Appendix B

Solutions

The following set of solutions covers most of the problems given in Appendix A. The solutions are given in compact form to save space.

Complex vectors

1.1 The extremal values of the square of the corresponding time-harmonic vector are found from

$$\frac{d}{dt}[\mathbf{A}(t) \cdot \mathbf{A}(t)] = \frac{d}{dt}[\mathsf{a}_1^2 \cos^2 \omega t + \mathsf{a}_2^2 \sin^2 \omega t] = 0,$$

which reduces to $(\mathsf{a}_1^2 - \mathsf{a}_2^2) \sin \omega t \cos \omega t = 0$. For a circularly polarized vector $(\mathsf{a}_1^2 = \mathsf{a}_2^2)$ this is satisfied for all t, which means that it has discrete axes. Otherwise we have solutions $t = n\pi/2\omega$. For n even the axis occurs at $\pm\mathbf{a}_1$, for n odd at $\pm\mathbf{a}_2$.

1.2 The time-harmonic vector of $\mathbf{a}e^{j\phi}$ equals $\Re\{\mathbf{a}e^{j\omega(t+\phi/\omega)}\}$, which shifts the time origin by ϕ/ω but otherwise does not change the ellipse of the vector $\mathbf{a}$.

1.3 Assume first $\mathbf{a}$ is not circularly polarized. Write $\mathbf{b} = \mathbf{a}e^{j\theta}$ in axial representation, with θ such that $\mathbf{b} \cdot \mathbf{b} = |\mathsf{a}|^2$. Because $\mathbf{p}(\mathbf{a}) = \mathbf{p}(\mathbf{b})$, the axial vectors $\mathbf{b}_{\mathrm{re}}$ and $\mathbf{b}_{\mathrm{im}}$ satisfy

$$|\mathbf{p}(\mathbf{b})| = \frac{2|\mathbf{b}_{\mathrm{re}}|\,|\mathbf{b}_{\mathrm{im}}|}{|\mathbf{b}_{\mathrm{re}}|^2 + |\mathbf{b}_{\mathrm{im}}|^2} = \frac{2|\mathbf{b}_{\mathrm{re}}|\,|\mathbf{b}_{\mathrm{im}}|}{|\mathsf{a}|^2}.$$

(a) From the geometry we have $\tan(\psi/2) = |\mathbf{b}_{\mathrm{im}}|/|\mathbf{b}_{\mathrm{re}}|$ and

$$|\mathbf{p}(\mathbf{a})| = \frac{2\tan(\psi/2)}{1 + \tan^2(\psi/2)} = \sin\psi.$$

(b) The area of the ellipse can be written as

$$A = \pi|\mathbf{b}_{\mathrm{re}}|\ |\mathbf{b}_{\mathrm{im}}| = \pi|\mathbf{a}|^2|\mathbf{p}(\mathbf{a})|/2,$$

which equals the required relation.

These results are also valid in the limit when $\mathbf{a}$ is circularly polarized.

1.4 The polarization vector $\mathbf{p}(\mathbf{a})$ can be expanded in each of the cases as

(a) $\mathbf{p}(\mathbf{a}) = -\mathbf{u}_z 2\sin\alpha\cos\alpha = -\mathbf{u}_z \sin 2\alpha$. The axial ratio of the ellipse is $e = \tan\alpha$. It is seen that all polarizations are obtained because $|\mathbf{p}(\mathbf{a})|$ goes through all values between 0 and 1 when α is changed. Linear polarization ($e = 0$ or $\pm\infty$) corresponds to $\alpha = n\pi/2$ and circular ($e = \pm 1$ to $\alpha = n\pi/4$), n odd.

(b) Assuming $\mathbf{b}$ not linearly polarized, $\mathbf{p}(\mathbf{b})$ is parallel to $\mathbf{u}$ and we can write

$$\frac{\mathbf{b}\times\mathbf{b}^* + (\mathbf{u}\times\mathbf{b})\times(\mathbf{u}\times\mathbf{b}^*) - j\mathbf{b}\times(\mathbf{u}\times\mathbf{b}^*) - j\mathbf{b}^*\times(\mathbf{u}\times\mathbf{b})}{2j(|\mathbf{b}|^2 + j\mathbf{u}\cdot\mathbf{b}\times\mathbf{b}^*)}$$

$$= \frac{\mathbf{p}(\mathbf{b}) - \mathbf{u}}{1 - \mathbf{u}\cdot\mathbf{p}(\mathbf{b})} = -\mathbf{u} = \mathbf{p}(\mathbf{a}).$$

Thus, $\mathbf{a}$ is circularly polarized, which is easily checked: $\mathbf{a}\cdot\mathbf{a} = 0$. This is valid in the limit also when $\mathbf{b}$ is linearly polarized.

(c) Because $(\mathbf{b}\times\mathbf{b}^*)\times(\mathbf{b}\times\mathbf{b}^*)^* = 0$ and, hence, $\mathbf{p}(\mathbf{a}) = 0$, $\mathbf{a}$ is linearly polarized (or zero).

1.5 $\mathbf{a}\cdot\mathbf{p}(\mathbf{a}) = \mathbf{a}^*\cdot\mathbf{p}(\mathbf{a}) = 0$ in all cases. Expanding

$$[\mathbf{a}\times\mathbf{p}(\mathbf{a})]\times[\mathbf{a}\times\mathbf{p}(\mathbf{a})]^* = \mathbf{p}(\mathbf{a})[\mathbf{a}\times\mathbf{a}^*\cdot\mathbf{p}(\mathbf{a})] = j\mathbf{p}(\mathbf{a})(\mathbf{a}\cdot\mathbf{a}^*)[\mathbf{p}(\mathbf{a})\cdot\mathbf{p}(\mathbf{a})],$$

$$[\mathbf{a}\times\mathbf{p}(\mathbf{a})]\cdot[\mathbf{a}\times\mathbf{p}(\mathbf{a})]^* = (\mathbf{a}\cdot\mathbf{a}^*)[\mathbf{p}(\mathbf{a})\cdot\mathbf{p}(\mathbf{a})]$$

we have

$$\frac{[\mathbf{a}\times\mathbf{p}(\mathbf{a})]\times[\mathbf{a}\times\mathbf{p}(\mathbf{a})]^*}{j[\mathbf{a}\times\mathbf{p}(\mathbf{a})]\cdot[\mathbf{a}\times\mathbf{p}(\mathbf{a})]^*} = j\mathbf{p}(\mathbf{a})\frac{(\mathbf{a}\cdot\mathbf{a}^*)[\mathbf{p}(\mathbf{a})\cdot\mathbf{p}(\mathbf{a})]}{j(\mathbf{a}\cdot\mathbf{a}^*)[\mathbf{p}(\mathbf{a})\cdot\mathbf{p}(\mathbf{a})]} = \mathbf{p}(\mathbf{a}).$$

1.6 If $\mathbf{a}$ is not linearly polarized, $\mathbf{p}(\mathbf{a}) \neq 0$ and we can define a unit vector $\mathbf{n} = \mathbf{p}(\mathbf{a})/|\mathbf{p}(\mathbf{a})|$ normal to the plane of $\mathbf{a}$. If $\mathbf{b} = \mathbf{a}+\mathbf{n}\alpha$, from $\mathbf{b}\cdot\mathbf{b} = 0$ we can solve for α and have two possibilities: $\mathbf{b} = \mathbf{a}\pm j\mathbf{n}\sqrt{\mathbf{a}\cdot\mathbf{a}}$. This is also valid for the linearly polarized case when $\mathbf{n}$ is any unit vector in the plane orthogonal to $\mathbf{a}$.

