Chapter 3

Line Integrals, Surface Integrals, and Volume Integrals

3-1 Differential Length, Area, and Volume

In this section, we shall give a brief review of the differential quantities to be used in vector analysis and, particularly, their notation. A differential length, in general, will be denoted by $d\boldsymbol{\ell}$. It is the same as the *total differential* of the position vector $d\mathbf{R}_{\mathrm{p}}$. In OCS,

$$d\boldsymbol{\ell} = \sum_i h_i \, dv_i \, \hat{u}_i. \tag{3.1}$$

For a cell with its center located at (v_1, v_2, v_3) and bounded by six surfaces located at $v_i \pm dv_i/2$ with $i = 1, 2, 3$, the *vector differential area* of the three surfaces located at $v_i + dv_i/2$ is then given by

$$\begin{aligned} d\mathbf{S}_i &= h_j \, dv_j \, \hat{u}_j \times h_k \, dv_k \, \hat{u}_k \,\big|_{v_i+dv_i/2} \\ &= h_j \, h_k \, dv_j \, dv_k \, \hat{u}_i \,\big|_{v_i+dv_i/2}, \end{aligned} \tag{3.2}$$

where (i, j, k) follows the cyclic order of $(1, 2, 3)$, $(2, 3, 1)$, and $(3, 1, 2)$. The vector differential areas of the other three surfaces are

$$d\mathbf{S}_i = -h_j \, h_k \, dv_j \, dv_k \, \hat{u}_i \,\big|_{v_i-dv_i/2} \,. \tag{3.3}$$

All of these vector areas are pointed away, or outward, from their surfaces. We should emphasize that the metric coefficients and the corresponding unit vectors are evaluated at the sites of these surfaces $v_i \pm dv_i/2$, not at the center of the cell. The differential volume of the cell is given by

$$\begin{aligned} dV &= h_i\, dv_i\, \hat{u}_i \cdot (h_j\, dv_j\, \hat{u}_j \times h_k\, dv_k\, \hat{u}_k) \\ &= h_i\, h_j\, h_k\, dv_i\, dv_j\, dv_k \\ &= h_1\, h_2\, h_3\, dv_1\, dv_2\, dv_3. \end{aligned} \tag{3.4}$$

3-2 Classification of Line Integrals

If we continuously change the position vector of a point in space in a certain specified manner, the locus of the point will trace a curve in space (Fig. 3-1). Let a typical point on the curve be denoted by $P(x, y, z)$ in the Cartesian coordinate system. If (x, y, z) are functions of a single parameter t, then as t varies, $x(t)$, $y(t)$, and $z(t)$ will vary accordingly. We call such a description the *parametric representation* of a curve. We assume that there is a one-to-one correspondence between t and (x, y, z). The vector differential length of the curve can now be written as

$$d\boldsymbol{\ell} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} = \left(\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right) dt. \tag{3.5}$$

It should be pointed out here that we use dx, dy, dz, and dt to denote the total differential of these variables, but dx/dt, dy/dt, and dz/dt are the derivatives of x, y, z with respect to t. As an example, let

$$\left.\begin{aligned} x &= a\cos\frac{2\pi}{T}t \\ y &= a\sin\frac{2\pi}{T}t \\ z &= \frac{b}{T}t \end{aligned}\right\}, \tag{3.6}$$

where a, b, T are constants and t is the parameter. As t varies, the locus of P describes a right-hand spiral, advancing in the positive z direction as t is increased. When t changes from 0 to T, the spiral starts at $(a, 0, 0)$ and ends at $(a, 0, b)$; therefore, b denotes the height of the spiral after making one complete turn, and

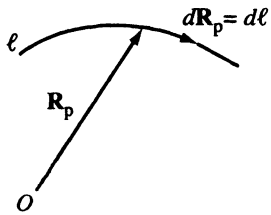


Figure 3-1 Curve in a three-dimensional space.

a denotes the radius of the circular projection of the spiral on the x–y plane. To calculate the length of the spiral, one starts with

$$(d\ell)^2 = d\boldsymbol{\ell} \cdot d\boldsymbol{\ell} = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right](dt)^2;$$

hence

$$d\ell = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2} dt. \tag{3.7}$$

The integral of (3.7) from $t = 0$ to T yields

$$L = \int_0^L d\ell = \frac{1}{T}\int_0^T \left[(2\pi a)^2 + b^2\right]^{1/2} dt$$
$$= \left[(2\pi a)^2 + b^2\right]^{1/2} = \left(c^2 + b^2\right)^{1/2}, \tag{3.8}$$

where c denotes the circumference of the projected circle. The pitch angle of the spiral is defined by

$$\alpha = \tan^{-1}\left(\frac{b}{c}\right). \tag{3.9}$$

Equation (3.8) represents the simplest form of a line integral. In general, we define the line integral of Type I as

$$I_1 = \int_c f(x, y, z)\, d\ell, \tag{3.10}$$

where c denotes the contour of the curve wherein the integration is executed.

