4

Electromagnetic Sources

4.1 INTRODUCTION

In this chapter, we introduce electromagnetic source representations. We begin by collecting expressions for the delta function in cylindrical and spherical coordinates. We follow with a discussion of time-harmonic representations of functions and vectors. We next introduce the electromagnetic model in the time domain, and then specialize to the time-harmonic case. We begin our study of electromagnetic sources with a consideration of the sheet current source. We then, in sequence, study the line source, the ring source, and the point source. Throughout, we shall utilize the Green's function method and the spectral representation method, developed in Chapters 2 and 3, respectively, in order to obtain alternative representations of the fields from various sources.

4.2 DELTA FUNCTION TRANSFORMATIONS

In what follows, we shall require delta function representations in two and three dimensions. Depending on the particulars of the analysis, it will be convenient to express the results in different coordinate systems. In particular, we consider rectangular, polar, cylindrical, and spherical coordinates.

We begin with a point source in two dimensions (Fig. 4-1), located at the point $Q(x', y')$. We represent this point source by $\delta(x - x')\delta(y - y')$

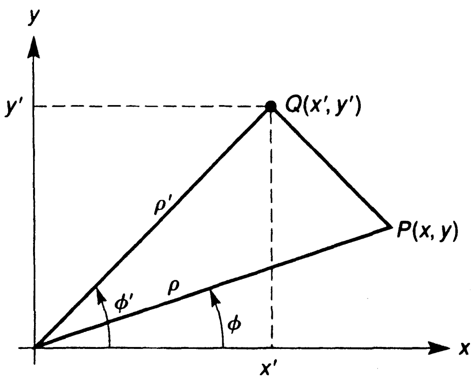

Fig. 4-1 Point source in two dimensions, located at (x', y').

and seek a corresponding representation in polar coordinates. Since the polar coordinate point corresponding to (x', y') is (ρ', ϕ'), we write

$$\delta(x - x')\delta(y - y') = f_1(\rho, \phi)\delta(\rho - \rho')\delta(\phi - \phi') \tag{4.1}$$

The function $f_1(\rho, \phi)$ allows for the possibility of an additional factor other than the delta functions that might be introduced by the Jacobian in the coordinate transformation. Integrating both sides of (4.1) over the xy-plane gives

$$1 = \int_0^{2\pi} \int_0^{\infty} f_1(\rho, \phi)\delta(\rho - \rho')\delta(\phi - \phi')\rho d\rho d\phi \tag{4.2}$$

Equation (4.2) is reduced to an identity by the choice

$$f_1(\rho, \phi) = \frac{1}{\rho} \tag{4.3}$$

Substitution into (4.1) gives

$$\delta(x - x')\delta(y - y') = \frac{\delta(\rho - \rho')\delta(\phi - \phi')}{\rho} \tag{4.4}$$

Equation (4.4) gives the polar representation of a point source in a plane as long as the source is at a location other than the origin. For a point source at the origin, we have $\delta(x)\delta(y)$. In transforming this representation to polar coordinates, we note that the origin in polar coordinates is given by $\rho = 0$, *independent of* ϕ. We say that the coordinate ϕ is *ignorable* at the origin [1] and write

$$\delta(x)\delta(y) = f_2(\rho)\delta(\rho) \tag{4.5}$$

Integrating (4.5) over the xy-plane gives

$$1 = \int_0^{2\pi} \int_0^{\infty} f_2(\rho)\delta(\rho)\rho d\rho d\phi$$
$$= \int_0^{\infty} [2\pi\rho f_2(\rho)]\delta(\rho)d\rho \tag{4.6}$$

Equation (4.6) is reduced to an identity by the choice

$$f_2(\rho) = \frac{1}{2\pi\rho} \tag{4.7}$$

Substitution into (4.5) gives

$$\delta(x)\delta(y) = \frac{\delta(\rho)}{2\pi\rho} \tag{4.8}$$

The proper transformations in three dimensions between rectangular and cylindrical coordinates require no further analysis since the z-coordinate remains the same in both systems. We have

$$\delta(x - x')\delta(y - y')\delta(z - z') = \frac{\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z')}{\rho} \tag{4.9}$$

$$\delta(x)\delta(y)\delta(z) = \frac{\delta(\rho)\delta(z)}{2\pi\rho} \tag{4.10}$$

In transformations to spherical coordinates, there are three cases of interest. We begin with a point source (Fig. 4-2) at the location $Q(x', y', z')$ and write

$$\delta(x-x')\delta(y-y')\delta(z-z') = f_3(r, \theta, \phi)\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') \tag{4.11}$$

Integrating both sides over all space gives

$$1 = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} f_3(r, \theta, \phi)\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')r^2 \sin\theta dr d\theta d\phi \tag{4.12}$$

Equation (4.12) is reduced to an identity by the choice

$$f_3(r, \theta, \phi) = \frac{1}{r^2 \sin\theta} \tag{4.13}$$

Therefore,

$$\delta(x - x')\delta(y - y')\delta(z - z') = \frac{\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')}{r^2 \sin\theta} \tag{4.14}$$

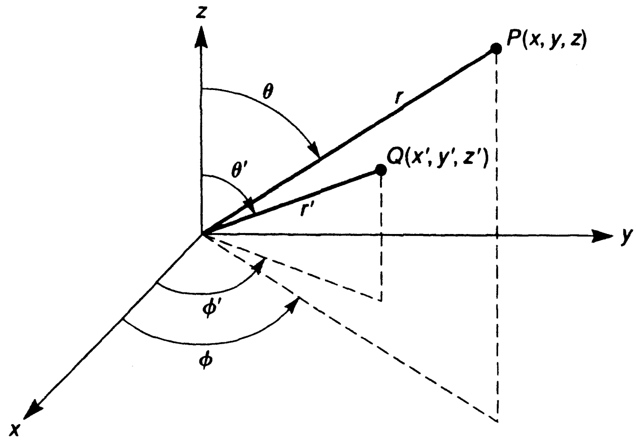

Fig. 4-2 Point source in three dimensions, located at (x', y', z').

In the case where the point source is located along the z-axis, the coordinate ϕ is ignorable. We have

$$\delta(x)\delta(y)\delta(z - z') = f_4(r, \theta)\delta(r - r')\delta(\theta) \tag{4.15}$$

Integrating over all space gives

$$\begin{aligned} 1 &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f_4(r, \theta)\delta(r - r')\delta(\theta)r^2 \sin\theta dr d\theta d\phi \\ &= \int_0^{\pi}\int_0^{\infty} [2\pi r^2 \sin\theta f_4(r, \theta)]\delta(r - r')\delta(\theta) dr d\theta \end{aligned} \tag{4.16}$$

Equation (4.16) is reduced to an identity by the choice

$$f_4(r, \theta) = \frac{1}{2\pi r^2 \sin\theta} \tag{4.17}$$

Substitution into (4.15) gives

$$\delta(x)\delta(y)\delta(z - z') = \frac{\delta(r - r')\delta(\theta)}{2\pi r^2 \sin\theta} \tag{4.18}$$

In the case where the point source is at the origin (0, 0, 0), the coordinates θ and ϕ are both ignorable. We have

$$\delta(x)\delta(y)\delta(z) = f_5(r)\delta(r) \tag{4.19}$$

Integrating over all space gives

$$\begin{aligned} 1 &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f_5(r)\delta(r)r^2 \sin\theta dr d\theta d\phi \\ &= \int_0^{\infty} [4\pi r^2 f_5(r)]\delta(r) dr \end{aligned} \tag{4.20}$$

Equation (4.20) is reduced to an identity by the choice

$$f_5(r) = \frac{1}{4\pi r^2} \tag{4.21}$$

Substitution into (4.19) gives

$$\delta(x)\delta(y)\delta(z) = \frac{\delta(r)}{4\pi r^2} \tag{4.22}$$

4.3 TIME-HARMONIC REPRESENTATIONS

In subsequent considerations of the electromagnetic model, we shall be dealing with quantities that vary harmonically with time t. The time-harmonic representation is useful in the determination of the cosinusoidal steady-state behavior of the electromagnetic fields. The representation can also be directly extended to give the response for more general forms of source input [2]. An excellent treatment of time-harmonic representations is given in [3]. We shall include herein only the major results. A time-harmonic function $f(t)$ has the form

$$f(t) = F_0 \cos(\omega t + \phi) \tag{4.23}$$

where ω is radian frequency and where F_0 and ϕ are real and time-independent. We write this function in terms of the real-part operator as follows:

$$f(t) = \mathrm{Re}\left(Fe^{i\omega t}\right) \tag{4.24}$$

where

$$F = F_0 e^{i\phi} \tag{4.25}$$

That (4.24) is equivalent to (4.23) can be demonstrated by substituting (4.25) into (4.24) and performing the real-part operation. The details are left to the reader. We may show the following relation for derivatives:

$$\frac{df}{dt} = \mathrm{Re}(i\omega F e^{i\omega t}) \tag{4.26}$$

The proof is straightforward and is left for the problems.

The real-part operator has the following useful properties [4]: For $z, z_1, z_2 \in \mathbf{C}$ and $a, t \in \mathbf{R}$,

$$\mathrm{Re}(z_1) + \mathrm{Re}(z_2) = \mathrm{Re}(z_1 + z_2) \tag{4.27}$$

$$\mathrm{Re}(az) = a\mathrm{Re}(z) \tag{4.28}$$

$$\frac{\partial}{\partial t}[\mathrm{Re}(z)] = \mathrm{Re}\left(\frac{\partial z}{\partial t}\right) \tag{4.29}$$

$$\int \mathrm{Re}(z)dt = \mathrm{Re}\left(\int z dt\right) \tag{4.30}$$

These real-part relationships are easily verified. The details are left for the problems. An additional relationship involving the real-part operator is the following: If $A, B \in \mathbf{C}$ and are time-independent, and if

$$\mathrm{Re}(Ae^{i\omega t}) = \mathrm{Re}(Be^{i\omega t}), \qquad \text{all } t \tag{4.31}$$

then

$$A = B \tag{4.32}$$

Indeed, let $t = 0$. Then $\mathrm{Re}(A) = \mathrm{Re}(B)$. Let $\omega t = \pi/2$. Then $\mathrm{Re}(iA) = \mathrm{Re}(iB)$, which implies that $\mathrm{Im}(A) = \mathrm{Im}(B)$. Since A and B are time-independent, the above equality of their real parts and their imaginary parts must be true for all t, and thus $A = B$.

