1. Space- and Time-Dependent Linear Fields

1.1 FORMULATION OF VECTOR FIELD AND SCALAR POTENTIAL PROBLEMS

Field equations describe implicitly the space- and time-dependent response of a linear field to a known excitation and pose a problem of determining field solutions subject to initial and boundary conditions. Since knowledge of symmetry (reciprocity) properties of a field frequently facilitates the determination of explicit field solutions, it is desirable to infer such properties from the general form of the field equations. For this purpose one considers certain auxiliary or adjoint problems, related to the original field problem in such a way as to reveal the space–time symmetry of the original field.

If field problems are phrased in terms of Green's functions, which describe the field response to "point-source excitation," the desired properties appear most succinctly as symmetries in these Green's functions. Relevant Green's functions for a number of classical linear fields are defined herein so as to emphasize common features in the description of any linear field and to expedite the application of generic mathematical solutions to several different fields. Since scalar, vector, or n-component fields are to be considered, the corresponding Green's functions assume scalar, dyadic, or matrix forms. Scalar Green's functions are most convenient for analysis, and hence wherever possible, dyadic and matrix Green's functions will be decomposed into their independent scalar components. Such a scalarization cannot always be effected.

Field equations may be phrased either as a system of first-order partial differential equations, or on elimination of some of the field variables, as higher-order equations. One advantage of the first-order formulation is its applicability in unchanged form to both homogeneous or inhomogeneous media (i.e., media whose constitutive parameters are respectively independent of, or dependent on, position). The second-order or "reduced" formulations are usually dependent upon whether they relate to homogeneous or inhomogeneous media. Both first- and second-order formulations will be presented for a number of linear fields.

In reduced formulations of a composite field, effects of one type of field are exhibited as a modification of the constitutive parameters of another type. For example, in an electromagnetic-plasma field, dynamical effects of charged particles in ionized media appear in a reduced electromagnetic formulation as a modification of the equivalent permittivity of the electromagnetic medium. Whereas the constitutive parameters in a first-order formulation are non-dispersive, the constitutive parameters in a reduced formulation exhibit spatial and temporal dispersion. Spatial and temporal dispersion in constitutive parameters obtains when the parameters depend, respectively, on spatial and temporal derivative operators, in addition to a possible dependence on space and time variables. It should be noted that it is also conventional to employ the term "dispersion" in connection with the frequency dependence of plane-wave constituents of a field. These two uses of the term are not necessarily equivalent and should be distinguished.

1.1a The Scalar Acoustic Field

General properties

A non-viscous uncharged fluid of mass density $n_0 m$ and static isotropic pressure p_0 provides a homogeneous or inhomogeneous background capable of supporting acoustic phenomena. In the linear regime the acoustic field is describable by small-amplitude variations of pressure $p = p(\mathbf{r}, t)$ and velocity $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ obeying the Euler field equations[1]:

$$\begin{aligned} \frac{1}{\gamma p_0}\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{v} &= -s, \\ \nabla p + n_0 m\frac{\partial \mathbf{v}}{\partial t} &= -\mathbf{f}. \end{aligned} \tag{1}$$

The excitation terms $s = s(\mathbf{r}, t)$ and $\mathbf{f} = \mathbf{f}(\mathbf{r}, t)$ represent the scalar particle source and the impressed vector force densities, respectively; γ is the specific heat ratio for the fluid, and ∇ is the spatial gradient operator. The pressure and velocity fields are uniquely defined by Eqs. (1) if one imposes on the excitation the requirement $s = 0 = \mathbf{f}$ for $t \leq t_1$, and on the fields both the initial condition,

$$p = 0 = \mathbf{v} \qquad \text{for } t \leq t_1, \tag{1a}$$

and the boundary condition,

$$p = \alpha\mathbf{v}\cdot\mathbf{n} \qquad \text{on } S. \tag{1b}$$

The unit vector $\mathbf{n}$ is normal to the surface S (if any) bounding the volume within which the field equations (1) are applicable, and α is a parameter characteristic of the boundary.

On multiplication of the two field equations (1) by p and $\mathbf{v}$, respectively, and addition, one derives the conservation-of-energy statement:

$$\nabla\cdot p\mathbf{v} = -\frac{\partial}{\partial t}\left(\frac{1}{2\gamma p_0}p^2 + \frac{n_0 m}{2}\mathbf{v}^2\right) - sp - \mathbf{f}\cdot\mathbf{v}. \tag{2}$$

The vector $p\mathbf{v}$ is identified as the instantaneous acoustic power flow per unit area at $\mathbf{r}, t$ and the terms in parentheses as the total stored energy density. The quantities $-sp$ and $-\mathbf{f}\cdot\mathbf{v}$ represent the power supplied per unit volume at $\mathbf{r}, t$ by the impressed source and force densities, respectively.

For homogeneous media one derives a second-order set of field equations that follow readily from Eq. (1) on elimination of $\mathbf{v}$ or p. For simplicity we omit the excitation terms and thereby obtain the homogeneous (wave) equations

$$\nabla^2 p - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}p = 0, \tag{3a}$$

$$\nabla\nabla\cdot\mathbf{v} - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\mathbf{v} = 0, \tag{3b}$$

where $a = (\gamma p_0/n_0 m)^{1/2} = (\gamma\kappa T_0/m)^{1/2}$ is the acoustic (thermal) speed, T_0 the background temperature, and κ the Boltzmann constant. The presence of source terms leads to an evident modification of Eqs. (3). The "longitudinal" character of the acoustic field becomes manifest on taking the divergence ($\nabla\cdot$) and curl ($\nabla\times$) of Eq. (3b). It is thereby apparent that in the source-free case $\nabla\cdot\mathbf{v}$ obeys a wave equation of the same form as Eq. (3a); the vorticity $\nabla\times\mathbf{v}$ remains constant in time and is, in fact, zero because of Eq. (1a), if the velocity field is bounded. As will be indicated in Sec. 1.2a, these statements imply that for acoustic propagation of a single plane wave, the only non-vanishing component of the velocity field is along the direction of propagation (i.e., is "longitudinal").

The solution of spatially homogeneous acoustic field problems can be based on either first-order or second-order field equations. A virtue of the latter is that since the acoustic field is basically scalar, the field description may be expressed in terms of a single scalar variable, which may be a pressure, a velocity potential, etc. Nevertheless, we shall employ a procedure based on the first-order field equations (1), since this has the advantages of (a) displaying in simple terms the field dependence on the excitations s and $\mathbf{f}$, (b) applying to both homogeneous and inhomogeneous media, and (c) permitting the use of a mathematical format common to all linear field problems.

Linearity of the field equations (1) implies that the pressure and velocity at any space–time point $\mathbf{r}, t$ can be expressed in terms of the excitations s and $\mathbf{f}$ as

$$\begin{aligned} p(\mathbf{r}, t) &= -\int G_{11}(\mathbf{r}, \mathbf{r}'; t, t')s(\mathbf{r}', t')\,d\mathbf{r}'\,dt' - \int \mathbf{G}_{12}(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{f}(\mathbf{r}', t')\,d\mathbf{r}'\,dt' \\ \mathbf{v}(\mathbf{r}, t) &= -\int \mathbf{G}_{21}(\mathbf{r}, \mathbf{r}'; t, t')s(\mathbf{r}', t')\,d\mathbf{r}'\,dt' - \int \mathcal{G}_{22}(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{f}(\mathbf{r}', t')\,d\mathbf{r}'\,dt', \end{aligned} \tag{4}$$

where $d\mathbf{r}'\,dt'$ is the differential space–time volume element and the integration extends over regions wherein the excitations are non-vanishing. From Eq. (4) one identifies $G_{11}(\mathbf{r}, \mathbf{r}'; t, t')$ and $\mathbf{G}_{21}(\mathbf{r}, \mathbf{r}'; t, t')$ as a scalar and a vector Green's function representing, respectively, the negative of the pressure and velocity at $\mathbf{r}, t$ due to a "unit" source† of fluid particles at $\mathbf{r}', t'$. Similarly, $\mathbf{G}_{12}(\mathbf{r}, \mathbf{r}'; t, t')$ and $\mathcal{G}_{22}(\mathbf{r}, \mathbf{r}'; t, t')$ are identifiable as vector and dyadic Green's functions representing (when dot product multiplied from the right by a unit vector $\mathbf{e}'$) the pressure and velocity, respectively, at $\mathbf{r}, t$ arising from a "unit" vector force density‡ acting in the direction $\mathbf{e}'$ at $\mathbf{r}', t'$.

The field representation in Eq. (4) reduces the problem of solving the field equations (1) to the determination of the four acoustic Green's functions G_{11}, $\mathbf{G}_{12}$, $\mathbf{G}_{21}$, and $\mathcal{G}_{22}$. The primary virtue of this reduction is that, in solution of the field problem for the Green's functions, complexities associated with the functional form of the excitations $s(\mathbf{r}, t)$ and $\mathbf{f}(\mathbf{r}, t)$ are eliminated. To ascertain the defining equations for the Green's functions, one substitutes the representation (4) into Eqs. (1), whence, in view of the arbitrariness of the excitations s and $\mathbf{f}$, one obtains§

$$\frac{1}{\gamma p_0}\frac{\partial}{\partial t}G_{11} + \nabla\cdot\mathbf{G}_{21} = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \qquad \frac{1}{\gamma p_0}\frac{\partial}{\partial t}\mathbf{G}_{12} + \nabla\cdot\mathcal{G}_{22} = 0,$$
$$\nabla G_{11} + n_0 m\frac{\partial}{\partial t}\mathbf{G}_{21} = 0, \qquad \nabla\mathbf{G}_{12} + n_0 m\frac{\partial}{\partial t}\mathcal{G}_{22} = \mathbf{1}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \tag{5}$$

subject to boundary and initial conditions

$$G_{11} = \alpha\mathbf{n}\cdot\mathbf{G}_{21} \quad \text{on } S, \qquad \mathbf{G}_{12} = \alpha\mathbf{n}\cdot\mathcal{G}_{22} \quad \text{on } S, \tag{5a}$$
$$G_{11} \equiv 0 \equiv \mathbf{G}_{21} \quad \text{for } t \leq t', \qquad \mathbf{G}_{12} \equiv 0 \equiv \mathcal{G}_{22} \quad \text{for } t \leq t', \tag{5b}$$

where $\mathbf{n}$ is the outward normal vector on the surfaces S (if any) bounding the region within which the field is defined.

Since the form of the boundary conditions (5a) may complicate the determination of the acoustic Green's functions, it is frequently desirable to introduce other Green's functions, defined by simpler boundary conditions such as $G_{11} = 0 = \mathbf{G}_{12}$ on S. In this event, occurring in the case of bounded regions, the representation (4) must be generalized by addition of surface integral terms arising from singular source distributions on the boundary surface S.

On elimination of $\mathbf{G}_{21}$ and $\mathbf{G}_{12}$ from Eqs. (5) one obtains, for a homogeneous medium, second-order partial differential equations defining the scalar

†A unit source has the space–time form $\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$, where $\delta(\mathbf{r} - \mathbf{r}')$ and $\delta(t - t')$ are, respectively, the three- and one-dimensional Dirac delta functions. The three-dimensional delta function is defined by $\delta(\mathbf{r}) = 0$ or ∞, depending on whether $\mathbf{r} \neq 0$ or $\mathbf{r} = 0$, and its volume integral is unity for $\mathbf{r}$ within the integration volume.

‡A unit vector force density at $\mathbf{r}', t'$ has the space–time form $\mathbf{e}'\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$, where $\mathbf{e}'$ is a unit vector.

§The unit dyadic $\mathbf{1}$ is defined by $\mathbf{1}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{1} = \mathbf{A}$, where $\mathbf{A}$ is an arbitrary vector.

Green's function $G_{11}(\mathbf{r}, \mathbf{r}'; t, t')$ and the dyadic Green's function $\mathscr{G}_{22}(\mathbf{r}, \mathbf{r}'; t, t')$:

$$\left(\nabla^2 - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\right)G_{11} = -n_0 m \frac{\partial}{\partial t}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \tag{6a}$$

$$\left(\nabla\nabla - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\mathbf{1}\right)\cdot\mathscr{G}_{22} = -\frac{1}{\gamma p_0}\frac{\partial}{\partial t}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')\mathbf{1}. \tag{6b}$$

Knowledge of the Green's functions defined in Eqs. (6) permits, on recourse to Eqs. (5), determination of $\mathbf{G}_{21}$ and $\mathbf{G}_{12}$ by time quadrature.

It is of interest to explore properties of acoustic Green's functions that can be inferred prior to the explicit solution of Eqs. (5). For example, the invariance of the form of Eqs. (5) to arbitrary linear (coordinate) displacements in space and time implies that, in an unbounded homogeneous region, the solutions of (5) are functions of the differences $\mathbf{r} - \mathbf{r}'$ and $t - t'$, i.e., that

$$G_{ij}(\mathbf{r}, \mathbf{r}'; t, t') = G_{ij}(\mathbf{r} - \mathbf{r}', t - t'), \tag{6c}$$

where $i, j = 1, 2$ distinguish the various scalar, vector, and dyadic Green's functions.

Additional properties of acoustic Green's functions can be inferred by relating the solutions of the field equations (1) to those of an "adjoint" problem. To this end, adjoint equations, as well as boundary and initial conditions, are chosen so as to evolve a reciprocity relation, Eq. (9), between the original and adjoint fields. Equations for the adjoint pressure and velocity fields $p^+ = p^+(\mathbf{r}, t)$ and $\mathbf{v}^+ = \mathbf{v}^+(\mathbf{r}, t)$, respectively, follow from the original field equations (1) on applying the reflection transformation $\partial/\partial t \to -\partial/\partial t$ and $\nabla \to -\nabla$:

$$\begin{aligned} -\frac{1}{\gamma p_0}\frac{\partial}{\partial t}p^+ - \nabla\cdot\mathbf{v}^+ &= -s^+, \\ -\nabla p^+ - n_0 m\frac{\partial \mathbf{v}^+}{\partial t} &= -\mathbf{f}^+. \end{aligned} \tag{7}$$

They are subject to boundary and initial conditions

$$p^+ = -\alpha\mathbf{n}\cdot\mathbf{v}^+ \qquad \text{on } S, \tag{7a}$$

$$p^+ = 0 = \mathbf{v}^+ \qquad \text{for } t \geq t_2, \tag{7b}$$

which display a similar spatial and temporal reflection, and to excitations $s^+ = s^+(\mathbf{r}, t)$ and $\mathbf{f}^+ = \mathbf{f}^+(\mathbf{r}, t)$, which vanish for $t > t_2$. The adjoint equations (7) differ from the original acoustic equations (1) in that they are "time reversed," have different excitations, and yield ingoing rather than outgoing wave solutions. Ingoing wave solutions have a functional form $F(|\mathbf{r}| + at)$ and may vanish for $t > t_2$, whereas outgoing solutions have the form $F(|\mathbf{r}| - at)$ and may vanish for $t < t_1$.

By appropriate multiplication of Eqs. (1) by p^+ and $\mathbf{v}^+$, of Eqs. (7) by p and $\mathbf{v}$, and addition of the resulting equations, one deduces the "reciprocity relation"

$$\nabla\cdot(p\mathbf{v}^+ + p^+\mathbf{v}) + \frac{1}{\gamma p_0}\frac{\partial}{\partial t}(pp^+) + n_0 m\frac{\partial}{\partial t}(\mathbf{v}\cdot\mathbf{v}^+)$$
$$= +ps^+ - p^+s - \mathbf{f}\cdot\mathbf{v}^+ + \mathbf{f}^+\cdot\mathbf{v} \qquad (8)$$

between solutions of the original and adjoint equations. On integration of Eq. (8) over the space–time region bounded by the spatial surface S and the times $t_1, t_2 > t_1$, and on use of the divergence theorem as well as the boundary and initial conditions (1a) and (1b) and (7a) and (7b), one derives the integral form of the reciprocity relation:

$$0 = \iiint d\mathbf{r}\int_{t_1}^{t_2} dt[ps^+ - p^+s - \mathbf{f}\cdot\mathbf{v}^+ + \mathbf{f}^+\cdot\mathbf{v}]. \qquad (9)$$

Linearity of the adjoint equations (7) implies that the adjoint field can be expressed in terms of adjoint Green's functions in a manner similar to Eqs. (4):

$$p^+(\mathbf{r}, t) = -\int G_{11}^+(\mathbf{r}, \mathbf{r}'; t, t')s^+(\mathbf{r}', t')\,d\mathbf{r}'\,dt' - \int \mathbf{G}_{12}^+(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{f}^+(\mathbf{r}', t')\,d\mathbf{r}'\,dt',$$

$$\mathbf{v}^+(\mathbf{r}, t) = -\int \mathbf{G}_{21}^+(\mathbf{r}, \mathbf{r}'; t, t')s^+(\mathbf{r}', t')\,d\mathbf{r}'\,dt' - \int \mathcal{G}_{22}^+(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{f}^+(\mathbf{r}', t')\,d\mathbf{r}'\,dt',$$
$$(10)$$

where the adjoint Green's functions, G_{ij}^+, distinguished by the superscript $^+$, play the same role in the adjoint field as the Green's functions in the original field. The defining equations for the adjoint Green's functions follow from Eqs. (7) by reflection of the original Green's function equations (5) and (6), together with boundary and initial conditions, in the manner indicated in Eqs. (7).

The reciprocity condition, Eq. (9), provides a relation between the fields p, $\mathbf{v}$ excited by s, $\mathbf{f}$ and the adjoint fields p^+, $\mathbf{v}^+$ excited by s^+, $\mathbf{f}^+$. If the various excitations are chosen to be of the "point-source" form,

$$s = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \qquad s^+ = \delta(\mathbf{r} - \mathbf{r}'')\delta(t - t''),$$
$$\mathbf{f} = 0, \qquad \mathbf{f}^+ = 0,$$

then, from Eqs. (4) and (10), the corresponding field solutions are

$$p = -G_{11}(\mathbf{r}, \mathbf{r}'; t, t'), \qquad p^+ = -G_{11}^+(\mathbf{r}, \mathbf{r}''; t, t''),$$
$$\mathbf{v} = -\mathbf{G}_{21}(\mathbf{r}, \mathbf{r}'; t, t'), \qquad \mathbf{v}^+ = -\mathbf{G}_{21}^+(\mathbf{r}, \mathbf{r}''; t, t'),$$

and hence one infers from Eq. (9) that

$$G_{11}(\mathbf{r}'', \mathbf{r}'; t'', t') = G_{11}^+(\mathbf{r}', \mathbf{r}''; t', t''). \qquad (11a)$$

Similarly, the point-source excitations may be chosen as

$$s = 0, \qquad s^+ = 0,$$
$$\mathbf{f} = \mathbf{e}'\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \qquad \mathbf{f}^+ = \mathbf{e}''\delta(\mathbf{r} - \mathbf{r}'')\delta(t - t''),$$

where $\mathbf{e}'$ and $\mathbf{e}''$ are unit vectors characterizing the directions of the point force densities at $\mathbf{r}'$ and $\mathbf{r}''$. Since, from Eqs. (4) and (10),

$$p = -\mathbf{G}_{12}(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{e}', \qquad p^+ = -\mathbf{G}_{12}^+(\mathbf{r}, \mathbf{r}''; t, t'')\cdot\mathbf{e}'',$$
$$\mathbf{v} = -\mathscr{G}_{22}(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{e}', \qquad \mathbf{v}^+ = -\mathscr{G}_{22}^+(\mathbf{r}, \mathbf{r}''; t, t'')\cdot\mathbf{e}'',$$

one deduces from Eq. (9) that

$$\begin{aligned}\mathbf{e}'\cdot\mathscr{G}_{22}(\mathbf{r}', \mathbf{r}''; t', t'')\cdot\mathbf{e}'' &= \mathbf{e}''\cdot\mathscr{G}_{22}^+(\mathbf{r}'', \mathbf{r}'; t'', t')\cdot\mathbf{e}',\\ \tilde{\mathscr{G}}_{22}(\mathbf{r}', \mathbf{r}''; t', t'') &= \mathscr{G}_{22}^+(\mathbf{r}'', \mathbf{r}'; t'', t'),\end{aligned} \tag{11b}$$

where $\tilde{\mathscr{G}}_{22}$ is the transposed dyadic to $\mathscr{G}_{22}$. The second equation in Eq. (11b) follows from the first on recognizing that a scalar is equal to its transpose and that $\mathbf{a}\cdot\mathscr{D}\cdot\mathbf{b} = \mathbf{b}\cdot\tilde{\mathscr{D}}\cdot\mathbf{a}$ where $\mathbf{a}$ and $\mathbf{b}$ are vectors and $\mathscr{D}$ is a dyadic. Correspondingly, with non-vanishing excitations

$$s = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \qquad \mathbf{f}^+ = \mathbf{e}''\delta(\mathbf{r} - \mathbf{r}'')\delta(t - t''),$$

or

$$\mathbf{f} = \mathbf{e}'\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \qquad s^+ = \delta(\mathbf{r} - \mathbf{r}'')\delta(t - t''),$$

one infers, respectively,

$$\mathbf{G}_{21}(\mathbf{r}'', \mathbf{r}'; t'', t') = \mathbf{G}_{12}^+(\mathbf{r}', \mathbf{r}''; t', t''), \tag{11c}$$

$$\mathbf{G}_{12}(\mathbf{r}'', \mathbf{r}'; t'', t') = \mathbf{G}_{21}^+(\mathbf{r}', \mathbf{r}''; t', t''). \tag{11d}$$

Other reciprocity properties of the acoustic Green's functions follow from the observations, manifest from a comparison of Eqs. (1) and (7), that the adjoint Green's functions are time reversed, and also, as a consequence of spatial reflection, have reversed velocity components:

$$G_{ij}^+(\mathbf{r}, \mathbf{r}'; t, t') = (-1)^{i+j}G_{ij}(\mathbf{r}, \mathbf{r}'; -t, -t'), \tag{12a}$$

whence, from relations (11a)–(11d), one finds

$$\tilde{G}_{ij}(\mathbf{r}', \mathbf{r}; t', t) = (-1)^{i+j}G_{ji}(\mathbf{r}, \mathbf{r}'; -t, -t'), \tag{12b}$$

with the same notational comments as in Eq. (6c). The reciprocity relations, Eqs. (12), applicable to a general class of acoustic field problems in both homogeneous and inhomogeneous media, frequently simplify the explicit determination of the acoustic Green's functions.

As can be inferred from Eqs. (6) and (5), or as can be verified by direct substitution into Eqs. (5), one can express all the desired acoustic Green's functions in a homogeneous medium in terms of a single scalar Green's function $g(\mathbf{r}, \mathbf{r}'; t, t')$ as follows:†

$$\begin{aligned}G_{11}(\mathbf{r}, \mathbf{r}'; t, t') &= n_0 m\frac{\partial}{\partial t}g(\mathbf{r}, \mathbf{r}'; t, t'),\\ \mathscr{G}_{22}(\mathbf{r}, \mathbf{r}'; t, t') &= \left(\frac{1}{\gamma p_0}\frac{\partial}{\partial t} + \frac{\nabla\times\nabla\times\mathbf{1}}{n_0 m(\partial/\partial t)}\right)g(\mathbf{r}, \mathbf{r}'; t, t'),\\ \mathbf{G}_{21}(\mathbf{r}, \mathbf{r}'; t, t') &= -\nabla g(\mathbf{r}, \mathbf{r}'; t, t') = \mathbf{G}_{12}(\mathbf{r}, \mathbf{r}'; t, t')\end{aligned} \tag{13a}$$

†Note that

$$\frac{1}{\partial/\partial t}f(t) \equiv \int_{-\infty}^{t} f(t)\,dt.$$

where $g(\mathbf{r}, \mathbf{r}'; t, t')$ satisfies the wave equation

$$\left(\nabla^2 - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\right)g(\mathbf{r}, \mathbf{r}'; t, t') = -\delta(\mathbf{r} - \mathbf{r}')\delta(t - t') \tag{13b}$$

plus boundary and initial conditions in keeping with (5a) and (5b). The ability to express all four acoustic Green's functions in terms of one scalar function is a general consequence of the scalar nature of the acoustic field but is a property not shared by general vector fields. In view of the reciprocity properties (12b) of the acoustic Green's functions, for a homogeneous and time-invariant medium, it follows from Eq. (13a) that

$$g(\mathbf{r}, \mathbf{r}'; t, t') = g(\mathbf{r}', \mathbf{r}; -t', -t). \tag{13c}$$

Conversely, it is evident that if Eq. (13c) obtains, relations (12b) follow.

Scalar Green's function for unbounded space

For an unbounded homogeneous region the scalar Green's function g displays the property (6c) that $g(\mathbf{r}, \mathbf{r}'; t, t') = g(\mathbf{r} - \mathbf{r}', t - t')$, and hence its defining equation (13b) reduces to

$$\left(\nabla^2 - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\right)g(\mathbf{r}, t) = -\delta(\mathbf{r})\delta(t), \tag{14}$$

wherein for simplicity we have chosen $\mathbf{r}' = 0 = t'$. A solution of Eq. (14) satisfying the initial (causality) condition of vanishing for $t \leq 0$ (or equivalently the outgoing wave condition) is

$$g(\mathbf{r}, t) = \frac{\delta[t - (r/a)]}{4\pi r}, \qquad r = |\mathbf{r}|. \tag{15a}$$

This solution can be inferred by considering a sphere of radius r, volume V, and surface S, centered at $r = 0$, whence the divergence theorem yields

$$\iiint\limits_V \nabla\cdot\nabla g\, dV = \iint\limits_S \frac{\partial g}{\partial r}\, dS = 4\pi r^2 \frac{\partial g}{\partial r},$$

and also

$$\iiint\limits_V g\, dV \to 0 \qquad \text{as } \mathbf{r} \to 0.$$

From the spherically symmetric equation (14), on volume integration about $\mathbf{r} = 0$, one then finds that

$$4\pi r^2 \left.\frac{\partial g}{\partial r}\right|_{r\to 0} = -\delta(t).$$

Since the general (spherically symmetric) solution of Eq. (14) satisfying the outgoing radiation condition has the functional form $g(\mathbf{r}, t) = F[t - (r/a)]/r$, one infers Eq. (15a), which we now write as

$$g(\mathbf{r}, \mathbf{r}'; t, t') = \frac{\delta[t - t' - (|\mathbf{r} - \mathbf{r}'|/a)]}{4\pi|\mathbf{r} - \mathbf{r}'|}, \tag{15b}$$

whence, on substitution into Eq. (13a), one obtains the various Green's functions for an unbounded homogeneous fluid.

If a point source $s(\mathbf{r}, t)$ at $\mathbf{r}' = 0$ has a time-dependent amplitude specified by $s(t)$ for $t \geq 0$, the corresponding (potential) field $\varphi(\mathbf{r}, t)$ may be obtained from Eq. (15b) on multiplication by $s(t')$ and integration over t' between the limits $t' = 0$ and $t' = t - r/a$. From the result

$$\varphi(\mathbf{r}, t) = \frac{s(t - r/a)}{4\pi r} U\left(t - \frac{r}{a}\right), \tag{15c}$$

where $U(\tau)$ is the unit step function, which equals 1 for $\tau > 0$ and 0 for $\tau < 0$, one observes that the time dependence of the field at $\mathbf{r}$ is the same as that of the source, but retarded by the time $\tau = r/a$ required for the field to propagate from the source to $\mathbf{r}$. Furthermore, the causality requirement $\varphi \equiv 0$ for $t < 0$ is seen to be sharpened to the condition $\varphi \equiv 0$ for $t < r/a$ (i.e., the first response at $\mathbf{r}$ is observed at a time τ after initiation of the excitation).

For a bounded region the scalar Green's function g cannot be expressed as simply as in Eq. (15b) but can be represented in terms of appropriate eigenfunctions of the region, as will be discussed in Sec. 1.4 and Chapter 2.

1.1b The Vector Electromagnetic Field

General properties

In a vacuum whose constitutive parameters are the dielectric constant ϵ_0 and the permeability μ_0, the vector electric field intensity $\mathbf{E}(\mathbf{r}, t)$ and magnetic field intensity $\mathbf{H}(\mathbf{r}, t)$ satisfy at any point the Maxwell equations,†

$$\begin{aligned} \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} &= -\mathbf{J}, \\ \nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} &= -\mathbf{M}, \end{aligned} \tag{16}$$

where the source excitations $\mathbf{J}(\mathbf{r}, t)$ and $\mathbf{M}(\mathbf{r}, t)$ are, respectively, the vector electric current density and magnetic current density.[2a] One associates electric and magnetic charge densities $\rho(\mathbf{r}, t)$ and $\rho_m(\mathbf{r}, t)$ with the above current densities via the continuity equations:

$$\begin{aligned} \nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t}, \\ \nabla \cdot \mathbf{M} &= -\frac{\partial \rho_m}{\partial t}. \end{aligned} \tag{16a}$$

At any time t, the field equations (16) are supplemented by the auxiliary equations

†The first-order Maxwell equations (16) are also valid for inhomogeneous media, wherein ϵ_0 and μ_0 are replaced by $\mathbf{r}$ dependent parameters.

$$\nabla\cdot\epsilon_0\mathbf{E} = \rho,$$
$$\nabla\cdot\mu_0\mathbf{H} = \rho_m, \tag{16b}$$

which make explicit the field dependence on the charge densities and which follow from Eq. (16) in view of Eqs. (16a) and (16c) after a divergence and time-quadrature operation. For uniqueness of the fields **E** and **H**, one requires that the excitations **J** and **M** vanish for $t < t_1$, and that the fields satisfy the initial (causality) condition

$$\mathbf{E} \equiv 0 \equiv \mathbf{H} \qquad \text{for } t \leq t_1, \tag{16c}$$

and the boundary condition

$$\mathbf{n}\times\mathbf{E} = \mathscr{Z}\cdot\mathbf{H} \tag{16d}$$

on the surface S (if any) bounding the field region. The unit vector **n** is the outward normal to the surface S, and $\mathscr{Z}$ is an appropriate "impedance" dyadic having components transverse to **n** and characteristic of the boundary surface.

On multiplication of the two Maxwell equations (16) by **E** and **H**, respectively, one readily derives the conservation-of-energy theorem:

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{\epsilon_0\mathbf{E}^2}{2} + \frac{\mu_0\mathbf{H}^2}{2}\right) - \mathbf{J}\cdot\mathbf{E} - \mathbf{M}\cdot\mathbf{H}. \tag{17}$$

One identifies $\mathbf{E}\times\mathbf{H}$ as the instantaneous electromagnetic power flow per unit area at $\mathbf{r}, t$; the term in parentheses as the total stored electromagnetic energy density; $-\mathbf{J}\cdot\mathbf{E}$ as the power per unit volume supplied by the electric current excitation; and $-\mathbf{M}\cdot\mathbf{H}$ as the power per unit volume supplied by the magnetic current excitation.

In the absence of excitation, and for spatially homogeneous media, one obtains on elimination of **H** or **E** from Eqs. (16) the second-order vector-field equations

$$\nabla\times\nabla\times\mathbf{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = 0,$$
$$\nabla\times\nabla\times\mathbf{H} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{H} = 0, \tag{18}$$

where $c = (\mu_0\epsilon_0)^{-1/2}$ is the speed of light in vacuum. The source-excited form of the second-order field equations follows readily from Eqs. (16) by the addition of equivalent source currents to the right-hand side of Eqs. (18), as does their counterpart for inhomogeneous media. In the absence of sources one observes from Eqs. (18) that $\nabla\cdot\mathbf{E} = 0 = \nabla\cdot\mathbf{H}$; this condition is later shown to imply that, in a uniform medium, there are no longitudinal components of **E** and **H** in the direction of propagation of a single plane-wave field.

Linearity of the field equations (16) implies that the fields can be expressed in terms of the excitation currents by the integral representations

$$\mathbf{E}(\mathbf{r}, t) = -\int \mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{J}(\mathbf{r}', t')\, d\mathbf{r}'\, dt' - \int \mathscr{G}_{12}(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{M}(\mathbf{r}', t')\, d\mathbf{r}'\, dt', \tag{19a}$$

$$\mathbf{H}(\mathbf{r}, t) = -\int \mathscr{G}_{21}(\mathbf{r}, \mathbf{r}'; t, t') \cdot \mathbf{J}(\mathbf{r}', t')\, d\mathbf{r}'\, dt' - \int \mathscr{G}_{22}(\mathbf{r}, \mathbf{r}'; t, t') \cdot \mathbf{M}(\mathbf{r}', t')\, d\mathbf{r}'\, dt', \tag{19b}$$

where the integrals are extended over four-dimensional space–time volume elements $d\mathbf{r}'\, dt'$ within which the currents $\mathbf{J}$ and $\mathbf{M}$ are non-vanishing. If current sources are present at infinity, their contributions $\mathbf{E}_{\text{inc}}$ and $\mathbf{H}_{\text{inc}}$ to Eqs. (19a) and (19b), respectively, must be indicated explicitly. From Eqs. (19), one readily identifies the dyadic components $\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t') \cdot \mathbf{e}'$ and $\mathscr{G}_{21}(\mathbf{r}, \mathbf{r}'; t, t') \cdot \mathbf{e}'$ as negatives of the vector electric and magnetic fields, respectively, at $\mathbf{r}, t$ produced by a unit electric current density† at $\mathbf{r}', t'$ in the direction $\mathbf{e}'$. Correspondingly, $\mathscr{G}_{12}(\mathbf{r}, \mathbf{r}'; t, t') \cdot \mathbf{e}'$ and $\mathscr{G}_{22}(\mathbf{r}, \mathbf{r}'; t, t') \cdot \mathbf{e}'$ are negatives of the electric and magnetic fields produced at $\mathbf{r}, t$ by a unit magnetic current density at $\mathbf{r}', t'$ in the direction $\mathbf{e}'$.

The four dyadic functions $\mathscr{G}_{ij}$ play the fundamental role of Green's functions for the electromagnetic field. Their defining equations are readily obtained on substitution of Eqs. (19) into the field equations (16), whence, on noting the arbitrariness of $\mathbf{J}$ and $\mathbf{M}$,

$$\epsilon_0 \frac{\partial \mathscr{G}_{11}}{\partial t} - \nabla \times \mathscr{G}_{21} = \mathbf{1}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'),$$

$$\nabla \times \mathscr{G}_{11} + \mu_0 \frac{\partial \mathscr{G}_{21}}{\partial t} = 0,$$

and (20)

$$\epsilon_0 \frac{\partial \mathscr{G}_{12}}{\partial t} - \nabla \times \mathscr{G}_{22} = 0,$$

$$\nabla \times \mathscr{G}_{12} + \mu_0 \frac{\partial \mathscr{G}_{22}}{\partial t} = \mathbf{1}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t').$$

Equations (20) are to be subject to the initial and boundary conditions

$$\mathscr{G}_{11} = 0 = \mathscr{G}_{21} \quad \text{for } t \leq t', \qquad \mathbf{n} \times \mathscr{G}_{11} = \mathscr{Z} \cdot \mathscr{G}_{21} \quad \text{on } S, \tag{20a}$$

$$\mathscr{G}_{12} = 0 = \mathscr{G}_{22} \quad \text{for } t \leq t', \qquad \mathbf{n} \times \mathscr{G}_{12} = \mathscr{Z} \cdot \mathscr{G}_{22} \quad \text{on } S, \tag{20b}$$

where $\mathbf{n}$ and $\mathscr{Z}$ are defined as in Eq. (16d). It is often desirable to simplify the Green's function problem by specification of simpler boundary conditions, for example,

$$\mathbf{n} \times \mathscr{G}_{11} = 0 \quad \text{on } S, \qquad \mathbf{n} \times \mathscr{G}_{12} = 0 = \mathbf{n} \times (\nabla \times \mathscr{G}_{22}) \quad \text{on } S. \tag{20c}$$

In this case the representation (19) must be generalized so as to include surface integrals representing the effect of "induced currents" arising from the difference between the field boundary conditions (16d) and the Green's function boundary conditions (20c).

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†A unit current density at $\mathbf{r}', t'$ in the direction $\mathbf{e}'$ has the space-time form

$$\mathbf{e}'\delta(\mathbf{r} - \mathbf{r}')\delta(t - t').$$

On elimination of the Green's functions $\mathscr{G}_{12}$ and $\mathscr{G}_{21}$ from the defining equations (20), one obtains for homogeneous media, or vacuum, the second-order partial differential equations

$$\nabla \times \nabla \times \mathscr{G}_{11} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathscr{G}_{11} = \mu_0 \frac{\partial}{\partial t}\mathbf{1}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{21a}$$

$$\nabla \times \nabla \times \mathscr{G}_{22} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathscr{G}_{22} = \epsilon_0 \frac{\partial}{\partial t}\mathbf{1}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \tag{21b}$$

for the electric and magnetic types of Green's functions $\mathscr{G}_{11}(\mathbf{r},\mathbf{r}';t,t')$ and $\mathscr{G}_{22}(\mathbf{r},\mathbf{r}';t,t')$. Knowledge of the Green's functions $\mathscr{G}_{11}$ and $\mathscr{G}_{22}$ permits determination of the remaining Green's functions $\mathscr{G}_{12}$ and $\mathscr{G}_{21}$ on use of Eqs. (20) and time quadrature.

If field symmetries exist, properties of the electromagnetic Green's functions can be inferred prior to their explicit determination. For example, in an unbounded, homogeneous, stationary region in which the field equations (20) are invariant under arbitrary linear space–time displacements, one readily infers that solutions of Eqs. (20) are functions of the differences $\mathbf{r}-\mathbf{r}'$ and $t-t'$, i.e.,

$$\mathscr{G}_{ij}(\mathbf{r},\mathbf{r}';t,t') = \mathscr{G}_{ij}(\mathbf{r}-\mathbf{r}';t-t') \tag{22}$$

for $i, j = 1, 2$.

Additional symmetry properties follow from consideration of an adjoint-field problem. The adjoint field is so defined as to permit the derivation of the reciprocity relation (24). With this intent, equations for adjoint electric and magnetic fields $\mathbf{E}^+ = \mathbf{E}^+(\mathbf{r},t)$ and $\mathbf{H}^+ = \mathbf{H}^+(\mathbf{r},t)$ are derived from the original field equations (16) by a temporal and spatial reflection transformation $\partial/\partial t \to -\partial/\partial t$ and $\nabla \to -\nabla$; the former effects a time reversal and the latter a reversal of the $\mathbf{E}/\mathbf{H}$ ratio for source-free fields, as can be inferred from Eqs. (23), (23a) and (23b). One thus obtains

$$\begin{aligned} -\epsilon_0 \frac{\partial \mathbf{E}^+}{\partial t} + \nabla \times \mathbf{H}^+ &= -\mathbf{J}^+, \\ -\nabla \times \mathbf{E}^+ - \mu_0 \frac{\partial \mathbf{H}^+}{\partial t} &= -\mathbf{M}^+, \end{aligned} \tag{23}$$

which are subject to "reflected" initial and boundary conditions

$$\mathbf{E}^+ = \mathbf{0} = \mathbf{H}^+ \qquad \text{for } t \geq t_2, \tag{23a}$$

$$\mathbf{n} \times \mathbf{E}^+ = -\tilde{\mathscr{Z}}\cdot\mathbf{H}^+ \qquad \text{on } S. \tag{23b}$$

and correspond to excitations $\mathbf{J}^+$ and $\mathbf{M}^+$ that vanish for $t > t_2$. The adjoint field differs from the original field defined by equations (16) in that it is "time reversed," the $\mathbf{E}/\mathbf{H}$ ratio is reversed, and the boundary is characterized by the transpose impedance dyadic $-\tilde{\mathscr{Z}}$. The adjoint field admits ingoing (advanced) wave solutions of the functional form $F(t + r/c)$ rather than the characteristic outgoing (retarded) wave solutions $F(t - r/c)$ of the original field, an inference from causality in Eq. (16c) and from its time-reversed analogue in Eq. (23a).

The adjoint and original fields are connected by a reciprocity relation,

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}^{+}+\mathbf{E}^{+}\times\mathbf{H})+\epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}^{+})+\mu_0\frac{\partial}{\partial t}(\mathbf{H}\cdot\mathbf{H}^{+})$$
$$=\mathbf{J}^{+}\cdot\mathbf{E}-\mathbf{J}\cdot\mathbf{E}^{+}+\mathbf{M}^{+}\cdot\mathbf{H}-\mathbf{M}\cdot\mathbf{H}^{+}, \tag{24}$$

derivable on suitable multiplication of Eqs. (16) by $\mathbf{E}^{+}, \mathbf{H}^{+}$, and of Eqs. (23) by $\mathbf{E}, \mathbf{H}$. On integration of Eq. (24) over the space–time domain bounded by the surface S and the times $t_1, t_2 > t_1$, and on use of the divergence theorem together with the initial and boundary conditions (16c) and (16d), and (23a) and (23b), one infers an integral form of the reciprocity relation:

$$0=\iiint d\mathbf{r}\int_{t_1}^{t_2} dt(\mathbf{J}^{+}\cdot\mathbf{E}-\mathbf{J}\cdot\mathbf{E}^{+}+\mathbf{M}^{+}\cdot\mathbf{H}-\mathbf{M}\cdot\mathbf{H}^{+}). \tag{25}$$

To display succinctly the relationship between the original and adjoint fields implicit in the reciprocity relation (25), it is first necessary to introduce adjoint dyadic Green's functions. Linearity of the adjoint equations (23) indicates that the adjoint fields are representable in a form similar to that in Eqs. (19):

$$\mathbf{E}^{+}(\mathbf{r},t)=-\int\mathscr{G}_{11}^{+}(\mathbf{r},\mathbf{r}';t,t')\cdot\mathbf{J}^{+}(\mathbf{r}',t')\,d\mathbf{r}'\,dt'$$
$$-\int\mathscr{G}_{12}^{+}(\mathbf{r},\mathbf{r}';t,t')\cdot\mathbf{M}^{+}(\mathbf{r}',t')\,d\mathbf{r}'\,dt', \tag{26a}$$

$$\mathbf{H}^{+}(\mathbf{r},t)=-\int\mathscr{G}_{21}^{+}(\mathbf{r},\mathbf{r}';t,t')\cdot\mathbf{J}^{+}(\mathbf{r}',t')\,d\mathbf{r}'\,dt'$$
$$-\int\mathscr{G}_{22}^{+}(\mathbf{r},\mathbf{r}';t,t')\cdot\mathbf{M}^{+}(\mathbf{r}',t')\,d\mathbf{r}'\,dt', \tag{26b}$$

where the adjoint Green's functions, distinguished by the superscript $^{+}$, have the same significance for the adjoint field as the Green's functions discussed under Eqs. (20) have for the original field. Defining equations for the adjoint Green's functions may be obtained by employing the same reflection transformations as in Eqs. (23), but we shall omit this step and proceed to infer properties of the adjoint Green's functions from the reciprocity relation (25).

To utilize the reciprocity relation (25) we consider a number of different choices of excitation for the original and adjoint fields. For example, if

$$\mathbf{J}=\mathbf{e}'\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'),\qquad \mathbf{J}^{+}=\mathbf{e}''\delta(\mathbf{r}-\mathbf{r}'')\delta(t-t''),$$
$$\mathbf{M}=0,\qquad \mathbf{M}^{+}=0,$$

there follows from Eqs. (19) and (26) that

$$\begin{aligned}\mathbf{E}(\mathbf{r},t)&=-\mathscr{G}_{11}(\mathbf{r},\mathbf{r}';t,t')\cdot\mathbf{e}', &\quad \mathbf{E}^{+}(\mathbf{r},t)&=-\mathscr{G}_{11}^{+}(\mathbf{r},\mathbf{r}'';t,t'')\cdot\mathbf{e}'',\\ \mathbf{H}(\mathbf{r},t)&=-\mathscr{G}_{21}(\mathbf{r},\mathbf{r}';t,t')\cdot\mathbf{e}', &\quad \mathbf{H}^{+}(\mathbf{r},t)&=-\mathscr{G}_{21}^{+}(\mathbf{r},\mathbf{r}'';t,t'')\cdot\mathbf{e}'',\end{aligned} \tag{27}$$

whence on substitution of the above fields and excitations into Eq. (25),

$$\mathbf{e}'\cdot\mathscr{G}_{11}^{+}(\mathbf{r}',\mathbf{r}'';t',t'')\cdot\mathbf{e}''=\mathbf{e}''\cdot\mathscr{G}_{11}(\mathbf{r}'',\mathbf{r}';t'',t')\cdot\mathbf{e}'$$

or

$$\mathscr{G}_{11}^{+}(\mathbf{r}', \mathbf{r}''; t', t'') = \tilde{\mathscr{G}}_{11}(\mathbf{r}'', \mathbf{r}'; t'', t'),$$

where $\tilde{\mathscr{G}}_{11}$ is the transpose of the dyadic $\mathscr{G}_{11}$. In a similar manner, from the point excitations

$$\mathbf{J} = 0, \qquad \mathbf{J}^{+} = 0,$$
$$\mathbf{M} = \mathbf{e}'\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \qquad \mathbf{M}^{+} = \mathbf{e}''\delta(\mathbf{r} - \mathbf{r}'')\delta(t - t''),$$

one infers

$$\mathscr{G}_{22}^{+}(\mathbf{r}', \mathbf{r}''; t', t'') = \tilde{\mathscr{G}}_{22}(\mathbf{r}'', \mathbf{r}'; t'', t'), \tag{28b}$$

and from the excitations

$$\mathbf{J} = \mathbf{e}'\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \qquad \mathbf{J}^{+} = 0,$$
$$\mathbf{M} = 0, \qquad \mathbf{M}^{+} = \mathbf{e}''\delta(\mathbf{r} - \mathbf{r}'')\delta(t - t''),$$

one obtains

$$\mathscr{G}_{12}^{+}(\mathbf{r}', \mathbf{r}''; t', t'') = \tilde{\mathscr{G}}_{21}(\mathbf{r}'', \mathbf{r}'; t'', t'). \tag{28c}$$

For the special case wherein $\mathscr{Z} = \tilde{\mathscr{Z}}$, one infers from Eqs. (23) that the adjoint Green's functions are identical to the original Green's functions defined in Eqs. (20) but with time, $\mathscr{G}_{11}/\mathscr{G}_{21}$ and $\mathscr{G}_{12}/\mathscr{G}_{22}$ ratios reversed. One thus obtains from Eqs. (28) as the reciprocity relations for electromagnetic Green's functions in a vacuum bounded by a surface on which $\mathscr{Z} = \tilde{\mathscr{Z}}$:

$$\mathscr{G}_{ij}(\mathbf{r}, \mathbf{r}'; t, t') = (-1)^{i+j}\tilde{\mathscr{G}}_{ji}(\mathbf{r}', \mathbf{r}; -t', -t), \tag{29}$$

where $i, j = 1, 2$. The relations (28) and (29) frequently facilitate explicit evaluation of the electromagnetic Green's functions. (See also Sec. 1.5b.)

Dyadic Green's functions in free space **(invariant evaluation)**

Electric and magnetic fields excited by prescribed sources may be expressed via Eqs. (19) in terms of the electromagnetic Green's functions for the regions containing these sources. This technique, employing Green's functions that are independent of the form and distribution of sources, is to be contrasted with the related classical method of scalar and vector potentials (both electric and magnetic) that are dependent on the form and distribution of sources. In the following we shall represent explicitly the dyadic Green's functions for a number of different regions, the form of representation depending on the symmetry properties of the region. From Eqs. (21) it is seen that explicit determination of the Green's functions $\mathscr{G}_{ij}$ is essentially concerned with inversion of the dyadic operator $[\nabla \times (\nabla \times \mathbf{1}) + \mathbf{1}(\partial^2/c^2\,\partial t^2)]$. In free space, the inversion is simple and may be effected by an analytical method equivalent to that of finding the classic scalar and vector potentials of a point source; in bounded spaces this method no longer applies and must be rephrased. Alternatively, as shown in this section, the inversion may be accomplished by an operator method that is applicable to both bounded and unbounded regions and constitutes a succinct, invariant procedure. The term "invariant" implies that the method and result are independent of the choice of coordinate system for the space in question.

Classical method

For free space (unbounded and homogeneous), the four electromagnetic dyadic Green's functions $\mathscr{G}_{ij}$ can be expressed more or less classically in terms of a single scalar Green's function. The result for $\mathscr{G}_{11}$ follows from the second-order wave equation (21a) with $\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t')$ subject to an initial (causality) condition of vanishing for $t < t'$ (with a thereby implied outgoing wave condition). On taking the divergence of the first of Eqs. (20), one observes that

$$\epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathscr{G}_{11} = \nabla \delta(\mathbf{r} - \mathbf{r}')\delta(t - t').$$

On substitution of this relation into Eq. (21a) and noting that $\nabla \times (\nabla \times \mathscr{G}_{11}) = \nabla\nabla \cdot \mathscr{G}_{11} - \nabla^2 \mathscr{G}_{11}$, one finds [see footnote on p. 7]

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathscr{G}_{11} = -\left(\mu_0 \frac{\partial}{\partial t}\mathbf{1} - \frac{\nabla\nabla}{\epsilon_0(\partial/\partial t)}\right)\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \tag{30a}$$

Introduction of a scalar Green's function $g(\mathbf{r}, \mathbf{r}'; t, t')$ via

$$\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t') = \left(\mu_0 \frac{\partial}{\partial t}\mathbf{1} - \frac{\nabla\nabla}{\epsilon_0(\partial/\partial t)}\right) g(\mathbf{r}, \mathbf{r}'; t, t') \tag{30b}$$

indicates that the expression (30b) satisfies Eq. (30a) provided $g(\mathbf{r}, \mathbf{r}'; t, t')$ vanishes for $t < t'$ (thus obeying an outgoing wave condition) and is defined by the scalar wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) g(\mathbf{r}, \mathbf{r}'; t, t') = -\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \tag{30c}$$

This scalar Green's function g, except for the presence of the light speed c rather than the sound speed a, is identical to that discussed in connection with Eq. (13b) and hence has the solution

$$g(\mathbf{r}, \mathbf{r}'; t, t') = \frac{\delta[t - t' - (|\mathbf{r} - \mathbf{r}'|/c)]}{4\pi|\mathbf{r} - \mathbf{r}'|}. \tag{31}$$

By duality to Eq. (30b), the dyadic Green's function $\mathscr{G}_{22}$ is expressible in terms of g in Eq. (31) as

$$\mathscr{G}_{22}(\mathbf{r}, \mathbf{r}'; t, t') = \left(\epsilon_0 \frac{\partial}{\partial t}\mathbf{1} - \frac{\nabla\nabla}{\mu_0(\partial/\partial t)}\right) g(\mathbf{r}, \mathbf{r}'; t, t'), \tag{32a}$$

and hence to satisfy Eqs. (20), the remaining electromagnetic Green's functions may be chosen as

$$\mathscr{G}_{12} = \nabla \times \mathbf{1}g = -\mathscr{G}_{21}. \tag{32b}$$

Operator method

An operator procedure for the determination of the Green's functions $\mathscr{G}_{ij}$ may be based on the dyadic operator identity $\nabla \times (\nabla \times \mathbf{1}) = \nabla\nabla - \nabla^2\mathbf{1}$, expressed in the suggestive operator form

$$\mathbf{1} = \mathbf{1}_L + \mathbf{1}_T = \frac{\nabla\nabla}{\nabla^2} - \frac{\nabla \times (\nabla \times \mathbf{1})}{\nabla^2}, \tag{33a}$$

where $\mathbf{1}_L$ and $\mathbf{1}_T$ are orthogonal unit dyads, "longitudinal" and "transverse" to the vector operator ∇, with the property $\mathbf{1}_L \cdot \mathbf{1}_T = 0$. The latter property implies that in a basis oriented along ∇, the matrix representative of Eq. (33a) is diagonal. In view of the unit dyads defined in Eq. (33a), the basic operator in Eqs. (21) may be represented as

$$\nabla \times (\nabla \times \mathbf{1}) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{1} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{\nabla\nabla}{\nabla^2} + \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{\nabla \times (\nabla \times \mathbf{1})}{\nabla^2}. \tag{33b}$$

Because of the orthogonality property of the unit dyads, one verifies, by direct (dot product) multiplication, the general inverse relation

$$\left[A\frac{\nabla\nabla}{\nabla^2} - B\frac{\nabla \times (\nabla \times \mathbf{1})}{\nabla^2}\right]^{-1} = \left[\frac{1}{A}\frac{\nabla\nabla}{\nabla^2} - \frac{1}{B}\frac{\nabla \times (\nabla \times \mathbf{1})}{\nabla^2}\right], \tag{33c}$$

where A and B are scalars, or scalar operators, that commute with ∇. Accordingly, one obtains from Eqs. (33a)–(33c) the desired inverse operator:

$$\begin{aligned}\left[\nabla \times (\nabla \times \mathbf{1}) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{1}\right]^{-1} &= \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)^{-1}\frac{\nabla\nabla}{\nabla^2} \\ &\quad + \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)^{-1}\frac{\nabla \times (\nabla \times \mathbf{1})}{\nabla^2} \\ &= \left[\mathbf{1} - \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)^{-1}\nabla\nabla\right]\left[-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)^{-1}\right].\end{aligned} \tag{33d}$$

The operator identities (33) are valid† for operation on any space–time function and, in particular, on the point-source function $\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$. In view of Eq. (30c), the inverse operator in the rightmost set of brackets of Eq. (33d), when operating on $\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$, is recognized as the scalar Green's function $g(\mathbf{r}, \mathbf{r}'; t, t')$. The dyadic Green's functions $\mathcal{G}_{11}$ and $\mathcal{G}_{22}$ then follow from Eqs. (21) and (33d) in the form already displayed in Eqs. (30b) and (32a), with the related Green's functions $\mathcal{G}_{21}$ and $\mathcal{G}_{12}$ given in Eq. (32b).

Field of an electric dipole current

As an application of the above free-space Green's functions, we consider the fields excited by the sudden creation at $\mathbf{r} = 0$ and $t = 0$ of an electric charge dipole of moment $\mathbf{p}(\mathbf{r}, t) = \mathbf{p}\delta(\mathbf{r})U(t)$, where $U(\alpha)$ is the Heaviside step function, which equals 1 for $\alpha > 0$ and 0 for $\alpha < 0$. This excitation gives rise to an impulsive electric current density $\mathbf{J}(\mathbf{r}, t) = \partial\mathbf{p}(\mathbf{r}, t)/\partial t = \mathbf{p}\,\delta(\mathbf{r})\,\delta(t)$ and

†The range and/or domain[5] of the relevant operators are determined by the boundary and initial conditions concomitant to Eqs. (21).

to space- and time-dependent electric and magnetic fields given by Eqs. (19) as

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\mathscr{G}_{11}(\mathbf{r}, 0; t, 0)\cdot\mathbf{p}, \\ \mathbf{H}(\mathbf{r}, t) &= -\mathscr{G}_{21}(\mathbf{r}, 0; t, 0)\cdot\mathbf{p}, \end{aligned} \tag{34}$$

where the relevant free-space Green's functions $\mathscr{G}_{11}$ and $\mathscr{G}_{21}$ follow from Eqs. (30b), (32b), and (31). To cast the latter in a form that more clearly distinguishes the nature of the various field contributions, we rewrite $\mathscr{G}_{11}$ of Eqs. (30b) and (31) as [note that $\int_{-\infty}^{t} \delta(\tau - r/c)\, d\tau = U(t - r/c)$]

$$\begin{aligned} \mathscr{G}_{11}(r, 0; t, 0) &= \mu_0 \frac{\delta'(t - r/c)}{4\pi r}\mathbf{1} - \nabla\nabla\frac{U(t - r/c)}{4\pi\epsilon_0 r} \\ &= \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{4\pi r}\left[(\mathbf{1} - \mathbf{r}_0\mathbf{r}_0)\frac{\delta'(t - r/c)}{c} + (\mathbf{1} - 3\mathbf{r}_0\mathbf{r}_0)\frac{\delta(t - r/c)}{r}\right] \\ &\quad + (\mathbf{1} - 3\mathbf{r}_0\mathbf{r}_0)\frac{U(t - r/c)}{4\pi\epsilon_0 r^3}, \end{aligned} \tag{34a}$$

and $\mathscr{G}_{21}$ of Eqs. (32b) and (31) as

$$\begin{aligned} \mathscr{G}_{21}(\mathbf{r}, 0; t, 0) &= -\nabla \times \mathbf{1}\frac{\delta(t - r/c)}{4\pi r} \\ &= \left[\frac{\delta(t - r/c)}{r} + \frac{\delta'(t - r/c)}{c}\right]\frac{\mathbf{r}_0 \times \mathbf{1}}{4\pi r}, \end{aligned} \tag{34b}$$

where $\delta'(\alpha) = d\delta(\alpha)/d\alpha$. The first two terms in brackets in Eq. (34a) represent "radiation" and "near-field" contributions in the form of impulsive spherical fronts traveling outward at the speed of light; the third term, which exists only in the region $r < ct$ following the spherical front, is the static electric field of the charge dipole. The radiative term is transverse to $\mathbf{r}_0$ and decays like $1/r$, whereas the other terms have the familiar $1/r^2$ and $1/r^3$ decay. Correspondingly, in Eq. (34b) the magnetic field is seen to be non-static and in the form of impulsive spherical fronts moving outward with the speed of light. The $\mathbf{q}$ component of field produced by a dipole in the direction of $\mathbf{p}$ is readily obtained from the dyadics in Eqs. (34a) and (34b) by pre- and post-multiplication by $\mathbf{q}$ and $\mathbf{p}$, respectively; note that $\mathbf{r}_0 \times \mathbf{1}\cdot\mathbf{p} = \mathbf{r}_0 \times \mathbf{p}$.

Free space, dyadic Green's functions (transversely invariant)—Hertz potentials

The free-space, dyadic Green's functions in Eqs. (30b), (32a), and (32b) were evaluated in a ∇-oriented basis that is invariant to choice of a spatial vector coordinate basis. It is of interest to carry out a similar evaluation in a basic oriented along a constant unit vector $\mathbf{a}$ but invariant in the surface transverse to $\mathbf{a}$. Representations of Green's functions in such a basis are particularly convenient for the analysis of field problems containing planar-stratified scattering structures with axis of stratification along the preferred $\mathbf{a}$ direction. Although the spatial cross section transverse to $\mathbf{a}$ is of infinite extent in this consideration, the method is readily generalizable to cylindrical regions of arbitrary but finite cross section. As in the previous invariant representations, the dyadic Green's functions are still representable in terms of a single scalar Green's function, which, however, is different from the scalar Green's function defined in Eq. (30c). An additional feature of these dyadic Green's function representations is that they provide a simple Hertz potential representation of the electric and magnetic fields produced by current sources, and they lead to a clarification of certain singularities characteristic of such representations.

As has been shown, via both classical and operator methods, the dyadic Green's function $\mathscr{G}_{11}$ can be expressed as in Eq. (30b) by an invariant dyadic operator and a single scalar Green's function g. We shall utilize that invariant result and reexpress the dyadic operators $\mathbf{1}$ and $\nabla\nabla$ of Eq. (30b) in a basis oriented along an axial direction defined by the constant unit vector $\mathbf{a}$.