As an example, let c be a parabolic curve described by

$$y^2 = 2x \quad \text{in the plane } z = 0, \tag{3.11}$$

and the contour extends from $x = 0,\ y = 0$ to $x = \frac{3}{2},\ y = \sqrt{3}$, and the function in (3.10) is supposed to be $f(x, y) = xy$.

If we choose y as the parameter, then

$$d\boldsymbol{\ell} = dx\,\hat{x} + dy\,\hat{y} = y\,dy\,\hat{x} + dy\,\hat{y},$$
$$d\ell = (y^2 + 1)^{1/2}\, dy,$$
$$f(x, y) = \frac{1}{2}y^3;$$

thus,

$$I_1 = \int_c f(x, y)\, d\ell = \int_0^{\sqrt{3}} \frac{1}{2}\, y^3 (y^2 + 1)^{1/2}\, dy = \frac{28}{15}.$$

We purposely choose $f(x, y)$ and c in such a manner that the integral can be evaluated in a closed form in order to clearly illustrate the steps.

The second type of line integral is defined by

$$\mathbf{I}_2 = \int_c \mathbf{f}(x, y, z)\, d\ell, \tag{3.12}$$

where $\mathbf{f}$ is a vector function.

If we write $\mathbf{f}$ in its component form in the Cartesian coordinate system,

$$\mathbf{f}(x, y, z) = f_x(x, y, z)\,\hat{x} + f_y(x, y, z)\,\hat{y} + f_z(x, y, z)\,\hat{z}, \tag{3.13}$$

and because $\hat{x}$, $\hat{y}$, $\hat{z}$ are constant vectors, we can change (3.12) into the form

$$\mathbf{I}_2 = \hat{x}\int_c f_x\, d\ell + \hat{y}\int_c f_y\, d\ell + \hat{z}\int f_z\, d\ell. \tag{3.14}$$

The three integrals contained in (3.14) are of Type I, which can be evaluated according to the method described previously. Line integrals of Types III, IV, and V are defined by

$$\mathbf{I}_3 = \int_c f(x, y, z)\, d\boldsymbol{\ell}, \tag{3.15}$$

$$I_4 = \int_c \mathbf{f}(x, y, z) \cdot d\boldsymbol{\ell}, \tag{3.16}$$

$$\mathbf{I}_5 = \int_c \mathbf{f}(x, y, z) \times d\boldsymbol{\ell}. \tag{3.17}$$

Integrals of Type III can be resolved into three integrals, that is,

$$\mathbf{I}_3 = \hat{x}\int_c f\, dx + \hat{y}\int f\, dy + \hat{z}\int_c f\, dz. \tag{3.18}$$

The three scalar integrals in (3.18) can be evaluated by choosing a proper parameter for each integral. In fact, one can, for example, use x as a parameter for the first integral and express both y and z in terms of x. In the case of the spiral contour, if we let z be the parameter, then

$$x = a\cos\frac{2\pi}{b}z, \qquad y = a\sin\frac{2\pi}{b}z.$$

Integrals of Type IV can be converted to

$$I_4 = \int_c (f_x\, dx + f_y\, dy + f_z\, dz). \tag{3.19}$$

Again, each term in (3.19) can be evaluated by the parametric method. Integrals of Type V can be written in the form

$$\mathbf{I}_5 = \hat{x}\int_c (f_2\, dz - f_3\, dy) + \hat{y}\int (f_3\, dx - f_1\, dy) + \hat{z}\int_c (f_1\, dy - f_2\, dx). \tag{3.20}$$

