# Chapter 5

# Electromagnetic field solutions

In this chapter we consider solutions of the time-harmonic Maxwell equations for simple sources like the point source, line source and plane source in various media. Solutions for more complicated sources can be constructed from solutions corresponding to these simple sources.

## 5.1 The Green function

The field due to a source of unit amplitude plays a basic role in electromagnetics, because fields for arbitrary sources in linear media can be obtained by integrating such a field function. If the source is a unit point source, line source or plane source, the corresponding fields are called, respectively, the three-dimensional, two-dimensional or one-dimensional Green functions.

The equation for the three-dimensional scalar Green function in homogeneous medium is written as

$$L(\nabla)G(\mathbf{r}-\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'). \tag{5.1}$$

We shall only consider such linear operators $L(\nabla)$ which are polynomial functions of the operator $\nabla$. The minus sign in front of the delta function is a convention which saves us from a minus sign of the Green function. In two dimensions, the vector variable is $\boldsymbol{\rho}$ and in one dimension, the scalar coordinate $z$, instead of the three-dimensional position vector $\mathbf{r}$. The two- and one-dimensional Green functions will be treated as special cases of the three-dimensional problem.

Knowing the Green function, the field of any source in a homogeneous medium can be written as an integral:

$$f(\mathbf{r}) = -\int_V G(\mathbf{r}-\mathbf{r}')g(\mathbf{r}')dV', \tag{5.2}$$

because operating this equation by $L(\nabla)$ from the left and applying properties of the delta function, the equation

$$L(\nabla)f(\mathbf{r}) = g(\mathbf{r}) \tag{5.3}$$

results. Here, the order of the differential operator $L$ and the integration was simply interchanged, which is valid if fields are calculated outside the source region. Field computation inside the source region requires a knowledge of the Green function singularities, to be discussed in a subsequent section.

Since the Green function performs a linear mapping from the source function $\mathbf{g}(\mathbf{r})$ to the field function $\mathsf{f}(\mathbf{r})$, for a scalar field problem, the Green function is a scalar quantity, whereas for a vector problem a dyadic Green function is obviously required. The Green function is essentially dependent on the medium. It is very difficult to derive expressions for other than homogeneous media. Also, closed-form expressions have only been obtained for homogeneous bi-isotropic and uniaxially anisotropic media and not for more general media.

### 5.1.1 Green dyadics of polynomial operators

Let us consider, in general terms, the solution of dyadic equations of the type

$$\overline{\overline{L}}(\nabla)\cdot\overline{\overline{G}}(\mathbf{r}) = -\delta(\mathbf{r})\overline{\overline{I}}. \tag{5.4}$$

The dyadic operator $\overline{\overline{L}}(\nabla)$ is assumed to be a polynomial of $\nabla$,

$$\overline{\overline{L}}(\nabla) = \mathcal{L}_2 + \mathcal{L}_3\cdot\nabla + \mathcal{L}_4 : \nabla\nabla + ..., \tag{5.5}$$

where $\mathcal{L}_2$ is a constant dyadic, $\mathcal{L}_3$ a constant triadic, $\mathcal{L}_4$ a constant tetradic etc. The order of the operator is the highest power of $\nabla$ in the polynomial. In the following we consider some methods of reducing the dyadic equation (5.4) to a scalar equation.

Let us define the dyadic operator $\overline{\overline{L}}(\nabla)$ to be *complete* if there exists a vector $\mathbf{p}$ such that the algebraic dyadic $\overline{\overline{L}}(\mathbf{p})$ is complete. Otherwise $\overline{\overline{L}}(\nabla)$ is called a *planar* operator. For example, the operator $\nabla\nabla+k^2\overline{\overline{I}}$ is complete for $k\neq 0$, because $\det(\mathbf{pp}+k^2\overline{\overline{I}}) = k^4(\mathbf{p}\cdot\mathbf{p}+k^2)$ is not identically zero. On the other hand, $\nabla\nabla{}^\times_\times\overline{\overline{I}}$ is planar because $\det(\mathbf{pp}{}^\times_\times\overline{\overline{I}}) = 0$ for all $\mathbf{p}$. The rule that the dot product of two dyadics is complete only if the dyadics are complete, valid for algebraic dyadics, does not hold for operator dyadics. For example, the dot product of the planar operator $\nabla\nabla$ and the linear dyadic $\mathbf{rr}$ equals $4\overline{\overline{I}}$, which is complete. In the following, we only consider complete operators $\overline{\overline{L}}(\nabla)$.

All dyadic identities are valid for dyadic operators if the result can be understood as an operator. The following identity is essential in solving Green dyadic equations:

$$\overline{\overline{L}}^{(2)T}(\nabla)\cdot\overline{\overline{L}}(\nabla) = \overline{\overline{L}}(\nabla)\cdot\overline{\overline{L}}^{(2)T}(\nabla) = [\det\overline{\overline{L}}(\nabla)]\,\overline{\overline{I}}, \tag{5.6}$$

because the dyadic equation can be reduced to solving a scalar Green function. In fact, starting from a scalar problem of the form

$$\mathrm{det}\overline{\overline{L}}(\nabla)\, G(\mathbf{r}) = -\delta(\mathbf{r}), \tag{5.7}$$

multiplying both sides by the unit dyadic and inserting (5.6) results in the equation

$$[\mathrm{det}\overline{\overline{L}}(\nabla)]G(\mathbf{r})\overline{\overline{I}} = \overline{\overline{L}}^{(2)T}(\nabla)\cdot[\overline{\overline{L}}(\nabla)G(\mathbf{r})] =$$
$$\overline{\overline{L}}(\nabla)\cdot[\overline{\overline{L}}^{(2)T}(\nabla)G(\mathbf{r})] = -\delta(\mathbf{r})\overline{\overline{I}}. \tag{5.8}$$

Comparing this to (5.4) shows us that the Green dyadic corresponding to the operator $\overline{\overline{L}}(\nabla)$ can be written in terms of $G(\mathbf{r})$, the solution to the scalar equation (5.7) in the form

$$\overline{\overline{G}}(\mathbf{r}) = \overline{\overline{L}}^{(2)T}(\nabla)\, G(\mathbf{r}). \tag{5.9}$$

This conclusion presumes uniqueness, which is obtained by imposing Sommerfeld and Silver–Müller radiation conditions at infinity.

If, instead of $\overline{\overline{L}}(\nabla)$, the operator in (5.4) is of the form $\overline{\overline{L}}^{(2)T}(\nabla)$, the corresponding Green dyadic $\overline{\overline{F}}(\mathbf{r})$, i.e. the solution of

$$\overline{\overline{L}}^{(2)T}(\nabla)\cdot\overline{\overline{F}}(\mathbf{r}) = -\delta(\mathbf{r})\overline{\overline{I}}, \tag{5.10}$$

can be written as

$$\overline{\overline{F}}(\mathbf{r}) = \overline{\overline{L}}(\nabla)\, G(\mathbf{r}), \tag{5.11}$$

where $G(\mathbf{r})$ satisfies (5.7). The two dyadic Green functions $\overline{\overline{G}}$ and $\overline{\overline{F}}$ corresponding to the operators $\overline{\overline{L}}$ and $\overline{\overline{L}}^{(2)T}$ have the simple relation

$$\overline{\overline{G}}(\mathbf{r}) = \frac{1}{2}\overline{\overline{L}}^{T}(\nabla)\,{}^{\times}_{\times}\,\overline{\overline{F}}(\mathbf{r}), \tag{5.12}$$

as can be quite easily seen. Knowing $\overline{\overline{F}}$, the other Green dyadic $\overline{\overline{G}}$ can be directly written down.

Thus, instead of solving a dyadic equation (5.1) for nine scalar functions it is sufficient to solve a single scalar function from the equation (5.7), provided the operator $\overline{\overline{L}}(\nabla)$ is complete so that the polynomial operator $\mathrm{det}\overline{\overline{L}}(\nabla)$ is non-zero. However, the price is paid in the order of the operator. In general, if the order of $\overline{\overline{L}}$ is $n$, the order of $\overline{\overline{L}}^{(2)}$ is $2n$ and that of $\mathrm{det}\overline{\overline{L}}$, $3n$. For the Helmholtz operator $\overline{\overline{L}}(\nabla)$, of special interest here, $\mathrm{det}\overline{\overline{L}}(\nabla)$ turns out to be at most of the fourth order. If the fourth-order equation can be solved, the dyadic Green function is straightforwardly obtained through application of the dyadic operator.

The order may be reduced if the original operator dyadic can be written in terms of another *polynomial* dyadic operator $\overline{\overline{A}}(\nabla)$ in the form $\overline{\overline{L}}(\nabla) = \overline{\overline{A}}^{(2)T}(\nabla)$. If this can be done, the equation is of the form (5.10), with $\overline{\overline{A}}(\nabla)$ replacing $\overline{\overline{L}}(\nabla)$, and the operator in (5.7) is in this case $\det\overline{\overline{A}}(\nabla)$, which is generally of lower order than $\det\overline{\overline{L}}(\nabla)$.

### 5.1.2 Examples of operators

Let us consider some operator triples $\overline{\overline{L}}(\nabla)$, $\overline{\overline{L}}^{(2)T}(\nabla)$ and $\det\overline{\overline{L}}(\nabla)$, appearing in electromagnetic problems.

In the first example, the order of all three operators happens to be the same:

$$\overline{\overline{L}}(\nabla) = \nabla\nabla + k^2\overline{\overline{I}}, \tag{5.13}$$

$$\overline{\overline{L}}^{(2)}(\nabla) = k^2(\nabla\nabla{}^{\times}_{\times}\overline{\overline{I}} + k^2\overline{\overline{I}}), \tag{5.14}$$

$$\det\overline{\overline{L}}(\nabla) = k^4(\nabla^2 + k^2). \tag{5.15}$$

The second example involves a constant dyadic $\overline{\overline{\alpha}}$ in a dyadic operator of the first order:

$$\overline{\overline{L}}(\nabla) = \nabla \times \overline{\overline{I}} + \overline{\overline{\alpha}}, \tag{5.16}$$

$$\overline{\overline{L}}^{(2)}(\nabla) = \nabla\nabla + (\nabla \cdot \overline{\overline{\alpha}}) \times \overline{\overline{I}} - \nabla(\overline{\overline{\alpha}}{}^{\times}_{\cdot}\overline{\overline{I}}) + \overline{\overline{\alpha}}^{(2)}, \tag{5.17}$$

$$\det\overline{\overline{L}}(\nabla) = \nabla\nabla : \overline{\overline{\alpha}} + \overline{\overline{\alpha}}^{(2)} : (\nabla \times \overline{\overline{I}}) + \det\overline{\overline{\alpha}}. \tag{5.18}$$

If $\overline{\overline{\alpha}}$ is symmetric, one term can be seen to drop off from the expressions $\overline{\overline{L}}^{(2)T}$ and $\det\overline{\overline{L}}$. If $\overline{\overline{\alpha}}$ is antisymmetric, the operator $\overline{\overline{L}}(\nabla)$ is planar and $\det\overline{\overline{L}}(\nabla) = 0$.

The third example involves a constant dyadic $\overline{\overline{\alpha}}$ in a dyadic operator of the second order:

$$\overline{\overline{L}}(\nabla) = \nabla\nabla + k^2\overline{\overline{\alpha}}, \tag{5.19}$$

$$\overline{\overline{L}}^{(2)}(\nabla) = k^2(\nabla\nabla{}^{\times}_{\times}\overline{\overline{\alpha}} + k^2\overline{\overline{\alpha}}^{(2)}), \tag{5.20}$$

$$\det\overline{\overline{L}}(\nabla) = k^4(\nabla\nabla : \overline{\overline{\alpha}}^{(2)} + k^2\det\overline{\overline{\alpha}}). \tag{5.21}$$

All operators are seen to be of the second order.

The fourth example involves two constant dyadics $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$:

$$\overline{\overline{L}}(\nabla) = \nabla\nabla{}^{\times}_{\times}\overline{\overline{\beta}} + k^2\overline{\overline{\alpha}}, \tag{5.22}$$

$$\overline{\overline{L}}^{(2)}(\nabla) = (\nabla\nabla : \overline{\overline{\beta}}^{(2)})\nabla\nabla + k^2(\nabla\nabla{}^{\times}_{\times}\overline{\overline{\beta}}){}^{\times}_{\times}\overline{\overline{\alpha}} + k^4\overline{\overline{\alpha}}^{(2)}, \tag{5.23}$$

$$\det\overline{\overline{L}}(\nabla) = k^2 \left[(\nabla\nabla : \overline{\overline{\beta}}^{(2)})(\nabla\nabla : \overline{\overline{\alpha}}) + k^2\nabla\nabla : (\overline{\overline{\alpha}}^{(2)} {}^{\times}_{\times} \overline{\overline{\beta}}) + k^4\det\overline{\overline{\alpha}}\right]. \tag{5.24}$$

It is seen to lead to fourth-order operators unless $\overline{\overline{\beta}}$ is linear or $\overline{\overline{\alpha}}$ antisymmetric.

As the last example we consider a composite operator:

$$\overline{\overline{L}}(\nabla) = \overline{\overline{A}}(\nabla) \cdot \overline{\overline{\beta}} \cdot \overline{\overline{B}}(\nabla) + \overline{\overline{\alpha}}, \tag{5.25}$$

$$\overline{\overline{L}}^{(2)}(\nabla) = \overline{\overline{A}}^{(2)} \cdot \overline{\overline{\beta}}^{(2)} \cdot \overline{\overline{B}}^{(2)} + (\overline{\overline{A}} \cdot \overline{\overline{\beta}} \cdot \overline{\overline{B}}) {}^{\times}_{\times} \overline{\overline{\alpha}} + \overline{\overline{\alpha}}^{(2)}, \tag{5.26}$$

$$\det\overline{\overline{L}}(\nabla) = (\det\overline{\overline{A}})(\det\overline{\overline{\beta}})(\det\overline{\overline{B}}) + (\overline{\overline{A}}^{(2)} \cdot \overline{\overline{\beta}}^{(2)} \cdot \overline{\overline{B}}^{(2)}) : \overline{\overline{\alpha}} + (\overline{\overline{A}} \cdot \overline{\overline{\beta}} \cdot \overline{\overline{B}}) : \overline{\overline{\alpha}}^{(2)} + \det\overline{\overline{\alpha}}. \tag{5.27}$$

The orders of these derived operators depend on the original operators $\overline{\overline{A}}(\nabla)$ and $\overline{\overline{B}}(\nabla)$. If $\overline{\overline{L}}(\nabla)$ is the Helmholtz operator of a bianisotropic medium, the operators $\overline{\overline{A}}(\nabla)$ and $\overline{\overline{B}}(\nabla)$ are of the form (5.16), which leads to fourth-order operators $\overline{\overline{L}}^{(2)}(\nabla)$ and $\det\overline{\overline{L}}(\nabla)$.

