Chapter 4

Vector Analysis in Space

4-1 Symbolic Vector and Symbolic Vector Expressions

In this chapter, the most important one in the book, we introduce a new method in treating vector analysis called the *symbolic vector method.* The main advantages of this method are that (1) the differential expressions of the three key functions in vector analysis are derived based on one basic formula, (2) all of the integral theorems in vector analysis are deduced from one generalized theorem, (3) the commonly used vector identities are found by an algebraic method without performing any differentiation, and (4) two differential operators in the curvilinear coordinate system, called the *divergence operator* and the *curl operator* and distinct from the operator used for the gradient, are introduced. The technical meanings of the terms "divergence," "curl," and "gradient" will be explained shortly. Note that the nomenclature for some technical terms introduced in this chapter differs from the original one used in [6].

Because vector algebra is the germ of the method, we will review several essential topics covered in Chapter 1. In vector algebra, there are various products, such as

$$\begin{gathered} a\mathbf{b}, \quad \mathbf{a}\cdot\mathbf{b}, \quad \mathbf{a}\times\mathbf{b}, \quad c(\mathbf{a}\cdot\mathbf{b}), \quad c(\mathbf{a}\times\mathbf{b}), \\ \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}), \quad \mathbf{c}\times(\mathbf{a}\times\mathbf{b}), \quad (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}). \end{gathered} \tag{4.1}$$

All of them have well-defined meanings in vector algebra. Here, we treat the scalar and vector quantities a, $\mathbf{a}$, b, $\mathbf{b}$, c, $\mathbf{c}$, $\mathbf{d}$ as functions of position, and they are assumed to be distinct from each other. For purposes of identification, the functions listed in (4.1) will be referred to as *vector expressions*. A quantity like **ab** is not a vector expression, although it is a well-defined quantity in dyadic analysis, a subject already introduced in Chapter 1. For the time being, we are dealing with vector expressions only. In one case, a dyadic quantity will be involved, and its implication will be explained. All of the vector expressions listed in (4.1) are linear with respect to a single function, that is, the distributive law holds true. For example, if $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$, then

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{c}_1 + \mathbf{c}_2) \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{c}_1 \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{c}_2 \cdot (\mathbf{a} \times \mathbf{b}). \tag{4.2}$$

The important identities in vector algebra are listed here:

$$a\mathbf{b} = \mathbf{b}a, \tag{4.3}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}, \tag{4.4}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \tag{4.5}$$

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \tag{4.6}$$

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}. \tag{4.7}$$

The proofs of (4.6)–(4.7) are found in Section 1-2.

Now, if one of the vectors in (4.1) is replaced by a symbolic vector denoted by ∇, such as $a\nabla$, $\nabla \cdot \mathbf{b}$, $\mathbf{a} \cdot \nabla$, $\nabla \times \mathbf{b}$, $\mathbf{a} \times \nabla$, $c\nabla \cdot \mathbf{b}$, $c\mathbf{a} \cdot \nabla$, $\mathbf{c} \cdot (\nabla \times \mathbf{b})$, and so on, these expressions will be called *symbolic vector expressions* or *symbolic expressions* for short. The symbol ∇ is designated as the *symbolic vector*, or *S vector* for short. Besides ∇, a symbolic expression contains other functions, either scalar or/and vector. Thus, $c(\nabla \times \mathbf{b})$ contains one scalar, one vector, and the S vector. In general, a symbolic expression will be denoted by $T(\nabla)$. More specifically, if there is a need to identify the functions contained in $T(\nabla)$ besides ∇, we will use, for example, $T(\nabla, \widetilde{a}, \widetilde{b})$, which shows that there are two functions, $\widetilde{a}$ and $\widetilde{b}$, besides ∇. These functions may be both scalar, both vector, or one each. We use a tilde over these letters to indicate such options. The symbolic expression so created is *defined by*

$$T(\nabla) = \lim_{\Delta V \to 0} \frac{\sum_i T(\hat{n}_i)\, \Delta S_i}{\Delta V}, \tag{4.8}$$

where ΔS_i denotes a typical elementary area (scalar) of a surface enclosing the volume ΔV of a cell and $\hat{n}_i$ denotes the outward unit vector from ΔS_i. The running index i in (4.8) corresponds to the number of surfaces enclosing ΔV. For a cell bounded by six coordinate surfaces, i goes from 1 to 6. Because the definition of $T(\nabla)$ is independent of the choice of the coordinate system, (4.8) is *invariant to the coordinate system*. The expression on the right side of (4.8), from the analytical point of view, represents *the integral-differential transform of the symbolic expression $T(\nabla)$, or simply the functional transform of $T(\nabla)$*. By choosing the proper measures of ΔS_i and ΔV in a certain coordinate system, one can find the differential expression of $T(\nabla)$ based on (4.8).

There are several important characteristics of (4.8) that must be emphasized. In the first place, because all symbolic expressions are generated by well-defined vector expressions, they are, in general, involving simple multiplication, scalar product, and vector product. For example, if we replace the vector function $\mathbf{d}$ in the vector expression $a\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})$ by ∇, we would create a symbolic expression of the form

$$T(\nabla) = a\mathbf{b}\cdot(\mathbf{c}\times\nabla), \tag{4.9}$$

where a is a scalar function, $\mathbf{b}$ and $\mathbf{c}$ are vector functions; then, we have a multiplication between a and the rest of $T(\nabla)$; a scalar product between $\mathbf{b}$ and $(\mathbf{c}\times\nabla)$; and finally a vector product between $\mathbf{c}$ and ∇, or the S vector. The vector product $\mathbf{c}\times\nabla$ is not a true product because ∇ is a symbolic vector, or a dummy vector, not a function. However, the expression leads to a function $T(\hat{n}_j)$ in (4.8) in the form

$$T(\hat{n}_j) = a\mathbf{b}\cdot(\mathbf{c}\times\hat{n}_j). \tag{4.10}$$

This is a well-defined function in which $\mathbf{c}\times\hat{n}_j$ is truly a vector product. This function is used to find the differential expression of $T(\nabla)$ based on the right-side term of (4.8). This description consists of the most important concept in the method of symbolic vector: *the multiplication, the scalar product, and the vector product contained in* $\mathrm{T}(\nabla)$ *are executed in the function* $\mathrm{T}(\hat{\mathrm{n}}_\mathrm{j})$.

Another characteristic of (4.8) deals with the algebraic property of $T(\nabla)$. Because $T(\hat{n})$ is a well-defined function for a given $T(\nabla)$, the vector identities applicable to $T(n_i)$ are shared by $T(\nabla)$. In the previous example,

$$T(\hat{n}_i) = a\mathbf{b}\cdot(\mathbf{c}\times\hat{n}_i) = -\mathbf{b}\cdot(\hat{n}_i\times\mathbf{c})a = a\hat{n}_i\cdot(\mathbf{b}\times\mathbf{c}), \tag{4.11}$$

which implies

$$T(\nabla) = a\mathbf{b}\cdot(\mathbf{c}\times\nabla) = -\mathbf{b}\cdot(\nabla\times\mathbf{c})a = a\nabla\cdot(\mathbf{b}\times\mathbf{c}). \tag{4.12}$$

This property can be stated in a lemma:

Lemma 4.1. For any symbolic expression $T(\nabla)$ generated from a valid vector expression, we can treat the symbolic vector ∇ in that expression as a vector and all of the algebraic identities in vector algebra are applicable.

For example, we have the vector identities listed in (4.3) to (4.7); then, according to Lemma 4.1, the following relations hold true:

$$a\nabla = \nabla a, \tag{4.13}$$

$$\nabla\cdot\mathbf{a} = \mathbf{a}\cdot\nabla, \tag{4.14}$$

$$\nabla\times\mathbf{a} = -\mathbf{a}\times\nabla, \tag{4.15}$$

$$\mathbf{b}\cdot(\mathbf{a}\times\nabla) = \nabla\cdot(\mathbf{b}\times\mathbf{a}) = \mathbf{a}\cdot(\nabla\times\mathbf{b}), \tag{4.16}$$

$$\begin{aligned}\nabla\times(\mathbf{a}\times\mathbf{b}) &= (\mathbf{b}\times\mathbf{a})\times\nabla \\ &= (\nabla\cdot\mathbf{b})\mathbf{a} - (\nabla\cdot\mathbf{a})\mathbf{b} \\ &= (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}.\end{aligned} \tag{4.17}$$

4-2 Differential Formulas of the Symbolic Expression in the Orthogonal Curvilinear Coordinate System for Gradient, Divergence, and Curl

Orthogonal coordinate systems are the most useful systems in formulating problems in physics and engineering. We will therefore derive the relevant differential expressions in OCS first based on the method of symbolic vector and treat the formulation for the general curvilinear system later. Actually, it is more efficient to do GCS first and consider OCS as special cases. For expository purposes, however, it is desirable to follow the proposed order because most readers are likely to have some acquaintance with vector analysis in orthogonal systems from a course in college physics or calculus. The mathematics in GCS would distract from the main feature of the new method at this stage.

To evaluate the integral-differential expression that defines $T(\nabla)$ in (4.8) in the OCS, we consider a volume bounded by six coordinate surfaces; then the running index in (4.8) goes from $i = 1$ to $i = 6$. Because $T(\hat{n}_i)$ is linear with respect to $\hat{n}_i$, we can combine $\hat{n}_i$ with ΔS_i to form a vector differential area. If that area corresponds to a segment $\hat{n}_i \, \Delta S$ on a coordinate surface with v_i being constant, then

$$\hat{n}_i \, \Delta S = h_j h_k \, \Delta v_j \, \Delta v_k \, \hat{u}_i = \frac{\Omega}{h_i} \Delta v_j \, \Delta v_k \, \hat{u}_i, \tag{4.18}$$

where $i, j, k = 1, 2, 3$ in cyclic order and $\Omega = h_1 h_2 h_3$. The h_i's are the metric coefficients of any, yet unspecified, orthogonal curvilinear system. For a cell bounded by six surfaces located at $v_i \pm \Delta v_i/2$, its volume, denoted by ΔV, is

$$\Delta V = \Omega \, \Delta v_i \, \Delta v_j \, \Delta v_k.$$

We can separate the sum in (4.8) into two groups:

$$T(\nabla) = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{3} \left[T\left(\frac{\Omega}{h_i} \hat{u}_i\right)_{v_i + \Delta v_i/2} - T\left(\frac{\Omega}{h_i} \hat{u}_i\right)_{v_i - \Delta v_i/2} \right] \Delta v_j \, \Delta v_k}{\Omega \, \Delta v_i \, \Delta v_j \, \Delta v_k}.$$

The limit yields

$$T(\nabla) = \frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i} T\left(\frac{\Omega}{h_i} \hat{u}_i\right). \tag{4.19}$$

Equation (4.19) is the differential expression for $T(\nabla)$ in any OCS. In (4.19), the summation goes from $i = 1$ to $i = 3$ and the metric coefficients and hence the parameter Ω are, in general, functions of v_i, the coordinate variables. Equation (4.19) is perhaps the most important formula in the method of symbolic vector when it is applied to an OCS.

We now consider the three simplest but also most basic forms of $T(\nabla)$, in which $T(\nabla)$ contains only one function besides ∇. There are only three possibilities, namely,

$$
\begin{aligned}
T_1(\nabla) &= \nabla f \quad \text{or} \quad f\nabla, \\
T_2(\nabla) &= \nabla \cdot \mathbf{F} \quad \text{or} \quad \mathbf{F} \cdot \nabla, \\
T_3(\nabla) &= \nabla \times \mathbf{F} \quad \text{or} \quad -\mathbf{F} \times \nabla,
\end{aligned}
$$

where f denotes a scalar function and $\mathbf{F}$, a vector function. They are the same as the ones in (4.13), (4.14), and (4.15). The differential expressions of these functions can be found based on (4.19).

1. **Gradient of a scalar function**

When $T(\nabla) = T_1(\nabla) = \nabla f = f\nabla$, (4.19) yields

$$
\begin{aligned}
\nabla f &= \frac{1}{\Omega}\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i f\right) \\
&= \frac{1}{\Omega}\sum_i \left[\frac{\Omega}{h_i}\hat{u}_i \frac{\partial f}{\partial v_i} + f\frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i\right)\right].
\end{aligned} \tag{4.20}
$$

The last sum vanishes because of the closed surface theorem stated by (2.62) in OCS, namely,

$$
\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i\right) = 0. \tag{4.21}
$$

Hence

$$
\nabla f = \sum_i \frac{\hat{u}_i}{h_i}\frac{\partial f}{\partial v_i}. \tag{4.22}
$$

The differential expression thus derived is called the *gradient* of the scalar function f. It is a vector function and it will be denoted by ∇f, where ∇ is a differential operator defined by

$$
\nabla = \sum_i \frac{\hat{u}_i}{h_i}\frac{\partial}{\partial v_i}. \tag{4.23}
$$

It will be called a *gradient operator*; thus,

$$
\nabla f = \sum_i \frac{\hat{u}_i}{h_i}\frac{\partial f}{\partial v_i}. \tag{4.24}
$$

The linguistic notation used by some authors is grad f. The symbol ∇ is also called *del* or *nabla* or *Hamilton operator*.

