $[A]$ (i.e., $t = 7$) translates to an error in the third decimal place for (the norm of) the solution vector when the matrix condition number is 10^4. Alternatively, if $\text{Cond}(\mathcal{A}) = 1$, the solution vector error is of the same order (i.e., to the same decimal place) as the matrix error itself.

1.11.5 Some Matrix Definitions

In Table 1.1 we give the mathematical and descriptive definitions of matrices often encountered in numerical analysis.

Additional definitions can be found in several numerical mathematics books (e.g., see [19] and [20]). As can be understood, the operators which generated the matrices in Table 1.1 also carry the same definition. That is, an operator is referred to as Hermitian or self-adjoint if the resulting matrix is also Hermitian.

TABLE 1.1 Definitions of Matrices Often Encountered in Numerical Analysis

Mathematical Statement	Descriptive Statement
$[A]^T = [A]$	Symmetric
$[A] = [A^*]^T$	Hermitian (self-adjoint)
$[A] = -[A^*]^T$	Skew Hermitian
$[A]^{-1} = [A^*]^T$	Unitary
$[A^T][A] = [I]$ $[I]$ = identity matrix	Orthogonal
$[A^*]^T[A] = [A][A^*]^T$	Normal
$A_{ij} > 0$	Positive-definite
$A_{ij} \geq 0$	Nonnegative or positive semidefinite
$A_{ij} = 0$ for $i \neq j$	Diagonal
$A_{ij} = 0$ for $i > j$	Upper triangular
$A_{ij} = 0$ for $i \geq j$	Strictly upper triangular
$A_{ij} > \sum_{\substack{j=1 \\ i \neq j}}^{N} \lvert A_{ij} \rvert$ for all i	Diagonally dominant
$\{u\}^T[A]\{u\} > 0, \{u\} \neq 0$	Positive-definite
$\{u\}^T[A]\{u\} \geq 0, \{u\} \neq 0$	Nonnegative
$(\{u\}^T[A]\{u\})(\{v\}^T[A]\{v\}) < 0$ $\{u\} \neq 0, \{v\} \neq 0$	Indefinite

TABLE 1.2 Some Common Relationships Between Operators and Eigenvalues

Operator Type	Eigenvalue (λ) Properties
Hermitian	Real eigenvalues and >0
Unitary	Eigenvalues on unit circle
Skew Hermitian	Eigenvalues on imaginary axis
Positive semidefinite	Eigenvalues ≥ 0
Positive-definite	Eigenvalues >0
Indefinite	Some eigenvalues are >0 and some are <0

Given the nature of the matrix or operator, we can immediately make a statement about the eigenvalues of that operator. Some of the most common relationships between operators and eigenvalues are given in Table 1.2.

1.11.6 Comparison of Solution Methods and Their Convergence

The Rayleigh–Ritz and Galerkin's methods are standard solution approaches for solving differential equations arising in practical engineering problems. Both methods project a continuous space onto a finite separable Hilbert space.[5] The mathematical problem is then rephrased to seek a discrete solution set whose entries are the coefficients of the expansion. The premise of each method is summarized in Table 1.3. The third entry in the table is referred to as the *Least Squares Method* and is generally more robust than the other two, as will be noted later.

It was shown in Section 1.11.3 that Galerkin's method is equivalent to the Rayleigh–Ritz approach when the first variation is set to zero. At the heart of this statement is the assumption that the operator $\mathcal{L}$ (or the resultant discrete matrix $\mathcal{A}$) is positive-definite. Unfortunately, in most practical problems in electromagnetics, particularly as k_0 becomes larger, the operators $\mathcal{L}u = \nabla u^2 - k_0^2 u$ and $\mathcal{L}\mathbf{u} = \nabla \times (\mu_r^{-1} \nabla \times \mathbf{u}) - k_0^2 \epsilon_r \mathbf{u}$ do not guarantee positive-definiteness. That is, if the operator is not positive-definite, the Rayleigh–Ritz method fails to ensure minimization of the functional since a global stationary point may not exist. However, the application of Galerkin's method to yield a discrete system does not require that the operator is positive-definite or even symmetric.[6] The resultant solution is simply a stationary point which is not guaranteed to be a minimum.