On decomposition of the gradient as $\nabla = \nabla_t + (\mathbf{a}\cdot\nabla)\mathbf{a}$, one reexpresses the unit dyadic operator on the right of Eq. (30b) by use of the identity

$$\nabla_t^2\mathbf{1} = \nabla_t^2\mathbf{aa} + \nabla_t\nabla_t + (\nabla \times \mathbf{a})(\nabla \times \mathbf{a}), \tag{35a}$$

where ∇_t is the invariant component of ∇ transverse to $\mathbf{a}$ and $\nabla_t^2 = \nabla^2 - (\mathbf{a}\cdot\nabla)^2$.† Equation (35a) may be verified by noting that $\mathbf{a}, \nabla_t, \nabla \times \mathbf{a} = \nabla_t \times \mathbf{a}$, are orthogonal vectors with magnitudes $1, \sqrt{\nabla_t^2}, \sqrt{\nabla_t^2}$, respectively, whence formally

$$\mathbf{1} = \mathbf{aa} + \frac{\nabla_t\nabla_t}{\nabla_t^2} + \frac{(\nabla \times \mathbf{a})(\nabla \times \mathbf{a})}{\nabla_t^2}. \tag{35b}$$

To reexpress the dyadic $\nabla\nabla$, one employs the identities

$$\begin{aligned} \nabla\nabla &= \nabla_t\nabla_t + (\nabla_t\mathbf{a} + \mathbf{a}\nabla_t)(\mathbf{a}\cdot\nabla) + \mathbf{aa}(\mathbf{a}\cdot\nabla)^2 \\ [\nabla \times (\nabla \times \mathbf{a})][\nabla \times (\nabla \times \mathbf{a})] &= \nabla_t\nabla_t(\mathbf{a}\cdot\nabla)^2 - (\nabla_t\mathbf{a} + \mathbf{a}\nabla_t)(\mathbf{a}\cdot\nabla)\nabla_t^2 \\ &\quad + \mathbf{aa}\nabla_t^2\nabla_t^2, \end{aligned}$$

which, on multiplication by ∇_t^2 and addition, yield

$$\nabla_t^2\nabla\nabla = (\mathbf{aa}\nabla_t^2 + \nabla_t\nabla_t)\nabla^2 - [\nabla \times (\nabla \times \mathbf{a})][\nabla \times (\nabla \times \mathbf{a})]. \tag{36}$$

From Eqs. (35a) and (36) one thus derives for application to the dyadic operator of Eq. (30b):

$$\begin{aligned} \nabla_t^2\left(\mu_0\frac{\partial}{\partial t}\mathbf{1} - \frac{\nabla\nabla}{\epsilon_0(\partial/\partial t)}\right) &= -\frac{1}{\epsilon_0(\partial/\partial t)}(\mathbf{aa}\nabla_t^2 + \nabla_t\nabla_t)\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \\ &\quad + \mu_0\frac{\partial}{\partial t}(\nabla \times \mathbf{a})(\nabla \times \mathbf{a}) \\ &\quad + \frac{[\nabla \times (\nabla \times \mathbf{a})][\nabla \times (\nabla \times \mathbf{a})]}{\epsilon_0(\partial/\partial t)}. \end{aligned} \tag{37}$$

From Eq. (30b) one obtains, on introducing a scalar function $\mathscr{S}(\mathbf{r}, \mathbf{r}'; t, t')$

†Note that dot-product multiplication of Eq. (35a) from the right by a vector $\mathbf{A}$ yields an expression for $\nabla_t^2\mathbf{A}$, with $(\nabla \times \mathbf{a})\cdot\mathbf{A} \equiv \nabla\cdot(\mathbf{a} \times \mathbf{A})$.

defined by $\nabla_t^2 \mathscr{S} = g$, the alternative representation of the free-space dyadic Green's function $\mathscr{G}_{11}$:

$$\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t') = \nabla_t^2\left(\mu_0 \frac{\partial}{\partial t}\mathbf{1} - \frac{\nabla\nabla}{\epsilon_0(\partial/\partial t)}\right)\mathscr{S}(\mathbf{r}, \mathbf{r}'; t, t'). \tag{38}$$

Substituting the operator identity (37) into Eq. (38), and noting that one may put $\nabla = -\nabla'$ when operating on a function of $\mathbf{r} - \mathbf{r}'$ such as $\mathscr{S}(\mathbf{r}, \mathbf{r}'; t, t')$, one finds

$$\begin{aligned}\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t') = {} & \frac{1}{\epsilon_0(\partial/\partial t)}\left(\mathbf{aa} + \frac{\nabla_t\nabla_t}{\nabla_t^2}\right)\delta(\mathbf{r} - \mathbf{r}')\,\delta(t - t') \\ & + \left[\mu_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{a})(\nabla' \times \mathbf{a}) \right. \\ & \left. - \frac{[\nabla \times (\nabla \times \mathbf{a})][\nabla' \times (\nabla' \times \mathbf{a})]}{\epsilon_0(\partial/\partial t)}\right]\mathscr{S}(\mathbf{r}, \mathbf{r}'; t, t'),\end{aligned} \tag{38a}$$

where $\mathscr{S}$ is a free-space scalar Green's function defined by

$$\nabla_t^2\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathscr{S}(\mathbf{r}, \mathbf{r}'; t, t') = \delta(\mathbf{r} - \mathbf{r}')\,\delta(t - t') \tag{38b}$$

and satisfies an initial condition of vanishing for $t < t'$. It is of interest to note that the first term on the right of Eq. (38a) contains a singular Green's function term $\bar{G}$ which, when evaluated in a ρ, φ, z coordinate system, has the form

$$\bar{G} = \frac{\delta(\mathbf{r} - \mathbf{r}')\,\delta(t - t')}{\nabla_t^2},$$

which for $\rho' = 0$ may be written more conventionally as

$$\nabla_t^2 \bar{G} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}\bar{G} = \delta(\boldsymbol{\rho})\,\delta(z - z')\,\delta(t - t'),$$

whence the ρ-dependent part is the static-line-source Green's function $(1/2\pi)\ln(\alpha\rho)$. On restoring $\boldsymbol{\rho}' \neq 0$, one has, explicitly,

$$\bar{G} = \left[\frac{1}{2\pi}\ln \alpha|\boldsymbol{\rho} - \boldsymbol{\rho}'|\right]\delta(z - z')\,\delta(t - t'), \tag{38c}$$

where α is a constant; this term is non-vanishing (and singular) only on the plane $z = z'$ and at $t = t'$.

Just as in Eq. (30b), the expression for $\mathscr{G}_{11}$ in Eq. (38a) requires only one scalar Green's function $\mathscr{S}$ (if $z \neq z', t \neq t'$). From Eqs. (31) and (38b) one has

$$\nabla_t^2 \mathscr{S} = -\frac{\delta[\tau - (|\mathbf{r} - \mathbf{r}'|/c)]}{4\pi|\mathbf{r} - \mathbf{r}'|}, \tag{39a}$$

where $\tau = t - t'$. In a cylindrical ρ, φ, z coordinate system with z in the direction $\mathbf{a}$, one finds, if $\mathbf{r}' = 0$ and hence $|\mathbf{r} - \mathbf{r}'| = \sqrt{\rho^2 + z^2}$, that

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}\mathscr{S} = -\frac{\delta[\tau - (\sqrt{\rho^2 + z^2}/c)]}{4\pi\sqrt{\rho^2 + z^2}}.$$

Multiplication by ρ and integration from 0 to ρ yields

$$\frac{\partial}{\partial\rho}\mathscr{S}(r, 0; \tau) = \frac{-c}{4\pi\rho}U\left(\tau - \frac{|z|}{c}\right)U\left(\frac{r}{c} - \tau\right), \tag{39b}$$

where the Heaviside unit function $U(\alpha)$ equals 1 for $\alpha > 0$ and 0 for $\alpha < 0$. In many applications (see Sec. 5.2), only $\nabla_t^2\mathscr{S}$ or $\nabla_t\mathscr{S}$, rather than $\mathscr{S}$ itself, needs to be known; in such cases, only the results in Eqs. (39) are necessary to determine the field.

A similar development leads to representation of $\mathscr{G}_{21}$, $\mathscr{G}_{12}$, and $\mathscr{G}_{22}$ and also provides an alternative to the free-space representation of Eqs. (32a), (32b), and (31). By Eq. (35a) one first reexpresses the dyadic $\nabla \times \mathbf{1}$ of Eq. (32b) by use of the identity

$$\begin{aligned}\nabla \times \mathbf{1}\nabla_t^2 &= \nabla \times \mathbf{aa}\nabla_t^2 + \nabla \times \nabla_t\nabla_t + (\nabla \times \nabla \times \mathbf{a})(\nabla \times \mathbf{a}) \\ &= -(\nabla \times \mathbf{a})(\nabla \times \nabla \times \mathbf{a}) + (\nabla \times \nabla \times \mathbf{a})(\nabla \times \mathbf{a})\end{aligned} \tag{40}$$

where in the second line we have employed the vector relations $\nabla \times (\nabla \times \mathbf{a}) = \nabla_t(\mathbf{a}\cdot\nabla) - \nabla_t^2\mathbf{a}$, $\nabla = \nabla_t + \mathbf{a}(\mathbf{a}\cdot\nabla)$, and $\mathbf{a} \times \nabla_t = -\nabla \times \mathbf{a}$, with $\mathbf{a}$ the unit vector along the symmetry direction. From Eqs. (32b) and (30c) one finds that

$$\nabla_t^2\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathscr{G}_{21}(\mathbf{r}, \mathbf{r}'; t, t') = \nabla_t^2\nabla \times \mathbf{1}\delta(\mathbf{r} - \mathbf{r}')\,\delta(t - t')$$

and hence, by Eqs. (40) and (38b),

$$\begin{aligned}\mathscr{G}_{21}(\mathbf{r}, \mathbf{r}'; t, t') = &-[(\nabla \times \mathbf{a})(\nabla' \times \nabla' \times \mathbf{a}) \\ &+ (\nabla \times \nabla \times \mathbf{a})(\nabla' \times \mathbf{a})]\mathscr{S}(\mathbf{r}, \mathbf{r}'; t, t')\end{aligned} \tag{41}$$

is the desired alternative representation of the magnetic-type dyadic Green's function (with $\nabla' = -\nabla$) in unbounded free space. As before, the Green's functions $\mathscr{G}_{12} = -\mathscr{G}_{21}$ of Eq. (41) and $\mathscr{G}_{22}$ may be obtained from $\mathscr{G}_{11}$ in Eq. (38a) on the duality replacements $\epsilon_0 \longleftrightarrow \mu_0$.

To exhibit explicitly the electric and magnetic fields produced by an electric current density $\mathbf{J}(\mathbf{r}, t)$, one utilizes Eqs. (19), (20), (38), and (41) to obtain (with restrictions noted below)

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= -\mu_0\frac{\partial}{\partial t}\nabla \times \mathbf{a}\,\Pi''(\mathbf{r}, t) + \nabla \times (\nabla \times \mathbf{a})\,\Pi'(\mathbf{r}, t), \\ \mathbf{H}(\mathbf{r}, t) &= \nabla \times (\nabla \times \mathbf{a})\,\Pi''(\mathbf{r}, t) + \epsilon_0\frac{\partial}{\partial t}\nabla \times \mathbf{a}\,\Pi'(\mathbf{r}, t),\end{aligned} \tag{42a}$$

where the scalar functions

$$\begin{aligned}\Pi''(\mathbf{r}, t) &= \int \nabla' \times \mathbf{a}\mathscr{S}(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{J}(\mathbf{r}', t')\,d\mathbf{r}'\,dt', \\ \Pi'(\mathbf{r}, t) &= \frac{1}{\epsilon_0(\partial/\partial t)}\int \nabla' \times (\nabla' \times \mathbf{a})\mathscr{S}(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{J}(\mathbf{r}', t')\,d\mathbf{r}'\,dt'\end{aligned} \tag{42b}$$

are Hertz potentials associated with the electric current density $\mathbf{J}$ [recall that

$\nabla' \times (\nabla' \times \mathbf{a})\mathscr{S} \equiv \nabla' \times [\nabla' \times (\mathbf{a}\mathscr{S})]$, etc.]. From Eqs. (42a) one observes that since $\nabla \times \mathbf{a}\,\Pi'$ is a vector transverse to **a**, the Hertz potential Π' does not contribute an **a** component of magnetic field and Π'' does not contribute an **a** component of electric field; for this reason Π' and Π'' are frequently termed E-mode and H-mode potentials, respectively, with respect to **a**. The *electric-field representation* in Eq. (42a) is not valid in planes transverse to **a** containing the source $\mathbf{J}(\mathbf{r}, t)$, since in the transition from Eqs. (38a) to (42a), singular contributions in the indicated planar region from the first term of Eq. (38a) have been omitted. This omission and consequent restriction on the region of applicability of the Hertzian potential representation of the electric field in Eq. (42a) should be recognized. Note also that for the magnetic-field representation in Eq. (42a), which is applicable even in source regions, $\epsilon_0(\partial/\partial t)\,\Pi'$ and not the potential Π' need be calculated.

The significance of the somewhat peculiar potential function in Eq. (39b) becomes evident on calculation of the fields radiated by an impulsive point electric current element $\mathbf{J}(\mathbf{r}, t) = \mathbf{y}_0 p\,\delta(\mathbf{r})\,\delta(t)$, where p is the electric dipole moment strength and $\mathbf{y}_0$ is perpendicular to $\mathbf{a} \equiv \mathbf{z}_0$. The portion of the x component of magnetic field, contributed via Eq. (42a) by the potential Π' in Eq. (42b), is found to be [note that $\delta'(\alpha) = (d/d\alpha)\delta(\alpha)$, and sgn $\alpha = \pm 1$ for $\alpha \gtrless 0$]

$$H'_x = p\,\frac{x^2 - y^2}{4\pi\rho^4}\left[(\operatorname{sgn} z)\,\delta\!\left(t - \frac{|z|}{c}\right) - \frac{z}{r}\,\delta\!\left(t - \frac{r}{c}\right)\right] + p\,\frac{y^2 z}{4\pi\rho^2 r^2}\left[\frac{\delta(t - r/c)}{r} + \frac{1}{c}\,\delta'\!\left(t - \frac{r}{c}\right)\right], \tag{43a}$$

and the portion contributed by Π'' is

$$H''_x = p\,\frac{y^2 - x^2}{4\pi\rho^4}\left[(\operatorname{sgn} z)\,\delta\!\left(t - \frac{|z|}{c}\right) - \frac{z}{r}\,\delta\!\left(t - \frac{r}{c}\right)\right] + p\,\frac{x^2 z}{4\pi\rho^2 r^2}\left[\frac{\delta(t - r/c)}{r} + \frac{1}{c}\,\delta'\!\left(t - \frac{r}{c}\right)\right], \tag{43b}$$

whence the total x component becomes

$$H_x = H'_x + H''_x = p\,\frac{z}{4\pi r^2}\left[\frac{\delta(t - r/c)}{r} + \frac{1}{c}\,\delta'\!\left(t - \frac{r}{c}\right)\right], \tag{43c}$$

where

$$r^2 = x^2 + y^2 + z^2 = \rho^2 + z^2.$$

As shown in Fig. 1.1.1, the constituent fields H'_x and H''_x are seen to comprise a combination of plane and spherical wavefronts, whereas the composite field H_x exhibits only the spherical wavefronts expected for a localized source. The decomposition of H_x into H'_x and H''_x corresponds to a similar decomposition of the source function $\mathbf{J}(\mathbf{r}, t)$ into portions that radiate only E-mode fields (Π') with respect to z, and only H-mode fields (Π''), respectively. These equivalent

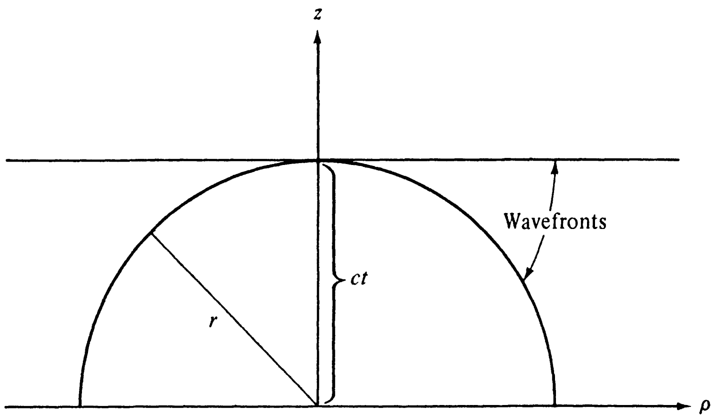

FIG. 1.1.1 Wavefronts generated by E-mode or H-mode potentials.

source distributions occupy the whole plane $z = 0$ and each gives rise to a plane-wave response in addition to the spherical wave, with the former canceled when the composite field is calculated. The E- and H-mode contributions to the y component of magnetic field behave in a similar manner:

$$H_y' = -H_y'' = \frac{pxy}{4\pi\rho^2}\left[\frac{\operatorname{sgn} z}{\rho^2}\delta\left(t - \frac{|z|}{c}\right) - \frac{z}{r}\left(\frac{1}{r^2} + \frac{1}{\rho^2}\right)\delta\left(t - \frac{r}{c}\right)\right.$$
$$\left. - \frac{z}{cr^2}\delta'\left(t - \frac{r}{c}\right)\right], \tag{44a}$$

but the total y component vanishes:

$$H_y = H_y' + H_y'' = 0. \tag{44b}$$

The z component of magnetic field is contributed entirely by the H-mode potential Π'' in Eqs. (42),

$$H_z \equiv H_z'' = -\frac{px}{4\pi r^2}\left[\frac{\delta(t - r/c)}{r} + \frac{1}{c}\delta'\left(t - \frac{r}{c}\right)\right], \tag{45}$$

and in contrast to the components H_x'' and H_y'' transverse to the guide (symmetry) axis z, does not contain the plane-wave constituent. The results in Eqs. (43c), (44b), and (45) are, respectively, the same as the x, y, and z components of $\mathbf{H} = -p\mathscr{G}_{21}\cdot\mathbf{y}_0$ obtained from the formulation in Eq. (34b) (noting that $\mathbf{r}_0 r = \mathbf{x}_0 x + \mathbf{y}_0 y + \mathbf{z}_0 z$).

In contrast to the magnetic-field expressions, the transverse electric field, calculated on substitution of Eq. (39b) into Eqs. (42), exhibits a singular behavior in the source plane $z = 0$. For the above y-directed electric current element, the contributions E_x' and E_x'' from the E- and H-mode potentials Π' and Π'', respectively, are found to be given by

$$E'_x = \frac{pxy}{4\pi\rho^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\left\{\frac{2\delta(t-|z|/c)}{\rho^2} + \frac{\delta(t-r/c)}{\rho^2 r^4}[\rho^4 - 2z^2(2\rho^2+z^2)] - \frac{z^2}{r^3 c}\,\delta'\left(t-\frac{r}{c}\right)\right\} + \frac{3pxy}{4\pi\epsilon_0 r^5}\,U\left(t-\frac{r}{c}\right) - \frac{pxy}{\pi\epsilon_0\rho^4}\,\delta(z), \tag{46a}$$

$$E''_x = \frac{pxy}{4\pi\rho^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\left\{-\frac{2\delta(t-|z|/c)}{\rho^2} + \frac{2\delta(t-r/c)}{\rho^2} + \frac{1}{rc}\,\delta'\left(t-\frac{r}{c}\right)\right\}, \tag{46b}$$

whence $E_x = E'_x + E''_x$ becomes

$$E_x = \frac{pxy}{4\pi r^3}\sqrt{\frac{\mu_0}{\epsilon_0}}\left[\frac{3\delta(t-r/c)}{r} + \frac{1}{c}\,\delta'\left(t-\frac{r}{c}\right)\right] + \frac{3pxy}{4\pi\epsilon_0 r^5}\,U\left(t-\frac{r}{c}\right) - \frac{pxy}{\pi\epsilon_0\rho^4}\,\delta(z). \tag{47}$$

One observes that both the E-mode and H-mode contributions contain the plane and spherical wavefronts depicted in Fig. 1.1.1. The E-mode portion has in addition the static dipole field given by the second term in Eq. (46a) and in the source plane $z = 0$, the last term in Eq. (46a) behaves singularly. This singularity in the total field expression (47) is spurious since it destroys the continuity of E_x across the $z = 0$ plane, and its presence represents a slight deficiency in the Hertz potential formulation. Cancellation of the singularity is assured by retention of the operator term $(\nabla_t\nabla_t/\nabla_t^2)$ contained in the exact Eq. (38a) but omitted in the formulation of Eqs. (42). [See the result obtained on pre- and postmultiplication of this term by $-p\mathbf{x}_0$ and $\mathbf{y}_0$, respectively, noting the footnote to Eq. (38a).] The above calculation demonstrates that care must be exercised in the use of the Hertz potential formulas (42) in planes containing sources and oriented perpendicular to **a**. However, the fields computed exterior to the source planes can be employed on these planes if necessary continuity requirements are satisfied [i.e., if the singularity can be isolated and ignored, as in Eq. (47)]. One verifies readily that with the last term omitted, E_x in Eq. (47) agrees with the expression obtained from the alternative formula, $-p\mathbf{x}_0\cdot\mathscr{G}_{11}\cdot\mathbf{y}_0$, in Eq. (34a).

Dyadic Green's functions for bounded cylindrical regions

For the case of a uniform waveguide of arbitrary (but non-varying) cross section transverse to the guide axis **a**, and bounded by perfectly conducting walls s, the electric-field Green's function $\mathscr{G}_{11}$ implicitly defined by Eqs. (21a) must satisfy a boundary condition

$$\mathbf{v}\times\mathscr{G}_{11} = 0 \qquad \text{on } s, \tag{48}$$

$\mathbf{v}$ being the outward normal vector at the walls. The representation in Eq. (38a) of the free-space dyadic $\mathscr{G}_{11}$ satisfies an outgoing wave condition in all directions but is inadequate to satisfy the requirement in Eq. (48). However, as derivable from the inverse-operator derivation, the representations in Eqs. (38a) and (41) are still valid inverses if two different scalar Green's functions $\mathscr{S}'$ and $\mathscr{S}''$ are inserted into these expressions:

$$
\begin{aligned}
\mathscr{G}_{11} = {} & \frac{1}{\epsilon_0(\partial/\partial t)}\left(\mathbf{aa} + \frac{\nabla_t\nabla_t}{\nabla_t^2}\right)\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') \\
& + \mu_0\frac{\partial}{\partial t}(\nabla\times\mathbf{a})(\nabla'\times\mathbf{a})\mathscr{S}''(\mathbf{r},\mathbf{r}';t,t') \\
& - \frac{[\nabla\times(\nabla\times\mathbf{a})][\nabla'\times(\nabla'\times\mathbf{a})]\mathscr{S}'(\mathbf{r},\mathbf{r}';t,t')}{\epsilon_0(\partial/\partial t)}
\end{aligned}
\tag{49a}
$$

$$
\begin{aligned}
\mathscr{G}_{21} = {} & -[(\nabla\times\mathbf{a})(\nabla'\times\nabla'\times\mathbf{a})\mathscr{S}'(\mathbf{r},\mathbf{r}';t,t') \\
& + (\nabla\times\nabla\times\mathbf{a})(\nabla'\times\mathbf{a})\mathscr{S}''(\mathbf{r},\mathbf{r}';t,t')],
\end{aligned}
\tag{49b}
$$

where both $\mathscr{S}'$ and $\mathscr{S}''$ still satisfy Eq. (38b) but are distinguished by their boundary conditions, deducible from the requirement (48):

$$
\begin{aligned}
&\frac{\partial}{\partial \nu}\mathscr{S}'' = 0 && \text{on } s, \\
&\nabla_t^2\mathscr{S}' = 0 = \mathscr{S}' && \text{on } s.
\end{aligned}
\tag{49c}
$$

Compatibility of the conditions $\nabla_t^2\mathscr{S}' = 0$ and $\mathscr{S}' = 0$ on s can be inferred from the homogeneous form of Eq. (38b). Initial conditions of vanishing for $t \leq t'$ and hence outgoing wave conditions at $\pm\infty$ on the guide axis are also required.

The Green's functions in Eqs. (49) can be utilized for the representation of the fields excited by an electric current density $\mathbf{J}$ in a uniform waveguide with perfectly conducting walls. The field representations in Eqs. (42) are still applicable but, now, instead of the expressions in Eqs. (42b), one finds for the electric-type Hertz potentials,

$$
\begin{aligned}
\Pi''(\mathbf{r},t) &= \int \nabla'\times\mathbf{a}\mathscr{S}''(\mathbf{r},\mathbf{r}';t,t')\cdot\mathbf{J}(\mathbf{r}',t')\,d\mathbf{r}'\,dt', \\
\Pi'(\mathbf{r},t) &= \frac{1}{\epsilon_0(\partial/\partial t)}\int \nabla'\times(\nabla'\times\mathbf{a})\,\mathscr{S}'(\mathbf{r},\mathbf{r}';t,t')\cdot\mathbf{J}(\mathbf{r}',t')\,d\mathbf{r}'\,dt'.
\end{aligned}
\tag{50}
$$

It should be noted that for linear stationary systems, the scalar Green's functions $\mathscr{S}'$ and $\mathscr{S}''$ both have the form

$$\mathscr{S}(\mathbf{r},\mathbf{r}';t,t') = \mathscr{S}(\mathbf{r},\mathbf{r}';t-t'), \tag{51a}$$

and since, in view of the reciprocity property (29) of the Green's function $\mathscr{G}_{11}$, $\mathscr{S}(\mathbf{r},\mathbf{r}';t,t') = \mathscr{S}(\mathbf{r}',\mathbf{r};-t',-t)$, one has

$$\mathscr{S}(\mathbf{r},\mathbf{r}';t-t') = \mathscr{S}(\mathbf{r}',\mathbf{r};t-t'). \tag{51b}$$

The above reciprocity properties permit one to anticipate the general forms that $\mathscr{S}'$ and $\mathscr{S}''$ must assume:

$$
\begin{aligned}
\mathscr{S}'(\mathbf{r},\mathbf{r}';t,t') &= \sum_\alpha A_\alpha\Phi_\alpha(\mathbf{r},t)\Phi_\alpha(\mathbf{r}',-t'), \\
\mathscr{S}''(\mathbf{r},\mathbf{r}';t,t') &= \sum_\beta B_\beta\psi_\beta(\mathbf{r},t)\psi_\beta(\mathbf{r}',-t'),
\end{aligned}
\tag{51c}
$$

and Eq. (38b) implies that for $\mathbf{r} \neq \mathbf{r}', t \neq t'$ the mode functions Φ_α and ψ_β must satisfy

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi_\alpha = 0, \qquad \Phi_\alpha = 0 \text{ on } s,$$
$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi_\beta = 0, \qquad \frac{\partial \psi_\beta}{\partial \nu} = 0 \text{ on } s, \tag{51d}$$

with the amplitude constants A_α and B_β being determined by the singularity of Eq. (38b) at $\mathbf{r} = \mathbf{r}', t = t'$. In the steady state, representations of this type will be exhibited in Secs. 2.3 and 5.2.

In a manner similar to the derivations in Eqs. (38a) and (41), or by duality, one infers that the general solutions of Eqs. (20) and (21b) for $\mathscr{G}_{22}$ and $\mathscr{G}_{12}$ in a uniform waveguide of perfectly conducting walls have the form

$$\begin{aligned}\mathscr{G}_{22} = &+\frac{1}{\mu_0(\partial/\partial t)}\left(\mathbf{aa} + \frac{\nabla_t\nabla_t}{\nabla_t^2}\right)\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t')\\ &+ \epsilon_0\frac{\partial}{\partial t}(\nabla\times\mathbf{a})(\nabla'\times\mathbf{a})\mathscr{S}'(\mathbf{r},\mathbf{r}';t,t')\\ &- \frac{[\nabla\times(\nabla\times\mathbf{a})][\nabla'\times(\nabla'\times\mathbf{a})]\mathscr{S}''(\mathbf{r},\mathbf{r}';t,t')}{\mu_0(\partial/\partial t)},\end{aligned} \tag{52a}$$

$$\begin{aligned}\mathscr{G}_{12} = &(\nabla\times\mathbf{a})[\nabla'\times(\nabla'\times\mathbf{a})]\mathscr{S}''(\mathbf{r},\mathbf{r}';t,t')\\ &+ [\nabla\times(\nabla\times\mathbf{a})](\nabla'\times\mathbf{a})\mathscr{S}'(\mathbf{r},\mathbf{r}';t,t'),\end{aligned} \tag{52b}$$

whence, by Eq. (19b), the electric and magnetic fields produced by a magnetic current density $\mathbf{M}(\mathbf{r}, t)$ are

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \nabla\times(\nabla\times\mathbf{a})\,\Pi'(\mathbf{r},t) - \mu_0\frac{\partial}{\partial t}\nabla\times\mathbf{a}\,\Pi''(\mathbf{r},t),\\ \mathbf{H}(\mathbf{r},t) &= \epsilon_0\frac{\partial}{\partial t}\nabla\times\mathbf{a}\,\Pi'(\mathbf{r},t) + \nabla\times(\nabla\times\mathbf{a})\,\Pi''(\mathbf{r},t),\end{aligned} \tag{53a}$$

where

$$\begin{aligned}\Pi'(\mathbf{r},t) &= -\int\nabla'\times\mathbf{a}\mathscr{S}'(\mathbf{r},\mathbf{r}';t,t')\cdot\mathbf{M}(\mathbf{r}',t')\,d\mathbf{r}'\,dt',\\ \Pi''(\mathbf{r},t) &= \frac{1}{\mu_0(\partial/\partial t)}\int\nabla'\times(\nabla'\times\mathbf{a})\mathscr{S}''(\mathbf{r},\mathbf{r}';t,t')\cdot\mathbf{M}(\mathbf{r}',t')\,d\mathbf{r}'\,dt'.\end{aligned} \tag{53b}$$

Since the field expressions in Eqs. (53a) and (42a) are identical, it is manifest that the Hertz potentials in the presence of both electric and magnetic current densities $\mathbf{J}$ and $\mathbf{M}$ are given by superposition of Eqs. (50) and (53b). There is a restriction in the range of applicability of the magnetic-field representation in Eq. (53a) to planes transverse to $\mathbf{a}$ not containing the source $\mathbf{M}(\mathbf{r}, t)$; as observed previously, this results from a singularity in $\mathscr{G}_{22}$ noted in Eq. (52a) but not included in Eq. (53a).

1.1c The Plasma Field (One-Component Fluid Model)

General properties

A plasma, comprising a system of charged particles in thermal motion, exhibits collective wave phenomena that in certain ranges may be approximately described in terms of a homogeneous, collisionless, fluid model. We restrict our considerations to the case wherein only plasma electrons are assumed mobile. The associated plasma field, linearized and suitably averaged, is describable by an electric field $\mathbf{E}(\mathbf{r}, t)$, a magnetic field $\mathbf{H}(\mathbf{r}, t)$, an electron pressure $p(\mathbf{r}, t)$, and an average electron velocity $\mathbf{v}(\mathbf{r}, t)$ obeying the field equations[3]

$$
\begin{aligned}
&\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} && && -n_0 q \mathbf{v} && = -\mathbf{J},\\
&\nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} && && && = -\mathbf{M},\\
& && \frac{1}{\gamma p_0}\frac{\partial p}{\partial t} + && \nabla\cdot\mathbf{v} && = -s,\\
&n_0 q \mathbf{E} && + \;\; \nabla p && + n_0 m\left(\frac{\partial \mathbf{v}}{\partial t} - \omega_c \mathbf{b}_0 \times \mathbf{v}\right) && = -\mathbf{f}.
\end{aligned}
\tag{54}
$$

As before, ϵ_0 is the vacuum dielectric constant, μ_0 the vacuum permeability, $-n_0 q$ and $n_0 m$ are the background electron charge and mass densities, p_0 is the background electron pressure (in a cold plasma $p_0 \sim n_0 T_0 = 0$), and γ is the specific-heat ratio for electrons. The plasma is polarized by a static magnetic field $\mathbf{B}_0 = B_0 \mathbf{b}_0$, thereby introducing an electron "cyclotron" (gyro) frequency $\omega_c = qB_0/m$ into the field description. The field is assumed to be excited externally by an electric current density $\mathbf{J}(\mathbf{r}, t)$, a magnetic current density $\mathbf{M}(\mathbf{r}, t)$, an electron source density $s(\mathbf{r}, t)$, and a force density $\mathbf{f}(\mathbf{r}, t)$. It is to be observed that the first two rows of Eqs. (54) comprise the conventional Maxwell equations appropriate to a charged fluid of electric current density $-n_0 q\mathbf{v}$ with applied current densities $\mathbf{J}$ and $\mathbf{M}$. The latter two rows constitute the Euler equations for a charged inviscid fluid, including a Lorentz force density $-n_0 q(\mathbf{E} + \mathbf{v} \times \mathbf{B}_0)$ and applied excitations s and $\mathbf{f}$ [see Eqs. (1) and (16)]. The presence of $\mathbf{v}$ dependent and $\mathbf{E}, \mathbf{H}$ dependent terms in the Maxwell and Euler equations, respectively, produces coupling between the electromagnetic and dynamical fields considered separately in Secs. 1.1a and 1.1b. The arrangement of Eq. (54) anticipates transition to the matrix equation (57) and to the abstract formulation in Sec. 1.1d.

To assure a unique description of the fields $\mathbf{E}, \mathbf{H}, p$, and $\mathbf{v}$ defined by Eqs. (54), one imposes the additional requirements that the excitations $\mathbf{J}, \mathbf{M}, s$, and $\mathbf{f}$ vanish for $t \leq t_1$ and that the fields satisfy the initial conditions

$$\mathbf{E} \equiv \mathbf{H} \equiv p \equiv \mathbf{v} \equiv 0 \qquad \text{for } t \leq t_1 \tag{54a}$$

and the boundary conditions

$$\left.\begin{aligned} \mathbf{n} \times \mathbf{E} &= \mathscr{Z} \cdot \mathbf{H} \\ p &= \alpha \mathbf{v} \cdot \mathbf{n} \end{aligned}\right\} \text{ on } S, \tag{54b}$$

where $\mathbf{n}$ is the outward unit vector normal to the surface S (if any) bounding the region within which the field is defined, while α and $\mathscr{Z}$ are boundary ("impedance") parameters for the dynamical and electromagnetic fields, respectively.

An energy-conservation theorem is readily inferred on suitably multiplying Eqs. (54) by $\mathbf{E}$, $\mathbf{H}$, p, and $\mathbf{v}$, respectively, and adding, whence one obtains

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H} + p\mathbf{v}) = &-\frac{\partial}{\partial t}\left(\frac{\epsilon_0 \mathbf{E}^2}{2} + \frac{\mu_0 \mathbf{H}^2}{2} + \frac{p^2}{2\gamma p_0} + \frac{n_0 m \mathbf{v}^2}{2}\right) \\ &- \mathbf{J} \cdot \mathbf{E} - \mathbf{M} \cdot \mathbf{H} - sp - \mathbf{f} \cdot \mathbf{v} \end{aligned} \tag{55}$$

as a natural generalization of Eqs. (2) and (17). The vector $(\mathbf{E} \times \mathbf{H} + p\mathbf{v})$ is identified as the total instantaneous plasma power flow per unit area at any space–time point $\mathbf{r}, t$ in the plasma field. Correspondingly, the time derivative on the right of Eq. (55) reveals the total instantaneous plasma field energy density, while the remaining terms display the instantaneous power per unit volume supplied to the field by the excitations $\mathbf{J}$, $\mathbf{M}$, s, and $\mathbf{f}$. The latter are evident combinations of previously defined electromagnetic and acoustic power and energy densities.

The linearity of the plasma field Eqs. (54) implies a corresponding linear dependence of the fields $\mathbf{E}$, $\mathbf{H}$, p, and $\mathbf{v}$ on the excitations $\mathbf{J}$, $\mathbf{M}$, s, and $\mathbf{f}$. Thus, in an evident matrix generalization of Eqs. (4) and (19), the fields at any space–time point $\mathbf{r}, t$ can be represented as

$$\begin{pmatrix} \mathbf{E}(\mathbf{r}, t) \\ \mathbf{H}(\mathbf{r}, t) \\ p(\mathbf{r}, t) \\ \mathbf{v}(\mathbf{r}, t) \end{pmatrix} = -\iiiint \begin{pmatrix} \mathscr{G}_{11} & \mathscr{G}_{12} & \mathbf{G}_{13} & \mathscr{G}_{14} \\ \mathscr{G}_{21} & \mathscr{G}_{22} & \mathbf{G}_{23} & \mathscr{G}_{24} \\ \mathbf{G}_{31} & \mathbf{G}_{32} & G_{33} & \mathbf{G}_{34} \\ \mathscr{G}_{41} & \mathscr{G}_{42} & \mathbf{G}_{43} & \mathscr{G}_{44} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{J}(\mathbf{r}', t') \\ \mathbf{M}(\mathbf{r}', t') \\ s(\mathbf{r}', t') \\ \mathbf{f}(\mathbf{r}', t') \end{pmatrix} d\mathbf{r}'\, dt', \tag{56}$$

where the integrals are to be extended over all space–time volume elements $d\mathbf{r}'\, dt'$ within which the excitations are finite. The scalar, vector, or dyadic nature of the Green's function matrix elements $G_{\alpha\beta} \equiv G_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; t, t')$ and the appropriate definition of the matrix product in Eq. (56) is determined by the scalar or vector character of the field components on the left side of the equation. The Green's function $G_{\alpha\beta}$ represents the negative of a field produced at $\mathbf{r}, t$ by a unit current applied at $\mathbf{r}', t'$. For example, $-\mathscr{G}_{11} \cdot \mathbf{e}$ is the vector electric field produced at $\mathbf{r}, t$ by a unit electric current density at $\mathbf{r}', t'$ in the direction $\mathbf{e}$, etc.

The field representation in Eq. (56) reduces the plasma field problem to one of determining the Green's functions $G_{\alpha\beta}$. The defining equations for these Green's functions are obtained on substitution of Eq. (56) into Eqs. (54). For example, the Green's functions $G_{\alpha 1}$ excited by a point dyadic electric current density $\mathscr{J} = \mathbf{1}\, \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t')$ are defined by

$$\begin{pmatrix} \epsilon_0 \dfrac{\partial}{\partial t}\mathbf{1} & -\nabla \times \mathbf{1} & 0 & -n_0 q \mathbf{1} \\ \nabla \times \mathbf{1} & \mu_0 \dfrac{\partial}{\partial t}\mathbf{1} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\gamma p_0}\dfrac{\partial}{\partial t} & \nabla \\ n_0 q \mathbf{1} & 0 & \nabla & n_0 m\left(\dfrac{\partial}{\partial t}\mathbf{1} - \omega_c \mathbf{b}_0 \times \mathbf{1}\right) \end{pmatrix} \cdot \begin{pmatrix} \mathscr{G}_{11} \\ \mathscr{G}_{21} \\ \mathbf{G}_{31} \\ \mathscr{G}_{41} \end{pmatrix} = \begin{pmatrix} \mathscr{J} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{57}$$

where in conformity with Eq. (56) we have employed a matrix notation for the basic field operator of Eqs. (54). For uniqueness, Eqs. (57) are to be subject to initial and boundary conditions which typically are of the form [see Eqs. (54a) and (54b)]

$$G_{\alpha 1} \equiv 0 \qquad \text{for } t \leq t' \tag{57a}$$

and

$$\left.\begin{aligned} \mathbf{n} \times \mathscr{G}_{11} &= \mathscr{Z}\cdot\mathscr{G}_{21} \\ \mathbf{G}_{31} &= \alpha \mathbf{n}\cdot\mathscr{G}_{41} \end{aligned}\right\} \quad \text{on } S. \tag{57b}$$

Note that, despite the dot-product multiplication on the left of Eq. (57), the product of the 43 vector matrix element and the vector $\mathbf{G}_{31}$ is simple (i.e., $\nabla\mathbf{G}_{31}$).

Symmetry and reciprocity properties of the plasma Green's functions are derivable from an adjoint problem in the manner illustrated in Secs. 1.1a and 1.1b. To avoid repetitiousness we merely state the result of such a derivation:

$$G_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; t, t') = (-1)^{\alpha+\beta}\widetilde{G}_{\beta\alpha}(\mathbf{r}', \mathbf{r}; -t', -t), \tag{58}$$

where $\tilde{}$ denotes the transpose operation for dyadics . The reciprocity property 58 generalizes and contains the previously obtained properties (12) and (29) for the acoustic and electromagnetic fields.

Dyadic Green's functions for an unbounded, isotropic, electron plasma

Closed-form expressions for the Green's functions $G_{\alpha\beta}$ of an unbounded plasma with no magnetic field ($\omega_c = 0$) can be derived by generalization of the inverse-operator procedure employed in Eqs. (33) et seq. for free-space Green's functions. In this case one finds that the overall plasma field (i.e., the $G_{\alpha\beta}$) can be expressed or scalarized in terms of two distinctive scalar Green's functions characteristic of wave propagation at light and acoustic speeds. One first solves Eq. (57) for $\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t')$ by elimination of $\mathscr{G}_{21}$, $\mathbf{G}_{31}$, and $\mathscr{G}_{41}$, whence one obtains

$$\left\{\epsilon_0 \frac{\partial}{\partial t}\left[1 + \frac{\omega_p^2}{(\partial^2/\partial t^2) - a^2\nabla^2}\right]\frac{\nabla\nabla}{\nabla^2} - \frac{(\partial^2/\partial t^2) + \omega_p^2 - c^2\nabla^2}{\mu_0(\partial/\partial t)}\frac{\nabla \times (\nabla \times \mathbf{1})}{c^2\nabla^2}\right\}\cdot\mathscr{G}_{11}$$
$$= \delta(\mathbf{r} - \mathbf{r}')\,\delta(t - t'), \tag{59a}$$

where an operational form† has been employed for simplicity of expression. Utilizing the inverse relation (33c), one can invert the bracketed operator in Eq. (59a) by writing in successive steps:

$$\mathscr{G}_{11} = -\frac{\nabla^2 - (1/a^2)(\partial^2/\partial t^2)\nabla\nabla}{\epsilon_0(\partial/\partial t)}\frac{}{\nabla^2}g_a - \mu_0\frac{\partial}{\partial t}\frac{\nabla \times (\nabla \times \mathbf{1})}{\nabla^2}g_c, \tag{59b}$$

$$\mathscr{G}_{11} = \mu_0\frac{\partial}{\partial t}\mathbf{1}g_c - \frac{\nabla\nabla}{\epsilon_0(\partial/\partial t)}\left[g_a + \frac{\partial^2/\partial t^2}{\nabla^2}\left(\frac{g_c}{c^2} - \frac{g_a}{a^2}\right)\right], \tag{59c}$$

$$\mathscr{G}_{11} = \mu_0\frac{\partial}{\partial t}\mathbf{1}g_c - \frac{\nabla\nabla}{\epsilon_0(\partial/\partial t)}\left[g_a + \frac{\partial^2/\partial t^2}{(\partial^2/\partial t^2) + \omega_p^2}(g_c - g_a)\right], \tag{59d}$$

where g_a and g_c are two different scalar Green's functions defined in Eq. (60b). Finally, on rearranging the g_a and g_c terms, one rewrites Eq. (59d) as

$$\mathscr{G}_{11} = \left(\mu_0\frac{\partial}{\partial t}\mathbf{1} - \frac{\nabla\nabla}{\epsilon_0}\frac{\partial/\partial t}{(\partial^2/\partial t^2) + \omega_p^2}\right)g_c - \frac{\nabla\nabla}{\epsilon_0}\frac{\omega_p^2}{(\partial/\partial t)[(\partial^2/\partial t^2) + \omega_p^2]}g_a, \tag{60a}$$

where

$$g_u = \frac{-1}{\nabla^2 - (1/u^2)[(\partial^2/\partial t^2) + \omega_p^2]}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \qquad u = a, c, \tag{60b}$$

with $c = 1/\sqrt{\mu_0\epsilon_0}$, $a = \sqrt{\gamma p_0/n_0 m}$, and $\omega_p = \sqrt{n_0 q^2/m\epsilon_0}$. Equations (60b) define operationally the two scalar Green's functions, in terms of which an isotropic one-component plasma field can be represented. It should be noted that when the plasma frequency $\omega_p \to 0$, the expression for $\mathscr{G}_{11}$ in Eq. (60a) reduces to that for the electromagnetic Green's function $\mathscr{G}_{11}$ given in Eq. (30b).

In an unbounded medium the outgoing solution $g_u(\mathbf{r}, \mathbf{r}'; t, t')$ of Eq. (60b) can be determined in the form (writing $\mathbf{r} - \mathbf{r}' = \mathbf{r}$ and $t - t' = t$)

$$g_u(\mathbf{r}, t) = \int_{-\infty}^{+\infty}\frac{\exp\{-i[\omega t - \sqrt{\omega^2 - \omega_p^2}(r/u)]\}}{4\pi r}\frac{d\omega}{2\pi}. \tag{61a}$$

This generalization of the $\omega_p = 0$ case,

$$\frac{\delta[t - (r/u)]}{4\pi r} = \int_{-\infty}^{+\infty}\frac{\exp\{-i[\omega t - \omega(r/u)]\}}{4\pi r}\frac{d\omega}{2\pi},$$

follows from the recognition that in a temporal Fourier representation of Eq. (60b) (in which $\partial/\partial t$ is replaced by $-i\omega$), the cases $\omega_p = 0$ and $\omega_p \neq 0$ differ only by the replacement $\omega \to \sqrt{\omega^2 - \omega_p^2}$ (see Sec. 1.5e). The integral in Eq. (61a) is tabulated as[4]

$$g_u(\mathbf{r}, t) = \frac{\delta[t - (r/u)]}{4\pi r} - \frac{\omega_p}{u}\frac{J_1[\omega_p\sqrt{t^2 - (r^2/u^2)}]}{\sqrt{t^2 - (r^2/u^2)}}\frac{U[t - (r/u)]}{4\pi}, \tag{61b}$$

where $J_1(x)$ is the first-order Bessel function, and the step function $U(x)$ equals 1 or 0 depending on whether $x > 0$ or < 0, respectively.

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†One can multiply this equation by $[(\partial^2/\partial t^2) - a^2\nabla^2]\nabla^2$ to remove the inverse operators.

The scalar field g_u of Eq. (61b) is seen to be formed of a spherical wave front identical with that in a medium having $\omega_p = 0$, followed by a dispersion-dominated trailing "wake" that persists indefinitely. Dispersion implies that the various harmonic wave constituents in Eq. (61a), required to synthesize the field, travel at different speeds. At observation times sufficiently long to permit replacement of the Bessel function by its large-argument approximation, $J_1(\alpha) \sim (2/\pi\alpha)^{1/2} \cos[\alpha - (3\pi/4)]$, the field may be interpreted in terms of wave packets, or bundles of plane waves, of appropriate wavenumber and frequency; this aspect is discussed in detail in Sec. 1.6.

On observing from Eq. (61b) that since $\mathscr{G}_{11}$ must vanish for $t < (r/u) = \tau$, one evaluates†

$$\frac{1}{(\partial^2/\partial t^2) + \omega_p^2} \delta(t - \tau) = \frac{\sin \omega_p(t - \tau)}{\omega_p} U(t - \tau) \tag{62a}$$

and obtains, on multiplication by $g_u(\mathbf{r}, t)$ and integration,

$$\frac{1}{(\partial^2/\partial t^2) + \omega_p^2} g_u(\mathbf{r}, t) = \int_{-\infty}^{t} \frac{\sin \omega_p(t - \tau)}{\omega_p} g_u(\mathbf{r}, \tau)\, d\tau. \tag{62b}$$

On substitution of Eqs. (61b) and (62) into the operational solution in Eq. (59b), one then finds that

$$\begin{aligned}
\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t') = {} & \frac{\mu_0}{4\pi} \mathbf{1} \frac{\partial}{\partial t} \left[\frac{\delta[t - (r/c)]}{r} - \frac{\omega_p}{c} \frac{J_1[\omega_p \sqrt{t^2 - (r^2/c^2)}]}{\sqrt{t^2 - (r^2/c^2)}} U\left(t - \frac{r}{c}\right) \right] \\
& - \frac{1}{4\pi\epsilon_0} \nabla\nabla \left\{ \left[\frac{\cos \omega_p[t - (r/c)]}{r} \right. \right. \\
& \left. \left. - \frac{\omega_p}{c} \int_{r/c}^{t} \cos \omega_p(t - \tau) \frac{J_1(\omega_p \sqrt{\tau^2 - (r^2/c^2)})}{\sqrt{\tau^2 - (r^2/c^2)}}\, d\tau \right] U\left(t - \frac{r}{c}\right) \right\} \\
& - \frac{1}{4\pi\epsilon_0} \nabla\nabla \left\{ \left[\frac{1 - \cos \omega_p[t - (r/a)]}{r} \right. \right. \\
& \left. \left. - \frac{\omega_p}{a} \int_{r/a}^{t} [1 - \cos \omega_p(t - \tau)] \frac{J_1(\omega_p \sqrt{\tau^2 - (r^2/a^2)})}{\sqrt{\tau^2 - (r^2/a^2)}}\, d\tau \right] U\left(t - \frac{r}{a}\right) \right\}.
\end{aligned} \tag{63}$$

Equation (63) displays explicitly the electric field produced by a suddenly created electric dipole at $t = 0 = \mathbf{r}$ as a superposition of two spherical wavefields traveling at speeds c and a, each associated with a dispersive wake.

Reduced formulation of plasma field

It is of interest to note that the first-order form (54) of the plasma field equations can be recast into a higher-order form on elimination of the field variables p and $\mathbf{v}$. One obtains by this reduction an electromagnetic phrasing of the plasma field equations as

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†As is verified by the operation of $(\partial^2/\partial t^2) + \omega_p^2$ on both sides of Eq. (62a).

$$\epsilon\left(\nabla, \frac{\partial}{\partial t}\right) \frac{\partial}{\partial t}\cdot \mathbf{E} - \nabla \times \mathbf{H} = -\mathbf{J}',$$
$$\nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\mathbf{M}, \tag{64}$$

where

$$\epsilon\left(\nabla, \frac{\partial}{\partial t}\right) = \epsilon_0\left(\mathbf{1} + \frac{\omega_p^2}{(\partial^2/\partial t^2)\mathbf{1} + a^2\nabla\nabla}\right)$$

is an equivalent dielectric constant operator for the isotropic ($\omega_c = 0$) plasma and $\mathbf{J}'$ represents an equivalent electric current density involving a somewhat complicated operator expression in $\mathbf{J}$, $\mathbf{f}$, and s. The dependence of the equivalent dielectric constant on ∇ and $\partial/\partial t$ is frequently referred to as a spatial and temporal dispersion property; this is to be contrasted with the non-dispersive character of the parameters in the first-order formulation. (See Sec. 1.5).

An alternative fluid dynamic rephrasing of the plasma field equations in terms of p and $\mathbf{v}$ can be obtained on elimination of the field variables $\mathbf{E}$ and $\mathbf{H}$ from Eqs. (54).

1.1d General Linear Field (Abstract Formulation)

The commonality in analytical procedures employed to describe the acoustic, electromagnetic, and the one-component plasma fields in Secs, 1.1a–1.1c, suggests their applicability to any linear field that is invariant under spatial and temporal displacements. This observation finds its most succinct expression when the field equations for a linear field are represented in the operator form

$$L\Psi = -\Phi \tag{65}$$

where $L = L(\nabla, \partial/\partial t)$ is a linear operator descriptive of the field equations, $\Psi = \Psi(\mathbf{r}, t)$ is a wavevector characterizing the field variables, and $\Phi = \Phi(\mathbf{r}, t)$ is a wavevector describing the excitation. Since L is a (unbounded) derivative operator, it is necessary for uniqueness to state that Ψ lies in a prescribed domain $\mathscr{D}_L$ of the operator L—a remark that is equivalent to the statement of initial and boundary conditions on the elements of Ψ. We list below the matrix forms taken by the operator L and the wavevectors Ψ and Φ for the fields considered in Secs. 1.1a–1.1c.

Acoustic field

$$L \to \begin{pmatrix} \dfrac{1}{\gamma p_0}\dfrac{\partial}{\partial t} & \nabla \\ \nabla & n_0 m \dfrac{\partial}{\partial t}\mathbf{1} \end{pmatrix},$$
$$\Psi \to \begin{pmatrix} p \\ \mathbf{v} \end{pmatrix}, \qquad \Phi \to \begin{pmatrix} s \\ \mathbf{f} \end{pmatrix}. \tag{66}$$

Electromagnetic field

$$L \to \begin{pmatrix} \epsilon_0 \dfrac{\partial}{\partial t} \mathbf{1} & -\nabla \times \mathbf{1} \\ \nabla \times \mathbf{1} & \mu_0 \dfrac{\partial}{\partial t} \mathbf{1} \end{pmatrix},$$

$$\Psi \to \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \qquad \Phi \to \begin{pmatrix} \mathbf{J} \\ \mathbf{M} \end{pmatrix}. \tag{67}$$

One-component plasma field

$$L \to \begin{pmatrix} \epsilon_0 \dfrac{\partial}{\partial t} \mathbf{1} & -\nabla \times \mathbf{1} & 0 & -n_0 q \mathbf{1} \\ \nabla \times \mathbf{1} & \mu_0 \dfrac{\partial}{\partial t} \mathbf{1} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\gamma p_0} \dfrac{\partial}{\partial t} & \nabla \\ n_0 q \mathbf{1} & 0 & \nabla & n_0 m \left(\dfrac{\partial}{\partial t} \mathbf{1} - \omega_c \mathbf{b}_0 \times \mathbf{1} \right) \end{pmatrix},$$

$$\Psi \to \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ p \\ \mathbf{v} \end{pmatrix}, \qquad \Phi \to \begin{pmatrix} \mathbf{J} \\ \mathbf{M} \\ s \\ \mathbf{f} \end{pmatrix}. \tag{68}$$

The matrix–wavevector product $L\Psi$ in Eq. (65) requires modest care in evaluation, as the matrix elements of the operator L are scalars, vectors, or dyadics, while the wavevector elements are scalars or vectors. Proper identification of the product is evident on comparison of Eq. (66) with Eq. (1), of Eq. (67) with Eq. (16), and of Eq. (68) with Eq. (54).

To effect combinatorial operations on Eq. (65) one defines the inner product of two wavevectors and is led thereby to the concept of an adjoint operator.[5] The inner product of two wavevectors Ψ^+ and Ψ is defined as the four-dimensional space–time integral

$$(\Psi^+, \Psi) \equiv \int \Psi^+(\mathbf{r}, t) \cdot \Psi(\mathbf{r}, t) \, d\mathbf{r} \, dt; \tag{69a}$$

for example, in the case of the real electromagnetic field described by the wavevector in Eq. (67), this relation takes the more explicit form

$$(\Psi^+, \Psi) = \int [\mathbf{E}^+ \cdot \mathbf{E} + \mathbf{H}^+ \cdot \mathbf{H}] \, d\mathbf{r} \, dt. \tag{69b}$$

If the inner product of the wavevectors Ψ^+ and $L\Psi$ can be related to the inner product of the wavevectors $L^+\Psi^+$ and Ψ as follows,

$$(\Psi^+, L\Psi) = (L^+\Psi^+, \Psi), \tag{70}$$

then the operator L^+ is said to be the adjoint of the operator L for Ψ and Ψ^+ lying in suitable domains $\mathscr{D}_L$ and $\mathscr{D}_{L^+}$, respectively. The operator L is to be identified as one of the matrix derivative operators shown in Eqs. (66–68); Eq. (70) then constitutes an integration-by-parts theorem in a four-dimensional space–time volume wherein the boundary contributions vanish by restricting Ψ and Ψ^+ to suitable domains.

The definition of the adjoint operator L^+ in Eq. (70) permits the introduction of an adjoint problem

$$L^+\Psi^+ = -\Phi^+, \tag{71}$$

where for uniqueness Ψ^+ is subject to appropriate boundary and initial conditions (i.e., Ψ^+ lies in a prescribed domain $\mathscr{D}_{L^+}$) and where the wavevector Φ^+ is arbitrary. The wavevector Ψ of the original problem in Eq. (65) is not unrelated to the wavevector Ψ^+ of the adjoint problem in Eq. (71). In fact, in view of Eq. (70), it is evident that

$$(\Psi^+, \Phi) = (\Phi^+, \Psi), \tag{72}$$

an adjointness relation that is equivalent to and generalizes the previously encountered adjointness relations in Eqs. (9) and (25).

One introduces a Green's function operator G for the linear field Ψ by

$$\Psi = -G\Phi, \qquad \text{whence } LG = 1, \tag{73a}$$

and, correspondingly, an adjoint Green's function G^+ for the adjoint field Ψ^+ by

$$\Psi^+ = -G^+\Phi^+, \qquad \text{whence } L^+G^+ = 1. \tag{73b}$$

Equations (73) constitute succinct defining equations for the Green's functions of a general linear field. Matrix representatives of G and G^+ are denoted by

$$G \rightarrow (G_{ij}(\mathbf{r}, \mathbf{r}'; t, t')) \qquad \text{and} \qquad G^+ \rightarrow (G_{ij}^+(\mathbf{r}, \mathbf{r}'; t, t')) \tag{73c}$$

with the matrix elements G_{ij} and G_{ij}^+ identifiable as scalars, vectors, or dyadics; see, for example, the representations in Eqs. (4), (19), and (56). In view of the adjointness relation (72), the adjoint Green's function G^+ is related to the original Green's function G by

$$(G^+\Phi^+, \Phi) = (\Phi^+, G\Phi), \tag{74a}$$

whence one infers from Eq. (69a) and the arbitrariness of the excitation wavevectors Φ and Φ^+ the matrix element relations

$$G_{ij}^+(\mathbf{r}, \mathbf{r}'; t, t') = \widetilde{G}_{ji}(\mathbf{r}', \mathbf{r}; t', t), \tag{74b}$$

which generalizes previous results derived in Eqs. (11) and (28); note in Eq. (74b) that the transpose symbol is necessary only if G_{ij} is a dyadic element.

The relation (74b) can be cast as a reciprocity relation involving only the original Green's function G if the adjoint G^+ is expressible in terms of G. Such a possibility exists if the original field operator L possesses symmetry properties under a time-reversal transformation in an appropriate domain. For example, if the time-reversed field operator $L(\nabla, -\partial/\partial t)$ is denoted by $\hat{L}$, then for all the operators L^+ considered above, $L^+ = T\hat{L}T$, where T is a diagonal operator with ± 1 matrix elements such that $T^{-1} = T$. It follows that

$$G^+ = T\hat{G}T, \tag{75}$$

where $\hat{G}$ is defined in a suitable domain by $\hat{L}\hat{G} = 1$ and has a matrix representative

$$\hat{G}_{ij}(\mathbf{r}, \mathbf{r}'; t, t') = G_{ij}(\mathbf{r}, \mathbf{r}'; -t, -t'). \tag{76}$$

From Eqs. (74b), (75), and (76) one then infers in an appropriate domain the reciprocity property

$$G_{ij}(\mathbf{r}, \mathbf{r}'; t, t') = (-1)^{i+j}\tilde{G}_{ji}(\mathbf{r}, \mathbf{r}'; -t', -t) \tag{77}$$

for the original field, a result that is a generalization of the previously noted relations (12), (29), and (58).

1.2 PLANE-WAVE FIELD REPRESENTATIONS

Closed-form solutions for the space- and time-dependent fields excited by arbitrary, space–time distributed sources are not generally possible. Although a number of formal solutions for such problems have been obtained in Sec. 1.1 via operator or equivalent techniques, their explicit evaluation frequently requires a complicated integration process, depending on the space–time distribution of the sources. The determination of field solutions for free-space sources of harmonic plane-wave form is, however, much simpler to effect, because the operator analysis becomes essentially algebraic. Thus, in suitable regions, if the source distributions can be analyzed into their plane-wave constituents, the corresponding field response can generally be ascertained by algebraic techniques and the desired space–time fields evaluated by synthesis (integration) of the constituent plane-wave responses. This analysis and synthesis procedure provides a very effective methodology for the calculation of power flow, asymptotic evaluation of far fields, etc., for appropriate linear fields. In the following, attention will be concentrated on plane-wave analyses of fields, and their Green's functions; the corresponding space- and time-dependent fields are derivable therefrom via transform relations to be stated.

Fields in linear, homogeneous, stationary, unbounded regions are invariant under space–time translations and hence are representable as superpositions of plane waves of the form $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$. The vector wavenumber $\mathbf{k}$ and angular frequency ω, which characterize the wave periodicities along the spatial coordinate $\mathbf{r}$ and time t, can be so selected that such plane waves constitute a complete set capable of representing relatively arbitrary space–time functions. The mathematical basis for such a representation is provided by the four-dimensional Fourier-integral theorem,[6] whereby an integrable space–time function $F(\mathbf{r}, t)$ is representable as

$$F(\mathbf{r}, t) = \int F(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} \tag{1a}$$

with the regular transform amplitude $F(\mathbf{k}, \omega)$ given by

$$F(\mathbf{k}, \omega) = \int F(\mathbf{r}, t) e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\, d\mathbf{r}\, dt. \tag{1b}$$

Each of the fourfold integrals in Eqs. (1a) and (1b) extends from $-\infty$ to $+\infty$, with $d\mathbf{k}$ and $d\mathbf{r}$ denoting volume elements in $\mathbf{k}$ and $\mathbf{r}$ space, respectively. For notational simplicity the same symbol, but with different arguments, designates the function $F(\mathbf{r}, t)$ and its transform $F(\mathbf{k}, \omega)$.

The Fourier transforms (1a) and (1b) can be combined into the more succinct form of a "completeness relation," which constitutes a plane-wave representation of the four-dimensional space–time delta function:

$$\delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t') = \int e^{i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - \omega(t-t')]} \frac{d\mathbf{k}\, d\omega}{(2\pi)^4}\,. \tag{2a}$$

The transform relations (1) are recoverable from Eq. (2a), as is evident on multiplication of the latter by $F(\mathbf{r}', t')$ and integration over all space–time volume elements $d\mathbf{r}'\, dt'$. The transform relations (1) also imply an "orthogonality" property

$$(2\pi)^4\, \delta(\mathbf{k} - \mathbf{k}')\, \delta(\omega - \omega') = \int e^{-i[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r} - (\omega-\omega')t]}\, d\mathbf{r}\, dt. \tag{2b}$$

In the domain of integrable functions, the range of integration in Eqs. (1a) and (2a) spans all real frequencies ω and real wavenumbers $\mathbf{k}$. In many physical problems there appear non-Fourier-integrable, but bounded, "causal" functions that vanish for t less than some finite time and that may be finite as t approaches infinity. To assure the existence of the transform (1b) of such a causal function, it is sufficient to shift slightly the ω contour of integration from the real ω axis into the $\mathrm{Im}\, \omega > 0$ region. Equations (1) and (2) so modified constitute the Fourier–Laplace integral theorem. That Eqs. (1) provide a complete representation of a causal function $F(\mathbf{r}, t)$, vanishing for $t < 0$, may be verified by contour deformation of the ω integral path in Eq. (1a) into the upper half-plane $\mathrm{Im}\, \omega > 0$ where, in view of the regularity property of $F(\mathbf{k}, \omega)$ in this region, the integral vanishes by Cauchy's theorem.

In a linear, homogeneous, stationary, unbounded medium, Green's functions possess an $\mathbf{r} - \mathbf{r}'$, $t - t'$ space–time dependence and hence, by Eq. (2), are representable in the form

$$g(\mathbf{r}, \mathbf{r}'; t, t') = \int g(\mathbf{k}, \omega) e^{i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - \omega(t-t')]} \frac{d\mathbf{k}\, d\omega}{(2\pi)^4}\,. \tag{3}$$

A characteristic (eigen) feature of such a plane-wave representation (basis) is that it algebraizes (diagonalizes) the derivative operators ∇ and $\partial/\partial t$ in field equations that are invariant to space–time translations. In such a basis

$$\nabla \equiv i\mathbf{k} \qquad \text{and} \qquad \frac{\partial}{\partial t} \equiv -i\omega. \tag{4}$$

This algebraic property implies that the representation (3) reduces space- and time-dependent Green's function problems in linear, homogeneous, stationary, unbounded regions to simple algebraic problems of determining $g(\mathbf{k}, \omega)$. In passive lossless systems it can be anticipated that $g(\mathbf{k}, \omega)$ will possess singularities for real values of both ω and $\mathbf{k}$, since impulsive excitation of such systems generates waves that are propagated to infinity. Alternatively stated, since natural plane-wave solutions of the source-free field equations exist for certain real ω and $\mathbf{k}$, the Green's function $g(\mathbf{k}, \omega)$ must exhibit "resonances" or singularities at the corresponding ω and $\mathbf{k}$. Accordingly, to define uniquely integrals of the form (3) it will be desirable to distinguish between the analytic representation of $g(\mathbf{k}, \omega)$ in the half-plane $\operatorname{Im} \omega > 0$ (to assure causality) and the singular, but integrable, representations for $\operatorname{Im} \omega = 0$, with $\mathbf{k}$ real in both instances.

