5

Electromagnetic Boundary Value Problems

5.1 INTRODUCTION

In this chapter, we consider electromagnetic boundary value problems. We apply the concepts developed in the first four chapters, concentrating on the application of the mathematical ideas to a representative set of electromagnetic examples. Our principal objective is *structure*. Once the reader understands how the concepts in linear spaces, coupled with the theories of Green's functions and spectral expansions, can be applied to examples, it should become apparent how the methods are used to approach the study of electromagnetic propagation, scattering, and diffraction in an organized, logical manner.

We begin by extending the Green's function method to three dimensions. We next consider the case where the three-dimensional geometry is independent of one spatial coordinate so that the problem reduces to two dimensions. We then present a series of examples. We shall find that we may construct solutions in two and three dimensions by using combinations of the one-dimensional Green's functions and one-dimensional spectral representations discussed in Chapters 2 and 3, respectively.

One of the important solution characteristics that emerges in the examples is the fact that there exist alternative representations for the solutions. In particular, we exhibit alternative representations for the fields in a parallel plate waveguide and the fields scattered by a perfectly conducting cylinder. These solutions not only have important physical interpretations, but also are useful in different portions of the frequency spectrum.

5.2 SLP1 EXTENSION TO THREE DIMENSIONS

We begin by considering the negative Laplacian operator $L = -\nabla^2$ on a three-dimensional closed and bounded region V. This region is surrounded by a surface S whose parts may or may not be contiguous. For example (Fig. 5-1), the surface S might consist of an external surface S_e and two internal surfaces S_1 and S_2 with

$$S = S_e + S_1 + S_2$$

In this case, the region V consists of the volume internal to S_e but external to S_1 and S_2. By convention, the unit normal vector $\hat{n}$ points outward from V. Our interest is in the three-dimensional partial differential equation

$$L_\lambda u = f \tag{5.1}$$

where f is a real function and where

$$L_\lambda = L - \lambda = -\nabla^2 - \lambda, \qquad \lambda \in \mathbf{R} \tag{5.2}$$

The functional dependence of u and f is

$$u = u(\mathbf{r})$$

$$f = f(\mathbf{r})$$

where $\mathbf{r} \in V$. In Cartesian coordinates, for example, $u(\mathbf{r})$ stands for $u(x, y, z)$. Let $u(\mathbf{r})$ and $v(\mathbf{r})$ be members of a Hilbert space $\mathcal{H}$ with inner product

$$\langle u, v \rangle = \int_V u(\mathbf{r})v(\mathbf{r})dV \tag{5.3}$$

for all $u, v \in \mathcal{H}$. The three-dimensional problem involving (5.1) can be stated as follows: Given the partial differential equation in (5.1) and a suitable boundary condition involving $u(\mathbf{r})$ and its normal derivative on the surface S, determine $u(\mathbf{r})$ throughout V. We shall require that $u(\mathbf{r})$ have the following specification on S:

$$B(u) = \alpha_1 u\big|_S + \alpha_2 \nabla u\big|_S \cdot \hat{n} = \alpha \tag{5.4}$$

where the coefficients $\alpha, \alpha_1, \alpha_2$ are real and where $\hat{n}$ is the outgoing normal from the surface S. Our notation $u\big|_S$ indicates the function $u(\mathbf{r})$ evaluated at points on S. In addition, $\nabla u|_S \cdot \hat{n}$ indicates the normal derivative of $u(\mathbf{r})$ evaluated at points on S. The condition in (5.4) has two important special cases. If $\alpha_2 = 0$ and $\alpha_1 = 1$, we have

$$B(u) = u|_S = \alpha \tag{5.5}$$

Equation (5.1), coupled with boundary condition (5.5), is called the *Dirichlet problem*, and (5.5) is referred to as the *inhomogeneous Dirichlet boundary condition* [1]. (If $\alpha = 0$, the boundary condition is *homogeneous*.) If $\alpha_1 = 0$ and $\alpha_2 = 1$, we have

$$B(u) = \nabla u|_S \cdot \hat{n} = \alpha \tag{5.6}$$

Equation (5.1), coupled with boundary condition (5.6), is called the *Neumann problem*, and (5.6) is referred to as the *inhomogeneous Neumann boundary condition*. The general case in (5.4) is called the *mixed problem*. We point out that it is perfectly reasonable to have one type of boundary condition (Dirichlet, Neumann, or mixed) on a portion of the surface S and a different type of boundary condition (Dirichlet, Neumann, or mixed) on the remainder.

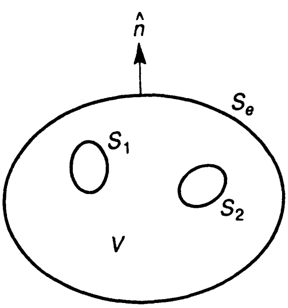

Fig. 5-1 Three-dimensional region V surrounded by surface S. In this case, $S = S_e + S_1 + S_2$.

We recognize the above collection of problems as a three-dimensional extension to SLP1, considered in Section 2.4. Most of the concepts developed in one dimension in Section 2.4 carry over to the present case, as we shall now demonstrate.

The operator $L = -\nabla^2$ has a formal adjoint. For $u, v \in \mathcal{H}$, we form

$$\langle Lu, v\rangle = \int_V (-\nabla^2 u) v \, dV \tag{5.7}$$

In the one-dimensional case, the adjoint was found by integrating by parts twice. The extension to three dimensions can be obtained by integrating by parts over all three coordinates comprising V. A more direct and compact method, however, employs *Green's theorem* [2]. In the case of the Laplacian operator, Green's theorem is given by

$$\int_V (-\nabla^2 u) v dV = \int_V u(-\nabla^2 v) dV + \int_S (-v\nabla u + u\nabla v) \cdot \hat{n} dS \tag{5.8}$$

We write this result in inner product notation as

$$\langle Lu, v\rangle = \langle u, L^* v\rangle + J(u, v)\Big|_S \tag{5.9}$$

where the conjunct $J(u, v)\Big|_S$ is given by

$$J(u, v)\Big|_S = \int_S (-v\nabla u + u\nabla v) \cdot \hat{n} dS \tag{5.10}$$

The operator L^* produced by Green's theorem in (5.9) is the formal adjoint to L. We observe in (5.8) that

$$L^* = L \tag{5.11}$$

and therefore, the operator $L = -\nabla^2$ is formally self-adjoint.

As was the case in Chapter 2, we shall assume initially that the boundary condition on u is homogeneous, $B(u) = 0$. We now choose the boundary condition on v to be that condition $B^*(v) = 0$ which, when coupled with the boundary condition on u, results in the vanishing of the conjunct, viz.

$$J(u, v)\Big|_S = 0 \tag{5.12}$$

In general, the boundary conditions associated with v are different from those associated with u. When they are the same, however, the operator L is self-adjoint.

EXAMPLE 5.1 Consider the homogeneous Dirichlet problem. Then, $B(u) = u\big|_S = 0$, and (5.12) gives

$$0 = \int_S v\nabla u \cdot \hat{n} dS$$

To satisfy this relationship, we choose $B^*(v) = v\big|_S = 0$. Since the boundary condition on v is identical to the boundary condition on u, the operator L for the Dirichlet problem is self-adjoint.

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EXAMPLE 5.2 Consider the homogeneous Neumann problem. Then, $B(u) = \nabla u\big|_S \cdot \hat{n} = 0$, and (5.12) gives

$$0 = \int_S u \nabla v \cdot \hat{n} dS$$

To satisfy this relationship, we choose $B^*(v) = \nabla v\big|_S \cdot \hat{n} = 0$. Since the boundary condition on v is identical to the boundary condition on u, the operator L for the Neumann problem is self-adjoint.

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To produce the solution to (5.1), we define two auxiliary problems: the Green's function problem and the adjoint Green's function problem. The Green's function problem is defined as follows:

$$L_\lambda g(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{5.13}$$

$$B(g) = 0 \tag{5.14}$$

where L_λ is defined in (5.2). We note that, by definition, the boundary condition on g is identical to the homogeneous boundary condition on u. The adjoint Green's function problem is defined as follows:

$$L_\lambda h(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{5.15}$$

$$B^*(h) = 0 \tag{5.16}$$

We note that, by definition, the boundary condition on h is identical to the boundary condition on v.

In the same manner as in Section 2.4, the solution to (5.1) is obtained by taking the inner product of $L_\lambda u$ with h, viz.

$$\langle L_\lambda u, h\rangle = \langle u, L_\lambda h\rangle + J(u, h)\Big|_S \tag{5.17}$$

where the integrations are with respect to the unprimed coordinates. Substitution of (5.1) and (5.15) into (5.17) gives

$$u(\mathbf{r}') = \langle f, h\rangle - J(u, h)\Big|_S \tag{5.18}$$

or, explicitly,

$$u(\mathbf{r}') = \int_V f(\mathbf{r}) h(\mathbf{r}, \mathbf{r}') dV + \int_S \left[h(\mathbf{r}, \mathbf{r}') \nabla u(\mathbf{r}) - u(\mathbf{r}) \nabla h(\mathbf{r}, \mathbf{r}')\right] \cdot \hat{n} dS \tag{5.19}$$

We note that (5.19) is the solution to (5.1), provided that we can determine the adjoint Green's function $h(\mathbf{r}, \mathbf{r}')$.

In a manner similar to the Green's function method developed in Section 2.4, we can show that it is never necessary to find the adjoint Green's function directly. Indeed, we form

$$\langle L_\lambda g(\mathbf{r}, \mathbf{r}'), h(\mathbf{r}, \mathbf{r}'')\rangle = \langle g(\mathbf{r}, \mathbf{r}'), L_\lambda h(\mathbf{r}, \mathbf{r}'')\rangle + J(g, h)\Big|_S \tag{5.20}$$

We are given the boundary conditions on g. We choose the boundary conditions on h so that

$$J(g, h)\Big|_S = 0 \tag{5.21}$$

Then, substitution of (5.13), (5.15), and (5.21) into (5.20) gives

$$h(\mathbf{r}', \mathbf{r}'') = g(\mathbf{r}'', \mathbf{r}')$$

or, with a change in variables,

$$h(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}', \mathbf{r}) \tag{5.22}$$

Therefore, the adjoint Green's function is given simply by interchanging $\mathbf{r}$ and $\mathbf{r}'$ in the expression for the Green's function $g(\mathbf{r}, \mathbf{r}')$. In cases where L is self-adjoint, the boundary conditions on h are the same as those on g, and we must have

$$h(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}', \mathbf{r}) \tag{5.23}$$

Therefore, the Green's function is symmetric. For the self-adjoint case, we may substitute (5.23) into (5.19) to obtain

$$u(\mathbf{r}') = \int_V f(\mathbf{r})g(\mathbf{r}, \mathbf{r}')dV + \int_S \left[g(\mathbf{r}, \mathbf{r}')\nabla u(\mathbf{r}) - u(\mathbf{r})\nabla g(\mathbf{r}, \mathbf{r}')\right] \cdot \hat{n}dS \tag{5.24}$$

For the case presently under consideration, where the boundary conditions on u are homogeneous, the term involving the conjunct in (5.24) vanishes, and we are left with

$$u(\mathbf{r}') = \int_V f(\mathbf{r})g(\mathbf{r}, \mathbf{r}')dV \tag{5.25}$$

To extend the results to the inhomogeneous case, we simply apply the inhomogeneous boundary conditions to (5.24), with the result that some of the terms in the conjunct will survive. The final step in the solution involves the interchange of the primed and unprimed coordinates, in the same manner as in Section 2.4. We demonstrate these concepts in the following example.

EXAMPLE 5.3 Consider a rectangular box (Fig. 5-2) with dimensions a, b, c. It is required to find the solution to $-\nabla^2 u = f$ in the region V inside the box, where it is given that u satisfies the Dirichlet condition $B(u) = u\big|_S = 0$ on the boundary. The formulation of the problem is as follows:

$$-\nabla^2 u = f, \qquad \text{in } V \tag{5.26}$$

$$u(0, y, z) = u(a, y, z) = 0 \tag{5.27}$$

$$u(x, 0, z) = u(x, b, z) = 0 \tag{5.28}$$

$$u(x, y, 0) = u(x, y, c) = 0 \tag{5.29}$$

We know that the operator $L = -\nabla^2$ with Dirichlet boundary conditions is self-adjoint. We may therefore use (5.25) rather than (5.19), viz.

$$u(\mathbf{r}') = \int_V f(\mathbf{r})g(\mathbf{r}, \mathbf{r}')dV \tag{5.30}$$

where we require the solution to

$$\left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) g(x, y, z, x', y', z') = \delta(x - x')\delta(y - y')\delta(z - z') \tag{5.31}$$

with

$$g\Big|_{x=0} = g\Big|_{x=a} = 0 \tag{5.32}$$

$$g\Big|_{y=0} = g\Big|_{y=b} = 0 \tag{5.33}$$

$$g\Big|_{z=0} = g\Big|_{z=c} = 0 \tag{5.34}$$

where we have chosen the boundary conditions on g to be identical to the boundary conditions on u. We begin the solution to (5.31) by invoking the spectral representation of $\delta(x - x')$. As we found in Problem 3.1, this representation produces the orthonormal eigenfunctions

$$u_m(x) = \sqrt{\frac{2}{a}} \sin \frac{m\pi x}{a} \tag{5.35}$$

and leads to the Fourier sine series, viz.

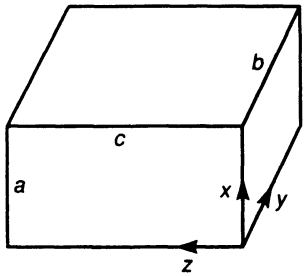

Fig. 5-2 Rectangular box problem.

$$g(x, y, z, x', y', z') = \sum_{m=1}^{\infty} \alpha_m(y, z, x', y', z')u_m(x) \tag{5.36}$$

where

$$\alpha_m(y, z, x', y', z') = \int_0^a g(x, y, z, x', y', z')u_m(x)dx \tag{5.37}$$

In a manner similar to (3.24) and (3.25), we note that (5.37) transforms the Green's function g into the coefficient α_m, viz.

$$g \Longrightarrow \alpha_m \tag{5.38}$$

Also, (5.36) provides the inverse transformation of α_m into g, viz.

$$g \Longleftarrow \alpha_m \tag{5.39}$$

Since the operator

$$L_x = -\frac{\partial^2}{\partial x^2}$$

is self-adjoint, (3.27) and (3.28) give

$$L_x g \Longrightarrow \left(\frac{m\pi}{a}\right)^2 \alpha_m \tag{5.40}$$

We also easily establish that

$$\int_0^a \delta(x - x')u_m(x)dx = u_m(x')$$

so that

$$\delta(x - x') \Longrightarrow u_m(x') \tag{5.41}$$

Using (5.38), (5.40), and (5.41) to transform (5.31), we obtain

$$-\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \left(\frac{m\pi}{a}\right)^2\right]\alpha_m(y, z, x', y', z') = u_m(x')\delta(y - y')\delta(z - z') \tag{5.42}$$

Equation (5.42) is a partial differential equation whose solution yields α_m. To solve (5.42), we invoke the spectral representation of $\delta(y - y')$. The operator $(-\partial^2/\partial y^2)$ with boundary conditions

$$\alpha_m(0, z, x', y', z') = \alpha_m(b, z, x', y', z') = 0$$

results in orthonormal eigenfunctions that we may again obtain from the results in Problem 3.1, viz.

$$v_n(y) = \sqrt{\frac{2}{b}} \sin \frac{n\pi y}{b} \tag{5.43}$$

These eigenfunctions lead to a Fourier sine series representation of α_m, viz.

$$\alpha_m(y, z, x', y', z') = \sum_{n=1}^{\infty} \beta_{mn}(z, x', y', z') v_n(y) \tag{5.44}$$

where

$$\beta_{mn}(z, x', y', z') = \int_0^b \alpha_m(y, z, x', y', z') v_n(y) dy \tag{5.45}$$

Since the operator

$$L_y = -\frac{\partial^2}{\partial y^2}$$

is self-adjoint, we proceed in the same manner as in the above treatment of L_x. With respect to the transformation given in (5.45), we have

$$\alpha_m \Longrightarrow \beta_{mn}$$

$$L_y \alpha_m \Longrightarrow \left(\frac{n\pi}{b}\right)^2 \beta_{mn}$$

$$\delta(y - y') \Longrightarrow v_n(y')$$

We use these relations to transform (5.42), with the result

$$-\left(\frac{d^2}{dz^2} - \gamma_{mn}^2\right) \beta_{mn}(z, x', y', z') = u_m(x') v_n(y') \delta(z - z') \tag{5.46}$$

where

$$\gamma_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{5.47}$$

We let

$$\beta_{mn}(z, x', y', z') = \hat{\beta}_{mn}(z, z') u_m(x') v_n(y') \tag{5.48}$$

and obtain the ordinary differential equation

$$\left(-\frac{d^2}{dz^2}+\gamma_{mn}^2\right)\hat{\beta}_{mn}=\delta(z-z') \tag{5.49}$$

We associate the following boundary conditions with (5.49):

$$\hat{\beta}_{mn}\Big|_{z=0}=\hat{\beta}_{mn}\Big|_{z=c}=0 \tag{5.50}$$

It is easy to show that satisfaction of (5.50) satisfies (5.34). The Green's function problem posed by (5.49) with boundary conditions in (5.50) can be solved by the standard Green's function methods developed in Chapter 2. The details are left for the problems. The results are as follows:

$$\hat{\beta}_{mn}(z,z')=\frac{1}{\gamma_{mn}\sinh\gamma_{mn}c}\begin{cases}\sinh\gamma_{mn}(c-z')\sinh\gamma_{mn}z, & z<z'\\ \sinh\gamma_{mn}(c-z)\sinh\gamma_{mn}z' & z>z'\end{cases} \tag{5.51}$$

Substitution of (5.51) into (5.48), (5.48) into (5.44), and (5.44) into (5.36) yields the required Green's function, viz.

$$g=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{u_m(x)v_n(y)u_m(x')v_n(y')}{\gamma_{mn}\sinh\gamma_{mn}c}\cdot\begin{cases}\sinh\gamma_{mn}(c-z')\sinh\gamma_{mn}z, & z<z'\\ \sinh\gamma_{mn}(c-z)\sinh\gamma_{mn}z', & z>z'\end{cases} \tag{5.52}$$

We note in (5.52) that the Green's function is symmetric, $g(\mathbf{r},\mathbf{r}')=g(\mathbf{r}',\mathbf{r})$, as predicted by the self-adjoint property of the negative Laplacian operator. Substitution of (5.52) into (5.30), followed by an interchange of the primed and unprimed coordinates, yields the final solution, viz.

$$u(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{u_m(x)v_n(y)}{\gamma_{mn}\sinh\gamma_{mn}c}\int_0^b v_n(y')\int_0^a u_m(x')\Bigg[\sinh\gamma_{mn}(c-z)\int_0^z \cdot f(x',y',z')\sinh\gamma_{mn}z'dz'+\sinh\gamma_{mn}z \cdot\int_z^c f(x',y',z')\sinh\gamma_{mn}(c-z')dz'\Bigg]dx'dy' \tag{5.53}$$

In the production of the Green's function, we chose to begin with a spectral expansion over the x-coordinate, followed by a spectral expansion over the y-coordinate. In a manner similar to many of the multiple-dimension cases considered in Chapter 4, alternate representations are possible. Other forms could be obtained by expanding spectrally over y and z or over x and z. Another possibility is to expand spectrally over all three coordinates.