### References

LINDELL, I.V. (1973). On the theory of Green's dyadics of polynomial operators. *Helsinki University of Technology, Radio Laboratory Report* **S58**.

## 5.2 Green functions for homogeneous media

### 5.2.1 Isotropic medium

The dyadic Green function for the isotropic homogeneous space with parameters $\epsilon$, $\mu$ is one of the most useful mathematical concepts in formulating electromagnetic problems. In the isotropic case, the Green function, originally a function of a vector variable, $\overline{\overline{G}}(\mathbf{r} - \mathbf{r}')$ is a function of a scalar variable only, $\overline{\overline{G}}(D)$, where $D$ is the distance between the two points $\mathbf{r}$ and $\mathbf{r}'$, in real space a positive number $D = |\mathbf{r} - \mathbf{r}'|$. This property is due to the fact that there is no preferred direction in the space, i.e. the Green function is invariant to all rotations of the space.

*The scalar Green function*

Consider the basic scalar Helmholtz operator problem in isotropic space:

$$H(\nabla)G(D) = (\nabla^2 + k^2)G(D) = -\delta(\mathbf{r} - \mathbf{r}'). \tag{5.28}$$

The solution satisfying the Sommerfeld radiation condition is of the form

$$G(D) = \frac{e^{-jkD}}{4\pi D}, \tag{5.29}$$

where the distance function $D$ is denoted by

$$D(\mathbf{r}, \mathbf{r}') = \sqrt{(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}. \tag{5.30}$$

We may check the result (5.29) by writing

$$\nabla \cdot \nabla G(\mathbf{r} - \mathbf{r}') = \nabla \cdot \left[ -(jk + \frac{1}{D}) G \nabla D \right] =$$

$$\left[ \frac{1}{D^2} + (jk + \frac{1}{D})^2 \right] \nabla D \cdot \nabla D - (jk + \frac{1}{D}) G \nabla^2 D, \tag{5.31}$$

and for $D \neq 0$,

$$\nabla D = \frac{\mathbf{r} - \mathbf{r}'}{D}, \quad \nabla^2 D = \frac{2}{D}. \tag{5.32}$$

These substituted in (5.31) gives us $\nabla \cdot \nabla G(\mathbf{r} - \mathbf{r}') = -k^2 G$ for either branch of $D$ and any $\mathbf{r}'$, which may also be a complex vector.

At the point with $D = 0$ the Green function is singular, and so also is the term $\nabla^2 G(D)$. This latter singularity is obviously more pronounced and is thus responsible for producing the delta function on the right-hand side of (5.28). The nature of the singularity can be checked by multiplying $(\nabla^2 + k^2) G(\mathbf{r} - \mathbf{r}')$ with an arbitrary function $f(\mathbf{r})$, continuous at $\mathbf{r} = \mathbf{r}'$, and integrating over the real space. Considering first only real vectors $\mathbf{r}'$, we can put $\mathbf{r}' = 0$ for simplicity and without losing anything from generality. Thus, $D = 0$ at the point $\mathbf{r} = 0$ and the integrand vanishes everywhere except at the origin. Surrounding this by a small sphere with radius $r_o$, we obtain, applying partial integration in the small volume $V_o$ with the surface $S_o$,

$$\int_{V_o} f(\mathbf{r}) \nabla^2 \frac{e^{-jkD}}{D} \, dV \to \int_{V_o} \nabla \cdot \left( f(\mathbf{r}) \nabla \frac{1}{r} \right) dV - \int_{V_o} (\nabla f(\mathbf{r})) \cdot \nabla \frac{1}{r} \, dV \to$$

$$f(0) \oint_{S_o} \mathbf{u}_r \cdot \nabla \frac{1}{r} dS + \nabla f(0) \cdot \int_{V_o} \frac{\mathbf{u}_r}{r^2} dV = -f(0) \oint_{S_o} \frac{dS}{r_o^2} = -4\pi f(0). \tag{5.33}$$

The last volume integral disappears because of radial symmetry. Thus, from the definition of the delta function, the solution (5.29) for (5.28) has been established for $\mathbf{r}' = 0$ and, through analytical continuation, for any $\mathbf{r}'$.

*The Green dyadic*

The Green dyadic in an isotropic and homogeneous medium satisfies the vector Helmholtz equation

$$\overline{\overline{H}}(\nabla)\cdot\overline{\overline{G}}(D) = -\delta(\mathbf{r}-\mathbf{r}')\overline{\overline{I}}, \tag{5.34}$$

where the Helmholtz operator is defined as

$$\overline{\overline{H}}(\nabla) = \nabla\nabla{}^{\times}_{\times}\overline{\overline{I}} + k^2\overline{\overline{I}}. \tag{5.35}$$

We may now apply the previous method for the solution of (5.34). In fact, identifying $\overline{\overline{H}}(\nabla)$ with $k^{-2}\overline{\overline{L}}^{(2)}(\nabla)$ of (5.14), the solution can be written as $\overline{\overline{G}} = k^2\overline{\overline{F}}$ with $\overline{\overline{F}}$ defined in (5.11) with the operator $\overline{\overline{L}}(\nabla)$ defined in (5.13) and the scalar equation in (5.15). The scalar Green function is given by (5.29) divided by $k^4$, whence the dyadic Green function is

$$\overline{\overline{G}}(D) = (\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla)G(D). \tag{5.36}$$

(5.36) is a very useful result and it forms the basis for integral equation formulation of electromagnetic problems, as first done by LEVINE and SCHWINGER (1950). The Green dyadic is usually given in this form without performing the differentiations, because in field computation the gradients can often be simplified through partial integrations. The Green dyadic appears highly singular owing to the double differentiation of the $1/D$ dependent scalar Green function. It is important to know how fields can be calculated inside a source region where the distance function becomes zero. The singularity will be studied in a subsequent section.

The electromagnetic fields due to and outside a current function $\mathbf{J}(\mathbf{r})$ can be written as integrals of the Green dyadic:

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu\int_V \overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')dV', \tag{5.37}$$

$$\mathbf{H}(\mathbf{r}) = \int_V \nabla G(\mathbf{r}-\mathbf{r}')\times\mathbf{J}(\mathbf{r}')dV'. \tag{5.38}$$

From duality, the fields from a magnetic current $\mathbf{J}_m(\mathbf{r})$ are

$$\mathbf{H}(\mathbf{r}) = -j\omega\epsilon\int_V \overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}_m(\mathbf{r}')dV'. \tag{5.39}$$

$$\mathbf{E}(\mathbf{r}) = -\int_V \nabla G(\mathbf{r}-\mathbf{r}')\times\mathbf{J}_m(\mathbf{r}')dV'. \tag{5.40}$$

### 5.2.2 Bi-isotropic medium

A slightly different method can be applied to finding the Green dyadic for the bi-isotropic medium with the parameters $\epsilon$, $\mu$, $\kappa$ and $\chi$. The Helmholtz operator applicable in this case can be written in product form

$$\overline{\overline{H}}(\nabla) = -(\nabla\times\overline{\overline{I}}-j\omega\xi\overline{\overline{I}})\cdot(\nabla\times\overline{\overline{I}}+j\omega\zeta\overline{\overline{I}})+\omega^2\mu\epsilon\overline{\overline{I}} = -\overline{\overline{L}}_+(\nabla)\cdot\overline{\overline{L}}_-(\nabla), \quad (5.41)$$

in terms of two first-order dyadic operators

$$\overline{\overline{L}}_\pm(\nabla) = \nabla\times\overline{\overline{I}} \mp k_\pm\overline{\overline{I}}, \qquad k_\pm = k_o(\sqrt{n^2-\chi^2}\pm\kappa), \quad (5.42)$$

which are both of the form (5.16) with $\overline{\overline{\alpha}} = \mp k_\pm\overline{\overline{I}}$.

The solution is obtained by first solving the scalar Green function $G(D)$ satisfying

$$\det(\overline{\overline{H}})G(D) = -\det(\overline{\overline{L}}_+)\det(\overline{\overline{L}}_-)G(D) = -\delta(\mathbf{r}-\mathbf{r}'). \quad (5.43)$$

Because the operator is factorized in terms of two determinant operators

$$\det(\overline{\overline{L}}_\pm) = \mp k_\pm(\nabla^2+k_\pm^2), \quad (5.44)$$

we first try to find the solution for $G$ in terms of a linear combination

$$G(D) = A_+G_+(D) + A_-G_-(D) \quad (5.45)$$

of solutions for two second-order equations

$$(\nabla^2+k_\pm)G_\pm(D) = -\delta(\mathbf{r}-\mathbf{r}'), \qquad G_\pm(D) = \frac{e^{-jk_\pm D}}{4\pi D}. \quad (5.46)$$

By substitution we have a condition for the coefficients $A_+$ and $A_-$:

$$(A_+ + A_-)\nabla^2\delta + (A_+k_-^2 + A_-k_+^2)\delta = \frac{1}{k_+k_-}\delta. \quad (5.47)$$

Requiring that the coefficients of both $\nabla^2\delta$ and $\delta$ vanish, we have two equations which can be solved to give

$$A_\pm = \mp\frac{1}{k_+k_-}\frac{1}{k_+^2-k_-^2}. \quad (5.48)$$

This inserted in (5.45) gives us the scalar Green function:

$$G(D) = -\frac{1}{k_+k_-}\frac{1}{k_+^2-k_-^2}[G_+(D)-G_-(D)]. \quad (5.49)$$

Finally, we can write for the Green dyadic corresponding to the bi-isotropic medium the expression

$$\overline{\overline{G}}(D) = \overline{\overline{H}}^{(2)T} G_e(D) = \overline{\overline{L}}_+{}^{(2)T} \cdot \overline{\overline{L}}_-{}^{(2)T} G(D), \tag{5.50}$$

$$\overline{\overline{L}}_\pm{}^{(2)T} = \nabla\nabla \pm k_\pm \nabla \times \overline{\overline{I}} + k_\pm^2 \overline{\overline{I}}. \tag{5.51}$$

Substituting the operators, we have for the final expression

$$\overline{\overline{G}}(D) = \frac{1}{k_+ + k_-}\left[\nabla\nabla(\frac{G_+}{k_+} + \frac{G_-}{k_-})+\right.$$

$$\left.\nabla(G_+ - G_-) \times \overline{\overline{I}} + (k_+ G_+ + k_- G_-)\overline{\overline{I}}\right]. \tag{5.52}$$

The Green dyadic involves two scalar Green functions (5.46), each with a different exponential factor. Together they represent a combination of two radially propagating electromagnetic wave components with two different phase velocities. It is easy to check that for the non-chiral limiting case $k_+ \to k_- \to k = k_o\sqrt{n^2 - \chi^2}$, the bi-isotropic Green dyadic reduces to one similar to the isotropic space Green dyadic (5.36).

The same result could be obtained by splitting the unit source into two self-dual parts and computing the corresponding self-dual fields using the corresponding isotropic Green dyadics, whence adding up the results for the total field, the Green dyadic (5.52) can be identified.

### 5.2.3 Anisotropic medium

The Green dyadic problem for any bianisotropic media can also be reduced to a scalar Green function problem, but the equation will be of the fourth order in general, whence it is not solvable in terms of special functions. In some special cases, however, the problem can be reduced to solving second-order equations. In the non-isotropic case, the space is not invariant to all rotations, whence we cannot write the Green functions as functions of a single scalar variable $D$. Instead, they must be written as functions of the vector variable $\mathbf{r} - \mathbf{r}'$. Let us restrict, for simplicity, the problem to anisotropic media with $\overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$.

*General anisotropic medium*

The Green dyadic problem for the general anisotropic medium corresponding to the electric Helmholtz operator is

$$\overline{\overline{H}}_e(\nabla) \cdot \overline{\overline{G}}_e(\mathbf{r} - \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')\overline{\overline{I}}, \qquad \overline{\overline{H}}_e(\nabla) = \nabla\nabla {}^{\times}_{\times} \overline{\overline{\mu}}^{-1} + \omega^2 \overline{\overline{\epsilon}} \tag{5.53}$$

and the electric field corresponding to a current source $\mathbf{J}(\mathbf{r})$ can be written as

$$\mathbf{E}(\mathbf{r}) = j\omega \int_V \overline{\overline{G}}_e(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')dV'. \tag{5.54}$$

The method of reducing (5.53) to a scalar equation presented in the previous section can be applied in a straightforward manner. The operator $\overline{\overline{H}}_e(\nabla)$ is of the form (5.22), whence the Green dyadic can be expressed in the form (5.9) with (5.23) inserted:

$$\overline{\overline{G}}_e(\mathbf{r}-\mathbf{r}') =$$

$$\left[(\nabla\nabla:\overline{\overline{\mu}}^{(-2)})\nabla\nabla+\omega^2(\nabla\nabla{}^{\times}_{\times}\overline{\overline{\mu}}^{-1}){}^{\times}_{\times}\overline{\overline{\epsilon}}+\omega^4\overline{\overline{\epsilon}}^{(2)}\right]^T G_e(\mathbf{r}-\mathbf{r}'). \tag{5.55}$$

The scalar Green function satisfies

$$\mathrm{det}\overline{\overline{H}}_e(\nabla)\; G_e(\mathbf{r}-\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'), \tag{5.56}$$

$$\mathrm{det}\overline{\overline{H}}_e(\nabla) = \omega^2(\nabla\nabla:\frac{\overline{\overline{\mu}}}{\mathrm{det}\overline{\overline{\mu}}})(\nabla\nabla:\overline{\overline{\epsilon}})+\omega^4\nabla\nabla:(\overline{\overline{\epsilon}}^{(2)}{}^{\times}_{\times}\overline{\overline{\mu}}^{-1})+\omega^6\mathrm{det}\overline{\overline{\epsilon}}. \tag{5.57}$$

Unfortunately, in general, this is a partial differential equation of fourth order and there are no special functions to express its general solution. For some special dyadics $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$ the fourth-order operator $\mathrm{det}\overline{\overline{H}}_e(\nabla)$ can be factorized to a product of two second-order operators. Let us study this possibility closer. Because of missing first- and third-order terms, the scalar operator (5.57) can be written in the following form after some algebraic manipulations:

$$\mathrm{det}\overline{\overline{H}}_e(\nabla) = \omega^2[\overline{\overline{\epsilon}}:\nabla\nabla+\alpha][\overline{\overline{\mu}}^{(-2)}:\nabla\nabla+\frac{\omega^4}{\alpha}\mathrm{det}\overline{\overline{\epsilon}}]-$$

$$\frac{\omega^2}{\alpha}(\alpha\overline{\overline{\mu}}^{-1}-\omega^2\overline{\overline{\epsilon}}^{(2)})^{(2)}:\nabla\nabla. \tag{5.58}$$

This expression equals (5.57) for any finite and non-zero value of the parameter $\alpha$.