1.7 For a circularly polarized we can write $\mathbf{b} = \mathbf{c} = \mathbf{a}/2$. Applying the result of the previous problem, we can directly write

$$\mathrm{b} = \frac{1}{2}(\mathrm{a} + j\mathrm{n}\sqrt{\mathrm{a}\cdot\mathrm{a}}\,), \qquad \mathrm{c} = \frac{1}{2}(\mathrm{a} - j\mathrm{n}\sqrt{\mathrm{a}\cdot\mathrm{a}}\,),$$

because $|\mathrm{b}| = |\mathrm{c}|$, as can be readily checked.

1.9 Because $\mathbf{a} \times (\mathbf{b} \times \mathbf{b}^*) = \mathbf{b}(\mathbf{a} \cdot \mathbf{b}^*) - \mathbf{b}^*(\mathbf{a} \cdot \mathbf{b}) = 0$, either $\mathbf{b} \times \mathbf{b}^* = 0$, whence $\mathbf{b}$ is linearly polarized, or $\mathbf{a}$ is a multiple of $\mathbf{b} \times \mathbf{b}^*$, which is known to be a linearly polarized vector.

1.11 For a circularly polarized, $\mathbf{p}(\mathbf{a}) = \mathbf{u}$ is a real unit vector. Thus, $\mathbf{a}_3 = j|\mathbf{a}|^2\mathbf{p}(\mathbf{a}) = j|\mathbf{a}|^2\mathbf{u}$ and $\mathbf{u} \times \mathbf{a} = (\mathbf{a} \times \mathbf{a}^*) \times \mathbf{a}/j|\mathbf{a}|^2 = j\mathbf{a}$. The reciprocal basis is

$$\mathrm{a}_1' = \frac{\mathrm{a}^* \times (j|\mathrm{a}|^2\mathrm{u})}{-|\mathrm{a}|^4} = \frac{\mathrm{a}^*}{|\mathrm{a}|^2}, \quad \mathrm{a}_2' = \frac{(j|\mathrm{a}|^2\mathrm{u}) \times \mathrm{a}}{-|\mathrm{a}|^4} = \frac{\mathrm{a}}{|\mathrm{a}|^2},$$

$$\mathrm{a}_3' = \frac{\mathrm{a} \times \mathrm{a}^*}{-|\mathrm{a}|^4} = -j\frac{\mathrm{u}}{|\mathrm{a}|^2}.$$

1.13 Writing $\mathbf{k}_{\mathrm{re}} + j\mathbf{k}_{\mathrm{im}}$, we have by separating the real and imaginary parts of $\mathbf{k} \cdot \mathbf{k} = k_o^2$,

$$\mathbf{k}_{\mathrm{re}} \cdot \mathbf{k}_{\mathrm{re}} - \mathbf{k}_{\mathrm{im}} \cdot \mathbf{k}_{\mathrm{im}} = k_o^2, \qquad \mathbf{k}_{\mathrm{re}} \cdot \mathbf{k}_{\mathrm{im}} = 0.$$

Thus, the vectors $\mathbf{k}_{\mathrm{re}}$ and $\mathbf{k}_{\mathrm{im}}$ are orthogonal and $|\mathbf{k}_{\mathrm{re}}| > |\mathbf{k}_{\mathrm{im}}|$. The equation $|\mathbf{k}_{\mathrm{re}}|^2 - |\mathbf{k}_{\mathrm{im}}|^2 = k_o^2$ is that of a hyperbola. In parametrized form, the most general solution for $\mathbf{k}$ is

$$\mathbf{k} = k_o(\mathbf{u}\cosh\theta + j\mathbf{v}\sinh\theta),$$

where $\mathbf{u}$ and $\mathbf{v}$ are any two real orthogonal unit vectors.

Dyadics

2.1 Consider mapping of a vector $\mathbf{b}$:

$$[\mathbf{a} \times \overline{\overline{I}} - \overline{\overline{I}} \times \mathbf{a}] \cdot \mathbf{b} = \mathbf{a} \times (\overline{\overline{I}} \cdot \mathbf{b}) - \overline{\overline{I}} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{b} = 0$$

Since the result is zero for any vector $\mathbf{b}$, the dyadic in square brackets must be zero, which gives the identity.

2.2 Using the result of the previous problem, we can write

$$(\mathbf{a}\times\overline{\overline{I}}):(\overline{\overline{I}}\times\mathbf{b})=\sum_i\sum_j[(\mathbf{a}\times\mathbf{u}_i)\mathbf{u}_i]:[\mathbf{u}_j(\mathbf{u}_j\times\mathbf{b})]$$

$$=\sum_i\sum_j[(\mathbf{a}\times\mathbf{u}_i)\cdot\mathbf{u}_j][\mathbf{u}_i\cdot(\mathbf{u}_j\times\mathbf{b})]$$

$$=\mathbf{a}\cdot\sum_i\sum_j(\mathbf{u}_i\times\mathbf{u}_j)(\mathbf{u}_i\times\mathbf{u}_j)\cdot\mathbf{b}=\mathbf{a}\cdot(\overline{\overline{I}}{}^\times_\times\overline{\overline{I}})\cdot\mathbf{b}=\mathbf{a}\cdot(2\overline{\overline{I}})\cdot\mathbf{b}=2\mathbf{a}\cdot\mathbf{b}.$$

2.3 It follows directly from

$$(\mathbf{a}\times\mathbf{cd}):(\mathbf{ef}\times\mathbf{b})=\mathbf{a}\cdot(\mathbf{c}\times\mathbf{e})(\mathbf{d}\times\mathbf{f})\cdot\mathbf{b}=\mathbf{a}\cdot(\mathbf{cd}{}^\times_\times\mathbf{ef})\cdot\mathbf{b}$$

