Chapter 2

Coordinate Systems

2-1 General Curvilinear System (GCS)

In the general curvilinear coordinate system (GCS), the coordinate variables will be denoted by ω^i with $i = 1, 2, 3$. The total differential of a position vector is defined by

$$d\mathbf{R}_\mathrm{p} = \sum_i \frac{\partial \mathbf{R}_\mathrm{p}}{\partial \omega^i} \, d\omega^i. \tag{2.1}$$

The geometrical interpretation of (2.1) is shown in Fig. 2-1. The vector coefficient $\partial \mathbf{R}_\mathrm{p}/\partial \omega^i$ is a measure of the change of $\mathbf{R}_\mathrm{p}$ due to a change of ω^i only; it will be denoted by

$$\mathbf{p}_i = \frac{\partial \mathbf{R}_\mathrm{p}}{\partial \omega^i} \, . \tag{2.2}$$

The vectors $\mathbf{p}_i$ with $i = 1, 2, 3$ are designated as the primary vectors. They are, in general, not orthogonal to each other. If the system is orthogonal, we will specifically use the term *orthogonal curvilinear system*. Orthogonal linear system is the same as the rectangular system. In terms of the primary vectors,

$$d\mathbf{R}_\mathrm{p} = \sum_i \mathbf{p}_i \, d\omega^i. \tag{2.3}$$

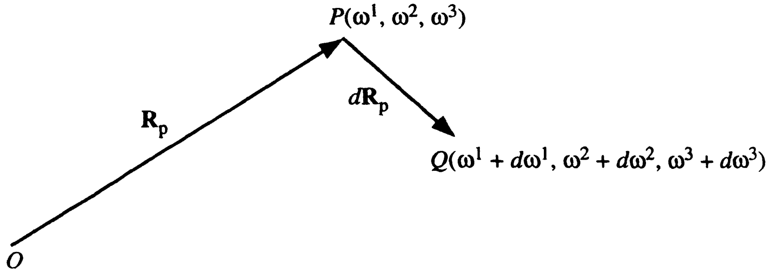

Figure 2-1 Position vector and its total differential.

The primary vectors are not necessarily of unit length, nor of the dimension of length. The three differential vectors $\mathbf{p}_i \, d\omega^i$ define a differential volume given by

$$dV = \mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3) \, d\omega^1 \, d\omega^2 \, d\omega^3 = \Lambda \, d\omega^1 \, d\omega^2 \, d\omega^3, \tag{2.4}$$

where

$$\Lambda = \mathbf{p}_i \cdot (\mathbf{p}_j \times \mathbf{p}_k) \tag{2.5}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order.

We now introduce three reciprocal vectors defined by

$$\mathbf{r}^i = \frac{1}{\Lambda} (\mathbf{p}_j \times \mathbf{p}_k) \tag{2.6}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order. They are called reciprocal vectors because

$$\mathbf{p}_i \cdot \mathbf{r}^j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.7}$$

The primary vectors can be expressed in terms of the reciprocal vectors in the form

$$\mathbf{p}_i = \Lambda \mathbf{r}^j \times \mathbf{r}^k, \tag{2.8}$$

which can be verified by means of (2.6). The reciprocal systems of vectors were originally introduced by Gibbs [4] without giving a nomenclature to the two systems of vectors. In Stratton's book [5], he designates our primary vectors as the unitary vectors and our reciprocal vectors as the reciprocal unitary vectors. His notations for our $\mathbf{p}_i$ and $\mathbf{r}^i$ are, respectively, $\mathbf{a}_i$ and $\mathbf{a}^i$. Superscript and subscript indices are used, following a tradition in tensor analysis. Incidentally, it should be mentioned that in Stratton's book, the relation described by (2.8) was inadvertently written, in our notation,

$$\mathbf{p}_i = \frac{\mathbf{r}^j \times \mathbf{r}^k}{\Lambda} \, .$$

A vector function $\mathbf{F}$ can be expressed either in terms of the primary vectors or the reciprocal vectors. We let

$$\mathbf{F} = \sum_i f_i \mathbf{r}^i = \sum_j g^j \mathbf{p}_j. \tag{2.9}$$

On account of the relations described by (2.7), one finds

$$f_i = \mathbf{p}_i \cdot \mathbf{F}, \tag{2.10}$$
$$g^j = \mathbf{r}^j \cdot \mathbf{F}. \tag{2.11}$$

f_i with $i = 1, 2, 3$ are designated as the primary components of $\mathbf{F}$, and g^j with $j = 1, 2, 3$ as the reciprocal components of $\mathbf{F}$. In tensor analysis, f_i are called the *covariant components* and g^j, the *contravariant components*. By substituting (2.10) and (2.11) into (2.9), we can write (2.9) in the form

$$\mathbf{F} = \sum_i \mathbf{F} \cdot \mathbf{p}_i \mathbf{r}^i = \sum_j \mathbf{F} \cdot \mathbf{r}^j \mathbf{p}_j$$

or

$$\mathbf{F} = \sum_i \mathbf{r}^i \mathbf{p}_i \cdot \mathbf{F} = \sum_j \mathbf{p}_j \mathbf{r}^j \cdot \mathbf{F}. \tag{2.12}$$

In the language of dyadic analysis, we denote

$$\sum_i \mathbf{r}^i \mathbf{p}_i = \sum_j \mathbf{p}_j \mathbf{r}^j = \bar{\bar{I}}, \tag{2.13}$$

where $\bar{\bar{I}}$ is an idemfactor, such that

$$\mathbf{F} = \mathbf{F} \cdot \bar{\bar{I}} = \bar{\bar{I}} \cdot \mathbf{F}. \tag{2.14}$$

This is an alternative representation of $\bar{\bar{I}}$ defined by (1.96).