Obviously, in order for the operator $\mathrm{det}\overline{\overline{H}}_e(\nabla)$ to be of factorized form, the last term in (5.58) should be zero. Because the symmetric part of the dyadic which double-dot multiplies the $\nabla\nabla$ operator must then vanish, the dyadic itself must be antisymmetric. But since it is of the form $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}}$, it cannot be strictly planar like the antisymmetric dyadic, whence we conclude that the dyadic itself must vanish. Thus, we have found the condition for factorizing the determinant operator by requiring that there exist a scalar $\alpha$ such that the following condition between the $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ dyadics be valid:

$$(\alpha\overline{\overline{\mu}}^{-1} - \omega^2\overline{\overline{\epsilon}}^{(2)})^{(2)} = 0. \tag{5.59}$$

This implies that the dyadic inside the brackets must be a linear dyadic, or there exist vectors **a**, **b** and a second scalar $\beta$ such that we can write

$$\overline{\overline{\epsilon}}^{-1} \cdot \overline{\overline{\mu}} = \beta\overline{\overline{I}} + \mathbf{ab}. \tag{5.60}$$

This condition resembles that for a medium which was earlier called affinely uniaxial. In fact, if either $\overline{\overline{\epsilon}}$ or $\overline{\overline{\mu}}$ is a symmetric dyadic, the medium parameter dyadics can be obtained from those of a uniaxial medium through an affine transformation.

*Affinely isotropic medium*

Affinely isotropic media with symmetric parameter dyadics $\overline{\overline{\epsilon}} = \epsilon_o\overline{\overline{\epsilon}}_r$, $\overline{\overline{\mu}} = \mu_o\overline{\overline{\mu}}_r$ satisfying the condition $\overline{\overline{\mu}}_r = \gamma\overline{\overline{\epsilon}}_r$ also satisfy the condition (5.59) with

$$\alpha = \omega^2\epsilon_o^2\gamma\mathrm{det}\overline{\overline{\epsilon}}_r, \tag{5.61}$$

and, thus, the corresponding scalar Helmholtz operator can be factorized. To avoid solving fourth-order differential equations at all, we can write the original dyadic Helmholtz operator $\overline{\overline{H}}_e(\nabla)$ in the form $\overline{\overline{A}}^{(2)}(\nabla)$ with a suitable polynomial operator $\overline{\overline{A}}(\nabla)$. In fact, comparing (5.53) and (5.20), we have

$$\overline{\overline{H}}_e(\nabla) = \frac{1}{k_o^2\mu_o\gamma^2\mathrm{det}\overline{\overline{\epsilon}}_r}(\nabla\nabla + k_o^2\gamma\overline{\overline{\epsilon}}_r{}^{(2)})^{(2)}. \tag{5.62}$$

Thus, the Green dyadic must be of the form $\overline{\overline{A}}^T(\nabla)G_e(\mathbf{r}-\mathbf{r}')$, or

$$\overline{\overline{G}}_e(\mathbf{r}-\mathbf{r}') = \frac{1}{k_o\gamma\sqrt{\mu_o\mathrm{det}\overline{\overline{\epsilon}}_r}}(\nabla\nabla + k_o^2\gamma\overline{\overline{\epsilon}}_r{}^{(2)})G_e(\mathbf{r}-\mathbf{r}'), \tag{5.63}$$

and the scalar Green function $G_e(\mathbf{r}-\mathbf{r}')$ satisfies the equation

$$\mathrm{det}\overline{\overline{A}}(\nabla)G_e = \frac{1}{k_o^3\gamma^3(\mu_o\mathrm{det}\overline{\overline{\epsilon}}_r)^{3/2}}\mathrm{det}[\nabla\nabla + k_o^2\gamma\overline{\overline{\epsilon}}_r{}^{(2)}]G_e =$$

$$\frac{k_o}{\mu_o^{3/2}\sqrt{\mathrm{det}\overline{\overline{\epsilon}}_r}}(\nabla\nabla : \overline{\overline{\epsilon}}_r + k_a^2)G_e = -\delta(\mathbf{r}-\mathbf{r}'), \tag{5.64}$$

with $k_a = k_o\sqrt{\gamma\mathrm{det}\overline{\overline{\epsilon}}_r}$.

The scalar function $G_e(\mathbf{r}-\mathbf{r}')$ can be obtained through an affine transformation of the isotropic Green dyadic by writing $\nabla_a = \overline{\overline{\epsilon}}_r{}^{1/2}\cdot\nabla$ and $\mathbf{r}_a = \overline{\overline{\epsilon}}_r{}^{-1/2}\cdot\mathbf{r}$, which transform (5.64) into

$$(\nabla_a^2 + k_a^2)G_e = -\frac{\mu_o^{3/2}\sqrt{\det\overline{\overline{\epsilon}}_r}}{k_o}\delta(\mathbf{r}-\mathbf{r}') = -\frac{\mu_o^{3/2}}{k_o}\delta(\mathbf{r}_a-\mathbf{r}_a'). \qquad (5.65)$$

The solution for the scalar Green function is, obviously,

$$G_e = \frac{\mu_o^{3/2}}{k_o}\frac{e^{-jk_aD_a}}{4\pi D_a}, \qquad (5.66)$$

$$D_a = \sqrt{(\mathbf{r}_a-\mathbf{r}_a')\cdot(\mathbf{r}_a-\mathbf{r}_a')} = \sqrt{\overline{\overline{\epsilon}}_r{}^{-1}:(\mathbf{r}-\mathbf{r})(\mathbf{r}-\mathbf{r}')}. \qquad (5.67)$$

Thus, the Green dyadic for the affine isotropic medium is obtained through substitution in (5.63):

$$\overline{\overline{G}}_e = \mu_o\sqrt{\det\overline{\overline{\epsilon}}_r}(\frac{1}{k_a^2}\nabla\nabla + \overline{\overline{\epsilon}}_r^{-1})\frac{e^{-jk_aD_a}}{4\pi D_a}. \qquad (5.68)$$

As a check, setting $\overline{\overline{\mu}}_r = \overline{\overline{I}}$, $\overline{\overline{\epsilon}}_r = \overline{\overline{I}}$, we have $k_a = k_o$ and $D_a = D$, whence the free-space Green dyadic can be seen to arise as a special case of (5.68).

*Reciprocal uniaxial medium*

It is also possible to find the Green dyadic of a reciprocal uniaxial medium in terms of special functions. Let us consider a uniaxial dielectric medium with a symmetric permittivity dyadic with an axis defined by the unit vector $\mathbf{v}$, defining the electric Helmholtz operator:

$$\overline{\overline{H}}_e(\nabla) = \mu_o^{-1}(\nabla\nabla\,{}^{\times}_{\times}\,\overline{\overline{I}} + k_o^2\overline{\overline{\epsilon}}_r), \qquad \overline{\overline{\epsilon}}_r = \epsilon_t(\overline{\overline{I}} - \mathbf{v}\mathbf{v}) + \epsilon_v\mathbf{v}\mathbf{v}. \qquad (5.69)$$

Taking $\alpha = k_o^2\epsilon_o\epsilon_t\epsilon_v$ in (5.58), the determinant of the Helmholtz operator can be written in factorized form as

$$\det\overline{\overline{H}}_e(\nabla) = \frac{k_o^2}{\mu_o^3}[\overline{\overline{\epsilon}}_r : \nabla\nabla + k_t^2\epsilon_v][\nabla^2 + k_t^2], \qquad k_t = k_o\sqrt{\epsilon_t}. \qquad (5.70)$$

The corresponding scalar Green function can be written in terms of the scalar Green functions of the two operators in integral form. It does not, however, seem to be possible to obtain any closed-form expression for the scalar Green function.

Nevertheless, the dyadic Green function can be and has been solved after considerable algebraic effort (LINDELL 1974; CHEN 1983; WEIGLHOFER 1990), details of which must be omitted here

$$\overline{\overline{G}}_e(\mathbf{r}-\mathbf{r}') = \mu_o \left( \frac{1}{k_t^2} \nabla\nabla + \epsilon_v \overline{\overline{\epsilon}}_r{}^{-1} \right) G(D_e) + \mu_o \nabla \times \frac{j\tau \mathbf{v}\mathbf{q}}{4\pi k_t}, \tag{5.71}$$

$$G(D_e) = \frac{e^{-jk_o D_e}}{4\pi D_e}, \qquad D_e = \sqrt{\epsilon_v \overline{\overline{\epsilon}}_r{}^{-1} : (\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}')}, \tag{5.72}$$

$$\tau = e^{-jk_t D_e} - e^{-jk_t |\mathbf{r}-\mathbf{r}'|}, \tag{5.73}$$

$$\mathbf{q} = \mathbf{v} \times \nabla \ln(k_o \rho) = \frac{\mathbf{v} \times \mathbf{r}}{(\mathbf{v} \times \mathbf{r})^2}. \tag{5.74}$$

The solution appears singular on the axis $\boldsymbol{\rho} = (\overline{\overline{I}} - \mathbf{v}\mathbf{v}) \cdot \mathbf{r} = 0$ because the function $\mathbf{q}$ is proportional to $1/\rho$. However, the function $\tau$ can be shown to vanish like $\rho^2$ so that the result is actually continuous on the axis.

The Green dyadic corresponding to the more general co-uniaxial medium with $\mu\overline{\overline{I}}$ replaced by $\overline{\overline{\mu}} = \mu_o(\mu_t \overline{\overline{I}}_t + \mu_v \mathbf{v}\mathbf{v})$ can be written in the same form as (5.71) with just a small change in the definition of $\tau$ (WEIGLHOFER 1990):

$$\tau = e^{-jk_t D_e} - e^{-jk_t D_m}, \qquad D_m = \sqrt{\mu_v \overline{\overline{\mu}}_r{}^{-1} : (\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}')}. \tag{5.75}$$

The resulting expression (5.71) can be checked for the special case $\mu_t = \gamma\epsilon_t$, $\mu_v = \gamma\epsilon_v$, which corresponds to an affine isotropic medium. In this case, we have $D_e = D_m$, whence $\tau = 0$ and the curl term vanishes in (5.71). The remaining expression can be seen to coincide exactly with that of (5.68), if we note that $k_a D_a = k_t D_e$ and $k_a^2 = k_t^2 \epsilon_v$.

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## 5.3 Special Green functions

From the Green function corresponding to the homogeneous three-dimensional space we can derive Green functions corresponding to some special cases by integrating the three-dimensional Green function. In particular, we consider the two- and one-dimensional Green functions as well as the half-space Green function limited by a PEC or PMC plane boundary.

### 5.3.1 Two-dimensional Green function

When sources, boundaries and media are independent of one Cartesian space coordinate, the fields also have the same independence and the problem is two dimensional. The basic source is a constant line source of unit amplitude and the corresponding field is called the two-dimensional Green function. Its functional form can be obtained either by solving the two-dimensional Helmholtz equation or through integration of the three-dimensional Green function.

The scalar Green function for an isotropic medium is the solution for (5.28) in two dimensions (the $x$–$y$ plane):

$$(\nabla_t^2 + k^2)G_2(D) = -\delta(\boldsymbol{\rho} - \boldsymbol{\rho}'), \tag{5.76}$$

and it can be written as

$$G_2(D) = \frac{1}{4j}H_o^{(2)}(kD), \tag{5.77}$$

where $H_o^{(2)}$ is the Hankel function of second kind and zero order. The distance in two dimensions is

$$D = \sqrt{(\boldsymbol{\rho} - \boldsymbol{\rho}') \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')}. \tag{5.78}$$

The function (5.77) can be obtained from (5.29) by integrating along the $z$ coordinate from $-\infty$ to $+\infty$:

$$G_2(\rho) = \int_{-\infty}^{\infty} \frac{e^{-jk\sqrt{\rho^2+z^2}}}{4\pi\sqrt{\rho^2 + z^2}} dz = \frac{1}{2\pi} \int_{1}^{\infty} \frac{e^{-jk\rho\tau}}{\sqrt{\tau^2 - 1}} d\tau, \tag{5.79}$$

and applying the integral identity (3.388.4) from GRADSHTEYN and RYZHIK (1980),

$$\int_{1}^{\infty} (x^2 - 1)^{\nu - 1} e^{-j\mu x} dx = -j\frac{\sqrt{\pi}}{2} \left(\frac{2}{\mu}\right)^{\nu - 1/2} \Gamma(\nu) H_{1/2-\nu}^{(2)}(\mu), \tag{5.80}$$

with $\nu = 1/2$ and $\mu = k\rho$, $\Gamma(1/2) = \sqrt{\pi}$.

The two-dimensional Green function satisfies the radiation condition in two dimensions for $\rho \to \infty$. This condition requires that the function behave like the Hankel function of second kind for large argument values:

$$H_n^{(2)}(k\rho) \to \sqrt{\frac{2}{\pi k\rho}} e^{-jk\rho} e^{j(\pi/4 + n\pi/2)}, \tag{5.81}$$

i.e. vanish like $1/\sqrt{\rho}$ and propagate outwards like a cylindrical wave with the propagation factor $k = \omega\sqrt{\mu\epsilon}$.

