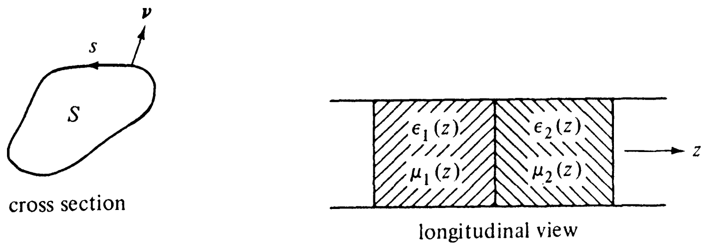
2. Network Formalism for Time-Harmonic Electromagnetic Fields in Uniform and Spherical Waveguide Regions

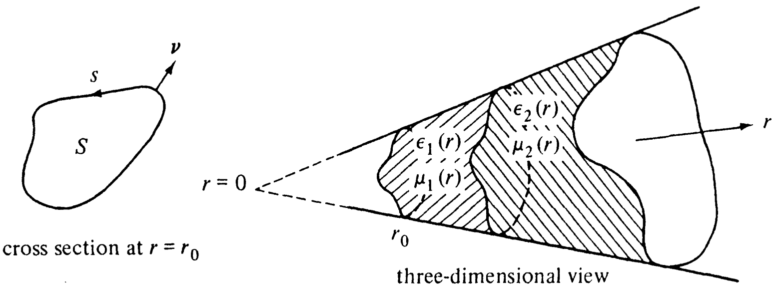
2.1 INTRODUCTION

The general theory of time-dependent and time-harmonic linear fields excited by prescribed sources in an infinite homogeneous medium has been given in Secs. 1.1–1.4 primarily for the simple case of unbounded media. The present chapter examines in greater detail the application of these general concepts to the time-harmonic electromagnetic field in waveguides with perfectly conducting walls bounding planes transverse to the rectilinear coordinate z or spherical surfaces transverse to the radial coordinate r. The boundaries for the former uniform waveguide configuration are defined by $f(\boldsymbol{\rho}) = 0$, where $\boldsymbol{\rho}$ is the vector coordinate transverse to z, and for the latter non-uniform waveguide configuration by $F(\theta, \phi) = 0$, where θ and ϕ are angular spherical coordinates; uniform and non-uniform waveguides are distinguished by whether or not the cross sections along the axial direction are constant. The media filling these regions are assumed to be isotropic but may vary abruptly or continuously with z and r, respectively, as schematized in Fig. 2.1.1. As noted in Sec. 1.4 for uniform regions, the waveguide concept is useful even for unbounded cross sections, especially when the medium parameters vary along the axial direction.

The basic procedure for obtaining field solutions is essentially the same as in Sec. 1.4 and utilizes representations of field variables and their sources in terms of a complete set of vector eigenfunctions. Since the orthogonality properties of the eigenfunctions involve only field components transverse to the transmission coordinate [see Eq. (1.4.30) for uniform regions], it is possible to seek modal expansions of the independent transverse fields $\mathbf{E}_t$ and $\mathbf{H}_t$, and to derive therefrom the dependent longitudinal components. A complete represen-



(a) Uniform



(b) Non-uniform (spherical)

FIG. 2.1.1 Waveguide regions.

tation in terms of E(TM) and H(TE) modes† is utilized in Secs. 2.2b and 2.5b for uniform and spherical waveguides, respectively, to represent the transverse fields and also their equivalent sources. General properties of vector-mode functions in isotropic regions are discussed; their explicit evaluation for various cross-sectional configurations is deferred to Chapter 3, and generalizations to anisotropic regions are considered in Chapters 7 and 8. It is shown that on use of modal representations and orthogonality properties, equations satisfied by the transverse vector field may be reduced to z- or r-dependent scalar transmission-line equations for the modal amplitudes, analogous to those in Eq. (1.4.15) or (1.4.36b).

As noted in Sec. 1.1b, calculation of vector fields is often facilitated through use of scalar potentials. For transversely unbounded uniform regions, this scalarization has been derived, for example, in Eqs. (1.1.38) [see also Eqs. (1.1.49)]. In Sec. 2.3, scalarization of vector fields in transversely bounded uniform regions is achieved by introduction of complete orthogonal sets of

†The abbreviations TM and TE stand for transverse magnetic and transverse electric, respectively; in uniform regions, $H_z \equiv 0$ for the former and $E_z \equiv 0$ for the latter, whereas in spherical regions, correspondingly, $H_r \equiv 0$ and $E_r \equiv 0$. See Sec. 8.2 for a general derivation and further discussion of E- and H-mode decompositions.

scalar-mode functions; the latter are then used to yield modal expansions for the scalar potentials. Although this procedure of formulating vector fields in terms of scalar potentials is more circuitous than that in Sec. 1.1b, it has the advantage of furnishing the actual potential solutions. As in Chapter 1, it is recognized that calculations of fields excited by arbitrary source distributions are simplified via dyadic Green's functions. On introduction of one-dimensional transmission-line Green's functions, with reciprocity properties similar to those of three-dimensional dyadic Green's functions, one may develop modal solutions for the latter, both for piecewise homogeneous media and for media exhibiting continuous z variation. These derivations form the substance of Sec. 2.3.

The transmission-line equations formulated in Sec. 2.2 and utilized in Sec. 2.3 are solved in Sec. 2.4 for the special case of media with piecewise homogeneous z dependence (continuous z variation is treated in Secs. 3.3 and 3.5). Emphasis is placed on network schematization of excitation, transmission and reflection processes, and on network methods for calculating transmission line voltages and currents; the latter represent the amplitude coefficients in modal expansions of the transverse electric and magnetic fields, respectively. Section 2.4 concludes with a brief summary of the relation between modes and resonant transmission lines, a subject to be explored more fully in Sec. 3.3.

Fields in spherical waveguide regions are analyzed in a manner similar to the above, with the presentations in Secs. 2.5, 2.6, and 2.7 paralleling those in Secs. 2.2, 2.3, and 2.4, respectively.

2.2 DERIVATION OF TRANSMISSION-LINE EQUATIONS IN UNIFORM REGIONS

2.2a The Transverse Field Equations

In this section, the transformation of the inhomogeneous, steady-state Maxwell vector field equations into the scalar transmission-line equations for a typical mode is summarized for the case of a uniform waveguide depicted in Fig. 2.1.1a. The procedure is similar to that given in Sec. 1.4 for isotropic linear fields in homogeneous unbounded regions, but it is generalized here to admit regions possessing transverse boundaries and longitudinal stratification. The steady-state electromagnetic vector fields excited by a specified electric current distribution $\mathbf{J}(\mathbf{r})$ and magnetic current distribution $\mathbf{M}(\mathbf{r})$ are defined by the field equations

$$\begin{aligned}\nabla \times \mathbf{E}(\mathbf{r}) &= -j\omega\mu\mathbf{H}(\mathbf{r}) - \mathbf{M}(\mathbf{r}),\\ \nabla \times \mathbf{H}(\mathbf{r}) &= j\omega\epsilon\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}).\end{aligned} \tag{1}$$

On the perfectly conducting boundary of the uniform waveguide, the tangential component of the electric field must vanish, i.e.,

$$\boldsymbol{\nu} \times \mathbf{E} = 0 \qquad \text{on } s, \tag{1a}$$

where s denotes the curve bounding the transverse cross section S and $\mathbf{v}$ is a unit vector normal to s and lying in the plane of the cross section. The vanishing of the tangential component of $\mathbf{E}$ on s also implies the vanishing on s of the normal component of $\mathbf{H}$. For a region with infinite cross section, condition (1a) is replaced by a "radiation condition" which requires that, for any source distribution contained in a finite region, the field solution at infinity comprises only "outgoing" waves. The boundary conditions on the longitudinal (z) termini of the region are left open for the moment and will be taken into account in the subsequent solution of the transmission-line equations. A harmonic time-dependence $\exp(+j\omega t)$ is assumed,† whence to obtain the solution for fields with arbitrary time variation, a temporal synthesis as described in Sec. 1.4 must be carried out as well. For the present discussion, the scalar permittivity ϵ and the permeability μ of the medium may both be z dependent.

As noted in Sec. 1.4, orthogonality conditions for modes guided along z involve only field components transverse to z. It is therefore desirable to eliminate the dependent components E_z and H_z from Eqs. (1) and derive field equations for the independent transverse components. To this end, one takes vector and scalar products of Eqs. (1) with the longitudinal unit vector $\mathbf{z}_0$. Thus,

$$\begin{aligned} j\omega\mu\mathbf{H} \times \mathbf{z}_0 + \mathbf{M} \times \mathbf{z}_0 = \mathbf{z}_0 \times (\nabla \times \mathbf{E}) &= -\frac{\partial}{\partial z}\mathbf{E} + \nabla E_z \\ &= -\frac{\partial}{\partial z}\mathbf{E}_t + \nabla_t E_z \end{aligned} \tag{2a}$$

and

$$-j\omega\mu H_z - M_z = \mathbf{z}_0 \cdot (\nabla \times \mathbf{E}) = -\nabla_t \cdot (\mathbf{z}_0 \times \mathbf{E}), \tag{2b}$$

where the transverse gradient operator $\nabla_t = \nabla - \mathbf{z}_0\,\partial/\partial z$. Similarly, for the second of Eqs. (1), one has, by duality,

$$j\omega\epsilon\mathbf{z}_0 \times \mathbf{E} + \mathbf{z}_0 \times \mathbf{J} = \nabla_t H_z - \frac{\partial}{\partial z}\mathbf{H}_t, \tag{3a}$$

$$j\omega\epsilon E_z + J_z = \nabla_t \cdot (\mathbf{H} \times \mathbf{z}_0). \tag{3b}$$

Upon substituting for E_z in Eq. (2a) from Eq. (3b), one obtains

$$\begin{aligned} -\frac{\partial}{\partial z}\mathbf{E}_t &= j\omega\mu\mathbf{H}_t \times \mathbf{z}_0 - \frac{1}{j\omega\epsilon}(\nabla_t\nabla_t \cdot \mathbf{H}_t \times \mathbf{z}_0 - \nabla_t J_z) + \mathbf{M}_t \times \mathbf{z}_0 \\ &= j\omega\mu\left(\mathbf{1} + \frac{\nabla_t\nabla_t}{k^2}\right) \cdot (\mathbf{H}_t \times \mathbf{z}_0) + \mathbf{M}_{te} \times \mathbf{z}_0, \end{aligned} \tag{4a}$$

and, by duality,

$$-\frac{\partial}{\partial z}\mathbf{H}_t = j\omega\epsilon\left(\mathbf{1} + \frac{\nabla_t\nabla_t}{k^2}\right) \cdot (\mathbf{z}_0 \times \mathbf{E}_t) + \mathbf{z}_0 \times \mathbf{J}_{te}, \tag{4b}$$

where the equivalent transverse electric and magnetic current distributions are given, respectively, by

†An $\exp(j\omega t)$ dependence is customary in applications involving electrical networks and in this chapter is chosen for this reason.

$$\mathbf{J}_{te} = \mathbf{J}_t - \mathbf{z}_0 \times \frac{\nabla_t M_z}{j\omega\mu} = \mathbf{J}_t + \frac{\nabla_t \times \mathbf{M}_z}{j\omega\mu} \tag{5a}$$

$$\mathbf{M}_{te} = \mathbf{M}_t + \mathbf{z}_0 \times \frac{\nabla_t J_z}{j\omega\epsilon} = \mathbf{M}_t - \frac{\nabla_t \times \mathbf{J}_z}{j\omega\epsilon}. \tag{5b}$$

The quantity $k = \omega(\mu\epsilon)^{1/2}$ is the wavenumber in the region and $\mathbf{1}$ is a unit dyadic such that $\mathbf{1} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{1} = \mathbf{A}$.

The transverse field equations (4) and (5), which admit a z-dependent ϵ and μ, provide the basis for the treatment of field problems in uniform waveguides.† They are completely descriptive of the total field equations (1), since from Eqs. (2b) and (3b), the longitudinal components are derivable from the transverse components as

$$j\omega\epsilon E_z = \nabla_t \cdot (\mathbf{H}_t \times \mathbf{z}_0) - J_z \tag{6a}$$

$$j\omega\mu H_z = \nabla_t \cdot (\mathbf{z}_0 \times \mathbf{E}_t) - M_z. \tag{6b}$$

The boundary condition (1a), requiring the vanishing of the *total* tangential electric field on the perfectly conducting guide walls, can be restated in terms of the *transverse* field components as

$$\left.\begin{aligned} \boldsymbol{\nu} \times \mathbf{E}_t &= 0 \\ \nabla_t \cdot (\mathbf{H}_t \times \mathbf{z}_0) &= 0 \end{aligned}\right\} \quad \text{on } s, \tag{7}$$

where the second relation follows from Eq. (6a) upon assuming that $J_z = 0$ on s. This restriction, which requires the vanishing *on the boundary* of the z component of the *applied* electric current source, is of no practical consequence since an applied tangential electric current source on a perfectly conducting surface is "short-circuited" and cannot radiate a finite field.

2.2b Modal Representation of the Fields and Their Sources

As noted in Sec. 1.4c for the special case of free space, the vector electromagnetic field equations can be transformed into ordinary scalar differential equations on representation of the fields in terms of a complete orthonormal set of "guided" eigenfunctions. Although in Sec. 1.4c, this representation was performed in terms of Ψ eigenvectors indicative of both the electric- and magnetic-field vectors, it could equally well have been effected by individual vector representations of the electric field $\mathbf{E}$ in terms of eigenvectors $\mathbf{e}(\boldsymbol{\rho})$, of the magnetic field $\mathbf{H}$ in terms of eigenvectors $\mathbf{h}(\boldsymbol{\rho})$, etc. This latter procedure will be followed for field representations in the bounded regions considered in this chapter. For a perfectly conducting waveguide filled with a homogeneous, isotropic medium, a possible complete eigenvector set comprises both E(TM) mode functions $\mathbf{e}'(\boldsymbol{\rho})$, $\mathbf{h}'(\boldsymbol{\rho})$ and H(TE) mode functions $\mathbf{e}''(\boldsymbol{\rho})$, $\mathbf{h}''(\boldsymbol{\rho})$; these mode functions are defined below in Eqs. (10) and in Sec. 2.3a; justification of the E- and H-mode decompositions is treated more generally in Sec. 8.2. In terms of

†For an analogous treatment of waveguides filled with anisotropic media or with media possessing transverse variation, see Sec. 8.2.

the indicated mode functions, a representation of the independent transverse fields is given as [1,2]

$$\mathbf{E}_t(\mathbf{r}) = \sum_i V_i'(z)\mathbf{e}_i'(\boldsymbol{\rho}) + \sum_i V_i''(z)\mathbf{e}_i''(\boldsymbol{\rho}), \tag{8a}$$

$$\mathbf{H}_t(\mathbf{r}) = \sum_i I_i'(z)\mathbf{h}_i'(\boldsymbol{\rho}) + \sum_i I_i''(z)\mathbf{h}_i''(\boldsymbol{\rho}), \tag{8b}$$

$$\mathbf{J}_{te}(\mathbf{r}) = \sum_i i_i'(z)\mathbf{e}_i'(\boldsymbol{\rho}) + \sum_i i_i''(z)\mathbf{e}_i''(\boldsymbol{\rho}), \tag{8c}$$

$$\mathbf{M}_{te}(\mathbf{r}) = \sum_i v_i'(z)\mathbf{h}_i'(\boldsymbol{\rho}) + \sum_i v_i''(z)\mathbf{h}_i''(\boldsymbol{\rho}), \tag{8d}$$

where i is in general a double index, and

$$\mathbf{h}_i = \mathbf{z}_0 \times \mathbf{e}_i. \tag{8e}$$

In view of Eqs. (6) and (8) one obtains the longitudinal-field representations [by Eqs. (10), only E modes contribute to the representation of E_z, while only H modes contribute to the representation of H_z]

$$j\omega\epsilon E_z(\mathbf{r}) + J_z(\mathbf{r}) = \sum_i I_i'(z)\nabla_t \cdot \mathbf{e}_i'(\boldsymbol{\rho}), \tag{9a}$$

$$j\omega\mu H_z(\mathbf{r}) + M_z(\mathbf{r}) = \sum_i V_i''(z)\nabla_t \cdot \mathbf{h}_i''(\boldsymbol{\rho}). \tag{9b}$$

The specific form of the transverse vector eigenfunctions $\mathbf{e}_i$ and $\mathbf{h}_i$ is dependent on the shape of the guide cross-section and is, in general, defined by the following z-independent equations [see Eqs. (8.2.25)]:

$$\begin{aligned} &\nabla_t\nabla_t \cdot \mathbf{e}_i' = -k_{ti}'^2\mathbf{e}_i' \qquad &&\nabla_t\nabla_t \cdot \mathbf{h}_i'' = -k_{ti}''^2\mathbf{h}_i'', \\ &\nabla_t\nabla_t \cdot \mathbf{h}_i' = 0, &&\nabla_t\nabla_t \cdot \mathbf{e}_i'' = 0, \end{aligned} \tag{10}$$

subject, in accord with Eqs. (7), to the boundary conditions on the curve s with normal $\boldsymbol{\nu}$ bounding the transverse cross section:

$$\boldsymbol{\nu} \times \mathbf{e}_i' = 0 = \nabla_t \cdot (\mathbf{h}_i' \times \mathbf{z}_0), \quad \boldsymbol{\nu} \times \mathbf{e}_i'' = 0 = \nabla_t \cdot (\mathbf{h}_i'' \times \mathbf{z}_0) \qquad \text{on } s. \tag{10a}$$

One notes from Eqs. (10a) that the vector mode functions introduced in the representations (8a) and (8b) satisfy individually the appropriate boundary conditions (7) on the transverse electromagnetic fields. Moreover, since *applied* electric and magnetic currents have no tangential or normal components, respectively, at a perfectly conducting surface, the representations for the source currents in Eqs. (8c) and (8d) are likewise meaningful for realizable source current distributions on the boundary.

Upon applying the following transverse form of Green's theorem,†

$$\begin{aligned} &\iint_S dS[\mathbf{A} \cdot \nabla_t\nabla_t \cdot \mathbf{B} - \mathbf{B} \cdot \nabla_t\nabla_t \cdot \mathbf{A}] \\ &= \oint_s ds[(\mathbf{A} \cdot \boldsymbol{\nu})(\nabla_t \cdot \mathbf{B}) - (\mathbf{B} \cdot \boldsymbol{\nu})(\nabla_t \cdot \mathbf{A})], \end{aligned} \tag{11a}$$

†Equation (11a) is obtained by applying the divergence theorem in the transverse cross section to the expression

$$\nabla_t \cdot [\mathbf{A}\nabla_t \cdot \mathbf{B} - \mathbf{B}\nabla_t \cdot \mathbf{A}] = \mathbf{A} \cdot \nabla_t\nabla_t \cdot \mathbf{B} - \mathbf{B} \cdot \nabla_t\nabla_t \cdot \mathbf{A}$$

where A and B are suitably continuous transverse vector functions, to the vector mode functions defined in Eqs. (10), one deduces the orthogonality conditions over the cross-sectional domain S (normalization to unity is assumed):

$$\iint_S \mathbf{e}'_i \cdot \mathbf{e}'^*_j \, dS = \delta_{ij} = \iint_S \mathbf{e}''_i \cdot \mathbf{e}''^*_j \, dS; \qquad \iint_S \mathbf{e}'_i \cdot \mathbf{e}''^*_i \, dS = 0, \tag{11b}$$

and similarly for the $\mathbf{h}_i$ functions. The asterisk denotes the complex conjugate,† and the Kronecker delta is defined as follows: $\delta_{ij} = 0, i \neq j$; $\delta_{ii} = 1$. In view of these orthonormality properties, the mode amplitudes in Eqs. (8) are determined as follows:

$$V_i(z) = \iint_S \mathbf{E}_t(\mathbf{r}) \cdot \mathbf{e}^*_i(\boldsymbol{\rho}) \, dS, \qquad I_i(z) = \iint_S \mathbf{H}_t(\mathbf{r}) \cdot \mathbf{h}^*_i(\boldsymbol{\rho}) \, dS, \tag{12a}$$

$$v_i(z) = \iint_S \mathbf{M}_{te}(\mathbf{r}) \cdot \mathbf{h}^*_i(\boldsymbol{\rho}) \, dS, \quad i_i(z) = \iint_S \mathbf{J}_{te}(\mathbf{r}) \cdot \mathbf{e}^*_i(\boldsymbol{\rho}) \, dS, \tag{12b}$$

where the distinguishing $'$ and $''$ have been omitted since the equations apply to both mode types. Utilizing the equivalent current definitions in Eq. (5) and employing the vector integration-by-parts formula (divergence theorem in two dimensions)

$$\iint_S dS \nabla_t f \cdot \mathbf{A} = -\iint_S dS f \nabla_t \cdot \mathbf{A} + \oint_s ds f(\mathbf{A} \cdot \boldsymbol{\nu}), \tag{13}$$

with f and A suitably continuous scalar and vector functions, one may reexpress the integrals of Eqs. (12b). The contribution to the gradient integrals from the bounding contour s vanishes in view of the boundary condition $\mathbf{h}_i \cdot \boldsymbol{\nu} = 0$ [Eq. (10a)] and the specification $J_z = 0$ on s, so Eqs. (12b) become

$$v_i(z) = \iint_S \mathbf{M}(\mathbf{r}) \cdot \mathbf{h}^*_i(\boldsymbol{\rho}) \, dS + Z^*_i \iint_S \mathbf{J}(\mathbf{r}) \cdot \mathbf{e}^*_{zi}(\boldsymbol{\rho}) \, dS, \tag{14a}$$

$$i_i(z) = \iint_S \mathbf{J}(\mathbf{r}) \cdot \mathbf{e}^*_i(\boldsymbol{\rho}) \, dS + Y^*_i \iint_S \mathbf{M}(\mathbf{r}) \cdot \mathbf{h}^*_{zi}(\boldsymbol{\rho}) \, dS, \tag{14b}$$

where

$$Y''_i \mathbf{h}''_{zi}(\boldsymbol{\rho}) \equiv z_0 \frac{\nabla_t \cdot \mathbf{h}''_i(\boldsymbol{\rho})}{j\omega\mu}, \qquad \mathbf{h}'_{zi} \equiv 0, \tag{14c}$$

$$Z'_i \mathbf{e}'_{zi}(\boldsymbol{\rho}) \equiv z_0 \frac{\nabla_t \cdot \mathbf{e}'_i(\boldsymbol{\rho})}{j\omega\epsilon}, \qquad \mathbf{e}''_{zi} \equiv 0. \tag{14d}$$

The vanishing of $\mathbf{h}'_{zi}$ (for E modes) and of $\mathbf{e}''_{zi}$ (for H modes) follows directly from Eqs. (10). The introduction of the characteristic impedance and admittance Z'_i and Y''_i [defined explicitly in Eqs. (15)] serves to highlight in a physical sense the contributions of the various integrals as either voltages or currents. It is to be noted that the formulations in Eqs. (14) do not require differentiability of J_z and M_z in the cross section S as implied in Eqs. (5) and (12b).

†Since the mode functions may be complex, although k'^2_{ti} and k''^2_{ti} are real, the orthogonality condition involves the complex conjugate function.

