Chapter 6

Source equivalence

From the uniqueness property of electromagnetic fields it is clear that two different fields always require two different sources. However, the reverse is not true, i.e. there may well be different sources which give rise to the same fields within a certain region, V. Such sources $\mathbf{J}_1$, $\mathbf{J}_2$ will be called *equivalent with respect to* V and denoted by $\mathbf{J}_1 \sim \mathbf{J}_2$. Equivalence of sources with respect to a region of interest is obviously an important property, because if an original complicated source can be shown to be equivalent with some simpler source, the problem can be simplified. The original source can also be a secondary source like a polarization source arising in a polarizable medium.

Source equivalence is often applied without recognition. For example, multipole expansions and surface sources based on Huygens' principle are equivalent to certain physical sources, because they give the same field in the space outside of these sources. Let us restrict ourselves here to time-harmonic sources and fields.

6.1 Non-radiating sources

If two sources are equivalent with respect to a region V, their difference obviously produces the null field in V. A non-null source like this is called non-radiating (NR) with respect to V. Fields in regular regions, whose medium parameters are analytic functions of position and whose interfaces and boundary surfaces have continuous tangents, can be shown to be analytic functions of position (MORREY and NIRENBERG 1957). This means that they can be analytically continued within the regular regions and, hence, the null-field actually extends to all of the regular region containing the NR region.

6.1.1 Electric sources in isotropic medium

Let us consider sources and their fields in a homogeneous isotropic infinite medium. Bounded electric NR sources produce the null field outside their support. Any two equivalent electric sources differ from one another through an NR electric source. The most general form of a time-harmonic NR electric source can be shown to be (DEVANEY and WOLF 1973)

$$\mathbf{J}^{NR} = \overline{\overline{H}}(\nabla) \cdot \mathbf{f}, \tag{6.1}$$

where $\overline{\overline{H}}(\nabla)$ is the Helmholtz operator

$$\overline{\overline{H}}(\nabla) = \nabla\nabla \overset{\times}{\times} \overline{\overline{I}} + k^2 = -\nabla \times (\nabla \times \overline{\overline{I}}) + k^2\overline{\overline{I}} \tag{6.2}$$

and $\mathbf{f}$ is a vector function whose support defines the complement of the region of null field. $\mathbf{J}^{NR}$ defined by (6.1) obviously has the same support as $\mathbf{f}$ and thus produces the null field outside this support.

Let us study the necessity and sufficiency of this statement. First, the necessity is seen from the form of the Helmholtz equation for the electric field,

$$\overline{\overline{H}}(\nabla) \cdot \mathbf{E} = j\omega\mu\mathbf{J}^{NR}. \tag{6.3}$$

In the NR case, the electric field vanishes outside a certain region, which is the support of the vector $\mathbf{E}$. Thus, dividing (6.3) by $j\omega\mu$, we see that the NR source function must necessarily be of the form (6.1) with $\mathbf{f} = \mathbf{E}/j\omega\mu$.

To show the sufficiency, let us substitute the expression (6.1) in the Helmholtz equation (6.3) and write it in the form

$$\overline{\overline{H}}(\nabla) \cdot (\mathbf{E} - j\omega\mu\mathbf{f}) = 0. \tag{6.4}$$

Because $\mathbf{f}$ vanishes at infinity, $\mathbf{E} - j\omega\mu\mathbf{f}$ satisfies the radiation condition of $\mathbf{E}$ implying that the solution is unique and since there are no source terms on the right-hand side, we arrive at the null solution

$$\mathbf{E} - j\omega\mu\mathbf{f} = 0. \tag{6.5}$$

Thus, the electric field, and consequently the magnetic field, vanish outside the support of $\mathbf{f}$, i.e. the region where $\mathbf{f} \neq 0$, whence every source of the form (6.1) is an NR source.

Special sources

Although the form (6.1) is the most general one for the NR source, it is often helpful to recognize special forms of NR sources. For example, the gradient of any scalar function ϕ of bounded support happens to be an NR source, because we can write

$$\mathbf{J}^{NR} = \nabla\phi = \overline{\overline{H}}(\nabla) \cdot \left(\frac{\nabla\phi}{k^2}\right). \tag{6.6}$$

Thus, irrotational sources do not excite fields beyond the source itself. Actually the function ϕ need not be of bounded support, it suffices that its gradient be of bounded support.

As an example of such a source we may consider radiation from an exploding atomic bomb in a homogeneous atmosphere or outside atmospheres in space. (This example is actually not one with time-harmonic sources, but serves as a demonstration of the same principle with a more general time dependence.) Such an electron flow corresponds to a radially symmetric current function $\mathbf{J}(\mathbf{r}) = \mathbf{u}_r J(r)$ and it can be written as a gradient of another function $\mathbf{J}(\mathbf{r}) = \nabla \int_{r_o}^{r} J(r)dr$, which is of bounded support if the function $J(r)$ is. Thus the radially symmetric source is obviously non-radiating. The EMP (electromagnetic pulse) arising from a nuclear detonation in the air is due to the inhomogeneity of the atmosphere, which breaks the radial symmetry. Thus, non-radiating sources are not just mathematical oddities but can really exist in the physical world (to a certain accuracy, at least).

Another special form of NR source is obtained from (6.1) by expanding the double curl operation and deleting the non-radiating gradient term :

$$\mathbf{J}^{NR} = (\nabla^2 + k^2)\mathbf{f}. \tag{6.7}$$

As a consequence we may note that the two sources

$$\mathbf{J}_1 = \frac{\partial^2 \mathbf{f}}{\partial z^2} + k^2\mathbf{f}, \qquad \mathbf{J}_2 = -\nabla_t^2 \mathbf{f} \tag{6.8}$$

are equivalent: $\mathbf{J}_1 \sim \mathbf{J}_2$, because they radiate the same fields and their difference does not radiate beyond the sources.

The possibility of NR sources poses an awkward problem to remote sensing of electromagnetic sources: it is theoretically impossible to determine the source by measuring the radiation in such a part of the space which does not include all the sources. Of course, knowing the field $\mathbf{E}(\mathbf{r})$ in all space, its source can be obtained uniquely by substituting the field in the left-hand side of the Helmholtz equation, whence the right-hand side will give the source function. Because any NR source with the same support can be added to the original source without changing the measurable field outside the sources, the problem of source determination becomes inherently non-unique. Thus, additional information about the source is necessary for its identification. Alternatively, we may look for the simplest possible source, for example, the one which has the minimum source energy. To find this, we have to require that a certain energy norm functional of the source function be minimized.

6.1.2 Sources in bianisotropic media

Combined electric and magnetic sources

Above we considered NR electric sources $\mathbf{J}^{NR}$ in isotropic media. A similar expression for the most general magnetic NR source $\mathbf{J}_m^{NR}$ can be written from duality. More generally, a combination of electric and magnetic sources $(\mathbf{J}, \mathbf{J}_m)^{NR}$ can be non-radiating. The form of the most general electromagnetic NR source for the general bianisotropic medium can be obtained by writing the Maxwell equations in the form

$$\overline{\overline{L}}(\nabla) \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} =$$

$$\begin{pmatrix} -j\omega\overline{\overline{\epsilon}} & \nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}} \\ -\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\zeta}} & -j\omega\overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{J} \\ \mathbf{J}_m \end{pmatrix}, \tag{6.9}$$

whence in analogy to the electric NR source we can write

$$\begin{pmatrix} \mathbf{J} \\ \mathbf{J}_m \end{pmatrix}^{NR} = \overline{\overline{L}}(\nabla) \cdot \begin{pmatrix} \mathbf{f}_e \\ \mathbf{f}_m \end{pmatrix} =$$

$$\begin{pmatrix} -j\omega\overline{\overline{\epsilon}} \cdot \mathbf{f}_e + (\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot \mathbf{f}_m \\ -(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \mathbf{f}_e - j\omega\overline{\overline{\mu}} \cdot \mathbf{f}_m \end{pmatrix}, \tag{6.10}$$

where $\mathbf{f}_e$ and $\mathbf{f}_m$ are two vector functions of bounded support V.

The proof goes as in the electric case: the necessity is obtained from (6.9) because if $\mathbf{E}$ and $\mathbf{H}$ are null outside V, the right side gives sources which thus must be of the form (6.10) with $\mathbf{f}_e$, $\mathbf{f}_m$ vanishing outside V. For sufficiency, (6.10) substituted in (6.9) results in

$$\overline{\overline{L}}(\nabla) \cdot \begin{pmatrix} \mathbf{E} - \mathbf{f}_e \\ \mathbf{H} - \mathbf{f}_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{6.11}$$

whence from uniqueness we have $\mathbf{E} = \mathbf{f}_e$ and $\mathbf{H} = \mathbf{f}_m$, which vanish outside V.

Electric and magnetic sources

The most general expression for the electric NR source in a bianisotropic medium is obtained from the previous (6.10) by setting the magnetic source equal to zero:

$$\begin{pmatrix} \mathbf{J}^{NR} \\ 0 \end{pmatrix} = \overline{\overline{L}}(\nabla) \cdot \begin{pmatrix} \mathbf{f}_e \\ \mathbf{f}_m \end{pmatrix} = \begin{pmatrix} -j\omega\overline{\overline{\epsilon}} \cdot \mathbf{f}_e + (\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot \mathbf{f}_m \\ -(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \mathbf{f}_e - j\omega\overline{\overline{\mu}} \cdot \mathbf{f}_m \end{pmatrix}, \tag{6.12}$$

from which, solving for the function $\mathbf{f}_m$ in the form

$$\mathbf{f}_m = -\frac{1}{j\omega}\overline{\overline{\mu}}^{-1} \cdot (\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \mathbf{f}_e, \tag{6.13}$$

and substituting this in one of the equations of (6.12), the resulting expression for the most general electric NR source in a bianisotropic medium is obtained:

$$\mathbf{J}^{NR} = [-j\omega\overline{\overline{\epsilon}} - \frac{1}{j\omega}(\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot \overline{\overline{\mu}}^{-1} \cdot (\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}})] \cdot \mathbf{f}_e. \tag{6.14}$$

The corresponding expression for the most general magnetic NR source is obtained correspondingly by starting from $\mathbf{J} = 0$,

$$\mathbf{J}_m^{NR} = [j\omega\overline{\overline{\mu}} + \frac{1}{j\omega}(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \overline{\overline{\epsilon}}^{-1} \cdot (\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}})] \cdot \mathbf{f}_m. \tag{6.15}$$

Setting $\overline{\overline{\epsilon}} = \epsilon\overline{\overline{I}}$, $\overline{\overline{\mu}} = \mu\overline{\overline{I}}$ and $\overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$ we immediately obtain the NR sources in isotropic spaces,

$$\mathbf{J}^{NR} = -j\omega\epsilon\mathbf{f}_e - \frac{1}{j\omega\mu}\nabla \times (\nabla \times \mathbf{f}_e), \tag{6.16}$$

$$\mathbf{J}_m^{NR} = -j\omega\mu\mathbf{f}_m - \frac{1}{j\omega\epsilon}\nabla \times (\nabla \times \mathbf{f}_m), \tag{6.17}$$

of which (6.16) can be recognized as equivalent to the previous expression of DEVANEY and WOLF (1973), with the substitution $\mathbf{f}_e = j\omega\mu\mathbf{f}$.

