Differential Forms in Electromagnetics. Ismo V. Lindell
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Multivectors

1.1 THE GRASSMANN ALGEBRA

The exterior algebra associated with differential forms is also known as the Grassmann algebra. Its originator was Hermann Grassmann (1809–1877), a German mathematician and philologist who mainly acted as a high-school teacher in Stettin (presently Szczecin in Poland) without ever obtaining a university position.[1] His father, Justus Grassmann, also a high-school teacher, authored two textbooks on elementary mathematics, *Raumlehre* (*Theory of the Space*, 1824) and *Trigonometrie* (1835). They contained footnotes where Justus Grassmann anticipated an algebra associated with geometry. In his view, a parallelogram was a geometric product of its sides whereas a parallelepiped was a product of its height and base parallelogram. This must have had an effect on Hermann Grassmann's way of thinking and eventually developed into the algebra carrying his name.

In the beginning of the 19th century, the classical analysis based on Cartesian coordinates appeared cumbersome for many simple geometric problems. Because problems in planar geometry could also be solved in a simple and elegant way in terms of complex variables, this inspired a search for a three-dimensional complex analysis. The generalization seemed, however, to be impossible.

To show his competence for a high-school position, Grassmann wrote an extensive treatise(over 200 pages), *Theorie der Ebbe und Flut* (*Theory of Tidal Movement*, 1840). There he introduced a geometrical analysis involving addition and differentiation of oriented line segments (Strecken), or vectors in modern language. By

[1]This historical review is based mainly on reference 15. See also references 22, 37 and 39.

generalizing the idea given by his father, he defined the geometrical product of two vectors as the area of a parallelogram and that of three vectors as the volume of a parallelepiped. In addition to the geometrical product, Grassmann defined also a linear product of vectors (the dot product). This was well before the famous day, Monday October 16, 1843, when William Rowan Hamilton (1805-1865) discovered the four-dimensional complex numbers, the quaternions.

During 1842–43 Grassmann wrote the book *Lineale Ausdehnungslehre* (*Linear Extension Theory*, 1844), in which he generalized the previous concepts. The book was a great disappointment: it hardly sold at all, and finally in 1864 the publisher destroyed the remaining stock of 600 copies. *Ausdehnungslehre* contained philosophical arguments and thus was extremely hard to read. This was seen from the fact that no one would write a review of the book. Grassmann considered algebraic quantities which could be numbers, line segments, oriented areas, and so on, and defined 16 relations between them. He generalized everything to a space of n dimensions, which created more difficulties for the reader.

The geometrical product of the previous treatise was renamed as outer product. For example, in the outer product ab of two vectors (line segments) a and b the vector a was moved parallel to itself to a distance defined by the vector b, whence the product ab defined a parallelogram with an orientation. The orientation was reversed when the order was reversed: $ab = -ba$. If the parallelogram ab was moved by the vector c, the product abc gave a parallelepiped with an orientation. The outer product was more general than the geometric product, because it could be extended to a space of n dimensions. Thus it could be applied to solving a set of linear equations without a geometric interpretation.

During two decades the scientific world took the *Ausdehnungslehre* with total silence, although Grassmann had sent copies of his book to many well-known mathematicians asking for their comments. Finally, in 1845, he had to write a summary of his book by himself.

Only very few scientists showed any interest during the 1840s and 1850s. One of them was Adhemar-Jean-Claude de Saint-Venant, who himself had developed a corresponding algebra. In his article "Sommes et différences géométriques pour simplifier la mecanique" (Geometrical sums and differences for the simplification of mechanics, 1845), he very briefly introduced addition, subtraction, and differentiation of vectors and a similar outer product. Also, Augustin Cauchy had in 1853 developed a method to solve linear algebraic equations in terms of anticommutative elements ($ij = -ji$), which he called "clefs algébraiques" (algebraic keys). In 1852 Hamilton obtained a copy of Grassmann's book and expressed first his admiration which later turned to irony ("the greater the extension, the smaller the intention"). The afterworld has, however, considered the *Ausdehnungslehre* as a first classic of linear algebra, followed by Hamilton's book *Lectures on Quaternions* (1853).

During 1844–1862 Grassmann authored books and scientific articles on physics, philology (he is still a well-known authority in Sanscrit) and folklore (he published a collection of folk songs). However, his attempts to get a university position were not succesful, although in 1852 he was granted the title of Professor. Eventually, Grassmann published a completely rewritten version of his book, *Vollständige Aus-*

$a = \frac{dH}{dy} - \frac{dG}{dz}$		
$b = \frac{dF}{dz} - \frac{dH}{dx}$	(A)	$\mathbf{B} = \nabla \times \mathbf{A}$
$c = \frac{dG}{dx} - \frac{dF}{dy}$		
$P = c\frac{dy}{dt} - b\frac{dz}{dt} - \frac{dF}{dt} - \frac{d\psi}{dx}$		
$Q = a\frac{dz}{dt} - c\frac{dx}{dt} - \frac{dG}{dt} - \frac{d\psi}{dy}$	(B)	$\mathbf{E} = \mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$
$R = b\frac{dx}{dt} - a\frac{dy}{dt} - \frac{dH}{dt} - \frac{d\psi}{dz}$		
$X = vc - wb$		
$Y = wa - uc$	(C)	$\mathbf{F} = \mathbf{J} \times \mathbf{B}$
$Z = ub - va$		
$a = \alpha + 4\pi A$		
$b = \beta + 4\pi B$	(D)	$\mathbf{B} = \mu_o \mathbf{H} + \mathbf{M}$
$c = \gamma + 4\pi C$		
$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz}$		
$4\pi v = \frac{d\alpha}{dz} - \frac{d\gamma}{dx}$	(E)	$\mathbf{J} = \nabla \times \mathbf{H}$
$4\pi w = \frac{d\beta}{dx} - \frac{d\alpha}{dy}$		
$\mathfrak{D} = \frac{1}{4\pi} K \mathfrak{E}$	(F)	$\mathbf{D} = \epsilon \mathbf{E}$
$\mathfrak{K} = C \mathfrak{E}$	(G)	$\mathbf{J}_c = \sigma \mathbf{E}$
$\mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}}$	(H)	$\mathbf{J} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$
$u = p + \frac{df}{dt}$		
$v = q + \frac{dg}{dt}$	(H*)	$\mathbf{J} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$
$w = r + \frac{dh}{dt}$		
$\mathfrak{C} = (C + \frac{1}{4\pi} K \frac{d}{dt}) \mathfrak{E}$	(I)	$\mathbf{J} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}$
$u = CP + \frac{1}{4\pi} K \frac{dP}{dt}$		
$v = CQ + \frac{1}{4\pi} K \frac{dQ}{dt}$	(I*)	$\mathbf{J} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}$
$w = CR + \frac{1}{4\pi} K \frac{dR}{dt}$		
$\rho = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$	(J)	$\varrho = \nabla \cdot \mathbf{D}$
$\sigma = lf + mg + nh + l'f' + m'g' + n'h'$	(K)	$\varrho_s = \mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2)$
$\mathfrak{B} = \mu \mathfrak{H}$	(L)	$\mathbf{B} = \mu \mathbf{H}$

Fig. 1.1 *The original set of equations (A)–(L) as labeled by Maxwell in his Treatise (1873), with their interpretation in modern Gibbsian vector notation. The simplest equations were also written in vector form.*

dehnungslehre (*Complete Extension Theory*), on which he had started to work in 1854. The foreword bears the date 29 August 1861. Grassmann had it printed on his own expense in 300 copies by the printer Enslin in Berlin in 1862 [29]. In its preface he complained the poor reception of the first version and promised to give his arguments in Euclidean rigor in the present version.[2] Indeed, instead of relying on philosophical and physical arguments, the book was based on mathematical theorems. However, the reception of the second version was similar to that of the first one. Only in 1867 Hermann Hankel wrote a comparative article on the Grassmann algebra and quaternions, which started an interest in Grassmann's work. Finally there was also growing interest in the first edition of the *Ausdehnungslehre*, which made the publisher release a new printing in 1879, after Grassmann's death. Toward the end of his life, Grassmann had, however, turned his interest from mathematics to philology, which brought him an honorary doctorate among other signs of appreciation.

Although Grassmann's algebra could have become an important new mathematical branch during his lifetime, it did not. One of the reasons for this was the difficulty in reading his books. The first one was not a normal mathematical monograph with definitions and proofs. Grassmann gave his views on the new concepts in a very abstract way. It is true that extended quantities (Ausdehnungsgrösse) like multivectors in a space of n dimensions were very abstract concepts, and they were not easily digestible. Another reason for the poor reception for the Grassmann algebra is that Grassmann worked in a high school instead of a university where he could have had a group of scientists around him. As a third reason, we might recall that there was no great need for a vector algebra before the the arrival of Maxwell's electromagnetic theory in the 1870s, which involved interactions of many vector quantites. Their representation in terms of scalar quantites, as was done by Maxwell himself, created a messy set of equations which were understood only by a few scientist of his time (Figure 1.1).

After a short success period of Hamilton's quaternions in 1860-1890, the vector notation created by J. Willard Gibbs (1839–1903) and Oliver Heaviside (1850–1925) for the three-dimensional space overtook the analysis in physics and electromagnetics during the 1890s. Einstein's theory of relativity and Minkowski's space of four dimensions brought along the tensor calculus in the early 1900s. William Kingdon Clifford (1845–1879) was one of the first mathematicians to know both Hamilton's quaternions and Grassmann's analysis. A combination of these presently known as the Clifford algebra has been applied in physics to some extent since the 1930's [33,54]. Élie Cartan (1869–1951) finally developed the theory of differential forms based on the outer product of the Grassmann algebra in the early 1900s. It was adopted by others in the 1930s. Even if differential forms are generally applied in physics, in electromagnetics the Gibbsian vector algebra is still the most common method of notation. However, representation of the Maxwell equations in terms of differential forms has remarkable simple form in four-dimensional space-time (Figure 1.2).

[2] This book was only very recently translated into English [29] based on an edited version which appeared in the collected works of Grassmann.

$$\begin{array}{rcl} \mathbf{d} \wedge \mathbf{\Psi} & = & \gamma \\ \mathbf{d} \wedge \mathbf{\Phi} & = & 0 \\ \mathbf{\Psi} & = & \overline{\overline{\mathsf{M}}}|\mathbf{\Phi} \end{array}$$

Fig. 1.2 *The two Maxwell equations and the medium equation in differential-form formalism. Symbols will be explained in Chapter 4.*

Grassmann had hoped that the second edition of *Ausdehnungslehre* would raise interest in his contemporaries. Fearing that this, too, would be of no avail, his final sentences in the foreword were addressed to future generations [15, 75]:

> ... But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another seventeen years or even longer, without entering into actual development of science, still that time will come when it will be brought forth from the dust of oblivion, and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me in a position (which I have up to now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich further these ideas, nevertheless there will come a time when these ideas, perhaps in a new form, will rise anew and will enter into living communication with contemporary developments. For truth is eternal and divine, and no phase in the development of the truth divine, and no phase in the development of truth, however small may be region encompassed, can pass on without leaving a trace; truth remains, even though the garments in which poor mortals clothe it may fall to dust.
>
> Stettin, 29 August 1861

1.2 VECTORS AND DUAL VECTORS

1.2.1 Basic definitions

Vectors are elements of an n-dimensional vector space denoted by $\mathbb{E}_1(n)$, and they are in general denoted by boldface lowercase Latin letters $\mathbf{a}$, $\mathbf{b}$,... Most of the analysis is applicable to any dimension n but special attention is given to three-dimensional Euclidean (Eu3) and four-dimensional Minkowskian (Mi4) spaces (these concepts will be explained in terms of metric dyadics in Section 2.5). A set of linearly independent vectors $\{\mathbf{e}_i\} = \mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ forms a basis if any vector $\mathbf{a}$ can be uniquely expressed in terms of the basis vectors as

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i, \tag{1.1}$$

where the a_i are scalar coefficients (real or complex numbers).

Dual vectors are elements of another n-dimensional vector space denoted by $\mathbb{F}_1(n)$, and they are in general denoted by boldface Greek letters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$,... A dual vector basis is denoted by $\{\varepsilon_i\} = \varepsilon_1, ..., \varepsilon_n$. Any dual vector $\boldsymbol{\alpha}$ can be uniquely expressed in terms of the dual basis vectors as

$$\boldsymbol{\alpha} = \sum_{i=1}^{n} \alpha_i \varepsilon_i, \tag{1.2}$$

with scalar coefficients α_i. Many properties valid for vectors are equally valid for dual vectors and conversely. To save space, in obvious cases, this fact is not explicitly stated.

Working with two different types of vectors is one factor that distinguishes the present analysis from the classical Gibbsian vector analysis [28]. Vector-like quantities in physics can be identified by their nature to be either vectors or dual vectors, or, rather, multivectors or dual multivectors to be discussed below. The disadvantage of this division is, of course, that there are more quantities to memorize. The advantage is, however, that some operation rules become more compact and valid for all space dimensions. Also, being a multivector or a dual multivector is a property similar to the dimension of a physical quantity which can be used in checking equations with complicated expressions. One could include additional properties to multivectors, not discussed here, which make one step forward in this direction. In fact, multivectors could be distinguished as being either true or pseudo multivectors, and dual multivectors could be distinguished as true or pseudo dual multivectors [36]. This would double the number of species in the zoo of multivectors.

Vectors and dual vectors can be given geometrical interpretations in terms of arrows and sets of parallel planes, and this can be extended to multivectors and dual multivectors. Actually, this has given the possibility to geometrize all of physics [58]. However, our goal here is not visualization but developing analytic tools applicable to electromagnetic problems. This is why the geometric content is passed by very quickly.

1.2.2 Duality product

The vector space[3] $\mathbb{E}_1$ and the dual vector space $\mathbb{F}_1$ can be associated so that every element of the dual vector space $\mathbb{F}_1$ defines a linear mapping of the elements of the vector space $\mathbb{E}_1$ to real or complex numbers. Similarly, every element of the vector space $\mathbb{E}_1$ defines a linear mapping of the elements of the dual vector space $\mathbb{F}_1$. This mutual linear mapping can be expressed in terms of a symmetric product called the duality product (inner product or contraction) which, following Deschamps [18], is denoted by the sign $|$

$$\boldsymbol{\alpha},\ \mathbf{a} \rightarrow \boldsymbol{\alpha}|\mathbf{a} = \mathbf{a}|\boldsymbol{\alpha}. \tag{1.3}$$

A vector $\mathbf{a}$ and a dual vector $\boldsymbol{\alpha}$ can be called orthogonal (or, rather, annihilating) if they satisfy $\mathbf{a}|\boldsymbol{\alpha} = 0$. The vector and dual vector bases $\{\mathbf{e}_i\}$, $\{\varepsilon_i\}$ are called

[3]When the dimension n is general or has an agreed value, iwe write $\mathbb{E}_1$ instead of $\mathbb{E}_1(n)$ for simplicity.