2.4 Start from the expression

$$(\mathbf{a}\times\mathbf{uu}){}^\times_\times(\mathbf{b}\times\mathbf{vv})=[(\mathbf{a}\times\mathbf{u})\times(\mathbf{b}\times\mathbf{v})](\mathbf{u}\times\mathbf{v})$$

$$=[\mathbf{ba}\cdot(\mathbf{u}\times\mathbf{v})+(\mathbf{a}\times\mathbf{b})\cdot\mathbf{uv}](\mathbf{u}\times\mathbf{v})$$

$$=\mathbf{ba}\cdot(\mathbf{uu}{}^\times_\times\mathbf{vv})-\mathbf{vv}\times\mathbf{uu}\cdot(\mathbf{a}\times\mathbf{b})$$

Replace **uu** and **vv** by $\overline{\overline{I}}$:

$$(\mathbf{a}\times\overline{\overline{I}}){}^\times_\times(\mathbf{b}\times\overline{\overline{I}})=\mathbf{ba}\cdot(\overline{\overline{I}}{}^\times_\times\overline{\overline{I}})-\overline{\overline{I}}\times\overline{\overline{I}}\cdot(\mathbf{a}\times\mathbf{b})$$

$$=2\mathbf{ba}-\overline{\overline{I}}\times(\mathbf{a}\times\mathbf{b})=2\mathbf{ba}-\mathbf{ba}+\mathbf{ab}=\mathbf{ab}+\mathbf{ba}$$

2.5 Start from the expression

$$(\mathbf{cd}{}^\times_\times\mathbf{ef})\cdot(\mathbf{a}\times\mathbf{b})=(\mathbf{c}\times\mathbf{e})[(\mathbf{d}\times\mathbf{f})\cdot(\mathbf{a}\times\mathbf{b})]$$

$$=(\mathbf{c}\times\mathbf{e})[(\mathbf{d}\cdot\mathbf{a})(\mathbf{f}\cdot\mathbf{b})-(\mathbf{d}\cdot\mathbf{b})(\mathbf{f}\cdot\mathbf{a})]$$

$$=(\mathbf{cd}\cdot\mathbf{a})\times(\mathbf{ef}\cdot\mathbf{b})-(\mathbf{cd}\cdot\mathbf{b})\times(\mathbf{ef}\cdot\mathbf{a})$$

Replace **cd** and **ef** by $\overline{\overline{A}}$:

$$(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}})\cdot(\mathbf{a}\times\mathbf{b})=2(\overline{\overline{A}}\cdot\mathbf{a})\times(\overline{\overline{A}}\cdot\mathbf{b}).$$

2.8 One can immediately see that $\alpha=0$ and all linear dyadics $\overline{\overline{A}}$ make one set of solutions. For $\alpha\neq0$, denoting $\overline{\overline{B}}=2\overline{\overline{A}}/\alpha$ we have a new problem $\overline{\overline{B}}{}^{(2)}=\overline{\overline{B}}$. Its solutions $\overline{\overline{B}}\neq0$ must be complete dyadics. In fact, assuming $\overline{\overline{B}}$ planar, $\overline{\overline{B}}{}^{(2)}$ must be linear, whence $\overline{\overline{B}}$ is also linear implying $B = \overline{\overline{B}}^{(2)} = 0$, a contradiction. Since $\det\overline{\overline{B}} \neq 0$ for a complete dyadic, taking the determinant gives us $\det\overline{\overline{B}}^{(2)} = \det^2\overline{\overline{B}} = \det\overline{\overline{B}}$, or $\det\overline{\overline{B}} = 1$. Hence, $\overline{\overline{B}}^{(2)} = \overline{\overline{B}}^{-1\,T} = \overline{\overline{B}}$. Because $\overline{\overline{B}}$ is assumed symmetric, it must be a square root of the unit dyadic, i.e., of the form $\overline{\overline{I}} - 2(\mathbf{ab}/\mathbf{a}\cdot\mathbf{b})$ with $\mathbf{a}\cdot\mathbf{b} \neq 0$.

2.10 The invariants are

$$\mathrm{tr}\overline{\overline{D}} = 2\alpha + \beta, \quad \mathrm{spm}\overline{\overline{D}} = \overline{\overline{D}}^{(2)} : \overline{\overline{I}} = 2\alpha\beta + \alpha^2, \quad \det\overline{\overline{D}} = \alpha^2\beta,$$

whence the Cayley-Hamilton equation can be written as

$$\lambda^3 - (2\alpha + \beta)\lambda^2 + (2\alpha\beta + \alpha^2)\lambda - \alpha^2\beta = (\lambda - \alpha)^2(\lambda - \beta) = 0.$$

Thus, the eigenvalues are $\lambda_1 = \beta$ and $\lambda_{2,3} = \alpha$. The eigenvector $\mathbf{a}_1 = u$ and the other two $\mathbf{a}_{2,3}$ are any vectors orthogonal to $\mathbf{u}$.

2.11 Because the dyadic $\overline{\overline{G}}$ consists of a transverse part and a multiple of $\mathbf{uu}$, it is obvious that one of the eigenvectors is $\mathbf{u}$ and the rest are orthogonal to $\mathbf{u}$. The first eigenvalue is $\lambda_1 = \beta$ and $\mathbf{a}_1 = \mathbf{u}$. Thus, we are left with the two-dimensional eigenvalue problem

$$e^{\overline{\overline{J}}\theta}\cdot\mathbf{a} = (\overline{\overline{I}}_t\cos\theta + \overline{\overline{J}}\sin\theta)\cdot\mathbf{a} = \frac{\lambda}{R}\mathbf{a}.$$

Because we have

$$\det(e^{\overline{\overline{J}}\theta}) = 0, \quad \mathrm{tr}(e^{\overline{\overline{J}}\theta}) = 2\cos\theta,$$

$$(e^{\overline{\overline{J}}\theta})\overset{\times}{\times}(e^{\overline{\overline{J}}\theta}) = 2\mathbf{uu}, \qquad \mathrm{spm}(e^{\overline{\overline{J}}\theta}) = 2,$$

the Cayley-Hamilton equation reads

$$\lambda^3 - \lambda^2 R(2\cos\theta) + \lambda R^2 = 0$$

and the solutions $\lambda \neq 0$ are $\lambda_{2,3} = Re^{\pm j\theta}$. The eigenvectors satisfy

$$R(e^{\overline{\overline{J}}\theta} - e^{\pm j\theta}\overline{\overline{I}}_t)\cdot\mathbf{a}_{2,3} = R\sin\theta(\overline{\overline{J}} \mp j\overline{\overline{I}}_t)\cdot\mathbf{a}_{2,3} = 0.$$

This equals $\mathbf{u}\times\mathbf{a}_{2,3} = \pm j\mathbf{a}_{2,3}$, whence the eigenvalues are circularly polarized ($\mathbf{a}_{2,3}\cdot\mathbf{a}_{2,3} = 0$), with a_2 right-hand and $\mathbf{a}_3$ left-hand rotation with respect to the $\mathbf{u}$ direction.

