Chapter 6

Vector Analysis of Transport Theorems

6-1 Helmholtz Transport Theorem

Thus far, we have been dealing only with functions of position, that is, functions that are dependent on spatial variables only. In many engineering and physical problems, the quantities involved are functions of both space and time. Examples are the induced voltage in a moving coil of an electric generator and the transport of fluid in a channel. The mathematical formulation of these problems requires a knowledge of vector analysis involving a moving surface or a moving body. One of the fundamental theorems in this area is the Helmholtz transport theorem, named after the renowned German scientist Hermann Ludwig Ferdinand von Helmholtz (1821–1894). The theorem deals with the time rate of change of a surface integral of Type IV stated by (3.70), in which the domain of integration and the integrand are functions of both space and time. The quantity under consideration is defined by

$$\begin{aligned} I &= \frac{d}{dt} \iint_{S(\mathbf{R},t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\iint_{S_2(\mathbf{R},\, t+\Delta t)} \mathbf{F}(\mathbf{R}, t + \Delta t) \cdot d\mathbf{S} - \iint_{S_1(\mathbf{R},t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} \right], \end{aligned} \tag{6.1}$$

where $\mathbf{F}(\mathbf{R}, t)$ is an abbreviated notation for $\mathbf{F}(x_1, x_2, x_3, t)$. The x_i's denote the coordinate variables of the position vector where the function $\mathbf{F}$ is defined, and t is the time variable. The domain of integration changes from $S_1(\mathbf{R}, t)$ to $S_2(\mathbf{R}, t + \Delta t)$ in a small time interval Δt, as shown in Fig. 6-1. To evaluate the

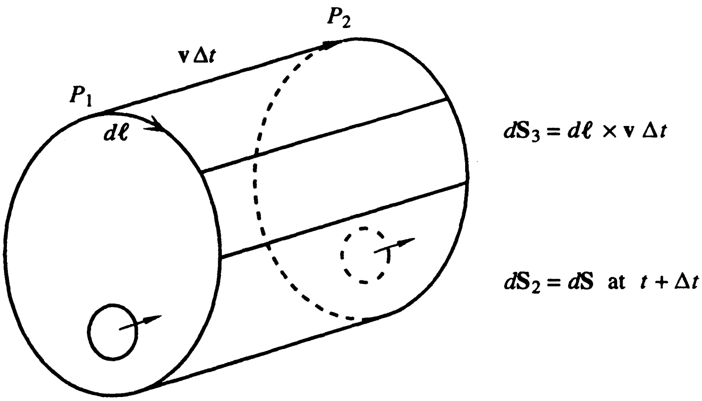

Figure 6-1 Moving surface at two different instants.

limiting value of the difference of the two surface integrals contained in (6.1), we first expand the integrand $\mathbf{F}(\mathbf{R},\ t + \Delta t)$ in a Taylor series with respect to t:

$$\mathbf{F}(\mathbf{R}, t + \Delta t) = \mathbf{F}(\mathbf{R}, t) + \frac{\partial \mathbf{F}(\mathbf{R}, t)}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \mathbf{F}(\mathbf{R}, t)}{\partial t^2}(\Delta t)^2 + \cdots . \tag{6.2}$$

Upon substituting (6.2) into (6.1), we have

$$\begin{aligned} I &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \iint_{S_2(\mathbf{R},\ t+\Delta t)} \left[\mathbf{F}(\mathbf{R}, t) + \frac{\partial \mathbf{F}(\mathbf{R}, t)}{\partial t}\Delta t + \cdots \right] \cdot d\mathbf{S} \right. \\ &\quad \left. - \iint_{S_1(\mathbf{R},t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} \right\} \\ &= \iint_{S(\mathbf{R},t)} \frac{\partial \mathbf{F}(\mathbf{R}, t)}{\partial t} \cdot d\mathbf{S} + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\iint_{S_2(\mathbf{R},\ t+\Delta t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} \right. \\ &\quad \left. - \iint_{S_1(\mathbf{R},t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} \right] . \end{aligned} \tag{6.3}$$