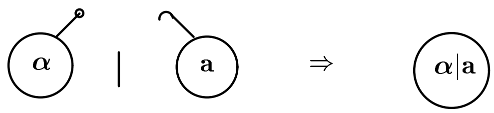

Fig. 1.3 *Hook and eye serve as visual aid to distinguish between vectors and dual vectors. The hook and the eye cancel each other in the duality product.*

reciprocal [21, 28] (dual in [18]) to one another if they satisfy

$$\boldsymbol{\varepsilon}_i|\mathbf{e}_j = \mathbf{e}_j|\boldsymbol{\varepsilon}_i = \delta_{ij}. \tag{1.4}$$

Here δ_{ij} is the Kronecker symbol, $\delta_{ij} = 0$ when $i \neq j$ and $= 1$ when $i = j$. Given a basis of vectors or dual vectors the reciprocal basis can be constructed as will be seen in Section 2.4. In terms of the expansions (1.1), (1.2) in the reciprocal bases, the duality product of a vector $\mathbf{a}$ and a dual vector $\boldsymbol{\alpha}$ can be expressed as

$$\boldsymbol{\alpha}|\mathbf{a} = \sum_{i,j}(\alpha_i\boldsymbol{\varepsilon}_i)|(a_j\mathbf{e}_j) = \sum_i \alpha_i a_i. \tag{1.5}$$

The duality product must not be mistaken for the scalar product (dot product) of the vector space, denoted by $\mathbf{a}\cdot\mathbf{b}$, to be introduced in Section 2.5. The elements of the duality product are from two different spaces while those of the dot product are from the same space.

To distinguish between different quantities it is helpful to have certain suggestive mental aids, for example, hooks for vectors and eyes for dual vectors as in Figure 1.3. In the duality product the hook of a vector is fastened to the eye of the dual vector and the result is a scalar with neither a hook nor an eye left free. This has an obvious analogy in atoms forming molecules.

1.2.3 Dyadics

Linear mappings from a vector to a vector can be conveniently expressed in the coordinate-free dyadic notation. Here we consider only the basic notation and leave more detailed properties to Chapter 2. Dyadic product of a vector $\mathbf{c}$ and a dual vector $\boldsymbol{\gamma}$ is denoted by $\mathbf{c}\boldsymbol{\gamma}$. The "no-sign" dyadic multiplication originally introduced by Gibbs [28, 40] is adopted here instead of the sign $\otimes$ preferred by the mathematicians. Also, other signs for the dyadic product have been in use since Gibbs,— for example, the colon [53].

The dyadic product can be defined by considering the expression

$$\mathbf{b} = \mathbf{c}(\boldsymbol{\gamma}|\mathbf{a}) = (\mathbf{c}\boldsymbol{\gamma})|\mathbf{a}, \tag{1.6}$$

which extends the associative law (order of the two multiplications as shown by the brackets). The dyad $\mathbf{c}\boldsymbol{\gamma}$ acts as a linear mapping from a vector $\mathbf{a}$ to another vector $\mathbf{b}$.

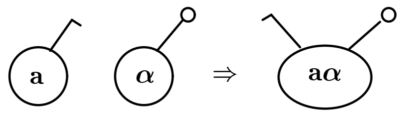

Fig. 1.4 *Dyadic product (no sign) of a vector and a dual vector in this order produces an object which can be visualized as having a hook on the left and an eye on the right.*

Similarly, the dyadic product $\boldsymbol{\gamma}\mathbf{c}$ acts as a linear mapping from a dual vector $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta} = \boldsymbol{\gamma}(\mathbf{c}|\boldsymbol{\alpha}) = (\boldsymbol{\gamma}\mathbf{c})|\boldsymbol{\alpha}. \tag{1.7}$$

The dyadic product $\mathbf{a}\boldsymbol{\alpha}$ can be pictured as an ordered pair of quantities glued back-to-back so that the hook of the vector $\mathbf{a}$ points to the left and the eye of the dual vector $\boldsymbol{\alpha}$ points to the right (Figure 1.4).

Any linear mapping within each vector space $\mathbb{E}_1$ and $\mathbb{F}_1$ can be performed through dyadic polynomials, or dyadics in short. Whenever possible, dyadics are denoted by capital sans-serif characters with two overbars, otherwise by standard symbols with two overbars:

$$\mathbf{b} = \overline{\overline{\mathsf{A}}}|\mathbf{a}, \quad \overline{\overline{\mathsf{A}}} = \sum \mathbf{c}_i\boldsymbol{\gamma}_i = \mathbf{c}_1\boldsymbol{\gamma}_1 + \cdots + \mathbf{c}_n\boldsymbol{\gamma}_n, \tag{1.8}$$

$$\boldsymbol{\beta} = \overline{\overline{\mathsf{A}}}^T|\boldsymbol{\alpha}, \quad \overline{\overline{\mathsf{A}}}^T = \sum \boldsymbol{\gamma}_i\mathbf{c}_i = \boldsymbol{\gamma}_1\mathbf{c}_1 + \cdots + \boldsymbol{\gamma}_n\mathbf{c}_n. \tag{1.9}$$

Here, ^{T} denotes the transpose operation: $(\mathbf{c}\boldsymbol{\gamma})^T = \boldsymbol{\gamma}\mathbf{c}$. Mapping of a vector by a dyadic can be pictured as shown in Figure 1.5.

Let us denote the space of dyadics of the type $\overline{\overline{\mathsf{A}}}$ above by $\mathbb{E}_1\mathbb{F}_1$ (short for $\mathbb{E}_1 \times \mathbb{F}_1$) and, that of the type $\overline{\overline{\mathsf{A}}}^T$ by $\mathbb{F}_1\mathbb{E}_1$ ($\mathbb{F}_1 \times \mathbb{E}_1$). An element of the space $\mathbb{E}_1\mathbb{F}_1$ maps the vector space $\mathbb{E}_1$ onto itself (from the right, from the left it maps the space $\mathbb{F}_1$ onto itself). If a given dyadic $\overline{\overline{\mathsf{A}}}$ maps the space $\mathbb{E}_1$ onto itself, i.e., any vector basis $\{\mathbf{e}_i\}$ to another vector basis $\{\mathbf{e}'_i\}$, the dyadic is called *complete* and there exists a unique inverse dyadic $\overline{\overline{\mathsf{A}}}^{-1}$. The dyadic is incomplete if it maps $\mathbb{E}_1$ only to a subspace of $\mathbb{E}_1$. Such a dyadic does not have a unique inverse. The dimensions of the dyadic spaces $\mathbb{F}_1\mathbb{E}_1$ and $\mathbb{E}_1\mathbb{F}_1$ are n^2.

The dyadic product does not commute. Actually, as was seen above, the transpose operation ^{T} maps dyadics $\mathbb{E}_1\mathbb{F}_1$ to another space $\mathbb{F}_1\mathbb{E}_1$. There are no concepts like symmetry and antisymmetry applicable to dyadics in these spaces. Later we will encounter other dyadic spaces $\mathbb{E}_1\mathbb{E}_1$, $\mathbb{F}_1\mathbb{F}_1$ containing symmetric and antisymmetric dyadics.

The unit dyadic $\overline{\overline{\mathsf{I}}}$ maps any vector to itself: $\overline{\overline{\mathsf{I}}}|\mathbf{a} = \mathbf{a}$. Thus, it also maps any $\mathbb{E}_1\mathbb{F}_1$ dyadic to itself: $\overline{\overline{\mathsf{I}}}|\overline{\overline{\mathsf{A}}} = \overline{\overline{\mathsf{A}}}$. Because any vector $\mathbf{a}$ can be expressed in terms of a basis $\{\mathbf{e}_i\}$ and its reciprocal dual basis $\{\boldsymbol{\varepsilon}_i\}$ as

$$\mathbf{a} = \sum \mathbf{e}_i(\boldsymbol{\varepsilon}_i|\mathbf{a}) = \sum(\mathbf{e}_i\boldsymbol{\varepsilon}_i)|\mathbf{a}, \tag{1.10}$$

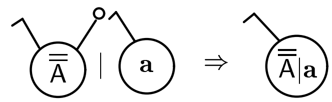

Fig. 1.5 *Dyadic $\overline{\overline{\mathsf{A}}}$ maps a vector $\mathbf{a}$ to the vector $\overline{\overline{\mathsf{A}}}|\mathbf{a}$.*

the unit dyadic can be expanded as

$$\overline{\overline{\mathsf{I}}} = \sum \mathbf{e}_i \varepsilon_i = \mathbf{e}_1\varepsilon_1 + \mathbf{e}_2\varepsilon_2 + \cdots + \mathbf{e}_n\varepsilon_n. \tag{1.11}$$

The form is not unique because we can choose one of the reciprocal bases $\{\mathbf{e}_i\}$, $\{\varepsilon_j\}$ arbitrarily. The transposed unit dyadic

$$\overline{\overline{\mathsf{I}}}^T = \sum \varepsilon_i \mathbf{e}_i = \varepsilon_1\mathbf{e}_1 + \varepsilon_2\mathbf{e}_2 + \cdots + \varepsilon_n\mathbf{e}_n \tag{1.12}$$

serves as the unit dyadic for the dual vectors satisfying $\overline{\overline{\mathsf{I}}}^T|\boldsymbol{\alpha} = \boldsymbol{\alpha}$ for any dual vector $\boldsymbol{\alpha}$. We can also write $\boldsymbol{\alpha}|\overline{\overline{\mathsf{I}}} = \boldsymbol{\alpha}$ and $\mathbf{a}|\overline{\overline{\mathsf{I}}}^T = \mathbf{a}$.

Problems

1.2.1 Given a basis of vectors $\{\mathbf{a}_i\}$ and a basis of dual vectors $\{\boldsymbol{\beta}_j\}$, find the basis of dual vectors $\{\boldsymbol{\alpha}_j\}$ dual to $\{\mathbf{a}_i\}$ in terms of the basis $\{\boldsymbol{\beta}_j\}$.

1.2.2 Show that, in a space of n dimensions, any dyadic $\overline{\overline{\mathsf{A}}}$ can be expressed as a sum of n dyads $\mathbf{a}_i\boldsymbol{\alpha}_i$.

1.3 BIVECTORS

1.3.1 Wedge product

The wedge product (outer product) between any two elements $\mathbf{a}$ and $\mathbf{b}$ of the vector space $\mathbb{E}_1$ and elements $\boldsymbol{\alpha}, \boldsymbol{\beta}$ of the dual vector space $\mathbb{F}_1$ is defined to satisfy the anticommutative law:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}, \quad \boldsymbol{\alpha} \wedge \boldsymbol{\beta} = -\boldsymbol{\beta} \wedge \boldsymbol{\alpha}. \tag{1.13}$$

Anticommutativity implies that the wedge product of any element with itself vanishes:

$$\mathbf{a} \wedge \mathbf{a} = 0, \quad \boldsymbol{\alpha} \wedge \boldsymbol{\alpha} = 0. \tag{1.14}$$

Actually, (1.14) implies (1.13), because we can expand

$$(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} + \mathbf{b}) = \mathbf{a} \wedge \mathbf{a} + \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} + \mathbf{b} \wedge \mathbf{b}$$

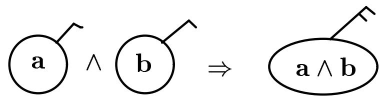

Fig. 1.6 *Visual aid for forming the wedge product of two vectors. The bivector has a double hook and, the dual bivector, a double eye.*

$$= \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} = 0. \tag{1.15}$$

A scalar factor can be moved outside the wedge product:

$$\mathbf{a} \wedge (\lambda \mathbf{b}) = \lambda(\mathbf{a} \wedge \mathbf{b}). \tag{1.16}$$

Wedge product between a vector and a dual vector is not defined.

1.3.2 Basis bivectors

The wedge product of two vectors is neither a vector nor a dyadic but a bivector,[4] or 2-vector, which is an element of another space $\mathbb{E}_2$. Correspondingly, the wedge product of two dual vectors is a dual bivector, an element of the space $\mathbb{F}_2$. A bivector can be visualized by a double hook as in Figure 1.6 and, a dual bivector, by a double eye. Whenever possible, bivectors are denoted by boldface Roman capital letters like $\mathbf{A}$, and dual bivectors are denoted by boldface Greek capital letters like $\mathbf{\Phi}$. However, in many cases we have to follow the classical notation of the electromagnetic literature.

A bivector of the form $\mathbf{a} \wedge \mathbf{b}$ is called a simple bivector [33]. General elements of the bivector space $\mathbb{E}_2$ are linear combinations of simple bivectors,

$$\mathbf{A} = \mathbf{a}_1 \wedge \mathbf{b}_1 + \mathbf{a}_2 \wedge \mathbf{b}_2 + \cdots = \sum \mathbf{a}_i \wedge \mathbf{b}_i. \tag{1.17}$$

The basis elements in the spaces $\mathbb{E}_2$ and $\mathbb{F}_2$ can be expanded in terms of the respective basis elements of $\mathbb{E}_1$ and $\mathbb{F}_1$. The basis bivectors and dual bivectors are denoted by lowercase letters with double indices as

$$\mathbf{e}_{ij} = \mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_{ji}, \tag{1.18}$$

$$\boldsymbol{\varepsilon}_{ij} = \boldsymbol{\varepsilon}_i \wedge \boldsymbol{\varepsilon}_j = -\boldsymbol{\varepsilon}_{ji}. \tag{1.19}$$

Due to antisymmetry of the wedge product, the bi-index ij has some redundancy since the basis elements with indices of the form ii are zero and the elements corresponding to the bi-index ij equal the negative of those with the bi-index ji. Thus, instead of n^2, the dimension of the spaces $\mathbb{E}_2(n)$ and $\mathbb{F}_2(n)$ is only $n(n-1)/2$. For the two-, three-, and four-dimensional vector spaces, the respective dimensions of the bivector spaces are one, three, and six.

[4]Note that, originally, J.W. Gibbs called complex vectors of the form $\mathbf{a} + j\mathbf{b}$ bivectors. This meaning is still occasionally encountered in the literature [9].