2. **Divergence of a vector function**

When $T(\nabla) = T_2(\nabla) = \nabla \cdot \mathbf{F} = \mathbf{F} \cdot \nabla$, (4.19) yields

$$
\nabla \cdot \mathbf{F} = \frac{1}{\Omega}\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u} \cdot \mathbf{F}\right) = \frac{1}{\Omega}\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}F_i\right). \tag{4.25}
$$

The differential expression thus derived is called the *divergence* of the vector function **F**. There is another functional form of this function that can be obtained from the second term of (4.25). We split the derivative with respect to v_i into two terms, just as the splitting in (4.20), that is,

$$\frac{1}{\Omega}\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}\cdot\mathbf{F}\right) = \frac{1}{\Omega}\sum_i\left[\frac{\Omega}{h_i}\hat{u}_i\cdot\frac{\partial\mathbf{F}}{\partial v_i} + \mathbf{F}\cdot\frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i\right)\right]. \tag{4.26}$$

The second term vanishes as a result of (4.21); hence

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i}\cdot\frac{\partial\mathbf{F}}{\partial v_i}. \tag{4.27}$$

Now, we introduced a divergence operator, denoted by $\boldsymbol{\nabla}$, and defined by

$$\boldsymbol{\nabla} = \sum_i \frac{\hat{u}_i}{h_i}\cdot\frac{\partial}{\partial v_i}. \tag{4.28}$$

Then,

$$\sum_i \frac{\hat{u}_i}{h_i}\cdot\frac{\partial\mathbf{F}}{\partial v_i} = \boldsymbol{\nabla}\mathbf{F}. \tag{4.29}$$

It can be verified that by evaluating the derivatives of **F** with respect to v_i, taking into due consideration that the unit vectors associated with **F** are functions of the coordinate variables and they can be expressed in terms of various $\hat{u}_i$'s according to (2.58) and (2.61), the function at the left of (4.29) reduces to (4.25) as it should be. The operand of a divergence operator must be a vector. Later on, we will show that it can also be applied to a dyadic. Comparing the divergence operator with the gradient operator defined by (4.23), we see that there is a dot, the scalar product symbol, between $\hat{u}_i / h_i$ and the partial derivative sign. It is this morphology that prompts us to use the symbol $\boldsymbol{\nabla}$ for the divergence operator. We must emphasize that there is no analytical relation between the gradient operator ∇ and the divergence operator $\boldsymbol{\nabla}$. They are distinct operators. In the history of vector analysis, there are two well-established notations for the divergence. One is the linguistic notation denoted by div **F**. The connotation of this notation is obvious. Another notation is $\nabla\cdot\mathbf{F}$, which was due to Gibbs, one of the founders of vector analysis. Unfortunately, some later authors treat $\nabla\cdot\mathbf{F}$ as the scalar product or "formal" scalar product between ∇ and **F** in the rectangular coordinate system, which is not a correct interpretation. The contradictions that resulted from the improper use of ∇ are discussed in the last chapter of this book. The evidence and the logic described therein strengthens our decision to adopt $\boldsymbol{\nabla}\mathbf{F}$ as the new notation for the divergence. In the original edition of this book [7], we kept Gibbs's notations for the divergence and the curl, a function to be introduced shortly. The use of Gibbs's notations does impede the understanding of the symbolic expressions $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$. We have therefore made a bold move in this edition by abandoning a long-established tradition. With this much discussion of the new operational notation for the divergence, we consider the last case of the triad.

3. **Curl of a vector function**

When $T(\nabla) = T_3(\nabla) = \nabla \times \mathbf{F} = -\mathbf{F} \times \nabla$, (4.19) yields

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{\Omega}\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i \times \mathbf{F}\right) \\ &= \frac{1}{\Omega}\left[\frac{\partial}{\partial v_1}(h_2h_3\hat{u}_1 \times \mathbf{F}) + \frac{\partial}{\partial v_2}(h_1h_3\hat{u}_2 \times \mathbf{F})\right. \\ &\quad \left. + \frac{\partial}{\partial v_3}(h_1h_2\hat{u}_3 \times \mathbf{F})\right] \\ &= \frac{1}{\Omega}\left[\frac{\partial}{\partial v_1} h_2h_3(F_2\hat{u}_3 - F_3\hat{u}_2) + \frac{\partial}{\partial v_2} h_1h_3(-F_1\hat{u}_3 + F_3\hat{u}_1)\right. \\ &\quad \left. + \frac{\partial}{\partial v_3} h_1h_2(F_1\hat{u}_2 - F_2\hat{u}_1)\right].\end{aligned} \tag{4.30}$$

Each term in (4.30) can be split into two parts; for example,

$$\frac{1}{\Omega}\frac{\partial}{\partial v_1}(h_2h_3F_2\hat{u}_3) = \frac{1}{\Omega}\left[h_2F_2\frac{\partial(h_3\hat{u}_3)}{\partial v_1} + h_2\hat{u}_3\frac{\partial(h_2F_2)}{\partial v_1}\right].$$

As a result of (2.56), one finds that all the terms containing the derivatives of $h_i\hat{u}_i$ cancel each other. The remaining terms are given by

$$\nabla \times \mathbf{F} = \frac{1}{\Omega}\sum_i h_i\hat{u}_i\left[\frac{\partial(h_kF_k)}{\partial v_j} - \frac{\partial(h_jF_j)}{\partial v_k}\right], \tag{4.31}$$

where $i, j, k = 1, 2, 3$ in cyclic order. This function is called the *curl* of $\mathbf{F}$. Like the divergence, we can find an operational form of this function. Using the first line of (4.30), we have

$$\frac{1}{\Omega}\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u} \times \mathbf{F}\right) = \frac{1}{\Omega}\sum_i\left[\frac{\Omega}{h_i}\hat{u}_i \times \frac{\partial \mathbf{F}}{\partial v_i} - \mathbf{F} \times \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i\right)\right].$$

The last term vanishes because of (4.21); hence

$$\nabla \times \mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i}. \tag{4.32}$$

Now we introduce a curl operator, denoted by $\nabla\!\!\!\!\nabla$, and defined by

$$\nabla\!\!\!\!\nabla = \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i}. \tag{4.33}$$

The operand of this operator must be a vector. The curl of a vector function $\mathbf{F}$ therefore can be written in the form

$$\sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i} = \nabla\!\!\!\!\times \mathbf{F}. \tag{4.34}$$

It can be verified that by evaluating the derivatives of $\mathbf{F}$ with respect to v_i, we can recover the differential expression of $\nabla\!\!\!\!\times \mathbf{F}$ given by (4.31). The linguistic notation for the curl is simply curl $\mathbf{F}$, used mostly in English-speaking countries, and rot $\mathbf{F}$ in Germany. Gibbs's notation for this function is $\nabla \times \mathbf{F}$. As with the divergence, some authors treat $\nabla \times \mathbf{F}$ as a vector product or a "formal" vector product between ∇ and $\mathbf{F}$, which is a misinterpretation. The new notation avoids this possibility. The location of the cross sign $(\times)$ in the left side of (4.34) suggests to us to adopt the notation $\nabla\!\!\!\!\times$ for this function. In summary, we have derived the differential expressions of three basic functions in vector analysis in the OCS based on a symbolic expression defined by (4.8); they are

$$\nabla f = \sum_i \frac{\hat{u}_i}{h_i} \frac{\partial f}{\partial v_i} \quad \text{(gradient)}, \tag{4.35}$$

$$\nabla \mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial \mathbf{F}}{\partial v_i} = \frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} F_i \right) \quad \text{(divergence)}, \tag{4.36}$$

$$\begin{aligned} \nabla\!\!\!\!\times \mathbf{F} &= \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i} \\ &= \frac{1}{\Omega} \sum_i h_i \hat{u}_i \left[\frac{\partial (h_k F_k)}{\partial v_j} - \frac{\partial (h_j F_j)}{\partial v_k} \right] \quad \text{(curl)}, \end{aligned} \tag{4.37}$$

where $\Omega = h_1 h_2 h_3$. In the rectangular system, all the metric coefficients are equal to unity; hence

$$\nabla f = \sum_i \hat{x}_i \frac{\partial f}{\partial x_i}, \tag{4.38}$$

$$\nabla \mathbf{F} = \sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} = \sum_i \frac{\partial F_i}{\partial x_i}, \tag{4.39}$$

$$\nabla\!\!\!\!\times \mathbf{F} = \sum_i \hat{x}_i \times \frac{\partial \mathbf{F}}{\partial x_i} = \sum_i \hat{x}_i \left(\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} \right). \tag{4.40}$$

4-3 Invariance of the Differential Operators

From the definition of $T(\nabla)$ given by (4.8), the differential expressions evaluated from that formula should be independent of the coordinate system in which the differential expressions are derived, such as the gradient, the divergence, and the curl. It is desirable, however, to show analytically that such an invariance is indeed true.

We consider the expressions for the divergence in two orthogonal curvilinear systems:

$$\nabla \mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial \mathbf{F}}{\partial v_i} \tag{4.41}$$

and

$$\nabla' \mathbf{F}' = \sum_j \frac{\hat{u}'_j}{h'_j} \cdot \frac{\partial \mathbf{F}'}{\partial v'_j} . \tag{4.42}$$

All the primed functions and the operator ∇' are defined with respect to v'_j. Because $\mathbf{F}$ and $\mathbf{F}'$ are the same vector function expressed in two different systems, it is sufficient to show the invariance of the two divergence operators. The total differential of a position vector is given by

$$d\mathbf{R}_{\mathrm{p}} = \sum_i h_i \hat{u}_i \, dv_i = \sum_j h'_j \hat{u}'_j \, dv'_j . \tag{4.43}$$

Thus,

$$h'_k \, dv'_k = \sum_i h_i (\hat{u}_i \cdot \hat{u}'_k) \, dv_i , \qquad k = 1, 2, 3$$

or

$$h'_j \, dv'_j = \sum_i h_i (\hat{u}_i \cdot \hat{u}'_j) \, dv_i , \qquad j = 1, 2, 3. \tag{4.44}$$

Hence

$$\frac{\partial v'_j}{\partial v_i} = \frac{h_i}{h'_j} (\hat{u}_i \cdot \hat{u}'_j). \tag{4.45}$$

Now,

$$\nabla = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} = \sum_{i,j} \frac{\hat{u}_i}{h_i} \cdot \frac{\partial v'_j}{\partial v_i} \frac{\partial}{\partial v'_j} . \tag{4.46}$$

In view of (4.45), we obtain

$$\nabla = \sum_{i,j} \frac{\hat{u}_i}{h'_j} (\hat{u}_i \cdot \hat{u}'_j) \cdot \frac{\partial}{\partial v'_j} , \tag{4.47}$$

but

$$\sum_i \hat{u}_i (\hat{u}_i \cdot \hat{u}'_j) = \bar{\bar{I}} \cdot \hat{u}'_j = \hat{u}'_j , \tag{4.48}$$

where $\bar{\bar{I}}$ denotes the idemfactor introduced in Chapter 1, now in terms of the dyads $\hat{u}_i \hat{u}_i$. Another interpretation of (4.48) is to write

$$\sum_i \hat{u}_i(\hat{u}_i \cdot \hat{u}'_j) = \sum_i \hat{u}_i \cos\alpha_{ij} = \hat{u}'_j, \tag{4.49}$$

where $\cos\alpha_{ij}$ denotes the directional cosines between $\hat{u}_i$ and $\hat{u}'_j$. Substituting (4.48) into (4.47), we obtain

$$\nabla = \sum_j \frac{\hat{u}'_j}{h'_j} \cdot \frac{\partial}{\partial v'_i} = \nabla'. \tag{4.50}$$

The proofs of the invariance of ∇ and $\nabla\!\!\!\!\nabla$ follow the same steps.

The invariance of the differential operators also ascertains the nature of these functions. To show their characteristics, let the primed and the unprimed coordinates represent two rectangular coordinate systems rotating with respect to each other as the ones formed in Section 1-3. The invariance of the gradient operator means

$$\sum_i \hat{x}_i \frac{\partial f}{\partial x_i} = \sum_j \hat{x}'_j \frac{\partial f}{\partial x'_j}.$$

By taking a scalar product of this equation with $\hat{x}'_k$, we find

$$\frac{\partial f}{\partial x'_k} = \sum_i a_{ki} \frac{\partial f}{\partial x_i}, \qquad k = 1, 2, 3.$$

These relations show that the components of the gradient obey the rules of transform of a polar vector.

The invariance of the divergence operator means

$$\sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} = \sum_j \hat{x}'_j \cdot \frac{\partial \mathbf{F}'}{\partial x'_j},$$

or

$$\sum_i \frac{\partial F_i}{\partial x_i} = \sum_j \frac{\partial F'_j}{\partial x'_j}.$$

The divergence of a vector function, therefore, is an invariant scalar.

The invariance of the curl operator has a more intricate implication. In the first place, the invariance of $\nabla\!\!\!\!\nabla\, \mathbf{F}$ means

$$\sum_i \hat{x}_i \left(\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} \right) = \sum_p \hat{x}'_p \left(\frac{\partial F'_r}{\partial x'_q} - \frac{\partial F'_q}{\partial x'_r} \right),$$

where $i, j, k = 1, 2, 3$ and $p, q, r = 1, 2, 3$ in cyclic order. By taking the scalar product of the previous equation with $\hat{x}'_\ell$, we obtain

$$\frac{\partial F'_n}{\partial x'_m} - \frac{\partial F'_m}{\partial x'_n} = \sum_i a_{\ell i} \left(\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} \right), \qquad \ell = 1, 2, 3,$$

where ℓ, m, n as well as $i, j, k = 1, 2, 3$ in cyclic order. If we denote

$$\frac{\partial F'_n}{\partial x'_m} - \frac{\partial F'_m}{\partial x'_n} = C'_\ell,$$
$$\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} = C_i,$$

then

$$C'_\ell = \sum_i a_{\ell i} C_i,$$

which shows C'_ℓ and C_i transform like a polar vector. On the other hand, if we denote

$$\frac{\partial F'_n}{\partial x'_m} - \frac{\partial F'_m}{\partial x'_n} = C'_{mn},$$
$$\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} = C_{jk},$$

then

$$\begin{aligned}
C'_{mn} &= \frac{\partial F'_n}{\partial x'_m} - \frac{\partial F'_m}{\partial x'_n} \\
&= \sum_j \frac{\partial F'_n}{\partial x_j}\frac{\partial x_j}{\partial x'_m} - \sum_j \frac{\partial F'_m}{\partial x_j}\frac{\partial x_j}{\partial x'_n} \\
&= \sum_j a_{mj}\frac{\partial}{\partial x_j}\sum_k a_{nk}F_k - \sum_j a_{nj}\frac{\partial}{\partial x_j}\sum_k a_{mk}F_k \\
&= \sum_{j,k}(a_{mj}a_{nk} - a_{nj}a_{mk})\frac{\partial F_k}{\partial x_j} \\
&= \sum_{j,k} a_{mj}a_{nk}\left(\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k}\right).
\end{aligned}$$

Hence

$$C'_{mn} = \sum_{j,k} a_{mj}a_{nk}C_{jk}.$$

The preceding relation shows that C'_{mn} and C_{jk} transform according to the rule of an antisymmetric tensor in a three-dimensional space. The tensor is antisymmetric because

$$C'_{mn} = -C'_{nm}, \qquad C_{jk} = -C_{kj}$$

and

$$C'_{mm} = 0, \qquad C_{jj} = 0.$$

In summary, the curl of $\mathbf{F}$ is basically an antisymmetric tensor, but its three distinct vector components C_i and C'_ℓ also transform like a polar vector. Its property therefore resembles an axial vector. However, one must not treat $\nabla \mathbf{F}$ as the vector product between ∇ and $\mathbf{F}$, which is a misleading interpretation; it is fully explained in Chapter 8.