All six terms in (3.20) can be evaluated in the same way. Thus, if the functions f and $\mathbf{f}$ and the differential lengths $d\ell$ and $d\boldsymbol{\ell}$ are expressed in a Cartesian system, we have a systematic method to evaluate all different types of line integrals. In many cases, the curve under consideration may correspond to the intersection of two surfaces represented by

$$z = F_1(x, y), \tag{3.21}$$

$$z = F_2(x, y). \tag{3.22}$$

In that case, we can eliminate z between (3.21) and (3.22) so that

$$F_1(x, y) = F_2(x, y) \tag{3.23}$$

and then solve for x in terms of y to yield

$$x = F_3(y), \tag{3.24}$$

$$z = F_1[F_3(y), y]. \tag{3.25}$$

It is obvious that y can be used as the parameter for the curve. In many problems, it is sometimes rather difficult to find the explicit form of F_3 unless F_1 and F_2 are relatively simple functions.

If the integrals and the contour c are described in an orthogonal system other than a Cartesian system, then

$$d\ell = \left[\sum_{i=1}^{3}(h_i\, dv_i)^2\right]^{1/2} \tag{3.26}$$

and

$$d\boldsymbol{\ell} = \sum_{i=1}^{3} h_i\, dv_i\, \hat{u}_i. \tag{3.27}$$

A scalar function f is then assumed to be a function of (v_1, v_2, v_3), and a vector function $\mathbf{f}$ would be a function of both the v_i's and $\hat{u}_i$'s. Integrals of Types I and IV can be evaluated by expressing the v_i's and h_i's in terms of a single parameter, as was done previously. The integrands of integrals of Types II, III, and V contain $\hat{u}_i$'s that are, in general, not constant vectors, so they cannot be removed to the outside of these integrals. For these cases, we can transform the $\hat{u}_i$'s in terms of $\hat{x}$, $\hat{y}$, and $\hat{z}$ in the form

$$\hat{u}_i = \cos\alpha_i\, \hat{x} + \cos\beta_i\, \hat{y} + \cos\gamma_i\, \hat{z}, \qquad i = 1, 2, 3. \tag{3.28}$$

Then

$$\begin{aligned} \mathbf{f} = \sum_{i=1}^{3} f_i\, \hat{u}_i &= \sum_{i=1}^{3} f_i \left(\cos\alpha_i\, \hat{x} + \cos\beta_i\, \hat{y} + \cos\gamma_i\, \hat{z}\right) \\ &= f_x\, \hat{x} + f_y\, \hat{y} + f_z\, \hat{z}, \end{aligned} \tag{3.29}$$

where

$$f_x = \sum_{i=1}^{3} f_i\, \cos\alpha_i, \tag{3.30}$$

$$f_y = \sum_{i=1}^{3} f_i \cos \beta_i, \tag{3.31}$$

$$f_z = \sum_{i=1}^{3} f_i \cos \gamma_i. \tag{3.32}$$

Afterwards, the unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ can be removed to the outside of the integrals, and the remaining scalar integrals can be evaluated by the parametric method. In a later section, we will introduce a relatively simple method to find the transformation of the unit vectors from one system to another like (3.28).

3-3 Classification of Surface Integrals

A surface in a three-dimensional space, in general, is characterized by a governing equation

$$F(x, y, z) = 0, \tag{3.33}$$

in which we can select any two variables as independent and the remaining one will be a dependent variable. We assume that we can convert (3.33) into the explicit form

$$S: \qquad z = f(x, y). \tag{3.34}$$

For two neighboring points located on S, the total differential of the displacement vector between the two adjacent points can be written in the form

$$d\mathbf{R}_p = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}. \tag{3.35}$$

Only two of the Cartesian variables are independent because of the constraint stated by (3.34). If the same surface can be described by the coordinates (v_1, v_2, v_3) with unit vectors $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ and metric coefficients $(h_1, h_2, 1)$ in a Dupin system, then the total differential of a displacement on the surface can be written as (Fig. 3-2)

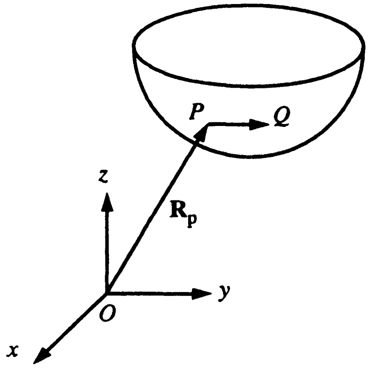
$$d\mathbf{R}_p = h_1\,dv_1\,\hat{u}_1 + h_2\,dv_2\,\hat{u}_2 \tag{3.36}$$

with $dv_3 = 0$.