Upon inserting the modal representations (8) into the transverse field equations (4), interchanging the order of summation and differentiation, making use of Eqs. (10), and equating like coefficients of the mode functions $\mathbf{e}_i$ and $\mathbf{h}_i$, one obtains the desired transmission-line equations for the E- and H-mode amplitudes as [see Eq. (1.4.35b)]

$$-\frac{dV_i}{dz} = j\kappa_i Z_i I_i + v_i, \tag{15a}$$

$$-\frac{dI_i}{dz} = j\kappa_i Y_i V_i + i_i, \tag{15b}$$

where the modal characteristic impedance Z_i (admittance Y_i) and the modal propagation constant κ_i are defined as follows:

E modes:

$$Z_i' = \frac{1}{Y_i'} = \frac{\kappa_i'}{\omega\epsilon}, \qquad \kappa_i' = \sqrt{k^2 - k_{ti}'^2} = -j\sqrt{k_{ti}'^2 - k^2}, \tag{15c}$$

H modes:

$$Z_i'' = \frac{1}{Y_i''} = \frac{\omega\mu}{\kappa_i''}, \qquad \kappa_i'' = \sqrt{k^2 - k_{ti}''^2} = -j\sqrt{k_{ti}''^2 - k^2}. \tag{15d}$$

$k^2 = \omega^2\mu\epsilon$, and both μ and ϵ may be functions of z. The form of Eqs. (15a) and (15b) permits identification of V_i and I_i as transmission-line voltage and current,[3] respectively. The choice of sign on the square roots in Eqs. (15) assures the damping of non-propagating modes (κ_i imaginary) away from the source region for the assumed time dependence exp $(+j\omega t)$. The evaluation of the source voltage v_i and current i_i amplitudes follows directly from the *specified* electric and magnetic source currents $\mathbf{J}$ and $\mathbf{M}$ via Eqs. (14a) and (14b). Solutions of Eqs. (15a) and (15b) for various stratifications and terminations in the z domain are discussed in Secs. 2.4 and 3.3b.

2.3 SCALARIZATION AND MODAL REPRESENTATION OF DYADIC GREEN'S FUNCTIONS IN UNIFORM REGIONS

Solutions for the vector electromagnetic field excited by prescribed sources in a uniform waveguide region bounded by perfectly conducting walls (if any) and filled with a transversely homogeneous material follow from the representations in Eqs. (2.2.8) and (2.2.9); the vector-mode functions are evaluated from Eqs. (2.2.10) and the modal amplitudes from Eqs. (2.2.15) subject to appropriate boundary conditions in the z domain. Solution of the vector eigenvalue problems in Eqs. (2.2.10) is facilitated by introduction of scalar-mode functions. The scalarization achieved in this manner may be utilized to define E- and H-mode (Hertz) potentials from which the electromagnetic fields themselves can be derived. For point-source excitation, these potentials are equivalent to the scalar

Green's functions that have been introduced in Eqs. (1.1.38) and (1.1.49) without intervention of an eigenfunction expansion. The procedure discussed below yields explicit modal representations for these functions, or equivalently for the customarily defined Hertz potentials related to them, and thereby *solves* the scalar potential problems. We first express vector-mode functions in terms of scalar-mode functions, and then scalarize the overall field representation.

2.3a Mode Functions

In representing the transverse electric vector field $\mathbf{E}_t$ in Eq. (2.2.8a) in terms of two independent vector mode sets $\{\mathbf{e}'_i\}$ and $\{\mathbf{e}''_i\}$, use has been made of a theorem which states that any *transverse* vector can be decomposed into two parts, one of which is solenoidal and the other of which is irrotational.[4] The vector set $\{\mathbf{e}'_i\}$ is irrotational (i.e., $\nabla_t \times \mathbf{e}'_i = 0$ in S), while the vector set $\{\mathbf{e}''_i\}$ is solenoidal (i.e., $\nabla_t \cdot \mathbf{e}''_i = 0$ in S) [see also Eqs. (2.2.10)]. In view of these properties, the vector-mode functions $\mathbf{e}'_i$ and $\mathbf{e}''_i$ can be represented as gradients and curls of scalar functions Φ_i and ψ_i as follows:

$$\mathbf{e}'_i(\boldsymbol{\rho}) = -\frac{\nabla_t \Phi_i(\boldsymbol{\rho})}{k'_{ti}}, \tag{1a}$$

$$\mathbf{e}''_i(\boldsymbol{\rho}) = -\frac{\nabla_t \psi_i(\boldsymbol{\rho})}{k''_{ti}} \times \mathbf{z}_0, \tag{1b}$$

and, consequently,

$$\mathbf{h}'_i(\boldsymbol{\rho}) = -\mathbf{z}_0 \times \frac{\nabla_t \Phi_i(\boldsymbol{\rho})}{k'_{ti}}, \tag{1c}$$

$$\mathbf{h}''_i(\boldsymbol{\rho}) = -\frac{\nabla_t \psi_i(\boldsymbol{\rho})}{k''_{ti}}. \tag{1d}$$

By Eqs. (1) and (2.2.10) the mode functions Φ_i and ψ_i are defined by the two scalar eigenvalue problems†

$$\nabla_t^2 \Phi_i + k'^2_{ti} \Phi_i = 0 \qquad \text{in } S, \tag{2a}$$

$$\left.\begin{aligned} \Phi_i &= 0 && \text{on } s \text{ if } k'_{ti} \neq 0, \\ \frac{\partial \Phi_i}{\partial s} &= 0 && \text{on } s \text{ if } k'_{ti} = 0 \qquad \text{(TEM mode)}, \end{aligned}\right. \tag{2b}$$

and

$$\nabla_t^2 \psi_i + k''^2_{ti} \psi_i = 0 \qquad \text{in } S, \tag{2c}$$

$$\frac{\partial \psi_i}{\partial \nu} = 0 \qquad \text{on } s. \tag{2d}$$

†The vector-mode functions for the TEM (transverse electromagnetic) case are determined via $\mathbf{e}'_0(\boldsymbol{\rho}) = \mathbf{h}'_0(\boldsymbol{\rho}) \times \mathbf{z}_0 = -\nabla_t \Phi_0(\boldsymbol{\rho})$, where $\Phi_0(\boldsymbol{\rho})$ is the solution of Eq. (2a) with $k'_{ti} = 0$, with the normalization $\iint_S e'^2_0(\boldsymbol{\rho})\, dS = 1$.

Specific solutions of Eqs. (2) for various guide cross sections are tabulated in Chapter 3.[2]†

2.3b Fields in Source-Free, Homogeneous Regions

Using Eqs. (1) and assuming interchangeability of summation and differentiation operations, one may write Eqs. (2.2.8a) and (2.2.8b) as

$$\mathbf{E}_t(\mathbf{r}) = -\nabla_t V'(\mathbf{r}) - \nabla_t V''(\mathbf{r}) \times \mathbf{z}_0, \tag{3a}$$

$$\mathbf{H}_t(\mathbf{r}) \times \mathbf{z}_0 = -\nabla_t I'(\mathbf{r}) - \nabla_t I''(\mathbf{r}) \times \mathbf{z}_0, \tag{3b}$$

where the potential functions $V'(\mathbf{r})$, $I'(\mathbf{r})$ and $V''(\mathbf{r})$, $I''(\mathbf{r})$ are defined as follows:

$$V'(\mathbf{r}) = \sum_i V_i'(z)\frac{\Phi_i(\boldsymbol{\rho})}{k_{ti}'}, \qquad V''(\mathbf{r}) = \sum_i V_i''(z)\frac{\psi_i(\boldsymbol{\rho})}{k_{ti}''}, \tag{4a}$$

$$I'(\mathbf{r}) = \sum_i I_i'(z)\frac{\Phi_i(\boldsymbol{\rho})}{k_{ti}'}, \qquad I''(\mathbf{r}) = \sum_i I_i''(z)\frac{\psi_i(\boldsymbol{\rho})}{k_{ti}''}. \tag{4b}$$

The potential functions in Eqs. (4) are related to the Hertz potentials Π' and Π'' [Eqs. (1.1.42) and (1.1.53)] by

$$\Pi'(\mathbf{r}) = \frac{I'(\mathbf{r})}{j\omega\epsilon}, \qquad \Pi''(\mathbf{r}) = \frac{V''(\mathbf{r})}{j\omega\mu}. \tag{5}$$

From Eqs. (3) and (2.2.6), the electromagnetic fields can be expressed at any source-free point where ϵ and μ are *non-variable* as[5]

$$\mathbf{E}(\mathbf{r}) = \nabla \times \nabla \times [\mathbf{z}_0\Pi'(\mathbf{r})] - j\omega\mu\nabla \times [\mathbf{z}_0\Pi''(\mathbf{r})], \tag{6a}$$

$$\mathbf{H}(\mathbf{r}) = j\omega\epsilon\nabla \times [\mathbf{z}_0\Pi'(\mathbf{r})] + \nabla \times \nabla \times [\mathbf{z}_0\Pi''(\mathbf{r})]. \tag{6b}$$

The two independent functions $I'(\mathbf{r})$ and $V''(\mathbf{r})$ in Eq. (5) suffice to determine the total fields via Eqs. (6). In a source-free region, $V'(\mathbf{r})$ and $I''(\mathbf{r})$ are obtainable from $I'(\mathbf{r})$ and $V''(\mathbf{r})$, respectively, by differentiation with respect to z, as is evident from the transmission-line equations (2.2.15). Thus,

$$V'(\mathbf{r}) = \sum_i \frac{1}{-j\kappa_i' Y_i'}\frac{dI_i'(z)}{dz}\frac{\Phi_i(\boldsymbol{\rho})}{k_{ti}'} = \frac{1}{-j\omega\varepsilon}\frac{\partial}{\partial z}I'(\mathbf{r}), \tag{7a}$$

and, similarly,

$$I''(\mathbf{r}) = \frac{1}{-j\omega\mu}\frac{\partial}{\partial z}V''(\mathbf{r}). \tag{7b}$$

Equations (2) and (2.2.15) may be used to verify that in a source-free, homogeneous region, the Hertz potentials Π' and Π'', given by Eqs. (4) and (5), satisfy the wave equations

$$(\nabla^2 + k^2)\frac{\Pi'}{\Pi''} = 0. \tag{8}$$

†It should be pointed out that the scalar eigenfunctions Φ_i and ψ_i, like the vector eigenfunctions $\mathbf{e}_i'$ and $\mathbf{e}_i''$, each form an orthonormal set (see Sec. 3.2). Normalization of these scalar eigenfunctions differs from that used in Reference 2. The relation between the eigenfunctions here and those in Reference 2 is the following:

$$k_{ti}'[\Phi_i]_{\text{ref.2}} = \Phi_i, \qquad k_{ti}''[\psi_i]_{\text{ref.2}} = \psi_i.$$

2.3c Green's Functions for the Transmission-Line Equations

To obtain explicit solutions for the Hertz potentials in source regions, it is necessary to relate the modal coefficients in Eqs. (4) to their excitations. For this purpose, it is convenient to introduce modal Green's functions that are one-dimensional scalar analogues of the dyadic Green's functions defined in Sec. 1.1b. In view of the linearity of the transmission-line equations (2.2.15), one can obtain the voltage and current solutions at any point z by superposing separate contributions from suitable point voltage and current generators distributed along points z'. In analogy with Eqs. (1.1.19), one thereby finds[3]

$$V(z) = -\int dz'\, T^V(z, z')v(z') - \int dz'\, Z(z, z')i(z'), \tag{9a}$$

$$I(z) = -\int dz'\, Y(z, z')v(z') - \int dz'\, T^I(z, z')i(z'), \tag{9b}$$

where the mode subscript i has been omitted. Equations (9) reduce the problem to that of determining $T^V(z, z')$, $Y(z, z')$ and $Z(z, z')$, $T^I(z, z')$, whose significance as modal Green's functions is evident: $-T^V(z, z')$ and $-Z(z, z')$ are the component voltage responses at z due, respectively, to a unit voltage and current source (generator) at the point z', while $-Y(z, z')$ and $-T^I(z, z')$ are the corresponding current responses to the same excitations. Thus, if in Eqs. (2.2.15), one sets $v(z) = -\delta(z - z')$ and $i(z) = 0$, there results

$$-\frac{d}{dz} T^V(z, z') = j\kappa Z Y(z, z') - \delta(z - z'), \tag{10a}$$

$$-\frac{d}{dz} Y(z, z') = j\kappa Y T^V(z, z'), \tag{10b}$$

and, if $v = 0, i = -\delta(z - z')$,

$$-\frac{d}{dz} Z(z, z') = j\kappa Z T^I(z, z'), \tag{10c}$$

$$-\frac{d}{dz} T^I(z, z') = j\kappa Y Z(z, z') - \delta(z - z'), \tag{10d}$$

subject to as-yet-unspecified boundary conditions at the z terminations.

The modal Green's functions defined in Eqs. (10) satisfy reciprocity properties when κ and Z are either constant or z dependent. We consider a given terminated transmission line to be excited by two separate source distributions: the first, $v(z)$, $i(z)$, giving rise to $V(z)$, $I(z)$; and the second, $\hat{v}(z)$, $\hat{\imath}(z)$, giving rise to $\hat{V}(z)$, $\hat{I}(z)$. Both sets satisfy transmission-line equations:

$$-\frac{dV}{dz} = j\kappa Z I + v, \tag{11a}$$

$$-\frac{dI}{dz} = j\kappa Y V + i, \tag{11b}$$

and

$$-\frac{d\hat{V}}{dz} = j\kappa Z\hat{I} + \hat{v}, \tag{11c}$$

$$-\frac{d\hat{I}}{dz} = j\kappa Y\hat{V} + \hat{i}. \tag{11d}$$

Upon multiplying Eqs. (11a)–(11d) by $\hat{I}, \hat{V}, I, V$, respectively, subtracting the sum of the resulting Eqs. (11a) and (11d) from the sum of Eqs. (11b) and (11c), and integrating over z between the limits z_1 and z_2, one obtains

$$(\hat{V}I - \hat{I}V)_{z_1}^{z_2} = \int_{z_1}^{z_2} dz(v\hat{I} + \hat{i}V - i\hat{V} - \hat{v}I). \tag{12}$$

Both sets of voltages and currents satisfy the same terminal conditions at z_1 and z_2:

$$V(z_{1,2}) = \mp Z(z_{1,2})I(z_{1,2}), \qquad \hat{V}(z_{1,2}) = \mp Z(z_{1,2})\hat{I}(z_{1,2}), \tag{13}$$

where $Z(z_{1,2})$ are terminal impedances† (see Sec. 2.4 for details on the network interpretation of the transmission-line equations). Thus, the left-hand side of Eq. (12), expressing the difference between the values at z_2 and z_1 of the bracketed quantity, vanishes and one obtains the reciprocity relation

$$\int_{z_1}^{z_2} dz(v\hat{I} + \hat{i}V - i\hat{V} - \hat{v}I) = 0. \tag{14}$$

To apply the reciprocity condition (14) to the modal Green's functions defined in Eqs. (10), one selects the following special source distributions:

(a) $$v = \hat{v} = 0, \quad i = -\delta(z - z'), \quad \hat{i} = -\delta(z - z''); \\ V \to Z(z, z'), \quad \hat{V} \to Z(z, z''),$$

(b) $$i = \hat{i} = 0, \quad v = -\delta(z - z'), \quad \hat{v} = -\delta(z - z''); \\ I \to Y(z, z'), \quad \hat{I} \to Y(z, z''),$$

(c) $$v = \hat{i} = 0, \quad i = -\delta(z - z'), \quad \hat{v} = -\delta(z - z''); \\ I \to T^I(z, z'), \quad \hat{V} \to T^V(z, z''),$$

whence one obtains the following reciprocity theorems:

$$\begin{aligned} &\text{(a)} && Z(z'', z') = Z(z', z''), \\ &\text{(b)} && Y(z'', z') = Y(z', z''), \\ &\text{(c)} && T^I(z'', z') = -T^V(z', z''), \end{aligned} \tag{15}$$

which bear evident similarities to those for the dyadic Green's functions listed in Eqs. (1.1.29).

In view of the reciprocity relation (15c) between T^I and T^V one deduces from Eqs. (10) the important fact that the general solution for the voltage and current in a source-free region can be expressed *solely* in terms of *either* $Y(z, z')$ or $Z(z, z')$. Suppose we have found $Y(z, z')$; then T^V is obtained from Eq. (10b).

†For clarity, it is recalled that in this section, Z, $Z(z_\alpha)$, and $Z(z, z')$ denote, respectively, the characteristic impedance, the terminating impedance at z_α, and the voltage Green's function for the ith mode.

Because of the reciprocity theorem, a knowledge of T^V implies the knowledge of T^I, which in turn determines $Z(z, z')$ via Eq. (10d), provided that $z \neq z'$ (i.e., away from the source). Thus, all the required information is contained in $Y(z, z')$; an alternative statement applies for $Z(z, z')$. Actually, one can also determine the voltage and current in a source-free region from either T^I or T^V; however, as is shown in Sec. 2.4d, the *basic* Green's functions are the transfer impedance $Z(z, z')$ and the transfer admittance $Y(z, z')$, from which T^I and T^V are derivable. Because of the fundamental role played by the current (i.e., the E_z field component) in the case of E modes, it is usually convenient to determine E-mode solutions from $Y(z, z')$; by duality, the Green's function $Z(z, z')$ is usually more convenient for H-mode quantities.

2.3d Modal Representations of the Dyadic Green's Functions in a Piecewise Homogeneous Medium

As shown in Sec. 1.1, the electromagnetic fields radiated by point current excitations are conveniently expressed in terms of dyadic Green's functions. In this section we derive modal solutions for the dyadic Green's functions in regions whose properties are constant along the z direction and show how the dyadic Green's functions can be related to scalar Green's functions.

From Eqs. (6) one notes that the electromagnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ exterior to source regions can be expressed in terms of the scalar potential functions $I'(\mathbf{r})$ and $V''(\mathbf{r})$ defined in Eqs. (4). If the assumed sources are electric and magnetic current elements situated at the point $\mathbf{r}'$,

$$\mathbf{J}(\mathbf{r}) = \mathbf{J}^0 \delta(\mathbf{r} - \mathbf{r}'), \qquad \mathbf{M}(\mathbf{r}) = \mathbf{M}^0 \delta(\mathbf{r} - \mathbf{r}'), \tag{16}$$

where $\mathbf{J}^0$ and $\mathbf{M}^0$ are arbitrarily oriented constant vectors, then the modal representations for I' and V'' in Eqs. (4) can be simplified. Consider first the E-mode current $I_i'(z)$ occurring in the representation for the E-mode current potential $I'(\mathbf{r})$ in Eq. (4b). Upon recalling the definitions for the transmission-line Green's functions $Y_i(z, z')$ and $T_i^I(z, z')$ in Eq. (9b), one notes that for a point source

$$I_i'(z, z') = -Y_i'(z, z') v_i'(z') - T_i'^I(z, z') i_i'(z'), \tag{17}$$

where the dependence of $I_i'(z)$ on z' has been indicated explicitly and the subscripts have been inserted to highlight the modal character of the various quantities (for a network interpretation, see Fig. 2.4.4). It will be desirable to have $T_i'^I(z, z')$ expressed in terms of $Y_i'(z, z')$. From Eqs. (15c), (10b), and (15b), one finds that

$$T_i^I(z, z') = -T_i^V(z', z) = \frac{1}{j\kappa_i Y_i} \frac{d}{dz'} Y_i(z', z) = \frac{1}{j\kappa_i Y_i} \frac{d}{dz'} Y_i(z, z'). \tag{18}$$

Since $\kappa_i' Y_i' = \omega\epsilon$ for E modes [see Eq. (2.2.15)], one obtains, instead of Eq. (17),

$$I_i'(z, z') = -\left[v_i'(z') + \frac{1}{j\omega\epsilon} i_i'(z') \frac{d}{dz'} \right] Y_i'(z, z'). \tag{19}$$

In a similar manner, one can show that the H-mode voltages $V_i''(z)$, occurring in the representation of the voltage potential function $V''(\mathbf{r})$ in Eq. (4a), can be expressed in a manner dual to that in Eq. (19):

$$V_i''(z, z') = -\left[i_i''(z') + \frac{1}{j\omega\mu} v_i''(z') \frac{d}{dz'}\right] Z_i''(z, z'). \tag{20}$$

Since $\delta(\mathbf{r} - \mathbf{r}') = \delta(\boldsymbol{\rho} - \boldsymbol{\rho}')\delta(z - z')$ in Eq. (16), the source terms v_i and i_i defined in terms of $\mathbf{J}$ and $\mathbf{M}$ by Eqs. (2.2.14), take on the following simple form:

$$v_i(z) = v_i(z')\delta(z - z'), \qquad i_i(z) = i_i(z')\delta(z - z'), \tag{21}$$

$$v_i(z') = \mathbf{h}_i^*(\boldsymbol{\rho}') \cdot \mathbf{M}^0 + Z_i^* \mathbf{e}_{zi}^*(\boldsymbol{\rho}') \cdot \mathbf{J}^0, \tag{21a}$$

$$i_i(z') = \mathbf{e}_i^*(\boldsymbol{\rho}') \cdot \mathbf{J}^0 + Y_i^* \mathbf{h}_{zi}^*(\boldsymbol{\rho}') \cdot \mathbf{M}^0. \tag{21b}$$

Upon substituting the scalar mode functions via Eqs. (1), one finds that for E modes,

$$\begin{aligned} &-\left[v_i'(z') + \frac{1}{j\omega\epsilon} i_i'(z') \frac{d}{dz'}\right] \\ &= \left[(\mathbf{z}_0 \times \nabla_t') \frac{\Phi_i^*(\boldsymbol{\rho}')}{k_{ti}'}\right] \cdot \mathbf{M}^0 - \frac{1}{j\omega\epsilon}\left[\left(\mathbf{z}_0 \nabla_t'^2 - \nabla_t' \frac{\partial}{\partial z'}\right) \frac{\Phi_i^*(\boldsymbol{\rho}')}{k_{ti}'}\right] \cdot \mathbf{J}^0, \end{aligned} \tag{22}$$

where ∇_t' denotes differentiation with respect to the primed coordinates $\boldsymbol{\rho}'$. In view of the vector identities

$$\mathbf{z}_0 \times \nabla_t' \varphi = -\nabla' \times (\mathbf{z}_0 \varphi) \rightarrow -(\nabla' \times \mathbf{z}_0)\varphi \tag{23a}$$

and

$$\begin{aligned} \left(\nabla_t' \frac{\partial}{\partial z'} - \mathbf{z}_0 \nabla_t'^2\right)\varphi &= \left(\nabla' \frac{\partial}{\partial z'} - \mathbf{z}_0 \nabla'^2\right)\varphi \\ &= \nabla'(\nabla' \cdot \mathbf{z}_0 \varphi) - \nabla'^2(\mathbf{z}_0 \varphi) \rightarrow (\nabla' \times \nabla' \times \mathbf{z}_0)\varphi, \end{aligned} \tag{23b}$$

where φ is a scalar function of $\boldsymbol{\rho}'$, one obtains the following concise expression for $I'(\mathbf{r})$ after substituting Eqs. (19)–(23) into Eq. (4b):

$$I'(\mathbf{r}) = (\nabla' \times \nabla' \times \mathbf{z}_0)\mathscr{S}'(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}^0 - j\omega\epsilon(\nabla' \times \mathbf{z}_0)\mathscr{S}'(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}^0, \tag{24}$$

where

$$j\omega\epsilon\mathscr{S}'(\mathbf{r}, \mathbf{r}') = \sum_i \frac{\Phi_i(\boldsymbol{\rho})\Phi_i^*(\boldsymbol{\rho}')}{k_{ti}'^2} Y_i'(z, z'). \tag{24a}$$

The meaning of the operations $(\nabla' \times \mathbf{z}_0)$ and $(\nabla' \times \nabla' \times \mathbf{z}_0)$ is defined in Eqs. (23a) and (23b), respectively. Equations (24) evidently are valid only when $k_{ti}' \neq 0$ (i.e., any possible TEM modes are excluded).† If the waveguide structure

†The interchange of operations of summation and differentiation, assumed valid in deriving Eqs. (24) from Eqs. (4), may not be permissible in certain problems involving continuous spectra of eigenfunctions (see Sec. 5.2b). [Similar remarks apply to Eqs. (25).] In these instances, the above expressions are to be considered as formal and must be properly interpreted [see Eq. (1.1.38) for related comments pertaining to the operator $1/\nabla_t^2$].

can support one or more TEM modes, the contribution to the radiated fields from these modes must be taken into account separately [see footnote to Eq. (2b)].