1.2a The Acoustic Field

As noted in Sec. 1.1a, acoustic fields excited in linear homogeneous regions are expressible in terms of a scalar Green's function defined, as in Eq. (1.1.13b), by

$$\left(\nabla^2 - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\right) g(\mathbf{r}, \mathbf{r}'; t, t') = -\delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t'), \tag{5}$$

subject to the initial condition $g \equiv 0$ for $t \leq t'$. In the case of an unbounded region this implies for given t, because of the finite propagation speed a, the boundary condition $g \equiv 0$ as $|\mathbf{r} - \mathbf{r}'| \to \infty$ [see Eq. (1.1.15c)]. In view of the algebraic property (4) and the representation in Eq. (3) one infers from Eq. (5) the simple transform solution

$$g(\mathbf{k}, \omega) = \frac{1}{k^2 - (\omega^2/a^2)}, \qquad \operatorname{Im} \omega > 0, \tag{6a}$$

where the constraint $\operatorname{Im} \omega > 0$ assures in this domain the analyticity of $g(\mathbf{k}, \omega)$, in keeping with the causal nature of the Green's function. To continue $g(\mathbf{k}, \omega)$ onto the real ω axis (in the limit $\operatorname{Im} \omega \to 0+$), one first observes that a real singular function such as $1/x$, undefined at $x = 0$, can be made unique by the following limiting process:

$$\frac{1}{x} = \lim_{\epsilon \to 0+} \frac{1}{x - i\epsilon} = \lim_{\epsilon \to 0} \left(\frac{x}{x^2 + \epsilon^2} + i\frac{\epsilon}{x^2 + \epsilon^2}\right) = P\frac{1}{x} + \pi i \delta(x),\dagger \tag{6b}$$

where P denotes the "principal part" and serves in the sense of Cauchy to exclude the singular point $x = 0$. Equation (6b) is to be interpreted as a "distribution" which is given conventional meaning on multiplication by a suitable function of x and integration over x.[7] In accordance with Eq. (6b), one continues $g(\mathbf{k}, \omega)$ as

†The relation $\pi\delta(x) = \lim_{\epsilon \to 0} [\epsilon/(x^2 + \epsilon^2)]$ can be verified on integrating both sides of Eq. (6b) about $x = 0$.

$$g(\mathbf{k}, \omega) = P\frac{1}{k^2 - (\omega^2/a^2)} + \pi i \delta\left(k^2 - \frac{\omega^2}{a^2}\right), \qquad \operatorname{Im} \omega = 0. \tag{6c}$$

Knowledge of the amplitude $g(\mathbf{k}, \omega)$ permits, via Eq. (3), the evaluation of the space- and time-dependent Green's function†

$$g(\mathbf{r}, t) = \begin{cases} \displaystyle\iint \frac{e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}}{k^2 - (\omega^2/a^2)} \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} = \frac{\delta[t - (r/a)]}{4\pi r}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{7}$$

where for simplicity $\mathbf{r}, t$ in Eqs. (7) denote $\mathbf{r} - \mathbf{r}', t - t'$, and where the result of the integration is known from the alternative evaluation in Eq. (1.1.15a). It should be noted that pole singularities of the integrand in Eq. (7), or equivalently of $g(\mathbf{k}, \omega)$, occur at those values of $\mathbf{k}, \omega$ that satisfy a "dispersion equation" $k^2 - (\omega^2/a^2) = 0$, a "resonance" relation that will be considered below in more detail.

The various acoustic Green's functions $G_{11}(\mathbf{r}, \mathbf{r}'; t, t')$, $\mathscr{G}_{22}(\mathbf{r}, \mathbf{r}'; t, t')$, etc., are derivable from $g(\mathbf{r}, \mathbf{r}'; t, t')$, as previously noted in Eqs. (1.1.13a). It readily follows from these equations together with Eqs. (4) and (6c) that in the $\mathbf{k}, \omega$ transform space, the acoustic Green's functions have the form (for $\operatorname{Im} \omega = 0$),

$$\begin{aligned} G_{11}(\mathbf{k}, \omega) &= \pi\omega n_0 m\, \delta\left(k^2 - \frac{\omega^2}{a^2}\right) + P\frac{1}{j(\omega/\gamma p_0) + (k^2/j\omega n_0 m)} \\ \mathscr{G}_{22}(\mathbf{k}, \omega) &= \left[\pi\frac{\omega}{\gamma p_0}\, \delta\left(k^2 - \frac{\omega^2}{a^2}\right) + P\frac{1}{j\omega n_0 m + [k^2/(j\omega/\gamma p_0)]}\right]\mathbf{1}_L + \frac{1}{j\omega n_0 m}\mathbf{1}_T \\ \mathbf{G}_{12}(\mathbf{k}, \omega) &= \mathbf{G}_{21}(\mathbf{k}, \omega) = j\mathbf{k}\left[-\pi j\delta\left(k^2 - \frac{\omega^2}{a^2}\right) + P\frac{1}{k^2 - (\omega^2/a^2)}\right], \end{aligned} \tag{8}$$

where the delta functions and the principal value symbol P are to be omitted when $\operatorname{Im} \omega \neq 0$.‡ The unit dyadics $\mathbf{1}_L$ and $\mathbf{1}_T$ are defined by

$$\mathbf{1}_L = \frac{\mathbf{k}\mathbf{k}}{k^2}, \qquad \mathbf{1}_T = \frac{-\mathbf{k} \times (\mathbf{k} \times \mathbf{1})}{k^2}, \qquad \mathbf{1} = \mathbf{1}_L + \mathbf{1}_T, \tag{8a}$$

which are longitudinal and transverse, respectively, to the direction $\mathbf{k}_0 = \mathbf{k}/k$ of plane-wave propagation. The significance of these transformed Green's functions is enhanced if one first rewrites the acoustic field equations (1.1.1) in transform space as

$$\begin{aligned} j\frac{\omega}{\gamma p_0} p(\mathbf{k}, \omega) - j\mathbf{k}\cdot\mathbf{v}(\mathbf{k}, \omega) &= -s(\mathbf{k}, \omega), \\ -j\mathbf{k}p(\mathbf{k}, \omega) + j\omega n_0 m\mathbf{v}(\mathbf{k}, \omega) &= -\mathbf{f}(\mathbf{k}, \omega), \end{aligned} \tag{9}$$

†It should be remarked that the ω integration in Eqs. (7) yields an additional term $\delta[t + (r/a)]/4\pi r$, which vanishes for $t > 0$ [see Eq. (1.3.25)].

‡The notation $j = -i$ will be employed occasionally to emphasize network interpretations and corresponds for any harmonic constituent to a time dependence $\exp(j\omega t)$; the half-plane $\operatorname{Im} \omega > 0$ in the i notation becomes $\operatorname{Im} \omega < 0$ in the j notation. Also note that $G_{11}(\mathbf{k}, \omega) = G_{11}(-\mathbf{k}, \omega)$, $\mathscr{G}_{22}(\mathbf{k}, \omega) = \mathscr{G}_{22}(-\mathbf{k}, \omega)$, etc.

whence their algebraic solution can be expressed in terms of the transformed acoustic Green's functions in Eqs. (8) as

$$\begin{aligned} p(\mathbf{k}, \omega) &= -G_{11}(\mathbf{k}, \omega)s(\mathbf{k}, \omega) - \mathbf{G}_{12}(\mathbf{k}, \omega)\cdot\mathbf{f}(\mathbf{k}, \omega), \\ \mathbf{v}(\mathbf{k}, \omega) &= -\mathbf{G}_{21}(\mathbf{k}, \omega)s(\mathbf{k}, \omega) - \mathscr{G}_{22}(\mathbf{k}, \omega)\cdot\mathbf{f}(\mathbf{k}, \omega). \end{aligned} \tag{10}$$

That Eqs. (10) are indeed the solution of Eqs. (9) in an unbounded space can be verified by direct inversion of Eqs. (9), a fact that is already evident from Eqs. (1.1.4) and (1.1.6).

A network schematization of the acoustic field equations in $\mathbf{k}, \omega$ transform space provides a pictorial view of interrelations among the field variables and also may be used to calculate dispersion properties (see p. 2) and power radiation. If one introduces the characteristic $\mathbf{k}_0, \mathbf{T}_0', \mathbf{T}_0'' = \mathbf{k}_0 \times \mathbf{T}_0'$ unit vector coordinate system shown in Fig. 1.2.1, vector fields can be resolved into longitudinal (L) and transverse (T) components as

$$\begin{aligned} \mathbf{v}(\mathbf{k}, \omega) &= v_L\mathbf{k}_0 + v_{T'}\mathbf{T}_0' + v_{T''}\mathbf{T}_0'', \\ \mathbf{f}(\mathbf{k}, \omega) &= f_L\mathbf{k}_0 + f_{T'}\mathbf{T}_0' + f_{T''}\mathbf{T}_0'', \end{aligned}$$

whence Eqs. (9) can be separated into longitudinal equations,

$$\begin{aligned} j\frac{\omega}{\gamma p_0}\hat{p} - kv_L &= -\hat{s}, \\ k\hat{p} + j\omega n_0 m v_L &= -f_L \end{aligned} \tag{11a}$$

and transverse equations

$$\begin{aligned} j\omega n_0 m v_{T'} &= -f_{T'}, \\ j\omega n_0 m v_{T''} &= -f_{T''}, \end{aligned} \tag{11b}$$

where $\hat{p} = -jp(\mathbf{k}, \omega)$ and $\hat{s} = -js(\mathbf{k}, \omega)$. As depicted in Fig. 1.2.2, Eqs. (11) can be schematized as a steady-state network whose elements comprise a "capacitance" $1/\gamma p_0$, an "inductance" $n_0 m$, and an ideal transformer of turns ratio k. The source terms $\hat{s}$ and f_α play the role of an "applied current" generator with infinite shunt impedance and "applied voltage" generators with zero series impedance, respectively, while $\hat{p}$ and v_α act as the "voltage" across the capacitance $1/\gamma p_0$ and the "currents" through the inductances $n_0 m$. The inde-

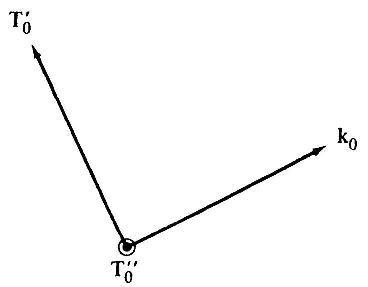


FIG. 1.2.1 $\mathbf{k}_0, \mathbf{T}_0', \mathbf{T}_0''$ coordinate system.

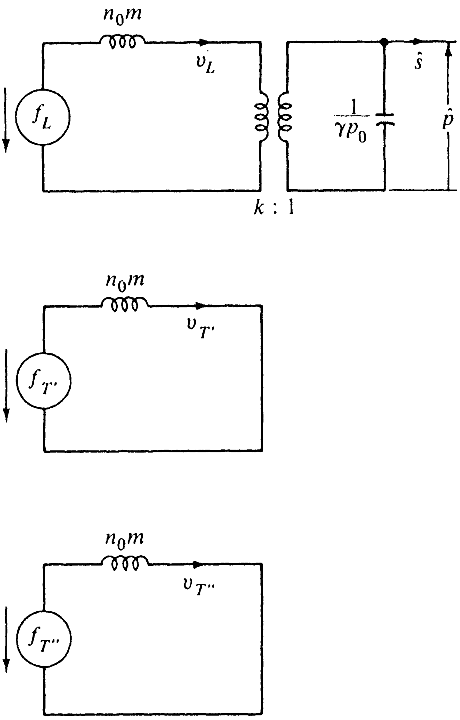


FIG. 1.2.2 Acoustic network.

pendence or uncoupling of the longitudinal (L) and transverse (T) circuits is manifest.

In the absence of excitation (i.e., $\hat{s} = 0 = f_\alpha$), it is apparent from Eqs. (11), or the longitudinal network picture, that non-vanishing source-free acoustic fields are possible for those $\mathbf{k}, \omega$ that satisfy the "resonance condition" (total mesh impedance $= 0$)

$$j\omega n_0 m + \frac{k^2}{j\omega/\gamma p_0} = 0, \tag{12a}$$

or, equivalently, the "dispersion equation" [secular determinant $= 0$ as in Eq. (44)]

$$\left(k^2 - \frac{\omega^2}{a^2}\right) = \left(k + \frac{\omega}{a}\right)\left(k - \frac{\omega}{a}\right) = 0, \qquad a = \sqrt{\frac{\gamma p_0}{n_0 m}}. \tag{12b}$$

These permitted real values of $\mathbf{k}, \omega$ evidently characterize two plane-wave fields traveling in $\pm\mathbf{k}_0$ directions with acoustic speed a. The pressure p and longitudinal velocity v_L are the only non-vanishing field components associated with these source-free waves. From Eqs. (11), or the network, one infers for the wave structure (characteristic impedance) of these longitudinal waves,

$$\frac{p}{v_L} = \pm\sqrt{\frac{n_0 m}{1/\gamma p_0}} \tag{12c}$$

with the $\pm$ signs referring to waves traveling in the $\pm\mathbf{k}_0$ directions, respectively.

Steady-state power radiated by acoustic source

At each point $\mathbf{r}, t$ the total power density supplied by a source $s(\mathbf{r}, t)$ to an acoustic field is equal to $-p(\mathbf{r}, t)s(\mathbf{r}, t)$, as derived in Eq. (1.1.2). Thus, in a non-dissipative unbounded medium the total acoustic *energy* supplied to the field by a distributed source $s(\mathbf{r}, t)$ is

$$-\int p(\mathbf{r}, t)s(\mathbf{r}, t)\, d\mathbf{r}\, dt = -\int p(\mathbf{k}, \omega)s^*(\mathbf{k}, \omega)\frac{d\mathbf{k}\, d\omega}{(2\pi)^4}, \tag{13a}$$

where the right-hand member follows by use of Eqs. (1a) and (2), plus the observation that $s(-\mathbf{k}, -\omega) = s^*(\mathbf{k}, \omega)$ for $\mathbf{k}, \omega$ real. The total *power* radiated at frequency ω_0 by a harmonic source of the form $s(\mathbf{r})e^{j\omega_0 t}$ is the real part of the energy (13a) delivered to the field provided the source transform $s(\mathbf{k}, \omega) = s(\mathbf{k})2\pi\delta(\omega - \omega_0)$, obtained from Eq. (1b), is substituted into Eq. (13a),

$$P_{\text{rad}}(\omega) = -\text{Re}\int p(\mathbf{k}, \omega)s^*(k)\frac{d\mathbf{k}}{(2\pi)^3} = \text{Re}\int G_{11}(\mathbf{k}, \omega)|s(\mathbf{k})|^2\frac{d\mathbf{k}}{(2\pi)^3}, \tag{13b}$$

where $\omega = \omega_0$ in Eq. (13b). It is apparent that only the real (resistive) part of the Green's function $G_{11}(\mathbf{k}, \omega)$, given in Eqs. (8), is necessary for calculation of the radiated power in an unbounded space.

For the special case of a complex harmonic point source of particles $s(\mathbf{r}) = S\delta(\mathbf{r})$ [i.e., $s(\mathbf{k}) = S$], the radiated acoustic power at frequency ω is, by Eqs. (13b) and (8),

$$\begin{aligned} P_{\text{rad}}(\omega) &= \int \omega n_0 m\pi\delta\left(k^2 - \frac{\omega^2}{a^2}\right)|S|^2\frac{d\mathbf{k}}{(2\pi)^3} \\ &= \frac{1}{4\pi}\sqrt{\frac{n_0 m}{1/\gamma p_0}}\,|Sk_a|^2, \qquad k_a = \frac{\omega}{a}, \end{aligned} \tag{14}$$

where the integration over $\mathbf{k}$ has been effected in a spherical coordinate system wherein $d\mathbf{k} = 4\pi k^2\, dk$ and it has been noted that

$$\delta\left(k^2 - \frac{\omega^2}{a^2}\right) = \frac{\delta(k - \omega/a)}{2k} \qquad \text{for } 0 < k < \infty.$$

[Note that $\delta(ax) = |a|^{-1}\delta(x)$].

In a similar manner, an applied force density $\mathbf{f}(\mathbf{r}, t)$ of harmonic form $\mathbf{f}(\mathbf{r})e^{j\omega_0 t}$ will radiate at frequency $\omega = \omega_0$ the acoustic power

$$P_{\text{rad}}(\omega) = \text{Re}\int \mathbf{f}^*(\mathbf{k})\cdot\mathscr{G}_{22}(\mathbf{k}, \omega)\cdot\mathbf{f}(\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^3}, \tag{15}$$

where $\mathbf{f}(\mathbf{k})$ is the spatial Fourier transform of $\mathbf{f}(\mathbf{r})$. For an unbounded medium, wherein only longitudinal waves can be radiated, Eq. (15) reduces by Eqs. (8) to

$$P_{\text{rad}}(\omega) = \text{Re}\int G_{22L}(\mathbf{k}, \omega)|f_L(\mathbf{k})|^2\frac{d\mathbf{k}}{(2\pi)^3}, \tag{16}$$

where the subscript L denotes the longitudinal component. The example of a point harmonic force density $\mathbf{f}(\mathbf{r}) = \mathbf{F}\delta(\mathbf{r})$, of complex vector amplitude $\mathbf{F}$, leads by Eqs. (16) and (8) to a radiated power of the form

$$\begin{aligned} P_{\mathrm{rad}}(\omega) &= \int \pi \frac{\omega}{\gamma p_0} \delta\left(k^2 - \frac{\omega^2}{a^2}\right) |\mathbf{F}\cdot\mathbf{k}_0|^2 \frac{d\mathbf{k}}{(2\pi)^3} \\ &= \pi \frac{\omega}{\gamma p_0} \int_0^\infty \frac{k^2\, dk}{(2\pi)^2} \int_0^\pi \sin\theta\, d\theta \left[|F|^2 \cos^2\theta \frac{\delta(k - \omega/a)}{2k} \right] \\ &= \frac{1}{12\pi} \sqrt{\frac{1/\gamma p_0}{n_0 m}}\, |Fk_a|^2, \qquad k_a = \frac{\omega}{a}, \end{aligned} \tag{17}$$

where for the integration a spherical coordinate system is employed in which the polar angle θ is the angle between $\mathbf{F}$ and $\mathbf{k}_0$.

1.2b The Electromagnetic Field

As indicated in Eqs. (1.1.31) and (1.1.32), the electromagnetic fields in an unbounded region are derivable from a scalar Green's function defined by

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) g(\mathbf{r}, \mathbf{r}'; t, t') = -\delta(\mathbf{r} - \mathbf{r}')\delta(t - t') \tag{18}$$

subject to the boundary condition $g \equiv 0$ as $|\mathbf{r} - \mathbf{r}'| \to \infty$ at any finite time t, and to the initial condition $g \equiv 0$ for $t \leq t'$. Paralleling the discussion in Sec. 1.2a of the almost identical equation for the acoustic field, one observes that, in the $\mathbf{k}, \omega$ transform space, the solution of Eq. (18) is

$$g(\mathbf{k}, \omega) = \begin{cases} \dfrac{1}{k^2 - (\omega^2/c^2)}, & \operatorname{Im} \omega > 0, \\[2ex] P\dfrac{1}{k^2 - (\omega^2/c^2)} + \pi i \delta\left(k^2 - \dfrac{\omega^2}{c^2}\right), & \operatorname{Im} \omega = 0, \end{cases} \tag{19}$$

whence in a space–time representation [see note to Eq. (7)],

$$g(\mathbf{r}, t) = \begin{cases} \displaystyle\int \frac{e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}}{k^2 - (\omega^2/c^2)} \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} = \frac{\delta[t - (r/c)]}{4\pi r}, & t > 0, \\[2ex] 0, & t < 0, \end{cases} \tag{20}$$

where $\mathbf{r}, t$ hereafter denote $\mathbf{r} - \mathbf{r}', t - t'$, respectively. The electromagnetic Green's functions $\mathscr{G}_{11}(\mathbf{r}, t)$, $\mathscr{G}_{22}(\mathbf{r}, t)$, etc. are expressible, as shown in Eqs. (1.1.30)–(1.1.32), in terms of $g(\mathbf{r}, t)$. In $\mathbf{k}, \omega$ space these relations become, for $\operatorname{Im} \omega = 0$,

$$\begin{aligned} \mathscr{G}_{11}(\mathbf{k}, \omega) &= \frac{\mathbf{1}_L}{j\omega\epsilon_0} + \left[\pi\omega\mu_0\delta\left(k^2 - \frac{\omega^2}{c^2}\right) + P\frac{1}{j\omega\epsilon_0 + (k^2/j\omega\mu_0)}\right]\mathbf{1}_T, \\ \mathscr{G}_{22}(\mathbf{k}, \omega) &= \frac{\mathbf{1}_L}{j\omega\mu_0} + \left[\pi\omega\epsilon_0\delta\left(k^2 - \frac{\omega^2}{c^2}\right) + P\frac{1}{j\omega\mu_0 + (k^2/j\omega\epsilon_0)}\right]\mathbf{1}_T, \\ \mathscr{G}_{12}(\mathbf{k}, \omega) &= -\mathscr{G}_{21}(\mathbf{k}, \omega) = -j\mathbf{k} \times \mathbf{1}\left[-\pi j\delta\left(k^2 - \frac{\omega^2}{c^2}\right) + P\frac{1}{k^2 - (\omega^2/c^2)}\right], \end{aligned} \tag{21}$$

where the ("resistive and conductive") delta functions and P symbols are to be omitted for Im $\omega \neq 0$, where the longitudinal and transverse dyads $\mathbf{1}_L$ and $\mathbf{1}_T$, respectively, are defined in Eq. (8a), and where $j = -i$.†

The role played by the transformed Green's functions of Eqs. (21) in the description of the electromagnetic field becomes evident if one rewrites the Maxwell equations (1.1.16) in $\mathbf{k}, \omega$ space as

$$\begin{aligned} j\omega\epsilon_0 \mathbf{E}(\mathbf{k}, \omega) + j\mathbf{k} \times \mathbf{H}(\mathbf{k}, \omega) &= -\mathbf{J}(\mathbf{k}, \omega), \\ -j\mathbf{k} \times \mathbf{E}(\mathbf{k}, \omega) + j\omega\mu_0 \mathbf{H}(\mathbf{k}, \omega) &= -\mathbf{M}(\mathbf{k}, \omega). \end{aligned} \tag{22}$$

The solution of these transformed Maxwell equations is evidently the transform of the space- and time-dependent solution given by Eqs. (1.1.19). In view of the $\mathbf{r} - \mathbf{r}'$ and $t - t'$ dependence of the Green's function for a homogeneous, stationary, unbounded medium, the transform of Eqs. (1.1.19) yields, either by direct transformation or by the Fourier convolution theorem[6], the simple relation

$$\begin{aligned} \mathbf{E}(\mathbf{k}, \omega) &= -\mathscr{G}_{11}(\mathbf{k}, \omega)\cdot\mathbf{J}(\mathbf{k}, \omega) - \mathscr{G}_{12}(\mathbf{k}, \omega)\cdot\mathbf{M}(\mathbf{k}, \omega), \\ \mathbf{H}(\mathbf{k}, \omega) &= -\mathscr{G}_{21}(\mathbf{k}, \omega)\cdot\mathbf{J}(\mathbf{k}, \omega) - \mathscr{G}_{22}(\mathbf{k}, \omega)\cdot\mathbf{M}(\mathbf{k}, \omega). \end{aligned} \tag{23}$$

On decomposition of the vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{J}$, and $\mathbf{M}$ into components along the characteristic coordinates $\mathbf{k}_0$, $\mathbf{T}_0'$, $\mathbf{T}_0''$ shown in Fig. 1.2.1,

$$\mathbf{E}(\mathbf{k}, \omega) = E_L \mathbf{k}_0 + E_{T'}\mathbf{T}_0' + E_{T''}\mathbf{T}_0'', \text{ etc.},$$

Eqs. (22) separate into the longitudinal equations

$$j\omega\epsilon_0 E_L = -J_L, \qquad j\omega\mu_0 H_L = -M_L \tag{24a}$$

and the transverse equations

$$\begin{aligned} j\omega\epsilon_0 E_{T'} - k\hat{H}_{T''} &= -J_{T'}, & j\omega\epsilon_0 E_{T''} - k\hat{H}_{T'} &= -J_{T''}, \\ kE_{T'} + j\omega\mu_0 \hat{H}_{T''} &= -\hat{M}_{T''}, & kE_{T''} + j\omega\mu_0 \hat{H}_{T'} &= -\hat{M}_{T'}, \end{aligned} \tag{24b}$$

where

$$\hat{H}_{T''} \equiv +jH_{T''}, \quad \hat{M}_{T''} \equiv +jM_{T''}, \quad \hat{H}_{T'} \equiv -jH_{T'}, \quad \hat{M}_{T'} \equiv -jM_{T'}. \tag{24c}$$

The transformed electromagnetic equations (24) can be schematized by the network shown in Fig. 1.2.3, whose elements comprise "inductances" μ_0, "capacitances" ϵ_0, and ideal transformers with turns ratio $k:1$. The electric field components E_α ($\alpha = L, T', T''$) appear as "voltages" across the capacitances ϵ_0, the magnetic field components $\hat{H}_\alpha$ are "currents" through the inductances μ_0, and J_α and $\hat{M}_\alpha$ act as "applied currents" and "applied voltages." The resonant nature of the transverse circuits, and their independence or decoupling from the longitudinal circuits, pictorializes the transverse nature of the electromagnetic field.

Source-free electromagnetic fields in the form of plane waves $\exp[-j(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ are possible only for those values of $\mathbf{k}, \omega$ that admit non-vanishing solutions of the homogeneous ($\mathbf{J} = 0 = \mathbf{M}$) field equations (24). Thus, transverse-wave solutions of Eqs. (24b) are possible whenever

†Note that in the j notation the relevant half-plane Im $\omega > 0$ in Eqs. (19) becomes Im $\omega < 0$.

$$j\omega\epsilon_0 + \frac{k^2}{j\omega\mu_0} = 0 \tag{25a}$$

or

$$k^2 - \frac{\omega^2}{c^2} = 0, \tag{25b}$$

and longitudinal solutions only for $\omega = 0$. Equations (25) constitute a dispersion relation [determinant of Eqs. (24b) = 0] for electromagnetic waves, or equivalently a condition for resonance (total "admittance" at P equal 0) of the transverse networks illustrated in Fig. 1.2.3. From Eqs. (24b), or the network, one ascertains that the wave fields have the ratio (characteristic impedance)

$$\frac{E_{T'}}{H_{T''}} = \pm\sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{-E_{T''}}{H_{T'}}, \tag{26}$$

with $\pm$ signs distinguishing waves traveling in $\pm\mathbf{k}_0$ directions, respectively.

Steady-state power radiated by electric and magnetic currents in free space

Calculation of the electromagnetic energy radiated by electric or magnetic current sources in a non-dissipative unbounded region can be readily effected in a $\mathbf{k}, \omega$ basis. Since the power per unit volume supplied by a real electric

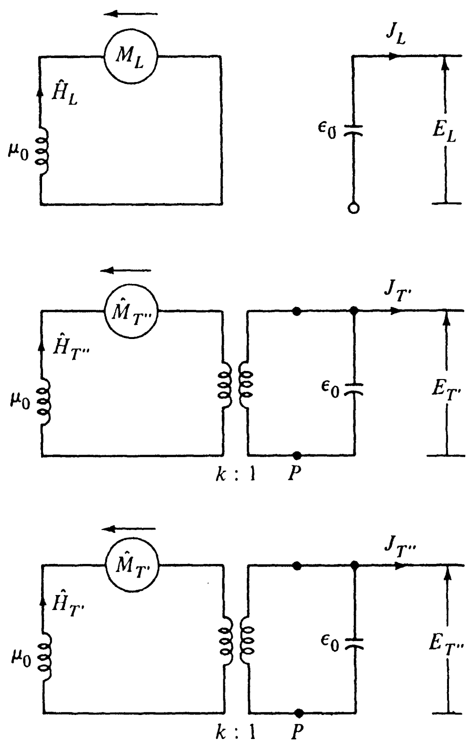


FIG. 1.2.3 Electromagnetic network.

current density $\mathbf{J}(\mathbf{r}, t)$ to the field at $\mathbf{r}, t$ is $-\mathbf{E}(\mathbf{r}, t)\cdot\mathbf{J}(\mathbf{r}, t)$, the total *energy* delivered to the field over an infinite time interval is

$$-\int \mathbf{E}(\mathbf{r}, t)\cdot\mathbf{J}(\mathbf{r}, t)\, d\mathbf{r}\, dt = -\int \mathbf{E}(\mathbf{k}, \omega)\cdot\mathbf{J}^*(\mathbf{k}, \omega)\frac{d\mathbf{k}\, d\omega}{(2\pi)^4}, \tag{27}$$

where, as in the analogous acoustic relation (13a), the right-hand member follows from the left by Parseval's theorem (Sec. 5.2d), or directly on use of Eqs. (1a), the analogue of Eqs. (2), and the observation that $\mathbf{J}(-\mathbf{k}, -\omega) = \mathbf{J}^*(\mathbf{k}, \omega)$ for real ω, $\mathbf{k}$. In the case of a harmonic current source of the form $\mathbf{J}(\mathbf{r})e^{j\omega_0 t}$, for which correspondingly $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})e^{j\omega_0 t}$, the average real *power* delivered to the field, and hence radiated at the frequency $\omega = \omega_0$, is [see Eq. (13a) at seq.]

$$P_{\text{rad}} = -\operatorname{Re}\int \mathbf{J}^*(\mathbf{k})\cdot\mathbf{E}(\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^3}, \tag{28a}$$

$$= \operatorname{Re}\int \mathbf{J}^*(\mathbf{k})\cdot\mathscr{G}_{11}(\mathbf{k}, \omega)\cdot\mathbf{J}(\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^3}, \tag{28b}$$

$$= \int R(\mathbf{k}, \omega)|J_T(\mathbf{k})|^2\frac{d\mathbf{k}}{(2\pi)^3}. \tag{28c}$$

$\mathbf{J}(\mathbf{k})$ and $\mathbf{E}(\mathbf{k})$ are the (root mean square) complex Fourier spatial transforms of $\mathbf{E}(\mathbf{r})$ and $\mathbf{J}(\mathbf{r})$. Equation (28b) follows by Eq. (23), and, as indicated in Eqs. (21), $\operatorname{Re}\mathscr{G}_{11}(\mathbf{k}, \omega) = R(\mathbf{k}, \omega)\mathbf{1}_T$ defines the real (resistive) component $R(\mathbf{k}, \omega)$ of the impedance dyadic.

For a point harmonic source of electric current density $\mathbf{J}(\mathbf{r}) = \mathbf{J}^0\delta(\mathbf{r}) = I\mathbf{l}\delta(\mathbf{r})$, where $\mathbf{J}^0 = I\mathbf{l}$ is the "vector current moment," the power radiated in the form of transverse waves is, by Eqs. (28c) and (21) [note comments under Eq. (14)],

$$\begin{aligned} P_{\text{rad}} &= \frac{\pi}{2}\sqrt{\frac{\mu_0}{\epsilon_0}}\int_0^\pi 2\pi\sin\theta\, d\theta\int_0^\infty \frac{k^2\, dk}{(2\pi)^3}|Il\sin\theta|^2\delta\left(k - \frac{\omega}{c}\right) \\ &= \frac{1}{6\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\,|Ik_0 l|^2, \qquad k_0 = \frac{\omega}{c}, \qquad l = |\mathbf{l}|, \end{aligned} \tag{29}$$

where the integration has been performed in a spherical coordinate system with volume element $d\mathbf{k} = 2\pi\sin\theta k^2\, d\theta\, dk$, θ being the angle between $\mathbf{k}$ and the source direction $\mathbf{l}$. For a point magnetic current source $\mathbf{M}(\mathbf{r}) = \mathbf{M}^0\delta(\mathbf{r}) = \hat{I}\mathbf{l}\delta(\mathbf{r})$, the average radiated power leads in a similar manner to the dual results

$$\begin{aligned} P_{\text{rad}} &= \operatorname{Re}\int \mathbf{M}^*(\mathbf{k})\cdot\mathscr{G}_{22}(\mathbf{k}, \omega)\cdot\mathbf{M}(\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^3} \\ &= \int G(\mathbf{k}, \omega)|\mathbf{M}_T(\mathbf{k})^2|\frac{d\mathbf{k}}{(2\pi)^3} \\ &= \frac{1}{6\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\,|\hat{I}k_0 l|^2, \qquad k_0 = \frac{\omega}{c}, \end{aligned} \tag{30}$$

where $G(\mathbf{k}, \omega)\mathbf{1}_T = \mathrm{Re}\,\mathscr{G}_{22}(\mathbf{k}, \omega)$ is the real (conductive) part of the admittance dyadic $\mathscr{G}_{22}$ given in Eq. (21).

1.2c The Plasma Field

For the one-component plasma field described in Sec. 1.1c, the overall Green's function problem, as defined in Eqs. (1.1.56) and (1.1.57), requires the evaluation of a large number of subsidiary Green's functions $G_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; t, t')$. Although for an isotropic ($\omega_c = 0$) plasma it is possible to relate all such Green's functions to two scalar Green's functions of the form given implicitly by Eqs. (5) and (18), we shall instead discuss the direct solution of the Green's function in Eqs. (1.1.57) for the special case $\omega_c = 0$. An identical but more complicated procedure exists for $\omega_c \neq 0$.

For an unbounded plasma medium, one introduces the $\mathbf{k}, \omega$ Green's function transform $G_{\alpha\beta}(\mathbf{k}, \omega)$ by

$$G_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; t, t') = \int G_{\alpha\beta}(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r}'' - \omega t'')} \frac{d\mathbf{k}\, d\omega}{(2\pi)^4}, \tag{31}$$

where $\mathbf{r}'' = \mathbf{r} - \mathbf{r}'$, $t'' = t - t'$. Hence in $\mathbf{k}, \omega$ space the defining equation (1.1.57) for the $G_{\alpha 1}$ becomes, when $\omega_c = 0$,

$$\begin{pmatrix} -i\omega\epsilon_0\mathbf{1} & -i\mathbf{k}\times\mathbf{1} & 0 & -n_0 q\mathbf{1} \\ +i\mathbf{k}\times\mathbf{1} & -i\omega\mu_0\mathbf{1} & 0 & 0 \\ 0 & 0 & -i\dfrac{\omega}{\gamma p_0} & i\mathbf{k} \\ n_0 q\mathbf{1} & 0 & i\mathbf{k} & -i\omega n_0 m\mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} \mathscr{G}_{11} \\ \mathscr{G}_{21} \\ \mathbf{G}_{31} \\ \mathscr{G}_{41} \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{32}$$

with similar equations for $G_{\alpha 2}$, $G_{\alpha 3}$, and $G_{\alpha 4}$, except that the unit source term on the right is in the second, third, and fourth row, respectively. Inverting Eq. (32), one can ascertain the various $G_{\alpha\beta}(\mathbf{k}, \omega)$. For instance (with $j = -i$ and $\mathrm{Im}\,\omega \neq 0$),

$$\begin{aligned} \mathscr{G}_{11}(\mathbf{k}, \omega) = {} & \frac{\mathbf{1}_L}{j\omega\epsilon_0 + \{n_0^2 q^2/[j\omega n_0 m + k^2/(j\omega/\gamma p_0)]\}} \\ & + \frac{\mathbf{1}_T}{j\omega\epsilon_0 + [k^2/(j\omega\mu_0)] + [n_0^2 q^2/(j\omega n_0 m)]}, \end{aligned} \tag{33}$$

where the longitudinal and transverse dyads $\mathbf{1}_L$ and $\mathbf{1}_T$ are defined in Eq. (8a). The physical significance of the transformed Green's functions $G_{\alpha\beta}(\mathbf{k}, \omega)$ is manifest on transformation of Eq. (1.1.56), whereby, for the case of a linear, homogeneous, stationary, unbounded medium, applicability of Eq. (1.1.22) and the convolution integral permits simple interpretations. In particular, it is apparent that $\mathscr{G}_{11}(\mathbf{k}, \omega)$ yields the electric field $\mathbf{E}(\mathbf{k}, \omega)$ excited only by an electric current density $\mathbf{J}(\mathbf{k}, \omega)$,

$$\mathbf{E}(\mathbf{k}, \omega) = -\mathscr{G}_{11}(\mathbf{k}, \omega)\cdot\mathbf{J}(\mathbf{k}, \omega), \tag{34}$$

whence one observes that $\mathscr{G}_{11}$ is a generalization of the electric Green's function $\mathscr{G}_{11}$ of Sec. 1.2b.

One can express Eq. (33) in an alternative form that makes explicit the dependence of the dyadic Green's function $\mathscr{G}_{11}$ on two scalar Green's functions. Thus, in the notation of Sec. 1.1c,

$$\mathscr{G}_{11}(\mathbf{k}, \omega) = j\omega\mu_0\left[\left(\mathbf{1} - \frac{k^2c^2}{\omega^2 - \omega_p^2}\mathbf{1}_L\right)G_{eo}(\mathbf{k}, \omega) + \frac{\omega_p^2}{\omega^2}\frac{k^2c^2}{\omega^2 - \omega_p^2}G_{ea}(\mathbf{k}, \omega)\mathbf{1}_L\right], \tag{35}$$

where the "optic-type" scalar Green's function transform G_{eo} is defined in the j notation by

$$G_{eo}(\mathbf{k}, \omega) = \begin{cases} \dfrac{1}{k^2 - [(\omega^2 - \omega_p^2)/c^2]}, & \operatorname{Im}\omega < 0, \\[2ex] P\dfrac{1}{k^2 - [(\omega^2 - \omega_p^2)/c^2]} - \pi j\delta\left(k^2 - \dfrac{\omega^2 - \omega_p^2}{c^2}\right), & \operatorname{Im}\omega = 0. \end{cases} \tag{35a}$$

The "acoustic-type" scalar Green's function G_{ea} is similarly defined but with c, the speed of light, replaced by a, the acoustic speed. The representation in Eq. (35a) of G_{eo} implies that the corresponding representation of the real (resistive) part of the dyadic Green's function $\mathscr{G}_{11}(\mathbf{k}, \omega)$ for $\operatorname{Im}\omega = 0$ is

$$\operatorname{Re}\mathscr{G}_{11}(\mathbf{k}, \omega) = \pi\omega\mu_0\left[\frac{\omega_p^2}{\omega^2}\frac{c^2}{a^2}\delta\left(k^2 - \frac{\omega^2 - \omega_p^2}{a^2}\right)\mathbf{1}_L + \delta\left(k^2 - \frac{\omega^2 - \omega_p^2}{c^2}\right)\mathbf{1}_T\right], \tag{36}$$

while the imaginary part is the principal part of that shown in Eq. (33).

The transformed plasma field equations (1.1.54) in $\mathbf{k}, \omega$ space, when expressed in terms of the components of $\mathbf{E}, \mathbf{H}, p, \mathbf{v}$ and $\mathbf{J}, \mathbf{M}, s, \mathbf{f}$ along the coordinate axes $\mathbf{k}_0, \mathbf{T}_0', \mathbf{T}_0''$ shown in Fig. 1.2.1, constitute 10 scalar equations for the 10 unknown field components of $\mathbf{E}, \mathbf{H}, p$, and $\mathbf{v}$. These equations are readily inferred from Eqs. (1.1.54) on use of the operator algebraizations $\partial/\partial t = j\omega$ and $\nabla = -j\mathbf{k}$. A network schematization of the resulting equations is shown in Fig. 1.2.4. This isotropic one-component plasma network, which represents a coupling of the electromagnetic and acoustic networks of Figs. 1.2.2 and 1.2.3, respectively, contains the same circuit elements as the latter networks with the addition of an ideal coupling transformer of turns ratio n_0q; the "voltages" and "currents" are similarly defined. The characteristic independence of the longitudinal and transverse networks of Fig. 1.2.4, together with their evident resonant nature, implies that both longitudinal and transverse waves can be propagated independently in an isotropic plasma.

Source-free plasma fields having the form of plane waves $\exp[-j(\mathbf{k}\cdot\mathbf{r}-\omega t)]$ are possible for those values of $\mathbf{k}, \omega$ for which there exist finite solutions of the $\mathbf{k}, \omega$ transformed equations (1.1.54) with $\mathbf{J} = \mathbf{M} = s = \mathbf{f} = 0$. The various possibilities are displayed by the zeros of the determinant of these equations or,

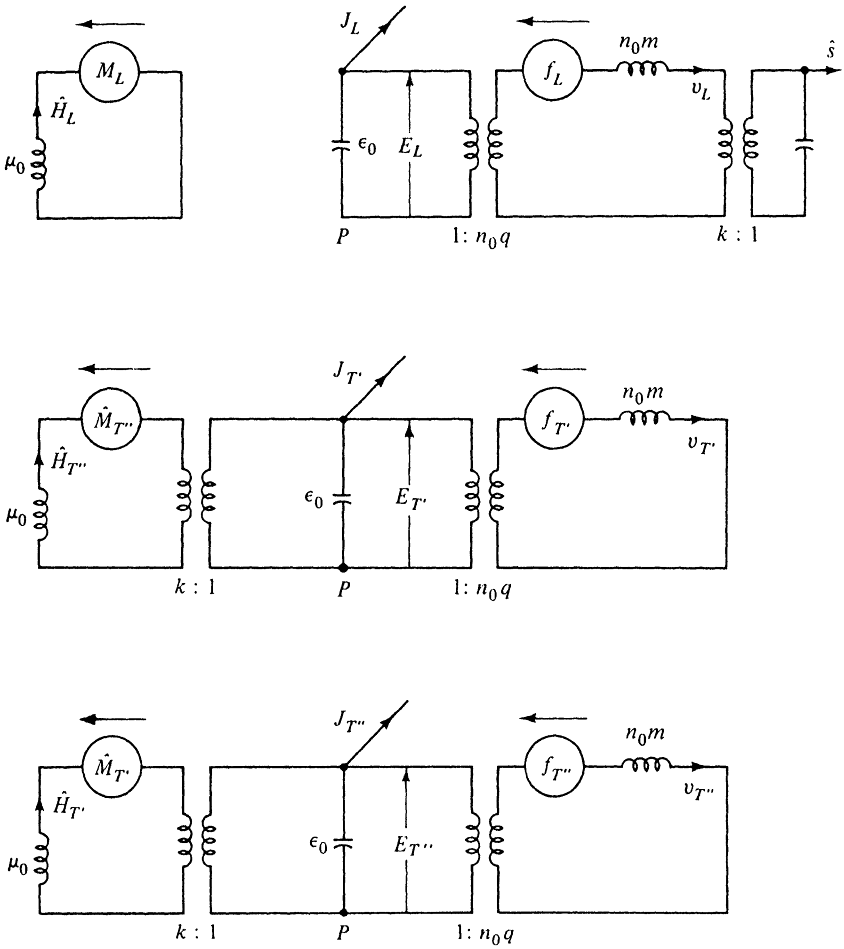


FIG. 1.2.4 One-component-plasma network.

equivalently, of the determinant of the matrix in Eq. (32). These yield the following dispersion relations:

$$j\omega\epsilon_0 + \frac{n_0^2q^2}{j\omega n_0 m + [k^2/(j\omega/\gamma p_0)]} = 0$$

or

$$k^2 - \frac{\omega^2 - \omega_p^2}{a^2} = 0 \tag{37a}$$

for the longitudinal waves, and

$$j\omega\epsilon_0 + \frac{k^2}{j\omega\mu_0} + \frac{n_0^2q^2}{j\omega n_0 m} = 0$$

or

$$k^2 - \frac{\omega^2 - \omega_p^2}{c^2} = 0 \tag{37b}$$

for the transverse waves. The dispersion relations (37) also characterize the pole singularities of the Green's function $\mathscr{G}_{11}$ of Eq. (33) and the zeros (resonances) of the total admittances at terminal plane P of the longitudinal and transverse networks of Fig. 1.2.4. The field-component ratios that characterize the field structure of these waves can be inferred from the transformed Eqs. (1.1.54), or from the resonant voltages and currents of the plasma network of Fig. 1.2.4, but will not be given explicitly at this point (see Chapter 8).

Steady-state power radiated by electric currents in an unbounded plasma

Calculation of the plasma field energy [i.e., the electromagnetic plus acoustic energy defined in Eq. (1.1.55)] radiated by sources in a plasma medium is relatively simple in $\mathbf{k}, \omega$ space. For example, in the non-dissipative case, the average real power delivered in electromagnetic form to the plasma by an electric current density $\mathbf{J}(\mathbf{r}) \exp(j\omega t)$ and thereby radiated as plasma field energy per unit time is [see Eqs. (28) and (35)]

$$\begin{aligned} P_{\text{rad}} &= -\text{Re} \int \mathbf{J}^*(\mathbf{k}) \cdot \mathbf{E}(\mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^3} \\ &= \text{Re} \int \mathbf{J}^*(\mathbf{k}) \cdot \mathscr{G}_{11}(\mathbf{k}, \omega) \cdot \mathbf{J}(\mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^3}, \end{aligned} \tag{38}$$

wherein $\mathbf{J}(\mathbf{k})$ and $\mathbf{E}(\mathbf{k})$ are (root mean square) spatial Fourier transforms of the harmonic electric current density and electric field, respectively. This result generalizes the purely electromagnetic relation (28), but it should be noted that the radiated plasma field power is both electromagnetic and acoustic [see Eq. (1.1.55)]. For a one-component isotropic plasma, $\text{Re}\, \mathscr{G}_{11}(\mathbf{k}, \omega)$ is given in Eq. (36).

The plasma power radiated into a one-component isotropic medium by a point electric current dipole of harmonic current density $\mathbf{J}(\mathbf{r}, t) = I\mathbf{l}\delta(\mathbf{r}) \exp(j\omega t)$ is calculated via Eqs. (38) and (36). Employing a spherical coordinate system in $\mathbf{k}$ space with $d\mathbf{k} = 2\pi k^2 \sin\theta \, d\theta \, dk$, and with θ as the polar angle between the vector $\mathbf{k}$ and the dipole direction $\mathbf{l}$, one obtains from Eqs. (38) and (36),

$$\begin{aligned} P_{\text{rad}} &= \pi\omega\mu_0 \int \left[\frac{\omega_p^2}{a^2} \frac{c^2}{a^2} \delta\left(k^2 - \frac{\omega^2 - \omega_p^2}{a^2}\right) |Il \cos\theta|^2 \right. \\ &\qquad \left. + \delta\left(k^2 - \frac{\omega^2 - \omega_p^2}{c^2}\right) |Il \sin\theta|^2 \right] 2\pi \sin\theta \frac{k^2 \, dk \, d\theta}{(2\pi)^3} \\ &= \frac{1}{6\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} |Ik_0 l|^2 \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \left[\frac{1}{2} \left(\frac{c}{a}\right)^3 \frac{\omega_p^2}{\omega^2} + 1 \right], \qquad \omega > \omega_p \\ P_{\text{rad}} &= 0, \qquad \omega < \omega_p \end{aligned} \tag{39}$$

where $k_0 = \omega/c$. The large first term in the brackets represents the portion of the power radiated in the form of longitudinal waves; the second term represents the transverse-wave contribution. For $\omega_p \to 0$, the expression in Eqs. (39) reduces to similar ones derived previously in Eq. (29) for the electromagnetic field.

The result in Eq. (39) is not realistic because of the assumed infinitesimal length of the radiating element. Results for a finite radiator can be obtained in a similar manner from Eq. (38) on use of an appropriate $\mathbf{J}(\mathbf{k})$.[8]

1.2d General Linear Field

Plane-wave representations exist for a general class of fields in a linear. homogeneous, stationary (time independent), and unbounded medium. As discussed in Sec. 1.1d, such a field is describable by a general linear field equation (1.1.65) and leads to a Green's function $G(\mathbf{r}, \mathbf{r}'; t, t')$ defined by

$$L\left(\nabla, \frac{\partial}{\partial t}\right)G(\mathbf{r}, \mathbf{r}'; t, t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \tag{40}$$

with G subject to a causality condition (outward propagation). Since, in general, L is representable by a square matrix whose elements are either scalar, vector, or dyadic operators, the Green's function G is likewise represented as a square matrix whose elements are subsidiary Green's functions of $\mathbf{r}$, $\mathbf{r}'$ and t, t'. A unit matrix is implicit in the right-hand delta-function term of Eq. (40). The invariance of Eq. (40) under arbitrary space–time displacements, manifest in the independence of L on the coordinates $\mathbf{r}$, t, implies the existence of plane-wave representations as in Eqs. (1) and leads via the property (4) to the transformed equation

$$L(\mathbf{k}, \omega)G(\mathbf{k}, \omega) = 1 \tag{41}$$

in $\mathbf{k}, \omega$ space. As in Eq. (3), $G(\mathbf{k}, \omega)$ is the Fourier–Laplace transform of the Green's function, and for simplicity we denote $L(\mathbf{k}, \omega) \equiv L(i\mathbf{k}, -i\omega)$.

The significance of $G(\mathbf{k}, \omega)$ is evident from the transform of the Green's function field relationship (1.1.73), which constitutes the solution of the general linear field equation (1.1.65). Thus, on use of the convolution theorem, one obtains

$$\Psi(\mathbf{k}, \omega) = -G(\mathbf{k}, \omega)\Phi(\mathbf{k}, \omega), \tag{42}$$

where $\Psi(\mathbf{k}, \omega)$ and $\Phi(\mathbf{k}, \omega)$, as in Eq. (1b), are the Fourier–Laplace transforms of $\Psi(\mathbf{r}, t)$ and $\Phi(\mathbf{r}, t)$. It is to be noted that $G(\mathbf{k}, \omega)$ is representable by a square matrix, with algebraic elements, given explicitly via Eq. (41) as the inverse

$$G(\mathbf{k}, \omega) = L^{-1}(\mathbf{k}, \omega) \tag{43}$$

of the known matrix $L(\mathbf{k}, \omega)$. The singularity properties of $G(\mathbf{k}, \omega)$ in the complex $\mathbf{k}, \omega$ planes determine the dispersion properties of plane waves characteristic of the source-free fields within the given medium. For example, singularities of $G(\mathbf{k}, \omega)$ occur at those values of $\mathbf{k}$, ω for which

$$\det L(\mathbf{k}, \omega) = 0. \tag{44}$$

Explicit knowledge of the transformed Green's function $G(\mathbf{k}, \omega)$ from Eq. (43) permits by Eq. (1a) the determination of the space- and time-dependent Green's function of Eq. (40) as

$$G(\mathbf{r}, \mathbf{r}'; t, t') = \int L^{-1}(\mathbf{k}, \omega) e^{i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]} \frac{d\mathbf{k}\, d\omega}{(2\pi)^4}. \tag{45}$$

By Eq. (42) the corresponding solution of the general linear field problem in Eq. (1.1.65) becomes

$$\Psi(\mathbf{r}, t) = -\int L^{-1}(\mathbf{k}, \omega)\Phi(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \frac{d\mathbf{k}\, d\omega}{(2\pi)^4}. \tag{46}$$

Although Eq. (46) provides a formal solution of the field problem in Eq. (1.1.65), it requires explicit evaluation of the product $L^{-1}\Phi$ and a fourfold integration over $\mathbf{k}$ and ω.

In many instances the above-indicated procedure can be simplified considerably by utilizing, ab initio, symmetry properties, with respect to a time or space direction, of the unbounded region under consideration. These simplifications, discussed in Secs. 1.3 and 1.4, are also applicable to (stationary, homogeneous) bounded regions.

1.3 GUIDED-WAVE (OSCILLATORY) REPRESENTATIONS IN TIME

As described in Sec. 1.2, the plane waves $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$, $-\infty < (k_x, k_y, k_z, \omega) < \infty$, possess four-dimensional space–time orthogonality properties characteristic of the spatial and temporal symmetry properties of an unbounded, linear, homogeneous, stationary medium. However, since ω and $\mathbf{k}$ are as yet unrelated, these plane waves do not represent solutions of the source-free field equations, nor are they characteristic for the polarization structure of the field. Such plane waves permit a complete algebraization of the space–time operators in a linear field and thereby reduce a field problem in an unbounded region to one of simple algebraic matrix inversion and evaluation of four-dimensional Fourier transforms. The time-guided fields, $\Psi_\alpha(\mathbf{k}) \exp[i\mathbf{k}\cdot\mathbf{r} - \omega_\alpha(\mathbf{k})t]$, $-\infty < k_x, k_y, k_z < \infty$, in this section have orthogonality properties characteristic both of the three-dimensional spatial volume containing the field and of the polarization structure (i.e., they are characteristic solutions of the source-free field equations, but they are not orthogonal along the time coordinate). Guided waves in time permit a representation of a linear field in terms of characteristic oscillations, require evaluation of only three-dimensional Fourier transforms, and are useful for solving radiation and initial-value problems. Such representations may also be generalized to accommodate media whose constitutive parameters vary with time.

In an unbounded region the mathematical basis for these guided-wave representations is provided by two transform theorems. The three-dimensional Fourier integral theorem yields a representation of an integrable wave function $\Psi(\mathbf{r}, t)$ in the form†

†Note that functions and their transforms are represented by the same symbols and distinguished, where necessary, by their arguments.

$$\Psi(\mathbf{r}, t) = \int \Psi(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3}, \tag{1a}$$

where

$$\Psi(\mathbf{k}, t) = \int \Psi(\mathbf{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}, \tag{1b}$$

and, as in the previous section, $d\mathbf{k}$ and $d\mathbf{r}$ denote coordinate-independent volume elements in the $\mathbf{k}$ and $\mathbf{r}$ spaces, respectively. In a homogeneous bounded space, eigenfunctions $\Phi(\mathbf{r})$ rather than $\exp(i\mathbf{k}\cdot\mathbf{r})$ would appear in Eqs. (1). Independently of the function Ψ, the Fourier transforms in Eqs. (1a) and (1b) can be combined into a "completeness relation," or three-dimensional delta-function representation, as

$$\delta(\mathbf{r} - \mathbf{r}') = \int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{d\mathbf{k}}{(2\pi)^3}. \tag{2a}$$

To assure completeness, the integration is extended over all real points in the three-dimensional $\mathbf{k}$ space. Correspondingly, any integrable function of $\mathbf{r} - \mathbf{r}'$, such as a Green's function $G(\mathbf{r}, \mathbf{r}'; t, t')$ in an unbounded homogeneous medium, is representable as

$$G(\mathbf{r}, \mathbf{r}'; t, t') = \int G(\mathbf{k}; t, t') e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{d\mathbf{k}}{(2\pi)^3}. \tag{2b}$$

One also infers from the transform relations (1) the "orthogonality" property,

$$(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') = \int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}\, d\mathbf{r}. \tag{2c}$$

In Eqs. (1), the transformed wavefunction $\Psi(\mathbf{k}, t)$, which is a one-column matrix or vector in a finite (n)-dimensional polarization space, can be further represented for given $\mathbf{k}$ as a superposition of eigenvectors $\Psi_\alpha(\mathbf{k})$:

$$\Psi(\mathbf{k}, t) = \sum_\alpha \Psi_\alpha(\mathbf{k}) a_\alpha(\mathbf{k}, t), \qquad \alpha = 1, 2, \ldots, n, \tag{3a}$$

where the coefficients a_α are given in general by the weighted, Hermitean (i.e., complex-conjugate) scalar product of the vector Ψ and an adjoint vector Ψ_α^+ as

$$a_\alpha(\mathbf{k}, t) = \frac{(W^+ \Psi_\alpha^+(\mathbf{k}), \Psi(\mathbf{k}, t))}{2N_\alpha(\mathbf{k})}, \tag{3b}$$

with W^+ as the "weighting" operator. For given $\mathbf{k}$ the Ψ_α and Ψ_α^+ comprise a complete biorthogonal set of base vectors (in polarization space) with the properties

$$(W^+ \Psi_\alpha^+, \Psi_\beta) = 2N_\alpha \delta_{\alpha\beta} = \begin{cases} 2N_\alpha, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases} \tag{3c}$$

N_α being a normalization constant. The transforms (3a) and (3b) may be synthesized, independently of $\Psi(\mathbf{k}, t)$, into a completeness relation for the polarization space,

$$1 = \sum_{\alpha} \frac{\Psi_\alpha(\mathbf{k})\, W^+ \Psi_\alpha^+(\mathbf{k})}{2N_\alpha(\mathbf{k})}, \tag{4}$$

where the unit operator 1 on the left is represented by a unit matrix in polarization space.

The transforms in Eqs. (1) and (3) provide a representation of the wavefunction $\Psi(\mathbf{r}, t)$ in both geometrical and polarization spaces as

$$\Psi(\mathbf{r}, t) = \int \sum_{\alpha} a_\alpha(\mathbf{k}, t)\Psi_\alpha(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3}, \tag{5a}$$

where

$$a_\alpha(\mathbf{k}, t) = \int \frac{(W^+\Psi^+(\mathbf{k}), \Psi(\mathbf{r}, t))}{2N_\alpha(\mathbf{k})} e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}. \tag{5b}$$

Alternative to these transforms is the completeness relation,

$$1\delta(\mathbf{r} - \mathbf{r}') = \int \sum_{\alpha} \frac{\Psi_\alpha(\mathbf{k})\, W^+ \Psi_\alpha^+(\mathbf{k})}{2N_\alpha(\mathbf{k})} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{d\mathbf{k}}{(2\pi)^3}, \tag{6}$$

from which one observes that the $\Psi_\alpha(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$ constitute a complete set of characteristic fields for the representation of solutions to a general linear field problem (see Secs. 1.1d and 1.2d) in an unbounded medium. In particular, one infers from Eq. (6) that a general (matrix) Green's function for a linear field is representable as

$$G(\mathbf{r}, \mathbf{r}'; t, t') = \int \sum_{\alpha} G_\alpha(\mathbf{k}; t, t') \frac{\Psi_\alpha(\mathbf{k})\, \Psi_\alpha^+(\mathbf{k})}{2N_\alpha(\mathbf{k})} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{d\mathbf{k}}{(2\pi)^3}, \tag{7}$$

where the coefficients G_α are to be determined from the equation defining G, as we shall illustrate below.

1.3a General Linear Field

A general field in a linear, homogeneous, and stationary medium is described, as in Sec. 1.1d, by a linear operator $L(\nabla, \partial/\partial t)$. If the operator L is decomposed into two components, one depending only on $\partial/\partial t$, the other only on ∇, the defining equation of the Green's function in Eq. (1.1.73a) can be put into the form

$$i\left[M(\nabla) + \frac{W}{i}\frac{\partial}{\partial t}\right]G(\mathbf{r}, \mathbf{r}'; t, t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \tag{8}$$

subject to the initial condition $G = 0$ for $t \leq t'$, and to appropriate boundary conditions. As noted in Sec. 1.1d, the Green's function G is in general representable as a matrix whose elements may be scalar, vector, or dyadic functions. Examples of the component operators M and W can be inferred from the $L = iM + W(\partial/\partial t)$ operators listed in Sec. 1.1d.

A representation of the Green's function G in the form shown in Eq. (7) requires the knowledge of characteristic vectors Ψ_α. To ascertain the latter one seeks solutions of the source-free field equation $L\Psi = 0$ with a time dependence

$\exp(-i\omega_\alpha t)$. In view of the representation of the operator L in Eq. (8), the eigenvectors Ψ_α and the associated eigenfrequencies ω_α are evidently described in coordinate space by

$$M(\nabla)\Psi_\alpha(\mathbf{r}) = \omega_\alpha W\Psi_\alpha(\mathbf{r}). \tag{9}$$

For a translationally invariant, unbounded medium, one can choose $\Psi_\alpha(\mathbf{r}) = \Psi_\alpha(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$,† and hence the eigenvalue problem in Eq. (9) takes the algebraic form

$$M(\mathbf{k})\Psi_\alpha(\mathbf{k}) = \omega_\alpha(\mathbf{k})W\Psi_\alpha(\mathbf{k}), \tag{10a}$$

where $M(\mathbf{k}) \equiv M(i\mathbf{k})$ is a matrix operator with algebraic elements and $\Psi_\alpha(\mathbf{k})$ is a multicomponent vector in polarization space. The corresponding adjoint eigenvalue problem,‡

$$M^+(\mathbf{k})\Psi_\alpha^+(\mathbf{k}) = \omega_\alpha^*(\mathbf{k})W^+\Psi_\alpha^+(\mathbf{k}) \tag{10b}$$

is associated with the conjugate eigenvalue ω_α^* and with adjoint operators M^+ and W^+, whose off-diagonal elements are transposed conjugates to those of M and W. By means of Hermitean products of vectors in polarization space, one forms from Eqs. (10) a relation connecting the solutions Ψ_α^+ of Eq. (10b) and Ψ_β of Eq. (10a),

$$(M^+\Psi_\alpha^+, \Psi_\beta) - (\Psi_\alpha^+, M\Psi_\beta) = \omega_\alpha(W^+\Psi_\alpha^+, \Psi_\beta) - \omega_\beta(\Psi_\alpha^+, W\Psi_\beta). \tag{10c}$$

Since by definition of the adjoint operators [see Eqs. (1.1.69)] the Hermitean products on both the left and right are equal, one derives the biorthogonality properties,

$$(\Psi_\alpha^+, W\Psi_\beta) = 2N_\alpha\delta_{\alpha\beta}, \tag{11}$$

which relation has already been noted in Eq. (3c). The associated completeness relation in the polarization space is

$$1 = \sum_\alpha \frac{\Psi_\alpha(\mathbf{k})\, W^+\Psi_\alpha^+(\mathbf{k})}{2N_\alpha(\mathbf{k})}, \tag{12}$$

as anticipated in Eq. (4). For the special case $M^+ = M$ and $W^+ = W$, one observes that $\Psi_\alpha^+ = \Psi_{\alpha^*}$, where Ψ_{α^*} is a solution of Eq. (10a) corresponding to the eigenvalue $\omega_\alpha = \omega_\alpha^*$.

