Chapter 1

Complex vectors

Complex vectors are vectors whose components can be complex numbers. They were introduced by the famous American physicist J. WILLARD GIBBS, sometimes called the 'Maxwell of America', at about the same period in the 1880's as the real vector algebra, in a privately printed but widely circulated pamphlet *Elements of vector analysis.* Gibbs called these complex extensions of vectors 'bivectors' and they were needed, for example, in his analysis of time-harmonic optical fields in crystals. In a later book compiled by Gibbs's student WILSON in 1909, the text reappeared in extended form, but with only few new ideas (GIBBS and WILSON 1909). Thenceforth, complex vectors have been treated mainly in books on electromagnetics in the context of time-harmonic fields. Instead of a full application of complex vector algebra, the analyses, however, mostly made use of trigonometric function calculations. As will be seen in this chapter, complex vector algebra offers a simple method for the analysis of time-harmonic fields. In fact, it is possible to use many of the rules known from real vector algebra, although not all the conclusions. Properties of the ellipse of time-harmonic vectors can be seen to be directly obtainable through operations on complex vectors.

1.1 Notation

As mentioned above, complex vector formalism is applied in electromagnetics when dealing with time-harmonic field quantities. A time-harmonic field vector $\mathbf{F}(t)$, or 'sinusoidal field' is any real vector function of time t that satisfies the differential equation

$$\frac{d^2}{dt^2}\mathbf{F}(t) + \omega^2\mathbf{F}(t) = 0. \tag{1.1}$$

A general solution can be expressed in terms of two constant real vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ in the form

$$\mathbf{F}(t) = \mathbf{F}_1 \cos\omega t + \mathbf{F}_2 \sin\omega t. \tag{1.2}$$

The complex vector formalism can be used to replace the time-harmonic vectors provided the angular frequency ω is constant. There are certain advantages to this change in notation and, of course, the disadvantage that some new concepts and formulas must be learned. The main bulk of formulas, however, is the same as for real vectors. As an advantage, in using complex vector algebra, work with trigonometric formulas can be avoided, and the formulas look much simpler.

The complex vector $\mathbf{f}$ is defined as a combination of two real vectors, $\mathbf{f}_{\mathrm{re}}$ the real part, and $\mathbf{f}_{\mathrm{im}}$ the imaginary part of $\mathbf{f}$:

$$\mathbf{f} = \mathbf{f}_{\mathrm{re}} + j\mathbf{f}_{\mathrm{im}}. \tag{1.3}$$

The subscripts re and im can be conceived as operators, giving the real and, respectively, the imaginary parts of a complex vector.

The essential point in the complex vector formalism lies in the one-to-one correspondence with the time-harmonic vectors $\mathbf{f} \leftrightarrow \mathbf{F}(t)$. In fact, there are two mappings which give a unique time-harmonic vector for a given complex vector and vice versa. They are:

$$\mathbf{f} \to \mathbf{F}(t): \quad \mathbf{F}(t) = \Re\{\mathbf{f}e^{j\omega t}\} = \mathbf{f}_{\mathrm{re}}\cos\omega t - \mathbf{f}_{\mathrm{im}}\sin\omega t, \tag{1.4}$$

$$\mathbf{F}(t) \to \mathbf{f}: \quad \mathbf{f} = \mathbf{F}(0) - j\mathbf{F}(\pi/2\omega) = \mathbf{F}_1 - j\mathbf{F}_2. \tag{1.5}$$

Thus, for the two representations (1.2) and (1.3) we can see the correspondences $\mathbf{f}_{\mathrm{re}} = \mathbf{F}_1$ and $\mathbf{f}_{\mathrm{im}} = -\mathbf{F}_2$.

The mappings (1.4), (1.5) are each other's inverses, as is easy to show. For example, let us insert (1.4) into (1.5):

$$\mathbf{f} = \Re\{\mathbf{f}e^{0}\} - j\Re\{\mathbf{f}e^{j\pi/2}\} = \mathbf{f}_{\mathrm{re}} + j\mathbf{f}_{\mathrm{im}}, \tag{1.6}$$

which results in the identity $\mathbf{f} = \mathbf{f}$.

It is important to note that there always exists a time-harmonic counterpart to a complex vector whatever its origin. In fact, in analysis, there arise complex vectors, which do not represent a time-harmonic field quantity, for example the wave vector $\mathbf{k}$ or the Poynting vector $\mathbf{P}$. We can, however, always define a time-harmonic vector through (1.4), maybe lacking physical content but helpful in forming a mental picture.

A time-harmonic vector $\mathbf{F}(t) = \mathbf{F}_1\cos\omega t + \mathbf{F}_2\sin\omega t$ traces an ellipse in space, which may reduce to a line segment or a circle. This is seen from the following reasoning.

- If $\mathbf{F}_1 \times \mathbf{F}_2 = 0$, the vectors are parallel or at least one of them is a null vector. Hence, $\mathbf{F}(t)$ is either a null vector or moves along a line and is called *linearly polarized* (LP).

- If $\mathbf{F}_1 \times \mathbf{F}_2 \neq 0$, the vectors define a plane, in which the vector $\mathbf{F}(t)$ rotates. Forming the auxiliary vectors $\mathbf{b} = \mathbf{F}_1 \times (\mathbf{F}_1 \times \mathbf{F}_2)$ and $\mathbf{c} = \mathbf{F}_2 \times (\mathbf{F}_1 \times \mathbf{F}_2)$, we can easily see that the equation $(\mathbf{b} \cdot \mathbf{F}(t))^2 + (\mathbf{c} \cdot \mathbf{F}(t))^2 = |\mathbf{F}_1 \times \mathbf{F}_2|^4$ is satisfied. This is a second order equation, whose solution $\mathbf{F}(t)$ is obviously finite for all t, whence the curve it traces is an ellipse.