For the H-mode potential function $V''(\mathbf{r})$ in Eq. (4a) one obtains by analogous considerations the dual representation

$$V''(\mathbf{r}) = j\omega\mu(\nabla' \times \mathbf{z}_0)\mathscr{S}''(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}^0 + (\nabla' \times \nabla' \times \mathbf{z}_0)\mathscr{S}''(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}^0, \quad (25)$$

where

$$j\omega\mu\mathscr{S}''(\mathbf{r}, \mathbf{r}') = \sum_i \frac{\psi_i(\boldsymbol{\rho})\psi_i^*(\boldsymbol{\rho}')}{k_{ti}''^2} Z_i''(\mathbf{z}, \mathbf{z}'). \quad (25a)$$

ψ_i are the scalar H-mode functions defined in Eqs. (2).

Upon substituting the representations for $I'(\mathbf{r})$ and $V''(\mathbf{r})$ from Eqs. (24) and (25) into Eqs. (6), one obtains the desired formulation for the electromagnetic fields observed at $\mathbf{r}$ due to vector point-source excitations of electric and magnetic currents at $\mathbf{r}'$ as in Eq. (16) [see Eq. (3a)]:

$$\mathbf{E}(\mathbf{r}, \mathbf{r}') = -\mathscr{Z}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}^0 - \mathscr{T}_e(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}^0, \quad (26a)$$

$$\mathbf{H}(\mathbf{r}, \mathbf{r}') = -\mathscr{T}_m(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}^0 - \mathscr{Y}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}^0 \quad (26b)$$

where $\mathscr{Z}$, $\mathscr{Y}$ and $\mathscr{T}_e$, $\mathscr{T}_m$ are the dyadic impedance, admittance, and electric and magnetic transfer functions, respectively [see Eqs. (1.1.49) with $\mathbf{r} \neq \mathbf{r}'$][6]:

$$-j\omega\epsilon\mathscr{Z}(\mathbf{r}, \mathbf{r}') = (\nabla \times \nabla \times \mathbf{z}_0)(\nabla' \times \nabla' \times \mathbf{z}_0)\mathscr{S}'(\mathbf{r}, \mathbf{r}') + k^2(\nabla \times \mathbf{z}_0)(\nabla' \times \mathbf{z}_0)\mathscr{S}''(\mathbf{r}, \mathbf{r}') \quad (27a)$$

$$-j\omega\mu\mathscr{Y}(\mathbf{r}, \mathbf{r}') = (\nabla \times \nabla \times \mathbf{z}_0)(\nabla' \times \nabla' \times \mathbf{z}_0)\mathscr{S}''(\mathbf{r}, \mathbf{r}') + k^2(\nabla \times \mathbf{z}_0)(\nabla' \times \mathbf{z}_0)\mathscr{S}'(\mathbf{r}, \mathbf{r}'), \quad (27b)$$

$$\mathscr{T}_e(\mathbf{r}, \mathbf{r}') = (\nabla \times \nabla \times \mathbf{z}_0)(\nabla' \times \mathbf{z}_0)\mathscr{S}'(\mathbf{r}, \mathbf{r}') + (\nabla \times \mathbf{z}_0)(\nabla' \times \nabla' \times \mathbf{z}_0)\mathscr{S}''(\mathbf{r}, \mathbf{r}'), \quad (27c)$$

$$-\mathscr{T}_m(\mathbf{r}, \mathbf{r}') = (\nabla \times \nabla \times \mathbf{z}_0)(\nabla' \times \mathbf{z}_0)\mathscr{S}''(\mathbf{r}, \mathbf{r}') + (\nabla \times \mathbf{z}_0)(\nabla' \times \nabla' \times \mathbf{z}_0)\mathscr{S}'(\mathbf{r}, \mathbf{r}'), \quad (27d)$$

where $k^2 = \omega^2\mu\epsilon = \text{constant}$. Via Eqs. (27), the dyadic Green's functions are expressed in terms of scalar functions $\mathscr{S}'$ and $\mathscr{S}''$ in what appears to be a fundamental form. The symmetry inherent in the expressions is to be noted. In Eqs. (32b) and (33b) the functions $-\nabla_t'^2\mathscr{S}'$ and $-\nabla_t'^2\mathscr{S}''$ are shown to be scalar Green's functions that satisfy Eqs. (36) and (37). Since from Eqs. (15), $Y_i'(z, z') = Y_i'(z', z)$ and $Z_i''(z, z') = Z_i''(z', z)$, it follows from the modal representations for $\mathscr{S}'$ and $\mathscr{S}''$ in Eqs. (24a) and (25a), respectively, that for *real* Φ_i and ψ_i,†

†Although not always convenient, the mode functions Φ_i and ψ_i in regions bounded either by perfectly conducting walls, or else unbounded, can always be chosen real. Only such regions, wherein k_{ti}^2 is real, are considered above.

$$\mathscr{S}'(\mathbf{r}, \mathbf{r}') = \mathscr{S}'(\mathbf{r}', \mathbf{r}), \qquad \mathscr{S}''(\mathbf{r}, \mathbf{r}') = \mathscr{S}''(\mathbf{r}', \mathbf{r}), \tag{28}$$

whence, from Eqs. (27),

$$\mathscr{Z}(\mathbf{r}, \mathbf{r}') = \tilde{\mathscr{Z}}(\mathbf{r}', \mathbf{r}), \ \mathscr{Y}(\mathbf{r}, \mathbf{r}') = \tilde{\mathscr{Y}}(\mathbf{r}', \mathbf{r}), \ \mathscr{T}_e(\mathbf{r}, \mathbf{r}') = -\tilde{\mathscr{T}}_m(\mathbf{r}', \mathbf{r}), \tag{29}$$

where the tilde (˜) denotes the transposed dyadics. We therefore verify the time-harmonic form of the reciprocity conditions derived previously in Eqs. (1.1.29), where for ease of identification we have changed the notation to

$$\mathscr{G}_{11} = \mathscr{Z}, \quad \mathscr{G}_{12} = \mathscr{T}_e, \quad \mathscr{G}_{21} = \mathscr{T}_m, \quad \mathscr{G}_{22} = \mathscr{Y}. \tag{30}$$

To include also the point $\mathbf{r} = \mathbf{r}'$, Eqs. (27) must be modified as in Eq. (1.1.38) or (1.1.49).

Equations (24) and (25) simplify considerably for the case of longitudinal sources,

$$\mathbf{J}^0 = \mathbf{z}_0 J^0, \qquad \mathbf{M}^0 = \mathbf{z}_0 M^0. \tag{31}$$

From Eq. (23a) one notes that $(\nabla' \times \mathbf{z}_0)\varphi \cdot \mathbf{z}_0 = 0$, while from Eq. (23b), $(\nabla' \times \nabla' \times \mathbf{z}_0)\varphi \cdot \mathbf{z}_0 = -\nabla_t'^2 \varphi$. One may write

$$I'(\mathbf{r}) = J^0 G'(\mathbf{r}, \mathbf{r}'), \tag{32a}$$

where, in view of $\nabla_t'^2 \Phi_i^*(\boldsymbol{\rho}') = -k_{ti}'^2 \Phi_i^*(\boldsymbol{\rho}')$ or $\nabla_t^2 \Phi_i(\boldsymbol{\rho}) = -k_{ti}'^2 \Phi_i(\boldsymbol{\rho})$,

$$\begin{aligned} G'(\mathbf{r}, \mathbf{r}') &\equiv -\nabla_t'^2 \mathscr{S}'(\mathbf{r}, \mathbf{r}') = -\nabla_t^2 \mathscr{S}'(\mathbf{r}, \mathbf{r}') \\ &= \frac{1}{j\omega\epsilon} \sum_i \Phi_i(\boldsymbol{\rho}) \Phi_i^*(\boldsymbol{\rho}') Y_i'(z, z'). \end{aligned} \tag{32b}$$

Similarly, one writes

$$V''(\mathbf{r}) = M^0 G''(\mathbf{r}, \mathbf{r}'), \tag{33a}$$

with

$$\begin{aligned} G''(\mathbf{r}, \mathbf{r}') &\equiv -\nabla_t'^2 \mathscr{S}''(\mathbf{r}, \mathbf{r}') = -\nabla_t^2 \mathscr{S}''(\mathbf{r}, \mathbf{r}') \\ &= \frac{1}{j\omega\mu} \sum_i \psi_i(\boldsymbol{\rho}) \psi_i^*(\boldsymbol{\rho}') Z_i''(z, z'). \end{aligned} \tag{33b}$$

One notes from Eqs. (32) and (33) that a longitudinal electric current source excites only E modes along z while a longitudinal magnetic current source excites only H modes. The fields are now determined by the following simplified form of Eqs. (26):

$$\mathbf{E}(\mathbf{r}, \mathbf{r}') = \frac{J^0}{j\omega\epsilon}(\nabla \times \nabla \times \mathbf{z}_0) G'(\mathbf{r}, \mathbf{r}') - M^0(\nabla \times \mathbf{z}_0) G''(\mathbf{r}, \mathbf{r}'), \tag{34a}$$

$$\mathbf{H}(\mathbf{r}, \mathbf{r}') = J^0(\nabla \times \mathbf{z}_0) G'(\mathbf{r}, \mathbf{r}') + \frac{M^0}{j\omega\mu}(\nabla \times \nabla \times \mathbf{z}_0) G''(\mathbf{r}, \mathbf{r}'). \tag{34b}$$

We show now that G' and G'' are scalar Green's functions satisfying, subject to appropriate boundary conditions, the scalar wave equation with an inhomogeneous term $-\delta(\mathbf{r} - \mathbf{r}')$. Let the operator $(\nabla^2 + k^2)$ act on G' as represented in Eq. (32b) and assume that the operations of summation and differentiation can be interchanged. Then, since $\nabla_t^2 \Phi_i = -k_{ti}'^2 \Phi_i$ and $\kappa_i'^2 = k^2 - k_{ti}'^2$,

$$\left(\nabla_t^2 + \frac{\partial^2}{\partial z^2} + k^2\right)G'(\mathbf{r}, \mathbf{r}') = \frac{1}{j\omega\epsilon}\sum_i \mathbf{\Phi}_i(\boldsymbol{\rho})\mathbf{\Phi}_i^*(\boldsymbol{\rho}')\left(\frac{d^2}{dz^2} + \kappa_i'^2\right)Y_i'(z, z') \tag{35a}$$

$$= -\delta(z - z')\sum_i \mathbf{\Phi}_i(\boldsymbol{\rho})\mathbf{\Phi}_i^*(\boldsymbol{\rho}')$$

$$= -\delta(z - z')\delta(\boldsymbol{\rho} - \boldsymbol{\rho}') = -\delta(\mathbf{r} - \mathbf{r}'). \tag{35b}$$

The transition from Eq. (35a) to Eq. (35b) follows via the differential equation for $Y_i'(z, z')$ obtained on elimination of $T_i^Y(z, z')$ from Eqs. (10a) and (10b), while the identification of the mode function series as $\delta(\boldsymbol{\rho} - \boldsymbol{\rho}')$ is discussed in Chapter 3 [see, for example, Eq. (3.2.17a)]. Thus, the *E*-mode function G' is a scalar three-dimensional Green's function which satisfies the inhomogeneous wave equation

$$(\nabla^2 + k^2)G'(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{36}$$

subject on the perfectly conducting waveguide boundary s, to the same boundary condition as $\mathbf{\Phi}_i(\boldsymbol{\rho})$ [see Eq. (2b)],

$$G'(\mathbf{r}, \mathbf{r}') = 0, \qquad \mathbf{r} \text{ on } s. \tag{36a}$$

The boundary conditions on G' in the z domain will depend on stratification along the z coordinate. For example, across a dielectric interface at $z = z_1$, the transverse electric and magnetic fields are continuous, so the voltage and current in each mode are continuous [see Eqs. (2.2.8a) and (2.2.8b)]. Since $Y_i'(z, z')$ represents a current, continuity of $Y_i'(z, z')$ across z_1 implies from Eq. (32b) that $G'(\mathbf{r}, \mathbf{r}')$ is likewise continuous across z_1. From the transmission-line equations, the mode voltage is proportional to $(1/\kappa_i' Y_i')(d/dz)Y_i'(z, z')$, and since $\kappa_i' Y_i' = \omega\epsilon$, continuity of voltage implies via Eq. (32b) that $(1/\epsilon)(\partial/\partial z)G'(\mathbf{r}, \mathbf{r}')$ must likewise be continuous at z_1.† Thus, we find that G' and $(1/\epsilon)(\partial G'/\partial z)$ are required to be continuous across a dielectric interface. Similarly, if the region is terminated at z_1 in a perfectly conducting plane on which the transverse electric field vanishes, each modal voltage vanishes and requires that $\partial G'/\partial z = 0$ at z_1, while for an unterminated z domain, a "radiation condition" requiring an outward flow of energy is appropriate. The modal representation for G' in Eq. (32b) thus constitutes the solution of the Green's function problem posed in Eq. (36) subject to the above-discussed boundary conditions.

By analogous considerations, one shows that the *H*-mode Green's function G'' in Eq. (33b) satisfies the inhomogeneous wave equation

$$(\nabla^2 + k^2)G''(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'), \tag{37}$$

subject on the perfectly conducting waveguide boundary s to the same condition as $\psi_i(\boldsymbol{\rho})$ [see Eq. (2d)],

†ϵ and μ in Eqs. (24a), (25a), (32b), and (33b) have constant values appropriate to the medium containing the source point z'; in Eqs. (24), (25), (27), and (34), ϵ and μ have constant values appropriate to the medium containing the observation point [see also Eqs. (38), (40), and (42)]. These remarks are relevant for analysis of media with piecewise constant ϵ and μ.

$$\frac{\partial G''}{\partial \nu} = 0 \qquad \text{on } s. \tag{37a}$$

The boundary conditions satisfied by G'' in the z domain are dual to those on G'. At an interface plane $z = z_1$, G'' and $(1/\mu)(\partial G''/\partial z)$ must be continuous, while at a perfectly conducting plane, $G'' = 0$.† It follows from Eqs. (32b), (33b), (36), and (37) that the scalar functions $\mathscr{S}'(\mathbf{r}, \mathbf{r}')$ and $\mathscr{S}''(\mathbf{r}, \mathbf{r}')$ satisfy the time-harmonic form of the differential equation (1.1.38b), subject to boundary conditions identical with those on G' and G'', respectively. The recovery of $\mathscr{S}'$ and $\mathscr{S}''$ from G' and G'', respectively, requires the inversion of Eqs. (32b) and (33b). For $k_{ti}^2 \neq 0$, this inversion is accomplished readily in a basis wherein $-\nabla_t^2 \rightarrow k_{ti}'^2$ or $k_{ti}''^2$, and leads directly to the representations in Eqs. (24a) and (25a).

2.3e Modal Representations of the Dyadic Green's Functions in an Inhomogeneous Medium

The formulas derived in Sec. 2.3d apply to homogeneous media and must be modified if ϵ and μ are functions of z. In this instance, the results of Secs. 2.2, 2.3a, 2.3b, and 2.3c remain valid with the exception of Eqs. (6), which should be written at a source-free point as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{j\omega\epsilon(z)}(\nabla \times \nabla \times \mathbf{z}_0)I'(\mathbf{r}) - (\nabla \times \mathbf{z}_0)V''(\mathbf{r}), \tag{38a}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{j\omega\mu(z)}(\nabla \times \nabla \times \mathbf{z}_0)V''(\mathbf{r}) + (\nabla \times \mathbf{z}_0)I'(\mathbf{r}), \tag{38b}$$

with $I'(\mathbf{r})$ and $V''(\mathbf{r})$ defined in Eqs. (4). As regards the results in Sec. 2.3d, one notes from the method of derivation that Eqs. (19)–(23) still apply provided that ϵ and μ are replaced by $\epsilon(z')$ and $\mu((z')$, respectively. It then follows that Eq. (24) should be written as

$$I'(\mathbf{r}) = -L_1'\mathscr{S}_d' \cdot \mathbf{M}^0 + \frac{1}{j\omega\epsilon(z')} L_2'\mathscr{S}_d' \cdot \mathbf{J}^0, \tag{39}$$

where the vector operators L_1' and L_2' are defined as

$$L_1' \equiv \nabla' \times \mathbf{z}_0, \qquad L_2' \equiv \nabla' \times \nabla' \times \mathbf{z}_0, \tag{39a}$$

and

$$\mathscr{S}_d' = \sum_i \frac{\Phi_i(\boldsymbol{\rho})\Phi_i^*(\boldsymbol{\rho}')}{k_{ti}'^2} Y_i'(z, z'). \tag{39b}$$

Dual considerations apply to Eq. (25).

With the above modifications, the dyadic Green's functions in Eqs. (27) are now written in the following form:

$$\mathscr{Z}(\mathbf{r}, \mathbf{r}') = \frac{1}{\omega^2\epsilon(z)\epsilon(z')} L_2L_2'\mathscr{S}_d' + L_1L_1'\mathscr{S}_d'', \tag{40a}$$

†See the preceding footnote.

$$\mathscr{Y}(\mathbf{r}, \mathbf{r}') = \frac{1}{\omega^2 \mu(z)\mu(z')} L_2 L_2' \mathscr{S}_d'' + L_1 L_1' \mathscr{S}_d', \tag{40b}$$

$$\mathscr{T}_e(\mathbf{r}, \mathbf{r}') = \frac{1}{j\omega\epsilon(z)} L_2 L_1' \mathscr{S}_d' + \frac{1}{j\omega\mu(z')} L_1 L_2' \mathscr{S}_d'', \tag{40c}$$

$$-\mathscr{T}_m(\mathbf{r}, \mathbf{r}') = \frac{1}{j\omega\mu(z)} L_2 L_1' \mathscr{S}_d'' + \frac{1}{j\omega\epsilon(z')} L_1 L_2' \mathscr{S}_d', \tag{40d}$$

where

$$L_1 \equiv \nabla \times \mathbf{z}_0, \quad L_2 \equiv \nabla \times \nabla \times \mathbf{z}_0, \quad \mathscr{S}_d'' = \sum_i \frac{\psi_i(\boldsymbol{\rho})\psi_i^*(\boldsymbol{\rho}')}{k_{ti}''^2} Z_i''(z, z'). \tag{40e}$$

It is readily verified that these more general expressions satisfy, as they must, the reciprocity relations (29).

The modal Green's functions $Y_i'(z, z')$ and $Z_i''(z, z')$ are defined in Eqs. (10). Because $\kappa(z) = [\omega^2 \mu(z)\epsilon(z) - k_{ti}^2]^{1/2}$ is now variable, the characteristic impedances $Z_i(z)$ and admittances $Y_i(z)$ are also functions of z, so the associated transmission lines are non-uniform.† On elimination of T_i^V and T_i^I from Eqs. (10a), (10b) and (10c), (10d), respectively, one finds that the modal Green's functions satisfy the following second-order differential equations [note from Eqs. (2.2.15) that $\kappa_i'(z)Y_i'(z) = \omega\epsilon(z)$, $\kappa_i''(z)Z_i''(z) = \omega\mu(z)$]:

$$[D_\epsilon^2(z) + \kappa_i'^2(z)]Y_i'(z, z') = -j\omega\epsilon(z')\delta(z - z'), \tag{41a}$$

$$[D_\mu^2(z) + \kappa_i''^2(z)]Z_i''(z, z') = -j\omega\mu(z')\delta(z - z'), \tag{41b}$$

where

$$D_\alpha^2(z) = \alpha(z)\frac{d}{dz}\frac{1}{\alpha(z)}\frac{d}{dz}, \qquad \alpha = \epsilon \text{ or } \mu. \tag{41c}$$

The boundary conditions at the endpoints of the transmission line are phrased as in Eq. (13). Note that the E-mode terminal impedance is given via Eqs. (10a) and (10b) by $[(d/dz)Y_i'(z, z')/-j\kappa_i' Y_i' Y_i'(z, z')]_{z_{1,2}}$; the spatially varying characteristic impedance here should not be confused with the terminal impedance in Sec. 2.3c. At a junction between two transmission lines with parameters $\kappa_{i1}(z)$, $Z_{i1}(z)$ and $\kappa_{i2}(z)$, $Z_{i2}(z)$, respectively, the voltage and current are continuous. Thus, from Eqs. (10), $Y_i'(z, z')$, $[1/\epsilon(z)](d/dz)Y_i'(z, z')$, and $Z_i''(z, z')$, $[1/\mu(z)]$ $(d/dz)Z_i''(z, z')$ are continuous across the junction point. (See also Secs. 3.3a and 3.3b for a solution of the non-uniform transmission-line equations.)

If the sources are longitudinal, Eqs. (40) simplify and lead to expressions analogous to those in Sec. 2.3d. In fact, one obtains expressions similar to Eqs. (34):

†Although the waveguide region is geometrically uniform in that successive geometrical cross sections transverse to z are identical, an electrical non-uniformity is introduced by the longitudinal variability of the medium constants. Consequently, the network representation discussed in Sec. 2.4 involves non-uniform transmission lines representative of the z behavior of a typical mode (see also Secs. 3.2d and 3.3b).

$$\mathbf{E}(\mathbf{r}, \mathbf{r}') = \frac{J^0}{j\omega\epsilon(z)} L_2 G'(\mathbf{r}, \mathbf{r}') - M^0 L_1 G''(\mathbf{r}, \mathbf{r}'), \tag{42a}$$

$$\mathbf{H}(\mathbf{r}, \mathbf{r}') = J^0 L_1 G'(\mathbf{r}, \mathbf{r}') + \frac{M^0}{j\omega\mu(z)} L_2 G''(\mathbf{r}, \mathbf{r}'), \tag{42b}$$

where

$$G'(\mathbf{r}, \mathbf{r}') = \frac{1}{j\omega\epsilon(z')} \sum_i Y_i'(z, z')\Phi_i(\boldsymbol{\rho})\Phi_i^*(\boldsymbol{\rho}') = -\frac{1}{j\omega\epsilon(z')} \nabla_t'^2 \mathscr{S}_d', \tag{43a}$$

$$G''(\mathbf{r}, \mathbf{r}') = \frac{1}{j\omega\mu(z')} \sum_i Z_i''(z, z')\psi_i(\boldsymbol{\rho})\psi_i^*(\boldsymbol{\rho}') = -\frac{1}{j\omega\mu(z')} \nabla_t'^2 \mathscr{S}_d''. \tag{43b}$$

The differential equations for the scalar Green's functions G' and G'' are now in view of Eqs. (41):

$$[\mathscr{D}_\epsilon^2(z) + \nabla_t^2 + k^2(z)]G'(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'), \quad k^2(z) = \omega^2\mu(z)\epsilon(z), \tag{44a}$$

$$[\mathscr{D}_\mu^2(z) + \nabla_t^2 + k^2(z)]G''(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'), \tag{44b}$$

where

$$\mathscr{D}_\alpha^2(z) = \alpha(z)\frac{\partial}{\partial z}\frac{1}{\alpha(z)}\frac{\partial}{\partial z}. \tag{44c}$$

It may also be verified that the Green's function $G'(\mathbf{r}, \mathbf{r}')/\sqrt{\epsilon(z)}$ satisfies the wave equation with the modified wavenumber $\bar{k}(z)$:

$$[\nabla^2 + \bar{k}^2(z)]\frac{G'(\mathbf{r}, \mathbf{r}')}{\sqrt{\epsilon(z)}} = -\frac{\delta(\mathbf{r} - \mathbf{r}')}{\sqrt{\epsilon(z')}}, \qquad \bar{k}^2(z) = k^2(z) - \sqrt{\epsilon(z)}\frac{d^2}{dz^2}\frac{1}{\sqrt{\epsilon(z)}}, \tag{45}$$

with a dual relation applicable to $G''(\mathbf{r}, \mathbf{r}')/\sqrt{\mu(z)}$. Corresponding equations for $\mathscr{S}_d'$ and $\mathscr{S}_d''$ follow on use of Eqs. (43). The conditions satisfied by G' and G'' on the transverse and longitudinal boundaries of the region are the same as those deduced in connection with Eqs. (36) and (37). These boundary conditions, in conjunction with Eqs. (44), render the specification of G' and G'' unique. The modal representations in Eqs. (43) constitute solutions for G' and G'' and are directly deducible from a z-transmission analysis. Alternative representations of the solution for G' and G'' can be constructed by the procedure discussed in Sec. 3.3c.