In the $\mathbf{k}$-transform space the Green's function equation (8) becomes, via the representation in Eq. (2b), the ordinary matrix differential equation,

$$\left[M(\mathbf{k}) + \frac{W}{i}\frac{d}{dt}\right]G(\mathbf{k}; t, t') = \frac{\delta(t-t')}{i}. \tag{13}$$

In view of the completeness relation (12), the transformed matrix Green's function $G(\mathbf{k}; t, t')$ can be represented as

†For notational simplicity the same symbol is employed for $\Psi_\alpha(\mathbf{r})$ and $\Psi_\alpha(\mathbf{k})$. The latter occurs most frequently and will usually be abbreviated as Ψ_α.

‡From Eq. (10c) one infers that the adjoint eigenvalue $\omega_\alpha^+ = \omega_\alpha^*$ provided there exists a Ψ_α and Ψ_α^+ such that $(W^+\Psi_\alpha^+, \Psi_\alpha) \neq 0$.

$$G(\mathbf{k}; t, t') = \sum_\alpha \frac{\Psi_\alpha(\mathbf{k})}{2N_\alpha}(\Psi_\alpha^+(\mathbf{k}), WG(\mathbf{k}; t, t'))$$
$$= \sum_\alpha \frac{\Psi_\alpha(\mathbf{k})\Psi_\alpha^+(\mathbf{k})}{2N_\alpha} G_\alpha(\mathbf{k}; t, t'), \tag{14}$$

where the G_α are scalar coefficients defined by $\Psi_\alpha^+ G_\alpha = (\Psi_\alpha^+, WG)$. Substitution of Eq. (14) into Eq. (2b) then provides a representation of the space- and time-dependent Green's function

$$G(\mathbf{r}, \mathbf{r}'; t, t') = \int \sum_\alpha G_\alpha(\mathbf{k}; t, t') \frac{\Psi_\alpha(\mathbf{k})\Psi_\alpha^+(\mathbf{k})}{2N_\alpha} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{d\mathbf{k}}{(2\pi)^3}, \tag{15}$$

as in Eq. (7). To determine the coefficients G_α, one forms the Hermitean product of both sides of Eq. (13) with $\Psi_\alpha^+(\mathbf{k})$, and using Eq. (14) obtains as the defining equation for G_α,

$$\left[\frac{d}{dt} + i\omega_\alpha(\mathbf{k})\right] G_\alpha(\mathbf{k}; t, t') = \delta(t - t')$$

subject to the initial condition $G_\alpha = 0$ for $t \leq t'$. Since $(d/dt)U(t - t') = \delta(t - t')$, where U is the Heaviside unit function, its solution is

$$G_\alpha(\mathbf{k}; t, t') = \begin{cases} e^{-i\omega_\alpha(\mathbf{k})(t-t')} & \text{for } t > t', \\ 0 & \text{for } t < t'. \end{cases} \tag{16}$$

Hence, from Eq. (15), one infers that the oscillatory space- and time-dependent Green's function has the form

$$G(\mathbf{r}, \mathbf{r}'; t, t') = \begin{cases} \displaystyle\int \sum_\alpha \frac{\Psi_\alpha(\mathbf{k})\Psi_\alpha^+(\mathbf{k})}{2N_\alpha(\mathbf{k})} e^{i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - \omega_\alpha(\mathbf{k})(t-t')]} \frac{d\mathbf{k}}{(2\pi)^3}, & t > t', \\ 0, & t < t'. \end{cases} \tag{17}$$

It should be noted that Eq. (17) also may be obtained from Eq. (1.2.45) by contour integration in the ω plane.

Equation (17) is the desired oscillatory representation of the Green's function of a linear field and satisfies the causality requirement of vanishing before a certain time and being outward going thereafter. Each of the component oscillations in the representation is a plane wave that can be viewed as either outward or inward going, depending on the point $\mathbf{r}$ of observation. The presence of both outgoing and ingoing plane waves in the representation of an overall outward radiating field, although correct, is somewhat redundant and raises the question as to whether it is possible to identify and eliminate the ingoing contributions.

A plane-wave oscillation of wavenumber $\mathbf{k}$ has the form $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t)]$, with the frequency $\omega(\mathbf{k})$ either positive or negative because of the time reversibility of the plane-wave fields under consideration. At a fixed observation point $\mathbf{r}$, a positive $\omega(\mathbf{k})$ implies the plane-wave oscillation is outgoing in the $\mathbf{r}$ direction if $\mathbf{k}$ is such that $\mathbf{k}\cdot\mathbf{r} > 0$ and ingoing if $\mathbf{k}\cdot\mathbf{r} < 0$; a converse statement is made if $\omega(\mathbf{k})$ is negative. These observations permit one to identify an outgoing wave at $\mathbf{r}$ by a frequency $\omega_+(\mathbf{k})$ which is positive if $\mathbf{k}\cdot\mathbf{r} > 0$ and negative if $\mathbf{k}\cdot\mathbf{r} < 0$; a frequency $\omega_-(\mathbf{k})$, defined as negative if $\mathbf{k}\cdot\mathbf{r} > 0$ and positive if $\mathbf{k}\cdot\mathbf{r} < 0$, distinguishes an ingoing wave at $\mathbf{r}$. Such frequency definitions are characterized by the property $\omega_\pm(\mathbf{k}) = -\omega_\pm(-\mathbf{k})$.

With the $\omega_\pm(\mathbf{k})$ identification of outgoing and ingoing wave contributions at $\mathbf{r}$ it is possible to obtain a representation alternative to that in Eq. (17) by subtracting therefrom the ingoing wave contributions. Distinguishing the $\omega_\pm(\mathbf{k})$ oscillations in the α sum of Eq. (17) by corresponding subscripts $\alpha_\pm$, one subtracts the ingoing α_- oscillations which satisfy the source-free equation (9). This subtraction must be carried out for all t in order not to introduce a singularity into G and thereby violate Eq. (8) [i.e., the subtracted part must satisfy the homogeneous equation (9)]. In this subtraction process, contributions from modes with $\omega_\alpha = 0$ (if they exist) are weighted by a factor of $\frac{1}{2}$, since these modes are, so to speak, both inward and outward going at $\mathbf{r}$. The resulting Green's function is no longer causal, since it is now finite for $t < t'$ in contrast to that in Eq. (17). For $t > t'$, however, it is identical with the causal Green's function in Eq. (17) and moreover contains only outgoing α_+ contributions. Performing the subtraction, one obtains as an alternative (outgoing) oscillatory representation of the Green's function of the linear field

$$G(\mathbf{r}, \mathbf{r}'; t, t') = \int \sum_{\alpha_+} \frac{\Psi_\alpha(\mathbf{k})\Psi_\alpha^+(\mathbf{k})}{2\epsilon_\alpha N_\alpha(\mathbf{k})} e^{i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - \omega_\alpha(\mathbf{k})(t-t')]} \frac{d\mathbf{k}}{(2\pi)^3}, \qquad t > t', \tag{18}$$

where

$$\epsilon_\alpha = \begin{cases} 2 & \text{for } \omega_\alpha = 0, \\ 1 & \text{for } \omega_\alpha \neq 0. \end{cases}$$

The representation in Eq. (18) is somewhat simpler to evaluate than that in Eq. (17) because of the omission of α_- contributions. Although Eqs. (17) and (18) yield identical results for $t > t'$, the representation in Eq. (18) suffers from the presence of a singularity at $t = t'$ [see the example of Eq. (25b)] which arises from the discontinuous $\mathbf{k}$ dependence of the $\omega_\pm(\mathbf{k})$ frequency definitions. This singularity is not troublesome if the Green's function representation in Eq. (18) is employed to represent fields produced by sources that vanish before the time t of observation of the field; a spurious contribution is introduced if the source acts beyond the observation time, and this may be difficult to identify. A somewhat similar singularity at certain z arises in connection with Hertz potential representations, as noted in Sec. 1.1c and at the end of Sec. 1.3c.

The representations in Eqs. (17) and (18) are in a more useful form than that in (1.2.45), not only because they require one less integration but also because the $\Psi_\alpha(\mathbf{k})$ are simpler to obtain than the elements of the inverse matrix $L^{-1}(\mathbf{k}, \omega)$. In the following we shall determine the oscillatory (plane-wave) eigenvectors $\Psi_\alpha(\mathbf{k})$ and eigenfrequencies $\omega_\alpha(\mathbf{k})$ for a number of linear fields. The resulting information permits one to obtain oscillatory representations of

Green's functions for these fields, in the form of Eqs. (17) or (18), with the explicit space–time dependence determined by integration over $\mathbf{k}$. We shall normally define the eigenfrequencies $\omega(\mathbf{k})$ as continuously dependent on $\mathbf{k}$; the discontinuously defined $\omega_{\pm}(\mathbf{k})$ frequencies shall only be employed in connection with the representation in Eq. (18).

1.3b The Acoustic Field

For the homogeneous linear acoustic field of Sec. 1.1, the oscillatory eigenvalue problem in Eq. (9), which defines source-free solutions with harmonic time dependence $\exp(-i\omega t)$, has the form

$$\begin{aligned} \nabla\cdot\mathbf{v} &= \frac{i\omega}{\gamma p_0}\,p, \\ \nabla p &= i\omega n_0 m\mathbf{v} \end{aligned} \tag{19}$$

where $p = p(\mathbf{r}, \omega)$ and $\mathbf{v} = \mathbf{v}(\mathbf{r}, \omega)$ denote the complex amplitudes of pressure and velocity at the point $\mathbf{r}$, and as in Eq. (1.1.1), $n_0 m$ and $1/\gamma p_0$ are the acoustic parameters. In an unbounded region the steady-state oscillatory solutions of Eqs. (19) are representable as

$$\begin{pmatrix} p(\mathbf{r}, \omega) \\ \mathbf{v}(\mathbf{r}, \omega) \end{pmatrix} = \begin{pmatrix} p(\mathbf{k}, \omega) \\ \mathbf{v}(\mathbf{k}, \omega) \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{20a}$$

where for notational simplicity, the same symbols p and $\mathbf{v}$ are employed for the $\mathbf{r}$- and $\mathbf{k}$-dependent fields. In the wavevector notation of the previous section, Eq. (20a) becomes

$$\Psi(\mathbf{r}, \omega) = \Psi(\mathbf{k}, \omega)e^{i\mathbf{k}\cdot\mathbf{r}} \tag{20b}$$

with the associated operators $M = M^+$ and $W = W^+$ of Eqs. (9) and (10) following from Eqs. (19) as

$$iM \rightarrow \begin{pmatrix} 0 & \nabla \\ \nabla & 0 \end{pmatrix}, \qquad W \rightarrow \begin{pmatrix} \dfrac{1}{\gamma p_0} & 0 \\ 0 & n_0 m\mathbf{1} \end{pmatrix}. \tag{21}$$

In Eq. (21) the 11 elements of M and W are recognized as scalars, the 12 and 21 elements as vectors, and the 22 elements are dyadics.

One infers from Eqs. (19), or Eqs. (10a) and (21), that eigenoscillations of the form (20) exist only for frequencies $\omega_\alpha(\mathbf{k})$ corresponding to the vanishing of $\det[M(\mathbf{k}) - \omega_\alpha(\mathbf{k})W]$, wherein one notes that $\nabla \equiv i\mathbf{k}$. There are four such frequencies, ordered as follows:

$$\omega_{\pm 1} = \pm ka, \qquad \omega_2 = 0, \quad \omega_3 = 0, \tag{22}$$

where $k = |\mathbf{k}|$ and $a = (\gamma p_0/n_0 m)^{1/2}$ is the acoustic speed. The frequencies (22) are seen to be the source-free resonant frequencies of the acoustic network depicted in Fig. 1.2.2. In the $\mathbf{k}_0, \mathbf{T}_0', \mathbf{T}_0''$ basis shown in Fig. 1.2.1, the eigenvector solutions $\Psi_\alpha(\mathbf{k}, \omega) \equiv \Psi_\alpha = \Psi_\alpha^+$ of Eqs. (19), corresponding to the eigenfrequencies ω_α noted in Eq. (22), are

$$\Psi_{\pm 1} \to \begin{pmatrix} \sqrt{\gamma p_0} \\ \pm \dfrac{\mathbf{k}_0}{\sqrt{n_0 m}} \end{pmatrix}, \qquad \Psi_2 \to \begin{pmatrix} 0 \\ \dfrac{\mathbf{T}_0'}{\sqrt{n_0 m}} \end{pmatrix}, \qquad \Psi_3 \to \begin{pmatrix} 0 \\ \dfrac{\mathbf{T}_0''}{\sqrt{n_0 m}} \end{pmatrix}, \tag{23a}$$

with orthonormality properties as in Eq. (11) defined in terms of the normalization constants

$$N_{\pm 1} = 1, \qquad 2N_2 = 1, \qquad 2N_3 = 1. \tag{23b}$$

Oscillatory representation of acoustic Green's function

The time-dependent (matrix) Green's function for the acoustic field may be evaluated from the oscillatory representation in either Eq. (17) or (18). Equation (17) includes both outgoing and ingoing waves and yields, with $\mathbf{r} - \mathbf{r}'$ and $t - t'$ denoted for simplicity by $\mathbf{r}$ and t, for $t > 0$:

$$G(\mathbf{r}; t) = \int \left(\frac{\Psi_1 \Psi_1}{2} e^{-ikat} + \frac{\Psi_{-1}\Psi_{-1}}{2} e^{+ikat} + \Psi_2\Psi_2 + \Psi_3\Psi_3 \right) e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3}, \tag{24a}$$

whereas Eq. (18), with only outgoing waves, yields for $t > 0$,

$$G(\mathbf{r}; t) = \int \frac{1}{2} (\Psi_1\Psi_1 e^{-i\omega_{+1}t} + \Psi_2\Psi_2 + \Psi_3\Psi_3) e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3}, \tag{24b}$$

with $\omega_\pm$ definitions noted in the discussion preceding Eq. (18). On substitution of the appropriate eigenfrequencies and eigenvectors from Eqs. (22) and (23) into Eq. (24a), one obtains for the 11 matrix element in successive steps

$$\begin{aligned} G_{11}(\mathbf{r}; t) &= \int \frac{\gamma p_0}{2} (e^{-i\omega_{+1}t} + e^{-i\omega_{-1}t}) e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3} \\ &= n_0 m \frac{\partial}{\partial t} \int (e^{-ikat} - e^{ikat}) e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{k}a}{-2ik(2\pi)^3}. \end{aligned}$$

On doing the θ and φ integrations in a polar coordinate system wherein $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$ and $d\mathbf{k} = k^2 \sin\theta \, d\theta \, d\varphi \, dk$, one finds

$$\begin{aligned} G_{11}(\mathbf{r}; t) &= n_0 m \frac{\partial}{\partial t} \int_0^\infty [e^{ik(r-at)} + e^{-ik(r-at)} - e^{ik(r+at)} - e^{-ik(r+at)}] \frac{d(ka)}{8\pi^2 r} \\ &= n_0 m \frac{\partial}{\partial t} \left[\frac{\delta[t - (r/a)]}{4\pi r} - \frac{\delta[t + (r/a)]}{4\pi r} \right] \\ &= n_0 m \frac{\partial}{\partial t} \frac{\delta[t - (r/a)]}{4\pi r} \qquad \text{for } t > 0. \end{aligned} \tag{25a}$$

Note in Eq. (25a) that the $\delta[t + (r/a)]$ term, contributed by the ingoing waves in the representation, vanishes for $t > 0$. For the alternative representation in Eq. (24b), involving only outgoing waves, one employs a similar procedure in the integration over the polar angle θ, but with the $\omega_\pm(\mathbf{k})$ definitions discussed in connection with Eq. (18); one notes that the frequency $\omega_{+1}(\mathbf{k})$ becomes $\omega_{+1} = ka$ for $0 < \theta < \pi/2$ and $\omega_{+1} = -ka$ for $\pi/2 < \theta < \pi$. Thus, one deduces from Eq. (24b) the same result as in Eq. (25a):

$$
\begin{aligned}
G_{11}(\mathbf{r}; t) &= \int \frac{\gamma p_0}{2} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{+1} t)} \frac{d\mathbf{k}}{(2\pi)^3} \\
&= \frac{\gamma p_0}{2}\left[\int_0^{\pi/2} e^{i(kr\cos\theta - kat)} \frac{k^2 \sin\theta \, d\theta \, dk}{(2\pi)^2} + \int_{\pi/2}^{\pi} e^{i(kr\cos\theta + kat)} \frac{k^2 \sin\theta \, d\theta \, dk}{(2\pi)^2}\right] \\
&= n_0 m \frac{\partial}{\partial t}\int_0^\infty [e^{ik(r-at)} + e^{-ik(r-at)} - e^{-ikat} - e^{ikat}] \frac{d(ka)}{8\pi^2 r} \\
&= n_0 m \frac{\partial}{\partial t}\left[\frac{\delta[t-(r/a)]}{4\pi r} - \frac{\delta(t)}{4\pi r}\right] \\
&= n_0 m \frac{\partial}{\partial t}\frac{\delta[t-(r/a)]}{4\pi r} \qquad \text{for } t > 0;
\end{aligned} \tag{25b}
$$

the $\delta(t)$ contribution in the next-to-last equation is spurious and arises from our ingoing–outgoing decomposition. The results in Eqs. (25a) and (25b) are in the form anticipated in the acoustic G_{11} Green's function result in Eq. (1.1.13a); the other acoustic Green's functions can be similarly evaluated from the remaining matrix elements of G in Eqs. (24).

1.3c The Electromagnetic Field

Source-free electromagnetic oscillations with time dependence $\exp(-i\omega t)$ are determined by the steady-state electromagnetic field equations

$$
\begin{aligned}
\nabla \times \mathbf{H} &= -i\omega\epsilon_0 \mathbf{E}, \\
\nabla \times \mathbf{E} &= i\omega\mu_0 \mathbf{H},
\end{aligned} \tag{26}
$$

subject to appropriate conditions at the boundary (if any) of the field region. Equations (26) pose an eigenvalue problem in the form of Eq. (9), wherein $\mathbf{E} = \mathbf{E}(\mathbf{r}, \omega)$ and $\mathbf{H} = \mathbf{H}(\mathbf{r}, \omega)$ denote the complex electric- and magnetic-field amplitudes, and ϵ_0 and μ_0 are the customary electromagnetic vacuum parameters. For free space the steady-state solutions of Eq. (26) are spatially representable as

$$
\begin{pmatrix} \mathbf{E}(\mathbf{r}, \omega) \\ \mathbf{H}(\mathbf{r}, \omega) \end{pmatrix} = \begin{pmatrix} \mathbf{E}(\mathbf{k}, \omega) \\ \mathbf{H}(\mathbf{k}, \omega) \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{27a}
$$

or in the eigenvector notation of Sec. 1.3a as

$$
\Psi(\mathbf{r}, \omega) = \Psi(\mathbf{k}, \omega) e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{27b}
$$

where in view of Eq. (26), the associated operators $M = M^+$ and $W = W^+$ are†

$$
iM \to \begin{pmatrix} 0 & -\nabla \times \mathbf{1} \\ \nabla \times \mathbf{1} & 0 \end{pmatrix}, \qquad W \to \begin{pmatrix} \epsilon_0 & 0 \\ 0 & \mu_0 \end{pmatrix} \mathbf{1}. \tag{28}
$$

Equations (26) have eigensolutions of the spatial form (27), whereby $\nabla \equiv i\mathbf{k}$, only for the six eigenfrequencies

†Note that $M = M^+$, since ∇/i is Hermitian and $\mathbf{a} \times \mathbf{1} = -\widetilde{\mathbf{a} \times \mathbf{1}}$ is antisymmetric.

$$\begin{aligned} \omega_{\pm 1} &= \omega_{\pm 2} = \pm\, kc, \\ \omega_3 &= \omega_4 = 0, \end{aligned} \tag{29}$$

where $k = |\mathbf{k}|$ and $c = (\mu_0 \epsilon_0)^{-1/2}$ is the speed of light in vacuum. The frequencies (29), some of which are doubly degenerate, are recognizable as the source-free resonances of the electromagnetic network depicted in Fig. 1.2.3. The associated resonant fields, which also follow from Eq. (26), are characterized by the eigenvectors $\Psi_\alpha \equiv \Psi_\alpha(\mathbf{k}, \omega)$, where in the $\mathbf{k}_0$, $\mathbf{T}_0'$, $\mathbf{T}_0''$ basis of Fig. 1.2.1,

$$\Psi_{\pm 1} \to \begin{pmatrix} \mathbf{T}_0'/\sqrt{\epsilon_0} \\ \pm \mathbf{T}_0''/\sqrt{\mu_0} \end{pmatrix}, \quad \Psi_{\pm 2} \to \begin{pmatrix} \mathbf{T}_0''/\sqrt{\epsilon_0} \\ \mp \mathbf{T}_0'/\sqrt{\mu_0} \end{pmatrix},$$

$$\Psi_3 \to \begin{pmatrix} \mathbf{k}_0/\sqrt{\epsilon_0} \\ 0 \end{pmatrix}, \quad \Psi_4 \to \begin{pmatrix} 0 \\ \mathbf{k}_0/\sqrt{\mu_0} \end{pmatrix}. \tag{30a}$$

These possess the $\mathbf{k}$ independent orthonormality properties in Eq. (11) with the normalization constants

$$N_{\pm 1} = 1 = N_{\pm 2}, \qquad 2N_3 = 1 = 2N_4. \tag{30b}$$

Oscillatory representation of electromagnetic Green's function

An oscillatory representation of the time-dependent (matrix) Green's function of the electromagnetic field is provided by Eq. (18). On use therein of the known eigenfrequencies (29) and eigenvectors (30) one obtains, since $\Psi_\alpha^+ = \Psi_\alpha$ in free space, for $t > t'$ (i.e., omitting a singularity at $t = t'$):

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}'; t, t') = \int \frac{1}{2}\Big[&(\Psi_1 \Psi_1 + \Psi_2 \Psi_2) e^{-i\omega_{+1}(t-t')} \\ &+ \Psi_3 \Psi_3 + \Psi_4 \Psi_4 \Big] e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{d\mathbf{k}}{(2\pi)^3}, \end{aligned} \tag{31}$$

whose 11 matrix element is identically the impedance dyadic Green's function $\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t')$ of Sec. 1.1b (with $\mathbf{r} - \mathbf{r}'$, $t - t'$ written as $\mathbf{r}, t$):

$$\begin{aligned} \mathscr{G}_{11}(\mathbf{r}, t) &= \int \left[\left(\frac{\mathbf{T}_0'\mathbf{T}_0'}{2\epsilon_0} + \frac{\mathbf{T}_0''\mathbf{T}_0''}{2\epsilon_0} \right) e^{-i\omega_{+1}t} + \frac{\mathbf{k}_0\mathbf{k}_0}{2\epsilon_0} \right] e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3} \\ &= \mathbf{1} \int \frac{e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{+1}t)}}{2\epsilon_0} \frac{d\mathbf{k}}{(2\pi)^3} - \int \mathbf{k}_0\mathbf{k}_0 \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{2\epsilon_0} (e^{-i\omega_{+1}t} - 1) \frac{d\mathbf{k}}{(2\pi)^3} && \text{(32a)} \\ &= \left(\mathbf{1}\mu_0 \frac{\partial}{\partial t} - \frac{\nabla\nabla}{\epsilon_0(\partial/\partial t)} \right) \int \frac{e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{+1}t)}}{-2ik/c} \frac{d\mathbf{k}}{(2\pi)^3}, && \text{(32b)} \end{aligned}$$

$$\mathscr{G}_{11}(\mathbf{r}, t) = \left(\mathbf{1}\mu_0 \frac{\partial}{\partial t} - \frac{\nabla\nabla}{\epsilon_0(\partial/\partial t)} \right) \frac{\delta(t - r/c)}{4\pi r}, \qquad t > 0. \tag{32c}$$

In successive steps we have utilized in Eqs. (32) the identities $\mathbf{1} - \mathbf{k}_0\mathbf{k}_0 = \mathbf{T}_0'\mathbf{T}_0' + \mathbf{T}_0''\mathbf{T}_0''$, $\nabla\nabla = -k^2\mathbf{k}_0\mathbf{k}_0$, $\partial/\partial t = -ikc$, $(\partial/\partial t)^{-1} = \int_0^t dt$, and the ω integrated result of the integral representation in Eq. (1.2.20) [see Eq. (1.3.25)].

The representation in Eq. (32c) is seen to be the previously obtained dyadic Green's function of Eqs. (1.1.30b) and (1.1.31). The other electromagnetic dyadic Green's functions are determined from the remaining matrix elements of Eq. (31) in a similar manner.

The noted singularity of oscillatory (time-guided) representations (31) of space- and time-dependent Green's functions should be contrasted with analogous properties of space-guided representations discussed in Secs. 1.1b and 1.4. In the space-guided formulation, a z-guided mode is excited by a distribution of sources with appropriate temporal and transverse spatial periodicities in a plane transverse to z; modal excitation by sources spatially confined to a plane $z = z'$ is determined on replacing such sources by equivalent distributions occupying the entire $z = z'$ plane [see Eqs. (1.1.43) and (1.4.15)]. A time-guided mode, on the other hand, is excited by sources distributed throughout space with appropriate spatial periodicity $\mathbf{k}$; modal excitation by spatially and temporally confined sources is determined on replacing such sources by equivalent volume distributions. When a simplified representation of either the space-guided [see Eqs. (1.1.42)] or time-guided [see Eq. (18)] Green's functions is used, spurious contributions are introduced into the former on the source plane $z = z' = 0$ [Eq. (1.1.47)] and into the latter at $t = t' = 0$ [see Eq. (25b)]. In both instances, these anomalies are evidently confined to hyperplanes containing the source and oriented perpendicular to the guiding axis in the four-dimensional $\mathbf{r}, t$ space.

1.3d The Plasma Field

As can be inferred from the time-dependent field equations of Sec. 1.1c, source-free harmonic oscillations of an isotropic one-component fluid model of a plasma are defined by

$$\begin{aligned} \nabla \times \mathbf{H} \qquad\qquad + n_0 q \mathbf{v} &= -i\omega\epsilon_0 \mathbf{E}, \\ \nabla \times \mathbf{E} \qquad\qquad\qquad\qquad &= i\omega\mu_0 \mathbf{H}, \\ -\nabla \cdot \mathbf{v} &= -i\frac{\omega}{\gamma p_0} p, \\ n_0 q \mathbf{E} \qquad + \nabla p \qquad\qquad &= i\omega n_0 m \mathbf{v}, \end{aligned} \tag{33}$$

subject to appropriate boundary conditions. Equations (33) constitute an eigenvalue problem of the type shown in Eq. (9), wherein ω plays the role of an eigenvalue and the associated eigenvector components are the electric-field intensity $\mathbf{E} = \mathbf{E}(\mathbf{r}, \omega)$, the magnetic-field intensity $\mathbf{H} = \mathbf{H}(\mathbf{r}, \omega)$, the linearized pressure $p = p(\mathbf{r}, \omega)$, and the linearized velocity $\mathbf{v} = \mathbf{v}(\mathbf{r}, \omega)$. In an unbounded, stationary, homogeneous plasma, steady-state solutions of Eqs. (33) are representable in the form of a 10 (scalar)-component wavevector as

$$\begin{pmatrix} \mathbf{E}(\mathbf{r}, \omega) \\ \mathbf{H}(\mathbf{r}, \omega) \\ p(\mathbf{r}, \omega) \\ \mathbf{v}(\mathbf{r}, \omega) \end{pmatrix} = \begin{pmatrix} \mathbf{E}(\mathbf{k}, \omega) \\ \mathbf{H}(\mathbf{k}, \omega) \\ p(\mathbf{k}, \omega) \\ \mathbf{v}(\mathbf{k}, \omega) \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{34a}$$

or, in wavevector notation, as

$$\Psi(\mathbf{r}, \omega) = \Psi(\mathbf{k}, \omega) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{34b}$$

Substitution of Eqs. (34) into (33) permits one to rewrite the latter in the abstract form of Eq. (10), wherein the operators $M(\mathbf{k}) = M^{+}(\mathbf{k})$ and $W = W^{+}$ are

$$\begin{gathered} M \rightarrow \begin{pmatrix} 0 & -\mathbf{k} \times \mathbf{1} & 0 & i n_0 q \mathbf{1} \\ \mathbf{k} \times \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{k} \\ -i n_0 q \mathbf{1} & 0 & \mathbf{k} & 0 \end{pmatrix}, \\ W \rightarrow \begin{pmatrix} \epsilon_0 \mathbf{1} & 0 & 0 & 0 \\ 0 & \mu_0 \mathbf{1} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\gamma p_0} & 0 \\ 0 & 0 & 0 & n_0 m \mathbf{1} \end{pmatrix}. \end{gathered} \tag{35}$$

Eigenvector solutions of Eqs. (10) or (33) exist for eigenfrequencies $\omega = \omega_\alpha(\mathbf{k})$ defined by the following dispersion relations:

$$\begin{aligned} \omega_{\pm 1} = \omega_{\pm 2} &= \pm \sqrt{\omega_p^2 + k^2 c^2}, \\ \omega_{\pm 3} &= \pm \sqrt{\omega_p^2 + k^2 a^2}, \\ \omega_4 = \omega_5 = \omega_6 = \omega_7 = 0, \qquad & \text{where } c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \quad \text{and} \quad a = \sqrt{\frac{\gamma p_0}{n_0 m}}. \end{aligned} \tag{36}$$

The first four frequencies define optic-type oscillations, the $\omega_{\pm 3}$ are the plasma oscillations, and the latter four are static zero-frequency oscillations† of longitudinal and transverse types. These 10 frequencies characterize the free resonances of the source-free plasma network displayed in Fig. 1.2.4. The corresponding resonant fields are described by eigenvectors $\Psi(\mathbf{k}, \omega_\alpha(\mathbf{k})) = \Psi_\alpha(\mathbf{k}) \equiv \Psi_\alpha$ of the type shown in Eq. (34). In the $\mathbf{k}_0, \mathbf{T}_0', \mathbf{T}_0''$ basis illustrated in Fig. 1.2.1, the Ψ_α have the form:

†It is assumed that $k \neq 0$ for the zero-frequency oscillation ω_5, which incidentally need not satisfy Poisson's equation. The case $k = 0$ necessitates a special consideration of the constant positive background required for charge neutrality.

$$\Psi_{\pm 1} \to \begin{pmatrix} \dfrac{\mathbf{T}_0'}{\sqrt{\epsilon_0}} \\ \dfrac{kc}{\omega_{\pm 1}} \dfrac{\mathbf{T}_0''}{\sqrt{\mu_0}} \\ 0 \\ \dfrac{-i\omega_p}{\omega_{\pm 1}} \dfrac{\mathbf{T}_0'}{\sqrt{n_0 m}} \end{pmatrix}, \quad \Psi_{\pm 2} \to \begin{pmatrix} \dfrac{\mathbf{T}_0''}{\sqrt{\epsilon_0}} \\ \dfrac{-kc}{\omega_{\pm 2}} \dfrac{\mathbf{T}_0'}{\sqrt{\mu_0}} \\ 0 \\ \dfrac{-i\omega_p}{\omega_{\pm 2}} \dfrac{\mathbf{T}_0''}{\sqrt{n_0 m}} \end{pmatrix},$$

$$\Psi_{\pm 3} \to \begin{pmatrix} \dfrac{\omega_p}{\omega_{\pm 3}} \dfrac{\mathbf{k}_0}{\sqrt{\epsilon_0}} \\ 0 \\ \dfrac{-ika}{\omega_{\pm 3}} \sqrt{\gamma p_0} \\ \dfrac{-i\mathbf{k}_0}{\sqrt{n_0 m}} \end{pmatrix}, \quad \Psi_4 \to \begin{pmatrix} 0 \\ \dfrac{\mathbf{k}_0}{\sqrt{\mu_0}} \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_5 \to \begin{pmatrix} \dfrac{\mathbf{k}_0}{\sqrt{\epsilon_0}} \\ 0 \\ i\dfrac{\omega_p}{ka} \sqrt{\gamma p_0} \\ 0 \end{pmatrix}, \tag{37a}$$

$$\Psi_6 \to \begin{pmatrix} 0 \\ \dfrac{i\mathbf{T}_0'}{\sqrt{\mu_0}} \\ 0 \\ \dfrac{kc}{\omega_p} \dfrac{\mathbf{T}_0''}{\sqrt{n_0 m}} \end{pmatrix}, \quad \Psi_7 \to \begin{pmatrix} 0 \\ \dfrac{i\mathbf{T}_0''}{\sqrt{\mu_0}} \\ 0 \\ -\dfrac{kc}{\omega_p} \dfrac{\mathbf{T}_0'}{\sqrt{n_0 m}} \end{pmatrix}.$$

These eigenvectors possess the orthogonality properties in Eq. (11) with normalization constants

$$N_{\pm 1} = N_{\pm 2} = N_{\pm 3} = 1, \quad 2N_4 = 1, \quad 2N_5 = 1 + \frac{\omega_p^2}{k^2 a^2},$$

$$2N_6 = 2N_7 = 1 + \frac{k^2 c^2}{\omega_p^2}. \tag{37b}$$

Oscillatory representation of plasma Green's function

As shown in Eqs. (18), the (matrix) Green's function $G(\mathbf{r}, \mathbf{r}'; t, t')$ of the plasma field is expressible in terms of the oscillatory eigenvectors in Eq. (37a). As a particular example, the dyadic element $\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t')$, which represents the negative of the vector electric field at $\mathbf{r}, t$ produced by a point electric current source at $\mathbf{r}', t'$ is given by [we denote $\mathbf{r} - \mathbf{r}'$, $t - t'$ by $\mathbf{r}, t$ and we employ the $\omega_\pm(\mathbf{k})$ frequencies defined in connection with the discussion of Eq. (18)]

$$\mathscr{G}_{11}(\mathbf{r}, t) = \frac{1}{2}\int\left[\Psi_1\Psi_1 e^{-i\omega_1 t} + \Psi_2\Psi_2 e^{-i\omega_2 t} + \Psi_3\Psi_3 e^{-i\omega_3 t}\right.$$

$$\left. + \frac{\Psi_5\Psi_5}{1 + (\omega_p^2/k^2a^2)}\right] e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3}$$

$$= \frac{1}{2\epsilon_0}\int\left[(\mathbf{T}_0'\mathbf{T}_0' + \mathbf{T}_0''\mathbf{T}_0'')e^{-i\omega_1 t}\right.$$

$$\left. + \mathbf{k}_0\mathbf{k}_0\frac{\omega_p^2}{\omega_p^2 + k^2a^2}(e^{-i\omega_3 t} - 1) + \mathbf{k}_0\mathbf{k}_0\right] e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3}$$

$$= \frac{1}{2}\int\left[\mathbf{1}\mu_0\frac{\partial}{\partial t} + \frac{\nabla\nabla}{\epsilon_0}\frac{\partial}{\partial t}\frac{1}{k^2c^2}\right]\frac{e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_1 t)}}{-i\omega_1}\frac{c^2 d\mathbf{k}}{(2\pi)^3}$$

$$-\frac{\nabla\nabla}{\epsilon_0}\frac{\omega_p^2}{\partial/\partial t}\int\frac{e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_3 t)}}{ik^2\omega_3}\frac{d\mathbf{k}}{(2\pi)^3}. \tag{38}$$

On integration over polar and azimuthal angles in $\mathbf{k}$ space and on change of variable from k to $\omega = \sqrt{\omega_p^2 + k^2u^2}$, as required, with $u = a$ or c, one obtains

$$\mathscr{G}_{11}(\mathbf{r}, t) = \left(\mathbf{1}\mu_0\frac{\partial}{\partial t} - \frac{\nabla\nabla}{\epsilon_0}\frac{\partial/\partial t}{(\partial^2/\partial t^2) + \omega_p^2}\right) g_c(\mathbf{r}, t)$$

$$- \frac{\nabla\nabla}{\epsilon_0}\frac{\omega_p^2}{(\partial/\partial t)[(\partial^2/\partial t^2) + \omega_p^2]} g_a(\mathbf{r}, t), \tag{39a}$$

where

$$g_u(\mathbf{r}, t) = \int_{-\infty}^{+\infty}\frac{\exp\left[-i(\omega t - \sqrt{\omega^2 - \omega_p^2}\,r/u)\right]}{4\pi r}\frac{d\omega}{2\pi}, \qquad \operatorname{Im}\omega > 0, \tag{39b}$$

with $u = c$ or a. A known Fourier transformation [see Eqs. (1.1.61)] indicates that the expressions (39) for $\mathscr{G}_{11}(\mathbf{r}, t)$ are identical for $t \neq t'$ to those previously derived in Eqs. (1.1.59)–(1.1.61).

1.4 GUIDED-WAVE REPRESENTATIONS IN SPACE

As we have seen, fields in linear, homogeneous, and stationary media can be represented in alternative ways. The $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ plane-wave representations of Sec. 1.2 reduce a field problem in an unbounded medium to a simple algebraic problem for the $\mathbf{k}$, ω-dependent field amplitudes. Similarly, the oscillatory $\exp(i\mathbf{k}\cdot\mathbf{r})$ representations of Sec. 1.3 reduce a field problem to an ordinary differential equation problem for the time- and k-dependent field amplitudes. In this section we shall introduce another representation well suited to the solution of field problems in media with a symmetry axis along some spatial direction, say z. The guided waves $\exp(i\mathbf{k}_t\cdot\boldsymbol{\rho} - i\omega t)\exp(i\kappa z)$ considered below are plane waves with real transverse (to z) wavenumber $\mathbf{k}_t$ and frequency ω. They possess orthogonality properties in time and in the cross section defined by the radius vector $\boldsymbol{\rho}$ transverse to z; they are field solutions in the medium and are distinguished by wavenumbers $\kappa = \kappa(\mathbf{k}_t, \omega)$ that are characteristic both of the cross-sectional shape and the polarization structure of the field. Such wave fields permit a representation of a linear field in terms of characteristic guided waves with $\exp(i\kappa z)$ dependence, require evaluations of three-dimensional transforms (two dimensional in space and one dimensional in time), and are well adapted to the solution of boundary-value problems in uniform stratified regions.

In a uniform region whose cross section transverse to z is unbounded and described by a radial vector $\boldsymbol{\rho}$, an integrable wave function $\Psi(\mathbf{r}, t)$ is representable by means of the three-dimensional Fourier integral theorem as†

$$\Psi(\boldsymbol{\rho}, t; z) = \iiint \Psi(\mathbf{k}_t, \omega; z) e^{i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)} \frac{d\mathbf{k}_t d\omega}{(2\pi)^3}, \tag{1a}$$

where

$$\Psi(\mathbf{k}_t, \omega; z) = \iiint \Psi(\boldsymbol{\rho}, t; z) e^{-i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)} d\boldsymbol{\rho}\, dt \tag{1b}$$

and $d\mathbf{k}_t d\omega$ denotes the "volume" element in $\mathbf{k}_t$, ω space while $d\boldsymbol{\rho} dt$ is the "volume" element in $\boldsymbol{\rho}$, t space; $\mathbf{k}_t$ is the wavevector component of $\mathbf{k}$ transverse to z. In a homogeneous, transversely *bounded* waveguide, $\exp(i\mathbf{k}_t \cdot \boldsymbol{\rho})$ in Eqs. (1) would be replaced by a suitable transverse eigenfunction $\Phi_\alpha(\boldsymbol{\rho})$ (see Chapter 3). The completeness relation equivalent to the transform relations (1a) and (1b) is

$$\delta(\boldsymbol{\rho} - \boldsymbol{\rho}')\delta(t - t') = \iiint e^{i\mathbf{k}_t \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} e^{-i\omega(t-t')} \frac{d\mathbf{k}_t d\omega}{(2\pi)^3}, \tag{2a}$$

where for completeness the integration is to be extended over all real wavenumbers $\mathbf{k}_t$ and frequencies ω from $-\infty$ to $+\infty$. From the transform relations (1), one also infers the orthogonality property

$$(2\pi)^3 \delta(\mathbf{k}_t - \mathbf{k}_t')\delta(\omega - \omega') = \int e^{-i(\mathbf{k}_t - \mathbf{k}_t') \cdot \boldsymbol{\rho} + i(\omega - \omega')t} d\boldsymbol{\rho}\, dt. \tag{2b}$$

A Green's function in the basis of Eq. (2a) would be represented as

$$G(\mathbf{r}, \mathbf{r}'; t, t') = \iiint G(\mathbf{k}_t, \omega; z, z') e^{i\mathbf{k}_t \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} e^{-i\omega(t-t')} \frac{d\mathbf{k}_t d\omega}{(2\pi)^3}. \tag{3}$$

Singularities of $G(\mathbf{k}_t, \omega; z, z')$ are frequently encountered on the real ω-integration path in Eq. (3); to satisfy causality (i.e., $G \equiv 0$ for $t < t'$), one may deform this path by analytic continuation into the region $\operatorname{Im} \omega > 0$ of the complex ω plane. As in the oscillatory representation of Sec. 1.3, the transformed wavefunction $\Psi(\mathbf{k}_t, \omega; z)$ of Eqs. (1) is a one-column matrix (vector) in an n-dimensional "polarization" space and is representable in terms of a set of modes or eigenvectors $\Psi_\alpha(\mathbf{k}_t, \omega)$ as

$$\Psi(\mathbf{k}_t, \omega; z) = \sum_\alpha \Psi_\alpha(\mathbf{k}_t, \omega) a_\alpha(\mathbf{k}_t, \omega; z), \tag{4a}$$

†Note that functions and their transforms are represented by the same symbol and distinguished, where necessary, only by their arguments.

where α is a multicomponent summation index distinguishing the αth mode. The amplitude a_α is determined as the weighted Hermitian inner product of the wave vector Ψ and an adjoint eigenvector $\Psi_\alpha^+(\mathbf{k}_t, \omega)$ as

$$a_\alpha(\mathbf{k}_t, \omega; z) = \frac{(\Gamma^+ \Psi_\alpha^+(\mathbf{k}_t, \omega), \Psi(\mathbf{k}_t, \omega; z))}{2N_\alpha(\mathbf{k}_t, \omega)}, \tag{4b}$$

Γ being a "weight operator. The eigenfunctions Ψ_α and Ψ_α^+ constitute a complete biorthogonal set in polarization space and possess the orthogonality properties

$$(\Gamma^+\Psi_\alpha^+, \Psi_\beta) = 2N_\alpha \delta_{\alpha\beta}, \tag{4c}$$

where N_α is the normalization constant and $\delta_{\alpha\beta}$ is the Kronecker delta, which vanishes when the mode indices α and β distinguish modes with different κ_α and κ_β. If the κ_α are n-fold degenerate, the various wavevectors $\Psi_{\alpha i}$ and their adjoints $\Psi_{\alpha j}^+$ with $i, j = 1, 2, \cdots, n$ should be chosen to satisfy the orthogonality properties

$$(\Gamma^+\Psi_{\alpha i}^+, \Psi_{\alpha j}) = 2N_{\alpha i}\delta_{ij}. \tag{4d}$$

In polarization space the completeness relation equivalent to the transforms in Eqs. (4a) and (4b) is provided by the unit-operator representation

$$1 = \sum_\alpha \frac{\Psi_\alpha(\mathbf{k}_t, \omega)\Gamma^+\Psi_\alpha^+(\mathbf{k}_t, \omega)}{2N_\alpha(\mathbf{k}_t, \omega)}. \tag{5}$$

The function and polarization space transforms in Eqs. (1) and (4) permit a representation of the overall space-time field $\Psi(\mathbf{r}, t)$ as

$$\Psi(\boldsymbol{\rho}, t; z) = \int \sum_\alpha \Psi_\alpha(\mathbf{k}_t, \omega) a_\alpha(\mathbf{k}_t, \omega; z) e^{i(\mathbf{k}_t\cdot\boldsymbol{\rho} - \omega t)} \frac{d\mathbf{k}_t\, d\omega}{(2\pi)^3} \tag{6a}$$

with

$$a_\alpha(\mathbf{k}_t, \omega; z) = \int \frac{(\Gamma^+\Psi_\alpha^+(\mathbf{k}_t, \omega), \Psi(\boldsymbol{\rho}, t; z))}{2N_\alpha(\mathbf{k}_t, \omega)} e^{-i(\mathbf{k}_t\cdot\boldsymbol{\rho} - \omega t)} d\boldsymbol{\rho}\, dt. \tag{6b}$$

The corresponding completeness relation in function and polarization space is

$$1\delta(\boldsymbol{\rho} - \boldsymbol{\rho}')\delta(t - t') = \int \sum_\alpha \frac{\Psi_\alpha(\mathbf{k}_t, \omega)\Gamma^+\Psi_\alpha^+(\mathbf{k}_t, \omega)}{2N_\alpha(\mathbf{k}_t, \omega)} e^{i\mathbf{k}_t\cdot(\boldsymbol{\rho} - \boldsymbol{\rho}')} e^{-i\omega(t - t')} \frac{d\mathbf{k}_t\, d\omega}{(2\pi)^3}, \tag{7}$$

which states that the set of guided mode fields $\Psi_\alpha(\mathbf{k}_t, \omega) e^{i(\mathbf{k}_t\cdot\boldsymbol{\rho} - \omega t)}$ is capable of completely representing the solution of a general linear field problem in a transversely unbounded medium. In particular, as we shall demonstrate, the (matrix) Green's function in a transversely unbounded medium is representable as

$$G(\mathbf{r}, \mathbf{r}'; t, t') = \int \sum_\alpha G(\mathbf{k}_t, \omega; z, z') \frac{\Psi_\alpha(\mathbf{k}_t, \omega)\Psi_\alpha^+(\mathbf{k}_t, \omega)}{2N_\alpha(\mathbf{k}_t, \omega)} e^{i\mathbf{k}_t\cdot(\boldsymbol{\rho} - \boldsymbol{\rho}')} e^{-i\omega(t - t')} \frac{d\mathbf{k}_t\, d\omega}{(2\pi)^3}. \tag{8}$$

1.4a General Linear Field

As noted in Sec. 1.1d, a general linear field in a homogeneous, stationary medium is describable by a linear operator $L(\nabla, \partial/\partial t)$. In a medium displaying symmetry along the z direction, the operator L may be decomposed into two components, one depending only on $\partial/\partial z$ and the other on ∇_t, $\partial/\partial t$, where $\nabla_t = \nabla - z_0(\partial/\partial z)$ is the component of the vector derivative ∇ transverse to z. The defining equation (1.1.73a) for the Green's function $G(\mathbf{r}, \mathbf{r}'; t, t')$ of a general linear field is thus expressible in the form

$$-i\left[K\left(\nabla_t, \frac{\partial}{\partial t}\right) - \frac{\Gamma}{i}\frac{\partial}{\partial z}\right]G(\boldsymbol{\rho}, \boldsymbol{\rho}', t, t'; z, z') = 1\delta(\boldsymbol{\rho} - \boldsymbol{\rho}')\delta(t - t')\delta(z - z') \quad (9)$$

and in an unbounded region is subject to causality (or equivalently, an outgoing wave condition as $|z - z'| \rightarrow \infty$). The component operators K and Γ may be inferred from the L operators noted in Sec. 1.1d and will be listed below for a number of fields. The elements of the matrix representatives of these operators are scalars, vectors, or dyadics, and so, correspondingly, are those of the Green's function G.

To represent in the form (8) the Green's function G defined by Eq. (9), one must first ascertain the characteristic field vectors Ψ_α. The latter are defined as solutions of the source-free field equations $L\Psi_\alpha = 0$ for fields Ψ_α with a z dependence $\exp(i\kappa_\alpha z)$. Introducing this z dependence, and noting the decomposition of the operator L in Eq. (9), one obtains from $L\Psi_\alpha = 0$ the eigenvalue equation

$$K\left(\nabla_t, \frac{\partial}{\partial t}\right)\Psi_\alpha(\boldsymbol{\rho}, t) = \kappa_\alpha \Gamma \Psi_\alpha(\boldsymbol{\rho}, t) \quad (10)$$

defining the eigenvectors Ψ_α and eigenvalues κ_α. For the case of an unbounded homogeneous (translationally invariant) cross section transverse to z, $\Psi_\alpha(\boldsymbol{\rho}, t) = \Psi_\alpha(\mathbf{k}_t, \omega)\exp[i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)]$ and hence in $\mathbf{k}_t$, ω space the eigenvalue problem in Eq. (10) takes the form

$$K(\mathbf{k}_t, \omega)\Psi_\alpha(\mathbf{k}_t, \omega) = \kappa_\alpha(\mathbf{k}_t, \omega)\Gamma\Psi_\alpha(\mathbf{k}_t, \omega), \quad (11a)$$

where $K(\mathbf{k}_t, \omega)$ is the algebraic matrix operator $K(\nabla_t, \partial/\partial t)$ with the substitutions $\nabla_t = i\mathbf{k}_t$ and $\partial/\partial t = -i\omega$, and $\Psi_\alpha(\mathbf{k}_t, \omega)$ is a multicomponent vector in an appropriate "polarization"space. The corresponding adjoint eigenvalue problem† is

$$K(\mathbf{k}_t, \omega)^+ \Psi_\alpha^+(\mathbf{k}_t, \omega) = \kappa_\alpha(\mathbf{k}_t, \omega)^* \Gamma^+ \Psi_\alpha^+(\mathbf{k}_t, \omega) \quad (11b)$$

and possesses complex-conjugate eigenvalues κ_α^* and adjoint eigenvectors Ψ_α^+ characteristic of adjoint operators K^+ and Γ^+, whose off-diagonal matrix elements are the transposed conjugates of the elements of K and Γ. From Eqs.

†The adjoint eigenvalue $\kappa_\alpha^+ = \kappa_\alpha^*$, provided there exists a Ψ_α and Ψ_α^+ such that the inner product $(\Gamma^+\Psi_\alpha^+, \Psi_\alpha) \neq 0$. In dissipative systems it is sometimes more convenient to define the inner product without the conjugate operation.

(11), on forming appropriate Hermitian inner products of the vectors Ψ_α and Ψ_β^+, one finds that

$$(\kappa_\beta - \kappa_\alpha)(\Psi_\beta^+, \Gamma\Psi_\alpha) = (K^+\Psi_\beta^+, \Psi_\alpha) - (\Psi_\beta^+, K\Psi_\alpha) = 0,$$

from which one infers in the $\mathbf{k}_t, \omega$ polarization space the previously noted biorthogonality property in Eq. (4c) and the abstract completeness relation (5),

$$1 = \sum_\alpha \frac{\Psi_\alpha(\mathbf{k}_t, \omega)\Gamma^+\Psi_\alpha^+(\mathbf{k}_t, \omega)}{2N_\alpha(\mathbf{k}_t, \omega)}. \tag{12}$$

For the Hermitian case wherein $K = K^+$ and $\Gamma = \Gamma^+$, one infers from Eqs. (11) that $\Psi_\alpha^+ = \Psi_{\alpha^*}$, where Ψ_{α^*} is the eigenvector with the eigenvalue κ_α^*.

In the $\mathbf{k}_t$, ω transform space the Green's function equation (9) becomes, on use of the representation in Eq. (3), the matrix ordinary differential equation

$$-i\left[K(\mathbf{k}_t, \omega) - \frac{\Gamma}{i}\frac{d}{dz}\right]G(\mathbf{k}_t, \omega; z, z') = 1\delta(z - z'), \tag{13}$$

where for a region unbounded in the z direction, the Green's function satisfies the boundary condition $G \to 0$ as $|z - z'| \to \infty$, provided that $\operatorname{Im}\omega > 0$. This $\mathbf{k}_t$, ω transformed Green's function can be further represented via Eq. (12) as

$$\begin{aligned} G(\mathbf{k}_t, \omega; z, z') &= \sum_\alpha \frac{\Psi_\alpha(\mathbf{k}_t, \omega)}{2N_\alpha(\mathbf{k}_t, \omega)}(\Psi_\alpha^+(\mathbf{k}_t, \omega), \Gamma G(\mathbf{k}_t, \omega; z, z')) \\ &= \sum_\alpha \frac{\Psi_\alpha(\mathbf{k}_t, \omega)\Psi_\alpha^+(\mathbf{k}_t, \omega)}{2N_\alpha(\mathbf{k}_t, \omega)} G_\alpha(\mathbf{k}_t, \omega; z, z'), \end{aligned} \tag{14}$$

where the scalar amplitudes G_α are defined by $\Psi_\alpha^+ G_\alpha = (\Psi_\alpha^+, \Gamma G)$. Equation (14), when employed in Eq. (3), leads to the space- and time-dependent Green's function representation anticipated in Eq. (8). The determination of the amplitude coefficients G_α involves transformation of Eq. (13) with respect to the basis Ψ_α. Thus, Hermitian inner-product multiplication of Eq. (13) by $\Psi_\alpha^+(\mathbf{k}_t, \omega)$ leads, in view of Eq. (14), to

$$\left[\frac{d}{dz} - i\kappa_\alpha(\mathbf{k}_t, \omega)\right]G_\alpha(\mathbf{k}_t, \omega; z, z') = \delta(z - z'), \tag{15}$$

subject to $G_\alpha \to 0$ as $|z - z'| \to \infty$, provided that $\operatorname{Im}\kappa_\alpha \neq 0$, as results from $\operatorname{Im}\omega > 0$. Equation (15) admits solutions

$$G_\alpha(\mathbf{k}_t, \omega; z, z') = \begin{cases} e^{i\kappa_\alpha(z-z')} & z > z', \\ 0, & z < z', \end{cases} \tag{16a}$$

for modes κ_α, with $\operatorname{Im}\kappa_\alpha > 0$, that characterize waves transporting energy (or decaying) in the $+z$ direction, and

$$G_\alpha(\mathbf{k}_t, \omega; z, z') = \begin{cases} 0, & z > z', \\ -e^{i\kappa_\alpha(z-z')} & z < z', \end{cases} \tag{16b}$$

for modes κ_α, with $\operatorname{Im}\kappa_\alpha < 0$, carrying energy (or decaying) in the $-z$ direction. Indices $\alpha > 0$ or $\alpha < 0$ will be employed to identify the proper κ_α to be associated with waves traveling in the $+z$ or $-z$ directions, respectively. These identifications pose difficulties in complex media; in simple passive media, waves carrying power in the $+z$ or $-z$ directions are distinguished by group speeds $(\partial\kappa_\alpha/\partial\omega)^{-1} > 0$ or <0, respectively (see Sec. 1.6c). The proper identification may be effected, as noted above and in Eq. (3), by analytic continuation into the region $\operatorname{Im}\omega > 0$ with subsequent selection of $\operatorname{Im}\kappa_\alpha \gtrless 0$ for the $\pm z$ traveling waves. Alternatively, for real $\mathbf{k}_t$, ω, the identification may be accomplished on assigning small loss to the medium, thereby removing the singularities of $G(\mathbf{k}_t, \omega; z, z')$ from the integration paths, and then passing to the lossless limit.

Substituting Eqs. (16) into Eqs. (14), one obtains with the aid of Eq. (3) the desired Green's function representation,

$$G(\mathbf{r}, \mathbf{r}'; t, t') = \int \sum_{\alpha \gtrless 0} \frac{\Psi_\alpha(\mathbf{k}_t, \omega)\Psi_\alpha^+(\mathbf{k}_t, \omega)}{\pm 2N_\alpha(\mathbf{k}_t, \omega)} e^{i[\mathbf{k}_t\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')-\omega(t-t')]} e^{i\kappa_\alpha(z-z')} \frac{d\mathbf{k}_t\, d\omega}{(2\pi)^3}, \tag{17}$$

where the $\alpha \gtrless 0$ and $\pm$ designations correspond, respectively, to the regions $z \gtrless z'$, and the triple integration ranges over all real values of $\mathbf{k}_t$ and ω from $-\infty$ to $+\infty$. The guided-wave representation in Eq. (17) is to be contrasted with the oscillatory Green's function representation of Eq. (1.3.17). In the latter representation the causality requirement of vanishing for $t < t'$ is evidently satisfied; in the guided-wave case causality requires the proper identification of the $+$ and $-$ (i.e., $\alpha \gtrless 0$) traveling waves.

The guided-wave representation in Eq. (17) becomes explicit with a knowledge of the eigenmode vectors Ψ_α and Ψ_α^+ together with their normalizations N_α. Their evaluation is exhibited for a number of linear fields in the following subsections (see also Secs. 8.3 and 8.4).

1.4b The Acoustic Field

In a linear acoustic field, the eigenvalue problem in Eq. (10), which characterizes source-free guided-wave solutions with $\exp(i\kappa z)$ space dependence, can be written, via Eqs. (1.1.1), in the form

$$\begin{aligned} \frac{1}{\gamma p_0}\frac{\partial p}{\partial t} + \nabla_t\cdot\mathbf{v}_t &= -i\kappa\mathbf{v}\cdot\mathbf{z}_0, \\ \nabla_t p + n_0 m\frac{\partial \mathbf{v}}{\partial t} &= -i\kappa p\mathbf{z}_0, \end{aligned} \tag{18}$$

where $p = p(\boldsymbol{\rho}, t; \kappa)$ and $\mathbf{v} = \mathbf{v}(\boldsymbol{\rho}, t; \kappa)$ are the complex amplitudes of the pressure and velocity fields, and where $\mathbf{v} = \mathbf{v}_t + v_z\mathbf{z}_0$, $\nabla = \nabla_t + (\partial/\partial z)\mathbf{z}_0$, and $\mathbf{r} = \boldsymbol{\rho} + z\mathbf{z}_0$ represent vector decompositions transverse and longitudinal to the chosen symmetry direction $\mathbf{z}_0$. In a homogeneous transversely unbounded medium, guided-wave solutions of Eqs. (18) are distinguished by a subscript α and represented as

$$\begin{pmatrix} p(\mathbf{r}, t) \\ \mathbf{v}(\mathbf{r}, t) \end{pmatrix} = \begin{pmatrix} p_\alpha(\mathbf{k}_t, \omega) \\ \mathbf{v}_\alpha(\mathbf{k}_t, \omega) \end{pmatrix} e^{i(\mathbf{k}_t\cdot\boldsymbol{\rho}-\omega t)} e^{i\kappa_\alpha z}, \tag{19a}$$

where for simplicity of notation, the same symbol is employed for a field and its $\mathbf{k}_t$, ω-dependent amplitude. In the wavevector notation of Sec. 1.4a, one writes Eq. (19a) as

$$\Psi(\mathbf{r}, t) = \Psi_\alpha(\mathbf{k}_t, \omega)e^{i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)}e^{i\kappa_\alpha z}, \tag{19b}$$

whence Eqs. (18) can be written in the form

$$L\Psi = -i(K - \kappa_\alpha \Gamma)\Psi_\alpha = 0, \tag{20a}$$

where

$$-iK\left(\nabla_t, \frac{\partial}{\partial t}\right) \rightarrow \begin{pmatrix} \dfrac{1}{\gamma p_0}\dfrac{\partial}{\partial t} & \nabla_t \\ \nabla_t & n_0 m \dfrac{\partial}{\partial t}\mathbf{1} \end{pmatrix}, \quad K(\mathbf{k}_t, \omega) \rightarrow \begin{pmatrix} \dfrac{\omega}{\gamma p_0} & -\mathbf{k}_t \\ -\mathbf{k}_t & \omega n_0 m \mathbf{1} \end{pmatrix},$$

$$\Gamma \rightarrow \begin{pmatrix} 0 & \mathbf{z}_0 \\ \mathbf{z}_0 & 0 \end{pmatrix} \tag{20b}$$

are the operator components of Eqs. (10) and (11a); it should be noted that $K = K^+$, $\Gamma = \Gamma^+$, and thus $\Psi_\alpha^+ = \Psi_{\alpha^*}$. Since in the $\mathbf{k}_t$, ω basis the biorthogonality property in Eq. (4c) takes the form

$$\begin{aligned}(\Psi_{\alpha^*}(\mathbf{k}_t, \omega), \Gamma\Psi_\beta(\mathbf{k}_t, \omega)) &\equiv p_{\alpha^*}^*(\mathbf{k}_t, \omega)v_{z\beta}(\mathbf{k}_t, \omega) + v_{z\alpha^*}^*(\mathbf{k}_t, \omega)p_\beta(\mathbf{k}_t, \omega) \\ &= 2N_\alpha \delta_{\alpha\beta},\end{aligned} \tag{21}$$

and evidently involves only the independent field components p and v_z, it is frequently convenient to elide the dependent field component $\mathbf{v}_t$ and denote the abbreviated acoustic field wavevector Ψ as

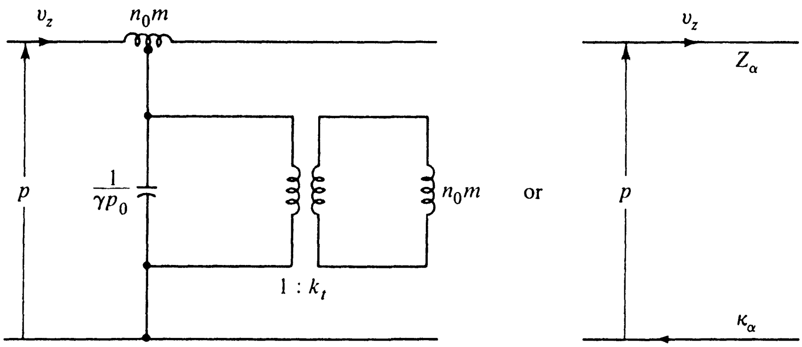
$$\bar{\Psi}(\mathbf{r}, t) \rightarrow \begin{pmatrix} p(\mathbf{r}, t) \\ v_z(\mathbf{r}, t)\mathbf{z}_0 \end{pmatrix}, \tag{22}$$

For eigenvectors of the form (19), one notes that $\nabla_t \equiv i\mathbf{k}_t$ and $\partial/\partial t \equiv -i\omega$. Hence, for prescribed $\mathbf{k}_t$, ω, non-vanishing solutions of Eqs. (18) or (20a) are found to exist only for the two wavenumbers (eigenvalues) given by the zeros of det L:

$$\kappa_{\pm 1} = \pm\sqrt{\frac{\omega^2}{a^2} - k_t^2}, \qquad \text{where } a = \sqrt{\frac{\gamma p_0}{n_0 m}}. \tag{23}$$

These are recognizable as the two propagation wavenumbers in the transmission-line network depicted in Fig. 1. 4. 1; the per-unit-length elements of the line comprise a series "inductance" $n_0 m$ and a shunt "capacitance" $1/\gamma p_0$ coupled by an ideal transformer of turns ratio $1 : k_t$ to an "inductance" $n_0 m$. The transmission-line interpretation becomes evident on elimination of the dependent field variable $\mathbf{v}_t$ from Eqs. (18), whence in the $\mathbf{k}_t$, ω basis the z-dependent acoustic equations (18), in which $i\kappa = \partial/\partial z$, may be expressed in the form

$$\begin{aligned}\frac{\partial v_z}{\partial z} &= i\left(\frac{\omega}{\gamma p_0} - \frac{k_t^2}{\omega n_0 m}\right)p = i\kappa Y p, \\ \frac{\partial p}{\partial z} &= i\omega n_0 m v_z \qquad\quad = i\kappa Z v_z,\end{aligned} \tag{24a}$$



(a) Per-unit-length network (b) Per-unit-length continuous line

FIG. 1.4.1 Acoustic transmission-line network.

where

$$\kappa = \sqrt{\frac{\omega^2}{a^2} - k_t^2} \qquad \text{and} \qquad Z = \frac{\omega n_0 m}{\kappa} = \frac{1}{Y}. \tag{24b}$$

κ and Z are the propagation constant and characteristic impedance, and p and v_z play the role of generalized transmission-line "voltage" and "current," respectively. To the eigenvalues $\kappa_{\pm 1}$ in Eq. (23) there correspond eigenvectors $\Psi_{\pm 1}$ of the form (19b), normalized to $2N_{\pm 1}$ in accordance with the biorthogonality property in Eq.(21); these follow from Eqs. (18) or (11a) and (20) as

$$\Psi_{\pm 1} \rightarrow \begin{pmatrix} 1 \\ (\kappa_{\pm 1}\mathbf{z}_0 + \mathbf{k}_t)/\omega n_0 m \end{pmatrix}, \qquad N_{\pm 1} = \frac{\kappa_{\pm 1}}{\omega n_0 m} = \pm Y \tag{25}$$

and display the field structure of the two possible guided eigenwaves for given $\mathbf{k}_t$ and ω.