TABLE 1.3 Mathematical Statement of Solution Methods

Method	Mathematical Statement (subject to boundary conditions)
Rayleigh–Ritz	minimize $\frac{1}{2}\langle \mathcal{L}u, u\rangle - \langle f, u\rangle$
Galerkin	solve $\langle \mathcal{L}u, u_j\rangle - \langle f, u_j\rangle = 0$
Least Squares	minimize $Q(u) = \langle \mathcal{L}u - f, \mathcal{L}u - f\rangle$

[5]Hilbert space refers to a linear space where a given interproduct has been defined and which is complete with respect to this interproduct.

[6]Because of the complex ϵ_r and μ_r, in electromagnetics, the operators may be symmetric but not Hermitian (i.e., self-adjoint). From Section 1.11.5, Hermitian operators have positive and real eigenvalues and are therefore positive.

In some cases, the problem statement is that of minimizing a functional and consequently the Rayleigh–Ritz procedure is the natural method of choice. Examples include problems associated with system energy minimizations and resonance computations. However, in solving the Helmholtz or vector wave equations (subject to given boundary conditions), the minimization of the functional must simultaneously imply a solution of those equations. For those cases, Galerkin's method is the appropriate approach for constructing the linear system. However, introduction of an appropriate variational functional can simplify the problem statement when dealing with boundary conditions other than Neumann-type (also referred to as natural conditions). As seen from the derivation given in Section 1.11.1, the presence of the boundary integral term provides a direct means for imposing the Dirichlet, impedance, or other type of boundary conditions. In the case of Galerkin's method, the boundary terms are introduced after application of the divergence theorem. Finally, we note that in some cases a variational (minimization) statement of the boundary value problem may not be possible. In those cases, Galerkin's method is the only approach for constructing a linear system of equations.

When dealing with operators that are not positive-definite, apart from the breakdown of the functional minimization process, most iterative linear system solvers also break down. Specifically, convergence of the approximate solution $\tilde{u}$ to the exact solution u cannot be proven [21], [22] without invoking positive-definiteness. When in doubt, a positive-definite operator (and a corresponding positive-definite matrix) can be generated by instead solving the differential equation

$$\mathcal{L}^a \mathcal{L} u = \mathcal{L}^a f \tag{1.182}$$

where $\mathcal{L}^a$ satisfies

$$\langle \mathcal{L} u, v \rangle = \langle u, \mathcal{L}^a v \rangle \tag{1.183}$$

and is referred to as the *adjoint* operator. Clearly, (1.182) is obtained by multiplying the right and left hand sides of $\mathcal{L}u = f$ by $\mathcal{L}^a$. The new operator $\mathcal{P}$, where

$$\mathcal{P} u = g \tag{1.184}$$

($g = \mathcal{L}^a f$) is now positive-definite and self-adjoint. The corresponding matrix system resulting from (1.184) is of the form

$$[A^*]^T [A]\{u\} = [A^*]^T \{f\} \tag{1.185}$$

or

$$[B]\{u\} = \{g\}$$

where $[B] = [A^*]^T [A]$. It is thus seen that the desired property of positive-definiteness comes at the price of squaring the matrix condition number. As is well known, large matrix conditions lead to less accurate solutions and slower convergence when an iterative solver is used.

It should be remarked that minimization of the functional for the *Least Squares Method* is equivalent to solving the differential equation (1.182). Consequently, the Least Squares Method leads to positive-definite systems at the expense of squaring the matrix condition. Also, the Least Squares Method minimizes the square of the norm (as $\tilde{u} \to u$), viz.

$$\lim_{N\to\infty} \|\mathcal{L}\tilde{u} - f\|^2 \to 0 \tag{1.186}$$

whereas Galerkin's method minimizes the norm

$$\lim_{N\to\infty} \|\mathcal{L}\tilde{u} - f\| \to 0 \tag{1.187}$$

Again, it is important to note that nothing can be said about convergence unless the operator is positive-definite.