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6.2 Equivalent electric and magnetic sources

In the previous section we obtained the most general expression for the NR combination of electric and magnetic sources, $(\mathbf{J}, \mathbf{J}_m)^{NR}$. Thus, we may state that since the field due to $\mathbf{J}$ is cancelled by that of $\mathbf{J}_m$ in the NR pair, outside their supports, the sources $\mathbf{J}$ and $-\mathbf{J}_m$ corresponding to the same NR pair are equivalent: $(\mathbf{J}, 0) \sim (0, -\mathbf{J}_m)$. This shows us that there exists a magnetic source equivalent to a given electric source and conversely. Actually there exists an infinity of equivalent sources because we can always add an NR source to a given equivalent source.

6.2.1 Bianisotropic medium

Starting from the electric source $\mathbf{J}$ and setting $\mathbf{f}_m = 0$ in (6.10), we may solve the magnetic source whose negative is equivalent to the electric source:

$$\mathbf{J}_m^{eq} = (\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \mathbf{f}_e = -\frac{1}{j\omega}(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \overline{\overline{\epsilon}}^{-1} \cdot \mathbf{J}. \tag{6.18}$$

Correspondingly, starting from a magnetic source $\mathbf{J}_m$ and setting $\mathbf{f}_e = 0$ in (6.10), we may solve for the equivalent electric source:

$$\mathbf{J}^{eq} = -(\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot \mathbf{f}_m = \frac{1}{j\omega}(\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot \overline{\overline{\mu}}^{-1} \cdot \mathbf{J}_m. \tag{6.19}$$

To see the equivalence, let us denote the fields due to $\mathbf{J} \neq 0$, $\mathbf{J}_m = 0$ by $\mathbf{E}_e$, $\mathbf{H}_e$:

$$\begin{pmatrix} -j\omega\overline{\overline{\epsilon}} & \nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}} \\ -\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\zeta}} & -j\omega\overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_e \\ \mathbf{H}_e \end{pmatrix} = \begin{pmatrix} \mathbf{J} \\ 0 \end{pmatrix}. \tag{6.20}$$

Denoting the fields due to $\mathbf{J} = 0$, $\mathbf{J}_m = \mathbf{J}_m^{eq} \neq 0$ by $\mathbf{E}_m$, $\mathbf{H}_m$ and substituting the magnetic source $\mathbf{J}_m^{eq}$ from (6.18) in (6.9), we can write this in the form

$$\begin{pmatrix} -j\omega\overline{\overline{\epsilon}} & \nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}} \\ -\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\zeta}} & -j\omega\overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_m - \frac{1}{j\omega}\overline{\overline{\epsilon}}^{-1} \cdot \mathbf{J} \\ \mathbf{H}_m \end{pmatrix} = \begin{pmatrix} \mathbf{J} \\ 0 \end{pmatrix}, \tag{6.21}$$

similar to those for the electric source $\mathbf{J}$, whence from uniqueness we have the following relation between the two sets of fields:

$$\mathbf{E}_m - \frac{1}{j\omega}\overline{\overline{\epsilon}}^{-1} \cdot \mathbf{J} = \mathbf{E}_e, \qquad \mathbf{H}_m = \mathbf{H}_e. \tag{6.22}$$

Thus, the magnetic fields of the original and equivalent sources are equal not only outside the sources but everywhere, whereas the electric fields coincide outside the sources only. The converse is true if we start from a magnetic source and replace it by the equivalent electric source (6.19). It is easy to see that there cannot exist complete equivalence of electric and magnetic sources so that $\mathbf{E}_e = \mathbf{E}_m$ and $\mathbf{H}_e = \mathbf{H}_m$ would be satisfied everywhere.

Of course, a magnetic source equivalent to an electric NR source obtained in the foregoing manner must be NR. To verify this, we substitute the expression of an NR source (6.14) in (6.18) and write the result as

$$\mathbf{J}_m^{eq} = -\frac{1}{j\omega}(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \overline{\overline{\epsilon}}^{-1} \cdot \mathbf{J}^{NR} =$$

$$-\frac{1}{j\omega}(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \overline{\overline{\epsilon}}^{-1} \cdot [-j\omega\overline{\overline{\epsilon}} - \frac{1}{j\omega}(\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot \overline{\overline{\mu}}^{-1} \cdot (\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}})] \cdot \mathbf{f}_e =$$

$$[j\omega\overline{\overline{\mu}} + \frac{1}{j\omega}(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \overline{\overline{\epsilon}}^{-1} \cdot (\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}})] \cdot \mathbf{f}_m, \tag{6.23}$$

with

$$\mathbf{f}_m = \frac{1}{j\omega}\overline{\overline{\mu}}^{-1} \cdot (\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \mathbf{f}_e. \tag{6.24}$$

Obviously, (6.23) is of the form (6.15), which means that the equivalent source of an electric NR source is really a magnetic NR source.

6.2.2 Isotropic medium

For the isotropic medium we can write the equivalent sources (6.18), (6.19) simply as

$$\mathbf{J}^{eq} = \frac{\nabla \times \mathbf{J}_m}{j\omega\mu}, \qquad \mathbf{J}_m^{eq} = -\frac{\nabla \times \mathbf{J}}{j\omega\epsilon}. \tag{6.25}$$

There are many interesting and important consequences from these expressions. Let us itemize a few of them.

- If $\nabla \times \mathbf{J} = 0$, the equivalent magnetic source is $\mathbf{J}_m^{eq} = 0$ and, hence, the electromagnetic field vanishes outside the original source. Thus, irrotational sources do not radiate. Such a source can be expressed as the gradient of a scalar function, which is an example of a class of NR sources, as was seen in the previous section.

- If $\mathbf{J}$ is a constant vector function within a volume bounded by a surface S, the corresponding $\mathbf{J}_m^{eq}$ is non-zero only on S. For example, if $\mathbf{J}$ is a small axially directed cylindrical source, the equivalent $\mathbf{J}_m$ is a magnetic surface current flowing around the cylinder perpendicular to the axis. This leads to equivalence of a dipole and a loop, which is considered in more detail below.

- Applying the equivalence condition twice we can see that the source $\mathbf{J}$ is equivalent to another electric source $\mathbf{J}^{eq} = \nabla \times \nabla \times \mathbf{J}/k^2$. For a cylindrical dipole this is a double surface current source. The source $\mathbf{J} - \mathbf{J}^{eq}$ is an example of an NR source, because its expression is of the general NR form (6.1).

- Because the source $\nabla(\nabla \cdot \mathbf{J})$ is irrotational and does not radiate, we conclude that the electric source $\mathbf{J}^{eq} = -\nabla^2 \mathbf{J}/k^2$ is also equivalent with the source $\mathbf{J}$.

Dipole and loop equivalence

As an example let us consider the electric current equivalent to the magnetic dipole current function

$$\mathbf{J}_m(\mathbf{r}) = \mathbf{u} I_m L \delta(\mathbf{r}). \tag{6.26}$$

The equivalent electric current is, from (6.25),

$$\mathbf{J}^{eq}(\mathbf{r}) = \frac{I_m L}{j\omega\mu} \nabla \delta(\mathbf{r}) \times \mathbf{u}. \tag{6.27}$$

An idea of this function is obtained if the magnetic dipole is considered to be the limit of a cylindrical constant magnetic volume current function parallel to the z coordinate:

$$\mathbf{J}_m(\mathbf{r}) = \mathbf{u}_z J_m U(a - \rho) U(h^2 - z^2). \tag{6.28}$$

Here, $U(x)$ is the Heaviside step function and a, $2h = L$ are the radius and length of the cylinder, respectively, with $J_m = I_m/\pi a^2$. The corresponding equivalent electric current can be written as

$$\mathbf{J}^{eq}(\mathbf{r}) = \mathbf{u}_\phi \frac{J_m}{j\omega\mu} \delta(\rho - a) U(h^2 - z^2), \tag{6.29}$$

which is easily pictured as a surface current flowing on the cylindrical surface $\rho = a$ perpendicular to the z axis. The total current on the surface is

$$I^{eq} = \frac{J_m L}{j\omega\mu} = \frac{I_m L}{j\omega\mu} \frac{1}{A}, \tag{6.30}$$

$A = \pi a^2$ being the cross-sectional area of the cylindrical dipole.

This expression is also valid for other than circular cross sections of the dipole. If fields are considered at distances large enough and the dipole is small with respect to the wavelength in all dimensions, the cylindrical surface current can be concentrated to a loop with the current I. Thus, magnetic dipole and electric current loop are equivalent sources.

From duality, we can also write the magnetic loop current equivalent to an electric dipole current I:

$$I_m^{eq} = -\frac{IL}{j\omega\epsilon A}. \tag{6.31}$$

The equivalent source of the constant cylindrical magnetic source can be applied to get an idea how an equivalent source of a general current source can be formed. In fact, any magnetic current function $\mathbf{J}_m(\mathbf{r})$ can be split into dipoles small enough that the current density of each of them can be considered constant, whence the equivalent sources for each dipole can be pictured as forming a distribution of surface currents in space which obviously result in a continuous equivalent magnetic volume current source. For example, a line magnetic current source can be replaced by a current tube with surface current flowing transverse to the original magnetic line current. Also, a magnetic surface current can be replaced by a double electric surface current, consisting of two surface currents flowing in opposite directions on each side of the original magnetic surface current in the transverse direction.

6.2.3 Chiral medium

As a final special case let us apply the previous expressions for the chiral medium with the parameter dyadics

$$\overline{\overline{\epsilon}} = \epsilon\,\overline{\overline{I}}, \quad \overline{\overline{\mu}} = \mu\,\overline{\overline{I}}, \quad \overline{\overline{\xi}} = -\overline{\overline{\zeta}} = -j\kappa\sqrt{\mu_o\epsilon_o}\,\overline{\overline{I}}. \tag{6.32}$$

Defining the self-dual sources as

$$\mathbf{J}_\pm = \frac{1}{2}[\mathbf{J} \pm \frac{1}{j\eta}\mathbf{J}_m], \tag{6.33}$$

we can write from (6.10) for the most general NR sources the expressions

$$\mathbf{J}_\pm^{NR} = (\nabla \times \overline{\overline{I}} \mp k_\pm\overline{\overline{I}}) \cdot \mathbf{f}_\pm, \tag{6.34}$$

with

$$k_\pm = k \pm \kappa k_o, \qquad \mathbf{f}_\pm = \frac{1}{2}[\mathbf{f}_m \mp \frac{1}{j\eta}\mathbf{f}_e]. \tag{6.35}$$

From (6.34) we see that, in the chiral medium, the NR sources $\mathbf{J}_\pm^{NR}$ are uncoupled. Thus, a combined NR source always consists of two self-dual sources which are individually NR sources.