All the above relations reduce to those in Sec. 2.3d when ϵ and μ are constant.

2.4 SOLUTION OF UNIFORM TRANSMISSION-LINE EQUATIONS (NETWORK ANALYSIS)

2.4a Source-Free Case

As shown in Secs. 2.2 and 2.3, the solution of an electromagnetic field problem in a uniform waveguide region requires knowledge of the (vector or scalar) eigenfunctions in the transverse domain, and of the modal voltage and current amplitudes along the transmission coordinate z. Mode functions for various transverse cross-sectional configurations are given in Chapter 3. This section is concerned with solutions of the transmission-line equations (2.2.15) or (2.3.10) when the medium filling the waveguide region is piecewise constant along z.[2,3] Continuous z variation leads to Eqs. (2.3.41) which are studied in detail in Sec. 3.3.

As a preliminary, we shall consider a region that contains no sources. Thus, $v = i = 0$ in Eq. (2.2.15) and omitting the modal subscript i, one obtains the *homogeneous* (*source-free*) transmission-line equations

$$\begin{aligned} -\frac{d}{dz} V(z) &= j\kappa Z I(z), \\ -\frac{d}{dz} I(z) &= j\kappa Y V(z). \end{aligned} \tag{1}$$

Since κ and Z are assumed to be constant, the two first-order differential equations (1) are reduced to the following second-order equation for either V or I:

$$\left(\frac{d^2}{dz^2} + \kappa^2\right)\begin{matrix} V(z) \\ I(z) \end{matrix} = 0, \qquad \kappa, Z \text{ constant.} \tag{2}$$

The solution of Eqs. (2) may be expressed either in traveling-wave or standing-wave form. For a *traveling-wave* description one writes

$$V(z) = V_{\text{inc}}(z')e^{-j\kappa(z-z')} + V_{\text{refl}}(z')e^{+j\kappa(z-z')}, \tag{3a}$$

$$ZI(z) = V_{\text{inc}}(z')e^{-j\kappa(z-z')} - V_{\text{refl}}(z')e^{+j\kappa(z-z')}, \tag{3b}$$

$V_{\text{inc}}(z')$ and $V_{\text{refl}}(z')$ are the (generally complex) amplitudes of the incident and reflected voltage waves at a point z' on the transmission line. The first term on the right-hand side of Eq. (3a) represents a wave traveling in the $+z$ direction [for the assumed time dependence $\exp(+j\omega t)$]; the second term represents a wave traveling in the $-z$ direction. Equation (3b) follows from Eq. (3a) in view of Eqs. (1). If one defines the voltage reflection coefficient $\Gamma(z)$ as

$$\Gamma(z) = \frac{V_{\text{refl}}(z)}{V_{\text{inc}}(z)}, \tag{4}$$

one may rewrite Eqs. (3) as

$$V(z) = V_{\text{inc}}(z')[e^{-j\kappa(z-z')} + \Gamma(z')e^{+j\kappa(z-z')}], \tag{5a}$$

$$I(z) = YV_{\text{inc}}(z')[e^{-j\kappa(z-z')} - \Gamma(z')e^{+j\kappa(z-z')}]. \tag{5b}$$

It is evident from Eqs. (3a) and (4) that the reflection coefficients at two points z and z' on the transmission line are related as follows:

$$\Gamma(z) = \Gamma(z')e^{+j2\kappa(z-z')}. \tag{6}$$

In the *standing-wave* description one expresses the voltage and current solutions as

$$V(z) = V(z')\cos\kappa(z - z') - jZI(z')\sin\kappa(z - z'), \tag{7a}$$

$$I(z) = I(z')\cos\kappa(z - z') - jYV(z')\sin\kappa(z - z'). \tag{7b}$$

The duality between the solutions in Eqs. (7a) and (7b) is to be noted. By defining the absolute impedance $Z(z)$ [admittance $Y(z)$] and the relative (normalized) impedance $Z'(z)$ [admittance $Y'(z)$] at the point z as†

$$Z(z) = \frac{V(z)}{I(z)} = \frac{1}{Y(z)}, \qquad Z'(z) = \frac{V(z)}{ZI(z)} = \frac{1}{Y'(z)}, \tag{8}$$

one may write Eqs. (7) as

$$V(z) = V(z')\,[\cos \kappa(z - z') - jY'(z') \sin \kappa(z - z')], \tag{9a}$$

$$I(z) = I(z')\,[\cos \kappa(z - z') - jZ'(z') \sin \kappa(z - z')]. \tag{9b}$$

Upon taking the ratio of Eqs. (9a) and (9b), one finds for the relation between impedances at two points on a transmission line,

$$Z'(z) = \frac{1 + jZ'(z') \cot \kappa(z - z')}{Z'(z') + j \cot \kappa(z - z')}. \tag{10}$$

From Eqs. (3) the relation between the standing-wave and traveling-wave formulations is evidently

$$\begin{aligned} V(z) &= V_{\text{inc}}(z) + V_{\text{refl}}(z), \\ ZI(z) &= V_{\text{inc}}(z) - V_{\text{refl}}(z), \end{aligned} \tag{11a}$$

and

$$\begin{aligned} V_{\text{inc}}(z) &= \tfrac{1}{2}\,[V(z) + ZI(z)], \\ V_{\text{refl}}(z) &= \tfrac{1}{2}\,[V(z) - ZI(z)]. \end{aligned} \tag{11b}$$

The relation between the impedance and reflection coefficient at z then follows readily as

$$Z'(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} = \frac{1}{Y'(z)}, \qquad \Gamma(z) = \frac{Z'(z) - 1}{Z'(z) + 1}. \tag{12}$$

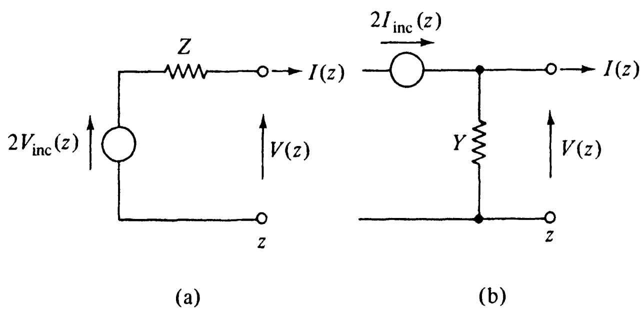


FIG. 2.4.1 Network representation of an incident wave.

†The prime in this section denotes normalized quantities and is not to be confused with the prime denoting E-mode quantities in Secs. 2.2 and 2.3. Also $Z(z)$ and $Y(z)$ denote herein the input impedance and admittance at the point z, while Z and Y represent the constant characteristic impedance and admittance, respectively.

The first of Eqs. (11b) permits the network representation of an incident wave at the point z in terms of either a (zero internal impedance) voltage generator of strength $2V_{\text{inc}}(z)$ or a (infinite internal impedance) current generator of strength $2I_{\text{inc}}(z) = 2YV_{\text{inc}}(z)$ shown in Figs. 2.4.1(a) and 2.4.1(b), respectively. Positive directions for voltage and current are assumed as indicated.

2.4b Point Source on an Infinite Transmission Line

In Sec. 2.4a we have obtained the solution of the homogeneous transmission-line equations with as-yet-unspecified boundary conditions. To obtain a solution of the inhomogeneous equations (2.2.15) requires the ability to describe source regions on the transmission line. As pointed out in Sec. 2.3c, the description of source regions is simplified by first considering point-source excitations and then obtaining the total response by use of the superposition theorem. The representation of the distributed voltage and current sources $v(z)$ and $i(z)$ in terms of Point sources is accomplished as follows:

$$v(z) = \int v(z')\delta(z - z')\,dz', \qquad i(z) = \int i(z')\delta(z - z')\,dz', \tag{13}$$

where the integration over z' extends over the source region. If the voltage and the current at z due to point sources $v(z')\delta(z - z')$ and $i(z')\delta(z - z')$ at z' are denoted by $V(z, z')$ and $I(z, z')$, respectively, one obtains the total voltage and current by superposition as†

$$V(z) = \int V(z, z')\,dz', \qquad I(z) = \int I(z, z')\,dz'. \tag{14}$$

In view of Eqs. (14), the basic problem is that of finding the point-source responses $V(z, z')$ and $I(z, z')$. The modal voltage and current sources $v(z)$ and $i(z)$ are known in terms of the *specified* electromagnetic source currents [see Eqs. (2.2.14)]. The problem can be reduced further by introduction of modal Green's functions as in Eqs. (2.3.9), but this aspect is deferred for the present.

The transmission-line equations appropriate to the indicated point-source excitations are

$$\begin{aligned} -\frac{d}{dz}V(z, z') &= j\kappa Z I(z, z') + v\delta(z - z'), \\ -\frac{d}{dz}I(z, z') &= j\kappa Y V(z, z') + i\delta(z - z'), \end{aligned} \tag{15}$$

where the notation $v(z') \equiv v$, $i(z') \equiv i$ has been adopted. By integrating Eqs. (15) over z between the limits $z' - \alpha$ and $z' + \alpha$, where $\alpha \to 0$, one obtains the following discontinuity relations for $V(z, z')$ and $I(z, z')$ at $z = z'$,

$$[V(z, z')]_{z'-\alpha}^{z'+\alpha} = -v, \qquad [I(z, z')]_{z'-\alpha}^{z'+\alpha} = -i. \tag{16}$$

†Note that the functions $V(z, z')$ and $I(z, z')$ are related to the modal Green's functions defined in Eqs. (2.3.9); $V(z, z') = -T^V(z, z')v(z') - Z(z, z')i(z')$ and $I(z, z') = -Y(z, z')v(z') - T^I(z, z')i(z')$ [see also Eq. (2.3.17)].

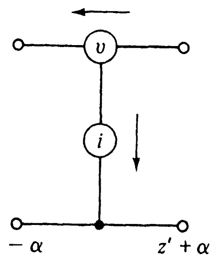


FIG. 2.4.2. Network representation for point-source excitation

It is assumed in Eq. (16), subject to subsequent verification, that $V(z, z')$ and $I(z, z')$, although discontinuous, are bounded at z'. The discontinuity relations (16) admit the simple network representation shown in Fig. 2.4.2, where the point-source voltage and current generators have zero and infinite internal impedance, respectively. In a strict network sense the voltage generator in Fig. 2.4.2 is to be interpreted as two generators of strength $v/2$, located on each side of the current generator.

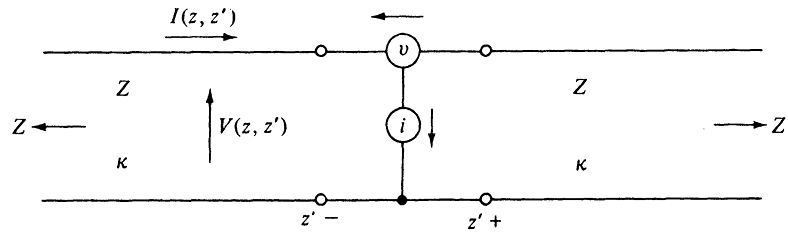


FIG. 2.4.3 Point source in an infinite transmission line.

To illustrate the use of the network in Fig. 2.4.2, we consider a simple example of a point source situated in an infinite transmission line (Fig. 2.4.3). (The general problem of arbitrary terminations is treated in Sec. 2.4c.) Since the line is infinitely long with no reflections, waves travel *away* from the source and boundary conditions at $z = \pm\infty$ are provided thereby. By inspection, the solution is written as

$$V(z, z') = \begin{cases} V(z'+, z')e^{-j\kappa(z-z')}, & z > z', \qquad (17a) \\ V(z'-, z')e^{+j\kappa(z-z')}, & z < z', \qquad (17b) \end{cases}$$

and, similarly for $I(z, z')$,

$$I(z, z') = \begin{cases} I(z'+, z')e^{-j\kappa(z-z')}, & z > z', \qquad (17c) \\ I(z'-, z')e^{+j\kappa(z-z')}, & z < z'. \qquad (17d) \end{cases}$$

To determine the amplitude of the voltage and current at $z'-$ and $z'+$, one notes that the impedance looking away from the source in either direction is equal to Z. Thus, the voltage generator v operates into a series impedance $2Z$, while the current generator i operates into a shunt admittance $2Y$. By superposing the responses from v and i one has

$$V(z'\pm, z') = -\tfrac{1}{2}(\pm v + Zi), \qquad I(z'\pm, z') = -\tfrac{1}{2}(\pm i + Yv), \tag{18}$$

so that anywhere on the transmission line

$$V(z, z') = -\tfrac{1}{2}[u(z, z')v + Zi]e^{-j\kappa|z-z'|}, \tag{19a}$$

$$I(z, z') = -\tfrac{1}{2}[u(z, z')i + Yv]e^{-j\kappa|z-z'|}, \tag{19b}$$

where

$$u(z, z') = \begin{cases} 1, & z > z', \\ -1, & z < z'. \end{cases} \tag{19c}$$

2.4c Excitation of General Transmission-Line Network by a Point Source

We consider now the general problem of a point source in a transmission-line system terminated by arbitrary, but prescribed, impedances. The terminating impedances represent the boundary conditions on the z termini of the transmission line and are determined directly from the corresponding boundary conditions on the electromagnetic fields. As a typical example we consider the network problem shown in Fig. 2.4.4, where $\overleftarrow{Z}_T$ and $\overrightarrow{Z}_T$ are the terminating

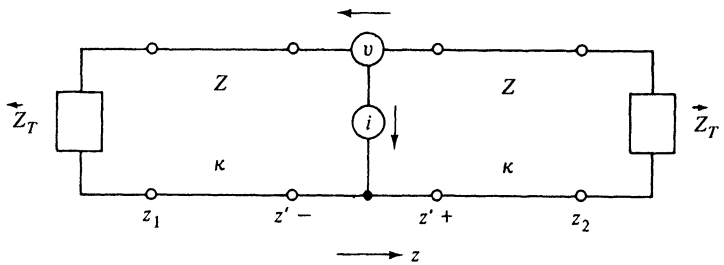


FIG. 2.4.4 Network representation for general point-source-excited transmission line.

impedances at the end points z_1 and z_2, respectively, and Z and κ are the (constant) characteristic impedance and propagation constant of the line. To the left and right, respectively, of the terminating points z_1 and z_2, the transmission-line system may be relatively arbitrary and is described by the impedances $\overleftarrow{Z}_T$ and $\overrightarrow{Z}_T$, which suffice to characterize the effect of the terminating structure on the transmission-line behavior in the interval $z_1 < z < z_2$. In passing from the transmission line with constants Z, κ to a line with constants Z_1, κ_1 connected at, say, z_2, one makes use of the relations in Sec. 2.4a, together with the continuity of the voltage and current across a direct junction between transmission lines. The latter property follows directly from the continuity of the transverse electric and magnetic fields across a (source-free) dielectric interface [see Eqs. (2.2.8a) and (2.2.8b) and recall the orthogonality of the vector-mode functions and the fact that the mode functions are independent of the parameters $\epsilon_{1,2}$ and $\mu_{1,2}$ descriptive of media 1 and 2 separated by an interface $z =$ constant].

The solution of the above network problem is accomplished by first finding the voltages and currents at the generator terminals, and determining therefrom the voltages and currents anywhere in the interval $z_1 < z < z_2$ by means of the homogeneous transmission-line solutions Eqs. (5) or (9). The generator circuit shown in Fig. 2.4.5 is representative of a general problem wherein $\overrightarrow{Z}(z')$ and

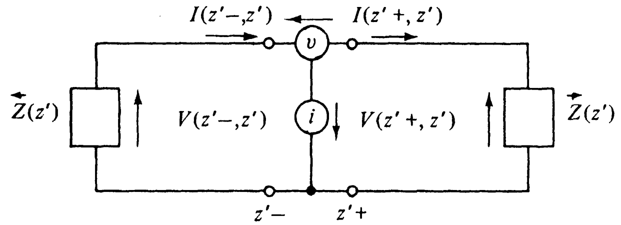


FIG. 2.4.5 Generator circuit for general transmission-line problem.

$\overleftarrow{Z}(z')$ are the impedances seen looking to the right and to the left, respectively, from the generator terminals. One computes $\overrightarrow{Z}(z')$ and $\overleftarrow{Z}(z')$ in terms of the given terminations $\overset{\rightleftarrows}{Z}_T$ of Fig. 2.4.4 by means of Eq. (10).† By conventional network analysis employing the superposition of the responses from v and i, the solution of the network problem in Fig. 2.4.5 is seen to be the following:

$$V(z'+, z') = \frac{-v\overrightarrow{Z}(z')}{\overleftrightarrow{Z}(z')} - \frac{i}{\overleftrightarrow{Y}(z')}, \tag{20a}$$

$$V(z'-, z') = \frac{v\overleftarrow{Z}(z')}{\overleftrightarrow{Z}(z')} - \frac{i}{\overleftrightarrow{Y}(z')}, \tag{20b}$$

$$I(z'+, z') = \frac{-v}{\overleftrightarrow{Z}(z')} - \frac{i\overrightarrow{Y}(z')}{\overleftrightarrow{Y}(z')}, \tag{20c}$$

$$I(z'-, z') = \frac{-v}{\overleftrightarrow{Z}(z')} + \frac{i\overleftarrow{Y}(z')}{\overleftrightarrow{Y}(z')}, \tag{20d}$$

where

$$\overleftrightarrow{Z}(z') = \overrightarrow{Z}(z') + \overleftarrow{Z}(z'), \qquad \overleftrightarrow{Y}(z') = \overrightarrow{Y}(z') + \overleftarrow{Y}(z'). \tag{20e}$$

The determination of V and I at any point z on a transmission line with constants κ and Z follows from corresponding values at z' via either Eqs. (5) or (9). Thus, from Eqs. (5), (11b), and (20), one obtains in the traveling-wave

†In view of the choice of positive directions for voltage and current noted in Fig. 2.4.1, $Z' \equiv \overrightarrow{Z}'$ in Eq. (10) represents a normalized impedance looking to the right. To apply Eq. (10) to an impedance $\overleftarrow{Z}'$ looking to the left, a minus sign must be placed before every Z' (i.e., $Z' \rightarrow -\overleftarrow{Z}'$).

description for $z > z'$,

$$V(z, z') = -\frac{1}{2}\left[v\frac{\vec{Z}(z') + Z}{\overleftrightarrow{Z}(z')} + Zi\frac{\vec{Y}(z') + Y}{\overleftrightarrow{Y}(z')}\right][e^{-j\kappa(z-z')} + \vec{\Gamma}(z')e^{+j\kappa(z-z')}], \tag{21a}$$

$$I(z, z') = -\frac{1}{2}\left[i\frac{\vec{Y}(z') + Y}{\overleftrightarrow{Y}(z')} + Yv\frac{\vec{Z}(z') + Z}{\overleftrightarrow{Z}(z')}\right][e^{-j\kappa(z-z')} - \vec{\Gamma}(z')e^{+j\kappa(z-z')}], \tag{21b}$$

while for $z < z'$,†

$$V(z, z') = -\frac{1}{2}\left[-v\frac{\overleftarrow{Z}(z') + Z}{\overleftrightarrow{Z}(z')} + Zi\frac{\overleftarrow{Y}(z') + Y}{\overleftrightarrow{Y}(z')}\right][e^{+j\kappa(z-z')} + \overleftarrow{\Gamma}(z')e^{-j\kappa(z-z')}], \tag{21c}$$

$$I(z, z') = -\frac{1}{2}\left[-i\frac{\overleftarrow{Y}(z') + Y}{\overleftrightarrow{Y}(z')} + Yv\frac{\overleftarrow{Z}(z') + Z}{\overleftrightarrow{Z}(z')}\right][e^{+j\kappa(z-z')} - \overleftarrow{\Gamma}(z')e^{-j\kappa(z-z')}]. \tag{21d}$$

The reflection coefficients $\vec{\Gamma}$ and $\overleftarrow{\Gamma}$ looking to the right and left, respectively, are obtained in terms of the corresponding impedances $\vec{Z}'$ and $\overleftarrow{Z}'$ as in Eq. (12).

In the standing-wave description, one has, from Eqs. (9) and (20) for $z > z'$,

$$V(z, z') = -\left[\frac{v\vec{Z}(z')}{\overleftrightarrow{Z}(z')} + \frac{i}{\overleftrightarrow{Y}(z')}\right][\cos\kappa(z - z') - j\vec{Y}'(z')\sin\kappa(z - z')], \tag{22a}$$

$$I(z, z') = -\left[\frac{i\vec{Y}(z')}{\overleftrightarrow{Y}(z')} + \frac{v}{\overleftrightarrow{Z}(z')}\right][\cos\kappa(z - z') - j\vec{Z}'(z')\sin\kappa(z - z')], \tag{22b}$$

and, for $z < z'$,

$$V(z, z') = -\left[-\frac{v\overleftarrow{Z}(z')}{\overleftrightarrow{Z}(z')} + \frac{i}{\overleftrightarrow{Y}(z')}\right][\cos\kappa(z - z') + j\overleftarrow{Y}'(z')\sin\kappa(z - z')], \tag{22c}$$

†The traveling-wave description for $z < z'$ has the form (incident wave in the $-z$ direction)

$$V(z) = \overleftarrow{V}_{\text{inc}}(z')[e^{+j\kappa(z-z')} + \overleftarrow{\Gamma}(z')e^{-j\kappa(z-z')}],$$
$$-I(z) = Y\overleftarrow{V}_{\text{inc}}(z')[e^{+j\kappa(z-z')} - \overleftarrow{\Gamma}(z')e^{-j\kappa(z-z')}],$$

so

$$V_{\text{inc}}(z') = \tfrac{1}{2}[V(z') - ZI(z')].$$

$$I(z, z') = -\left[-\frac{i\overleftarrow{Y}(z')}{\overleftrightarrow{Y}(z')} + \frac{v}{\overleftrightarrow{Z}(z')}\right][\cos \kappa(z - z') + j\overleftarrow{Z}'(z') \sin \kappa(z - z')]. \tag{22d}$$

Equations (21) and (22) are general solutions of the point source-excited transmission-line problem. For special cases, appreciable simplification is achieved. The case of an infinite transmission line, for which $\overrightarrow{Z}(z') = \overleftarrow{Z}(z') = Z$ [i.e., $\overrightarrow{\Gamma}(z') = \overleftarrow{\Gamma}(z') = 0$; see Eq. (12)] has already been discussed [Eqs. (19)]. If the transmission line is terminated on the right but extends undisturbed to infinity on the left, then $\overleftarrow{Z}(z') = Z$, and $\overleftarrow{\Gamma}(z') = 0$. One finds for the traveling-wave description, particularly appropriate in this case:

$$\left.\begin{aligned} V(z, z') &= -\tfrac{1}{2}(v + Zi)[e^{-j\kappa(z-z')} + \overrightarrow{\Gamma}(z')e^{+j\kappa(z-z')}] \qquad (23a)\\ I(z, z') &= -\tfrac{1}{2}(i + Yv)[e^{-j\kappa(z-z')} - \overrightarrow{\Gamma}(z')e^{+j\kappa(z-z')}] \qquad (23b)\end{aligned}\right\} z > z';$$

$$\left.\begin{aligned} V(z, z') &= -\left[-v\frac{Z}{\overrightarrow{Z}(z') + Z} + i\frac{1}{\overrightarrow{Y}(z') + Y}\right]e^{+j\kappa(z-z')} \qquad (23c)\\ I(z, z') &= -\left[-i\frac{Y}{\overrightarrow{Y}(z') + Y} + v\frac{1}{\overrightarrow{Z}(z') + Z}\right]e^{+j\kappa(z-z')} \qquad (23d)\end{aligned}\right\} z < z'.$$

Additional special terminations for this case are

1. Short circuit at z_2: $\Gamma(z_2) = -1, Z(z_2) = 0$,

whence, from Eq. (10),

$$\overrightarrow{Z}'(z') = j \tan \kappa(z_2 - z'), \tag{24a}$$

and

2. Open circuit at z_2: $\Gamma(z_2) = +1, Z(z_2) = \infty$,

whence

$$\overrightarrow{Z}'(z') = -j \cot \kappa(z_2 - z'). \tag{24b}$$

The total voltage $V(z)$ and current $I(z)$ excited by a distributed source $v(z)$ and $i(z)$ are obtained from $V(z, z')$ and $I(z, z')$ by integration as in Eq. (14).