The primary vectors $\mathbf{p}_i$ defined by (2.2), being used to represent $d\mathbf{R}_\mathrm{p}$ in (2.3), can be found if the scalar relations between the curvilinear coordinate variables ω^i, $i = 1, 2, 3$, and the rectangular variables x_j, $j = 1, 2, 3$, are known or given. In terms of the rectangular unit vectors $\hat{x}_j$ and the differentials dx_j, the differential position vector can be written as

$$d\mathbf{R}_\mathrm{p} = \sum_j \hat{x}_j \, dx_j. \tag{2.15}$$

By equating it to (2.3), we have

$$\sum_i \mathbf{p}_i \, d\omega^i = \sum_j \hat{x}_j \, dx_j. \tag{2.16}$$

Thus,

$$\mathbf{p}_i = \sum_j \hat{x}_j \frac{\partial x_j}{\partial \omega^i}, \qquad i = 1, 2, 3. \tag{2.17}$$

This is the explicit expression of $\mathbf{p}_i$ in terms of $\hat{x}_j$ and the derivative of x_j with respect to ω^i. Later, we will illustrate the application of (2.17) to determine $\mathbf{p}_i$ for many commonly used orthogonal curvilinear systems.

To determine the parameter Λ defined by (2.5), it is convenient to introduce the coefficients α_{ij} defined by

$$\mathbf{p}_i \cdot \mathbf{p}_j = \alpha_{ij}, \qquad i, j = 1, 2, 3. \tag{2.18}$$

It is obvious that $\alpha_{ij} = \alpha_{ji}$, thus we have only six distinct coefficients. By means of (2.17), we find

$$\alpha_{ij} = \sum_k \frac{\partial x_k}{\partial \omega^i} \frac{\partial x_k}{\partial \omega^j}. \tag{2.19}$$

When $i = j$,

$$\alpha_{ii} = \sum_k \left(\frac{\partial x_k}{\partial \omega^i} \right)^2. \tag{2.20}$$

These coefficients can now be used to determine the parameter Λ. By definition,

$$\Lambda = \mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3). \tag{2.21}$$

The vector function $\mathbf{p}_2 \times \mathbf{p}_3$ in (2.21) can be expressed in terms of $\mathbf{r}^i$ with $i = 1, 2, 3$. According to (2.12) and (2.14),

$$\mathbf{p}_2 \times \mathbf{p}_3 = \sum_i [(\mathbf{p}_2 \times \mathbf{p}_3) \cdot \mathbf{r}^i]\mathbf{p}_i. \tag{2.22}$$

The reciprocal vectors $\mathbf{r}^i$ in (2.22) can be changed to

$$\mathbf{r}^i = \frac{1}{\Lambda} \mathbf{p}_j \times \mathbf{p}_k \tag{2.23}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order. Thus, (2.22) becomes

$$\begin{aligned} \mathbf{p}_2 \times \mathbf{p}_3 = \frac{1}{\Lambda}\Big[&(\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_2 \times \mathbf{p}_3)\mathbf{p}_1 \\ &+ (\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_3 \times \mathbf{p}_1)\mathbf{p}_2 \\ &+ (\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_1 \times \mathbf{p}_2)\mathbf{p}_3 \Big]. \end{aligned} \tag{2.24}$$

The scalar products in (2.24) can be simplified using

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

Thus,

$$\begin{aligned} (\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_2 \times \mathbf{p}_3) &= (\mathbf{p}_2 \cdot \mathbf{p}_2)(\mathbf{p}_3 \cdot \mathbf{p}_3) - (\mathbf{p}_2 \cdot \mathbf{p}_3)(\mathbf{p}_3 \cdot \mathbf{p}_2) \\ &= \alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}. \end{aligned}$$

Similarly,

$$(\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_3 \times \mathbf{p}_1) = \alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33},$$

$$(\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_1 \times \mathbf{p}_2) = \alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}.$$

Hence by taking the scalar product of $\mathbf{p}_1$ with (2.24), we obtain

$$\begin{aligned}\Lambda &= \mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3) \\ &= \frac{1}{\Lambda}\Big[\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) \\ &\quad + \alpha_{12}(\alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33}) \\ &\quad + \alpha_{13}(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31})\Big]\end{aligned}$$

or

$$\Lambda = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}^{1/2} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{vmatrix}^{1/2}. \tag{2.25}$$

We take the positive square root of the determinant as the proper expression for Λ.

Before we leave this section, a theorem involving the sum of the derivative of the vectors $\Lambda \mathbf{r}^i$ should be presented. It is known in geometry that the total vectorial area of a closed surface vanishes, that is,

$$\oiint_S d\mathbf{s} = 0. \tag{2.26}$$

If we consider a small volume ΔV bounded by six coordinate surfaces located at $\omega^i \pm \Delta\omega^i/2$ with $i = 1, 2, 3$ in GCS, then

$$\begin{aligned}\Delta V &= \Lambda\, \Delta\omega^1\, \Delta\omega^2\, \Delta\omega^3, \\ \Delta \mathbf{S}_i &= \Lambda \mathbf{r}^i\, \Delta\omega^j\, \Delta\omega^k\end{aligned}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order. The differential form of (2.26) can be obtained by taking the limit of the following identity:

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_S d\mathbf{s} = 0$$

or

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_{i=1}^{3} \left[(\Lambda \mathbf{r}^i)_{\omega^i + \Delta\omega^i/2} - (\Lambda \mathbf{r}^i)_{\omega^i - \Delta\omega^i/2}\right] \Delta\omega^j\, \Delta\omega^k = 0,$$

which yields

$$\sum_i \frac{\partial}{\partial \omega^i}(\Lambda \mathbf{r}^i) = 0. \tag{2.27}$$

Equation (2.27) is a very useful theorem; it will be designated as the *closed surface theorem.*

2-2 Orthogonal Curvilinear System (OCS)