2.12 We have first

$$\mathbf{b}\cdot\mathbf{b}^* = \mathbf{a}\cdot\overline{\overline{R}}^T\cdot\overline{\overline{R}}\cdot\mathbf{a}^* = \mathbf{a}\cdot\mathbf{a}^*,$$

whence the magnitude of the vector is not changed. Secondly, because of

$$\overline{\overline{R}}^{-1} = \frac{\overline{\overline{R}}^{(2)T}}{\mathrm{det}\overline{\overline{R}}} = \overline{\overline{R}}^{(2)T} = \overline{\overline{R}}^T,$$

we have

$$\mathbf{b}\times\mathbf{b}^* = (\overline{\overline{R}}\cdot\mathbf{a})\times(\overline{\overline{R}}\cdot\mathbf{a}^*) = \overline{\overline{R}}^{(2)}\cdot(\mathbf{a}\times\mathbf{a}^*) = \overline{\overline{R}}\cdot(\mathbf{a}\times\mathbf{a}^*),$$

and thus

$$(\mathbf{b}\times\mathbf{b}^*)\cdot(\mathbf{b}\times\mathbf{b}^*)^* = (\mathbf{a}\times\mathbf{a}^*)\cdot\overline{\overline{R}}^T\cdot\overline{\overline{R}}\cdot(\mathbf{a}\times\mathbf{a}^*)^* = (\mathbf{a}\times\mathbf{a}^*)\cdot(\mathbf{a}\times\mathbf{a}^*)^*,$$

which tells us that the form of the ellipse has not changed.

2.13 Since the dyadic consists of orthogonal axial and transverse parts, both of these can be treated separately. The square-root of the axial part is obviously $\pm\sqrt{\beta}\mathbf{uu}$ since its square is $\beta\mathbf{uu}$. The transverse part gives correspondingly $\pm\sqrt{R}e^{\overline{\overline{J}}\theta/2}$, whence the following four dyadics (double signs are not related)

$$[\overline{\overline{G}}(\beta,R,\theta)]^{1/2} = \pm\sqrt{\beta}\mathbf{uu}\pm\sqrt{R}e^{\overline{\overline{J}}\theta/2}$$

are square roots of the gyrotropic dyadic.

2.14 For each case we apply the fact that if $\mathbf{a}\cdot\overline{\overline{A}}\cdot\mathbf{b}=0$ for all $\mathbf{a}$, $\mathbf{b}$, then $\overline{\overline{A}}=0$.

(a) $\overline{\overline{A}}:(\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b})=0$ for all vectors $\mathbf{a}$, $\mathbf{b}$ implies $\overline{\overline{A}}:(\mathbf{ab}+\mathbf{ba}) = \mathbf{a}\cdot(\overline{\overline{A}}+\overline{\overline{A}}^T)\cdot\mathbf{b}=0$, whence $\overline{\overline{A}}=-\overline{\overline{A}}^T$ and $\overline{\overline{A}}$ is antisymmetric.

(b) $\overline{\overline{A}}:(\mathbf{ab}-\mathbf{ba}) = \mathbf{a}\cdot(\overline{\overline{A}}-\overline{\overline{A}}^T)\cdot\mathbf{b}=0$, whence $\overline{\overline{A}}=\overline{\overline{A}}^T$ and $\overline{\overline{A}}$ is symmetric.

(c) $\overline{\overline{A}}:(\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b})^* = \overline{\overline{A}}:(\mathbf{ab}^*+\mathbf{ba}^*)=0$ and $\overline{\overline{A}}:(\mathbf{a}+j\mathbf{b})(\mathbf{a}+j\mathbf{b})^* = -j\overline{\overline{A}}:(\mathbf{ab}^*-\mathbf{ba}^*)=0$ imply $\overline{\overline{A}}:\mathbf{ab}^*=0$ for all vectors $\mathbf{a}$, $\mathbf{b}^*$, whence $\overline{\overline{A}}=0$.

Field equations

3.1 The Helmholtz operator can be written as

$$-\mu\overline{\overline{H}}_e = (\nabla \times \overline{\overline{I}} - j\omega\xi\overline{\overline{I}}) \cdot (\nabla \times \overline{\overline{I}} + j\omega\zeta\overline{\overline{I}}) - \omega^2\mu\epsilon\overline{\overline{I}}$$

$$= (\nabla \times \overline{\overline{I}})^2 + j\omega(\zeta - \xi)\nabla \times \overline{\overline{I}} - \omega^2(\mu\epsilon - \xi\zeta)\overline{\overline{I}}$$

This is a polynom of $\nabla \times \overline{\overline{I}}$ and can be written in the form

$$-\mu\overline{\overline{H}}_1(\nabla) \cdot \overline{\overline{H}}_2(\nabla) = (\nabla \times \overline{\overline{I}} + \alpha\overline{\overline{I}}) \cdot (\nabla \times \overline{\overline{I}} + \beta\overline{\overline{I}}).$$

$$= (\nabla \times \overline{\overline{I}})^2 + (\alpha + \beta)\nabla \times \overline{\overline{I}} + \alpha\beta\overline{\overline{I}}$$

The coefficients α and β can be solved by comparing the expressions and the result is

$$\alpha, \beta = j\omega\left(\frac{\zeta - \xi}{2} \pm \sqrt{\frac{(\zeta + \xi)^2}{4} + \mu\epsilon}\right)$$

In the reciprocal chiral (Pasteur) medium with $\zeta = -\xi = j\kappa\sqrt{\mu_o\epsilon_o}$ the coefficients reduce to

$$\alpha, \beta = jk_o(\kappa \pm \sqrt{\mu_r\epsilon_r}).$$

3.2 From the π network with shunt admittances Y_1 and series impedance Z_2 with loading impedance η_o we have the input admittance

$$Y_{in} = Y_1 + \frac{1}{Z_2 + \frac{1}{Y_1 + 1/\eta_o}}$$

When this is equated with $1/\eta_o$, we can solve for Z_2 in the form

$$Z_2 = \frac{2\eta_o^2 Y_1}{1 - \eta_o^2 Y_1^2}.$$

Substituting $Z_2 = jk_o t(\mu_r - 1)\eta_o$, $Y_1 = jk_o t(\epsilon_r - 1)/\eta_o$, we have

$$k_o t_2(\mu_r - 1) = \frac{2k_o t_1(\epsilon_r - 1)}{1 + k_o^2 t_1^2(\epsilon_r - 1)^2}.$$

3.3 Because of the lossless character, the medium six-dyadic M is hermitian, which implies that all its eigenvalues are real. The positive energy function requires that M is positive definite (PD), whence its eigenvalues are positive. Also, its diagonal elements $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ must be PD. From matrix algebra we know that the inverse M^{-1} must be PD, whence its diagonal dyadics are PD. Solving for the inverse dyadics by solving two vector equations, we arrive at the required condition that

$$\overline{\overline{\epsilon}},\ \ \overline{\overline{\mu}},\ \ \overline{\overline{\epsilon}} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}},\ \ \ \overline{\overline{\mu}} - \overline{\overline{\zeta}} \cdot \overline{\overline{\epsilon}}^{-1} \cdot \overline{\overline{\xi}}$$

must all be positive definite dyadics.

