Appendix E

Vector Analysis in the Special Theory of Relativity

To study the theory of relativity, the most efficient mathematical tool is multidimensional tensor analysis with a dimension greater than three. Within the realm of the special theory of relativity, the subject can be treated by ordinary vector analysis in three dimensions. In fact, this is what Einstein did in his original work published in 1905. In this appendix, we shall first follow this approach and then show how the same result is obtained by a four-dimensional analysis.

Based on the experimental evidence that the velocity of light is independent of the status of source, moving or stationary, that emits the light signal, Einstein postulated the doctrine in his special theory of relativity that for two coordinate systems in relative motion with a constant velocity of separation $\mathbf{v} = v\hat{z}$, the space and the time variables in the two systems must obey the Lorentz transform stated by the following relations:

$$x = x', \tag{E.1}$$

$$y = y', \tag{E.2}$$

$$z = \gamma(z' + vt'), \tag{E.3}$$

$$t = \gamma\left(t' + \frac{v}{c^2}z'\right); \tag{E.4}$$

the reverse transforms of (E.3) and (E.4) are

$$z' = \gamma(z - vt) \tag{E.5}$$

and

$$t' = \gamma\left(t - \frac{v}{c^2}z\right), \tag{E.6}$$

where v is the velocity of separation in the z or z' direction between the two systems, $\gamma = 1/(1-\beta^2)^{1/2}$, $\beta = v/c$, and c is the velocity of light in free space. Another principle contained in Einstein's theory is the invariance of Maxwell's equations in the two coordinate systems, that is,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{E.7}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{E.8}$$

$$\nabla \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{E.9}$$

$$\nabla \mathbf{D} = \rho, \tag{E.10}$$

$$\nabla \mathbf{B} = 0, \tag{E.11}$$

and

$$\nabla' \times \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'}, \tag{E.12}$$

$$\nabla' \times \mathbf{H}' = \mathbf{J}' + \frac{\partial \mathbf{D}'}{\partial t'}, \tag{E.13}$$

$$\nabla' \mathbf{J}' = -\frac{\partial \rho'}{\partial t'}, \tag{E.14}$$

$$\nabla' \mathbf{D}' = \rho', \tag{E.15}$$

$$\nabla' \mathbf{B}' = 0, \tag{E.16}$$

where the unprimed operators and the unprimed functions are defined with respect to $(x, y, z; t)$ and the primed ones with respect to $(x', y', z'; t')$. The Lorentz transform assures us the relation that when

$$x^2 + y^2 + z^2 - c^2t^2 = 0, \tag{E.17}$$

then

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0. \tag{E.18}$$

(E.17) or (E.18) corresponds to "equation of motion" of the propagation of the light signal. In fact, these are two of the equations used to derive (E.3) and (E.4). With this background we can find the transform between the field vectors. Let us consider first the x-component of (E.7) pertaining to Faraday's law in the rectangular systems, that is,

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}. \tag{E.19}$$

The derivatives with respect to y, z, and t can be converted to the derivatives with respect to y', z', and t' with the aid of (E.1) to (E.6). Thus, (E.19) can be written as

$$\frac{\partial E_z}{\partial y'} - \left(\frac{\partial E_y}{\partial z'}\frac{\partial z'}{\partial z} + \frac{\partial E_y}{\partial t'}\frac{\partial t'}{\partial z}\right) = -\left(\frac{\partial B_x}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial B_x}{\partial z'}\frac{\partial z'}{\partial t}\right) \tag{E.20}$$

or

$$\frac{\partial E_z}{\partial y'} - \gamma\frac{\partial E_y}{\partial z'} + \frac{\gamma v}{c^2}\frac{\partial E_y}{\partial t'} = -\gamma\frac{\partial B_x}{\partial t'} + \gamma v\frac{\partial B_x}{\partial z'};$$

hence

$$\frac{\partial E_z}{\partial y'} - \frac{\partial}{\partial z'}\left[\gamma\left(E_y + vB_z\right)\right] = -\frac{\partial}{\partial t'}\left[\gamma\left(B_x + \frac{v}{c^2}E_y\right)\right]. \tag{E.21}$$