4-4 Differential Formulas of the Symbolic Expression in the General Curvilinear System

The integral-differential expression that defines $T(\nabla)$ in (4.8) will now be evaluated to obtain a differential expression for $T(\nabla)$ in GCS, which was introduced in Chapter 2. We consider a volume bounded by six coordinate surfaces; then the running index in (4.8) goes from $i = 1$ to $i = 6$. Because $T(\hat{n}_i)$ is linear with respect to $\hat{n}_i$, we can combine ΔS_i with $\hat{n}_i$ to form a vector differential area. If that area corresponds to a segment $\hat{n}_i\, \Delta S_i$ on a coordinate surface with v_i being constant in GCS, then

$$\hat{n}_i\, \Delta S_i = \Delta \mathbf{S}_i = \mathbf{p}_j \times \mathbf{p}_k\, \Delta\omega^j\, \Delta\omega^k = \Lambda \mathbf{r}^i\, \Delta\omega^j\, \Delta\omega^k. \tag{4.51}$$

For a cell bounded by six surfaces located at $\omega^i \pm (\Delta\omega^i/2)$ with $i = 1, 2, 3$, its volume would be equal to

$$\Delta V = \Lambda\, \Delta\omega^i\, \Delta\omega^j\, \Delta\omega^k.$$

We can separate the surface sum in (4.8) into two groups:

$$T(\nabla) = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{3} \left[T(\Lambda \mathbf{r}^i)_{\omega^i+\Delta\omega^i/2} - T(\Lambda \mathbf{r}^i)_{\omega^i-\Delta\omega^i/2}\right] \Delta\omega^j\, \Delta\omega^k}{\Lambda\, \Delta\omega^i\, \Delta\omega^j\, \Delta\omega^k}.$$

The limit yields

$$T(\nabla) = \frac{1}{\Lambda} \sum_{i=1}^{3} \frac{\partial}{\partial \omega^i} T(\Lambda \mathbf{r}^i). \tag{4.52}$$

Equation (4.52) is the differential expression for $T(\nabla)$ in GCS. From now on, it is understood that the summation index i goes from 1 to 3 unless specified otherwise. We now consider the three symbolic expressions ∇f, $\nabla \cdot \mathbf{F}$, and $\nabla \cdot \mathbf{F}$ in GCS.

Because $T(\Lambda \mathbf{r}^i)$ is linear with respect to $\Lambda \mathbf{r}^i$, there are also three possible combinations of $\Lambda \mathbf{r}^i$ with the remaining part of $T(\Lambda \mathbf{r}^i)$. In a rather compact notation, we can write $T(\Lambda \mathbf{r}^i)$ for the three cases in the form

$$T(\Lambda \mathbf{r}^i) = \Lambda \mathbf{r}^i * \tilde{f}, \tag{4.53}$$

where $*$ represents either a null (absent) when $\tilde{f}$ is a scalar function, or a dot (scalar product symbol), or a cross (vector product symbol) when $\tilde{f}$ is a vector function. We list these cases in Table 4-1. Substituting (4.53) into (4.52), we obtain

$$T(\nabla) = \frac{1}{\Lambda} \sum_i \frac{\partial}{\partial \omega^i} (\Lambda \mathbf{r}^i * \tilde{f}) = \frac{1}{\Lambda} \sum_i \left[\Lambda \mathbf{r}^i * \frac{\partial \tilde{f}}{\partial \omega^i} + \frac{\partial (\Lambda \mathbf{r}^i)}{\partial \omega^i} * \tilde{f} \right]. \tag{4.54}$$

Table 4-1: The Three Simplest Forms of $T(\Lambda\mathbf{r}^i)$

Case	$T(\nabla, \tilde{f})$	$T(\Lambda\mathbf{r}^i)$	$*$	$\tilde{f}$
1	$\nabla f = f\nabla$	$\Lambda\mathbf{r}^i f$	null	f, scalar
2	$\nabla\cdot\mathbf{F} = \mathbf{F}\cdot\nabla$	$\Lambda\mathbf{r}^i\cdot\mathbf{F}$	$\cdot$	$\mathbf{F}$, vector
3	$\nabla\times\mathbf{F} = -\mathbf{F}\times\nabla$	$\Lambda\mathbf{r}^i\times\mathbf{F}$	$\times$	$\mathbf{F}$, vector

The second term in (4.54) vanishes as a result of the closed surface theorem stated by (2.27), hence

$$T(\nabla) = \sum_i \mathbf{r}^i * \frac{\partial \tilde{f}}{\partial \omega^i}. \tag{4.55}$$

We now treat the cases listed in Table 4-1 individually.

1. **The gradient of a scalar function**

   In this case, we have

   $$T(\nabla) = \nabla f = f\nabla, \qquad \tilde{f} = f,$$

   and $*$ is null or absent in (4.55); then

   $$\nabla f = \sum_i \mathbf{r}^i \frac{\partial f}{\partial \omega^i}. \tag{4.56}$$

   This function is the *gradient* of f in GCS. The gradient operator is now defined by

   $$\nabla = \sum_i \mathbf{r}^i \frac{\partial}{\partial \omega^i} \quad \text{(gradient operator).} \tag{4.57}$$

   Thus,

   $$\sum_i \mathbf{r}^i \frac{\partial f}{\partial \omega^i} = \nabla f. \tag{4.58}$$

   In summary, the gradient of a scalar function f in GCS is represented by

   $$\nabla f = \lim_{\Delta V\to 0} \frac{\sum_i (\hat{n}_i f)\,\Delta S_i}{\Delta V} = \sum_i \mathbf{r}^i \frac{\partial f}{\partial \omega^i}. \tag{4.59}$$

2. **The divergence**

   In this case, we have

   $$T(\nabla) = \nabla\cdot\mathbf{F} = \mathbf{F}\cdot\nabla, \qquad T(\Lambda\mathbf{r}^i) = \Lambda\mathbf{r}^i\cdot\mathbf{F}, \qquad * = \cdot, \qquad \tilde{f} = \mathbf{F}.$$

   Substituting these quantities into (4.55), we obtain

   $$\nabla\cdot\mathbf{F} = \sum_i \mathbf{r}^i \cdot \frac{\partial \mathbf{F}}{\partial \omega^i}. \tag{4.60}$$

The divergence operator now has the form

$$\nabla = \sum_i \mathbf{r}^i \cdot \frac{\partial}{\partial \omega^i} \quad \text{(divergence operator)}. \tag{4.61}$$

Hence

$$\sum_i \mathbf{r}^i \cdot \frac{\partial \mathbf{F}}{\partial \omega^i} = \nabla \mathbf{F}. \tag{4.62}$$

In summary, the divergence of a vector function $\mathbf{F}$ in GCS is represented by

$$\nabla \mathbf{F} = \lim_{\Delta V \to 0} \frac{\sum_i (\hat{n}_i \cdot \mathbf{F})\, \Delta S_i}{\Delta V} = \sum_i \mathbf{r}^i \cdot \frac{\partial \mathbf{F}}{\partial \omega^i}. \tag{4.63}$$

It will be recalled that the operational form of this function was originally derived from the following expression:

$$\nabla \mathbf{F} = \frac{1}{\Lambda} \sum_i \frac{\partial}{\partial \omega^i} (\Lambda \mathbf{r}^i \cdot \mathbf{F}). \tag{4.64}$$

According to (2.11),

$$\mathbf{r}^i \cdot \mathbf{F} = g^i,$$

where g^i denotes the reciprocal components of $\mathbf{F}$ in GCS; hence the divergence of $\mathbf{F}$ can be written in the form

$$\nabla \mathbf{F} = \frac{1}{\Lambda} \sum_i \frac{\partial}{\partial \omega^i} (\Lambda g^i). \tag{4.65}$$

The differentiations are now applied to the scalar functions Λg^i; it is no longer applied to the full vector function $\mathbf{F}$ as in (4.62) or (4.63).

3. **The curl**

In this case, we have

$$T(\nabla) = \nabla \times \mathbf{F} = -\mathbf{F} \times \nabla, \qquad T(\Lambda \mathbf{r}^i) = \Lambda \mathbf{r}^i \times \mathbf{F}, \qquad * = \times, \qquad \tilde{f} = \mathbf{F}.$$

Then

$$\nabla \times \mathbf{F} = \sum_i \mathbf{r}^i \times \frac{\partial \mathbf{F}}{\partial \omega^i}.$$

The curl operator now has the form

$$\nabla = \sum_i \mathbf{r}^i \times \frac{\partial}{\partial \omega^i} \quad \text{(curl operator)}. \tag{4.66}$$

Hence

$$\sum_i \mathbf{r}^i \times \frac{\partial \mathbf{F}}{\partial \omega^i} = \nabla \mathbf{F}. \tag{4.67}$$

The curl of a vector function in GCS is therefore represented by

$$\nabla\!\!\!\!\nabla\, \mathbf{F} = \lim_{\Delta V \to 0} \frac{\sum_i (\hat{n}_i \times \mathbf{F})\, \Delta S_i}{\Delta V} = \sum_i \mathbf{r}^i \times \frac{\partial \mathbf{F}}{\partial \omega^i}. \tag{4.68}$$

The operational form of curl **F** was originally derived from the following expression:

$$\nabla\!\!\!\!\nabla\, \mathbf{F} = \frac{1}{\Lambda} \sum_i \frac{\partial}{\partial \omega^i} (\Lambda \mathbf{r}^i \times \mathbf{F}). \tag{4.69}$$

The vector product in (4.69) can be expressed in terms of the primary components of **F** in GCS. We start with

$$\mathbf{F} = \sum_j f_j \mathbf{r}^j, \tag{4.70}$$

where f_j denotes a primary component of **F** as stated by (2.10); then

$$\mathbf{r}^i \times \mathbf{F} = \sum_j f_j \mathbf{r}^i \times \mathbf{r}^j, \qquad i = 1, 2, 3. \tag{4.71}$$

More specifically,

$$\begin{aligned}
\mathbf{r}^1 \times \mathbf{F} &= f_2 \mathbf{r}^1 \times \mathbf{r}^2 + f_3 \mathbf{r}^1 \times \mathbf{r}^3 = \frac{1}{\Lambda}(f_2 \mathbf{p}_3 - f_3 \mathbf{p}_2), \\
\mathbf{r}^2 \times \mathbf{F} &= f_1 \mathbf{r}^2 \times \mathbf{r}^1 + f_3 \mathbf{r}^2 \times \mathbf{r}^3 = \frac{1}{\Lambda}(-f_1 \mathbf{p}_3 + f_3 \mathbf{p}_1), \\
\mathbf{r}^3 \times \mathbf{F} &= f_1 \mathbf{r}^3 \times \mathbf{r}^1 + f_2 \mathbf{r}^3 \times \mathbf{r}^2 = \frac{1}{\Lambda}(f_1 \mathbf{p}_2 - f_2 \mathbf{p}_1),
\end{aligned}$$

where $\mathbf{p}_i$ and $\mathbf{f}_i$ denote the primary vectors and the primary components, respectively, of **F**. Substituting these expressions into (4.69), we have

$$\nabla\!\!\!\!\nabla\, \mathbf{F} = \frac{1}{\Lambda} \left[\frac{\partial}{\partial \omega^1}(f_2 \mathbf{p}_3 - f_3 \mathbf{p}_2) + \frac{\partial}{\partial \omega^2}(f_3 \mathbf{p}_1 - f_1 \mathbf{p}_3) + \frac{\partial}{\partial \omega^3}(f_1 \mathbf{p}_2 - f_2 \mathbf{p}_1) \right]. \tag{4.72}$$

The derivative of the first term in (4.72) consists of two parts:

$$\frac{\partial}{\partial \omega^1}(f_2 \mathbf{p}_3) = f_2 \frac{\partial \mathbf{p}_3}{\partial \omega^1} + \mathbf{p}_3 \frac{\partial f_2}{\partial \omega^1}. \tag{4.73}$$

The derivative of the last term in (4.72) gives

$$\frac{\partial}{\partial \omega^3}(-f_2 \mathbf{p}_1) = -f_2 \frac{\partial \mathbf{p}_1}{\partial \omega^3} - \mathbf{p}_1 \frac{\partial f_2}{\partial \omega^3}. \tag{4.74}$$

According to (2.2),

$$\mathbf{p}_3 = \frac{\partial \mathbf{R}_p}{\partial \omega^3} \quad \text{and} \quad \mathbf{p}_1 = \frac{\partial \mathbf{R}_p}{\partial \omega^1};$$

hence

$$\frac{\partial^2 \mathbf{R}_p}{\partial \omega^1 \, \partial \omega^3} = \frac{\partial \mathbf{p}_3}{\partial \omega^1} = \frac{\partial \mathbf{p}_1}{\partial \omega^3}.$$

The first two terms at the right sides of (4.73) and (4.74) are therefore equal and opposite in sign. Six terms in (4.69) involving the derivatives of the primary vectors cancel each other in this manner; the net result yields

$$\nabla\times\mathbf{F} = \frac{1}{\Lambda}\left[\mathbf{p}_1\left(\frac{\partial f_3}{\partial \omega^2} - \frac{\partial f_2}{\partial \omega^3}\right) + \mathbf{p}_2\left(\frac{\partial f_1}{\partial \omega^3} - \frac{\partial f_3}{\partial \omega^1}\right) + \mathbf{p}_3\left(\frac{\partial f_2}{\partial \omega^1} - \frac{\partial f_1}{\partial \omega^2}\right)\right],$$

or

$$\nabla\times\mathbf{F} = \frac{1}{\Lambda}\sum_i \mathbf{p}_i\left(\frac{\partial f_k}{\partial \omega^j} - \frac{\partial f_j}{\partial \omega^k}\right) \tag{4.75}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order. The expressions for ∇f, $\nabla\cdot\mathbf{F}$, and $\nabla\times\mathbf{F}$ given by (4.60), (4.65), and (4.75) have previously been derived by Stratton [5, p. 44] [based on the variation of f (total differential of f) for ∇f], Gauss's theorem for $\nabla\cdot\mathbf{F}$, and Stokes's theorem for $\nabla\times\mathbf{F}$. We have not yet touched upon these theorems. Our derivation is based on only one formula, namely, the differential expression of the symbolic expression $T(\nabla)$ as stated by (4.8). In summary, the three differential operators in GCS have the forms

$$\nabla = \sum_i \mathbf{r}^i \frac{\partial}{\partial \omega^i} \quad \text{(gradient operator)},$$

$$\nabla\cdot = \sum_i \mathbf{r}^i \cdot \frac{\partial}{\partial \omega^i} \quad \text{(divergence operator)},$$

$$\nabla\times = \sum_i \mathbf{r}^i \times \frac{\partial}{\partial \omega^i} \quad \text{(curl operator)}.$$

The three operators can be condensed into one formula:

$$\nabla * = \sum_i \mathbf{r}^i * \frac{\partial}{\partial \omega^i}, \tag{4.76}$$

where $*$ represents a null, a dot, or a cross. We would like to emphasize that these operators are independent of each other; in other words, they are distinctly different differential operators, and they are invariant with respect to the choice of the coordinate system. We leave the proof in GCS as an exercise for the reader. For the three functions, they can be written in a compact form

$$\nabla * \tilde{f} = \sum_i \mathbf{r}^i * \frac{\partial \tilde{f}}{\partial \omega^i}, \tag{4.77}$$

where $\tilde{f}$ can be a scalar (for the gradient) or vector (for the divergence or curl). This completes our presentation of the differential expressions of the three key functions in their most general form.