As another implication we may state that a plus source and a minus source cannot be equivalent if they both radiate, because their difference is an NR source only if the plus and minus components are individually NR sources. This means that the sources $\mathbf{J}$ and $\mathbf{J}_m$ are equivalent in a chiral medium only if both their plus and minus components are equivalent.

The equivalent source expressions (6.18), (6.19) reduce to the following form for equivalent sources in chiral medium:

$$\mathbf{J}_m^{eq} = -\frac{1}{j\omega\epsilon}(\nabla \times \overline{\overline{I}} - \kappa k_o \overline{\overline{I}}) \cdot \mathbf{J}, \tag{6.36}$$

$$\mathbf{J}^{eq} = \frac{1}{j\omega\mu}(\nabla \times \overline{\overline{I}} - \kappa k_o \overline{\overline{I}}) \cdot \mathbf{J}_m. \tag{6.37}$$

These expressions mean that, unlike for the isotropic medium, an irrotational source is not necessarily NR.

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6.3 Multipole sources

The multipole expansion of a bounded electromagnetic source is one example of an equivalent source producing the same field as the original source in the region outside the original source and the multipole. Since the expansion involves an infinite sum, even though it represents the true field analytically, the series may diverge and thus give numerical trouble for points close to the source or the multipole. The greatest advantage of the expansion is obtained for such sources which can effectively be approximated through just a couple of terms in the expansion or for field points sufficiently far from the source.

The multipole expansion consists of a series of point sources, which are located at the same point, usually determined by symmetry or some other obvious choice. The location of the multipole can, however, be selected on the basis of obtaining the simplest approximation for the expansion. Through this process, the best location will in general be found to be in complex space.

6.3.1 Delta expansions

It is known that slowly varying functions can be approximated effectively through the Taylor series, in terms of power functions. On the other hand, it is possible to express properties of highly localized functions in terms of delta function expansions of the form

$$f(x) \sim f_0\delta(x) + f_1\delta'(x) + f_2\delta''(x) + ... + f_n\delta^{(n)} + ... \tag{6.38}$$

Because this expansion does not mean that values for the function on the left-hand side can be obtained from the series on the right-hand side, the equivalence sign $\sim$ has been applied. The significance of the delta expansion lies in the fact that the integral of the product of $f(x)$ and an analytic function $g(x)$ can be expressed as

$$\int_{-\infty}^{\infty} f(x)g(x)dx = f_0 g(0) - f_1 g'(0) + f_2 g''(0) + ... + (-1)^n f_n g^{(n)}(0) + ..., \tag{6.39}$$

following from the properties of the delta function and its derivatives:

$$\int_{-\infty}^{\infty} g(x)\delta^{(n)}(x)dx = (-1)^n g^{(n)}(0) \qquad n = 0, 1, 2, ... \tag{6.40}$$

For more exact definitions in terms of distributions the reader is advised to consult a recent book by VAN BLADEL (1991).

The expansion (6.39) resembles the one obtained when the function $g(x)$ is expanded in a Taylor series within the integral:

$$g(x) = g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 + ... + \frac{1}{n!}g^{(n)}(0)x^n + ..., \tag{6.41}$$

resulting in the following integral expression:

$$\int_{-\infty}^{\infty} f(x)g(x)dx = g(0)\int f(x)dx + g'(0)\int xf(x)dx +$$

$$... + g^{(n)}(0)\int \frac{1}{n!}x^n f(x)dx + ... \tag{6.42}$$

Identifying terms in (6.39) and (6.42) gives us coefficients for the delta expansion of $f(x)$, whence (6.38) can be written as

$$f(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\delta^{(n)}(x) \int_{-\infty}^{\infty} f(x')(x')^n dx'. \tag{6.43}$$

It can be noted that the two function sets, $\{\delta^{(m)}(x)\}$ and $\{x^n\}$ are biorthogonal, because they satisfy

$$\int_{-\infty}^{\infty} x^n \delta^{(m)}(x)dx = (-1)^m m! \delta_{mn} \ . \tag{6.44}$$

Thus, the nth coefficient in the delta series expansion (6.38) can be simply obtained through multiplying by x^n, integrating and applying the orthogonality (6.44).

The orthogonality in one dimension can be generalized into three dimensions. In fact, the n-adic $\mathbf{rr}...\mathbf{r}$ and m-adic $\nabla\nabla...\nabla\delta(\mathbf{r})$ can be shown to be orthogonal, for $m \neq n$, when integrated over the whole three-dimensional space. This property is easily understood because, for example, $\nabla\nabla\mathbf{r} = \nabla\overline{\overline{I}} = 0$ everywhere and $\nabla(\mathbf{rr}) = 0$ at the origin. Similarly, all expressions with different numbers of ∇ and $\mathbf{r}$ vanish at the origin. On the other hand, $\nabla\mathbf{r} = \overline{\overline{I}}$ and also all other expressions with the same number of ∇ and $\mathbf{r}$ give nonzero results.

This property can be applied when expanding a function of $\mathbf{r}$ as a delta series. To write the result in coordinate-independent form we adopt the following n-dot product ⓝ between two n-ads (polyads of rank n):

$$\mathbf{a}_1\mathbf{a}_2...\mathbf{a}_n \; ⓝ \; \mathbf{b}_1\mathbf{b}_2...\mathbf{b}_n = (\mathbf{a}_1 \cdot \mathbf{b}_1)(\mathbf{a}_2 \cdot \mathbf{b}_2)...(\mathbf{a}_n \cdot \mathbf{b}_n). \tag{6.45}$$

Special cases are the dot product of vectors ($n = 1$) and double-dot product of dyadics ($n = 2$).

With this, we can write

$$f(\mathbf{r}) = \mathcal{F}_0\delta(\mathbf{r}) + \mathcal{F}_1 \cdot \nabla\delta(\mathbf{r}) + ... + \mathcal{F}_n \; ⓝ \; \underbrace{\nabla\nabla..\nabla}_{n}\delta(\mathbf{r}) + ..., \tag{6.46}$$

where the coefficients $\mathcal{F}_n$ are n-ads. Their expressions can be found from orthogonality, through multiplying the expression by the m-ad $\mathbf{rr}...\mathbf{r}$ and integrating over the whole space.

6.3.2 Multipole expansion

A multipole expansion of a given source is just a delta series representation of the source function. As can be well understood, the expansion is practical only if the source is localized enough. For calculating the far field, the higher order derivatives of the delta function give rise to more rapidly decreasing fields, whence the series converges more rapidly than for the near field. Because there also exist NR point sources, the multipole series is non-unique. NR multipoles can always be added to make the series look different. Of course, it is of interest to obtain the simplest possible form.

Let us write the delta expansion for a concentrated current source function $\mathbf{J}(\mathbf{r})$ in the form

$$\mathbf{J}(\mathbf{r}) = \delta(\mathbf{r})\mathcal{P}_0(0) - \nabla\delta(\mathbf{r}) \cdot \mathcal{P}_1(0) + \frac{1}{2!}\nabla\nabla\delta(\mathbf{r}) : \mathcal{P}_2(0) - ...$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \nabla\nabla...\nabla\delta(\mathbf{r}) \; ⓝ \; \mathcal{P}_n(0), \tag{6.47}$$

with n operators ∇ in the nth term. The quantities $\mathcal{P}_n$ are $n+1$-adics, i.e. polynomials of $n+1$-ads: $\mathcal{P}_0$ is a vector, $\mathcal{P}_1$ a dyadic etc. They represent the moments of the current function:

$$\mathcal{P}_n(0) = \int_V \mathbf{r}'\mathbf{r}'...\mathbf{r}'\mathbf{J}(\mathbf{r}')dV', \tag{6.48}$$

with n vector $\mathbf{r}'$ multiplicants.

The multipole considered so far is located at the origin. The expansion is changed if the point of the multipole is moved from the origin. Taking $\mathbf{r} = \mathbf{a}$ as the point of the multipole, we can write another expansion:

$$\mathbf{J}(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \nabla\nabla...\nabla\delta(\mathbf{r}-\mathbf{a}) \; ⓝ \; \mathcal{P}_n(\mathbf{a}), \tag{6.49}$$

with current moments

$$\mathcal{P}_n(\mathbf{a}) = \int_V (\mathbf{r}'-\mathbf{a})(\mathbf{r}'-\mathbf{a})...(\mathbf{r}'-\mathbf{a})\mathbf{J}(\mathbf{r}')dV'. \tag{6.50}$$

The location of the expansion can be chosen at will. Usually the choice is made in some obvious way, by taking a point of symmetry or the origin of the coordinate system. However, by a proper choice of the multipole location the convergence of the expansion can be improved. In general, such a point $\mathbf{a}$ will be in complex space.

The dipole term in the current function is $\delta(\mathbf{r}-\mathbf{a})\mathcal{P}_0(\mathbf{a})$, with the current moment

$$\mathcal{P}_0(\mathbf{a}) = \int_V \mathbf{J}(\mathbf{r}')dV' = j\omega\mathbf{p}_e. \tag{6.51}$$

Because the dipole moment vector $\mathbf{p}_e$ does not depend on $\mathbf{a}$, the vector moment $\mathcal{P}_0(\mathbf{a})$ is actually independent of the location of the multipole. For a current function small in volume and not varying too rapidly, the dipole term is the most dominant.

The dyadic moment of the multipole series can be written as

$$\mathcal{P}_1(\mathbf{a}) = \int_V (\mathbf{r}' - \mathbf{a})\mathbf{J}(\mathbf{r}')dV' = \mathcal{P}_1(0) - \mathbf{a}\mathcal{P}_0(0) =$$

$$\frac{j\omega}{2}\overline{\overline{Q}}_e - \mathbf{p}_m \times \overline{\overline{I}} - j\omega\mathbf{a}\mathbf{p}_e, \tag{6.52}$$

where the vector $\mathbf{p}_m$ denotes the magnetic dipole moment and the symmetric dyadic $\overline{\overline{Q}}_e$ the electric quadrupole moment in common notation. It is seen that the expression (6.52) is changed if the vector $\mathbf{a}$ is changed. However, if the electric dipole moment is zero, $\mathcal{P}_1(\mathbf{a})$ is independent of $\mathbf{a}$. More generally, it can be stated that the lowest moment in the multipole series does not depend on the position.

6.3.3 Dipole approximation of a multipole source

Let us consider 'simple' sources, which can be well approximated by a couple of first terms in the multipole series. If the electric dipole term is non-zero, we can optimize the location $\mathbf{a}$ of the multipole by defining it so that the dyadic moment $\mathcal{P}_1(\mathbf{a})$ is minimized. This can be made more exact by defining a norm in the space of dyadics, for example,

$$\| \overline{\overline{A}} \| = \sqrt{\overline{\overline{A}} : \overline{\overline{A}}^*}, \tag{6.53}$$

which is called the Erhard Schmidt norm in mathematics.