The complex representation of time-harmonic functions can be extended to vectors. Indeed, let $\mathcal{E}(t)$ be a real, time-varying vector with time-harmonic components, viz.

$$\mathcal{E}(t) = \hat{x}E_x\cos(\omega t+\phi_x)+\hat{y}E_y\cos(\omega t+\phi_y)+\hat{z}E_z\cos(\omega t+\phi_z) \tag{4.33}$$

where $\hat{x}$, $\hat{y}$, $\hat{z}$ are unit vectors in the three Cartesian coordinate directions and $E_x, E_y, E_z, \phi_x, \phi_y, \phi_z$ are real and time-independent. In the same manner as in the above treatment of complex functions, complex vectors can be written in terms of the real-part operator, as follows:

$$\mathcal{E}(t) = \mathrm{Re}(\mathbf{E}e^{i\omega t}) \tag{4.34}$$

where

$$\mathbf{E} = \hat{x}E_xe^{i\phi_x} + \hat{y}E_ye^{i\phi_y} + \hat{z}E_ze^{i\phi_z} \tag{4.35}$$

The equivalence of (4.34) and (4.33) can be verified by substituting (4.35) into (4.34) and performing the real-part operation. The details are left for the reader.

4.4 THE ELECTROMAGNETIC MODEL

The macroscopic model for the behavior of electromagnetic fields is given by the following set of equations:

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} - \mathcal{M} \tag{4.36}$$

$$\nabla \times \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J} \tag{4.37}$$

$$\nabla \cdot \mathcal{D} = \rho \tag{4.38}$$

$$\nabla \cdot \mathcal{B} = \rho_m \tag{4.39}$$

$$\nabla \cdot \mathcal{J} = -\frac{\partial \rho}{\partial t} \tag{4.40}$$

$$\nabla \cdot \mathcal{M} = -\frac{\partial \rho_m}{\partial t} \tag{4.41}$$

where the symbols are defined as follows:

$\mathcal{E}$ electric field intensity (volts/meter)

$\mathcal{H}$ magnetic field intensity (amps/meter)

$\mathcal{D}$ electric flux density (coulombs/meter^2)

$\mathcal{B}$ magnetic flux density (webers/meter^2)

$\mathcal{J}$ electric current density (amps/meter^2)

$\mathcal{M}$ magnetic current density (volts/meter^2)

ρ electric charge density (coulombs/meter^3)

ρ_m magnetic charge density (webers/meter^3)

In performing dimensional analyses, it is helpful to have the following equalities:

coulomb=amp · second

weber=volt · second

ohm=volt/amp

All field and source quantities vary with both space and time. Typically, for the electric field, we have $\mathcal{E}(x, y, z, t)$, which we write in shorthand notation as $\mathcal{E}(\mathbf{r}, t)$. The magnetic charge $\rho_m(\mathbf{r}, t)$ and the magnetic current $\mathcal{M}(\mathbf{r}, t)$ have not been shown to exist in nature, but can be defined as *equivalent* sources in equivalence theorems involving the electric fields [5]. We shall assume that all quantities vary time-harmonically. Typically,

$$\mathcal{E}(\mathbf{r}, t) = \text{Re}\left[\mathbf{E}(\mathbf{r}, \omega)e^{i\omega t}\right] \tag{4.42}$$

Then, in (4.36), we have

$$\nabla \times \left[\text{Re}(\mathbf{E}e^{i\omega t})\right] = -\frac{\partial}{\partial t}\left[\text{Re}(\mathbf{B}e^{i\omega t})\right] - \text{Re}(\mathbf{M}e^{i\omega t}) \tag{4.43}$$

Using the real-part operator relationships in (4.26)–(4.29), we obtain

$$\mathrm{Re}\left[\nabla \times (\mathbf{E}e^{i\omega t})\right] = -\mathrm{Re}\left[(i\omega\mathbf{B} + \mathbf{M})e^{i\omega t}\right] \tag{4.44}$$

But, by a well-known vector identity,

$$\nabla \times (\mathbf{E}e^{i\omega t}) = (\nabla \times \mathbf{E})e^{i\omega t} \tag{4.45}$$

Substituting (4.45) into (4.44) and applying (4.31) and (4.32), we obtain

$$\nabla \times \mathbf{E} = -i\omega\mathbf{B} - \mathbf{M} \tag{4.46}$$

A similar procedure in (4.37)–(4.41) yields

$$\nabla \times \mathbf{H} = i\omega\mathbf{D} + \mathbf{J} \tag{4.47}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{4.48}$$

$$\nabla \cdot \mathbf{B} = \rho_m \tag{4.49}$$

$$\nabla \cdot \mathbf{J} = -i\omega\rho \tag{4.50}$$

$$\nabla \cdot \mathbf{M} = -i\omega\rho_m \tag{4.51}$$

We remark that, in keeping with usual practice, we have used the same symbols for electric charge density in both the time and frequency domains, and similarly for magnetic charge density. Which domain we are considering will be clear in context.

In a *simple medium*, the electric flux density $\mathbf{D}$ is simply related to the electric field intensity $\mathbf{E}$ by

$$\mathbf{D} = \epsilon\mathbf{E} \tag{4.52}$$

where ϵ is the permittivity of the medium in farads/meter. Similarly, for the magnetic field,

$$\mathbf{B} = \mu\mathbf{H} \tag{4.53}$$

where μ is the permeability of the medium in henrys/meter. The basic units of farads can be deduced from the units of $\mathbf{D}$ and $\mathbf{E}$ in (4.52), and similarly for the units of henrys in (4.53).

Equations (4.46)–(4.53) constitute the *electromagnetic model* in simple media. We shall use the model in what follows to develop representations of electromagnetic sources. In Chapter 5, we shall use the model to develop solutions to some example boundary value problems.

4.5 THE SHEET CURRENT SOURCE

Consider a planar electric current sheet in the xy-plane (Fig. 4-3). We assume linear polarization in the x-direction and no variations in either x or y. The mathematical representation of this current sheet involves the electric current density $\mathbf{J}(z)$ in amps/m^2, given by

$$\mathbf{J}(z) = \hat{x} J_{s0} \delta(z) \tag{4.54}$$

where $\hat{x}$ is a unit vector in the x-direction and J_{s0} is a constant surface current density in amps/m. We write Maxwell's curl equations by substituting (4.52) into (4.47) and (4.53) into (4.46) to give

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H} - \mathbf{M} \tag{4.55}$$

$$\nabla \times \mathbf{H} = i\omega\epsilon\mathbf{E} + \mathbf{J} \tag{4.56}$$

In general, the permittivity is complex and is given by [6]

$$\epsilon = \epsilon_d + \frac{\sigma}{i\omega} \tag{4.57}$$

where σ is the conductivity in mhos/m and ϵ_d is the permittivity of a perfect dielectric ($\sigma = 0$). The complex permittivity accounts for losses in the medium. For the case where the losses are negligible, the complex permittivity reduces to the perfect dielectric permittivity ϵ_d.

Since the source, given by (4.54), varies only with z, and since there are no scattering objects present, we conclude that $\partial/\partial x = \partial/\partial y = 0$. Substituting this result and (4.54) into Maxwell's equations, setting $\mathbf{M} = 0$, and expanding in Cartesian coordinates, we obtain

$$-\frac{dH_y}{dz} = J_{s0}\delta(z) + i\omega\epsilon E_x \tag{4.58}$$

$$\frac{dH_x}{dz} = i\omega\epsilon E_y \tag{4.59}$$

$$0 = i\omega\epsilon E_z \tag{4.60}$$

$$\frac{dE_y}{dz} = i\omega\mu H_x \tag{4.61}$$

$$\frac{dE_x}{dz} = -i\omega\mu H_y \tag{4.62}$$

$$0 = i\omega\mu H_z \tag{4.63}$$

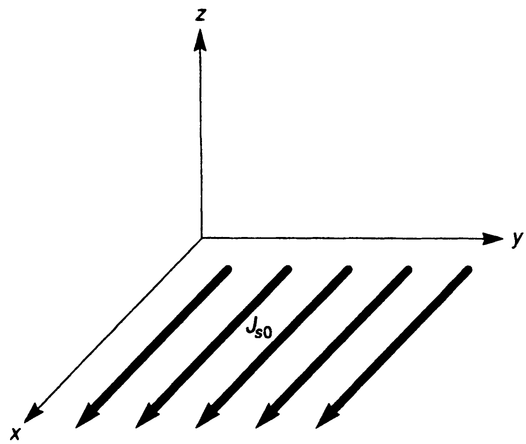

Fig. 4-3 Electric current sheet located in the xy-plane and linearly polarized in the x-direction. Current sheet extends over entire xy-plane.

From (4.60) and (4.63), we find that

$$E_z = H_z = 0 \tag{4.64}$$

The remaining equations decouple into two independent sets. The first set is given by

$$\frac{dE_x}{dz} = -i\omega\mu H_y \tag{4.65}$$

$$-\frac{dH_y}{dz} = J_{s0}\delta(z) + i\omega\epsilon E_x \tag{4.66}$$

The second set is given by

$$\frac{dE_y}{dz} = i\omega\mu H_x \tag{4.67}$$

$$\frac{dH_x}{dz} = i\omega\epsilon E_y \tag{4.68}$$

Note that the first set contains E_x, H_y, and J_{s0}, while the second set contains E_y, H_x, and no sources. Since the second set is source-free throughout all space and is not coupled in any manner to the first set, we must conclude that the only solution to (4.67) and (4.68) is the trivial solution, viz.

$$E_y = H_x = 0 \tag{4.69}$$

The problem is therefore completely described by (4.65) and (4.66), together with appropriate conditions as $z \to \pm\infty$. We take the derivative of (4.65) with respect to z and substitute (4.66) to obtain the following set:

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = i\omega\mu J_{s0}\delta(z) \tag{4.70}$$

$$H_y = -\frac{1}{i\omega\mu}\frac{dE_x}{dz} \tag{4.71}$$

where the wavenumber k is given by

$$k = k_d\sqrt{1 - iS} \tag{4.72}$$

with

$$S = \frac{\sigma}{\omega\epsilon_d} \tag{4.73}$$

and

$$k_d = \omega\sqrt{\mu\epsilon_d} \tag{4.74}$$

The wavenumber k_d is the wavenumber that would be present in a perfect dielectric ($\sigma = 0$). In the engineering literature, S is called the *loss tangent* [7]. We note that, if the second-order, linear, ordinary differential equation in (4.70) can be solved for the electric field E_x, the magnetic field H_y can be found by the simple differentiation indicated in (4.71). For limiting conditions, we demand that, for $k \in \mathbf{C}$,

$$\lim_{z\to\pm\infty} E_x = 0 \tag{4.75}$$

To solve the differential equation in (4.70), we let

$$E_x = -i\omega\mu J_{s0} g \tag{4.76}$$

Substitution into (4.70) yields

$$-\frac{d^2 g}{dz^2} - k^2 g = \delta(z) \tag{4.77}$$

with limiting conditions

$$\lim_{z\to\pm\infty} g = 0 \tag{4.78}$$

We recognize this as a Green's function problem in SLP3. In general, the Green's function is associated with the delta-function source $\delta(z - \zeta)$.