- The special case of a *circularly polarized* (CP) vector is obtained, when $|\mathbf{F}(t)|^2 = \mathbf{F}_1^2 \cos^2 \omega t + \mathbf{F}_2^2 \sin^2 \omega t + \mathbf{F}_1 \cdot \mathbf{F}_2 \sin 2\omega t$ is constant for all t. Taking $t = 0$ and $t = \pi/2\omega$ gives $\mathbf{F}_1^2 = \mathbf{F}_2^2$, which leads to the second condition $\mathbf{F}_1 \cdot \mathbf{F}_2 = 0$.

Thus, to every complex vector $\mathbf{f}$ there corresponds an ellipse just as for every real vector there corresponds an arrow in space. The real and imaginary parts $\mathbf{f}_{re}$, $\mathbf{f}_{im}$ both lie on the ellipse. $\mathbf{f}_{re}$ equals the time origin value and is called *the phase vector of the ellipse.* The direction of rotation of $\mathbf{F}(t)$ on the ellipse equals that of $\mathbf{f}_{im}$ turned towards $\mathbf{f}_{re}$ in the shortest way. A complex vector which is *not linearly polarized* (NLP) has a handedness of rotation, which depends on the direction of aspect. The rotation is right handed when looked at in a direction $\mathbf{u}$ (a real vector) such that $\mathbf{f}_{im} \times \mathbf{f}_{re} \cdot \mathbf{u}$ is a positive number and, conversely, left handed if it is negative.

An LP vector must be represented by a double-headed arrow (infinitely thin ellipse), which is in distinction with the one-headed arrow representation of real vectors. The difference is of course due to the fact that the time-harmonic vector (1.2) oscillates between its two extremities.

The complex conjugate of a complex vector $\mathbf{f}$, denoted by $\mathbf{f}^*$, is defined by

$$\mathbf{f}^* = (\mathbf{f}_{re} + j\mathbf{f}_{im})^* = \mathbf{f}_{re} - j\mathbf{f}_{im}. \tag{1.7}$$

From (1.4) we can see that $\mathbf{f}^*$ corresponds to the time-dependent vector $\mathbf{F}(-t)$, or it rotates in the opposite direction along the same ellipse as $\mathbf{f}(t)$.

The complex vector $\mathbf{f}$ is LP if and only if $\mathbf{f}_{re} \times \mathbf{f}_{im} = 0$. This is equivalent with the following condition:

$$\mathbf{f} \text{ is LP} \quad \Leftrightarrow \quad \mathbf{f} \times \mathbf{f}^* = 0. \tag{1.8}$$

The corresponding condition for the CP vector is

$$\mathbf{f} \text{ is CP} \quad \Leftrightarrow \quad \mathbf{f} \cdot \mathbf{f} = 0. \tag{1.9}$$

In fact, (1.9) implies that $\mathbf{f}_{re}^2 = \mathbf{f}_{im}^2$ and $\mathbf{f}_{re} \cdot \mathbf{f}_{im} = 0$, which is equivalent with the CP property of the corresponding time-harmonic vector, as was seen above.

Every LP vector can be written as a multiple of a real unit vector $\mathbf{u}$ in the form $\mathbf{f} = \alpha\mathbf{u}$. Every CP vector $\mathbf{f}$ can be written in terms of two orthogonal real unit vectors $\mathbf{u}$, $\mathbf{v}$ in the form $\mathbf{f} = \alpha(\mathbf{u} + j\mathbf{v})$. In these expressions α is a complex scalar, in general.

1.2 Complex vector identities

The algebra of complex vectors obeys many of the rules known from the real vector algebra, but not all. For example, the implication $\mathbf{a}\cdot\mathbf{a} = 0 \ \Rightarrow \ \mathbf{a} = 0$ is not valid for complex vectors. To be more confident in using identities of real vector algebra, the following theorem appears useful:

all multilinear identities valid for real vectors are also valid for complex vectors.

A multilinear function F of vector arguments $\mathbf{a}_1, \ \mathbf{a}_2, ...$ is a function which is linear in every argument, or the following is valid for $i = 1 \cdots n$:

$$F(\mathbf{a}_1, \mathbf{a}_2, ..., (\alpha\mathbf{a}_i' + \beta\mathbf{a}_i''), ..., \mathbf{a}_n) =$$

$$\alpha F(\mathbf{a}_1, ..., \mathbf{a}_i', ..., \mathbf{a}_n) + \beta F(\mathbf{a}_1, ..., \mathbf{a}_i'', ...\mathbf{a}_n). \tag{1.10}$$

A multilinear identity is of the form

$$F(\mathbf{a}_1, ..., \mathbf{a}_n) = 0 \ \text{ for all } \mathbf{a}_i, \ i = 1...n. \tag{1.11}$$

Now, if the identity is valid for real vectors $\mathbf{a}_i$ and the function does not involve a conjugation operation, from the linearity property (1.10) we can show that it must be valid for complex vectors $\mathbf{a}_i$ as well. In fact, taking $\alpha = 1, \beta = j$, the identity is obviously valid if the real vector $\mathbf{a}_i$ is replaced by the complex vector $\mathbf{a}_i' + j\mathbf{a}_i''$. This can be repeated for every i and, thus, all vectors $\mathbf{a}_i$ can be complex in the identity (1.11). As an example of a trilinear identity we might write

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = 0 \ \text{ for all } \ \mathbf{a}, \ \mathbf{b}, \ \mathbf{c}. \tag{1.12}$$

Also, all non-linear identities which can be derived from multilinear identities are valid for complex vectors, like $\mathbf{a} \times \mathbf{a} = 0$ for all vectors $\mathbf{a}$. The conjugation operation can be introduced by inserting conjugated complex vectors in multilinear identities. Thus, the identity

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a} \cdot \mathbf{b}^*|^2, \tag{1.13}$$

can be obtained from the real quadrilinear identity

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}), \tag{1.14}$$

with the substitution $\mathbf{c} = \mathbf{a}^*$, $\mathbf{d} = \mathbf{b}^*$. The absolute value for a complex vector is defined by

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}^*. \tag{1.15}$$

All implications that can be derived from identities are valid for complex vectors if they are valid for real vectors. The basis for these is the null vector property:

$$|\mathbf{a}| = 0 \;\Leftrightarrow\; \mathbf{a} = 0, \tag{1.16}$$

which can be shown to be valid by expanding $|\mathbf{a}|^2 = |\mathbf{a}_{\mathrm{re}} + j\mathbf{a}_{\mathrm{im}}|^2$.