2.4d Green's Functions for Transmission-Line Equations

The point-source excitation discussed above can be simplified on introduction of unit amplitude generators of voltage and current to represent the excitations v and i employed in Figs. 2.4.2–2.4.5. The transmission-line voltages and currents excited by these point sources constitute the transmission-line (modal) Green's functions $Z(z, z')$, $Y(z, z')$, $T^I(z, z')$, and $T^V(z, z')$ introduced in Sec. 2.3c and defined in Eqs. (2.3.10). We shall evaluate the modal Green's functions by two alternative procedures, the first of which is based on the network representation of the transmission-line problem, while the second is inferred directly from the differential equations (2.3.10).

The network procedure is illustrated in the following calculation of $Y(z, z')$. Since $Y(z, z')$ is defined as the current at a point z on a transmission line excited at z' by a series voltage source of amplitude $v = -1$ (see Sec. 2.3c), the pertinent network problem is shown in Fig. 2.4.6. The general solution for the voltage and current on a point-source-excited and arbitrarily terminated

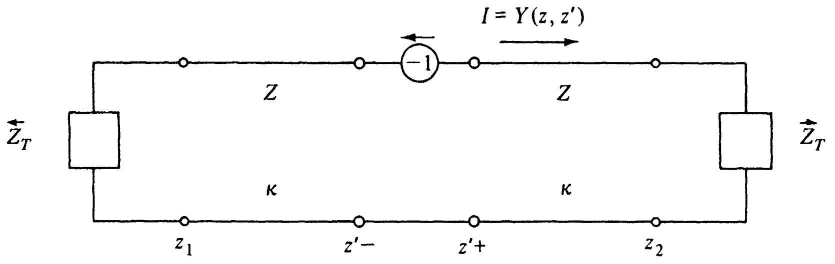


FIG. 2.4.6 Network problem for the determination of $Y(z, z')$.

transmission line of propagation constant κ and characteristic impedance Z has been given in Eqs. (21) and (22), and is readily specialized to the determination of $Y(z, z')$. In particular, the standing-wave representation yields

$$Y(z, z') = \begin{cases} Y(z', z')\,[\cos\kappa(z - z') - j\overrightarrow{Z}'(z')\sin\kappa(z - z')], & z > z', \quad (25a) \\ Y(z', z')\,[\cos\kappa(z - z') + j\overleftarrow{Z}'(z')\sin\kappa(z - z')], & z < z', \quad (25b) \end{cases}$$

where

$$Y(z', z') = \frac{1}{\overleftrightarrow{Z}(z')} = \frac{1}{\overrightarrow{Z}(z') + \overleftarrow{Z}(z')}. \qquad (25c)$$

It is to be noted that the evaluation of $Y(z', z')$ is source independent and depends only on the impedances seen to the right and left of the point z'.

It is desirable on occasion to evaluate $\overleftrightarrow{Z}$ not at the source point z' but at some other point z_0 at which its determination is simpler. The relation between $\overleftrightarrow{Z}(z')$ and $\overleftrightarrow{Z}(z_0)$ may be inferred from Eq. (10) (see also the footnote to Fig. 2.4.5), which yields

$$\frac{1}{\overleftrightarrow{Z}(z')} = \frac{[\cos\kappa(z' - z_0) + j\overleftarrow{Z}'(z_0)\sin\kappa(z' - z_0)]}{\overleftrightarrow{Z}(z_0)}$$
$$\times\ [\cos\kappa(z' - z_0) - j\overrightarrow{Z}'(z_0)\sin\kappa(z' - z_0)]. \qquad (26)$$

For $z > z'$, the term inside the brackets in Eq. (25a) can be written as $I(z)/I(z')$ [see Eq. (9b)], whence the second term inside the brackets in Eq. (26) is written as $I(z')/I(z_0)$. Upon substituting Eq. (26) for $Y(z', z')$ into Eq. (25a) and interpreting the resulting ratio $I(z)/I(z_0)$, one obtains the desired reformulation for $z > z'$:

$$Y(z, z') = \frac{[\cos \kappa(z' - z_0) + j\overleftarrow{Z}'(z_0) \sin \kappa(z' - z_0)]}{\overleftrightarrow{Z}(z_0)}$$

$$\times [\cos \kappa(z - z_0) - j\overrightarrow{Z}'(z_0) \sin \kappa(z - z_0)]. \tag{27a}$$

In view of the reciprocity relation $Y(z, z') = Y(z', z)$ [Eq. (2.3.15)], the corresponding expression valid for $z < z'$ follows as

$$Y(z, z') = \frac{[\cos \kappa(z - z_0) + j\overleftarrow{Z}'(z_0) \sin \kappa(z - z_0)]}{\overleftrightarrow{Z}(z_0)}$$

$$\times [\cos \kappa(z' - z_0) - j\overrightarrow{Z}'(z_0) \sin \kappa(z' - z_0)]. \tag{27b}$$

Equations (27a) and (27b), can be subsumed into a single expression valid for all z as

$$Y(z, z') = \frac{[\cos \kappa(z_< - z_0) + j\overleftarrow{Z}'(z_0) \sin \kappa(z_< - z_0)]}{\overleftrightarrow{Z}(z_0)}$$

$$\times [\cos \kappa(z_> - z_0) - j\overrightarrow{Z}'(z_0) \sin \kappa(z_> - z_0)], \tag{28a}$$

where $z_<$ stands for z when $z < z'$, and for z' when $z > z'$; the converse holds for $z_>$.

By duality, the Green's function $Z(z, z')$ is obtained as

$$Z(z, z') = \frac{[\cos \kappa(z_< - z_0) + j\overleftarrow{Y}'(z_0) \sin \kappa(z_< - z_0)]}{\overleftrightarrow{Y}(z_0)}$$

$$\times [\cos \kappa(z_> - z_0) - j\overrightarrow{Y}'(z_0) \sin \kappa(z_> - z_0)], \tag{28b}$$

and by virtue of its definition as the voltage excited on a transmission line by a point current source of amplitude $i = -1$, evidently represents the voltage solution for the network problem shown in Fig. 2.4.7. Since

$$\cos \kappa(z - z_0) \mp j\overset{\rightleftarrows}{Z}'(z_0) \sin \kappa(z - z_0)$$

$$= \tfrac{1}{2} [1 + \overset{\rightleftarrows}{Z}'(z_0)]\{e^{\mp j\kappa(z-z_0)} - \overset{\rightleftarrows}{\Gamma}(z_0)\, e^{\pm j\kappa(z-z_0)}\}, \tag{29a}$$

where

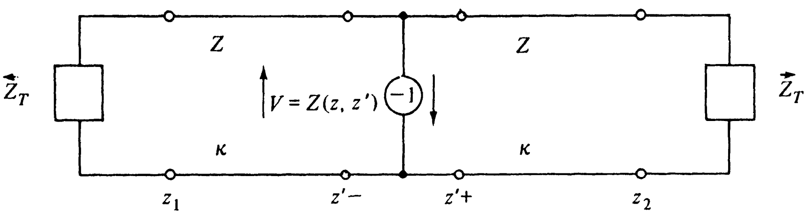


FIG. 2.4.7 Network problem for the determination of $Z(z, z')$.

$$\overset{\rightleftarrows}{\Gamma}(z_0) = \frac{\overset{\rightleftarrows}{Z}'(z_0) - 1}{\overset{\rightleftarrows}{Z}'(z_0) + 1}, \tag{29b}$$

one may write Eq. (28a) in the alternative traveling-wave form,

$$\begin{aligned} Y(z, z') = \frac{1}{\overset{\leftrightarrows}{Z}(z_0)} & \frac{e^{+j\kappa(z_< - z_0)} - \overleftarrow{\Gamma}(z_0)\, e^{-j\kappa(z_< - z_0)}}{1 - \overleftarrow{\Gamma}(z_0)} \\ & \times \frac{e^{-j\kappa(z_> - z_0)} - \overrightarrow{\Gamma}(z_0)\, e^{+j\kappa(z_> - z_0)}}{1 - \overrightarrow{\Gamma}(z_0)}. \end{aligned} \tag{29c}$$

Similarly, one has, for Eq. (28b),

$$\begin{aligned} Z(z, z') = \frac{1}{\overset{\leftrightarrows}{Y}(z_0)} & \frac{e^{+j\kappa(z_< - z_0)} + \overleftarrow{\Gamma}(z_0)\, e^{-j\kappa(z_< - z_0)}}{1 + \overleftarrow{\Gamma}(z_0)} \\ & \times \frac{e^{-j\kappa(z_> - z_0)} + \overrightarrow{\Gamma}(z_0)\, e^{+j\kappa(z_> - z_0)}}{1 + \overrightarrow{\Gamma}(z_0)}. \end{aligned} \tag{29d}$$

Alternative to the network derivation of the admittance and impedance Green's functions is an evaluation based on their defining differential equations (2.3.10) and on the source-free wave solutions thereof, each of the latter satisfying the boundary condition to the left or right of the source point. By eliminating T^V from Eqs. (2.3.10a) and (2.3.10b), one obtains a second-order differential equation for $Y(z, z')$ in the form

$$\left(\frac{d^2}{dz^2} + \kappa^2\right) Y(z, z') = -j\kappa Y \delta(z - z'), \tag{30}$$

valid for constant κ and Z. The boundary conditions at the endpoints z_1 and z_2 of the z domain distinguish a unique solution of Eq. (30), and are stated in terms of the terminating "logarithmic derivatives" or normalized impedances (i.e., ratios of voltage to current):

$$\left[\frac{(d/dz)\, Y(z, z')}{(-j\kappa Y)\, Y(z, z')}\right]_{z_1, z_2}, \tag{30a}$$

as seen from Eq. (2.3.10).

The construction of the Green's function of a second-order differential equation can be carried out by a well-established mathematical formalism (see also Sec. 3.3) which is conveniently and significantly viewed from a network standpoint. For all points $z \neq z'$, $Y(z, z')$ must satisfy the homogeneous equation (30) (i.e., it is a source-free wave solution of the transmission-line equations). Since $Y(z, z')$ is actually a (normalized) current on a transmission line, then for $z > z'$, a homogeneous solution that satisfies the boundary conditions at z_2 can be written from Eq. (9b) as

$$\overrightarrow{I}(z) = \cos \kappa(z - z_0) - j\overrightarrow{Z}'(z_0) \sin \kappa(z - z_0), \tag{31a}$$

where an arbitrary point z_0 has been introduced as a reference, and a normalization $\vec{I}(z_0) = 1$ has been adopted. The homogeneous network problem for which $\vec{I}(z)$ represents a solution is shown in Fig. 2.4.8; it should be stressed that the above wave solution makes no reference to the presence of a source.

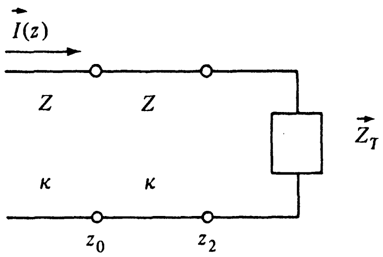


FIG. 2.4.8 Homogeneous network problem for $\vec{I}(z)$.

In an entirely analogous manner, one obtains for $z < z'$ a homogeneous solution which satisfies the boundary conditions at z_1 as

$$\overleftarrow{I}(z) = \cos\kappa(z - z_0) + j\overleftarrow{Z}'(z_0)\sin\kappa(z - z_0). \tag{31b}$$

Again, $\overleftarrow{I}(z_0) = 1$. In the notation of Figs. 2.4.6 and 2.4.8, the homogeneous network problem is that shown in Fig. 2.4.9. The normalized impedances (logarithmic derivatives) at z_0 are related bilinearly to those specified at the endpoints z_1 and z_2 [see Eq. (10), with footnote on p. 208].

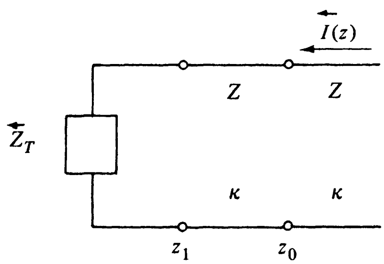


FIG. 2.4.9 Homogeneous network problem for $\overleftarrow{I}(z)$.

To construct the Green's function, one imposes the following requirements on $Y(z, z')$:

1. It must have a z dependence of the form $\vec{I}(z)$ for $z > z'$, and $\overleftarrow{I}(z)$ for $z < z'$.

2. From the symmetry condition $Y(z, z') = Y(z', z)$ [Eq. (2.3.15)], $Y(z, z')$ depends symmetrically on $\vec{I}$ and $\overleftarrow{I}$.

3. From the defining differential equation (30),† or the network picture

†From Eq. (30) one infers by integration over an infinitesimal interval centered at z' that $(d/dz)Y(z, z')$ has a jump discontinuity, with $Y(z, z')$ bounded and continuous, at $z = z'$.

(Fig. 2.4.6), it is evident that $Y(z, z')$ is continuous at $z = z'$ and has a jump in its derivative (i.e., the voltage) at $z = z'$.

From the above it follows that $Y(z, z')$ must be of the form

$$Y(z, z') = \begin{cases} A\,\overrightarrow{I}(z)\overleftarrow{I}(z'), & z > z', \qquad (32a) \\ A\,\overrightarrow{I}(z')\overleftarrow{I}(z), & z < z', \qquad (32b) \end{cases}$$

or, in the notation of Eq. (28),

$$Y(z, z') = A\,\overleftarrow{I}(z_<)\overrightarrow{I}(z_>). \tag{33}$$

A is a constant *independent* of z (or z') and must have the correct magnitude to satisfy the discontinuity requirement on $dY(z, z')/dz$ at z'. Since A is not a function of z (or z'), we may choose the source point at $z' = z_0$ so as to permit a particularly simple evaluation of A. Since $\overleftarrow{I}(z_0) = \overrightarrow{I}(z_0) = 1$, a simple network evaluation of the current at $z = z_0$ yields (see Fig. 2.4.6, with $z' = z_0$)

$$Y(z_0, z_0) = \frac{1}{\overleftrightarrow{Z}(z_0)} = A, \tag{34}$$

and the Green's function $Y(z, z')$ is given by

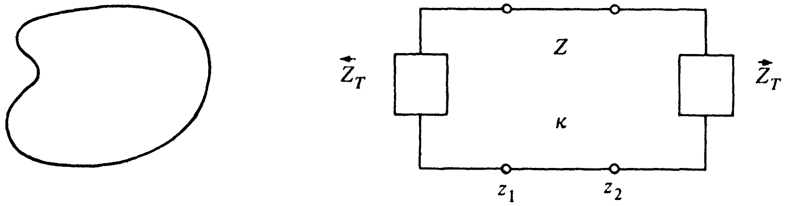
$$Y(z, z') = \frac{\overleftarrow{I}(z_<, z_0)\overrightarrow{I}(z_>, z_0)}{\overleftrightarrow{Z}(z_0)}, \tag{35}$$

where the dependence on z_0 has been explicitly exhibited in $\overleftarrow{I}$ and $\overrightarrow{I}$. Equation (35) is identical with Eq. (28). It is to be noted that the evaluation of A in Eq. (34) by a simple network scheme is equivalent to calculating the reciprocal of the Wronskian of the two solutions $\overleftarrow{I}$ and $\overrightarrow{I}$. (See Sec. 3.3b.)

2.4e Resonance Properties of Terminated Transmission Lines

In the preceding sections we have evaluated the response of a terminated transmission line to prescribed excitation. This involves the determination of a modal Green's function $Y(z, z')$ or $Z(z, z')$ from which the desired response can be inferred. The form of these Green's functions makes evident the interesting possibility of obtaining a response even in the absence of excitation. Such a circumstance characterizes a resonance and is achieved whenever $Y(z, z')$ or $Z(z, z')$ becomes infinite, either for some mode wavenumber k_t with fixed k, or for some k with k_t fixed.

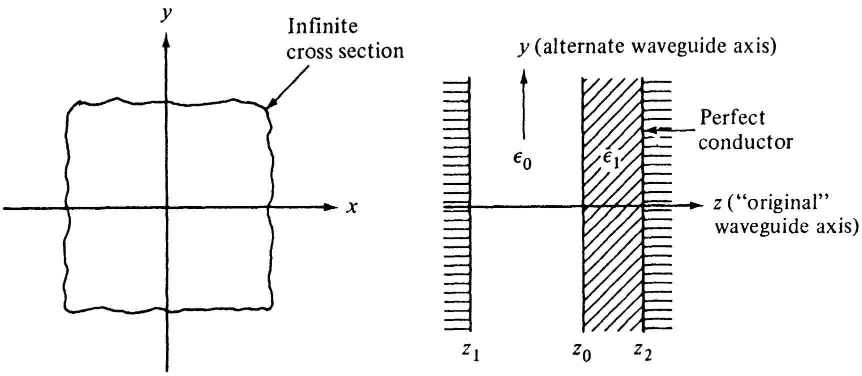
Since a terminated transmission line is representative of a cavity or of a transversely viewed waveguide, resonant situations provide information about the complete sets of modes in cavities or in waveguides. In the former case we consider a waveguide of finite cross section so terminated as to constitute a (closed) cavity resonator (see Fig. 2.4.10). In the latter, the original waveguide must be of infinite extent along at least one of its cross-sectional directions and terminated in the longitudinal direction; it may then be viewed alternatively as an infinitely long waveguide with axis along the infinitely extended cross-sec-



(a) Waveguide cross section (k_t prescribed)

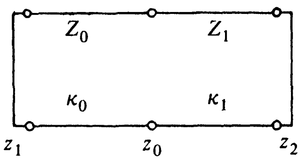
(b) Terminated transmission line

FIG. 2.4.10 Cavity problem.



(a) View in xy plane

(b) Longitudinal view



(c) Network picture

FIG. 2.4.11 "Transverse resonance" problem.

tional direction. The special case of a parallel-plate waveguide containing a stratified dielectric is depicted in Fig. 2.4.11. In each event the resonances characterize the possible modes either of oscillation or of propagation that are capable of being excited in such a structure. More generally and mathematically, it can be shown that the singularities (poles or branch points and associated branch cuts) of modal Green's functions distinguish the complete spectrum of possible modes in cavities or waveguides, and the residues at the poles and/or suitable branch-cut contributions, characterize the corresponding mode functions. (See Sec. 3.3a.)

By inspection of Eqs. (27)–(29), it is apparent that the modal Green's functions become infinite at the zeros of the total impedance $\overleftrightarrow{Z}(z_0)$, or of the total admittance $\overleftrightarrow{Y}(z_0)$. One observes that the impedances $\overleftarrow{Z}(z_0)$ and $\overrightarrow{Z}(z_0)$ seen looking to the left and right, respectively, of the point z_0 are functions of the propagation constant $\kappa = (k^2 - k_t^2)^{1/2}$ and hence depend implicitly on the free-space wavenumber $k = \omega/c$ (c = velocity of light in the waveguide medium) and mode wavenumber k_t. The values $\kappa = \kappa_r$ for which

$$\overleftrightarrow{Z}(z_0) = 0 \qquad \text{or} \qquad \overleftarrow{Z}(z_0, \kappa) = -\overrightarrow{Z}(z_0, \kappa) \tag{36}$$

are independent of the choice of z_0 and distinguish the resonances of the transmission-line system (i.e., the wavenumbers κ_r for which the line current or voltage may be finite even with vanishing excitation). One distinguishes between real and complex values of κ_r; the former obtain in reactive (nondissipative) systems, the latter in lossy systems.

The resonant frequencies of modes in a cavity, formed by a uniform guide terminated at both ends, may be determined from the equivalent modal network. For a specified cross-sectional mode wavenumber k_t, the values of κ_r at which $\overleftrightarrow{Z}(z_0)$ vanishes determine the resonant angular frequencies ω_r of the given mode k_t in the cavity. For a non-dissipative cavity, ω_r is real and hence the resonance is undamped (in time). For a dissipative or loaded cavity, ω_r is complex and the resonance is damped, the ratio of the real and imaginary parts of κ_r being indicative of the Q (quality factor) of the particular resonance.

The modes in an infinite uniform waveguide may be inferred from the resonances of an appropriate "transverse network" if the guide cross section has the appropriate symmetry. In this application, one prescribes the frequency ω and determines mode wavenumbers k_t corresponding to the resonant values κ_r of the terminated network representative of the transversely viewed waveguide under consideration; this is the so-called "transverse resonance" procedure for mode evaluations. The k_t so determined are the propagation constants of the modes capable of propagating along the axis of the waveguide in question. The relation between the problem of resonances on a terminated transmission-line network and the eigenvalue problem for modes in the waveguide cross section, of which the network is representative, are treated in more detail in the Sec. 3.3a.

A resonant situation provides information not only about the resonant frequencies of a network but also about the corresponding source-free voltage and current distribution that can exist at these frequencies. Although the distributions $\overrightarrow{I}(z)$ and $\overleftarrow{I}(z)$ in Eqs. (31a) and (31b) arise from sources to the left or right, respectively, of the network terminations, it is evident from Eq. (36) that at resonance, the wave solutions $\overrightarrow{I}(z)$ and $\overleftarrow{I}(z)$ are identical and satisfy the boundary conditions at both terminations of the network. These source-free resonant solutions characterize the eigensolutions or mode functions of the eigenvalue problems mentioned above (see Sec. 3.3b).

2.5 DERIVATION OF TRANSMISSION-LINE EQUATIONS IN SPHERICAL REGIONS

To describe fields in the spherical waveguide region of Fig. 2.1.1(b), the vector field equations (2.2.1) are conveniently expressed in spherical polar coordinates (r, θ, ϕ). The electric and magnetic field components $\mathbf{E}_t$ and $\mathbf{H}_t$ transverse to the radial direction r are independent field quantities from which the longitudinal components E_r and H_r may be determined. The field equations may therefore be reduced to equations for just the transverse fields. On introduction of a complete set of transverse vector eigenfunctions, one may derive modal representations for $\mathbf{E}_t$ and $\mathbf{H}_t$, with r-dependent mode amplitudes that satisfy spherical transmission-line equations. The procedure is directly analogous to that in Sec. 2.2 for uniform waveguide regions with the direction z replaced by r.