This source would produce a Green's function $g(z, \zeta)$. In the case in (4.77), $\zeta = 0$ and we obtain $g(z, 0)$. The solution to this Green's function problem has been given in Example 2.20, viz.

$$g(z,0) = \frac{e^{-ik|z|}}{2ik}, \qquad \mathrm{Im}(k) < 0 \tag{4.79}$$

Substitution into (4.76) gives

$$E_x(z) = -\frac{\omega\mu}{2k} J_{s0} e^{-ik|z|} \tag{4.80}$$

We may normalize this result by letting

$$J_{s0} = -\frac{2k}{\omega\mu} \tag{4.81}$$

We then have

$$E_x(z) = e^{-ik|z|} \tag{4.82}$$

Substitution of this result into (4.71) yields the accompanying magnetic field, viz.

$$H_y(z) = \frac{1}{\eta} \begin{cases} e^{-ikz}, & z > 0 \\ -e^{ikz}, & z < 0 \end{cases} \tag{4.83}$$

where the intrinsic impedance η is given by

$$\eta = \frac{\omega\mu}{k} \tag{4.84}$$

The solution to (4.77) with limiting conditions in (4.78) has been obtained by the Green's function method. Alternately, the solution can also be obtained by spectral methods. From the result in Example 3.4, the spectral representation of the delta function for the operator $-d^2/dz^2$ with limiting conditions given by (4.78) is given by

$$\delta(z-\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\beta(z-\zeta)} d\beta \tag{4.85}$$

This spectral representation defines the Fourier transform. Applying this transform to the Green's function, we have

$$g(z,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\beta, 0) e^{i\beta z} d\beta \tag{4.86}$$

$$G(\beta, 0) = \int_{-\infty}^{\infty} g(z, 0)e^{-i\beta z}dz \tag{4.87}$$

Taking the Fourier transform of (4.77) and rearranging gives

$$G(\beta, 0) = \frac{1}{\beta^2 - k^2} \tag{4.88}$$

Taking the inverse Fourier transform yields the alternative solution form

$$g(z, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta \tag{4.89}$$

Substituting into (4.76) and applying (4.81), we find that

$$E_x(z) = -\frac{ik}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\beta z}}{k^2 - \beta^2} d\beta \tag{4.90}$$

Since the solution to (4.70) is unique, we may equate the two results in (4.82) and (4.90) to give the following useful relationship:

$$e^{-ik|z|} = \frac{ik}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta, \qquad \mathrm{Im}(k) < 0 \tag{4.91}$$

We remark that the result in (4.91) can also be obtained by contour integration techniques, as we show in the following example.

EXAMPLE 4.1 We shall show that

$$\int_{-\infty}^{\infty} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta = \frac{\pi}{ik} e^{-ik|z|}, \qquad \mathrm{Im}(k) < 0$$

Consider

$$\oint_{C_R + C_1} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta \tag{4.92}$$

around a closed contour (Fig. 4-4) along the real axis from $-R$ to R and a semicircle of radius R through the upper half-plane. We constrain R such that $R > |k|$. The denominator of (4.92) has simple poles at $\beta = \pm k$. We have $\mathrm{Im}(k) < 0$. It can be shown that this selection of the sign of the imaginary part, together with the definition of k in (4.72), implies that $\mathrm{Re}(k) > 0$. The details are left for the problems. Therefore, the pole at $\beta = -k$ is enclosed by the contour, and the residue theorem gives

$$\oint_{C_R + C_1} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta = 2\pi i \mathrm{Res}\left[\frac{e^{i\beta z}}{\beta^2 - k^2}; -k\right] \tag{4.93}$$

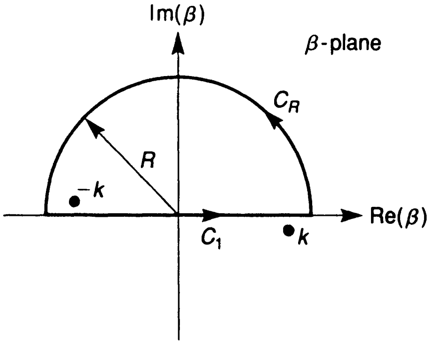

Fig. 4-4 Contour for the evaluation of the contour integral in (4.92).

Evaluating the residue and splitting the contour integral into two pieces gives

$$\int_{-R}^{R} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta + \int_{C_R} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta = \frac{\pi}{ik} e^{-ikz} \tag{4.94}$$

We now show that

$$\lim_{R\to\infty} \left| \int_{C_R} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta \right| = 0, \qquad z > 0 \tag{4.95}$$

Indeed, on C_R, let

$$\beta = Re^{i\theta} \tag{4.96}$$

Then,

$$\begin{aligned}
\left| \int_{C_R} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta \right| &= \left| \int_0^{\pi} \frac{e^{izR\cos\theta} e^{-zR\sin\theta} iRe^{i\theta} d\theta}{R^2 e^{i2\theta} - k^2} \right| \\
&\leq \frac{R}{R^2 - |k|^2} \int_0^{\pi} e^{-zR\sin\theta} d\theta \\
&\leq \frac{R\pi}{R^2 - |k|^2}
\end{aligned} \tag{4.97}$$

In the last inequality, we have used the fact that $zR\sin\theta > 0, 0 < \theta < \pi$ to bound the integrand with unity. Therefore, in the limit as $R \to \infty$, the integral around C_R approaches zero. Taking the limit in (4.94) gives

$$\int_{-\infty}^{\infty} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta = \frac{\pi}{ik} e^{-ikz}, \qquad z > 0 \tag{4.98}$$

Equation (4.98) gives the result for $z > 0$. For $z < 0$, we close the contour C_1 along a semi-circle through the lower half-plane. The details are left for the problems.

■

4.6 THE LINE SOURCE

Consider a line current source located along the z-axis (Fig. 4-5) and extending from $z = -\infty$ to $z = \infty$. We represent the current density associated with this source by

$$\mathbf{J}(\rho) = \hat{z} I_0 \delta(x)\delta(y) \tag{4.99}$$

where I_0 is a constant current in amps. We begin our study of the fields produced by this current source by considering the problem in cylindrical coordinates. From (4.8), the cylindrical coordinate representation of the current $\mathbf{J}(\rho)$ is given by

$$\mathbf{J}(\rho) = \hat{z} I_0 \frac{\delta(\rho)}{2\pi\rho} \tag{4.100}$$

Since the current source is independent of ϕ and z, and since there are no scattering objects, we must have $\partial/\partial\phi = \partial/\partial z = 0$. Maxwell's curl equations in (4.55) and (4.56) can therefore be written in cylindrical coordinates by

$$E_\rho = H_\rho = 0 \tag{4.101}$$

$$\frac{dH_z}{d\rho} = -i\omega\epsilon E_\phi \tag{4.102}$$

$$\frac{1}{\rho}\left[\frac{d}{d\rho}(\rho E_\phi)\right] = -i\omega\mu H_z \tag{4.103}$$

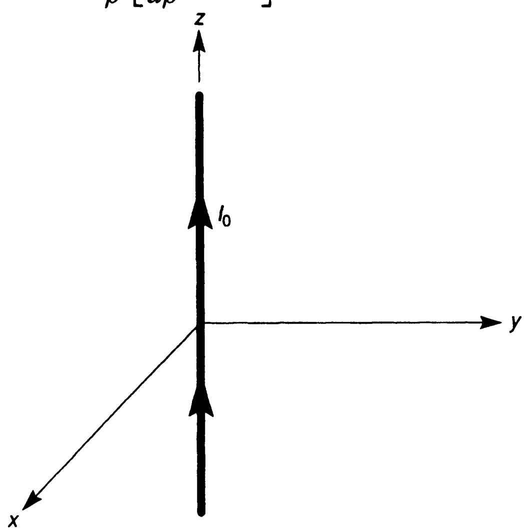

Fig. 4-5 Electric line current located along the z-axis and extending from $z = -\infty$ to $z = \infty$.

$$\frac{dE_z}{d\rho} = i\omega\mu H_\phi \tag{4.104}$$

$$\frac{1}{\rho}\left[\frac{d}{d\rho}(\rho H_\phi)\right] = J_z + i\omega\epsilon E_z \tag{4.105}$$

where

$$J_z = I_0 \frac{\delta(\rho)}{2\pi\rho} \tag{4.106}$$

We note that (4.102) and (4.103) are source-free and independent of (4.104) and (4.105). Therefore,

$$H_z = E_\phi = 0 \tag{4.107}$$

We conclude that (4.104) and (4.105), together with appropriate boundary and/or limiting conditions, completely characterize the problem. We multiply (4.104) by ρ, take the derivative with respect to ρ, multiply by $1/\rho$, and then divide by $i\omega\mu$ to obtain

$$\frac{1}{i\omega\mu\rho}\left[\frac{d}{d\rho}\left(\rho\frac{dE_z}{d\rho}\right)\right] = \frac{1}{\rho}\left[\frac{d}{d\rho}(\rho H_\phi)\right] \tag{4.108}$$