Knowledge of the guided-wave characteristics in Eqs. (23) and (25) permits an explicit representation of the acoustic Green's functions defined by Eqs. (9) and (20). Thus via Eq. (17), the z-guided-wave representation of the acoustic Green's function G_{11} of Eq. (1.1.4) is given by

$$G_{11}(\mathbf{r}, \mathbf{r}'; t, t') = \int \frac{\omega n_0 m}{2\sqrt{(\omega^2/a^2) - k_t^2}} e^{i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)} e^{\pm i\sqrt{(\omega^2/a^2) - k_t^2}\, z} \frac{d\mathbf{k}_t\, d\omega}{(2\pi)^3}, \quad z \gtrless 0, \tag{26}$$

where the integration extends over all real $\mathbf{k}_t$, ω from $-\infty$ to $+\infty$, and for simplicity we have set $\boldsymbol{\rho}' = z' = t' = 0$. Equation (26) provides a representation alternative to that in Eq. (1.3.25a), but when integrated is identical to the closed form result in that equation. It should be noted that Eq. (26) also follows from the $\mathbf{k}, \omega$ representation of G_{11} in Eqs. (1.2.7) and (1.2.8) by integration over the z component κ of the wavevector $\mathbf{k} = \mathbf{k}_t + \kappa \mathbf{z}_0$ (see Sec. 1.5c).

1.4c The Electromagnetic Field

Source-free guided waves in an electromagnetic field have an exp $(i\kappa z)$ dependence and satisfy the homogeneous form of the Maxwell equations (1.1.16) with $\partial/\partial z \equiv i\kappa$:

$$\begin{aligned} \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla_t \times \mathbf{H} &= i\kappa \mathbf{z}_0 \times \mathbf{H}, \\ \nabla_t \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} &= -i\kappa \mathbf{z}_0 \times \mathbf{E}. \end{aligned} \tag{27}$$

Equations (27) constitute an eigenvalue problem of the type shown in Eq. (10), with $\mathbf{E} = \mathbf{E}(\boldsymbol{\rho}, t; \kappa)$ and $\mathbf{H} = \mathbf{H}(\boldsymbol{\rho}, t; \kappa)$ as the eigenvector amplitudes of the electric and magnetic fields, and κ as the eigenvalue. As before, $\nabla = \nabla_t + i\kappa\mathbf{z}_0$ and $\mathbf{r} = \boldsymbol{\rho} + z\mathbf{z}_0$ represent vector decompositions transverse and longitudinal to the guiding direction z. In a transversely unbounded, homogeneous medium, eigenwave solutions of Eq. (27) are representable in the form

$$\begin{pmatrix} \mathbf{E}(\mathbf{r}, t) \\ \mathbf{H}(\mathbf{r}, t) \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\alpha(\mathbf{k}_t, \omega) \\ \mathbf{H}_\alpha(\mathbf{k}_t, \omega) \end{pmatrix} e^{i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)} e^{i\kappa_\alpha z}, \tag{28a}$$

where the subscript α distinguishes different eigensolutions with a fixed $\mathbf{k}_t, \omega$ dependence. For notational convenience, we employ the same symbol for a field as for its $\mathbf{k}_t, \omega$ amplitude; ambiguity is avoided by specifying, where necessary, the arguments of the field function.

In the Ψ wavevector notation of Sec. 1.4a, one rewrites Eq. (28a) as

$$\Psi(\mathbf{r}, t) = \Psi_\alpha(\mathbf{k}_t, \omega) e^{i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)} e^{i\kappa_\alpha z}, \tag{28b}$$

whence Eqs. (27) with $\kappa = \kappa_\alpha$ take the general form

$$L\Psi_\alpha = -i(K - \kappa_\alpha \Gamma)\Psi_\alpha = 0 \tag{29a}$$

with the component operators K and Γ of Eqs. (10) and (11a) defined by

$$-iK\left(\nabla_t, \frac{\partial}{\partial t}\right) \to \begin{pmatrix} \epsilon_0 \frac{\partial}{\partial t} \mathbf{1} & -\nabla_t \times \mathbf{1} \\ \nabla_t \times \mathbf{1} & \mu_0 \frac{\partial}{\partial t} \mathbf{1} \end{pmatrix}, \quad K(\mathbf{k}_t, \omega) \to \begin{pmatrix} \omega\epsilon_0 \mathbf{1} & \mathbf{k}_t \times \mathbf{1} \\ -\mathbf{k}_t \times \mathbf{1} & \omega\mu_0 \mathbf{1} \end{pmatrix},$$

$$\Gamma \to \begin{pmatrix} 0 & -\mathbf{z}_0 \times \mathbf{1} \\ \mathbf{z}_0 \times \mathbf{1} & 0 \end{pmatrix} \tag{29b}$$

The operators $K = K^+$ and $\Gamma = \Gamma^+$ are Hermitian (since $\mathbf{a} \times \mathbf{1} = \mathbf{1} \times \mathbf{a} = -\widetilde{\mathbf{a} \times \mathbf{1}}$), and hence $\Psi_\alpha^+ = \Psi_{\alpha^*}$.† In the $\mathbf{k}_t, \omega$ basis, the biorthogonality property (4c) of the eigenvectors Ψ_α takes the specific form

†K is Hermitian in the transversely unbounded case and, with suitable transverse boundary conditions, also in the bounded case. In this case the adjoint eigenvector $\Psi_\alpha^+ = \Psi_{\alpha^*}$, where Ψ_{α^*} is the eigenvector with eigenvalue κ_α^*.

$$(\Psi_{\alpha^*}, \Gamma\Psi_\beta) = \mathbf{E}^*_{\alpha^*}(\mathbf{k}_t, \omega)\cdot\mathbf{H}_\beta(\mathbf{k}_t, \omega) \times \mathbf{z}_0 + \mathbf{H}^*_{\alpha^*}(\mathbf{k}_t, \omega)\cdot\mathbf{z}_0 \times \mathbf{E}_\beta(\mathbf{k}_t, \omega) = 2N_\alpha\delta_{\alpha\beta} \tag{30}$$

and involves only transverse electric- and magnetic-field components. It is thereby implied that the transverse components $\mathbf{E}_t$, $\mathbf{H}_t$ are the independent components of the electromagnetic field; accordingly, one frequently employs an abbreviated notation for the transverse electromagnetic field wavevector,

$$\Psi(\mathbf{r}, t) \to \begin{pmatrix}\mathbf{E}_t(\mathbf{r}, t)\\ \mathbf{H}_t(\mathbf{r}, t)\end{pmatrix}, \tag{31}$$

whose eigenvectors are represented as in Eq. (28b) and possess the same biorthogonality property as in Eq. (30).

In a transversely unbounded and homogeneous medium, electromagnetic eigenvectors of the form (28) are defined as the non-vanishing resonant solutions of Eqs. (27) with $\nabla_t \equiv i\mathbf{k}_t$ and $\partial/\partial t \equiv -i\omega$. For prescribed $\mathbf{k}_t, \omega$, such solutions exist only for the four degenerate eigenvalues (the zeros of det L)

$$\kappa_{\pm 1} = \pm\sqrt{\frac{\omega^2}{c^2} - k_t^2} = \kappa_{\pm 2}, \qquad \text{where } c = \frac{1}{\sqrt{\mu_0\epsilon_0}}. \tag{32}$$

The existence of four eigenvalues is commensurate with the fact that there are four independent components of the transverse electromagnetic field. In a $\mathbf{k}_t, \omega$ basis, the dependent longitudinal field components E_z, H_z may be eliminated via the relations

$$-\omega\epsilon_0 E_z = \mathbf{k}_t\cdot\mathbf{H}_t \times \mathbf{z}_0 \qquad \text{and} \qquad -\omega\mu_0 H_z = \mathbf{k}_t\cdot\mathbf{z}_0 \times \mathbf{E}_t, \tag{33}$$

derivable from Eqs. (27). On restoring $\partial/\partial z = i\kappa$ in Eqs. (27), one thereby obtains in a $\mathbf{k}_t, \omega$ basis the following defining equations for the z-dependent transverse field amplitudes $\mathbf{E}_t$, $\mathbf{H}_t$:

$$\begin{aligned}\frac{\partial \mathbf{E}_t}{\partial z} &= \left(i\omega\mu_0\mathbf{1} + \frac{\mathbf{k}_t\mathbf{k}_t}{i\omega\epsilon_0}\right)\cdot(\mathbf{H}_t \times \mathbf{z}_0),\\ \frac{\partial \mathbf{H}_t}{\partial z} &= \left(i\omega\epsilon_0\mathbf{1} + \frac{\mathbf{k}_t\mathbf{k}_t}{i\omega\mu_0}\right)\cdot(\mathbf{z}_0 \times \mathbf{E}_t).\end{aligned} \tag{34}$$

The form of the bracketed dyadics in Eq. (34) suggests the following transverse vector decompositions along the directions $\mathbf{k}_{t0}$ and $\mathbf{k}_{t0} \times \mathbf{z}_0$ ($\mathbf{k}_{t0}$ is the unit vector in the direction of $\mathbf{k}_t$):

$$\begin{aligned}\mathbf{E}_t(\mathbf{k}_t, \omega; z) &= E_t'(z)\mathbf{k}_{t0} + E_t''(z)\mathbf{k}_{t0} \times \mathbf{z}_0,\\ \mathbf{H}_t(\mathbf{k}_t, \omega; z) &= H_t'(z)\mathbf{z}_0 \times \mathbf{k}_{t0} + H_t''(z)\mathbf{k}_{t0},\end{aligned} \tag{35a}$$

which, on substitution into Eqs. (34), uncouple the latter into two independent sets of scalar equations:

$$\begin{aligned}\frac{\partial E_t'}{\partial z} &= \left(i\omega\mu_0 + \frac{k_t^2}{i\omega\epsilon_0}\right)H_t', &\qquad \frac{\partial E_t''}{\partial z} &= i\omega\mu_0 H_t'',\\ \frac{\partial H_t'}{\partial z} &= i\omega\epsilon_0 E_t', &\qquad \frac{\partial H_t''}{\partial z} &= \left(i\omega\epsilon_0 + \frac{k_t^2}{i\omega\mu_0}\right)E_t''.\end{aligned} \tag{35b}$$

Equations (35b) are recognized as transmission-line equations for the independent fields specified by the amplitudes E_t', H_t' and E_t'', H_t'', and can be schematized by the networks shown in Fig. 1.4.2. The amplitudes E_t', H_t' distinguish the "voltage" and "current" of an E mode (with $H_z = 0$) on a transmission line, depicted in Fig. 1.4.2a, whose series and shunt elements per differential length are expressible in terms of an "inductance" μ_0, a "capacitance" ϵ_0, and an ideal transformer of turns ratio k_t. A corresponding interpretation of the amplitudes E_t'', H_t'' is depicted in the H-mode (with $E_z = 0$) network of Fig. 1.4.2b. The equivalent smoothed "two-wire-line" schematizations are also shown in Fig. 1.4.2. and are expressible in terms of characteristic impedances Z', Z'' and propagation wavenumbers κ', κ'' that may be inferred from the networks of Fig. 1.4.2, or Eqs. (35b), as

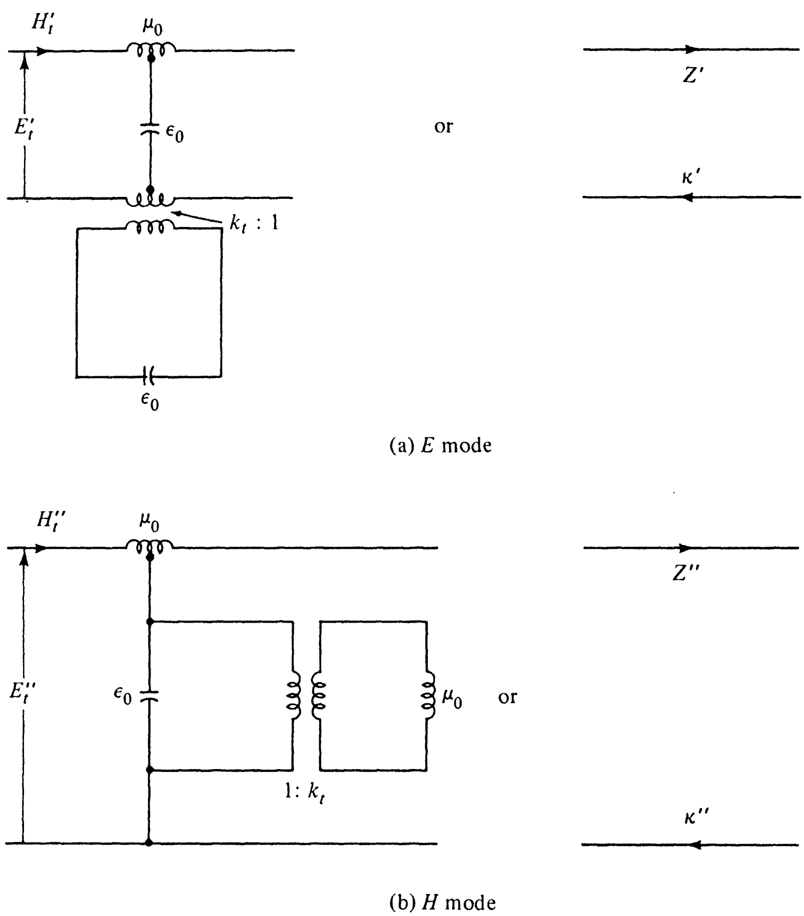


FIG. 1.4.2 Electromagnetic transmission-line networks.

$$Z' = \frac{\kappa'}{\omega\epsilon_0} = \frac{1}{Y'}, \qquad Z'' = \frac{\omega\mu_0}{\kappa''} = \frac{1}{Y''},$$
$$\kappa' = \sqrt{\omega^2\mu_0\epsilon_0 - k_t^2} = \kappa''. \tag{36a}$$

The relations (36a) permit the rewriting of Eqs. (35b) in the alternative transmission-line form:

$$\frac{\partial E_t'}{\partial z} = i\kappa' Z' H_t' \qquad \frac{\partial E_t''}{\partial z} = i\kappa'' Z'' H_t'',$$
$$\frac{\partial H_t'}{\partial z} = i\kappa' Y' E_t' \qquad \frac{\partial H_t''}{\partial z} = i\kappa'' Y'' E_t''. \tag{36b}$$

They readily permit the evaluation of the four independent guided-mode wavevectors Ψ_α as (note $\Psi_\alpha^+ = \Psi_{\alpha^*}$)†

$$\Psi_{\pm 1} \longrightarrow \begin{pmatrix} \pm Z'\mathbf{k}_{t0} - \dfrac{k_t}{\omega\epsilon_0}\mathbf{z}_0 \\ \mathbf{z}_0 \times \mathbf{k}_{t0} \end{pmatrix}, \qquad \Psi_{\pm 2} \longrightarrow \begin{pmatrix} \pm Z''\mathbf{k}_{t0} \times \mathbf{z}_0 \\ \mathbf{k}_{t0} - \dfrac{k_t}{\kappa_{\pm 1}}\mathbf{z}_0 \end{pmatrix}, \tag{37a}$$

with normalization constants

$$N_{\pm 1} = \pm Z' = \frac{\kappa_{\pm 1}}{\omega\epsilon_0}, \qquad N_{\pm 2} = \pm Z'' = \frac{\omega\mu_0}{\kappa_{\pm 2}} \tag{37b}$$

and eigenvalues $\kappa_{\pm 1}$ and $\kappa_{\pm 2}$, as shown in Eq. (32). The wavevectors (37a), which display the field structures of the two independent guided waves for prescribed $\mathbf{k}_t$, ω, satisfy the biorthogonality property in Eq. (30).

With the knowledge of the wavevectors Ψ_α and eigenvalues κ_α from Eqs. (37) and (32), one can employ Eq. (17) to obtain a guided-wave representation of the electromagnetic-field Green's functions. Thus, by Eq. (17), the matrix electromagnetic Green's function G of Eq. (1.1.19) may be represented as

$$G(\mathbf{r}, \mathbf{r}'; t, t') = \int \left[\frac{\Psi_{\pm 1}\Psi_{\pm 1^*}}{2\kappa/\omega\epsilon_0} + \frac{\Psi_{\pm 2}\Psi_{\pm 2^*}}{2\omega\mu_0/\kappa}\right] e^{i(\mathbf{k}_t\cdot\boldsymbol{\rho} - \omega t)}\, e^{\pm i\kappa z} \frac{d\mathbf{k}_t d\omega}{(2\pi)^3} \tag{38a}$$

where the $\pm$ signs refer to $z \gtrless 0$, respectively, and where, for simplicity, we have set $\boldsymbol{\rho}' = t' = z' = 0$. To evaluate the dyadic Green's function $\mathscr{G}_{11}$, one substitutes the first-row elements of Ψ_α from Eq. (37a) into Eq. (38a). Noting that the bracketed term in the integrand of Eq. (38a) yields‡

$$[\quad]_{11} = \frac{1}{2}\left[\frac{\omega\epsilon_0}{\kappa}\mathbf{k}_{t0}\mathbf{k}_{t0} \mp \frac{k_t}{\omega\epsilon_0}(\mathbf{k}_{t0}\mathbf{z}_0 + \mathbf{z}_0\mathbf{k}_{t0}) + \frac{k_t^2}{\omega\epsilon_0\kappa}\mathbf{z}_0\mathbf{z}_0 + \frac{\omega\mu_0}{\kappa}(\mathbf{k}_{t0} \times \mathbf{z}_0)(\mathbf{k}_{t0} \times \mathbf{z}_0)\right],$$

†See Eq. (8.2.10a).

‡Note that the ijth element of the matrix $\Psi_\alpha\Psi_{\alpha^*}$ is the product of the ith-row element of Ψ_α and the conjugate of the jth-row element of Ψ_{α^*}; for lossless guides $\Psi_{\alpha^*}^* = \Psi_\alpha$. The ijth element is recognized as $(\varphi_i, \Psi_\alpha)(\Psi_{\alpha^*}, \varphi_j)$, where φ_i and φ_j are "unit wavevectors" and the Hermitian product is implied.

one obtains

$$\mathscr{G}_{11} = \left(\mu_0 \frac{\partial}{\partial t} \mathbf{1} - \frac{\nabla\nabla}{\epsilon_0(\partial/\partial t)}\right) \int \frac{e^{i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)}\, e^{\pm i\kappa z}}{-2i\kappa} \frac{d\mathbf{k}_t\, d\omega}{2(\pi)^3}, \qquad z \gtrless 0, \tag{38b}$$

where we have taken $\nabla = ik_t\mathbf{k}_{t0} \pm i\kappa\mathbf{z}_0$ and $\partial/\partial t = -i\omega$ terms outside the integral and where $\kappa = \kappa' = \kappa''$ by Eq. (36a) or (32). Equation (38b) provides a guided-wave representation alternative to the oscillatory representation in Eq. (1.3.32b) but is identical to the integrated result shown in Eq. (1.3.32c). One notes that the integral in Eq. (38b) follows from the $\mathbf{k}, \omega$ integral of Eq. (1. 2. 20) by residue integration over the κ components of the wavevector $\mathbf{k} = \mathbf{k}_t + \kappa\mathbf{z}_0$(see Sec 1.5c).

1.5 REDUCED ELECTROMAGNETIC FIELD EQUATIONS

In Secs. 1.1–1.4 the discussion of linear fields has been based primarily on a first-order formulation of the field equations. However, much of the technical literature on electromagnetic wave propagation in homogeneous or inhomogeneous media is based on a "reduced" formulation wherein non-electromagnetic variables have been eliminated from the first-order equations. The reduced field equations are characterized by constitutive parameters (permittivity† and permeability) that depend in general on the derivative operators ∇ and $\partial/\partial t$, and for inhomogeneous but stationary media on the spatial coordinate $\mathbf{r}$ as well. An example is provided by an ionized plasma medium whose electromagnetic properties are described in the cold-fluid approximation by a temporally dispersive (i.e., $(\partial/\partial t)$-dependent) dielectric tensor; inclusion of temperature effects introduces an additional spatial dispersion (∇ dependence), as noted in Eqs. (1.1.64).

The reduced field equations may be solved in terms of Green's functions $\mathbf{G}_{ij}$ that are identical to, but fewer in number than, those discussed in Sec. 1.1d, since the $_{ij}$ now refer only to a reduced number of field variables. Thus, in the time-dependent case, the electric field $\mathbf{E}(\mathbf{r}, t)$ excited solely by an electric current density $\mathbf{J}(\mathbf{r}, t)$ in an inhomogeneous medium is given by

$$\mathbf{E}(\mathbf{r}, t) = -\int \mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t') \cdot \mathbf{J}(\mathbf{r}', t')\, d\mathbf{r}'\, dt',$$

which implies that a reduced field equation for this case is of the form

$$\mathscr{Y}(\nabla, \frac{\partial}{\partial t}; \mathbf{r}) \cdot \mathbf{E}(\mathbf{r}, t) = -\mathbf{J}(\mathbf{r}, t),$$

where the derivative (admittance) operator $\mathscr{Y}$ manifestly has the Green's function $\mathscr{G}_{11}$ as its inverse. Alternatively, the reduced field equations can be expressed in terms of both the electric and magnetic-field variables $\mathbf{E}$ and $\mathbf{H}$,

†Since medium properties may be spatially variable, we frequently use the term "permittivity" instead of "dielectric constant."

as, for example, in Eqs. (1.1.64), and will involve dispersive permittivity and permeability operators $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$, respectively, instead of the admittance $\mathscr{Y}$.

Because of the dispersive nature of the constitutive parameters, energy-storage and power-flow relations in a reduced field description are expressed differently than in the first-order formulation of Eq. (1.1.55). To identify energy-density and power-flow expressions, it is necessary to distinguish between harmonic fields of the form exp $[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ and wavepackets that are superpositions of such harmonic fields. A number of energy theorems can be inferred from the dispersive properties (i. e., from the ω and $\mathbf{k}$ derivatives) of the admittance operator $\mathscr{Y}$ for the harmonic fields. Because time-harmonic fields represent an important class of problems, the presentation emphasizes this regime. Reciprocity properties for reduced time-harmonic fields are similar to those obtained in previous sections and do not require separate treatment.

It has been noted that certain results for spatially inhomogeneous or bounded media can be derived by straightforward modification of those for the homogeneous case. These aspects will be pursued in more detail in the present section. In addition to, or instead of, the radiation (outgoing energy) condition at infinity employed for unbounded regions, electromagnetic fields on boundaries or interfaces must satisfy other boundary conditions. Also, in the presence of medium inhomogeneities, the reciprocity properties of the field are modified from their form in an unbounded homogeneous space. These aspects are considered in Sec. 1.5b.

Propagation in plane-stratified media or in regions bounded by cylindrical, perfectly conducting walls can be analyzed by simple modification of the guided-wave representations of Sec. 1.4 (see Secs. 2.1–2.3 et seq.). For the vector electromagnetic field, this entails a scalarized description carried out with respect to a "waveguide axis" z parallel to the boundary walls or to the direction of stratification. The resulting scalar Green's function problems, stated in Eqs. (1.1.38b) and (1.1.49), are not tied to a special choice of z as the preferred direction and it is often convenient to analyze these scalar problems in alternative "guided" representations with respect to coordinates transverse to z. An introduction to the theory of alternative representations is given in Sec. 1.5c. A more general treatment will be given in Sec 3.3c, and specific applications may be found in Secs. 5.4a and 5.6a.

1.5a Energy Density, Power Flow, and Group Velocity for the Electromagnetic Field

Energy-conservation theorems are important for characterization of energy storage and transport properties in a field. In a first-order field formulation, wherein the constitutive parameters for a medium are non-dispersive, conservation theorems take on a relatively simple form, as is illustrated for a plasma medium in Eq.(1.1.55). In a reduced formulation, wherein non-electromagnetic variables have been eliminated, one is led to vacuum-like electromagnetic field equations but with a permittivity $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\nabla, \partial/\partial t; \mathbf{r})$ and permeability $\boldsymbol{\mu} = \boldsymbol{\mu}(\nabla, \partial/\partial t; \mathbf{r})$ which are both anisotropic, spatially and temporally dispersive, and for a spatially inhomogeneous (but stationary) medium, dependent on $\mathbf{r}$ as well. The reduced formulation is exemplified by Eqs. (1.1.64) for the special case of a warm isotropic homogeneous plasma fluid. Because of the dispersive nature of the $\boldsymbol{\epsilon}, \boldsymbol{\mu}$ parameters, energy-conservation theorems for non-monochromatic reduced fields require care in their interpretation. For example, the difficulty with interpreting $\epsilon|E|^2/2$ as a stored (positive) electric energy density in an isotropic but dispersive medium may be appreciated by recalling that, for an isotropic cold plasma under time-harmonic conditions, the equivalent permittivity $\epsilon = \epsilon_0(1 - \omega_p^2/\omega^2)$ may be negative for certain frequency ranges.

Energy-transport properties of a field are investigated most naturally for time-dependent conditions since during a finite interval one may then follow a bundle of energy (wavepacket) excited by a source. As noted in Sec. 1.3, source-excited, time-dependent fields in unbounded regions may be represented rigorously as integral superpositions of oscillatory plane waves with different wavenumber $\mathbf{k}$. The evaluation of these representations by asymptotic techniques will be discussed in Sec. 1.6, where it is found that the field energy is localized in regions wherein the plane waves constituting a wavepacket interfere constructively. The condition for constructive interference leads to the conclusion that, in certain regions, a field behaves *locally* like a plane wave $\Psi \sim \exp(i\psi)$, where $\psi = [\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t]$, and that its energy is transported with a group velocity $\mathbf{v}_g = \nabla_k\omega(\mathbf{k})$ given by the gradient in $\mathbf{k}$ space of the frequency $\omega = \omega(\mathbf{k})$ determined from the dispersion equation for the medium [see Eq. (1.3.10a)]. The direction of $\mathbf{v}_g$ is called the *ray* direction, and in an anisotropic medium this differs generally from the direction of the wavevector $\mathbf{k}$ along which equiphase surfaces (wavefronts) $\psi =$ constant advance with a phase speed ω/k. Relations governing the detailed behavior of rays, wavepackets, wavefronts, etc., will be considered in Secs. 1.6 and 1.7 both for time-dependent and for time-independent (harmonic) conditions. In the present section, energy-transport properties are inferred not from specific integral representations of the field, but from general conservation theorems derivable from reduced electromagnetic field equations.

In the following we limit our considerations to a reduced electromagnetic field description with only temporally dispersive, space-dependent, dyadic parameters $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$; the effects of spatial dispersion can be included but for the most part are omitted for simplicity of presentation. Medium anisotropy is retained to emphasize the distinction between phase- and energy-propagation characteristics. Since applications to dispersive media in this book relate primarily to anisotropic plasmas, the permeability μ is usually assumed to be a scalar function, thereby simplifying some of the manipulations and also exhibiting within a single formula expressions appropriate to tensor and scalar media. The extension to media with dyadic permeability is easily accomplished.

The reduced time-dependent Maxwell equations descriptive of the electromagnetic field excited by prescribed electric and magnetic current distributions $\mathbf{J}(\mathbf{r}, t)$ and $\mathbf{M}(\mathbf{r}, t)$ in a temporally dispersive, spatially inhomogeneous, anisotropic medium are given by

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t) - \mathbf{M}(\mathbf{r}, t). \\ \nabla \times \mathbf{H}(\mathbf{r}, t) &= \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t). \end{aligned} \tag{1a}$$

The constitutive relations are assumed to be of the form

$$\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\mu}\left(\frac{\partial}{\partial t}, \mathbf{r}\right)\cdot\mathbf{H}(\mathbf{r}, t), \qquad \mathbf{D}(\mathbf{r}, t) = \boldsymbol{\epsilon}\left(\frac{\partial}{\partial t}, \mathbf{r}\right)\cdot\mathbf{E}(\mathbf{r}, t), \tag{1b}$$

which are applicable in the absence of spatial dispersion and can equally well be phrased in terms of linear integral operators. For only electric current excitation (i.e., $\mathbf{M} = 0$), elimination of $\mathbf{H}$ from Eqs. (1) leads to†

$$\left[\frac{\partial}{\partial t}\boldsymbol{\epsilon} + \nabla \times \left(\frac{1}{(\partial/\partial t)\boldsymbol{\mu}}\cdot(\nabla \times \mathbf{1})\right)\right]\cdot\mathbf{E}(\mathbf{r}, t) = -\mathbf{J}(\mathbf{r}, t) \tag{2a}$$

or

$$\mathscr{Y}\left(\nabla, \frac{\partial}{\partial t}; \mathbf{r}\right)\cdot\mathbf{E}(\mathbf{r}, t) = -\mathbf{J}(\mathbf{r}, t) \tag{2b}$$

where the spatially and temporally dispersive dyadic (admittance) operator $\mathscr{Y}$ evidently characterizes the reduced electromagnetic field. As in Sec. 1.1, the solution of Eqs. (2) is expressible as

$$\mathbf{E}(\mathbf{r}, t) = -\int \mathscr{G}_{11}(\mathbf{r}, \mathbf{r}'; t, t')\cdot\mathbf{J}(\mathbf{r}', t')\, d\mathbf{r}'\, dt' \tag{3}$$

in terms of a dyadic Green's function $\mathscr{G}_{11}$ which is the inverse of the admittance operator $\mathscr{Y}$. The desired energy-transport properties of the field are deducible from the dispersive properties of the operator $\mathscr{Y}$.

Energy density and power flow

Before ascertaining the equation of energy transport, we shall first attempt to identify energy-density and power-flow expressions for the field. From the source-free ($\mathbf{J} = 0 = \mathbf{M}$) form of the field equations (1a), one derives the well-known relation

$$\nabla\cdot(\mathbf{E} \times \mathbf{H}) = -\left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}\right). \tag{4}$$

$\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the instantaneous density of electromagnetic power flow at $\mathbf{r}, t$ (i.e., the Poynting vector), but it is difficult to interpret the expression in parentheses on the right of Eq. (4) as the time derivative of an instantaneous

†Note that $1/(\partial/\partial t)$ is the integral operator $\int dt$ and $1/\boldsymbol{\mu}$ is the inverse dyadic $\boldsymbol{\mu}^{-1}$.

energy density W. This difficulty stems from the temporally dispersive nature of the constitutive parameters in the relations

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t}\left[\boldsymbol{\epsilon}\left(\frac{\partial}{\partial t}, \mathbf{r}\right)\cdot\mathbf{E}\right], \qquad \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}\left[\boldsymbol{\mu}\left(\frac{\partial}{\partial t}, \mathbf{r}\right)\cdot\mathbf{H}\right], \tag{5}$$

in consequence of which it is not possible to identify an energy density W for general time-dependent fields. An energy interpretation on a time average basis is, however, possible for monochromatic steady-state fields of the form

$$\mathbf{E}(\mathbf{r}, t) = \sqrt{2}\ \mathrm{Re}\ [\bar{\mathbf{E}}(\mathbf{r})e^{-i\omega t}], \qquad \mathbf{H}(\mathbf{r}, \mathrm{t}) = \sqrt{2}\ \mathrm{Re}\ [\bar{\mathbf{H}}(\mathbf{r})e^{-i\omega t}] \tag{6}$$

where $\bar{\mathbf{E}}(\mathbf{r})$ and $\bar{\mathbf{H}}(\mathbf{r})$ are conventional root-mean-square (rms) complex time-independent field amplitudes, whence the relations (5) become†

$$\begin{aligned} \frac{\partial \mathbf{D}}{\partial t} &= -\sqrt{2}\ \mathrm{Re}\ [i\omega\boldsymbol{\epsilon}(\omega, \mathbf{r})\cdot\bar{\mathbf{E}}e^{-i\omega t}], \\ \frac{\partial \mathbf{B}}{\partial t} &= -\sqrt{2}\ \mathrm{Re}\ [i\omega\boldsymbol{\mu}(\omega, \mathbf{r})\cdot\bar{\mathbf{H}}e^{-i\omega t}]. \end{aligned} \tag{7}$$

With this complex amplitude notation the time average ‡ of Eq. (4) can be expressed as

$$\nabla\cdot\bar{\mathbf{S}} = -\frac{\overline{\partial W}}{\partial t}, \tag{8}$$

where

$$\begin{aligned} \bar{\mathbf{S}} &= \mathrm{Re}\,\bar{\mathbf{E}} \times \bar{\mathbf{H}}^*, \\ \frac{\overline{\partial W}}{\partial t} &= -\frac{i\omega}{2}[\bar{\mathbf{E}}^* \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^+) \cdot \bar{\mathbf{E}} + \bar{\mathbf{H}}^* \cdot (\boldsymbol{\mu} - \boldsymbol{\mu}^+) \cdot \bar{\mathbf{H}}] \end{aligned} \tag{8a}$$

and $\boldsymbol{\epsilon}^+$ and $\boldsymbol{\mu}^+$ denote the transposed conjugate (Hermitian adjoint) dyadics to $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$. Although the expression for $\overline{\partial W}/\partial t$ in Eq. (8a) represents dimensionally the time rate of increase of energy per unit volume, it does not permit an identification of an average energy density $\bar{W}$. For a medium with a real scalar permeability $\boldsymbol{\mu} = \boldsymbol{\mu}^+ = \mu\mathbf{1}$,

$$\frac{\overline{\partial W}}{\partial t} = \bar{\mathbf{E}}^* \cdot \boldsymbol{\sigma} \cdot \bar{\mathbf{E}} \tag{9}$$

represents conventionally the power dissipation per unit volume in terms of a medium conductivity dyadic§ $\boldsymbol{\sigma} = -i\omega(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^+)/2$. If the medium is lossy,

†Note in Eq. (5) that $\partial/\partial t$ commutes with $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ in stationary media.

‡In complex notation the time average of the product of harmonic time-dependent quantities AB is $(1/T)\int_0^T AB\,dt = \mathrm{Re}\,\bar{A}\bar{B}^* = \frac{1}{2}(\bar{A}\bar{B}^* + \bar{A}^*\bar{B})$, where $T = 2\pi/\omega$ is the oscillation period.

§Losses due to conduction currents $\bar{\mathbf{J}}_c(\mathbf{r}, \omega) = \boldsymbol{\sigma}(\mathbf{r}, \omega)\cdot\bar{\mathbf{E}}(\mathbf{r}, \omega)$ are expressed in terms of the conductivity dyadic $\boldsymbol{\sigma}$ included in the dielectric tensor $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_1 + (\boldsymbol{\sigma}/-i\omega)$. It follows from Eq. (10) that $\boldsymbol{\epsilon}_1$ represents the Hermitian part of $\boldsymbol{\epsilon}$ (i.e., $\boldsymbol{\epsilon}_1 = \boldsymbol{\epsilon}_1^+$). Since $\boldsymbol{\sigma}/-i\omega$ is then anti-Hermitian, one concludes that the conductivity dyadic is also Hermitian (i.e., $\boldsymbol{\sigma} = \boldsymbol{\sigma}^+$).

energy is converted into heat and the expression on the right-hand side of Eq. (9) must be positive. This condition applies more generally to Eq. (8a), and for arbitrary fields this is assured only if each of the terms is positive, whence the Hermitian tensors $-i(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^+)$ and $-i(\boldsymbol{\mu} - \boldsymbol{\mu}^+)$ must be positive definite. For a lossless medium with a real permeability, the average energy density remains constant and the consequent vanishing of the right-hand side of Eq. (8a) requires that (if $\boldsymbol{\mu} = \mu\mathbf{1}$)

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^+, \qquad \mu = \text{real}. \tag{10}$$

Thus, the dyadics representative of the constitutive parameters in a lossless anisotropic medium must be Hermitian. If the medium is isotropic, the constitutive parameters are real but may be negative for certain frequencies.

To calculate the average stored electric energy density in a lossless, dispersive medium, it is necessary to consider non-monochromatic fields. Generalizing Eqs (6), we introduce

$$\mathbf{E}(\mathbf{r}, t) = \sqrt{2}\ \mathrm{Re}\,[\bar{\mathbf{E}}(\mathbf{r}; t)e^{-i\omega t}], \tag{11a}$$

$$\mathbf{D}(\mathbf{r}, t) = \sqrt{2}\ \mathrm{Re}\,[\bar{\mathbf{D}}(\mathbf{r}; t)e^{-i\omega t}], \tag{11b}$$

where $\bar{\mathbf{E}}$ and $\bar{\mathbf{D}}$ are complex rms functions of t that vary slowly during a time interval equal to the period $T = 2\pi/\omega$. Then over the period T, the time average of the first term on the right of Eq. (4) yields

$$\begin{aligned}\overline{\frac{\partial W_e}{\partial t}} &= \frac{1}{2T}\int_0^T (\bar{\mathbf{E}}e^{-i\omega t} + \bar{\mathbf{E}}^*e^{i\omega t}) \cdot \frac{\partial}{\partial t}(\bar{\mathbf{D}}e^{-i\omega t} + \bar{\mathbf{D}}^*e^{i\omega t})\,dt \\ &\cong \frac{1}{2T}\int_0^T \left[\bar{\mathbf{E}}^*e^{i\omega t} \cdot \frac{\partial}{\partial t}(\bar{\mathbf{D}}e^{-i\omega t}) + \bar{\mathbf{E}}e^{-i\omega t} \cdot \frac{\partial}{\partial t}(\bar{\mathbf{D}}^*e^{i\omega t})\right] dt,\end{aligned} \tag{12}$$

and since $t = 0$ is an arbitrary reference point, the result in Eq. (12) refers to a typical period. Contributions arising from terms in the integrand containing factors $\exp(\pm i2\omega t)$ have been ignored since they are small compared to the remaining ones. This is verified upon assuming, for example, that during the relevant time interval, $\bar{\mathbf{E}}$ and $\bar{\mathbf{D}}$ may be represented as† [9a]

$$\bar{\mathbf{E}}(\mathbf{r}; t) = \bar{\mathbf{E}}_1(\mathbf{r})e^{-i\omega_1 t}, \qquad \bar{\mathbf{D}}(\mathbf{r}; t) = \bar{\mathbf{D}}_1(\mathbf{r})e^{-i\omega_1 t} \tag{13}$$

with the restriction $\omega_1 \ll \omega$ imposed to ensure the slow variation of these functions over the period T. Integration then shows that the ignored terms are $O(\omega_1/\omega)$ relative to the retained contribution, thereby justifying their omission. From Eq. (13) one has

†For a more general derivation utilizing Fourier integral representations of the field, see V. L. Ginzburg, *The Propagation of Electromagnetic Waves in Plasmas*, Pergamon Press, New York, 1964, App. B. Alternatively, in evaluating $(\partial/\partial t)\boldsymbol{\epsilon}(\partial/\partial t)\cdot\bar{\mathbf{E}}(\mathbf{r}; t)\exp(-i\omega t)$ to obtain a result as in Eq. (14), one should also compare the more general procedure in Eq. (27a) below and the footnote thereto.

$$\frac{\partial}{\partial t}(\bar{\mathbf{D}}e^{-i\omega t}) = -i(\omega + \omega_1)\boldsymbol{\epsilon}(\omega + \omega_1) \cdot \bar{\mathbf{E}}e^{-i\omega t} \tag{14}$$

since $\bar{\mathbf{D}} \exp(-i\omega t)$ corresponds to a time-harmonic field with frequency $(\omega + \omega_1)$ which can be expressed in terms of $\bar{\mathbf{E}} \exp(-i\omega t)$ via Eq. (7). Upon expanding the factor $-i(\omega + \omega_1)\boldsymbol{\epsilon}(\omega + \omega_1)$ about ω up to the linear term $O(\omega_1)$, one finds

$$\frac{\partial}{\partial t}(\bar{\mathbf{D}}e^{-i\omega t}) = \left\{-i\omega\boldsymbol{\epsilon}(\omega) \cdot \bar{\mathbf{E}} + \frac{\partial}{\partial\omega}[\omega\boldsymbol{\epsilon}(\omega)] \cdot \frac{\partial\bar{\mathbf{E}}}{\partial t}\right\}e^{-i\omega t} \tag{15}$$

which evidently generalizes the monochromatic result obtained in Eq. (7). Substitution into Eq. (12) then yields for the dominant contribution,

$$\begin{aligned}\frac{\partial \overline{W_e}}{\partial t} &\cong \frac{-i\omega}{2}\bar{\mathbf{E}}^* \cdot [\boldsymbol{\epsilon}(\omega) - \boldsymbol{\epsilon}^+(\omega)] \cdot \bar{\mathbf{E}} \\ &\quad + \frac{1}{2}\left\{\bar{\mathbf{E}}^* \cdot \frac{\partial}{\partial\omega}[\omega\boldsymbol{\epsilon}(\omega)] \cdot \frac{\partial\bar{\mathbf{E}}}{\partial t} + \frac{\partial\bar{\mathbf{E}}^*}{\partial t} \cdot \frac{\partial}{\partial\omega}[\omega\boldsymbol{\epsilon}^+(\omega)] \cdot \bar{\mathbf{E}}\right\}.\end{aligned} \tag{16}$$

A directly analogous result is obtained for the magnetic energy provided that $\bar{\mathbf{E}}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}^+$ are replaced by $\bar{\mathbf{H}}$, $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^+$, respectively. For the lossless case $\boldsymbol{\epsilon}(\omega) = \boldsymbol{\epsilon}^+(\omega)$, Eq. (16) reduces to

$$\frac{\partial \overline{W_e}}{\partial t} = \frac{\partial}{\partial t}\left\{\frac{1}{2}\bar{\mathbf{E}}^* \cdot \frac{\partial}{\partial\omega}[\omega\boldsymbol{\epsilon}(\omega)] \cdot \bar{\mathbf{E}}\right\}. \tag{17}$$

Thus, the average stored energy density $\bar{W}$ in a lossless, dispersive, electrically anisotropic medium with a scalar permeability μ may be identified as †

$$\bar{W} = \frac{1}{2}\left\{\bar{\mathbf{E}}^* \cdot \frac{\partial}{\partial\omega}[\omega\boldsymbol{\epsilon}(\omega)] \cdot \bar{\mathbf{E}} + \frac{\partial}{\partial\omega}(\omega\mu)|\bar{\mathbf{H}}|^2\right\}, \tag{18}$$

where $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$ are to be taken as space- and weakly time-dependent amplitudes of the harmonic field $\bar{\mathbf{E}}(\mathbf{r}, t) \exp(-i\omega t)$ and $\bar{\mathbf{H}}(\mathbf{r}, t) \exp(-i\omega t)$, respectively. Since the stored energy is positive, $\partial(\omega\mu)/\partial\omega$ must be positive and the dyadic $(\partial/\partial\omega)[\omega\boldsymbol{\epsilon}(\omega)]$ must be positive definite. Moreover, the average energy stored by a given electromagnetic field in a physical medium is at least as great as that in vacuum, since energy has to be expended to bring about the polarization effects which are expressed via the constitutive parameters. Thus, $\bar{W} \geq \bar{W}_0$, where $\bar{W}_0$ is the stored energy in vacuum, with $\mu = \mu_0$, $\boldsymbol{\epsilon} = \mathbf{1}\epsilon_0$ (μ_0, ϵ_0 constant). These considerations lead to the sharper condition[9b]

$$\bar{W} - \bar{W}_0 = \frac{1}{2}\left\{\bar{\mathbf{E}}^* \cdot \left[\frac{\partial}{\partial\omega}(\omega\boldsymbol{\epsilon}) - \mathbf{1}\epsilon_0\right] \cdot \bar{\mathbf{E}} + \left[\frac{\partial}{\partial\omega}(\omega\mu) - \mu_0\right]|\bar{\mathbf{H}}|^2\right\} \geq 0. \tag{19}$$

where $\mathbf{1}$ is the unit dyadic.

†Since W_e varies slowly over the period T, one has $\overline{W_e} \approx W_e$ in $0 \leq t \leq T$ and $\overline{\partial W_e/\partial t} \approx \partial \overline{W_e}/\partial t$. In the latter relation, the time derivative relates to a time scale long compared to T.

In a lossless, cold electron plasma subjected to a steady external magnetic field $\mathbf{H}_0$ along the z axis of a rectangular coordinate system, one has $\mu = \mu_0$, while the $\boldsymbol{\epsilon}$ tensor has the form

$$\boldsymbol{\epsilon} \to \begin{pmatrix} \epsilon_1 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix}, \tag{20}$$

where

$$\frac{\epsilon_1}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}, \quad \frac{\epsilon_2}{\epsilon_0} = -\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2 - \omega_c^2}, \quad \frac{\epsilon_z}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \tag{20a}$$

and

$$\omega_p^2 = \frac{e^2 N}{m\epsilon_0}, \qquad \omega_c = \frac{eB_0}{m},$$

where $-e$ and m represent the electronic charge and mass, respectively. Only the electrons are considered mobile in this simple, zero temperature, collisionless description of a plasma, and ω_p and ω_c denote the plasma and cyclotron frequencies of the electrons, respectively. To derive the form of the permittivity matrix in Eq. (20), one notes for a time-harmonic field $\bar{\mathbf{E}} \exp(-i\omega t)$ that electrons with harmonic velocity $\bar{\mathbf{v}} \exp(-i\omega t)$ in a static magnetic field $\mathbf{B}_0 = B_0\mathbf{z}_0$ obey the dynamical equation [see Eq. (1.1.54) with $p = 0$ and $\partial/\partial t = -i\omega$]

$$-i\omega m\bar{\mathbf{v}} = -e(\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \mathbf{B}_0), \tag{21}$$

whence

$$(i\omega\mathbf{1} + \omega_c\mathbf{z}_0 \times \mathbf{1}) \cdot m\bar{\mathbf{v}} = e\bar{\mathbf{E}}. \tag{21a}$$

From the dyadic identities

$$(i\omega\mathbf{1} + \omega_c\mathbf{z}_0 \times \mathbf{1}) \cdot (i\omega\mathbf{1} - \omega_c\mathbf{z}_0 \times \mathbf{1}) = -(\omega^2 - \omega_c^2)\mathbf{1}_t - \omega^2\mathbf{z}_0\mathbf{z}_0,$$
$$-(\mathbf{z}_0 \times \mathbf{1}) \cdot (\mathbf{z}_0 \times \mathbf{1}) = \mathbf{1} - \mathbf{z}_0\mathbf{z}_0 = \mathbf{1}_t,$$

and Eq. (21a), the electron velocity components $\bar{\mathbf{v}}_t$ and $\bar{\mathbf{v}}_z$, transverse and along the magnetic field direction, contribute the current densities

$$\begin{aligned} -Ne\bar{\mathbf{v}}_t &= i\omega\epsilon_0\left[\frac{\omega_p^2\mathbf{1}_t + i(\omega_c/\omega)\omega_p^2\mathbf{z}_0 \times \mathbf{1}}{\omega^2 - \omega_c^2}\right]\bar{\mathbf{E}}_t, \\ -Ne\bar{\mathbf{v}}_z &= i\omega\epsilon_0\frac{\omega_p^2}{\omega^2}\bar{\mathbf{E}}_z. \end{aligned} \tag{22}$$

Since the permittivity dyadic for the plasma medium may be defined by the relation

$$-Ne\bar{\mathbf{v}} = -i\omega(\boldsymbol{\epsilon} - \epsilon_0\mathbf{1}) \cdot \bar{\mathbf{E}},$$

one infers from Eqs. (22) and (20a) that

$$\boldsymbol{\epsilon} = \epsilon_1\mathbf{1}_t + i\epsilon_2\mathbf{z}_0 \times \mathbf{1}_t + \epsilon_z\mathbf{z}_0\mathbf{z}_0,$$

which is the dyadic form of the matrix (20). For $[\partial(\omega\boldsymbol{\epsilon})/\partial\omega - \mathbf{1}\epsilon_0]$ to be positive semidefinite [see Eq. (19)], the principal minors of the permittivity matrix in Eq. (20) must be non-negative, whence

$$\frac{\partial}{\partial\omega}\left(\omega\frac{\epsilon_1}{\epsilon_0}\right) \geq 1, \quad \frac{\partial}{\partial\omega}\left(\omega\frac{\epsilon_z}{\epsilon_0}\right) \geq 1, \quad \left[\frac{\partial}{\partial\omega}\left(\omega\frac{\epsilon_1}{\epsilon_0}\right) - 1\right]^2 \geq \left[\frac{\partial}{\partial\omega}\left(\omega\frac{\epsilon_2}{\epsilon_0}\right)\right]^2. \tag{23}$$

It is easily checked that these conditions are satisfied by the constitutive parameters in Eq. (20a), although the elements ϵ_1, ϵ_2, and ϵ_z may themselves be negative.

Average energy transport (group velocity)

For weakly anharmonic fields a kinetic description of the average flow of energy throughout parts of a field can be obtained in terms of wavepackets moving along characteristic trajectories with well-defined group velocity. It will be shown that in an homogeneous, non-spatially-dispersive medium an anharmonic field is characterized by a time-averaged Poynting vector $\bar{\mathbf{S}}$ [Eq. (8a)] and a time-averaged energy density $\bar{W}$ [Eq. (18)] that are related by

$$\bar{\mathbf{S}} = \bar{W}\mathbf{v}_g, \tag{24a}$$

where $\mathbf{v}_g$ is a group velocity characteristic of the energy flow (or of the type of wavepacket) at the point in question. In this case we shall find that one can infer an average energy theorem of the form (8). In view of Eq. (24a), it follows that $\nabla \cdot \bar{\mathbf{S}} = \mathbf{v}_g \cdot \nabla\bar{W}$ provided $\mathbf{v}_g$ is space independent, and Eq. (8) can be written as

$$\frac{\partial}{\partial t}\bar{W} + \mathbf{v}_g \cdot \nabla\bar{W} = 0, \tag{24b}$$

whence

$$\frac{d}{dt}\bar{W} = 0 \qquad \text{if } \mathbf{v}_g = \frac{d\mathbf{r}}{dt}. \tag{24c}$$

Equation (24c) implies that the average energy density $\bar{W}$ is unchanged when referred to a point moving with a wavepacket along a linear trajectory $\mathbf{r} = \mathbf{r}(t)$ with the as-yet-undetermined velocity $\mathbf{v}_g$. Since Eqs. (24) refer to an anharmonic field, $\bar{W}$ and hence $\bar{\mathbf{S}}$ are weakly space- and time-dependent even in a homogeneous medium; we shall show that they are expressible in terms of both the admittance operator $\mathscr{Y}$ of Eqs. (2) and the anharmonic field amplitudes. For inhomogeneous media, Eqs. (24) require generalization, as will be shown in Sec. 1.7.

To explore energy-transport properties under conditions for which Eqs. (24) are applicable, we turn to a consideration of anharmonic fields in lossless dispersive homogeneous media. In accordance with Eq. (2), the electric field $\mathbf{E}$

in an homogeneous medium is described at source-free points by an admittance operator $\mathscr{Y}$ such that

$$\mathscr{Y}\left(\nabla, \frac{\partial}{\partial t}\right) \cdot \mathbf{E}(\mathbf{r}, t) = 0. \tag{25}$$

We seek a slightly anharmonic (wavepacket) solution of Eq. (25) as the real part of

$$\mathbf{E}(\mathbf{r}, t) = \bar{\mathbf{E}}(\mathbf{r}, t)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{26}$$

where for prescribed $\mathbf{k}$ both the weakly space- and time-dependent rms amplitude $\bar{\mathbf{E}}(\mathbf{r}, t)$ and the frequency $\omega = \omega(\mathbf{k})$ are to be determined from Eq. (25). In the following, $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$ are weakly dependent on both $\mathbf{r}$ and t; they should be distinguished from the fields $\bar{\mathbf{E}} = \bar{\mathbf{E}}(\mathbf{r}; t)$ and $\bar{\mathbf{H}} = \bar{\mathbf{H}}(\mathbf{r}; t)$ in Eqs. (11) et seq., which depend arbitrarily on $\mathbf{r}$. From Eq. (25) and the footnote to Eqs. (8) the total average stored energy density at source-free points does not vary with time [i.e., the average power density Re $(\mathbf{E}^* \cdot \mathscr{Y} \cdot \mathbf{E}) = 0$ by Eq. (9) with Re $\mathscr{Y} = \sigma = 0$], whence

$$\mathbf{E}^*(\mathbf{r}, t) \cdot \mathscr{Y}\left(\nabla, \frac{\partial}{\partial t}\right) \cdot \mathbf{E}(\mathbf{r}, t) + \mathbf{E}(\mathbf{r}, t) \cdot \mathscr{Y}^*\left(\nabla, \frac{\partial}{\partial t}\right) \cdot \mathbf{E}^*(\mathbf{r}, t) = 0,$$

or on substitution of (26) and use of the differentiation rule † (ω, $\mathbf{k}$ are real),

$$\bar{\mathbf{E}}^* \cdot \mathscr{Y}\left(i\mathbf{k} + \nabla, -i\omega + \frac{\partial}{\partial t}\right) \cdot \bar{\mathbf{E}} + \bar{\mathbf{E}} \cdot \mathscr{Y}^*\left(i\mathbf{k} + \nabla, -i\omega + \frac{\partial}{\partial t}\right) \cdot \bar{\mathbf{E}}^* = 0. \tag{27a}$$

For a weakly space- and time-dependent amplitude $\bar{\mathbf{E}} \equiv \bar{\mathbf{E}}(\mathbf{r}, t)$, one can expand the operator $\mathscr{Y}$ of Eq. (27a) in a power series‡ about the point $i\mathbf{k}$, $-i\omega$, whence, for a lossless medium, wherein $\mathscr{Y}(i\mathbf{k}, -i\omega) = -i\mathscr{B}(\mathbf{k}, \omega)$ and $\mathscr{B} = \tilde{\mathscr{B}}$ is real, one infers from Eq.(27a) that§

$$\frac{\partial}{\partial t}\left[\bar{\mathbf{E}}^* \cdot \frac{\partial}{\partial \omega}\mathscr{B}(\mathbf{k}, \omega) \cdot \bar{\mathbf{E}}\right] - \nabla \cdot \nabla_k[\bar{\mathbf{E}}^* \cdot \mathscr{B}(\mathbf{k}, \omega) \cdot \bar{\mathbf{E}}] = 0, \tag{27b}$$

where ∇_k acts only on $\mathscr{B}(\mathbf{k}, \omega)$ and ∇ only on $\bar{\mathbf{E}}$ and $\bar{\mathbf{E}}^*$. To identify the bracketed terms in Eq. (27b), one observes from Eqs. (2) for a lossless homogeneous medium with a scalar permeability [see Eq. (10)] that

$$\mathscr{B}(\mathbf{k}, \omega) = \omega\epsilon + \frac{\mathbf{k} \times (\mathbf{k} \times \mathbf{1})}{\omega\mu} \equiv \tilde{\mathscr{B}}, \tag{28a}$$

whence noting Eq.(18), one has as a generalization of a familiar network energy theorem:

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†An operator function $F(\nabla, \partial/\partial t)$ of the derivatives $\nabla, \partial/\partial t$ acting on a product of two space-time functions A, B can be expressed as $F(\nabla, \partial/\partial t)A(\mathbf{r}, t)B(\mathbf{r}, t) = F(\nabla' + \nabla, \partial/\partial t' + \partial/\partial t)A(\mathbf{r}, t)B(\mathbf{r}', t')]_{r=r', t=t'}$, which permits separate differentiation of the two factors.

‡$\mathscr{Y}(i\mathbf{k} + \nabla, -i\omega + \partial/\partial t) = \mathscr{Y}(i\mathbf{k}, -i\omega) + i(\partial/\partial t)(\partial/\partial\omega)\mathscr{Y}(i\mathbf{k}, -i\omega) - i\nabla\cdot\nabla_k\mathscr{Y}(i\mathbf{k}, -i\omega) + \cdots$.

§Note the relation for the transpose of the scalar $\mathbf{a}\cdot\mathscr{F}\cdot\mathbf{b} = \mathbf{b}\cdot\tilde{\mathscr{F}}\cdot\mathbf{a}$.

$$\bar{\mathrm{E}}^* \cdot \frac{\partial \mathscr{B}}{\partial \omega} \cdot \bar{\mathrm{E}} = \bar{\mathrm{E}}^* \cdot \left[\frac{\partial(\omega\epsilon)}{\partial\omega} - \frac{\mathbf{k} \times (\mathbf{k} \times \mathbf{1})}{\omega^2\mu^2}\frac{\partial}{\partial\omega}(\omega\mu)\right] \cdot \bar{\mathrm{E}}$$

$$= \bar{\mathrm{E}}^* \cdot \frac{\partial}{\partial\omega}(\omega\epsilon) \cdot \bar{\mathrm{E}} + \bar{\mathrm{H}}^* \cdot \frac{\partial}{\partial\omega}(\omega\mu)\bar{\mathrm{H}} = 2\bar{W}, \qquad (28\text{b})$$

since $\mathbf{k} \times \bar{\mathrm{E}} = \omega\mu\bar{\mathrm{H}}$. Similarly, by Eq. (28a) one finds in Eq. (27b), if ϵ is non-spatially dispersive, that

$$-\nabla \cdot \nabla_k[\bar{\mathrm{E}}^* \cdot \mathscr{B} \cdot \bar{\mathrm{E}}] = \nabla \cdot \nabla_k\left[\frac{(\mathbf{k} \times \bar{\mathrm{E}}^*) \cdot (\mathbf{k} \times \bar{\mathrm{E}})}{\omega\mu}\right]$$

$$= \nabla \cdot \left[\frac{\bar{\mathrm{E}}^* \times (\mathbf{k} \times \bar{\mathrm{E}}) + \bar{\mathrm{E}} \times (\mathbf{k} \times \bar{\mathrm{E}}^*)}{\omega\mu}\right]$$

$$= \nabla \cdot [\bar{\mathrm{E}}^* \times \bar{\mathrm{H}} + \bar{\mathrm{E}} \times \bar{\mathrm{H}}^*] = 2\nabla \cdot \bar{\mathrm{S}}, \qquad (28\text{c})$$

where we have used the identities $\nabla_k \mathbf{k} = \mathbf{1}$ and $\nabla_k(\mathbf{A} \cdot \mathbf{B}) = (\nabla_k \mathbf{A}) \cdot \mathbf{B} + (\nabla_k \mathbf{B} \cdot \mathbf{A})$. In view of the relations (28), Eq. (27b) assumes the form (8):

$$\frac{\partial \bar{W}}{\partial t} + \nabla \cdot \bar{\mathrm{S}} = 0.$$

This conservation of average energy flow for the anharmonic field in Eq. (26) is also applicable more generally to fields other than the electromagnetic.