The expressions of the three key functions in OCS given by (4.35) to (4.37) can now be treated as the special case of the formulas in GCS. We let

$$\mathbf{p}_i = h_i\,\hat{u}_i, \qquad i = 1, 2, 3 \tag{4.78}$$

according to (2.28), where h_i and $\hat{u}_i$ denote, respectively, the metric coefficients and the unit vectors in OCS. The unit vectors are orthogonal to each other; thus,

$$\mathbf{p}_i \cdot \mathbf{p}_j = \begin{cases} h_i^2, & i = j \\ 0, & i \neq j \end{cases}$$

$$\mathbf{p}_i \times \mathbf{p}_j = \begin{cases} h_i h_j\,\hat{u}_k, & i \neq j \neq k \\ 0, & i = j \end{cases}$$

for $(i, j, k) = (1, 2, 3)$ in cyclic order. The parameter Λ reduces to

$$\Lambda = \mathbf{p}_i \cdot (\mathbf{p}_j \times \mathbf{p}_k) = h_i h_j h_k = \Omega.$$

The reciprocal vectors $\mathbf{r}^i$ become

$$\mathbf{r}^i = \frac{1}{\Lambda}\mathbf{p}_j \times \mathbf{p}_k = \frac{h_j h_k}{\Omega}\,\hat{u}_i = \frac{\hat{u}_i}{h_i}. \tag{4.79}$$

The differential expression of the symbolic expression in OCS is then given by

$$T(\nabla\!\!\!\!{\scriptstyle *}\;) = \frac{1}{\Omega}\sum_i \frac{\partial}{\partial\omega^i}\,T\left(\frac{\Omega}{h_i}\,\hat{u}_i\right). \tag{4.80}$$

We still use ω^i to denote the coordinate variables in OCS. The operational form of the three key functions previously described by

$$\nabla\!\!\!\!{\scriptstyle *}\;\tilde{f} = \sum_i \mathbf{r}^i * \frac{\partial \tilde{f}}{\partial\omega^i} \tag{4.81}$$

reduces to

$$\nabla\!\!\!\!{\scriptstyle *}\;\tilde{f} = \sum_i \frac{\hat{u}_i}{h_i} * \frac{\partial \tilde{f}}{\partial\omega^i}, \tag{4.82}$$

or more specifically,

$$\nabla f = \sum_i \frac{\hat{u}_i}{h_i}\,\frac{\partial f}{\partial\omega^i}, \tag{4.83}$$

$$\nabla\!\!\!\!{\scriptstyle\cdot}\;\mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial \mathbf{F}}{\partial\omega^i}, \tag{4.84}$$

$$\nabla\!\!\!\!{\scriptstyle\times}\;\mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial\omega^i}. \tag{4.85}$$

To find the component form of $\nabla\!\!\!\!{\scriptstyle\cdot}\;\mathbf{F}$ and $\nabla\!\!\!\!{\scriptstyle\times}\;\mathbf{F}$ in OCS, we let

$$\mathbf{F} = \sum_i F_i\,\hat{u}_i. \tag{4.86}$$

The primary and the reciprocal components of $\mathbf{F}$, in view of (2.10) and (2.11), are related to F_i by

$$f_i = h_i F_i, \qquad i = 1, 2, 3, \tag{4.87}$$

$$g_i = \frac{1}{h_i} F_i, \qquad i = 1, 2, 3. \tag{4.88}$$

The component forms of $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ as given by (4.65) and (4.75) respectively become

$$\nabla \cdot \mathbf{F} = \frac{1}{\Omega} \sum_i \frac{\partial}{\partial \omega^i} \left(\frac{\Omega}{h_i} F_i \right), \tag{4.89}$$

$$\nabla \times \mathbf{F} = \frac{1}{\Omega} \sum_i h_i\, \hat{u}_i \left[\frac{\partial (h_j f_j)}{\partial \omega^k} - \frac{\partial (h_k f_k)}{\partial \omega^j} \right]. \tag{4.90}$$

In the special case of an orthogonal linear system or the rectangular system $\hat{u}_i = \hat{x}_i$, $\omega^i = x_i$, $h_i = 1$ for $i = 1, 2, 3$, and $\Omega = 1$. Equations (4.83), (4.89), and (4.90) are the commonly used formulas for these functions in devising physical problems.

4-5 Alternative Definitions of Gradient and Curl

To distinguish these functions in a conceptual manner, we propose some names for the integral-differential expressions of these functions based on a "physical" model. Thus, the quantity $\sum_i \hat{n}_i f \Delta S_i$ in the second term of (4.59) will be identified as the total *directional radiance* of f from the volume cell ΔV, or *radiance* for short; the gradient is then a measure of radiance per unit volume. The quantity $\sum_i \hat{n}_i \cdot \mathbf{F} \Delta S_i$ in the second term of (4.63) has a well-established name used by many authors as the *total flux* of $\mathbf{F}$ from ΔV, or *flux* for short; the divergence is then a measure of flux per unit volume. For the vector quantity $\sum_i \hat{n}_i \times \mathbf{F} \Delta S_i$ in the second term of (4.68), we propose the name of the total *shear* of $\mathbf{F}$ around the enclosing volume ΔV, or *shear* for short; the curl is then a measure of shear per unit volume. From the mathematical point of view, there is no need to invoke this physical model. It is proposed here merely as an aid to distinguish these functions.

The expressions for the gradient and the curl that have been derived by the method of symbolic vector can be derived alternatively by two different approaches.

In the defining integral-differential expression for ∇f, let the shape of ΔV be a flat cell of uniform thickness Δs and area ΔA at the broad surfaces, as shown in Fig. 4-1. The outward normal unit vector is denoted by $\hat{s}$. By taking a scalar product of the third term of (4.59) with $\hat{s}$, we obtain

$$\hat{s} \cdot \nabla f = \lim_{\Delta V \to 0} \frac{\sum_i f(\hat{s} \cdot \hat{n}_i) \Delta S_i}{\Delta V} = \lim_{\substack{\Delta V \to 0 \\ \Delta s \to 0}} \frac{\sum_i f(\hat{s} \cdot \hat{n}_i) \Delta S_i}{\Delta A \, \Delta s}. \tag{4.91}$$

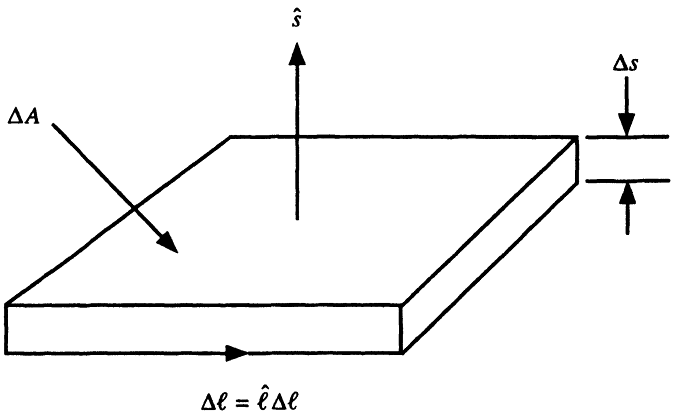

Figure 4-1 Thin flat volume of uniform thickness Δs and area ΔA.

The scalar product $\hat{s} \cdot \hat{n}_i$ vanishes for all the side surfaces because $\hat{s}$ is perpendicular to $\hat{n}_i$ therein. The only contributions result from the top and the bottom surfaces where $\hat{n}_i = \pm\hat{s}$ and $\Delta S_i = \Delta A$; thus, we obtain

$$\hat{s} \cdot \nabla f = \lim_{\Delta s \to 0} \frac{\left[f\left(s + \dfrac{\Delta s}{2} \right) - f\left(s + \dfrac{\Delta s}{2} \right) \right]}{\Delta s} = \frac{\partial f}{\partial s}, \tag{4.92}$$

where $s \pm \Delta s/2$ correspond to the locations of the broad surfaces along s, and the center of the flat cell is located at s. Equation (4.92) can be treated as an alternative definition of the component of the gradient in an arbitrary direction s.

By the rule of chain differentiation,

$$\frac{\partial f}{\partial s} = \sum_i \frac{\partial \omega^i}{\partial s} \frac{\partial f}{\partial \omega^i}, \tag{4.93}$$

where ω^i denotes one of the coordinate variables in GCS. Equation (4.93) can be interpreted as the scalar product between

$$\nabla f = \sum_i \mathbf{r}^i \frac{\partial f}{\partial \omega^i} \tag{4.94}$$

and

$$\hat{s} = \sum_j \mathbf{p}_j \frac{\partial \omega^j}{\partial s}, \tag{4.95}$$

where we have used the relation

$$\hat{s}\, ds = \sum \mathbf{p}_j \, d\omega^j \tag{4.96}$$

to obtain (4.95). $\mathbf{p}_j$ and $\mathbf{r}^i$ are, respectively, the primary and reciprocal vectors in GCS. Equation (4.93), therefore, is the same as

$$\frac{\partial f}{\partial s} = \hat{s} \cdot \nabla f,$$

previously derived by means of (4.91).

The same model can be used to find a typical component of the curl of $\mathbf{F}$ in GCS. Let us assume that $\hat{s}$ represents the unit normal in the direction of $\mathbf{r}^1$ in GCS, that is,

$$\hat{s} = \frac{\mathbf{r}^1}{(\mathbf{r}^1 \cdot \mathbf{r}^1)^{1/2}} = \hat{r}_1. \tag{4.97}$$

By taking the scalar product of the second term of (4.68) with $\hat{r}_1$, we obtain

$$\hat{r}_1 \cdot \nabla \mathbf{F} = \lim_{\Delta V \to 0} \frac{\sum_i \hat{r}_1 \cdot (\hat{n}_i \times \mathbf{F})\, \Delta S_i}{\Delta V} = \lim_{\substack{\Delta A \to 0 \\ \Delta s \to 0}} \frac{\sum_i (\hat{r}_1 \times \hat{n}_i) \cdot \mathbf{F}\, \Delta S_i}{\Delta A\, \Delta s}. \tag{4.98}$$

The area of the broad surface, ΔA, corresponds to

$$\Delta A = \left[(\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_2 \times \mathbf{p}_3)\right]^{1/2} \Delta\omega^2\, \Delta\omega^3.$$

Because

$$\mathbf{r}^1 = \frac{1}{\Lambda} \mathbf{p}_2 \times \mathbf{p}_3,$$

it is evident that

$$\Delta A = \Lambda[\mathbf{r}^1 \cdot \mathbf{r}^1]^{1/2}\, \Delta\omega^2\, \Delta\omega^3. \tag{4.99}$$

In (4.98), the only contributions come from the side surfaces where

$$\hat{r}_1 \times \hat{n}_i = \hat{\ell}_j \quad \text{and} \quad \Delta S_i = \Delta s\, \Delta\ell_j.$$

$\hat{\ell}_j\, \Delta\ell_j$ represents a segment of the contour around the periphery of the broad surface. Equation (4.98) then reduces to

$$\hat{r}_1 \cdot \nabla \mathbf{F} = \lim_{\Delta A \to 0} \frac{\sum_j \mathbf{F} \cdot \Delta\boldsymbol{\ell}_j}{\Delta A}. \tag{4.100}$$

By considering a contour formed by $\mathbf{p}_2\, \Delta\omega^2$ and $\mathbf{p}_3\, \Delta\omega^3$ with center located at ω^2 and ω^3, we obtain

$$\hat{r}_1 \cdot \nabla \mathbf{F} = \lim_{\substack{\Delta\omega^2 \to 0 \\ \Delta\omega^3 \to 0}} \left[\frac{(\mathbf{F} \cdot \mathbf{p}_2)_{\omega^3 - \Delta\omega^3/2} - (\mathbf{F} \cdot \mathbf{p}_2)_{\omega^3 + \Delta\omega^3/2}}{\Delta\omega^3} \right.$$
$$\left. + \frac{(\mathbf{F} \cdot \mathbf{p}_3)_{\omega^2 + \Delta\omega^2/2} - (\mathbf{F} \cdot \mathbf{p}_3)_{\omega^2 - \Delta\omega^2/2}}{\Delta\omega^2} \right] \frac{1}{\Lambda(\mathbf{r}^1 \cdot \mathbf{r}^1)^{1/2}}$$
$$= \frac{1}{\Lambda(\mathbf{r}^1 \cdot \mathbf{r}^1)^{1/2}} \left[\frac{\partial(\mathbf{p}_3 \cdot \mathbf{F})}{\partial\omega^2} - \frac{\partial(\mathbf{p}_2 \cdot \mathbf{F})}{\partial\omega^3} \right]$$

or

$$\mathbf{r}^1 \cdot \nabla \mathbf{F} = \frac{1}{\Lambda}\left(\frac{\partial f_3}{\partial \omega^2} - \frac{\partial f_2}{\partial \omega^3}\right), \tag{4.101}$$

where f_2 and f_3 denote two of the primary components of $\mathbf{F}$ in GCS. In general, we find

$$\mathbf{r}^i \cdot \nabla \mathbf{F} = \frac{1}{\Lambda}\left(\frac{\partial f_k}{\partial \omega^j} - \frac{\partial f_j}{\partial \omega^k}\right), \tag{4.102}$$

where $(i, j, k) = (1, 2, 3)$ in cyclic order. Equation (4.102) represents one component of (4.75) because

$$\sum_i \mathbf{p}_i(\mathbf{r}^i \cdot \nabla \mathbf{F}) = \nabla \mathbf{F} = \sum_i \frac{\mathbf{p}_i}{\Lambda}\left(\frac{\partial f_k}{\partial \omega^j} - \frac{\partial f_j}{\partial \omega^k}\right).$$

4-6 The Method of Gradient

Once the differential expressions of certain functions are available in terms of different coordinate variables, we can derive many relations from them by taking advantage of the invariance property of these functions. The method of gradient is based on this principle. We will use an example to illustrate this method.