The wedge product of two vector expansions

$$\mathbf{a} = \sum a_i \mathbf{e}_i, \quad \mathbf{b} = \sum b_j \mathbf{e}_j \tag{1.20}$$

gives the bivector expansion

$$\mathbf{a} \wedge \mathbf{b} = \sum a_i \mathbf{e}_i \wedge \sum b_j \mathbf{e}_j = \sum_{i,j} a_i b_j \mathbf{e}_{ij} \tag{1.21}$$

$$= a_1 b_2 \mathbf{e}_{12} + a_2 b_1 \mathbf{e}_{21} + a_1 b_3 \mathbf{e}_{13} + a_3 b_1 \mathbf{e}_{31} + \cdots. \tag{1.22}$$

Because of the redundancy, we can reduce the number of bi-indices ij by ordering, i.e., restricting to indices satisfying $i < j$:

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} = & \sum_{i<j} (a_i b_j - a_j b_i) \mathbf{e}_{ij} \\ & = (a_1 b_2 - a_2 b_1)\mathbf{e}_{12} + (a_1 b_3 - a_3 b_1)\mathbf{e}_{13} + \cdots + (a_1 b_n - a_n b_1)\mathbf{e}_{1n} \\ & +(a_2 b_3 - a_3 b_2)\mathbf{e}_{23} + (a_2 b_4 - a_4 b_2)\mathbf{e}_{24} + \cdots + (a_2 b_n - a_n b_2)\mathbf{e}_{2n} \\ & + \cdots + (a_{n-1} b_n - a_n b_{n-1})\mathbf{e}_{(n-1)n}. \end{aligned} \tag{1.23}$$

Euclidean and Minkowskian bivectors For a more symmetric representation, cyclic ordering of the bi-indices is often preferred in the three-dimensional Euclidean Eu3 space:

$$\mathbf{a} \wedge \mathbf{b} = (a_1 b_2 - a_2 b_1)\mathbf{e}_{12} + (a_2 b_3 - a_3 b_2)\mathbf{e}_{23} + (a_3 b_1 - a_1 b_3)\mathbf{e}_{31}. \tag{1.24}$$

The four-dimensional Minkowskian space Mi4 can be understood as Eu3 with an added dimension corresponding to the index 4. In this case, the ordering is usually taken cyclic in the indices 1,2,3 and the index 4 is written last as

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} = & (a_1 b_2 - a_2 b_1)\mathbf{e}_{12} + (a_2 b_3 - a_3 b_2)\mathbf{e}_{23} + (a_3 b_1 - a_1 b_3)\mathbf{e}_{31} \\ & +(a_1 b_4 - a_4 b_1)\mathbf{e}_{14} + (a_2 b_4 - a_4 b_2)\mathbf{e}_{24} + (a_3 b_4 - a_4 b_3)\mathbf{e}_{34}. \end{aligned} \tag{1.25}$$

More generally, expressing Minkowskian vectors $\mathbf{a}_{\mathrm{M}}$ and dual vectors $\boldsymbol{\alpha}_{\mathrm{M}}$ as

$$\mathbf{a}_{\mathrm{M}} = \mathbf{a} + \mathbf{e}_4 a_4, \quad \boldsymbol{\alpha}_{\mathrm{M}} = \boldsymbol{\alpha} + \boldsymbol{\varepsilon}_4 \alpha_4, \tag{1.26}$$

where $\mathbf{a}$ and $\boldsymbol{\alpha}$ are vector and dual vector components in the Euclidean Eu3 space, the wedge product of two Minkowskian vectors can be expanded as

$$\mathbf{a}_{\mathrm{M}} \wedge \mathbf{b}_{\mathrm{M}} = (\mathbf{a} + \mathbf{e}_4 a_4) \wedge (\mathbf{b} + \mathbf{e}_4 b_4) = \mathbf{a} \wedge \mathbf{b} + (\mathbf{a} b_4 - \mathbf{b} a_4) \wedge \mathbf{e}_4. \tag{1.27}$$

Thus, any bivector or dual bivector in the Mi4 space can be naturally expanded in the form

$$\mathbf{A}_{\mathrm{M}} = \mathbf{A} + \mathbf{a} \wedge \mathbf{e}_4, \quad \boldsymbol{\Phi}_{\mathrm{M}} = \boldsymbol{\Phi} + \boldsymbol{\alpha} \wedge \boldsymbol{\varepsilon}_4, \tag{1.28}$$

where $\mathbf{A}$, $\mathbf{a}$, $\boldsymbol{\Phi}$, and $\boldsymbol{\alpha}$ denote the respective Euclidean bivector, vector, dual bivector, and dual vector components.

For two-dimensional vectors the dimension of the bivectors is 1 and all bivectors can be expressed as multiples of a single basis element $\mathbf{e}_{12}$. Because for the three-dimensional vector space the bivector space has the dimension 3, bivectors have a close relation to vectors. In the Gibbsian vector algebra, where the wedge product is replaced by the cross product, bivectors are identified with vectors. In the four-dimensional vector space, bivectors form a six-dimensional space, and they can be represented in terms of a combination of a three-dimensional vector and bivector, each of dimension 3.

In terms of basis bivectors, respective expansions for the general bivector $\mathbf{A} = \sum_{i,j} A_{ij}\mathbf{e}_{ij}$ in spaces of dimension $n = 2, 3$, and 4 can be, respectively, written as

$$\mathbf{A} = A_{12}\mathbf{e}_{12}, \tag{1.29}$$

$$\mathbf{A} = A_{12}\mathbf{e}_{12} + A_{23}\mathbf{e}_{23} + A_{31}\mathbf{e}_{31}, \tag{1.30}$$

$$\mathbf{A} = A_{12}\mathbf{e}_{12} + A_{23}\mathbf{e}_{23} + A_{31}\mathbf{e}_{31} + A_{14}\mathbf{e}_{14} + A_{24}\mathbf{e}_{24} + A_{34}\mathbf{e}_{34}. \tag{1.31}$$

Similar expansions apply for the dual bivectors $\mathbf{\Phi} = \sum \Phi_{ij}\varepsilon_{ij}$. It can be shown that any bivector $\mathbf{A}$ in the case $n = 3$ can be expressed in the form of a simple bivector $\mathbf{A} = \mathbf{a} \wedge \mathbf{b}$ in terms of two vectors $\mathbf{a}, \mathbf{b}$. The proof is left as an exercise. This decomposition is not unique since, for example, we can write $(\mathbf{a} + \lambda\mathbf{b}) \wedge \mathbf{b}$ instead of $\mathbf{a} \wedge \mathbf{b}$ with any scalar λ without changing the bivector. On the other hand, for $n = 4$, any bivector can be expressed as a sum of two simple bivectors, in the form $\mathbf{A} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge \mathbf{d}$ in terms of four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Again, this representation is not unique. The proof can be based on separating the fourth dimension as was done in (1.28).

1.3.3 Duality product

The duality product of a vector and a dual vector is straightforwardly generalized to that of a bivector and a dual bivector by defining the product for the reciprocal basis bivectors and dual bivectors as

$$\varepsilon_{12}|\mathbf{e}_{12} = 1, \quad \varepsilon_{12}|\mathbf{e}_{13} = 0, \quad \varepsilon_{13}|\mathbf{e}_{13} = 1, ... \tag{1.32}$$

and more generally

$$\varepsilon_J|\mathbf{e}_K = \delta_{JK}, \quad J = \{ij\},\ K = \{k\ell\}. \tag{1.33}$$

Here, J and K are ordered bi-indices $(i < j,\ k < \ell)$ and the symbol δ_{JK} has the value 1 only when both $ij = k\ell$, otherwise it is zero. Thus, we can write

$$\delta_{JK} = \delta_{\{ij\}\{k\ell\}} = \delta_{ik}\delta_{j\ell}. \tag{1.34}$$

The corresponding definition for nonordered indices J, K has to take also into account that $\delta_{JK} = -1$ when $ij = \ell k$, in which case (1.34) is generalized to

$$\delta_{JK} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}. \tag{1.35}$$

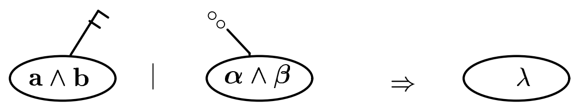

Fig. 1.7 *Duality product of a bivector* $\mathbf{a}\wedge\mathbf{b}$ *and a dual bivector* $\boldsymbol{\alpha}\wedge\boldsymbol{\beta}$ *gives the scalar* $\lambda = (\mathbf{a}\wedge\mathbf{b})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})$.

Bivectors and dual bivectors can be pictured as objects with a respective double hook and double eye. In the duality product the double hook is fastened to the double eye to make an object with no free hooks or eyes, a scalar, Figure 1.7. The duality product of a bivector and a dual bivector can be expanded in terms of duality products of vectors and dual vectors. The expansion is based on the bivector identity

$$\begin{aligned}(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})|(\mathbf{a}\wedge\mathbf{b}) &= (\boldsymbol{\alpha}|\mathbf{a})(\boldsymbol{\beta}|\mathbf{b}) - (\boldsymbol{\alpha}|\mathbf{b})(\boldsymbol{\beta}|\mathbf{a}) \\ &= \det\begin{pmatrix} (\boldsymbol{\alpha}|\mathbf{a}) & (\boldsymbol{\alpha}|\mathbf{b}) \\ (\boldsymbol{\beta}|\mathbf{a}) & (\boldsymbol{\beta}|\mathbf{b}) \end{pmatrix}\end{aligned} \tag{1.36}$$

which can be easily derived by first expanding the vectors and dual vectors in terms of the basis vectors $\mathbf{e}_i$ and the reciprocal dual basis vectors $\boldsymbol{\varepsilon}_j$. From the form of (1.36) it can be seen that all expressions change sign when $\mathbf{a}$ and $\mathbf{b}$ or $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are interchanged. By arranging terms, the identity (1.36) can be rewritten in two other forms

$$(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})|(\mathbf{a}\wedge\mathbf{b}) = \boldsymbol{\alpha}|(\mathbf{ab}-\mathbf{ba})|\boldsymbol{\beta} = \mathbf{a}|(\boldsymbol{\alpha\beta}-\boldsymbol{\beta\alpha})|\mathbf{b}. \tag{1.37}$$

Here we have introduced the dyadic product of two vectors, $\mathbf{ab}$, and two dual vectors, $\boldsymbol{\alpha\beta}$, which are elements of the respective spaces $\mathbb{E}_1\mathbb{E}_1 (= \mathbb{E}_1 \times \mathbb{E}_1)$ and $\mathbb{F}_1\mathbb{F}_1 (= \mathbb{F}_1 \times \mathbb{F}_1)$. Any sum of dyadic products $\sum \mathbf{a}_i\mathbf{b}_i$ serves as a mapping of a dual vector to a vector $\mathbb{F}_1 \to \mathbb{E}_1$ as $\sum \mathbf{a}_i\mathbf{b}_i|\boldsymbol{\gamma} = \mathbf{c}$. In analogy to the double-dot product in the Gibbsian dyadic algebra [28,40], we can define the double-duality product $||$ between two dyadics, elements of the spaces $\mathbb{E}_1\mathbb{E}_1$ and $\mathbb{F}_1\mathbb{F}_1$ or $\mathbb{E}_1\mathbb{F}_1$ and $\mathbb{F}_1\mathbb{E}_1$:

$$\begin{aligned}(\boldsymbol{\alpha}|\mathbf{a})(\boldsymbol{\beta}|\mathbf{b}) &= (\boldsymbol{\alpha\beta})||(\mathbf{ab}) = (\mathbf{ab})||(\boldsymbol{\alpha\beta}) \\ &= (\mathbf{a}\boldsymbol{\beta})||(\boldsymbol{\alpha}\mathbf{b}) = (\boldsymbol{\alpha}\mathbf{b})||(\mathbf{a}\boldsymbol{\beta}).\end{aligned} \tag{1.38}$$

The result is a scalar. For two dyadics $\overline{\overline{\mathsf{A}}}, \overline{\overline{\mathsf{B}}} \in \mathbb{E}_1\mathbb{F}_1$ or $\in \mathbb{F}_1\mathbb{E}_1$ the double-duality product satisfies

$$\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{B}}}^T = \overline{\overline{\mathsf{B}}}^T||\overline{\overline{\mathsf{A}}} = \overline{\overline{\mathsf{A}}}^T||\overline{\overline{\mathsf{B}}} = \overline{\overline{\mathsf{B}}}||\overline{\overline{\mathsf{A}}}^T. \tag{1.39}$$

The identity (1.36) can be rewritten in the following forms:

$$\begin{aligned}(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})|(\mathbf{a}\wedge\mathbf{b}) &= (\boldsymbol{\alpha\beta})||(\mathbf{ab}-\mathbf{ba}) = (\boldsymbol{\alpha\beta}-\boldsymbol{\beta\alpha})||(\mathbf{ab}) \\ &= \frac{1}{2}(\boldsymbol{\alpha\beta}-\boldsymbol{\beta\alpha})||(\mathbf{ab}-\mathbf{ba}).\end{aligned} \tag{1.40}$$

For two antisymmetric dyadics

$$\overline{\overline{\mathsf{A}}} = \sum(\mathbf{a}_i\mathbf{b}_i - \mathbf{b}_i\mathbf{a}_i), \quad \overline{\overline{\mathsf{B}}} = \sum(\boldsymbol{\alpha}_j\boldsymbol{\beta}_j - \boldsymbol{\beta}_j\boldsymbol{\alpha}_j) \tag{1.41}$$

this can be generalized as

$$\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{B}}} = 2\sum_{i,j}(\mathbf{a}_i \wedge \mathbf{b}_i)|(\boldsymbol{\alpha}_j \wedge \boldsymbol{\beta}_j). \tag{1.42}$$

The double-duality product of a symmetric and an antisymmetric dyadic always vanishes, as is seen from (1.39).

1.3.4 Incomplete duality product

Because the scalar $(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})|(\mathbf{a} \wedge \mathbf{b})$ is a linear function of the dual vector $\boldsymbol{\alpha}$, there is must exist a vector $\mathbf{c}$ such that we can write

$$(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})|(\mathbf{a} \wedge \mathbf{b}) = \boldsymbol{\alpha}|\mathbf{c}. \tag{1.43}$$

Since $\mathbf{c}$ is a linear function of the dual vector $\boldsymbol{\beta}$ and the bivector $\mathbf{a}\wedge\mathbf{b}$, we can express it in the form of their product as

$$\mathbf{c} = \boldsymbol{\beta}\rfloor(\mathbf{a} \wedge \mathbf{b}), \tag{1.44}$$

where $\rfloor$ is the sign of the incomplete duality product. Another way to express the same scalar quantity is

$$\mathbf{c}|\boldsymbol{\alpha} = -(\mathbf{a} \wedge \mathbf{b})|(\boldsymbol{\beta} \wedge \boldsymbol{\alpha}) = -((\mathbf{a} \wedge \mathbf{b})\lfloor\boldsymbol{\beta})|\boldsymbol{\alpha}, \tag{1.45}$$

which defines another product sign $\lfloor$. The relation between the two incomplete products is, thus,

$$\boldsymbol{\beta}\rfloor(\mathbf{a} \wedge \mathbf{b}) = -(\mathbf{a} \wedge \mathbf{b})\lfloor\boldsymbol{\beta}. \tag{1.46}$$

More generally, we can write

$$\boldsymbol{\alpha}\rfloor\mathbf{A} = -\mathbf{A}\lfloor\boldsymbol{\alpha}, \quad \mathbf{a}\rfloor\boldsymbol{\Phi} = -\boldsymbol{\Phi}\lfloor\mathbf{a}, \tag{1.47}$$

when $\mathbf{a}$ is a vector, $\boldsymbol{\alpha}$ is a dual vector, $\mathbf{A}$ is a bivector and $\boldsymbol{\Phi}$ is a dual bivector. The horizontal line points to the quantity which has lower grade. In these cases it is either a vector or a dual vector. In an incomplete duality product of a dual vector and a bivector one hook and one eye are eliminated as is shown in Figure 1.8.

The defining formulas can be easily memorized from the rules

$$\mathbf{a}|(\mathbf{b}\rfloor\boldsymbol{\Phi}) = (\mathbf{a} \wedge \mathbf{b})|\boldsymbol{\Phi}, \quad (\boldsymbol{\Phi}\lfloor\mathbf{b})|\mathbf{a} = \boldsymbol{\Phi}(\mathbf{b} \wedge \mathbf{a}), \tag{1.48}$$

valid for any two vectors $\mathbf{a}, \mathbf{b}$ and a dual bivector $\boldsymbol{\Phi}$. The dual form is

$$\boldsymbol{\alpha}|(\boldsymbol{\beta}\rfloor\mathbf{A}) = (\boldsymbol{\alpha} \wedge \boldsymbol{\beta})|\mathbf{A}, \quad (\mathbf{A}\lfloor\boldsymbol{\alpha})|\boldsymbol{\beta} = \mathbf{A}|(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}). \tag{1.49}$$

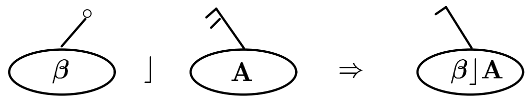

Fig. 1.8 *The incomplete duality product between a dual vector* $\boldsymbol{\beta}$ *and a bivector* $\mathbf{A}$ *gives a vector* $\boldsymbol{\beta}\rfloor\mathbf{A}$.