The GCS degenerates to the orthogonal curvilinear system (OCS) when the primary vectors are mutually perpendicular to each other. In this case, we let

$$\mathbf{p}_i = \frac{\partial \mathbf{R}_\mathrm{p}}{\partial \omega^i} = h_i \hat{u}_i, \qquad i = 1, 2, 3, \tag{2.28}$$

where $\hat{u}_i$ denote the unit vectors along the coordinates ω^i and h_i, the metric coefficients in the OCS. For a specific system, such as the spherical coordinate system with coordinate variables R, θ, ϕ, we use the notations $\hat{R}$, $\hat{\theta}$, and $\hat{\phi}$ to denote the unit vectors in that system. $\hat{u}_i$ are used when the orthogonal curvilinear system is arbitrary or unspecified. For the rectangular system, which is a special case of OCS, $h_i = 1$, and $\omega^i = x_i$, or more specifically, x, y, z. The reciprocal vectors in OCS now become

$$\mathbf{r}^j = \frac{\mathbf{p}_k \times \mathbf{p}_i}{\Lambda} = \frac{h_k h_i \hat{u}_j}{\Omega} = \frac{\hat{u}_j}{h_j}, \tag{2.29}$$

where $\Omega = h_i h_j h_k = h_1 h_2 h_3$. The metric coefficients h_i can be found if we know the relations between ω^i and x_j. In the rectangular system,

$$\frac{\partial \mathbf{R}_\mathrm{p}}{\partial x_j} = \hat{x}_j. \tag{2.30}$$

By the chain rule of differentiation, we find

$$h_i \hat{u}_i = \frac{\partial \mathbf{R}_\mathrm{p}}{\partial \omega^i} = \sum_j \frac{\partial \mathbf{R}_\mathrm{p}}{\partial x_j} \frac{\partial x_j}{\partial \omega^i} = \sum_j \frac{\partial x_j}{\partial \omega^i} \hat{x}_j. \tag{2.31}$$

In (2.31), the summation with respect to j goes from $j = 1$ to $j = 3$; that labeling will be omitted henceforth. Thus,

$$h_i^2 = \sum_j \left(\frac{\partial x_j}{\partial \omega^i} \right)^2, \tag{2.32}$$

and the positive square root of (2.32) yields

$$h_i = \left[\sum_j \left(\frac{\partial x_j}{\partial \omega^i} \right)^2 \right]^{1/2}. \tag{2.33}$$

Equation (2.33) can be used to determine h_i when the relations between x_j, the dependent variables, and ω^i, the independent variables, are known. Another expression of h_i is sometimes useful when the roles of x_j and ω^i are interchanged with x_j as independent variables and ω^i as dependent variables. By definition,

$$d\mathbf{R}_\mathrm{p} = \sum_j \hat{x}_j \, dx_j = \sum_i h_i \hat{u}_i \, d\omega^i; \tag{2.34}$$

hence

$$d\omega^i = \frac{1}{h_i}\sum_j \hat{u}_i \cdot \hat{x}_j\, dx_j; \tag{2.35}$$

thus,

$$\frac{\partial \omega^i}{\partial x_j} = \frac{1}{h_i}\hat{u}_i \cdot \hat{x}_j. \tag{2.36}$$

Let

$$\frac{\hat{u}_i}{h_i} = \sum_j c_{ij}\hat{x}_j. \tag{2.37}$$

Then

$$c_{ij} = \frac{1}{h_i}\hat{u}_i \cdot \hat{x}_j = \frac{\partial \omega^i}{\partial x_j}\,. \tag{2.38}$$

Equation (2.37) therefore becomes

$$\frac{\hat{u}_i}{h_i} = \sum_j \frac{\partial \omega^i}{\partial x_j}\hat{x}_j; \tag{2.39}$$

hence

$$\frac{1}{h_i^2} = \sum_j \left(\frac{\partial \omega^i}{\partial x_j}\right)^2. \tag{2.40}$$

We take the positive square root of (2.40) to be the expression for $1/h_i$:

$$\frac{1}{h_i} = \left[\sum_j \left(\frac{\partial \omega^i}{\partial x_j}\right)^2\right]^{1/2}. \tag{2.41}$$

In contrast to (2.33), ω^i's are now the dependent variables and x_j, the independent variables. Unlike functions with one independent variable,

$$\frac{\partial \omega^i(x, y, z)}{\partial x} \neq 1 \Big/ \frac{\partial x(\omega^i, \omega^j, \omega^k)}{\partial \omega^i},$$

while

$$\frac{dy(x)}{dx} = 1 \Big/ \frac{dx(y)}{dy}\,.$$

The list that follows shows the metric coefficients of some commonly used OCS, and the relations between (v_1, v_2, v_3) and (x, y, z), based on which these metric coefficients are derived by means of (2.33).

Rectangular
Coordinate variables: (x, y, z)
Metric coefficients: $(1, 1, 1)$.

Cylindrical
Coordinate variables: (r, ϕ, z)
Metric coefficients: $(1, r, 1)$
Relations: $x = r\cos\phi,\ \ y = r\sin\phi,\ \ z = z.$

Spherical
Coordinate variables: (R, θ, ϕ)
Metric coefficients: $(1, R, R\sin\theta)$
Relations: $x = R\sin\theta\cos\phi,\ \ y = R\sin\theta\sin\phi,\ \ z = R\cos\theta.$

Elliptical Cylinder
Coordinate variables: (η, ξ, z)
Metric coefficients:

$$\left[c\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{1/2},\ c\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{1/2},\ 1\right]$$

Relations: $x = c\eta\xi,\ \ y = c\left[(1-\eta^2)(\xi^2-1)\right]^{1/2},\ \ z = z.$

Parabolic Cylinder
Coordinate variables: (η, ξ, z)
Metric coefficients:

$$\left[(\eta^2+\xi^2)^{1/2}, (\eta^2+\xi^2)^{1/2}, 1\right]$$

Relations: $x = \frac{1}{2}(\eta^2-\xi^2),\ \ y = \eta\xi,\ \ z = z.$

Prolate Spheroidal
Coordinate variables: (η, ξ, ϕ)
Metric coefficients:

$$\left[c\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{1/2},\ c\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{1/2},\ c(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}\right]$$