The partial derivatives of (3.35) and (3.36) with respect to v_1 and v_2 are

$$\frac{\partial \mathbf{R}_p}{\partial v_1} = h_1\hat{u}_1 = \frac{\partial x}{\partial v_1}\hat{x} + \frac{\partial y}{\partial v_1}\hat{y} + \frac{\partial z}{\partial v_1}\hat{z}, \tag{3.37}$$

$$\frac{\partial \mathbf{R}_p}{\partial v_2} = h_2\hat{u}_2 = \frac{\partial x}{\partial v_2}\hat{x} + \frac{\partial y}{\partial v_2}\hat{y} + \frac{\partial z}{\partial v_2}\hat{z}. \tag{3.38}$$



$P(x, y, z)$; $P(v_1, v_2)$

$Q(x + dx, y + dy, z + dz)$;

$Q(v_1 + dv_1, v_2 + dv_2)$

$$d\mathbf{R}_p = h_1\, dv_1\, \hat{u}_1 + h_2\, dv_2\, \hat{u}_2 = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

Figure 3-2 Total differential of the position vector on a surface $z = f(x, y)$ where $v_3 =$ constant.

Let the vector differential area of the surface be denoted by $d\mathbf{S}$; then

$$\begin{aligned} d\mathbf{S} &= h_1\, dv_1\, \hat{u}_1 \times h_2\, dv_2\, \hat{u}_2 \\ &= h_1\, h_2\, dv_1\, dv_2\, \hat{u}_3 \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial x}{\partial v_1} & \dfrac{\partial y}{\partial v_1} & \dfrac{\partial z}{\partial v_1} \\ \dfrac{\partial x}{\partial v_2} & \dfrac{\partial y}{\partial v_2} & \dfrac{\partial z}{\partial v_2} \end{vmatrix} dv_1\, dv_2 = \mathbf{J}\, dv_1\, dv_2. \end{aligned} \tag{3.39}$$

The determinant in (3.39), denoted by $\mathbf{J}$, results from $h_1\, \hat{u}_1 \times h_2\, \hat{u}_2$. For convenience, we will call it the vector Jacobian of transformation between (x, y, z) and (v_1, v_2). If we write

$$\mathbf{J} = J_x\, \hat{x} + J_y\, \hat{y} + J_z\, \hat{z}, \tag{3.40}$$

then

$$J_x = \begin{vmatrix} \dfrac{\partial y}{\partial v_1} & \dfrac{\partial z}{\partial v_1} \\ \dfrac{\partial y}{\partial v_2} & \dfrac{\partial z}{\partial v_2} \end{vmatrix} \tag{3.41}$$

is the Jacobian of transformation between (y, z) and (v_1, v_2). Alternatively, it may be denoted by

$$J_x = \frac{\partial\, (y, z)}{\partial\, (v_1, v_2)}\,. \tag{3.42}$$

Similarly,

$$J_y = \frac{\partial (z, x)}{\partial (v_1, v_2)} \tag{3.43}$$

and

$$J_z = \frac{\partial (x, y)}{\partial (v_1, v_2)} \; . \tag{3.44}$$

Now, let us consider the case where the rectangular variables (x, y) are selected as (v_1, v_2); then

$$\mathbf{J}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & \dfrac{\partial z}{\partial x} \\ 0 & 1 & \dfrac{\partial z}{\partial y} \end{vmatrix} = -\frac{\partial z}{\partial x}\hat{x} - \frac{\partial z}{\partial y}\hat{y} + \hat{z}. \tag{3.45}$$

The subscript 1 attached to $\mathbf{J}_1$ means that this is our first choice or first case. From (3.45), we can determine the unit normal vector $\hat{u}_3$, namely,

$$\hat{u}_3 = \frac{\mathbf{J}_1}{|\mathbf{J}_1|} = \frac{-\dfrac{\partial z}{\partial x}\hat{x} - \dfrac{\partial z}{\partial y}\hat{y} + \hat{z}}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} \; . \tag{3.46}$$

The directional cosines of $\hat{u}_3$ are therefore given by

$$\cos \alpha_3 = \frac{-\dfrac{\partial z}{\partial x}}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} \; , \tag{3.47}$$

$$\cos \beta_3 = \frac{-\dfrac{\partial z}{\partial y}}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} \; , \tag{3.48}$$

$$\cos \gamma_3 = \frac{1}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} \; . \tag{3.49}$$