Two important, although simple looking theorems can be obtained from vector identities:

$$\mathbf{a} \times \mathbf{b} = 0 \text{ and } \mathbf{a} \neq 0 \;\Rightarrow\; \exists \alpha,\; \mathbf{b} = \alpha \mathbf{a}. \tag{1.17}$$

$$\mathbf{a} \cdot \mathbf{b} = 0 \text{ and } \mathbf{a} \neq 0 \;\Rightarrow\; \exists \mathbf{c},\; \mathbf{b} = \mathbf{c} \times \mathbf{a}. \tag{1.18}$$

(1.17) follows from the identity

$$\mathbf{a}^* \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a}^* \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a}^*)\mathbf{b}, \tag{1.19}$$

from which $\mathbf{b}$ can be solved. Correspondingly, (1.18) is obtained from

$$\mathbf{a} \times (\mathbf{a}^* \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{a}^* - (\mathbf{a} \cdot \mathbf{a}^*)\mathbf{b}. \tag{1.20}$$

As a consequence, from (1.17), (1.18) we see that the theorem

$$\mathbf{a} \times \mathbf{b} = 0 \text{ and } \mathbf{a} \cdot \mathbf{b} = 0 \;\Rightarrow\; \mathbf{a} = 0 \text{ or } \mathbf{b} = 0, \tag{1.21}$$

valid for real vectors, is not valid for complex vectors. In fact, assuming $|\mathbf{a}| \neq 0$ gives us either $\mathbf{b} = 0$ or $\mathbf{a}$ is CP, which implies $\mathbf{b}$ is CP. A theorem corresponding to (1.21) for complex vectors is the following one:

$$\mathbf{a} \times \mathbf{b} = 0 \text{ and } \mathbf{a} \cdot \mathbf{b}^* = 0 \;\Rightarrow\; \mathbf{a} = 0 \text{ or } \mathbf{b} = 0, \tag{1.22}$$

as can be readily verified from (1.17), (1.18).

1.3 Parallel and perpendicular vectors

For real vectors $\mathbf{a}, \mathbf{b}$ there exist the geometrical concepts of parallel vectors for $\mathbf{a} \times \mathbf{b} = 0$ and perpendicularity for $\mathbf{a} \cdot \mathbf{b} = 0$. Although the geometrical content will be different, it is helpful to define parallelity and perpendicularity with these same equations for complex vectors. This leads, however, to the existence of vectors which are perpendicular to themselves, namely the CP vectors.

A complex vector $\mathbf{b}$ is parallel to a non-null vector $\mathbf{a}$ if there exists a complex scalar α such that $\mathbf{b} = \alpha\mathbf{a}$, as implied by (1.17). Let us denote $\alpha = \lambda e^{j\theta}$ with real λ, θ, and $\lambda > 0$. It is easy to see from the definition (1.4) that if we have the correspondence

$$\mathbf{a} \quad \leftrightarrow \quad \mathbf{A}(t), \tag{1.23}$$

then we also have

$$\lambda e^{j\theta}\mathbf{a} \quad \leftrightarrow \quad \lambda\mathbf{A}(t + \frac{\theta}{\omega}). \tag{1.24}$$

Thus, the magnitude of the ellipse is multiplied by the factor λ and the phase of the ellipse is shifted by θ/ω. The form of the ellipse as well as its axial directions and sense of rotation are the same for parallel vectors. Parallel vectors are said to have the same polarization.

The geometrical content of perpendicular complex vectors is more difficult to express. Let us find the most general vector $\mathbf{b}$ perpendicular to a given non-null vector $\mathbf{a}$. Obviously, if $\mathbf{a}$ is LP, or parallel to a real unit vector $\mathbf{u}$: $\mathbf{a} = \alpha\mathbf{u}$, $\mathbf{b}$ may be any vector in the plane perpendicular to $\mathbf{u}$, or of the form $\mathbf{u} \times \mathbf{c}$.

For an NLP vector $\mathbf{a}$ there exists a real unit vector $\mathbf{n}$ satisfying $\mathbf{n}\cdot\mathbf{a} = 0$, which is normal to the ellipse of $\mathbf{a}$. Writing $\mathbf{b} = \beta\mathbf{n}+\mathbf{b}_a$ with $\mathbf{b}_a$ in the plane of $\mathbf{a}$, we see that β may have an arbitrary complex value. The problem is to find $\mathbf{b}_a$, which must also be perpendicular to $\mathbf{a}$. From the identity $\mathbf{b}_a \times (\mathbf{n} \times \mathbf{a}) = \mathbf{n}(\mathbf{b}_a \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b}_a \cdot \mathbf{n})$, whose right-hand side vanishes, and (1.17) we see that there must exist a scalar α such that $\mathbf{b}_a = \alpha\mathbf{n} \times \mathbf{a}$.