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5.3 SLP1 IN TWO DIMENSIONS

In many of the interesting problems in electromagnetics, the assumption is made that the fields are independent of one of the three spatial coordinates, with the result that the problem to be solved is two-dimensional. To solve two-dimensional problems, we modify the Green's function method developed for three dimensions in the previous section. The starting point is again the application of Green's theorem to the negative Laplacian operator, as in (5.8), viz.

$$\int_V (-\nabla^2 u) v dV = \int_V u(-\nabla^2 v) dV + \int_S (-v\nabla u + u\nabla v) \cdot \hat{n} dS \quad (5.54)$$

In the two-dimensional case, the Laplacian is two-dimensional. For example, if the problem is independent of z, we would write the negative Laplacian as $-\nabla^2_{xy}$. In addition, the volume V and the surface S become *degenerate*, in the sense that the integration over one of the coordinates involved in both the volume and the surface integrals is trivial. For example, if the problem is independent of z, the integrations over z in all three integrals in (5.54) will cancel. We illustrate these ideas in the following example.

EXAMPLE 5.4 Consider a rectangular cylinder (Fig. 5-3) with cross-sectional dimensions a and b. It is required to find the solution to $-\nabla^2_{xy} u = f$ in the region V interior to the cylinder, where it is assumed that f is independent of z. Since the geometry is also independent of z, the solution u will be z-independent. Suppose it is given that homogeneous Dirichlet boundary conditions apply on the surfaces bounding the cylinder, except for the surface at $y = b$, where it is given that the inhomogeneous boundary condition

$$u|_{y=b} = \alpha$$

applies, where α is a real constant. We state the problem as follows:

$$-\nabla^2_{xy} u = f \quad (5.55)$$

$$u(0, y) = u(a, y) = u(x, 0) = 0 \quad (5.56)$$

$$u(x, b) = \alpha \quad (5.57)$$

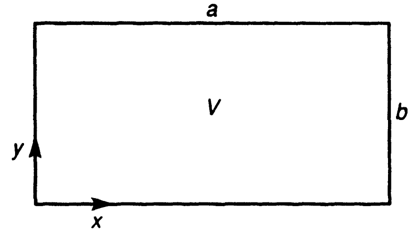

Fig. 5-3 Rectangular cylinder problem.

The associated Green's function problem is as follows:

$$-\nabla_{xy}^2 g = \delta(x - x')\delta(y - y') \tag{5.58}$$

$$g|_{x=0} = g|_{x=a} = g|_{y=0} = g|_{y=b} = 0 \tag{5.59}$$

where, consistent with the discussion associated with (5.13) and (5.14), we have chosen the boundary conditions associated with g to be the same as the homogeneous form of the boundary conditions associated with u. The homogeneous Dirichlet boundary condition case associated with (5.55) is a self-adjoint problem. We therefore formulate in terms of the Green's function $g(x, y, x', y')$. In this case, (5.54) becomes

$$\int_V (-\nabla^2 u) g dV = \int_V u(-\nabla^2 g) dV + \int_S (-g\nabla u + u\nabla g) \cdot \hat{n} dS \tag{5.60}$$

The surface integral S consists of integrals over the surfaces bounding the cylinder at $x = 0$, $x = a$, $y = 0$, and $y = b$, plus the integrals over the cylinder cross sections at $z \to -\infty$ and $z \to \infty$. Application of the boundary conditions specified in (5.56) and (5.59) reduces all surface integrals to zero, except over the surface at $y = b$ and over the surfaces at $z \to -\infty$ and $z \to \infty$. Consider the integral at $z \to \infty$. We have

$$\nabla u \cdot \hat{n} = \frac{\partial u}{\partial z}$$

$$\nabla g \cdot \hat{n} = \frac{\partial g}{\partial z}$$

By hypothesis, there are no variations with respect to z, and these partial derivatives vanish. The integral at $z \to -\infty$ vanishes in a similar manner. The only remaining surface integral contribution is over the surface at $y = b$, and (5.60) reduces to the following:

$$\int_{-\infty}^{\infty}\int_0^b\int_0^a (-\nabla^2 u) g dx dy dz = \int_{-\infty}^{\infty}\int_0^b\int_0^a u(-\nabla^2 g) dx dy dz$$
$$+ \alpha \int_{-\infty}^{\infty}\int_0^a (\nabla g \cdot \hat{y})\Big|_{y=b} dx dz \tag{5.61}$$

Since the integrands in all three integrals are independent of z, the z-integrations cancel and we have

$$\int_0^b\int_0^a (-\nabla_{xy}^2 u) g dx dy = \int_0^b\int_0^a u(-\nabla_{xy}^2 g) dx dy + \alpha \int_0^a \frac{\partial g}{\partial y}\Big|_{y=b} dx \tag{5.62}$$

where we have used $\partial/\partial z = 0$ to reduce the Laplacian to two dimensions. Substituting (5.55) and (5.58) and performing the delta function integration, we obtain

$$u(x', y') = \int_0^b\int_0^a f(x, y) g(x, y, x', y') dx dy - \alpha \int_0^a \frac{\partial g(x, b, x', y')}{\partial y} dx \tag{5.63}$$

To complete the solution, we must solve the Green's function problem given in (5.58) and (5.59). In the same manner as in Example 5.3, the spectral representation in the x-direction leads to a Fourier sine series, viz.

$$g(x, y, x', y') = \sum_{m=1}^{\infty} \alpha_m(y, x', y')u_m(x) \tag{5.64}$$

$$\alpha_m(y, x', y') = \int_0^a g(x, y, x', y')u_m(x)dx \tag{5.65}$$

where

$$u_m(x) = \sqrt{\frac{2}{a}} \sin \frac{m\pi x}{a} \tag{5.66}$$

With respect to the transformation given in (5.65), we have

$$g \Longrightarrow \alpha_m$$

Using the same procedure as in (5.37)–(5.41), we obtain

$$-\frac{\partial^2 g}{\partial x^2} \Longrightarrow \left(\frac{m\pi}{a}\right)^2 \alpha_m$$

$$\delta(x - x') \Longrightarrow u_m(x')$$

Using these relations to transform (5.58), we obtain

$$\left[-\frac{d^2}{dy^2} + \left(\frac{m\pi}{a}\right)^2\right] \alpha_m(y, x', y') = u_m(x')\delta(y - y') \tag{5.67}$$

We associate the following boundary conditions with (5.67):

$$\alpha_m(0, x', y') = \alpha_m(b, x', y') = 0 \tag{5.68}$$

Application of these Dirichlet boundary conditions satisfies the boundary condition requirements in (5.59) at $y = 0$ and $y = b$. The solution to (5.67) is available immediately from the result previously obtained in (5.51), viz.

$$\alpha_m = \frac{u_m(x')}{\frac{m\pi}{a} \sinh \frac{m\pi b}{a}} \begin{cases} \sinh \frac{m\pi(b-y')}{a} \sinh \frac{m\pi y}{a}, & y < y' \\ \sinh \frac{m\pi(b-y)}{a} \sinh \frac{m\pi y'}{a}, & y > y' \end{cases} \tag{5.69}$$

Substitution in (5.64) yields the Green's function

$$g(x, y, x', y') = \frac{2}{a} \sum_{m=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a}}{\frac{m\pi}{a} \sinh \frac{m\pi b}{a}} \begin{cases} \sinh \frac{m\pi(b-y')}{a} \sinh \frac{m\pi y}{a}, & y < y' \\ \sinh \frac{m\pi(b-y)}{a} \sinh \frac{m\pi y'}{a}, & y > y' \end{cases} \tag{5.70}$$

Equation (5.70) gives the Green's function required in the first integral in (5.63). In the second integral in (5.63), we require $\partial g/\partial y$ evaluated at $y = b$. Performing the required differentiation in (5.70) yields

$$\frac{\partial g(x, b, x', y')}{\partial y} = -\frac{2}{a}\sum_{m=1}^{\infty} \sin\frac{m\pi x}{a} \sin\frac{m\pi x'}{a} \frac{\sinh\frac{m\pi y'}{a}}{\sinh\frac{m\pi b}{a}} \tag{5.71}$$

Substitution of (5.70) and (5.71) into (5.63), followed by an interchange of the prime and unprimed coordinates, yields the following result:

$$u(x, y) = \alpha\int_0^a \frac{2}{a}\sum_{m=1}^{\infty} \sin\frac{m\pi x}{a} \sin\frac{m\pi x'}{a} \frac{\sinh\frac{m\pi y}{a}}{\sinh\frac{m\pi b}{a}} dx' + \frac{2}{a}\int_0^b\int_0^a dx'dy' f(x', y')$$

$$\cdot \sum_{m=1}^{\infty} \frac{\sin\frac{m\pi x}{a} \sin\frac{m\pi x'}{a}}{\frac{m\pi}{a}\sinh\frac{m\pi b}{a}} \begin{cases} \sinh\frac{m\pi(b-y')}{a}\sinh\frac{m\pi y}{a}, & y < y' \\ \sinh\frac{m\pi(b-y)}{a}\sinh\frac{m\pi y'}{a}, & y > y' \end{cases} \tag{5.72}$$

It is easy to show that (5.72) satisfies the boundary conditions at $x = 0$, $x = a$, and $y = 0$. The details are left for the reader. We now show that (5.72) satisfies the inhomogeneous boundary condition at $y = b$ required by (5.57). Indeed, at $y = b$, the second term vanishes and we have

$$u(x, b) = \alpha\int_0^a \frac{2}{a}\sum_{m=1}^{\infty} \sin\frac{m\pi x}{a} \sin\frac{m\pi x'}{a} dx' = \alpha\int_0^a \delta(x - x')dx' = \alpha \tag{5.73}$$

where we have used the spectral representation of the delta function

$$\delta(x - x') = \frac{2}{a}\sum_{m=1}^{\infty} \sin\frac{m\pi x}{a} \sin\frac{m\pi x'}{a}$$

It is also important to show that the solution in (5.72) satisfies the original differential equation in (5.55). The details are left for the Problems.

■

5.4 SLP2 AND SLP3 EXTENSION TO THREE DIMENSIONS

In this section, we consider complex f, complex λ, and admit the possibility of unbounded regions. We produce an SLP2 and SLP3 extension to three dimensions. We again confine our attention to the three-dimensional negative Laplacian operator and consider the partial differential equation

$$L_\lambda u = f \tag{5.74}$$

where

$$L_\lambda = L - \lambda, \qquad \lambda \in \mathbf{C} \tag{5.75}$$

and where L is the Laplacian operator

$$L = -\nabla^2 \tag{5.76}$$

Define the three-dimensional inner product

$$\langle f, g\rangle = \int_V f(\mathbf{r})\overline{g}(\mathbf{r})dV \tag{5.77}$$

The Green's function problem associated with (5.74) is given by

$$L_\lambda g = \delta(\mathbf{r} - \mathbf{r}') \tag{5.78}$$

Extending our analysis in Sections 2.5 and 2.6, we solve (5.74) by taking the inner product with the adjoint Green's function $h(\mathbf{r}, \mathbf{r}')$, as follows:

$$\langle L_\lambda u, h\rangle = \int_V \left[\left(-\nabla^2 - \lambda\right) u\right]\overline{h}dV = \int_V (-\nabla^2 u)\overline{h}dV + \int_V (-\lambda u)\overline{h}dV \tag{5.79}$$

To produce the adjoint operator L_λ^* and the conjunct $J(u, h)$, we again use *Green's theorem*. In the case of the Laplacian operator, we have

$$\int_V (-\nabla^2 u)\overline{h}dV = \int_V u(-\nabla^2\overline{h})dV + \int_S \left(-\overline{h}\nabla u + u\nabla\overline{h}\right)\cdot \hat{n}dS \tag{5.80}$$

where S is the surface bounding V and $\hat{n}$ is the unit normal to S in the direction outward from V. But,

$$\int_V u(-\nabla^2\overline{h})dV = \int_V u(\overline{-\nabla^2 h})dV = \langle u, -\nabla^2 h\rangle \tag{5.81}$$

and

$$\int_V (-\lambda u)\overline{h}dV = \int_V u(\overline{-\overline{\lambda} h})dV = \langle u, -\overline{\lambda} h\rangle \tag{5.82}$$

Substituting (5.81) in (5.80) and then (5.80) and (5.82) into (5.79), we obtain

$$\langle L_\lambda u, h\rangle = \langle u, L_\lambda^* h\rangle + J(u, h)\Big|_S \tag{5.83}$$

where

$$L_\lambda^* u = -\nabla^2 - \overline{\lambda} \tag{5.84}$$

$$J(u,h)\Big|_S = -\int_S \left(\overline{h}\nabla u - u\nabla\overline{h}\right)\cdot\hat{n}\,dS \tag{5.85}$$

We note that (5.83) is the three-dimensional extension of (2.134). We may solve for u in (5.83) by considering the adjoint Green's function problem given by

$$L_\lambda^* h = \delta(\mathbf{r}-\mathbf{r}') \tag{5.86}$$

Substitution of (5.74) and (5.86) into (5.83) yields

$$u(\mathbf{r}') = \langle f, h\rangle - J(u,h)\Big|_S \tag{5.87}$$

or, explicitly,

$$u(\mathbf{r}') = \int_V f(\mathbf{r})\overline{h}(\mathbf{r},\mathbf{r}')dV + \int_S \left[\overline{h}(\mathbf{r},\mathbf{r}')\nabla u(\mathbf{r}) - u(\mathbf{r})\nabla\overline{h}(\mathbf{r},\mathbf{r}')\right]\cdot\hat{n}\,dS \tag{5.88}$$

We note that (5.88) is the solution to (5.74), provided that we can determine the conjugate adjoint Green's function $\overline{h}(\mathbf{r},\mathbf{r}')$. Taking the complex conjugate of both sides of (5.86), we obtain a partial differential equation for $\overline{h}$, viz.

$$\overline{L_\lambda^* h(\mathbf{r},\mathbf{r}')} = L_\lambda\overline{h}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \tag{5.89}$$

As in Chapter 2, we can show that it is never necessary to find the conjugate adjoint Green's function directly. Indeed, we form

$$\langle L_\lambda g(\mathbf{r},\mathbf{r}'), h(\mathbf{r},\mathbf{r}'')\rangle = \langle g(\mathbf{r},\mathbf{r}'), L_\lambda^* h(\mathbf{r},\mathbf{r}'')\rangle + J(g,h)\Big|_S \tag{5.90}$$

We are given the boundary conditions on g. We choose the boundary conditions on $\overline{h}$ so that

$$J(g,h)\Big|_S = 0 \tag{5.91}$$

Then, substitution of (5.78), (5.86), and (5.91) into (5.90) gives

$$\overline{h}(\mathbf{r}',\mathbf{r}'') = g(\mathbf{r}'',\mathbf{r}')$$

or, with a change in variables,

$$\overline{h}(\mathbf{r},\mathbf{r}') = g(\mathbf{r}',\mathbf{r}) \tag{5.92}$$

Therefore, the conjugate adjoint Green's function is given simply by interchanging $\mathbf{r}$ and $\mathbf{r}'$ in the expression for the Green's function $g(\mathbf{r}, \mathbf{r}')$. In cases where the Green's function is symmetric, $g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}', \mathbf{r})$ and

$$\bar{h}(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}', \mathbf{r}) \qquad \text{(symmetric case)} \tag{5.93}$$

We shall find that the Green's function is symmetric in many of the interesting cases to follow. It is certainly symmetric in cases where the operator L is self-adjoint. In addition, we shall find symmetry, as we have previously found in Chapter 2, when examining many problems containing limit points and limit circles. For the symmetric Green's function case, we may substitute (5.93) into (5.88) to obtain

$$u(\mathbf{r}') = \int_V f(\mathbf{r})g(\mathbf{r}, \mathbf{r}')dV + \int_S \left[g(\mathbf{r}, \mathbf{r}')\nabla u(\mathbf{r}) - u(\mathbf{r})\nabla g(\mathbf{r}, \mathbf{r}')\right] \cdot \hat{n} dS \tag{5.94}$$

We shall summarize the steps for solving (5.74) by the above-described extension to the one-dimensional Green's function method. We distinguish two cases, dependent on whether or not the Green's function is symmetric.