It now remains to prove the energy-flux relation (24a). We observe from the source-free field equation (25) for a lossless homogeneous medium and the wavepacket solution in Eq. (26) that

$$\mathscr{B}(\mathbf{k}, \omega) \cdot \bar{\mathrm{E}} = 0, \qquad (29\text{a})$$

where $\mathscr{B}(\mathbf{k}, \omega)$ is defined in Eq. (28a). Non-vanishing solutions $\bar{\mathrm{E}}$ of Eq. (29a) exist for those frequencies $\omega = \omega_a(\mathbf{k})$ that satisfy the determinental requirement, or dispersion equation, $\det \mathscr{B}(\mathbf{k}, \omega_a) = 0$. For an arbitrary incremental change $\delta\mathbf{k}$, it follows from Eq. (29a) that

$$\delta[\bar{\mathrm{E}}^* \cdot \mathscr{B}(\mathbf{k}, \omega_a) \cdot \bar{\mathrm{E}}] = \bar{\mathrm{E}}^* \cdot \delta\mathscr{B}(\mathbf{k}, \omega_a) \cdot \bar{\mathrm{E}} = 0,$$

and since (with $\delta\omega_a = \delta\mathbf{k} \cdot \nabla_k\omega_a$)

$$\delta\mathscr{B}(\mathbf{k}, \omega_a) = \delta\mathbf{k} \cdot \nabla_k\mathscr{B}(\mathbf{k}, \omega_a) + \delta\mathbf{k} \cdot \nabla_k\omega_a\frac{\partial\mathscr{B}}{\partial\omega}, \qquad (29\text{b})$$

one infers from the arbitrariness of $\delta\mathbf{k}$ that

$$\nabla_k[\bar{\mathrm{E}}^* \cdot \mathscr{B} \cdot \bar{\mathrm{E}}] + \nabla_k\omega_a\left[\bar{\mathrm{E}}^* \cdot \frac{\partial\mathscr{B}}{\partial\omega} \cdot \bar{\mathrm{E}}\right], \qquad (29\text{c})$$

whence from Eqs. (28b) and (28c) one obtains the result asserted in Eq. (24a),

$$\bar{\mathrm{S}} = \bar{W}\mathbf{v}_g, \qquad \mathbf{v}_g = \nabla_k\omega_a \qquad (30)$$

where $\mathbf{v}_g$ is the group velocity. Since the vanishing of $\det \mathscr{B}(\mathbf{k}, \omega)$ occurs for different dispersion relations $\omega = \omega_a(\mathbf{k})$, the energy flux relation in Eq. (30) and hence the group velocity $\mathbf{v}_g = \nabla_k\omega_a$ are characteristic of different types $(a = 1, 2, \ldots)$ of wavepackets.

1.5b Boundary Conditions, Uniqueness, and Reciprocity Relations for the Electromagnetic Field

Boundary conditions and uniqueness

In a material medium the field equations may be expressed either in the first-order form discussed in Sec. 1.1d or in the reduced electromagnetic form noted, for example, in Eqs. (1.1.64) or (1). In the presence of an externally applied magnetic field, the permittivity and permeability are generally dyadic parameters. For time-harmonic fields, which vary as $\exp(-i\omega t)$, and for a spatially inhomogeneous medium with non-spatially-dispersive parameters, the permittivity and permeability will be expressed as complex dyadics $\boldsymbol{\epsilon}(\mathbf{r}, \omega)$ and $\boldsymbol{\mu}(\mathbf{r}, \omega)$. Under such conditions, one obtains for the steady-state form of the reduced Maxwell equations (1) descriptive of the electromagnetic field excited by prescribed electric and magnetic current distributions $\mathbf{J}$ and $\mathbf{M}$ in a temporally dispersive anisotropic medium:

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mathbf{B}(\mathbf{r}) - \mathbf{M}(\mathbf{r}), \qquad \nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\mathbf{D}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \tag{31a}$$

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\mu}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r}), \qquad \mathbf{D}(\mathbf{r}) = \boldsymbol{\epsilon}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}), \tag{31b}$$

where the ω dependence and the factor $\exp(-i\omega t)$ have been suppressed. Special cases of lossless media, isotropic media with $\boldsymbol{\epsilon} = \mathbf{1}\epsilon$, $\boldsymbol{\mu} = \mathbf{1}\mu$, etc. are contained within this formulation.

Across a regular surface S of a medium discontinuity, the electromagnetic field vectors must satisfy the conditions

$$\boldsymbol{\nu} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s, \qquad \boldsymbol{\nu} \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{M}_s, \tag{32a}$$

$$\boldsymbol{\nu} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = \eta_m, \qquad \boldsymbol{\nu} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \eta, \tag{32b}$$

where the subscripts 1 and 2 distinguish quantities in regions 1 and 2 of Fig. 1.5.1. On the interface, $\mathbf{J}_s$ and $\mathbf{M}_s$ are, respectively, electric and magnetic surface current distributions, η and η_m are electric and magnetic surface charge

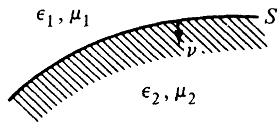


FIG. 1.5.1 Interface between two media.

densities, and $\boldsymbol{\nu}$ is a unit vector normal to S and pointing into medium 2. If neither medium is perfectly conducting, no surface currents or charges are induced and $\mathbf{J}_s = \mathbf{M}_s = \eta = \eta_m = 0$; however, if *external* surface current distributions are placed on the interface, these must be taken into account. When medium 2 is a perfect electric conductor, $\mathbf{E}_2$ and $\mathbf{H}_2$ vanish everywhere inside this medium and induced electric currents and charges exist on the surface. Thus, from Eqs. (32),

$$\mathbf{H}_1 \times \boldsymbol{\nu} = \mathbf{J}_s, \quad \boldsymbol{\nu} \times \mathbf{E}_1 = 0, \quad \boldsymbol{\nu} \cdot \mathbf{B}_1 = 0, \quad -\mathbf{D}_1 \cdot \boldsymbol{\nu} = \eta. \tag{33}$$

While the tangential electric field vanishes at the surface of a perfect conductor, one notes from Eq. (32a) that a finite field can be generated next to the surface by *placing* thereon a magnetic surface current distribution. In fact, if the magnetic current distribution is $\mathbf{M}_s$, the tangential electric field generated by it in the immediate vicinity of a perfectly conducting surface is given by

$$\mathbf{v} \times \mathbf{E}_t = \mathbf{M}_s. \tag{33a}$$

Although physically realizable electromagnetic sources can be described solely in terms of electric charges and currents, the use of equivalent magnetic currents is frequently a convenient artifice; for example, a linear magnetic current element in an isotropic medium is equivalent to a circular electric current flowing around a path of vanishingly small radius in the plane perpendicular to the element. Since Eqs. (32) and (33) apply to any time-harmonic field, the Fourier inversion $f(\mathbf{r}, t) = \int_{-\infty}^{\infty} f(\mathbf{r}, \omega)e^{-i\omega t}\, d\omega$ implies that they are valid also for the time-dependent field.

For an unbounded region it is necessary to know the field behavior on a surface at infinity. If all sources are contained in a finite region, the field behavior at large distances from the sources must meet the physical requirement that energy travel *away* from the source region (i.e., the field solution comprises only "outgoing" waves). This requirement, originated by Sommerfeld, constitutes a boundary condition on a surface at infinity and is referred to as the "radiation condition."[10] More specifically, the transverse fields in a spherically diverging wave in homogeneous medium decay like $1/r$ at large distances r from the source region and behave locally like plane waves traveling outward in the r direction: thus each field component transverse to r behaves like $\exp(ik_n r)/r$. This requirement can be phrased mathematically as†

$$\lim_{r\to\infty} r\left(\frac{\partial A}{\partial r} - ik_n A\right) = 0 \tag{34a}$$

where A stands for any field component transverse to r, and $k_n = \omega\sqrt{\mu\epsilon}$ approaches a constant as $r \to \infty$. In view of the local plane-wave character, the relation between the transverse electric and magnetic fields as $r \to \infty$ is simply

$$\mathbf{r}_0 \times \mathbf{E} \to \sqrt{\frac{\mu}{\epsilon}}\,\mathbf{H}, \tag{34b}$$

where $\mathbf{r}_0$ is a unit vector pointing in the $\mathbf{r}$ direction and $\sqrt{\mu/\epsilon}$ is the impedance of unbounded space.

For two-dimensional fields that are independent of a rectilinear coordinate, say x, the outgoing waves excited by source distributions contained in a bounded

†The present discussion is restricted to the case of isotropic media; the *energy*-radiation condition applies also to anisotropic regions, but its imposition is more complicated since the directions of phase- and energy-propagation are generally different (see Secs. 1.6 and 1.7).

region transverse to x are cylindrical [i.e., each field component behaves like $\exp(ik_n\rho)/\sqrt{\rho}$, where ρ is the radial variable in a plane transverse to the x coordinate]. In this instance, one obtains as the two-dimensional analogue of Eq. (34a),

$$\lim_{\rho\to\infty} \sqrt{\rho}\left(\frac{\partial A}{\partial \rho} - ik_n A\right) = 0, \tag{34c}$$

where A stands for any transverse field component. Equation (34b) obtains also, with $\mathbf{r}_0$ replaced by $\boldsymbol{\rho}_0$.

Equations (34a) and (34c) apply to non-dissipative media, wherein k_n is real. If one assumes the medium to be slightly lossy (as all physical media are), then k_n has a positive imaginary part and the "outgoing wave" condition at infinity can be replaced by the simpler requirement that all fields excited by sources in a finite region vanish at infinity. This simple condition is valid also for anisotropic and inhomogeneous media and forces the choice of a decaying dependence as $r \to \infty$. For non-harmonic time dependence, the radiation condition is subsumed under the causality condition required for time-dependent fields. On analytic continuation into the upper half of the complex ω plane, the causal requirement Im $\omega > 0$ (see Sec. 1.2) may also be used to select the decaying field solution.

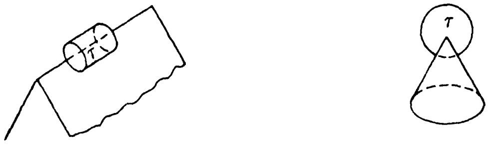
In the discussion of the boundary conditions in Eqs. (32) and (33), it was implied that the discontinuity surface S is regular (i.e., it possesses no sharp points, edges, or corners). If geometrical singularities exist on an electrically impenetrable surface, some field components may become infinite in the neighborhood of these singularities. To explore permissible types of divergence in an electromagnetic field, one employs the following energy argument: The energy inside a bounded volume containing any physical set of sources is finite. The singular electromagnetic field behavior near a surface discontinuity on an impenetrable surface such as a perfect conductor is caused by *induced* surface currents whose energy content cannot be larger than that of the true sources. Thus, one imposes the condition that the electric and magnetic energy stored in any volume surrounding the geometrical singularities is finite. In an isotropic non-dispersive medium this requirement can be phrased as†

$$\int_\tau [\epsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2]\, d\tau = \text{finite}, \tag{35}$$

where τ is a finite volume surrounding the singularity.

Equation (35) puts a definite limitation on the growth of any field component. For example, for the edge singularity in Fig. 1.5.2(a) one chooses a cylindrical volume centered on the edge; if the edge is described by a regular curve, the height of the cylinder is chosen so small that the portion of the edge contained therein is essentially straight. Upon employing cylindrical coordinates, ρ, φ, z, with z directed along the edge, one has $d\tau = \rho\, d\rho\, d\varphi\, dz$. If it is assumed that

†See Sec. 1.5a for the effects of dispersion and anisotropy.



(a) Edge (b) Tip

FIG. 1.5.2 Surface singularities.

each field component behaves like $f(\rho)\psi(\varphi, z)$ as the edge is approached (i.e., as $\rho \rightarrow 0$†), then condition (35) can be phrased in terms of the ρ behavior alone since the fields are regular functions of φ and z. The pertinent portion of Eq. (35) can be written as (R = radius of cylindrical volume)

$$\int_0^R |f(\rho)|^2 \rho \, d\rho = \text{finite}, \tag{36}$$

from which one infers that $|f(\rho)|^2$ can behave at worst like $\rho^{-2(1-\alpha)}$ as $\rho \rightarrow 0$, where α is an arbitrarily small positive quantity. Thus, we infer the "edge condition" that

no component of **E** or **H** can grow more rapidly than $\rho^{-1+\alpha}$, $\qquad \alpha > 0$, (37)

where ρ is the distance from the edge.

Similarly, for a tip singularity as in Fig. 1.5.2(b), one surrounds the tip by a spherical volume of radius R. In a spherical (r, θ, φ) coordinate system centered at the tip, the volume element $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\varphi$. If one assumes that the behavior of each field component can be described in the form $g(r)\psi(\theta, \varphi)$ as $r \rightarrow 0$, one can again write the pertinent portion of Eq. (35) as

$$\int_0^R |g(r)|^2 r^2 \, dr = \text{finite}, \tag{38}$$

i.e., $|g(r)|^2$ behaves at most like $r^{-3+2\alpha}$, $\alpha > 0$. Thus, one finds that near a conical tip,

no component of **E** or **H** can grow more rapidly than $r^{-(3/2)+\alpha}$, $\qquad \alpha > 0$, (39)

where r is the distance from the tip. The requirements (37) and (39) constitute additional boundary conditions to be imposed on the fields in the presence of surface singularities in order to assure a unique solution of the field problem.

When solving the Maxwell field equations subject to all the above-stated boundary conditions, one is assured that only one (unique) solution is possible. The following uniqueness statement, given without proof, is relevant for a volume of space τ bounded by a surface S (which may be multiply connected) and occupied by a "physical" medium having some dissipation: The steady-state electromagnetic field inside τ is determined uniquely if [2b] (1) the sources (if any) are specified in τ, and (2) either the tangential electric field or the tangential

†This behavior can be shown to hold rigorously for a straight edge (see Sec. 6.5).

magnetic field is specified on the various parts of S. Included in condition (2) is the case where the tangential electric field on the boundary is linearly related to the tangential magnetic field via [see Eq. (1.1.16d)]

$$\mathbf{v} \times \mathbf{E} = \mathscr{Z} \cdot \mathbf{H}, \qquad \mathbf{v} = \text{normal unit vector directed into boundary}, \tag{40}$$

where $\mathscr{Z}$ is an impedance dyadic with non-vanishing components perpendicular to $\mathbf{v}$ (i.e., $\mathbf{v} \cdot \mathscr{Z} = \mathscr{Z} \cdot \mathbf{v} = 0$). Equation (40) constitutes an "impedance boundary condition" that can be employed to approximate field behavior on the surface of highly lossy or "reactive" regions. The above conditions remain valid when the medium is anisotropic. If a portion of S lies at infinity and all sources are confined to a finite region, uniqueness obtains when the field behavior at infinity is expressed in terms of the previously stated radiation condition. For surfaces with singularities, condition (35) must be imposed, in addition.

If the fields are time dependent, their values at time $t > t_1$ are determined uniquely by their initial values at the reference time t_1. The above-stated conditions on S must also be satisfied for $t \geq t_1$.

Reciprocity relations

Reciprocity relations for the electromagnetic field in vacuum and in a homogeneous plasma have been considered in Secs. 1.1b and 1.1c. Apart from the assumption of medium homogeneity, that derivation involves all field constituents rather than only the electromagnetic field, as in the reduced formulation leading to Eqs.(31). Furthermore, the results have been given for the time-dependent case so that their time-harmonic form, although obtainable without difficulty, is not stated explicitly. We shall now derive various reciprocity conditions for the time-harmonic electromagnetic field in spatially varying anisotropic media directly from Eqs. (31) and compare the results with those given in Sec. 1.1.

In a given region τ bounded by the surface S, consider two sets of field solutions $\mathbf{E}, \mathbf{H}$ and $\hat{\mathbf{E}}, \hat{\mathbf{H}}$, corresponding to excitations by the sources $\mathbf{J}, \mathbf{M}$, and $\hat{\mathbf{J}}, \hat{\mathbf{M}}$, respectively. The medium in the first problem is assumed to be described by the space-dependent dyadic permittivity $\boldsymbol{\epsilon}(\mathbf{r})$ and permeability $\boldsymbol{\mu}(\mathbf{r})$, with $\hat{\boldsymbol{\epsilon}}(\mathbf{r})$ and $\hat{\boldsymbol{\mu}}(\mathbf{r})$ representing the corresponding quantities in the second problem. At the real steady frequency ω, each set of fields is therefore a solution of the Maxwell equations†

$$\nabla \times \mathbf{E} = i\omega\boldsymbol{\mu} \cdot \mathbf{H} - \mathbf{M}, \qquad \nabla \times \mathbf{H} = -i\omega\boldsymbol{\epsilon} \cdot \mathbf{E} + \mathbf{J}, \tag{41a}$$

$$\nabla \times \hat{\mathbf{E}} = i\omega\hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{H}} - \hat{\mathbf{M}}, \qquad \nabla \times \hat{\mathbf{H}} = -i\omega\hat{\boldsymbol{\epsilon}} \cdot \hat{\mathbf{E}} + \hat{\mathbf{J}}. \tag{41b}$$

Upon calculating the expressions $\nabla \cdot (\mathbf{E} \times \hat{\mathbf{H}}) = (\hat{\mathbf{H}} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \hat{\mathbf{H}})$ and $\nabla \cdot (\hat{\mathbf{E}} \times \mathbf{H})$, taking their difference, integrating over τ, and invoking the divergence theorem, one finds

†Quantities distinguished by ^ are related to the adjoint field in Sec. 1.1 on identifying $\hat{\mathbf{J}} = -\mathbf{J}^{+}$ and $\hat{\mathbf{M}} = -\mathbf{M}^{+}$.

$$\oint_s [\mathbf{n} \times \mathbf{E} \cdot \hat{\mathbf{H}} - \mathbf{n} \times \hat{\mathbf{E}} \cdot \mathbf{H}]\, dS$$
$$- i\omega \int_\tau [\hat{\mathbf{H}} \cdot \boldsymbol{\mu} \cdot \mathbf{H} - \mathbf{H} \cdot \hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{H}} - \mathbf{E} \cdot \hat{\boldsymbol{\epsilon}} \cdot \hat{\mathbf{E}} + \hat{\mathbf{E}} \cdot \boldsymbol{\epsilon} \cdot \mathbf{E}]\, d\tau$$
$$= \int_\tau [\mathbf{H} \cdot \hat{\mathbf{M}} + \hat{\mathbf{E}} \cdot \mathbf{J} - \hat{\mathbf{H}} \cdot \mathbf{M} - \mathbf{E} \cdot \hat{\mathbf{J}}]\, d\tau. \tag{42}$$

The reciprocity statement follows if the right-hand side of Eq. (42) can be equated to zero. Since generally $\hat{\mathbf{H}} \cdot \boldsymbol{\mu} \cdot \mathbf{H} = \mathbf{H} \cdot \tilde{\boldsymbol{\mu}} \cdot \hat{\mathbf{H}}$, etc., the volume integral on the left-hand side vanishes if the medium in the second problem is the "transpose" of that in the first, i.e., if

$$\hat{\boldsymbol{\epsilon}}(\mathbf{r}) = \tilde{\boldsymbol{\epsilon}}(\mathbf{r}), \qquad \hat{\boldsymbol{\mu}}(\mathbf{r}) = \tilde{\boldsymbol{\mu}}(\mathbf{r}). \tag{43a}$$

If the boundary conditions on S are given by the general impedance relation (40) and the corresponding requirement $\boldsymbol{\nu} \times \hat{\mathbf{E}} = \hat{\mathscr{Z}} \cdot \hat{\mathbf{H}}$ on S for the second problem, then the identification

$$\hat{\mathscr{Z}}(\mathbf{r}) = \tilde{\mathscr{Z}}(\mathbf{r}) \tag{43b}$$

assures the vanishing of the surface integral.† Thus, if Eqs. (43) apply, the fields $\mathbf{E}$, $\mathbf{H}$ and $\hat{\mathbf{E}}$, $\hat{\mathbf{H}}$ are related as follows:

$$\int_\tau [\mathbf{H} \cdot \hat{\mathbf{M}} + \hat{\mathbf{E}} \cdot \mathbf{J} - \hat{\mathbf{H}} \cdot \mathbf{M} - \mathbf{E} \cdot \hat{\mathbf{J}}]\, d\tau = 0. \tag{44}$$

The reciprocity relations in Eq. (44) are clarified by special choices of the excitation. If one selects $\mathbf{M} = \hat{\mathbf{M}} = 0$, $\mathbf{J} = \mathbf{J}^\circ \delta(\mathbf{r} - \mathbf{r}')$, $\hat{\mathbf{J}} = \hat{\mathbf{J}}^\circ \delta(\mathbf{r} - \hat{\mathbf{r}}')$, where $\mathbf{J}^\circ$ and $\hat{\mathbf{J}}^\circ$ are constant vectors of equal magnitude, then

$$\mathbf{J}^\circ \cdot \hat{\mathbf{E}}(\mathbf{r}') = \hat{\mathbf{J}}^\circ \cdot \mathbf{E}(\hat{\mathbf{r}}'). \tag{45a}$$

This relation states that in a region described by the parameters $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$, $\mathscr{Z}$, the electric-field component excited at $\hat{\mathbf{r}}'$ in the direction of $\hat{\mathbf{J}}^\circ$ by an electric current element $\mathbf{J}^\circ$ at $\mathbf{r}'$ is identical with the electric-field component excited at $\mathbf{r}'$ in the direction of $\mathbf{J}^\circ$ by an electric current element $\hat{\mathbf{J}}^\circ$ at $\hat{\mathbf{r}}'$ in a region of the same geometrical configuration described by parameters $\tilde{\boldsymbol{\epsilon}}$, $\tilde{\boldsymbol{\mu}}$, $\tilde{\mathscr{Z}}$. Similarly, for $\mathbf{J} = \hat{\mathbf{J}} = 0$, $\mathbf{M} = \mathbf{M}^\circ \delta(\mathbf{r} - \mathbf{r}')$, $\hat{\mathbf{M}} = \hat{\mathbf{M}}^\circ \delta(\mathbf{r} - \hat{\mathbf{r}}')$, where $\mathbf{M}^\circ$ and $\hat{\mathbf{M}}^\circ$ are constant vectors of equal magnitude,

$$\mathbf{M}^\circ \cdot \hat{\mathbf{H}}(\mathbf{r}') = \hat{\mathbf{M}}^\circ \cdot \mathbf{H}(\hat{\mathbf{r}}'), \tag{45b}$$

while for $\mathbf{J} = \hat{\mathbf{M}} = 0$, $\quad \hat{\mathbf{J}} = \hat{\mathbf{J}}^\circ \delta(\mathbf{r} - \hat{\mathbf{r}}')$, $\quad \mathbf{M} = \mathbf{M}^\circ \delta(\mathbf{r} - \mathbf{r}')$,

$$\mathbf{M}^\circ \cdot \hat{\mathbf{H}}(\mathbf{r}') = -\hat{\mathbf{J}}^\circ \cdot \mathbf{E}(\hat{\mathbf{r}}'). \tag{45c}$$

Although the above reciprocity relations generally involve fields in the "transposed" problem, they pertain to fields in the *same* medium provided the

†If a portion of S lies at infinity, the vanishing of the surface integral there is assured by the radiation condition. If S is a perfectly conducting surface containing surface singularities such as edges or tips, one surrounds these singularities by the closed surface S' and applies the edge condition to secure vanishing of the surface-integral contribution from S' as τ' shrinks to zero.

anisotropy, if any, is of the symmetric type [i.e., $\boldsymbol{\epsilon}(\mathbf{r}) = \tilde{\boldsymbol{\epsilon}}(\mathbf{r})$, $\boldsymbol{\mu}(\mathbf{r}) = \tilde{\boldsymbol{\mu}}(\mathbf{r})$, $\mathscr{X}(\mathbf{r}) = \tilde{\mathscr{X}}(\mathbf{r})$]. The latter constraints are, of course, satisfied in isotropic regions. Simplifications occur also in lossless regions wherein $\boldsymbol{\epsilon} = \tilde{\boldsymbol{\epsilon}}^* \equiv \boldsymbol{\epsilon}^+$, $\boldsymbol{\mu} = \boldsymbol{\mu}^+$, $\mathscr{X} = -\mathscr{X}^+$ [see Eq. (10)]. If the complex conjugate of Eqs. (41b) is employed, with $\hat{\boldsymbol{\epsilon}}$ and $\hat{\boldsymbol{\mu}}$ replaced by $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$, respectively, and the preceding derivation followed with minor modification, one finds in view of the above lossless relation that

$$\begin{aligned} &\hat{\mathbf{J}}^{\circ *} \cdot \mathbf{E}(\hat{\mathbf{r}}') = -\mathbf{J}^\circ \cdot \hat{\mathbf{E}}^*(\mathbf{r}'), \qquad \hat{\mathbf{M}}^{\circ *} \cdot \mathbf{H}(\hat{\mathbf{r}}') = -\mathbf{M}^\circ \cdot \hat{\mathbf{H}}^*(\mathbf{r}'), \\ &\hat{\mathbf{J}}^{\circ *} \cdot \mathbf{E}(\hat{\mathbf{r}}') = -\mathbf{M}^\circ \cdot \hat{\mathbf{H}}^*(\mathbf{r}'), \end{aligned} \tag{45d}$$

with all fields and sources referring to the same lossless region. It should be noted that the results in Eq. (45d) generalize the reciprocity statements in Sec. 1.1b for time-dependent fields, in vacuum.

As in Eqs. (1.1.19) with $d\mathbf{r}' \equiv d\tau'$, one may define time-harmonic Green's functions that represent electromagnetic fields arising from arbitrary harmonic current distributions,

$$\mathbf{E}(\mathbf{r}) = -\int_\tau \mathscr{G}_{11}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\, d\tau' - \int_\tau \mathscr{G}_{12}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}(\mathbf{r}')\, d\tau', \tag{46a}$$

$$\mathbf{H}(\mathbf{r}) = -\int_\tau \mathscr{G}_{21}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\, d\tau' - \int_\tau \mathscr{G}_{22}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}(\mathbf{r}')\, d\tau', \tag{46b}$$

where $\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}')$, $\mathscr{G}_{22}(\mathbf{r}, \mathbf{r}')$ and $\mathscr{G}_{12}(\mathbf{r}, \mathbf{r}')$ or $\mathscr{G}_{21}(\mathbf{r}, \mathbf{r}')$ are the dyadic electric, magnetic, and "transfer" Green's functions, respectively. The interpretation of the various $\mathscr{G}_{ij}$ is the same as in Eq. (1.1.19), with evident modifications to accommodate the time-harmonic regime. It follows from Eqs. (31), with $\mathbf{M} = 0$, $\mathbf{J}(\mathbf{r}) = \mathbf{J}^\circ\delta(\mathbf{r} - \mathbf{r}')$, that $\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}')$ satisfies the vector differential equation

$$\nabla \times [\boldsymbol{\mu}^{-1}(\mathbf{r}) \cdot \nabla \times \mathscr{G}_{11}(\mathbf{r}, \mathbf{r}')] - \omega^2\boldsymbol{\epsilon}(\mathbf{r}) \cdot \mathscr{G}_{11}(\mathbf{r}, \mathbf{r}') = -i\omega\mathbf{1}\delta(\mathbf{r} - \mathbf{r}'), \tag{47a}$$

were $\boldsymbol{\mu}^{-1}$ is the inverse dyadic defined so that $\boldsymbol{\mu}^{-1} \cdot \boldsymbol{\mu} = \boldsymbol{\mu} \cdot \boldsymbol{\mu}^{-1} = \mathbf{1}$, and the constant vector $\mathbf{J}^\circ$ has been elided. By dual considerations one finds that

$$\nabla \times [\boldsymbol{\epsilon}^{-1}(\mathbf{r}) \cdot \nabla \times \mathscr{G}_{22}(\mathbf{r}, \mathbf{r}')] - \omega^2\boldsymbol{\mu}(\mathbf{r}) \cdot \mathscr{G}_{22}(\mathbf{r}, \mathbf{r}') = -i\omega\mathbf{1}\delta(\mathbf{r} - \mathbf{r}'). \tag{47b}$$

Equations (47) constitute a generalization of Eqs. (1.1.21), for the harmonic case $\partial/\partial t = -i\omega$, to inhomogeneous and anisotropic media.

As in Sec. 1.1b, the reciprocity conditions in Eqs. (45) can be phrased concisely in terms of the above dyadic Green's functions. In view of the definition of $\mathscr{G}_{11}$ in Eq. (46a) and similarly of $\hat{\mathscr{G}}_{11}$, Eq. (45a) can be written as

$$\mathbf{J}^\circ \cdot \hat{\mathscr{G}}_{11}(\mathbf{r}', \hat{\mathbf{r}}') \cdot \hat{\mathbf{J}}^\circ = \hat{\mathbf{J}}^\circ \cdot \mathscr{G}_{11}(\hat{\mathbf{r}}', \mathbf{r}') \cdot \mathbf{J}^\circ = \mathbf{J}^\circ \cdot \tilde{\mathscr{G}}_{11}(\hat{\mathbf{r}}', \mathbf{r}') \cdot \hat{\mathbf{J}}^\circ, \tag{48}$$

from which

$$\hat{\mathscr{G}}_{11}(\mathbf{r}', \hat{\mathbf{r}}') = \tilde{\mathscr{G}}_{11}(\hat{\mathbf{r}}', \mathbf{r}'). \tag{49a}$$

In a directly analogous manner one notes from Eqs. (45b, c) and the above that

$$\hat{\mathscr{G}}_{22}(\mathbf{r}', \hat{\mathbf{r}}') = \tilde{\mathscr{G}}_{22}(\hat{\mathbf{r}}', \mathbf{r}'), \tag{49b}$$

$$-\hat{\mathscr{G}}_{21}(\mathbf{r}', \hat{\mathbf{r}}') = \tilde{\mathscr{G}}_{12}(\hat{\mathbf{r}}', \mathbf{r}'). \tag{49c}$$

From Maxwell's equations and Eqs. (49) the *total* electromagnetic field in a source-free region can be inferred from the knowledge of *any one* of the dyadic Green's functions. Suppose that we know $\mathscr{G}_{11}(\mathbf{r}, \mathbf{r}')$. From the Maxwell field equations one obtains $\mathscr{G}_{21}(\mathbf{r}, \mathbf{r}')$ in terms of $\nabla \times \mathscr{G}_{11}(\mathbf{r}, \mathbf{r}')$. Equation (49c) then yields $\mathscr{G}_{12}(\mathbf{r}, \mathbf{r}')$, and the Maxwell field equations finally lead to $\mathscr{G}_{22}(\mathbf{r}, \mathbf{r}')$ in terms of $\nabla \times \mathscr{G}_{12}(\mathbf{r}, \mathbf{r}')$.

In a region with symmetric anisotropy $\boldsymbol{\epsilon} = \tilde{\boldsymbol{\epsilon}}, \boldsymbol{\mu} = \tilde{\boldsymbol{\mu}}, \mathscr{X} = \tilde{\mathscr{X}}$, or in an isotropic medium, the Green's functions on the left-hand side of Eqs. (49) are replaced by $\mathscr{G}_{11}(\mathbf{r}', \hat{\mathbf{r}}'), \mathscr{G}_{22}(\mathbf{r}', \hat{\mathbf{r}}'), \mathscr{G}_{21}(\mathbf{r}', \hat{\mathbf{r}}')$, respectively, thereby making the reciprocity statement applicable in one and the same region. The resulting relations are of the same form as in Eqs. (1.1.29) or (1.1.58), on ignoring the time dependence. These considerations apply also to lossless regions with $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^+, \boldsymbol{\mu} = \boldsymbol{\mu}^+$, and $\mathscr{X} = -\mathscr{X}^+$, wherein one replaces the Green's functions on the left-hand side of Eqs. (49) by $-\mathscr{G}_{11}^*(\mathbf{r}', \hat{\mathbf{r}}')$, $-\mathscr{G}_{22}^*(\mathbf{r}', \hat{\mathbf{r}}')$, and $\mathscr{G}_{21}^*(\mathbf{r}', \hat{\mathbf{r}}')$, respectively.

Reciprocity conditions analogous to the above are also found to be satisfied by modal Green's functions which represent the voltages or currents excited by unit strength point current or voltage generators on a transmission line. These results are derived in Sec. 2.3c and listed in Eqs. (2.3.15).

1.5c Alternative Representations

When z is a symmetry axis, the guided-wave approach in Sec. 1.4 leads naturally to a field representation in terms of eigenfunctions characteristic of the cross-sectional domain transverse to z. If the cross-sectional region is describable simply in terms of a separable (u, v) coordinate system, it is possible to deduce alternative representations involving eigenfunctions in the (u, z) or (v, z) coordinates. Since alternative field representations usually have different convergence properties, their availability aids in the evaluation of formal field solutions for various parameter regimes. The basic theory underlying the determination of alternative representations is given in Sec. 3.3, and detailed applications to electromagnetic or acoustic fields may be found in Chapter 5. At this time, we make certain observations within the context of problems discussed in this chapter.

Field representations developed in Sec. 1.4 [see also Eqs. (1.1.49) et seq.] for homogeneous unbounded regions can readily be modified to accommodate regions that contain inhomogeneities along the wave-guide axis z, or are bounded in the directions transverse to z. Although the reduction of vector field problems to equivalent scalar problems in such regions usually requires assignment of a special role to the symmetry axis z, the component scalar problems are not necessarily so restricted. For the electromagnetic field, this aspect may be inferred from Eqs. (1.1.38b) and (1.1.49c), wherein the scalar potential functions $\mathscr{S}'$ and $\mathscr{S}''$ satisfy differential equations which, subject to stated boundary conditions, may be solved in any convenient representation. The same statement evidently applies to the acoustic field which is derivable from the scalar Green's function defined in Eq. (1.1.13b), subject to suitable boundary conditions.

A number of alternative field representations have already been given in connection with the development of the plane-wave, time-guided, and space-guided wave representations of Green's functions in Secs. 1.2, 1.3, and 1.4, respectively. It has been noted that the time-guided representations in Eq. (1.3.2b) involve eigenfunctions in the $\mathbf{r}$ domain; they can be derived from the plane-wave integral representations in Eq. (1.2.3) on performing the ω integration by deforming the integration contour in the complex ω plane about the ω singularities of the Green's function $G(\mathbf{k}, \omega)$, and invoking the residue theorem. Alternatively, the space-guided representations in Eq. (1.4.3) involve eigenfunctions in the (ρ, t)domain and result from those in Eq. (1.2.3) on doing the k_z integration; one deforms the integration contour in the complex k_z plane about the k_z singularities of $G(\mathbf{k}, \omega)$, with subsequent use of the residue theorem. It is evident from these considerations that the plane-wave representation is quite general (although not necessarily most convenient for field evaluations) since either of the guided representations may be derived from it. To pass from one guided representation to another, one first constructs the fourfold integral representation of a field in terms of the Green's function $G(\mathbf{k}, \omega)$, and then eliminates the undesired integration variable in the manner noted. By an alternative procedure, based on "characteristic" Green's functions, guided-wave representations are derived without intervention of an additional integration.

To illustrate these techniques, we consider the one-dimensional version of the scalar potential problem for the isotropic plasma field defined in Eq. (1.1.60b):

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{u^2}\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right)\right]g(z, z'; t, t') = -\delta(z - z')\delta(t - t'), \tag{50}$$

subject to the causality requirement $g \equiv 0$ for $t < t'$. By the plane-wave integral representation [see Eq. (1.2.31)]

$$g(z, z'; t, t') = \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} G(k, \omega)e^{ik(z-z')-i\omega(t-t')}\,dk\,d\omega, \tag{51}$$

where

$$G(k, \omega) = \frac{1}{k^2 - (\omega^2 - \omega_p^2)/u^2}. \tag{51a}$$

To satisfy causality, the integration path in the complex ω plane runs with $\operatorname{Im}\omega > 0$ above the pole singularities at $\omega = \pm\omega_1$, $\omega_1 = (k^2u^2 + \omega_p^2)^{1/2}$. In the complex k plane, pole singularities lie at $k = \pm k_1 = \pm(\omega^2 - \omega_p^2)^{1/2}/u$, $\operatorname{Im} k_1 > 0$, on opposite sides of the integration path $\operatorname{Im} k = 0$. On deforming the path into the lower half of the complex ω plane, where $\exp[-i\omega(t - t')]$ decays for $t > t'$, one obtains on use of the residue theorem the following reduction of Eq. (51):†

$$g = \frac{u^2}{2\pi}\int_{-\infty}^{\infty} e^{ik(z-z')}\frac{\sin\omega_1(t-t')}{\omega_1}\,dk, \qquad \omega_1(k) = \sqrt{k^2u^2+\omega_p^2}. \tag{52}$$

Alternatively, on deforming the k-plane path into the upper and lower half-planes for $(z - z') > 0$ and $(z - z') < 0$, respectively, one finds

$$g = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}\frac{e^{ik_1|z-z'|}}{-2ik_1}\,d\omega, \qquad k_1(\omega) = \frac{\sqrt{\omega^2-\omega_p^2}}{u}. \tag{53}$$

The representations in Eqs. (52) and (53) may be obtained directly on use of the time-guided and space-guided formulations of Secs. 1.3 and 1.4, respectively. From the one-dimensional form of Eqs. (1.3.2), utilizing normalized eigenfunctions $\Phi_k(z)$ in the z domain,

$$\delta(z-z') = \int_{-\infty}^{\infty}\Phi_k(z)\Phi_k^*(z')\,dk, \qquad \Phi_k(z) = \frac{e^{ikz}}{\sqrt{2\pi}} \tag{54}$$

$$g(z,z';t,t') = \int_{-\infty}^{\infty} g_t(t,t';\lambda_{tk})\Phi_k(z)\Phi_k^*(z')\,dk, \tag{55}$$

where the notation and normalization conforms with the more general discussion in Sec. 3.3. On substitution into Eq. (50), g_t is found to satisfy the equation

$$\left(\frac{d^2}{dt^2}+\lambda_{tk}\right)g_t(t,t';\lambda_{tk}) = u^2\delta(t-t'), \qquad \lambda_{tk} = \omega_1^2(k), \tag{56}$$

which thus identifies g_t as the one-dimensional Green's function in the time domain. Subject to $g_t = 0$ for $t < t'$, the solution of Eq. (56) is

$$g_t(t,t';\lambda_{tk}) = u^2\frac{\sin\sqrt{\lambda_{tk}}\,(t-t')}{\sqrt{\lambda_{tk}}}, \tag{57}$$

whence Eq. (55) yields the same result as Eq. (52). g_t is the analogue of the Green's function G_α defined in Eq. (1.3.16); in the present discussion the difference between them arises from use of the second-order form of the field equations in Eqs. (50).

To obtain the space-guided representation, we employ the one-dimensional form of Eqs. (1.4.2) and (1.4.3),

$$\delta(t-t') = \int_{-\infty}^{\infty}\Phi_\omega(t)\Phi_\omega^*(t')\,d\omega, \qquad \Phi_\omega(t) = \frac{e^{-i\omega t}}{\sqrt{2\pi}}, \tag{58}$$

$$g(z,z';t,t') = \int_{-\infty}^{\infty} g_z(z,z';\lambda_{z\omega})\Phi_\omega(t)\Phi_\omega^*(t')\,d\omega. \tag{59}$$

On substitution into Eq. (50), there results the reduced form

$$\left(\frac{d^2}{dz^2}+\lambda_{z\omega}\right)g_z(z,z';\lambda_{z\omega}) = -\delta(z-z'), \qquad \lambda_{z\omega} = k_1^2(\omega), \tag{60}$$

†This integral is tabulated and yields the closed-form solution as[11]

$$g = \frac{u}{2}J_0\left(\frac{\omega_p}{u}\sqrt{u^2(t-t')^2-(z-z')^2}\right)U[u(t-t')-|z-z'|].$$

whence subject to a radiation condition at $|z - z'| \to \infty$ and in view of $\mathrm{Im}\sqrt{\lambda_{z\omega}} > 0$, the one-dimensional Green's function g_z in the z domain is given by

$$g_z(z, z'; \lambda_{z\omega}) = \frac{\exp(i\sqrt{\lambda_{z\omega}}\,|z - z'|)}{-2i\sqrt{\lambda_{z\omega}}}, \tag{61}$$

so Eqs. (59) and (53) are equivalent. It has thus been demonstrated that appropriate reduction of the plane-wave integral representation in Eq. (51) yields either the time- or space-guided representations, whence, by reconstructing Eq.(51), it is possible to derive one guided representation from the other. Comparison of Eqs. (51), (55), and (59) shows that the reconstruction involves the representation of g_t in Eq. (57), or g_z in Eq. (61), in terms of t- or z-dependent eigenfunctions:

$$-\frac{g_t(t, t'; \lambda_{tk})}{u^2} = \int_{-\infty}^{\infty} \frac{\Phi_\omega(t)\Phi_\omega^*(t')}{\omega^2 - \lambda_{tk}}\, d\omega, \tag{62a}$$

$$g_z(z, z'; \lambda_{z\omega}) = \int_{-\infty}^{\infty} \frac{\Phi_k(z)\Phi_k^*(z')}{k^2 - \lambda_{z\omega}}\, dk, \tag{62b}$$

which forms suggest that there is an intimate relation between Green's functions and spectral representations. This aspect is given further attention in Sec.3.3.

It is significant to observe that either of Eqs. (55) and (59) may also be derived from the other without need of the intermediary relation (51). In the complex ω plane, the space-guided representation in Eqs. (59) and (61) has branch-point singularities at $\omega = \pm\omega_p$. If the corresponding Riemann surface is chosen so that everywhere on the top sheet $\mathrm{Im}\sqrt{\lambda_{z\omega}} > 0$, branch cuts are drawn as shown in Fig. 1.5.3 (see Sec. 5.3b for discussion of Riemann surfaces).

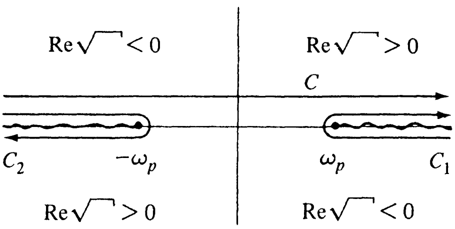


FIG. 1.5.3 Contours in complex ω-plane and behavior of $\mathrm{Re}\sqrt{\omega^2 - \omega_p^2}$ on top sheet of Riemann surface where $\mathrm{Im}\sqrt{\omega^2 - \omega_p^2} > 0$.

For $t > t'$, the exponential behavior of the integrand at infinity in the lower half of the complex ω plane is one of decay, so the contour C can be deformed into the contours C_1 and C_2 around the branch cuts. On changing variables to $-\omega$, one may combine the integral over C_2 with that over C_1; a further change of variable from ω to $\sqrt{\lambda_{tk}}$, with reference to Fig. 1.5.3, leads to Eq. (55).

The preceding considerations will be generalized and phrased succinctly in terms of "characteristic" Green's functions in Sec. 3.3.

1.6 RAY-OPTIC APPROXIMATIONS OF INTEGRAL REPRESENTATIONS

The exact alternative forms for the space- and time-dependent Green's functions developed in Secs. 1.2–1.4 generally lead to integral representations that cannot be evaluated in terms of known functions for arbitrary observation points in space and time. However, they permit useful approximations to be derived when the range of observations is suitably restricted. Concern in this section is with the field behavior at "large" observation distances from the source, in which regions the representation integrals are amenable to asymptotic evaluation by the method of saddle points described in detail in Chapter 4. The principal contributions to an integral will be shown to arise from the vicinity of isolated critical points—stationary (saddle) points, singularities, and endpoints—in the integration interval. One thereby obviates the need for precise evaluation of the integral over the entire integration range. The contribution from each critical point implies a wave process with distinct physical characteristics that are to be examined in detail in Secs. 1.6a–1.6c.

Saddle-point contributions are found to describe wavepackets (narrow bundles of plane waves with limited $\mathbf{k}$, ω values) that move along certain trajectories, called rays, in space–time. The rays are straight in a homogeneous medium but curved when the medium exhibits spatial variation. Energy in a wavepacket is preserved along its space–time trajectory. Both the wavevector $\mathbf{k}$ and the frequency $\omega(\mathbf{k})$ in the plane-wave bundle remain constant in time when the medium is homogeneous, but only ω is invariant in the presence of spatial inhomogeneities. These conclusions may be inferred either from the oscillatory or the guided-wave representations in Secs. 1.6a and 1.6b, respectively; their separate examination reveals distinctive features that illuminate further the propagation mechanism ascribed to each asymptotic solution.

While the saddle-point condition may be stated simply, the location of saddle points by analytical means is generally quite involved when the medium has any but the simplest dispersive properties. However, the $(\mathbf{k}, \omega)$ dispersion surfaces representing the plane-wave dispersion relations are useful for graphically locating the saddle points since the latter may be found thereon by simple geometrical construction. This construction provides the space–time rays determining the spatial and temporal location of a wavepacket. It is performed in four-dimensional $(\mathbf{k}, \omega)$ space when the dispersion surface has no symmetries whatever, but reduces to a plot in three or two dimensions, respectively, when the medium is gyrotropically anisotropic (as in a magnetoplasma) or isotropic. Use of dispersion surfaces also facilitates the charting of space–time rays in an inhomogeneous medium and reveals the intimate connection between propagation and radiation processes under time-dependent and time-harmonic conditions. The conventional rays of geometrical optics emerge as special cases of the space–time rays.

The wavepacket description of field phenomena at distant observation points applies provided that multiple wavepackets, if they exist, are fully formed and well separated in wavenumber and frequency. There are, however, certain regions in space–time, called transition regions, wherein individual wavepackets either are not yet fully developed or interact strongly with adjacent packets. In integral field representations the transition regions are revealed by a confluence of critical points in the integrand. The simple integration procedure applying to isolated critical points is then invalidated; the necessary modifications in the method are indicated in Sec. 1.6c.

1.6a Oscillatory Integral Representations

Homogeneous media

Oscillatory representations of time-dependent Green's functions in an unbounded, stationary, homogeneous, isotropic, or anisotropic medium lead to plane-wave integrals of the form

$$I(\mathbf{r}, t) = \int A(\mathbf{k}) e^{i\psi(\mathbf{r},t;\mathbf{k})}\, d\mathbf{k}, \qquad \psi(\mathbf{r}, t; \mathbf{k}) = \mathbf{k} \cdot \mathbf{r} - \omega(\mathbf{k})t, \tag{1}$$

where $d\mathbf{k} = dk_x\, dk_y\, dk_z$ and the integration extends over the infinite volume in $\mathbf{k}$ space. In general, the overall solution may comprise several integrals of the form shown in Eq. (1), one for each mode type, distinguished by different $\omega_\alpha(\mathbf{k})$ and $A_\alpha(\mathbf{k})$ (see Sec. 1.3).

Since the integral in Eq. (1) cannot generally be evaluated in closed form, it is relevant to discuss approximation procedures. For integrals whose integrands contain a large parameter, one of the most useful procedures is the method of saddle points. This method, and its relation to the stationary-phase method, is discussed in detail in Chapter 4 and is applicable if the distance r from the source point $\mathbf{r} = 0$ to the observation point $\mathbf{r}$ is chosen sufficiently large. Under these circumstances, the major contribution to the integral arises from the vicinity of saddle (stationary) points $\mathbf{k}_s$ defined implicitly by $\nabla_k \psi = 0$, where $\nabla_k = \mathbf{x}_0(\partial/\partial k_x) + \mathbf{y}_0(\partial/\partial k_y) + \mathbf{z}_0(\partial/\partial k_z)$, or

$$\mathbf{v}_g(\mathbf{k}) \equiv \nabla_k \omega(\mathbf{k}) = \frac{\mathbf{r}}{t}, \qquad \text{at } \mathbf{k}_s = \mathbf{k}_s(\mathbf{r}, t). \tag{2}$$

Near $\mathbf{k}_s$, the (slowly varying) amplitude function may be approximated by $A(\mathbf{k}) \approx A(\mathbf{k}_s)$ but the phase requires power-series expansion up to the quadratic term in $(\mathbf{k} - \mathbf{k}_s)$(the linear term is absent in view of the saddle-point condition):

$$\psi(\mathbf{r}, t; \mathbf{k}) = \psi(\mathbf{r}, t; \mathbf{k}_s) + \tfrac{1}{2}[(\mathbf{k} - \mathbf{k}_s) \cdot \nabla_{k_s}]^2 \psi(\mathbf{r}, t; \mathbf{k}_s) + \cdots, \tag{3a}$$

$$= \psi(\mathbf{r}, t; \mathbf{k}_s) - \tfrac{1}{2}t[(\mathbf{k} - \mathbf{k}_s) \cdot \nabla_{k_s}]^2 \omega(\mathbf{k}_s) + \cdots, \tag{3b}$$

where it is understood that ∇_{k_s} operates only on ψ or ω. The resulting integral is evaluated on use of the formula [see Eq. (4.7.5)]

$$\int \exp\left\{-i\frac{t}{2}\left[(\mathbf{k} - \mathbf{k}_s) \cdot \nabla_{k_s}\right]^2 \omega\right\} d\mathbf{k} = (2\pi)^{3/2} \frac{e^{-i(\pi/4)\sigma}}{t^{3/2} Q^{1/2}}, \tag{4}$$

where $Q = |\bar{R}_1 \bar{R}_2 \bar{R}_3|^{-1}$ is the absolute value of the determinant of the matrix $\mathcal{Q}$, with

$$\mathcal{Q} = \begin{vmatrix} \dfrac{\partial^2\omega}{\partial k_x^2} & \dfrac{\partial^2\omega}{\partial k_x \partial k_y} & \dfrac{\partial^2\omega}{\partial k_x \partial k_z} \\ \dfrac{\partial^2\omega}{\partial k_y \partial k_x} & \dfrac{\partial^2\omega}{\partial k_y^2} & \dfrac{\partial^2\omega}{\partial k_y \partial k_z} \\ \dfrac{\partial^2\omega}{\partial k_z \partial k_x} & \dfrac{\partial^2\omega}{\partial k_z \partial k_y} & \dfrac{\partial^2\omega}{\partial k_z^2} \end{vmatrix}_{\mathbf{k}=\mathbf{k}_s} = |\nabla_k \nabla_k \omega(\mathbf{k}_s)|. \tag{4a}$$

Here $\sigma = \sum_{j=1}^{3} \operatorname{sgn} \bar{R}_j$, with $\bar{R}_j$ denoting the reciprocal of the elements (eigenvalues) in the diagonalized form of the matrix $\mathcal{Q}$, and $\operatorname{sgn} \bar{R}_j = \pm 1$ for $\bar{R}_j \gtrless 0$. Thus, one finds for the major contribution to the integral in Eq. (1) for large r [and hence, via Eq. (2), also for large t] and real $\psi(\mathbf{r}, t; \mathbf{k}_s)$(lossless medium):

$$I(\mathbf{r}, t) \sim A(\mathbf{k}_s) e^{i\psi(\mathbf{r},t;\mathbf{k}_s)} \frac{(2\pi)^{3/2}}{(t^3 Q)^{1/2}} e^{-i(\pi/4)\sigma}. \tag{5}$$

$\mathbf{k}_s$ is specified implicitly in Eq. (2) and if this equation has several solutions $\mathbf{k}_{si}$, the single term in Eq. (5) is replaced by a sum over i. Since $I(\mathbf{r}, t)$ is real, there exists for each saddle-point solution $\mathbf{k}_s$, $\omega(\mathbf{k}_s)$ another solution $-\mathbf{k}_s$, $\omega(-\mathbf{k}_s) = -\omega(\mathbf{k}_s)$; the sum over i contains these conjugate pairs as well as different saddle-point species. Additional contributions to the integral may arise from singularities in the integrand (see Sec. 1.6c and Chapter 4) but are ignored for the present.

The approximate asymptotic solution in Eq. (5) has an interesting physical interpretation. Evidently, the field at the space-time point $(\mathbf{r}, t)$ is established by a plane wave $\exp[i\mathbf{k}_s \cdot \mathbf{r} - i\omega(\mathbf{k}_s)t]$ whose amplitude is not $A(\mathbf{k}_s)$, as in the integrand of Eq. (1), but is modified by the last factor in Eq. (5). This modification, noted from the manner of evaluation of the integral, arises from constructive interference of a "packet" or "bundle" of oscillatory plane waves of amplitude $A(\mathbf{k})$ whose wavenumbers lie in a small $\mathbf{k}$-space interval $\delta\mathbf{k} = (\mathbf{k} - \mathbf{k}_s)$ about the saddle-point value $\mathbf{k}_s$. The average phase ψ of the waves in this packet is determined by the central wavenumber $\mathbf{k}_s$, while the composite amplitude changes according to Eq. (5). From Eq. (2), $\mathbf{k}_s$ is constant if $(\mathbf{r}/t)$ is constant so that an observer moving with the wavepacket at the constant velocity $\mathbf{v}_g = (\mathbf{r}/t)$ sees a fixed wavenumber and frequency $\omega(\mathbf{k}_s)$. Stated alternatively, a trajectory on which the frequency ω of a wavepacket remains constant is defined in the $(\mathbf{r}, \mathbf{k})$ phase space by the parametric relations

$$\frac{d\mathbf{r}}{dt} = \nabla_k \omega(\mathbf{k}), \qquad \frac{d\mathbf{k}}{dt} = 0, \qquad \left[\frac{d\omega(\mathbf{k})}{dt} = 0\right]. \tag{6}$$

These trajectory equations for a homogeneous medium are modified, as noted in Eqs. (17a) and (19), in the presence of spatial inhomogeneities. Since the field energy is localized in the moving wavepacket, $\mathbf{v}_g = \nabla_k \omega$ is identified as the group velocity, or energy-transport velocity; the latter is generally different from the phase velocity,

$$\mathbf{v}_p = \frac{\omega}{|\mathbf{k}|} \mathbf{n}_0, \tag{7}$$

with which the equiphase surfaces $\psi =$ constant advance along the normal direction $\mathbf{n}_0$. Only in the dispersionless case $\omega(\mathbf{k}) = kc$ does $\mathbf{v}_g$ equal $\mathbf{v}_p$, with c denoting a $\mathbf{k}$-independent wave-propagation speed. When the observer is at rest at the observation point $\mathbf{r}$, the saddle-point wavenumber $\mathbf{k}_s$ in Eq. (2) changes with time so that the response in Eq. (5) describes a distinct wavepacket at each observation time t.

The amplitude change due to plane-wave interference within a wavepacket centered at $\mathbf{k}_s$ can be deduced in simple physical terms.[12-14] We shall assume that $I(\mathbf{r}, t)$ in Eq. (1) is normalized so that the total energy at any time t is given by

$$W(t) = \int |I(\mathbf{r}, t)|^2 \, d\mathbf{r}, \tag{8a}$$

where the integration is over the entire physical space. By Parseval's theorem [or direct substitution from Eq. (1) with use of Eq. (1.3.2a)], one finds for the energy at $t = 0$,

$$W(0) = (2\pi)^3 \int |A(\mathbf{k})|^2 \, d\mathbf{k}, \tag{8b}$$

so the initial energy represented by waves in the above wavenumber interval $\delta\mathbf{k}$ is $\Delta W = (2\pi)^3 |A(\mathbf{k}_s)|^2 \Delta\mathbf{k}$, where $\Delta\mathbf{k}$ is the volume element corresponding to $\delta\mathbf{k}$. In the absence of dissipation the energy cannot change with time; hence we shall assume that ΔW remains constant and eventually describes the energy in the wavepacket when the latter has formed at $(\mathbf{r}, t)$ values validating Eq. (5). At a particular time thereafter, let the wavepacket occupy the spatial region $\Delta\mathbf{r}$, with corresponding energy $\Delta W = |I|^2 \Delta\mathbf{r}$, whence by equating the above expressions for ΔW,

$$|I| = (2\pi)^{3/2} |A(\mathbf{k}_s)| \sqrt{\frac{\Delta\mathbf{k}}{\Delta\mathbf{r}}}. \tag{9}$$

Although $\Delta\mathbf{k}$ remains constant, $\Delta\mathbf{r}$ changes with time, since individual plane waves within the packet have slightly different group velocities. The mapping from $\mathbf{r}$ to $\mathbf{k}$ space is accomplished via

$$\Delta\mathbf{r} = J\left(\frac{x, y, z}{k_x, k_y, k_z}\right) \Delta\mathbf{k}, \tag{10}$$

where J, the Jacobian of the transformation, can be evaluated from $\mathbf{r} = \mathbf{v}_g(\mathbf{k})t$ as given in Eq. (2). Since $x = v_{gx}(\mathbf{k})t = t\partial\omega(\mathbf{k})/\partial k_x$, etc., one finds $|J| = t^3 Q$, with Q defined in connection with Eq. (4), thereby establishing agreement of the results in Eqs. (5) and (9).

The preceding field solution is valid for those values of $(\mathbf{r}, t)$ at which wavepackets are distinct and fully developed. In "transition regions" of $(\mathbf{r}, t)$ space where these conditions are not satisfied, alternative solutions must be derived by the considerations of Sec. 1.6c.

Dispersion surfaces and space-time rays

Calculation of the transient fields from Eq. (5) requires knowledge of the saddle-point wavenumbers $\mathbf{k}_s(\mathbf{r}, t)$ specified implicitly by the condition (2). However, the (plane-wave) dispersion relation $\omega = \omega(\mathbf{k})$ is generally so complicated that Eq. (2) cannot be solved explicitly for the various $\mathbf{k}_s$. Hence, it is useful to employ graphical procedures based (for lossless media) on the real dispersion surface $\omega = \omega(k_x, k_y, k_z)$, or more generally on the implicit relation

$$f(k_x, k_y, k_z, \omega) = 0. \tag{11}$$

In the most general case, f is a hypersurface in the four-dimensional $(\mathbf{k}, \omega)$ space but simplifies when the dispersion relation has certain symmetries. In an isotropic medium, wave-propagation properties are independent of the direction of propagation, whence $\omega(\mathbf{k}) = \omega(k)$, k being the magnitude of the wavevector. Equation (11) then reduces to

$$f(k, \omega) = 0, \tag{11a}$$

which can be plotted in a two-dimensional (k, ω) frame. In a magnetoactive medium, such as a plasma rendered anisotropic by the application of a steady external magnetic field $\mathbf{H}_0 = \mathbf{z}_0 H_0$, plane-wave propagation characteristics do not depend on direction in a plane transverse to $\mathbf{H}_0$. The dispersion surface thus takes the form

$$f(k_\rho, k_z, \omega) = 0, \tag{11b}$$

which, with k_ρ representing the wavenumber component transverse to z, can be plotted in a three-dimensional (k_ρ, k_z, ω) coordinate system.

Graphical determination of the saddle points $\mathbf{k}_s(\mathbf{r}, t)$ satisfying Eq. (2) can be effected with the aid either of the dispersion surface $f(\mathbf{k}, \omega) = 0$ in the four-dimensional $(\mathbf{k}, \omega)$ space, or of its $\omega = \omega(\mathbf{k}) =$ constant projections in three-dimensional $\mathbf{k}$ space. In the former instance, the desired saddle points are located at those points $[\mathbf{k}_s(\mathbf{r}, t), \omega(\mathbf{k}_s)]$ of the dispersion surface whereon the four-vector normal is parallel to the space–time four-vector $(\mathbf{r}/t, 1)$, provided that the coordinate axes k_x, k_y, k_z, ω are chosen parallel to $x, y, z, -t$, respectively. In the latter case, the $\mathbf{k}_s$ distinguish points of the constant $\omega(\mathbf{k})$ projections at which the normal gradient vector is equal to $\mathbf{r}/t$ in both direction and magnitude. The proof of the former statement follows from the observations that the four-vector normal to the dispersion surface $f(\mathbf{k}, \omega) = 0$ is given by the four-dimensional gradient $\Box f \equiv (\nabla_k f, \partial f/\partial\omega)$ and is perpendicular to the tangential four-vector $(d\mathbf{k}, d\omega) = (d\mathbf{k}, \nabla_k \omega \cdot d\mathbf{k})$,† whence on forming the scalar product of the two vectors, one infers that

$$df = 0 = \nabla_k f + \frac{\partial f}{\partial \omega} \nabla_k \omega; \tag{12}$$

†For $(\mathbf{k}, \omega)$ and $(\mathbf{k} + d\mathbf{k}, \omega + d\omega)$ on the dispersion surface, $(d\mathbf{k}, d\omega)$ is a tangential vector element.

thus, the four-vector normal $\Box f$ is parallel to the four-vector $(\nabla_k \omega, -1)$ which, by Eq. (2) and because of the opposite orientation of the ω and t axes, must be made parallel to $(\mathbf{r}/t, 1)$. The proof of the latter statement above follows from the fact that the normal gradient vector to the surface $\omega =$ constant is the three-vector $\nabla_k \omega$ which, by Eq. (2), must be set equal to $\mathbf{r}/t$. The vector $(\mathbf{r}/t, 1) = (\mathbf{v}_g, 1)$, the group-velocity vector in four-dimensional space, defines a "space–time ray" trajectory $\mathbf{r} = \mathbf{r}(t)$ descriptive of the spatial and temporal location of a moving wavepacket.