Thus, the most general vector $\mathbf{b}$ perpendicular to $\mathbf{a}$ can be written as

$$\mathbf{b} = \beta\mathbf{n} + \alpha\mathbf{n} \times \mathbf{a}, \quad \text{with} \quad \mathbf{n}\cdot\mathbf{a} = 0. \tag{1.25}$$

The corresponding time-harmonic vector $\mathbf{B}(t)$ is easily seen to be a sum of an LP vector along $\mathbf{n}$ plus an NLP vector in the plane of $\mathbf{a}$, which is obtained from a vector parallel to $\mathbf{a}$ rotated by $\pi/2$ in its plane. The most general $\mathbf{b}$ vector can be seen to lie on an elliptic cylinder, whose cross section is the ellipse of $\mathbf{b}_a$. It is also easy to see that there exist vectors orthogonal to a given vector $\mathbf{a}$ of any ellipticity, because we can obtain ellipses with every axial ratio by cutting the elliptic cylinder with planes of different orientations. In particular, the LP vector $\mathbf{b}$ is a multiple of $\mathbf{n}$, whereas the two CP vectors

$$\mathbf{b}_\pm = \alpha(\mathbf{n} \times \mathbf{a} \pm j\sqrt{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{n}), \tag{1.26}$$

are obtained from (1.25) with the CP condition $\mathbf{b}\cdot\mathbf{b} = 0$.

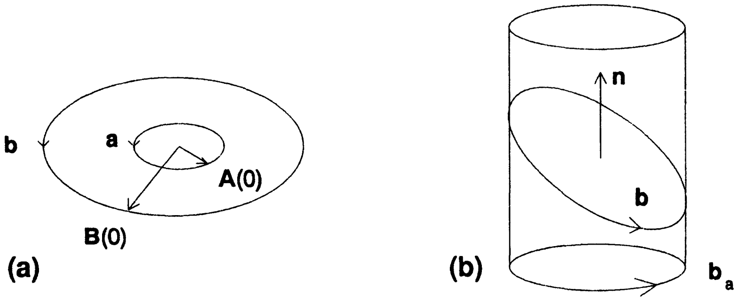

Fig. 1.1 (a) Parallel complex vectors **a** and **b**. (b) Elliptic cylinder construction of a vector **b** perpendicular to a given vector **a**. The projection vector $\mathbf{b}_a$ is parallel to $\mathbf{n}\times\mathbf{a}$, where **n** is a real unit vector normal to **a**.

Any vector **b** can be written as

$$\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} - \frac{\mathbf{a}\times(\mathbf{a}\times\mathbf{b})}{\mathbf{a}\cdot\mathbf{a}}. \tag{1.27}$$

This gives a decomposition of a vector **b** into parts parallel and perpendicular to a vector **a**, and it is used in real vector algebra. Although (1.27) is also valid for complex vectors, it fails when **a** is CP. A more practical decomposition theorem is the following one:

$$\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}^*}\mathbf{a}^* - \frac{\mathbf{a}\times(\mathbf{a}^*\times\mathbf{b})}{\mathbf{a}\cdot\mathbf{a}^*}, \tag{1.28}$$

which splits the vector **b** into vectors parallel to $\mathbf{a}^*$ and perpendicular to **a**. This decomposition has the property of *power orthogonality* of its parts. In fact, writing (1.28) $\mathbf{b} = \mathbf{b}_{co} + \mathbf{b}_{cr}$ as respective terms, where $\mathbf{b}_{co}$ is the co-polarized part and $\mathbf{b}_{cr}$ the cross-polarized part of the vector **b** with respect to the vector **a**, we can write

$$|\mathbf{b}|^2 = |\mathbf{b}_{co}|^2 + |\mathbf{b}_{cr}|^2. \tag{1.29}$$

The co-polarized component is thus parallel not to **a** but to $\mathbf{a}^*$. The definition is needed, for example, in antenna theory when reception of an incoming wave with the field vector **E** is considered with the polarization match factor

$$p(\mathbf{h},\mathbf{E}) = \frac{|\mathbf{h}\cdot\mathbf{E}|^2}{(\mathbf{h}\cdot\mathbf{h}^*)(\mathbf{E}\cdot\mathbf{E}^*)} = 1 - \frac{|\mathbf{h}\times\mathbf{E}^*|^2}{(\mathbf{h}\cdot\mathbf{h}^*)(\mathbf{E}\cdot\mathbf{E}^*)}, \tag{1.30}$$

which tells us how well the polarization of the incoming field can be received by an antenna with the effective length vector **h**. It is seen that only the co-polarized component of $\mathbf{E}$ with respect to $\mathbf{h}$ contributes to the value of $p(\mathbf{h}, \mathbf{E})$ and complete polarization match $p(\mathbf{h}, \mathbf{E}) = 1$ is obtained for $\mathbf{h} \times \mathbf{E}^* = 0$, or when $\mathbf{h}$ and $\mathbf{E}^*$ are parallel vectors. On the other hand, there is a total mismatch $p(\mathbf{h}, \mathbf{E}) = 0$ for perpendicular vectors $\mathbf{h}, \mathbf{E}$, or when the incoming field is cross polarized with respect to the antenna vector $\mathbf{h}$.

As an example of the polarization match factor, let us consider radar reflection from an orthogonal plane. The far field of an antenna has the same polarization as its effective length vector $\mathbf{h}$. Reflection from the surface does not change the polarization of the field (but its handedness is changed!), whence the polarization of the field coming back to the antenna is also that of $\mathbf{h}$. Because the polarization match factor is independent of the magnitude of the field, it equals $p(\mathbf{h}, \mathbf{h})$ in this case. It is seen that for an LP antenna $\mathbf{h} \times \mathbf{h}^* = 0$ and $p(\mathbf{h}, \mathbf{h}) = 1$, or there is no polarization mismatch between the antenna and the incoming field. On the other hand, if $\mathbf{h}$ is CP, we have $p(\mathbf{h}, \mathbf{h}) = 0$, or there is complete mismatch. A CP radar does not see reflections from an orthogonal plane, or other circularly symmetric obstacles, wheras an LP antenna receives the best possible signal.