The x-component of (E.12) reads

$$\frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} = -\frac{\partial B_x'}{\partial t'}. \tag{E.22}$$

The matching of (E.21) and (E.22) yields

$$E_z' = E_z, \tag{E.23}$$

$$E_y' = \gamma\left(E_y + vB_x\right), \tag{E.24}$$

$$B_x' = \gamma\left(B_x + \frac{v}{c^2}E_y\right). \tag{E.25}$$

By working, similarly, on the other equations and combining the resultant equations, we find

$$\mathbf{E}' = \bar{\bar{\gamma}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{E.26}$$

$$\mathbf{B}' = \bar{\bar{\gamma}} \cdot \left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right), \tag{E.27}$$

$$\mathbf{H}' = \bar{\bar{\gamma}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}), \tag{E.28}$$

$$\mathbf{D}' = \bar{\bar{\gamma}} \cdot \left(\mathbf{D} + \frac{1}{c^2}\mathbf{v} \times \mathbf{H}\right), \tag{E.29}$$

$$\mathbf{J}' = \gamma\bar{\bar{\gamma}}^{-1}(\mathbf{J} - \rho\mathbf{v}), \tag{E.30}$$

$$\rho' = \gamma\left(\rho - \frac{1}{c^2}\mathbf{v} \cdot \mathbf{J}\right), \tag{E.31}$$

where $\mathbf{v} = v\hat{z}$ and the dyadics $\bar{\bar{\gamma}}$ and $\bar{\bar{\gamma}}^{-1}$ are defined by

$$\bar{\bar{\gamma}} = \gamma\left(\hat{x}\hat{x} + \hat{y}\hat{y}\right) + \hat{z}\hat{z},$$

$$\bar{\bar{\gamma}}^{-1} = \frac{1}{\gamma}\left(\hat{x}\hat{x} + \hat{y}\hat{y}\right) + \hat{z}\hat{z};$$

so

$$\gamma \bar{\bar{\gamma}}^{-1} = \hat{x}\hat{x} + \hat{y}\hat{y} + \gamma \hat{z}\hat{z}$$

and

$$\bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}}^{-1} = \bar{\bar{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} .$$

(E.26) to (E.29) are the transforms of the field vectors defined in the two coordinate systems. These are the equations based on which the problems involving moving media can be formulated. When $v^2/c^2 \ll 1$,

$$\gamma \doteq 1,$$

the following relations hold true

$$\mathbf{E}'^{*} = \mathbf{E} + \mathbf{v} \times \mathbf{B} , \tag{E.32}$$

$$\mathbf{B}'^{*} = \mathbf{B} , \tag{E.33}$$

$$\mathbf{H}'^{*} = \mathbf{H} - \mathbf{v} \times \mathbf{D} , \tag{E.34}$$

$$\mathbf{D}'^{*} = \mathbf{D} , \tag{E.35}$$

$$\mathbf{J}'^{*} = \mathbf{J} - \rho \mathbf{v} , \tag{E.36}$$

$$\rho'^{*} = \rho . \tag{E.37}$$

The symbol * means that these expressions are approximate under the condition $v^2/c^2 \ll 1$.