Nonsymmetric Green's Function Case

1. Write the solution in the form given by (5.88).
2. Substitute the boundary conditions for u on the surface S into (5.88).
3. Substitute the conjugate adjoint boundary conditions for $\bar{h}$ on the surface S into (5.88).
4. Solve the Green's function problem given by (5.78) with boundary conditions on S the same as the homogeneous form of the boundary conditions on u.
5. Obtain the conjugate adjoint Green's function $\bar{h}$ through (5.92) and substitute into (5.88).
6. Interchange the variables $\mathbf{r}$ and $\mathbf{r}'$ in (5.88).

Symmetric Green's Function Case

1. Write the solution in the form given by (5.94).
2. Substitute the boundary conditions for u on the surface S into (5.94).
3. Substitute the boundary conditions for g on the surface S into (5.94).

4. Solve the Green's function problem given by (5.78) with boundary conditions on S the same as the homogeneous form of the boundary conditions on u and substitute into (5.94).
5. Interchange the variables $\mathbf{r}$ and $\mathbf{r}'$ in (5.94).

5.5 THE PARALLEL PLATE WAVEGUIDE

In this section, we consider the propagation of electromagnetic waves in a parallel plate waveguide. We consider a waveguide of uniform cross section with no scattering objects.

Suppose that the waveguide is formed from two parallel, perfectly conducting plates (Fig. 5-4), separated by a distance d and extending from $-\infty$ to ∞ in the y-direction and the z-direction. Assume that the medium between the parallel plates is free space. We shall choose the source of the electromagnetic field to be independent of y. Since the structure is also independent of y, we must have

$$\frac{\partial}{\partial y} = 0 \tag{5.95}$$

We begin with Maxwell's curl equations, given in (4.55) and (4.56). If ϵ_0 is the permittivity of free space, we have

$$\nabla \times \mathbf{H} = \mathbf{J} + i\omega\epsilon_0 \mathbf{E} \tag{5.96}$$

$$\nabla \times \mathbf{E} = -\mathbf{M} - i\omega\mu_0 \mathbf{H} \tag{5.97}$$

We expand these two equations in Cartesian coordinates and group them into two sets as follows:

Set 1: TM_z

$$\frac{\partial H_y}{\partial z} = -J_x - i\omega\epsilon_0 E_x \tag{5.98}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -M_y - i\omega\mu_0 H_y \tag{5.99}$$

$$\frac{\partial H_y}{\partial x} = J_z + i\omega\epsilon_0 E_z \tag{5.100}$$

Set 2: TE_z

$$\frac{\partial E_y}{\partial z} = M_x + i\omega\mu_0 H_x \tag{5.101}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + i\omega\epsilon_0 E_y \tag{5.102}$$

$$\frac{\partial E_y}{\partial x} = -M_z - i\omega\mu_0 H_z \tag{5.103}$$

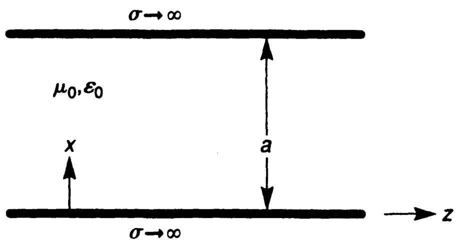

Fig. 5-4 Parallel plate waveguide.

We note that the transverse magnetic (TM_z) set is not coupled to the transverse electric (TE_z) set. It is therefore possible to excite one set independent of the other by appropriate selection of the **J** and **M** sources. We produce second-order partial differential equations governing each set by the following procedure. We differentiate (5.98) with respect to z, (5.100) with respect to x, add the result, and substitute (5.99) to obtain the following:

Set 1: TM_z

$$(\nabla_{xz}^2 + k^2)H_y = i\omega\epsilon_0 M_y + \frac{\partial J_z}{\partial x} - \frac{\partial Je_x}{\partial z} \tag{5.104}$$

$$E_x = -\frac{1}{i\omega\epsilon_0}\left(\frac{\partial H_y}{\partial z} + J_x\right) \tag{5.105}$$

$$E_z = \frac{1}{i\omega\epsilon_0}\left(\frac{\partial H_y}{\partial x} - J_z\right) \tag{5.106}$$

where

$$\nabla_{xz}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \tag{5.107}$$

and

$$k^2 = \omega^2\mu_0\epsilon_0 \tag{5.108}$$

A similar procedure applied to (5.101)–(5.103) yields

Set 2: TE_z

$$(\nabla_{xz}^2 + k^2)E_y = i\omega\mu_0 J_y - \frac{\partial M_z}{\partial x} + \frac{\partial M_x}{\partial z} \tag{5.109}$$

$$H_x = \frac{1}{i\omega\mu_0}\left(\frac{\partial E_y}{\partial z} - M_x\right) \tag{5.110}$$

$$H_z = -\frac{1}{i\omega\mu_0}\left(\frac{\partial E_y}{\partial x} + M_z\right) \tag{5.111}$$

We remark that the sources M_y, J_x, J_z excite the TM_z set, whereas the sources J_y, M_x, M_z excite the TE_z set. We shall consider the TM_z case.

To excite the TM_z fields only, we set $J_y = M_x = M_z = 0$ since these sources excite TE_z modes. We are left with three options to excite the TM_z modes, namely, M_y, J_x, or J_z. Since M_y enters into (5.104) in the least complicated manner of the three, we choose M_y and set $J_x = J_z = 0$. This choice results in $E_y = H_x = H_z = 0$ and

$$(\nabla_{xz}^2 + k^2)H_y = i\omega\epsilon_0 M_y \tag{5.112}$$

$$E_x = -\frac{1}{i\omega\epsilon_0}\frac{\partial H_y}{\partial z} \tag{5.113}$$

$$E_z = \frac{1}{i\omega\epsilon_0}\frac{\partial H_y}{\partial x} \tag{5.114}$$

At present, the only restriction on the source M_y is that it is independent of y. Associated with the Laplacian operator in (5.112) are the following boundary and limiting conditions on the field $H_y(x, z)$:

$$\lim_{z\to\pm\infty} H_y(x, z) = 0 \tag{5.115}$$

$$\frac{\partial H_y(0, z)}{\partial x} = \frac{\partial H_y(a, z)}{\partial x} = 0 \tag{5.116}$$

Equation (5.116) arises from the fact that E_z is tangential to the perfectly conducting waveguide surfaces at $x = 0$ and $x = a$, and therefore must vanish. Substituting this information in (5.114) yields (5.116). The Green's function problem associated with (5.112), (5.115), and (5.116) is as follows:

$$-(\nabla_{xz}^2 + k^2)g = \delta(x - x')\delta(z - z') \tag{5.117}$$

$$\lim_{z\to\pm\infty} g(x, z, x', z') = 0 \tag{5.118}$$

$$\frac{\partial g(0, z, x', z')}{\partial x} = \frac{\partial g(a, z, x', z')}{\partial x} = 0 \tag{5.119}$$

To solve for the TM_z fields, we shall solve (5.112) by the Green's function method given in Section 5.4. In the present case, however, the problem is independent of y. We introduce this simplification by beginning with (5.83), which gives, in this case,

$$\langle L_{k^2} H_y, h\rangle = \langle H_y, L_{k^2}^* h\rangle + J(H_y, h)\Big|_S \tag{5.120}$$

Explicitly,

$$\int_V \left[\left(-\nabla^2 - k^2\right) H_y\right]\bar{h}dV = \int_V H_y \left(-\nabla^2 - k^2\right)\bar{h}dV$$
$$+ \int_S \left(H_y \nabla\bar{h} - \bar{h}\nabla H_y\right) \cdot \hat{n}dS \qquad (5.121)$$

We anticipate, and will verify, that the Green's function will be symmetric, and use (5.93) to write

$$\int_V \left[\left(-\nabla^2 - k^2\right) H_y\right] g dV = \int_V H_y \left(-\nabla^2 - k^2\right) g dV$$
$$+ \int_S (H_y \nabla g - g\nabla H_y) \cdot \hat{n}dS \qquad (5.122)$$

The volume V is that region between the parallel plates. The surface S consists of the surfaces of the two parallel plates and the cross-sectional planar surfaces at $z \to \pm\infty$ and $y \to \pm\infty$. The surface integral contributions at $y \to \pm\infty$ vanish since

$$\nabla g \cdot \hat{n}\Big|_{y\to\pm\infty} = \pm\frac{\partial g}{\partial y}\Big|_{y\to\pm\infty} = 0$$

$$\nabla H_y \cdot \hat{n}\Big|_{y\to\pm\infty} = \pm\frac{\partial H_y}{\partial y}\Big|_{y\to\pm\infty} = 0$$

where we have used (5.95). The contributions on all remaining surfaces vanish with application of (5.115), (5.116), (5.118), and (5.119). The result is

$$\int_V \left[\left(-\nabla^2 - k^2\right) H_y\right] g dV = \int_V H_y \left(-\nabla^2 - k^2\right) g dV \qquad (5.123)$$

Because of the absence of y-variations, the dy portion of the volume integrals cancel and the del-operator reduces to ∇_{xz}, with the result

$$\int_{-\infty}^{\infty}\int_0^a \left[\left(-\nabla_{xz}^2 - k^2\right) H_y\right] g dx dz = \int_{-\infty}^{\infty}\int_0^a H_y \left(-\nabla_{xz}^2 - k^2\right) g dx dz \qquad (5.124)$$

Substituting (5.112) and (5.117) and performing the right-side integration, we obtain

$$H_y(x', z') = -i\omega\epsilon_0 \int_A M_y(x, z) g(x, z, x', z') dx dz$$

where A indicates the area occupied by the source. An interchange of primed and unprimed coordinates gives

$$H_y(x, z) = -i\omega\epsilon_0 \int_A M_y(x', z')g(x, z, x', z')dx'dz' \tag{5.125}$$

where we have again assumed that the Green's function is symmetric. Equation (5.125) gives the H_y-field everywhere inside the parallel plates, provided we can solve for the Green's function g, which we consider next.

To solve for the Green's function, defined by (5.117)–(5.119), we expand in terms of the spectral representation over the x-coordinate, viz.

$$g(x, z, x', z') = \sum_{n=0}^{\infty} \beta_n(z, x', z')u_n(x) \tag{5.126}$$

where the normalized eigenfunction $u_n(x)$ is given by

$$u_n(x) = \sqrt{\frac{\epsilon_n}{a}} \cos \frac{n\pi x}{a} \tag{5.127}$$

and ϵ_n is Neumann's number. The coefficient β_n is given by

$$\beta_n(z, x', z') = \int_0^a g(x, z, x', z')u_n(x)dx \tag{5.128}$$

With respect to the transformation in (5.128), we have

$$g \Longrightarrow \beta_n$$

Using the procedure in (5.37)–(5.41), we have

$$-\frac{\partial^2 g}{\partial x^2} \Longrightarrow \left(\frac{n\pi}{a}\right)^2 \beta_n$$

$$\delta(x - x') \Longrightarrow u_n(x')$$

Using these relations to transform (5.117), we obtain

$$\left(\frac{d^2}{dz^2} + k_z^2\right) \beta_n(z, x', z') = -u_n(x')\delta(z - z') \tag{5.129}$$

where

$$k_z^2 = k^2 - \left(\frac{n\pi}{a}\right)^2 \tag{5.130}$$

Let

$$\gamma_n = \frac{\beta_n}{u_n(x')} \tag{5.131}$$

With this definition, (5.129) becomes

$$\left(\frac{d^2}{dz^2} + k_z^2\right)\gamma_n = -\delta(z - z') \tag{5.132}$$

We associate the following limiting conditions with (5.132):

$$\lim_{z \to \pm\infty} \gamma_n = 0 \tag{5.133}$$

These conditions are consistent with those in (5.118). The solution to this one-dimensional Green's function differential equation has been obtained previously in Example 2.20. Applied to (5.132), we find that

$$\gamma_n = \frac{e^{-ik_z|z-z'|}}{2ik_z}, \qquad \operatorname{Im}(k_z) < 0 \tag{5.134}$$

Substituting (5.134) into (5.131) and the result into (5.126) gives

$$g(x, z, x', z') = \sum_{n=0}^{\infty} \left(\frac{\epsilon_n}{a}\right) \frac{e^{-ik_z|z-z'|}}{2ik_z} \cos\frac{n\pi x}{a} \cos\frac{n\pi x'}{a} \tag{5.135}$$

We note that the Green's function is symmetric, as anticipated. Substituting this result into (5.125) gives the magnetic field H_y, viz.

$$H_y(x, z) = -i\omega\epsilon_0 \sum_{n=0}^{\infty} \left[\int_A M_y(x', z') \left(\frac{\epsilon_n}{a}\right) \frac{e^{-ik_z|z-z'|}}{2ik_z} \cos\frac{n\pi x'}{a} dx'dz'\right] \cos\frac{n\pi x}{a} \tag{5.136}$$

The electric fields E_x and E_z associated with (5.136) can be calculated from (5.113) and (5.114), respectively.

We examine the modal structure of (5.136) by specializing the source M_y as follows:

$$M_y(x', z') = M_{sy}(x')\delta(z' - \ell) \tag{5.137}$$

where M_{sy} is a magnetic surface current in volts/m. Substituting (5.137) into (5.136) and performing the indicated z'-integration, we obtain

$$H_y(x, z) = -i\omega\epsilon_0 \sum_{n=0}^{\infty} B_n \frac{e^{-ik_z|z-\ell|}}{2ik_z} \sqrt{\frac{\epsilon_n}{a}} \cos\frac{n\pi x}{a} \tag{5.138}$$

where B_n is a *modal coefficient*, given by

$$B_n = \int_0^a M_{sy}(x')\sqrt{\frac{\epsilon_n}{a}}\cos\frac{n\pi x'}{a}dx' \tag{5.139}$$

The representation of the H_y-field in (5.138) shows the decomposition of the field into the familiar TEM ($n = 0$) and TM ($n > 0$) modes described in the undergraduate texts ([3],[4], for example). By making different choices of M_{sy}, we may adjust the coefficient B_n associated with each mode. For example, we may excite only the TEM mode by choosing

$$M_{sy} = \sqrt{\frac{1}{a}}M_0 \tag{5.140}$$

where M_0 is a constant. This choice gives

$$B_n = \begin{cases} 0, & n \neq 0 \\ M_0, & n = 0 \end{cases} \tag{5.141}$$

Substituting into (5.138), we obtain

$$H_y(x,z) = -i\omega\epsilon_0\sqrt{\frac{1}{a}}M_0\frac{e^{-ik|z-\ell|}}{2ik} \tag{5.142}$$

which is a pure TEM wave traveling away from the source location $z = \ell$. For $z < \ell$, the wave travels right to left; for $z > \ell$, the wave travels left to right. If we are interested specifically in the region $z > \ell$, we may select the constant M_0 to produce a unit left-to-right TEM wave. Indeed, the choice

$$M_0 = \left(\frac{-i\omega\epsilon_0 e^{ik\ell}}{2i\sqrt{a}k}\right)^{-1} \tag{5.143}$$

produces

$$H_y(x,z) = e^{-ikz}, \qquad z > \ell \tag{5.144}$$

In (5.126), we chose to expand the Green's function in a spectral expansion over the x-coordinate. This expansion led to the Fourier cosine series, and produced a solution for the magnetic field H_y in terms of the waveguide modes. An *alternative representation* is possible. We shall begin by expanding the Green's function in a spectral expansion over z, rather than x. In (5.117), the spectral representation of the delta function associated with $-\partial^2/\partial z^2$ subject to the limiting condition in (5.118) leads to the Fourier transform pair

$$G(x, k_z, x', z') = \int_{-\infty}^{\infty} g(x, z, x', z')e^{-ik_z z}dz \tag{5.145}$$

$$g(x, z, x', z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x, k_z, x', z')e^{ik_z z}dk_z \tag{5.146}$$

We represent the transform pair by

$$g(x, z, x', z') \Longleftrightarrow G(x, k_z, x', z') \tag{5.147}$$

and easily find that

$$-\frac{\partial^2 g}{\partial z^2} \Longleftrightarrow k_z^2 G \tag{5.148}$$

$$\delta(z - z') \Longleftrightarrow e^{-ik_z z'} \tag{5.149}$$

Applying (5.145) and (5.147)–(5.149) to (5.117), we obtain

$$-\left(\frac{d^2}{dx^2} + k_x^2\right) G = e^{-ik_z z'}\delta(x - x') \tag{5.150}$$

where

$$k_x = \left(k^2 - k_z^2\right)^{1/2} \tag{5.151}$$

We let

$$\hat{G} = Ge^{ik_z z'} \tag{5.152}$$

and obtain

$$-\left(\frac{d^2}{dx^2} + k_x^2\right) \hat{G} = \delta(x - x') \tag{5.153}$$

with boundary conditions inferred from (5.119), viz.

$$\left.\frac{d\hat{G}}{dx}\right|_{x=0} = \left.\frac{d\hat{G}}{dx}\right|_{x=a} = 0 \tag{5.154}$$

The solution to this differential equation and associated boundary conditions has been obtained previously in Example 2.13, viz.

$$\hat{G} = -\frac{1}{k_x \sin k_x a}\begin{cases} \cos k_x x \cos k_x(a - x'), & x < x' \\ \cos k_x x' \cos k_x(a - x) & x > x' \end{cases} \tag{5.155}$$

Using (5.152) and taking the inverse Fourier transform, we have

$$g(x, z, x', z') = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik_z(z-z')}dk_z}{k_x \sin k_x a} \cdot \begin{cases} \cos k_x x \cos k_x (a - x'), & x < x' \\ \cos k_x x' \cos k_x (a - x) & x > x' \end{cases} \tag{5.156}$$

This representation of the Green's function is an alternative to the representation given in (5.135), which we repeat here for convenience, viz.

$$g(x, z, x', z') = \sum_{n=0}^{\infty} \left(\frac{\epsilon_n}{a}\right) \frac{e^{-ik_z|z-z'|}}{2ik_z} \cos\frac{n\pi x}{a} \cos\frac{n\pi x'}{a} \tag{5.157}$$

Although these two representations lead to the same Green's function $g(x, z, x', z')$, their forms are quite different. In (5.157), the cross-sectional waveguide modes are displayed explicitly. In (5.156), we find no explicit modal display. However, (5.156) is the starting point for constructing waveguide *ray representations* [5]. These ray representations are particularly useful in cases where the frequency is so high that a large number of modes can propagate in the waveguide. In addition, Felsen and Kamel have shown that the Green's function forms in (5.156) and (5.157) can be combined to produce what are called *hybrid ray-mode formulations*. The hybrid forms effectively exhibit the useful features in both the modal and ray formulations. The details can be found in [5].