3.4 Same as above when $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ are replaced by $\overline{\overline{\epsilon}} - \epsilon_o \overline{\overline{I}}$ and $\overline{\overline{\mu}} - \mu_o \overline{\overline{I}}$, respectively. However, because the sign $>$ was replaced by $\geq$, PD (positive definite) must be replaced by PSD (positive semidefinite). In fact, for the vacuum, the equality sign must apply.

3.5 Consider the energy expression of the problem 3.3. Setting $\mathbf{H} = 0$ and $\mathbf{E} = 0$ we conclude that $\epsilon > 0$ and $\mu > 0$, respectively, corresponding to $\epsilon \geq \epsilon_o$ and $\mu \geq \mu_o$ of the problem 3.4. Denoting

$$\mathbf{a} = \sqrt{\epsilon}\mathbf{E}, \quad \mathbf{b} = \sqrt{\mu}\mathbf{H},$$

we can write the inequality $W > 0$ as

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 \pm 2|\chi_r|\Re\{\mathbf{a} \cdot \mathbf{b}^*\} \pm 2|\kappa_r|\Im\{\mathbf{a} \cdot \mathbf{b}^*\} > 0,$$

which should be valid for all vectors $\mathbf{a}$, $\mathbf{b}$. Here we have denoted $\xi = (\chi_r - j\kappa_r)\sqrt{\mu\epsilon}$ and $\zeta = (\chi_r + j\kappa_r)\sqrt{\mu\epsilon}$. Choosing $\mathbf{a} = e^{j\alpha}\mathbf{b}$, the inequality becomes

$$|\chi_r| \cos\alpha + |\kappa_r| \sin\alpha < 1,$$

which is valid for all real angles α if

$$\chi_r^2 + \kappa_r^2 < 1, \quad \Rightarrow \quad \chi^2 + \kappa^2 < \mu_r \epsilon_r.$$

In the sharper case of the problem 3.4, the inequality is replaced by

$$\chi^2 + \kappa^2 \leq (\mu_r - 1)(\epsilon_r - 1).$$

These conditions can be readily obtained as special cases of the results of the two previous problems. In fact, if $\overline{\overline{\epsilon}} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}}$ is PD and $(\overline{\overline{\epsilon}} - \epsilon_o \overline{\overline{I}}) - \overline{\overline{\xi}} \cdot (\overline{\overline{\mu}} - \mu_o \overline{\overline{I}})^{-1} \cdot \overline{\overline{\zeta}}$ is PSD, in the bi-isotropic case these reduce to the respective conditions $\epsilon\mu - \xi\zeta > 0$ and $(\epsilon - \epsilon_o)(\mu - \mu_o) - \xi\zeta \geq 0$, which coincide with the results derived with a change of symbols.

3.6 The condition of losslessness is $\mathbf{n}\cdot\Re\{\mathbf{E}\times\mathbf{H}^*\} = 0$ for all possible fields on the surface. Evaluating

$$-\mathbf{n}\cdot\mathbf{E}\times\mathbf{H}^* = \mathbf{H}^*\cdot\overline{\overline{Z}}_s\cdot\mathbf{H} = Z_1|\mathbf{H}|^2 + Z_2\mathbf{n}\cdot\mathbf{H}\times\mathbf{H}^*$$

$$= (Z_1 + jZ_2\mathbf{n}\cdot\mathbf{p}(\mathbf{H}))|\mathbf{H}|^2 = 0.$$

The real part of the bracketed term must be null for any field $\mathbf{H}$. Writing $Z_1 = R_1 + jX_1$ and $Z_2 = R_2 + jX_2$, we have

$$R_1 - X_2\mathbf{n}\cdot\mathbf{p}(\mathbf{H}) = 0.$$

Because the real vector $\mathbf{p}(\mathbf{H})$ can have any direction, we must have separately $R_1 = 0$ and $X_2 = 0$. The lossless surface impedance dyadic is of the form

$$\overline{\overline{Z}}_s = jX_1\overline{\overline{I}} + R_2\mathbf{n}\times\overline{\overline{I}},$$

where X_1 and R_2 have real values.

Field transformations

4.1 (a) $\det\mathsf{T}(\alpha) = \frac{1}{1-\sin 2\alpha}(1 - 2\sin\alpha\cos\alpha) = 1.$

(b) Inverting the $\mathsf{T}(\alpha)$ matrix changes the signs of $\sin\alpha$ and $\cos\alpha$ in $\mathsf{T}(\alpha)$, which is tantamount to replacing α by $\alpha+\pi$.

(c) Denoting $\overline{\overline{\tau}} = \overline{\overline{\xi}} + \overline{\overline{\zeta}}$, the transformation formulas are

$$(1-\sin 2\alpha)\overline{\overline{\epsilon}}_d = \overline{\overline{\epsilon}} + 2\frac{\overline{\overline{\mu}}}{\eta_o^2}\cos^2\alpha - 2\sqrt{2}\frac{\overline{\overline{\tau}}}{\eta_o}\cos\alpha,$$

$$(1-\sin 2\alpha)\overline{\overline{\mu}}_d = \overline{\overline{\mu}} + 2\overline{\overline{\epsilon}}\eta_o^2\sin^2\alpha - 2\sqrt{2}\overline{\overline{\tau}}\eta_o\sin\alpha,$$

$$(1-\sin 2\alpha)\overline{\overline{\tau}}_d = (1+\sin 2\alpha)\overline{\overline{\tau}} - \sqrt{2}\overline{\overline{\epsilon}}\eta_o\sin\alpha - \sqrt{2}\frac{\overline{\overline{\mu}}}{\eta_o}\cos\alpha$$

$$\overline{\overline{\xi}}_d - \overline{\overline{\zeta}}_d = \overline{\overline{\xi}} - \overline{\overline{\zeta}}$$

(d) From the previous equations it is seen that if $\overline{\overline{\tau}} = 0$, in general we have $\overline{\overline{\tau}}_d \neq 0$, which means that a reciprocal medium is transformed to a nonreciprocal medium.

(e) $\overline{\overline{\mu}}_d = \overline{\overline{\mu}}$ when $\sin\alpha = 0$, in which case the transformation formulas read

$$\overline{\overline{\epsilon}}_d = \overline{\overline{\epsilon}} + 2\frac{\overline{\overline{\mu}}}{\eta_o^2} \mp 2\sqrt{2}\frac{\overline{\overline{\tau}}}{\eta_o},$$

$$\overline{\overline{\mu}}_d = \overline{\overline{\mu}},$$

$$\overline{\overline{\tau}}_d = \overline{\overline{\tau}} \mp \sqrt{2}\frac{\overline{\overline{\mu}}}{\eta_o}$$

$$\overline{\overline{\xi}}_d - \overline{\overline{\zeta}}_d = \overline{\overline{\xi}} - \overline{\overline{\zeta}}$$

The double sign corresponds to $\cos\alpha = \pm 1$.