For a vector problem in two dimensions the Green dyadic can be formed in much the same way as in three dimensions. In fact, the same dyadic formalism can be applied, but in this case the scalar Green function is given by (5.77):

$$\overline{\overline{G}}_2(D) = (\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla)G_2(D) = \mathbf{u}_z\mathbf{u}_z G_2(D) + (\overline{\overline{I}}_t + \frac{1}{k^2}\nabla_t\nabla_t)G_2(D). \tag{5.82}$$

The gradients are in two dimensions because they operate on a function depending on two space dimensions only.

(5.82) shows us that the problem is split into two parts, because $\mathbf{u}_z\mathbf{u}_z G_2$ gives a $z$ directed electric field for a $z$ directed current, whereas the last term gives a transverse field for a transverse current. Thus, the field problem has two components that do not couple to one another. Because the magnetic field is orthogonal to a two-dimensional electric field if it is $z$ directed or transversal ($\mathbf{E} \cdot \nabla_t \times \mathbf{E} = 0$ for $\mathbf{E} = \mathbf{u}_z E_z$ or $\mathbf{E} = \mathbf{E}_t$), we can speak of splitting the problem into transverse electric (TE) and transverse magnetic (TM) field problems.

### 5.3.2 One-dimensional Green function

The one-dimensional problem involves plane parallel boundaries and sources that are constant on parallel planes. Thus, the fields are also constant on parallel planes and depend only on the normal ($z$) coordinate.

The one-dimensional scalar Green function satisfies the equation

$$(\frac{d^2}{dz^2} + k^2)G_1(z - z') = -\delta(z - z'). \tag{5.83}$$

The solution which represents a plane wave propagating outwards from the plane $z = z'$ is

$$G_1(z - z') = \frac{1}{2jk}e^{-jk|z-z'|}, \tag{5.84}$$

which is also directly obtained by integrating the three-dimensional Green function in the $x - y$ plane

$$G_1(z) = \int_0^{2\pi}\int_0^{\infty} G(\sqrt{\rho^2 + z^2})\rho d\rho d\phi = \frac{1}{2}\int_0^{\infty} e^{-jk\sqrt{\rho^2+z^2}}d(\sqrt{\rho^2 + z^2}). \tag{5.85}$$

The corresponding Green dyadic in one space dimension is

$$\overline{\overline{G}}_1(z - z') = (\overline{\overline{I}} + \mathbf{u}_z\mathbf{u}_z\frac{1}{k^2}\frac{d^2}{dz^2})G_1(z - z') =$$

$$\overline{\overline{I}}_t G_1(z - z') - \mathbf{u}_z\mathbf{u}_z\frac{1}{k^2}\delta(z - z'). \tag{5.86}$$

It is easy to solve the electric field due to and outside of a constant transverse surface current source $\mathbf{J}(\mathbf{r}) = \mathbf{J}_s\delta(z)$ in the form

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu\int\overline{\overline{G}}_1(z - z')\cdot\mathbf{J}_s(z')dz' = -\frac{\eta}{2}\mathbf{J}_s e^{-jk|z|}. \tag{5.87}$$

The electric field is continuous across the surface current layer and outside the layer it is always tangential to the plane. A normal current source $\mathbf{J}(z) = \mathbf{u}_z J(z)$ does not produce any field outside the current layer, but, instead, a field is generated inside the layer.

The magnetic field experiences a step discontinuity across the surface, as is observed by taking the curl of the electric field expression above:

$$\mathbf{H}(\mathbf{r}) = \pm\frac{1}{2}(\mathbf{u}_z \times \mathbf{J}_s)e^{-jk|z|}. \tag{5.88}$$

This means that a $z$ directed current component in the sheet does not produce a magnetic field.

### 5.3.3 Half-space Green function

Green functions can also be expressed for problems involving inhomogeneous media, provided the field can be solved for the general source. A number of such problems has been treated in a book by TAI (1971). As a simple example, let us consider the problem of half space bounded by a PEC or PMC plane at $\mathbf{u} \cdot \mathbf{r} = 0$.

We can apply the two reflection transformations to construct the corresponding Green dyadics in terms of the free-space Green dyadic. Because the field $\mathbf{E}(\mathbf{r})$ due to the source $\mathbf{J}(\mathbf{r})$ can be made to satisfy the PMC and PEC boundary conditions by adding the respective transformed fields $\pm\overline{\overline{C}} \cdot \mathbf{E}(\overline{\overline{C}} \cdot \mathbf{r})$, the total field can be written as

$$\mathbf{E}_{\pm}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) \pm \overline{\overline{C}} \cdot \mathbf{E}(\overline{\overline{C}} \cdot \mathbf{r}). \tag{5.89}$$

From this, it is easy to form the corresponding Green dyadics. In fact, we can write

$$\overline{\overline{G}}_{\pm}(\mathbf{r}, \mathbf{r}') = \overline{\overline{G}}(\mathbf{r} - \mathbf{r}') \pm \overline{\overline{C}} \cdot \overline{\overline{G}}(\overline{\overline{C}} \cdot \mathbf{r} - \mathbf{r}'), \tag{5.90}$$

where the upper signs refer to the PMC and the lower signs to the PEC boundary.

To check the electric field boundary conditions for $\mathbf{r} = \overline{\overline{C}} \cdot \mathbf{r} = \boldsymbol{\rho}$, we can write

$$\overline{\overline{G}}_{\pm}(\boldsymbol{\rho}, \mathbf{r}') = \overline{\overline{A}}_{\pm} \cdot \overline{\overline{G}}(\boldsymbol{\rho} - \mathbf{r}'), \qquad \overline{\overline{A}}_{+} = 2\overline{\overline{I}}_t, \qquad \overline{\overline{A}}_{-} = 2\mathbf{u}\mathbf{u}, \tag{5.91}$$

whence $\mathbf{u}_z \cdot \overline{\overline{G}}_{+} = 0$, which is the PMC condition and $\mathbf{u}_z \times \overline{\overline{G}}_{-} = 0$, which is the PEC condition. It is also seen that the Green dyadic is not a function of the difference $\mathbf{r} - \mathbf{r}'$, as for homogeneous media. Applying suitable reflection transformations, Green dyadics for problems involving more than one conducting or magnetically conducting plane can be constructed. Green dyadics corresponding to plane parallel material interfaces are discussed in Chapter 7.

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## 5.4 Singularity of the Green dyadic

To be able to apply the Green function concept to integral equations, its singularity must be known exactly. Basically the question is how to calculate fields inside the source region where the source and field points coincide. Let us study two simple cases for which the fields within the source can be easily determined.

### 5.4.1 Constant volume current

For a constant electric current source $\mathbf{J}(\mathbf{r}) = \mathbf{J}_o$ filling the whole space, the resulting electromagnetic field is obviously also constant and the curl terms in the Maxwell equations vanish. Thus, the fields in an isotropic medium are

$$\mathbf{E}_o(\mathbf{r}) = -\frac{1}{j\omega\epsilon}\mathbf{J}_o, \qquad \mathbf{H}_o(\mathbf{r}) = 0. \tag{5.92}$$

For a constant magnetic current $\mathbf{J}_{mo}$ we can write the fields from duality:

$$\mathbf{E}_o(\mathbf{r}) = 0, \qquad \mathbf{H}_o(\mathbf{r}) = -\frac{1}{j\omega\mu}\mathbf{J}_{mo}. \tag{5.93}$$

These expressions can be generalized for the anisotropic medium

$$\mathbf{E}_o(\mathbf{r}) = -\frac{1}{j\omega}\overline{\overline{\epsilon}}^{-1}\cdot\mathbf{J}_o, \quad \mathbf{H}_o(\mathbf{r}) = -\frac{1}{j\omega}\overline{\overline{\mu}}^{-1}\cdot\mathbf{J}_{mo}, \tag{5.94}$$

and even for the bianisotropic medium

$$\mathbf{E}_o(\mathbf{r}) = \frac{1}{j\omega}(\overline{\overline{\epsilon}} - \overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}})^{-1}\cdot(-\mathbf{J}_o + \overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\mathbf{J}_{mo}), \tag{5.95}$$

$$\mathbf{H}_o(\mathbf{r}) = \frac{1}{j\omega}(\overline{\overline{\mu}} - \overline{\overline{\zeta}}\cdot\overline{\overline{\epsilon}}^{-1}\cdot\overline{\overline{\xi}})^{-1}\cdot(-\mathbf{J}_{mo} + \overline{\overline{\zeta}}\cdot\overline{\overline{\epsilon}}^{-1}\cdot\mathbf{J}_o). \tag{5.96}$$

When studying the singularity of the isotropic Green dyadic it is of interest to find the electric field $\mathbf{E}_\delta$ at a point $\mathbf{r}$ within a small element of constant current $\mathbf{J}_o$, zero outside a small volume $V_\delta$ bounded by a surface $S_\delta$. The field can be obtained by subtracting from $\mathbf{E}_o$ in (5.92) the field of the complementary problem: constant current $\mathbf{J}_o$ filling the whole space except the volume $V_\delta$. The field in the cavity can be determined with the isotropic Green dyadic without any knowledge of its singularities since the field point is now outside the source region.

Let us find the field at the origin $\mathbf{r} = 0$ inside the cavity, for simplicity of notation. Taking the limit of a small volume $V_\delta$ with its largest measure satisfying $k\delta \to 0$, we obtain after evaluating an integral for the cavity field

$$\mathbf{E}_o - \mathbf{E}_\delta = -j\omega\mu \int\limits_{V-V_\delta} (\overline{\overline{I}} + \frac{1}{k^2}\nabla'\nabla')\frac{e^{-jkr'}}{4\pi r'} dV' \cdot \mathbf{J}_o \to -\frac{\mathbf{J}_o}{j\omega\epsilon} + \frac{\overline{\overline{L}} \cdot \mathbf{J}_o}{j\omega\epsilon}, \quad (5.97)$$

$$\overline{\overline{L}} = \int\limits_{V-V_\delta} \nabla'\nabla' G(r')dV' = -\oint\limits_{S_\delta} \mathbf{n}'\nabla' G(r')dS', \quad (5.98)$$

with $\mathbf{n}$ denoting the unit normal vector pointing outside of $V_\delta$. The surface integral at infinity vanishes if we assume losses however small, because the exponential decay of the Green function is faster than the algebraic growth of the integrand. Since the term $-\mathbf{J}_o/j\omega\epsilon$ equals $\mathbf{E}_o$, the last term in (5.97) must be $-\mathbf{E}_\delta$ and we can write for the field inside the current element the expression

$$\mathbf{E}_\delta = -\frac{1}{j\omega\epsilon}\overline{\overline{L}} \cdot \mathbf{J}_o = \oint\limits_{S_d} \frac{\mathbf{n}' \cdot \mathbf{J}_o}{j\omega\epsilon}\nabla' \frac{1}{4\pi r'} dS'. \quad (5.99)$$

This can be interpreted as the electric field due to a static surface charge $\varrho_s(\mathbf{r}')$ distributed on the small surface $S_\delta$:

$$\varrho_s(\mathbf{r}') = \frac{\mathbf{n}' \cdot \mathbf{J}_o}{j\omega}. \quad (5.100)$$

The field $\mathbf{E}_\delta$ depends on the charge distribution, which again depends on the current vector $\mathbf{J}_o$ and the form of the surface $S_\delta$. The result is best expressed through the dyadic $\overline{\overline{L}}$, called the depolarization dyadic (STRATTON 1941).

If the volume $V_\delta$ is not small, or, the equivalent, $k$ is not small, the depolarization dyadic should be written as $\overline{\overline{L}}(k)$, an analytic function of $k$, which can be expanded in a Taylor series as

$$\overline{\overline{L}}(k) = \overline{\overline{L}} + k\overline{\overline{L}}_1 + k^2\overline{\overline{L}}_2 + \ldots \quad (5.101)$$

with

$$\overline{\overline{L}}_1 = 0, \quad \overline{\overline{L}}_2 = \frac{k^2}{8\pi}\oint\limits_{S_\delta} \mathbf{n}'\mathbf{u}'_r dS', \ldots \quad (5.102)$$

Because $\overline{\overline{L}}_2$ and the following terms contain powers of $kr'$ in the integrand, the integrals are orders of magnitude smaller than that of $\overline{\overline{L}}$.

The dyadic $\overline{\overline{L}}$ is symmetric, because it is the volume integral of a symmetric dyadic. Further, it satisfies the condition

$$\overline{\overline{L}}:\overline{\overline{I}} = \frac{1}{4\pi}\oint_{S_\delta}\frac{\mathbf{n}'\cdot\mathbf{u}_{r'}}{r'^2}dS' = \frac{1}{4\pi}\int_0^{4\pi}d\Omega' = 1. \tag{5.103}$$

Because the solid angle of a surface element $dS'$ seen from the origin is $d\Omega' = \mathbf{n}'\cdot\mathbf{u}'_r dS'/r'^2$, the integral in this equation gives the solid angle of the surface $S_\delta$ as seen from the field point $\mathbf{r} = 0$ and it equals $4\pi$. The dyadic $\overline{\overline{L}}$ in general depends on the position of the origin within the volume $V_\delta$ except for ellipsoidal volumes for which it is actually independent of that position. Beyond that, $\overline{\overline{L}}$ is only dependent on the form of the surface $S_\delta$ and not on its size, although it must be small so that the dyadic $\overline{\overline{L}}(k)$ can be approximated by $\overline{\overline{L}}$.

It is intuitively clear that in the case of a non-constant current distribution $\mathbf{J}(\mathbf{r})$, the field due to and within a small volume element $V_\delta$ of this current is given predominantly by the value of the current density at the volume and not by its derivatives. This is because the largest correction term obtained from the Taylor expansion for the current function $\mathbf{J}(\mathbf{r})$ is $\mathbf{r}_\delta\cdot\nabla\mathbf{J}(0)$, where $\mathbf{r}_\delta$ is the largest distance within the volume $V_\delta$, and it is negligible for small $|\mathbf{r}_\delta|$, whereas the basic term is independent of $|\mathbf{r}_\delta|$. Of course, for strongly changing current functions $V_\delta$ must be small enough. The field at a zero of the current function cannot be obtained in this manner.

### 5.4.2 Constant planar current sheet

Another canonical source is the constant surface current $\mathbf{J}_{so}$ on the plane $\mathbf{u}\cdot\mathbf{r} = 0$ with $\mathbf{u}\cdot\mathbf{J}_{so} = 0$. The electric field is everywhere tangential and continuous across the surface, as is seen from the expression (5.87):

$$\mathbf{E} = -\frac{\eta}{2}\mathbf{J}_{so}. \tag{5.104}$$

This can be interpreted in terms of a current source across a vector transmission line of dyadic impedance $\eta\overline{\overline{I}}_t$ so that the loading impedance is $(\eta/2)\overline{\overline{I}}_t$, whence (5.104) describes the resulting vector voltage.