It is known that the relationships between (x, y), two of the rectangular variables, and (r, ϕ), two of the cylindrical variables, are

$$x = r\cos\phi, \tag{4.103}$$

$$y = r\sin\phi, \tag{4.104}$$

$$r = \left(x^2 + y^2\right)^{1/2}, \tag{4.105}$$

and

$$\phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{4.106}$$

By taking the gradient of (4.103) and (4.104) in the rectangular system for x and y on the left sides of these two equations, and in the cylindrical coordinate system on the right sides, we obtain

$$\hat{x} = \cos\phi\,\hat{r} - \sin\phi\,\hat{\phi}, \tag{4.107}$$

$$\hat{y} = \sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}. \tag{4.108}$$

By doing the same for (4.105) and (4.106), but in reverse order, we obtain

$$\hat{r} = \frac{x}{\left(x^2 + y^2\right)^{1/2}}\hat{x} + \frac{y}{\left(x^2 + y^2\right)^{1/2}}\hat{y} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}, \tag{4.109}$$

$$\frac{\hat{\phi}}{r} = \frac{-y}{x^2 + y^2}\hat{x} + \frac{x}{x^2 + y^2}\hat{y},$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}. \tag{4.110}$$

These relations can be derived by a geometrical method, but the method of gradient is straightforward, particularly if the orthogonal system is a more complicated one compared to the cylindrical and spherical coordinate systems. Equations (4.107)–(4.110), together with the unit vector $\hat{z}$, can be tabulated in a matrix form, as shown in Table 4-2. The table can be used in both directions. Horizontally, it gives

$$\hat{r} = \cos\phi\, \hat{x} + \sin\phi\, \hat{y}, \tag{4.111}$$

$$\hat{\phi} = -\sin\phi\, \hat{x} + \cos\phi\, \hat{y}, \tag{4.112}$$

which are the same as (4.109) and (4.110). Vertically, it yields

$$\hat{x} = \cos\phi\, \hat{r} - \sin\phi\, \hat{\phi}, \tag{4.113}$$

$$\hat{y} = \sin\phi\, u_r + \cos\phi\, \hat{\phi}, \tag{4.114}$$

which can be derived algebraically by solving for $\hat{x}$ and $\hat{y}$ from (4.109) and (4.110) in terms of $\hat{r}$ and $\hat{\phi}$. Each coefficient in Table 4-2 corresponds to the scalar product of the two unit vectors in the intersecting column and row; thus, $\hat{r} \cdot \hat{x} = \cos\phi$, $\hat{\phi} \cdot \hat{x} = -\sin\phi$, and so on. For this reason, the same table is applicable to the transformation of the scalar components of a vector in the two systems. Because

$$\mathbf{f} = f_x\hat{x} + f_y\hat{y} + f_z\hat{z} = f_r\hat{r} + f_\phi\hat{\phi} + f_z\hat{z},$$

it follows that

$$f_x = \hat{x} \cdot \mathbf{f} = (\hat{x} \cdot \hat{r}) f_r + (\hat{x} \cdot \hat{\phi}) f_\phi = \cos\phi\, f_r - \sin\phi\, f_\phi \tag{4.115}$$

and

$$f_y = \hat{y} \cdot \mathbf{f} = \left(\hat{y} \cdot \hat{r}\right) f_r + \left(\hat{y} \cdot \hat{\phi}\right) f_\phi = \sin\phi\, f_r + \cos\phi\, f_\phi. \tag{4.116}$$

These relations are of the same form as (4.107) and (4.108). The transformations of the unit vectors of the orthogonal system reviewed in Section 2-1, and the unit vectors in the rectangular system, are listed in Appendix A, including the cylindrical system just described.

Table 4-2: Transportation of Unit Vectors

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{r}$	$\cos\phi$	$\sin\phi$	0
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0
$\hat{z}$	0	0	1

As another example, let us consider the problem of relating $(\hat{R}, \hat{\theta}, \hat{\phi})$ to $(\hat{R}, \hat{\alpha}, \hat{\beta})$ of another spherical system in which the polar angle α is measured from the x axis, and the azimuthal angle β is measured with respect to the x–y plane; thus,

$$x = R\sin\theta\cos\phi = R\cos\alpha, \tag{4.117}$$

$$y = R\sin\theta\sin\phi = R\sin\alpha\cos\beta, \tag{4.118}$$

$$z = R\cos\theta = R\sin\alpha\sin\beta. \tag{4.119}$$

We are seeking the relationships between $(\hat{\alpha}, \hat{\beta})$ and $(\hat{\theta}, \hat{\phi})$. The metric coefficients of the two systems are $(1, R, R\sin\theta)$ and $(1, R, R\sin\alpha)$. By taking the gradient of $\sin\theta\cos\phi = \cos\alpha$ in the two systems, we obtain

$$\frac{1}{R}\frac{\partial}{\partial\theta}(\sin\theta\cos\phi)\,\hat{\theta} + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}(\sin\theta\cos\phi)\,\hat{\phi} = \frac{1}{R}\frac{\partial}{\partial\alpha}(\cos\alpha)\,\hat{\alpha}. \tag{4.120}$$

Hence $-\sin\alpha\,\hat{\alpha} = \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$, or

$$\hat{\alpha} = \frac{-1}{\left(1-\sin^2\theta\cos^2\phi\right)^{1/2}}\left(\cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right). \tag{4.121}$$

By taking the gradient of

$$\cot\beta = \tan\theta\sin\phi = \left(\frac{y}{z}\right), \tag{4.122}$$

we obtain

$$\hat{\beta} = \frac{-1}{\left(1-\sin^2\theta\cos^2\phi\right)^{1/2}}\left(\sin\phi\,\hat{\theta} + \cos\theta\cos\phi\,\hat{\phi}\right). \tag{4.123}$$

From (4.121) and (4.123), we can solve for $\hat{\theta}$ and $\hat{\phi}$ in terms of $\hat{\alpha}$ and $\hat{\beta}$. An alternative method is to use (4.119) and the relation

$$\tan\phi = \tan\alpha\cos\beta\left(= \frac{y}{x}\right), \tag{4.124}$$

and repeat the same operations; we then obtain

$$\hat{\theta} = \frac{-1}{\left(1-\sin^2\alpha\sin^2\beta\right)^{1/2}}\left(\cos\alpha\sin\beta\,\hat{\alpha} + \cos\beta\,\hat{\beta}\right) \tag{4.125}$$

and

$$\hat{\phi} = \frac{1}{\left(1-\sin^2\alpha\sin^2\beta\right)^{1/2}}\left(\cos\beta\,\hat{\alpha} - \cos\alpha\sin\beta\,\hat{\beta}\right). \tag{4.126}$$

The reader can verify these expressions by solving for $\hat{\theta}$ and $\hat{\phi}$ from (4.121) and (4.123) at the expense of a tedious calculation.

These relations are very useful in antenna theory when one is interested in finding the resultant field of two linear antennas placed at the origin, with one antenna pointed in the z direction and another one pointed in the x direction. In order to calculate the resultant distant field, the individual field must be expressed in a common coordinate system, say (R, θ, ϕ) in this case. Because the field of the x-directed antenna is proportional to $\hat{\alpha}$, (4.121) can be used to combine it with the field of the z-directed antenna, whose field is proportional to $\hat{\theta}$. In fact, it is this technical problem that motivated the author to formulate the method of gradient many years ago.

The method of gradient can also be used effectively to derive the expressions for the divergence operator and the curl operator in the orthogonal curvilinear system from their expressions in the rectangular system. In the rectangular system, the divergence operator and the curl operator are given respectively by

$$\nabla = \sum_i \hat{x}_i \cdot \frac{\partial}{\partial x_i}, \tag{4.127}$$

$$\boldsymbol{\nabla} = \sum_i \hat{x}_i \times \frac{\partial}{\partial x_i}. \tag{4.128}$$

Upon applying the method of gradient to the coordinate variables x_i with $i = (1, 2, 3)$, we obtain

$$\hat{x}_i = \nabla x_i = \sum_j \frac{\hat{u}_j}{h_j} \frac{\partial x_i}{\partial v_j}, \tag{4.129}$$

and by the chain rule of differentiation,

$$\frac{\partial}{\partial x_i} = \sum_k \frac{\partial v_k}{\partial x_i} \frac{\partial}{\partial v_k}. \tag{4.130}$$

We also have the relations

$$\frac{\partial v_i}{\partial v_j} = \sum_k \frac{\partial v_i}{\partial x_k} \frac{\partial x_k}{\partial v_j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4.131}$$

Upon substituting (4.129) and (4.130) into (4.127) and (4.128), and making use of (4.131), we find

$$\nabla = \sum_i \hat{x}_i \cdot \frac{\partial}{\partial x_i} = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i}, \tag{4.132}$$

$$\boldsymbol{\nabla} = \sum_i \hat{x}_i \times \frac{\partial}{\partial x_i} = \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i}. \tag{4.133}$$

This exercise shows again that the divergence operator and the curl operator, as with the del operator for the gradient, are invariant to the choice of the coordinate system, a property we have demonstrated before.

4-7 Symbolic Expressions with Two Functions and the Partial Symbolic Vectors

Symbolic expressions with two functions are represented by $T(\nabla, \widetilde{a}, \widetilde{b})$, where $\widetilde{a}$ and $\widetilde{b}$ both can be scalars, vectors, or one each. In this section, we will introduce a new method for finding the identities of these functions in terms of the individual functions $\widetilde{a}$ and $\widetilde{b}$ without using the otherwise tedious method in differential calculus.

Because of the invariance theorem, it is sufficient to use the differential expression of $T(\nabla, \widetilde{a}, \widetilde{b})$ in the rectangular system to describe this new method. In the rectangular system,

$$T(\nabla, \widetilde{a}, \widetilde{b}) = \sum_i \frac{\partial}{\partial x_i} T(\hat{x}_i, \widetilde{a}, \widetilde{b}). \tag{4.134}$$

We now introduce two partial symbolic vectors, denoted by ∇_a and ∇_b, which are defined by the following two equations:

$$T(\nabla_a, \widetilde{a}, \widetilde{b}) = \sum_i \frac{\partial}{\partial x_i} T(\hat{x}_i, \widetilde{a}, \widetilde{b})_{\widetilde{b}=\text{constant}}, \tag{4.135}$$

$$T(\nabla_b, \widetilde{a}, \widetilde{b}) = \sum_i \frac{\partial}{\partial x_i} T(\hat{x}_i, \widetilde{a}, \widetilde{b})_{\widetilde{a}=\text{constant}}. \tag{4.136}$$

In (4.135), $\widetilde{b}$ is considered to be constant, and in (4.136), $\widetilde{a}$ is considered to be constant. The process is similar to the partial differentiation of a function of two independent variables, that is,

$$\frac{\partial f(x, y)}{\partial x} = \left[\frac{df(x, y)}{dx}\right]_{y=\text{constant}}. \tag{4.137}$$

The name *partial symbolic vector* was chosen because of this analogy. It is obvious that Lemma 4.1 is also applicable to symbolic expressions defined with a partial symbolic vector, because, in general,

$$T(\nabla_a, \widetilde{a}, \widetilde{b}) = \lim_{\Delta V \to 0} \frac{\sum_i T(\hat{n}_i, \widetilde{a}, \widetilde{b})_{\widetilde{b}=c} \, \Delta S_i}{\Delta V} \tag{4.138}$$

and

$$T(\nabla_b, \widetilde{a}, \widetilde{b}) = \lim_{\Delta V \to 0} \frac{\sum_i T(\hat{n}_i, \widetilde{a}, \widetilde{b})_{\widetilde{a}=c} \, \Delta S_i}{\Delta V}. \tag{4.139}$$

We now introduce the second lemma in the method of symbolic vector.

Lemma 4.2. For a symbolic expression containing two functions, the following relation holds true:

$$T(\nabla, \widetilde{a}, \widetilde{b}) = T(\nabla_a, \widetilde{a}, \widetilde{b}) + T(\nabla_b, \widetilde{a}, \widetilde{b}). \tag{4.140}$$

The proof of this lemma follows directly from the definition of the expressions (4.135) and (4.136) or (4.138) and (4.139). This lemma may also be called the *decomposition theorem.*

Let us now apply both Lemma 4.1 and Lemma 4.2 to derive various vector identities without actually performing any differentiation. Because the steps involved are algebraic, most of the time we merely write down the intermediate steps, omitting the explanation.

The symbolic expression with two functions besides the symbolic vector are listed here; there are only eight possibilities.

Two scalars:

$$\nabla ab = a\nabla b = b\nabla a. \tag{4.141}$$

One scalar and one vector:

$$\nabla \cdot a\mathbf{b} = a\nabla \cdot \mathbf{b} = \nabla a \cdot \mathbf{b}, \tag{4.142}$$

$$\nabla \times a\mathbf{b} = a\nabla \times \mathbf{b} = \nabla a \times \mathbf{b}. \tag{4.143}$$

Two vectors:

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\nabla = (\mathbf{b} \cdot \mathbf{a})\nabla, \tag{4.144}$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \nabla) = \mathbf{b} \cdot (\nabla \times \mathbf{a}), \tag{4.145}$$

$$(\nabla \cdot \mathbf{a})\mathbf{b} = (\mathbf{a} \cdot \nabla)\mathbf{b} = \nabla \cdot \mathbf{ab}, \tag{4.146}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} = \nabla \cdot \mathbf{ba} - \nabla \cdot \mathbf{ab}, \tag{4.147}$$

$$(\nabla \times \mathbf{a}) \times \mathbf{b} = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\nabla = \nabla \cdot \mathbf{ba} - \nabla(\mathbf{a} \cdot \mathbf{b}). \tag{4.148}$$

1.

$$\begin{aligned} \nabla ab &= \nabla_a(ab) + \nabla_b(ab) \quad \text{[Lemma 4.2]} \\ &= b\nabla_a a + a\nabla_b b; \quad \text{[Lemma 4.1]} \end{aligned}$$

hence

$$\nabla(ab) = b\nabla a + a\nabla b. \tag{4.149}$$

2.

$$\begin{aligned} \nabla \cdot (a\mathbf{b}) &= \nabla_a \cdot (a\mathbf{b}) + \nabla_b \cdot (a\mathbf{b}) \\ &= (\nabla_a a) \cdot \mathbf{b} + a\nabla_b \cdot \mathbf{b}; \end{aligned}$$

hence

$$\nabla(a\mathbf{b}) = \mathbf{b} \cdot \nabla a + a\nabla \mathbf{b}. \tag{4.150}$$

3.

$$\begin{aligned} \nabla \times (a\mathbf{b}) &= \nabla_a \times (a\mathbf{b}) + \nabla_b \times (a\mathbf{b}) \\ &= (\nabla_a a) \times \mathbf{b} + a\nabla_b \times \mathbf{b}; \end{aligned}$$

hence

$$\nabla(a\mathbf{b}) = -\mathbf{b} \times \nabla a + a\nabla \mathbf{b}. \tag{4.151}$$

4.