Based on (3.39) and (3.45), we find

$$\begin{aligned} dS = |d\mathbf{S}| &= |\mathbf{J}_1|\, dv_1\, dv_2 \\ &= \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2} dx\, dy \\ &= \frac{1}{\cos\gamma_3}\, dx\, dy. \end{aligned} \tag{3.50}$$

Equation (3.50) can be used to find the area of a surface. As an example, let the surface be a portion of a parabola of revolution described by

$$S: \qquad z = \frac{1}{2}(x^2 + y^2), \qquad \frac{1}{2} \geq z \geq 0. \tag{3.51}$$

Then

$$\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2} = (x^2 + y^2 + 1)^{1/2}. \tag{3.52}$$

Hence

$$S = \iint_{S_1} dS = \iint_{S_1} (x^2 + y^2 + 1)^{1/2}\, dx\, dy, \tag{3.53}$$

where S_1 denotes the domain of integration with respect to (x, y), covering the projection of S in the x–y plane. For this particular example, it is convenient to convert (3.53) into an integral with respect to the cylindrical variables r and ϕ, that is,

$$S = \iint_{S_1} (1 + r^2)^{1/2}\, r\, dr\, d\phi = \int_0^{2\pi}\int_0^1 (1 + r^2)^{1/2}\, r\, dr\, d\phi = \frac{4\sqrt{2}\pi}{3}. \tag{3.54}$$

Here, we have used the transformation

$$dx\, dy = \frac{\partial(x, y)}{\partial(r, \phi)}\, dr\, d\phi, \tag{3.55}$$

where

$$\frac{\partial(x, y)}{\partial(r, \phi)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\[2ex] \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} \end{vmatrix} = r \tag{3.56}$$

with

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Equation (3.55) is a special case of (3.39) when it is applied to a plane surface corresponding to the x–y plane.

Returning now to the expression for **J** defined by (3.39), we can select either (y, z) or (z, x) as two alternative choices for (v_1, v_2); then

$$\mathbf{J}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial x}{\partial y} & 1 & 0 \\ \dfrac{\partial x}{\partial z} & 0 & 1 \end{vmatrix} = \hat{x} - \frac{\partial x}{\partial y}\hat{y} - \frac{\partial x}{\partial z}\hat{z}. \tag{3.57}$$

In this case, x is the dependent variable. The expression for the unit vector $\hat{u}_3$ is now given by

$$\hat{u}_3 = \frac{\mathbf{J}_2}{|\mathbf{J}_2|} = \frac{\hat{x} - \dfrac{\partial x}{\partial y}\hat{y} - \dfrac{\partial x}{\partial z}\hat{z}}{\left[1 + \left(\dfrac{\partial x}{\partial y}\right)^2 + \left(\dfrac{\partial x}{\partial z}\right)^2\right]^{1/2}} \tag{3.58}$$

and

$$d\mathbf{S} = \mathbf{J}_2\, dy\, dz = |\mathbf{J}_2|\, dy\, dz\, \hat{u}_3 = \frac{1}{\cos\alpha_3}\, dy\, dz\, \hat{u}_3.$$

Similarly, we have

$$\mathbf{J}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dfrac{\partial y}{\partial z} & 1 \\ 1 & \dfrac{\partial y}{\partial x} & 0 \end{vmatrix} = -\frac{\partial y}{\partial x}\hat{x} + \hat{y} - \frac{\partial y}{\partial z}\hat{z}, \tag{3.59}$$

$$\hat{u}_3 = \frac{\mathbf{J}_3}{|\mathbf{J}_3|} = \frac{-\dfrac{\partial y}{\partial x}\hat{x} + \hat{y} - \dfrac{\partial y}{\partial z}\hat{z}}{\left[\left(\dfrac{\partial y}{\partial x}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + 1\right]^{1/2}}, \tag{3.60}$$

$$d\mathbf{S} = \mathbf{J}_3\, dx\, dz = |\mathbf{J}_3|\, dx\, dz\, \hat{u}_3 = \frac{1}{\cos\beta_3}\, dx\, dz\, \hat{u}_3. \tag{3.61}$$