1.4 Axial representation

Polarization properties of a complex vector such as an electromagnetic field are often needed. For example, given a complex vector, how can we determine its axial directions and magnitudes? Usually, in books working first with complex vectors, the notation is suddenly changed to time dependent representation and the quantities needed are obtained through trigonometric function analysis. This is, however, unnecessary, because the same can be written down in complex vector notation quite simply. The procedure is based on the following simple facts.

(i) The vector $\mathbf{b} = e^{j\theta}\mathbf{a}$ has the same ellipse as $\mathbf{a}$ for real θ.

(ii) There exists θ real such that $\mathbf{b}_{\rm re} \cdot \mathbf{b}_{\rm im} = 0$.

(iii) If $\mathbf{b}_{\rm re} \cdot \mathbf{b}_{\rm im} = 0$, $\mathbf{b}_{\rm re}$ and $\mathbf{b}_{\rm im}$ lie on the axes of the ellipse of $\mathbf{b}$.

(i) was demonstrated above and (ii) defines an equation for θ, which obviously has solutions. (iii) can be shown to be true through a consideration of the corresponding time-harmonic vector $\mathbf{B}(t)$, because $|\mathbf{B}(t)|^2 =$

$|\mathbf{b}_{re}|^2 \cos^2 \omega t + |\mathbf{b}_{im}|^2 \sin^2 \omega t$ has the extremal values $|\mathbf{b}_{re}|^2$ and $|\mathbf{b}_{im}|^2$.

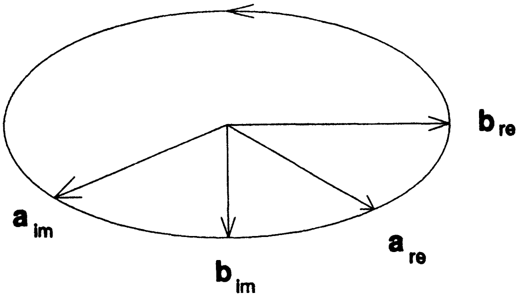

Fig. 1.2 Construction of the axis vectors of a complex vector **a**.

We are now ready to write *an axial decomposition* for any NCP vector **a**, from which the axes of its ellipse can be obtained. Obviously, a CP vector does not have any preferred axes, so it can be excluded. For an NCP vector, the scalar $\sqrt{\mathbf{a}\cdot\mathbf{a}}$ is non-zero and we can define another complex vector **b** through

$$\mathbf{b} = |\sqrt{\mathbf{a}\cdot\mathbf{a}}|\frac{\mathbf{a}}{\sqrt{\mathbf{a}\cdot\mathbf{a}}}. \tag{1.31}$$

The factor multiplying **a** is obviously of the form $e^{-j\theta}$, whence **b** is of the form (i) above implying that **b** and **a** have the same axial vectors. There is no need to solve for θ. Because $\mathbf{b}\cdot\mathbf{b} = |\mathbf{a}\cdot\mathbf{a}|$ is real, (ii) is also satisfied, and (iii) is valid, whence the real and imaginary parts of **b** are the axis vectors. Since $\mathbf{b}\cdot\mathbf{b}$ is positive and equals $\mathbf{b}_{re}^2 - \mathbf{b}_{im}^2$, the real part of the vector lies on the major axis and the imaginary part of the vector on the minor axis of the ellipse of **b** and, hence, **a**. The axial representation of complex vectors was probably first given by MÜLLER (1969) in his monograph on electromagnetic theory.

The axial decomposition (1.31) giving the major axis vectors defined by

$$\mathbf{b}_{re} = |\sqrt{\mathbf{a}\cdot\mathbf{a}}|\Re\left\{\frac{\mathbf{a}}{\sqrt{\mathbf{a}\cdot\mathbf{a}}}\right\}, \tag{1.32}$$

and the minor axis vectors by

$$\mathbf{b}_{im} = |\sqrt{\mathbf{a}\cdot\mathbf{a}}|\Im\left\{\frac{\mathbf{a}}{\sqrt{\mathbf{a}\cdot\mathbf{a}}}\right\}, \tag{1.33}$$

can easily be memorized and applied to simplify the analysis.

For example, we can write $\mathbf{a}\cdot\mathbf{a}^* = \mathbf{b}\cdot\mathbf{b}^* = |\mathbf{b}_{re}|^2 + |\mathbf{b}_{im}|^2$, to obtain a geometrical interpretation for the magnitude $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}^*}$ of a complex vector **a**, as the hypotenuse of the right triangle defined by the vectors $\mathbf{b}_{re}$ and $\mathbf{b}_{im}$ in Fig. 1.2.

1.5 Polarization vectors

As stated above, two parallel vectors have the same polarization. Thus, the polarization of a vector $\mathbf{a}$ consists of all its properties that are not changed when multiplied by a complex scalar α. Because this operation changes the magnitude and phase of the ellipse, the following are left as properties of polarization:

- plane of the ellipse (can be defined by its normal vector $\mathbf{n}$),
- direction of rotation on the plane (right hand in the direction $\mathbf{n}$),
- e, the axial ratio of the ellipse, which defines its form,
- axial directions on the plane of the ellipse (major axis along $\mathbf{u}_1$, minor axis along $\mathbf{u}_2$).

Because the complex vectors are defined by 3+3=6 real parameters and complex scalars by 2 real parameters, the definition of the polarization concept requires 4 real parameters. For example, the unit vector $\mathbf{n}$ takes 2 parameters to define, the axial ratio e one, and the direction of the major axis $\mathbf{u}_1$ on the plane one more parameter (angle on the plane). The minor axis direction is then obtained as $\mathbf{u}_2 = \mathbf{n} \times \mathbf{u}_1$.

The polarization of a complex vector is very often of more interest than the complex vector itself. As one example, in certain microwave ferrite devices, a piece of ferrite material should be positioned in the spot where the magnetic field is circularly polarized, whatever the magnitude of the field may be.