Substitution of (4.108) into (4.105) produces the following set:

$$\frac{1}{\rho}\left[\frac{d}{d\rho}\left(\rho\frac{dE_z}{d\rho}\right)\right] + k^2 E_z = i\omega\mu I_0 \frac{\delta(\rho)}{2\pi\rho} \tag{4.109}$$

$$H_\phi = \frac{1}{i\omega\mu}\frac{dE_z}{d\rho} \tag{4.110}$$

To solve the differential equation in (4.109), we let

$$g = -\frac{2\pi E_z}{i\omega\mu I_0} \tag{4.111}$$

and obtain

$$\frac{1}{\rho}\left[\frac{d}{d\rho}\left(\rho\frac{dg}{d\rho}\right)\right] + k^2 g = -\frac{\delta(\rho)}{\rho} \tag{4.112}$$

with limiting condition

$$\lim_{\rho\to\infty} g = 0 \tag{4.113}$$

To solve (4.112), we first consider a result from Example 2.21. In that example, we considered

$$\frac{1}{\rho}\left[\frac{d}{d\rho}\left(\rho\frac{dg}{d\rho}\right)\right] + k^2 g = -\frac{\delta(\rho-\rho')}{\rho} \tag{4.114}$$

with the limiting condition given in (4.113) and a finiteness condition at the origin. The solution was given in (2.184) and is repeated here with some trivial changes in notation, viz.

$$g(\rho,\rho') = \frac{\pi}{2i}\begin{cases} H_0^{(2)}(k\rho')J_0(k\rho), & \rho < \rho' \\ H_0^{(2)}(k\rho)J_0(k\rho'), & \rho > \rho' \end{cases} \tag{4.115}$$

Taking the limit as $\rho' \to 0$ yields the solution to (4.112), viz.

$$g(\rho, 0) = \frac{\pi}{2i}H_0^{(2)}(k\rho) \tag{4.116}$$

Substituting (4.116) into (4.111) and solving for E_z, we obtain

$$E_z = -\frac{\omega\mu I_0}{4}H_0^{(2)}(k\rho) \tag{4.117}$$

Substitution of this result into (4.110) gives

$$H_\phi = -\frac{ikI_0}{4}H_1^{(2)}(k\rho) \tag{4.118}$$

The solution to (4.114) with limiting condition (4.113) at infinity and a finiteness condition at the origin has been obtained by the Green's function method. Alternately, the solution can also be obtained by spectral methods. From Example 3.5, the spectral representation of the differential operator in (4.114) with the given limiting and finiteness conditions is given by

$$\frac{\delta(\rho-\rho')}{\rho} = \int_0^\infty J_0(\lambda\rho)J_0(\lambda\rho')\lambda d\lambda \tag{4.119}$$

As found in Example 3.5, this representation gives the Fourier–Bessel transform pair

$$F(\lambda) = \int_0^\infty f(\rho)J_0(\lambda\rho)\rho d\rho \tag{4.120}$$

$$f(\rho) = \int_0^\infty F(\lambda)J_0(\lambda\rho)\lambda d\lambda \tag{4.121}$$

Taking the Fourier–Bessel transform of both sides of (4.114) gives

$$(-\lambda^2 + k^2)G(\lambda,\rho') = -J_0(\lambda\rho') \tag{4.122}$$

where $G(\lambda, \rho')$ is the Fourier–Bessel transform of $g(\rho, \rho')$. Solving for G and taking the inverse Fourier–Bessel transform gives

$$g(\rho,\rho') = \int_0^\infty \frac{J_0(\lambda\rho)J_0(\lambda\rho')}{\lambda^2 - k^2}\lambda d\lambda \tag{4.123}$$

To produce the solution to (4.112), we take the limit as $\rho' \to 0$ and obtain

$$g(\rho, 0) = \int_0^\infty \frac{J_0(\lambda\rho)}{\lambda^2 - k^2}\lambda d\lambda \tag{4.124}$$

Substitution into (4.111) yields

$$E_z = -\frac{i\omega\mu I_0}{2\pi}\int_0^\infty \frac{J_0(\lambda\rho)}{\lambda^2 - k^2}\lambda d\lambda \tag{4.125}$$

Comparing (4.125) to (4.117), we obtain the following integral representation of the Hankel function [8]:

$$H_0^{(2)}(k\rho) = \frac{2i}{\pi}\int_0^\infty \frac{J_0(\lambda\rho)}{\lambda^2 - k^2}\lambda d\lambda \tag{4.126}$$

Taking the Fourier–Bessel transform gives the relation

$$\frac{1}{\lambda^2 - k^2} = \frac{\pi}{2i}\int_0^\infty H_0^{(2)}(k\rho)J_0(\lambda\rho)\rho d\rho \tag{4.127}$$

We have obtained two representations of the electric field E_z produced by a line current located along the z-axis. These representations are given by (4.117) and (4.125). Further representations can be obtained by considering the same problem in Cartesian coordinates. Since $\partial/\partial z = 0$, Maxwell's curl equations reduce to

$$\frac{\partial E_z}{\partial y} = -i\omega\mu H_x \tag{4.128}$$

$$\frac{\partial E_z}{\partial x} = i\omega\mu H_y \tag{4.129}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu H_z \tag{4.130}$$

$$\frac{\partial H_z}{\partial y} = i\omega\epsilon E_x \tag{4.131}$$

$$\frac{\partial H_z}{\partial x} = -i\omega\epsilon E_y \tag{4.132}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = I_0\delta(x)\delta(y) + i\omega\epsilon E_z \tag{4.133}$$

These six equations can be grouped into two independent sets, as follows:

Set 1: TM_z

$$\frac{\partial E_z}{\partial y} = -i\omega\mu H_x \tag{4.134}$$

$$\frac{\partial E_z}{\partial x} = i\omega\mu H_y \tag{4.135}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = I_0\delta(x)\delta(y) + i\omega\epsilon E_z \tag{4.136}$$

Set 2: TE_z

$$\frac{\partial H_z}{\partial y} = i\omega\epsilon E_x \tag{4.137}$$

$$\frac{\partial H_z}{\partial x} = -i\omega\epsilon E_y \tag{4.138}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu H_z \tag{4.139}$$

Set 1 is labeled TM_z since it contains no magnetic field component in the z-direction. In a similar manner, Set 2 is TE_z. We note that the two sets are not coupled and that Set 2 is source-free. Therefore, the only solution to Set 2 consists of the null fields, viz.

$$H_z = E_x = E_y = 0 \tag{4.140}$$

To solve for the TM_z fields, we differentiate (4.134) with respect to y, (4.135) with respect to x, add the result, and substitute (4.136) to obtain

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k^2 E_z = i\omega\mu I_0\delta(x)\delta(y) \tag{4.141}$$

$$H_x = -\frac{1}{i\omega\mu}\frac{\partial E_z}{\partial y} \tag{4.142}$$

$$H_y = \frac{1}{i\omega\mu}\frac{\partial E_z}{\partial x} \tag{4.143}$$

We consider (4.141), together with the limiting conditions

$$\lim_{x\to\pm\infty} E_z(x, y) = 0 \tag{4.144}$$

$$\lim_{y\to\pm\infty} E_z(x, y) = 0 \tag{4.145}$$

To reduce (4.141), we combine the spectral representation and Green's function methods. First, from Example 3.4, the spectral representation of $\partial^2/\partial x^2$ with limiting condition in (4.144) is given by

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_x(x-x')}dk_x \tag{4.146}$$

Multiplying both sides of (4.146) by $E_z(x', y)$ and integrating over $(-\infty, \infty)$ gives the *spatial Fourier transform* pair

$$E_z(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{E}_z(k_x, y)e^{ik_x x}dk_x \tag{4.147}$$

$$\hat{E}_z(k_x, y) = \int_{-\infty}^{\infty} E_z(x,y)e^{-ik_x x}dx \tag{4.148}$$

We therefore take the Fourier transform of both sides of (4.141) and produce

$$\frac{d^2\hat{E}_z}{dy^2} + k_y^2\hat{E}_z = i\omega\mu I_0\delta(y) \tag{4.149}$$

where

$$k_y = \sqrt{k^2 - k_x^2} \tag{4.150}$$

and where $E_z(x, y)$ and $\hat{E}_z(k_x, y)$ are Fourier transform pairs. We let

$$G_1 = -\frac{\hat{E}_z}{i\omega\mu I_0} \tag{4.151}$$

so that

$$-\left(\frac{d^2}{dy^2} + k_y^2\right)G_1 = \delta(y) \tag{4.152}$$

where

$$\lim_{y\to\pm\infty} G_1(k_x, y) = 0 \tag{4.153}$$

We have previously produced the solution to this Green's function problem in (4.79). In this case, we have

$$G_1 = \frac{e^{-ik_y|y|}}{2ik_y}, \qquad \text{Im}(k_y) < 0 \tag{4.154}$$

Substituting into (4.151), solving for $\hat{E}_z$, and taking the inverse Fourier transform, we obtain

$$E_z(x,y) = -\frac{\omega\mu I_0}{4\pi}\int_{-\infty}^{\infty} \frac{e^{-ik_y|y|}}{k_y}e^{ik_x x}dk_x \tag{4.155}$$

Equation (4.155) gives another form of solution to the line source problem. If we compare (4.117) and (4.155), we produce the following integral representation of the Hankel function:

$$H_0^{(2)}\left[k(x^2+y^2)^{1/2}\right] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i(k^2-k_x^2)^{1/2}|y|}}{(k^2-k_x^2)^{1/2}} e^{ik_x x} dk_x \tag{4.156}$$

where

$$\mathrm{Im}(k^2-k_x^2)^{1/2} < 0 \tag{4.157}$$

Taking the transform of both sides of (4.156) yields

$$\frac{e^{-i(k^2-k_x^2)^{1/2}|y|}}{(k^2-k_x^2)^{1/2}} = \frac{1}{2}\int_{-\infty}^{\infty} H_0^{(2)}\left[k(x^2+y^2)^{1/2}\right] e^{-ik_x x} dx \tag{4.158}$$

We note in (4.141), (4.144), and (4.145) that the x and y differential operators and their manifolds are identical. We therefore could have taken the Fourier transform with respect to y. The result can be immediately obtained by interchanging x with y and k_x with k_y.