The directional cosines of $\hat{u}_3$, therefore, can be expressed in several different forms. Our three different choices of (v_1, v_2) yield

$$\cos\alpha_3 = \frac{-\dfrac{\partial z}{\partial x}}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} = \frac{1}{\left[\left(\dfrac{\partial x}{\partial y}\right)^2 + \left(\dfrac{\partial x}{\partial z}\right)^2 + 1\right]^{1/2}}$$
$$= \frac{-\dfrac{\partial y}{\partial x}}{\left[\left(\dfrac{\partial y}{\partial x}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + 1\right]^{1/2}}, \tag{3.62}$$

$$\cos\beta_3 = \frac{-\dfrac{\partial z}{\partial y}}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} = \frac{-\dfrac{\partial x}{\partial y}}{\left[\left(\dfrac{\partial x}{\partial y}\right)^2 + \left(\dfrac{\partial x}{\partial z}\right)^2 + 1\right]^{1/2}}$$
$$= \frac{1}{\left[\left(\dfrac{\partial y}{\partial x}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + 1\right]^{1/2}}, \tag{3.63}$$

$$\cos\gamma_3 = \frac{1}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} = \frac{-\dfrac{\partial x}{\partial z}}{\left[\left(\dfrac{\partial x}{\partial y}\right)^2 + \left(\dfrac{\partial x}{\partial z}\right)^2 + 1\right]^{1/2}}$$
$$= \frac{-\dfrac{\partial y}{\partial z}}{\left[\left(\dfrac{\partial y}{\partial x}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + 1\right]^{1/2}}. \tag{3.64}$$

We remind the student that

$$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy} \tag{3.65}$$

for functions of single variables, such as $y = f(x)$, but

$$\frac{\partial y}{\partial x} \neq 1 \Big/ \frac{\partial x}{\partial y} \tag{3.66}$$

for functions of multiple variables. As an example, we consider the relations

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Then

$$r = \left(x^2 + y^2\right)^{1/2}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right),$$

so that

$$\frac{\partial x}{\partial r} = \cos\phi$$

and

$$\frac{\partial r}{\partial x} = \frac{x}{\left(x^2 + y^2\right)^{1/2}} = \cos\phi.$$

That is,

$$\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x},$$

while

$$\frac{\partial x}{\partial \phi} = -r\sin\phi, \qquad \frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r},$$

so that

$$\frac{\partial x}{\partial \phi} = r^2\frac{\partial \phi}{\partial x}.$$

One must therefore be very careful to distinguish between the dependent and independent variables.

Like the line integrals, there are five types of surface integrals. From now on, functions of space variables (x, y, z) or (v_1, v_2, v_3) will be denoted by $F(\mathbf{R}_\mathrm{p})$ or $\mathbf{F}(\mathbf{R}_\mathrm{p})$, where $\mathbf{R}_\mathrm{p}$ denotes the position vector. The five types of surface integrals are as follows:

Type I:

$$I_1 = \iint_S F(\mathbf{R}_\mathrm{p})\,dS, \tag{3.67}$$

Type II:

$$\mathbf{I}_2 = \iint_S \mathbf{F}(\mathbf{R}_\mathrm{p})\,dS, \tag{3.68}$$

Type III:

$$\mathbf{I}_3 = \iint_S F(\mathbf{R}_\mathrm{p})\,d\mathbf{S}, \tag{3.69}$$

Type IV:

$$I_4 = \iint_S \mathbf{F}(\mathbf{R}_p) \cdot d\mathbf{S}, \tag{3.70}$$

Type V:

$$\mathbf{I}_5 = \iint_S \mathbf{F}(\mathbf{R}_p) \times d\mathbf{S}, \tag{3.71}$$

where $F(\mathbf{R}_p)$ is a scalar function of position and $\mathbf{F}(\mathbf{R}_p)$ denotes a vector function.

We assume that the surface S can be described by a governing equation of the form

$$z = f(x, y). \tag{3.72}$$

The same surface can always be considered as a normal surface ($v_3 = 0$) in a proper Dupin system with parameters (v_1, v_2), $(\hat{u}_1, \hat{u}_2)$ and metric coefficients (h_1, h_2). Treating v_1 and v_2 as two independent variables, we can write

$$x = f_1(v_1, v_2), \tag{3.73}$$

$$y = f_2(v_1, v_2), \tag{3.74}$$

$$z = f(x, y) = f[f_1(v_1, v_2), f_2(v_1, v_2)] = f_3(v_1, v_2). \tag{3.75}$$

The functions $F(\mathbf{r}_p)$ and $\mathbf{F}(\mathbf{r}_p)$ contained in (3.67)–(3.71), therefore, can be changed into functions of (v_1, v_2) for the scalar function, and of (v_1, v_2) as well as $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ for the vector function. An integral of Type I can be transformed into