On the other hand, according to (5.88) the magnetic field is discontinuous at the surface satisfying

$$\mathbf{n}_1\times\mathbf{H}_1 + \mathbf{n}_2\times\mathbf{H}_2 = \mathbf{J}_{so}, \tag{5.105}$$

which actually is the interface condition for the magnetic field valid at any interface. Because of antisymmetry, we must have $\mathbf{H}_2 = -\mathbf{H}_1$ and, hence,

$$\mathbf{n}_1 \times \mathbf{H}_1 = \mathbf{n}_2 \times \mathbf{H}_2 = \frac{1}{2}\mathbf{J}_{so}. \tag{5.106}$$

Also, in the middle of the current sheet the magnetic field must be zero. Thus there are two limiting values for the tangential magnetic field when the field point approaches the plane from each side and their average value is zero. This can be assumed to be valid also on curved surfaces if the tangent to the surface is continuous. From duality we have corresponding properties for fields around a magnetic surface current.

Let us now consider the field inside the constant planar current element $\mathbf{J}_{so}$ on a small disk $A_\delta$ on the $x$–$y$ plane $A$ as due to two parts: (1) the constant surface current on the whole $A$ minus (2) the field inside a hole $A_\delta$ in the plane surface current. As in the volume current case, we consider the expression for the field $\mathbf{E}_o - \mathbf{E}_\delta$ due to the surface current in a decreasing hole $A_\delta$ with the rim curve $C_\delta$:

$$\mathbf{E}_o - \mathbf{E}_\delta = -j\omega\mu \int\limits_{A-A_\delta} [\overline{\overline{I}} + \frac{1}{k^2}\nabla'\nabla']G(|\boldsymbol{\rho} - \boldsymbol{\rho}'|)dS' \cdot \mathbf{J}_{so} =$$

$$-\frac{\eta}{2}\mathbf{J}_{so} - \frac{1}{j\omega\epsilon}\oint\limits_{C_\delta}(\mathbf{n}' \cdot \mathbf{J}_{so})\nabla' G(|\boldsymbol{\rho} - \boldsymbol{\rho}'|)dC', \tag{5.107}$$

where $\mathbf{n}$ denotes the unit normal vector at the rim point $\boldsymbol{\rho}$ pointing outwards from the disk $A_\delta$.

The term $-(\eta/2)\mathbf{J}_{so} = \mathbf{E}_o$ can be identified as the field due to the total surface current, whence the last term equals $-\mathbf{E}_\delta$. When the maximum measure of the disk satisfies $k\delta \to 0$, we can apply static approximation and write

$$\mathbf{E}_\delta = -\nabla \oint\limits_{C_\delta} \frac{(\mathbf{n}' \cdot \mathbf{J}_{so}/j\omega)}{4\pi\epsilon|\boldsymbol{\rho} - \boldsymbol{\rho}'|}dC'. \tag{5.108}$$

This, again, can be interpreted as the field due to a static line charge along the rim curve $C_\delta$:

$$\varrho_c(\boldsymbol{\rho}') = \frac{\mathbf{n}' \cdot \mathbf{J}_{so}}{j\omega}. \tag{5.109}$$

It is seen that the integral for $\mathbf{E}_\delta$ diverges as the area $A_\delta$ grows smaller, because the integrand grows like $1/|\boldsymbol{\rho} - \boldsymbol{\rho}'|$.

### 5.4.3 Singularity for a volume source

When trying to determine fields inside a continuous volume source $\mathbf{J}(\mathbf{r})$, the singularity of the Green function can be expressed in terms of previous analysis. In fact, enclosing the field point within an infinitesimal volume $V_\delta$, the field can be expressed in two parts: (1) from the volume current surrounding $V_\delta$ and (2) from the local current in $V_\delta$. The latter was seen to be affected only by the shape of $V_\delta$ and the density of the current and not the size of the element or the total current integral. It was also seen to be dependent on the location of the field point within $V_\delta$, in general.

This division in two parts can be transferred into the Green dyadic itself. The cavity field (1) can be computed in terms of the non-singular part of the Green dyadic, which is called the *principal value* term and denoted by $PV_\delta$. The field (2) within the current element can be given as (5.99). Thus, the Green dyadic can be written in two terms corresponding to these two fields

$$\overline{\overline{G}}(D) = PV_\delta \overline{\overline{G}}(D) - \frac{1}{k^2}\overline{\overline{L}}\delta(\mathbf{r}-\mathbf{r}'). \tag{5.110}$$

The dyadic $\overline{\overline{L}}$ depends on the geometrical form of the infinitesimal volume and the position of the field point within the volume. $\overline{\overline{L}}$ may be called the normalized dyadic solid angle of the volume $V_\delta$. In the following list, expressions of $\overline{\overline{L}}$ dyadics for some simple volume geometries, are given according to YAGHJIAN (1980):

- for any point within the sphere or for the centre point within any Platonic polyhedron, $\overline{\overline{L}} = \frac{1}{3}\overline{\overline{I}}$;
- for any point within an ellipsoid, $\overline{\overline{L}} = A(L_x\mathbf{u}_x\mathbf{u}_x + L_y\mathbf{u}_y\mathbf{u}_y + L_z\mathbf{u}_z\mathbf{u}_z)$, where the $L_i$ are the depolarizing factors (STRATTON 1941) and the axes of the ellipsoid are the coordinate axes, $A = 1/(L_x + L_y + L_z)$;
- for a parallelepiped $\overline{\overline{L}} = (\Omega_x\mathbf{u}_x\mathbf{u}_x + \Omega_y\mathbf{u}_y\mathbf{u}_y + \Omega_z\mathbf{u}_z\mathbf{u}_z)/4\pi$, where $\Omega_i$ is the solid angle subtended at the centre point by sides perpendicular to the axis $i$, $\Omega_x + \Omega_y + \Omega_z = 4\pi$;
- for a pillbox of arbitrary cross section, thin in the $z$ direction, $\overline{\overline{L}} = \mathbf{u}_z\mathbf{u}_z$;
- for a circular cylinder of any cross section, long in the $z$ direction, $\overline{\overline{L}} = \overline{\overline{I}}_t/2$;
- for a finite circular cylinder, axis in the $z$ direction, $\overline{\overline{L}} = (1 - \cos\theta)\mathbf{u}_z\mathbf{u}_z + \frac{1}{2}\cos\theta(\overline{\overline{I}} - \mathbf{u}_z\mathbf{u}_z)$, where $\theta$ is half of the angle subtended by the bottom of the cylinder at the centre point.

*Volume integral equations*

The singularity of the Green dyadic gives a kind of internal boundary condition which can be applied for the construction of volume integral equations for electromagnetic fields. In fact, consider a dielectric object as an example with permittivity $\epsilon = \epsilon_o \epsilon_r$. The polarization in the object can be represented by the polarization current $\mathbf{J}_p = j\omega\epsilon_o(\epsilon_r - 1)\mathbf{E}$. The field inside the object comes from two parts: from the incident field $\mathbf{E}_i$ from outside the object and the field from the polarization current of the object, which can be written as an integral

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_i(\mathbf{r}) - j\omega\mu_o \int_V \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{J}_p(\mathbf{r}')dV' =$$

$$\mathbf{E}_i(\mathbf{r}) + k_o^2 PV_\delta \int_V (\epsilon_r - 1)\overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}')dV' - (\epsilon_r - 1)\overline{\overline{L}} \cdot \mathbf{E}(\mathbf{r}). \quad (5.111)$$

This defines an integral equation for the unknown field $\mathbf{E}(\mathbf{r})$ inside the obstacle volume $V$. For a spherical volume $V_\delta$ we have $\overline{\overline{L}} = \overline{\overline{I}}/3$ and the volume integral equation can be written as

$$\mathbf{E}_i(\mathbf{r}) = \frac{\epsilon_r + 2}{3}\mathbf{E}(\mathbf{r}) - k_o^2 PV_\delta \int_V (\epsilon_r - 1)\overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}')dV'. \quad (5.112)$$

### 5.4.4 Singularity for a surface source

Because the field inside a small surface current element grows infinite as the element size decreases, the previous principal value decomposition does not work. Actually, this can be seen from the fact that since in a surface current the volume current density becomes infinite, in the volume field integral, the term containing the $\overline{\overline{L}}$ dyadic becomes infinite. Since the infinity is cancelled by another infinity in the principal value integral, there should be a better means to take out the singularity from the integral. Such a method is called the *finite part* method (DAVIES and DAVIES, 1989).

Mathematically it amounts to subtracting from the current function $\mathbf{J}_s(\mathbf{r}')$ another function, $\mathbf{J}_{so}(\mathbf{r}')$, causing the same singularity, but whose contribution can be computed separately. As such a function we might take the constant planar surface current flowing on the plane $A$ tangential to the surface $S$ at the field point $\mathbf{r}$, with the current of the field point $\mathbf{J}_{so}(\mathbf{r}') = \mathbf{J}_s(\mathbf{r})$, constant in $\mathbf{r}'$. Thus, we can write

$$\mathbf{E}(\mathbf{r}) = -\frac{\eta}{2}\mathbf{J}_s(\mathbf{r}) - j\omega\mu FP \int_S \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{J}_s(\mathbf{r}')dS', \quad (5.113)$$

$$FP\int_S \overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}_s(\mathbf{r}')dS' = \int_{S+A} \overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot[\mathbf{J}_s(\mathbf{r}')-\mathbf{J}_s(\mathbf{r})]dS'. \quad (5.114)$$

The last integral means an integration on the surface $S$ of the source $\mathbf{J}_s(\mathbf{r}')$ plus another integration on the tangent plane $A$ at $\mathbf{r}'=\mathbf{r}$ of the constant source $-\mathbf{J}_s(\mathbf{r})$. With this as an interpretation we can define the Green dyadic in the form

$$\overline{\overline{G}}(\mathbf{r}-\mathbf{r}') = FP\overline{\overline{G}}(\mathbf{r}-\mathbf{r}') + \frac{1}{2jk}\overline{\overline{I}}\delta(\mathbf{r}-\mathbf{r}'). \quad (5.115)$$

Of course, it is numerically inconvenient to integrate along two surfaces. This is why the Green dyadic is normally discarded in surface integral equations where the unknown is the surface current function. Integral equations for surface sources will be discussed in more detail in Chapter 6.

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## 5.5 Complex source point Green function

Complex space source points have been adopted to electromagnetic theory because, as first shown by DESCHAMPS in 1971, Gaussian beams can be simply approximated through a field arising from a point source in complex space. In fact, if the source point vector $\mathbf{r}'$ of the distance function $D$ in (5.30) is made complex, the Green function (5.29) continues to satisfy the Helmholtz equation (5.28), provided the delta function of complex argument is correctly interpreted. Let us study the condition $D = 0$ in some detail.

### 5.5.1 Complex distance function

The condition $D = \sqrt{(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')} = 0$ is not only satisfied at the complex point $\mathbf{r} = \mathbf{r}'$, but there are also points $\mathbf{r}$ in real space where the distance function vanishes. Denoting $\mathbf{r}' = \mathbf{a} + j\mathbf{b}$ they can be found from the equation $D^2 = (\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{a}) - \mathbf{b}\cdot\mathbf{b} - 2j(\mathbf{r}-\mathbf{a})\cdot\mathbf{b} = 0$. In fact, equating real and imaginary parts we have the two conditions

$$|\mathbf{r}-\mathbf{a}| = |\mathbf{b}|, \qquad (\mathbf{r}-\mathbf{a})\cdot\mathbf{b} = 0. \tag{5.116}$$

The first of these represents a sphere centred at $\mathbf{a}$ with radius $|\mathbf{b}|$, and the second one, a plane orthogonal to the vector $\mathbf{b}$ going through the point $\mathbf{r} = \mathbf{a}$. Thus, the points $\mathbf{r}$ satisfying $D = 0$ lie on a circle with radius $|\mathbf{b}|$, centred at $\mathbf{a}$ and perpendicular to $\mathbf{b}$. If $\mathbf{b} \to 0$, the circle obviously shrinks to the point $\mathbf{r} = \mathbf{a}$. So we see that, instead of a point of singularity, the complex-source-point Green function actually has a circle of singularity in real space.

It can also be called the *branch circle* of the distance function $D$, because, if the value of the function is traced along a curve which passes through the circle of singularity, it does not return to the original value but differs from that by the sign. This is clearly seen if we consider points close to the branch circle by taking $\mathbf{b} = \mathbf{u}_z b$ and

$$\mathbf{r} = \mathbf{a} + \mathbf{u}_x b + \mathbf{q}, \qquad \mathbf{q} = q(\mathbf{u}_x \cos\psi + \mathbf{u}_z \sin\psi). \tag{5.117}$$

If $q \ll |b|$, we have the approximation

$$D \approx \sqrt{2bq}\, e^{-j\psi/2}. \tag{5.118}$$

Thus, when $\psi$ is changed by $2\pi$, the distance function $D$ does not return to its original value but changes its sign.

The branch circle corresponds to the branch point of a function of one variable in the complex plane. This function can be made unique by introducing branch cut lines and Riemann surfaces. Similarly, the distance function $D$ and any functions of $D$ can be made unique by a *branch cut surface*, which can be any surface with a rim along the branch circle. The branch cut surface can also extend to infinity. In analogy with Riemann sheets we can also think of having two real spaces glued together along the branch circle, each with a unique value for the distance function $D$. Moving the point through the branch cut surface moves $\mathbf{r}$ between the two spaces. The idea of two real spaces to interpret two branches of a function is not new, actually it was applied in a classical diffraction paper by SOMMERFELD (1896). Following this idea, the point source in real space can be interpreted as actually being an infinitesimally small circular source which separates two real spaces, one with positive and the other one with negative distances $D$.