In (6.3), $S_1(\mathbf{R}, t)$ is the same as $S(\mathbf{R}, t)$; the subscripts 1 and 2 are used to identify the location of the surface at time t and at a later time $t + \Delta t$, respectively, as shown in Fig. 6-1. A point P_1 at the contour of S_1 is displaced to a point P_2 at the contour of S_2 during the time interval Δt. The displacement is equal to $\mathbf{v}\Delta t$, where $\mathbf{v}$ denotes the velocity of motion at that location, which may vary continuously from one location to another around the contour. For example, when a circular loop spins around its diagonal axis, the linear velocity varies around its circumference. The two surface integrals within the brackets of (6.3) can be written in the form

$$\begin{aligned}\iint_{S_2(\mathbf{R},\ t+\Delta t)} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S} - \iint_{S_1(\mathbf{R},t)} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S} \\ = \oiint_{S_1+S_2+S_3} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S} - \iint_{S_3} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S}_3,\end{aligned} \tag{6.4}$$

where S_3 denotes the lateral surface swept by the displacement vector $\mathbf{v}\Delta t$ as S_1 moves to S_2. It is observed that $d\mathbf{S}_1$ is pointed into the volume bounded by S_1, S_2, and S_3, while $d\mathbf{S}_2$ is pointed outward. The closed surface integral in (6.4) can be transformed to a volume integral, that is,

$$\oiint_{\mathrm{S}} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S} = \iiint_{\mathrm{V}} \nabla\, \mathbf{F}(\mathbf{R},t)\, dV. \tag{6.5}$$

In (6.4) and (6.5),

$$d\mathbf{S}_3 = d\boldsymbol{\ell} \times (\mathbf{v}\,\Delta t)\,, \tag{6.6}$$

$$dV = (\mathbf{v}\,\Delta t)\cdot d\mathbf{S}. \tag{6.7}$$

By making use of the mean-value theorem in calculus, the surface integral in (6.4) evaluated on S_3 and the volume integral in (6.5) can be written in the following form:

$$\begin{aligned}-\iint_{S_3} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S}_3 &= \Delta t \oint_{\mathrm{L}} \mathbf{F}(\mathbf{R},t)\cdot(\mathbf{v}\times d\boldsymbol{\ell}) \\ &= -\Delta t \oint_{\mathrm{L}} [\mathbf{v}\times\mathbf{F}(\mathbf{R},t)]\cdot d\boldsymbol{\ell},\end{aligned} \tag{6.8}$$

$$\iiint_{\mathrm{V}} \nabla\, \mathbf{F}(\mathbf{R},t)\, dV = \Delta t \iint_{\mathrm{S}} [\mathbf{v}\nabla\, \mathbf{F}(\mathbf{R},t)]\cdot d\mathbf{S}. \tag{6.9}$$

Equation (6.4) now becomes

$$\begin{aligned}&\iint_{S_2(\mathbf{R},\ t+\Delta t)} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S} - \iint_{S_1(\mathbf{R},t)} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S} \\ &\quad = \Delta t \left\{ \iint_{\mathrm{S}} [\mathbf{v}\nabla\, \mathbf{F}(\mathbf{R},t)]\cdot d\mathbf{S} - \oint_{\mathrm{L}} [\mathbf{v}\times\mathbf{F}(\mathbf{R},t)]\cdot d\boldsymbol{\ell} \right\} \\ &\quad = \Delta t \left\{ \iint_{\mathrm{S}} \{\mathbf{v}\nabla\, \mathbf{F}(\mathbf{R},t) - \nabla\,[\mathbf{v}\times\mathbf{F}(\mathbf{R},t)]\}\cdot d\mathbf{S} \right\}.\end{aligned} \tag{6.10}$$

In (6.10), the line integral has been converted to a surface integral by means of the Stokes theorem. Equation (6.3), after taking the limit with respect to Δt, yields

$$\begin{aligned}\frac{d}{dt}\iint_{S(\mathbf{R},t)} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S} = \iint_{S(\mathbf{R},t)} \Bigg\{ \frac{\partial \mathbf{F}(\mathbf{R},t)}{\partial t} + \mathbf{v}\nabla\, \mathbf{F}(\mathbf{R},t) \\ - \nabla\,[\mathbf{v}\times\mathbf{F}(\mathbf{R},t)] \Bigg\}\cdot d\mathbf{S},\end{aligned}$$

or simply,

$$\frac{d}{dt}\iint_{S} \mathbf{F}\cdot d\mathbf{S} = \iint_{S}\left[\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\nabla\ \mathbf{F} - \nabla\,(\mathbf{v}\times\mathbf{F})\right]\cdot d\mathbf{S}, \tag{6.11}$$

which represents the Helmholtz transport theorem [12] using the modern notation of vector analysis first formulated by Lorentz [13], and reiterated by Sommerfeld [14]. Equation (6.11) can be cast in a different form by making use of identity (4.159) for $\nabla\ (\mathbf{v}\times\mathbf{F})$, which yields

$$\frac{d}{dt}\iint_{S} \mathbf{F}\cdot d\mathbf{S} = \iint_{S}\left[\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{F} + \mathbf{F}\nabla\mathbf{v} - \mathbf{F}\cdot\nabla\mathbf{v}\right]\cdot d\mathbf{S}. \tag{6.12}$$

This version of the Helmholtz transport theorem is used by Candel and Poinsot [15] in formulating a problem in gas dynamics.