Relations:

$$x = c\left[(1-\eta^2)(\xi^2-1)\right]^{1/2}\cos\phi,$$
$$y = c\left[(1-\eta^2)(\xi^2-1)\right]^{1/2}\sin\phi,$$
$$z = c\eta\xi.$$

Oblate Spheroidal
Coordinate variables: (ξ, η, ϕ)
Metric coefficients:

$$\left[c\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{1/2}, c\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{1/2}, c\xi\eta\right]$$

Relations:

$$x = c\xi\eta\cos\phi,$$
$$y = c\xi\eta\sin\phi,$$
$$z = c\left[\left(\xi^2-1\right)\left(1-\eta^2\right)\right]^{1/2}.$$

Bipolar Cylinders
Coordinate variables: (η, ξ, z)
Metric coefficients:

$$\left[\frac{a}{\cosh\xi - \cos\eta}, \frac{a}{\cosh\xi - \cos\eta}, 1\right]$$

Relations:

$$x = \frac{a\sinh\xi}{\cosh\xi - \cos\eta},$$
$$y = \frac{a\sin\eta}{\cosh\xi - \cos\eta},$$
$$z = z.$$

In this list, for the case of the elliptical cylinder, the governing equations for the elliptical cylinder and the conformal hyperbolic cylinder are

$$\frac{x^2}{c^2\xi^2} + \frac{y^2}{c^2\left(\xi^2-1\right)} = 1, \qquad \infty > \xi \geq 1, \tag{2.42}$$

$$\frac{x^2}{c^2\eta^2} - \frac{y^2}{c^2\left(1-\eta^2\right)} = 1, \qquad 1 \geq \eta \geq -1, \tag{2.43}$$

where c denotes half of the focal distance between the foci of the ellipse. For the prolate spheroid, the governing equations are

$$\frac{z^2}{c^2\xi^2} + \frac{r^2}{c^2\left(\xi^2-1\right)} = 1, \qquad 2\pi \geq \phi \geq 0, \quad \infty > \xi \geq 1, \tag{2.44}$$

$$\frac{z^2}{c^2\eta^2} - \frac{r^2}{c^2\left(1-\eta^2\right)} = 1, \qquad 2\pi \geq \phi \geq 0, \quad 1 \geq \eta \geq -1, \tag{2.45}$$

where $r^2 = x^2 + y^2$. Equation (2.44) represents an ellipse of revolution revolving around the z axis, which is the major axis. The conformal hyperboloid is represented by (2.45). For the oblate spheroid, the governing equations are

$$\frac{r^2}{c^2\xi^2} + \frac{z^2}{c^2\left(\xi^2 - 1\right)} = 1, \qquad 2\pi \geq \phi \geq 0, \quad \infty > \xi \geq 1, \tag{2.46}$$

$$\frac{r^2}{c^2\eta^2} - \frac{z^2}{c^2\left(1 - \eta^2\right)} = 1, \qquad 2\pi \geq \phi \geq 0, \quad 1 \geq \eta \geq -1. \tag{2.47}$$

Equation (2.46) represents an oblate spheroid generated by revolving an ellipse around the z axis, which is the minor axis in this case, and (2.47) is the equation for the conformal hyperboloid.

For the bipolar cylinders, the governing equations are

$$\left(x - a \,\coth\, \xi\right)^2 + y^2 = a^2 \,\operatorname{csch}^2\, \xi, \tag{2.48}$$

$$(y - a \,\cot\, \eta)^2 + x^2 = a^2 \,\csc^2\, \eta, \tag{2.49}$$

with $\infty > \xi > -\infty$ and $2\pi > \eta > 0$, and where a denotes half of the distance between two pivoting points from which these circles are generated. In fact, (2.48) and (2.49) can be derived by considering a conformal transformation between the complex variables $x + jy$ and $\eta + j\xi$ in the form

$$\frac{(x + jy) - a}{(x + jy) + a} = e^{j(\eta + j\xi)}, \tag{2.50}$$

which is called a *bilinear transformation* in the theory of complex variables.

In the complex (x, y) plane, the numbers $c_1 = (x - a) + jy$ and $c_2 = (x + a) + jy$ are shown graphically in Fig. 2-2, where we assume a to be real. These numbers can also be written in the form

$$\begin{aligned} c_1 &= \left[(x - a)^2 + y^2\right]^{1/2} e^{j\alpha_1}, \qquad & \alpha_1 &= \tan^{-1} \frac{y}{x - a}, \\ c_2 &= \left[(x + a)^2 + y^2\right]^{1/2} e^{j\alpha_2}, \qquad & \alpha_2 &= \tan^{-1} \frac{y}{x + a}. \end{aligned} \tag{2.51}$$

Equation (2.50) is therefore equivalent to

$$\frac{c_1}{c_2} = \left[\frac{(x - a)^2 + y^2}{(x + a)^2 + y^2}\right]^{1/2} e^{j(\alpha_1 - \alpha_2)} = e^{-\xi} e^{j\eta}; \tag{2.52}$$

hence

$$\left[\frac{(x - a)^2 + y^2}{(x + a)^2 + y^2}\right]^{1/2} = e^{-\xi}, \tag{2.53}$$

$$\tan^{-1} \frac{y}{x - a} - \tan^{-1} \frac{y}{x + a} = \eta. \tag{2.54}$$

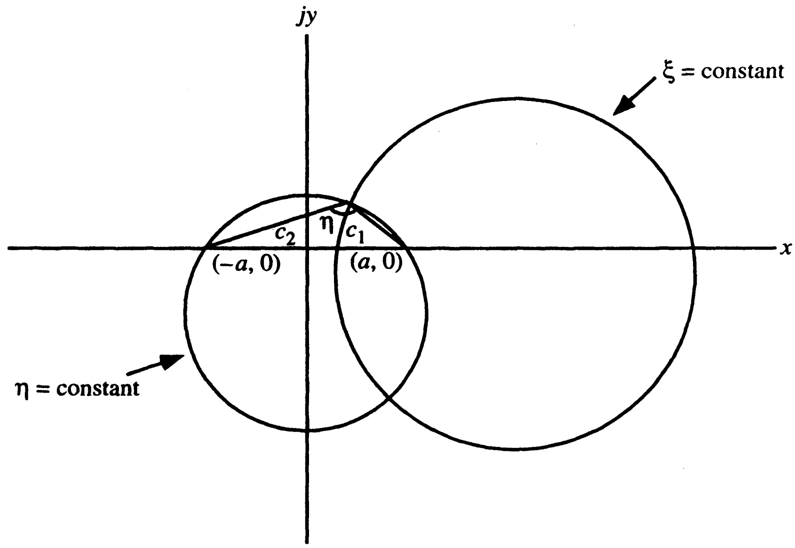

Figure 2-2 Locus of the complex number $x + jy$ resulting from a bilinear transformation.