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \nabla_a(\mathbf{a} \cdot \mathbf{b}) + \nabla_b(\mathbf{a} \cdot \mathbf{b}).$$

In view of Lemma 4.1,

$$\begin{aligned}\nabla_b(\mathbf{a}\cdot\mathbf{b}) &= \mathbf{a}\times(\nabla_b\times\mathbf{b}) + (\mathbf{a}\cdot\nabla_b)\mathbf{b}\\ &= \mathbf{a}\times\nabla\mathbf{b} + \mathbf{a}\cdot\nabla\mathbf{b}.\end{aligned}$$

By interchanging the role of **a** and **b**, we obtain

$$\nabla_a(\mathbf{a}\cdot\mathbf{b}) = \mathbf{b}\times\nabla\mathbf{a} + \mathbf{b}\cdot\nabla\mathbf{a};$$

hence

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{a} + \mathbf{a}\times\nabla\mathbf{b} + \mathbf{b}\times\nabla\mathbf{a}. \tag{4.152}$$

We would like to point out that the first two terms in (4.152) involve two new functions in the form of $\nabla\mathbf{b}$ and $\nabla\mathbf{a}$. They are two dyadic functions corresponding to the gradient of two vector functions. In the rectangular system, $\nabla\mathbf{b}$ is defined by

$$\begin{aligned}\nabla\mathbf{b} &= \sum_i \hat{x}_i \frac{\partial\mathbf{b}}{\partial x_i}\\ &= \sum_i\sum_j \hat{x}_i \frac{\partial}{x_i}(\hat{x}_j b_j)\\ &= \sum_i\sum_j \hat{x}_i\hat{x}_j \frac{\partial b_j}{\partial x_i}.\end{aligned} \tag{4.153}$$

Then,

$$\mathbf{a}\cdot\nabla\mathbf{b} = \sum_i\sum_j a_i \frac{\partial b_j}{\partial x_i}\hat{x}_j = \sum_i a_i \frac{\partial\mathbf{b}}{\partial x_i}. \tag{4.154}$$

In an orthogonal curvilinear system,

$$\begin{aligned}\mathbf{a}\cdot\nabla\mathbf{b} &= \sum_i \frac{a_i}{h_i}\frac{\partial\mathbf{b}}{\partial v_i}\\ &= \sum_i \frac{a_i}{h_i}\frac{\partial}{\partial v_i}\sum_j b_j\hat{u}_j\\ &= \sum_i \frac{a_i}{h_i}\sum_j\left[b_j\frac{\partial\hat{u}_j}{\partial v_i} + \frac{\partial b_j}{\partial v_i}\hat{u}_j\right].\end{aligned} \tag{4.155}$$

The derivatives of $\hat{u}_j$ with respect to v_i can be expressed in terms of the unit vectors themselves and the derivatives of the metric coefficients with the aid of (2.59) and (2.61). The result yields

$$\mathbf{a}\cdot\nabla\mathbf{b} = \sum_i\sum_j \frac{a_i}{h_i}\frac{\partial b_j}{\partial v_i}\hat{u}_j + \mathbf{A}\times\mathbf{b}, \tag{4.156}$$

where

$$\mathbf{A} = \frac{1}{\Omega} \sum_i \left(a_k \frac{\partial h_k}{\partial v_j} - a_j \frac{\partial h_j}{\partial v_k} \right) h_i \hat{u}_i$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order, and $\Omega = h_1 h_2 h_3$. To obtain (4.156) by means of (4.154) through coordinate transformation would be a very complicated exercise.

5.

$$\begin{aligned} \nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \nabla_a \cdot (\mathbf{a} \times \mathbf{b}) + \nabla_b \cdot (\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{b} \cdot (\nabla_a \times \mathbf{a}) + \mathbf{a} \cdot (\mathbf{b} \times \nabla_b); \end{aligned}$$

hence

$$\nabla(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b}. \qquad (4.157)$$

6.

$$(\nabla \cdot \mathbf{a})\mathbf{b} = (\nabla_a \cdot \mathbf{a})\mathbf{b} + (\nabla_b \cdot \mathbf{a})\mathbf{b};$$

hence

$$\nabla(\mathbf{ab}) = \mathbf{b}\nabla\mathbf{a} + \mathbf{a} \cdot \nabla\mathbf{b}. \qquad (4.158)$$

It is seen that $(\nabla \cdot \mathbf{a})\mathbf{b}$ is not equal to $(\nabla \mathbf{a})\mathbf{b}$; rather, it is the divergence of a dyadic function **ab**.

7.

$$\begin{aligned} \nabla \times (\mathbf{a} \times \mathbf{b}) &= \nabla_a \times (\mathbf{a} \times \mathbf{b}) + \nabla_b \times (\mathbf{a} \times \mathbf{b}) \\ &= (\nabla_a \cdot \mathbf{b})\mathbf{a} - (\nabla_a \cdot \mathbf{a})\mathbf{b} + (\nabla_b \cdot \mathbf{b})\mathbf{a} - (\nabla_b \cdot \mathbf{a})\mathbf{b}; \end{aligned}$$

hence

$$\nabla(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla\mathbf{a} - \mathbf{b}\nabla\mathbf{a} + \mathbf{a}\nabla\mathbf{b} - \mathbf{a} \cdot \nabla\mathbf{b}. \qquad (4.159)$$

8.

$$\begin{aligned} (\nabla \times \mathbf{a}) \times \mathbf{b} &= (\nabla_a \times \mathbf{a}) \times \mathbf{b} + (\nabla_b \times \mathbf{a}) \times \mathbf{b} \\ &= (\nabla_a \times \mathbf{a}) \times \mathbf{b} + \mathbf{a}(\nabla_b \cdot \mathbf{b}) - \nabla_b(\mathbf{a} \cdot \mathbf{b}) \\ &= (\nabla_a \times \mathbf{a}) \times \mathbf{b} + \mathbf{a}(\nabla_b \cdot \mathbf{b}) - \mathbf{a} \times (\nabla_b \times \mathbf{b}) - (\mathbf{a} \cdot \nabla_b)\mathbf{b}; \end{aligned}$$

hence

$$(\nabla \times \mathbf{a}) \times \mathbf{b} = -\mathbf{b} \times \nabla\mathbf{a} + \mathbf{a}(\nabla\mathbf{b}) - \mathbf{a} \times \nabla\mathbf{b} - \mathbf{a} \cdot \nabla\mathbf{b}. \qquad (4.160)$$

Two vector identities can be conveniently derived by means of partial symbolic expressions.

9.

$$\mathbf{a} \times (\nabla_b \times \mathbf{b}) = \nabla_b(\mathbf{b} \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla_b)\mathbf{b}.$$

Hence

$$\mathbf{a} \times \nabla\mathbf{b} = (\nabla\mathbf{b}) \cdot \mathbf{a} - \mathbf{a} \cdot \nabla\mathbf{b}. \qquad (4.161)$$

10.

$$(\mathbf{a} \times \nabla_b) \times \mathbf{b} = \nabla_b(\mathbf{b} \cdot \mathbf{a}) - (\nabla_b \cdot \mathbf{b})\mathbf{a}.$$

Hence

$$\begin{aligned}(\mathbf{a} \times \nabla_b) \times \mathbf{b} &= (\nabla\mathbf{b}) \cdot \mathbf{a} - \mathbf{a}\nabla\mathbf{b} \\ &= \mathbf{a} \cdot \nabla\mathbf{b} - \mathbf{a}\nabla\mathbf{b} + \mathbf{a} \times \nabla\, \mathbf{b}.\end{aligned} \tag{4.162}$$

The second line of (4.162) results from (4.161).

It is interesting to observe that the partial symbolic expression $(\mathbf{a} \times \nabla_b) \times \mathbf{b}$ involves the products of $\mathbf{a}$ with the gradient, divergence, and curl of $\mathbf{b}$. The convenience of using the method of symbolic vector in deriving the vector identities is evident.

4-8 Symbolic Expressions with Double Symbolic Vectors

When a symbolic expression is created with a vector expression containing two vector functions $\mathbf{g}_1$ and $\mathbf{g}_2$ and a third function $\tilde{f}$ that can be scalar or vector, we can generate a symbolic expression of the form $T(\nabla, \mathbf{g}_2, \tilde{f})$ after $\mathbf{g}_1$ is replaced by a symbolic vector ∇. In the rectangular system,

$$T(\nabla, \mathbf{g}_2, \tilde{f}) = \sum_i \frac{\partial}{\partial x_i} T(\hat{x}_i, \mathbf{g}_2, \tilde{f}). \tag{4.163}$$

Now, if $\mathbf{g}_2$ is replaced by another symbolic vector ∇' in (4.163), we would obtain a symbolic expression with double S vectors whose differential form in the rectangular system will be given by

$$T(\nabla, \nabla', \tilde{f}) = \sum_i \sum_j \frac{\partial^2}{\partial x_i\, \partial x_j} T(\hat{x}_i, \hat{x}_j, \tilde{f}). \tag{4.164}$$

It is obvious that Lemma 4.1 also applies to $T(\nabla, \nabla', \tilde{f})$. Several distinct cases will be considered.

1. **Laplacian of a scalar function**

$$T(\nabla, \nabla', \tilde{f}) = (\nabla \cdot \nabla')f,$$

where f is a scalar function; then

$$\nabla \cdot \nabla' f = \sum_i \sum_j \frac{\partial^2}{\partial x_i\, \partial x_j} (\hat{x}_i \cdot \hat{x}_j) f = \sum_i \frac{\partial^2 f}{\partial x_i^2} = \nabla\nabla f. \tag{4.165}$$

Although we have arrived at this result using functions defined in the rectangular system, the result is applicable to any system because of the invariance theorem of the differential operators. The function $\nabla\nabla f$ is called the *Laplacian of the scalar function* f, in honor of the French mathematical physicist Pierre Simon Laplace (1749–1827). In the past, many authors used the notation $\nabla^2 f$ for this function, which is a compact form of the original notation $\nabla \cdot \nabla f$ used by Gibbs. We have completely abandoned Gibbs's notations in this new edition so that the "formal" scalar and vector products, discussed in Chapter 8, will not appear and interfere with the operations involved in the method of symbolic vector.

2. **Laplacian of a vector function**

$$T(\nabla, \nabla', \tilde{f}) = \nabla \cdot \nabla' \mathbf{F},$$

where $\mathbf{F}$ is a vector function; then,

$$\nabla \cdot \nabla' \mathbf{F} = \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} (\hat{x}_i \cdot \hat{x}_j) \mathbf{F} = \sum_i \frac{\partial^2 \mathbf{F}}{\partial x_i^2} = \nabla \nabla \mathbf{F}. \tag{4.166}$$

In this case, we encounter a dyadic function corresponding to the gradient of a vector function. The divergence of a dyadic is a vector function. In the rectangular coordinate system, we can write

$$\nabla \nabla \mathbf{F} = \sum_i \frac{\partial^2 \mathbf{F}}{\partial x_1^2} = \sum_i \sum_j \frac{\partial^2}{\partial x_i^2} F_j \hat{x}_j = \sum_j (\nabla \nabla F_j) \hat{x}_j. \tag{4.167}$$

In the orthogonal curvilinear system,

$$\nabla \nabla \mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} \sum_j \frac{\hat{u}_j}{h_j} \frac{\partial \mathbf{F}}{\partial v_j}, \tag{4.168}$$

where the operational form of the divergence and the gradient have been used. The derivatives of the dyadic function $\nabla \mathbf{F}$ can be simplified as follows:

$$\frac{\partial}{\partial v_i} \nabla \mathbf{F} = \sum_j \left[\frac{\hat{u}_j}{h_j} \frac{\partial^2 \mathbf{F}}{\partial v_i \, \partial v_j} + \frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \frac{\partial \mathbf{F}}{\partial v_j} \right].$$

Hence

$$\begin{aligned} \nabla \nabla \mathbf{F} &= \sum_i \sum_j \frac{\hat{u}_i}{h_i} \cdot \left[\frac{\hat{u}_j}{h_j} \frac{\partial^2 \mathbf{F}}{\partial v_i \, \partial v_j} + \frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \frac{\partial \mathbf{F}}{\partial v_j} \right] \\ &= \sum_i \sum_j \left\{ \left(\frac{\hat{u}_i}{h_i} \cdot \frac{\hat{u}_j}{h_j} \right) \frac{\partial^2 \mathbf{F}}{\partial v_i \, \partial v_j} + \frac{\hat{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \frac{\partial \mathbf{F}}{\partial v_j} \right\} \\ &= \sum_i \frac{1}{h_i^2} \frac{\partial^2 \mathbf{F}}{\partial v_i^2} + \sum_i \sum_j \left[\frac{\hat{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \frac{\partial \mathbf{F}}{\partial v_j} \right]. \end{aligned} \tag{4.169}$$

The expression of $\nabla \nabla \mathbf{F}$ will be used later to demonstrate an identity involving this function.

3. **The gradient of the divergence of a vector function**

$$T(\nabla, \nabla', \tilde{f}) = \nabla(\nabla' \cdot \mathbf{F}).$$

Then,

$$\begin{aligned}
\nabla(\nabla' \cdot \mathbf{F}) &= \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} \hat{x}_i(\hat{x}_j \cdot \mathbf{F}) \\
&= \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} \hat{x}_i F_j \\
&= \sum_i \hat{x}_i \frac{\partial}{\partial x_i} \sum_j \frac{\partial F_j}{\partial x_j} = \nabla\nabla \mathbf{F}.
\end{aligned} \tag{4.170}$$

In OCS,

$$\begin{aligned}
\nabla\nabla\mathbf{F} &= \sum_i \frac{\hat{u}_i}{h_i} \frac{\partial}{\partial v_i} \sum_j \frac{\hat{u}_j}{h_j} \cdot \frac{\partial \mathbf{F}}{\partial v_j} \\
&= \sum_i \sum_j \frac{\hat{u}_i}{h_i} \left[\frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \cdot \frac{\partial \mathbf{F}}{\partial v_j} + \frac{\hat{u}_j}{h_j} \cdot \frac{\partial^2 \mathbf{F}}{\partial v_i \, \partial v_j} \right].
\end{aligned} \tag{4.171}$$

We will leave it in this form for the time being.