In this section, we have studied formulations describing the propagation of the TEM and TM modes in a parallel plate waveguide. We leave the production of a modal series describing the TE modes for the problems. In the next section, we shall consider an obstacle in a parallel plate waveguide. We shall demonstrate the decomposition of the fields into incident, transmitted, and reflected waves.

5.6 IRIS IN PARALLEL PLATE WAVEGUIDE

In Section 5.5, we solved for the fields in a parallel plate waveguide. In this section, we add an iris to the waveguide interior. The waveguide has cross-sectional dimension a (Fig. 5-5), and contains an infinitesimally thin, perfectly conducting iris connected to the top plate at $x = a, z = 0$ and extending perpendicular to the plate and into the waveguide interior. The iris effectively divides the interior of the waveguide into two regions: Region 1, $z < 0$; Region 2, $z > 0$. The regions are connected electromagnetically

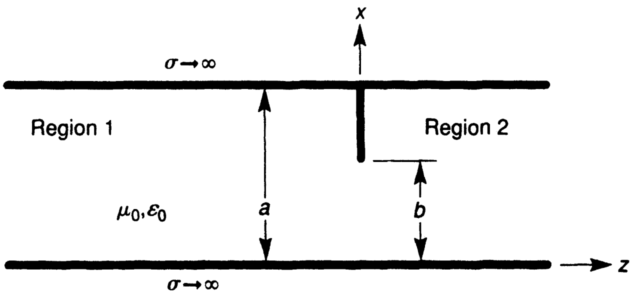

Fig. 5-5 Iris in parallel plate waveguide.

through an *aperture*, located at $z = 0, x \in (0, b)$. We note that the insertion of the iris does not change the y-independence of the waveguide geometry. Therefore, Maxwell's equations again separate into a TE_z set and a TM_z set, in the same manner as in Section 5.5. We shall cause excitation of the TM_z set by placing a constant magnetic sheet current source at $z = -d$ in Region 1, viz.

$$M_y = M_0 \delta(z + d) \tag{5.158}$$

Such a choice will produce a *TEM* wave incident from left to right in Region 1. As we shall discover in the ensuing analysis, this *TEM* wave, when encountering the iris, will cause *TEM* and *TM* waves to scatter from the iris into both Region 1 and Region 2. Using (5.112)–(5.114), we write the equations describing the TM_z fields in Region 1 and Region 2 as follows:

Region 1:

$$(\nabla_{xz}^2 + k^2) H_{y1} = i\omega\epsilon_0 M_y \tag{5.159}$$

$$E_{x1} = -\frac{1}{i\omega\epsilon_0} \frac{\partial H_{y1}}{\partial z} \tag{5.160}$$

$$E_{z1} = \frac{1}{i\omega\epsilon_0} \frac{\partial H_{y1}}{\partial x} \tag{5.161}$$

Region 2:

$$(\nabla_{xz}^2 + k^2) H_{y2} = 0 \tag{5.162}$$

$$E_{x2} = -\frac{1}{i\omega\epsilon_0} \frac{\partial H_{y2}}{\partial z} \tag{5.163}$$

$$E_{z2} = \frac{1}{i\omega\epsilon_0} \frac{\partial H_{y2}}{\partial x} \tag{5.164}$$

We first consider Region 1. Using the Green's function method and anticipating that the Green's function will be symmetric, we adapt (5.122) to the present case as follows:

$$\int_{V_1} \left[\left(-\nabla^2 - k^2\right) H_{y1}\right] g_1 dV = \int_{V_1} H_{y1} \left(-\nabla^2 - k^2\right) g_1 dV$$
$$+ \int_{S_1} \left(H_{y1} \nabla g_1 - g_1 \nabla H_{y1}\right) \cdot \hat{n} dS \tag{5.165}$$

where g_1 is the Green's function in Region 1, governed by

$$-(\nabla_{xz}^2 + k^2) g_1 = \delta(x - x')\delta(z - z') \tag{5.166}$$

with boundary and limiting conditions yet to be determined. The volume V_1 consists of Region 1. The surface S_1 consists of the following parts:

1. The surfaces of the two parallel plates in Region 1 ($z < 0, x = 0$ and $z < 0, x = a$).
2. The cross-sectional planar surfaces at $z \to -\infty$, $x \in (0, a)$ and $y \to \pm\infty$, $x \in (0, a)$.
3. The surface of the iris and aperture, $z = 0$, $x \in (0, a)$.

By the same reasoning as in the previous section, the surface integrals at $y \to \pm\infty$ vanish. In addition, we have the following boundary and limiting conditions governing the H_{y1}-fields:

$$\lim_{z \to -\infty} H_{y1} = 0 \tag{5.167}$$

$$\frac{\partial H_{y1}(0, z)}{\partial x} = \frac{\partial H_{y1}(a, z)}{\partial x} = 0 \tag{5.168}$$

$$\frac{\partial H_{y1}(x, 0)}{\partial z} = 0, \qquad x \in (b, a) \tag{5.169}$$

We choose the following boundary and limiting conditions for the Green's function g_1:

$$\lim_{z \to -\infty} g_1(x, z, x', z') = 0 \tag{5.170}$$

$$\frac{\partial g_1(0, z, x', z')}{\partial x} = \frac{\partial g_1(a, z, x', z')}{\partial x} = 0 \tag{5.171}$$

$$\frac{\partial g_1(x, 0, x', z')}{\partial z} = 0, \qquad x \in (0, a) \tag{5.172}$$

We note that the Green's function boundary and limiting conditions are the same as those associated with the H_{y1}-field, with one important exception. In (5.169), the boundary condition on H_{y1} is over the iris, whereas in

(5.172), the boundary condition is over the iris *and* the aperture. We therefore have a Green's function problem (Fig. 5-6) for a parallel plate waveguide extending from $z \to -\infty$ and terminating in a perfect conductor at $z = 0$. Before commenting on this choice, we substitute (5.167)–(5.172) into (5.165) and obtain

$$\int_{V_1} \left[\left(-\nabla^2 - k^2\right) H_{y1}\right] g_1 dV = \int_{V_1} H_{y1} \left(-\nabla^2 - k^2\right) g_1 dV - \int_{-\infty}^{\infty} \int_0^b \left(g_1 \frac{\partial H_{y1}}{\partial z}\right)\bigg|_{z=0} dx dy \tag{5.173}$$

Because of the invariance with y, the integrations with respect to y cancel in all terms. In addition, the del-operator reduces to ∇_{xz} and we have

$$\int_{-\infty}^{0} \int_0^a \left[\left(-\nabla_{xz}^2 - k^2\right) H_{y1}\right] g_1 dx dz = \int_{-\infty}^{0} \int_0^a H_{y1} \left(-\nabla_{xz}^2 - k^2\right) g_1 dx dz - \int_0^b \left(g_1 \frac{\partial H_{y1}}{\partial z}\right)\bigg|_{z=0} dx \tag{5.174}$$

Substituting (5.159) and (5.166), doing the delta-function integrations, and rearranging, we obtain

$$H_{y1}(x', z') = -i\omega\epsilon_0 \int_A M_y(x, z) g_1(x, z, x', z') dx dz + \int_0^b g_1(x, 0, x', z') \frac{\partial H_{y1}(x, 0)}{\partial z} dx$$

Anticipating the symmetry of the Green's function, we interchange the primed and unprimed coordinates and then substitute (5.158) and (5.160) to obtain

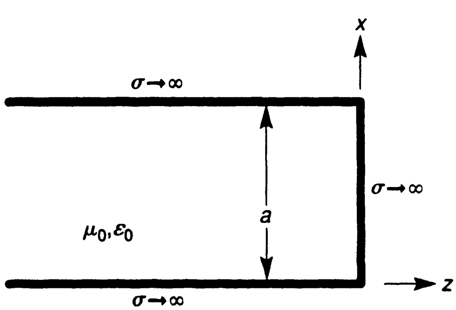

Fig. 5-6 Green's function problem for Region 1.

$$H_{y1}(x,z) = -i\omega\epsilon_0 M_0 \int_0^a g_1(x,z,x',-d)dx' - i\omega\epsilon_0 \cdot \int_0^b g_1(x,z,x',0)E_{x1}(x',0)dx' \tag{5.175}$$

Our choice of the boundary condition in (5.172) deserves some comment. If we had chosen this condition to be the same as that governing the H-field derivative in (5.169), we would have produced a Green's function problem with an unspecified boundary condition over $x \in (0,b)$, $z = 0$. Our choice completes the specification of the Green's function in Region 1, and has the added advantage of simplifying the problem solution by eliminating a portion of the surface integral.

We now consider Region 2. In a manner similar to our treatment in Region 1, we obtain

$$\int_{V_2} \left[\left(-\nabla^2 - k^2\right) H_{y2}\right] g_2 dV = \int_{V_2} H_{y2}\left(-\nabla^2 - k^2\right) g_2 dV + \int_{S_2} \left(H_{y2}\nabla g_2 - g_2 \nabla H_{y2}\right) \cdot \hat{n} dS \tag{5.176}$$

where g_2 is the Green's function in Region 2, governed by

$$-(\nabla_{xz}^2 + k^2)g_2 = \delta(x-x')\delta(z-z') \tag{5.177}$$

The volume V_2 consists of Region 2. The surface S_2 consists of the following parts:

1. The surfaces of the two parallel plates in Region 2 ($z > 0, x = 0$ and $z > 0, x = a$).
2. The cross-sectional planar surfaces at $z \to \infty$, $x \in (0,a)$ and $y \to \pm\infty$, $x \in (0,a)$.
3. The surface of the iris and aperture, $z = 0$, $x \in (0,a)$.

Again, the surface integrals at $y \to \pm\infty$ vanish. In addition, we have the following boundary and limiting conditions governing the H_{y2}-fields:

$$\lim_{z\to\infty} H_{y2} = 0 \tag{5.178}$$

$$\frac{\partial H_{y2}(0,z)}{\partial x} = \frac{\partial H_{y2}(a,z)}{\partial x} = 0 \tag{5.179}$$

$$\frac{\partial H_{y2}(x,0)}{\partial z} = 0, \qquad x \in (b,a) \tag{5.180}$$

We choose the following boundary and limiting conditions for the Green's function g_2:

$$\lim_{z\to\infty} g_2(x,z,x',z') = 0 \tag{5.181}$$

$$\frac{\partial g_2(0,z,x',z')}{\partial x} = \frac{\partial g_2(a,z,x',z')}{\partial x} = 0 \tag{5.182}$$

$$\frac{\partial g_2(x,0,x',z')}{\partial z} = 0, \qquad x \in (0,a) \tag{5.183}$$

We therefore have a Green's function problem (Fig. 5-7) for a parallel plate waveguide extending to $z \to \infty$ and terminating in a perfect conductor at $z = 0$. We substitute (5.178)–(5.183) into (5.176) and obtain

$$\int_{V_2} \left[\left(-\nabla^2 - k^2\right) H_{y2}\right] g_2 dV = \int_{V_2} H_{y2}\left(-\nabla^2 - k^2\right) g_2 dV + \int_{-\infty}^{\infty}\int_0^b \left(g_2 \frac{\partial H_{y2}}{\partial z}\right)\bigg|_{z=0} dxdy \tag{5.184}$$

Again, the integrations with respect to y cancel in all terms and the del-operator reduces to ∇_{xz}. We have

$$\int_0^{\infty}\int_0^a \left[\left(-\nabla_{xz}^2 - k^2\right) H_{y2}\right] g_2 dxdz = \int_0^{\infty}\int_0^a H_{y2}\left(-\nabla_{xz}^2 - k^2\right) g_2 dxdz + \int_0^b \left(g_2 \frac{\partial H_{y2}}{\partial z}\right)\bigg|_{z=0} dx \tag{5.185}$$

Substituting (5.162) and (5.177), doing the delta-function integrations, and rearranging, we obtain

$$H_{y2}(x',z') = -\int_0^b g_2(x,0,x',z')\frac{\partial H_{y2}(x,0)}{\partial z}dx$$

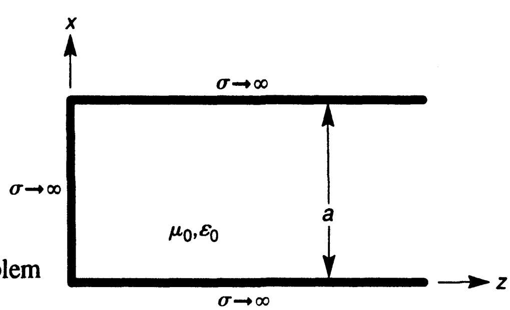

Fig. 5-7 Green's function problem for Region 2.