It is not difficult to deduce (2.48) from (2.53), and (2.49) from (2.54). From this discussion, we see that the locus of constant ξ, which is a circle, corresponds to a constant ratio of the magnitude of c_1 and c_2, and the locus of constant η is also a circle conformal to the circle of constant ξ. The fact that they are conformal is because (2.50) is a conformal transformation in the theory of complex numbers.

2-3 Derivatives of Unit Vectors in OCS

Equation (2.28) states

$$\frac{\partial \mathbf{R}_p}{\partial \omega^i} = h_i \, \hat{u}_i, \qquad i = 1, 2, 3. \tag{2.55}$$

Let us change the variables ω^i to v_i for OCS; then,

$$\frac{\partial \mathbf{R}_p}{\partial v_j} = h_j \, \hat{u}_j, \qquad \frac{\partial \mathbf{R}_p}{\partial v_k} = h_k \, \hat{u}_k.$$

Hence

$$\frac{\partial \left(h_j \, \hat{u}_j\right)}{\partial v_k} = \frac{\partial \left(h_k \, \hat{u}_k\right)}{\partial v_j} \tag{2.56}$$

because both are equal to $\partial^2 \mathbf{R}_p / \partial v_j \, \partial v_k$. We assume that all of the first and second derivatives of $\mathbf{R}_p$ do exist. Equation (2.56) can be written in the form

$$h_j \frac{\partial \hat{u}_j}{\partial v_k} + \frac{\partial h_j}{\partial v_k} \hat{u}_j = h_k \frac{\partial \hat{u}_k}{\partial v_j} + \frac{\partial h_k}{\partial v_j} \, \hat{u}_k. \tag{2.57}$$

Because

$$\hat{u}_j \cdot \hat{u}_j = 1,$$

hence

$$\hat{u}_j \cdot \frac{\partial \hat{u}_j}{\partial v_k} = 0.$$

$\partial \hat{u}_j/\partial v_k$ is therefore perpendicular to $\hat{u}_j$. In the (v_j, v_k) plane, it is parallel to $\hat{u}_k$; similarly, $\partial \hat{u}_k/\partial v_j$ is parallel to $\hat{u}_j$. Thus, we can write

$$\frac{\partial \hat{u}_j}{\partial v_k} = \alpha\, \hat{u}_k, \qquad \frac{\partial \hat{u}_k}{\partial v_j} = \beta \hat{u}_j.$$

Equation (2.57) can now be put in the form

$$\left(\frac{\partial h_k}{\partial v_j} - \alpha h_j\right) \hat{u}_k = \left(\frac{\partial h_j}{\partial v_k} - \beta h_k\right) \hat{u}_j.$$

Because $\hat{u}_k$ and $\hat{u}_j$ are independent and orthogonal to each other, this equation can be satisfied only if

$$\frac{\partial h_k}{\partial v_j} = \alpha h_j, \qquad \frac{\partial h_j}{\partial v_k} = \beta h_k;$$

hence

$$\frac{\partial \hat{u}_j}{\partial v_k} = \frac{1}{h_j} \frac{\partial h_k}{\partial v_j} \hat{u}_k \tag{2.58}$$

and

$$\frac{\partial \hat{u}_k}{\partial v_j} = \frac{1}{h_k} \frac{\partial h_j}{\partial v_k} \hat{u}_j. \tag{2.59}$$

Equations (2.58) and (2.59) hold true for j and $k = 1, 2, 3$ with $j \neq k$.

The derivative $\partial \hat{u}_i/\partial v_i$ can be found by considering the relationship among the three orthogonal unit vectors $\hat{u}_i$, $\hat{u}_j$, and $\hat{u}_k$ in a right-hand system:

$$\hat{u}_i = \hat{u}_j \times \hat{u}_k, \qquad i, j, k = \begin{cases} 1, 2, 3 & \text{or} \\ 2, 3, 1 & \text{or} \\ 3, 1, 2. \end{cases} \tag{2.60}$$

The coordinate variables of the system are denoted by (v_1, v_2, v_3). The partial derivative of (2.60) with respect to v_i yields

$$\frac{\partial \hat{u}_i}{\partial v_i} = \hat{u}_j \times \frac{\partial \hat{u}_k}{\partial v_i} + \frac{\partial \hat{u}_j}{\partial v_i} \times \hat{u}_k.$$

In view of (2.58) or (2.59), this equation is equivalent to

$$\begin{aligned} \frac{\partial \hat{u}_i}{\partial v_i} &= \hat{u}_j \times \frac{1}{h_k} \frac{\partial h_i}{\partial v_k} \hat{u}_i + \frac{1}{h_j} \frac{\partial h_i}{\partial v_j} \hat{u}_i \times \hat{u}_k \\ &= -\left(\frac{1}{h_k} \frac{\partial h_i}{\partial v_k} \hat{u}_k + \frac{1}{h_j} \frac{\partial h_i}{\partial v_j} \hat{u}_j\right). \end{aligned} \tag{2.61}$$

Equations (2.58) and (2.59) are very important formulas that will be used frequently in subsequent sections. It can be easily verified that as a result of (2.61),

$$\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i\right) = 0, \tag{2.62}$$

where $\Omega = h_1 h_2 h_3$. Equation (2.62) can be derived readily from the general closed surface theorem stated in (2.27) by letting $\Lambda = \Omega$ and $\mathbf{r}^i = \hat{u}_i / h_i$. The derivation of this theorem for the OCS appears to be more complicated than for the GCS. However, the relations between the derivatives of the unit vectors give us a deeper understanding of the vector relations in OCS.