Bac cab rule Applying (1.36), we obtain an important expansion rule for two vectors $\mathbf{b}$, $\mathbf{c}$ and a dual vector $\boldsymbol{\alpha}$, which, followed by its dual, can be written as

$$\boldsymbol{\alpha}\rfloor(\mathbf{b}\wedge\mathbf{c}) = \mathbf{b}(\boldsymbol{\alpha}|\mathbf{c}) - \mathbf{c}(\boldsymbol{\alpha}|\mathbf{b}) = -(\mathbf{b}\wedge\mathbf{c})\lfloor\boldsymbol{\alpha}, \tag{1.50}$$

$$\mathbf{a}\rfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) = \boldsymbol{\beta}(\mathbf{a}|\boldsymbol{\gamma}) - \boldsymbol{\gamma}(\mathbf{a}|\boldsymbol{\beta}) = -(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})\lfloor\mathbf{a}. \tag{1.51}$$

These identities can be easily memorized from their similarity to the "bac-cab" rule familiar for Gibbsian vectors:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) = -(\mathbf{b}\times\mathbf{c})\times\mathbf{a}. \tag{1.52}$$

The incomplete duality product often corresponds to the cross product in Gibbsian vector algebra.

Decomposition of vector The bac-cab rule for Gibbsian vectors, (1.52), can be used to decompose any vector to components parallel and orthogonal to a given vector. Equation (1.50) defines a similar decomposition for a vector $\mathbf{b}$ in a component parallel to a given vector $\mathbf{a}$ and orthogonal to a given dual vector $\boldsymbol{\alpha}$ provided these satisfy the condition $\mathbf{a}|\boldsymbol{\alpha}\neq 0$,

$$\mathbf{b} = \mathbf{b}_{\|} + \mathbf{b}_{\perp}, \quad \mathbf{a}\wedge\mathbf{b}_{\|} = 0, \quad \boldsymbol{\alpha}|\mathbf{b}_{\perp} = 0. \tag{1.53}$$

In fact, writing (1.50) with $\mathbf{c}=\mathbf{a}$ as

$$\mathbf{b} = \frac{\mathbf{b}|\boldsymbol{\alpha}}{\mathbf{a}|\boldsymbol{\alpha}}\mathbf{a} + \frac{\boldsymbol{\alpha}\rfloor(\mathbf{b}\wedge\mathbf{a})}{\mathbf{a}|\boldsymbol{\alpha}}, \tag{1.54}$$

we can identify the components as

$$\mathbf{b}_{\|} = \frac{\mathbf{b}|\boldsymbol{\alpha}}{\mathbf{a}|\boldsymbol{\alpha}}\mathbf{a}, \quad \mathbf{b}_{\perp} = \frac{\boldsymbol{\alpha}\rfloor(\mathbf{b}\wedge\mathbf{a})}{\mathbf{a}|\boldsymbol{\alpha}}. \tag{1.55}$$

1.3.5 Bivector dyadics

Dyadic product $\mathbf{C}\boldsymbol{\Gamma}$ of a bivector $\mathbf{C}$ and a dual bivector $\boldsymbol{\Gamma}$ is defined as a linear mapping from a bivector $\mathbf{A}$ to a multiple of the bivector $\mathbf{C}$ ($\mathbb{E}_2\rightarrow\mathbb{E}_2$):

$$(\mathbf{C}\boldsymbol{\Gamma})|\mathbf{A} = \mathbf{C}(\boldsymbol{\Gamma}|\mathbf{A}). \tag{1.56}$$

More generally, bivector dyadics of the form $\overline{\overline{\mathsf{A}}} = \sum \mathbf{C}_i \mathbf{\Gamma}_i$ are elements of the space $\mathbb{E}_2\mathbb{F}_2$. The dual counterpart is the dyadic $\sum \mathbf{\Gamma}_i \mathbf{C}_i$ of the space $\mathbb{F}_2\mathbb{E}_2$. In terms of reciprocal bivector and dual bivector bases $\{\mathbf{e}_{ik}\}$, $\{\boldsymbol{\varepsilon}_{j\ell}\}$, any bivector dyadic $\overline{\overline{\mathsf{A}}}$ can be expressed as

$$\overline{\overline{\mathsf{A}}} = \sum A_{IJ}\mathbf{e}_I\boldsymbol{\varepsilon}_J, \tag{1.57}$$

where I and J are two (ordered) bi-indices. Expanding any bivector $\mathbf{C}$ as

$$\mathbf{C} = \sum_{i<j} \mathbf{e}_{ij}(\boldsymbol{\varepsilon}_{ij}|\mathbf{C}) = \sum_{i<j}(\mathbf{e}_{ij}\boldsymbol{\varepsilon}_{ij})|\mathbf{C}, \tag{1.58}$$

we can identify the unit bivector dyadic denoted by $\overline{\overline{\mathsf{I}}}^{(2)}$, which maps a bivector to itself as

$$\overline{\overline{\mathsf{I}}}^{(2)} = \sum_{i<j} \mathbf{e}_{ij}\boldsymbol{\varepsilon}_{ij} = \mathbf{e}_{12}\boldsymbol{\varepsilon}_{12} + \mathbf{e}_{13}\boldsymbol{\varepsilon}_{13} + \mathbf{e}_{23}\boldsymbol{\varepsilon}_{23} + \cdots \tag{1.59}$$

The indices here are ordered, but we can also write it in terms of nonordered indices as

$$\overline{\overline{\mathsf{I}}}^{(2)} = \frac{1}{2}\sum_{i,j} \mathbf{e}_{ij}\boldsymbol{\varepsilon}_{ij} = \frac{1}{2}[\mathbf{e}_{12}\boldsymbol{\varepsilon}_{12} + \mathbf{e}_{21}\boldsymbol{\varepsilon}_{21} + \mathbf{e}_{13}\boldsymbol{\varepsilon}_{13} + \mathbf{e}_{31}\boldsymbol{\varepsilon}_{31} + \cdots]. \tag{1.60}$$

The bivector unit dyadic $\overline{\overline{\mathsf{I}}}^{(a)}$ maps any bivector to itself, and its dual equals its transpose,

$$\overline{\overline{\mathsf{I}}}^{(2)}|(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}\wedge\mathbf{b}, \quad \overline{\overline{\mathsf{I}}}^{(2)T}|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}) = \boldsymbol{\alpha}\wedge\boldsymbol{\beta}. \tag{1.61}$$

Double-wedge product We can introduce the double-wedge product $\,^\wedge_\wedge$ between two dyadic products of vectors and dual vectors,

$$(\mathbf{a}\boldsymbol{\alpha})\,^\wedge_\wedge(\mathbf{b}\boldsymbol{\beta}) = (\mathbf{a}\wedge\mathbf{b})(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}), \tag{1.62}$$

and extend it to two dyadics as

$$\overline{\overline{\mathsf{A}}}\,^\wedge_\wedge\overline{\overline{\mathsf{B}}} = \sum(\mathbf{a}_i\boldsymbol{\alpha}_i)\,^\wedge_\wedge\sum(\mathbf{b}_j\boldsymbol{\beta}_j) = \sum_{i,j}(\mathbf{a}_i\wedge\mathbf{b}_j)(\boldsymbol{\alpha}_i\wedge\boldsymbol{\beta}_j). \tag{1.63}$$

Double-wedge product of two dyadics of the space $\mathbb{E}_1\mathbb{F}_1$ is in the space $\mathbb{E}_2\mathbb{F}_2$. From the anticommutation property of the wedge product, $\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$, it follows that the double-wedge product of two dyadics commutes:

$$\begin{aligned}\overline{\overline{\mathsf{A}}}\,^\wedge_\wedge\overline{\overline{\mathsf{B}}} &= \sum(\mathbf{a}_i\boldsymbol{\alpha}_i)\,^\wedge_\wedge\sum(\mathbf{b}_j\boldsymbol{\beta}_j) = \sum(\mathbf{a}_i\wedge\mathbf{b}_j)(\boldsymbol{\alpha}_i\wedge\boldsymbol{\beta}_j)\\ &= \sum(\mathbf{b}_i\wedge\mathbf{a}_j)(\boldsymbol{\beta}_i\wedge\boldsymbol{\alpha}_j) = \overline{\overline{\mathsf{B}}}\,^\wedge_\wedge\overline{\overline{\mathsf{A}}}.\end{aligned} \tag{1.64}$$

$^\wedge_\wedge$ resembles its Gibbsian counterpart $^\times_\times$, the double-cross product [28, 40], which is also commutative. Expanding the bivector unit dyadic as

$$\overline{\overline{\mathsf{I}}}^{(2)} = \frac{1}{2}\sum_{i,j}(\mathbf{e}_i\wedge\mathbf{e}_j)(\boldsymbol{\varepsilon}_i\wedge\boldsymbol{\varepsilon}_j) = \frac{1}{2}\sum(\mathbf{e}_i\boldsymbol{\varepsilon}_i)\,^\wedge_\wedge\sum(\mathbf{e}_j\boldsymbol{\varepsilon}_j), \tag{1.65}$$

we obtain an important relation between the unit dyadics in vector and bivector spaces:

$$\overline{\overline{\mathsf{I}}}^{(2)} = \frac{1}{2}\overline{\overline{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{I}}}. \tag{1.66}$$

The transpose of this identity gives the corresponding relation in the space of dual vectors and bivectors.

Problems

1.3.1 From linear independence of the basis bivectors show that the condition $\mathbf{a}\wedge\mathbf{b} = 0$ with $\mathbf{a} \neq 0$ implies the existence of a scalar λ such that $\mathbf{b} = \lambda\mathbf{a}$.

1.3.2 Prove (1.36) (a) through the basis bivectors and dual bivectors and (b) by considering the possible resulting scalar function made up with the four different duality products $\boldsymbol{\alpha}|\mathbf{a}$, $\boldsymbol{\beta}|\mathbf{a}$, $\boldsymbol{\alpha}|\mathbf{b}$ and $\boldsymbol{\beta}|\mathbf{b}$.

1.3.3 Show that a bivector of the form $\mathbf{A} = A_{12}\mathbf{a}_1 \wedge \mathbf{a}_2 + A_{23}\mathbf{a}_2 \wedge \mathbf{a}_3 + A_{31}\mathbf{a}_3 \wedge \mathbf{a}_1$ can be written as a simple bivector, in the form $\mathbf{A} = \mathbf{b} \wedge \mathbf{c}$. This means that for $n = 3$ any bivector can be expressed as a simple bivector. *Hint*: The representation is not unique. Try $\mathbf{A} = (\alpha\mathbf{a}_1 + \beta\mathbf{a}_2) \wedge (\gamma\mathbf{a}_2 + \delta\mathbf{a}_3)$ and find the scalar coefficients $\alpha \cdots \delta$.

1.3.4 Show that for $n = 4$ any bivector $\mathbf{A}$ can be written as a sum of two simple bivectors, in the form $\mathbf{A} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge \mathbf{d}$.

1.3.5 Prove the rule (1.90).

1.3.6 Prove that if a dual vector $\boldsymbol{\alpha}$ satisfies $\mathbf{a}|\boldsymbol{\alpha} = 0$ for some vector $\mathbf{a}$, there exists a dual bivector $\boldsymbol{\Phi}$ such that $\boldsymbol{\alpha}$ can be expressed in the form $\boldsymbol{\alpha} = \mathbf{a}\rfloor\boldsymbol{\Phi}$.

1.4 MULTIVECTORS

Further consecutive wedge multiplications of vectors $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \cdots$ give trivectors, quadrivectors, and multivectors of higher grade. This brings about a problem of notation because there does not exist enough different character types for each multivector and dual multivector species. To emphasize the grade of the multivector, we occasionally denote p-vectors and dual p-vectors by superscripts as $\mathbf{a}^p$ and $\boldsymbol{\gamma}^p$. However, in general, trivectors and quadrivectors will be denoted similarly as either vectors or bivectors, by boldface Latin letters like $\mathbf{a}$, $\mathbf{A}$. Dual trivectors and quadrivectors will be denoted by the corresponding Greek letters like $\boldsymbol{\alpha}$, $\boldsymbol{\Phi}$.

1.4.1 Trivectors

Because the wedge product is associative, it is possible to leave out the brackets in defining the trivector,

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}. \tag{1.67}$$

From the antisymmetry of the wedge product, we immediately obtain the permutation rule

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} &= \mathbf{b}\wedge\mathbf{c}\wedge\mathbf{a} = \mathbf{c}\wedge\mathbf{a}\wedge\mathbf{b} = -\mathbf{a}\wedge\mathbf{c}\wedge\mathbf{b} \\ &= -\mathbf{c}\wedge\mathbf{b}\wedge\mathbf{a} = -\mathbf{b}\wedge\mathbf{a}\wedge\mathbf{c}. \end{aligned} \tag{1.68}$$

This means that the trivector is invariant to cyclic permutations and otherwise changes sign. The space of trivectors is $\mathbb{E}_3$, and that of dual trivectors, $\mathbb{F}_3$. Because of antisymmetry, the product vanishes when two vectors in the trivector product are the same. More generally, if one of the vectors is a linear combination of the other two, the trivector product vanishes:

$$\mathbf{a}\wedge\mathbf{b}\wedge(\alpha\mathbf{a}+\beta\mathbf{b}) = -\alpha\mathbf{a}\wedge\mathbf{a}\wedge\mathbf{b} + \beta\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{b} = 0. \tag{1.69}$$

The general trivector can be expressed as a sum of trivector products,

$$\mathbf{Q} = \sum \mathbf{a}_i\wedge\mathbf{b}_i\wedge\mathbf{c}_i = \mathbf{a}_1\wedge\mathbf{b}_1\wedge\mathbf{c}_1 + \mathbf{a}_2\wedge\mathbf{b}_2\wedge\mathbf{c}_2 + \cdots \tag{1.70}$$

The maximum number of linearly independent terms in this expansion equals the dimension of the trivector space which is $n(n-1)(n-2)/3!$. When $n=2$, the dimension of the space of trivectors is zero. In fact, there exist no trivectors based on the two-dimensional vector space $\mathbb{E}_1(2)$ because three vectors are always linearly dependent. In the three-dimensional space the trivectors form a one-dimensional space because they all are multiples of a single trivector $\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3 = \mathbf{e}_{123}$. In the four-dimensional vector space trivectors form another space of four dimensions. In particular, in the Minkowskian space Mi4 of vectors, trivectors $\mathbf{k}_{\mathbf{M}}$ can be expressed in terms of three-dimensional Euclidean trivectors $\mathbf{k}$ and bivectors $\mathbf{K}$ as

$$\mathbf{k}_{\mathbf{M}} = \mathbf{k} + \mathbf{e}_4\wedge\mathbf{K}. \tag{1.71}$$

As a memory aid, trivectors can be pictured as objects with triple hooks and dual trivectors as ones with triple eyes. In the duality product the triple hook is fastened to the triple eye and the result is a scalar with no free hooks or eyes. Geometrical interpretation will be discussed in Section 1.5.