Anticipating the symmetry of the Green's function, we interchange the primed and unprimed coordinates and substitute (5.163) to obtain

$$H_{y2}(x, z) = i\omega\epsilon_0 \int_0^b g_2(x, z, x', 0)E_{x2}(x', 0)dx' \tag{5.186}$$

We next consider the determination of the Green's functions. In Region 1, we wish to solve (5.166) with boundary conditions given by (5.170)–(5.172). As we found in Example 3.2, the spectral representation of $(-\partial^2/\partial x^2)$ with Neumann boundary conditions given in (5.171) results in the Fourier cosine series. We therefore expand the Green's function

$$g_1(x, z, x', z') = \sum_{n=0}^{\infty} \alpha_n(z, x', z')u_n(x) \tag{5.187}$$

where $u_n(x)$ is the orthonormal eigenfunction

$$u_n(x) = \sqrt{\frac{\epsilon_n}{a}} \cos \frac{n\pi x}{a} \tag{5.188}$$

and where

$$\alpha_n(z, x', z') = \int_0^a g_1(x, z, x', z')u_n(x)dx \tag{5.189}$$

Equation (5.189) defines the transformation

$$g_1 \Longrightarrow \alpha_n \tag{5.190}$$

Using the procedure in (5.37)–(5.41), we establish that

$$-\frac{\partial^2 g_1}{\partial x^2} \Longrightarrow \left(\frac{n\pi}{a}\right)^2 \alpha_n \tag{5.191}$$

$$\delta(x - x') \Longrightarrow u_n(x') \tag{5.192}$$

We use (5.190)–(5.192) to transform (5.166), with the result

$$-\left(\frac{d^2}{dz^2} + k_z^2\right) \alpha_n(z, x', z') = u_n(x')\delta(z - z') \tag{5.193}$$

where

$$k_z^2 = k^2 - \left(\frac{n\pi}{a}\right)^2 \tag{5.194}$$

We define

$$\beta_n(z, x', z') = \frac{\alpha_n(z, x', z')}{u_n(x')} \tag{5.195}$$

and obtain

$$-\left(\frac{d^2}{dz^2} + k_z^2\right) \beta_n(z, x', z') = \delta(z - z') \tag{5.196}$$

We assign the boundary and limiting conditions

$$\lim_{z \to -\infty} \beta_n(z, x', z') = 0 \tag{5.197}$$

$$\frac{d\beta_n(0, x', z')}{dz} = 0, \qquad x \in (0, a) \tag{5.198}$$

Satisfaction of these two conditions results in the satisfaction of the conditions in (5.170) and (5.172). The differential equation in (5.196) with conditions in (5.197) and (5.198) is solved by the standard Green's function methods developed in Chapter 2. The details are left for the problems. The result is

$$\beta_n(z, x', z') = \frac{1}{ik_z} \begin{cases} e^{ik_z z} \cos k_z z', & z < z' \\ e^{ik_z z'} \cos k_z z, & z > z' \end{cases} \tag{5.199}$$

where

$$\mathrm{Im}(k_z) < 0$$

Substituting (5.199) into (5.195), solving for α_n, and substituting the result into (5.187) gives

$$g_1(x, z, x', z') = \sum_{n=0}^{\infty} \frac{\epsilon_n}{ik_z a} \cos \frac{n\pi x}{a} \cos \frac{n\pi x'}{a} \begin{cases} e^{ik_z z} \cos k_z z', & z < z' \\ e^{ik_z z'} \cos k_z z, & z > z' \end{cases} \tag{5.200}$$

To obtain the Green's function g_2 for Region 2, we note that the geometry in Fig. 5-7 can be obtained from Fig. 5-6 by reflection through the $z = 0$ plane. We therefore can obtain g_2 from g_1 by replacing z by $-z$ and z' by $-z'$, with the result

$$g_2(x, z, x', z') = \sum_{n=0}^{\infty} \frac{\epsilon_n}{ik_z a} \cos \frac{n\pi x}{a} \cos \frac{n\pi x'}{a} \begin{cases} e^{-ik_z z} \cos k_z z', & z > z' \\ e^{-ik_z z'} \cos k_z z, & z < z' \end{cases} \tag{5.201}$$

We note that the anticipated symmetry of the Green's functions occurs for both g_1 and g_2. In the solution for the fields, we shall need the Green's functions with $z' = 0$. We obtain

$$g_1(x, z, x', 0) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{ik_z a} e^{ik_z z} \cos\frac{n\pi x}{a} \cos\frac{n\pi x'}{a} \tag{5.202}$$

$$g_2(x, z, x', 0) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{ik_z a} e^{-ik_z z} \cos\frac{n\pi x}{a} \cos\frac{n\pi x'}{a} \tag{5.203}$$

We also need the Green's function in Region 1 evaluated with $z' = -d$. Confining our interest to regions to the right of the source, we obtain

$$g_1(x, z, x', -d) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{ik_z a} \cos\frac{n\pi x}{a} \cos\frac{n\pi x'}{a} e^{-ik_z d} \cos k_z z, \quad -d < z < 0 \tag{5.204}$$

We note that in (5.202)–(5.204), care must be taken to select the proper cases for the Green's functions g_1 and g_2. In (5.202), $z < z'$; in (5.203), $z > z'$; and, in (5.204), $z > z' = -d$.

The magnetic current source in Region 1, given by (5.158), has been chosen to produce a *TEM* wave incident from left to right. We may show this by substituting (5.204) into the first term in (5.175) to give

$$H_{y1}(x, z) = -\frac{M_0}{\eta} e^{-ikd} \cos kz - i\omega\epsilon_0 \int_0^b g_1(x, z, x', 0) E_{x1}(x', 0) dx' \tag{5.205}$$

As a normalization, we choose

$$M_0 = -2e^{ikd} \tag{5.206}$$

and obtain

$$H_{y1}(x, z) = \frac{2\cos kz}{\eta} - i\omega\epsilon_0 \int_0^b g_1(x, z, x', 0) E_{x1}(x', 0) dx' \tag{5.207}$$

This normalization is used to produce a unit-strength incident electric field. Indeed, consider the limiting case where the size of the aperture shrinks to zero. We have

$$\lim_{b\to 0} H_{y1}(x, z) = \frac{1}{\eta}\left(e^{ikz} + e^{-ikz}\right)$$

Substituting into (5.160) gives

$$\lim_{b\to 0} E_{x1}(x, z) = e^{ikz} - e^{-ikz}$$

Therefore, in the limit, we produce a unit-strength electric field, incident from left to right and reflecting from a perfect conductor at $z = 0$.

As a final step in the problem formulation, we note in (5.186) and (5.207) that

$$E_{x1}(x', 0) = E_{x2}(x', 0), \qquad z' \in (0, b) \tag{5.208}$$

which is a statement that the tangential electric field is continuous in the aperture. We symbolize this aperture field by $E_A(x')$ and write (5.186) and (5.207) as

$$H_{y1}(x, z) = \frac{2\cos kz}{\eta} - i\omega\epsilon_0 \int_0^b g_1(x, z, x', 0)E_A(x')dx' \tag{5.209}$$

$$H_{y2}(x, z) = i\omega\epsilon_0 \int_0^b g_2(x, z, x', 0)E_A(x')dx' \tag{5.210}$$

Substitution of (5.202) and (5.203) gives

$$H_{y1}(x, z) = \frac{2\cos kz}{\eta} - \frac{k}{\eta}\sum_{n=0}^{\infty}\frac{\epsilon_n}{k_z a}e^{ik_z z}\cos\frac{n\pi x}{a}\int_0^b E_A(x')\cos\frac{n\pi x'}{a}dx' \tag{5.211}$$

$$H_{y2}(x, z) = \frac{k}{\eta}\sum_{n=0}^{\infty}\frac{\epsilon_n}{k_z a}e^{-ik_z z}\cos\frac{n\pi x}{a}\int_0^b E_A(x')\cos\frac{n\pi x'}{a}dx' \tag{5.212}$$

These two expressions give the magnetic fields everywhere in the two regions. We note, however, that the electric field E_A in the aperture is as yet unknown. We have, however, one additional boundary condition that we have not utilized, namely, the continuity of the tangential magnetic field in the aperture. We invoke this continuity by equating (5.211) and (5.212) in the aperture and obtain

$$1 = \sum_{n=0}^{\infty}\frac{\epsilon_n}{a}\frac{k}{k_z}\cos\frac{n\pi x}{a}\int_0^b E_A(x')\cos\frac{n\pi x'}{a}dx' \tag{5.213}$$

or

$$1 = \int_0^b E_A(x')\mathcal{G}(x, x')dx' \tag{5.214}$$

where

$$\mathcal{G}(x, x') = \sum_{n=0}^{\infty}\frac{\epsilon_n}{a}\frac{k}{k_z}\cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a} \tag{5.215}$$

Expression (5.214) is an integral equation whose solution yields the aperture field E_A. Once E_A is known, the result can be substituted into

(5.211) and (5.212) to yield the magnetic fields. The corresponding electric fields can be obtained by substitution of these results into (5.160), (5.161), (5.163), and (5.164).

Unfortunately, the integral equation in (5.214) cannot be solved analytically. A popular method for finding an approximate solution for the aperture field E_A is the Method of Moments (MOM), introduced in Section 1.8. Although the approximate solution to (5.214) is beyond the central theme of this book, a few comments are in order.

The kernel $\mathcal{G}(x, x')$ for the integral equation in (5.214) is logarithmically singular. Therefore, care must be taken in dealing with the limit as $x' \to x$. For a discussion of the issues involved, the reader is referred to [6]–[8]. The series contained in $\mathcal{G}(x, x')$ is slowly converging. For methods to speed the convergence, the reader is referred to [6]–[10]. Finally, the aperture field possesses an edge singularity [6],[11] in its behavior as $x \to b$. This singularity must be considered in evaluations involving MOM. For a discussion, the reader is referred to [6],[8],[12].

5.7 APERTURE DIFFRACTION

We next consider the classic problem of diffraction by an aperture in a perfectly conducting screen. A perfectly conducting screen (Fig. 5-8) divides unbounded empty space into two regions: Region 1, $y < 0$; Region 2, $y > 0$. An aperture interrupts the screen at $y = 0, x \in (-a/2, a/2)$. Electromagnetic fields are excited by a z-directed magnetic current source M_z in Region 1. It is assumed that the source and the geometry are z-independent, so that

$$\frac{\partial}{\partial z} = 0 \tag{5.216}$$

Expanding Maxwell's curl equations in Cartesian coordinates and invoking (5.216), we have

$$\frac{\partial H_z}{\partial y} = i\omega\epsilon_0 E_x \tag{5.217}$$

$$\frac{\partial H_z}{\partial x} = -i\omega\epsilon_0 E_y \tag{5.218}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega\epsilon_0 E_z \tag{5.219}$$

$$\frac{\partial E_z}{\partial y} = -i\omega\mu_0 H_x \tag{5.220}$$

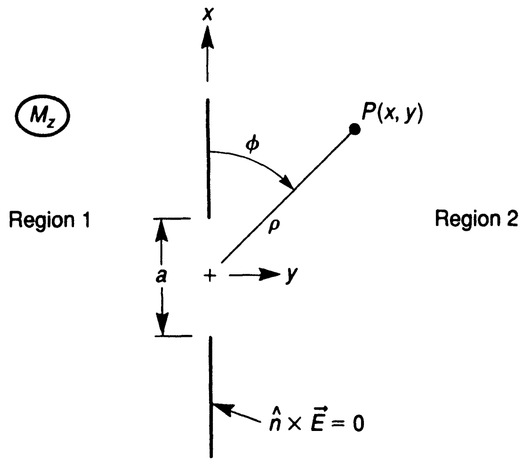

Fig. 5-8 Aperture in a perfectly conducting screen.

$$\frac{\partial E_z}{\partial x} = i\omega\mu_0 H_y \tag{5.221}$$

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = M_z + i\omega\mu_0 H_z \tag{5.222}$$

The set comprised of (5.217), (5.218), and (5.222) is excited by M_z and is decoupled from the unexcited set comprised of (5.219)–(5.221). We therefore have $H_x = H_y = E_z = 0$. Differentiating (5.217) with respect to y, (5.218) with respect to x, adding, and then substituting (5.222) gives the following set:

$$\left(\nabla_{xy}^2 + k^2\right) H_z = i\omega\epsilon_0 M_z \tag{5.223}$$

$$E_x = \frac{1}{i\omega\epsilon_0}\frac{\partial H_z}{\partial y} \tag{5.224}$$

$$E_y = -\frac{1}{i\omega\epsilon_0}\frac{\partial H_z}{\partial x} \tag{5.225}$$

where k is given by (5.108). We next specialize the above equations to Regions 1 and 2 as follows:

Region 1:

$$\left(\nabla_{xy}^2 + k^2\right) H_{z1} = i\omega\epsilon_0 M_z \tag{5.226}$$

$$E_{x1} = \frac{1}{i\omega\epsilon_0}\frac{\partial H_{z1}}{\partial y} \tag{5.227}$$

$$E_{y1} = -\frac{1}{i\omega\epsilon_0}\frac{\partial H_{z1}}{\partial x} \tag{5.228}$$

Region 2:

$$\left(\nabla_{xy}^2 + k^2\right) H_{z2} = 0 \tag{5.229}$$

$$E_{x2} = \frac{1}{i\omega\epsilon_0}\frac{\partial H_{z2}}{\partial y} \tag{5.230}$$

$$E_{y2} = -\frac{1}{i\omega\epsilon_0}\frac{\partial H_{z2}}{\partial x} \tag{5.231}$$

We first consider Region 1. Using the Green's function method and anticipating symmetry of the Green's function, we adapt (5.165) to the present case as follows:

$$\begin{aligned}\int_{V_1}\left[\left(-\nabla^2 - k^2\right) H_{z1}\right] g_1 dV = &\int_{V_1} H_{z1}\left(-\nabla^2 - k^2\right) g_1 dV \\ &+ \int_{S_1} (H_{z1}\nabla g_1 - g_1 \nabla H_{z1}) \cdot \hat{n} dS\end{aligned} \tag{5.232}$$

where g_1 is the Green's function in Region 1, governed by

$$-\left(\nabla_{xy}^2 + k^2\right) g_1 = \delta(x - x')\delta(y - y') \tag{5.233}$$

with boundary and limiting conditions to be determined. The volume V_1 is Region 1. The surface S_1 consists of the following parts:

1. The surface of the screen and aperture at $y = 0$.
2. The planar surface at $y \to -\infty$.
3. The planar surfaces at $x \to \pm\infty$, $y < 0$.
4. The planar surfaces at $z \to \pm\infty$, $y < 0$.

By the same reasoning as in the treatment of the parallel plate waveguide, the surface integrals at $z \to \pm\infty$ vanish. In addition, we have the following boundary and limiting conditions governing the H_{z1}-fields:

$$\lim_{y\to-\infty} H_{z1} = 0 \tag{5.234}$$

$$\lim_{x\to\pm\infty} H_{z1} = 0, \qquad y \in (-\infty, 0) \tag{5.235}$$

$$\frac{\partial H_{z1}(x,0)}{\partial y} = 0, \qquad x \notin (-a/2, a/2) \tag{5.236}$$

Inspection of (5.227) indicates that the condition in (5.236) is equivalent to the vanishing of the tangential electric field on the surface of the screen. We choose the following boundary and limiting conditions for the Green's function g_1:

$$\lim_{y\to-\infty} g_1(x, y, x', y') = 0 \tag{5.237}$$

$$\lim_{x\to\pm\infty} g_1(x, y, x', y') = 0, \qquad y \in (-\infty, 0) \tag{5.238}$$

$$\frac{\partial g_1(x, 0, x', y')}{\partial y} = 0 \tag{5.239}$$

We note that the Green's function boundary and limiting conditions are the same as those associated with the H_{z1}-field, with one exception. At $y = 0$, the boundary condition on H_{z1} is over the screen, whereas the boundary condition on g_1 is over the screen *and* the aperture. Therefore, the Green's function problem is for the half-space $y < 0$ with a perfectly conducting surface at $y = 0$. We substitute (5.234)–(5.239) into (5.232) and obtain

$$\int_{V_1} \left[\left(-\nabla^2 - k^2\right) H_{z1}\right] g_1 dV = \int_{V_1} H_{z1} \left(-\nabla^2 - k^2\right) g_1 dV - \int_{-\infty}^{\infty}\int_{-a/2}^{a/2} g_1 \frac{\partial H_{z1}}{\partial y} dxdz \tag{5.240}$$

Because of the invariance with z, the integrations with respect to z cancel in all terms. In addition, the del-operator reduces to ∇_{xy} and we have

$$\int_{-\infty}^{0}\int_{-\infty}^{\infty} \left[\left(-\nabla_{xy}^2 - k^2\right) dH_{z1}\right] g_1 dxdy = \int_{-\infty}^{0}\int_{-\infty}^{\infty} H_{z1} \left(-\nabla_{xy}^2 - k^2\right) g_1 dxdy - \int_{-a/2}^{a/2} g_1 \frac{\partial H_{z1}}{\partial y} dx \tag{5.241}$$

Substituting (5.226) and (5.233), doing the delta-function integrations, and rearranging, we obtain

$$H_{z1}(x', y') = -i\omega\epsilon_0 \int_A M_z(x, y) g_1(x, y, x', y') dxdy + \int_{-a/2}^{a/2} g_1(x, 0, x', y') \frac{\partial H_{z1}(x, 0)}{\partial y} dx$$

Anticipating the symmetry of the Green's function, we interchange the primed and unprimed coordinates and substitute (5.227) to obtain

$$H_{z1}(x,y) = -i\omega\epsilon_0 \int_A M_z(x',y')g_1(x,y,x',y')dx'dy' + i\omega\epsilon_0 \int_{-a/2}^{a/2} g_1(x,y,x',0)E_{x1}(x',0)dx' \tag{5.242}$$

We now consider Region 2. In a manner similar to Region 1, we obtain

$$\int_{V_2} \left[\left(-\nabla^2 - k^2\right) H_{z2}\right] g_2 dV = \int_{V_2} H_{z2}\left(-\nabla^2 - k^2\right) g_2 dV + \int_{S_2} (H_{z2}\nabla g_2 - g_2\nabla H_{z2}) \cdot \hat{n} dS \tag{5.243}$$

where g_2 is the Green's function in Region 2, governed by

$$-\left(\nabla_{xy}^2 + k^2\right) g_2 = \delta(x-x')\delta(y-y') \tag{5.244}$$

with boundary and limiting conditions to be determined. The volume V_2 is Region 2. The surface S_2 consists of the following parts:

1. The surface of the screen and aperture at $y = 0$.
2. The planar surface at $y \to \infty$.
3. The planar surfaces at $x \to \pm\infty$, $y > 0$.
4. The planar surfaces at $z \to \pm\infty$, $y > 0$.