The position vector $\mathbf{a}$ can now be obtained by minimizing the non-negative real number

$$\nu(\mathbf{a}) = [\mathcal{P}_1(0) - \mathbf{a}\mathcal{P}_0(0)] : [\mathcal{P}_1(0) - \mathbf{a}\mathcal{P}_0(0)]^* =$$

$$\mathcal{P}_1(0) : \mathcal{P}_1^*(0) - \mathbf{a}\cdot\mathcal{P}_1^*(0)\cdot\mathcal{P}_0(0) - \mathbf{a}^*\cdot\mathcal{P}_1(0)\cdot\mathcal{P}_0^*(0) + (\mathbf{a}\cdot\mathbf{a}^*)\mathcal{P}_0(0)\cdot\mathcal{P}_0^*(0). \tag{6.54}$$

This can be done by writing $\mathbf{a} = \mathbf{a}_r + j\mathbf{a}_i$, differentiating $\nu(\mathbf{a})$ with respect to $\mathbf{a}_r$ and $\mathbf{a}_i$ and equating the results to zero. The same result is obtained if the vector $\mathbf{a}^*$ is considered independent of the vector $\mathbf{a}$ and differentiating ν with respect to $\mathbf{a}^*$ by keeping $\mathbf{a}$ constant. As a result, the following expression for $\mathbf{a}$ is obtained:

$$\mathbf{a} = \frac{\mathcal{P}_1(0)\cdot\mathcal{P}_0^*(0)}{\mathcal{P}_0(0)\cdot\mathcal{P}_0^*(0)} = \frac{\int \mathbf{r}'\mathbf{J}(\mathbf{r}')dV' \cdot \int \mathbf{J}^*(\mathbf{r}')dV'}{\int \mathbf{J}(\mathbf{r}')dV' \cdot \int \mathbf{J}^*(\mathbf{r}')dV'}. \tag{6.55}$$

If the phase of the current function is constant, $\mathbf{a}$ is a real vector, otherwise it is complex, in general.

To see that the choice of the vector $\mathbf{a}$ in (6.55) really decreases the norm of the dyadic moment, we can write after some algebra

$$||\mathcal{P}_1(\mathbf{a})|| = \left\| \mathcal{P}_1(0) \cdot \left(\overline{\overline{I}} - \frac{\mathcal{P}_0(0)\mathcal{P}_0^*(0)}{\mathcal{P}_0(0)\cdot\mathcal{P}_0^*(0)} \right) \right\| =$$

$$||\mathcal{P}_1(0)|| \sqrt{1 - \frac{|\mathcal{P}_1(0)\cdot\mathcal{P}_0^*(0)|^2}{||\mathcal{P}_1(0)||^2|\mathcal{P}_0(0)|^2}}, \tag{6.56}$$

showing us that the norm decreases unless $\mathcal{P}_1(0)\cdot\mathcal{P}_0^*(0) = 0$, in which case the square root equals unity and the norm remains the same.

If the original current function has constant polarization $\mathbf{J}(\mathbf{r}) = \mathbf{u}J(\mathbf{r})$ with $\mathbf{u}\cdot\mathbf{u}^* = 1$, the vector $\mathcal{P}_0(0)$ is parallel to $\mathbf{u}$ and $\mathcal{P}_1(0)$ is of the form $\mathbf{qu}$. Substituting in the above expression we see that both the square root and $\mathcal{P}_1(\mathbf{a})$ vanish. In this case transfer of the multipole from the origin to the point $\mathbf{a}$ is really worth while.

Cubic current source

As an example, let us consider radiation from a current wave of constant polarization, $\mathbf{J}(\mathbf{r}) = \mathbf{u}Je^{-jkx}$, non-zero inside and zero outside the cube defined by $0 < x, y, z < L$. The polarization of the current is given by the complex unit vector $\mathbf{u}$ satisfying $\mathbf{u}\cdot\mathbf{u}^* = 1$. Substituting $\mathbf{J}(\mathbf{r})$ in (6.48) we have, after integration, for the first two moments

$$\mathcal{P}_0(0) = \mathbf{u}JL^3\frac{\sin\tau}{\tau}, \qquad \mathcal{P}_1(0) = j\mathbf{u}_x\mathbf{u}\frac{L^3J}{k}\left(\cos\tau - \frac{\sin\tau}{\tau}\right), \tag{6.57}$$

with $\tau = kL/2$.

The position of the multipole can be obtained from the expression (6.55):

$$\mathbf{a} = \mathbf{u}_x\frac{j}{k}\left(\frac{kL}{2}\cot\frac{kL}{2} - 1\right). \tag{6.58}$$

From the expression (6.52) it is easily seen that $\mathcal{P}_1(\mathbf{a}) = 0$, or moving the dipole moment (6.57) to the point $\mathbf{r} = \mathbf{a}$ makes the second multipole term vanish. The optimum point of the multipole is seen to be at an imaginary distance from the orgin. When the side of the cube is small, $kL \ll 1$, (6.58) is approximated by

$$\mathbf{a} \approx -j\mathbf{u}_x\frac{kL^2}{12} = -j\mathbf{u}_x\frac{(kL)^2}{24\pi}\lambda. \tag{6.59}$$

The shift is small when compared with the wavelength. Hence, for small current sources the optimum point of the approximating dipole is almost at the centre of the cube.

The advantage for choosing the complex point (6.58) for the multipole follows when the size of the cube is not very small. This is seen by comparing the radiation patterns. Assuming that the current is polarized in the direction of the z axis: $\mathbf{u} = \mathbf{u}_z$, the exact pattern function can be obtained by integration:

$$F(\theta,\phi) = \left| \sin\theta \left(\frac{\sin[\tau(1-\sin\theta\cos\phi)]}{\tau(1-\sin\theta\cos\phi)} \right) \cdot \left(\frac{\sin[\tau\sin\theta\sin\phi]}{\tau\sin\theta\sin\phi} \right) \left(\frac{\sin[\tau\cos\theta]}{\tau\cos\theta} \right) \right|. \tag{6.60}$$

The approximating by a dipole at the origin gives us simply

$$F(\theta,\phi) \approx |\sin\theta|, \tag{6.61}$$

whereas by putting the dipole at $\mathbf{r} = \mathbf{a}$ of (6.58) leads to

$$F(\theta,\phi) \approx \left| \sin\theta e^{-(1-\tau\cot\tau)(1-\sin\theta\cos\phi)} \right|. \tag{6.62}$$

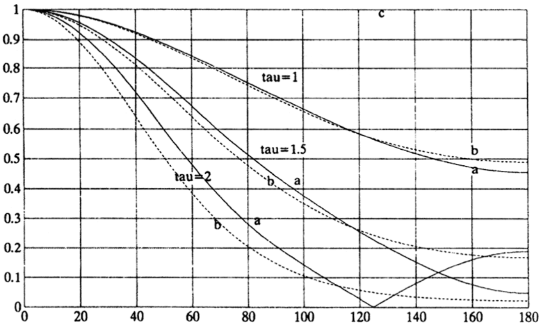

Fig. 6.1 Radiation pattern for the cubic current wave in three approximations: (a) exact, (b) dipole at a complex point and (c) dipole at the origin. The parameter τ equals $kL/2$, where L is the side of the cube.

Expanding the pattern expressions (6.60), (6.61) and (6.62) in Taylor series in powers of $\tau = kL/2$, it is seen that (6.60) and (6.62) coincide up to terms of the second order,

$$F(\theta,\phi) \approx \left| \sin\theta \left(1 - \frac{\tau^2}{3}(1-\sin\theta\cos\phi) \right) \right|, \tag{6.63}$$

whereas (6.61) gives only the first-order term. This difference is clearly seen from the graphical representation of (6.60), (6.61) and (6.62). The increase in directivity of a radiator when shifted in a complex direction can be explained in the same way as the Gaussian beam emerging from a point source in complex space.

6.3.4 Electric and magnetic dipole approximation

Writing the dyadic moment (6.52) in the equivalent form

$$\mathcal{P}_1(\mathbf{a}) = \frac{j\omega}{2}(\overline{\overline{Q}}_e - \mathbf{a}\mathbf{p}_e - \mathbf{p}_e\mathbf{a}) - (\mathbf{p}_m - \frac{j\omega}{2}\mathbf{a} \times \mathbf{p}_e) \times \overline{\overline{I}}, \tag{6.64}$$

we see how moving the multipole changes the magnetic dipole and the electric quadrupole terms. If the source is not well approximated by a single electric dipole, a combination of electric and magnetic dipoles may be tried. This leads to minimization of the norm of the electric quadrupole term, i.e. the symmetric part of $\mathcal{P}_1$ in a manner similar to the previous one. The equation for **a** now becomes

$$(\overline{\overline{Q}}_e - \mathbf{a}\mathbf{p}_e - \mathbf{p}_e\mathbf{a}) \cdot \mathbf{p}_e^* = 0, \tag{6.65}$$

and its solution can be written as

$$\mathbf{a} = (\overline{\overline{I}} - \frac{\mathbf{p}_e\mathbf{p}_e^*}{2|\mathbf{p}_e|^2}) \cdot \overline{\overline{Q}}_e \cdot \frac{\mathbf{p}_e^*}{|\mathbf{p}_e|^2}. \tag{6.66}$$

The resulting quadrupole dyadic takes the form

$$\overline{\overline{Q}}_e(\mathbf{a}) = (\overline{\overline{I}} - \frac{\mathbf{p}_e\mathbf{p}_e^*}{|\mathbf{p}_e|^2}) \cdot \overline{\overline{Q}}_e \cdot (\overline{\overline{I}} - \frac{\mathbf{p}_e^*\mathbf{p}_e}{|\mathbf{p}_e|^2}) = (\overline{\overline{Q}}_e {}^{\times}_{\times} \frac{\mathbf{p}_e\mathbf{p}_e}{|\mathbf{p}_e|^2}) {}^{\times}_{\times} \frac{\mathbf{p}_e^*\mathbf{p}_e^*}{|\mathbf{p}_e|^2}, \tag{6.67}$$

and its norm is obviously less than or equal to that of $\overline{\overline{Q}}_e$. The magnetic dipole moment is changed to

$$\mathbf{p}_m(\mathbf{a}) = \mathbf{p}_m - \frac{j\omega}{2}\mathbf{a} \times \mathbf{p}_e = \mathbf{p}_m + \frac{j\omega}{2|\mathbf{p}_e|^2}\mathbf{p}_e \times \overline{\overline{Q}}_e \cdot \mathbf{p}_e. \tag{6.68}$$

Thus, it is seen that when moving the multipole from the origin to the point (6.66), the electric dipole is unchanged but the magnetic dipole changes from $\mathbf{p}_m$ to $\mathbf{p}_m(\mathbf{a})$, while the electric quadrupole is minimized. From duality, we could write similar formulas also for the magnetic current $\mathbf{J}_m$, in which case the magnetic dipole is the dominating term and the electric dipole changes when the position of the multipole is moved.

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6.4 Huygens' sources

Huygens' sources are surface sources equivalent to physical sources, whether primary or secondary, existing behind the surface. To apply the equivalence, the region behind the surface with Huygens' sources can be replaced by any medium, because there will be no electromagnetic interaction across that surface. Thus, a part of electromagnetic problem can be separated by terminating the fields in the Huygens source.