These transforms have been derived with a rather tedious procedure involving altogether nine scalar differential equations. Many of the details are not shown here. A more elegant method is to recast the Lorentz transform into a pseudo-real orthogonal transform, a method due to Sommerfeld [45]. According to that method, we let

$$\gamma = \frac{1}{\left(1 - \beta^2\right)^{1/2}} = \cos \eta , \tag{E.38}$$

then

$$\sin \eta = \left(1 - \cos^2 \eta\right)^{1/2} = \frac{i\beta}{\left(1 - \beta^2\right)^{1/2}} . \tag{E.39}$$

Because

$$\beta = \frac{v}{c} < 1,$$

η must be an imaginary angle, hence the term "pseudo-real." Now, we introduce the four-dimensional coordinate variables (x_1, x_2, x_3, x_4) defined by

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict,$$

and similarly for x'_j with $j = 1, 2, 3, 4$; then (E.1) to (E.6) can be written as

$$\begin{aligned} x_1 &= x'_1 , \\ x_2 &= x'_2 , \\ x_3 &= x'_3 \cos\eta - x'_4 \sin\eta = a_{33}x'_3 + a_{43}x'_4 , \\ x_4 &= x'_4 \cos\eta + x'_3 \sin\eta = a_{34}x'_3 + a_{44}x'_4 , \end{aligned} \tag{E.40}$$

where

$$\begin{aligned} a_{33} &= \cos\eta, & a_{34} &= \sin\eta, \\ a_{43} &= -\sin\eta, & a_{44} &= \cos\eta. \end{aligned} \tag{E.41}$$

The "directional cosines" between the two sets of axes can be tabulated as follows:

$$\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x'_1 & 1 & 0 & 0 & 0 \\ x'_2 & 0 & 1 & 0 & 0 \\ x'_3 & 0 & 0 & a_{33} & a_{34} \\ x'_4 & 0 & 0 & a_{43} & a_{44} \end{array} . \tag{E.42}$$

As with the three-dimensional coefficients, they satisfy the orthogonal relations

$$\sum_{j=1}^{4} a_{ij}a_{kj} = \delta_{ik} \tag{E.43}$$

and

$$|a_{ij}| = 1.$$

The field vectors in Maxwell's equations can now be formulated as vectors or tensors in a four-dimensional manifold. We first define a four-potential vector, denoted by **P**, as

$$\mathbf{P} = \sum_{i=1}^{4} P_i \hat{x}_i$$

with $P_1 = A_1$, $P_2 = A_2$, $P_3 = A_3$, $P_4 = i\phi/c$, where A_i with $i = 1, 2, 3$ are the components of the vector potential **A** and ϕ is the dynamic scalar potential. The two functions have been introduced previously in Section 4-10. We define the components of the curl of **P** as

$$\nabla_{mn}\mathbf{P} = \frac{\partial P_n}{\partial x_m} - \frac{\partial P_m}{\partial x_n}, \qquad m, n = 1, 2, 3, 4. \tag{E.44}$$

They are functions with two indices and there are 12 of them, but because

$$\nabla_{mn}\mathbf{P} = -\nabla_{nm}\mathbf{P} \tag{E.45}$$

and

$$\nabla_{mm}\mathbf{P} = 0,$$

there are actually only six distinct components. These components or functions will be denoted by F_{mn} and are designated as the *six-vectors*. The components of the field vectors **B** and **E** can now be treated as six vectors. For example,

$$B_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = \frac{\partial P_3}{\partial x_2} - \frac{\partial P_2}{\partial x_3} = \nabla_{23}\mathbf{P} = F_{23}\,, \tag{E.46}$$

$$E_1 = -\frac{\partial \phi}{\partial x_1} - \frac{\partial A_1}{\partial t} = ic\left(\frac{\partial P_4}{\partial x_1} - \frac{\partial P_1}{\partial x_4}\right) = ic\nabla_{14}\mathbf{P} = icF_{14},$$

or

$$\frac{-iE_1}{c} = \nabla_{14}\mathbf{P} = F_{14}\,. \tag{E.47}$$

The six functions $(B_i, -iE_i/c)$ with $i = 1, 2, 3$ form the components of a 4×4 antisymmetric tensor, which is shown here:

$$\left[F_{ij}\right] = \begin{bmatrix} 0 & B_3 & -B_2 & \dfrac{-i}{c}E_1 \\ -B_3 & 0 & B_1 & \dfrac{-i}{c}E_2 \\ B_2 & B_1 & 0 & \dfrac{-i}{c}E_3 \\ \dfrac{i}{c}E_1 & \dfrac{i}{c}E_2 & \dfrac{i}{c}E_3 & 0 \end{bmatrix}. \tag{E.48}$$