1.4.2 Basis trivectors

A set of basis trivectors $\{\mathbf{e}_{ijk}\}$ can be constructed from basis vectors and bivectors as

$$\begin{aligned}\mathbf{e}_{ijk} &= \mathbf{e}_{ij}\wedge\mathbf{e}_k = \mathbf{e}_i\wedge\mathbf{e}_{jk} = \mathbf{e}_i\wedge\mathbf{e}_j\wedge\mathbf{e}_k \\ &= \mathbf{e}_{jki} = \mathbf{e}_{kij} = -\mathbf{e}_{ikj} = -\mathbf{e}_{kji} = -\mathbf{e}_{jik}. \end{aligned} \tag{1.72}$$

Similar expressions can be written for the dual case. The trivector and dual trivector bases are reciprocal if the underlying vector and dual vector bases are reciprocal:

$$\begin{aligned}\varepsilon_{ijk}|\mathbf{e}_{rst} &= \mathbf{e}_{rst}|\varepsilon_{ijk} = \delta_{\{rst\}\{ijk\}} = \delta_{ri}\delta_{sj}\delta_{tk} \\ &= (\mathbf{e}_r|\varepsilon_i)(\mathbf{e}_s|\varepsilon_j)(\mathbf{e}_t|\varepsilon_k). \end{aligned} \tag{1.73}$$

Here ijk and rst are ordered tri-indices.

Basis expansion gives us a possibility to prove the following simple theorem: If $\mathbf{a} \neq 0$ is a vector and $\mathbf{A}$ a bivector which satisfies $\mathbf{a} \wedge \mathbf{A} = 0$, there exists another vector $\mathbf{b}$ such that we can write $\mathbf{A} = \mathbf{a} \wedge \mathbf{b}$. In fact, choosing a basis with $\mathbf{e}_1 = \mathbf{a}$ and expressing $\mathbf{A} = \sum A_{ij}\mathbf{e}_{ij}$ with $i < j$ gives us

$$\mathbf{a} \wedge \mathbf{A} = \sum_{1<i<j} A_{ij}\mathbf{e}_1 \wedge \mathbf{e}_i \wedge \mathbf{e}_j = 0. \tag{1.74}$$

Because $\mathbf{e}_1 \wedge \mathbf{e}_i \wedge \mathbf{e}_j$ are linearly independent trivectors for $1 < i < j$, this implies $A_{ij} = 0$ when $i \neq 1$. Thus, the bivector $\mathbf{A}$ can be written in the form

$$\mathbf{A} = A_{12}\mathbf{e}_1 \wedge \mathbf{e}_2 + A_{13}\mathbf{e}_1 \wedge \mathbf{e}_3 + \cdots = \mathbf{a} \wedge \mathbf{b}, \quad \mathbf{b} = \sum A_{1j}\mathbf{e}_j. \tag{1.75}$$

1.4.3 Trivector identities

From the permutational property (1.72) we can derive the following basic identity for the duality product of a trivector and a dual trivector:

$$(\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma})|(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \det \begin{pmatrix} \boldsymbol{\alpha}|\mathbf{a} & \boldsymbol{\alpha}|\mathbf{b} & \boldsymbol{\alpha}|\mathbf{c} \\ \boldsymbol{\beta}|\mathbf{a} & \boldsymbol{\beta}|\mathbf{b} & \boldsymbol{\beta}|\mathbf{c} \\ \boldsymbol{\gamma}|\mathbf{a} & \boldsymbol{\gamma}|\mathbf{b} & \boldsymbol{\gamma}|\mathbf{c} \end{pmatrix}. \tag{1.76}$$

As a simple check, it can be seen that the right-hand side vanishes whenever two of the vectors or dual vectors are scalar multiples of each other. Introducing the triple duality product between triadic products of vectors and dual vectors as

$$(\mathbf{abc})|||(\boldsymbol{\alpha\beta\gamma}) = (\mathbf{a}|\boldsymbol{\alpha})(\mathbf{b}|\boldsymbol{\beta})(\mathbf{c}|\boldsymbol{\gamma}) = (\boldsymbol{\alpha\beta\gamma})|||(\mathbf{abc}), \tag{1.77}$$

(1.76) can expressed as

$$\begin{aligned}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma})&|(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \\ &= (\boldsymbol{\alpha\beta\gamma})|||(\mathbf{abc} + \mathbf{bca} + \mathbf{cab} - \mathbf{acb} - \mathbf{cba} - \mathbf{bac}) \\ &= (\boldsymbol{\alpha\beta\gamma} + \boldsymbol{\beta\gamma\alpha} + \boldsymbol{\gamma\alpha\beta} - \boldsymbol{\alpha\gamma\beta} - \boldsymbol{\beta\alpha\gamma} - \boldsymbol{\gamma\beta\alpha})|||(\mathbf{abc}),\end{aligned} \tag{1.78}$$

which can be easily memorized. Triadics or polyadics of higher order will not otherwise be applied in the present analysis. Other forms are obtained by applying (1.40):

$$(\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma})|(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \tag{1.79}$$

$$\begin{aligned} &= (\boldsymbol{\alpha} \wedge \boldsymbol{\beta})|\Big((\mathbf{a} \wedge \mathbf{b})\mathbf{c} + (\mathbf{b} \wedge \mathbf{c})\mathbf{a} + (\mathbf{c} \wedge \mathbf{a})\mathbf{b}\Big)|\boldsymbol{\gamma} \\ &= \boldsymbol{\alpha}|\Big(\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) + \mathbf{b}(\mathbf{c} \wedge \mathbf{a}) + \mathbf{c}(\mathbf{a} \wedge \mathbf{b})\Big)|(\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) \\ &= (\mathbf{a} \wedge \mathbf{b})|\Big((\boldsymbol{\alpha} \wedge \boldsymbol{\beta})\boldsymbol{\gamma} + (\boldsymbol{\beta} \wedge \boldsymbol{\gamma})\boldsymbol{\alpha} + (\boldsymbol{\gamma} \wedge \boldsymbol{\alpha})\boldsymbol{\beta}\Big)|\mathbf{c}, \\ &= \mathbf{a}|\Big(\boldsymbol{\alpha}(\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) + \boldsymbol{\beta}(\boldsymbol{\gamma} \wedge \boldsymbol{\alpha}) + \boldsymbol{\gamma}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})\Big)|(\mathbf{b} \wedge \mathbf{c}). \end{aligned} \tag{1.80}$$

The sum expressions involve dyadic products of vectors and bivectors or dual vectors and dual bivectors.

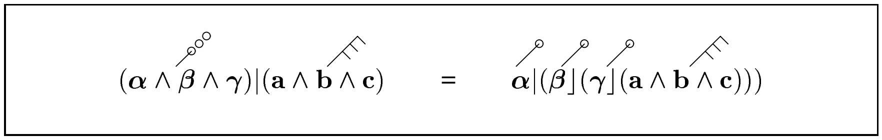

Fig. 1.9 *Visualization of a trivector product rule.*

Incomplete duality product In terms of (1.78), (1.80) we can expand the incomplete duality products between a dual bivector and a trivector defined by

$$(\boldsymbol\alpha\wedge\boldsymbol\beta\wedge\boldsymbol\gamma)|(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}) = \boldsymbol\alpha|((\boldsymbol\beta\wedge\boldsymbol\gamma)\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}), \tag{1.81}$$

as

$$(\boldsymbol\beta\wedge\boldsymbol\gamma)\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}) = \Big(\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b})\Big)|(\boldsymbol\beta\wedge\boldsymbol\gamma). \tag{1.82}$$

A visualization of incomplete products arising from a trivector product is seen in Figure 1.9. Equation (1.82) shows us that $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = 0$ implies linear dependence of the vector triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, because we then have

$$\mathbf{a}((\mathbf{b}\wedge\mathbf{c})|(\boldsymbol\beta\wedge\boldsymbol\gamma)) + \mathbf{b}((\mathbf{c}\wedge\mathbf{a})|(\boldsymbol\beta\wedge\boldsymbol\gamma)) + \mathbf{c}((\mathbf{a}\wedge\mathbf{b})|(\boldsymbol\beta\wedge\boldsymbol\gamma)) = 0, \tag{1.83}$$

valid for any dual vectors $\boldsymbol\beta, \boldsymbol\gamma$. Thus, in a space of three dimensions, three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ can make a basis only if $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} \neq 0$. Similarly, from

$$(\boldsymbol\alpha\wedge\boldsymbol\beta\wedge\boldsymbol\gamma)|(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}) = (\boldsymbol\alpha\wedge\boldsymbol\beta)|\Big(\boldsymbol\gamma\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})\Big), \tag{1.84}$$

we can expand the incomplete duality product between a dual vector and a trivector as

$$\boldsymbol\gamma\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}) = \Big((\mathbf{a}\wedge\mathbf{b})\mathbf{c} + (\mathbf{b}\wedge\mathbf{c})\mathbf{a} + (\mathbf{c}\wedge\mathbf{a})\mathbf{b}\Big)|\boldsymbol\gamma. \tag{1.85}$$

Different forms for this rule can be found through duality and multiplying from the right. Equation (1.85) shows us that $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = 0$ also implies linear dependence of the three bivectors $(\mathbf{a}\wedge\mathbf{b})$, $(\mathbf{b}\wedge\mathbf{c})$, $(\mathbf{c}\wedge\mathbf{a})$.

The trivector unit dyadic $\overline{\overline{\mathsf{I}}}^{(3)}$ maps any trivector to itself, and its dual equals its transpose,

$$\begin{aligned} \overline{\overline{\mathsf{I}}}^{(3)}|(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}) &= \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}, && (1.86)\\ \overline{\overline{\mathsf{I}}}^{(3)T}|(\boldsymbol\alpha\wedge\boldsymbol\beta\wedge\boldsymbol\gamma) &= \boldsymbol\alpha\wedge\boldsymbol\beta\wedge\boldsymbol\gamma. && (1.87)\end{aligned}$$

The basis expansion is

$$\overline{\overline{\mathsf{I}}}^{(3)} = \sum_{i<j<k} \mathbf{e}_{ijk}\boldsymbol\varepsilon_{ijk} = \mathbf{e}_{123}\boldsymbol\varepsilon_{123} + \mathbf{e}_{124}\boldsymbol\varepsilon_{124} + \mathbf{e}_{134}\boldsymbol\varepsilon_{134} + \cdots \tag{1.88}$$

The trivector unit dyadic can be shown to satisfy the relations

$$\overline{\overline{\mathsf{I}}}^{(3)} = \frac{1}{3}\overline{\overline{\mathsf{I}}}^{(2)}{}^\wedge_\wedge\overline{\overline{\mathsf{I}}} = \frac{1}{3}\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{I}}}^{(2)} = \frac{1}{3!}\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{I}}}. \tag{1.89}$$

Bac cab rule A useful bac cab formula is obtained from (1.50) when replacing the vector $\mathbf{c}$ by a bivector $\mathbf{C}$. The rule (1.50) is changed to

$$\boldsymbol{\alpha}\rfloor(\mathbf{b}\wedge\mathbf{C}) = \mathbf{b}\wedge(\boldsymbol{\alpha}\rfloor\mathbf{C}) + \mathbf{C}(\boldsymbol{\alpha}|\mathbf{b}) = (\mathbf{b}\wedge\mathbf{C})\lfloor\boldsymbol{\alpha}, \tag{1.90}$$

while its dual (1.51) is changed to

$$\mathbf{a}\rfloor(\boldsymbol{\beta}\wedge\boldsymbol{\Gamma}) = \boldsymbol{\beta}\wedge(\mathbf{a}\rfloor\boldsymbol{\Gamma}) + \boldsymbol{\Gamma}(\mathbf{a}|\boldsymbol{\beta}) = (\boldsymbol{\beta}\wedge\boldsymbol{\Gamma})\lfloor\mathbf{a}. \tag{1.91}$$

Note the difference in sign in (1.50) and (1.90) and their duals (this is why we prefer calling them "bac cab rules" instead of "bac-cab rules" as is done for Gibbsian vectors [14]). The proof of (1.90) and some of its generalizations are left as an exercise. A futher generalization to multivectors will be given in Section 1.4.8. The rule (1.90) gives rise to a decomposition rule similar to that in (1.53). In fact, given a bivector $\mathbf{B}$ we can expand it in components parallel to another bivector $\mathbf{A}$ and orthogonal to a dual vector $\boldsymbol{\alpha}$ as

$$\mathbf{B} = \mathbf{B}_{\|} + \mathbf{B}_{\perp}, \quad \mathbf{a}\wedge\mathbf{B}_{\|} = 0, \quad \boldsymbol{\alpha}\rfloor\mathbf{B}_{\perp} = 0. \tag{1.92}$$

The components can be expressed as

$$\mathbf{B}_{\|} = -\frac{\mathbf{a}\wedge(\boldsymbol{\alpha}\rfloor\mathbf{B})}{\boldsymbol{\alpha}|\mathbf{a}}, \quad \mathbf{B}_{\perp} = \frac{\boldsymbol{\alpha}\rfloor(\mathbf{a}\wedge\mathbf{B})}{\boldsymbol{\alpha}|\mathbf{a}}. \tag{1.93}$$

1.4.4 p-vectors

Proceeding in the same manner as before, multiplying p vectors or dual vectors, quantities called p-vectors or dual p-vectors, elements of the space $\mathbb{E}_p$ or $\mathbb{F}_p$, are obtained. The dimension of these two spaces equals the binomial coefficient

$$C_p^n = \begin{pmatrix} n \\ p \end{pmatrix} = \frac{n(n-1)..(n-p+1)}{p!} = \frac{n!}{p!(n-p)!}, \quad n \geq p. \tag{1.94}$$

The following table gives the dimension C_p^n corresponding to the grade p of the multivector and dimension n of the original vector space:

p	$n=2$	$n=3$	$n=4$
5	0	0	0
4	0	0	1
3	0	1	4
2	1	3	6
1	2	3	4
0	1	1	1

One may note that the dimension of $\mathbb{E}_p$ is largest for $p = n/2$ when n is even and for $p = (n \pm 1)/2$ when n is odd. The dimension C_p^n is the same when p is replaced by $n-p$. This is why the spaces $\mathbb{E}_p$ and $\mathbb{E}_{n-p}$ can be mapped onto one another. A mapping of this kind is denoted by * (Hodge's star operator) in mathematical literature. Here we express it through Hodge's dyadics $\overline{\overline{\mathsf{H}}}_p$. If $\mathbf{x}^p$ is an element of $\mathbb{E}_p$, $^*\mathbf{x}^p$ or $\overline{\overline{\mathsf{H}}}_p|\mathbf{x}^p$ is an element of $\mathbb{E}_{n-p}$. More on Hodge's dyadics will be found in Chapter 2.