Polarization can be represented most naturally in terms of a normalized complex vector $\mathbf{u}$ with

$$\mathbf{a} = \alpha \mathbf{u}. \tag{1.34}$$

For NCP vectors, α can be defined as $\sqrt{\mathbf{a} \cdot \mathbf{a}}$, whence $\mathbf{u}$ is a complex unit vector satisfying $\mathbf{u} \cdot \mathbf{u} = 1$. For CP vectors, however, this breaks down. As another possibility we could try to define α as the real and positive number $\sqrt{\mathbf{a} \cdot \mathbf{a}^*}$, whence $\mathbf{u}$ is another complex unit vector satisfying $\mathbf{u} \cdot \mathbf{u}^* = 1$, but this $\mathbf{u}$ is no longer a representation of polarization because it contains the phase information of $\mathbf{a}$.

p vector representation

A very useful way to present the polarization is by two real vectors $\mathbf{p}$ and $\mathbf{q}$ to be described next. $\mathbf{p}$ is defined as the following non-linear real vector function of $\mathbf{a}$:

$$\mathbf{p}(\mathbf{a}) = \frac{\mathbf{a} \times \mathbf{a}^*}{j \mathbf{a} \cdot \mathbf{a}^*}. \tag{1.35}$$

This vector has the following properties.

1. $[\mathbf{p}(\mathbf{a})]^* = \mathbf{p}(\mathbf{a})$, or it is a real vector.

2. $\mathbf{p}(\mathbf{a}^*) = -\mathbf{p}(\mathbf{a})$, or a change in the direction of rotation changes the direction of $\mathbf{p}$.

3. $\mathbf{p}(\alpha\mathbf{a}) = \mathbf{p}(\mathbf{a})$, or $\mathbf{p}$ is independent of the magnitude and phase of $\mathbf{a}$. However, for $\mathbf{a} = 0$ the $\mathbf{p}$ vector is indeterminate.

4. $|\mathbf{p}(\mathbf{a})| = 2e/(e^2+1)$, where e is the ellipticity (axial ratio) of $\mathbf{a}$. Hence, $\mathbf{p}(\mathbf{a}) = 0 \Leftrightarrow$ $\mathbf{a}$ is LP and $|\mathbf{p}(\mathbf{a})| = 1 \Leftrightarrow$ $\mathbf{a}$ is CP. Otherwise the length of $\mathbf{p}$ is between 0 and 1.

5. $\mathbf{p}(\mathbf{a}) = 2\mathbf{a}_{\mathrm{im}} \times \mathbf{a}_{\mathrm{re}}/\mathbf{a}\cdot\mathbf{a}^*$, whence for NLP vectors $\mathbf{p}(\mathbf{a})$ points in the positive normal direction of the $\mathbf{a}$ ellipse. The rotation of $\mathbf{a}$ is right handed when looking in the direction $\mathbf{p}(\mathbf{a})$.

6. $\mathbf{p}(\mathbf{a}\cos\theta + \mathbf{n}\times\mathbf{a}\sin\theta) = \mathbf{p}(\mathbf{a})$ for $\mathbf{n} = \mathbf{p}(\mathbf{a})/|\mathbf{p}(\mathbf{a})|$ and θ real. This makes sense for NLP vectors only and means that the ellipse may be rotated in its plane by any angle θ without changing its $\mathbf{p}$ vector. Thus, it is not sufficient to represent the polarization by the $\mathbf{p}$ vector only.

Although the real vector function $\mathbf{p}(\mathbf{a})$ does not carry all the polarization information of $\mathbf{a}$, it is useful in analysing elliptic polarizations. In fact, it gives us the following information about $\mathbf{a}$:

- whether it is an LP vector or not;

- for NLP vectors, the plane of polarization, sense of rotation and ellipticity.

It *does not* give the following information:

- direction, magnitude or phase of an LP vector;

- for NLP vectors, the magnitude or phase or axial directions on the plane of polarization.

It is seen that the $\mathbf{p}$ vector provides least information for LP vectors and most information for CP vectors, for which in fact the polarization is totally known.

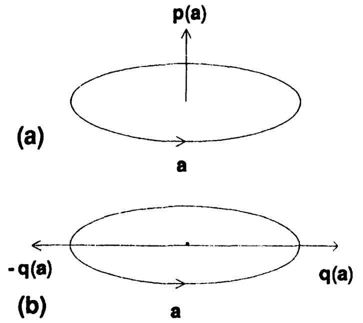

Fig. 1.3 Polarization vectors $\mathbf{p}(\mathbf{a})$ and $\mathbf{q}(\mathbf{a})$ of a complex vector $\mathbf{a}$.

q vector representation

The complementary representation is the $\mathbf{q}(\mathbf{a})$ vector defined as follows:

$$\mathbf{q}(\mathbf{a}) = \frac{|\mathbf{a}\cdot\mathbf{a}|}{\mathbf{a}\cdot\mathbf{a}^*}\frac{\Re\{\mathbf{a}/\sqrt{\mathbf{a}\cdot\mathbf{a}}\}}{|\Re\{\mathbf{a}/\sqrt{\mathbf{a}\cdot\mathbf{a}}\}|}. \tag{1.36}$$

In fact, (1.36) defines a pair of real vectors because of the two branches of the square-root function. Hence, we may depict it as a double-headed arrow. The vector function $\mathbf{q}(\mathbf{a})$ has the following properties.

1. $[\mathbf{q}(\mathbf{a})]^* = \mathbf{q}(\mathbf{a})$, or it is a real function.

2. $\mathbf{q}(\mathbf{a}^*) = \mathbf{q}(\mathbf{a})$, or the sense of rotation has no effect on $\mathbf{q}$.

3. $\mathbf{q}(\alpha\mathbf{a}) = \mathbf{q}(\mathbf{a})$, or the magnitude and phase of $\mathbf{a}$ have no effect on the $\mathbf{q}$ vector. For a null vector $\mathbf{q}$ is indeterminate.