The geometrical principles underlying the graphical method for locating the saddle points are relatively simple. However, in a general medium described by the dispersion equation (11), application of this procedure is complicated by the necessity of dealing with normals to four-dimensional surfaces, or with three-dimensional surfaces, when finding the projection of the normal onto hyperplanes $\omega =$ constant. Simplification occurs when the medium anisotropy is of the magnetoactive type since the dispersion surface [Eq. (11b)] can then be plotted in a three-dimensional frame and projections onto planes $\omega =$ constant reduce to curves in the (k_ρ, k_z) wavenumber plane. These considerations are illustrated for electromagnetic wave propagation in a cold electron plasma rendered uniaxially anisotropic by imposition of an infinitely strong external magnetic field H_0 along the z axis. The dispersion equation in this instance (see Sec. 7.2), with c denoting the speed of light in vacuum,

$$f(k_\rho, k_z, \omega) = k_\rho^2 + \left(1 - \frac{\omega_p^2}{\omega^2}\right)k_z^2 - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) = 0, \tag{13}$$

describes a surface whose intersections with planes $\omega =$ constant yield a family of ellipses when $|\omega|$ exceeds the plasma frequency ω_p, and a family of hyperbolas when $|\omega| < \omega_p$. For simplicity, we consider only the $|\omega| > \omega_p$ portion of the surface, as shown in Fig. 1.6.1(a). It is convenient for dimensional reasons to deal with plots of $(\mathbf{k}c, \omega)$ and $(\mathbf{r}, ct)$, so the normal to the dispersion surface is now defined as $\bar{\Box} f \equiv (c^{-1}\nabla_k f, \partial f/\partial\omega)$. To locate the saddle point corresponding to the group-velocity vector $\mathbf{V}_g = (\mathbf{r}/t, c)$ [see Fig. 1.6.1(b)] we search on the dispersion surface, plotted in the coordinate frame described previously, for normals $\bar{\Box} f$ parallel to $\mathbf{V}_g$ and ascertain thereby the projection $c\mathbf{k}_s$ [Fig. 1.6.1(a)].

Alternatively, one may plot the family of wavenumber curves for various ω as in Fig. 1.6.1(c), and the family of group speed curves $v_g(\mathbf{k}) = v_g(k, \bar{\theta}) = |\nabla_k \omega| = [(\partial\omega/\partial k)^2 + (\partial\omega/k\partial\bar{\theta})^2]^{1/2}$ as in Fig. 1.6.1(d), with k and $\bar{\theta}$ denoting the magnitude and orientation angle of $\mathbf{k}$ in cylindrical polar coordinates. Location of the saddle point is achieved by use of the projected vector $\mathbf{v}_g = \mathbf{r}/t$ in Fig. 1.6.1(b). In Fig. 1.6.1(c), the required codirectionality of the vectors $\nabla_k \omega$ and $\mathbf{r}/t$ is enforced by locating those points on the wavenumber curves having normals parallel to $\mathbf{r}$; one thereby obtains the curve A, described parametrically by $k = k(\bar{\theta})$, on which the saddle-point value $k_s = k(\bar{\theta}_s)$ lies. Note that the vector $\nabla_k \omega$ on a curve $\omega = \omega_i$ points toward the curve described by a value

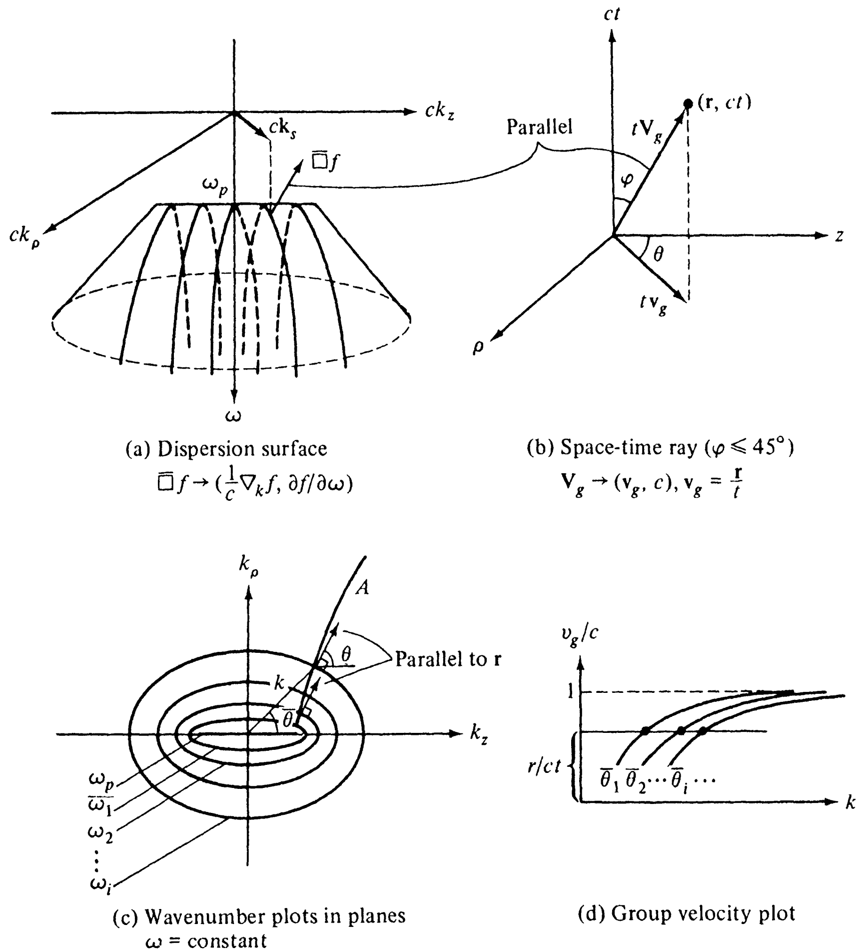


FIG. 1.6.1 Graphical methods for locating the saddle points $\mathbf{k}_s$ on an anisotropic dispersion surface.

$\omega > \omega_i$. The magnitude requirement $v_g = |\nabla_k \omega| = r/t$ is imposed by the construction in Fig. 1.6.1(d), and the saddle point $\mathbf{k}_s$ is that point on A in Fig. 1.6.1(c) which is compatible with the k versus $\bar{\theta}$ values deduced from Fig. 1.6.1(d).

One observes from Fig. 1.6.1(c) that in an anisotropic medium, the wavevector $\mathbf{k}$ and the group velocity vector $\mathbf{v}_g$ generally are nonparallel, so phase and energy propagate along different directions. This distinction is illustrated in Fig. 1.6.2. Let us recall that a wavepacket comprises a bundle of plane waves, whose wave vectors lie within a small cone in $\mathbf{k}$ space, that propagates in the direction of constructive wave interference. Wavefronts advance along the direction of the wavevector $\mathbf{k}$ with a speed given by $v_p = \omega/k$ [see Eq. (7)]; this quantity may be plotted to furnish the phase velocity

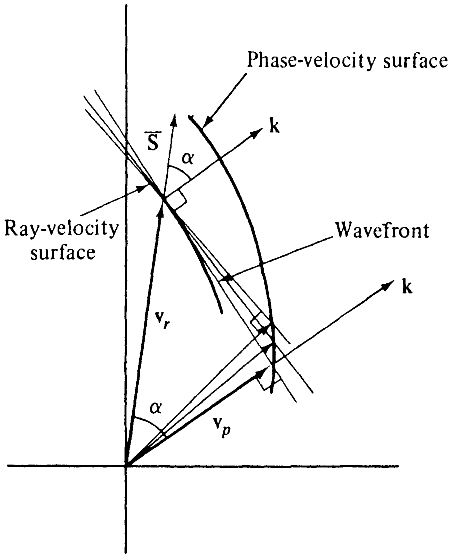


FIG. 1.6.2 Phase- and ray-velocity surfaces.

surfaces, one of which is shown in part in Fig. 1.6.2 for a typical member of the wavenumber diagrams in Fig. 1.6.1(c). Since the vector $\mathbf{v}_p t$ yields the spatial displacement of a wavefront from its location at a reference time $t = 0$, the phase velocity surface may be used to construct the wavefront configuration after a unit time interval.[15] The phase fronts in a wavepacket located at the origin of the $\mathbf{v}_p t$ plot at $t = 0$ will have moved to the locations shown in Fig. 1.6.2 at $t = 1$; their intersections locate the interference maximum and hence the new position of the wavepacket, as shown by the "ray-velocity" vector $\mathbf{v}_r$ in Fig. 1.6.2 along which the energy propagates. Unless the phase-velocity surface is spherical as in an isotropic medium, the phase- and ray-velocity vectors are displaced by an angle α and $v_r = v_p/\cos\alpha$ represents the speed of propagation of the wavefront along the direction of the ray. By carrying out the wavefront construction for all points on the phase-velocity surface, one generates a new surface, the ray-velocity surface or simply "ray surface," which constitutes the envelope of the wavefronts. The energy-transport, or ray, direction corresponding to a given direction of $\mathbf{v}_p$ is inferred by drawing a perpendicular plane at the endpoint of $\mathbf{v}_p$, determining its point of tangency on the ray surface, and drawing a vector from the origin to the contact point, as shown in Fig. 1.6.2. Conversely, to the ray direction specified by a given point on the ray surface corresponds a wavevector direction given by the normal to the surface.

For the special case of an isotropic medium, the two procedures described in connection with Fig. 1.6.1 are equivalent since the relevant dispersion equation (11a) requires only a two-dimensional plot in (k, ω) space. Because of the rotational symmetry of the dispersion surface about the ω axis, the wavenumber plots in Fig. 1.6.1(c) are now circular and therefore the curve A degenerates into a radial straight line. Since $\mathbf{v}_g(\mathbf{k}) = \mathbf{v}_g(k)$, the family of curves in Fig. 1.6.1(d) collapses into a single $\bar{\theta}$-independent curve. For illustration, consider a cold electron plasma having the dispersion relation

$$f(k, \omega) = \omega^2 - (kc)^2 - \omega_p^2 = 0, \tag{14}$$

whence the dispersion surface and space–time ray plots are those in Fig. 1.6.3(a) and (b), with the group-velocity curve given in Fig. 1.6.3(c). The determination of k_s follows as previously. In the present instance, it is also possible (and conventional) to use instead of Fig. 1.6.3(c) the dispersion curve in Fig. 1.6.3(a); by direct imposition of the saddle-point condition $d\omega/dk = r/t$, one locates k_s on drawing a tangent having the slope r/t. The dispersion surface can also be employed to provide a pictorial representation for the amplitude variation noted in Eq. (5). In the discussion following Eqs. (8), it is emphasized that the energy in a wavepacket described by a fixed wavenumber deviation $\Delta\mathbf{k}$ from $\mathbf{k}_s$ remains constant. For the isotropic problem in Fig. 1.6.3, this feature may be illustrated by letting the wavepacket under consideration be characterized by the wave-

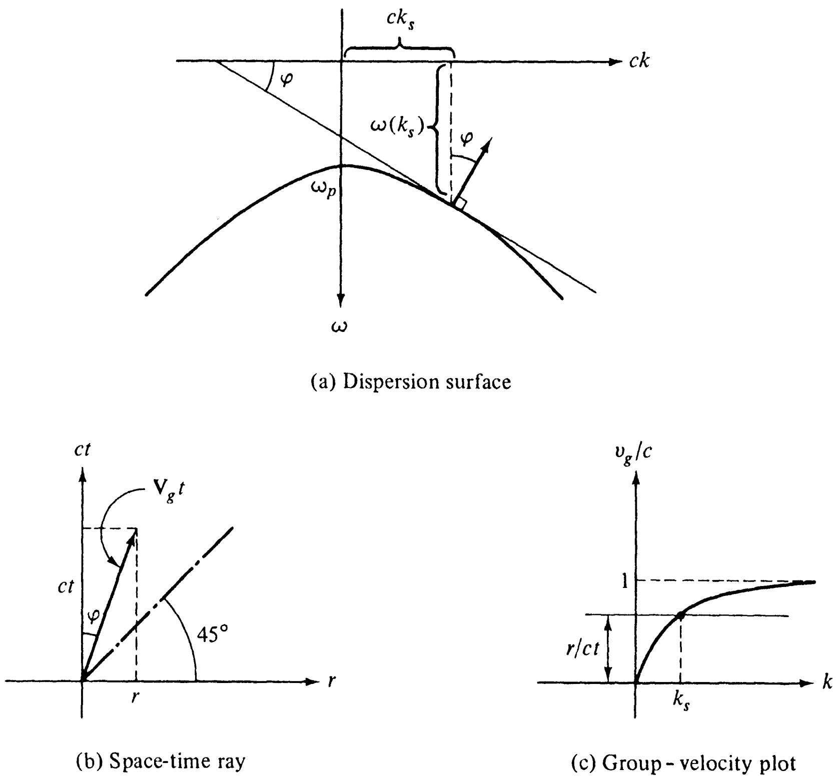
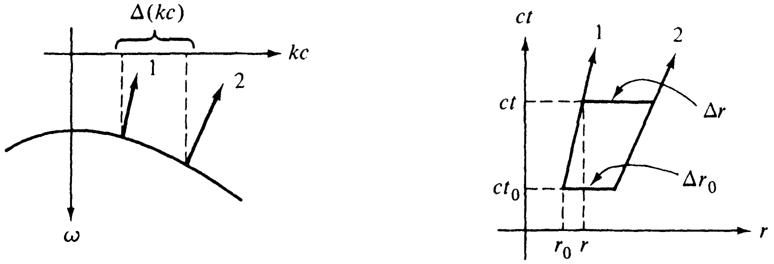


FIG. 1.6.3 Graphical methods for locating the saddle points $\mathbf{k}_s$ for an isotropic medium.



(a) Dispersion surface (b) Space-time ray tube

FIG. 1.6.4 Conservation of energy in a space-time ray tube.

number interval shown in Fig. 1.6.4(a), with corresponding limiting rays 1 and 2 as indicated. In the configuration space of Fig. 1.6.4(b), these rays form a space–time ray tube, and since this ray tube describes the same wavepacket with fixed wavenumber spread $\Delta(kc)$, the energy in the ray tube remains constant. The constancy of the energy $\Delta W = |I|^2 \Delta r$ implies that the field amplitude $|I(r, t)|$ at t is related to its value $|I(r_0, t_0)|$ at t_0 by the square root of the ratio of the corresponding ray tube cross sections in a plane t = constant:

$$|I(r, t)| = |I(r_0, t_0)| \sqrt{\frac{\Delta(r_0)}{\Delta(r)}}. \tag{15}$$

This relation continues to hold for more general dispersion surfaces characterized by three- or four-dimensional ($\mathbf{k}c, \omega$) plots. It is equivalent to that given in Eq. (9) since the expression for the energy, $|I(r_0, t_0)|^2 \Delta r_0$, is also proportional to $|A(k_s)|^2 \Delta k$. As is evident from Fig. 1.6.4, for the same wavenumber interval $\Delta(kc)$, a more strongly curved dispersion surface provides a more rapidly diverging ray tube, thereby leading to a more rapidly decreasing amplitude, as expressed by the factor $Q^{-1/2}$ in Eq. (5).

Weakly inhomogeneous media

When the properties of a medium vary with spatial position, it is no longer possible to represent the space- and time-dependent field by the plane-wave superposition in $\mathbf{k}$ space as in Eq. (1), since $A(\mathbf{k}) \exp [i\mathbf{k} \cdot \mathbf{r} - i\omega(\mathbf{k})t]$ is then no longer an eigenfunction in the $\mathbf{r}, t$ domain. Although a valid representation may be achieved through superposition of functions of the form $A(\mathbf{k}, t) \exp (i\mathbf{k} \cdot \mathbf{r})$, the calculation of $A(\mathbf{k}, t)$ is complicated since the functions $\exp (i\mathbf{k} \cdot \mathbf{r})$ do not form an orthogonal set in $\mathbf{r}$ space in the presence of spatial inhomogeneity. However, for "sufficiently slow" medium variations, one may synthesize the field solution approximately by a spectrum of *local* plane waves whose amplitude A and frequency ω are dependent not only on the wavevector $\mathbf{k}$ but also on the average position coordinate $\bar{\mathbf{r}}$, with the latter playing the role of a slow parameter descriptive of the local properties in the medium. For a medium with one-dimensional inhomogeneity, the validity of such an approximate integral representation is substantiated by use of WKB methods (see Sec. 3.5 and 1.6b); for two- and three-dimensional variations, conclusions obtained therefrom are confirmed by the procedure described in Sec. 1.7e.

As suggested by the field representation (1) for a homogeneous medium, it is possible to account for slow variations in the medium properties by first considering integrals appropriate to a piecewise constant medium. One is then led to consider integrals of the form

$$I(\mathbf{r}, t; \bar{\mathbf{r}}) = \int A(\mathbf{k}, \bar{\mathbf{r}}) e^{i[\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k}, \bar{\mathbf{r}})t]}\, d\mathbf{k} \tag{16}$$

where the range of observation points $\mathbf{r}$ is restricted to a volume τ centered on $\mathbf{r} = \bar{\mathbf{r}}$, τ being small enough to render the coordinate dependent dispersion equation $\omega = \omega(\mathbf{k}, \mathbf{r}) \approx \omega(\mathbf{k}, \bar{\mathbf{r}})$ essentially constant. For fixed $\mathbf{r}, \bar{\mathbf{r}}$, and t, the major contribution to $I(\mathbf{r}, t; \bar{\mathbf{r}})$ in Eq. (16) arises from those points $\mathbf{k} = \mathbf{k}_s(\mathbf{r}, t; \bar{\mathbf{r}})$ that satisfy the saddle-point condition

$$\frac{\mathbf{r}}{t} = \nabla_k \omega(\mathbf{k}, \bar{\mathbf{r}}). \tag{17}$$

Just as for the corresponding condition (5) in a homogeneous medium, there is an interesting physical interpretation of Eq. (17) for $\mathbf{r}, t$ regions wherein wavepackets are fully developed. An observer moving on a trajectory $\mathbf{r} = \mathbf{r}(t)$ in τ with the velocity

$$\mathbf{v}_g = \frac{d\mathbf{r}}{dt} = \frac{\mathbf{r}}{t} = \nabla_k \omega(\mathbf{k}, \bar{\mathbf{r}})$$

sees a wavenumber $\mathbf{k} = \mathbf{k}_s$ dependent on $\bar{\mathbf{r}}$. Since $\bar{\mathbf{r}}$ takes on a different value $\bar{\mathbf{r}}_1$ in a volume τ_1 adjacent to τ, the relevant $\mathbf{k}_s$ values are given by Eq. (17) with $\bar{\mathbf{r}}$ and τ replaced by $\bar{\mathbf{r}}_1$ and τ_1, and similarly for other volume elements τ_i. But since $\omega(\mathbf{k}, \bar{\mathbf{r}})$ depends weakly on $\bar{\mathbf{r}}$ [i.e., $\omega(\mathbf{k}, \bar{\mathbf{r}}_i) \approx \omega(\mathbf{k}, \mathbf{r})$ for $\mathbf{r}$ in τ_i], the wavenumber $\mathbf{k}$ seen by an observer moving with velocity $d\mathbf{r}/dt$ is specified by

$$\frac{d\mathbf{r}}{dt} = \nabla_k \omega(\mathbf{k}, \mathbf{r}). \tag{17a}$$

If the observer moves with a wavepacket on a phase-space trajectory $\mathbf{k} = k[\mathbf{r}(t)] = \mathbf{k}(t), \mathbf{r} = \mathbf{r}(t)$, along which the frequency $\omega(\mathbf{k}, \mathbf{r})$ of the wavepacket is conserved, then one infers from the constancy of the frequency

$$\frac{d}{dt}\omega(\mathbf{k}, \mathbf{r}) = 0 = \nabla_k \omega \cdot \frac{d\mathbf{k}}{dt} + \nabla\omega \cdot \frac{d\mathbf{r}}{dt}, \tag{18}$$

and hence, using Eq. (17a),

$$\frac{d\mathbf{k}}{dt} = -\nabla\omega(\mathbf{k}, \mathbf{r}). \tag{19}$$

For the homogeneous case $\omega = \omega(\mathbf{k})$, Eqs. (17a) and (19) manifestly reduce to Eqs. (6).

The significance of phase-space coordinates and of constant-frequency trajectories in the propagation of a wavepacket in an inhomogeneous medium is discussed further in Sec. 1.7a.

The variability of $\mathbf{k}_s$ on the ray trajectory complicates the use of graphical methods, as discussed in connection with Fig. 1.6.1. For media plane-stratified along z, where $\nabla\omega = \mathbf{z}_0(\partial\omega/\partial z)$, Eq. (19) implies constancy of the component $\mathbf{k}_t$ transverse to z, thereby permitting application of the graphical procedure in Fig. 1.6.9 for charting the ray paths; relevant $\mathbf{k}_s$ values are those corresponding to rays passing through the prescribed point $(\mathbf{r}, t)$. For medium variability in two or three dimensions, Eq. (19) implies constancy of the component $\mathbf{k}_t$ transverse to the $\mathbf{r}$-dependent direction of $\nabla\omega$; for comments concerning the graphical construction in this case, see remarks following Eq. (1.7.27).

1.6b Guided-wave Integral Representations

Field representations in terms of waves guided along a rectilinear spatial coordinate z are useful for stratified media whose properties vary continuously or abruptly along the z direction. Guided-wave integral representations of the field have the form [see Eq. (1.4.3)]

$$I(\mathbf{r}, t) = \int_{-\infty+i\gamma}^{\infty+i\gamma} I(\mathbf{r}, \omega)e^{-i\omega t}\, d\omega, \qquad \gamma > 0, \tag{20}$$

where

$$I(\mathbf{r}, \omega) = \iint_{-\infty}^{\infty} F(\mathbf{k}_t, \omega; z)e^{i\mathbf{k}_t\cdot\boldsymbol{\rho}}\, d\mathbf{k}_t. \tag{21}$$

γ in Eq. (20) is chosen large enough to ensure that the integration path in the complex ω plane lies above all the singularities of $I(\mathbf{r}, \omega)$, as required by causality. The transverse wavenumber $\mathbf{k}_t = \mathbf{x}_0 k_x + \mathbf{y}_0 k_y$ in Eq. (21) ranges over all real values in the plane transverse to z. The separation into $\mathbf{k}_t$ and ω integrations emphasizes the utility of guided-wave field representations for the study of time-harmonic excitation. If ω_0 is the harmonic oscillation frequency, linearity and invariability of the medium properties with time require that the frequency dependence of F has the form† $F(\mathbf{k}_t, \omega; z) = \bar{F}(\mathbf{k}_t, \omega_0; z)\delta(\omega - \omega_0)$, thereby rendering the ω integration in Eq. (20) superfluous and identifying $I(\mathbf{r}, \omega_0)$ as the time-harmonic response function.

Since even in a homogeneous medium, the guided-wave representation provides information that may be compared and contrasted with that deduced from the oscillatory representation in Sec. 1.6a, the homogeneous case is considered first and furnishes the basis for subsequent study of inhomogeneous media.

Homogeneous media (time-harmonic case)

In a homogeneous medium, the z dependence of F is in the form of plane waves, so $I(\mathbf{r}, \omega)$ is written typically as

†For this form of F, $\gamma = 0$ in Eq. (20). If $\gamma > 0$, one may use $F(\mathbf{k}_t, \omega; z) = \bar{F}(\mathbf{k}_t, \omega_0; z)/(\omega - \omega_0)$, close the integration contour in the lower half of the complex ω plane when $t > 0$, and evaluate via the residue theorem.

$$I(\mathbf{r}, \omega) = \int A(\mathbf{k}_t, \omega) e^{i\psi(\mathbf{r};\mathbf{k}_t,\omega)}\, d\mathbf{k}_t, \quad \psi(\mathbf{r}; \mathbf{k}_t, \omega) = \mathbf{k}_t \cdot \boldsymbol{\rho} + \kappa(\mathbf{k}_t, \omega) z, \tag{22}$$

where the longitudinal propagation constant $k_z \equiv \kappa = \kappa(\mathbf{k}_t, \omega)$ follows from a plane-wave dispersion equation $f(\mathbf{k}_t, k_z, \omega) = 0$. For large values of $\mathbf{r} = \boldsymbol{\rho} + \mathbf{z}_0 z$, the integral can be approximated by the saddle-point technique discussed in Chapter 4. The saddle points $\mathbf{k}_{ts}$ are defined implicitly by $\nabla_{k_t}\psi = 0$, where $\nabla_{k_t} = \nabla_k - \mathbf{z}_0(\partial/\partial k_z) = \mathbf{x}_0(\partial/\partial k_x) + \mathbf{y}_0(\partial/\partial k_y)$, or on suppression of the ω dependence,

$$\frac{\boldsymbol{\rho}}{z} = -\nabla_{k_t}\kappa(\mathbf{k}_t) \qquad \text{at } \mathbf{k}_{ts} = \mathbf{k}_{ts}(\boldsymbol{\rho}, z). \tag{23}$$

By proceeding as in Eqs. (3) and (4), one may write for the $\mathbf{k}_{ts}$ contribution to the integral in Eq. (22),

$$I(\mathbf{r}, \omega) \sim A(\mathbf{k}_{ts}) e^{i[\mathbf{k}_{ts}\cdot\boldsymbol{\rho} + \kappa(\mathbf{k}_{ts})z]} \frac{2\pi e^{i(\pi/4)[\operatorname{sgn} \hat{R}_1 + \operatorname{sgn} \hat{R}_2]}}{z\bar{Q}^{1/2}} \tag{24}$$

where $\bar{Q} = |\hat{R}_1 \hat{R}_2|^{-1}$ is the absolute value of the determinant of the matrix $\bar{\mathcal{Q}}$, with

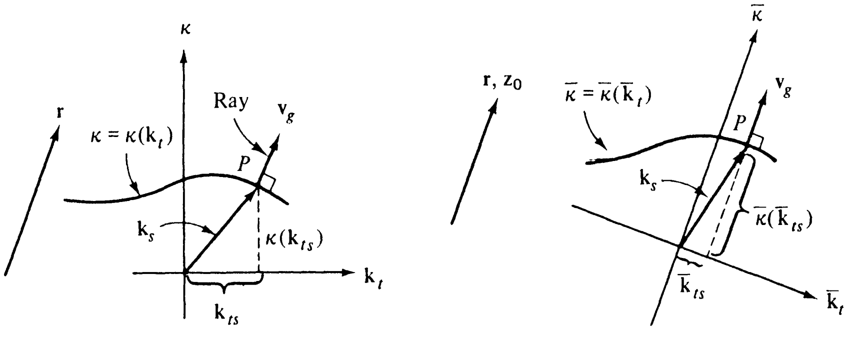
$$\bar{\mathcal{Q}} = \begin{bmatrix} \dfrac{\partial^2 \kappa}{\partial k_x^2} & \dfrac{\partial^2 \kappa}{\partial k_x \partial k_y} \\ \dfrac{\partial^2 \kappa}{\partial k_y \partial k_x} & \dfrac{\partial^2 \kappa}{\partial k_y^2} \end{bmatrix}_{\mathbf{k}_t = \mathbf{k}_{ts}} \tag{24a}$$

and $(1/\hat{R}_{1,2})$ are the elements in the diagonalized form of the matrix $\bar{\mathcal{Q}}$.

It is useful to write the result in Eq. (24) in an invariant form that is more directly descriptive of wave propagation in a homogeneous medium.[13] To this end, one assumes first that the z axis of the coordinate system is rotated so as to coincide with the radius vector $\mathbf{r}$, whence $\boldsymbol{\rho} = 0$ and $z = r$. The saddle-point condition (23) in the rotated $(\bar{\mathbf{k}}_t, \bar{\kappa})$ coordinate system then selects $\bar{\mathbf{k}}_{ts}$ points on the (wavenumber) dispersion surface $\bar{\kappa} = \bar{\kappa}(\bar{\mathbf{k}}_t)$ at which the normal vector is parallel to z (Fig. 1.6.5). At such points P, the diagonalized form of $\bar{\mathcal{Q}}$ is equal to $(1/\bar{R}_1 \bar{R}_2)$, where $\bar{R}_1$ and $\bar{R}_2$ are the principal radii of curvature of the wavenumber surface at P. The following quantities are invariant under coordinate rotation: the vector $\mathbf{r}$ from the origin in physical space to the observation point, the wave vector $\mathbf{k}_s$ from the origin in wavenumber space to a point P of the wavenumber surface whereon the normal is parallel to $\mathbf{r}$, and the principal radii of curvature at P. Thus, one may write Eq. (24) in the original fixed-coordinate system as

$$I(\mathbf{r}, \omega) \sim A(\mathbf{k}_{ts}) e^{i\mathbf{k}_s\cdot\mathbf{r}} \frac{2\pi\, e^{\,i(\pi/4)(\operatorname{sgn} \bar{R}_1 + \operatorname{sgn} \bar{R}_2)} |\bar{R}_1 \bar{R}_2|^{1/2}}{r}. \tag{25}$$

If several points P on the surface satisfy the above requirements, each contributes an expression of the form (25) to $I(\mathbf{r}, \omega)$ (see subsequent remarks concerning the radiation condition). Singularities in the integrand of Eq. (22) may also contribute to the asymptotic value of the integral but are ignored for the present.



(a) Original axes (b) Rotated axes

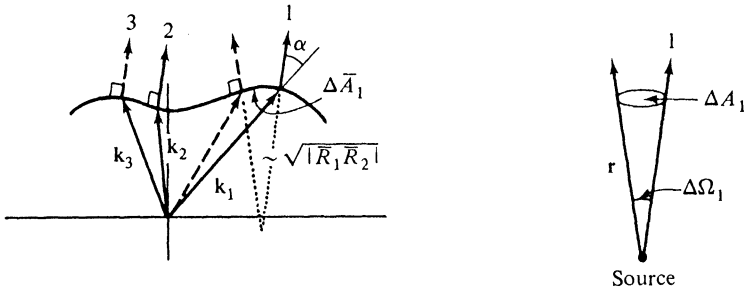
FIG. 1.6.5 Graphical location of saddle point.

It is useful to recall that a wavenumber surface in **k** space, corresponding to a fixed oscillation frequency ω, is one of a family of $\omega = \omega(\mathbf{k})$ dispersion surfaces descriptive of the space–time dispersion relation in four dimensions. The normal vector to the surface $\omega =$ constant is given by $\nabla_k \omega(\mathbf{k})$, identified previously in Eq. (2) as the group-velocity vector $\mathbf{v}_g$ in the direction of energy transport, the *ray* direction. By allowing for a slight frequency spread in a time-harmonic signal, one may still assign significance to the group velocity in the present discussion. Thus, Eq. (23) may be interpreted as defining a ray trajectory, $\boldsymbol{\rho}/z =$ constant, along which travels the energy emitted continuously by the source. Since the medium is homogeneous, the rays are straight lines, whence, from Eq. (23),

$$\mathbf{k}_{ts} = \text{const.}, \qquad \kappa(\mathbf{k}_{ts}) = \text{const.} \Rightarrow \mathbf{k}_s = \text{const.} \tag{26}$$

The graphical location of the saddle points in Fig. 1.6.5 is precisely the same as in Fig. 1.6.1(c). Only those points $\mathbf{k}_s$ on the **k** surface may be included in (25) for which the orientation and direction of $\nabla_k \omega$ coincide with that of **r**. Such points satisfy a radiation condition requiring outflow of energy along the radial direction. For electromagnetic wave propagation in a non-spatially-dispersive, anisotropic dielectric (e.g., a cold magnetoplasma), identification of permitted points is simplified by the recognition that the angle between $\mathbf{k}_s$ and $\mathbf{v}_g$ (codirectional with **r**) must not exceed 90° [see Eq. (1.7.53a)].

As in the analogous result in Eq. (5) for the time-dependent case, the time-harmonic field is established locally by a plane wave $\exp(i\mathbf{k}_s \cdot \mathbf{r})$ whose amplitude $A(\mathbf{k}_{ts})$ in Eq. (22) is modified by the last factor in Eq. (25). The latter arises from the interference of plane waves having wavenumbers within a narrow cone $\Delta\mathbf{k}$ in **k** space. Such a cone near $\mathbf{k}_1$ is shown in Fig. 1.6.6(a) and intercepts an area element $\Delta\bar{A}_1$ on the wavenumber surface; the corresponding ray tube, drawn perpendicular to the surface on the boundaries of $\Delta\bar{A}_1$, has its "vertex" at a distance proportional to $\sqrt{|\bar{R}_1\bar{R}_2|}$, where $\bar{R}_1$ and $\bar{R}_2$ are the principal radii of curvature of the wavenumber surface at $\mathbf{k}_1$. The (constant)



(a) Wavenumber surface (b) Physical configuration

FIG. 1.6.6 Point source in an anisotropic medium.

energy contained in the plane-wave bundle described by $\Delta\mathbf{k}$ is propagated in space within the ray tube, whence it is evident that the energy density in space varies inversely with the ray-tube cross section, which, in turn, depends on the rate of divergence of the rays (see also Sec. 1.7b). From Fig. 1.6.6(a), a weakly curved surface at $\mathbf{k}_1$ gives rise to a slowly diverging ray cone with weak decay of energy density in the ray tube, whereas the decay is rapid for strong surface curvature. If $\Delta\Omega_1$ denotes the solid angle subtended by the ray cone at the source, then on comparing Figs. 1.6.6(a) and (b), $\Delta\Omega_1 = \Delta A_1/r^2 \propto \Delta\bar{A}_1/\bar{R}_1\bar{R}_2$, so the ray-tube cross section corresponding to a fixed $\Delta\bar{A}_1$ varies like $r^2/\bar{R}_1\bar{R}_2$. Since the field amplitude is proportional to the square root of the energy density, one verifies the behavior exhibited in Eq. (25).[13,14,16]

z-stratified media (time-harmonic case)

In a medium with variability along the z coordinate, the function F in the integrand of Eq. (21) cannot generally be expressed in closed form in terms of known functions. Closed-form expressions for F are obtainable for special z functions descriptive of the medium variation (see Sec. 5.9), but even under this circumstance the integral (21) is usually not calculable exactly. For slow medium variations, however, F may be approximated at almost all z by its local plane-wave (WKB) representation (Sec. 5.8d), whence Eq. (21) becomes

$$I(\mathbf{r},\omega) = \int A(\mathbf{k}_t,\omega;z)\exp\left\{i\left[\mathbf{k}_t\cdot\boldsymbol{\rho} + \int^z \kappa(\mathbf{k}_t,\omega;\zeta)\,d\zeta\right]\right\}d\mathbf{k}_t, \qquad (27)$$

where both the local longitudinal propagation constant κ, which follows from the z-dependent plane-wave dispersion equation $f(\mathbf{k}_t,\kappa,\omega;z) = 0$, and the local amplitude A depend weakly on z. At large observation distances $\mathbf{r} = \boldsymbol{\rho} + \mathbf{z}_0 z$, or for short wavelengths,† the principal contribution to the integral (27) arises from the vicinity of saddle points $\mathbf{k}_{ts}$ (see Chapter 4); these are defined implicitly

†In the time-harmonic problem, the relevant parameter is the normalized distance r/λ, where λ is the local wavelength. Largeness of r/λ can be secured either by sufficiently large r or sufficiently small λ. The latter choice is used in Sec. 1.7a.

on equating to zero the $\mathbf{k}_t$ derivative (∇_{k_t}) of the exponent in the integrand (ω dependence omitted):

$$\boldsymbol{\rho} = -\int^z \nabla_{k_t} \kappa(\mathbf{k}_t, \zeta)\, d\zeta \qquad \text{at } \mathbf{k}_t = \mathbf{k}_{ts}(\boldsymbol{\rho}, z). \tag{28}$$

Following the procedure described previously, one obtains for the $\mathbf{k}_{ts}$ contribution to the integral (27) at the observation point $(\boldsymbol{\rho}, z)$ the result in Eq. (24), provided that the product $\kappa(\mathbf{k}_{ts})z$ in the exponent and denominator of Eq. (24) is replaced by $\int^z \kappa(\mathbf{k}_{ts}, \zeta)\, d\zeta$.

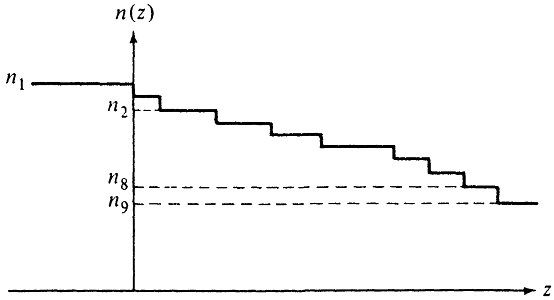
To facilitate location of the saddle points $\mathbf{k}_{ts}$, it is convenient, as in the case of the homogeneous medium [see Eqs. (23) and (26)], to interpret Eq. (28) as defining trajectories in $\mathbf{r}$ space. Of special interest are trajectories on which $\mathbf{k}_{ts}$ = constant. These describe propagation paths of local plane waves since in a homogeneous stratum at level z_i, a plane wave is described by the wavevector $\mathbf{k}(z_i) = \mathbf{k}_t + \mathbf{z}_0 \kappa(\mathbf{k}_t, z_i)$; phase continuity across an interface (Snell's law of refraction) leading to the next stratum z_j requires constancy of the tangential wavevector $\mathbf{k}_t$, whence $\mathbf{k}(z_j) = \mathbf{k}_t + \mathbf{z}_0 \kappa(\mathbf{k}_t, z_j)$. Thus, the differential form of Eq. (28),

$$\frac{d\boldsymbol{\rho}}{dz} = -\nabla_{k_t} \kappa(\mathbf{k}_t, z), \qquad \mathbf{k}_t = \text{constant}, \tag{29}$$

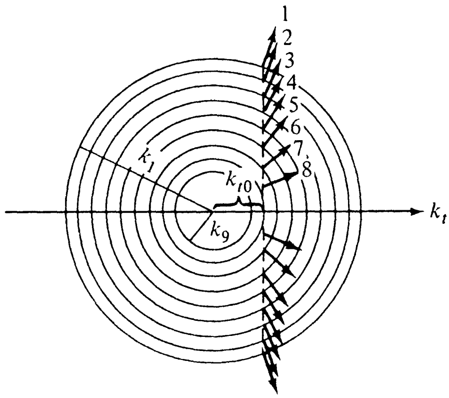
defines a family of curved trajectories along which plane-wave packets described by the parameter $\mathbf{k}_t$ propagate, and the saddle point $\mathbf{k}_{ts}$ selects the trajectory passing through a particular observation point $(\boldsymbol{\rho}, z)$.

Comparison of Eqs. (29) and (23) with Fig. 1.6.5 reveals that for a given value of $\mathbf{k}_t$, the local direction of the trajectory is along the normal to the (wavenumber) dispersion surface. Since the normal direction, determined in accord with the radiation condition, coincides with that of the energy flow, the curves of Eq. (29) characterize ray (energy-flow) trajectories. Their progress through the inhomogeneous medium can be charted by repeating the construction in Fig. 1.6.5(a), with $\mathbf{k}_t$ = constant, for the family of wavenumber plots. This procedure is illustrated in Fig. 1.6.7 for an isotropic medium wherein coincidence of ray and wavevector directions implies that each wavenumber plot is spherical. The medium is divided into thin locally homogeneous layers whose width is so chosen as to yield a good approximation to the given continuous profile [Fig. 1.6.7(a)]; the corresponding family of wavenumber diagrams is shown in Fig. 1.6.7(b). A reference value k_{t0} is assigned to the incident ray 1 in medium n_1, and the trajectory through the medium is plotted as in Fig. 1.6.7(c) by applying the condition $k_t = k_{t0}$ = constant to the ray construction in Fig. 1.6.7(b). One observes that the ray does not penetrate layer n_9 since $k_9 < k_{t0}$; this feature implies total reflection at the upper interface as shown in Fig. 1.6.7.(c) Additional details may be found in Sec. 5.8.

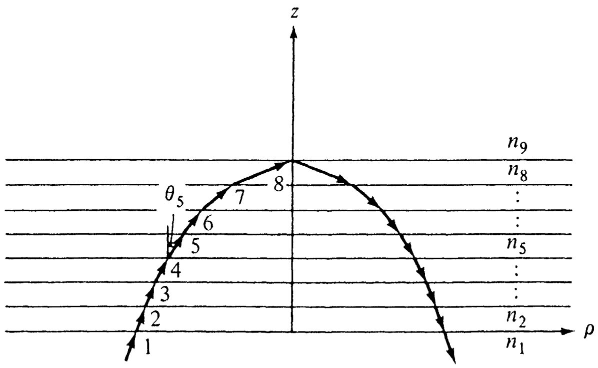
For excitation by a point source, all the rays pass through the source point $(\boldsymbol{\rho}', z') = (0, z')$; for each ray, the parameter k_t is specified in terms of the ray



(a) Refractive index profile



(b) Wavenumber plot
($k_i = k_0 n_i$, $k_0 = \frac{\omega}{c}$)



(c) Ray path in space
(constant k_{ts})

FIG. 1.6.7 Finely layered medium.

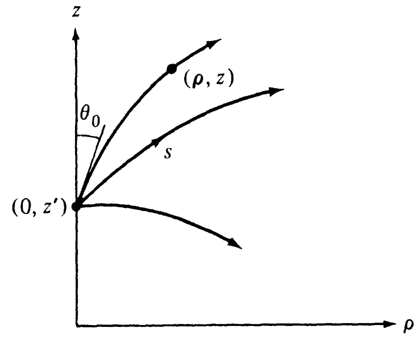


FIG. 1.6.8 Graphical determination of saddle point for an isotropic medium: $k_{ts} = k_0 n(z') \sin \theta_0$.

departure angle θ_0 as $k_t = k(z') \sin \theta_0 = k_0 n(z') \sin \theta_0$, where k_0 is a (constant) reference wavenumber and $n(z)$ is the refractive index. By selecting that ray which passes through the observation point $(\boldsymbol{\rho}, z)$ as in Fig. 1.6.8, one ascertains the saddle point $\mathbf{k}_{ts}$.

z-stratified media (transient case)

For excitation by a transient source, the representation integrals contain in addition to the $\mathbf{k}_t$ integration in Eq. (27) the ω integration in Eq. (20). Relevant saddle points $(\mathbf{k}_{ts}, \omega_s)$ are now determined from the conditions

$$\nabla_{k_t}\psi = \frac{\partial \psi}{\partial \omega} = 0, \qquad \psi = \mathbf{k}_t \cdot \boldsymbol{\rho} + \int^z \kappa(\mathbf{k}_t, \omega; \zeta)\, d\zeta - \omega t. \tag{30}$$

For propagation of local plane-wave fields, only those saddle points that yield real values of $\mathbf{k}_t$, κ, and ω are of interest. The requirement $\nabla_{k_t}\psi = 0$ has already been discussed in connection with the time-harmonic problem [Eqs. (28) and (29)]. $\partial\psi/\partial\omega = 0$ yields

$$t = \int^z \frac{\partial \kappa(\mathbf{k}_t, \omega; \zeta)}{\partial \omega}\, d\zeta \qquad \text{at } \omega_s = \omega_s(\mathbf{r}, t), \tag{31}$$

and, in differential form,

$$\frac{dz}{dt} = \frac{1}{\partial \kappa(\mathbf{k}_t, \omega; z)/\partial \omega} \qquad \text{at } \omega_s. \tag{32}$$

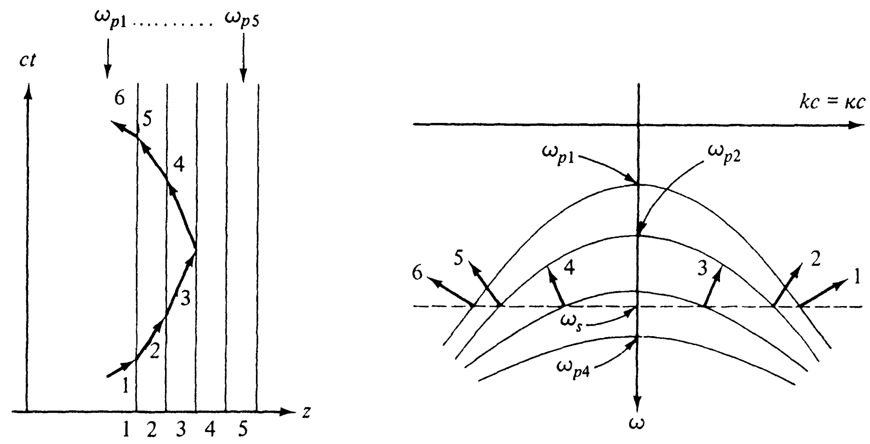
For an interpretation of the simultaneous saddle-point conditions in Eqs. (28) and (31), it is instructive to consider a homogeneous medium with κ independent of z so that the z integrations can be performed trivially. The problem is now the same as in Sec. 1.6a, but the solution here has been obtained by a guided-wave representation of the fields, whereas in Sec. 1.6a the representation involves oscillatory modes. In connection with the latter, it has been shown that the saddle-point condition in Eq. (2) can be satisfied geometrically by locating on the four-dimensional $(\mathbf{k}, \omega)$ dispersion surface those points $(\mathbf{k}_s, \omega_s)$ where the four-vector normal is parallel to the four-vector $(\mathbf{r}/t, 1)$, provided that the coordinate axes k_x, k_y, k_z, ω are chosen parallel to $x, y, z, -t$, respectively. Since the geometrical procedure is independent of the particular form of the dispersion equation [$\omega = \omega(\mathbf{k})$ in Sec. 1.6a and $\kappa = \kappa(\mathbf{k}_t, \omega)$ in the present case], it is to be expected that Eqs. (28) and (31) select the same points on the $(\mathbf{k}, \omega)$ dispersion surface as does Eq. (2). For proof, we observe that at the point $(\mathbf{k}, \omega)$, the slopes with respect to the z axis of projections of the normal vector $\Box f = (\nabla_{k_t} f, \partial f/\partial\kappa, \partial f/\partial\omega)$ onto the $(\boldsymbol{\rho}, z)$ and $(-t, z)$ hyperplanes are given by $\nabla_{k_t} f/(\partial f/\partial\kappa)$ and $-(\partial f/\partial\omega)/(\partial f/\partial\kappa)$, respectively. On the other hand, Eqs. (28) and (31), when specialized to a homogeneous medium, define a vector whose corresponding slopes are $-\nabla_{k_t}\kappa$ and $\partial\kappa/\partial\omega$, respectively. In view of the relations

$$\nabla_{k_t} f + \frac{\partial f}{\partial \kappa}\nabla_{k_t}\kappa = 0, \qquad \frac{\partial f}{\partial \omega} + \frac{\partial f}{\partial \kappa}\frac{\partial \kappa}{\partial \omega} = 0, \tag{33}$$

which follow from the dispersion equation $f(\mathbf{k}_t, \omega; \kappa(\mathbf{k}_t, \omega)) = 0$ on forming $df = 0$, the two vectors have the same direction. Since, at a given value of $(\mathbf{r}, t)$, the saddle points for either of the two guided-wave representations are deduced from the same points $(\mathbf{k}_s, \omega_s)$ on the dispersion surface, they describe the same wavepackets. To obtain the saddle points in the oscillatory formulation, one selects the projection $\mathbf{k}_s$, whereas for guided waves along z, the relevant projection is $(\mathbf{k}_{ts}, \omega_s)$.

From the preceding considerations and those in Sec. 1.6a, it is noted that in a homogeneous medium, a wavepacket moves along a straight-line ray path in space–time and preserves constancy of the total wavevector $\mathbf{k} = \mathbf{k}_s$ as well as the frequency $\omega = \omega_s$. When the medium is inhomogeneous, the wavepacket is described by the parameters $(\mathbf{k}_{ts}, \omega_s)$, with $\kappa(\mathbf{k}_{ts}, \omega_s; z)$ variable along z, so only the frequency, not the *total* wavevector $\mathbf{k}$, remains constant [see also Eq. (19)]. As noted from Eqs. (29) and (32), the space–time ray trajectories are now curved. When projected onto a hyperplane perpendicular to the time axis, the space–time rays generate the ray curves in Eq. (29), descriptive of the time-harmonic (i.e., ω = constant) field (Fig. 1.6.8). As noted previously, this relation permits the time-harmonic ray curves to be identified with paths of continuous energy transport in $\mathbf{r}$ space.

To track the space–time ray curves in an inhomogeneous medium, a graphical procedure analogous to that in Fig. 1.6.7 may be utilized. One constructs an appropriate sequence of dispersion surfaces for successive differential z elements along the path, and imposes the condition $(\mathbf{k}_{ts}, \omega_s)$ = constant. For illustration, we consider a wavepacket moving in the (z, ct) plane such that $\boldsymbol{\rho}$ and therefore $\mathbf{k}_{ts}$ equals zero. If the medium is an isotropic plasma, the dispersion equation for each local value of z_i [Fig. 1.6.9(a)] is given by Eq. (14) with ω_p replaced by $\omega_{pi} = \omega_p(z_i)$; the relevant portions of the corresponding dispersion curves are shown in Fig. 1.6.9(b). A wavepacket at a frequency ω_s is assumed to be incident from region 1 and is therefore characterized by ray segment 1. To effect the transition into region 2, whose properties differ slightly from those in region 1, one imposes the condition ω_s = constant to construct



(a) Ray paths in space-time (constant ω_s) (b) Dispersion surfaces

FIG. 1.6.9 Construction of space-time ray trajectories in an inhomogeneous medium.

ray segment 2, etc. (note that time always increases along a space-time ray path, in accord with causality). Since $\kappa(z)$ decreases with z in the present example, the space–time ray turns and the wavepacket is reflected; no energy at the selected frequency ω_s penetrates beyond layer 3.

With the saddle points determined implicitly from Eqs. (29) and (32), the behavior of the transient field can be ascertained from an asymptotic evaluation of the integrals in Eqs. (27) and (21) by techniques similar to those used for Eq. (1).

Transients in non-dispersive configurations (closed-form inversion of time-harmonic result)

The integration in Eq. (20) generally cannot be performed explicitly, so one must resort to asymptotic or other approximation procedures to evaluate the time-dependent field. An exception occurs for a class of non-dispersive problems in which the time-harmonic solution can be cast as a Laplace integral

$$I(\mathbf{r}, \omega) = \int_0^\infty e^{-s\tau} B(\mathbf{r}, \tau)\, d\tau, \qquad s = -i\omega, \tag{34}$$

where B is independent of s. If Eq. (34) applies, then it follows from the Fourier inversion of Eq. (20), with the causality requirement $I(\mathbf{r}, t) \equiv 0$ for $t < 0$, that

$$I(\mathbf{r}, \omega) = \frac{1}{2\pi} \int_0^\infty I(\mathbf{r}, t) e^{i\omega t}\, dt, \tag{35}$$

whence, from a comparison of Eqs. (34) and (35),

$$I(\mathbf{r}, t) = 2\pi B(\mathbf{r}, t). \tag{36}$$

An example included in the category of integrals (34) is the generic time-harmonic radiation integral of Eq. (5.3.14),

$$I(L, \omega) = \int_{\bar{P}} e^{ikL\cos(w-\alpha)} f(w)\, dw, \tag{37}$$

where $\bar{P}$ is the contour shown in Fig. 5.3.6b. The parameters L and α are assumed to be positive, with α restricted to the range $0 < \alpha < \pi/2$, and the function $f(w)$ is to be independent of $k = \omega/c$. Setting $\omega \to is$ in Eq. (37), with Re s sufficiently large, one may write

$$I(L, \omega) = \int_{i\infty}^{-i\infty} e^{-s(L/c)\cos w} f(w + \alpha)\, dw, \tag{38}$$

if it is assumed that the function $f(w)$ has no singularities in the strip $0 < |\mathrm{Re}\, w| < \pi/2$; if singularities are present, their effect may lead to additional contributions. Since $\exp[-s(L/c)\cos(w - \alpha)]$ decays in the strip $\cos(w_{\mathrm{real}} - \alpha) > 0$, the contour of integration can be shifted to achieve the representation (38) if Re s is large enough. The successive changes of variables $\beta = iw$ and $\tau = (L/c)\cosh\beta$ lead to the formulation

$$I(L, \omega) = -i\int_{L/c}^{\infty} e^{-s\tau} \frac{b(\tau)}{\sqrt{\tau^2 - (L/c)^2}}\, d\tau, \tag{39a}$$

where

$$b(\tau) = f\left[\alpha - i\cosh^{-1}\left(\frac{c\tau}{L}\right)\right] + f\left[\alpha + i\cosh^{-1}\left(\frac{c\tau}{L}\right)\right]. \tag{39b}$$

Equation (39a) is evidently in the form (34), with

$$B(L, \tau) = \begin{cases} 0, & \tau < \dfrac{L}{c}, \\[2ex] \dfrac{-ib(\tau)}{\sqrt{\tau^2 - (L^2/c^2)}}, & \tau > \dfrac{L}{c}. \end{cases} \tag{40}$$

If $v(w) = -if(w)$ is real for real values of w, then $v(w^*) = v^*(w)$(from the Schwartz reflection principle[17]) and $b(\tau)$ can be written as

$$-ib(\tau) = 2\,\mathrm{Re}\left\{-if\left[\alpha - i\cosh^{-1}\left(\frac{c\tau}{L}\right)\right]\right\}. \tag{41}$$

It may be noted that the formulation in Eq. (39a) is useful even when $f(w)$ is dependent on s. Although one cannot then perform the Laplace inversion in closed form, a series expansion of $f(w; s)$ in powers of $1/s$, or of s, permits the derivation of asymptotic results in the time domain, applicable immediately or long after the arrival of the first response, respectively.

1.6c Diffraction and Transition Phenomena

The characterization of the far-zone field in terms of distinct wavepackets as carried out in Secs. 1.6a and 1.6b applies at almost all space-time points $(\mathbf{r}, t)$ but fails in "transition regions" wherein individual wavepackets either are not yet fully developed or are strongly modified by interaction with other wavepackets. The former regime obtains near the time of arrival of the initial disturbance, or wavefront, traveling at the highest propagation speed c in the medium. Since the field vanishes before the first response arrives, the field variables and (or) their derivatives must behave discontinuously across the wavefront. Wavefront field variations are synthesized primarily by very high frequency waves, for which dispersive effects are negligible. In contrast, wavepackets emerge from well-ordered dispersive wavetrains only at sufficiently long observation distances behind the wavefront or, for a stationary observer, at sufficiently long observation times after arrival of the initial response. In the analytical treatment, the transition region between the wavefront and wavepacket regimes is characterized by $\mathbf{k}_s, \omega_s \to \infty$.

Another class of transition phenomena occurs when two or more wavepackets have the same local wavenumber, frequency, and group velocity, thereby providing strong interaction that destroys the independent existence of each. These transition regions in $(\mathbf{r}, t)$ space for transient problems, or in $\mathbf{r}$ space for time-harmonic problems, are characterized by a confluence of saddle points and (or) singularities (critical points) in the integral representations of the field. Isolated pole or branch-point singularities near saddle points have been ignored in Secs. 1.6a and 1.6b, but may also provide asymptotic field contributions that can be interpreted in terms of distinct wave processes. These singularities are relevant when intercepted during path deformations required for the asymptotic evaluation of integrals by the saddle-point method (see Chapter 4)

The presence of transition regions can usually be discerned by a divergence in the simple saddle-point calculation of the amplitude of the affected wave constituents. For example, transition regions in $\mathbf{r}$ space for which $\bar{R}_{1,2} \to \infty$ in Eq. (25) correspond to inflection points on the dispersion curve and signify the coalescence of two saddle points. This divergence does not imply unlimited growth of the field but rather the inadequacy of the simple asymptotic formula for a particular wave type. For description of the far field valid at *all* observation points, one employs more complicated uniform asymptotic approximations, given in Chapter 4 for the case of integrals containing adjacent critical points. The somewhat different characterization of transition effects near a wavefront is discussed at the end of this section.

The wave processes mentioned above are classified conveniently as primary and diffraction (secondary) effects, with the former representing dominant contributions and the latter distinguishing corrections thereto. The identification of a particular wave type with a critical point depends on the integral representation employed; in one representation, a primary field may arise from a saddle point, whereas in another, it may be attributed to a singularity. Primary and diffracted asymptotic wave types are characterized conveniently in terms of rays. A list of various ray species, their mechanism of excitation, and their use in constructing the solution of a time-harmonic diffraction problem are illustrated in Sec. 1.7c. Since no detailed discussion of transient propagation in dispersive media is given elsewhere in this book, some aspects concerning the initial formation and subsequent interaction of wavepackets will be considered at this time. The interaction problem is treated first since it relates more directly to the discussion in Secs. 1.6a and 1.6b.

Transient and signal propagation in a magnetoplasma (interaction between wavepackets)

We assume that the time-harmonic field has been approximated asymptotically in the far zone so that $I(\mathbf{r}, \omega)$ in Eq. (20) is known explicitly. In simplified form, the time-dependent field then requires evaluation of the integral $I(\mathbf{r}, t)$:

$$I(\mathbf{r}, t) = \int_{-\infty+i\gamma}^{\infty+i\gamma} f(\omega)e^{iq(\omega)}\,d\omega, \qquad q(\omega) = \xi(\omega)r - \omega t, \tag{42}$$

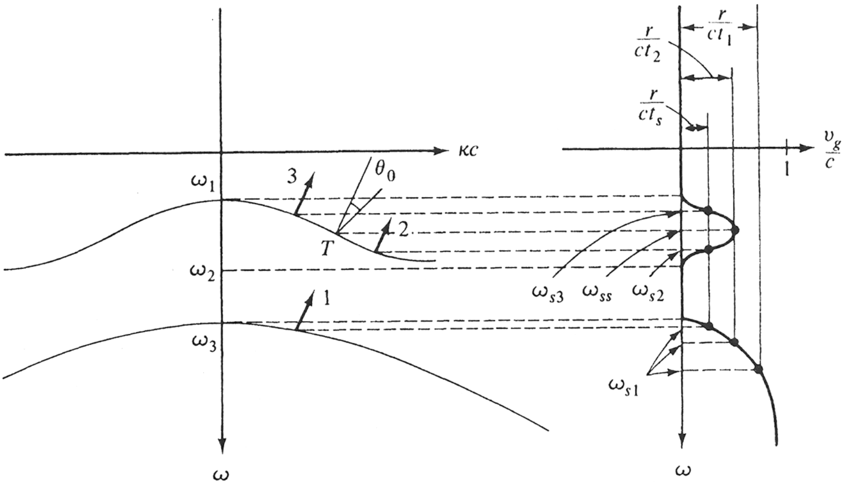
where f is an amplitude, r a distance coordinate, and ξ a modified wavenumber equal to k or $k\cos\alpha$ in an isotropic or anisotropic medium, respectively, with α denoting the angle between $\mathbf{k}$ and the ray vector. Although Eq. (42) implies a homogeneous medium, no essential complication arises when slow inhomogeneities, in the form of weak dependence of f and ξ on r, are present. The saddle-point contributions to Eq. (42) at $\omega = \omega_{si}$, treated analogously in Sec. 1.6b, furnish the result [see Eq. (4.2.1) et seq.]

$$I(\mathbf{r}, t) \sim \sum_i \sqrt{\frac{2\pi}{r|\xi''(\omega_{si})|}}\, f(\omega_{si}) \exp\left\{i\left[q(\omega_{si}) + \frac{\pi}{4}\operatorname{sgn}\xi''(\omega_{si})\right]\right\}, \tag{43}$$

where the saddle points $\omega_{si}(r, t)$ are determined implicitly by $q'(\omega_{si}) = 0 = (r/t) - 1/\xi'(\omega_{si})$, with the prime denoting the derivative with respect to ω.† As noted previously, each i-term in Eq. (43) represents a wavepacket with central frequency ω_{si}, wavenumber $\xi(\omega_{si})$, and group speed $[1/\xi'(\omega_{si})]$. The saddle points ω_{si} may be located graphically as shown in Fig. 1.6.10(a) for a multibranched dispersion surface (representative, for example, of extraordinary wave propagation across the magnetic field in a cold magnetoplasma; see Secs. 8.3a and 8.3b). This should be contrasted with the analogous construction in Fig. 1.6.3, where the saddle-point variable is the wavenumber k rather than the frequency ω. With r fixed, one observes from Fig. 1.6.10(b) that a single saddle point contributes for observation times $r/c < t < t_s$, while three saddle points contribute for $t > t_s$ [see also Fig. 1.6.10(c)]. The time t_s distinguishes the arrival of a wavepacket of frequency ω_{ss} traveling at the maximum group speed corresponding to the inflection point T of the upper branch of the plot in Fig. 1.6.10(a).

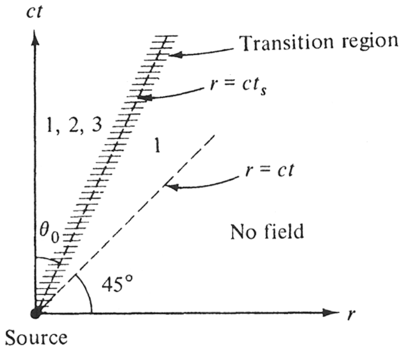
One observes from Fig. 1.6.10(a) or 1.6.10(b) that the wavepackets characterized by rays 2 and 3 interact strongly when $t \approx t_s$. In fact, distinct

†Since $I(\mathbf{r}, t)$ must be real, saddle points occur in pairs at $(\pm\omega_{si})$, the contribution from $(-\omega_{si})$ being the complex conjugate of that from $(+\omega_{si})$. Only the latter contributions are included in Eq. (43) [see the remarks following Eq. (5)]. Note also, that by expanding $\xi(\omega)$ about ω_s and inverting, one shows readily that $(d\xi(\omega)/d\omega)_{\omega_s} = (d\omega(\xi)/d\xi)_{\xi_s}^{-1}$, with $\omega_s = \omega(\xi_s)$ and $\xi_s = \xi(\omega_s)$, thereby providing alternative expressions for the group velocity.