We now have three representations of the electric field from the line current source along the z-axis, given by (4.117), (4.125), and (4.155). A fourth representation can be obtained by taking the Fourier transforms of (4.141) with respect to x *and* y to obtain

$$\left(-k_x^2 - k_y^2 + k^2\right)\tilde{E}_z = i\omega\mu I_0 \tag{4.159}$$

where $E_z(x, y)$ and $\tilde{E}_z(k_x, k_y)$ are two-dimensional Fourier transform pairs. Solving for $\tilde{E}_z$ and taking the two-dimensional inverse transform gives

$$E_z(x, y) = i\omega\mu I_0 \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_y y)}}{k^2-k_x^2-k_y^2} dk_x dk_y \tag{4.160}$$

Comparing (4.117) and (4.160), we obtain the following double-integral representation of the Hankel function:

$$H_0^{(2)}\left[k(x^2+y^2)^{1/2}\right] = \frac{1}{i\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_y y)}}{k^2-k_x^2-k_y^2} dk_x dk_y \tag{4.161}$$

Taking the two-dimensional transform, we obtain

$$\frac{1}{k^2-k_x^2-k_y^2} = \frac{i}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_0^{(2)}\left[k(x^2+y^2)^{1/2}\right] e^{-i(k_x x + k_y y)} dx dy \tag{4.162}$$

We next move the line source away from the z-axis (Fig. 4-6) by defining a current

$$\begin{aligned}\mathbf{J} &= \hat{z} I_0 \delta(x - x')\delta(y - y') \\ &= \hat{z} I_0 \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi') \end{aligned} \tag{4.163}$$

where we have used (4.4) to make the delta function coordinate transformation. We expand Maxwell's curl equations in cylindrical coordinates and obtain

$$\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} = i\omega\epsilon E_\rho \tag{4.164}$$

$$\frac{\partial H_z}{\partial \rho} = -i\omega\epsilon E_\phi \tag{4.165}$$

$$\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho E_\phi) - \frac{\partial E_\rho}{\partial \phi}\right] = -i\omega\mu H_z \tag{4.166}$$

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} = -i\omega\mu H_\rho \tag{4.167}$$

$$\frac{\partial E_z}{\partial \rho} = i\omega\mu H_\phi \tag{4.168}$$

$$\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho H_\phi) - \frac{\partial H_\rho}{\partial \phi}\right] = J_z + i\omega\epsilon E_z \tag{4.169}$$

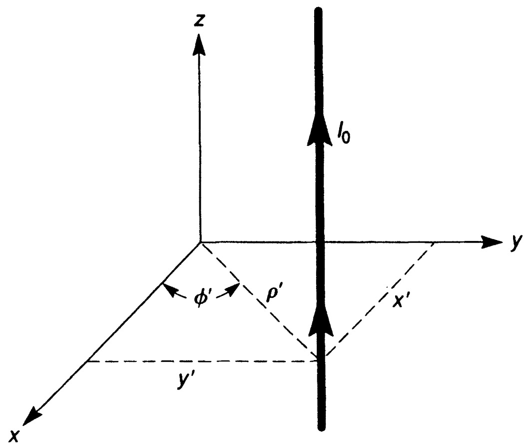

Fig. 4-6 Electric line current located at (x', y') and extending from $z = -\infty$ to $z = \infty$.

where we have used $\partial/\partial z = 0$ in making the expansions. We have arranged the six equations above such that the first three are TE_z and the second three are TM_z. We note that the TE_z and TM_z fields are not coupled, and that the TE_z fields are source-free. We therefore conclude that the TE_z fields are zero. For the TM_z set, we multiply (4.168) by ρ and differentiate with respect to ρ to give

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial E_z}{\partial \rho}\right) = i\omega\mu \frac{\partial}{\partial \rho}(\rho H_\phi) \tag{4.170}$$

Next, we differentiate (4.167) with respect to ϕ to give

$$\frac{1}{\rho}\frac{\partial^2 E_z}{\partial \phi^2} = -i\omega\mu \frac{\partial H_\rho}{\partial \phi} \tag{4.171}$$

Using (4.170) and (4.171) in (4.169) to eliminate H_ϕ and H_ρ yields the following:

$$\nabla^2_{\rho\phi} E_z + k^2 E_z = i\omega\mu I_0 \frac{\delta(\rho - \rho')\delta(\phi - \phi')}{\rho} \tag{4.172}$$

where, from (4.167) and (4.168), we have

$$H_\phi = \frac{1}{i\omega\mu}\frac{\partial E_z}{\partial \rho} \tag{4.173}$$

$$H_\rho = -\frac{1}{i\omega\mu\rho}\frac{\partial E_z}{\partial \phi} \tag{4.174}$$

and where

$$\nabla^2_{\rho\phi} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right)\right] + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} \tag{4.175}$$

The boundary and limiting conditions associated with E_z are as follows:

$$\lim_{\rho \to \infty} E_z = 0 \tag{4.176}$$

$$\lim_{\rho \to 0} E_z = \text{finite} \tag{4.177}$$

$$E_z \Big|_{\phi=\phi_0} = E_z \Big|_{\phi=\phi_0+2\pi} \tag{4.178}$$

$$\frac{\partial E_z}{\partial \phi}\Big|_{\phi=\phi_0} = \frac{\partial E_z}{\partial \phi}\Big|_{\phi=\phi_0+2\pi} \tag{4.179}$$

where ϕ_0 is any fixed angle. Let

$$g = -\frac{E_z}{i\omega\mu I_0} \tag{4.180}$$

Substitution of (4.180) and (4.175) into (4.172) yields the two-dimensional Green's function problem

$$\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right)\right] + \frac{1}{\rho^2}\frac{\partial^2 g}{\partial\phi^2} + k^2 g = -\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') \tag{4.181}$$

$$\lim_{\rho\to\infty} g = 0 \tag{4.182}$$

$$\lim_{\rho\to 0} g = \text{finite} \tag{4.183}$$

$$g\Big|_{\phi=\phi_0} = g\Big|_{\phi=\phi_0+2\pi} \tag{4.184}$$

$$\frac{\partial g}{\partial\phi}\Big|_{\phi=\phi_0} = \frac{\partial g}{\partial\phi}\Big|_{\phi=\phi_0+2\pi} \tag{4.185}$$

In order to separate the ρ-operator from the ϕ-operator, we multiply both sides of (4.181) by ρ^2 and obtain

$$\rho\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right)\right] + \frac{\partial^2 g}{\partial\phi^2} + (k\rho)^2 g = -\rho\delta(\rho-\rho')\delta(\phi-\phi') \tag{4.186}$$

A perhaps more descriptive way of writing (4.186) is as follows:

$$(L_\rho + L_\phi)g = \rho\delta(\rho-\rho')\delta(\phi-\phi') \tag{4.187}$$

where

$$L_\rho = -\rho\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)\right] - (k\rho)^2 \tag{4.188}$$

$$L_\phi = -\frac{\partial^2}{\partial\phi^2} \tag{4.189}$$

The operator L_ρ, with boundary and limiting conditions given in (4.182) and (4.183), is called the *Kantorovich–Lebedev* operator and leads to the Kantorovich–Lebedev transform considered in Example 3.6. The operator L_ϕ with periodic boundary conditions given in (4.184) and (4.185) produces the complex Fourier series considered in Problem 3.2. To solve for the Green's function $g(\rho, \phi, \rho', \phi')$, we have the choice of applying a complex

Fourier series expansion or the Kantorovich–Lebedev transform. We shall consider both choices in turn. We begin by choosing the complex Fourier series.

Using the complex Fourier expansion in Problem 3.2, we expand the Green's function as follows:

$$g(\rho, \phi, \rho', \phi') = \sum_{n=-\infty}^{\infty} a_n(\rho, \rho', \phi')\sqrt{\frac{1}{2\pi}}e^{in\phi} \tag{4.190}$$

The coefficient a_n is given by

$$\begin{aligned} a_n(\rho, \rho', \phi') &= \int_0^{2\pi} g(\rho, \phi, \rho', \phi')\sqrt{\frac{1}{2\pi}}e^{-in\phi}d\phi \\ &= \langle g, u_n \rangle \end{aligned} \tag{4.191}$$

where u_n is the normalized eigenfunction

$$u_n = \sqrt{\frac{1}{2\pi}}e^{in\phi} \tag{4.192}$$

and where we have defined the complex inner product

$$\langle u, v \rangle = \int_0^{2\pi} u(\phi)\overline{v}(\phi)d\phi \tag{4.193}$$

We symbolize the transform from g to a_n given in (4.191) by

$$g \Longrightarrow a_n \tag{4.194}$$

Since the operator L_ϕ is self-adjoint, we use the procedure in (3.24)–(3.27) and find that

$$L_\phi g \Longrightarrow n^2 a_n \tag{4.195}$$

Also,

$$\delta(\phi - \phi') \Longrightarrow \sqrt{\frac{1}{2\pi}}e^{-in\phi'} \tag{4.196}$$

Using these results to transform (4.187), we obtain

$$(L_\rho + n^2)a_n = \rho\delta(\rho - \rho')\sqrt{\frac{1}{2\pi}}e^{-in\phi'} \tag{4.197}$$

Substituting (4.188) and dividing both sides by ρ^2 gives

$$-\left\{\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)\right]+k^2-\frac{n^2}{\rho^2}\right\}b_n=\frac{\delta(\rho-\rho')}{\rho} \tag{4.198}$$

where

$$b_n=\frac{a_n}{\sqrt{\frac{1}{2\pi}}e^{-in\phi'}} \tag{4.199}$$

For limiting conditions associated with b_n, we choose

$$\lim_{\rho\to\infty} b_n=0 \tag{4.200}$$

$$\lim_{\rho\to 0} b_n=\text{ finite} \tag{4.201}$$

The reader should verify that this choice of limiting conditions is consistent with the limiting conditions associated with g given in (4.182) and (4.183).