4. $|\mathbf{q}(\mathbf{a})| = (1-e^2)/(1+e^2)$, whence $\mathbf{p}^2 + \mathbf{q}^2 = 1$ and $\mathbf{p}$, $\mathbf{q}$ is a complementary pair of vectors. $\mathbf{q}(\mathbf{a}) \to 0$ for $\mathbf{a}$ approaching CP and $|\mathbf{q}(\mathbf{a})| = 1$ for $\mathbf{a}$ LP. Otherwise, the magnitude of $\mathbf{q}$ is between 0 and 1.

5. If $\mathbf{a} = e^{j\theta}\mathbf{b}$ with $\mathbf{b}\cdot\mathbf{b} > 0$, $\mathbf{q}(\mathbf{a}) = \pm\mathbf{b}_{\rm re}|\mathbf{q}(\mathbf{a})|/|\mathbf{b}_{\rm re}|$, or $\mathbf{q}(\mathbf{a})$ is directed along the major axis of $\mathbf{a}$.

6. For $\mathbf{a} = \alpha\mathbf{u}_1 + j\beta\mathbf{u}_2$ with real α, β, $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_1\cdot\mathbf{u}_2 = 0$, we have $\mathbf{q}(\mathbf{a}) = \pm|\mathbf{q}(\mathbf{a})|\mathbf{u}_1$, or the direction of the minor axis of the ellipse does not affect on the $\mathbf{q}$ vector.

Unit vector representation

Because real vectors have three parameters, it is not possible to represent polarization, requiring four parameters, by either of the $\mathbf{p}$ and $\mathbf{q}$ vectors alone. Some information (like the ellipticity of the complex vector) is shared by both vectors, in other respects they are complementary. It is possible to form a pair of real unit vectors as a combination of the two real vectors:

$$\mathbf{u}_{\pm}(\mathbf{a}) = \mathbf{p}(\mathbf{a}) \pm \mathbf{q}(\mathbf{a}), \tag{1.37}$$

which together exactly represent the polarization of the complex vector $\mathbf{a}$. The subscript + or − is not essential, because the pair $\pm\mathbf{q}$ is not ordered. The properties of $\mathbf{u}_{\pm}(\mathbf{a})$ can be listed as follows.

- For a LP we have $\mathbf{u}_{-} = -\mathbf{u}_{+}$. Thus, a pair of opposite unit vectors gives us the polarization of the LP vector.
- For a CP, $\mathbf{u}_{-} = \mathbf{u}_{+}$. Coinciding unit vectors give the plane and sense of polarization of the CP vector.
- In the general case, there is an angle ψ between the unit vectors. From $\mathbf{u}_{-} + \mathbf{u}_{+} = 2\mathbf{p}(\mathbf{a})$ the plane and sense of polarization as well as the ellipticity are obtained, whereas $\mathbf{u}_{+} - \mathbf{u}_{-} + = 2\mathbf{q}(\mathbf{a})$ gives us the direction of the major axis. The ellipticity and the angle ψ have the relation $e = \cos(\psi/2)/(1 + \sin\psi/2)) = \tan[(\pi - \psi)/4]$.

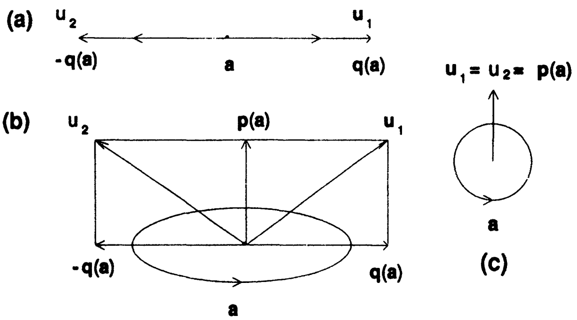

Fig. 1.4 Unit vector pair $\mathbf{u}_1 = \mathbf{u}_+(\mathbf{a})$, $\mathbf{u}_2 = \mathbf{u}_-(\mathbf{a})$ representation of the complex vector $\mathbf{a}$ in (a) linearly polarized, (b) elliptically polarized and (c) circularly polarized cases.

In fact, any NCP complex vector can be written in the form

$$\mathbf{a} = \alpha\left[\left(1 + \sqrt{\frac{1 - \mathbf{u}_+ \cdot \mathbf{u}_-}{2}}\right)(\mathbf{u}_+ - \mathbf{u}_-) + j\mathbf{u}_+ \times \mathbf{u}_-\right], \tag{1.38}$$

or in the equivalent form

$$\mathbf{a} = 2\alpha[(1+|\mathbf{q}(\mathbf{a})|)\mathbf{q}(\mathbf{a}) + j\mathbf{q}(\mathbf{a}) \times \mathbf{p}(\mathbf{a})], \tag{1.39}$$

where

$$\alpha = \frac{\mathbf{a}\cdot\mathbf{a}}{((1+|\mathbf{q}|)^2 - \mathbf{p}^2)\,\mathbf{q}^2}. \tag{1.40}$$

These expressions are not valid for CP vectors $\mathbf{a}$, for which $\mathbf{u}_+ = \mathbf{u}_-$ and $\mathbf{q} = 0$. Equation (1.39) can, however, be extended to CP vectors if we let $\mathbf{q} \to 0$ so that $\alpha\mathbf{q} = \mathbf{c}$ is finite, whence

$$\mathbf{a} = \mathbf{c} + j\mathbf{c} \times \mathbf{p}. \tag{1.41}$$

If only vectors $\mathbf{a}$ on a certain plane orthogonal to $\mathbf{n}$ are considered, the direction of $\mathbf{p}(\mathbf{a})$ is fixed on the line $\pm\mathbf{n}$. Then, one single unit vector is enough to represent the polarization of $\mathbf{a}$, since the other one can be obtained from $\mathbf{u}_+ + \mathbf{u}_- = 2\mathbf{p}$. The unit vector pair corresponds to two points on a unit sphere. This is closely related to the well-known Poincaré sphere representation of plane polarized vectors, where only a single point is used. In fact, if a direction on the plane is chosen, from which the angle ϕ is measured, the double point representation can be transformed to a single point representation by mapping the points (ϕ, θ) and $(\phi + \pi, \theta)$ onto the same point $(2\phi, \theta)$. What results, is the Poincaré sphere with just one point on it. The Poincaré sphere representation has the disadvantage that it can only be used for polarizations on a fixed plane. On the other hand, the polarization match factor $p(\mathbf{a}, \mathbf{b})$ can be given a geometrical interpretation (DESCHAMPS 1951).