(f) Requiring in the bi-isotropic case $\overline{\overline{\tau}}_d = \tau_d \overline{\overline{I}} = 0$ and denoting $\tau = \chi\sqrt{\mu_o\epsilon_o}$, $\eta = \sqrt{\mu/\epsilon}$, we have the equation

$$\frac{\chi}{\sqrt{\mu\epsilon}} = \sqrt{2}\frac{\eta^{-1}\sin\alpha + \eta\cos\alpha}{1 + \sin 2\alpha}$$

for the angle α. The right-hand side can be seen to obtain continuously all values from $-\infty$ at $\alpha = -\pi/4$ to $+\infty$ at $\alpha = 3\pi/4$. Thus, there is a unique solution α in the interval $-\pi/4 < \alpha < 3\pi/4$ corresponding to any value of χ and a transformation from any nonreciprocal bi-isotropic medium to a reciprocal bi-isotropic medium exists.

4.2 Writing

$$\nabla \cdot [\overline{\overline{\epsilon}}_r \cdot \nabla\phi(\mathbf{r})] = (\overline{\overline{A}} \cdot \nabla) \cdot (\overline{\overline{A}} \cdot \nabla)\phi(\mathbf{r}),$$

where $\overline{\overline{A}} = \overline{\overline{\epsilon}}_r{}^{1/2}$ is a real, symmetric and positive definite dyadic, we denote $\overline{\overline{A}} \cdot \nabla = \nabla'$, whence $\mathbf{r}' = \overline{\overline{A}}^{-1} \cdot \mathbf{r}$. Writing also

$$\psi(\mathbf{r}') = \phi(\mathbf{r}) = \phi(\overline{\overline{A}} \cdot \mathbf{r}'),$$

the Poisson equation becomes

$$\nabla'^2\psi(\mathbf{r}') = -\frac{Q}{\epsilon_o}\delta(\mathbf{r}) = -\frac{Q}{\epsilon_o \mathrm{det}\overline{\overline{A}}}\delta(\mathbf{r}').$$

This is an equation for a point charge in an isotropic medium with the permittivity $(\mathrm{det}\overline{\overline{A}})\epsilon_o$. The solution for the potential is known:

$$\psi(\mathbf{r}') = \frac{Q}{4\pi\epsilon_o r' \mathrm{det}\overline{\overline{A}}},$$

$$r' = \sqrt{\mathbf{r}' \cdot \mathbf{r}'} = \sqrt{(\overline{\overline{A}}^{-1} \cdot \mathbf{r}) \cdot (\overline{\overline{A}}^{-1} \cdot \mathbf{r})} = \sqrt{\overline{\overline{\epsilon}}_r{}^{-1} : \mathbf{rr}}$$

Thus, the potential in the original medium can be written as

$$\phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_o r' \sqrt{\mathrm{det}\overline{\overline{\epsilon}}_r}} = \frac{Q}{4\pi\epsilon_o D}, \qquad D(\mathbf{r}) = \sqrt{\overline{\overline{\epsilon}}_r{}^{(2)} : \mathbf{rr}}.$$

Surfaces of constant potential are ellipsoids $\bar{\bar{\epsilon}}_r^{-1} : \mathbf{rr} = \text{const}$. The electric field is orthogonal to these surfaces:

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_o D^3}\bar{\bar{\epsilon}}_r^{(2)} \cdot \mathbf{r},$$

but the flux density vector $\mathbf{D}$ is not parallel to $\mathbf{E}$:

$$\mathbf{D}(\mathbf{r}) = \bar{\bar{\epsilon}}_r \epsilon_o \cdot \mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi D^3}(\det\bar{\bar{\epsilon}}_r)\mathbf{r}.$$

Actually, the flux density vector is radially directed as in an isotropic medium.

Let us check the solution. Noting that $\nabla \cdot \mathbf{r} = 3$, when $\mathbf{r} \neq 0$, we have

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \frac{Q}{4\pi}[\frac{3}{D^3} - 3\frac{\bar{\bar{\epsilon}}_r^{(2)} \cdot \mathbf{r}}{D^5} \cdot \mathbf{r}] = 0.$$

Thus, the source must be at $\mathbf{r} = 0$. The singularity of the source is found when integrating over any small volume V containing the origin, with unit vector $\mathbf{n}$ normal to its surface S:

$$\int_V \nabla \cdot \mathbf{D}(\mathbf{r}) dV = \oint_S \mathbf{n} \cdot \mathbf{D}(\mathbf{r}) dS = \frac{Q}{4\pi}\det\bar{\bar{\epsilon}}_r \oint_S \frac{\mathbf{n} \cdot \mathbf{r}}{(\bar{\bar{\epsilon}}_r^{(2)} : \mathbf{rr})^{3/2}} dS$$

Now we can show through the same transformation that

$$\det\bar{\bar{\epsilon}}_r \oint_S \frac{\mathbf{n} \cdot \mathbf{r}}{(\bar{\bar{\epsilon}}_r^{(2)} : \mathbf{rr})^{3/2}} dS = \oint_{S'} \frac{\mathbf{n}' \cdot \mathbf{r}'}{(\mathbf{r}' \cdot \mathbf{r}')^{3/2}} dS' = \oint d\Omega' = 4\pi,$$

because $dS' = r'^2 d\Omega' / \mathbf{n}' \cdot \mathbf{u}'_r$. This leads to

$$\int_V \nabla \cdot \mathbf{D}(\mathbf{r}) dV = Q,$$

whence the source is the point charge, $\nabla \cdot \mathbf{D} = Q\delta(\mathbf{r})$ to which any nonradiating point source can be added.