In practice, Huygens' sources are often not known because they depend on the fields at the surface, which are not known in general. However, applying surface integral equations, Huygens' sources can be computed. Also, if sufficient knowledge of the problem exists, they can be approximated and radiation fields computed with reasonable accuracy. For example, radiation from aperture antennas like reflectors, lenses or slots in a metallic plane is usually calculated through a straightforward approximation of the Huygens source.

6.4.1 The truncated problem

Let us consider an electromagnetic differential equation problem in the form

$$\mathsf{Lf} = (\mathsf{L}_\nabla + \mathsf{L}_o)\mathsf{f} = \mathsf{g}, \tag{6.69}$$

where the linear operator L and the field f and source g quantities can be associated with any version of the electromagnetic problem. For, example, the Maxwell equations in bianisotropic media lead to the definitions

$$\mathsf{L}_\nabla = \begin{pmatrix} 0 & \nabla \times \overline{\overline{I}} \\ -\nabla \times \overline{\overline{I}} & 0 \end{pmatrix}, \qquad \mathsf{L}_o = -j\omega \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}, \tag{6.70}$$

$$\mathsf{f} = \begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix}, \qquad \mathsf{g} = \begin{pmatrix} \mathbf{J}(\mathbf{r}) \\ \mathbf{J}_m(\mathbf{r}) \end{pmatrix}. \tag{6.71}$$

Let us now truncate the problem by introducing a pulse function $P_V(\mathbf{r})$, where V refers to some region in space of sufficient regularity

$$P_V(\mathbf{r}) = 1, \quad \mathbf{r} \in V, \qquad P_V(\mathbf{r}) = 0, \quad \mathbf{r} \notin V. \tag{6.72}$$

Multiplying (6.69) by the pulse function P_V, we have $(\mathsf{Lf})P_V = \mathsf{g}P_V$, which means that the source has been truncated and those parts of the source which lie outside V have been removed. Because L_∇ is a first-order differential operator, we can write

$$\mathsf{L}(\mathsf{f}P_V) = (\mathsf{Lf})P_V + (\mathsf{L}_\nabla P_V)\mathsf{f} = \mathsf{g}P_V + (\mathsf{L}_\nabla P_V)\mathsf{f}. \tag{6.73}$$

This is actually a theorem equivalent to Huygens' principle and the last term equals Huygens' source.

6.4.2 Huygens' principle

The equation (6.73) can be interpreted as follows: the source $\mathsf{g}P_V$ is the truncated source and $\mathsf{f}P_V$ the truncated field, equal to the original field inside V and zero outside V. Hence, the term $(\mathsf{L}_\nabla P_V)\mathsf{f}$ represents an equivalent source, which radiates inside V the same field as the removed source $\mathsf{g} - \mathsf{g}P_V$. Outside V it is obviously equivalent to the source $-\mathsf{g}P_V$, because it cancels the field due to the source $\mathsf{g}P_V$. The source $(\mathsf{L}_\nabla P_V)\mathsf{f}$ appears only at the boundary surface S of the volume V.

Let us be more specific with the operator. Taking the definitions (6.70), (6.71), we can write the Huygens source term in bianisotropic medium from (6.73) as

$$(\mathsf{L}_\nabla P_V)\mathsf{f} = \begin{pmatrix} 0 & \nabla P_V \times \overline{\overline{I}} \\ -\nabla P_V \times \overline{\overline{I}} & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{n} \times \mathbf{H}\delta_S \\ -\mathbf{n} \times \mathbf{E}\delta_S \end{pmatrix}, \tag{6.74}$$

where $\mathbf{n}$ is the unit normal on S pointing into the volume V. Here we denote by $\delta_S(\mathbf{r}) = \mathbf{n} \cdot \nabla P_V(\mathbf{r})$ the surface delta function.

Writing (6.73) in explicit form in this case results in the equation

$$\begin{pmatrix} -j\omega\overline{\overline{\epsilon}} & \nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}} \\ -\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\zeta}} & -j\omega\overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}P_V \\ \mathbf{H}P_V \end{pmatrix} =$$

$$\begin{pmatrix} \mathbf{J}P_V \\ \mathbf{J}_m P_V \end{pmatrix} + \begin{pmatrix} \mathbf{n}\times\mathbf{H}\delta_S \\ -\mathbf{n}\times\mathbf{E}\delta_S \end{pmatrix}. \tag{6.75}$$

The solution can be written in terms of Green dyadics, functions of $\mathbf{r}$ and $\mathbf{r}'$, corresponding to the homogeneous bianisotropic medium, even if we do not know their analytic expressions. $V+$ denotes a volume containing V and S,

$$\begin{pmatrix} \mathbf{E}P_V \\ \mathbf{H}P_V \end{pmatrix} = \int_{V+} \begin{pmatrix} \overline{\overline{G}}_{ee} & \overline{\overline{G}}_{em} \\ \overline{\overline{G}}_{me} & \overline{\overline{G}}_{mm} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{J}P_V + \mathbf{n}'\times\mathbf{H}\delta_S \\ \mathbf{J}_m P_V - \mathbf{n}'\times\mathbf{E}\delta_S \end{pmatrix} dV'. \tag{6.76}$$

In the isotropic medium we know the expressions of the scalar and dyadic Green functions $G(\mathbf{r}-\mathbf{r}')$, $\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')$, whence we can write

$$\begin{pmatrix} \mathbf{E}P_V \\ \mathbf{H}P_V \end{pmatrix} =$$

$$\int_{V+} \begin{pmatrix} -j\omega\mu\overline{\overline{G}} & \nabla'G\times\overline{\overline{I}} \\ -\nabla'G\times\overline{\overline{I}} & -j\omega\epsilon\overline{\overline{G}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{J}P_V + \mathbf{n}'\times\mathbf{H}\delta_S \\ \mathbf{J}_m P_V - \mathbf{n}'\times\mathbf{E}\delta_S \end{pmatrix} dV'. \tag{6.77}$$

Let us introduce, for convenience, the following shorthand notation:

$$\mathcal{E}[\mathbf{J}|\mathbf{J}_m]_V = -j\omega\mu\int_V \mathbf{J}(\mathbf{r}')\cdot\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')dV' -$$

$$\int_V \mathbf{J}_m(\mathbf{r}')\cdot(\nabla'\times\overline{\overline{G}}(\mathbf{r}-\mathbf{r}'))dV', \tag{6.78}$$

where V is any volume. The function $\mathcal{E}[\mathbf{J}|\mathbf{J}_m]_V$ gives a vector function of $\mathbf{r}$, which equals the electric field produced by those parts of the electric current function $\mathbf{J}(\mathbf{r})$ and the magnetic current function $\mathbf{J}_m(\mathbf{r})$ which lie inside the volume V. For surface sources, the corresponding notation is

$$\mathcal{E}[\mathbf{J}_s|\mathbf{J}_{ms}]_S = -j\omega\mu\int_S \mathbf{J}_s(\mathbf{r}')\cdot\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')dS' -$$

$$\int_S \mathbf{J}_{ms}(\mathbf{r}')\cdot(\nabla'\times\overline{\overline{G}}(\mathbf{r}-\mathbf{r}'))dS', \tag{6.79}$$

and the expression gives us the field from those parts of the surface source functions which lie on the surface S.

With this notation, Huygens' source problem can be conveniently written as

$$\mathbf{E}(\mathbf{r})P_V(\mathbf{r}) = \mathcal{E}[\mathbf{J}|\mathbf{J}_m]_V + \mathcal{E}[\mathbf{n}\times\mathbf{H}| - \mathbf{n}\times\mathbf{E}]_S. \tag{6.80}$$

The interpretation is dependent on the location of the field point $\mathbf{r}$. In fact we have three cases.

1. $\mathbf{r}$ is inside the volume V. The left-hand side of (6.80) gives us $\mathbf{E}(\mathbf{r})$, which arises from those parts of the sources $\mathbf{J}$, $\mathbf{J}_m$ that are contained in V plus the surface sources $\mathbf{J}_s = \mathbf{n}\times\mathbf{H}$, $\mathbf{J}_{ms} = -\mathbf{n}\times\mathbf{E}$ on S, the boundary of V. Thus, for points $\mathbf{r}$ inside V, the surface sources are equivalent to the parts of volume sources outside V.

2. $\mathbf{r}$ is outside the volume V. The left-hand side of (6.80) is now zero. With respect to the space outside V, the sources inside V together with the surface sources are NR.

3. $\mathbf{r}$ is at the surface S. This case will be considered later through limit processes. For a smooth surface at $\mathbf{r}$, the left-hand side of (6.80) gives us $\mathbf{E}(\mathbf{r})/2$. We can picture the field point being in the middle of the surface source, whence only half of it is cancelling the field from the interior sources, thus resulting in half of the original field. This excludes points on S where the tangent is discontinuous (wedge, apex), because the normal vector $\mathbf{n}$ is not uniquely defined at those points.

Corresponding formulas can be written for the magnetic field from duality:

$$\mathbf{H}(\mathbf{r})P_V(\mathbf{r}) = \mathcal{H}[\mathbf{J}|\mathbf{J}_m]_V + \mathcal{H}[\mathbf{n}\times\mathbf{H}| - \mathbf{n}\times\mathbf{E}]_S, \tag{6.81}$$

with similar definitions

$$\mathcal{H}[\mathbf{J}|\mathbf{J}_m]_V = \int_V \mathbf{J}(\mathbf{r}')\cdot\nabla'\times\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')dV' - j\omega\epsilon\int_V \mathbf{J}_m(\mathbf{r}')\cdot\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')dV', \tag{6.82}$$

$$\mathcal{H}[\mathbf{J}_s|\mathbf{J}_{ms}]_S = \int_S \mathbf{J}_s(\mathbf{r}')\cdot\nabla'\times\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')dS' - j\omega\epsilon\int_S \mathbf{J}_{ms}(\mathbf{r}')\cdot\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')dS'. \tag{6.83}$$

6.4.3 Consequences of Huygens' principle

Let us consider some implications of the previous principle.

Analyticity of field functions

The representations (6.80), (6.81) give field functions $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$, which are analytic at points $\mathbf{r}$ outside sources and the boundary S. To see this, we may note that the Green dyadic is an analytic function when $\mathbf{r}$ is not in the domain of integration. Thus, the field functions are continuous and bounded functions together with their derivatives. The field functions cannot take infinite values at such points. They might be infinite, however, at the boundary S, or inside the source region if the source function has some discontinuity.

A consequence of the analyticity is that the field functions can be continued analytically along a path whose points are in the region of analyticity. Also, all analytic functions of fields are analytic. For example, if the electric field or one of its components vanishes in some small part of the region of analyticity, it vanishes in all of it. Thus, the field is transverse electric in the whole region of analyticity, not just in a part of it.

Equivalence of sources

From (6.80) and (6.81) we see that a Huygens source on the surface S containing the volume V is actually equivalent to the volume sources outside V, with respect to V. If all volume sources are in V, the Huygens source on S is equivalent to no sources and, thus, is NR in V. This means that the electric surface current is then equivalent to the negative of the magnetic surface current with respect to V.