The duality product can be generalized for any p-vectors and dual p-vectors through the reciprocal p-vector and dual p-vector bases satisfying the orthogonality

$$\varepsilon_J|\mathbf{e}_K = \delta_{JK}. \tag{1.95}$$

Here $J = \{j_1 j_2 \cdots j_p\}$ and $K = \{k_1 k_2 \cdots k_p\}$ are two ordered p-indices with $j_1 < j_2 ... < j_p$, $k_1 < k_2 ... < k_p$ and

$$\delta_{JK} = \delta_{j_1 k_1}\delta_{j_2 k_2}\cdots\delta_{j_p k_p}. \tag{1.96}$$

The p-vector unit dyadic is defined through p unit dyadics $\overline{\overline{\mathsf{I}}}$ as

$$\overline{\overline{\mathsf{I}}}^{(p)} = \frac{1}{p!}\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\cdots{}^\wedge_\wedge\overline{\overline{\mathsf{I}}}. \tag{1.97}$$

If $\mathbf{x}^p$ is a p-vector and $\mathbf{y}^q$ is a q-vector, from the anticommutativity rule of the wedge product we can obtain the general commutation rule

$$\mathbf{x}^p \wedge \mathbf{y}^q = (-1)^{pq}\mathbf{y}^q \wedge \mathbf{x}^p. \tag{1.98}$$

This product commutes unless both p and q are odd numbers, in which case it anti-commutes. For example, if $\mathbf{y}^q$ is a bivector ($q=2$), it commutes with any p-vector $\mathbf{x}^p$ in the wedge product.

1.4.5 Incomplete duality product

The incomplete duality product can be defined to a p-vector and a dual q-vector when $p \neq q$. The result is a $p-q$ vector if $p > q$ and a dual $q-p$ vector if $p < q$. In mathematics such a product is known as contraction, because it reduces the grade of the multivector or dual multivector. In the multiplication sign $\lfloor$ or $\rfloor$ the short line points to the multivector or dual multivector with smaller p or q. In reference 18 the duality product sign $|$ was defined to include also the incomplete duality product. Actually, since the different multiplication signs $\wedge$, $|$, $\lfloor$, $\rfloor$ apply in different environments, we could replace all of them by a single sign and interprete the operation with respect to the grades of the multiplicants. However, since this would make the analysis more vulnerable to errors, it does not appear wise to simplify the notation too much.

Considering a dual q-vector $\boldsymbol{\alpha}^q$ and a q-vector of the form $\mathbf{a}^p \wedge \mathbf{b}^{q-p}$ with $q > p$, their duality product can be expressed as

$$\boldsymbol{\alpha}^q|(\mathbf{a}^p \wedge \mathbf{b}^{q-p}) = (\boldsymbol{\alpha}^q\lfloor\mathbf{a}^p)|\mathbf{b}^{q-p}, \tag{1.99}$$

which defines the incomplete duality product of a p-vector and a dual q-vector. The dual case is

$$\mathbf{a}^q|(\boldsymbol{\alpha}^p \wedge \boldsymbol{\beta}^{q-p}) = (\mathbf{a}^q\lfloor\boldsymbol{\alpha}^p)|\boldsymbol{\beta}^{q-p}. \tag{1.100}$$

Again, we can use the memory aid by inserting p hooks in the p-vector $\mathbf{a}^p$ and q eyes in the dual q-vector $\boldsymbol{\alpha}^q$, as was done in Figure 1.9. If $p < q$, in the incomplete duality product p hooks are caught by p eyes and the object is left with $q-p$ free eyes, which makes it a dual $(q-p)$-vector $\boldsymbol{\alpha}^q \lfloor \mathbf{a}^p$.

From the symmetry of the duality product we can write

$$\begin{aligned} \boldsymbol{\alpha}_q|(\mathbf{a}_p \wedge \mathbf{b}_{q-p}) &= (\mathbf{a}_p \wedge \mathbf{b}_{q-p})|\boldsymbol{\alpha}_q \\ &= (-1)^{p(q-p)}(\mathbf{b}_{p-q} \wedge \mathbf{a}_p)|\boldsymbol{\alpha}_q \\ &= \mathbf{b}_{q-p}|((-1)^{p(q-p)}\mathbf{a}_p \rfloor \boldsymbol{\alpha}_q). \end{aligned} \tag{1.101}$$

Thus, we obtain the important commutation relations

$$\boldsymbol{\alpha}_q \lfloor \mathbf{a}_p = (-1)^{p(q-p)}\mathbf{a}_p \rfloor \boldsymbol{\alpha}_q, \qquad \mathbf{a}_q \lfloor \boldsymbol{\alpha}_p = (-1)^{p(q-p)}\boldsymbol{\alpha}_p \rfloor \mathbf{a}_q, \tag{1.102}$$

which can be remembered so that the power of -1 is the smaller of p and q multiplied by their difference. It is seen that the incomplete duality product is antisymmetric only when the smaller index p is odd and the larger index q is even. In all other cases it is symmetric.

1.4.6 Basis multivectors

In forming incomplete duality products between different elements of multivector spaces, it is often helpful to work with expansions, in which cases incomplete duality products of different basis elements are needed. Most conveniently, they can be expressed as a set of rules which can be derived following the example of

$$i \neq j, \quad (\varepsilon_{ij} \lfloor \mathbf{e}_i)|\mathbf{e}_k = \varepsilon_{ij}|\mathbf{e}_{ik} = \delta_{jk} = \varepsilon_j|\mathbf{e}_k \quad \Rightarrow \quad \varepsilon_{ij} \lfloor \mathbf{e}_i = \varepsilon_j. \tag{1.103}$$

Here we can use the antisymmetric property of the wedge product as $\varepsilon_{ij} = -\varepsilon_{ji}$ and the commutation rule (1.102) to obtain other variants:

$$i \neq j, \quad \varepsilon_{ij} \lfloor \mathbf{e}_i = \varepsilon_j, \quad \varepsilon_{ij} \lfloor \mathbf{e}_j = -\varepsilon_i, \tag{1.104}$$
$$i \neq j, \quad \varepsilon_i \rfloor \mathbf{e}_{ij} = -\mathbf{e}_j, \quad \varepsilon_j \rfloor \mathbf{e}_{ij} = \mathbf{e}_i. \tag{1.105}$$

The dual cases can be simply written by excanging vectors and dual vectors:

$$i \neq j, \quad \mathbf{e}_{ij} \lfloor \varepsilon_i = \mathbf{e}_j, \quad \mathbf{e}_{ij} \lfloor \varepsilon_j = -\mathbf{e}_i, \tag{1.106}$$
$$i \neq j, \quad \mathbf{e}_i \rfloor \varepsilon_{ij} = -\varepsilon_j, \quad \mathbf{e}_j \rfloor \varepsilon_{ij} = \varepsilon_i. \tag{1.107}$$

Generic formulas for trivectors and quadrivectors can be written as follows when $\mathbf{e}_{ijk} \neq 0$ and $\mathbf{e}_{ijk\ell} \neq 0$:

$$\varepsilon_{ijk} \lfloor \mathbf{e}_{ij} = \mathbf{e}_{ij} \rfloor \varepsilon_{ijk} = \varepsilon_k, \tag{1.108}$$
$$\varepsilon_{ijk} \lfloor \mathbf{e}_i = \mathbf{e}_i \rfloor \varepsilon_{ijk} = \varepsilon_{jk}, \tag{1.109}$$
$$\varepsilon_{ijk\ell} \lfloor \mathbf{e}_i = -\mathbf{e}_i \rfloor \varepsilon_{ijk\ell} = \varepsilon_{jk\ell}, \tag{1.110}$$
$$\varepsilon_{ijk\ell} \lfloor \mathbf{e}_{ij} = \mathbf{e}_{ij} \rfloor \varepsilon_{ijk\ell} = \varepsilon_{k\ell}, \tag{1.111}$$
$$\varepsilon_{ijk\ell} \lfloor \mathbf{e}_{ijk} = -\mathbf{e}_{ijk} \rfloor \varepsilon_{ijk\ell} = \varepsilon_\ell, \tag{1.112}$$

which can be transformed to other forms using antisymmetry and duality. A good memory rule is that we can eliminate the first indices in the expressions of the form $(\varepsilon_{ijk\ell} \neq 0, \mathbf{e}_{ijk\ell} \neq 0)$

$$\varepsilon_{ijk\ell} \rfloor \mathbf{e}_{ijk} = \varepsilon_{jk\ell} \rfloor \mathbf{e}_{jk} = \varepsilon_{k\ell} \rfloor \mathbf{e}_{k} = \varepsilon_{\ell}, \tag{1.113}$$

and the last indices in the form

$$\varepsilon_{ijk} \lfloor \mathbf{e}_{ijk\ell} = -\varepsilon_{ijk} \lfloor \mathbf{e}_{\ell ijk} = -\varepsilon_{ij} \lfloor \mathbf{e}_{\ell ij} = -\varepsilon_{i} \lfloor \mathbf{e}_{\ell i} = -\mathbf{e}_{\ell}. \tag{1.114}$$

If $J = j_1 j_2 \cdots j_p$ is a p-index and $\mathbf{e}_J^p, \mathbf{e}_{K(J)}^{n-p}$ are a basis p-vector and its complementary basis $n-p$ vector,

$$\mathbf{e}_J^p = \mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_p}, \tag{1.115}$$

$$\mathbf{e}_{K(J)}^{n-p} = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{j_1-1} \wedge \mathbf{e}_{j_1+1} \wedge \cdots \wedge \mathbf{e}_{j_p-1} \wedge \mathbf{e}_{j_p+1} \wedge \cdots \wedge \mathbf{e}_n, \tag{1.116}$$

we can derive the relations

$$\mathbf{e}_J^p \wedge \mathbf{e}_{K(J)}^{n-p} = (-1)^{\sigma(J)} \mathbf{e}_N, \quad \mathbf{e}_{K(J)}^{n-p} \wedge \mathbf{e}_J^p = (-1)^{p(n-p)} (-1)^{\sigma(J)} \mathbf{e}_N, \tag{1.117}$$

where we denote

$$\sigma(J) = \sum_{i=1}^{p} (j_i - i) = (j_1 - 1) + (j_2 - 2) + \cdots + (j_p - p). \tag{1.118}$$

Details are left as an exercise. Equation (1.117) implies the rules

$$\varepsilon_N \rfloor \mathbf{e}_J^p = (-1)^{\sigma(J)} \varepsilon_{K(J)}, \tag{1.119}$$

$$\varepsilon_N \rfloor \mathbf{e}_{K(J)}^{n-p} = (-1)^{p(n-p)} (-1)^{\sigma(J)} \varepsilon_J^p, \tag{1.120}$$

$$\mathbf{e}_N \rfloor (\varepsilon_N \rfloor \mathbf{e}_J^p) = (-1)^{\sigma(P)} \mathbf{e}_N \rfloor \varepsilon_{K(J)}^{n-p} = (-1)^{p(n-p)} \mathbf{e}_J^p, \tag{1.121}$$

$$\mathbf{e}_N \rfloor (\varepsilon_N \rfloor \mathbf{e}_{K(J)}^{n-p}) = (-1)^{p(n-p)} (-1)^{\sigma(P)} \mathbf{e}_N \rfloor \varepsilon_J^p = (-1)^{p(n-p)} \mathbf{e}_{K(J)}^{n-p}, \tag{1.122}$$

which can be easily written in their dual form.

Example As an example of applying the previous formulas let us expand the dual quadrivector $\varepsilon_{k\ell} \wedge (\varepsilon_{ijrs} \rfloor \mathbf{e}_{ij})$ in the case $n = 4$. From (1.111) we have for $\varepsilon_{ijrs} \neq 0$

$$\varepsilon_{k\ell} \wedge (\varepsilon_{ijrs} \rfloor \mathbf{e}_{ij}) = \varepsilon_{k\ell} \wedge \varepsilon_{rs} = \varepsilon_{k\ell rs}, \tag{1.123}$$

which is a multiple of ε_{1234}. Let us assume ordered pairs $k < \ell$ and $i < j$ and $r \neq s$. Now, obviously, ε_{ijrs} vanishes unless i and j are different from r and s. Thus, the result vanishes unless $k = i$ and $\ell = j$ and for $\varepsilon_{ijrs} \neq 0$ the result can be written as

$$\varepsilon_{k\ell} \wedge (\varepsilon_{ijrs} \rfloor \mathbf{e}_{ij}) = \delta_{\{k\ell\}\{ij\}} \varepsilon_{k\ell rs} = (\varepsilon_{k\ell} | \mathbf{e}_{ij}) \varepsilon_{ijrs}. \tag{1.124}$$

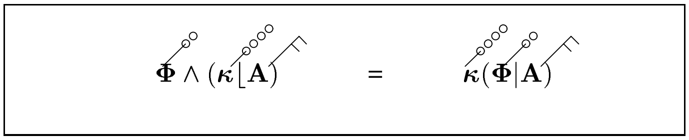

Fig. 1.10 *Visualization of the rule (1.125) valid for $n = 4$. One can check that the quantities on each side is a dual quadrivector.*

Let us generalize this result. Since in a four-dimensional space any dual quadrivector is a multiple of ε_{1234}, we can replace ε_{ijrs} by an arbitrary quadrivector $\boldsymbol{\kappa} = \kappa\varepsilon_{1234}$. Further, we can multiply the equation by a scalar A_{ij} and sum over i, j to obtain an arbitrary bivector $\mathbf{A} = \sum A_{ij}\mathbf{e}_{ij}$. Similarly, we can multiply both sides by the scalar $\Phi_{k\ell}$ and sum, whence an arbitrary dual bivector $\mathbf{\Phi} = \sum \Phi_{k\ell}\varepsilon_{k\ell}$ will arise. Thus, we arrive at the dual quadrivector identity

$$\mathbf{\Phi} \wedge (\boldsymbol{\kappa}\lfloor\mathbf{A}) = \boldsymbol{\kappa}(\mathbf{\Phi}|\mathbf{A}) \tag{1.125}$$

valid for any dual bivector $\mathbf{\Phi}$, dual quadrivector $\boldsymbol{\kappa}$ and bivector $\mathbf{A}$ in the four-dimensional space. A visualization of this is seen in Figure 1.10.