Because the material derivative of **F** or the total time derivative of **F** is defined by

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial t} + \sum_{i=1}^{3}\frac{\partial \mathbf{F}}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{F}, \tag{6.13}$$

(6.12) can be written in the form

$$\frac{d}{dt}\iint_{S} \mathbf{F}\cdot d\mathbf{S} = \iint_{S}\left[\frac{d\mathbf{F}}{dt} + \mathbf{F}\nabla\mathbf{v} - \mathbf{F}\cdot\nabla\mathbf{v}\right]\cdot d\mathbf{S}. \tag{6.14}$$

This form of the Helmholtz theorem is found in the treatment by Truesdell and Toupin [16].

6-2 Maxwell Theorem and Reynolds Transport Theorem

Two related theorems can now be derived from the Helmholtz transport theorem, although in the original works these two theorems were formulated independently of the Helmholtz theorem. In the Helmholtz theorem, if we let

$$\mathbf{F} = \nabla\,\mathbf{f} \tag{6.15}$$

and then convert the surface integral into a line integral, we find, noting that $\nabla\,\mathbf{F} = 0$ in view of (4.178),

$$\frac{d}{dt}\oint_{S} \mathbf{f}\cdot d\boldsymbol{\ell} = \oint_{L}\left(\frac{\partial \mathbf{f}}{\partial t} - \mathbf{v}\times\nabla\,\mathbf{f}\right)\cdot d\boldsymbol{\ell}. \tag{6.16}$$

This is the statement of the Maxwell theorem originally found in his great work on electromagnetic theory [17, 18].

In the Helmholtz theorem, if the surface is a closed one, we obtain

$$\frac{d}{dt}\oiint \mathbf{F}\cdot d\mathbf{S} = \oiint\left(\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\nabla\ \mathbf{F}\right)\cdot d\mathbf{S}. \tag{6.17}$$

The closed surface integral of $\nabla (\mathbf{v} \times \mathbf{F})$ vanishes because it is equal to a volume integral of $\nabla \nabla (\mathbf{v} \times \mathbf{F})$, which vanishes identically because of (4.179).

As a consequence of the Gauss theorem, (6.17) can be changed into the form

$$\frac{d}{dt} \iiint \nabla \mathbf{F} \, dV = \iiint \left[\frac{\partial}{\partial t} \nabla \mathbf{F} + \nabla (\mathbf{v} \nabla \mathbf{F}) \right] dV. \tag{6.18}$$

Now, if we let $\nabla \mathbf{F} = \rho$, a scalar function, then we obtain the Reynolds transport theorem [19], namely,

$$\frac{d}{dt} \iiint \rho \, dV = \iiint \left[\frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{v}) \right] dV, \tag{6.19}$$

where we identify $\mathbf{v}$ as the velocity of the fluid with density ρ. Because

$$\nabla (\rho \mathbf{v}) = \rho \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \rho, \tag{6.20}$$

and the total time derivative, or the material derivative, of ρ is given by

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho,$$

(6.19) can be written in the form

$$\frac{d}{dt} \iiint \rho \, dV = \iiint \left[\frac{d\rho}{dt} + \rho \nabla \mathbf{v} \right] dV. \tag{6.21}$$

It should be mentioned that in the original work of Reynolds, (6.19) was derived by evaluating the total time derivative of $\iiint \rho \, dV$,

$$\frac{d}{dt} \iiint \rho \, dV = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\iiint_{V(t+\Delta t)} \rho(t + \Delta t) \, dV - \iiint_{V(t)} \rho(t) \, dV \right],$$

in a manner very similar to the derivation of the Helmholtz theorem.

From the preceding discussion, it is seen that the Helmholtz transport theorem can be considered as the principal transport theorem; both the Maxwell theorem and the Reynolds theorem can be treated as lemmas of that theorem.