With respect to the outside V, however, Huygens' sources are equivalent to the negative of the volume sources in V. Changing the direction of $\mathbf{n}$ in Huygens' sources is tantamount to changing the reference volume from V to its outside and the signs of the surface sources. In this case we see that the principle that volume sources are equivalent to Huygens' surface sources is also valid when the volume is the outside a closed surface. If V is bounded by two closed surfaces S_1 and S_2, Huygens' priciple applies when the surface sources are taken on both surfaces S_1 and S_2.

Use of Huygens' principle

Huygens' principle is usually applied in one of two ways: to simplify complicated problems through suitable approximations or to derive integral equations for the problem. If we have a complicated problem with boundaries and material interfaces and we are interested in fields radiated outside this system, we may enclose it within a closed surface S and use the approximation method. The fields radiated outside S can be calculated from Huygens' sources depending on the tangential electric and magnetic fields on S. Usually these are not known exactly, but in certain cases they can be approximated. For example, when S is at a PEC surface with an aperture of small extent, the tangential electric field is zero everywhere on S except at the aperture. Also, if the surface contains the aperture of a large reflector antenna, we can approximate the surface fields by calculating them through geometrical optics procedures.

6.4.4 Surface integral equations

Huygens' principle also allows one to define integral equations for the unknown surface sources by letting the field point approach the surface S. Because of the singularity of the Green dyadic the approach must be made with care. Let us consider the limiting cases when the field point approaches the surface S containing the electric and magnetic surface currents $\mathbf{J}_s$, $\mathbf{J}_{ms}$, from the two normal directions $\mathbf{n}_1$, $\mathbf{n}_2$. From the interface conditions

$$\mathbf{n}_1 \times \mathbf{H}_1 + \mathbf{n}_2 \times \mathbf{H}_2 = \mathbf{J}_s, \tag{6.84}$$

$$\mathbf{n}_1 \times \mathbf{E}_1 + \mathbf{n}_2 \times \mathbf{E}_2 = -\mathbf{J}_{ms}, \tag{6.85}$$

we see that the tangential magnetic and electric fields are not continuous through the surface unless the corresponding surface source is zero. To build an equation for the surface sources it is not useful to consider limiting values of fields but fields exactly at the surface.

It was seen that the Green dyadic is too singular to let the fields be calculated inside a surface source as we wish to do here. Even the principal value method does not work, because it does not exist. The finite part method does the job, but it involves additional integration.

To find converging field integrals we can modify definitions of the surface source functions (6.79), (6.83) by breaking the double gradient through partial integration. In fact, we may write for an open surface S, when the field point $\mathbf{r}$ is not on S

$$\int_S \overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}_s(\mathbf{r}')dS' = \int_S G\mathbf{J}_s dS' + \frac{1}{k^2}\int_S (\nabla\nabla G)\cdot\mathbf{J}_s dS' =$$

$$\int_S G\mathbf{J}_s dS' - \frac{1}{k^2}\int_S (\nabla' G)(\nabla'\cdot\mathbf{J}_S)dS' + \frac{1}{k^2}\int_S \nabla'\cdot(\mathbf{J}_s\nabla' G)dS', \tag{6.86}$$

and the last term can be reduced to a line integral around the boundary curve C of S. For a closed surface, this curve reduces to a point and produces zero if the field point is outside the surface. For a point on the surface the integral is responsible for the singularity. Discarding this term we may give new definitions for the functions (6.79), (6.83), valid for *closed surfaces S*:

$$\mathcal{E}[\mathbf{J}_s|\mathbf{J}_{ms}]_S = -j\omega\mu \int_S G(\mathbf{r}-\mathbf{r}')\mathbf{J}_s(\mathbf{r}')dS' -$$

$$\frac{1}{j\omega\epsilon}\int_S \nabla' G(\mathbf{r}-\mathbf{r}')\nabla'\cdot\mathbf{J}_s(\mathbf{r}')dS' + \int_S \nabla' G(\mathbf{r}-\mathbf{r}')\times\mathbf{J}_{ms}(\mathbf{r}')dS', \quad (6.87)$$

$$\mathcal{H}[\mathbf{J}_s|\mathbf{J}_{ms}]_S = -j\omega\epsilon \int_S G(\mathbf{r}-\mathbf{r}')\mathbf{J}_{ms}(\mathbf{r}')dS' -$$

$$\frac{1}{j\omega\mu}\int_S \nabla' G(\mathbf{r}-\mathbf{r}')\nabla'\cdot\mathbf{J}_{ms}(\mathbf{r}')dS' - \int_S \nabla' G(\mathbf{r}-\mathbf{r}')\times\mathbf{J}_s(\mathbf{r}')dS'. \quad (6.88)$$

These field functions are still singular when the field point $\mathbf{r}$ is on the surface S, but the singularity is of lower order and can be removed by removing a circular disk of area A_δ whose measure δ is small. The limit $\delta \to 0$ is again called the principal value of the integral. It is not difficult to check that there is the following relation between the PV and limiting values of the field function,

$$\lim_{\mathbf{r}\to S}\mathcal{E}[\mathbf{J}_s|\mathbf{J}_{ms}]_S = PV\mathcal{E}[\mathbf{J}_s|\mathbf{J}_{ms}]_S + \frac{\mathbf{m}}{2j\omega\epsilon}\nabla\cdot\mathbf{J}_s - \frac{\mathbf{m}}{2}\times\mathbf{J}_{ms}, \quad (6.89)$$

$$\lim_{\mathbf{r}\to S}\mathcal{H}[\mathbf{J}_s|\mathbf{J}_{ms}]_S = PV\mathcal{H}[\mathbf{J}_s|\mathbf{J}_{ms}]_S + \frac{\mathbf{m}}{2j\omega\mu}\nabla\cdot\mathbf{J}_{ms} + \frac{\mathbf{m}}{2}\times\mathbf{J}_s, \quad (6.90)$$

when the field point $\mathbf{r}$ approaches S in the direction of the unit normal vector $\mathbf{m}$, which may be either $\mathbf{n} = \nabla P_V(\mathbf{r})$ or its opposite.

To form the surface integral equations, we take Huygens' expressions of the fields and let the field point approach the surface S. Substituting the limit in terms of the PV integral above, we have

$$\frac{1}{2}\mathbf{E}(\mathbf{r}) = \mathcal{E}[\mathbf{J}|\mathbf{J}_m]_V + PV\mathcal{E}[\mathbf{n}\times\mathbf{H}| -\mathbf{n}\times\mathbf{E}]_S, \quad (6.91)$$

$$\frac{1}{2}\mathbf{H}(\mathbf{r}) = \mathcal{H}[\mathbf{J}|\mathbf{J}_m]_V + PV\mathcal{H}[\mathbf{n}\times\mathbf{H}|\mathbf{n}\times\mathbf{E}]_S. \quad (6.92)$$

These equations presume that the tangent surface is continuous at the field point $\mathbf{r}$, so that $\mathbf{n} = \nabla P_V$ is continuous.

Because tangential fields on a closed surface S are enough to determine the fields inside S through Huygens' principle, it is sufficient to make integral equations for the tangential field components. Operating (6.91) and (6.92) by $\mathbf{n}\times$ we have

$$\frac{1}{2}\mathbf{n}\times\mathbf{E}(\mathbf{r}) = \mathbf{n}\times\mathcal{E}[\mathbf{J}|\mathbf{J}_m]_V + \mathbf{n}\times PV\mathcal{E}[\mathbf{n}\times\mathbf{H}| -\mathbf{n}\times\mathbf{E}]_S, \quad (6.93)$$

$$\frac{1}{2}\mathbf{n}\times\mathbf{H}(\mathbf{r}) = \mathbf{n}\times\mathcal{H}[\mathbf{J}|\mathbf{J}_m]_V + \mathbf{n}\times PV\mathcal{H}[\mathbf{n}\times\mathbf{H}| - \mathbf{n}\times\mathbf{E}]_S, \qquad (6.94)$$

either of which represents a relation between the tangential components of the electric and magnetic field on any closed surface whose inside is a homogeneous isotropic space. It is customary to call (6.93) EFIE, the electric field integral equation, and (6.94), MFIE or the magnetic field integral equation.

It can be easily deduced that, although (6.93) and (6.94) are two vector equations for two vector unknowns, the tangential field vectors on S cannot be solved from them uniquely. In fact, these equations depend only on those volume sources that are inside S, whereas the true solutions certainly must also depend on sources which are outside S. Thus, (6.93) and (6.94) must actually contain the same information, like the equations $2x - y = 0$ and $2y - 4x = 0$. To solve the fields on the surface, something more must be known.

For example, if one of the tangential fields is known on on S, either of (6.93), (6.94) can be used to solve the other tangential field and then, applying Huygens' principle (6.80), (6.81) the fields can be calculated at any point in V though integration. For example, if $\mathbf{n}\times\mathbf{E}$ is known on S, the procedure would be

- $\mathbf{n}\times\mathbf{E}$ is known on S,
- $\mathbf{n}\times\mathbf{H}$ can be computed from either EFIE or MFIE on S,
- $\mathbf{E}$ and $\mathbf{H}$ can be determined in V from (6.80), (6.81).

The integral equation is a substitute for formulating the problem in terms of the Helmholtz equation plus boundary conditions. The advantage of the integral equation formalism lies in the more compact domain of the equation: surface instead of volume, which in radiation problems extends to infinity.

6.4.5 Integral equations for bodies with impedance surface

Considering scattering of electromagnetic waves from a PEC obstacle, the surface S can be taken to coincide with the surface of the obstacle. To apply the previous procedure, the PEC boundary condition has $\mathbf{n}\times\mathbf{E} = 0$ on S, whence the tangential magnetic field can be solved from either the EFIE or the MFIE equation. Assuming the incident fields $\mathcal{E}[\mathbf{J}|\mathbf{J}_m]_V = \mathbf{E}^i$, $\mathcal{H}[\mathbf{J}|\mathbf{J}_m]_V = \mathbf{H}^i$ known, the respective EFIE and MFIE equations read

$$\mathbf{n}\times\mathbf{E}^i = \mathbf{n}\times[j\omega\mu PV\int_S G(\mathbf{n}'\times\mathbf{H})dS' +$$

$$\frac{1}{j\omega\epsilon}PV\int_S(\nabla' G)\nabla'\cdot(\mathbf{n}'\times\mathbf{H})dS'], \tag{6.95}$$

$$\mathbf{n}\times\mathbf{H}^i=\frac{1}{2}\mathbf{n}\times\mathbf{H}-\mathbf{n}\times PV\int_S(\mathbf{n}'\times\mathbf{H})\times\nabla' G dS'. \tag{6.96}$$

The normal unit vector $\mathbf{n}$ points to the region where the fields are calculated, i.e. out of the conducting body. The latter (MFIE) equation is also called the Maue integral equation, although, as pointed out by YAGHJIAN and WOODWORTH (1989), MAUE (1949) was not the first to derive this equation.