1.4.7 Generalized bac cab rule

The bac cab rules (1.50) and (1.90) can be generalized to the following identity, which, however, does not have the mnemonic "bac cab" form:

$$(\mathbf{a}^q \wedge \mathbf{a}^p)\lfloor\boldsymbol{\alpha} = (\mathbf{a}^q\lfloor\boldsymbol{\alpha}) \wedge \mathbf{a}^p + (-1)^q\mathbf{a}^q \wedge (\mathbf{a}^p\lfloor\boldsymbol{\alpha}). \tag{1.126}$$

Here, $\mathbf{a}^q$ is a q-vector, $\mathbf{a}^p$ is a p-vector, and $\boldsymbol{\alpha}$ is a dual vector. For the special case $q = 1$ we have the rule

$$(\mathbf{a} \wedge \mathbf{a}^p)\lfloor\boldsymbol{\alpha} = (\mathbf{a}|\boldsymbol{\alpha})\mathbf{a}^p - \mathbf{a} \wedge (\mathbf{a}^p\lfloor\boldsymbol{\alpha}). \tag{1.127}$$

When $p + q > n$, the left-hand side vanishes and we have

$$(\mathbf{a}^q\lfloor\boldsymbol{\alpha}) \wedge \mathbf{a}^p = (-1)^{pq}(\mathbf{a}^p\lfloor\boldsymbol{\alpha}) \wedge \mathbf{a}^q, \quad p + q > n. \tag{1.128}$$

The proof of (1.126) is left as a problem. Let us make a few checks of the identity. The inner consistency can be checked by changing the order of all wedge multiplications with associated sign changes. Equation (1.126), then, becomes

$$\begin{aligned} &(-1)^{pq}(\mathbf{a}^p \wedge \mathbf{a}^q)\lfloor\boldsymbol{\alpha} \\ &= \; (-1)^{p(q-1)}\mathbf{a}^p \wedge (\mathbf{a}^q\lfloor\boldsymbol{\alpha}) + (-1)^q(-1)^{q(p-1)}(\mathbf{a}^p\lfloor\boldsymbol{\alpha}) \wedge \mathbf{a}^q, \end{aligned} \tag{1.129}$$

which can be seen to reduce to (1.126) with q and p interchanged. As a second check we choose $\mathbf{a}^q = \mathbf{c}$ and $\mathbf{a}^p = \mathbf{b}$ as vectors with $p = q = 1$. Equation (1.126) then gives us the bac cab rule (1.50), because with proper change of product signs we arrive at

$$\begin{aligned}(\mathbf{c}\wedge\mathbf{b})\lfloor\boldsymbol{\alpha} &= \boldsymbol{\alpha}\rfloor(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{c}|\boldsymbol{\alpha})\mathbf{b} - \mathbf{c}(\mathbf{b}|\boldsymbol{\alpha}) \\ &= \mathbf{b}(\boldsymbol{\alpha}|\mathbf{c}) - \mathbf{c}(\boldsymbol{\alpha}|\mathbf{b}). \qquad (1.130)\end{aligned}$$

Finally, choosing $\mathbf{a}^q = \mathbf{b}$, a vector and $\mathbf{a}^p = \mathbf{C}$, a bivector, with $q = 1, p = 2$, the rule (1.90) is seen to follow from (1.126):

$$\begin{aligned}(\mathbf{b}\wedge\mathbf{C})\lfloor\boldsymbol{\alpha} &= \boldsymbol{\alpha}\rfloor(\mathbf{b}\wedge\mathbf{C}) = (\mathbf{b}|\boldsymbol{\alpha})\mathbf{C} - \mathbf{b}\wedge(\mathbf{C}\lfloor\boldsymbol{\alpha}) \\ &= \mathbf{b}\wedge(\boldsymbol{\alpha}\rfloor\mathbf{C}) + \mathbf{C}(\boldsymbol{\alpha}|\mathbf{b}). \qquad (1.131)\end{aligned}$$

Thus, (1.126) could be called the mother of bac cab rules.

Equation (1.126) involves a single dual vector $\boldsymbol{\alpha}$ and it can be used to produce other useful rules by adding new dual vectors. In fact, multiplying (1.126) by a second dual vector as $\lfloor\boldsymbol{\beta}$ we obtain the following identity valid for $p, q \geq 2$:

$$\begin{aligned}&(\mathbf{a}^q\wedge\mathbf{a}^p)\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}) = \\ &\quad(\mathbf{a}^q\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}))\wedge\mathbf{a}^p - (-1)^q(\mathbf{a}^q\lfloor\boldsymbol{\alpha})\wedge(\mathbf{a}^p\lfloor\boldsymbol{\beta}) \\ &\quad+(-1)^q(\mathbf{a}^q\lfloor\boldsymbol{\beta})\wedge(\mathbf{a}^p\lfloor\boldsymbol{\alpha}) + \mathbf{a}^q\wedge(\mathbf{a}^p\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})). \qquad (1.132)\end{aligned}$$

A further multiplication of (1.132) by another dual vector as $\lfloor\boldsymbol{\gamma}$ gives

$$\begin{aligned}&(\mathbf{a}^q\wedge\mathbf{a}^p)\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) \\ &= (\mathbf{a}^q\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}))\wedge\mathbf{a}^p + (\mathbf{a}^q\lfloor\boldsymbol{\alpha})\wedge(\mathbf{a}^p\lfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})) \\ &+(\mathbf{a}^q\lfloor\boldsymbol{\beta})\wedge(\mathbf{a}^p\lfloor(\boldsymbol{\gamma}\wedge\boldsymbol{\alpha})) + (\mathbf{a}^q\lfloor\boldsymbol{\gamma})\wedge(\mathbf{a}^p\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})) \\ &+(-1)^q(\mathbf{a}^q\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}))\wedge(\mathbf{a}^p\lfloor\boldsymbol{\gamma}) + (-1)^q(\mathbf{a}^q\lfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}))\wedge(\mathbf{a}^p\lfloor\boldsymbol{\alpha}) \\ &+(-1)^p(\mathbf{a}^q\lfloor(\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}))\wedge(\mathbf{a}^p\lfloor\boldsymbol{\beta}) + (-1)^q\mathbf{a}^q\wedge(\mathbf{a}^p\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma})), \quad (1.133)\end{aligned}$$

which now requires $p, q \geq 3$.

Decomposition rule As an example of using the generalized bac cab rule (1.126) we consider the possibility of decomposing a p-vector $\mathbf{a}^p$ in two components as

$$\mathbf{a}^p = \mathbf{a}^p_{\|} + \mathbf{a}^p_{\perp}, \qquad (1.134)$$

with respect to a given vector $\mathbf{a}$ and a given dual vector $\boldsymbol{\alpha}$ assumed to satisfy $\mathbf{a}|\boldsymbol{\alpha} \neq 0$. Using terminology similar to that in decomposing vectors and bivectors as in (1.53) and (1.92), the component $\mathbf{a}^p_{\|}$ is called parallel to the vector $\mathbf{a}$, and the component $\mathbf{a}^p_{\perp}$ orthogonal to $\boldsymbol{\alpha}$. These concepts are defined by the respective conditions

$$\mathbf{a}\wedge\mathbf{a}^p_{\|} = 0, \quad \boldsymbol{\alpha}\rfloor\mathbf{a}^p_{\perp} = 0. \qquad (1.135)$$

Applying the bac cab rule (1.127), the decomposition can be readily written as

$$\mathbf{a}^p = \frac{\mathbf{a}\wedge(\mathbf{a}^p\lfloor\boldsymbol{\alpha})}{\mathbf{a}|\boldsymbol{\alpha}} - \frac{(\mathbf{a}\wedge\mathbf{a}^p)\lfloor\boldsymbol{\alpha}}{\mathbf{a}|\boldsymbol{\alpha}}, \qquad (1.136)$$

from which the two p-vector components can be identified as

$$\mathbf{a}^p_{\|} = \frac{\mathbf{a} \wedge (\mathbf{a}^p \lfloor \boldsymbol{\alpha})}{\mathbf{a}|\boldsymbol{\alpha}}, \qquad \mathbf{a}^p_{\perp} = -\frac{(\mathbf{a} \wedge \mathbf{a}^p)\lfloor \boldsymbol{\alpha}}{\mathbf{a}|\boldsymbol{\alpha}}. \tag{1.137}$$

Problems

1.4.1 Show that we can express

$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) = (\mathbf{a}|\boldsymbol{\alpha})(\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) \\ +(\mathbf{a}|\boldsymbol{\beta})(\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}) + (\mathbf{a}|\boldsymbol{\gamma})(\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}),$$

and

$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) \\ = \quad (\mathbf{a}|\boldsymbol{\alpha})(\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) - (\mathbf{a}\rfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}))|(\boldsymbol{\alpha}\rfloor(\mathbf{b}\wedge\mathbf{c})).$$

1.4.2 The bac cab rule can be written as

$$\boldsymbol{\alpha}\rfloor(\mathbf{a}\wedge\mathbf{b}) = (\mathbf{ab} - \mathbf{ba})|\boldsymbol{\alpha}.$$

Derive its generalizations

$$\boldsymbol{\alpha}\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}) = ((\mathbf{a}\wedge\mathbf{b})\mathbf{c} + (\mathbf{b}\wedge\mathbf{c})\mathbf{a} + (\mathbf{c}\wedge\mathbf{a})\mathbf{b})|\boldsymbol{\alpha}$$

and

$$\boldsymbol{\alpha}\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}) \\ = \quad ((\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})\mathbf{d} - (\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})\mathbf{a} + (\mathbf{c}\wedge\mathbf{d}\wedge\mathbf{a})\mathbf{b} - (\mathbf{d}\wedge\mathbf{a}\wedge\mathbf{b})\mathbf{c})|\boldsymbol{\alpha}$$

1.4.3 Show that if a bivector $\mathbf{A}$ in a four-dimensional space $n = 4$ satisfies $\mathbf{A}\wedge\mathbf{A} = 0$, it can be represented as a simple bivector, in the form $\mathbf{A} = \mathbf{a}\wedge\mathbf{b}$.

1.4.4 Derive (1.76)

1.4.5 Prove that the space of trivectors has the dimension $n(n-1)(n-2)/3!$.

1.4.6 Derive the general commutation rule (1.98).

1.4.7 Given a basis of vectors $\{\mathbf{e}_i\}$, $i = 1\cdots n$ and defining the complementary $(n-1)$-vectors $\mathbf{e}_{K(i)} = \mathbf{e}_1\wedge\cdots\mathbf{e}_{i-1}\wedge\mathbf{e}_{i+1}\cdots\wedge\mathbf{e}_n$ and the n-vector $\mathbf{e}_N = \mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_n$, prove the identities

$$\mathbf{e}_i\wedge\mathbf{e}_{K(i)} = (-1)^{i-1}\mathbf{e}_N, \qquad \mathbf{e}_{K(i)}\wedge\mathbf{e}_i = (-1)^{n-i}\mathbf{e}_N.$$

1.4.8 Defining the dual n-vector $\boldsymbol{\varepsilon}_N = \boldsymbol{\varepsilon}_1\wedge\cdots\wedge\boldsymbol{\varepsilon}_n$ corresponding to the basis reciprocal to $\{\mathbf{e}_i\}$, prove the identities

$$\boldsymbol{\varepsilon}_N\lfloor\mathbf{e}_i = (-1)^{i-1}\boldsymbol{\varepsilon}_{K(i)}, \qquad \boldsymbol{\varepsilon}_N\lfloor\mathbf{e}_{K(i)} = (-1)^{n-i}\boldsymbol{\varepsilon}_i.$$

1.4.9 Prove the identity

$$(\boldsymbol{\alpha}\rfloor\mathbf{a}_N)\wedge\mathbf{a}=(\boldsymbol{\alpha}|\mathbf{a})\mathbf{a}_N,$$

where $\mathbf{a}$ is a vector, $\boldsymbol{\alpha}$ is a dual vector, and $\mathbf{a}_N$ is an n-vector.

1.4.10 If $\mathbf{A}$ is a bivector and $\mathbf{a}\neq 0$ is a vector, show that vanishing of the trivector $\mathbf{a}\wedge\mathbf{A}=0$ implies the existence of a vector $\mathbf{b}$ such that we can write $\mathbf{A}=\mathbf{a}\wedge\mathbf{b}$.

1.4.11 Show that $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}=0$ implies a linear relation between the four vectors $\mathbf{a}\cdots\mathbf{d}$.

1.4.12 Derive (1.117) and (1.118).

1.4.13 Starting from the last result of problem 1.4.2 and applying (1.85), show that the bac cab rule (1.90) can be written as

$$\boldsymbol{\alpha}\rfloor(\mathbf{b}\wedge\mathbf{C})=\mathbf{b}\wedge(\boldsymbol{\alpha}\rfloor\mathbf{C})-\mathbf{C}(\boldsymbol{\alpha}|\mathbf{b}),$$

when $\mathbf{C}$ is a trivector, $\mathbf{b}$ is a vector, and $\boldsymbol{\alpha}$ is a dual vector. Write the special case for $n=3$.

1.4.14 Show that the identity in the previous problem can be written as

$$(\mathbf{b}\wedge\mathbf{C})\lfloor\boldsymbol{\alpha}=\mathbf{C}(\boldsymbol{\alpha}|\mathbf{b})-(\boldsymbol{\alpha}\rfloor\mathbf{C})\wedge\mathbf{b}$$

and check that the same form applies for $\mathbf{C}$ being a bivector as well as a trivector. Actually it works for a quadrivector as well, so it may be valid for any p-vector, although a proof was not yet found.

1.4.15 Starting from (1.82), derive the following rule between vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ and dual vectors $\boldsymbol{\beta},\boldsymbol{\gamma}$:

$$(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})=\mathbf{a}(\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})-(\mathbf{b}\wedge\mathbf{c})\lfloor\Big(\mathbf{a}\rfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})\Big),$$

whose dual can be expressed in terms of a bivector $\mathbf{A}$ and a dual bivector $\boldsymbol{\Gamma}$ as

$$\mathbf{A}\rfloor(\boldsymbol{\beta}\wedge\boldsymbol{\Gamma})=\boldsymbol{\beta}(\mathbf{A}|\boldsymbol{\Gamma})+\boldsymbol{\Gamma}\lfloor(\mathbf{A}\lfloor\boldsymbol{\beta}).$$

This is another bac cab rule.

1.4.16 Prove the identity

$$\mathbf{A}\wedge(\mathbf{B}\lfloor\boldsymbol{\alpha})+\mathbf{B}\wedge(\mathbf{A}\lfloor\boldsymbol{\alpha})=(\mathbf{A}\wedge\mathbf{B})\lfloor\boldsymbol{\alpha}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two bivectors and $\boldsymbol{\alpha}$ a dual vector.

1.4.17 Prove the identity

$$\mathbf{a}\wedge(\mathbf{e}_N\lfloor\boldsymbol{\Gamma})=-\mathbf{e}_N\lfloor(\boldsymbol{\Gamma}\lfloor\mathbf{a}),$$

where $\mathbf{a}$ is a vector, $\boldsymbol{\Gamma}$ is a dual bivector, and $\mathbf{e}_N$ is an n-vector.

1.4.18 Prove the identity

$$(\mathbf{a}\rfloor\mathbf{\Gamma})\rfloor\mathbf{A} = \mathbf{A}\lfloor(\mathbf{\Gamma}\lfloor\mathbf{a}) = (\mathbf{a}\wedge\mathbf{A})\lfloor\mathbf{\Gamma} - \mathbf{a}(\mathbf{A}|\mathbf{\Gamma})$$

where $\mathbf{A}$ is a bivector, $\mathbf{\Gamma}$ is a dual bivector, and $\mathbf{a}$ is a vector.

1.4.19 Starting from the expansion

$$(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\cdots\wedge\mathbf{a}_p)\lfloor\boldsymbol{\alpha} = -\boldsymbol{\alpha}|\sum_{i=1}^{p}(-1)^i\mathbf{a}_i\mathbf{a}_{K(i)},$$

with $\mathbf{a}_{K(i)} = \mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_{i-1}\wedge\mathbf{a}_{i+1}\wedge\cdots\wedge\mathbf{a}_p$, $1\leq p\leq n$, prove the identity

$$(\mathbf{a}\wedge\mathbf{a}^q)\lfloor\boldsymbol{\alpha} = (\mathbf{a}|\boldsymbol{\alpha})\mathbf{a}^q - \mathbf{a}\wedge(\mathbf{a}^q\lfloor\boldsymbol{\alpha})$$

where $\mathbf{a}^q$ is a q-vector, $\mathbf{a}$ is a vector, and $\boldsymbol{\alpha}$ is a dual vector, $1\leq q\leq n$.