1.11.7 Field Formulation Issues

The finite element method can assume various forms depending on the desired field quantity. Many applications prefer either a total or secondary electric field formulation. Other applications desire a result in terms of either the total or secondary magnetic field. Some applications can utilize a potential formulation. Thus, even though Maxwell's equations relate these various quantities, an accurate field computation often demands a particular formulation. The advantages and disadvantages of each of these formulations are discussed below.

The total electric field formulation is a very popular choice. This is because enforcement of the boundary conditions associated with perfect electric conductors (pec) is particularly easy. Since the tangential electric fields on a pec surface must vanish, the edges of the mesh associated with those surfaces are a priori set to zero. Three methods are commonly used in practice to enforce this condition. The first is accomplished by forcing a null field condition to zero out all entries of the matrix associated with that edge (except for the self-term which is set to unity), and by also setting the excitation entry to zero. Thus, as the unknown fields are solved, the edges lying on pec surfaces are forced to zero. The second method involves a preprocessing step where the edges associated with a pec surface are removed from the list of unknowns. Thus, the number of edges is greater than the number of unknowns and matrix entries for these pec edges are never computed. This approach has the advantage of reducing the order of the matrix and therefore reducing memory and compute cycle demands. The third method, useful when an iterative matrix solver is employed, involves forcing the unknowns associated with the pec edges to zero during each iteration.

Thus, use of a total electric field formulation and edge-based elements (see Chapter 2 for a discussion on edge-elements) reduces the order of the matrix and computational burden. However, this is not the only valid formulation and in certain circumstances, a scattered field or a magnetic field formulation may be preferred. Scattered (or secondary) field formulations are used to simplify the use of absorbing boundary conditions (see Chapters 4 and 6). However, they also have an added advantage when a boundary integral is used for mesh closure. Experience has shown that phase errors in the computed interior field tend to increase within the mesh at locations distant from boundaries on which boundary conditions are imposed. This is due to unavoidable numerical inaccuracies that increase as the effect of the boundary conditions propagate throughout the mesh. That is, previous errors in the adjoining field are incorporated and magnified as the field is evaluated at a more distant field point. Since boundary conditions always are enforced with total fields, the total field formulation enforces such conditions only on the boundaries of

the mesh and pec surfaces (for E-field formulations). For very large computational domains, significant distance can lie between a field point within the interior of the mesh and the mesh boundary. Hence, the potential for error propagation throughout the mesh. A scattered field formulation enforces the boundary conditions on the mesh boundary, pec surfaces, and dissimilar material interfaces. Therefore, the distance between a boundary condition and any interior field point is reduced and accordingly the phase error throughout the mesh may be also reduced. The scattered field formulation has the disadvantage of higher matrix order (i.e., more unknowns and equations) and explicit enforcement of the boundary conditions associated with pec surfaces and material discontinuities.

Magnetic field formulations are also possible and can be obtained by applying duality to the corresponding electric field equations. A total magnetic field formulation has the same advantages for a fictitious perfect magnetic conductor (pmc) as the total electric field has for pec surfaces. A scattered magnetic field formulation also can reduce phase error propagation in the same manner as the scattered electric field formulation.

Magnetic field formulations are preferred for applications where the desired result is the magnetic field within the computational domain. This is due to the fact that although Maxwell's equations relate the electric and magnetic fields, in practice one quantity cannot be accurately obtained from the other by numerical differentiation. This is due to the inherent error occurring when continuous derivatives are replaced with discrete differences. Rather, a computationally expensive integral expression is necessary for accurate field differentiation provided a suitable Green's function is available. Hence, an accurate solution demands a formulation consistent with the desired result.

Finally, some finite element practitioners utilize a potential formulation which employs the scalar or vector potentials as the unknown quantities (see Chapter 5). The use of this approach is related to the hybrid finite element-boundary integral method where the singularity of the integral equation associated with the boundary can be reduced with a potential formulation. However, if this reduction is present, we note that a numerical differentiation operation may be required to obtain the desired field quantity and this operation may lead to inaccuracy.

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