The equations can also be written in terms of the surface current $\mathbf{J}_s$ induced on the scatterer, by substituting $\mathbf{n}\times\mathbf{H}=\mathbf{J}_s$:

$$\mathbf{n}\times\mathbf{E}^i=\mathbf{n}\times[j\omega\mu PV\int_S G\mathbf{J}_s dS'+\frac{1}{j\omega\epsilon}PV\int_S(\nabla G)\nabla\cdot\mathbf{J}_s dS'], \tag{6.97}$$

$$\mathbf{n}\times\mathbf{H}^i=\frac{1}{2}\mathbf{J}_s-\mathbf{n}\times PV\int_S\mathbf{J}_s\times\nabla' G dS'. \tag{6.98}$$

The latter (MFIE) equation is often preferred because there is no need to differentiate the surface current function on S. However, if the obstacle is a thin body, there are numerical difficulties because the cross product within the integral tends to be a small quantity, which might give numerical error. Thus, the EFIE equation is more often applied for conducting wires and sheets.

Integral equations for the PMC scatterer can be written through the duality transformation. For an impedance surface with surface impedance $\overline{\overline{Z}}_s$ obeying the condition $\mathbf{E}_t=\overline{\overline{Z}}_s\cdot\mathbf{J}_s$, neither $\mathbf{n}\times\mathbf{E}$ nor $\mathbf{n}\times\mathbf{H}$ vanish on S. Instead, we may consider two-dimensional vector functions

$$\mathbf{A}=\mathbf{E}_t+\overline{\overline{Z}}_s\cdot\mathbf{J}_s,\qquad \mathbf{B}=\mathbf{E}_t-\overline{\overline{Z}}_s\cdot\mathbf{J}_s, \tag{6.99}$$

whence the boundary condition is $\mathbf{B}=0$ on S. The functions $\mathbf{A}$ and $\mathbf{B}$ can be taken as new unknowns,

$$\mathbf{E}_t=\frac{1}{2}(\mathbf{A}+\mathbf{B}),\qquad \mathbf{J}_s=-\mathbf{n}\times\mathbf{H}=\frac{1}{2}\overline{\overline{Y}}_s\cdot(\mathbf{A}-\mathbf{B}), \tag{6.100}$$

which inserted in the EFIE and MFIE equations together with setting $\mathbf{B}=0$, give rise to two equivalent equations, of which either one can be applied for solving for the unknown $\mathbf{A}$ and, hence, the tangential fields.

6.4.6 Integral equations for material bodies

For bodies with non-zero inner field, there is no boundary condition on S, but instead, interface conditions giving relations between the inner and outer fields. In fact, because of the continuity of the transverse fields through the interface, we may express Huygens' principle for both the inside and outside the body and obtain a pair of surface integral equations for the two field functions. This, however, can only be made for homogeneous bodies. For inhomogeneous bodies, the volume integral equation method given in Chapter 5 must be applied.

Let us denote the medium of the object by 1 and the outside medium by 2, and the Green functions corresponding to these two media are G_1 and G_2. If all volume sources are in the medium 2, the surface integral equations can be written from (6.93), (6.94) as

$$\frac{1}{2}\mathbf{n}_2 \times \mathbf{E} = \mathbf{n}_2 \times \mathcal{E}_2[\mathbf{J}|\mathbf{J}_m] + \mathbf{n}_2 \times PV\mathcal{E}_2[\mathbf{n}_2 \times \mathbf{H}| - \mathbf{n}_2 \times \mathbf{E}]_S, \tag{6.101}$$

$$\frac{1}{2}\mathbf{n}_2 \times \mathbf{H} = \mathbf{n}_2 \times \mathcal{H}_2[\mathbf{J}|\mathbf{J}_m] + \mathbf{n}_2 \times PV\mathcal{H}_2[\mathbf{n}_2 \times \mathbf{H}| - \mathbf{n}_2 \times \mathbf{E}]_S, \tag{6.102}$$

where $\mathbf{n}_2 = -\mathbf{n}_1$ points into the medium 2. The notation $\mathcal{E}_2$ corresponds to the medium 2 in the Green function. For the medium 1, we may write similar formulas except that there are no volume sources:

$$\frac{1}{2}\mathbf{n}_1 \times \mathbf{E} = \mathbf{n}_1 \times PV\mathcal{E}_1[\mathbf{n}_1 \times \mathbf{H}| - \mathbf{n}_1 \times \mathbf{E}]_S, \tag{6.103}$$

$$\frac{1}{2}\mathbf{n}_1 \times \mathbf{H} = \mathbf{n}_1 \times PV\mathcal{H}_1[\mathbf{n}_1 \times \mathbf{H}| - \mathbf{n}_1 \times \mathbf{E}]_S. \tag{6.104}$$

All these expressions assume a continuous tangent plane at the field point on the surface S.

From the two pairs of integral equations (6.101)-(6.104) we may choose two, one on each side of S and the resulting two integral equations constitute a true equation pair for the two unknown surface fields. So, the problem of scattering from dielectric bodies is more complicated than that from conducting bodies in this formulation.

Extensive work has been done on the question of obtaining uniqueness for the solutions of these surface equations so that NR surface sources interfering with the solution would be controlled. This subject will, however, not be touched here. A number of references is given in the report by YAGHJIAN and WOODWORTH (1989) on the conducting scatterer problem.

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6.5 TE/TM decomposition of sources

Electromagnetic problems are inherently vector problems and in this sense more complicated than, for example, the scalar problems encountered in acoustics. In some geometries there are eigenpolarizations which do not couple to each other, which makes it possible to reduce the vector problem into scalar problems. This presumes that, given an incoming field with one of the eigenpolarizations, the resulting field has the same polarization everywhere. Thus, the polarization of the field is known and only the amplitude of the field, a scalar quantity, remains to be solved.

As examples of problems for which such eigenpolarizations exist we may mention horizontally layered media, in which case the fields can be split into TE and TM fields with respect to the normal direction $\mathbf{n}$. Also problems with two-dimensional (cylindrical) structures, invariant in the axial direction $\mathbf{u}$, have TE and TM eigenpolarizations with respect to $\mathbf{u}$. Further, problems involving reciprocal uniaxial anisotropic media share the same property with respect to their axes. Thus, it would be of advantage to be able to split the source of the original problem into two parts, one of which radiates only a TE field and the other, only a TM field, with respect to a given direction $\mathbf{u}$ in space. Such a splitting is called TE/TM decomposition and will be considered in this section.

It is easily seen that an electric current of constant linear polarization $\mathbf{J}(\mathbf{r}) = \mathbf{u}J(\mathbf{r})$ gives rise to a TM field, as is evident from the field integral

$$\mathbf{H}(\mathbf{r}) = \int_V \nabla G(\mathbf{r}-\mathbf{r}') \times \mathbf{u}J(\mathbf{r}')dV'. \tag{6.105}$$

From duality we also know that a magnetic current of the same polarization $\mathbf{J}_m(\mathbf{r}) = \mathbf{u}J_m(\mathbf{r})$ gives rise to a field which has TE polarization outside of the sources. Replacing this by an equivalent electric current from (6.25), we see that the current of the form $\mathbf{u} \times \nabla g(\mathbf{r})$ gives rise to a TE field. Thus, if we can find functions $f(\mathbf{r})$, $g(\mathbf{r})$ and $\mathbf{J}^{NR}(\mathbf{r})$ such that our original source can be written in the form

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}f(\mathbf{r}) + \mathbf{u} \times \nabla g(\mathbf{r}) + \mathbf{J}^{NR}(\mathbf{r}), \tag{6.106}$$

where $\mathbf{J}^{NR}(\mathbf{r})$ is of the form (6.1) and does not radiate outside of the defined volume, the original source has been given a TE/TM decomposition of $\mathbf{J}(\mathbf{r})$.

6.5.1 Decomposition identity

The decomposition for a given source $\mathbf{J}$ can be formulated by applying the following identity:

$$\mathbf{J} = (\mathbf{J} - \frac{1}{k^2}\nabla \times \nabla \times \mathbf{J}) + \frac{1}{k^2}\mathbf{uu} \cdot (\nabla \times \nabla \times \mathbf{J}) - \frac{1}{k^2}\mathbf{u} \times \nabla(\mathbf{u} \cdot \nabla \times \mathbf{J}) +$$
$$\nabla[\frac{1}{k^2}(\mathbf{u} \cdot \nabla)(\mathbf{u} \cdot \mathbf{J})] - \left(\frac{\mathbf{u} \cdot \nabla}{k}\right)^2 \mathbf{J}, \tag{6.107}$$

where the argument of $\mathbf{J}$ is $\mathbf{r}$.

We see that, with the exception of the last term, the terms on the right-hand side are either of the form $\mathbf{u}f$, $\mathbf{u} \times \nabla g$ or NR type. The operator $(\mathbf{u} \cdot \nabla/k)^2$ does not destroy this property. To proceed, we have basically three alternatives.

1. Apply (6.107) recursively by substituting the right-hand side for $\mathbf{J}$ in the last term over and over again. This leads to a series representation of $\mathbf{J}$, which for a point source corresponds to a point (multipole) decomposition.

2. Move the last term of (6.107) to the left-hand side. This results in an ordinary differential equation for the current, which can be solved. For a point source this corresponds to a decomposition in terms of line sources.

3. Add the term $-(\nabla_t^2 + k^2)\mathbf{J}/k^2$ to both sides of (6.107). When combined with the last term, the result is an NR term and can be omitted. Thus, a two-dimensional partial differential equation for $\mathbf{J}$ is obtained, which can be solved. For a point source, this corresponds to a decomposition in terms of planar sources.

Thus, the identity (6.107) can be split into three equations for the TE, TM and NR components of the original current. The equations can be written in three different forms as given above and these will be considered separately.

6.5.2 Line source decomposition

To decompose a current $\mathbf{J}(\mathbf{r})$ in terms of line currents, let us identify from the identity (6.107) the TM, TE and NR components on the right- hand side and write equations for the corresponding current components:

$$[(\mathbf{u}\cdot\nabla)^2 + k^2]\mathbf{J}^{TM} = \mathbf{uu}\cdot\nabla\times\nabla\times\mathbf{J}, \tag{6.108}$$

$$[(\mathbf{u}\cdot\nabla)^2 + k^2]\mathbf{J}^{TE} = -(\mathbf{u}\times\nabla)(\mathbf{u}\times\nabla)\cdot\mathbf{J}. \tag{6.109}$$

The one corresponding to the NR currents is composed of the rest of the right-hand side in (6.107).