1.4.20 Show that by inserting $\mathbf{a}^q = \mathbf{b}\wedge\mathbf{a}^p$ in the previous identity and denoting the bivector $\mathbf{a}\wedge\mathbf{b} = \mathbf{A}$ we can derive the identity

$$(\mathbf{A}\wedge\mathbf{a}^p)\lfloor\boldsymbol{\alpha} = (\mathbf{A}\lfloor\boldsymbol{\alpha})\wedge\mathbf{a}^p + \mathbf{A}\wedge(\mathbf{a}^p\lfloor\boldsymbol{\alpha})$$

1.4.21 As a generalization of the two previous identities, we anticipate the identity

$$(\mathbf{a}^q\wedge\mathbf{a}^p)\lfloor\boldsymbol{\alpha} = (\mathbf{a}^q\lfloor\boldsymbol{\alpha})\wedge\mathbf{a}^p + (-1)^q\mathbf{a}^q\wedge(\mathbf{a}^p\lfloor\boldsymbol{\alpha})$$

to be valid. Assuming this and writing $\mathbf{a}^p = \mathbf{b}\wedge\mathbf{a}^{p-1}$, $\mathbf{a}^q\wedge\mathbf{b} = \mathbf{a}^{q+1}$, show that the identity takes the form

$$(\mathbf{a}^{q+1}\wedge\mathbf{a}^{p-1})\lfloor\boldsymbol{\alpha} = (\mathbf{a}^{q+1}\lfloor\boldsymbol{\alpha})\wedge\mathbf{a}^{p-1} + (-1)^{q+1}\mathbf{a}^{q+1}\wedge(\mathbf{a}^{p-1}\lfloor\boldsymbol{\alpha}).$$

From this we can conclude that, if the anticipated identity is valid for $q = 1$ and $q = 2$, it will be valid for any q.

1.4.22 Derive (1.132) and (1.133) in detail.

1.4.23 Prove the generalized form of (1.125) for general n,

$$\mathbf{\Phi}\wedge(\boldsymbol{\kappa}_N\lfloor\mathbf{A}) = \boldsymbol{\kappa}_N(\mathbf{\Phi}|\mathbf{A}),$$

when $\mathbf{A}$ is a bivector, $\mathbf{\Phi}$ is a dual bivector and $\boldsymbol{\kappa}_N$ is a dual n-vector.

1.4.24 Prove that if a dual bivector $\mathbf{\Phi}$ satisfies $\mathbf{a}\rfloor\mathbf{\Phi} = 0$, for some vector $\mathbf{a}$, there exists a dual trivector $\boldsymbol{\gamma}\in\mathbb{F}_3$ such that $\mathbf{\Phi}$ can be expressed in the form $\mathbf{\Phi} = \mathbf{a}\rfloor\boldsymbol{\gamma}$.

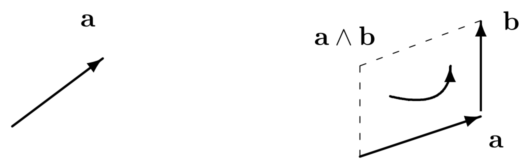

Fig. 1.11 *Geometric interpretation of a three-dimensional vector as a directed line segment and a bivector as an oriented area. The orientation is defined as a sense of circulation on the area.*

1.5 GEOMETRIC INTERPRETATION

Multivector algebra is based on the geometric foundations introduced by Grassmann, and its elements can be given geometric interpretations. While appealing to the eye, they do not very much help in problem solving. The almost naive algebraic memory aid with hooks and eyes appears to be sufficient in most cases for checking algebraic expressions against obvious errors. It is interesting to note that the article by Deschamps [18] introducing differential forms to electromagnetics did not contain a single geometric figure. On the other hand, there are splendid examples of geometric interpretation of multivectors and dual multivectors [58]. In the following we give a simplified overview on the geometric aspects.

1.5.1 Vectors and bivectors

A three-dimensional real vector is generally interpreted as an oriented line segment (arrow) in space. The corresponding interpretation for a bivector is an oriented area. For example, the bivector $\mathbf{a}\wedge\mathbf{b}$ defines a parallelogram defined by the vectors $\mathbf{a}$ and $\mathbf{b}$ (Figure 1.11). The order of the two vectors defines an orientation of the bivector as the sense of circulation around the parallelogram and the area of the parallelogram is proportional to the magnitude of the bivector. Change of order of $\mathbf{a}$ and $\mathbf{b}$ does not change the area but reverses the orientation. Parallel vectors defined by a linear relation of the form $\mathbf{a} = a\mathbf{b},\ a \neq 0$ give zero area, $\mathbf{a}\wedge\mathbf{b} = 0$. Orthogonality at this stage is only defined between a vector and a dual vector as $\mathbf{a}|\boldsymbol{\alpha} = 0$. Orthogonality of two vectors or two dual vectors is a concept which depends on the definition of a metric dyadic (see Section 2.5).

For example, taking two basis vectors $\mathbf{u}_x$ and $\mathbf{u}_y$, the wedge product of $\mathbf{a} = a\mathbf{u}_x$ and $\mathbf{b} = b\mathbf{u}_y$ gives the bivector

$$\mathbf{a}\wedge\mathbf{b} = ab\mathbf{u}_x\wedge\mathbf{u}_y, \tag{1.138}$$

which is a multiple of the basis bivector $\mathbf{u}_x\wedge\mathbf{u}_y$. Rotating the vectors by an angle θ as $\mathbf{a} = a(\mathbf{u}_x\cos\theta + \mathbf{u}_y\sin\theta)$ and $\mathbf{b} = b(\mathbf{u}_x\sin\theta - \mathbf{u}_y\cos\theta)$, the bivector becomes

$$\mathbf{a}\wedge\mathbf{b} = ab(\cos^2\theta + \sin^2\theta)\mathbf{u}_x\wedge\mathbf{u}_y = ab\mathbf{u}_x\wedge\mathbf{u}_y, \tag{1.139}$$

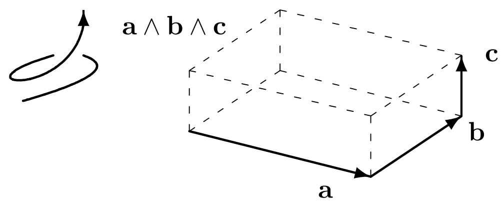

Fig. 1.12 *Geometric interpretation of a trivector as an oriented volume. Orientation is defined as handedness, which in this case is that of a right-handed screw.*

which coincides with the previous one. Thus, the bivector is invariant to the rotation. Also, it is invariant if we multiply $\mathbf{a}$ and divide $\mathbf{b}$ by the same real number λ.

1.5.2 Trivectors

Three real vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ define a parallelepiped. The trivector $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$ represents its volume V with orientation. In component form we can write

$$\begin{aligned} &\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} \\ &= [a_x(b_yc_z - b_zc_y) + a_y(b_zc_x - b_xc_z) + a_z(b_xc_y - b_yc_x)]\mathbf{u}_x\wedge\mathbf{u}_y\wedge\mathbf{u}_z \\ &= \det\begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}\mathbf{u}_x\wedge\mathbf{u}_y\wedge\mathbf{u}_z. \end{aligned} \tag{1.140}$$

The determinant of the 3×3 matrix defined by the components of the vector equals the Gibbsian scalar triple product $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}$ and it vanishes when the three vectors are linearly dependent. In this case the vectors lie in the same plane and the volume of the parallepiped is zero. The orientation of a given trivector $\mathbf{k}$ is given in terms of its handedness. Taken after one another, three nonplanar vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ define a screw in space which can be right or left handed, Figure 1.12. Changing the order of any two vectors changes the handedness of the trivector. Handedness of a trivector can also be determined with respect to a given reference dual trivector ε_{123}. If $\mathbf{k}|\varepsilon_{123}$ is positive, $\mathbf{k}$ has the same handedness as ε_{123}, otherwise it has the opposite handedness. If the coordinate system x, y, z is labeled as right handed, the trivector $\mathbf{u}_x\wedge\mathbf{u}_y\wedge\mathbf{u}_z$ corresponds to a unit cube with right-handed orientation. If the expression in the square brackets in (1.140) is positive, it represents the volume of a right-handed parallelepiped. With negative sign, its magnitude gives the volume of a left-handed parallelepiped. For complex vectors the simple geometric interpretation breaks. Complex vectors can be interpreted in terms of oriented ellipses [28,40], but bivectors and trivectors cannot be easily given a mental picture.

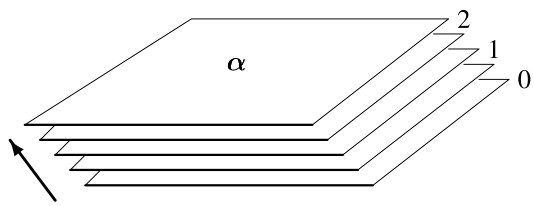

Fig. 1.13 *Geometric interpretation of a dual vector as a set of parallel planes with orientation and density.*

1.5.3 Dual vectors

Dual vectors are objects which map vectors onto scalars through the duality product. Elements of dual vectors can be represented in the three-dimensional Euclidean vector space by considering the set of vectors which are mapped onto the same scalar value. For example, the dual vector $\boldsymbol{\alpha} = \alpha \boldsymbol{\varepsilon}_1$ maps all vectors $\mathbf{r} = \sum x_i \mathbf{e}_i$ onto the scalar value a when the tip of the vector $\mathbf{r}$ lies on the plane $x_1 = a$. Thus, the dual vector $\boldsymbol{\alpha}$ is like a directed measuring tape with parallel planes labeled by the distance in the perpendicular direction of the tape. The magnitude ("length´´) of the dual vector is the density of planes (labeling by the inch or centimeter on the tape) and the direction of increasing scalar values represents the orientation of the dual vector. Of course, in the space of dual vectors, a dual vector appears as a directed line segment and a vector, which is the dual of a dual vector, as a set of parallel planes.

A field of vectors can be represented by field lines, which give the orientation of the vector at each point in the physical space. The magnitude of the vector at each position can be given by a set of surfaces corresponding to a constant value. The field lines need not be orthogonal to these surfaces. The dual vectors can then be represented by surfaces tangent to the dual-vector planes at each position, FigureForm153. The density of the planes gives the magnitude of the dual vector.

The sum of two vectors is represented through the well-known parallelogram construction. The sum of two dual vectors $\boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{\gamma}$ can be constructed as follows. Since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are represented by two families of parallel planes, let us consider two planes of each. Unless the planes are parallel, in which case the addition is simple, they cut in four common parallel lines which define two diagonal planes not parallel to either of the original ones. One of these new planes defines a set of parallel planes corresponding to the sum, and the other one the difference, of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The sum corresponds to the pair of planes whose positive orientation has positive components on both original sets of planes.

The duality product of a vector and a dual vector $\boldsymbol{\alpha}|\mathbf{a}$ equals the number of planes pierced by the vector $\mathbf{a}$. A good mental picture is obtained through the method of measuring a javelin throw in prewar Olympic games where the lines of constant

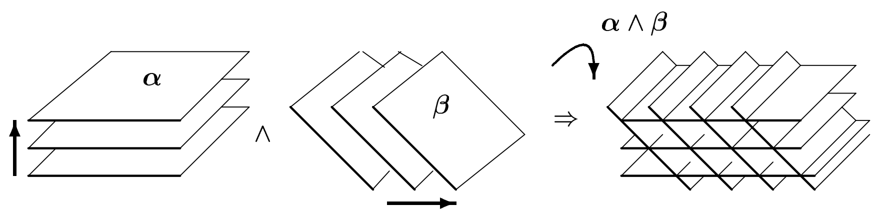

Fig. 1.14 *Geometric interpretation of a dual bivector as a set of parallel tubes with orientation (sense of rotation) and density.*

distance used to be parallel[5] to the line behind which the throw was done. If the vector is parallel to the planes, there is no piercing and the inner product is zero. In this case, the vector $\mathbf{a}$ and the dual vector $\boldsymbol{\alpha}$ can be called orthogonal. The number of planes pierced is largest when the vector is perpendicular to them.

A three-dimensional vector basis $\{\mathbf{e}_i\}$ can be formed by any three non-coplanar vectors. They need not have the same length or be perpendicular to one another. The associated dual basis $\{\varepsilon_j\}$ consists of three families of planes which cut the space in closed cells, that is, they do not form open tubes. The densities of the dual basis vectors depend on the vectors $\mathbf{e}_i$. Because, for example, we have $\varepsilon_2|\mathbf{e}_1 = 0$ and $\varepsilon_3|\mathbf{e}_1 = 0$, the vector $\mathbf{e}_1$ must be parallel to the common lines of the plane families ε_2 and ε_3. $\mathbf{e}_1$ is not orthogonal to the family of planes ε_1. However, the density of ε_1 is such that $\mathbf{e}_1$ pierces exactly one interval of planes.

1.5.4 Dual bivectors and trivectors

The bivectors in the dual space, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$, can be pictured as a family of tubes defined by two families of planes and the magnitude equals the density of tubes, Figure 1.14. The inner product $(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})|(\mathbf{a} \wedge \mathbf{b})$ represents the number of tubes enclosed by the parallelogram defined by the bivector $\mathbf{a} \wedge \mathbf{b}$.

The dual trivector $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma}$ represents the density of cells defined by three families of planes. The inner product $(\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma})|(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})$ gives the number of cells of the dual trivector $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma}$ enclosed by the parallelepiped of the trivector $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$. The result can also be a negative number if the handedness of the cells is opposite to that of the parallelepiped. Similar ideas can be applied to p-vectors and dual p-vectors in spaces of higher dimension, but their visualization must rely on analogies.

Problems

1.5.1 Find a geometrical meaning for summing two bivectors, $\mathbf{A} + \mathbf{B}$, in the three-dimensional space. *Hint*: Bivectors are oriented areas on a certain plane. The form of their contour line can be changed. Assume the planes of $\mathbf{A}$ and $\mathbf{B}$ cut

[5] In the 1930s this method was changed to one with circular lines.

along a line containing the vector $\mathbf{c}$. Then there exist vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{A} = \mathbf{a} \wedge \mathbf{c}$ and $\mathbf{B} = \mathbf{b} \wedge \mathbf{c}$.

1.5.2 Verify the geometric construction of the sum $\boldsymbol{\alpha} + \boldsymbol{\beta}$ and difference $\boldsymbol{\alpha} - \boldsymbol{\beta}$ of two dual vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

1.5.3 Interprete geometrically the bivector relation

$$\mathbf{a}_1 \wedge \mathbf{a}_2 = \mathbf{a}_1 \wedge (\mathbf{a}_1 + \mathbf{a}_2)$$

by considering the parallelograms defined by the bivectors on each side.

1.5.4 Interprete geometrically the planar bivector relation

$$\mathbf{a}_1 \wedge \mathbf{a}_2 + (\mathbf{a}_1 + \mathbf{a}_2) \wedge \mathbf{a}_3 = \mathbf{a}_1 \wedge \mathbf{a}_2 + \mathbf{a}_1 \wedge \mathbf{a}_3 + \mathbf{a}_2 \wedge \mathbf{a}_3$$

by considering the areas of the different triangles.