### 5.5.2 Point source in complex space

A point source in complex space may also be a source of confusion, because the delta function for complex arguments is not a very familiar concept. Intuitionally, we could imagine a function which is zero at all other points, real or complex, except 'the point' of the delta function. However, with this definition, we lose the Cauchy integral theorem that any path of integration can be changed without changing the value of the integral provided the function is analytic. Of course, the delta function is not analytic but can be obtained as a limit of a sequence of analytic functions. The Cauchy property is very important to maintain, because we often want to change the integration path to obtain the best convergence. Therefore it is important to define the delta function as a limit of an analytic function.

It is known that, for a real argument, the delta function can be defined as a limit of an analytic function in infinitely many ways. Perhaps the most popular analytic function is the Gaussian pulse function

$$\delta_\tau(x) = \sqrt{\frac{\tau}{\pi}} e^{-\tau x^2}, \tag{5.119}$$

for which the limit can be written as $\delta_\tau(x) \to \delta(x)$, as $\tau \to \infty$.

For complex $x = x_{\mathrm{re}} + jx_{\mathrm{im}}$ we can write (5.119) in the form

$$\delta_\tau(x) = \sqrt{\frac{\tau}{\pi}} e^{-\tau x_{\mathrm{re}}^2} e^{\tau x_{\mathrm{im}}^2} e^{-2j\tau x_{\mathrm{re}} x_{\mathrm{im}}}, \tag{5.120}$$

and in three dimensions as a product of three similar functions, or more generally as

$$\delta_\tau(\mathbf{r}-\mathbf{r}') = \left(\frac{\tau}{\pi}\right)^{3/2} e^{-\tau D^2}, \qquad D = \sqrt{(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}. \tag{5.121}$$

Studying (5.119) we can see what kind of function the delta sequence approaches as $\tau \to \infty$. The two lines $x_{\mathrm{re}} = \pm x_{\mathrm{im}}$ define four sectors in which the function (5.120) has a different behaviour. In regions $|x_{\text{re}}| > |x_{\text{im}}|$, the function approaches zero except at the boundary lines. On the other hand, in the other two sectors $|x_{\text{re}}| < |x_{\text{im}}|$ the amplitude and rate of oscillation of the function (5.120) grow without limit. For example, along the line $x_{\text{im}} = a$ the real part of the delta sequence function is

$$\Re\{\delta_\tau(x)\} = \sqrt{\frac{\tau}{\pi}} e^{-\tau x_{\text{re}}^2} e^{\tau a^2} \cos(2\tau a x_{\text{re}}). \tag{5.122}$$

The amplitude is bounded by the Gaussian function $e^{-\tau x_{\text{re}}^2}$ and the rate of oscillation grows without limit as $\tau a \to \infty$. In Figure 5.1 we see an example of the delta function $\delta_\tau(z - ja)$ for argument values on the real axis $z = x$.

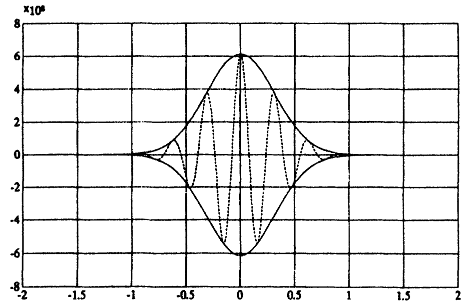


**Fig. 5.1** Approximation for the complex delta function at $a = 2$ for $\tau = 5$. The envelope gives the absolute value and dashed line the real part of $\delta_\tau(x - ja)$.

The analytic extension of the delta function is essential in extending integration of functions to the complex plane. Thus the integral

$$\int_{x_1}^{x_2} f(x)\delta(x - x_o)dx = f(x_o), \tag{5.123}$$

valid for real-valued parameters $x_1 < x_o < x_2$, can be extended to integration from $x_1$ to $x_2$ along a path on the complex $x$ plane. Also, if $x_o$ is complex, the delta function for real $x$ has a similar property. Finally, a point source in complex space at $\mathbf{r} = \mathbf{a} + j\mathbf{b}$ can be pictured as a spherical source in real space, centred at $\mathbf{r} = \mathbf{a}$ and with the radius $|\mathbf{b}|$. In fact, the limit of $\exp(-\tau(r^2 - b^2))\exp(2j\tau\mathbf{b}\cdot\mathbf{r})$ is zero for $r > |\mathbf{b}|$ and of infinite amplitude and rate of oscillation for $r < |\mathbf{b}|$.

### 5.5.3 Green function

Because the distance function $D$ for complex source point $\mathbf{r}' = \mathbf{a} + j\mathbf{b}$ vanishes on the branch circle, the Green function $G(\mathbf{r} - \mathbf{r}')$ (5.29) has a singularity at the branch circle in real space. Also, the function is two valued unless a branch cut surface is defined. This can be made with respect to the condition at infinity.

*Dipole mode branch cut*

Requiring the Sommerfeld radiation condition

$$\mathbf{u}_r \cdot \nabla G(\mathbf{r}-\mathbf{r}') + jkG(\mathbf{r}-\mathbf{r}') = o(1/r) \tag{5.124}$$

to be satisfied, the branch is defined by the condition $\Re\{D\} \geq 0$, corresponding to an outgoing wave solution radiating towards infinity. Conversely, choosing the condition $\Re\{D\} \leq 0$, we obtain the incoming wave solution radiating from infinity. The equality sign in both definitions corresponds to the branch cut surface separating these two solutions: $\Re\{D\} = 0$, or the equivalent,

$$\mathbf{b} \cdot (\mathbf{r}-\mathbf{a}) = 0, \qquad |\mathbf{r}-\mathbf{a}| \leq |\mathbf{b}|. \tag{5.125}$$

These conditions define the circular disk inside the branch circle.

We can actually picture the energy coming through the branch cut disk to the space with the outgoing field branch. Where does it come from? From the other space with the incoming field branch! Thus, the energy from the complex space point source circulates between these two branch spaces by passing through the branch cut disk and reflecting from the infinity. This is also valid for a point source in real space, where the energy obviously also passes from one (incoming field) space to the other (outgoing field) space through the source point. This choice of branch cut (5.125) defines the dipole mode of the complex space point source.

*Beam mode branch cut*

Another possibility is the beam mode in which the field comes from the left of the branch circle, passes through the circle and radiates to the right. Now the radiation condition above is only valid on the right-hand side, whereas on the left-hand side we must have the incoming wave condition. The corresponding condition for the distance function branch can be shown to be $\Im\{D\} \leq 0$, and the limiting case $\Im\{D\} = 0$ defines the branch cut surface:

$$\mathbf{b} \cdot (\mathbf{r}-\mathbf{a}) = 0, \quad |\mathbf{r}-\mathbf{a}| \geq |\mathbf{b}|, \tag{5.126}$$

which is the complement of the disk in the plane of the branch circle, i.e. the plane with a circular hole.

For large values of $|\mathbf{b}|$, the field defined by $\Im\{D\} \leq 0$ can be shown to satisfy the Gaussian beam condition, because assuming $\mathbf{b} = \mathbf{u}_z b$ we can write for paraxial points $z \gg |\rho|$ the approximation

$$D = \sqrt{(z - jb)^2 + \rho^2} \approx z - jb + \frac{\rho^2}{2(z^2 + b^2)}(z + jb). \tag{5.127}$$

The Green function can be written in this approximation as

$$G(\mathbf{r} - j\mathbf{b}) \approx \frac{e^{-kb}}{4\pi z} e^{-jkz} \exp\left(\frac{-jkz\rho^2}{2(z^2 + b^2)}\right) \exp\left(\frac{kb\rho^2}{2(z^2 + b^2)}\right). \tag{5.128}$$

In the last exponential factor it is seen that the function decreases as the Gaussian function of $\rho$ only if $b < 0$. Also, it is seen that the surface of constant phase is not a plane because of the term quadratic in $\rho$.

The surface of constant phase can be approximated by a spheroid sufficiently far from the origin and close to the $z$ axis, in which case $D$ can be approximated by $z$ in the denominator of the Green function. In fact, requiring that the real part of the distance function $D = \sqrt{\rho^2 + (z - jb)^2}$ be a constant $c$, we arrive at the equation of a spheroidal phase surface:

$$\frac{z^2}{c^2} + \frac{\rho^2}{c^2 + b^2} = 1. \tag{5.129}$$

This can be approximated by a sphere whose radius is a function of $z$ by fitting the two surfaces at $z = c$. The radius $R$ of the sphere can be written as

$$R \approx \frac{c^2 + b^2}{c} = \frac{z^2 + b^2}{z}. \tag{5.130}$$

The surface is a plane both at the origin and at infinity and has the minimum radius at $z = \pm b$.

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## 5.6 Plane waves

Plane waves are simple exponential function solutions of the Maxwell equations, which also serve as basic building bocks for Fourier-transformed problems. The plane wave field is known as soon as one of the four field vector functions is known, for example, the electric field vector $\mathbf{E}(\mathbf{r})$, which depends on two constant vectors $\mathbf{E}_o$, $\mathbf{k}$:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}}. \tag{5.131}$$

Substituting this in the Maxwell equations corresponding to a source-free region and in the medium relations shows us that the other three field vectors must have a similar exponential dependence on $\mathbf{r}$:

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_o e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{5.132}$$

$$\mathbf{D}(\mathbf{r}) = \mathbf{D}_o e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{5.133}$$

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_o e^{-j\mathbf{k}\cdot\mathbf{r}}. \tag{5.134}$$

All constant vectors are connected through the field and medium equations.

### 5.6.1 Dispersion equations

Let us consider conditions for the constant vectors corresponding to the general bianisotropic medium. From the medium equations we have

$$\mathbf{D}_o = \overline{\overline{\epsilon}} \cdot \mathbf{E}_o + \overline{\overline{\xi}} \cdot \mathbf{H}_o, \tag{5.135}$$

$$\mathbf{B}_o = \overline{\overline{\zeta}} \cdot \mathbf{E}_o + \overline{\overline{\mu}} \cdot \mathbf{H}_o. \tag{5.136}$$

Substituting these in the Maxwell equations gives us two vector relations for the three constant vectors $\mathbf{k}$, $\mathbf{E}_o$ and $\mathbf{H}_o$:

$$\mathbf{k} \times \mathbf{E}_o = \omega \mathbf{B}_o = \omega(\overline{\overline{\zeta}} \cdot \mathbf{E}_o + \overline{\overline{\mu}} \cdot \mathbf{H}_o), \tag{5.137}$$

$$\mathbf{k} \times \mathbf{H}_o = -\omega \mathbf{D}_o = -\omega(\overline{\overline{\epsilon}} \cdot \mathbf{E}_o + \overline{\overline{\xi}} \cdot \mathbf{H}_o). \tag{5.138}$$

Eliminating $\mathbf{E}_o$ or $\mathbf{H}_o$ results in the following relations:

$$\mathbf{H}_o = \overline{\overline{\mu}}^{-1} \cdot (\frac{\mathbf{k}}{\omega} \times \overline{\overline{I}} - \overline{\overline{\zeta}}) \cdot \mathbf{E}_o, \tag{5.139}$$

$$\mathbf{E}_o = -\overline{\overline{\epsilon}}^{-1} \cdot (\frac{\mathbf{k}}{\omega} \times \overline{\overline{I}} + \overline{\overline{\xi}}) \cdot \mathbf{H}_o. \tag{5.140}$$

Thus, from the knowledge of the vectors $\mathbf{k}$ and $\mathbf{E}_o$ of a plane wave field (5.131), all the other field vectors can be determined.

Substituting (5.139) and (5.140) in (5.137) and (5.138), respectively, we have the following two equations for the two vectors $\mathbf{k}$ and $\mathbf{E}_o$ or $\mathbf{k}$ and $\mathbf{H}_o$:

$$\overline{\overline{D}}_e(\frac{\mathbf{k}}{\omega}) \cdot \mathbf{E}_o = 0, \qquad \overline{\overline{D}}_m(\frac{\mathbf{k}}{\omega}) \cdot \mathbf{H}_o = 0, \tag{5.141}$$

$$\overline{\overline{D}}_e(\frac{\mathbf{k}}{\omega}) = (\frac{\mathbf{k}}{\omega} \times \overline{\overline{I}} + \overline{\overline{\xi}}) \cdot \overline{\overline{\mu}}^{-1} \cdot (\frac{\mathbf{k}}{\omega} \times \overline{\overline{I}} - \overline{\overline{\zeta}}) + \overline{\overline{\epsilon}}, \tag{5.142}$$

$$\overline{\overline{D}}_m(\frac{\mathbf{k}}{\omega}) = (\frac{\mathbf{k}}{\omega} \times \overline{\overline{I}} - \overline{\overline{\zeta}}) \cdot \overline{\overline{\epsilon}}^{-1} \cdot (\frac{\mathbf{k}}{\omega} \times \overline{\overline{I}} + \overline{\overline{\xi}}) + \overline{\overline{\mu}}. \tag{5.143}$$

The dyadics $\overline{\overline{D}}_e$, $\overline{\overline{D}}_m$ are called dispersion dyadics and (5.141) can be recognized as eigenvalue equations, although there is no obvious eigenvalue parameter. The dispersion dyadics are related to the Helmholtz operators $\overline{\overline{H}}_e(\nabla)$, $\overline{\overline{H}}_m(\nabla)$, defined in (3.58) and (3.59), by $\overline{\overline{D}}_e(\mathbf{k}/\omega) = \overline{\overline{H}}_e(-j\mathbf{k})/\omega^2$, $\overline{\overline{D}}_m(\mathbf{k}/\omega) = \overline{\overline{H}}_m(-j\mathbf{k})/\omega^2$.