4. **The curl of the curl of a vector function**

$$T(\nabla, \nabla', \tilde{f}) = \nabla \times (\nabla' \times \mathbf{F}).$$

Then,

$$\begin{aligned}
\nabla \times (\nabla' \times \mathbf{F}) &= \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} \hat{x}_i \times (\hat{x}_j \times \mathbf{F}) \\
&= \sum_i \frac{\partial}{\partial x_i} \left[\hat{x}_i \times \sum_j \frac{\partial}{\partial x_j} (\hat{x}_j \times \mathbf{F}) \right] \\
&= \sum_i \frac{\partial}{\partial x_i} [\hat{x}_i \times \nabla \mathbf{F}] = \nabla \nabla \mathbf{F}.
\end{aligned} \tag{4.172}$$

In OCS,

$$\begin{aligned}
\nabla \nabla \mathbf{F} &= \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i} \left(\sum_j \frac{\hat{u}_j}{h_j} \times \frac{\partial \mathbf{F}}{\partial v_j} \right) \\
&= \sum_i \sum_j \frac{\hat{u}_i}{h_i} \times \left[\frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \times \frac{\partial \mathbf{F}}{\partial v_j} + \frac{\hat{u}_j}{h_j} \times \frac{\partial^2 \mathbf{F}}{\partial v_i \, \partial v_j} \right].
\end{aligned} \tag{4.173}$$

Because $T(\nabla, \nabla', \tilde{f})$ obeys Lemma 4.1, we can also change $\nabla \times (\nabla' \times \mathbf{F})$ to

$$\nabla \times (\nabla' \times \mathbf{F}) = (\nabla \cdot \mathbf{F})\nabla' - (\nabla \cdot \nabla')\mathbf{F}. \tag{4.174}$$

In view of (4.166), (4.170), and (4.172), we have

$$\nabla\!\!\!\!\nabla\, \nabla\!\!\!\!\nabla\, \mathbf{F} = \nabla\nabla\!\!\!\!\nabla\, \mathbf{F} - \nabla\!\!\!\!\nabla\, \nabla\mathbf{F}. \tag{4.175}$$

This identity has been derived using functions defined in the rectangular coordinate system. However, in view of the invariance theorem, it is valid in any coordinate system. The misinterpretation of $\nabla\!\!\!\!\nabla\, \nabla\mathbf{F}$ in (4.175) in OCS has troubled many authors in the past (see Chapter 8); it is therefore desirable to prove (4.175) analytically to confirm our assertion that (4.175) is an identity in any coordinate system. By taking the difference between (4.173) and (4.171) and rearranging the terms, we find

$$\nabla\nabla\!\!\!\!\nabla\, \mathbf{F} - \nabla\!\!\!\!\nabla\, \nabla\!\!\!\!\nabla\, \mathbf{F} = \nabla\!\!\!\!\nabla\, \nabla\mathbf{F} + \sum_i \sum_j \frac{\partial \mathbf{F}}{v_j} \times \left[\frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i}\left(\frac{\hat{u}_j}{h_j}\right)\right], \tag{4.176}$$

where $\nabla\!\!\!\!\nabla\, \nabla\mathbf{F}$ corresponds to the function given by (4.169). The last term in (4.176) vanishes because

$$\sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i}\left(\frac{\hat{u}_j}{h_j}\right) = \nabla\!\!\!\!\nabla\, \nabla v_j = 0.$$

The curl of the gradient of any differentiable scalar function vanishes by considering $\nabla\!\!\!\!\nabla\, \nabla f$ in the rectangular coordinate system that yields

$$\nabla\!\!\!\!\nabla\, \nabla f = \sum_i \hat{x}_i \left[\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right)\right] = 0. \tag{4.177}$$

This theorem is treated later from the point of view of the method of symbolic vector. We have thus shown that (4.175) is indeed valid in OCS. The identity can be proved in GCS in a similar manner.

5. **The curl of the gradient of a scalar or a vector function**

$$T(\nabla\!\!\!\!\nabla, \nabla\!\!\!\!\nabla', \tilde{f}) = \nabla\!\!\!\!\nabla \times \nabla\!\!\!\!\nabla' \tilde{f}.$$

Then,

$$\begin{aligned} \nabla\!\!\!\!\nabla \times \nabla\!\!\!\!\nabla' \tilde{f} &= \sum_i \sum_j \frac{\partial^2}{\partial x_i\, \partial x_j} (\hat{x}_i \times \hat{x}_j \tilde{f}) \\ &= \sum_i \hat{x}_i \times \frac{\partial}{\partial x_i}\left[\sum_j \hat{x}_j \frac{\partial \tilde{f}}{\partial x_j}\right] = \nabla\!\!\!\!\nabla\, \nabla \tilde{f}. \end{aligned}$$

But

$$\hat{x}_i \times \hat{x}_j = \begin{cases} 0, & i = j, \\ -\hat{x}_j \times \hat{x}_i, & i \neq j; \end{cases}$$

hence

$$\nabla\!\!\!\!\nabla\, \nabla \tilde{f} = 0, \tag{4.178}$$

where $\tilde{f}$ can be a scalar or a vector. When $\tilde{f}$ is a scalar, we have already proved it as stated by (4.177).

6. The divergence of the curl of a vector function

$$T(\nabla, \nabla', \widetilde{f}) = \nabla \cdot (\nabla' \times \mathbf{F}).$$

Then,

$$\begin{aligned}\nabla \cdot (\nabla' \times \mathbf{F}) &= \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} [\hat{x}_i \cdot (\hat{x}_j \times \mathbf{F})] \\ &= \sum_i \hat{x}_i \cdot \frac{\partial}{\partial x_i} \left[\sum_j \hat{x}_j \times \frac{\partial \mathbf{F}}{\partial x_j} \right] \\ &= \nabla \nabla \mathbf{F}.\end{aligned}$$

But

$$\begin{aligned}\hat{x}_i \cdot (\hat{x}_j \times \mathbf{F}) &= \mathbf{F} \cdot (\hat{x}_i \times \hat{x}_j) \\ &= \begin{cases} 0, & i = j, \\ -\mathbf{F} \cdot (\hat{x}_j \times \hat{x}_i), & i \neq j. \end{cases}\end{aligned}$$

Hence

$$\nabla \nabla \mathbf{F} = 0. \tag{4.179}$$

When a symbolic expression consists of double S vectors and two functions, its definition in the rectangular system is

$$T(\nabla, \nabla', \widetilde{a}, \widetilde{b}) = \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} T(\hat{x}_i, \hat{x}_j, \widetilde{a}, \widetilde{b}). \tag{4.180}$$

To simplify (4.180), we can apply Lemma 4.2 repeatedly to an expression with a single S vector, that is,

$$\begin{aligned}T(\nabla, \nabla', \widetilde{a}, \widetilde{b}) &= T(\nabla, \nabla'_a, \widetilde{a}, \widetilde{b}) + T(\nabla, \nabla'_b, \widetilde{a}, \widetilde{b}) \\ &= T(\nabla_a, \nabla'_a, \widetilde{a}, \widetilde{b}) + T(\nabla_b, \nabla'_a, \widetilde{a}, \widetilde{b}) \\ &\quad + T(\nabla_a, \nabla'_b, \widetilde{a}, \widetilde{b}) + T(\nabla_b, \nabla'_b, \widetilde{a}, \widetilde{b}).\end{aligned} \tag{4.181}$$

As an example, let

$$T(\nabla, \nabla', \widetilde{a}, \widetilde{b}) = (\nabla \cdot \nabla')ab. \tag{4.182}$$

According to (4.165), this is equal to $\nabla \nabla(ab)$. Equation (4.181) yields

$$\nabla \nabla(ab) = b\nabla \nabla a + 2(\nabla a)(\nabla b) + a\nabla \nabla b. \tag{4.183}$$

The same answer can be obtained by applying (4.149) and (4.150) to $\nabla \nabla(ab)$ in succession.

4-9 Generalized Gauss Theorem in Space

The principal integral theorem involving a symbolic expression can be formulated based on the very definition of $T(\nabla)$, namely,

$$T(\nabla) = \lim_{\Delta V \to 0} \frac{\sum_i T(\hat{n}_i)\, \Delta S_i}{\Delta V}. \tag{4.184}$$

Equation (4.184) can be considered as a limiting form of a parent equation

$$T(\nabla) = \frac{\sum_i T(\hat{n}_i)\, \Delta S_i}{\Delta V} + \varepsilon, \tag{4.185}$$

where $\varepsilon \to 0$ as $\Delta V \to 0$. If ΔV_j denotes a typical cell in a volume V with an enclosing surface S, then for that cell, we can write

$$T(\nabla)\, \Delta V_j = \sum_i T(\hat{n}_{ij})\, \Delta S_{ij} + \varepsilon_j \Delta V_j, \tag{4.186}$$

where ΔS_{ij} denotes an elementary area of ΔV_j, $\hat{n}_{ij}$ being an outward normal unit vector. By taking the Riemann sum of (4.186) with respect to j, we obtain

$$\sum_j T(\nabla)\, \Delta V_j = \sum_j \sum_i T(\hat{n}_{ij})\, \Delta S_{ij} + \sum_j \varepsilon_j\, \Delta V_j. \tag{4.187}$$

Then, as $\Delta V_j \to 0$, $\varepsilon_j \to 0$. By assuming $T(\nabla)$ to be continuous throughout V, we obtain

$$\iiint_V T(\nabla)\, dV = \oint\!\!\!\oint_S T(\hat{n})\, dS. \tag{4.188}$$

The sign around the double integral means that the surface is closed. It is observed that the contributions of $T(\hat{n}_{ij})\, \Delta S_{ij}$ from two contacting surfaces of adjacent cells cancel each other. The only contribution results from the exterior surface where there are no neighboring cells. In (4.188), $\hat{n}$ denotes the outward unit normal vector to S. The same formula can be obtained by integrating the symbolic expression in the rectangular system

$$\iiint_V T(\nabla)\, dV = \iiint_V \sum_i \frac{\partial T(\hat{x}_i)}{\partial x_i}\, dV. \tag{4.189}$$

The integral involving the partial derivative of $T(\hat{x}_i)$ with respect to x_i can be reduced to the surface integral found in (4.188). The linearity of $T(\hat{x}_i)$ with respect to $\hat{x}_i$ is a key link in that reduction. In a later section, we will give a detailed treatment of a two-dimensional version of a similar problem for a surface to demonstrate this approach. The formula that we have derived will be designated the *generalized Gauss theorem*, which converts a volume integral of $T(\nabla)$, continuous throughout V, to a surface integral evaluated at the enclosing surface S. Many of the classical theorems in vector analysis can be readily derived from this generalized theorem by a proper choice of the symbolic expression $T(\nabla)$.

1. *Divergence theorem or Gauss theorem.* Let $T(\nabla) = \nabla \cdot \mathbf{F} = \nabla\mathbf{F}$. Upon substituting these quantities into (4.188), we obtain the divergence theorem, or the standard Gauss theorem, named in honor of the great mathematician Karl Friedrich Gauss (1777–1855):

$$\iiint_V \nabla\mathbf{F}\, dV = \oiint_S (\hat{n} \cdot \mathbf{F})\, dS. \tag{4.190}$$

2. *Curl theorem.* Let $T(\nabla) = \nabla \times \mathbf{F} = \nabla\mathbf{F}$; then $T(\hat{n}) = \hat{n} \times \mathbf{F}$. By means of (4.188), we obtain the curl theorem:

$$\iiint_V \nabla\mathbf{F}\, dV = \oiint_S (\hat{n} \times \mathbf{F})\, dS. \tag{4.191}$$

3. *Gradient theorem.* This theorem is obtained by letting $T(\nabla) = \nabla f = \nabla f$; then $T(\hat{n}) = \hat{n} f$; hence

$$\iiint_V \nabla f\, dV = \oiint_S f\hat{n}\, dS = \oiint_S f\, d\mathbf{S}. \tag{4.192}$$

4. *Hallén's formula.* If we let $T(\nabla) = (\nabla \cdot \mathbf{a})\mathbf{b}$, then $T(\hat{n}) = (\mathbf{n} \cdot \mathbf{a})\mathbf{b}$. Because $T(\nabla)$ consists of two functions $\mathbf{a}$ and $\mathbf{b}$, we can apply Lemma 4.2 to obtain

$$(\nabla \cdot \mathbf{a})\mathbf{b} = (\nabla_a \cdot \mathbf{a})\mathbf{b} + (\nabla_b \cdot \mathbf{a})\mathbf{b} = \mathbf{b}(\nabla_a \cdot \mathbf{a}) + (\mathbf{a} \cdot \nabla_b)\mathbf{b}.$$

The rearrangement of the various terms follows Lemma 4.1; thus,

$$(\nabla \cdot \mathbf{a})\mathbf{b} = \mathbf{b}(\nabla\mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b}. \tag{4.193}$$

By substituting (4.193) into (4.188), we obtain

$$\iiint_V [\mathbf{b}\nabla\mathbf{a} + \mathbf{a} \cdot \nabla\mathbf{b}]\, dV = \oiint_S (\hat{n} \cdot \mathbf{a})\mathbf{b}\, dS. \tag{4.194}$$

Equation (4.194), with $\mathbf{b}$ equal to $\mathbf{c}/r$, where $r = \left(x^2 + y^2 + z^2\right)^{1/2}$ and $\mathbf{c}$ is a constant vector that can be deleted from the resultant equation, was derived by Hallén [8], based on differential calculus carried out in a rectangular system. We designate (4.194) as *Hallén's formula* for convenient identification.

The three theorems stated by (4.190)–(4.192) are closely related. In fact, it is possible to derive the divergence theorem and the curl theorem based on the gradient theorem. The derivation is given in Appendix D. The relationship between several surface theorems to be derived in Chapter 5 is also shown in that appendix.

With the vector theorems and identities at our disposal, it is of interest to give an interpretation of the closed surface theorem (2.62) based on the gradient theorem, and to identify (2.63) as a vector identity.

According to (4.19), when $T(\nabla) = \nabla f$ and $f = \text{constant}$, we have

$$\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i\right) = 0, \tag{4.195}$$

which is the same as (2.62), originally proved with the aid of the relationships between the derivatives of the unit vectors. From the point of view of the gradient theorem given by

$$\iiint \nabla f \, dV = \oiint f \, d\mathbf{S}$$

when f = constant, we obtain

$$\oiint d\mathbf{S} = 0, \tag{4.196}$$

which is the closed surface theorem in the integral form. Equation (4.195) can therefore be considered as the differential form of the integral theorem for a closed surface.