The 3×3 minor in the upper left corner, or the adjoint of F_{44}, is recognized as the antisymmetric tensor of the axial vector $\mathbf{B} = \nabla\mathbf{A}$.
Another six-vector is defined by

$$\mathbf{G} = (\mathbf{H}, -ic\mathbf{D})\,. \tag{E.49}$$

In free space,

$$\mathbf{G} = \frac{1}{\mu_0}\mathbf{F} = \frac{1}{\mu_0}\nabla\mathbf{P}\,. \tag{E.50}$$

The 4×4 antisymmetric tensor of **G** is shown here:

$$\left[G_{ij}\right] = \begin{bmatrix} 0 & H_3 & -H_2 & -icD_1 \\ -H_3 & 0 & H_1 & -icD_2 \\ H_2 & H_1 & 0 & -icD_3 \\ icD_1 & icD_2 & icD_3 & 0 \end{bmatrix}. \tag{E.51}$$

It is possible to construct two four-dimensional dyadics using these tensors, but they are not necessary in this presentation.

We can now extend the rules of the transformation of 3 × 3 tensors in Chapter 1 for two orthogonal rectangular systems to the 4 × 4 antisymmetric tensors under consideration, that is,

$$f'_{ij} = \sum_{m=1}^{4} \sum_{n=1}^{4} a_{im} a_{jn} f_{mn}. \tag{E.52}$$

Applying (E.52) to **F** and **G** with a_{ij} given by (E.42), we find

$$\mathbf{E}' = \bar{\bar{\gamma}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{E.53}$$

$$\mathbf{B}' = \bar{\bar{\gamma}} \cdot \left(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right), \tag{E.54}$$

$$\mathbf{H}' = \bar{\bar{\gamma}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}), \tag{E.55}$$

$$\mathbf{D}' = \bar{\bar{\gamma}} \cdot \left(\mathbf{D} + \frac{\mathbf{v}}{c^2} \times \mathbf{H}\right). \tag{E.56}$$

They are the same as (E.26) to (E.29) obtained before by the classical method. The transform of the current density function **J** and the charge density function ρ can be found by applying the four-dimensional rule to the components of a four-current vector defined by

$$K_i = \left(J_1, J_2, J_3, ic\rho\right), \qquad i = 1, 2, 3, 4. \tag{E.57}$$

The transform is

$$K'_i = \sum_{j=1}^{4} a_{ij} K_j, \qquad i = 1, 2, 3, 4, \tag{E.58}$$

which yields

$$K'_1 = K_1; \qquad \text{hence } J'_1 = J_1, \tag{E.59}$$

$$K'_2 = K_2; \qquad \text{hence } J'_2 = J_2, \tag{E.60}$$

$$K'_3 = a_{33} K_3 + a_{34} K_4;$$

hence

$$\begin{aligned} J'_3 &= \gamma \left(J_3 - v\rho\right), \\ K'_4 &= a_{43} K_3 + a_{44} K_4 \end{aligned} \tag{E.61}$$

or

$$ic\rho' = -\frac{i\gamma v}{c} J_3 + \gamma \left(ic\rho\right);$$

hence

$$\rho' = \gamma \left(\rho - \frac{v}{c^2} J_3\right). \tag{E.62}$$

(E.53) to (E.62) are identical to (E.26) to (E.31). It is seen that the four-dimensional analysis to derive the transform of the field vectors is very elegant as long as we get used to the rather novel concept of pseudo-real representation of Lorentz transform in a four-dimensional space or manifold.