We now solve (4.198) by the Green's function method. This particular Green's function problem has been previously considered in Example 3.6. Using the results therein, we may immediately write

$$b_n=\frac{\pi}{2i}\begin{cases} H_n^{(2)}(k\rho')J_n(k\rho), & \rho<\rho' \\ H_n^{(2)}(k\rho)J_n(k\rho'), & \rho>\rho' \end{cases} \tag{4.202}$$

Substituting (4.202) into (4.199), solving for a_n, and substituting the result into (4.190) yields

$$g(\rho,\phi,\rho',\phi')=\frac{1}{4i}\sum_{n=-\infty}^{\infty}e^{in(\phi-\phi')}\begin{cases} H_n^{(2)}(k\rho')J_n(k\rho), & \rho<\rho' \\ H_n^{(2)}(k\rho)J_n(k\rho'), & \rho>\rho' \end{cases} \tag{4.203}$$

Finally, this result substituted into (4.180) gives

$$E_z=-\frac{\omega\mu I_0}{4}\sum_{n=-\infty}^{\infty}e^{in(\phi-\phi')}\begin{cases} H_n^{(2)}(k\rho')J_n(k\rho), & \rho<\rho' \\ H_n^{(2)}(k\rho)J_n(k\rho'), & \rho>\rho' \end{cases} \tag{4.204}$$

We recall from (4.117) that the electric field from a line source located at the origin is given by

$$E_z=-\frac{\omega\mu I_0}{4}H_0^{(2)}(k\rho)$$

A coordinate transformation $x \to x - x'$, $y \to y - y'$ gives, in cylindrical coordinates,

$$E_z = -\frac{\omega\mu I_0}{4} H_0^{(2)}(k(|\rho - \rho'|) \tag{4.205}$$

where

$$|\rho - \rho'| = \sqrt{(x - x')^2 + (y - y')^2} = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')} \tag{4.206}$$

Since (4.204) and (4.205) must yield the same result, we may equate them to give

$$H_0^{(2)}(k|\rho - \rho'|) = \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \begin{cases} H_n^{(2)}(k\rho')J_n(k\rho), & \rho < \rho' \\ H_n^{(2)}(k\rho)J_n(k\rho'), & \rho > \rho' \end{cases} \tag{4.207}$$

which is the *Addition Theorem* for the Hankel function.

We next obtain an alternative representation of the solution for the electric field by applying the Kantorovich–Lebedev transform to (4.186). If $g(\rho, \phi, \rho', \phi')$ and $G(\beta, \phi, \rho', \phi')$ are Kantorovich–Lebedev transform pairs, then the transform applied to (4.186) gives

$$\frac{d^2G}{d\phi^2} + \beta^2 G = -H_\beta^{(2)}(k\rho')\delta(\phi - \phi') \tag{4.208}$$

where we have used the transform in (3.157) and the derivative transform in (3.161). We define

$$\hat{G}(\beta, \phi, \phi') = \frac{G(\beta, \phi, \rho', \phi')}{H_\beta^{(2)}(k\rho')} \tag{4.209}$$

and obtain

$$\frac{d^2\hat{G}}{d\phi^2} + \beta^2\hat{G} = -\delta(\phi - \phi') \tag{4.210}$$

with the boundary conditions

$$\hat{G}(\beta, \phi_0, \phi') = \hat{G}(\beta, \phi_0 + 2\pi, \phi') \tag{4.211}$$

$$\frac{d\hat{G}(\beta, \phi_0, \phi')}{d\phi} = \frac{d\hat{G}(\beta, \phi_0 + 2\pi, \phi')}{d\phi} \tag{4.212}$$

From Problem 2.18, the solution to this Green's function problem is given by

$$\hat{G} = -\frac{\cos[\beta(|\phi - \phi'| - \pi)]}{2\beta \sin \pi\beta} \tag{4.213}$$

We substitute (4.213) into (4.209), solve for G, and use the inverse Kantorovich–Lebedev transform given in (3.159) to obtain

$$g(\rho, \phi, \rho', \phi') = -\frac{1}{8}\int_{i\infty}^{-i\infty} \frac{H_\beta^{(2)}(k\rho) H_\beta^{(2)}(k\rho') \cos[\beta(|\phi - \phi'| - \pi)]}{\sin \pi\beta} d\beta \tag{4.214}$$

Substituting into (4.180) and solving for E_z gives the electric field at (ρ, ϕ) caused by an electric line current source at (ρ', ϕ'), viz.

$$E_z = \frac{i\omega\mu I_0}{8}\int_{i\infty}^{-i\infty} \frac{H_\beta^{(2)}(k\rho) H_\beta^{(2)}(k\rho') \cos[\beta(|\phi - \phi'| - \pi)]}{\sin \pi\beta} d\beta \tag{4.215}$$

Equation (4.215) gives an alternative representation to (4.204) for the electric field. By comparison with (4.205), we obtain the following integral representation alternative to the Hankel function addition theorem in (4.207):

$$H_0^{(2)}(k|\rho - \rho'|) = \frac{1}{2i}\int_{i\infty}^{-i\infty} \frac{H_\beta^{(2)}(k\rho) H_\beta^{(2)}(k\rho') \cos[\beta(|\phi - \phi'| - \pi)]}{\sin \pi\beta} d\beta \tag{4.216}$$

For further discussion of the Kantorovich–Lebedev transform, the reader is referred to [9]. The transform is particularly useful in the solution to electromagnetic problems involving conducting wedges [10].

4.7 THE CYLINDRICAL SHELL SOURCE

Consider a circularly-cylindrical shell current source (Fig. 4-7), located symmetrically about the z-axis and extending over $z \in (-\infty, \infty)$. We represent the current by

$$\mathbf{J} = \hat{z} I_0 \frac{\delta(\rho - \rho')}{2\pi\rho} \tag{4.217}$$

The factor 2π has been included so that the total current is I_0, viz.

$$\int_0^{2\pi}\int_0^{\infty} I_0 \left[\frac{\delta(\rho - \rho')}{2\pi\rho}\right] \rho d\rho d\phi = I_0$$

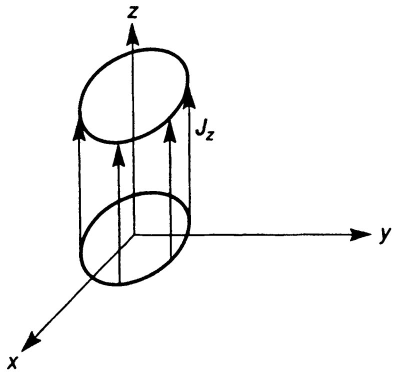

Fig. 4-7 Electric cylindrical shell current located symmetrically about the z-axis and extending from $z = -\infty$ to $z = \infty$.

This problem is independent of both ϕ and z. Therefore, using (4.109) and (4.110), we have

$$\frac{1}{\rho}\left[\frac{d}{d\rho}\left(\rho\frac{dE_z}{d\rho}\right)\right] + k^2 E_z = i\omega\mu I_0 \frac{\delta(\rho-\rho')}{2\pi\rho} \tag{4.218}$$

$$H_\phi = \frac{1}{i\omega\mu}\frac{dE_z}{d\rho} \tag{4.219}$$

Let

$$g = -\frac{2\pi E_z}{i\omega\mu I_0} \tag{4.220}$$

so that

$$\frac{1}{\rho}\left[\frac{d}{d\rho}\left(\rho\frac{dg}{d\rho}\right)\right] + k^2 g = -\frac{\delta(\rho-\rho')}{\rho} \tag{4.221}$$

with the limiting condition

$$\lim_{\rho\to\infty} g(\rho,\rho') = 0$$

and a finiteness condition at the origin. The solution to this Green's function problem has been given in (4.115), viz.

$$g(\rho,\rho') = \frac{\pi}{2i}\begin{cases} H_0^{(2)}(k\rho')J_0(k\rho), & \rho < \rho' \\ H_0^{(2)}(k\rho)J_0(k\rho'), & \rho > \rho' \end{cases} \tag{4.222}$$

Substituting into (4.220) and solving for E_z, we obtain

$$E_z = -\frac{\omega\mu I_0}{4}\begin{cases} H_0^{(2)}(k\rho')J_0(k\rho), & \rho < \rho' \\ H_0^{(2)}(k\rho)J_0(k\rho'), & \rho > \rho' \end{cases} \tag{4.223}$$

Note that

$$\lim_{\rho' \to 0} E_z = -\frac{\omega\mu I_0}{4} H_0^{(2)}(k\rho) \tag{4.224}$$

which reproduces the result for the line current at the origin, given in (4.117). The reader should compare the development of the cylindrical shell source in this section to the treatment in [11].

4.8 THE RING SOURCE

Consider a magnetic ring source $\mathbf{M}$, located symmetrically about the z-axis (Fig. 4-8). We describe the source by

$$\mathbf{M} = P_0 \frac{\delta(\rho - \rho')}{\rho} \delta(z - z')\hat{\phi} \tag{4.225}$$

where $\hat{\phi}$ is a unit vector in the ϕ-direction and P_0 is a magnetic current moment in volt · meters. Since the problem is symmetric in ϕ, we must have $\partial/\partial\phi = 0$. With this restriction, Maxwell's curl equations in cylindrical coordinates reduce to

$$-\frac{\partial H_\phi}{\partial z} = i\omega\epsilon E_\rho \tag{4.226}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi) = i\omega\epsilon E_z \tag{4.227}$$

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = -M_\phi - i\omega\mu H_\phi \tag{4.228}$$

$$\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = i\omega\epsilon E_\phi \tag{4.229}$$

$$\frac{\partial E_\phi}{\partial z} = i\omega\mu H_\rho \tag{4.230}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\phi) = -i\omega\mu H_z \tag{4.231}$$

We have arranged the six equations above so that the first three are TE_ϕ and the second three are TM_ϕ. We note that the TE_ϕ and TM_ϕ fields are not coupled, and that the TM_ϕ fields are source-free. We therefore conclude that the TM_ϕ fields are zero. For the TE_ϕ set, we differentiate (4.226) with respect to z and (4.227) with respect to ρ to obtain

$$-\frac{\partial^2 H_\phi}{\partial z^2} = i\omega\epsilon \frac{\partial E_\rho}{\partial z} \tag{4.232}$$

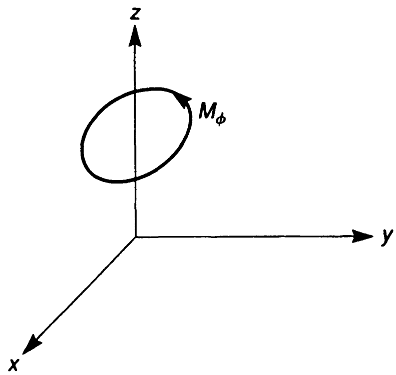

Fig. 4-8 Magnetic ring current source located symmetrically about the z-axis.

$$\frac{\partial}{\partial \rho}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi)\right] = i\omega\epsilon\frac{\partial E_z}{\partial \rho} \tag{4.233}$$

Subtracting (4.232) from (4.233) and substituting (4.228), we obtain

$$\frac{\partial}{\partial \rho}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi)\right] + \frac{\partial^2 H_\phi}{\partial z^2} + k^2 H_\phi = i\omega\epsilon M_\phi \tag{4.234}$$