$$I_1 = \iint_S F(v_1, v_2) \, |\mathbf{J}| \, dv_1 \, dv_2, \tag{3.76}$$

which can be evaluated by the parametric method. Thus, if we let $(v_1, v_2) = (x, y)$, (3.76) becomes

$$I_1 = \iint_{S_3} F(x, y) \frac{1}{\cos \gamma_3} \, dx \, dy, \tag{3.77}$$

where S_3 denotes the domain of integration on the x–y plane covered by the projection of S on that plane. The execution to carry out the integration is very similar to the problem of finding the area of a curved surface, except that the integrand contains an additional function.

An integral of Type II is equivalent to

$$\begin{aligned} \mathbf{I}_2 = \hat{x} &\iint F_x(v_1, v_2) \, |\mathbf{J}| \, dv_1 \, dv_2 \\ &+ \hat{y} \iint F_y(v_1, v_2) \, |\mathbf{J}| \, dv_1 \, dv_2 \\ &+ \hat{z} \iint F_z(v_1, v_2) \, |\mathbf{J}| \, dv_1 \, dv_2. \end{aligned} \tag{3.78}$$

The three scalar integrals in (3.78) are of Type I. However, it is not necessary to use the same set of (v_1, v_2) for these integrals.

An integral of Type III, in view of (3.39) and (3.40), is equivalent to

$$\mathbf{I}_3 = \iint F(v_1, v_2)\, \mathbf{J}\, dv_1\, dv_2 = \hat{x} \iint J_x\, F(v_1, v_2)\, dv_1\, dv_2 + \hat{y} \iint J_y\, F(v_1, v_2)\, dv_1\, dv_2 + \hat{z} \iint J_z\, F(v_1, v_2)\, dv_1\, dv_2. \tag{3.79}$$

The three scalar integrals in (3.79) are of Type I with different integrands.

An integral of Type IV is equivalent to

$$I_4 = \iint \left(J_x\, F_x + J_y\, F_y + J_z\, F_z\right) dS, \tag{3.80}$$

which belongs to Type I. Here, we have omitted the functional dependence of these functions and the Jacobians on (v_1, v_2).

Finally, an integral of Type V is equivalent to

$$\mathbf{I}_5 = \hat{x} \iint \left(J_z\, F_y - J_y\, F_z\right) dv_1\, dv_2 + \hat{y} \iint \left(J_x\, F_z - J_z\, F_x\right) dv_1\, dv_2 + \hat{z} \iint \left(J_y\, F_x - J_x\, F_y\right) dv_1\, dv_2. \tag{3.81}$$

All of the scalar integrals in (3.81) are of Type I. In essence, an integral of Type I is the basic one; all other types of integrals can be reduced to that type. The choice of (v_1, v_2) depends greatly on the exact nature of the problem. Many integrals resulting from the formulation of physical problems may not be evaluated in a closed form. In these cases, we can seek the help of a numerical method.

3-4 Classification of Volume Integrals

There are only two types of volume integrals:

Type I:

$$I_1 = \iiint_V F(\mathbf{R}_p)\, dV, \tag{3.82}$$

Type II:

$$\mathbf{I}_2 = \iiint_V \mathbf{F}(\mathbf{R}_p)\, dV, \tag{3.83}$$

where V denotes the domain of integration, which can be either bounded or covering the entire space. We now have three independent variables. In an orthogonal system, they are (v_1, v_2, v_3). An integral of Type I, when expressed in that system, becomes

$$I_1 = \iiint_V F(v_1, v_2, v_3)\, h_1\, h_2\, h_3\, dv_1\, dv_2\, dv_3. \tag{3.84}$$

The choice of the proper coordinate system depends greatly on the shape of V. From the point of view of the numerical method, we can always use a rectangular coordinate system to partition the region of integration.

An integral of Type II is equivalent to

$$\mathbf{I}_2 = \hat{x} \iiint F_x\, dV + \hat{y} \iiint F_y\, dV + \hat{z} \iiint F_z\, dV. \tag{3.85}$$

The three scalar integrals in (3.85) are of Type I. We will not discuss the actual evaluation of (3.84), as the method is described in many standard books on calculus.