In view of the definition of the curl operator given by (4.85), (2.63) is recognized as

$$\triangledown \left(\frac{\hat{u}_j}{h_j} \right) = 0, \qquad j = (1, 2, 3)\,. \tag{4.197}$$

Now,

$$\frac{\hat{u}_j}{h_j} = \nabla v_j;$$

hence (4.197) is equivalent to

$$\triangledown \, \nabla v_j = 0, \tag{4.198}$$

which is a valid identity according to (4.178). By applying the curl theorem to the function $\mathbf{F} = \nabla v_j$, we obtain

$$\iiint \triangledown \, \nabla v_j \, dV = \oiint \hat{n} \times \nabla v_j \, dS = \oiint \hat{n} \times \hat{n} \frac{\partial v_j}{\partial n} \, dS = 0. \tag{4.199}$$

Hence (4.198) may be considered as the differential form of the integral theorem stated by (4.199).

4-10 Scalar and Vector Green's Theorems

There are numerous theorems bearing the name of George Green (1793–1841). We first consider Green's theorem involving scalar functions. In the Gauss theorem, stated by (4.190), if we let

$$\mathbf{F} = a \nabla b, \tag{4.200}$$

where a and b are two scalar functions, then

$$\nabla \mathbf{F} = a \triangledown \nabla b + (\nabla a) \cdot (\nabla b), \tag{4.201}$$

which is obtained by (4.150). Upon substituting (4.201) into (4.190) with $\hat{n} \cdot \mathbf{F} = a(\hat{n} \cdot \nabla b)$, we obtain

$$\iiint_V [a\nabla\nabla b + (\nabla a) \cdot (\nabla b)]\, dV = \oiint_S a(\hat{n} \cdot \nabla b)\, dS. \tag{4.202}$$

Because $\hat{n} \cdot \nabla b$ is the scalar component of ∇b in the direction of the unit vector $\hat{n}$, it is equal to $\partial b/\partial n$; (4.202) is often written in the form

$$\iiint_V [a\nabla\nabla b + (\nabla a) \cdot (\nabla b)]\, dV = \oiint_S a\frac{\partial b}{\partial n}\, dS. \tag{4.203}$$

For convenience, we will designate it as the *first scalar Green's theorem*.

If we let $\mathbf{F} = a\nabla b - b\nabla a$, then it is obvious that

$$\begin{aligned}\iiint_V (a\nabla\nabla b - b\nabla\nabla a)\, dV &= \oiint_S \left(a\frac{\partial b}{\partial n} - b\frac{\partial a}{\partial n}\right) dS \\ &= \oiint \hat{n} \cdot (a\nabla b - b\nabla a)\, dS.\end{aligned} \tag{4.204}$$

Equation (4.204) will be designated as the *second scalar Green's theorem*. Both (4.203) and (4.204) involve scalar functions only.

Two theorems involving one scalar and one vector can be constructed from (4.202) and (4.204). We consider three equations of the form (4.202) with three different scalar functions b_i ($i = 1, 2, 3$). Then, by juxtaposing a unit vector $\hat{x}_i$ to each of these equations with $i = 1, 2, 3$ and summing the resultant equations, we obtain

$$\iiint [a\nabla\nabla\mathbf{b} + (\nabla a) \cdot (\nabla\mathbf{b})]\, dS = \oiint a(\hat{n} \cdot \nabla\mathbf{b})\, dS. \tag{4.205}$$

Similarly, by moving the function b to the posterior position in (4.204) and following the same procedure, we obtain

$$\iiint [a\nabla\nabla\mathbf{b} - (\nabla\nabla a)\mathbf{b}]\, dV = \oiint \hat{n} \cdot [a\nabla\mathbf{b} - (\nabla a)\mathbf{b}]\, dS. \tag{4.206}$$

Equation (4.205) is designated as the scalar–vector Green's theorem of the first kind and (4.206) as the second kind. Because of the invariance of the gradient and the divergence operators, these theorems, derived here using rectangular variables, are valid for functions defined in any coordinate system, including GCS. However, one must be careful to calculate $\nabla\mathbf{b}$, a dyadic, in a curvilinear system. In OCS,

$$\nabla\mathbf{b} = \sum_i \frac{\hat{u}_i}{h_i}\frac{\partial\mathbf{b}}{\partial v_i} = \sum_{i,j}\frac{\hat{u}_i}{h_i}\left[\frac{\partial b_j}{\partial v_i}\hat{u}_j + b_j\frac{\partial\hat{u}_j}{\partial v_i}\right], \tag{4.207}$$

which has appeared before in (4.155).

There are two vector Green's theorems, which are formulated, first, by letting

$$\mathbf{F} = \mathbf{a} \times \nabla\mathbf{b}. \tag{4.208}$$

In view of (4.157), we have

$$\nabla \mathbf{F} = \nabla \mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \nabla \mathbf{b} \tag{4.209}$$

and

$$\hat{n} \cdot \mathbf{F} = \hat{n} \cdot (\mathbf{a} \times \nabla \mathbf{b}). \tag{4.210}$$

Upon substituting (4.209) and (4.210) into the Gauss theorem, we obtain

$$\iiint_V [\nabla \mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \nabla \mathbf{b}]\, dV = \oiint_S \hat{n} \cdot (\mathbf{a} \times \nabla \mathbf{b})\, dS. \tag{4.211}$$

Equation (4.211) is designated as the *first vector Green's theorem*. By combining (4.211) with another equation of the same form as (4.211) with the roles of $\mathbf{a}$ and $\mathbf{b}$ interchanged, or by starting with $\mathbf{F} = \mathbf{a} \times \nabla \mathbf{b} - \mathbf{b} \times \nabla \mathbf{a}$, we obtain the *second vector Green's theorem*:

$$\iiint_V [\mathbf{b} \cdot \nabla \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \nabla \mathbf{b}]\, dV = \oiint_S \hat{n} \cdot (\mathbf{a} \times \nabla \mathbf{b} - \mathbf{b} \times \nabla \mathbf{a})\, dS. \tag{4.212}$$

The continuity of the function $\mathbf{F}$ imposed on the Gauss theorem is now carried over, for example, to the continuity of $\mathbf{a} \times \nabla \mathbf{b}$ in (4.212), and similarly for the other theorems.

4-11 Solenoidal Vector, Irrotational Vector, and Potential Functions

The main purpose of this book is to treat vector analysis based on a new symbolic method. The application of vector analysis to physical problems is not covered in this treatise. However, there are several topics in introductory courses on electromagnetics and hydrodynamics involving some technical terms in vector analysis that should be introduced in a book of this nature.

When the divergence of a vector function vanishes everywhere in the entire spatial domain, such a function is called a *solenoidal vector*, and it will be denoted by $\mathbf{F}_s$ in this section. If the curl of the same vector function also vanishes everywhere, it can be proved that the function under consideration must be a constant vector. Physically, when both the divergence and the curl of a vector vanish, it means that the field has no source. In general, a solenoidal field is characterized by

$$\nabla \cdot \mathbf{F}_s = 0, \tag{4.213}$$

$$\nabla \mathbf{F}_s = \mathbf{f}, \tag{4.214}$$

where we treat $\mathbf{f}$ as the source function responsible for producing the vector field.

When the curl of a vector vanishes but its divergence is nonvanishing, such a vector is called an *irrotational vector*, and it will be denoted by $\mathbf{F}_i$. Such a field vector is characterized by

$$\nabla \mathbf{F}_i = 0, \tag{4.215}$$

$$\nabla \mathbf{F}_i = f, \tag{4.216}$$

where the scalar function f is treated as the source function responsible for producing the field. In electromagnetics, $\mathbf{F}_s$ corresponds to the magnetic field in magnetostatics and $\mathbf{F}_i$ to the electric field in electrostatics. In hydrodynamics, $\mathbf{F}_i$ corresponds to the velocity field of a laminar flow, and $\mathbf{F}_s$ to that of a vortex.

In electrodynamics, the electric and magnetic fields are coupled, and they are both functions of space and time. Their relations are governed by Maxwell's equations. For example, in air, the system of equations is

$$\nabla \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \tag{4.217}$$

$$\nabla \mathbf{H} = J + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{4.218}$$

$$\nabla (\varepsilon_0 \mathbf{E}) = \rho, \tag{4.219}$$

$$\nabla (\mu_0 \mathbf{H}) = 0, \tag{4.220}$$

$$\nabla \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{4.221}$$

where $\mathbf{J}$ and ρ denote, respectively, the current density and the charge density functions responsible for producing the electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$, and μ_0 and ε_0 are two fundamental constants. It is seen that the magnetic field $\mathbf{H}$ is a solenoidal field, but the electric field is neither solenoidal nor irrotational, that is, $\nabla \mathbf{E} \neq 0$ and $\nabla \mathbf{E} \neq 0$.

The theoretical work in electrostatics and magnetostatics is to investigate the solutions of (4.213)–(4.214) and (4.215)–(4.216) under various boundary conditions of the physical problems. In electrodynamics, the theoretical work is to study the solutions of the differential equations such as (4.217)–(4.221) for various problems. In the case of electrostatics, in view of the vector identity (4.178), the electric field, now denoted by $\mathbf{E}$, can be expressed in terms of a scalar function V such that

$$\mathbf{E} = -\nabla V. \tag{4.222}$$

The negative sign in (4.222) is just a matter of tradition based on physical consideration; mathematically, it has no importance. The function V is called the *electrostatic potential function*. As a result of (4.219), we find that

$$\nabla \mathbf{E} = -\nabla \nabla V = \frac{\rho}{\varepsilon_0}, \tag{4.223}$$

where we have replaced the function f by ρ/ε_0, with ρ denoting the density function of a charge distribution and ε_0, a physical constant. The problem is now shifted to the study of the second-order partial differential equation

$$\nabla \nabla V = -\frac{\rho}{\varepsilon_0}, \tag{4.224}$$

which is called *Poisson's equation*. The operator $\nabla \nabla$, or div grad, is the Laplacian operator we introduced in Section 4.8.

In the case of magnetostatics, in view of identity (4.179), the magnetic field, now denoted by $\mathbf{H}$, replacing $\mathbf{F}_s$, can be expressed in terms of a vector function $\mathbf{A}$ such that

$$\mathbf{H} = \nabla\times\mathbf{A}. \tag{4.225}$$

$\mathbf{A}$ is called the *magnetostatic vector potential.* The function $\mathbf{f}$ in (4.214) corresponds to the density of a current distribution in magnetostatics, commonly denoted by $\mathbf{J}$. By taking the divergence of (4.214), we find that $\nabla\cdot\mathbf{f} = \nabla\cdot\mathbf{J} = 0$, which is true for a steady current. Upon substituting (4.225) into (4.214) with $\mathbf{F}_s$ and $\mathbf{f}$ replaced by $\mathbf{H}$ and $\mathbf{J}$, respectively, we obtain

$$\nabla\times\nabla\times\mathbf{A} = \mathbf{J}. \tag{4.226}$$

According to the Helmholtz theorem [9], in order to determine $\mathbf{A}$, one must impose a condition on the divergence of the vector function $\mathbf{A}$ in addition to (4.225). Because

$$\nabla\times\nabla\times\mathbf{A} = -\nabla\cdot\nabla\mathbf{A} + \nabla\nabla\cdot\mathbf{A}, \tag{4.227}$$

if we impose the condition

$$\nabla\cdot\mathbf{A} = 0, \tag{4.228}$$

then (4.226) becomes

$$\nabla\cdot\nabla\mathbf{A} = -\mathbf{J}. \tag{4.229}$$

The condition on the divergence of $\mathbf{A}$ so imposed upon is called the *gauge condition.* This condition must be compatible with the resultant differential equation for $\mathbf{A}$, (4.229). By taking the divergence of that equation, we observe that $\nabla\cdot\mathbf{A}$ must be equal to zero because $\nabla\cdot\mathbf{J} = 0$. Thus, the gauge condition so imposed is indeed compatible with (4.229). The analytical work in magnetostatics now rests on the study of the vector Poisson equation stated by (4.229) for various problems.

To solve the system of equations in electrodynamics such as the ones stated by (4.217)–(4.221), we let

$$\mu_0\mathbf{H} = \nabla\times\mathbf{A}, \tag{4.230}$$

because $\mathbf{H}$ is a solenoidal vector. The function $\mathbf{A}$ is called the *dynamic vector potential.* Upon substituting (4.230) into (4.217), we obtain

$$\nabla\times\left(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t}\right) = 0. \tag{4.231}$$

Hence $\mathbf{E} + (\partial\mathbf{A}/\partial t)$ is irrotational, so we can express it in terms of a dynamic scalar potential ϕ such that

$$\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t} = -\nabla\phi. \tag{4.232}$$

Upon substituting the expressions for $\mathbf{H}$ and $\mathbf{E}$ given by (4.230) and (4.232) into (4.218), we obtain

$$\boldsymbol{\nabla}\,\boldsymbol{\nabla}\,\mathbf{A} = \mu_0\mathbf{J} - \frac{1}{c^2}\left(\frac{\partial^2\mathbf{A}}{\partial t^2} + \nabla\frac{\partial\phi}{\partial t}\right), \tag{4.233}$$

where $c = (\mu_0\varepsilon_0)^{-1/2}$ is the velocity of light in free space.

In view of identity (4.227), we can impose a gauge condition on $\mathbf{A}$ such that

$$\nabla\mathbf{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}. \tag{4.234}$$

Then, (4.233) reduces to

$$\boldsymbol{\nabla}\,\nabla\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}, \tag{4.235}$$

which is called the *vector Helmholtz wave equation.* By taking the divergence of (4.235) and making use of (4.221) and (4.234), we find that ϕ satisfies the following scalar Helmholtz wave equation:

$$\boldsymbol{\nabla}\,\nabla\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}. \tag{4.236}$$

Once $\mathbf{A}$ and ϕ are known, the electromagnetic field vectors $\mathbf{E}$ and $\mathbf{H}$ can be found by using the following relations:

$$\mu_0\mathbf{H} = \boldsymbol{\nabla}\,\mathbf{A}, \tag{4.237}$$

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi. \tag{4.238}$$

The method of potentials in electrodynamics is a classical method. Another approach is to deal with the equations for $\mathbf{E}$ and $\mathbf{H}$ directly. Thus, by eliminating $\mathbf{E}$ or $\mathbf{H}$ between (4.217)–(4.218), we obtain

$$\boldsymbol{\nabla}\,\boldsymbol{\nabla}\,\mathbf{E} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = -\mu_0\frac{\partial\mathbf{J}}{\partial t}, \tag{4.239}$$

$$\boldsymbol{\nabla}\,\boldsymbol{\nabla}\,\mathbf{H} + \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = \nabla\times\mathbf{J}. \tag{4.240}$$

These are two basic equations that can be solved by the method of dyadic Green functions [3] or the vector Green's theorem [5].