Again, the surface integrals at $z \to \pm\infty$ vanish. In addition, we have the following boundary and limiting conditions governing the H_{z2}-fields:

$$\lim_{y\to\infty} H_{z2} = 0 \tag{5.245}$$

$$\lim_{x\to\pm\infty} H_{z2} = 0, \qquad y \in (0,\infty) \tag{5.246}$$

$$\frac{\partial H_{z2}(x,0)}{\partial y} = 0, \qquad x \notin (-a/2, a/2) \tag{5.247}$$

We choose the following boundary and limiting conditions for the Green's function g_2:

$$\lim_{y\to\infty} g_2(x,y,x',y') = 0 \tag{5.248}$$

$$\lim_{x\to\pm\infty} g_2(x,y,x',y') = 0, \qquad y \in (0,\infty) \tag{5.249}$$

$$\frac{\partial g_2(x,0,x',y')}{\partial y} = 0 \tag{5.250}$$

We therefore have a Green's function problem for a half-space $y > 0$ with a perfectly conducting surface at $y = 0$. We substitute (5.245)–(5.250) into (5.243) and obtain

$$\int_{V_2} \left[\left(-\nabla^2 - k^2 \right) H_{z2} \right] g_2 dV = \int_{V_2} H_{z2} \left(-\nabla^2 - k^2 \right) g_2 dV + \int_{-\infty}^{\infty} \int_{-a/2}^{a/2} \left(g_2 \frac{\partial H_{z2}}{\partial y} \right) \bigg|_{z=0} dx dz \tag{5.251}$$

Again, the integrations with respect to z cancel in all terms and the del-operator reduces to ∇_{xy}. We have

$$\int_0^{\infty} \int_{-\infty}^{\infty} \left[\left(-\nabla_{xz}^2 - k^2 \right) H_{z2} \right] g_2 dx dy = \int_0^{\infty} \int_{-\infty}^{\infty} H_{z2} \left(-\nabla_{xz}^2 - k^2 \right) g_2 dx dy + \int_{-a/2}^{a/2} \left(g_2 \frac{\partial H_{z2}}{\partial y} \right) \bigg|_{z=0} dx \tag{5.252}$$

Substituting (5.229) and (5.244), doing the delta-function integrations, and rearranging, we obtain

$$H_{z2}(x', y') = -\int_{-a/2}^{a/2} g_2(x, 0, x', y') \frac{\partial H_{z2}(x, 0)}{\partial y} dx$$

Anticipating the symmetry of the Green's function, we interchange the primed and unprimed coordinates and substitute (5.230) to obtain

$$H_{z2}(x, y) = -i\omega\epsilon_0 \int_{-a/2}^{a/2} g_2(x, y, x', 0) E_{x2}(x', 0) dx' \tag{5.253}$$

At this point in the development, we have two remaining tasks, namely, the specification of boundary conditions at points in the aperture and the selection of a specific magnetic current source. We first require that the tangential electric field in the aperture be continuous. We symbolize the aperture electric field by $E_A(x)$ and write

$$E_{x1}(x, 0) = E_{x2}(x, 0) = E_A(x), \qquad x \in (-a/2, a/2) \tag{5.254}$$

We next specialize the source M_z to be a line source located at $x' = \xi$, $y' = \eta$, viz.

$$M_z(x', y') = M_0 \delta(x' - \xi) \delta(y' - \eta) \tag{5.255}$$

We substitute (5.254) into both (5.242) and (5.253). Also, we substitute (5.255) into (5.242) and perform the indicated delta-function integrations to give

$$H_{z1}(x, y) = -i\omega\epsilon_0 M_0 g_1(x, y, \xi, \eta) + i\omega\epsilon_0 \int_{-a/2}^{a/2} g_1(x, y, x', 0) E_A(x') dx' \tag{5.256}$$

$$H_{z2}(x, y) = -i\omega\epsilon_0 \int_{-a/2}^{a/2} g_2(x, y, x', 0) E_A(x') dx' \tag{5.257}$$

Expressions (5.256) and (5.257) give the magnetic fields everywhere, provided that we know the Green's functions g_1 and g_2 and provided we can find the aperture electric field E_A. We shall derive the Green's functions subsequently. The aperture field E_A is obtained by requiring the tangential magnetic field in the aperture to be continuous, viz.

$$H_{z1}(x, 0) = H_{z2}(x, 0), \qquad x \in (-a/2, a/2) \tag{5.258}$$

Substituting (5.256) and (5.257) into (5.258) yields the integral equation

$$M_0 g_1(x, 0, \xi, \eta) = \int_{-a/2}^{a/2} \left[g_1(x, 0, x', 0) + g_2(x, 0, x', 0)\right] E_A(x') dx' \tag{5.259}$$

Once the Green's functions have been determined, an approximate solution to the integral equation (using Method of Moments, for example) yields an approximation to the aperture field E_A. The aperture field can then be substituted into (5.256) and (5.257) to give the magnetic fields everywhere. Once the magnetic fields are known, the electric fields can be obtained by differentiation in (5.227), (5.228), (5.230), and (5.231). We next consider the Green's functions g_1 and g_2.

We may determine g_2 from (5.244) and (5.248)–(5.250), which we reproduce here for convenience, viz.

$$\left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - k^2\right) g_2 = \delta(x - x')\delta(y - y') \tag{5.260}$$

$$\lim_{x \to \pm\infty} g_2(x, y, x', y') = 0, \qquad y \in (0, \infty) \tag{5.261}$$

$$\frac{\partial g_2(x, 0, x', y')}{\partial y} = 0 \tag{5.262}$$

$$\lim_{y \to \infty} g_2(x, y, x', y') = 0 \tag{5.263}$$

The spectral representation of $(-\partial^2/\partial x^2)$ with limiting conditions in (5.261) leads to the Fourier transform, as given in Example 3.4. In this case, we have

$$g_2(x, y, x', y') \Longleftrightarrow G_2(k_x, y, x', y') \tag{5.264}$$

$$-\frac{\partial^2 g_2}{\partial x^2} \Longleftrightarrow k_x^2 G_2 \tag{5.265}$$

$$\delta(x - x') \Longleftrightarrow e^{-ik_x x'} \tag{5.266}$$

Applying these relationships to (5.260) gives

$$\left(-\frac{d^2}{dy^2} - k_y^2\right) \hat{G}_2 = \delta(y - y') \tag{5.267}$$

where

$$k_y = \sqrt{k^2 - k_x^2} \tag{5.268}$$

$$G_2 = e^{-ik_x x'} \hat{G}_2 \tag{5.269}$$

The boundary and limiting conditions associated with (5.267) are as follows:

$$\frac{d\hat{G}_2(k_x, 0, x', y')}{dy} = 0 \tag{5.270}$$

$$\lim_{y \to \infty} \hat{G}_2(x, y, x', y') = 0 \tag{5.271}$$

Invoking these conditions is consistent with the conditions on g_2. The solution to (5.267) with the above associated conditions can be inferred from the Green's function problem discussed previously in (5.196)–(5.199). If in (5.199) we let $z \to -y$, $z' \to -y'$, and $k_z \to k_y$, we obtain

$$\hat{G}_2 = \frac{1}{ik_y} \begin{cases} e^{-ik_y y'} \cos k_y y, & y < y' \\ e^{-ik_y y} \cos k_y y', & y > y' \end{cases}$$

Expanding the cosine terms into exponentials gives the following useful alternate form:

$$\hat{G}_2 = \frac{1}{2ik_y} \left[e^{-ik_y |y - y'|} + e^{-ik_y (y + y')} \right] \tag{5.272}$$

Substituting this result into (5.269) and then taking the inverse Fourier transform, we have

$$g_2 = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{e^{-ik_y|y-y'|} + e^{-ik_y(y+y')}}{k_y} e^{ik_x(x-x')} dk_x \tag{5.273}$$

From (4.156), we have the following identity:

$$H_0^{(2)}\left(k\sqrt{x^2+y^2}\right) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-ik_y|y|} e^{ik_x x}}{k_y} dk_x \tag{5.274}$$

Therefore,

$$g_2 = \frac{1}{4i} \left\{ H_0^{(2)}\left[k\sqrt{(x-x')^2 + (y-y')^2}\right] + H_0^{(2)}\left[k\sqrt{(x-x')^2 + (y+y')^2}\right] \right\} \tag{5.275}$$

Referring to the development in Section 4.6, we recognize (5.275) as a description of the radiation from a line source located at $x = x'$, $y = y'$, plus a line source at the image location $x = x'$, $y = -y'$, with respect to the ground plane.

Consider g_1. We may obtain the solution for g_1 directly from the above solution for g_2 by replacing y by $-y$ and y' by $-y'$. Such replacement does not change the solution, and we therefore have

$$g_2 = g_1 \tag{5.276}$$

The above determination of the Green's functions completes the formulation of the problem.

In many applications, the source M_z is located at a distance far enough from the aperture so that a plane wave approximation can be invoked. To accomplish this, we consider the Green's function in the first term in (5.256), viz.

$$g_1(x, y, \xi, \eta) = \frac{1}{4i} \left\{ H_0^{(2)}\left[k\sqrt{(x-\xi)^2 + (y-\eta)^2}\right] + H_0^{(2)}\left[k\sqrt{(x-\xi)^2 + (y+\eta)^2}\right] \right\} \tag{5.277}$$

Rewriting this expression in cylindrical coordinates, we have

$$g_1(\rho, \phi, \rho', \phi') = \frac{1}{4i} \left\{ H_0^{(2)}\left[k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi')}\right] + H_0^{(2)}\left[k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi+\phi')}\right] \right\} \tag{5.278}$$

where (ρ', ϕ') marks the position of the line source (Fig. 5-9). We may locate the line source at a distance remote from the aperture by letting ρ' become very large, in which case

$$\left[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi \mp \phi')\right]^{1/2} \longrightarrow \rho' - \rho \cos(\phi \mp \phi') \tag{5.279}$$

Furthermore, using the large argument approximation for the Hankel function, given in Example 2.21, we have

$$H_0^{(2)} \left\{k\left[\rho' - \rho \cos(\phi \mp \phi')\right]\right\} \sim \sqrt{\frac{2i}{\pi k\rho'}} e^{-ik\rho'} e^{ik\rho \cos(\phi \mp \phi')} \tag{5.280}$$

Using (5.280) in (5.278), we obtain

$$\begin{aligned} g_1(\rho, \phi, \rho', \phi') &\sim \frac{1}{4i}\sqrt{\frac{2i}{\pi k\rho'}} e^{-ik\rho'} \left[e^{ik\rho \cos(\phi - \phi')} + e^{ik\rho \cos(\phi + \phi')}\right] \\ &= \frac{1}{4i}\sqrt{\frac{2i}{\pi k\rho'}} e^{-ik\rho'} 2e^{ikx \cos\phi'} \cos(ky \sin\phi') \end{aligned} \tag{5.281}$$

Let

$$M_0 = \left[\frac{-\omega\epsilon_0}{4}\sqrt{\frac{2i}{\pi k\rho'}} e^{-ik\rho'}\right]^{-1} \tag{5.282}$$

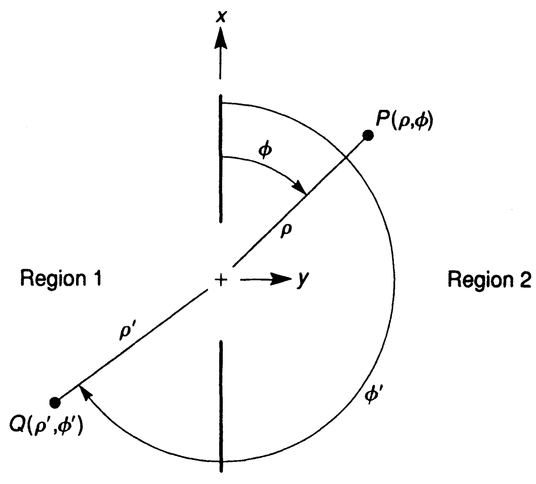

Fig. 5-9 Aperture in a perfectly conducting screen, cylindrical coordinates.

We substitute (5.275), (5.276), (5.281), and (5.282) into (5.256) and (5.257) and obtain

$$H_{z1}(x,y) = 2e^{ikx\cos\phi'}\cos(ky\sin\phi') + \frac{k}{2\eta}\int_{-a/2}^{a/2} H_0^{(2)}\left[k\sqrt{(x-x')^2+y^2}\right] E_A(x')dx' \tag{5.283}$$

$$H_{z2}(x,y) = -\frac{k}{2\eta}\int_{-a/2}^{a/2} H_0^{(2)}\left[k\sqrt{(x-x')^2+y^2}\right] E_A(x')dx' \tag{5.284}$$

where we have used

$$\omega\epsilon_0 = \frac{k}{\eta}$$

We note that in the limit as the aperture length approaches zero, we have

$$\lim_{a\to 0} H_{z1}(x,y) = 2e^{ikx\cos\phi'}\cos(ky\sin\phi') = e^{ikx\cos\phi'}\left(e^{iky\sin\phi'} + e^{-iky\sin\phi'}\right) \tag{5.285}$$

which represents a unit magnitude plane wave approaching the screen at angle ϕ' and reflecting according to Snell's law of reflection. Continuity of the tangential H-field in the aperture gives the integral equation

$$-\frac{2\eta}{k}e^{ikx\cos\phi'} = \int_{-a/2}^{a/2} H_0^{(2)}(k|x-x'|)E_A(x')dx' \tag{5.286}$$

which completes the problem formulation for the case of plane wave incidence.

Again, the integral equation in (5.286) cannot be inverted analytically. The aperture field E_A therefore must be determined approximately using numerical methods. We note, however, that for the aperture size a small enough, the Hankel function $H_0^{(2)}(k|x-x'|)$ can be approximated by $\ln|x-x'|$, in which case an analytical solution is possible in terms of Chebyshev polynomials. The resulting integral equation is considered in Problem 1.23. The details are given in [13],[14].

5.8 SCATTERING BY A PERFECTLY CONDUCTING CYLINDER

Consider an electric current J_z that excites a surface current on a perfectly conducting cylinder of uniform cross section (Fig. 5-10). We assume that the cylinder geometry and the source are independent of z, so that

$$\frac{\partial}{\partial z} = 0 \tag{5.287}$$

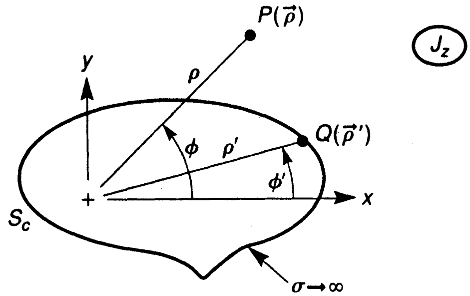

Fig. 5-10 Electric current J_z exciting a perfectly conducting cylinder.

We require the fields at a point $P(\rho)$. We begin with Maxwell's curl equations, viz.

$$\nabla \times \mathbf{H} = \hat{z} J_z + i\omega\epsilon_0 \mathbf{E} \tag{5.288}$$

$$\nabla \times \mathbf{E} = -i\omega\mu_0 \mathbf{H} \tag{5.289}$$

The curl operator in (5.288) and (5.289) is composed of components transverse to the z-direction, plus a z-component, viz.

$$\nabla = \nabla_t + \hat{z}\frac{\partial}{\partial z}$$

In this case, because of (5.287), we have

$$\nabla = \nabla_t \tag{5.290}$$

We divide the electric and magnetic fields into transverse and z-components, viz.

$$\mathbf{H} = \mathbf{H}_t + \hat{z} H_z \tag{5.291}$$

$$\mathbf{E} = \mathbf{E}_t + \hat{z} E_z \tag{5.292}$$

Substituting (5.290)–(5.292) into (5.288) and (5.289), we obtain

$$\nabla_t \times \mathbf{H}_t + \nabla_t \times \hat{z} H_z = \hat{z} J_z + i\omega\epsilon_0 \mathbf{E} \tag{5.293}$$

$$\nabla_t \times \mathbf{E}_t + \nabla_t \times \hat{z} E_z = -i\omega\mu_0 \mathbf{H} \tag{5.294}$$

Equating transverse components and z-components on either side of (5.293) and (5.294), we produce the following two sets:

Set 1: TM_z

$$\nabla_t \times \mathbf{H}_t = \hat{z}(J_z + i\omega\epsilon_0 E_z) \tag{5.295}$$

$$\nabla_t \times \hat{z} E_z = -i\omega\mu_0 \mathbf{H}_t \tag{5.296}$$

Set 2: TE_z

$$\nabla_t \times \mathbf{E}_t = -\hat{z} i\omega\mu_0 H_z \tag{5.297}$$

$$\nabla_t \times \hat{z} H_z = i\omega\epsilon_0 \mathbf{E}_t \tag{5.298}$$

Set 1 is excited by the source J_z, while Set 2 is unexcited. We therefore have $E_t = H_z = 0$. We take the curl of (5.296) and substitute (5.295) to obtain

$$\begin{aligned} -i\omega\mu_0 \hat{z}\left(J_z + i\omega\epsilon_0 E_z\right) &= \nabla_t \times \nabla_t \times \hat{z} E_z \\ &= \left[\nabla_t \left(\nabla_t \cdot \hat{z} E_z\right) - \nabla_t^2 (\hat{z} E_z)\right] \end{aligned} \tag{5.299}$$

where we have used a well-known vector identity to expand the double-curl. But,

$$\nabla_t \cdot \hat{z} E_z = 0 \tag{5.300}$$

and thus Set 1 becomes

$$\nabla_t^2 E_z + k^2 E_z = i\omega\mu_0 J_z \tag{5.301}$$

$$\mathbf{H}_t = -\frac{1}{i\omega\mu_0} \nabla_t \times \hat{z} E_z = \frac{1}{i\omega\mu_0} \hat{z} \times \nabla_t E_z \tag{5.302}$$

where k is defined in (5.108). The procedure is now to solve the partial differential equation in (5.301) to yield E_z. The result can then be substituted into (5.302) to produce the magnetic fields $\mathbf{H}_t$.