For the solution of (5.141) the dispersion dyadics must be be planar. This results in two conditions for the $\mathbf{k}$ vector called dispersion equations:

$$\det\overline{\overline{D}}_e(\frac{\mathbf{k}}{\omega}) = 0, \tag{5.144}$$

$$\det\overline{\overline{D}}_m(\frac{\mathbf{k}}{\omega}) = 0. \tag{5.145}$$

It can be shown that (5.144) and (5.145) are in fact the same equation. This is seen from the expansion of the following determinant function:

$$\det(\overline{\overline{A}} \cdot \overline{\overline{B}} + \overline{\overline{I}}) = \det\overline{\overline{A}}\,\det\overline{\overline{B}} + (\overline{\overline{A}} \cdot \overline{\overline{B}})^{(2)} : \overline{\overline{I}} + (\overline{\overline{A}} \cdot \overline{\overline{B}}) : \overline{\overline{I}} + 1 =$$

$$\det\overline{\overline{A}}\,\det\overline{\overline{B}} + \overline{\overline{A}}^{(2)} : \overline{\overline{B}}^{(2)T} + \overline{\overline{A}} : \overline{\overline{B}}^T + 1, \tag{5.146}$$

which is obviously invariant to the interchange of the two dyadics $\overline{\overline{A}}$ and $\overline{\overline{B}}$. Choosing

$$\overline{\overline{A}} = \overline{\overline{\epsilon}}^{-1} \cdot (\frac{\mathbf{k}}{\omega} \times \overline{\overline{I}} + \overline{\overline{\xi}}), \qquad \overline{\overline{B}} = \overline{\overline{\mu}}^{-1} \cdot (\frac{\mathbf{k}}{\omega} \times \overline{\overline{I}} - \overline{\overline{\zeta}}), \tag{5.147}$$

we see that the determinant functions in the equations (5.144), (5.145) can be written as multiples of the form (5.146) with $\overline{\overline{A}}$ and $\overline{\overline{B}}$ interchanged, whence (5.144) and (5.145) are the same equation. $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ have been tacitly assumed to be complete dyadics.

The dispersion equations (5.144) and (5.145) are only fourth-order algebraic equations for the wave vector $\mathbf{k}$, because higher order terms are of the form $(\mathbf{k} \times \overline{\overline{I}}) : \overline{\overline{S}}$ with a symmetric dyadic $\overline{\overline{S}}$ and, consequently, vanish. The dispersion equation can be visualized in $\mathbf{k}$ space by a set of wave vector surfaces. For the general bianisotropic medium there are four such surfaces, which may have complex values. In special cases, some of the surfaces may coincide. In fact, for anisotropic and bi-isotropic media the number of wave vector surfaces is two and for isotropic media just one. The dispersion equation in the general bianisotropic case is not very simple in appearance if the coefficients are expanded. In special cases, it can be essentially simplified.

### 5.6.2 Isotropic medium

Let us consider the simplest special case. The dispersion dyadics for an isotropic medium can be written as

$$\overline{\overline{D}}_e(\frac{\mathbf{k}}{\omega}) = \epsilon\overline{\overline{D}}, \quad \overline{\overline{D}}_m(\frac{\mathbf{k}}{\omega}) = \mu\overline{\overline{D}}, \quad \overline{\overline{D}} = \overline{\overline{I}} - \frac{1}{\omega^2\mu\epsilon}\mathbf{k}\mathbf{k}{}^{\times}_{\times}\overline{\overline{I}}, \tag{5.148}$$

and the dispersion equation simply reads

$$\mathbf{k} \cdot \mathbf{k} = \omega^2\mu\epsilon. \tag{5.149}$$

From the form of the dispersion dyadics (5.148) with (5.149) substituted

$$\overline{\overline{D}} = \overline{\overline{I}} - \frac{1}{\omega^2\mu\epsilon}(\mathbf{k}\cdot\mathbf{k}\overline{\overline{I}} - \mathbf{k}\mathbf{k}) = \frac{\mathbf{k}\mathbf{k}}{\omega^2\mu\epsilon}, \tag{5.150}$$

it is seen that any vector $\mathbf{E}_o$ satisfying the condition $\mathbf{k} \cdot \mathbf{E}_o = 0$ will satisfy (5.141) and be a possible electric field vector of a plane wave. In this case, the wave vector surfaces reduce to one single sphere.

### 5.6.3 Bi-isotropic medium

The bi-isotropic medium case is just slightly more complicated to handle but more general because it has two dispersion surfaces. The dispersion dyadic in (5.141) can be given the factorized form

$$\overline{\overline{D}}_e(\frac{\mathbf{k}}{\omega}) = \epsilon\overline{\overline{I}} + \frac{1}{\mu}[\xi\overline{\overline{I}} + \frac{\mathbf{k}}{\omega} \times \overline{\overline{I}}] \cdot [\zeta\overline{\overline{I}} - \frac{\mathbf{k}}{\omega} \times \overline{\overline{I}}] = \frac{1}{\omega^2\mu}\overline{\overline{D}}_+ \cdot \overline{\overline{D}}_-, \tag{5.151}$$

with

$$\overline{\overline{D}}_\pm = \mathbf{k}\times\overline{\overline{I}} \mp jk_\pm\overline{\overline{I}}, \qquad k_\pm = k_o n_\pm = k_o(\sqrt{n^2-\chi^2}\pm\kappa). \tag{5.152}$$

Thus, the dispersion equation is split into two equations

$$\det\overline{\overline{D}}_\pm = \det(\mathbf{k}\times\overline{\overline{I}} \mp jk_\pm\overline{\overline{I}}) = \mp jk_\pm(\mathbf{k}\cdot\mathbf{k}-k_\pm^2) = 0 \tag{5.153}$$

and the wave-vector surfaces consist of spheres with radii $k = k_+$ and $k = k_-$.

The electric field polarizations corresponding to the two eigenwaves are obtained from the dispersion dyadic (5.151) by substituting $\mathbf{k} = \mathbf{u}k_+$ or $\mathbf{k} = \mathbf{u}k_-$ with the unit vector $\mathbf{u}$ denoting the direction of propagation. Because the dyadics $\overline{\overline{D}}_-$ and $\overline{\overline{D}}_+$ commute, the two eigenpolarizations satisfy $\overline{\overline{D}}_+$ $\mathbf{E}_{o+} = 0$ and $\overline{\overline{D}}_-\cdot\mathbf{E}_{o-} = 0$, or the equivalent,

$$\mathbf{u}\times\mathbf{E}_{o+} = j\mathbf{E}_{o+}, \qquad \mathbf{u}\times\mathbf{E}_{o-} = -j\mathbf{E}_{o-}. \tag{5.154}$$

The fields are circularly polarized in the plane perpendicular to $\mathbf{u}$, right hand for $\mathbf{E}_{o+}$ and left hand for $\mathbf{E}_{o-}$, because $\mathbf{u}\cdot\mathbf{E}_{o\pm} = 0$ and we can write for their polarization vectors

$$\mathbf{p}(\mathbf{E}_{o\pm}) = \frac{\mathbf{E}_{o\pm}\times\mathbf{E}_{o\pm}^*}{j\mathbf{E}_{o\pm}\cdot\mathbf{E}_{o\pm}^*} = \frac{\mathbf{E}_{o\pm}\times(\mp j\mathbf{u}\times\mathbf{E}_{o\pm})^*}{j\mathbf{E}_{o\pm}\cdot\mathbf{E}_{o\pm}^*} = \pm\mathbf{u}. \tag{5.155}$$

Thus, plane waves propagating in a bi-isotropic medium are either right-hand circularly polarized with the wave vector $\mathbf{k}_+ = \mathbf{u}k_+ = \mathbf{u}k_o(\sqrt{n^2-\chi^2}+\kappa)$ or left-hand circularly polarized with the wave vector $\mathbf{k}_- = \mathbf{u}k_- = \mathbf{u}k_o(\sqrt{n^2-\chi^2}-\kappa)$. A linearly polarized field can be split into these two components and because their phase velocities are different, their sum is a linearly polarized vector whose direction is turned much like that of a wave propagating in an anisotropic medium. It is easily seen that the angle of rotation is proportional to $k_+ - k_- = 2k_o\kappa$ and the Tellegen parameter has no effect.

### 5.6.4 Anisotropic medium

For the anisotropic medium, the dispersion equation reads

$$\det\overline{\overline{D}}_e(\frac{\mathbf{k}}{\omega}) = \det(\overline{\overline{\epsilon}} - \frac{\mathbf{kk}}{\omega^2}{}^\times_\times\overline{\overline{\mu}}^{-1}) = 0, \tag{5.156}$$

which is biquadratic in $\mathbf{k}/\omega$:

$$(\overline{\overline{\mu}}:\frac{\mathbf{kk}}{\omega^2})(\overline{\overline{\epsilon}}:\frac{\mathbf{kk}}{\omega^2}) - \overline{\overline{\mu}}^{(2)T}{}^\times_\times\overline{\overline{\epsilon}}^{(2)}:\frac{\mathbf{kk}}{\omega^2} + \det\overline{\overline{\epsilon}}\,\det\overline{\overline{\mu}} = 0. \tag{5.157}$$

Writing $\mathbf{k} = \mathbf{u}k$ with $\mathbf{u}$ a unit vector, the biquadratic equation can be solved for $k = k(\mathbf{u})$ in an analytic form:

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\overline{\overline{\mu}}^{-1} {}^{\times}_{\times} \overline{\overline{\epsilon}}^{-1T} : \mathbf{uu} \pm$$

$$\frac{1}{2}\sqrt{(\overline{\overline{\mu}}^{-1} {}^{\times}_{\times} \overline{\overline{\epsilon}}^{-1T} : \mathbf{uu})^2 - (\overline{\overline{\mu}}^{-1} {}^{\times}_{\times} \overline{\overline{\mu}}^{-1} : \mathbf{uu})(\overline{\overline{\epsilon}}^{-1} {}^{\times}_{\times} \overline{\overline{\epsilon}}^{-1} : \mathbf{uu})}. \qquad (5.158)$$

There are two pairs of solutions corresponding to two waves propagating with the same velocity in opposite directions in each pair. This is seen from changing the sign of the direction of propagation, $\mathbf{u}$, which does not change the solutions. Thus, the wave vectors of an anisotropic medium form two surfaces symmetric with respect to the origin. Unlike the general bianisotropic medium, the anisotropic medium is reciprocal for plane waves in that they propagate with the same wave number in opposite directions.

As a special case we consider the dielectrically anisotropic medium with $\overline{\overline{\epsilon}} = \epsilon_o \overline{\overline{\epsilon}}_r$, $\overline{\overline{\mu}} = \mu_o \overline{\overline{I}}$, for which the solution (5.158) can be written by defining $n$ as the refraction factor for the plane wave, $n = k/k_o$, with $\overline{\overline{I}}_t = \overline{\overline{I}} - \mathbf{uu}$,

$$n^2 = \frac{\det\overline{\overline{\epsilon}}_r}{2\overline{\overline{\epsilon}}_r : \mathbf{uu}}\left(\overline{\overline{\epsilon}}_r{}^{-1} : \overline{\overline{I}}_t \mp \sqrt{(\overline{\overline{\epsilon}}_r^{-1} : \overline{\overline{I}}_t)^2 - 4(\overline{\overline{\epsilon}}_r : \mathbf{uu}/\det\overline{\overline{\epsilon}}_r)}\right). \qquad (5.159)$$

The polarization of the field vector $\mathbf{E}_o$ can be determined through (5.141) when the $\mathbf{k}$ vector is known. If the dyadic $\overline{\overline{D}}_e(\mathbf{k}/\omega)$ is strictly planar, there is a unique polarization for $\mathbf{E}_o$. Applying the dyadic identity $(\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{A}}) \times \mathbf{a} = 2(\overline{\overline{A}} \cdot \mathbf{a}) \times \overline{\overline{A}}$, we see that $\overline{\overline{D}}_e{}^{(2)} \times \mathbf{E}_o = 0$, whence there exists a vector $\mathbf{a}$ such that $\overline{\overline{D}}_e{}^{(2)} = \mathbf{a}\mathbf{E}_o$, from which the polarization of the eigenvector $\mathbf{E}_o$ can be determined by dot multiplication from the left.

This breaks down, however, if the dyadic $\overline{\overline{D}}_e$ is linear, because then we have $\mathbf{a} = 0$. In this case, there are two linearly independent polarizations corresponding to the same wave vector $\mathbf{k}$ and any linear combination of these will do. For this direction of $\mathbf{k}$ the wave vector surfaces coincide and the direction is called an *optical axis* of the medium. Such an axis may also be in a complex direction corresponding to a complex unit vector $\mathbf{u}$. Wave vectors $\mathbf{k}$ with complex direction vectors $\mathbf{u}$ correspond to inhomogeneous plane waves, whose planes of constant amplitude and phase are non-parallel.

### Uniaxial medium

Let us consider plane wave propagation in a uniaxial anisotropic medium with the symmetric uniaxial medium dyadics

$$\overline{\overline{\epsilon}} = \epsilon_o \overline{\overline{\epsilon}}_r = \epsilon_o[\epsilon_t(\overline{\overline{I}} - \mathbf{vv}) + \epsilon_v \mathbf{vv}], \qquad \overline{\overline{\mu}} = \mu_o \overline{\overline{I}}. \qquad (5.160)$$

Here, $\mathbf{v}$ is a real unit vector and defines the axis of the medium. It will be seen that it also defines the optical axis, as can be expected since there are no other special directions in the medium.

The refraction index $n$ can be solved as a function of the propagation direction $\mathbf{u}$ from the expression (5.159). In fact, writing

$$\frac{\overline{\overline{\epsilon}}_r : \mathbf{uu}}{\det\overline{\overline{\epsilon}}_r} = \frac{\epsilon_t(\mathbf{u}\times\mathbf{v})^2 + \epsilon_v(\mathbf{u}\cdot\mathbf{v})^2}{\epsilon_t^2\epsilon_v},$$

$$\overline{\overline{\epsilon}}_r{}^{-1} : \overline{\overline{I}}_t = \frac{2\epsilon_v - (\epsilon_v - \epsilon_t)(\mathbf{u}\times\mathbf{v})^2}{\epsilon_t\epsilon_v}, \tag{5.161}$$

we have from (5.159) for the general solution

$$n = \sqrt{\frac{2\epsilon_t\epsilon_v}{2\epsilon_v - (\epsilon_v - \epsilon_t)(\mathbf{u}\times\mathbf{v})^2 \pm (\epsilon_v - \epsilon_t)(\mathbf{u}\times\mathbf{v})^2}}. \tag{5.162}$$

From this we obtain the two solutions

$$n_1 = \sqrt{\epsilon_t} = \sqrt{\overline{\overline{\epsilon}}_r : \mathbf{ww}}, \tag{5.163}$$

$$n_2 = \sqrt{\frac{\epsilon_t\epsilon_v}{\epsilon_t(\mathbf{u}\times\mathbf{v})^2 + \epsilon_v(\mathbf{u}\cdot\mathbf{v})^2}} = \sqrt{\frac{\overline{\overline{\epsilon}}_r^{(2)} : \mathbf{ww}}{\overline{\overline{\epsilon}}_r : \mathbf{uu}}}. \tag{5.164}$$

In the last expressions we have introduced the unit vector $\mathbf{w}$ in

$$\mathbf{ww} = \frac{\mathbf{uu}{}^{\times}_{\times}\mathbf{vv}}{(\mathbf{u}\times\mathbf{v})^2}, \tag{5.165}$$

which is meaningful except for $\mathbf{u} = \mathbf{v}$, in which case the solutions $n_1$ and $n_2$ coincide.