2.5a The Transverse Field Equations

In the spherical coordinate system of Fig. 2.5.1, the location of a point P is specified by the radial distance r from the origin; the angle θ measures the inclination of the radius vector $\mathbf{r}$ with respect to the z axis, and the azimuthal

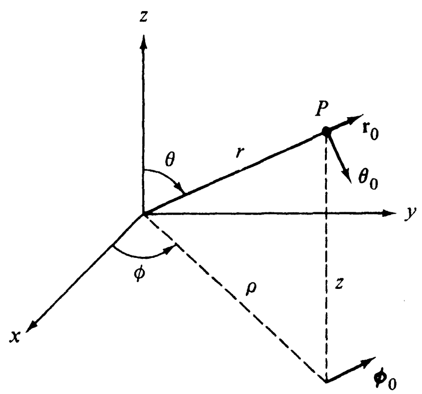


FIG. 2.5.1 Spherical coordinates.

angle ϕ locates the xy plane projection ρ of r with respect to the x axis. $\mathbf{r}_0$, $\boldsymbol{\theta}_0$, and $\boldsymbol{\phi}_0$ are defined as unit vectors along the directions of increasing r, θ, and ϕ, respectively, whence $\mathbf{r}_0 \times \boldsymbol{\theta}_0 = \boldsymbol{\phi}_0$. It will be convenient to employ two alternative representations of the operator ∇ in a spherical coordinate system[7]:

$$\nabla = \mathbf{r}_0 \frac{\partial}{\partial r} + {}_t\nabla = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \mathbf{r}_0 + \nabla_t, \tag{1}$$

where

$${}_t\nabla = \boldsymbol{\theta}_0 \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\phi}_0 \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi},$$

$$\nabla_t = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\boldsymbol{\theta}_0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\boldsymbol{\phi}_0. \tag{1a}$$

The first and second formulations are convenient for vector operations to the left and right, respectively, with their difference arising from the non-vanishing angular derivatives of the unit vectors in a curvilinear system.

To reduce the Maxwell field equations (2.2.1) to their transverse to **r** form, we take scalar and vector products with the radial unit vector $\mathbf{r}_0$ and find, after manipulations analogous to those leading to Eqs. (2.2.4),[8]

$$-\frac{\partial}{\partial r}(r\mathbf{E}_t) = j\omega\mu(r)\left[\mathbf{1} + \frac{1}{k^2(r)}{}_t\nabla\nabla_t\right] \cdot (\mathbf{H}_t \times \mathbf{r}) + \mathbf{M}_{te} \times \mathbf{r}, \tag{2a}$$

$$-\frac{\partial}{\partial r}(r\mathbf{H}_t) = j\omega\epsilon(r)\left[\mathbf{1} + \frac{1}{k^2(r)}{}_t\nabla\nabla_t\right] \cdot (\mathbf{r} \times \mathbf{E}_t) + \mathbf{r} \times \mathbf{J}_{te}, \tag{2b}$$

where the permittivity $\epsilon(r)$ and permeability $\mu(r)$ are permitted to be functions of r but not of θ or ϕ, $k^2 = \omega^2\mu\epsilon$, $\mathbf{r} = \mathbf{r}_0 r$, and $\mathbf{J}_{te}$, $\mathbf{M}_{te}$ are the equivalent transverse source distributions:

$$\mathbf{J}_{te} = \mathbf{J}_t - \frac{r}{j\omega\mu(r)}\mathbf{r} \times {}_t\nabla M_r, \qquad \mathbf{M}_{te} = \mathbf{M}_t + \frac{r}{j\omega\epsilon(r)}\mathbf{r} \times {}_t\nabla J_r. \tag{2c}$$

A subscript t distinguishes a vector transverse to r. The longitudinal (radial) field components are derivable from the transverse fields and the longitudinal source currents via

$$E_r = \frac{1}{j\omega\epsilon(r)}[\nabla_t \cdot (\mathbf{H}_t \times \mathbf{r}_0) - J_r], \qquad H_r = \frac{1}{j\omega\mu(r)}[\nabla_t \cdot (\mathbf{r}_0 \times \mathbf{E}_t) - M_r]. \tag{3}$$

If a perfectly conducting transverse boundary s described by the equation $f(\theta, \phi) = 0$ is present, the tangential electric field must vanish so that

$$\boldsymbol{\nu} \times \mathbf{E}_t = 0, \quad \nabla_t \cdot (\mathbf{H}_t \times \mathbf{r}_0) = 0 \qquad \text{on } s, \tag{4}$$

where $\boldsymbol{\nu}$ is a unit vector normal to s. Equations (2) and (3) are completely equivalent to the original field equations (2.2.1).

2.5b Modal Representation of the Fields and Their Sources

As in the uniform waveguide case in Sec. 2.2b, solutions of the source-free transverse field equations, individually satisfying the transverse boundary conditions, constitute a complete set of functions that may be employed to represent an arbitrary field. In view of the assumed variability of ϵ and μ with r only, a typical source-free solution may be written in the separable form

$$r\mathbf{E}_{ti} = V_i(r)\mathbf{e}_i(\boldsymbol{\rho}), \qquad r\mathbf{H}_{ti} = I_i(r)\mathbf{h}_i(\boldsymbol{\rho}), \tag{5}$$

where $\boldsymbol{\rho} = (\theta, \phi)$. Substitution into Eqs. (2) (with $\mathbf{J} = \mathbf{M} = 0$) yields separate equations for the radially and transversely dependent factors. The transverse vector dependence may be eliminated from these equations in a simple manner if: (1) operation on a vector by ${}_t\nabla\nabla_t \cdot$ produces a result proportional to the vector, and (2) $\mathbf{h}_i = \mathbf{r}_0 \times \mathbf{e}_i$. As in Eqs. (2.2.10), these two requirements may be made consistent by separately imposing the auxiliary conditions $\nabla_t \cdot (\mathbf{r}_0 \times \mathbf{e}_i) = 0$ or $\nabla_t \cdot (\mathbf{h}_i \times \mathbf{r}_0) = 0$, thereby splitting the mode set into two parts. It is then implied from Eq. (3) that $H_{ri} = 0$ and $E_{ri} = 0$, respectively, thus identifying the two sets as E and H modes with respect to r. This argument is again related to the theorem that permits decomposition of any vector into a curlless and a divergenceless part.[4] The curlless part may subsequently be expressed as the gradient of a scalar function, whereas the divergenceless part may be written as the curl of a longitudinal vector [see Eqs. (2.6.1) and Sec. 2.3a].

The mode functions $\mathbf{e}_i$ and $\mathbf{h}_i = \mathbf{r}_0 \times \mathbf{e}_i$ are chosen to satisfy the following eigenvalue problems and subsidiary conditions in the transverse cross section S:[9]

$$r^2{}_t\nabla\nabla_t \cdot \mathbf{e}_i' = -k_{ti}'^2 \mathbf{e}_i', \qquad \nabla_t \cdot (\mathbf{r}_0 \times \mathbf{e}_i') = 0, \tag{6a}$$

$$r^2{}_t\nabla\nabla_t \cdot \mathbf{h}_i'' = -k_{ti}''^2 \mathbf{h}_i'', \qquad \nabla_t \cdot (\mathbf{h}_i'' \times \mathbf{r}_0) = 0, \tag{6b}$$

where r^2 in $(r^2{}_t\nabla\nabla_t)$ has been included to make the operator dependent on the transverse coordinates only, and the single and double primes denote E and H modes, respectively. The boundary conditions on both mode sets are, from Eq. (4),

$$\boldsymbol{\nu} \times \mathbf{e}_i = 0 = \nabla_t \cdot (\mathbf{h}_i \times \mathbf{r}_0) \qquad \text{on } s. \tag{6c}$$

On use of Eq. (2.2.11a), one may verify that the vector eigenfunctions satisfy orthogonality relations analogous to those in Eq. (2.2.11b):

$$\iint_S \mathbf{e}_i' \cdot \mathbf{e}_j'^* \, d\Omega = \delta_{ij} = \iint_S \mathbf{e}_i'' \cdot \mathbf{e}_j''^* \, d\Omega; \qquad \iint_S \mathbf{e}_i' \cdot \mathbf{e}_i''^* \, d\Omega = 0, \tag{7}$$

and similarly for the $\mathbf{h}_i$, with $d\Omega = \sin\theta \, d\theta \, d\phi = dS/r^2$ denoting the angular surface element. The eigenfunctions are normalized by setting the integral equal to unity when $i = j$.

The complete set of vector eigenfunctions defined in Eqs. (6) may now be utilized to represent the electromagnetic fields and the excitation functions in Eqs. (2a) and (2b):

$$r\mathbf{E}_t(\mathbf{r}) = \sum_i V_i'(r)\mathbf{e}_i'(\boldsymbol{\rho}) + \sum_i V_i''(r)\mathbf{e}_i''(\boldsymbol{\rho}), \tag{8a}$$

$$r\mathbf{H}_t(\mathbf{r}) = \sum_i I_i'(r)\mathbf{h}_i'(\boldsymbol{\rho}) + \sum_i I_i''(r)\mathbf{h}_i''(\boldsymbol{\rho}), \qquad \mathbf{h}_i = \mathbf{r}_0 \times \mathbf{e}_i, \tag{8b}$$

$$r\mathbf{J}_{te}(\mathbf{r}) = \sum_i i_i'(r)\mathbf{e}_i'(\boldsymbol{\rho}) + \sum_i i_i''(r)\mathbf{e}_i''(\boldsymbol{\rho}), \tag{8c}$$

$$r\mathbf{M}_{te}(\mathbf{r}) = \sum_i v_i'(r)\mathbf{h}_i'(\boldsymbol{\rho}) + \sum_i v_i''(r)\mathbf{h}_i''(\boldsymbol{\rho}). \tag{8d}$$

Correspondingly, from Eq. (3),

$$rE_r + \frac{rJ_r}{j\omega\epsilon} = \sum_i Z_i' I_i' e_{ri}', \qquad rH_r + \frac{rM_r}{j\omega\mu} = \sum_i Y_i'' V_i'' h_{ri}'', \tag{8e}$$

with $Z_i e_{ri}$ and $Y_i h_{ri}$ defined below in Eqs. (10c) and (10d). When these expansions are substituted into Eqs. (2a) and (2b), and use is made of Eqs. (6a) and (6b) together with the orthogonality relations (7), one obtains spherical transmission-line equations for the modal voltage and current amplitudes V_i and I_i:

$$-\frac{dV_i}{dr} = j\kappa_i Z_i I_i + v_i, \qquad -\frac{dI_i}{dr} = j\kappa_i Y_i V_i + i_i, \tag{9}$$

where κ_i, Z_i, and Y_i are the modal propagation constant, characteristic impedance, and characteristic admittance, respectively. For the E modes,

$$\kappa_i'(r) = \sqrt{k^2(r) - \frac{k_{ti}'^2}{r^2}}, \qquad Z_i'(r) = \frac{1}{Y_i'(r)} = \frac{\kappa_i'(r)}{\omega\epsilon(r)}; \tag{9a}$$

for the H modes,

$$\kappa_i''(r) = \sqrt{k^2(r) - \frac{k_{ti}''^2}{r^2}}, \qquad Z_i''(r) = \frac{1}{Y_i''(r)} = \frac{\omega\mu(r)}{\kappa_i''(r)}, \tag{9b}$$

with $k^2(r) = \omega^2\mu(r)\epsilon(r)$. One observes that even when the medium is homogeneous (ϵ, μ constant), the propagation constant and characteristic impedance are r dependent (i.e., the transmission lines are non-uniform). This is a consequence of the non-uniformity of the transverse sections, which are not identical but only similar, at different radial distances. The source terms v_i and i_i in Eqs. (9) are calculated from the known currents $\mathbf{J}$ and $\mathbf{M}$ on inversion of Eqs. (8c) and (8d):

$$\frac{1}{r}v_i(r) = \iint_S \mathbf{M}(\mathbf{r}) \cdot \mathbf{h}_i^*(\boldsymbol{\rho})\, d\Omega + Z_i^* \iint_S \mathbf{J}(\mathbf{r}) \cdot \mathbf{e}_{ri}^*\, d\Omega, \tag{10a}$$

$$\frac{1}{r}i_i(r) = \iint_S \mathbf{J}(\mathbf{r}) \cdot \mathbf{e}_i^*(\boldsymbol{\rho})\, d\Omega + Y_i^* \iint_S \mathbf{M}(\mathbf{r}) \cdot \mathbf{h}_{ri}^*\, d\Omega, \tag{10b}$$

where

$$Z_i'\mathbf{e}_{ri}' = \mathbf{r}_0 \frac{\nabla_t \cdot \mathbf{e}_i'}{j\omega\epsilon} = \mathbf{r}_0 \frac{k_{ti}'\Phi_i}{j\omega\epsilon r}, \qquad \mathbf{e}_{ri}'' \equiv 0, \tag{10c}$$

$$Y_i''\mathbf{h}_{ri}'' = \mathbf{r}_0 \frac{\nabla_t \cdot \mathbf{h}_i''}{j\omega\mu} = \mathbf{r}_0 \frac{k_{ti}''\psi_i}{j\omega\mu r}, \qquad \mathbf{h}_{ri}' \equiv 0. \tag{10d}$$

The scalar mode functions Φ_i and ψ_i are defined in Sec. 2.6a. Solutions of the transmission line equations for various radial domains are given in Sec. 2.7.

The average power $\bar{S}$ transferred across a spherical surface with radius r is given by

$$\bar{S} = \text{Re}\int_S \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{r}_0\, dS = \text{Re}\left[\sum_i V_i'(r)I_i'^*(r) + \sum_i V_i''(r)I_i''^*(r)\right], \tag{11a}$$

where the last expression follows from Eqs. (8a) and (8b) and the orthogonality conditions (7). The total power is therefore equal to the sum of the contributions carried by the individual modes. When the observation point is moved to infinity ($r \to \infty$), and all sources (real or induced) are assumed to be confined within the region $r < a$, the contributing modes have $kr \gg k_{ti}$, whence $\kappa_i \to k$ $Z_i(r) \to \zeta = \sqrt{\mu/\epsilon}$, and $V_i(r) \to \zeta I_i(r)$ [see Eqs. (9) and Sec. 2.7]. Since modes with $k_{ti} \gg ka$ have strongly damped amplitudes, the mode series contains only a finite number of terms in the effective range. The total radiated power may then be expressed in terms of the "far-field" voltages as follows:

$$\bar{S} = \frac{1}{\zeta}\sum_i |V_i'(r)|^2 + \frac{1}{\zeta}\sum_i |V_i''(r)|^2, \qquad r \to \infty, \tag{11b}$$

which formulation is useful for the calculation of such quantities as the scattering cross section of an obstacle, or the transmission cross section of an aperture.

2.6 SCALARIZATION AND MODAL REPRESENTATION OF DYADIC GREEN'S FUNCTIONS IN SPHERICAL REGIONS

As for uniform waveguide regions, the vector-field and eigenvalue problems in Sec. 2.5 may be reduced to scalar problems through use of E- and H-mode (Hertz) potentials. The scalarization procedure and the definition of scalar Green's functions $G'(\mathbf{r}, \mathbf{r}')$, $G''(\mathbf{r}, \mathbf{r}')$, $\mathscr{S}'(\mathbf{r}, \mathbf{r}')$, $\mathscr{S}''(\mathbf{r}, \mathbf{r}')$ equivalent to these potentials are summarized below, in analogy with the discussion in Sec. 2.3.

2.6a Mode Functions

Since from the second of Eqs. (2.5.6a) and (2.5.6b), the vector set $\{\mathbf{e}'_i\}$ is irrotational ($\nabla_t \times \mathbf{e}'_i = 0$) while the vector set $\{\mathbf{e}''_i\}$ is solenoidal ($\nabla_t \cdot \mathbf{e}''_i = 0$), the mode functions $\mathbf{e}'_i$ and $\mathbf{e}''_i$ can be represented as gradients and curls of scalar functions Φ_i and ψ_i as follows:

$$\mathbf{e}'_i = -\frac{r_t\nabla\Phi_i}{k'_{ti}} = \mathbf{h}'_i \times \mathbf{r}_0, \qquad \mathbf{h}''_i = -\frac{r_t\nabla\psi_i}{k''_{ti}} = \mathbf{r}_0 \times \mathbf{e}''_i. \tag{1}$$

Then from the first of Eqs. (2.5.6a) and (2.5.6b) there result the following scalar eigenvalue problems subject to boundary conditions deduced from Eq. (2.5.6c):

$$r^2\nabla_t \cdot {}_t\nabla\Phi_i + k'^2_{ti}\Phi_i = 0 \qquad \text{in } S,$$

$$r^2\nabla_t \cdot {}_t\nabla = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{2}$$

$$\left.\begin{aligned} \Phi_i &= 0, \qquad k'_{ti} \neq 0 \\ \frac{\partial\Phi_i}{\partial s} &= 0, \qquad k'_{ti} = 0\dagger \end{aligned}\right\} \quad \text{on } s \tag{2a}$$

and

$$r^2\nabla_t \cdot {}_t\nabla\psi_i + k''^2_{ti}\psi_i = 0 \qquad \text{in } S, \tag{3}$$

$$\frac{\partial\psi_i}{\partial\nu} = 0 \qquad \text{on } s. \tag{3a}$$

2.6b Fields in Source-Free, Homogeneous Regions

As for uniform regions (Sec. 2.3b), the definition and modal representation of the scalar radial Hertzian (or Debye) potentials $\Pi'(\mathbf{r})$ and $\Pi''(\mathbf{r})$ follows directly on substitution of Eq. (1) into Eqs. (2.5.8) with the result that at any source-free point $\mathbf{r}$,[10-12]

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}'(\mathbf{r}) + \mathbf{E}''(\mathbf{r}), \qquad \mathbf{H}(\mathbf{r}) = \mathbf{H}'(\mathbf{r}) + \mathbf{H}''(\mathbf{r}). \tag{4}$$

†This case corresponds to a TEM mode ($E_r = H_r = 0$), which is treated separately.

The E-mode constituents are given by

$$\mathbf{E}'(\mathbf{r}) = \nabla \times \nabla \times [\mathbf{r}\Pi'(\mathbf{r})], \qquad \mathbf{H}'(\mathbf{r}) = j\omega\epsilon\nabla \times [\mathbf{r}\Pi'(\mathbf{r})], \tag{4a}$$

and the H-mode constituents by

$$\mathbf{E}''(\mathbf{r}) = -j\omega\mu\nabla \times [\mathbf{r}\Pi''(\mathbf{r})], \qquad \mathbf{H}''(\mathbf{r}) = \nabla \times \nabla \times [\mathbf{r}\Pi''(\mathbf{r})]. \tag{4b}$$

ϵ and μ have been assumed constant, for simplicity. In reducing these expressions, the following relations are useful:

$$\nabla \times \mathbf{r}_0 A = -\mathbf{r}_0 \times {}_t\nabla A, \qquad \nabla \times \nabla \times \mathbf{r}_0 A = {}_t\nabla \frac{\partial A}{\partial r} - \mathbf{r}_0 \nabla_t \cdot {}_t\nabla A. \tag{5}$$

The scalar potential functions have the modal representation

$$\Pi'(\mathbf{r}) = \frac{1}{j\omega\epsilon r} \sum_i \frac{I_i'(r)\Phi_i(\boldsymbol{\rho})}{k_{ti}'}, \tag{6a}$$

$$\Pi''(\mathbf{r}) = \frac{1}{j\omega\mu r} \sum_i \frac{V_i''(r)\psi_i(\boldsymbol{\rho})}{k_{ti}''}, \tag{6b}$$

and satisfy at any source-free point the scalar wave equation

$$(\nabla^2 + k^2) \begin{matrix}\Pi'(\mathbf{r}) \\ \Pi''(\mathbf{r})\end{matrix} = 0,$$

$$\nabla^2 A = \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \nabla_t \cdot {}_t\nabla\right) A = \left(\frac{1}{r}\frac{\partial^2}{\partial r^2} + \frac{1}{r}\nabla_t \cdot {}_t\nabla\right)(rA), \tag{7}$$

where $\nabla_t \cdot {}_t\nabla$ is defined in Eq. (2). As for uniform waveguides, one may alternatively define Π' and Π'' directly by Eq. (7) subject to appropriate boundary conditions, and then derive the electromagnetic fields from Eqs. (4); this sequence, reversed herein, yields in the process the modal solutions in Eqs. (6a) and (6b).

The boundary conditions satisfied by the potentials Π' and Π'' are easily inferred from the modal representations (6a) and (6b). On a transverse boundary, the E mode potential behaves as Φ_i, and across a radial boundary, I_i' and V_i' must be continuous [see Eqs. (2.5.8a) and (2.5.8b)]. Thus, if the region is bounded transversely by a perfectly conducting surface s described by the equation $f(\theta, \phi) = 0$, and comprises homogeneous spherical layers along the radial direction, then from Eqs. (2a) and (2.5.9),

$$\begin{aligned} &\Pi'(\mathbf{r}) = 0 \qquad \text{on } s, \\ &\epsilon\Pi' \text{ and } \frac{\partial}{\partial r}(r\Pi') \qquad \text{continuous at } r = r_a, \end{aligned} \tag{8a}$$

where r_a locates an interface between two different homogeneous regions. Similarly, for the H-mode potential,

$$\begin{aligned} &\frac{\partial \Pi''(\mathbf{r})}{\partial \nu} = 0 \qquad \text{on } s, \\ &\mu\Pi'' \text{ and } \frac{\partial}{\partial r}(r\Pi'') \qquad \text{continuous at } r = r_a, \end{aligned} \tag{8b}$$

where ν denotes the normal to s. For a perfectly conducting radial boundary at $r = r_a$, the vanishing of V_i'' and dI_i'/dr implies that these conditions reduce to $\partial(r\Pi')/\partial r = 0$ and $\Pi'' = 0$, respectively.