Anticipating the symmetry of the Green's function, we adapt (5.122) to the present case and obtain

$$\int_V g\left(\nabla^2 + k^2\right) E_z dV = \int_V E_z \left(\nabla^2 + k^2\right) g\, dV + \int_S \left(g \nabla E_z - E_z \nabla g\right) \cdot \hat{n}\, dS \tag{5.303}$$

The volume V consists of all space exterior to the cylinder. The surface S is the surface of the cylinder S_c plus the surface at infinity. By the same reasoning as in the case of the parallel plate waveguide, the surface integrals at $z \to \pm\infty$ vanish. The Green's function g is governed by

$$-\left(\nabla_t^2 + k^2\right) g = \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi') = \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \tag{5.304}$$

with boundary and/or limiting conditions to be determined. The conditions on E_z are as follows:

$$E_z |_{S_c} = 0 \tag{5.305}$$

$$\lim_{\rho \to \infty} E_z = 0 \tag{5.306}$$

We choose the following condition for the Green's function g:

$$\lim_{\rho \to \infty} g = 0 \tag{5.307}$$

Therefore, the Green's function problem is for two-dimensional free space. We substitute (5.305)–(5.307) into (5.303) and obtain

$$\int_V g\left(\nabla^2 + k^2\right) E_z dV = \int_V E_z \left(\nabla^2 + k^2\right) g dV + \int_{S_c} g \nabla E_z \cdot \hat{n} dS$$

We note that we did *not* require

$$g|_{S_c} = 0$$

Although such a requirement would eliminate the surface integral, we would be unable to find an analytical solution for the Green's function, except in the special case where the cross section is circular. (We shall consider the circular case subsequently.) Because of the invariance with z, the integrations with respect to z cancel in all terms. In addition, from (5.290), the del-operator reduces to ∇_t and we have

$$\int_A g\left(\nabla_t^2 + k^2\right) E_z dA = \int_A E_z \left(\nabla_t^2 + k^2\right) g dA + \int_{s_c} g \nabla_t E_z \cdot \hat{n} ds \tag{5.308}$$

where s_c is the arc-length integration around the cross section of the cylinder and A is the planar area external to s_c. We shall consider the ds integration in some detail subsequently. Substitution of (5.301) and (5.304) into (5.308) gives, after some rearrangement,

$$E_z(\rho') = -i\omega\mu_0 \int_A g(\rho, \rho') J_z(\rho) dA + \int_{s_c} g(\rho, \rho') \nabla_t E_z(\rho) \cdot n ds \tag{5.309}$$

However, from (5.302), we have

$$\hat{z} \times \nabla_t E_z = i\omega\mu_0 \mathbf{H}_t$$

so that

$$\hat{z} \times (\hat{z} \times \nabla_t E_z) = i\omega\mu_0 \hat{z} \times \mathbf{H}_t$$

But,

$$\hat{z} \times (\hat{z} \times \nabla_t E_z) = -\nabla_t E_z$$

where we have used the vector triple product identity and the fact that

$$\hat{z} \cdot \nabla_t E_z = 0$$

Taking the inner product with the normal vector $\hat{n}$ gives

$$\hat{n} \cdot \nabla_t E_z = i\omega\mu_0 \hat{n} \cdot \mathbf{H}_t \times \hat{z} = i\omega\mu_0 \hat{z} \cdot (\hat{n} \times \mathbf{H}_t) = i\omega\mu_0 J_{sz} \tag{5.310}$$

where J_{sz} is the equivalent surface current in the z-direction in amps/m. Substituting this result into (5.309) and interchanging the primed and unprimed coordinates, we obtain

$$E_z(\rho) = -i\omega\mu_0 \int_A g(\rho, \rho') J_z(\rho') dA' + i\omega\mu_0 \int_{s_c} g(\rho, \rho') J_{sz}(\rho') ds' \tag{5.311}$$

The Green's function problem given in (5.304) and (5.307) has been previously solved in (4.116), followed by the coordinate transformation indicated in (4.205). We have, including 2π from (4.10),

$$g(\rho, \rho') = \frac{1}{4i} H_0^{(2)} \left(k|\rho - \rho'|\right) \tag{5.312}$$

We shall specialize the source J_z to be a line source of strength I_0 amps, located at $\rho' = \rho_0$, viz.

$$J_z = I_0 \delta(\rho' - \rho_0) \tag{5.313}$$

Substituting (5.312) and (5.313) into (5.311), we have

$$E_z(\rho) = -\frac{i\omega\mu_0 I_0}{4i} H_0^{(2)}(k|\rho - \rho_0|) + \frac{i\omega\mu_0}{4i} \int_{s_c} H_0^{(2)} \left(k|\rho - \rho'|\right) J_{sz}(\rho') ds' \tag{5.314}$$

This equation gives the electric field E_z everywhere exterior to the cylinder, provided that we can determine the surface current J_{sz} on the surface of the cylinder. We accomplish this by forming an integral equation. We let ρ approach a general point on the surface of the cylinder $\rho \in s_c$. Since $E_z = 0$ on the cylinder surface, we have

$$I_0 H_0^{(2)} \left(k|\rho - \rho_0|\right) = \int_{s_c} H_0^{(2)} \left(k|\rho - \rho'|\right) J_{sz}(\rho') ds', \qquad \rho \in s_c \tag{5.315}$$

Equations (5.314) and (5.315) complete the formulation of the problem.

Considerable care must be taken in the evaluation of the arc-length integral in (5.314) and (5.315). In Cartesian coordinates, the differential element can be represented by

$$ds' = \left(dx'^2 + dy'^2\right)^{1/2} \tag{5.316}$$

We shall parametrize with respect to the polar angle ϕ' (Fig. 5-10), as follows:

$$ds' = \left[\left(\frac{dx'}{d\phi'}\right)^2 + \left(\frac{dy'}{d\phi'}\right)^2\right]^{1/2} d\phi' \tag{5.317}$$

We locate the origin of the coordinate system (Fig. 5-10) internal to s_c. Substituting (5.317) into (5.314) and (5.315), we produce

$$\begin{aligned} E_z(\rho) = &-\frac{\omega\mu_0 I_0}{4} H_0^{(2)}\left(k|\rho - \rho_0|\right) \\ &+ \frac{\omega\mu_0}{4}\int_0^{2\pi} H_0^{(2)}(k|\rho - \rho'|)\, J_{sz}(\rho')\left[\left(\frac{dx'}{d\phi'}\right)^2 + \left(\frac{dy'}{d\phi'}\right)^2\right]^{1/2} d\phi' \end{aligned} \tag{5.318}$$

and for $\rho \in s_c$,

$$\begin{aligned} I_0 H_0^{(2)}\left(k|\rho - \rho_0|\right) = &\int_0^{2\pi} H_0^{(2)}\left(k|\rho - \rho'|\right) J_{sz}(\rho') \\ &\cdot\left[\left(\frac{dx'}{d\phi'}\right)^2 + \left(\frac{dy'}{d\phi'}\right)^2\right]^{1/2} d\phi' \end{aligned} \tag{5.319}$$

As in the previous problems in this chapter, we must solve the integral equation to determine the unknown quantity under the integral, in this case, J_{sz}. In general, numerical methods must be employed to obtain an approximation to the solution. In the case where the cylindrical cross section is circular, however, we may invert the integral equation in (5.319) analytically. Indeed, consider a perfectly conducting circular cylinder of radius a (Fig. 5-11). In this case,

$$\begin{aligned} x' &= \rho' \cos\phi' \\ y' &= \rho' \sin\phi' \\ \frac{dx'}{d\phi'} &= -\rho' \sin\phi' \\ \frac{dy'}{d\phi'} &= \rho' \cos\phi' \end{aligned}$$

Substitution in (5.317) produces the usual cylindrical representation of the arc-length integration

$$ds' = \rho' d\phi' \tag{5.320}$$

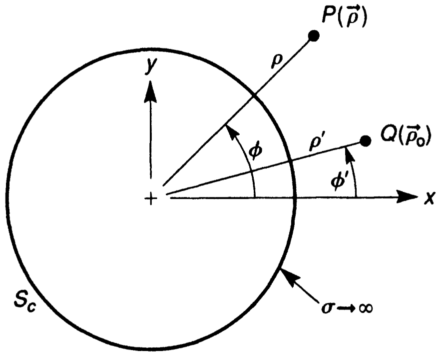

Fig. 5-11 Electric current J_z exciting a perfectly conducting circular cylinder.

We shall employ the addition theorem for the Hankel function from (4.207), viz.

$$H_0^{(2)}(k|\rho - \rho'|) = \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \begin{cases} H_n^{(2)}(k\rho')J_n(k\rho), & \rho < \rho' \\ H_n^{(2)}(k\rho)J_n(k\rho'), & \rho > \rho' \end{cases} \tag{5.321}$$

Substituting (5.320) and (5.321) into (5.319) and rearranging, we have

$$\sum_{n=-\infty}^{\infty} \left[I_0 e^{-in\phi_0} H_n^{(2)}(k\rho_0) J_n(ka) \right] e^{in\phi}$$
$$= \sum_{n=-\infty}^{\infty} \left[\int_0^{2\pi} J_{sz}(a,\phi') e^{-in\phi'} a\,d\phi' H_n^{(2)}(ka) J_n(ka) \right] e^{in\phi} \tag{5.322}$$

To obtain (5.322), since $\rho \in s_c$, we have chosen the $\rho < \rho'$ case in (5.321). We recognize each side of (5.322) as a complex Fourier series on $(0, 2\pi)$. We equate coefficients and rearrange to give

$$\int_0^{2\pi} J_{sz}(a,\phi') e^{-in\phi'} d\phi' = \frac{I_0 e^{-in\phi_0} H_n^{(2)}(k\rho_0)}{aH_n^{(2)}(ka)} \tag{5.323}$$

We shall use this result to find the electric field E_z by noting that, in (5.318),

$$\int_0^{2\pi} H_0^{(2)}(k|\rho - \rho'|)\, J_{sz}(\rho') a\,d\phi'$$
$$= \sum_{n=-\infty}^{\infty} e^{in\phi} H_n^{(2)}(k\rho) J_n(ka) \int_0^{2\pi} J_{sz}(a,\phi') e^{-in\phi'} a\,d\phi'$$
$$= I_0 \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi_0)} \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) H_n^{(2)}(k\rho_0) \tag{5.324}$$

Substituting this result and (5.320) into (5.314), we obtain the following expansion for the electric field E_z:

$$E_z(\rho) = -\frac{\omega\mu_0 I_0}{4}\left\{H_0^{(2)}\left(k|\rho - \rho_0|\right) - \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi_0)} \cdot \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) H_n^{(2)}(k\rho_0)\right\} \tag{5.325}$$

which is the classical ϕ-directed eigenfunction expansion for the electric field [15].

In the above example of the circular cylinder, we were able to invert the integral equation analytically. This event occurred because the surface S_c was a coordinate surface, in this case, $\rho = a$. In cases where the cylinder does not conform to a complete coordinate surface, the integral equation in (5.315) must be inverted numerically. We include a specific case, the rectangular cylinder, in the problems.

5.9 PERFECTLY CONDUCTING CIRCULAR CYLINDER

In the previous section, we derived the fields associated with scattering from a perfectly conducting circular cylinder by beginning with the conducting cylinder of arbitrary cross section. We obtained an integral equation in (5.319). For the case of circular cross section, we were able to invert the integral equation and obtain an expression for the electric field E_z in (5.325). It is, however, possible to proceed more directly. In this section, we derive the fields scattered from a perfectly conducting cylinder of circular cross section when the excitation is an electric current line source. We are able to verify the result obtained in (5.325). Next, we obtain an alternative representation, useful in describing scattering in the form of *creeping waves.*

We again consider the geometry in Fig. 5-11. The source is given explicitly in (4.163) and the differential equation describing the E_z-field in (4.180) and (4.181), which we repeat here for convenience, viz.

$$\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right)\right] + \frac{1}{\rho^2}\frac{\partial^2 g}{\partial\phi^2} + k^2 g = -\frac{\delta(\rho-\rho')\delta(\phi-\phi')}{\rho} \tag{5.326}$$

$$g = -\frac{E_z}{i\omega\mu_0 I_0} \tag{5.327}$$

From the results in Problem 3.2, we may expand the Green's function g in terms of the spectral representation with respect to ϕ, viz.

$$g(\rho, \phi, \rho', \phi') = \sum_{n=-\infty}^{\infty} a_n(\rho, \rho', \phi')\sqrt{\frac{1}{2\pi}}e^{in\phi} \tag{5.328}$$

We write this transformation

$$g \Longleftrightarrow a_n \tag{5.329}$$

and easily find that

$$-\frac{\partial^2 g}{\partial \phi^2} \Longleftrightarrow n^2 a_n \tag{5.330}$$

$$\delta(\phi - \phi') \Longleftrightarrow \sqrt{\frac{1}{2\pi}}e^{-in\phi'} \tag{5.331}$$

Applying (5.329)–(5.331) to (5.326), we obtain

$$\frac{1}{\rho}\left[\frac{d}{d\rho}\left(\rho\frac{db_n}{d\rho}\right)\right] + k^2 b_n - \frac{n^2}{\rho^2}b_n = -\frac{\delta(\rho - \rho')}{\rho} \tag{5.332}$$

where

$$a_n = \sqrt{\frac{1}{2\pi}}e^{-in\phi'}b_n \tag{5.333}$$

So far, the development is identical to that in (4.190)–(4.198), except that the boundary and limiting conditions are now

$$b_n\Big|_{\rho=a} = 0 \tag{5.334}$$

$$\lim_{\rho\to\infty} b_n = 0 \tag{5.335}$$

We write the solution for b_n as a linear combination of Bessel and Hankel functions, as follows:

$$b_n = \begin{cases} AJ_n(k\rho) + CH_n^{(2)}(k\rho), & \rho < \rho' \\ BH_n^{(2)}(k\rho) + DH_n^{(1)}(k\rho), & \rho > \rho' \end{cases} \tag{5.336}$$

The limiting condition in (5.335) results in $D = 0$. At $\rho = a$, we have

$$AJ_n(ka) + CH_n^{(2)}(ka) = 0$$

Solving for C and substituting into (5.336), we have

$$b_n = \begin{cases} A\left[J_n(k\rho) - c_n H_n^{(2)}(k\rho)\right], & \rho < \rho' \\ B H_n^{(2)}(k\rho), & \rho > \rho' \end{cases} \tag{5.337}$$

where

$$c_n = \frac{J_n(ka)}{H_n^{(2)}(ka)} \tag{5.338}$$

Invoking the continuity and jump conditions at $\rho = \rho'$ allows us to evaluate the coefficients A and B, as follows:

$$A = \frac{\pi}{2i} H_n^{(2)}(k\rho') \tag{5.339}$$

$$B = \frac{\pi}{2i}\left[J_n(k\rho') - c_n H_n^{(2)}(k\rho')\right] \tag{5.340}$$

where we have used the Bessel function identity

$$J_n H_n^{(2)\prime} - J_n{}' H_n^{(2)} = \frac{2}{i\pi x} \tag{5.341}$$

where the prime indicates differentiation with respect to x. Therefore,

$$b_n = \frac{\pi}{2i}\begin{cases} H_n^{(2)}(k\rho')\left[J_n(k\rho) - c_n H_n^{(2)}(k\rho)\right], & \rho < \rho' \\ H_n^{(2)}(k\rho)\left[J_n(k\rho') - c_n H_n^{(2)}(k\rho')\right], & \rho > \rho' \end{cases} \tag{5.342}$$

Substitution of (5.342) into (5.333) and the result into (5.328) gives

$$g = \frac{1}{4i}\sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')}\left[-c_n H_n^{(2)}(k\rho)H_n^{(2)}(k\rho') + \begin{cases} H_n^{(2)}(k\rho')J_n(k\rho), & \rho < \rho' \\ H_n^{(2)}(k\rho)J_n(k\rho'), & \rho > \rho' \end{cases}\right] \tag{5.343}$$

Use of (5.327) and the addition theorem given in (4.207) again produces the result in (5.325), which we display here for reference, viz.

$$E_z(\rho) = -\frac{\omega\mu_0 I_0}{4}\left\{H_0^{(2)}\left(k|\rho - \rho_0|\right) - \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi_0)} \cdot \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho)H_n^{(2)}(k\rho_0)\right\} \tag{5.344}$$

It is instructive to consider the important special case of a plane wave incident on the cylinder. In (5.344), the first term in the brackets is the incident field, given by

$$E_z^{inc} = -\frac{\omega_0 \mu_0 I_0}{4} H_0^{(2)}(k|\rho - \rho'|) \tag{5.345}$$

We expand $|\rho - \rho'|$ in cylindrical coordinates and obtain

$$|\rho - \rho'| = \left[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')\right]^{1/2} \tag{5.346}$$