Electromagnetic field solutions

5.1 Apply the formal procedure

$$\overline{\overline{G}}(\rho) = -\overline{\overline{H}}^{-1}(\nabla)\delta(\rho) = -\frac{\overline{\overline{H}}^{(2)T}(\nabla)}{\det\overline{\overline{H}}(\nabla)}\delta(\rho).$$

The Helmholtz operator can be written as

$$\overline{\overline{H}}(\nabla) = -\overline{\overline{L}}_+(\nabla)\cdot\overline{\overline{L}}_-(\nabla), \quad \overline{\overline{L}}_\pm(\nabla) = \nabla\times\overline{\overline{I}} \mp k_\pm\overline{\overline{I}},$$

The Green dyadic can be written as

$$\overline{\overline{G}}(\rho) = \overline{\overline{H}}^{(2)T}(\nabla)G(\rho) = \overline{\overline{L}}_+{}^{(2)T}(\nabla)\cdot\overline{\overline{L}}_-{}^{(2)T}(D)G(\rho),$$

$$\overline{\overline{L}}_\pm{}^{(2)T}(\nabla) = \nabla\nabla \pm k_\pm\nabla\times\overline{\overline{I}} + k_\pm^2\overline{\overline{I}}.$$

Because $\det\overline{\overline{L}}_\pm(\nabla) = \mp k_\pm(\nabla^2 + k_\pm^2)$, the scalar Green function obeys

$$G(\rho) = \frac{1}{\det\overline{\overline{L}}_+(\nabla)\det\overline{\overline{L}}_-(\nabla)}\delta(\rho)$$

$$= -\frac{1}{k_+k_-(k_-^2 - k_+^2)}\left(\frac{1}{\nabla^2 + k_+^2} - \frac{1}{\nabla^2 + k_-^2}\right)\delta(\rho)$$

Writing

$$G_\pm(\rho) = -\frac{1}{\nabla^2 + k_\pm^2}\delta(\rho), \quad \Rightarrow \quad (\nabla^2 + k_\pm^2)G_\pm(\rho) = -\delta(\rho),$$

whose solutions are known to be

$$G_\pm(\rho) = \frac{1}{4j}H_0^{(2)}(k_\pm\rho),$$

the Green dyadic can finally be written in the form

$$\overline{\overline{G}}(\rho) = \frac{\overline{\overline{L}}_+{}^{(2)T}(\nabla)\cdot\overline{\overline{L}}_-{}^{(2)T}(\nabla)}{k_+k_-(k_-^2 - k_+^2)}[G_+(\rho) - G_-(\rho)].$$

5.2 The Green dyadic expression for the bi-isotropic medium is

$$\overline{\overline{G}}(\mathbf{r}) = \frac{\overline{\overline{L}}_+{}^{(2)T}(\nabla)\cdot\overline{\overline{L}}_-{}^{(2)T}(\nabla)}{k_+k_-(k_-^2 - k_+^2)}[G_+(\mathbf{r}) - G_-(\mathbf{r})]$$

$$= \frac{1}{k_+ + k_-}\left[\nabla\nabla\frac{G_+(\mathbf{r})}{k_+} + \nabla\nabla\frac{G_-(\mathbf{r})}{k_-} + \nabla(G_+(\mathbf{r}) - G_-(\mathbf{r}))\times\overline{\overline{I}}\right.$$

$$\left. + k_+G_+(\mathbf{r}) + k_-G_-(\mathbf{r})\right], \qquad G_\pm(\mathbf{r}) = \frac{e^{-jk_\pm r}}{4\pi r}.$$

In analogy to the isotropic medium we can write

$$\nabla\nabla G_\pm(\mathbf{r}) = \mathrm{PV}\ \nabla\nabla G_\pm(\mathbf{r}) - \overline{\overline{L}}\delta(\mathbf{r}),$$

whence the singularity of the Green dyadic can be expressed as

$$\overline{\overline{G}}(\mathbf{r}) = \mathrm{PV}\overline{\overline{G}}(\mathbf{r}) - \frac{1}{k_+k_-}\overline{\overline{L}}\delta(\mathbf{r}).$$

5.3 The vector potential due to a dipole source whether in real or complex space is

$$\mathbf{A}(\mathbf{r}) = \mathbf{u}_y\mu ILG(D), \qquad D = \sqrt{(\mathbf{r} - j\mathbf{u}_z a)\cdot(\mathbf{r} - j\mathbf{u}_z a)}$$

and the electric field

$$\mathbf{E}(\mathbf{r}) = -j\omega[\mathbf{A}(\mathbf{r}) + \frac{1}{k^2}\nabla\nabla\cdot\mathbf{A}(\mathbf{r})].$$

In the far field ($kD \gg 1$) we have

$$\nabla\nabla G(D) \approx -k^2\frac{(\mathbf{r} - j\mathbf{u}_z a)(\mathbf{r} - j\mathbf{u}_z a)}{D^2}G(D),$$

so that $\mathbf{u}_y\cdot\nabla\nabla G(D)$ is of the order $(y/D)G(D) \ll G(D)$ close to the z axis. Thus, we can approximate

$$\mathbf{E}(\mathbf{r}) \approx -j\omega\mathbf{A}(\mathbf{r}) = -\mathbf{u}_y j\omega\mu ILG(D).$$

Inserting the expression (5.128) for $G(D)$ from the book, the electric field is seen to be a Gaussian beam with polarization $\mathbf{u}_y$ close to the z axis.

5.4 The fields of any plane wave satisfy the conditions

$$\mathbf{E}\cdot\mathbf{B} = \mathbf{E}\cdot\overline{\overline{\mu}}\cdot\mathbf{H} = \mu_t\mathbf{E}_t\cdot\mathbf{H}_t + \mu_v E_v H_v = 0,$$

$$\mathbf{H}\cdot\mathbf{D} = \mathbf{H}\cdot\overline{\overline{\epsilon}}\cdot\mathbf{E} = \epsilon_t\mathbf{E}_t\cdot\mathbf{H}_t + \epsilon_v E_v H_v = 0.$$

Eliminating the transverse fields we have

$$(\mu_t\epsilon_v - \epsilon_t\mu_v)E_v H_v = 0.$$

Thus, unless the bracketed term vanishes, we must have either $E_v = 0$ or $H_v = 0$. The bracketed term vanishes when $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ are multiples of one another, i.e., when the uniaxial medium is affine-isotropic. In this case the two spheroidal wave-number surfaces coincide and any combination of TE and TM waves is a possible plane wave.

5.5 The eigenvalue equation for the plane wave can be written as

$$\overline{\overline{D}}(\mathbf{k}) \cdot \mathbf{E} = 0,$$

$$\overline{\overline{D}}(\mathbf{k}) = (\mathbf{l} \quad \overline{\overline{I}} + jk\overline{\overline{\kappa}}_r)^2 + k^2\overline{\overline{I}} = \overline{\overline{D}}_+(\mathbf{k}) \cdot \overline{\overline{D}}_-(\mathbf{k}),$$

$$\overline{\overline{D}}_\pm(\mathbf{k}) = \mathbf{k} \times \overline{\overline{I}} + jk(\overline{\overline{\kappa}}_r \mp \overline{\overline{I}}), \qquad k = \omega\sqrt{\mu\epsilon}.$$

The two dyadics $\overline{\overline{D}}_\pm(\mathbf{k})$ obviously commute. In this case the problem can be split into two parts:

$$\overline{\overline{D}}_\pm(\mathbf{k}_\pm) \cdot \mathbf{E}_\pm = 0 \qquad \Rightarrow \qquad \det\overline{\overline{D}}_\pm(\mathbf{k}_\pm) = 0.$$