2.6c Modal Representations of the Dyadic Green's Functions

The potentials in Eqs. (6a) and (6b) may be represented in terms of modal Green's functions $Z_i(r, r')$, $Y_i(r, r')$, $T_i^I(r, r')$, and $T_i^V(r, r')$, defined in Eqs. (2.3.10) as voltage or current responses to either a voltage or current point generator. The considerations in Sec. 2.3c hold for variable κ_i and Z_i so that the results apply directly to the present discussion, with z, z' replaced by r, r', respectively. By proceeding as in Sec. 2.3d, one may show that the potential functions Π' and Π'', corresponding to excitation by point current elements $\mathbf{M}(\mathbf{r}) = \mathbf{M}^0\delta(\mathbf{r} - \mathbf{r}')$ and $\mathbf{J}(\mathbf{r}) = \mathbf{J}^0\delta(\mathbf{r} - \mathbf{r}')$, may be expressed in terms of auxiliary functions $\mathscr{S}'(\mathbf{r}, \mathbf{r}')$ and $\mathscr{S}''(\mathbf{r}, \mathbf{r}')$ for $r \neq r'$ as

$$j\omega\epsilon r\Pi'(\mathbf{r}, \mathbf{r}') = (\nabla' \times \nabla' \times \mathbf{r}_0')\mathscr{S}'(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}^0 - j\omega\epsilon(\nabla' \times \mathbf{r}_0')\mathscr{S}'(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}^0, \tag{9a}$$

$$j\omega\mu r\Pi''(\mathbf{r}, \mathbf{r}') = (\nabla' \times \nabla' \times \mathbf{r}_0')\mathscr{S}''(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}^0 + j\omega\mu(\nabla' \times \mathbf{r}_0')\mathscr{S}''(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}^0, \tag{9b}$$

with†

$$j\omega\epsilon\mathscr{S}'(\mathbf{r}, \mathbf{r}') = \sum_i \frac{\Phi_i(\boldsymbol{\rho})\Phi_i^*(\boldsymbol{\rho}')}{k_{ti}'^2} Y_i'(r, r'),$$

$$j\omega\mu\mathscr{S}''(\mathbf{r}, \mathbf{r}') = \sum_i \frac{\psi_i(\boldsymbol{\rho})\psi_i^*(\boldsymbol{\rho}')}{k_{ti}''^2} Z_i''(r, r'). \tag{9c}$$

In these equations, ∇' denotes differentiation with respect to the source point coordinates (r', θ', ϕ') and $\mathbf{r}_0'$ is the radial unit vector in the primed coordinate system which describes the orientation of the source vectors $\mathbf{J}^0$ and $\mathbf{M}^0$ (i.e., $\mathbf{J}^0 = \mathbf{r}_0'J_r^0 + \boldsymbol{\theta}_0'J_\theta^0 + \boldsymbol{\phi}_0'J_\phi^0$, etc.). It is to be noted that the vectors $\mathbf{J}^0$ and $\mathbf{M}^0$ are treated as *constant* and that the curl operations are to be interpreted as in Eq. (5), with the dot product carried out subsequently. The form of Eqs. (9a) and (9b) permits the dyadic Green's functions at a source-free point to be written symmetrically as follows:

$$\begin{aligned} -j\omega\epsilon\mathscr{Z}(\mathbf{r}, \mathbf{r}') &= (\nabla \times \nabla \times \mathbf{r}_0)(\nabla' \times \nabla' \times \mathbf{r}_0')\mathscr{S}'(\mathbf{r}, \mathbf{r}') \\ &\quad + k^2(\nabla \times \mathbf{r}_0)(\nabla' \times \mathbf{r}_0')\mathscr{S}''(\mathbf{r}, \mathbf{r}'), \end{aligned} \tag{10a}$$

$$\begin{aligned} -j\omega\mu\mathscr{Y}(\mathbf{r}, \mathbf{r}') &= (\nabla \times \nabla \times \mathbf{r}_0)(\nabla' \times \nabla' \times \mathbf{r}_0')\mathscr{S}''(\mathbf{r}, \mathbf{r}') \\ &\quad + k^2(\nabla \times \mathbf{r}_0)(\nabla' \times \mathbf{r}_0')\mathscr{S}'(\mathbf{r}, \mathbf{r}'), \end{aligned} \tag{10b}$$

with similar formulas resulting for $\mathscr{T}_e$ and $\mathscr{T}_m$. These results are directly analogous to those in Eqs. (2.3.27).

†It is implied that $k_{ti} \neq 0$. The case $k_{ti}' = 0$ arises in connection with the TEM mode, whose contribution must be included separately.

As in uniform regions, the preceding formulas simplify substantially when the sources are longitudinal (radial). One observes from Eqs. (9) that a radial electric current element excites only E modes and a radial magnetic current element only H modes, thereby making the fields excited separately by these sources derivable from a single scalar function. The potential functions Π' and Π'' are now expressed as follows:

$$\Pi'(\mathbf{r}) = J_r^0 \frac{G'(\mathbf{r}, \mathbf{r}')}{j\omega\epsilon r'}, \qquad \Pi''(\mathbf{r}) = M_r^0 \frac{G''(\mathbf{r}, \mathbf{r}')}{j\omega\mu r'}, \tag{11}$$

where, in view of Eqs. (2), (3), and (9c),

$$rr'G'(\mathbf{r}, \mathbf{r}') \equiv -r'^2\nabla_t' \cdot {}_t\nabla' \mathscr{S}'(\mathbf{r}, \mathbf{r}') = \sum_i \Phi_i(\rho)\Phi_i^*(\rho') \frac{Y_i'(r, r')}{j\omega\epsilon}, \tag{11a}$$

$$rr'G''(\mathbf{r}, \mathbf{r}') \equiv -r'^2\nabla_t' \cdot {}_t\nabla' \mathscr{S}''(\mathbf{r}, \mathbf{r}') = \sum_i \psi_i(\rho)\psi_i^*(\rho') \frac{Z_i''(r, r')}{j\omega\mu}. \tag{11b}$$

It may be verified that G' and G'' are scalar Green's functions which satisfy the inhomogeneous wave equations

$$(\nabla^2 + k^2)\begin{matrix} G'(\mathbf{r}, \mathbf{r}') \\ G''(\mathbf{r}, \mathbf{r}') \end{matrix} = -\delta(\mathbf{r} - \mathbf{r}'), \qquad \delta(\mathbf{r} - \mathbf{r}') = \frac{\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')}{r'^2 \sin\theta'}, \tag{11c}$$

subject to boundary conditions identical with those stated for Π' and Π'', respectively, in Eqs. (8a) and (8b). This follows on performing the operation $(\nabla^2 + k^2)$ on the modal expansions for G' and G'', recalling Eqs. (2), (3), and (2.5.9), and recognizing that $\sum_i \Phi_i(\rho)\Phi_i^*(\rho') = \delta(\theta - \theta')\delta(\phi - \phi')/\sin\theta'$ (see Eq. 3.3.32a).

2.7 SOLUTION OF SPHERICAL TRANSMISSION-LINE EQUATIONS (NETWORK ANALYSIS)

2.7a Source-Free and Source-Excited Transmission Lines

As noted in Sec. 2.6c, the solution of the transmission-line equations (2.5.9) is facilitated by introduction of the modal Green's functions $Z_i(r, r')$, $Y_i(r, r')$, $T_i^I(r, r')$, and $T_i^V(r, r')$ defined in Eqs. (2.3.10). Since details on non-uniform transmission lines are given in Secs. 3.3a and 3.3b, only a summary for spherical lines is presented here. We observe that the source-free equations for the voltage and the current are different, and that it is usually preferable [in view of the relations $\kappa_i' Y_i' = \omega\epsilon$, $\kappa_i'' Z_i'' = \omega\mu$ from Eqs. (2.5.9a) and (2.5.9b)] to solve these equations for the E-mode current and for the H-mode voltage, with the E-mode voltage and H-mode current derived from Eqs. (2.5.9). The E-mode current Green's function $Y_i'(r, r')$ and the H-mode voltage Green's function $Z_i''(r, r')$ satisfy the second-order differential equations [see Eqs. (2.3.10)]†

†One should not confuse the Green's function $Y_i'(r, r')$ with the characteristic admittance Y_i', and similarly for $Z_i''(r, r')$ and Z_i''.

$$\left[\kappa_i' Y_i' \frac{d}{dr} \frac{1}{\kappa_i' Y_i'} \frac{d}{dr} + \kappa_i'^2\right] Y_i'(r, r') = -j\kappa_i' Y_i' \delta(r - r'), \tag{1a}$$

$$\left[\kappa_i'' Z_i'' \frac{d}{dr} \frac{1}{\kappa_i'' Z_i''} \frac{d}{dr} + \kappa_i''^2\right] Z_i''(r, r') = -j\kappa_i'' Z_i'' \delta(r - r'), \tag{1b}$$

which simplify substantially in a homogeneous medium wherein $(\kappa_i' Y_i')$ and $(\kappa_i'' Z_i'')$ are constant. This reduction does not occur for $(\kappa_i'' Y_i'')$ or $(\kappa_i' Z_i')$, which enter into the equations for the H-mode current and E-mode voltage, respectively.

In a source-free region where μ and ϵ are constant, both I_i' and V_i'' are solutions of

$$\left[\frac{d^2}{dr^2} + \kappa_i^2(r)\right] \begin{matrix} I_i'(r) \\ V_i''(r) \end{matrix} = 0, \qquad \kappa_i(r) = k\sqrt{1 - \frac{k_{ti}^2}{(kr)^2}}. \tag{2}$$

A distinction arises in the use of $k_{ti} = k_{ti}'$ and k_{ti}'' for the E and H modes, respectively. These equations are solved in terms of the spherical Bessel functions [see also Eq. (5.9.3)]

$$j_p(kr), \quad n_p(kr), \quad h_p^{(1)}(kr), \quad h_p^{(2)}(kr), \quad k_{ti}^2 = p(p+1), \tag{3}$$

any two of which are linearly independent. The spherical Bessel function $z_p(kr)$ is related to the cylindrical Bessel function $Z_{p+1/2}(kr)$ as follows:

$$z_p(kr) = \sqrt{\frac{\pi kr}{2}}\, Z_{p+1/2}(kr). \tag{3a}$$

It is noted from Eqs. (2.5.9) that the spatial dependence of the E-mode voltage V_i' and the H-mode current I_i'' is given by the r derivatives of the functions in Eq. (3). The spherical Bessel functions have the following asymptotic behavior for reasonably large p and $kr \gg p$:

$$j_p(kr) \sim \sin\left(kr - \frac{p\pi}{2}\right), \qquad n_p(kr) \sim -\cos\left(kr - \frac{p\pi}{2}\right),$$

$$h_p^{(1,2)}(kr) \sim \mp j e^{\pm j(kr - p\pi/2)}, \qquad kr \gg p;† \tag{4a}$$

for $kr \ll p$, one has, to within constant factors,

$$j_p(kr) \sim \left(\frac{kr}{p}\right)^{p+1}, \quad n_p(kr) \sim \mp j h_p^{(1,2)}(kr) \sim -\left(\frac{p}{kr}\right)^p, \qquad kr \ll p. \tag{4b}$$

The expression for $j_p(kr)$ when p is arbitrary and $r \to 0$ is

$$j_p(kr) \sim \frac{\sqrt{\pi}}{\Gamma(p + \frac{3}{2})} \left(\frac{kr}{2}\right)^{p+1}, \tag{4c}$$

from which the first of Eqs. (4b) follows on employing Stirling's formula for the gamma function [Eq. (3.6.52b)].

Evidently, for observation points with $kr \gg p$, a mode characterized by the

†Here and in the following, the first and second of (1, 2) superscripts or subscripts refer to the upper and lower symbols, respectively.

index p propagates locally like a uniform plane wave in the radial direction, whereas no propagation takes place in the region $kr \ll p$. This behavior conforms with the transmission-line viewpoint since the propagation constant κ_i and the characteristic impedance Z_i reduce to their free-space values when $kr \gg k_{ti}$, but are imaginary when $kr \ll k_{ti}$ (below cutoff mode), with the changeover occurring in the vicinity of the turning point $kr = k_{ti}$ (see also Sec. 3.5c). Thus, depending on the location of the observation point, a given spherical mode may exhibit either propagating or evanescent characteristics. To amplify on these observations, consider a point source in free space located at $r = r'$ in a coordinate system not centered at the source; the equivalent network includes a point generator at r' on the non-uniform transmission line. As will be seen below in Eq. (11), the voltage and current in a mode with index p behave according to $j_p(kr)$ (or its derivative) when $r < r'$, and $h_p^{(2)}(kr)$ (or its derivative) when $r > r'$. Two regimes may now be distinguished: $p \ll kr'$ and $p \gg kr'$. When $p \ll kr'$, then $kr \gg p$ in the region $r > r'$ so that the mode field propagates undamped in the outward direction. Propagation obtains as well in the region $r' > r > p/k$ but changes to decay when $r < p/k$, with the latter inequalities to be understood in an approximate sense only. Alternatively, when $p \gg kr'$, the mode fields decay on both sides of the source but propagation takes place eventually when $r > p/k$. If various modes have the same amplitude at the source, the important ones in the radiation field are those with $p < kr'$. Modes with $p > kr'$ contribute primarily to energy storage in the neighborhood of the source region.

It may also be mentioned that in the radiation zone $kr \gg k_{ti}$, where the ith mode propagates locally like a radial plane wave, the associated electromagnetic field is transverse. This follows from Eqs. (4a) and (2.5.8) from which it is noted that $\mathbf{E}_{ti}$ and $\mathbf{H}_{ti}$ are $O(1/r)$, whereas E_{ri} and H_{ri} are $O(1/r^2)$ (the scalar and vector mode functions have no radial dependence). Thus, the well-known far-field behavior of a localized source distribution emerges naturally from the spherical transmission-line analysis.

Methods for solving the E- or H-mode equations (1) in the presence of sources are detailed in Secs. 3.3a and 3.3b. From the spherical Bessel functions in (3) and their Wronskian $[j_p(x)n_p'(x) - n_p(x)j_p'(x)] = 1$, one finds that source-free "standing-wave" solutions of Eqs. (1) are expressible in terms of the functions c and s of Eq. (3.3.18); for a radially homogeneous medium with constant ϵ and μ, these have the form

$$c(kr, kr_0) = j_p(kr)n_p'(kr_0) - n_p(kr)j_p'(kr_0), \tag{5a}$$

$$s(kr, kr_0) = \frac{1}{k}[n_p(kr)j_p(kr_0) - j_p(kr)n_p(kr_0)], \tag{5b}$$

where the prime denotes the derivative with respect to the argument. The H-mode input admittance of a line terminated at $r = r_{2,1}$, as seen from r_0, may then be determined from Eqs. (3.3.20) and (3.3.21) as

$$\mp\frac{j\omega\mu}{k}\overset{\rightleftarrows}{Y}''_i(r_0) =$$

$$\frac{[j'_p(x_0)n'_p(x_{2,1}) - n'_p(x_0)j'_p(x_{2,1})] \mp (j\omega\mu/k)\overset{\rightleftarrows}{Y}''_{Ti}[n'_p(x_0)j_p(x_{2,1}) - j'_p(x_0)n_p(x_{2,1})]}{[j_p(x_0)n'_p(x_{2,1}) - n_p(x_0)j'_p(x_{2,1})] \mp (j\omega\mu/k)\overset{\rightleftarrows}{Y}''_{Ti}[n_p(x_0)j_p(x_{2,1}) - j_p(x_0)n_p(x_{2,1})]}, \tag{6a}$$

with $x_\nu = kr_\nu$, $\nu = 1, 2$. $\overrightarrow{Y}''_{Ti}$ and $\overleftarrow{Y}''_{Ti}$ denote terminal admittances at the endpoints $r_2 > r_0$ and $r_1 < r_0$, respectively. The solution for the H-mode voltage Green's function $Z''_i(r, r')$ then follows from Eq. (3.3.22) as

$$Z''_i(r, r') = \frac{[c(x_<, x_0) + j\omega\mu\overleftarrow{Y}''_i(r_0)s(x_<, x_0)][c(x_>, x_0) - j\omega\mu\overrightarrow{Y}''_i(r_0)s(x_>, x_0)]}{\overleftarrow{Y}''_i(r_0) + \overrightarrow{Y}''_i(r_0)}. \tag{6b}$$

From Eq. (3.3.26b), the expression for the E-mode input impedances $\overset{\rightleftarrows}{Z}'_i(r_0)$ has the same form as Eq. (6a) except for the duality replacements $\mu \leftrightarrow \epsilon$, $Y \rightarrow Z$; the E-mode current Green's function $Y'_i(r, r') = j\omega\epsilon g'(r, r')$ then follows from Eq. (3.3.26).

An interesting feature of Eq. (6a) may be noted. When the region includes the origin (i.e., $r_1 = 0$), the input admittance or impedance reduces to [see Eq. (4b)]

$$\zeta\overleftarrow{Y}''_i(r_0) = \frac{\overleftarrow{Z}'_i(r_0)}{\zeta} = -j\frac{j'_p(x_0)}{j_p(x_0)}, \qquad \zeta = \sqrt{\frac{\mu}{\epsilon}}, \tag{7}$$

which expression is independent of the termination $\overleftarrow{Y}''_{Ti}$ or $\overleftarrow{Z}'_{Ti}$. Thus, no boundary condition need be specified at the singular point $r = 0$, the only requirement being the finiteness of the solutions for the voltage and current. As noted in Sec. 3.3b, this point is termed a "limit point" in the theory of differential equations.

An alternative formulation may be carried out in terms of traveling-wave functions as in Eqs. (3.3.28)–(3.3.30). For constant μ, ϵ, wave functions traveling in the $(+r)$ direction are given, for a time dependence exp $(j\omega t)$, by

$$\left.\begin{matrix} I'_{i+} \\ V''_{i+} \end{matrix}\right\} \propto h_p^{(2)}(kr), \qquad \left.\begin{matrix} I''_{i+} \\ V'_{i+} \end{matrix}\right\} \propto h_p'^{(2)}(kr); \tag{8a}$$

in the $-r$ direction,

$$\left.\begin{matrix} I'_{i-} \\ V''_{i-} \end{matrix}\right\} \propto h_p^{(1)}(kr), \qquad \left.\begin{matrix} I''_{i-} \\ V'_{i-} \end{matrix}\right\} \propto h_p'^{(1)}(kr). \tag{8b}$$

Thus, for matched (reflectionless) terminations,† the input impedances for the E and H modes are

$$\overrightarrow{\zeta}_i'(r) = j\zeta \frac{h_p'^{(2)}(kr)}{h_p^{(2)}(kr)}, \qquad \overleftarrow{\zeta}_i'(r) = -j\zeta \frac{h_p'^{(1)}(kr)}{h_p^{(1)}(kr)}, \qquad \zeta = \sqrt{\frac{\mu}{\epsilon}}, \tag{9a}$$

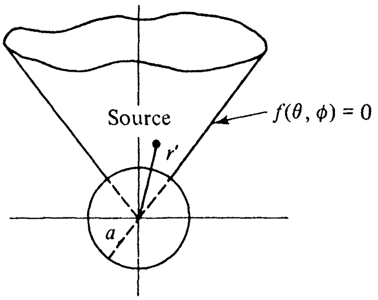
$$\overrightarrow{\zeta}_i''(r) = -j\zeta \frac{h_p^{(2)}(kr)}{h_p'^{(2)}(kr)}, \qquad \overleftarrow{\zeta}_i''(r) = j\zeta \frac{h_p^{(1)}(kr)}{h_p'^{(1)}(kr)}, \tag{9b}$$

and these quantities, together with the input impedances at $r_0 = r$ obtained from Eq. (6) and its dual, may be employed in the calculation of the reflection coefficients in Eqs. (3.3.28), etc.

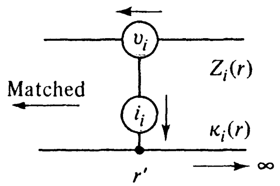
2.7b Special Terminations

Bilaterally matched region

Bilateral matching occurs in a suitably idealized spherical region wherein the field solution is comprised entirely of waves traveling away from the source (bilateral radiation condition); such a region constitutes the spherical analogue of a biinfinite uniform region. While the matched condition is automatically satisfied at $r \to \infty$ in an unbounded region, it does not obtain at the lower end-point $r = 0$, which introduces reflection. It is therefore necessary to shield the origin by a reflectionless termination, for example, a "perfectly absorbing" sphere having a radius a [Fig. 2.7.1(a)].‡ As in the analogous problem of propagation



(a) Physical configuration



(b) Equivalent network

FIG. 2.7.1 Bilaterally matched radial region.

on an angular transmission line (Sec. 3.4b), such a boundary, which must absorb completely *all* radially propagating modes, is generally non-physical but forms a useful prototype for subsequent considerations; it may be synthesized approxi-

†When observed for $kr \gg p$, a matched termination is taken to imply negligible reflected field; no separate identification of waves traveling along the $+r$ and $-r$ directions is possible when $p \geq kr$.

‡The present discussion concerns the radial domain only and remains valid in the presence of radially independent, perfectly conducting boundaries as shown in Fig. 2.7.1(a).

mately in sufficiently lossy regions. The modal network problem is sketched in Fig. 2.7.1(b), with the voltage and (or) current generators representing the excitation. The input impedances $\overset{\rightleftarrows}{Z}_i(r')$ are now given by their matched values $\overset{\rightleftarrows}{\zeta}_i(r')$, and the reflection coefficients $\overset{\rightleftarrows}{\Gamma}_i(r')$ equal zero. Equations (6) and their dual then yield, for the modal Green's functions,

$$\frac{Z_i''(r, r')}{\zeta} = Y_i'(r, r')\zeta = \frac{1}{2} h_p^{(1)}(kr_<)h_p^{(2)}(kr_>). \tag{10}$$

Homogeneous region, $0 < r < \infty$

When the source is located in an infinite homogeneous region, the boundary at $r = a$ is absent and the modal network problem is the one shown in Fig. 2.7.2. The input impedance seen looking toward $r = \infty$ is still equal to $\overrightarrow{\zeta}_i(r')$,

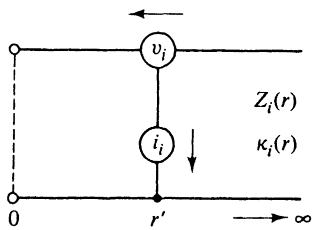


FIG. 2.7.2 Radial region $0 < r < \infty$.

but its value in the other direction must now be taken from Eq. (7). Then from Eq. (6b) and its dual, the modal Green's functions follow as

$$\frac{Z_i''(r, r')}{\zeta} = Y_i'(r, r')\zeta = j_p(kr_<)h_p^{(2)}(kr_>). \tag{11}$$

Semiinfinite homogeneous region, $0 < a \leq r < \infty$

When the obstacle in Fig. 2.7.1(a) is perfectly conducting, the expression in Eq. (11) must be modified to assure the vanishing of the voltage when $r = a$. This may be accomplished either from Eqs. (6a) and (6b), with $\overleftarrow{Y}_{Ti}'' = \infty$ (see Fig. 2.7.3, where a short circuit appears at $r = a$), or directly by inspection. The following result is obtained:

$$\frac{Z_i''(r, r')}{\zeta} = \left[j_p(kr_<) - \frac{j_p(ka)}{h_p^{(2)}(ka)} h_p^{(2)}(kr_<)\right] h_p^{(2)}(kr_>). \tag{12a}$$

For the E modes, the requirement $V_i'(a) = 0$ implies via Eq. (2.5.9) the vanishing of $(dI_i'/dr)_{r=a}$, whence

$$Y_i'(r, r')\zeta = \left[j_p(kr_<) - \frac{j_p'(ka)}{h_p'^{(2)}(ka)} h_p^{(2)}(kr_<)\right] h_p^{(2)}(kr_>). \tag{12b}$$

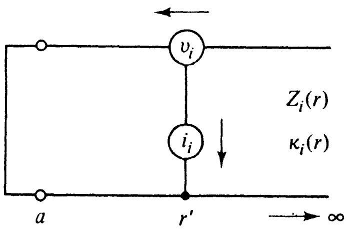


FIG. 2.7.3 Perfectly conducting sphere.

The expressions for $Z_i''(r, r')$ and $Y_i'(r, r')$ are no longer identical as in Eqs. (10) and (11) because the perfectly conducting boundary is not its own dual.

Composite region, $0 < r < \infty$

If the region $r > a$ is filled with a homogeneous medium having constitutive parameters ϵ, μ, while the region $0 < r < a$ is characterized by ϵ_1, μ_1, one has the network problem shown in Fig. 2.7.4. The continuity of the tangential components of **E** and **H** at $r = a$ is assured from the continuity of V_i and I_i

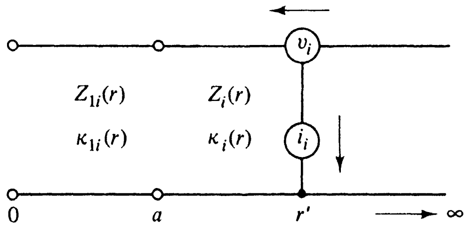


FIG. 2.7.4 Dielectric sphere.