1.6 Complex vector bases

In many practical cases it is necessary to expand a given vector, real or complex, in a base of complex vectors. For example, a wave travelling in the ionosphere is split into characteristic waves with different polarizations, whose propagation can be simply calculated. The propagation of a wave with general polarization must then be written in terms of these characteristic polarizations, after which it is easily computed. The characteristic polarizations are complex so that a decomposition theorem for an arbitrary vector $\mathbf{d}$ in terms of three given complex vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is needed, such as the following Gibbs' identity:

$$(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})\mathbf{d} = (\mathbf{d}\cdot\mathbf{a}\times\mathbf{b})\mathbf{c} + (\mathbf{d}\cdot\mathbf{b}\times\mathbf{c})\mathbf{a} + (\mathbf{d}\cdot\mathbf{c}\times\mathbf{a})\mathbf{b}. \tag{1.42}$$

Being a tetralinear identity, (1.42) is valid for all real as well as complex vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. It can be derived by expanding the expression $(\mathbf{a}\times\mathbf{b})\times$

$(\mathbf{c} \times \mathbf{d})$ in two ways and equating the results. The vector triple $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is called a base if $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \neq 0$, in which case (1.42) gives the decomposition theorem

$$\mathbf{d} = \mathbf{d} \cdot \mathbf{a}'\mathbf{a} + \mathbf{d} \cdot \mathbf{b}'\mathbf{b} + \mathbf{d} \cdot \mathbf{c}'\mathbf{c}, \tag{1.43}$$

with the reciprocal base vectors defined by $\mathbf{a}' = \mathbf{b}\times\mathbf{c}/J$, $\mathbf{b}' = \mathbf{c}\times\mathbf{a}/J$, $\mathbf{c}' = \mathbf{a} \times \mathbf{b}/J$, with $J = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.

As an example of longitudinal ionospheric propagation in the direction $\mathbf{u}$ (real unit vector), a base corresponding to characteristic polarizations can be formed with two CP vectors $\mathbf{a}$, $\mathbf{a}^*$ both orthogonal to $\mathbf{u}$, satisfying $\mathbf{u} \times \mathbf{a} = j\mathbf{a}$ and $\mathbf{u} \times \mathbf{a}^* = -j\mathbf{a}^*$. Because $\mathbf{p}(\mathbf{a}) = \mathbf{p}(\mathbf{u} \times \mathbf{a}) = \mathbf{u}$, $\mathbf{a}$ has right-hand and $\mathbf{a}^*$ left-hand polarization with respect to the direction of propagation $\mathbf{u}$. The vector triple is a base, since $\mathbf{u} \cdot \mathbf{a} \times \mathbf{a}^* = j\mathbf{a} \cdot \mathbf{a}^* \neq 0$. Hence, any field vector $\mathbf{E}$ can be expanded as

$$\mathbf{E} = \mathbf{a}\frac{\mathbf{a}^* \cdot \mathbf{E}}{\mathbf{a} \cdot \mathbf{a}^*} + \mathbf{a}^*\frac{\mathbf{a} \cdot \mathbf{E}}{\mathbf{a} \cdot \mathbf{a}^*} + \mathbf{u}\mathbf{u} \cdot \mathbf{E}. \tag{1.44}$$

A similar base can be generated from any NLP vector $\mathbf{a}$ as the triple $\mathbf{a}$, $\mathbf{a}^*$, $\mathbf{p}(\mathbf{a})$, because obviously $\mathbf{a} \times \mathbf{a}^* \cdot \mathbf{p}(\mathbf{a}) = j(\mathbf{a} \cdot \mathbf{a}^*)\mathbf{p}(\mathbf{a}) \cdot \mathbf{p}(\mathbf{a})$, which is nonzero for $\mathbf{p}(\mathbf{a}) \neq 0$.

An interesting and natural base can be generated from any NCP vector $\mathbf{a}$ through the following eigenvalue problem:

$$\mathbf{a} \times \mathbf{v} = \lambda\mathbf{v}. \tag{1.45}$$

The eigenvalue λ can be easily seen to have the values 0, $j\sqrt{\mathbf{a} \cdot \mathbf{a}}$, $-j\sqrt{\mathbf{a} \cdot \mathbf{a}}$ corresponding to the respective eigenvectors $\mathbf{v} = \mathbf{a}$, $\mathbf{v}_+$, $\mathbf{v}_-$, defined by

$$\mathbf{v}_\pm = \alpha_\pm \left(\mathbf{p}(\mathbf{a}) \mp j\frac{\mathbf{a} \times \mathbf{p}(\mathbf{a})}{\sqrt{\mathbf{a} \cdot \mathbf{a}}}\right), \tag{1.46}$$

with arbitrary coefficients $\alpha_\pm$. The vectors defined by (1.46) are easily seen to be CP vectors. If $\mathbf{a}$ is CP, the base does not exist. In fact, all three vectors tend to the same vector $\mathbf{a}$ as it approaches circular polarization. Also, for LP vectors $\mathbf{a}$ (1.46) does not seem to work because $\mathbf{p}(\mathbf{a}) = 0$. However, the limit exists as $\mathbf{a}$ approaches linear polarization, if the product $\alpha_\pm\mathbf{p}(\mathbf{a})$ is kept finite.

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