The plane wave case is produced by allowing the line source to be very far removed from the cylinder. Mathematically, $\rho' >> \rho$ and

$$\begin{aligned}|\rho - \rho'| &\cong \rho' \left[1 - 2\left(\frac{\rho}{\rho'}\right) \cos(\phi - \phi')\right]^{1/2} \\ &\cong \rho' - \rho \cos(\phi - \phi')\end{aligned} \tag{5.347}$$

where we have discarded terms in ρ/ρ' higher than first order, and where we have used the first two terms in the Taylor series expansion for $\sqrt{(1+x)}$. Substituting (5.347) into (5.345), and using the large argument approximation for the Hankel function given in Example 2.21, we obtain

$$E_z^{inc} = -\frac{i\omega\mu_0 I_0}{4i} \sqrt{\frac{2i}{\pi k \rho'}} e^{-ik[\rho' - \rho\cos(\phi - \phi')]} \tag{5.348}$$

We let the incident wave arrive from left to right along the x-axis (Fig. 5-11), so that $\phi' = \pi$ and

$$E_z^{inc} = -\frac{i\omega\mu_0 I_0}{4i} \sqrt{\frac{2i}{\pi k \rho'}} e^{-ik\rho'} e^{-ik\rho\cos\phi} \tag{5.349}$$

To produce a unit magnitude plane wave from left to right, we adjust the intensity I_0 as follows:

$$I_0 = \left[-\frac{i\omega\mu_0 I_0}{4i} \sqrt{\frac{2i}{\pi k \rho'}} e^{-ik\rho'}\right]^{-1} \tag{5.350}$$

so that

$$E_z^{inc} = e^{-ik\rho\cos\phi} = e^{-ikz} \tag{5.351}$$

In obtaining the fields for TM propagation between parallel plates in Section 5.5, we found that there was an alternative representation for the Green's function, useful at high frequencies. In the case under consideration here, we may again obtain a useful alternative representation. We begin by writing the differential equation describing the Green's function in (5.326) in the form that separates the ρ-operator from the ϕ-operator. This form is given in (4.187)–(4.189) and is repeated here for convenience, viz.

$$(L_\rho + L_\phi)g = \rho\delta(\rho - \rho')\delta(\phi - \phi') \tag{5.352}$$

where

$$L_\rho = -\rho\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)\right] - (k\rho)^2 \tag{5.353}$$

$$L_\phi = -\frac{\partial^2}{\partial\phi^2} \tag{5.354}$$

We require the spectral representation of the operator L_ρ. The Green's function problem associated with this spectral representation is

$$(L_\rho - \lambda)G = \rho\delta(\rho - \rho') \tag{5.355}$$

$$G\bigg|_{\rho=a} = 0 \tag{5.356}$$

$$\lim_{\rho\to\infty} G = 0 \tag{5.357}$$

The reader should carefully compare this problem to the problem in Example 3.6. The only difference is in the boundary condition at the lower end of the interval. In Example 3.6, we had a finiteness condition at $\rho = 0$. In this case, we have a Dirichlet condition at $\rho = a$. We still have the limit point case as $\rho \to \infty$, but the condition at $\rho = a$ is regular. We write the solution as

$$G = \begin{cases} AJ_\nu(k\rho) + CH_\nu^{(2)}(k\rho), & \rho < \rho' \\ BH_\nu^{(2)}(k\rho) + DH_\nu^{(1)}(k\rho), & \rho > \rho' \end{cases} \tag{5.358}$$

where

$$\nu = i\sqrt{\lambda} \tag{5.359}$$

in the same manner as in (3.136). Application of the limiting condition in (5.357) results in $D = 0$. From this point, the solution for the Green's function follows the development in (5.337)–(5.342). The result is

$$G = \frac{\pi}{2i} \begin{cases} H_\nu^{(2)}(k\rho')\left[J_\nu(k\rho) - c_\nu H_\nu^{(2)}(k\rho)\right], & \rho < \rho' \\ H_\nu^{(2)}(k\rho)\left[J_\nu(k\rho') - c_\nu H_\nu^{(2)}(k\rho')\right], & \rho > \rho' \end{cases} \tag{5.360}$$

where the branch cut in $\sqrt{\lambda}$ lies along the positive real axis and is explicitly determined by (3.143).

Our next step is to determine the spectral representation of $\rho\delta(\rho - \rho')$ by integrating the Green's function with respect to λ by the methods developed in Chapter 3. We first consider the case $\rho < \rho'$. We have

$$G = \frac{\pi}{2i} H_{i\sqrt{\lambda}}^{(2)}(k\rho') \left[J_{i\sqrt{\lambda}}(k\rho) - \frac{J_{i\sqrt{\lambda}}(ka)}{H_{i\sqrt{\lambda}}^{(2)}(ka)} H_{i\sqrt{\lambda}}^{(2)}(k\rho) \right] \tag{5.361}$$

where we have used (5.338) and (5.359). We define the branch cut associated with $\sqrt{\lambda}$ by using (3.143) and (3.144), and produce a cut along the positive-real axis in the λ-plane. Following the development in Example 3.6, we now investigate whether the branch cut in $\sqrt{\lambda}$ produces a branch cut in G. As we approach the positive- real axis from above and below, we have, respectively,

$$\lim_{\phi \to -2\pi} G = \frac{\pi}{2i} \frac{H_{-\tau}^{(2)}(k\rho')}{H_{-\tau}^{(2)}(ka)} \left[J_{-\tau}(k\rho) H_{-\tau}^{(2)}(ka) - J_{-\tau}(ka) H_{-\tau}^{(2)}(k\rho) \right] \tag{5.362}$$

$$\lim_{\phi \to 0} G = \frac{\pi}{2i} \frac{H_\tau^{(2)}(k\rho')}{H_\tau^{(2)}(ka)} \left[J_\tau(k\rho) H_\tau^{(2)}(ka) - J_\tau(ka) H_\tau^{(2)}(k\rho) \right] \tag{5.363}$$

where

$$\tau = i|\lambda|^{1/2} \tag{5.364}$$

But, from [16], we have

$$J_\tau = \frac{1}{2}\left[H_\tau^{(1)} + H_\tau^{(2)}\right] \tag{5.365}$$

$$J_{-\tau} = \frac{1}{2}\left[e^{i\pi\tau} H_\tau^{(1)} + e^{-i\pi\tau} H_\tau^{(2)}\right] \tag{5.366}$$

and, from (3.149),

$$H_{-\tau}^{(2)} = e^{-i\pi\tau} H_\tau^{(2)} \tag{5.367}$$

We substitute (5.365) into (5.363); in addition, we substitute (5.366) and (5.367) into (5.362). After some routine algebra, we find that

$$\lim_{\phi \to -2\pi} G = \lim_{\phi \to 0} G \tag{5.368}$$

Therefore, there is no branch cut in G along the positive- real axis. Since G has no branch cut singularities, the spectral representation of the delta function is given by (3.39), viz.

$$-\rho\delta(\rho - \rho') = \frac{1}{2\pi i} \oint G(\rho, \rho', \lambda) d\lambda \tag{5.369}$$

where the only possible singularities in G are poles. Our analysis of the pole contributions is based on the treatments in [17] and [18]. We write the expression for G given in (5.361) as

$$G = G_1 + G_2 \tag{5.370}$$

where

$$G_1 = \frac{\pi}{2i} H^{(2)}_{i\sqrt{\lambda}}(k\rho') J_{i\sqrt{\lambda}}(k\rho) \tag{5.371}$$

$$G_2 = -\frac{\pi}{2i} \frac{J_{i\sqrt{\lambda}}(ka)}{H^{(2)}_{i\sqrt{\lambda}}(ka)} H^{(2)}_{i\sqrt{\lambda}}(k\rho) H^{(2)}_{i\sqrt{\lambda}}(k\rho') \tag{5.372}$$

There are no poles contained in G_1; there are, however, a countably infinite number of simple poles [18] in G_2 whenever

$$H^{(2)}_{i\sqrt{\lambda}}(ka) = 0 \tag{5.373}$$

Therefore, by Cauchy's Theorem, only the second term in (5.361) contributes to the contour integral in (5.369), and we have

$$\rho\delta(\rho - \rho') = -\frac{1}{4} \oint \frac{J_{i\sqrt{\lambda}}(ka)}{H^{(2)}_{i\sqrt{\lambda}}(ka)} H^{(2)}_{i\sqrt{\lambda}}(k\rho) H^{(2)}_{i\sqrt{\lambda}}(k\rho') d\lambda \tag{5.374}$$

Using the residue theorem, we obtain

$$\rho\delta(\rho - \rho') = \frac{\pi}{2i} \sum_{p=1}^{\infty} J_{i\sqrt{\lambda_p}}(ka) H^{(2)}_{i\sqrt{\lambda_p}}(k\rho) H^{(2)}_{i\sqrt{\lambda_p}}(k\rho') \mathrm{Res}\left[\frac{1}{H^{(2)}_{i\sqrt{\lambda_p}}}; \lambda_p\right] \tag{5.375}$$

where

$$\mathrm{Res}[f(z); z]$$

signifies the residue of $f(z)$ evaluated at z, and where the sum is over the zeros evaluated in (5.373). Using the relationship between λ and ν in (5.359), we obtain

$$\rho\delta(\rho - \rho') = \sum_{p=1}^{\infty} \Phi_p(k\rho)\Phi_p(k\rho') \tag{5.376}$$

where

$$\Phi_p(k\rho) = \left\{ \pi i \frac{\nu_p J_{\nu_p}(ka)}{\frac{\partial}{\partial \nu}\left[H_\nu^{(2)}(ka)\right]_{\nu_p}} \right\}^{1/2} H_{\nu_p}^{(2)}(k\rho) \tag{5.377}$$

Expression (5.376) gives the required spectral representation for the delta function. We note that the result is symmetric with respect to ρ and ρ'. Therefore, the restriction $\rho < \rho'$ can be removed.

Using the methods in Chapter 3, we may develop the spectral representation in (5.376) into a Fourier expansion useful for solving (5.352). For $f(\rho) \in \mathcal{L}_2(a, \infty)$, we have

$$f(\rho) = \int_a^\infty f(\rho')\delta(\rho - \rho')d\rho' = \int_a^\infty f(\rho')\left[\rho'\delta(\rho - \rho')\right]\frac{d\rho'}{\rho'} \tag{5.378}$$

Since $\rho'\delta(\rho - \rho') = \rho\delta(\rho - \rho')$, we may substitute (5.376) and obtain

$$f(\rho) = \sum_{p=1}^{\infty} \alpha_p \Phi_p(k\rho) \tag{5.379}$$

where

$$\alpha_p = \int_a^\infty f(\rho)\Phi_p(k\rho)\frac{d\rho}{\rho} \tag{5.380}$$

We next use this Fourier expansion to solve (5.352). Let

$$g = \sum_{p=1}^{\infty} \alpha_p(\rho', \phi, \phi')\Phi_p(k\rho) \tag{5.381}$$

Substitution into (5.352) gives

$$\left(\frac{d^2}{d\phi^2} + \nu_p^2\right)\alpha_p = -\Phi_p(k\rho')\delta(\phi - \phi') \tag{5.382}$$

This Green's function problem with periodic boundary conditions has been solved in Problem 2.18. The result applied here is

$$\alpha_p = -\frac{\cos[\nu_p(|\phi - \phi'| - \pi)]}{2\nu_p \sin \nu_p \pi}\Phi_p(k\rho') \tag{5.383}$$

We substitute into (5.381) and find that

$$g = -\sum_{p=1}^{\infty} \frac{\cos[\nu_p(|\phi - \phi'| - \pi)]}{2\nu_p \sin \nu_p \pi} \Phi_p(k\rho)\Phi_p(k\rho') \tag{5.384}$$

Using (5.327), we produce the electric field

$$E_z = i\omega\mu_0 I_0 \sum_{p=1}^{\infty} \frac{\cos[\nu_p(|\phi - \phi'| - \pi)]}{2\nu_p \sin \nu_p \pi} \Phi_p(k\rho)\Phi_p(k\rho') \tag{5.385}$$

We again specialize to the case where a plane wave is incident from left to right along the x-axis (Fig. 5-11). Let

$$\phi' = \pi, \qquad -\pi < \phi \leq \pi \tag{5.386}$$

so that

$$\cos[\nu_p(|\phi - \phi'| - \pi)] = \cos \nu_p \phi \tag{5.387}$$

Let ρ' become large enough so that the Hankel function can be approximated by

$$H_{\nu_p}^{(2)}(k\rho') \sim \sqrt{\frac{2i}{\pi k\rho'}} i^{\nu_p} e^{-ik\rho'} \tag{5.388}$$

Then,

$$E_z = -\frac{\omega\mu_0\pi I_0}{2}\sqrt{\frac{2i}{\pi k\rho'}} e^{-ik\rho'} \sum_{p=1}^{\infty} i^{\nu_p} \frac{J_{\nu_p}(ka) H_{\nu_p}^{(2)}(k\rho) \cos \nu_p \phi}{\frac{\partial}{\partial \nu}[H_\nu^{(2)}(ka)]_{\nu_p} \sin \nu_p \pi} \tag{5.389}$$

To produce a unit plane wave incident, we use (5.350) and produce

$$E_z(\rho, \phi) = 2\pi \sum_{p=1}^{\infty} i^{\nu_p} \frac{J_{\nu_p}(ka) H_{\nu_p}^{(2)}(k\rho) \cos \nu_p \phi}{\frac{\partial}{\partial \nu}[H_\nu^{(2)}(ka)]_{\nu_p} \sin \nu_p \pi} \tag{5.390}$$

The reader may wish to compare this result with [17, eq. (129)] by using (5.365) and (5.373). In [17], James has used the classic residue series approach to produce the representation for the electric field that we give in (5.390). We have used the alternative spectral representation, a method also used in [18].

The alternative spectral representation is useful for obtaining solutions at high frequencies where summing the series in (5.344) requires a large number of terms for convergence. The alternative representation is particularly suited for the so-called *shadow region* [19] behind the cylinder, away from the side directly illuminated by the incoming plane wave. Here, the field is given in the form of *creeping waves* [17], [20]. For a thorough discussion of the zeros of the Hankel function needed in (5.373), the reader is referred to [18] and [21].

5.10 DYADIC GREEN'S FUNCTIONS

In the electromagnetic problems in this chapter, the geometry and source in each case have been independent of one coordinate dimension. These two-dimensional problems have been chosen as models to illustrate the use of spectral expansions and Green's functions. Indeed, many of the interesting and useful problems in electromagnetic theory can be modeled in two spatial dimensions. Additional two-dimensional examples directly using the methods developed in this book can be found in [7]–[9].

There are, however, many electromagnetic problems where it is not feasible to assume that the problem is independent of one spatial dimension. In these three-dimensional cases, the analysis in this book may be directly and elegantly extended using a dyadic form of Green's theorem. The dyadic method is presented in detail in the book by Tai [22].

Dyadic analysis is based on the formulation of dyadic spectral representations of the delta function and the derivation of problem-dependent dyadic Green's functions. The interested reader is referred to [22] for a description of the procedures, as well as application to some classical problems, such as waveguide propagation, scattering from cylinders, and interactions with plane stratified media. In addition, the book by Collin [23] provides a logical, systematic presentation of dyadic Green's functions and their use in electromagnetics.

Dyadic analysis can be applied to boundary value problems where the solution depends on inverting an integral equation. In these cases, the reader is cautioned that the analysis of the singularities associated with dyadic kernels in integral equations is a delicate matter. For a discussion, the reader is referred to [23],[24].

PROBLEMS

5.1. Using the Green's function method, show that (5.51) is 6the solution to (5.49) with the boundary conditions in (5.50).

5.2. Show that the solution in (5.72) satisfies the differential equation in (5.55).

5.3. Beginning with the TE_z equation set in (5.109)–(5.111), derive a modal series dual to the modal series describing the TM_z modes in (5.138).

5.4. Using the Green's function method, show that (5.199) is the solution to (5.196) with the boundary conditions in (5.197) and (5.198).

5.5. In the problem describing the scattering from a perfectly conducting cylinder given in Section 5.8, assume that the cylinder cross section is rectangular. For this specific case, specialize the expression for the electric field in (5.318) and the form of the integral equation in (5.319).

5.6. Consider a y-directed magnetic current source M_y above an impedance plane (Fig. 5-12). Let the current source be independent of y, so that

$$\frac{\partial}{\partial y} = 0$$

Assume that the boundary condition at the impedance plane is given by

$$\lim_{x \to 0} \left(\frac{E_z}{H_y} \right) = i\omega L$$

where $L > 0$ is the inductance of the impedance sheet.

(a) Show that the only nonzero field components are H_y, E_x, E_z.

(b) Using Green's theorem, formulate an expression for the magnetic field H_y. Solve explicitly for the Green's function by two methods:

1. Use a spectral expansion in z, followed by a closed form solution in x.
2. Use a spectral expansion in x, followed by a closed form solution in z. *Note*: This spectral expansion utilizes the impedance transform derived in Chapter 3.

(c) Specialize the solution to the case where the magnetic current source is a line source on the x-axis at a distance d above the impedance plane.

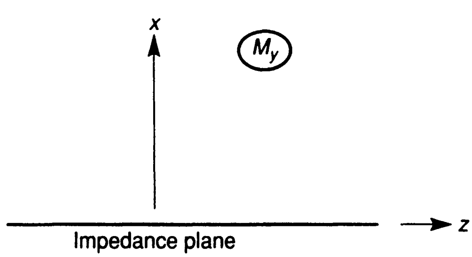

Fig. 5-12 Magnetic current M_y above an impedance plane.

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