Another identity that can be proved with the aid of (2.58) and (2.62) is

$$\sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i}\left(\frac{\hat{u}_j}{h_j}\right) = 0, \qquad j = 1, 2, 3. \tag{2.63}$$

Equations (2.61) and (2.63) will be used in the derivation of many important formulas. The interpretation of these two identities from the point of view of vector theorems will be discussed in Chapter 4.

2-4 Dupin Coordinate System

The Dupin coordinate system is an indispensable tool to treat vector analysis on a surface. In the general Dupin system, the coordinate variables will be denoted by (v_1, v_2, v_3) and the corresponding unit vectors by $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ with metric coefficients $(h_1, h_2, 1)$. The variables (v_1, v_2) are used to describe the coordinate lines on the surface, while v_3 denotes the normal distance measured linearly from the surface; hence $h_3 = 1$. For a right-hand system, the direction of $\hat{u}_3$ is determined by $\hat{u}_1 \times \hat{u}_2$. $\hat{u}_1$ and $\hat{u}_2$ are assumed to be orthogonal, and both are tangential to the surface. Figure 2-3 shows the disposition of these quantities. The total differential of the position vector measured from a point on the surface to a neighboring point in the space is then written as

$$d\mathbf{R}_p = h_1\, dv_1\, \hat{u}_1 + h_2\, dv_2\, \hat{u}_2 + dv_3\, \hat{u}_3. \tag{2.64}$$

When v_1 and v_2 have not yet been specified, we designate the system as the general Dupin system. The surface of a circular cylinder and that of a sphere are two of the simplest surfaces belonging to the Dupin system. The one-to-one correspondence of the variables, the unit vectors, and the metric coefficients is listed in Table 2-1.

Let us now consider a spheroidal surface described by

$$\frac{x^2 + y^2}{b^2} + \frac{z^2}{a^2} = 1$$

or

$$\frac{r^2}{b^2} + \frac{z^2}{a^2} = 1, \qquad 2\pi \geq \phi \geq 0, \tag{2.65}$$

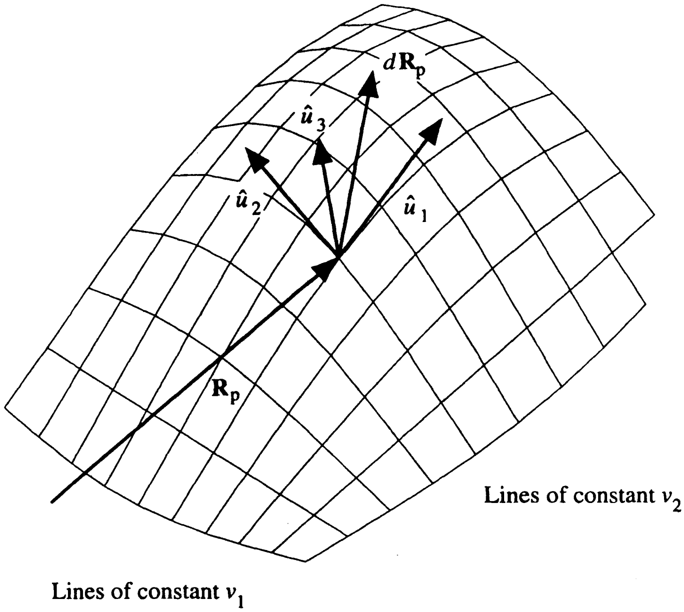

Figure 2-3 Dupin coordinate system, $d\mathbf{R}_p = h_1\, dv_1\, \hat{u}_1 + h_2\, dv_2\, \hat{u}_2 + dv_3\, \hat{u}_3$.

Table 2-1: Two Dupin coordinate systems

System	(v_1, v_2, v_3)	$(\hat{u}_1, \hat{u}_2, \hat{u}_3)$	(h_1, h_2, h_3)
Cylindrical surface	(ϕ, z, r)	$(\hat{\phi}, \hat{z}, \hat{r})$	$(r, 1, 1)$
Spherical surface	(θ, ϕ, R)	$(\hat{\theta}, \hat{\phi}, \hat{R})$	$(R, R\sin\theta, 1)$

where (ϕ, z, r) are the cylindrical variables. If we choose (ϕ, z) as (v_1, v_2), then

$$d\mathbf{R}_p = ds_1\, \hat{u}_1 + ds_2\, \hat{u}_2 + ds_3\, \hat{u}_3, \tag{2.66}$$

where

$$ds_1 = r\, d\phi = b\left[1 - \left(\frac{z}{a}\right)^2\right]^{1/2} d\phi = h_1\, dv_1,$$

$$ds_2 = [(dr)^2 + (dz)^2]^{1/2} = \left[1 + \left(\frac{dr}{dz}\right)^2\right]^{1/2} dz = h_2\, dv_2,$$

$$ds_3 = dv_3.$$

Hence

$$h_1 = b\left[1 - \left(\frac{z}{a}\right)^2\right]^{1/2},$$

$$h_2 = \left[1 + \left(\frac{dr}{dz}\right)^2\right]^{1/2} = \sec\,\alpha,$$

$\dfrac{dr}{dz} = \tan\,\alpha =$ slope of the tangent to the ellipse, (2.65), making an angle α with the z axis.