The determinants are expanded as

$$\det\overline{\overline{D}}_\pm(\mathbf{k}_\pm) = -jk\ \mathbf{k}_\pm\mathbf{k}_\pm : (\overline{\overline{\kappa}}_r \mp \overline{\overline{I}}) + jk^3\det(\overline{\overline{\kappa}}_r \mp \overline{\overline{I}}),$$

which leads to the wave-number equations

$$\mathbf{k}_\pm\mathbf{k}_\pm : (\overline{\overline{I}} \mp \overline{\overline{\kappa}}_r) = k^2\det(\overline{\overline{I}} \mp \overline{\overline{\kappa}}_r) = k^2(1 - \kappa_r^2)(1 \mp \kappa_r).$$

To find the rotationally symmetric eigensurfaces, denoting $\mathbf{k}_\pm = kN_\pm\mathbf{u}$ with the unit vector satisfying $\mathbf{u} \cdot \mathbf{u}_z = \cos\theta$, we can write for the refraction factors

$$N_\pm(\theta) = \sqrt{\frac{(1 - \kappa_r^2)(1 \mp \kappa_r)}{1 \pm \kappa_r \cos 2\theta}},$$

which defines two spheroids. The two spheroids have common points at $\theta = \pi/2$, because $N_+(\pi/2) = N_-(\pi/2) = \sqrt{1 - \kappa_r^2}$. This means that, instead of one or two optical axes, there exists an 'optical plane'. The axial refraction-factor values are $N_\pm(0) = 1 \mp \kappa_r$. When $|\kappa_r| \to 1$, the refraction factors tend to zero and the wavelengths become infinite.

The eigenvectors satisfy

$$N_\pm(\theta)\mathbf{u} \times \mathbf{E}_\pm = \mp j(\overline{\overline{I}} \mp \overline{\overline{\kappa}}_r) \cdot \mathbf{E}_\pm.$$

When $\mathbf{u} = \mathbf{u}_z$, the eigenvectors have no z component, as is easy to see by multiplying the equation by $\mathbf{u}_z\cdot$. Thus, in this case we have

$$\mathbf{E}_\pm = \frac{N_\pm(0)}{\mp j(1 \mp \kappa_r)}\mathbf{u}_z \times \mathbf{E}_\pm = \pm j\mathbf{u}_z \times \mathbf{E}_\pm,$$

and the eigenwaves are circularly polarized ($\mathbf{E}_\pm \cdot \mathbf{E}_\pm = 0$) with '+' corresponding to the right-hand polarization and '−' to the left-hand polarization. Eigenpolarizations for waves in other directions are elliptical and have more complicated expressions.

Source equivalence

6.1 The equivalent electric current in a chiral medium is

$$\mathbf{J}^{eq}(\mathbf{r}) = \frac{1}{j\omega\mu}[\nabla \times \mathbf{J}_m(\mathbf{r}) - \kappa k_o \mathbf{J}_m(\mathbf{r})]$$

$$= \frac{I_m L}{j\omega\mu}\nabla\delta(\mathbf{r}) \times \mathbf{u}_z + j\mathbf{u}_z \frac{\kappa_r}{\eta} I_m L \delta(\mathbf{r}).$$

The curl term corresponds to a loop current just like in a nonchiral medium and the second term corresponds to a dipole with $I = j\kappa_r I_m/\eta$.

6.2 A radial function satisfies $\nabla \times \mathbf{J} = 0$, which means that the equivalent magnetic current is zero. Thus, the radiation field outside the source is zero.

6.3 Writing

$$\nabla \times (\mathbf{u}_\varphi J_m(\rho)) = \mathbf{u}_z \frac{1}{\rho}[\rho J_m(\rho)]' = j\omega\mu \mathbf{J}(\mathbf{r}),$$

we have the solution in integral form as

$$J_m(\rho) = \frac{1}{\rho}\int_0^\rho \frac{j\omega\mu I}{2\pi}[\delta(\rho-a)-\delta(\rho-b)]d\rho = \frac{j\omega\mu I}{2\pi\rho}[U(\rho-a)-U(\rho-b)].$$

Thus, the equivalent magnetic volume current has the amplitude $j\omega\mu I/2\pi\rho$ in the region $a \le \rho \le b$.

6.4 From $\int \mathbf{R}(\mathbf{r})dV = 0$ we have the condition $\mathbf{P} = \int \mathbf{J}(\mathbf{r})dV$. Minimization of the norm of $\int \mathbf{r}\mathbf{R}(\mathbf{r})dV = \int \mathbf{r}\mathbf{J}(\mathbf{r})dV - \mathbf{a}\mathbf{P}$ is the same operation as minimizing the norm of the moment $\mathcal{P}_1(\mathbf{a})$, whence the resulting expression for $\mathbf{a}$ coincides with (6.55).

6.5 Truncating the region by the pulse function $P_V(\mathbf{r})$ gives us

$$\nabla \times \mathbf{E}_T = (\nabla \times \mathbf{E})P_V + (\nabla P_V) \times \mathbf{E} = \mathbf{J}_{mT} + \mathbf{J}_{mH},$$

$$\nabla \cdot \mathbf{D}_T = (\nabla \cdot \mathbf{D})P_V + (\nabla P_V) \cdot \mathbf{D} = \varrho_T + \varrho_H,$$

with the Huygens sources

$$\mathbf{J}_{mH} = (\nabla P_V)\times\mathbf{E} = \mathbf{n}\times\mathbf{E}\delta_S, \quad \varrho_H = (\nabla P_V)\cdot\mathbf{D} = \mathbf{n}\cdot\mathbf{D}\delta_S = \mathbf{n}\cdot\epsilon\mathbf{E}\delta_S.$$

The electric field can be written as a sum of the field from charges and magnetic currents. The component from the charge is familiar:

$$\mathbf{E}_c = -\nabla\phi = -\int \nabla G(\mathbf{r}-\mathbf{r}')\frac{\varrho(\mathbf{r}')}{\epsilon}dV'$$

The electric field from the magnetic current is dual to the magnetic field from an electric current:

$$\mathbf{E}_m = \int \nabla G(\mathbf{r}-\mathbf{r}') \times \mathbf{J}_m(\mathbf{r}')dV'.$$

Thus, the total field in the volume V is

$$\mathbf{E} = -\int_V \nabla G(\mathbf{r}-\mathbf{r}')\frac{\varrho(\mathbf{r}')}{\epsilon}dV' + \int_V \nabla G(\mathbf{r}-\mathbf{r}') \times \mathbf{J}_m(\mathbf{r}')dV'$$

$$-\oint_S \nabla G(\mathbf{r}-\mathbf{r}')\mathbf{n}' \cdot \mathbf{E}(\mathbf{r}')dS' + \oint_S \nabla G(\mathbf{r}-\mathbf{r}') \times (\mathbf{n}' \times \mathbf{E}(\mathbf{r}'))dS'$$