But,

$$\frac{\partial}{\partial \rho}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi)\right] = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial H_\phi}{\partial \rho}\right) - \frac{H_\phi}{\rho^2} \tag{4.235}$$

Substituting (4.235) into (4.234), we obtain

$$\nabla^2_{\rho z} H_\phi + \left(k^2 - \frac{1}{\rho^2}\right) H_\phi = i\omega\epsilon M_\phi \tag{4.236}$$

where

$$\nabla^2_{\rho z} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{\partial^2}{\partial z^2} \tag{4.237}$$

and

$$M_\phi = P_0\frac{\delta(\rho - \rho')}{\rho}\delta(z - z') \tag{4.238}$$

Once we have solved (4.236), the electric field components can be obtained from (4.226) and (4.227), viz.

$$E_\rho = -\frac{1}{i\omega\epsilon}\frac{\partial H_\phi}{\partial z} \tag{4.239}$$

$$E_z = \frac{1}{i\omega\epsilon\rho}\frac{\partial}{\partial \rho}(\rho H_\phi) \tag{4.240}$$

By suitably normalizing H_ϕ, we may obtain the following Green's function problem:

$$\left(L_\rho + L_z - k^2\right) g = \frac{\delta(\rho - \rho')}{\rho}\delta(z - z') \tag{4.241}$$

where

$$L_\rho = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2} \tag{4.242}$$

$$L_z = -\frac{\partial^2}{\partial z^2} \tag{4.243}$$

and where

$$g = -\frac{H_\phi}{i\omega\epsilon P_0} \tag{4.244}$$

The associated limiting conditions are

$$\lim_{\rho\to 0} g \text{ finite} \tag{4.245}$$

$$\lim_{\rho\to\infty} g = 0 \tag{4.246}$$

$$\lim_{z\to\pm\infty} g = 0 \tag{4.247}$$

As shown in Problem 3.5, the operator L_ρ in (4.242) with limiting conditions in (4.245) and (4.246) leads to the Fourier–Bessel transform of order one. As shown in Example 3.4, the operator L_z in (4.243) with limiting conditions in (4.247) leads to the Fourier transform. We therefore have the choice of applying either of these transforms. We shall consider each in turn, beginning with the Fourier transform.

Using the Fourier transform over the z-variable in (4.241), we obtain

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) - \left(k_\rho^2 - \frac{1}{\rho^2}\right) G = e^{-ik_z z'}\frac{\delta(\rho - \rho')}{\rho} \tag{4.248}$$

where $g(\rho, z, \rho', z')$ and $G(\rho, k_z, \rho', z')$ are Fourier transform pairs and

$$k_\rho^2 = k^2 - k_z^2 \tag{4.249}$$

We let

$$H = \frac{G}{e^{-ik_z z'}} \tag{4.250}$$

and obtain

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial H}{\partial\rho}\right) - \left(k_\rho^2 - \frac{1}{\rho^2}\right) H = \frac{\delta(\rho - \rho')}{\rho} \tag{4.251}$$

We may satisfy the limiting conditions in (4.245) and (4.246) by requiring

$$\lim_{\rho\to 0} H \text{ finite} \tag{4.252}$$

$$\lim_{\rho\to\infty} H = 0 \tag{4.253}$$

The solution to (4.251) with conditions in (4.252) and (4.253) has been previously considered in Problem 3.5. We find that

$$H = \frac{\pi}{2i}\begin{cases} H_1^{(2)}(k_\rho\rho')J_1(k_\rho\rho), & \rho < \rho' \\ H_1^{(2)}(k_\rho\rho)J_1(k_\rho\rho'), & \rho > \rho' \end{cases} \tag{4.254}$$

Substituting (4.254) into (4.250) and the result into (4.244), we obtain

$$H_\phi = -\frac{\omega\epsilon P_0}{4}\int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')}\begin{cases} H_1^{(2)}(k_\rho\rho')J_1(k_\rho\rho), & \rho < \rho' \\ H_1^{(2)}(k_\rho\rho)J_1(k_\rho\rho'), & \rho > \rho' \end{cases} \tag{4.255}$$

We may obtain an alternative representation by taking the Fourier–Bessel transform of order one in (4.241). From the results in Problem 3.5, we may write the following Fourier–Bessel transform pair:

$$\hat{G}(\lambda, z, \rho', z') = \int_0^\infty g(\rho, z, \rho', z')J_1(\lambda\rho)\rho d\rho \tag{4.256}$$

with inverse

$$g(\rho, z, \rho', z') = \int_0^\infty \hat{G}(\lambda, z, \rho', z')J_1(\lambda\rho)\lambda d\lambda \tag{4.257}$$

From Problem 3.5, we have the following Fourier–Bessel transform pair:

$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + \frac{1}{\rho^2}\right]g \Longleftrightarrow \lambda^2\hat{G} \tag{4.258}$$

Applying (4.256) and (4.258) in (4.241), we obtain

$$-\frac{d^2\hat{H}}{dz^2} - \beta^2\hat{H} = \delta(z - z') \tag{4.259}$$

where

$$\hat{H} = \frac{\hat{G}}{J_1(\lambda\rho')} \tag{4.260}$$

and

$$\beta = \sqrt{k^2 - \lambda^2} \tag{4.261}$$

To satisfy the limiting condition in (4.247), we require

$$\lim_{z \to \pm\infty} \hat{H} = 0 \tag{4.262}$$

We have previously considered this Green's function problem in Example 2.20. The result applied to (4.259) is as follows:

$$\hat{H} = \frac{e^{-i\beta|z-z'|}}{2i\beta}, \qquad \operatorname{Im}(\beta) < 0 \tag{4.263}$$

Substituting (4.263) into (4.260) and the result into (4.257), we obtain

$$g = \int_0^\infty \frac{e^{-i\beta|z-z'|}}{2i\beta} J_1(\lambda\rho') J_1(\lambda\rho) \lambda d\lambda \tag{4.264}$$

Finally, the magnetic field can be obtained by substituting (4.264) into (4.244), viz.

$$H_\phi = -i\omega\epsilon P_0 \int_0^\infty \frac{e^{-i\beta|z-z'|}}{2i\beta} J_1(\lambda\rho') J_1(\lambda\rho) \lambda d\lambda \tag{4.265}$$

4.9 THE POINT SOURCE

Consider an electric point source located at the origin (Fig. 4-9) and polarized in the z-direction. We assume that the source radiates in empty space and that there is a small amount of loss present so that we may apply limiting conditions on the fields as we approach $\pm\infty$ in x, y, and z. We may describe the source in terms of its current moment $I_0\ell$ as follows:

$$\mathbf{J} = \hat{z} I_0 \ell \delta(\mathbf{r}) \tag{4.266}$$

where the units of $I_0\ell$ are amp · meters and where

$$\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z) \tag{4.267}$$

The fields from such a source can be described conveniently by use of the magnetic vector potential **A**, as follows:

$$-\left(\nabla^2 + k^2\right) \mathbf{A} = \mu \mathbf{J} \tag{4.268}$$

The magnetic and electric fields are obtained from the vector potential by the following two relationships:

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \tag{4.269}$$

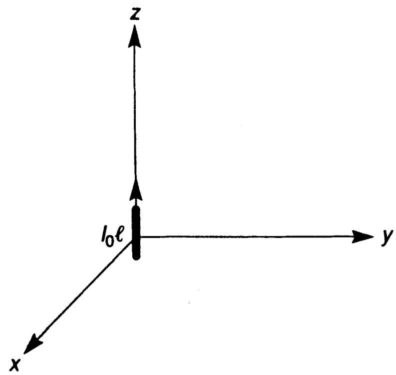

Fig. 4-9 Electric current point source located at the origin.

$$\mathbf{E} = -i\omega\left[\mathbf{A} + \frac{1}{k^2}\nabla(\nabla\cdot\mathbf{A})\right] \tag{4.270}$$

Since the current is in the z-direction, the magnetic vector potential is also z-directed, viz.

$$\mathbf{A} = \hat{z}A_z \tag{4.271}$$

The z-component of (4.268) yields a differential equation for determining the vector potential A_z, viz.

$$-\left(\nabla^2 + k^2\right)A_z = \mu I_0\ell\delta(\mathbf{r}) \tag{4.272}$$

We let

$$g = \frac{A_z}{\mu I_0\ell} \tag{4.273}$$

and obtain

$$-\left(\nabla^2 + k^2\right)g = \delta(\mathbf{r}) \tag{4.274}$$

We first consider the solution to the point source problem in Cartesian coordinates. Expanding (4.274), we have

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)g = \delta(x)\delta(y)\delta(z) \tag{4.275}$$

From the results in Exampe 3.4, the spectral representation of each of the three differential operators in (4.275), with limiting conditions at $\pm\infty$, yields the Fourier transform. Typically, for the operator $-\partial^2/\partial x^2$ with limiting conditions

$$\lim_{x\to\pm\infty} g = 0$$

we have

$$\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_x(x-x')}dk_x \tag{4.276}$$

Taking the Fourier transform in (4.275) over x, y, z, we obtain

$$-\left(-k_x^2 - k_y^2 - k_z^2 + k^2\right)G = 1$$

where $g \Longleftrightarrow G$ is a triple Fourier transform pair over (x, y, z), (k_x, k_y, k_z). Solving for G, we obtain

$$G = -\frac{1}{k^2 - k_x^2 - k_y^2 - k_z^2}$$

Taking the inverse transform over k_x, k_y, k_z yields

$$g = -\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(k_xx+k_yy+k_zz)}}{k^2 - k_x^2 - k_y^2 - k_z^2}dk_xdk_ydk_z \tag{4.277}$$

Using (4.273), we obtain, for the magnetic vector potential,

$$A_z = -\frac{\mu I_0\ell}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(k_xx+k_yy+k_zz)}}{k^2 - k_x^2 - k_y^2 - k_z^2}dk_xdk_ydk_z \tag{4.278}$$

An alternative form in Cartesian coordinates can be obtained by taking the Fourier transform over any two Cartesian variables. In (4.275), if we take the Fourier transform over x and y, we obtain

$$-\left(\frac{d^2}{dz^2} + k^2 - k_x^2 - k_y^2\right)\hat{G} = \delta(z) \tag{4.279}$$

where $g \Longleftrightarrow \hat{G}$ is a double Fourier transform pair over (x, y), (k_x, k_y). The one-dimensional Green's function problem in (4.279) is identical to that posed in (4.152) and (4.153). The solution is given by

$$\hat{G} = \frac{e^{-i\beta|z|}}{2i\beta}, \qquad \mathrm{Im}(\beta) < 0 \tag{4.280}$$

where

$$\beta = \sqrt{k^2 - k_x^2 - k_y^2} \tag{4.281}$$

Taking the inverse Fourier transform over k_x, k_y yields

$$g = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i\beta|z|}}{2i\beta}e^{i(k_xx+k_yy)}dk_xdk_y \tag{4.282}$$