The corresponding unit vectors are

$$\hat{u}_1 = \hat{\phi}, \tag{2.67}$$

$$\hat{u}_2 = \sin\alpha\,\hat{r} + \cos\alpha\,\hat{z}, \tag{2.68}$$

$$\hat{u}_3 = \cos\alpha\,\hat{r} - \sin\alpha\,\hat{z}. \tag{2.69}$$

The choice of (v_1, v_2) in a Dupin system is not unique. In the previous example, we can use (ϕ, r) as (v_1, v_2); then,

$$ds_2 = \sqrt{1 + \left(\frac{dz}{dr}\right)^2}\,dr = \csc\alpha\,dr = h_2\,dv_2. \tag{2.70}$$

Hence $h_2 = \csc\alpha$, but h_1, $\hat{u}_1$, and $\hat{u}_2$ remain the same. In the case of a spherical surface, we can use (z, x) as (v_1, v_2). In certain problems dealing with integrations, such a choice sometimes is desirable, particularly from the point of view of numerical calculations.

2-5 Radii of Curvature

For a surface described in the general Dupin coordinate system, there are two radii of curvature of the surface associated with two contours in the (v_1, v_3) plane and the (v_2, v_3) plane. These are two normal planes containing $(\hat{u}_1, \hat{u}_3)$ and $(\hat{u}_2, \hat{u}_3)$ at $P\,(v_1, v_2, v_3)$ (see Fig. 2-3). These radii of curvature are closely related to the metric coefficients h_1, h_2 and the rate of change of these coefficients with respect to v_3. Figure 2-4 shows a section of the contour C in the neighborhood of P, resulting from the intersection of the $(\hat{u}_2, \hat{u}_3)$ plane with the surface.

Referring to Fig. 2-4, we denote

$PQ = h_2\,dv_2$,

$OP = R_2$ (second principal radius of the curvature),

$PS = QT = dv_3$.

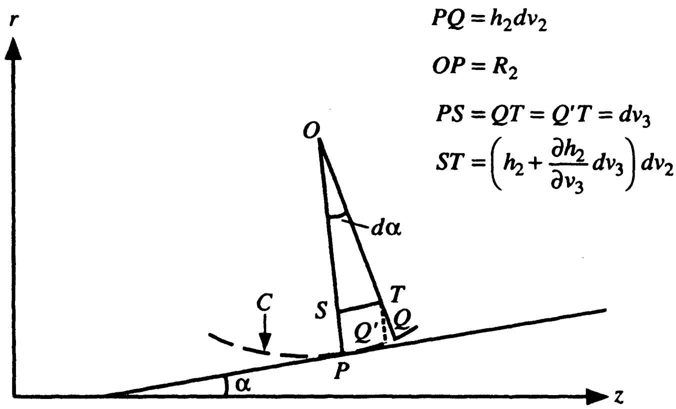

Figure 2-4 Radius of curvature of a surface in the plane containing $\hat{u}_2$ and $\hat{u}_3$ (r–z plane for the example illustrated in the text).

Then,

$$ST = \left(h_2 + \frac{\partial h_2}{\partial v_3} dv_3\right) dv_2,$$

$$Q'Q = PQ - ST = -\frac{\partial h_2}{\partial v_3} dv_3\, dv_2.$$

The triangles OPQ and $TQ'Q$ are similar; hence

$$\frac{PQ}{OP} = \frac{Q'Q}{QT}, \tag{2.71}$$

which yields

$$\frac{h_2}{R_2} = -\frac{\partial h_2}{\partial v_3} \tag{2.72}$$

or

$$\frac{1}{R_2} = -\frac{1}{h_2}\frac{\partial h_2}{\partial v_3}. \tag{2.73}$$

Equation (2.73) relates the second principal radius of curvature in the (v_2, v_3) plane at P to the metric coefficient h_2 and its derivative with respect to v_3 at that point. To find the expression of R_2 in terms of the shape of the curve, we have to know the governing equation for C. Let this equation be given in the form

$$r = f_1(z), \tag{2.74}$$

where z represents v_2; then,

$$ds_2 = \sqrt{(dr)^2 + (dz)^2} = \sqrt{1 + \left(\frac{dr}{dz}\right)^2}\, dz = h_2\, dv_2,$$

$$h_2 = \sqrt{1 + \left(\frac{dr}{dz}\right)^2} = \sqrt{1 + \tan^2\alpha} = \sec\alpha,$$

where α denotes the angle of inclination of the tangent at P made with the z axis. Now,

$$d\alpha = \frac{h_2\, dv_2}{R_2} = \frac{h_2\, dz}{R_2} ;$$

hence

$$\frac{1}{R_2} = \frac{1}{h_2}\frac{d\alpha}{dz} = \cos\alpha \frac{d\alpha}{dz} .$$

Because

$$\tan\ \alpha = \frac{dr}{dz} = r',$$

a differentiation of this equation with respect to z yields

$$\frac{1}{\cos^2\alpha}\frac{d\alpha}{dz} = \frac{dr'}{dz} = r''.$$

We have

$$\frac{1}{R_2} = (\cos^3\alpha)r'' = \frac{r''}{\left(1 + r'^2\right)^{3/2}} . \tag{2.75}$$

The derivation of this formula is found in many books on calculus. It is repeated here to show its relationship with the derivative of the relevant metric coefficient. Equation (2.75) shows that for a concave surface, $r'' > 0$, so R_2 is positive, and for a convex surface, R_2 is negative. Similarly, one finds that in the $(\hat{u}_1, \hat{u}_3)$ plane,

$$\frac{1}{R_1} = \frac{-1}{h_1}\frac{\partial h_1}{\partial v_3}, \tag{2.76}$$

where R_1 denotes the first principal radius of curvature. A formula similar to (2.75) can be derived if the governing equation of the curve in the (v_1, v_3) plane is known. The reciprocals of the two radii of curvature are called Gaussian curvatures of the surface in the two orthogonal planes.