The solutions for (6.108) and (6.109) can be written in terms of the one-dimensional Green function satisfying the transmission-line equation

$$(\frac{d^2}{dz^2} + k^2)G_1(z-z') = -\delta(z-z'), \tag{6.110}$$

$$G_1(z-z') = \frac{1}{2jk}e^{-jk|z-z'|}. \tag{6.111}$$

Taking $\mathbf{u} = \mathbf{u}_z$, we can write the solutions

$$\mathbf{J}^{TM}(\mathbf{r}) \sim -\mathbf{uu}\cdot\int_{-\infty}^{\infty} G_1(z-z')(\nabla\times\nabla\times\mathbf{J})dz', \tag{6.112}$$

$$\mathbf{J}^{TE}(\mathbf{r}) \sim \mathbf{uu}{}^{\times}_{\times}\nabla\nabla\cdot\int_{-\infty}^{\infty} G_1(z-z')\mathbf{J}dz'. \tag{6.113}$$

Here, under the integral sign, $\mathbf{J}$ depends on x, y, z', and ∇s in (6.112) also operate on these variables. Obviously, the current components are of the right form (6.106) to be able to produce TM and TE fields. A better form is obtained if the following identities are applied:

$$\mathbf{uu}\cdot(\nabla\times\nabla\times\mathbf{J}) = [(\mathbf{u}\cdot\nabla)^2+k^2]\mathbf{uu}\cdot\mathbf{J} - (\nabla^2+k^2)\mathbf{uu}\cdot\mathbf{J} + \mathbf{uu}\cdot\nabla\nabla_t\cdot\mathbf{J}, \tag{6.114}$$

$$-(\mathbf{uu}{}^{\times}_{\times}\nabla\nabla)\cdot\mathbf{J} = [(\mathbf{u}\cdot\nabla)^2 + k^2]\mathbf{J}_t - (\nabla^2 + k^2)\mathbf{J}_t + \nabla\nabla_t\cdot\mathbf{J} - \mathbf{uu}\cdot\nabla\nabla_t\cdot\mathbf{J}. \tag{6.115}$$

In these expressions, the subscript t denotes the vector component transversal to $\mathbf{u}$. Ignoring the NR terms in these expressions, the solutions (6.112), (6.113) can be written as

$$\mathbf{J}^{TM}(\mathbf{r}) \sim \mathbf{uu}\cdot\mathbf{J} - \mathbf{u}\nabla_t \cdot \int_{-\infty}^{\infty} \frac{d\mathbf{J}}{dz'} G_1(z-z')dz', \tag{6.116}$$

$$\mathbf{J}^{TE}(\mathbf{r}) \sim \mathbf{J}_t + \mathbf{u}\nabla_t \cdot \int_{-\infty}^{\infty} \frac{d\mathbf{J}}{dz'} G_1(z-z')dz'. \tag{6.117}$$

It is seen that the sum of these expressions equals $\mathbf{J}$, whence the NR currents discarded in the course of the analysis have in fact cancelled each other.

Decomposition of a dipole

Let us apply the decomposition of a dipole current $\mathbf{J}(\mathbf{r}) = \mathbf{v}IL\delta(\mathbf{r})$. Because the $\mathbf{u}$ directed component produces a TM field, we may only concentrate on a transversal dipole with $\mathbf{v}\cdot\mathbf{u} = 0$. From the integral expressions (6.112), (6.113) we immediately have the decomposition:

$$\mathbf{J}^{TM}(\mathbf{r}) \sim -\mathbf{u}IL[\mathbf{v}\cdot\nabla\delta(\boldsymbol{\rho})]\mathbf{u}\cdot\nabla G_1(z), \tag{6.118}$$

$$\mathbf{J}^{TE}(\mathbf{r}) \sim \mathbf{v}IL\delta(\mathbf{r}) - \mathbf{J}^{TM}(\mathbf{r}). \tag{6.119}$$

These expressions can be interpreted through a transmission-line analogy. In fact, a two-conductor transmission line with line separation L, parallel to $\mathbf{u}$ and in the plane of $\mathbf{v}$, can be characterized in space by the current density function

$$\mathbf{J}(\mathbf{r}) = -\mathbf{u}LI(z)\mathbf{v}\cdot\nabla\delta(\boldsymbol{\rho}). \tag{6.120}$$

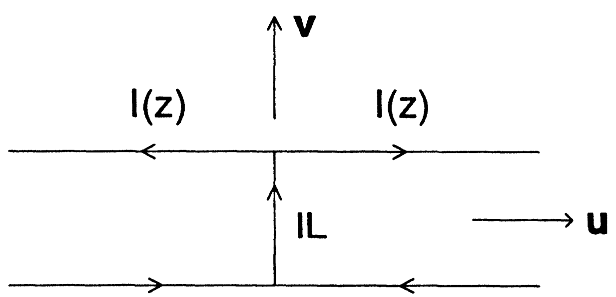

Fig. 6.2 TE component of dipole decomposition in line sources consists of the original dipole plus a transmission-line source.

6.5.3 Plane source decomposition

Another possibility for decomposing the current $\mathbf{J}(\mathbf{r})$ is in terms of planar currents. This can be done by again identifying from the identity (6.107) the TM, TE and NR components on the right-hand side and writing the following equations for the corresponding current components:

$$\nabla_t^2 \mathbf{J}^{TM} = -\mathbf{uu}\cdot\nabla\times(\nabla\times\mathbf{J}) = \nabla_t^2\mathbf{u}(\mathbf{u}\cdot\mathbf{J}) - \mathbf{uu}\cdot\nabla(\nabla_t\cdot\mathbf{J}), \quad (6.121)$$

$$\nabla_t^2 \mathbf{J}^{TE} = -(\mathbf{uu}{}^\times_\times\nabla\nabla)\cdot\mathbf{J} = \nabla_t^2\mathbf{J} - \nabla(\nabla_t\cdot\mathbf{J}) + \mathbf{uu}\cdot\nabla(\nabla_t\cdot\mathbf{J}). \quad (6.122)$$

The equation for the NR current is not of interest.

The solutions for (6.121) and (6.122) can be written in terms of the two-dimensional Green function satisfying the stationary plane current equation

$$\nabla_t^2 G_2(\boldsymbol{\rho}-\boldsymbol{\rho}') = -\delta(\boldsymbol{\rho}-\boldsymbol{\rho}'), \quad (6.123)$$

$$G_2(\boldsymbol{\rho}-\boldsymbol{\rho}') = -\frac{1}{2\pi}\ln(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|). \quad (6.124)$$

The solutions are

$$\mathbf{J}^{TM}(\mathbf{r}) \sim \mathbf{uu}\cdot\int_S(\nabla\times\nabla\times\mathbf{J})G_2(\boldsymbol{\rho}-\boldsymbol{\rho}')dS' =$$

$$\mathbf{uu}\cdot\mathbf{J} + \mathbf{uu}\cdot\nabla\int_S(\nabla_t\cdot\mathbf{J})G_2(\boldsymbol{\rho}-\boldsymbol{\rho}')dS', \quad (6.125)$$

$$\mathbf{J}^{TE}(\mathbf{r}) \sim -\mathbf{u}\times\int_S\nabla(\mathbf{u}\times\nabla\cdot\mathbf{J})G_2(\boldsymbol{\rho}-\boldsymbol{\rho}')dS' =$$

$$\mathbf{J}_t - \mathbf{uu}\cdot\nabla\int_S(\nabla_t\cdot\mathbf{J})G_2(\boldsymbol{\rho}-\boldsymbol{\rho}')dS', \quad (6.126)$$

Here, under the integral sign, $\mathbf{J}$ depends on the variables $\boldsymbol{\rho}'$, z, and ∇ also operates on the same variables. Again, the current components are of the right form (6.106) to produce TM and TE fields as seen in the expressions (6.121), (6.122). Also, the sum of the final expressions equals $\mathbf{J}$, whence the NR current terms discarded earlier have actually cancelled each other.

Decomposition of a dipole

Taking the dipole source $\mathbf{J} = \mathbf{v}IL\delta(\mathbf{r})$ with $\mathbf{v}\cdot\mathbf{u} = 0$, without losing the generality, we can write from (6.125), (6.126), neglecting the NR terms

$$\mathbf{J}^{TM}(\mathbf{r}) = \mathbf{u}IL\delta'(z)(\mathbf{v}\cdot\nabla G_2(\boldsymbol{\rho})) = \mathbf{u}IL\delta'(z)\frac{\mathbf{v}\cdot\boldsymbol{\rho}}{2\pi\rho^2} \sim$$

$$IL\delta(z)[\overline{\overline{I}} - 2\mathbf{u}_\rho\mathbf{u}_\rho]\cdot\frac{\mathbf{v}}{2\pi\rho^2} + \frac{1}{2}\mathbf{v}IL\delta(\mathbf{r}), \tag{6.127}$$

$$\mathbf{J}^{TE}(\mathbf{r}) = \mathbf{v}IL\delta(\mathbf{r}) - \mathbf{J}^{TM}(\mathbf{r}). \tag{6.128}$$

The last expression of the surface current term in (6.127) can be identified as a d.c. surface current flowing on a resistive sheet as excited by a dipole current source at the origin. The flow lines on the plane can be shown to be circles, each starting and ending at the origin. Thus, the TE component of the dipole consists of one-half of the original dipole plus the stationary surface current generated by the original dipole. The TM part consists of the other half of the original dipole plus the negative of the previous surface current. This decomposition in terms of planar currents was given in Fourier components in the classic paper by CLEMMOW in 1963.

6.5.4 Point source decomposition

The third possibility for decomposing a current $\mathbf{J}(\mathbf{r})$ can be made by applying (6.107) recursively, substituting $\mathbf{J}$ in the last term over and over again. The resulting series expansions can be written in the following forms:

$$\mathbf{J}^{TM} = \mathbf{u}\mathbf{u}\cdot\mathbf{J} + \frac{j}{k}\mathbf{u}\nabla_t\cdot\sum_{n=0}^{\infty}\left(\frac{\mathbf{u}\cdot\nabla}{jk}\right)^{2n+1}\mathbf{J}, \tag{6.129}$$

$$\mathbf{J}^{TE} = \mathbf{J}_t - \frac{j}{k}\mathbf{u}\nabla_t\cdot\sum_{n=0}^{\infty}\left(\frac{\mathbf{u}\cdot\nabla}{jk}\right)^{2n+1}\mathbf{J}. \tag{6.130}$$

These decompositions are limited in the region of the original source $\mathbf{J}(\mathbf{r})$, because only differential operations are involved.

Decomposition of a dipole

If these expressions are applied to a transversal dipole $\mathbf{J} = \mathbf{v}IL\delta(\mathbf{r})$, the following multipole expansions are obtained:

$$\mathbf{J}^{TM}(\mathbf{r}) = \mathbf{u}\frac{IL}{k}[\mathbf{v}\cdot\nabla\delta(\boldsymbol{\rho})][\frac{\delta'(z)}{k} - \frac{\delta^{(3)}(z)}{k^3} + \frac{\delta^{(5)}(z)}{k^5} - ...], \tag{6.131}$$

$$\mathbf{J}^{TE}(\mathbf{r}) = \mathbf{v}IL\delta(\mathbf{r}) - \mathbf{u}\frac{IL}{k}[\mathbf{v}\cdot\nabla\delta(\boldsymbol{\rho})][\frac{\delta'(z)}{k} - \frac{\delta^{(3)}(z)}{k^3} + \frac{\delta^{(5)}(z)}{k^5} - ...]. \tag{6.132}$$

It is not evident that the point decomposition will converge in all cases. In fact, comparing with the delta expansion (6.43), we see that the amplitude of the moments of the multipole increases as $n!/k^{n+1}$. However, the method can be useful, as was demonstrated by LINDELL (1988).

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