The solution 1 is independent of the wave direction $\mathbf{u}$ and can thus be called the *ordinary wave.* Its wavenumber surface is a sphere. The other solution 2 depends on the wave direction and can be called the *extraordinary wave.* Its wavenumber surface is a spheroid if $\epsilon_t$ and $\epsilon_v$ are positive real numbers, as can be easily demonstrated. The optical axis of the medium is obtained for $n_1 = n_2$, which occurs for $\mathbf{u} = \mathbf{v}$, in the direction of the axis of the uniaxial medium, as expected.

The field polarizations corresponding to the two possible wave vectors $\mathbf{k}_i = \mathbf{u}n_ik_o$, $i = 1, 2$ are obtained by forming the dyadics $\overline{\overline{D}}_e{}^{(2)}$ corresponding to the two solutions $n_i$. From the equation (5.141) applied to the uniaxial anisotropic medium we have for the two dispersion dyadics

$$\overline{\overline{D}}_{ei} = \overline{\overline{D}}_e(\frac{\mathbf{k}_i}{\omega}) = \epsilon_o\left(-n_i^2\overline{\overline{I}}{}^{\times}_{\times}\mathbf{uu} + \epsilon_t\overline{\overline{I}}_t + \epsilon_v\mathbf{vv}\right), \tag{5.166}$$

from which we can write

$$\overline{\overline{D}}_{e1}{}^{(2)} = \epsilon_o^2\epsilon_t(\epsilon_v - \epsilon_t)(\mathbf{u}\times\mathbf{v})^2\mathbf{ww}, \tag{5.167}$$

implying that $\mathbf{E}_1$ is parallel to $\mathbf{w}$, or $\mathbf{u}\times\mathbf{v}$, and

$$\overline{\overline{D}}_{e2}{}^{(2)} = \frac{\epsilon_o^4\epsilon_t(\epsilon_t+\epsilon_v)}{\overline{\overline{\epsilon}} : \mathbf{uu}}(\mathbf{u}\times\mathbf{v})^2\mathbf{ww}{}^{\times}_{\times}(\overline{\overline{\epsilon}}_r\cdot\mathbf{u})(\overline{\overline{\epsilon}}_r\cdot\mathbf{u}), \tag{5.168}$$

which means that $\mathbf{E}_2$ is parallel to the vector $\mathbf{w}\times(\overline{\overline{\epsilon}}_r\cdot\mathbf{u})$.

The two waves are seen to be orthogonally polarized, because $\mathbf{E}_1\cdot\mathbf{E}_2 = 0$, except for propagation along the optical axis, $\mathbf{u}=\mathbf{v}$, in which case the previous expressions fail and any vector orthogonal to $\mathbf{u}$ can represent the polarization just like in isotropic media. For propagation transverse to the axis, $\mathbf{u}\cdot\mathbf{v}=0$, the polarization of the wave 2 is seen to lie parallel to the $\mathbf{v}$ axis.

*Gyrotropic medium*

The expression (5.158) can also be applied to a gyrotropic medium, which is a generalization of the symmetric uniaxial medium. Assuming only dielectric gyrotropy such as in magnetoplasma, we consider the medium dyadics

$$\overline{\overline{\epsilon}} = \epsilon_o[\epsilon_t(\overline{\overline{I}} - \mathbf{vv}) + \epsilon_v\mathbf{vv} + \gamma\mathbf{v}\times\overline{\overline{I}}], \qquad \overline{\overline{\mu}} = \mu_o\overline{\overline{I}}. \tag{5.169}$$

The term $\gamma\mathbf{v}\times\overline{\overline{I}}$ is responsible for the gyrotropic effect.

Applying dyadic identities we can evaluate

$$\overline{\overline{\mu}}^{-1}{}^{\times}_{\times}\overline{\overline{\mu}}^{-1} : \mathbf{uu} = \frac{2}{\mu_o^2}, \qquad \overline{\overline{\epsilon}}^{-1}{}^{\times}_{\times}\overline{\overline{\epsilon}}^{-1} : \mathbf{uu} = \frac{2\epsilon_t(\mathbf{u}\times\mathbf{v})^2 + 2\epsilon_v(\mathbf{u}\cdot\mathbf{v})^2}{\epsilon_o^2\epsilon_v(\epsilon_t^2+\gamma^2)}, \tag{5.170}$$

$$\overline{\overline{\mu}}^{-1}{}^{\times}_{\times}\overline{\overline{\epsilon}}^{-1T} : \mathbf{uu} = \frac{1}{\mu_o\epsilon_o\epsilon_v}[\frac{\epsilon_t\epsilon_v(1+(\mathbf{u}\cdot\mathbf{v})^2)}{(\epsilon_t^2+\gamma^2)} + (\mathbf{u}\times\mathbf{v})^2]. \tag{5.171}$$

These substituted in (5.158) give us, after some algebra, the following expression for the two possible refraction indices:

$$n_\pm^2 = \frac{2\epsilon_v(\epsilon_t^2+\gamma^2)}{(\gamma^2-\epsilon_t(\epsilon_v-\epsilon_t))s^2 + 2\epsilon_t\epsilon_v \mp \sqrt{(\gamma^2+\epsilon_t(\epsilon_t-\epsilon_v))^2s^4 - 4\epsilon_v^2\gamma^2c^2}} =$$

$$\frac{(\gamma^2+\epsilon_t^2)s^2 + \epsilon_t\epsilon_v(1+c^2) \pm \sqrt{(\gamma^2+\epsilon_t(\epsilon_t-\epsilon_v))^2s^4 - 4\epsilon_v^2\gamma^2c^2}}{2(\epsilon_ts^2+\epsilon_vc^2)}. \tag{5.172}$$

Here we have denoted

$$s^2 = (\mathbf{u}\times\mathbf{v})^2, \qquad c = \mathbf{u}\cdot\mathbf{v}, \qquad s^2+c^2 = 1. \tag{5.173}$$

The famous Appleton–Hartree–Lassen formula for wave propagation in magnetoplasma is straightforwardly obtained if the parameters $\epsilon_t$, $\epsilon_v$ and $\gamma$ are given the expressions corresponding to the physics of the magnetoplasma. As a check, the previous result for the uniaxial medium is readily recovered by setting $\gamma = 0$. The expression (5.172) for $n$ can be applied for any direction of propagation $\mathbf{u}$, although the relation is not very simple. Let us consider two special cases.

*Longitudinal propagation*

For the wave propagating along the axis ($\mathbf{u} = \mathbf{v}$), we have, setting $s = 0$, $c = 1$ in (5.172), for the refraction indices of the two plane waves:

$$n_\pm^2 = \epsilon_t \pm j\gamma. \tag{5.174}$$

In comparison with (2.164), it is seen that these are two of the three eigenvalues of the gyrotropic dyadic $\overline{\overline{\epsilon}}_r$ itself.

The corresponding polarizations are easily seen from the dispersion dyadic by comparison with (2.165):

$$\overline{\overline{D}}_\pm(\frac{\mathbf{k}}{\omega}) = -\epsilon_o n_\pm^2 \mathbf{vv}{}^{\times}_{\times}\overline{\overline{I}} + \overline{\overline{\epsilon}} = \epsilon_o[\mp\gamma(\overline{\overline{I}} \mp j\mathbf{v}\times\overline{\overline{I}}) + (\epsilon_v \pm j\gamma)\mathbf{vv}] =$$

$$\epsilon_o[\mp j\gamma(\mathbf{w} \mp j\mathbf{w}\times\mathbf{v})(\mathbf{w} \pm j\mathbf{w}\times\mathbf{v}) + \epsilon_v\mathbf{vv}], \tag{5.175}$$

or, better still, by forming the dyadics

$$\overline{\overline{D}}_\pm{}^{(2)} = \mp j\epsilon_o^2\epsilon_v\gamma(\mathbf{w} \pm j\mathbf{v}\times\mathbf{w})(\mathbf{w} \pm j\mathbf{v}\times\mathbf{w}). \tag{5.176}$$

This time, the vector $\mathbf{w}$ is any unit vector satisfying $\mathbf{w}\cdot\mathbf{v} = 0$. Thus, the electric polarization vectors $\mathbf{E}_\pm$ are multiples of the two vectors

$$\mathbf{E}_\pm = E_\pm(\mathbf{w} \mp j\mathbf{v}\times\mathbf{w}), \tag{5.177}$$

which are CP vectors orthogonal to the direction of propagation $\mathbf{u} = \mathbf{v}$.

By forming the corresponding polarization vectors

$$\mathrm{p}(\mathbf{E}_\pm) = \frac{\mathbf{E}_\pm \times \mathbf{E}_\pm^*}{j\mathbf{E}_\pm \cdot \mathbf{E}_\pm^*} = \pm\mathbf{v}, \tag{5.178}$$

we see that of the two plane waves in longitudinal propagation, $\mathbf{E}_+$ has the right-hand and $\mathbf{E}_-$, the left-hand polarization.

*Transversal propagation*

Considering wave propagation transverse to the $\mathbf{v}$ axis with $\mathbf{u}\cdot\mathbf{v} = 0$, we have, by setting $c = 0$, $s^2 = 1$ in (5.172), for the refraction indices of the two possible plane waves:

$$n_1^2 = \epsilon_v, \quad n_2^2 = \epsilon_t + \frac{\gamma^2}{\epsilon_t}. \tag{5.179}$$

In this case, the wave 1 is not affected by the gyrotropy of the medium since the parameter $\gamma$ is absent in $n_1$. This wave is called the *ordinary wave* and the other one (2), the *extraordinary wave.* The polarizations $\mathbf{E}_1$, $\mathbf{E}_2$ are obtained from the dispersion dyadics

$$\overline{\overline{D}}_1(\frac{\mathbf{k}}{\omega}) = \overline{\overline{\epsilon}} - \epsilon_v\epsilon_o\mathbf{uu}{}^{\times}_{\times}\overline{\overline{I}} =$$

$$\epsilon_o[(\epsilon_t\mathbf{u} - \gamma\mathbf{u}\times\mathbf{v})\mathbf{u} + (\gamma\mathbf{u} + (\epsilon_t - \epsilon_v)\mathbf{u}\times\mathbf{v})(\mathbf{u}\times\mathbf{v})], \tag{5.180}$$

$$\overline{\overline{D}}_2(\frac{\mathbf{k}}{\omega}) = \frac{\epsilon_o}{\epsilon_t}[\epsilon_t^2\mathbf{uu} - (\gamma^2 + \epsilon_t(\epsilon_t - \epsilon_v))\mathbf{vv} - \gamma^2\mathbf{uu}{}^{\times}_{\times}\mathbf{vv} + \epsilon_t\gamma\mathbf{v}\times\overline{\overline{I}}] =$$

$$\frac{\epsilon_o}{\epsilon_t}[(\epsilon_t\mathbf{u} - \gamma\mathbf{u}\times\mathbf{v})(\epsilon_t\mathbf{u} + \gamma\mathbf{u}\times\mathbf{v}) + (\gamma^2 + \epsilon_t(\epsilon_t - \epsilon_v))\mathbf{vv}]. \tag{5.181}$$

For the ordinary wave 1, we have

$$\overline{\overline{D}}_1^{(2)} = \epsilon_o^2(\gamma^2 + \epsilon_t(\epsilon_t - \epsilon_v))\mathbf{vv}, \tag{5.182}$$

whence the corresponding polarization is in the direction of $(\mathbf{u}\times\mathbf{v})\times\mathbf{u} = \mathbf{v}$. This also explains the notion 'ordinary wave': the wave behaves like in an isotropic medium since the electric field is only affected by the dielectric coefficient along the axis $\mathbf{v}$, which equals $\epsilon_v\epsilon_o$.

For the extraordinary wave 2 we have from the dispersion dyadic after some algebra

$$\overline{\overline{D}}_2^{(2)} = -\epsilon_o^2\frac{\gamma^2 + \epsilon_t(\epsilon_t - \epsilon_v)}{\epsilon_t^2}(\epsilon_t\mathbf{u}\times\mathbf{v} + \gamma\mathbf{u})(\epsilon_t\mathbf{u}\times\mathbf{v} - \gamma\mathbf{u}), \tag{5.183}$$

from which the polarization of $\mathbf{E}_2$ is seen to be in the direction of $\epsilon_t\mathbf{u}\times\mathbf{v} - \gamma\mathbf{u}$. For $\gamma\to 0$, the medium becomes uniaxial and the polarization is along $\mathbf{u}\times\mathbf{v}$, which coincides with that of the ordinary wave in the uniaxial medium.

*Optical axes*

One may finally ask about the optical axes of the gyrotropic medium. To have only one solution for the refraction index, the square-root term must be equal to zero in (5.172), which gives us the equation

$$(\gamma^2 + \epsilon_t(\epsilon_t - \epsilon_v))^2 s^4 = 4\epsilon_v^2\gamma^2c^2. \tag{5.184}$$

Writing $s^2 = 1 - c^2$ leads us to the four solutions

$$c = \mathbf{u}\cdot\mathbf{v} = \pm\frac{\epsilon_v\gamma \pm \sqrt{(\epsilon_t^2 + \gamma^2)((\epsilon_t - \epsilon_v)^2 + \gamma^2)}}{\gamma^2 + \epsilon_t(\epsilon_t - \epsilon_v)}, \tag{5.185}$$

where the two double signs are independent of each other.

Real axis directions are obtained only if the expression above gives real values for $c$ and satisfies $|c| \leq 1$. For a lossless gyrotropic medium with hermitian $\bar{\bar{\epsilon}}$, this equation does not have a real solution, because $\epsilon_t$, $\epsilon_v$ are real and $\gamma$ is an imaginary number and, thus, $c$ becomes complex. However, if $\epsilon_t$ and $\epsilon_v$ are complex, there may exist a real optical axis. This happens, for example, in lossy magnetoplasma like the ionosphere, where the ordinary and extraordinary waves may have the same refraction factor for a certain frequency and direction of propagation.

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