[see Eqs. (2.5.8a) and (2.5.8b)], so the equivalent network involves only a simple junction. When the source is located outside the sphere, the transmission line is matched for $r > r'$ and reflection occurs only at $r = a$. The solution may then be constructed by adding to the bilaterally matched Green's functions in Eq. (10) the reflected wave contribution; there is no need to analyze the problem from the beginning. The situation is more involved when the source is located inside the sphere, in which instance reflections occur from both ends of the transmission line.

The E-mode current reflection coefficient $\overleftarrow{\Gamma}'_{Ii}(a+)$ seen slightly to the left of the point $r = a$ in Fig. 2.7.4 is given from Eqs. (3.3.29a) and (3.3.29c) by

$$\overleftarrow{\Gamma}'_{Ii}(a+) = \frac{[\overleftarrow{\zeta}'_i(a)/\overleftarrow{Z}'_i(a)] - 1}{[\overrightarrow{\zeta}'_i(a)/\overleftarrow{Z}'_i(a)] + 1}, \tag{13}$$

where the $\overleftrightarrow{\zeta}'_i(a)$ represent the matched impedances seen from $r = a$ in the transmission-line descriptive of the region exterior to the sphere,

$$\overleftarrow{\zeta}_i'(a) = -j\zeta \frac{h_p'^{(1)}(ka)}{h_p^{(1)}(ka)}, \quad \overrightarrow{\zeta}_i'(a) = j\zeta \frac{h_p'^{(2)}(ka)}{h_p^{(2)}(ka)}, \qquad k = \omega\sqrt{\mu\epsilon}, \quad \zeta = \sqrt{\frac{\mu}{\epsilon}}, \tag{13a}$$

while $\overleftarrow{Z}_i'(a)$ denotes the input impedance seen to the left from $r = a$ [see Eq. (7)],

$$\overleftarrow{Z}_i'(a) = -j\zeta_1 \frac{j_p'(k_1 a)}{j_p(k_1 a)}, \qquad k_1 = \omega\sqrt{\mu_1 \epsilon_1}, \quad \zeta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}. \tag{13b}$$

If the incident current is taken as $h_p^{(1)}(kr)$, the reflected current at $r = a$ has an amplitude $\overleftarrow{\Gamma}_{li}' h_p^{(1)}(ka)$, so its value at any point must be $[h_p^{(1)}(ka)/h_p^{(2)}(ka)]\overleftarrow{\Gamma}_{li}' h_p^{(2)}(kr)$. Upon combining this result with Eq. (10), one finds

$$Y_i'(r, r')\zeta = \frac{1}{2}\left[h_p^{(1)}(kr_<) + \frac{h_p^{(1)}(ka)}{h_p^{(2)}(ka)}\overleftarrow{\Gamma}_{li}'(a) h_p^{(2)}(kr_<)\right] h_p^{(2)}(kr_>), \qquad a \le r < \infty, \tag{14}$$

and an analogous expression for $Z_i''(r, r')$. The behavior of the modal Green's function inside the sphere may in this simple case be obtained by inspection from the requirements that the current remains finite at $r = 0$ and continuous at $r = a$:

$$Y_i'(r, r') = Y_i'(a, r')\frac{j_p(k_1 r)}{j_p(k_1 a)}, \qquad 0 \le r \le a. \tag{15}$$

The previously derived results may be recovered as special cases from Eq. (14). For the bilaterally matched configuration, $\overleftarrow{\Gamma}_{li}' = 0$. For the perfectly conducting sphere, obtained in the limit as $|\epsilon_1| \to \infty$, $\arg \epsilon_1 \to -\pi/2$, one has $\overleftarrow{Z}_i'(a) \to 0$, and, therefrom, Eq. (12b). Equation (11) follows on letting $a \to 0$.

PROBLEMS

1. A homogeneously filled uniform waveguide has a cross section S bounded by the curve $s = s_1 + s_2$. The following boundary conditions are assumed on the two portions s_1 and s_2 which comprise the waveguide boundary: s_1 is a perfect electric conductor ($\mathbf{E}_{\text{tan}} = 0$) while s_2 is a perfect magnetic conductor ($\mathbf{H}_{\text{tan}} = 0$).
   (a) Show that the fields in this waveguide may be decomposed into E and H modes, and formulate the eigenvalue problem for the scalar mode functions from which the vector modes may be derived.
   (b) Apply these results to the determination of the mode functions in a rectangular waveguide, one side wall of which is a perfect magnetic conductor while the remaining walls are perfect electric conductors.
2. The dyadic Green's functions $\mathscr{Z}(\mathbf{r}, \mathbf{r}')$ and $\mathscr{Y}(\mathbf{r}, \mathbf{r}')$ defined in Eqs. (2.3.27) are given in an unbounded homogeneous medium with constant ϵ, μ by

$$\sqrt{\frac{\epsilon}{\mu}}\,\mathscr{Z}(\mathbf{r}, \mathbf{r}') = \sqrt{\frac{\mu}{\epsilon}}\,\mathscr{Y}(\mathbf{r}, \mathbf{r}') = jk\left[\mathbf{1} + \frac{\nabla\nabla}{k^2}\right] G_f(\mathbf{r}, \mathbf{r}'), \tag{1a}$$

where $G_f = (4\pi|\mathbf{r}-\mathbf{r}'|)^{-1}\exp[-jk|\mathbf{r}-\mathbf{r}'|]$ [assumed time dependence $\exp(j\omega t)$]. $\mathbf{1}$ is the unit dyadic ($\mathbf{1}\cdot\mathbf{A}=\mathbf{A}\cdot\mathbf{1}=\mathbf{A}$). When $r\to\infty$, show that Eq. (1a) reduces to the far field relation

$$\sqrt{\frac{\epsilon}{\mu}}\,\mathscr{Z} = \sqrt{\frac{\mu}{\epsilon}}\,\mathscr{Y} \sim jk\frac{e^{-jkr+jk(\mathbf{r}\cdot\mathbf{r}')/r}}{r}(\mathbf{1}-\mathbf{r}_0\mathbf{r}_0), \qquad r \gg r', \tag{1b}$$

where $\mathbf{r}_0$ is a unit vector in the r direction.

3. (a) Show that for an $\exp(j\omega t)$ dependence, the free-space dyadic Green's functions in Eqs. (1) have for $r \neq r'$ the spherical mode representation

$$\zeta\mathscr{Y}(\mathbf{r},\mathbf{r}') = \sum_i \mathscr{H}_i'^{(1)}(\mathbf{r}_<)\mathscr{H}_i'^{(+)}(\mathbf{r}_>) + \sum_i \mathscr{H}_i''^{(1)}(\mathbf{r}_<)\mathscr{H}_i''^{(+)}(\mathbf{r}_>), \tag{2}$$

$$\frac{1}{\zeta}\mathscr{Z}(\mathbf{r},\mathbf{r}') = \sum_i \mathscr{E}_i'^{(1)}(\mathbf{r}_<)\mathscr{E}_i'^{(+)}(\mathbf{r}_>) + \sum_i \mathscr{E}_i''^{(1)}(\mathbf{r}_<)\mathscr{E}_i''^{(+)}(\mathbf{r}_>), \qquad \zeta = \sqrt{\frac{\mu}{\epsilon}}, \tag{3}$$

where $\mathbf{r}_< = \mathbf{r}$ when $r < r'$, $\mathbf{r}_< = \mathbf{r}'$ when $r > r'$, with the converse applying to $\mathbf{r}_>$. Also, if $h_p(kr)$ denotes the spherical Hankel function,

$$r\mathscr{H}_i'^{(+,-)}(\mathbf{r}) = h_p^{(2,1)}(kr)\mathbf{h}_i'(\theta,\phi), \qquad k_{ti}'^2 = p(p+1),$$

$$r\mathscr{H}_i''^{(+,-)}(\mathbf{r}) = h_p'^{(2,1)}(kr)\mathbf{h}_i''(\theta,\phi) - jh_p^{(2,1)}(kr)\zeta Y_i''\mathbf{h}_{ri}''(\theta,\phi), \tag{4a}$$

$$\mathscr{H}_i^{(1)}(\mathbf{r}) = \frac{1}{2}[\mathscr{H}_i^{(-)}(\mathbf{r}) + \mathscr{H}_i^{(+)}(\mathbf{r})],\ \mathscr{H}_i^{(2)}(\mathbf{r}) = \frac{1}{2j}[\mathscr{H}_i^{(-)}(\mathbf{r}) - \mathscr{H}_i^{(+)}(\mathbf{r})],$$

and

$$r\mathscr{E}_i'^{(+,-)}(\mathbf{r}) = h_p'^{(2,1)}(kr)\mathbf{e}_i'(\theta,\phi) - \frac{j}{\zeta}h_p^{(2,1)}(kr)Z_i'\mathbf{e}_{ri}'(\theta,\phi), \qquad k_{ti}''^2 = p(p+1),$$

$$r\mathscr{E}_i''^{(+,-)}(\mathbf{r}) = h_p^{(2,1)}(kr)\mathbf{e}_i''(\theta,\phi), \tag{4b}$$

$$\mathscr{E}_i^{(1)}(\mathbf{r}) = \frac{1}{2}[\mathscr{E}_i^{(-)}(\mathbf{r}) + \mathscr{E}_i^{(+)}(\mathbf{r})], \qquad \mathscr{E}_i^{(2)}(\mathbf{r}) = \frac{1}{2j}[\mathscr{E}_i^{(-)}(\mathbf{r}) - \mathscr{E}_i^{(+)}(\mathbf{r})].$$

The transverse and longitudinal mode functions are defined in Eqs. (2.6.1) and (2.5.10c, 2.5.10d), and their explicit form is given by Eqs. (3.4.63), (3.2.51b) and (3.4.79a).

(b) Apply Eqs. (1)-(3) to calculate the field of an electric current element with strength J located at r' on the z axis and oriented parallel to the x axis. By letting $r'\to\infty$, show that the identification

$$-\frac{j\omega\mu e^{-jkr'}J}{4\pi r'} \to 1 \tag{5a}$$

yields an incident plane wave of unit amplitude.

Note: Show first from Eq. (6.8.2a), with $r'\to\infty$ and $\theta = 0$, that

$$kre^{-jkr} = \sum_{n=0}^{\infty}(2n+1)(-j)^n j_n(kr). \tag{5b}$$

(c) What modification must be included in Eqs. (2)-(4) when $r = r'$ [see Eq. (2.5.8e)]?

(d) Show that the preceding formulas remain valid in the presence of conical boundaries provided that one employs the appropriate eigenfunctions in Sec. 3.4b.

4. Show that the normalized vector mode functions for the TEM mode in a biconical region bounded by the surface $\theta = \theta_{1,2}$ are as follows:

$$\mathbf{e}'_{0,0}(\theta, \phi) = \boldsymbol{\theta}_0 \frac{1}{\sin\theta}\left[2\pi \ln\left(\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}\right)\right]^{-1/2}, \qquad 0 < \theta_1 \leq \theta \leq \theta_2 < \pi, \tag{6}$$

$$\mathbf{h}'_{0,0} = \mathbf{r}_0 \times \mathbf{e}'_{0,0}, \qquad e'_{ri} = h'_{ri} = 0. \tag{6a}$$

5. Derive the modal Green's functions (radial transmission-line solutions) when the source in Fig. 2.7.4 is situated in the region $0 < r' < a$.

6. The electric field in an aperture of arbitrary shape perforating a perfectly conducting infinite screen of negligible thickness may be represented equivalently by a magnetic current distribution $\mathbf{M}(\mathbf{r}') = \mathbf{E}(\mathbf{r}') \times \mathbf{n}$ placed on the unperforated screen (Sec. 1.5b). $\mathbf{E}(\mathbf{r}')$ denotes the aperture electric field and $\mathbf{n}$ the unit normal pointing into the half-space region. The power $\bar{S}$ radiated into the half-space is given by Eq. (2.5.11b).

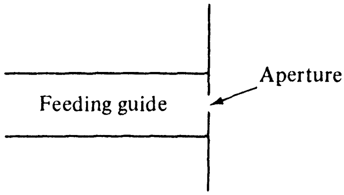


FIG. P2.1 Radiating aperture.

(a) Show that the voltages required in Eq. (2.5.11b) are given by:

$$|V_i(r)| = \left|\iint_A \mathbf{M}(\mathbf{r}') \cdot \mathscr{H}_i^{(1)}(\mathbf{r}')\, dS'\right|, \tag{7}$$

where $\mathscr{H}_i^{(1)}$ is defined in Eq. (4a) and A is the area of the aperture.

(b) Assume that the aperture is fed on one side by a waveguide propagating a single mode (Fig. P2.1); at a reference plane removed "many" guide wavelengths from the aperture, the voltage in this mode is given by

$$\hat{V} = \iint_A \mathbf{M}(\mathbf{r}') \cdot \hat{\mathbf{h}}(\mathbf{r}')\, dS', \tag{8}$$

where $\hat{\mathbf{h}}$ is the real transverse vector mode function for the dominant mode in the guide (see Sec. 2.3a). Assume that $\mathbf{M}$, and therefore $\hat{V}$, is real. Show that if $\hat{I}$ is the dominant mode current in the waveguide referred to the aperture plane, the radiation conductance G of the aperture as seen from the waveguide may be defined as

$$G = \frac{\operatorname{Re}\hat{I}}{\hat{V}} = \frac{1}{\zeta}\sum_i \frac{|V'_i(r)|^2}{\hat{V}^2} + \frac{1}{\zeta}\sum_i \frac{|V''_i(r)|^2}{\hat{V}^2}, \tag{9}$$

which expression is obtained from power flow considerations at $r \to \infty$ in the half-space, or by

$$G = \frac{\operatorname{Re} \hat{I}}{\hat{V}} = \frac{\operatorname{Re} \iint_A \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{n}\, dS}{\hat{V}^2}$$

$$= \frac{\iint_A dS \iint_A dS' \mathbf{M}(\mathbf{r}) \cdot [\operatorname{Re} \mathscr{Y}_h(\mathbf{r}, \mathbf{r}')] \cdot \mathbf{M}(\mathbf{r}')}{\hat{V}^2}, \tag{10}$$

which results from the total power flow calculated at the aperture. The half-space dyadic admittance $\mathscr{Y}_h(\mathbf{r}, \mathbf{r}')$ for $\mathbf{r}'$ on A is equal to twice the free space admittance $\mathscr{Y}(\mathbf{r}, \mathbf{r}')$ in Eq. (1a). Show that the aperture field satisfies the integral equation

$$(\operatorname{Re} \hat{I})\hat{\mathbf{h}}(\mathbf{r}) = \iint_A [\operatorname{Re} \mathscr{Y}_h(\mathbf{r}, \mathbf{r}')] \cdot \mathbf{M}(\mathbf{r}')\, dS' \tag{11a}$$

$$= \frac{1}{\zeta} \sum_i |V_i'(r)| \mathscr{H}_i'^{(1)}(\mathbf{r}) + \frac{1}{\zeta} \sum_i |V_i''(r)| \mathscr{H}_i''^{(1)}(\mathbf{r}), \quad \mathbf{r} \text{ in } A. \tag{11b}$$

Show also that the expressions for G are stationary (variational) with respect to small variations of $\mathbf{M}$ about the correct value specified in Eqs. (11), and that the result obtained for G by substituting an approximate "trial function" for $\mathbf{M}$ is larger than the true conductance. (See R. E. Collin, *Field Theory of Guided Waves*, McGraw-Hill Book Co., New York (1960), Chapter 8.)

7. Various elementary distributions of magnetic current as shown in Fig. P2.2 are placed on a perfectly conducting infinite plane. Consider two possible coordinate systems, both having the common origin shown in Fig. P2.2. In system I, the z axis is perpendicular to the plane whereas in system II, the z axis is taken along the horizontal axis in Fig. P2.2. Show that the following spherical E modes (E_{mn}) and H modes (H_{mn}) are excited by these arrangements [refer to Problem 6 of Chapter 3 for forms of the scalar mode functions]:

System I

(a) H_{11};

(b) H_{mn}, m odd, n odd (lowest: H_{11});

(c) H_{mn}, m even, n even (lowest: H_{02}, H_{22});

(d) E_{mn}, m odd, n even and H_{mn}, m, n odd (lowest: H_{11});

(e) E_{mn}, m even, n odd and H_{mn}, m, n even (lowest: E_{01}, H_{22});

(f) E_{mn}, m even, n odd and H_{mn}, m, n even (lowest: E_{01}, H_{22});

(g) E_{0n}, n odd (lowest: E_{01});

(h) H_{0n}, n even (lowest: H_{02}).

System II

(a) H_{01};

(b) H_{0n}, n odd (lowest: H_{01});

(c) H_{0n}, n even (lowest: H_{02});

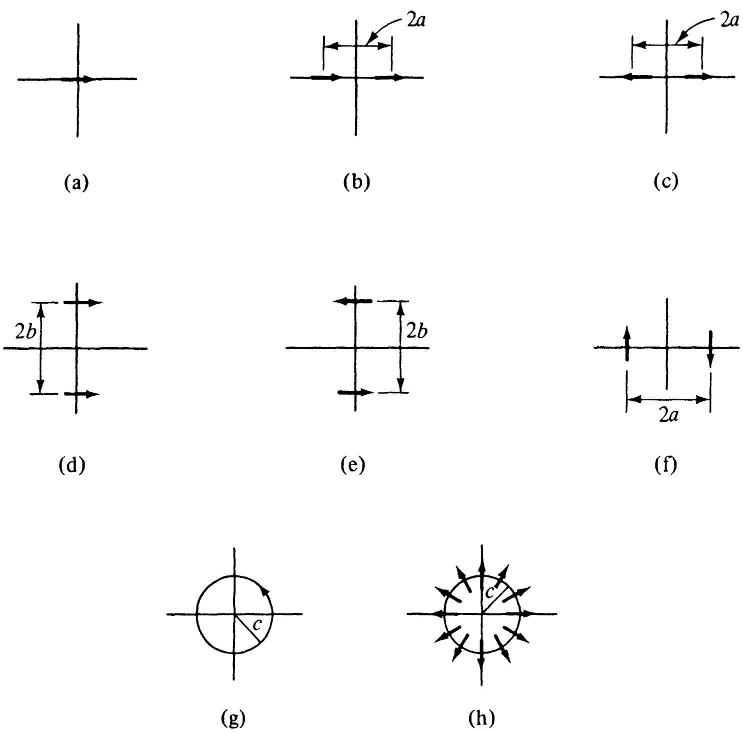


FIG. P2.2 Elementary source distributions: (a) single element; (b)-(f) various symmetrical arrangements of two elements; (g) ring current; (h) ring of radial elemets.

(d) E_{mn}, m, n even and H_{mn}, m even, n odd (lowest: H_{01});

(e) E_{mn}, m, n odd and H_{mn}, m odd, n even (lowest: E_{11}, H_{12});

(f) E_{1n}, n odd and H_{1n}, n even (lowest: E_{11}, H_{12});

(g) E_{mn}, m, n odd (lowest: E_{11});

(h) H_{mn}, m, n even (lowest: H_{02}, H_{22}).

The modes designated as "lowest" remain dominant when the dimensions a, b, c are small compared to the wavelength.

8. When the aperture in Problem 6 is small compared to the wavelength, the spherical mode series in Eq. (9) is rapidly convergent. If the field distribution in the aperture is representable in terms of one of the elementary source configurations in Fig. P2.2, the dominant contribution to the conductance G arises from the corresponding lowest mode(s).

(a) Rectangular slot terminating a rectangular waveguide:
A symmetrically placed rectangular slot having dimensions a', b' terminates a rectangular waveguide with dimensions a, b. When the waveguide is ex-

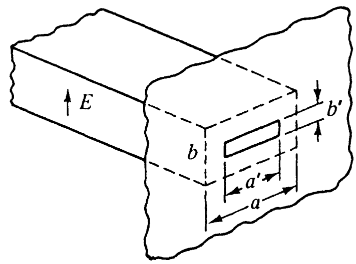


FIG. P2.3 Rectangular configuration.

cited in the dominant mode, a reasonable "guess value" for the induced aperture field is

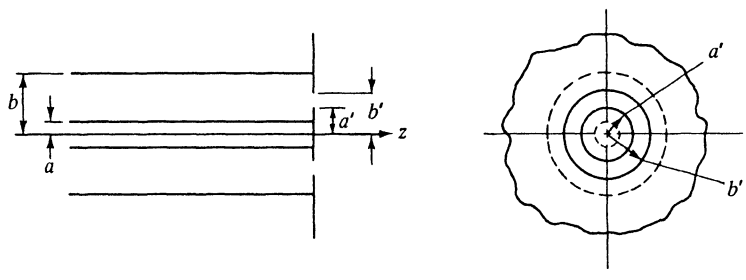
$$\mathbf{E}(\mathbf{r}') \times \mathbf{n} = \mathbf{M}(\mathbf{r}') = \mathbf{z}_0 \cos\frac{\pi z'}{a'}, \tag{12}$$

where the z axis has been chosen parallel to the a dimension of the guide. This source distribution excites predominantly the H_{01} spherical mode (see Problem 7). When the slot dimensions are small, show that

$$\frac{G}{\hat{Y}_0} \cong \frac{1}{\zeta \hat{Y}_0}\frac{|V''_{01}(r)|^2}{\hat{V}^2} \approx \frac{2\pi}{3} ab \frac{\lambda_g}{\lambda^3}\frac{[1-(a'/a)^2]^2}{[\cos(\pi a'/2a)]^2}\left\{1 + O\left[\left(\frac{a'}{\lambda}\right)^2\right]\right\}, \tag{13}$$

where $\hat{Y}_0$ and λ_g are the admittance and the guide wavelength of the dominant (H_{10}) mode in the rectangular waveguide, and λ is the free-space wavelength [for $\hat{Y}_0$ and $\lambda_g = 2\pi/\kappa$, see Eqs. (2.2.15d)].

(b) Annular slot terminating a coaxial waveguide:
A coaxial waveguide with dimensions a, b and excited in the TEM mode is terminated in a concentric annular slot having dimensions a', b' (Fig. P2.4).



(a) Side view (b) End view

FIG. P2.4 Coaxial configuration.

A simple trial function for the electric field in the slot is $\mathbf{E}(\mathbf{r}') = \boldsymbol{\rho}_0(1/\rho')$, the variation in the incident TEM mode, so that

$$\mathbf{M}(\mathbf{r}') = -\boldsymbol{\phi}_0 \frac{1}{\rho'}, \tag{14}$$

where ρ is the radial coordinate transverse to the guide axis which has been chosen coincident with z (in view of the rotational symmetry of the aperture field, this coordinate choice is natural). Show that for small values of (b'/λ) and (a'/λ),

$$\frac{G}{\hat{Y}_0} \approx \frac{|V'_{10}(r)|^2}{\hat{V}^2} \approx \frac{2 \ln (b/a)}{3[\ln (b'/a')]^2}\left[\left(\frac{\pi b'}{\lambda}\right)^2 - \left(\frac{\pi a'}{\lambda}\right)^2\right]^2. \tag{15}$$

Show that the complete spherical mode expansion for the conductance is [refer to Problem 6 of Chapter 3 for the scalar mode functions]:

$$\frac{G}{\hat{Y}_0} = \frac{\ln (b/a)}{\left[\int_{a'}^{b'} M(r')\, dr'\right]^2} \sum_{n=0}^{\infty} \frac{(4n+3)}{(2n+1)(2n+2)}\left[\frac{d}{d\theta} P_{2n+1}(\cos\theta)\right]^2_{\theta=\pi/2} \times \left[\int_{a'}^{b'} M(r') j_n(kr')\, dr'\right]^2. \tag{16}$$

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