As an illustration of the application of these formulas, we consider the equation of a paraboloidal surface defined by

$$r^2 = 4fz, \qquad 2\pi \geq \phi \geq 0, \tag{2.77}$$

where f denotes the focal length of the paraboloid and (r, ϕ, z) denote the cylindrical variables. The coordinates in the Dupin system for the surface are identified as $(v_1, v_2, v_3) = (\phi, z, v_3)$ with

$$h_2 = \sqrt{1 + \left(\frac{dr}{dz}\right)^2} = \sqrt{1 + r'^2}, \qquad h_1 = r, \qquad h_3 = 1.$$

For the surface under consideration,

$$r' = \frac{dr}{dz} = \sqrt{\frac{f}{z}}, \qquad r'' = -\frac{1}{2}\sqrt{f}\, z^{-3/2}.$$

Upon substituting r' and r'' into (2.75), we find

$$R_2 = -2f\left(1 + \frac{z}{f}\right)^{3/2}. \tag{2.78}$$

At $z = 0$, $R_2 = -2f$, and at $z = f$, $R_2 = -4\sqrt{2}\,f$, and so on. To find R_1, it is simpler to use (2.76) instead of finding the equation of the cross section in the $(\hat{u}_2, \hat{u}_3)$ plane. Now,

$$\frac{1}{R_1} = \frac{-1}{h_1}\frac{\partial h_1}{\partial v_3} = \frac{-1}{r}\frac{\partial r}{\partial v_3} = -\frac{\sin\beta}{r}, \tag{2.79}$$

where β denotes the angle between the normal to the surface $\hat{u}_3$ and the z axis, and

$$\tan\beta = -\frac{1}{r'}, \qquad \sin\beta = \frac{1}{\left(1 + r'^2\right)^{1/2}}.$$

Thus,

$$R_1 = -r\left(1 + r'^2\right)^{1/2} = -2f\left(1 + \frac{z}{f}\right)^{1/2}.$$

It can be proved that $-R_1$ is the distance measured from the point $P(r, z)$ on the parabola along the inward normal to its intersect with the z axis. At $z = 0$, $R_1 = -2f$, and at $z = f$, $R_1 = -2\sqrt{2}f = \frac{1}{2}R_2$. The relationship between R_1 and R_2, in general, is $R_1^3 = 4f^2 R_2$.

As an exercise, the reader may be interested to verify that for a spheroidal surface defined by

$$\left(\frac{r}{b}\right)^2 + \left(\frac{z}{a}\right)^2 = 1, \qquad 2\pi \geq \phi \geq 0,$$

$$R_2 = -\frac{a^2}{b}\left[1 + \frac{z^2}{a^2}\left(\frac{b^2}{a^2} - 1\right)\right]^{3/2} \tag{2.80}$$

and

$$R_1 = -b\left[1 + \frac{z^2}{a^2}\left(\frac{b^2}{a^2} - 1\right)\right]^{1/2}. \tag{2.81}$$

The relationship between R_1 and R_2 is $R_1^3 = (b^4/a^2)R_2$.

The two Gaussian curvatures that are defined by (2.73) and (2.76) are related to the rate of change of an elementary area of the curved surface. For an orthogonal Dupin coordinate system under consideration,

$$\Delta S_3 = h_1 h_2\, \Delta v_1\, \Delta v_2; \tag{2.82}$$

hence

$$\lim_{\Delta S_3 \to 0} \frac{1}{\Delta S_3} \frac{\partial \Delta S_3}{\partial v_3} = \frac{1}{h_1 h_2} \frac{\partial (h_1 h_2)}{\partial v_3}$$
$$= \frac{1}{h_1} \frac{\partial h_1}{\partial v_3} + \frac{1}{h_2} \frac{\partial h_2}{\partial v_3} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{2.83}$$

The sum of the two principal Gaussian curvatures is therefore equal to the decrease of the normal derivative of an elementary area per unit area. We will denote the sum of the two principal curvatures by J:

$$J = \frac{1}{R_1} + \frac{1}{R_2}, \tag{2.84}$$

and it will be designated as the *surface curvature*. It is convenient to define a mean radius of curvature for a surface by R_m, such that

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_m}; \tag{2.85}$$

then,

$$J = \frac{1}{R_m}. \tag{2.86}$$

There is a simple graphical method to determine R_m for a given pair of R_1 and R_2 based on (2.85). We erect two vertical lines with lengths equal to R_1 and R_2 on a graded paper and draw two lines from the tips of these two vertical lines to the bases, as shown in Fig. 2-5. The intersecting point of the two inclined lines yields R_m. The validity of this method is based on the relation

$$\frac{R_m}{R_1} + \frac{R_m}{R_2} = \frac{d_2}{d_1 + d_2} + \frac{d_1}{d_1 + d_2} = 1.$$

The same method applies to the case when $R_1 > 0$ and $R_2 < 0$ (a saddle surface), as shown in Fig. 2-6. In this case,

$$\frac{R_m}{R_1} + \frac{R_m}{R_2} = \frac{-D_2}{D_1} + \frac{D_1 + D_2}{D_1} = 1.$$

This simple graphical method applies to problems involving the sum of two reciprocals such as two resistances or two reactances in parallel and two optical lenses with different focal lengths aligned in cascade. All these quantities could be of the same sign or opposite signs. The method is a visible aid to an algebraic identity.

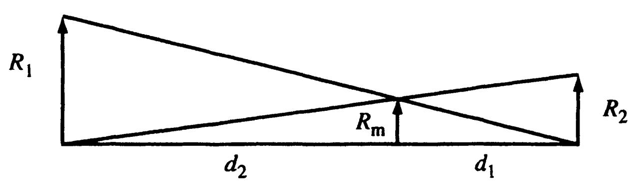

Figure 2-5 A graphical method to determine R_m for R_1, $R_2 > 0$.

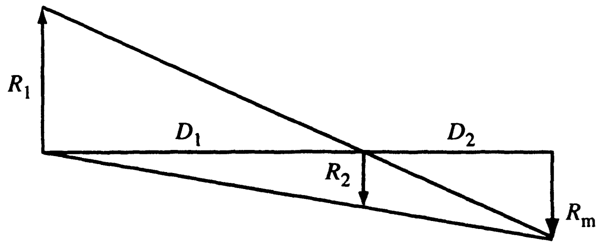

Figure 2-6 $R_1 > 0,\ R_2 < 0.$