Using (4.273), we obtain the magnetic vector potential A_z as follows:

$$A_z = \frac{\mu I_0 \ell}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i\beta|z|}}{2i\beta} e^{i(k_x x + k_y y)} dk_x dk_y \tag{4.283}$$

We next consider the point source problem in cylindrical coordinates. We note that, in cylindrical coordinates, the Green's function problem in (4.274) is a function of ρ and z and independent of ϕ. Expanding (4.274), we obtain

$$-\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial g}{\partial \rho}\right) - \frac{\partial^2 g}{\partial z^2} - k^2 g = \frac{\delta(\rho)\delta(z)}{2\pi\rho} \tag{4.284}$$

The spectral representation of the ρ-operator produces the Fourier–Bessel transform of order zero, while the spectral representation of the z-operator produces the Fourier transform. We shall derive three forms of solution in cylindrical coordinates. For the first form, we apply the Fourier–Bessel transform to (4.284) and obtain

$$-\frac{d^2 \tilde{G}}{dz^2} - \Gamma^2 \tilde{G} = \frac{\delta(z)}{2\pi} \tag{4.285}$$

where

$$\tilde{G}(\lambda, z) = \int_0^{\infty} g(\rho, z) J_0(\lambda\rho)\rho d\rho \tag{4.286}$$

and

$$\Gamma = \sqrt{k^2 - \lambda^2} \tag{4.287}$$

The solution to (4.285) with limiting conditions

$$\lim_{z \to \pm\infty} \tilde{G} = 0$$

is the same as (4.263), viz.

$$2\pi \tilde{G} = \frac{e^{-i\Gamma|z|}}{2i\Gamma}, \qquad \operatorname{Im}(\Gamma) < 0 \tag{4.288}$$

Taking the inverse Fourier–Bessel transform yields

$$g(\rho, z) = \frac{1}{2\pi} \int_0^{\infty} \frac{e^{-i\Gamma|z|}}{2i\Gamma} J_0(\lambda\rho)\lambda d\lambda \tag{4.289}$$

Using (4.273), we produce the magnetic vector potential

$$A_z = \frac{\mu I_0 \ell}{2\pi} \int_0^{\infty} \frac{e^{-i\Gamma|z|}}{2i\Gamma} J_0(\lambda\rho)\lambda d\lambda \tag{4.290}$$

The second form of solution is obtained by taking the Fourier transform of (4.284) with respect to z, with the result

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\hat{G}}{\partial\rho}\right) - \tau^2\hat{G} = \frac{\delta(\rho)}{2\pi\rho} \tag{4.291}$$

where

$$\hat{G}(\rho, k_z) = \int_{-\infty}^{\infty} g(\rho, z)e^{-ik_z z}dz \tag{4.292}$$

and

$$\tau = \sqrt{k^2 - k_z^2} \tag{4.293}$$

We now use (2.185) and write the solution to (4.291) as follows:

$$2\pi\hat{G} = \frac{\pi}{2i}H_0^{(2)}(\tau\rho) \tag{4.294}$$

Taking the inverse Fourier transform, we obtain

$$g = \frac{1}{8\pi i}\int_{-\infty}^{\infty} H_0^{(2)}(\tau\rho)e^{ik_z z}dk_z \tag{4.295}$$

Using (4.273), we have, for the magnetic vector potential,

$$A_z = \frac{\mu I_0 \ell}{8\pi i}\int_{-\infty}^{\infty} H_0^{(2)}(\tau\rho)e^{ik_z z}dk_z \tag{4.296}$$

The third form of solution is obtained by taking the Fourier–Bessel transform with respect to ρ *and* the Fourier transform with respect to z, with the result

$$(\lambda^2 + k_z^2 - k^2)\mathcal{G} = \frac{1}{2\pi}$$

Solving for $\mathcal{G}$ and taking the inverse Fourier–Bessel transform and the inverse Fourier transform gives

$$g = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{e^{ik_z z}J_0(\lambda\rho)}{\lambda^2 + k_z^2 - k^2}\lambda d\lambda dk_z \tag{4.297}$$

Finally, we consider the point source problem in spherical coordinates. We note that the Green's function problem posed by (4.274) is symmetric in spherical coordinates over both θ and ϕ. Therefore, the operator ∇^2 is given totally by its radial component. We therefore may write (4.274) as follows:

$$\frac{1}{r^2}\left[\frac{d}{dr}\left(r^2\frac{dg}{dr}\right)\right] + k^2 g = -\frac{\delta(r)}{4\pi r^2} \tag{4.298}$$

where we have used (4.22) for the spherical coordinate representation of the delta function at the origin. We have obtained the solution to this Green's function problem previously in Example 2.22. Using (2.201), we obtain

$$g = \frac{e^{-ikr}}{4\pi r} \tag{4.299}$$

Again using (4.273), we obtain the magnetic vector potential, as follows:

$$A_z = \frac{\mu I_0 \ell}{4\pi}\left(\frac{e^{-ikr}}{r}\right) \tag{4.300}$$

We may exhibit five identities from the alternative representations of the point source. Comparing (4.299) and (4.277), we obtain

$$\frac{e^{-ikr}}{r} = -\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(k_x x+k_y y+k_z z)}}{k^2-k_x^2-k_y^2-k_z^2}dk_x dk_y dk_z \tag{4.301}$$

From (4.299) and (4.282), we have

$$\frac{e^{-ikr}}{r} = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i\beta|z|}}{2i\beta}e^{i(k_x x+k_y y)}dk_x dk_y \tag{4.302}$$

where

$$\beta = \sqrt{k^2-k_x^2-k_y^2} \tag{4.303}$$

From (4.299) and (4.289), we have

$$\frac{e^{-ikr}}{r} = 2\int_0^{\infty}\frac{e^{-i\Gamma|z|}}{2i\Gamma}J_0(\lambda\rho)\lambda d\lambda \tag{4.304}$$

where

$$\Gamma = \sqrt{k^2-\lambda^2} \tag{4.305}$$

From (4.299) and (4.295), we have

$$\frac{e^{-ikr}}{r} = \frac{1}{2i}\int_{-\infty}^{\infty}H_0^{(2)}(\tau\rho)e^{ik_z z}dk_z \tag{4.306}$$

where

$$\tau = \sqrt{k^2-k_z^2} \tag{4.307}$$

From (4.299) and (4.297), we have

$$\frac{e^{-ikr}}{r} = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{ik_z z}J_0(\lambda\rho)}{\lambda^2+k_z^2-k^2}\lambda d\lambda dk_z \tag{4.308}$$

PROBLEMS

4.1. Given the time-harmonic representation of $f(t)$ in (4.23), show that

$$\frac{df}{dt} = \mathrm{Re}(i\omega F e^{i\omega t})$$

4.2. Verify the four relations for the real-part operator, given in (4.27)–(4.30). *Hint*: To prove the two relations for derivatives and integrals, begin with the basic definition of a derivative and a Riemann integral.

4.3. One of the important theorems of electromagnetic theory is the *principle of duality* [12],[13]. Using duality, make the necessary changes in (4.80)–(4.83) to obtain the fields produced by the magnetic sheet source

$$\mathbf{M}(z) = \hat{x} M_{s0} \delta(z)$$

where M_{s0} is a constant magnetic surface current density in volts/m.

4.4. From (4.72), the wavenumber with loss is given by

$$k = k_d \sqrt{1 - iS}$$

Show that the requirement $\mathrm{Im}(k) < 0$ implies that $\mathrm{Re}(k) > 0$. *Hint*: Write ik in terms of its real and imaginary parts, viz.

$$ik = \alpha + i\beta = ik_d\sqrt{1 - iS}$$

Solve for α and β by squaring both sides and discarding the extraneous root. Note that $\mathrm{Im}(k) < 0$ implies $\mathrm{Re}(\alpha) > 0$. From the sign of α, it is then possible to infer the sign of β.

4.5. In (4.98), we obtained

$$\int_{-\infty}^{\infty} \frac{e^{i\beta z}}{\beta^2 - k^2} d\beta = \frac{\pi}{ik} e^{-ikz}, \qquad z > 0$$

By contour integration and the calculus of residues, obtain the result for $z < 0$. *Hint*: In Example 4.1, we closed the contour on a semi-circle through the upper half of the β-plane. For $z < 0$, close the contour through the lower half of the β-plane.

4.6. An interesting variation [14] on the line source problem examined in Section 4.3 is the line source located at (x', y') parallel to the z-axis and polarized in the ρ-direction (Fig. 4-10). Such a source can be represented by

$$\mathbf{J} = \hat{\rho} I_0 \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi')$$

Show that the magnetic field radiated by this source is given by

$$H_z = -\frac{I_0}{4\rho'} \sum_{-\infty}^{\infty} n e^{in(\phi-\phi')} \begin{cases} H_n^{(2)}(k\rho')J_n(k\rho), & \rho < \rho' \\ H_n^{(2)}(k\rho)J_n(k\rho'), & \rho > \rho' \end{cases}$$

Show that, despite the presence in the sum of the multiplicative factor n, the series converges as $n \to \pm\infty$.

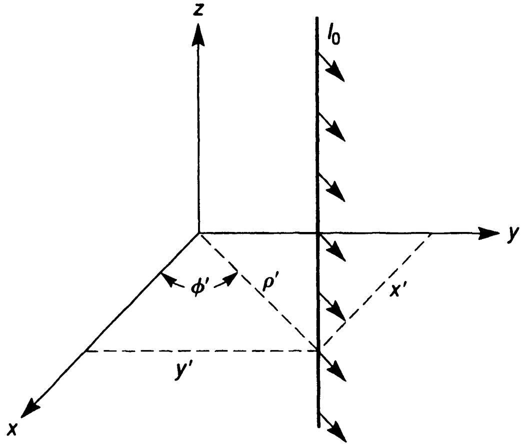

Fig. 4-10 Line source parallel to z-axis and ρ-polarized.

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