(a) Dispersion curve

(b) Group velocity
$(r/ct_s) = \tan\theta_0$



(c) Space-time regions reached by various ray species

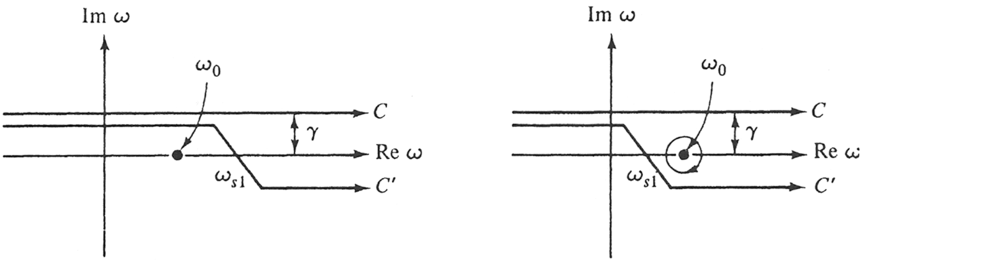
FIG. 1.6.10 Two-branch dispersion curve, group velocity, and space-time ray plots (inflection-point singularity).

wavepackets can be distinguished only at sufficiently long observation times after the arrival at t_s of the cluster of energy with central frequency ω_{ss}. The existence of a transition region near ω_{ss} is manifested by the vanishing of the curvature term $\xi''(\omega_{ss}) = 0$, thereby invalidating Eq. (43). The proper description of the field near ω_{ss} allows for the coalescence of the two adjacent saddle points $\omega_{s2,3}$ and leads to a representation in terms of Airy functions (see Sec. 4.2e), whence this portion of the transient behavior is sometimes called the Airy phase.[18] The enhancement of the transient field observed during the Airy phase is analogous to the enhancement of the time-harmonic field observed near a caustic (see Sec. 5.8d). Both phenomena describe a focusing of energy.

Another interesting transition phenomenon is associated with a pole singularity in the amplitude function $f(\omega)$ in Eq. (42), as occurs for a suddenly switched-on time-harmonic wave source $e^{-i\omega_0 t}U(t)$, where $U(t) = 0$ or 1 for $t < 0$ or $t > 0$, respectively. The Fourier transform of the source function (and hence $f(\omega)$) has a simple pole at $\omega = \omega_0$ which contributes to the integral in Eq. (42) the residue

$$I_r = -2\pi i[(\omega - \omega_0)f(\omega)]_{\omega_0} e^{iq(\omega_0)} U(t - t_0). \tag{44}$$

Equation (44) represents a time-harmonic signal which arrives at the observation point r at a time t_0 specified implicitly by $\omega_s(r, t_0) = \omega_0$. Such a signal appears when, as in Fig. 1.6.11(a) and (b), the original integration path C, on being deformed into the path C' through the saddle point, crosses the pole singularity at ω_0 [the contour C' is drawn for the condition $q''(\omega_s) < 0$ satisfied on the ω_{s1} branch in Fig. 1.6.10(a); see Sec. 4.4a for relevant details on saddle point



(a) Pole not intercepted ($t < t_0$) (b) Pole intercepted ($t > t_0$)

FIG. 1.6.11 Original path C in the presence of a pole and deformed path C' through the saddle point.

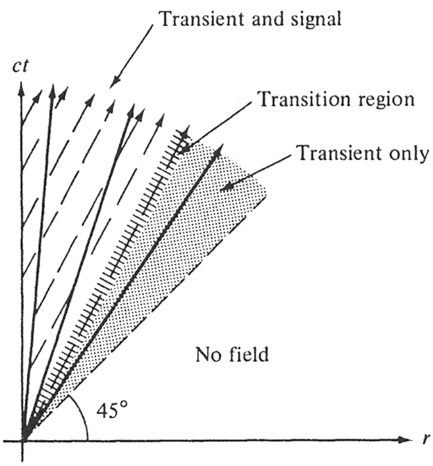


FIG. 1.6.12 Space-time ray plot for transient ($\rightarrow$) and time-harmonic ($--\rightarrow$) signal (pole singularity).

integration]. The time t_0 is required for the harmonic signal to traverse at its group speed $v_{g0} = 1/\xi'(\omega_0)$ the distance from the source to the observation point at r; the wave contribution (44) is maintained at subsequent observation times by the continual arrival of wavepackets at the group speed v_{g0}. The space–time ray diagram in Fig. 1.6.12 depicts both the time-harmonic signal in Eq. (44) and the transient terms in Eq. (43). Near the arrival time t_0, Eq. (43) fails since $f(\omega_0) \to \infty$; in this transition region characterized by adjacent pole and saddle point, the field is represented in terms of a Fresnel integral (Sec. 4.4a). The main signal in Eq. (44) may be regarded as the primary contribution, whereas the transients in Eq. (43) constitute a diffraction effect.[19]

Field behavior near a wavefront prior to formation of a wavepacket

The procedures of Secs. 1.6a and 1.6b fail for saddle-point values $\mathbf{k}_s, \omega_s \to \infty$. In view of the limiting form $\omega = kc$ of the dispersion relation as $k, \omega \to \infty$, the surface curvature term proportional to $d^2\omega/dk^2$ or $d^2k/d\omega^2$ in the denominators of Eqs. (5) or (43), respectively, tends to zero in this limit. As noted previously, the medium does not possess dispersive properties at frequencies characterizing the highest propagation speed c of the initial disturbance or wavefront, with a consequent inability for wavepacket formation. Since the fields on and near the wavefront are established by the high frequency wave components, it is to be expected that the high-frequency time-harmonic wave solution characterizes the initial transient response, and conversely. Let us assume that the field $I(\mathbf{r}, t)$ near the wavefront $t = r/c$ behaves as

$$I(\mathbf{r}, t) \sim a\left(t - \frac{r}{c}\right)^{\beta}, \qquad t \geq \frac{r}{c}, \tag{45}$$

where a is time independent and $\beta > -1$. Substitution of this approximate expression into Eq. (35), with the lower limit replaced by r/c since $I \equiv 0$ for $t < r/c$, may be justified for $\omega \to \infty$ [with Im $\omega > 0$; see Eq. (20)] since $\exp(i\omega t)$ then decays rapidly away from $t = r/c$, thereby localizing the effective integration range. Use of the gamma function

$$\Gamma(x) = \int_0^\infty v^{x-1} e^{-v}\, dv, \qquad \operatorname{Re} x > 0, \tag{46}$$

in Eq. (35) then yields

$$I(\mathbf{r}, \omega) \sim e^{ikr} \frac{a\Gamma(\beta + 1)}{2\pi(-i\omega)^{\beta+1}}, \qquad \omega \to \infty, \tag{47}$$

where $k \approx \omega/c$ and ω is now allowed to be real. Thus, the field behavior near $t = (r/c)$ is related to the time-harmonic behavior for $\omega \to \infty$ as in Eqs. (47) and (45). If $I(\mathbf{r}, \omega) \sim (2\pi)^{-1} a' \exp(ikr)$, where a' is independent of ω, the corresponding time-dependent function is $I(\mathbf{r}, t) \sim a'\delta[t - (r/c)]$. By including higher-order expansion coefficients in Eq. (45), one may generate higher-order terms in the asymptotic expansion (47), and conversely.[20] This aspect is considered further in Sec. 1.7e.

Even when higher-order terms in inverse powers of ω are included in Eq. (47), the resulting time-dependent field $I(\mathbf{r}, t)$ does not provide a transition to the propagation regime characterizable in terms of wavepackets since the (non-dispersive) dispersion relation $k \sim \omega/c$ is too restrictive. An improved formulation is obtained by retaining the next term in the high-frequency approximation,

$$k(\omega) \sim \frac{\omega}{c} - \frac{\omega_\alpha^2}{c\omega}, \qquad \omega \to \infty, \tag{48}$$

where ω_α is generally a characteristic frequency of the medium. Equation (48) accounts for the onset of dispersion, removes the infinity introduced by the vanishing of $d^2\omega/dk^2$ in Eq. (5) or $d^2k/d\omega^2$ in Eq. (43)(with $\xi \equiv k$), and therefore forms the basis for a uniform approximation which connects the wavefront with the wavepacket regimes. Insertion of Eq. (48) into Eq. (42), with use of the asymptotic behavior of the amplitude function $f \sim B(-i\omega)^\nu$, $\omega \to \infty$, where $B = \text{const.}$, yields via a known integral expression for the Bessel function,[21]

$$I(\mathbf{r}, t) \sim 2\pi B\left(\frac{\tau}{b}\right)^{-(1+\nu)/2} J_{-1-\nu}(2\sqrt{b\tau}), \qquad b = \frac{\omega_\alpha^2 r}{c}, \qquad \tau = t - \frac{r}{c} \tag{49}$$

valid for a small range of observation times at or near r/c; b is a large parameter if $\mathbf{r}$ is large. For very short observation times such that $b\tau \to 0$, the small argument approximation for the Bessel function reduces Eq. (49) to

$$I \sim 2\pi B \frac{\tau^{-(1+\nu)}}{\Gamma(-\nu)}, \tag{50}$$

which agrees with the result in Eq. (45) since in the present instance, $I(r, \omega) \sim B(-i\omega)^\nu \exp(ikr)$. For somewhat longer observation times such that $2\sqrt{b\tau} \gg 1$ (since b is large, the inequalities $b\tau \gg 1$ and $\tau \ll 1$ can be satisfied simultaneously), one finds on use of the large-argument approximation for the Bessel function [see. Eq. (4.2.22b)],

$$I \sim 2B\pi^{1/2} \frac{b^{(\nu+\frac{1}{2})/2}}{\tau^{(\nu+3/2)/2}} \cos\left(2\sqrt{b\tau} + \frac{\nu+1}{2}\pi - \frac{\pi}{4}\right). \tag{51}$$

With $\xi(\omega) \equiv k(\omega)$ given by Eq. (48), one may verify that the resulting expression for I in Eq. (43), added to its complex conjugate, agrees with the expression in Eq. (51). Thus, Eq. (49) provides the desired transition from the wavefront to the wavepacket behavior of the field. For a uniform asymptotic treatment valid for a larger range of τ, see Reference 22.

1.7 RAY-OPTIC APPROXIMATIONS FOR DIFFERENTIAL EQUATIONS

In Sec. 1.6, rigorous integral representations of the time-dependent and time-harmonic field in separable regions were approximated by asymptotic (saddle-point) techniques valid at distant observation points. These asymptotic approximations frequently assumed an invariant ray-optical form that suggests their applicability to classes of problems broader than the separable ones to which they refer. The validity of ray-optical approximations in non-separable inhomogeneous configurations is established in this section by an asymptotic procedure based directly on the differential field equations.

Modal (integral representation), as opposed to direct asymptotic, techniques are based on exact field solutions (although for a limited class of problems), so approximations employed in their evaluation can be validated systematically. Thus, as discussed in Sec. 1.6c, no new representation, but only a modified asymptotic technique, is required for calculating the field in transition regions. While the saddle-point procedure highlights wave interference and thereby wave-packet processes, the introduction of rays and trajectories is not essential to the asymptotic evaluation but does serve to clarify physical propagation mechanisms. In contrast, if one assumes the form of the asymptotic field at the outset as in the direct procedure, one can use the exact differential equations for the field to derive simplified equations for phase and amplitude functions in the assumed field representation. These simpler equations define ray trajectories and energy-conservation theorems descriptive of transport properties that play a direct role in the analysis. For this reason, the direct method is henceforth called the "ray method." Since it describes energy-transport phenomena, the ray method does not furnish initial values of a field constituent; in the vicinity of source or scattering regions, such values must be determined by other (e.g., modal) methods. Like the simple saddle-point method, the ray procedure fails in transition regions. A major modification is required to remedy this failure, the latter being attributable to inadequacy of the initially assumed asymptotic field form. This defect may be removed by recourse to boundary-layer techniques,[23] but these will not be discussed herein. As mentioned, the ray method is not limited to separable configurations and therefore has broad scope. Confidence in its validity is confirmed by comparisons with asymptotic fields derived from exact modal solutions of separable problems, and an effective methodology can be constructed by selective use of both procedures.

The basic features of the ray method have already been described; details of its application depend on the form of the time-dependent or time-harmonic equations defining the field. As may be surmised, the formulation is simplest for scalar fields, with additional complexity arising from polarization, anisotropy, etc., for vector fields. The time-harmonic case is analyzed more easily than the time-dependent case since the presence of dispersion in the latter complicates the structure of the field equations. Although the differences are most pronounced in the calculation of field amplitudes and polarizations, information concerning ray trajectories and phase progression is deduced from first-order partial differential equations having a common structure for all cases. These equations, their formal solution by the method of characteristics, and their trajectory interpretation are discussed in Sec. 1.7a. The presentation then proceeds to time-harmonic propagation of scalar fields in Sec. 1.7b, and to the treatment of time-harmonic vector fields in isotropic and anisotropic media in Sec. 1.7c. Ray constructs developed in Secs. 1.7b and 1.7c for propagation in unbounded media are extended in Sec. 1.7d by introduction of reflected, refracted, and diffracted rays to account for the presence of boundaries and scattering centers. The geometrical theory of diffraction, a ray theory for synthesizing high-frequency fields in the presence of composite objects in terms of simpler constituent ray fields, is also presented in Sec. 1.7d and applied to an illustrative example. In the time-dependent regime, the ray method leads directly from the differential field equations to wavepackets and their trajectories which have been deduced in Secs. 1.6a and 1.6b by asymptotic evaluation of modal integrals. Transient propagation in an isotropic plasma serves to illustrate the ray method for a simple case (Sec. 1.7e). The ray method also permits further elaboration of the relation between high-frequency time-harmonic fields and transient fields near an impinging wavefront.

1.7a Rays and the Theory of Characteristics

As noted in Sec. 1.6, ray methods are based on asymptotic field representations in terms of assumed local plane waves. In the abstract notation of Sec. 1.1d, one assumes to the lowest order of approximation (see Sec. 1.7e for a complete asymptotic expansion) that for large $\mathbf{r}$, t, a general linear space- and time-dependent field can be represented as (ν here should not be confused with the same symbol used on p. 123)

$$\Psi(\mathbf{r}, t) \sim \Psi_0(\mathbf{r}, t)e^{i\nu\psi(\mathbf{r},t)}, \tag{1}$$

where ν denotes the large parameter in the asymptotic development. Although some of the considerations below apply also to lossy media, we shall deal only with the lossless case unless specified otherwise. The wavevector $\Psi(\mathbf{r}, t)$ satisfies the first-order source-free field equations

$$L\left(\nabla, \frac{\partial}{\partial t}; \mathbf{r}, t\right)\Psi(\mathbf{r}, t) = 0, \tag{2}$$

wherein the explicit dependence of the operator L on $\mathbf{r}$ and t signifies the formal applicability of Eq. (2) to media with weak spatial and temporal inhomogeneities. On substitution of Eq. (1) into Eq. (2), one derives on retention of only the dominant term in ν the following first-order system of partial differential equations for the phase function ψ and the amplitude function Ψ_0:

$$L\left(i\nu\nabla\psi, i\nu\frac{\partial\psi}{\partial t}; \mathbf{r}, t\right)\Psi_0(\mathbf{r}, t) = 0. \tag{3}$$

Retention of $\mathbf{r}$ and t in Eq. (3) and below implies that these quantities are of $O(\nu)$; moreover, one may regard certain "natural frequencies" in the medium to be $O(\nu)$ (these aspects, unimportant for the general discussion, are clarified in Sec. 1.7e).

Since in a homogeneous medium, $\nu\psi = \mathbf{k}\cdot\mathbf{r} - \omega t$, where $\mathbf{k}$ and ω are the (constant) wavevector and wave frequency, respectively, the derivatives $\nabla\psi$ and $-\partial\psi/\partial t$ assume the roles of *local* wavenumber $\bar{\mathbf{k}}$ and frequency $\bar{\omega}$, respectively, normalized with respect to ν. From Eq. (3), for non-vanishing Ψ_0, the phase ψ must evidently satisfy the determinantal equation

$$\det L\left(i\nu\nabla\psi, i\nu\frac{\partial\psi}{\partial t}; \mathbf{r}, t\right) = 0, \tag{4a}$$

which, in terms of $\bar{\mathbf{k}}$ and $\bar{\omega}$, becomes the local (space- and time-dependent) *dispersion equation* [see Eq. (1.2.44) and the notational statement following Eq. (1.2.41)]

$$\det L(\bar{\mathbf{k}}, \bar{\omega}; \mathbf{r}, t) = 0, \qquad \bar{\mathbf{k}} = \nabla\psi, \qquad \bar{\omega} = -\frac{\partial\psi}{\partial t}, \tag{4b}$$

or, explicitly,

$$\bar{\omega} = \bar{\omega}(\bar{\mathbf{k}}, \mathbf{r}, t), \tag{4c}$$

where the same symbol has been used to denote $\bar{\omega}$ and its functional dependence on $\bar{\mathbf{k}}, \mathbf{r}, t$. The eigenvectors $\Psi_{0\alpha}(\mathbf{r}, t)$ of Eq. (3), corresponding to each solution $\bar{\omega}_\alpha(\bar{\mathbf{k}}, \mathbf{r}, t)$ of the dispersion equation (4b), then yield the polarization of each of the eigenwave constituents of the lowest-order wavevector Ψ_0. The amplitude $\Psi_0(\mathbf{r}, t) = \sum_\alpha A_\alpha(\mathbf{r}, t)\Psi_{0\alpha}(\mathbf{r}, t)$ is synthesized by superposition of the eigenwave solutions; its further determination requires use of a transport equation, the next in a series of equations obtained when $\Psi(\mathbf{r}, t)$ is expanded beyond the first term in Eq. (1) in a series of terms involving decreasing powers of ν.[24]

For time-harmonic wave propagation in a time-invariant but spatially inhomogeneous medium, the time dependence $\exp(-i\omega t)$ may be suppressed and the lowest-order approximation of the wavevector assumes the form (see Secs. 1.7b and 1.7c for a complete asymptotic expansion)

$$\Psi(\mathbf{r}) \sim \Psi_0(\mathbf{r})e^{ik_0\psi(\mathbf{r})}, \tag{5}$$

where the reference wavenumber $k_0 = \omega/c$ takes on the role of the large parameter. The time-harmonic field equations follow from Eq. (2) on removal of the time variable t and the replacement $(\partial/\partial t) \to -i\omega$. To a lowest order of approximation, the correspondingly modified Eq. (3) becomes

$$L(\nabla, \mathbf{r}; k_0)\Psi(\mathbf{r}) = 0, \qquad -iL(\nabla, \mathbf{r}; k_0) = M(\nabla) - k_0 cW(\mathbf{r}), \tag{6a}$$

where use has been made of the decomposition of the operator L stated in Eq. (1.3.8). Substitution of Ψ from Eq. (5) yields, to a first order of approximation,

$$\bar{L}(\nabla\psi, \mathbf{r})\Psi_0(\mathbf{r}) = 0, \qquad \bar{L}(\nabla\psi, \mathbf{r}) = M(\nabla\psi) + icW(\mathbf{r}). \tag{6b}$$

On introduction of the wavevector $\bar{\mathbf{k}} = \nabla\psi$, normalized with respect to k_0, the determinental equation for ψ becomes

$$\det \bar{L}(\bar{\mathbf{k}}, \mathbf{r}) = 0, \qquad \bar{\mathbf{k}} = \nabla\psi; \tag{6c}$$

this equation is commonly referred to as the *eiconal equation* of geometrical optics. For the special case of an isotropic medium, one has the explicit form [see Eqs. (18a) and (38b)]

$$\bar{\mathbf{k}} = \mathbf{s}\, n(\mathbf{r}), \qquad \mathbf{s} = \frac{\bar{\mathbf{k}}}{\bar{k}}, \tag{6d}$$

where n is the refractive index and s is a unit vector in the direction of $\nabla\psi$ perpendicular to the wavefront $\psi =$ constant. As in the time-dependent case, calculation from Eq. (6b) of the eigenvectors $\Psi_{0\alpha}(\mathbf{r})$ for each solution of the eiconal equation provides the polarization and amplitudes of the eigenwaves required for the synthesis of the wave vector $\Psi_0(\mathbf{r})$ in Eq. (5). Further properties of $\Psi_0(\mathbf{r})$ are inferred from a transport equation derived on use of an expansion of $\Psi(\mathbf{r})$ in decreasing powers of k_0.

Both the dispersion and eiconal equations (4b) and (6c) are first-order partial differential equations. They are reducible by the method of characteristics to first-order ordinary differential equations that can be integrated formally along special trajectories, the rays defined previously in Sec. 1.6. Consider the generic form

$$G\left(\frac{\partial\varphi}{\partial x_1}\cdots\frac{\partial\varphi}{\partial x_n}, x_1\cdots x_n\right) = 0, \tag{7}$$

defining the function $\varphi(x_1\cdots x_n)$, where $x_i, i = 1, \ldots, n$, represents either a space or time coordinate. By implicit differentiation, one observes on denoting $(\partial\varphi_i/\partial x)$ by ξ_i that

$$dG = 0 = \sum_{i=1}^{n}\left[\frac{\partial G}{\partial \xi_i}d\xi_i + \frac{\partial G}{\partial x_i}dx_i\right], \tag{7a}$$

which equation can be satisfied if

$$\frac{d\xi_i}{\partial G/\partial x_i} = -\frac{dx_i}{\partial G/\partial \xi_i}, \qquad i = 1, \ldots, n. \tag{7b}$$

By introducing a parameter s and defining the trajectory $x_i = x_i(s)$, $\xi_i = \xi_i(s)$ in ξ, x phase space, one may write Eq. (7b) in a form similar to Hamilton's canonical equations in mechanics,

$$\frac{dx_i}{ds} = \frac{\partial G}{\partial \xi_i}, \quad \frac{d\xi_i}{ds} = -\frac{\partial G}{\partial x_i}, \qquad i = 1, \ldots, n. \tag{8}$$

On these trajectories, the derivative of φ is given by

$$\frac{d\varphi}{ds} = \sum_{i=1}^{n}\frac{\partial\varphi}{\partial x_i}\frac{dx_i}{ds}, \tag{9}$$

whence φ can be determined by integration over s. (For a rigorous derivation, the reader should consult standard references on the theory of partial differential equations.[25,26])

To apply these results to the time-dependent dispersion relation, we let $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = t$, $\varphi \equiv \psi$, $G \equiv \bar{\omega} - \bar{\omega}(\bar{\mathbf{k}}, \mathbf{r}, t)$. From Eq. (4b) and one of the relations in Eq. (8), $dt/ds = -1$, so the time variable can be used as the parameter descriptive of the ray trajectories $\mathbf{r} = \mathbf{r}(t)$. Then, from Eq. (8),

$$\frac{d\mathbf{r}}{dt} = \nabla_k\bar{\omega}(\bar{\mathbf{k}}, \mathbf{r}, t), \quad \frac{d\bar{\mathbf{k}}}{dt} = -\nabla\bar{\omega}(\bar{\mathbf{k}}, \mathbf{r}, t), \tag{10a}$$

whence

$$\frac{d\bar{\omega}(\bar{\mathbf{k}}, \mathbf{r}, t)}{dt} = \frac{\partial\bar{\omega}(\bar{\mathbf{k}}, \mathbf{r}, t)}{\partial t}. \tag{10b}$$

Equation (10b) is not independent but follows from Eqs. (10a) and the derivative formula $(d\bar{\omega}/dt) = (\nabla\bar{\omega}\cdot d\mathbf{r}/dt) + (\nabla_k\bar{\omega}\cdot d\bar{\mathbf{k}}/dt) + (\partial\bar{\omega}/\partial t)$. As in Sec. 1.6, $\nabla_k\bar{\omega}$ represents the group velocity $\mathbf{v}_g$ of energy transport in the lossless medium. By Eq. (9), on integrating from the space–time point $(\mathbf{r}_1, t_1)$ to $(\mathbf{r}, t)$ along a ray,

$$\psi(\mathbf{r}, t) - \psi(\mathbf{r}_1, t_1) = \int_{(\mathbf{r}_1,t_1)}^{(\mathbf{r},t)} [\bar{\mathbf{k}}\cdot d\mathbf{r} - \bar{\omega}dt]. \tag{11}$$

For the time-harmonic problem in Eq. (6c), Eqs. (8) yield with $\mathbf{x} \equiv \mathbf{r}$ and $\varphi \equiv \psi$,

$$\frac{d\mathbf{r}}{ds} = \nabla_k G, \quad \frac{d\bar{\mathbf{k}}}{ds} = -\nabla G, \qquad G \equiv \det \bar{L}, \tag{12a}$$

or in an isotropic medium with $G = \bar{\mathbf{k}} - n(\mathbf{r})$ as in Eq. (6d),

$$\frac{d\mathbf{r}}{ds} = \mathbf{s}, \qquad \frac{d}{ds}(n\mathbf{s}) = \nabla n. \tag{12b}$$

By integrating along the ray trajectory $\mathbf{r}(s)$ defined by Eq. (12a) between the points $\mathbf{r}_1$ and $\mathbf{r}$, one solves for the phase function ψ via (9),

$$\psi(\mathbf{r}) - \psi(\mathbf{r}_1) = \int_{\mathbf{r}_1}^{\mathbf{r}} \bar{\mathbf{k}}\cdot\nabla_k G ds = \int_{\mathbf{r}_1}^{\mathbf{r}} \bar{\mathbf{k}}\cdot d\mathbf{r}. \tag{13}$$

The ray equations (10a) and (12), and consequently also Eqs. (11) and (13), simplify for special configurations. If the medium parameters do not vary in space, then $\bar{\omega} = \bar{\omega}(\bar{\mathbf{k}}, t)$, whence $\bar{\mathbf{k}}$ = constant on a space–time ray defined in Eq. (10a). If the medium parameters do not vary with time, then $\bar{\omega} = \bar{\omega}(\bar{\mathbf{k}}, \mathbf{r})$, whence $\bar{\omega}$ = constant on a space–time ray. If as a special case, the medium is merely plane stratified along z so that $\bar{\omega} = \bar{\omega}(\bar{\mathbf{k}}, z)$ then $\bar{\mathbf{k}}_t$, the component of $\bar{\mathbf{k}}$ transverse to z, is also constant; the resulting ray equation (10) is equivalent to Eqs. (1.6.29) and (1.6.32). Finally, if the medium parameters do not vary with either space or time so that $\bar{\omega} = \bar{\omega}(\bar{\mathbf{k}})$, then both $\bar{\mathbf{k}}$ and $\bar{\omega}$ are constant along a ray, as noted in Eq. (1.6.6). These constraints may be used for the graphical construction of ray trajectories in the manner depicted in Figs. 1.6.1, 1.6.7, and 1.6.9.

In the following sections and also in Chapters 5–8, detailed attention is given to the asymptotic evaluation of time-harmonic fields and to their ray-optical interpretation. Time-dependent fields in dispersive media are treated briefly (see Sec. 1.7e), but the Problems at the end of this chapter contain additional results.

1.7b Scalar Time-Harmonic Fields

Let us consider scalar fields exterior to source regions, for example, the acoustic field treated in Sec. 1.3b. Since the vectorial aspect of the (longitudinal) acoustic field is trivial, as $\mathbf{v}$ is derivable directly from p by differentiation [see

Eq. (1.3.19)], one needs to consider only the scalar pressure p and thereby avoids the use of the $p, \mathbf{v}$ wavevector formalism of Eq. (6a). In an inhomogeneous medium where the background density n_0, the static pressure p_0, and the temperature ($\propto p_0/n_0$) are functions of $\mathbf{r}$, one may verify from Eqs. (1.3.19) that the time-harmonic pressure p satisfies the wave equation

$$\left[\nabla^2 + k_0^2 n^2(\mathbf{r}) - \left(\sqrt{n_0(\mathbf{r})}\nabla^2 \frac{1}{\sqrt{n_0(\mathbf{r})}}\right)\right]\frac{p(\mathbf{r})}{\sqrt{n_0(\mathbf{r})}} = 0, \qquad k_0 = \frac{\omega}{a_0}, \tag{14}$$

where $n(\mathbf{r}) = a_0/a(\mathbf{r})$ is the refractive index (not to be confused with the background density n_0), $a(\mathbf{r}) = [\gamma p_0(\mathbf{r})/m n_0(\mathbf{r})]^{1/2}$ is the local wave-propagation speed, and a_0 is a reference speed in a medium with $n = 1$. If the background density varies sufficiently slowly, the last term in the brackets in Eq. (14) can be neglected, and one obtains the approximate wave equation

$$[\nabla^2 + k_0^2 n^2(\mathbf{r})]u(\mathbf{r}) = 0, \tag{15}$$

where $u = p/n_0^{1/2}$.

It is desired to construct a high-frequency asymptotic solution of the form (the $\sim$ signifies "asymptotically equal to")

$$u(\mathbf{r}) \sim e^{ik_0\psi(\mathbf{r})} \sum_{m=0}^{\infty} \frac{u_m(\mathbf{r})}{(ik_0)^m}. \tag{16}$$

In this development, $u_m(\mathbf{r})$ and $\psi(\mathbf{r})$ are assumed to be independent of the wavenumber k_0, which is chosen as the large parameter rather than the observation distance as in Sec. 1.6b. By substituting Eq. (16) into Eq. (15) and assuming that the differentiation can be performed on each term in the sum, one finds with $\bar{\mathbf{k}} \equiv \nabla\psi = n\mathbf{s}$,

$$\{(ik_0)^2[\bar{k}^2 - n^2] + (ik_0)[\nabla\cdot\bar{\mathbf{k}} + 2\bar{\mathbf{k}}\cdot\nabla] + \nabla^2\}\sum_{m=0}^{\infty}\frac{u_m(\mathbf{r})}{(ik_0)^m} \sim 0. \tag{17}$$

Since this expansion is to hold for arbitrary (though large) values of k_0, the coefficient of each power of k_0 must vanish independently. From the k_0^2 term,

$$\bar{k}^2 = n^2, \tag{18a}$$

from the k_0^1 term,

$$(\nabla\cdot\bar{\mathbf{k}} + 2\bar{\mathbf{k}}\cdot\nabla)u_0 = 0; \tag{18b}$$

and from the general term $k_0^{-\nu}$, $\nu = 0, 1, 2, \ldots,$ †

$$(\nabla\cdot\bar{\mathbf{k}} + 2\bar{\mathbf{k}}\cdot\nabla)u_m = -\nabla^2 u_{m-1}, \qquad m \geq 1. \tag{18c}$$

It is recalled that the wavevector $\bar{\mathbf{k}}$ is normalized with respect to k_0. In deriving Eqs. (18b) and (18c), the *transport equations* for the amplitude coefficients in the asymptotic expansion (16), use has been made of Eq. (18a), the *eiconal equation* of geometrical optics.[27] In view of the recursive character of the sys-

†In this formal development, Eq. (15) is regarded as exact. If Eq. (15) is only an approximate form derived, for example, from Eq. (14), the higher-order terms in the expansion must be reexamined.

tem of equations described by Eq. (18c), all the coefficients u_m, $m > 1$, can be derived in principle from the lowest-order coefficient u_0.

If the medium is non-dissipative, so that n^2 is positive real, the procedure for solving Eqs. (18) involves the following steps:

1. Determination of the ray trajectories (i.e., the curves parallel to the local time-averaged energy flux density vector, $\bar{\mathbf{S}}$).
2. Evaluation of the phase function ψ by integrating along a ray.
3. Calculation of the lowest-order amplitude u_0 by invoking conservation of energy in a tube of rays.

Steps 1 and 2 have already been formulated in Eqs. (12b) and (13) and are recapitulated below. Concerning step 3 we consider first the time-averaged energy flux density in a scalar harmonic field u, given typically by

$$\bar{\mathbf{S}} = \zeta \, \mathrm{Im}\,(u^* \nabla u), \tag{19}$$

where ζ is a real constant. The lowest-order solution of Eq. (16) in the high-frequency limit is the local plane-wave field

$$u \sim u_0 e^{ik_0\psi}. \tag{20}$$

This "geometric-optical" field† dominates the remaining terms in Eq. (16) if the relative variation in the refractive index, $|\nabla n|/n$, is small compared with the local wavelength $k_0 n$, i.e., if‡

$$\frac{|\nabla n|}{k_0 n^2} \ll 1. \tag{21}$$

Evidently, Eq. (20) is of the same form as Eq. (5). Wherever u_0 is slowly varying [an exception occurs in focal regions; see Eq. (36)], $\nabla u \sim ik_0 u \nabla\psi$, so that

$$\bar{\mathbf{S}} \sim \zeta k_0 |u_0|^2 \nabla\psi = \zeta k_0 |u_0|^2 \bar{\mathbf{k}}. \tag{22}$$

For the acoustic field, $u = p/n_0^{1/2}$, so that in view of Eq. (1.3.19),

$$\bar{\mathbf{S}} = \mathrm{Re}\,(p\mathrm{v}^*) = \frac{1}{a_0 m} |u_0|^2 \bar{\mathbf{k}}, \tag{22a}$$

which is evidently of the form given in Eq. (22) with $\zeta = (1/\omega m)$. This result for $\bar{\mathbf{S}}$ is used below for integration of the transport equation (18b).

It should be noted that the preceding (also subsequent) considerations are not affected when the series (16) is multiplied by $(ik_0)^{-\beta}$, $0 < \beta < 1$, so fields with fractional power dependence on k_0 may also be accommodated.

Ray trajectories

From Eq. (12b) [see also Eq. (22)], rays are tangent to the "ray vector" $n\mathrm{s}$, where s is the unit vector defined in Eqs. (6d). The resulting ray equation (12b), repeated for convenience, is

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†Although the present discussion does not deal with the propagation of light, it is suggested that the term "geometric-optical" be retained since the field so described obeys laws analogous to those in light optics.

‡More precisely, the condition $u_0(\mathbf{r}) \gg k_0^{-1} u_1(\mathbf{r})$ is satisfied if $[k_0 u_0(\mathbf{r}) n(\mathbf{r})]^{-1} \nabla^2 u_0(\mathbf{r}) \ll 1$ along the ray trajectory, with $u_0(\mathbf{r})$ given by Eq. (34) (see Problem section).

$$\frac{d}{ds} n \frac{d\mathbf{r}}{ds} = \nabla n, \qquad \text{with } \left(\frac{d\mathbf{r}}{ds}\right)^2 = 1. \tag{23}$$

In a homogeneous medium, for which $n =$ constant, Eq. (23) has the solution $\mathbf{r} = \mathbf{A}s + \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are constant vectors. The rays in this case are straight lines.

In an inhomogeneous medium where n varies continuously, the rays are smoothly curved. Information about a ray trajectory is obtained by examining the vector derivative $d\mathbf{s}/ds$, which defines a vector perpendicular to the ray curve; its magnitude is the local curvature of the ray,

$$\frac{d\mathbf{s}}{ds} = \frac{\boldsymbol{\tau}}{D}, \tag{24}$$

where $\boldsymbol{\tau}$ is a unit vector and D is the local radius of curvature along the ray. From Eq. (23) and since $\boldsymbol{\tau}\cdot\mathbf{s} = 0$, $\boldsymbol{\tau}\cdot\nabla n = \boldsymbol{\tau}\cdot d(n\mathbf{s})/ds = n\boldsymbol{\tau}\cdot d\mathbf{s}/ds = n/D > 0$. Thus, the ray bends toward the direction of increasing refractive index.

For a plane-stratified medium where $n(\mathbf{r}) = n(z)$, the x- and y-component forms of the ray equation (23) yield

$$\frac{d}{ds} n \frac{dx}{ds} = 0 = \frac{d}{ds} n \frac{dy}{ds}, \tag{25}$$

whence $dy/dx =$ constant along a ray [i.e., the ray is confined to a plane perpendicular to the x, y plane]. It follows from Eq. (25) that $n(z)(d\boldsymbol{\rho}/ds) =$ constant along the ray, where $\boldsymbol{\rho} = \mathbf{x}_0 x + \mathbf{y}_0 y$. On defining $\theta(z) = \sin^{-1}(d\rho/ds)$ as the angle between the ray and the z axis, one then finds that along a ray,†

$$n(z) \sin\theta(z) = a = \text{constant}, \tag{26}$$

which relation expresses Snell's law of refraction and assures that on a ray, each incremental change in wavevector direction is along the direction of maximum rate of change of the refractive index (i.e., along z). The same condition for determining ray paths has been given in Eq. (1.6.29), wherein this ray trajectory interpretation of the saddle-point condition has been used for the graphical construction in Figs. 1.6.7 and 1.6.8. An explicit equation for the ray trajectories follows from $d\rho/dz = \tan\theta$, which, in view of Eq. (26), leads to [see also Eqs. (1.6.28)]

$$\rho(z) = a \int_{z_1}^{z} \frac{d\zeta}{\sqrt{n^2(\zeta) - a^2}}. \tag{27}$$

where $\rho(z_1) = 0$ denotes an arbitrary reference point.

For arbitrary $n(\mathbf{r})$, the second of Eqs. (12b) implies that along the ray trajectory, increments in the wavevector occur in the variable direction of maximum rate of change of the local refractive index. This requirement may be used to adapt the graphical procedure in Fig. 1.6.7 to the case of arbitrary medium variation. The medium along the trajectory is again divided into locally homogeneous segments, with the interfaces between segments now chosen perpendicular to the local refractive index gradient.

†The ray parameter a should not be confused with the acoustic speeds a_0 or $a(\mathbf{r})$.

Phase functions

With ray trajectories computed from Eq. (23), one may find the phase function ψ via the integration shown in Eq. (13), or, in view of Eq. (6d),

$$\psi(\mathbf{r}) - \psi(\mathbf{r}_1) = \int_{\mathbf{r}_1}^{\mathbf{r}} n\,ds. \tag{28}$$

The phase integral in Eq. (28) defines the optical path length along the ray. Alternatively, Eq. (28) can be interpreted as the time required for the phase fronts to cover the distance between points $\mathbf{r}_1$ and $\mathbf{r}$ along the ray; on noting that $ds/dt = a_0/n$ is the local propagation speed, one may write

$$\psi(\mathbf{r}) - \psi(\mathbf{r}_1) = a_0 \int_{\mathbf{r}_1}^{\mathbf{r}} dt, \tag{29}$$

with a_0 representing the propagation speed in a medium with $n = 1$. In a regular region where only one ray passes through a given point, it may be shown that for points $\mathbf{r}_1$ and $\mathbf{r}$ along a ray, the optical length in Eq. (28) or the propagation time in Eq. (29) is less along the ray than along any other neighboring curve connecting the two points. This result is known variously as Fermat's principle, the principle of shortest optical path, or the principle of least time.[27]

In a homogeneous medium where $n =$ constant and the rays are straight lines, one has $\psi(\mathbf{r}) - \psi(\mathbf{r}_1) = n|\mathbf{r} - \mathbf{r}_1|$. Thus, the phase accumulation over a distance L along a ray in a homogeneous medium is given by the factor exp (ik_0L) characteristic of a *plane wave.* For a plane-stratified inhomogeneous medium as in Fig. 1.6.7, use of Eq. (27) and the condition $n \sin\theta = a$ yields

$$\begin{aligned}\psi(\mathbf{r}) - \psi(\mathbf{r}_1) &= \int_{\mathbf{r}_1}^{\mathbf{r}} n(\sin\theta\, d\rho + \cos\theta\, dz) \\ &= a(\rho - \rho_1) + \int_{z_1}^{z} \sqrt{n^2(z) - a^2}\,dz,\end{aligned} \tag{30}$$

an expression for the phase agreeing with that obtained by the saddle-point method [Eq. (1.6.24), with modifications noted after Eq. (1.6.28)]. The above ray solutions in inhomogeneous media are intimately related to the WKB approximation, as shown in Sec. 5.8.

Amplitude variation

To determine the amplitude dependence $u_0(\mathbf{r})$ of the field in Eq. (20), it is convenient to write Eq. (18b) in a somewhat different form. First, the equation is multiplied by the complex-conjugate function $u_0^*(\mathbf{r})$ to yield

$$|u_0|^2\nabla\cdot\bar{\mathbf{k}} + 2u_0^*\nabla u_0\cdot\bar{\mathbf{k}} = 0. \tag{31a}$$

Recalling that $\bar{\mathbf{k}}$ is real, and adding the complex conjugate of Eq. (31a), one then finds

$$|u_0|^2\nabla\cdot\bar{\mathbf{k}} + \nabla|u_0|^2\cdot\bar{\mathbf{k}} = 0 = \nabla\cdot(|u_0|^2\bar{\mathbf{k}}). \tag{31b}$$

Since this relation yields $\nabla\cdot\bar{\mathbf{S}} = 0$ [Eq. (22) with $\zeta k_0 =$ constant; for the acoustic-wave-propagation problem, see Eq. (22a)], one obtains by the divergence theorem of vector analysis the conservation-of-energy statement equivalent to the original Eq. (18b):

$$\oint_A \bar{\mathbf{S}} \cdot \mathbf{v} \, dA = 0, \tag{32}$$

where A is a closed surface and $\mathbf{v}$ a unit vector normal to A. If A is chosen as the surface area of a narrow ray tube as shown in Fig. 1.7.1 with $\bar{\mathbf{S}} \cdot \mathbf{v} = 0$ along the side walls of the tube, and $\bar{\mathbf{S}} \cdot \mathbf{v} = \pm \bar{S}_{1,2}$ over the infinitesimal end sections $dA_{1,2}$ then, from Eq. (32), $\bar{S}_1 dA_1 = \bar{S}_2 dA_2$, or

$$\bar{S}_2 = \bar{S}_1 \frac{dA_1}{dA_2}. \tag{33}$$

Thus, the magnitude of the flux density vector (i.e., the intensity) at $\mathbf{r}_2$ along a ray is related to that at $\mathbf{r}_1$ by the ratio of the infinitesimal areas cut out of the wavefronts by a narrow ray bundle, and the total power flow within such a ray tube remains constant. This energy-conservation statement generalizes those in Secs. 1.6a and 1.6b to media with arbitrary (though slow) refractive index variations. From Eqs. (6d), (22), and (23), the field magnitude $|u_0(\mathbf{r})|$, proportional to $[\bar{S}(\mathbf{r})/n(\mathbf{r})]^{1/2}$, at an arbitrary point $\mathbf{r}$ along a ray is related to $|u_0(\mathbf{r}_1)|$ at a reference point $\mathbf{r}_1$ on the same ray as follows:

$$|u_0(\mathbf{r})| = |u_0(\mathbf{r}_1)| \sqrt{\frac{n(\mathbf{r}_1)}{n(\mathbf{r})} \frac{dA(\mathbf{r}_1)}{dA(\mathbf{r})}}. \tag{34}$$

For example, for a spherically spreading disturbance in a homogeneous medium, the rays diverge radially from the origin $r = 0$ and the cross section of a conical ray tube is proportional to r^2. Thus, $dA(\mathbf{r}_1)/dA(\mathbf{r}) = (r_1/r)^2$, whence $|u_0(r)| = |u_0(r_1)|(r_1/r)$.

By an alternative calculation, Eq. (31b) may be written as

$$\frac{1}{|u_0|^2} \frac{d}{ds} |u_0|^2 = \frac{d}{ds} \ln |u_0|^2 = -\frac{1}{n} \nabla \cdot \bar{\mathbf{k}} = -\frac{1}{n} \nabla \cdot (n\mathbf{s}), \tag{35a}$$

so that by integration between the points $\mathbf{r}_1$ and $\mathbf{r}$ along a ray,

$$|u_0(\mathbf{r})| = |u_0(\mathbf{r}_1)| \exp \left[-\frac{1}{2} \int_{\mathbf{r}_1}^{\mathbf{r}} \frac{1}{n} \nabla \cdot (n\mathbf{s}) \, ds \right]. \tag{35b}$$

For the preceding example, $\mathbf{s}$ is equal to the radial unit vector $\mathbf{r}_0$, whence $\nabla \cdot \mathbf{s} = 2/r$ and the previous result follows.

It is evident from Eq. (34) that the formula fails when $dA(\mathbf{r}) \to 0$ (i.e., when the rays forming the ray tube converge on a line or point). This condition defines the location of caustics or foci, near which a more detailed analysis of the ray-optical field must be carried out (see Sec. 5.8d for the calculation required in a plane-stratified medium). Thus, the condition

$$dA(\mathbf{r}) \not\approx 0, \tag{36}$$

at each point $\mathbf{r}$ along the ray, must be added to the one in Eq. (21) to assure the validity of the ray-optical formulas.

By collecting results from Eqs. (28) and (34), one may write for the leading term in the asymptotic expansion of a particular constituent of the high-frequency field [Eq. (20)]:

$$u(\mathbf{r}) \sim u(\mathbf{r}_1)\left[\frac{n(\mathbf{r}_1)}{n(\mathbf{r})}\frac{dA(\mathbf{r}_1)}{dA(\mathbf{r})}\right]^{1/2} e^{ik_0[\psi(\mathbf{r})-\psi(\mathbf{r}_1)]}. \tag{37}$$

In this formula, $\mathbf{r}$ denotes the observation point along a given ray, $\mathbf{r}_1$ is a reference point on the same ray, $dA(\mathbf{r})$ and $dA(\mathbf{r}_1)$ are the ray-tube cross sections at $\mathbf{r}$ and $\mathbf{r}_1$, respectively, $n(\mathbf{r})$ and $n(\mathbf{r}_1)$ are the corresponding refractive indexes, while $\psi(\mathbf{r})$ and $\psi(\mathbf{r}_1)$ are the phase functions. The field at $\mathbf{r}$ is thus specified in terms of a known field at $\mathbf{r}_1$. For applications of these formulas to plane-stratified media, see Sec. 5.8.

1.7c Vector Time-Harmonic Fields

Isotropic media

In an isotropic medium with variable relative permittivity $\epsilon(\mathbf{r}) = n^2(\mathbf{r})$, where n is the refractive index, the steady-state electromagnetic field exterior to source regions is defined by the Maxwell field equations which take the form shown in (6a) with [see Eqs. (1.3.26) and (1.3.28)]

$$iM(\nabla) \to \begin{pmatrix} 0 & -\nabla \times \mathbf{1} \\ \nabla \times \mathbf{1} & 0 \end{pmatrix}, \quad W(\mathbf{r}) \to \begin{pmatrix} \epsilon_0 n^2(\mathbf{r})\mathbf{1} & 0 \\ 0 & \mu_0 \mathbf{1} \end{pmatrix},$$

$$\Psi(\mathbf{r}) \to \begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix}, \tag{38}$$

where ϵ_0 and μ_0 are the permittivity and permeability of vacuum and $c = (\mu_0\epsilon_0)^{-1/2}$. A time factor $\exp(-i\omega t)$ is suppressed. Use of the lowest-order approximation for the wavevector [Eq. (5)] then leads to Eq. (6b), written explicitly as

$$\bar{\mathbf{k}} \times \mathbf{H}_0 = -\frac{n^2}{\zeta}\mathbf{E}_0, \quad \bar{\mathbf{k}} \times \mathbf{E}_0 = \zeta\mathbf{H}_0, \quad \bar{\mathbf{k}} \equiv \nabla\psi, \quad \zeta = \sqrt{\frac{\mu_0}{\epsilon_0}}, \tag{38a}$$

whence the eiconal equation (6c) becomes

$$\bar{\mathbf{k}}^2 = n^2(\mathbf{r}), \qquad \text{or} \qquad \bar{\mathbf{k}} = \mathbf{s}\, n(\mathbf{r}), \tag{38b}$$

as noted in Eq. (6d). The lowest-order eigensolutions $\mathbf{E}(\mathbf{r}) \sim \mathbf{E}_0(\mathbf{r}) \exp[ik_0\psi(\mathbf{r})]$ and $\mathbf{H}(\mathbf{r}) \sim \mathbf{H}_0(\mathbf{r}) \exp[ik_0\psi(\mathbf{r})]$ constitute the geometric-optical field whose polarization properties are evident from Eq. (38a),

$$\mathbf{H}_0\cdot\bar{\mathbf{k}} \sim 0 \sim \mathbf{E}_0\cdot\bar{\mathbf{k}}, \qquad \mathbf{E}_0\cdot\mathbf{H}_0 \sim 0 \tag{38c}$$

i.e., the electric and magnetic field vectors are mutually perpendicular, lie in the equiphase surface $\psi =$ constant, and describe a local plane-wave field in an isotropic medium. After determination of the ray trajectories from Eq. (12b), the phase function ψ can be calculated from Eq. (13). The variation of the amplitudes $E_0(\mathbf{r}) \equiv |\mathbf{E}_0(\mathbf{r})|$ and $H_0(\mathbf{r})$ of the geometric-optical field may be determined from the energy-conservation law

$$\nabla \cdot \bar{\mathbf{S}} = 0, \qquad \bar{\mathbf{S}} = \mathrm{Re}\,(\mathbf{E} \times \mathbf{H}^*) \sim \frac{1}{\zeta} E_0^2 \bar{\mathbf{k}} = \bar{W} v \mathbf{s}, \tag{39}$$

where $\bar{\mathbf{S}}$ is the real time-averaged Poynting vector (power density), $\bar{W} = n^2 \epsilon_0 E_0^2 = (\zeta/c) H_0^2$ is the total average stored energy density, and $v = c/n$ is the local wave propagation speed (rms values for the field quantities are implied). Equation (39) has the same form as Eq. (31b); when applied to a ray tube as shown in Fig. 1.7.1, it yields the amplitude variation specified in Eq. (34). On combining the results for the amplitude and phase variation, one obtains

$$E(\mathbf{r}) \sim E(\mathbf{r}_1) \left[\frac{n(\mathbf{r}_1)\, dA(\mathbf{r}_1)}{n(\mathbf{r})\, dA(\mathbf{r})}\right]^{1/2} \exp\left(ik_0 \int_{\mathbf{r}_1}^{\mathbf{r}} n\, ds\right), \tag{40}$$

where $\mathbf{r}_1$ and $\mathbf{r}$ are two points along a ray defined in Eq. (12b). The direction of the field is determined from a transport equation for the polarization vector [Eq. (49)]. From the so-determined vector electric field at $\mathbf{r}$, one may calculate the vector magnetic field directly from the second equation in (38a).

As for the acoustic field in Sec. 1.7b, one may construct an asymptotic expansion for the high-frequency electromagnetic field in which the geometric-optical field of Eq. (40) et seq. constitutes the leading term.[27] Because of the relative simplicity of the time-harmonic electromagnetic field in an isotropic medium, instead of using the first-order form of the Maxwell field equations in Sec. 1.7a, it is more convenient to perform the calculation from the second-order form. This procedure is instructive even for the lowest-order contribution, since it yields the results in Eqs. (38b) and (39) somewhat more directly and simplifies the formulation of the transport equation for the polarization vectors. We deal explicitly with the electric field $\mathbf{E}$ which satisfies the vector wave equation

$$\nabla^2 \mathbf{E} + k_0^2 n^2 \mathbf{E} + 2\nabla(\mathbf{E} \cdot \nabla \ln n) = 0, \tag{41}$$

with the magnetic field determined from $\mathbf{H} = (ik_0\zeta)^{-1} \nabla \times \mathbf{E}$. As in Eq. (16), we assume that for large k_0,

$$\mathbf{E}(\mathbf{r}) \sim e^{ik_0\psi(\mathbf{r})} \sum_{m=0}^{\infty} \frac{\mathbf{E}_m(\mathbf{r})}{(ik_0)^m}, \tag{42}$$

where the amplitude coefficients $\mathbf{E}_m$ and the phase function ψ (and hence the refractive index n) are regarded as independent of k_0. As in Sec. 1.7d, subsequent considerations will still apply when Eq. (42) is multiplied by a factor containing a fractional power of k_0. Assuming the differentiability of the asymptotic representation, introducing the wavevector $\bar{\mathbf{k}} \equiv \nabla\psi$ normalized with respect to k_0, and substituting Eq. (42) into Eq. (41), one finds

$$e^{ik_0\psi} \sum_{m=0}^{\infty} \frac{1}{(ik_0)^m} \Big\{k_0^2 \mathbf{E}_m [n^2 - \bar{k}^2] + ik_0\Big[\mathbf{E}_m \nabla \cdot \bar{\mathbf{k}} + 2\bar{\mathbf{k}}\mathbf{E}_m \cdot \nabla \ln n + 2\bar{\mathbf{k}} \cdot \nabla \mathbf{E}_m\Big]$$
$$+ \nabla^2 \mathbf{E}_m + 2\nabla(\mathbf{E}_m \cdot \nabla \ln n)\Big\} = 0. \tag{43}$$

On equating to zero the coefficients of each power of k_0, one obtains from the coefficient of k_0^2 the eiconal equation (38b), while from the coefficient of k_0 and Eq. (38b),

$$\mathbf{E}_0 \nabla \cdot \bar{\mathbf{k}} + 2(\mathbf{E}_0 \cdot \nabla \ln n)\bar{\mathbf{k}} + 2(\bar{\mathbf{k}} \cdot \nabla)\mathbf{E}_0 = 0. \tag{44a}$$

As in Eq. (18c), the amplitude functions $\mathbf{E}_m$, $m \geq 1$, can be determined from $\mathbf{E}_{m-1}$ by the recursive system obtained from the coefficient of $k_0^{-\nu}$, $\nu \geq 0$, in Eq. (43):

$$[\nabla \cdot \bar{\mathbf{k}}\mathbf{1} + 2\bar{\mathbf{k}}(\nabla \ln n) + 2\bar{\mathbf{k}} \cdot \nabla \mathbf{1}] \cdot \mathbf{E}_m = -\nabla^2 \mathbf{E}_{m-1} - 2\nabla[(\nabla \ln n) \cdot \mathbf{E}_{m-1}], \tag{44b}$$

where $\mathbf{1}$ is the unit dyadic. Equations (44a) and (44b) constitute the transport equations for the electric-field amplitudes. As in Sec. 1.7a, we shall not consider the higher-order terms $m \geq 1$. The $m = 0$ contribution, the geometric-optical field, is specified by Eqs. (38a) and (38c). It represents a valid approximation when $E_1 \ll k_0 E_0$, which condition requires that the medium is slowly varying in the sense defined in Eq. (21).[28] To the order of accuracy of the geometric-optical approximation, the refractive index n may depend weakly on k_0, thereby including dispersive media within the scope of the lowest-order theory.†

In contrast to the abstract procedure, based on the first-order form of the field equations and leading to Eq. (6b), the present approach yields directly separate equations for the phase and amplitude variations in the geometric-optical field. To convert Eq. (44a) into the form (39), it is convenient to introduce the polarization vector $\boldsymbol{\beta}$ parallel to the electric-field direction,

$$\boldsymbol{\beta} = \frac{\mathbf{E}_0}{E_0} = \frac{\mathbf{E}_0}{(\mathbf{E}_0 \cdot \mathbf{E}_0^*)^{1/2}}. \tag{45}$$

On dot-product multiplication of Eq. (44a) by $\mathbf{E}_0^*$ and use of Eqs. (38b) and (38c), one finds

$$\frac{1}{2}(\mathbf{E}_0^* \cdot \mathbf{E}_0)\nabla \cdot \bar{\mathbf{k}} + n\mathbf{E}_0^* \cdot \frac{d}{ds}\mathbf{E}_0 = 0, \tag{46}$$

and, on adding to this equation its complex conjugate,

$$E_0^2 \nabla \cdot \bar{\mathbf{k}} + n\frac{d}{ds}E_0^2 = 0 = \nabla \cdot (E_0^2 \bar{\mathbf{k}}), \tag{47}$$

which expresses $\nabla \cdot \bar{\mathbf{S}} = 0$ as in Eq. (39).

In a similar manner, one may convert Eq. (44a) into a transport equation for the polarization vector $\boldsymbol{\beta}$ along a ray. Since

$$\frac{d\boldsymbol{\beta}}{ds} = \frac{d}{ds}\frac{\mathbf{E}_0}{E_0} = \frac{1}{E_0}\frac{d}{ds}\mathbf{E}_0 - \frac{\boldsymbol{\beta}}{E_0}\frac{d}{ds}E_0, \tag{48}$$

one finds, on use of Eqs. (47) and (44a), recalling that $\bar{\mathbf{k}} \cdot \nabla = n(d/ds)$,

†For a cold isotropic plasma, $n^2(\mathbf{r}) = 1 - [\omega_p^2(\mathbf{r})/\omega^2]$, where ω_p is the plasma frequency. Weak dependence on k_0 is taken to imply that ω_p/ω remains finite.

$$\frac{d\boldsymbol{\beta}}{ds} = -\mathrm{s}(\boldsymbol{\beta}\cdot\nabla \ln n). \tag{49}$$

Evidently, $\boldsymbol{\beta}$ is constant in homogeneous regions where $\nabla n = 0$ and also for fields polarized with the electric vector perpendicular to the direction of stratification in the medium. Inclusion of the polarization vector $\boldsymbol{\beta}(\mathbf{r})$ in Eq. (40) completes specification of the geometric-optical field in a medium with slow spatial variation.

Anisotropic media

In an inhomogeneous anisotropic dielectric (e.g., a crystal or a cold magnetoplasma) where the local wave-propagation properties depend on direction as well as location, the time-harmonic electromagnetic field satisfies the Maxwell field equations (6a), with

$$iM(\nabla) \to \begin{pmatrix} 0 & -\nabla \times \mathbf{1} \\ \nabla \times \mathbf{1} & 0 \end{pmatrix}, \quad W(\mathbf{r}) \to \begin{pmatrix} \epsilon_0 \boldsymbol{\epsilon}(\mathbf{r}) & 0 \\ 0 & \mu_0 \mathbf{1} \end{pmatrix}, \quad \Psi(\mathbf{r}) \to \begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix}, \tag{50}$$

where ϵ_0 and μ_0 are the permittivity and permeability of vacuum and $\boldsymbol{\epsilon}(\mathbf{r})$ is the permittivity dyadic (normalized to ϵ_0) which may depend weakly on $k_0 = \omega/c$, $c = (\mu_0\epsilon_0)^{-1/2}$. A time factor $\exp(-i\omega t)$ has been suppressed. The lowest-order field approximation in Eq. (5) satisfies Eq. (6b) written explicitly as

$$\bar{\mathbf{k}} \times \mathbf{H}_0 = -\frac{1}{\zeta}\boldsymbol{\epsilon}\cdot\mathbf{E}_0, \quad \bar{\mathbf{k}} \times \mathbf{E}_0 = \zeta \mathbf{H}_0, \quad \bar{\mathbf{k}} \equiv \nabla\psi, \quad \zeta = \sqrt{\frac{\mu_0}{\epsilon_0}}, \tag{51}$$

from which it follows that

$$\bar{\mathbf{k}}\cdot\mathbf{H}_0 = \bar{\mathbf{k}}\cdot\boldsymbol{\epsilon}\cdot\mathbf{E}_0 = \mathbf{H}_0\cdot\mathbf{E}_0 = \mathbf{H}_0\cdot\boldsymbol{\epsilon}\cdot\mathbf{E}_0 = 0. \tag{52}$$

Thus, $\mathbf{H}$ is perpendicular to $\mathbf{E}$ and $\bar{\mathbf{k}}$, but $\mathbf{E}$ generally is not perpendicular to $\bar{\mathbf{k}}$, so the real time-averaged power flow density vector $\bar{\mathbf{S}} = \mathrm{Re}(\mathbf{E} \times \mathbf{H}^*)$ is generally not parallel to $\bar{\mathbf{k}}$. In the absence of spatial dispersion (i.e., when $\boldsymbol{\epsilon}$ is independent of $\mathbf{k}$), the vector $\bar{\mathbf{S}}$ is parallel to the group-velocity (ray) vector $\mathbf{v}_g$ [see Eq. (1.5.24a)]. Since

$$\bar{\mathbf{S}} \sim \frac{1}{\zeta}\mathrm{Re}\,[\mathbf{E}_0 \times (\mathbf{k} \times \mathbf{E}_0^*)] \tag{53}$$

and therefore

$$\mathbf{k}\cdot\bar{\mathbf{S}} = \frac{1}{\zeta}|\mathbf{k} \times \mathbf{E}_0|^2 \geq 0, \tag{53a}$$

the angle between the ray and wavevector does not exceed 90°. This observation is important for locating saddle points by the graphical procedures of Secs. 1.6a and 1.6b. From Eqs. (6c) and (50), one finds the eiconal equation

$$\det[\bar{k}^2\mathbf{1} - \bar{\mathbf{k}}\bar{\mathbf{k}} - \boldsymbol{\epsilon}(\mathbf{r})] = 0, \tag{54}$$

which permits one to calculate the ray trajectories via Eq. (12a) and thence the phase function ψ via Eq. (13).

Graphical methods for finding the ray direction in a homogeneous or plane-stratified medium have been discussed in connection with Figs. 1.6.5 and 1.6.7, and other observations concerning rays and wavefronts in an anisotropic environment have been made in connection with Figs. 1.6.2 and 1.6.6. Conservation of energy in a tube of rays (Fig. 1.7.1) [inferred from the transport equation mentioned in connection with Eqs. (6)] can be used as in the isotropic case to

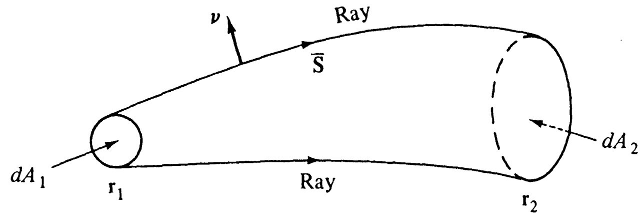


FIG. 1.7.1 Ray tube.

determine the amplitude variation $|\mathbf{E}_0(\mathbf{r})|$, but polarization effects must be found from Eq. (6b), $\bar{L}\Psi_0 = 0$. In the isotropic case $\epsilon(\mathbf{r}) = \mathbf{1}n^2(\mathbf{r})$, and Eq. (54) reduces to Eq. (6d), since in view of Eq. (52), $\mathbf{k}\cdot\mathbf{E}_0 \to 0$ in that limit.

The explicit construction of the field in an inhomogeneous anisotropic medium involves substantial difficulty even for the lowest-order (geometric-optical) approximation (for plane-stratified media, see References 15 and 28); further considerations are restricted to the homogeneous case $\epsilon =$ constant. In this instance, the rays are straight lines with

$$\psi(\mathbf{r}) = \mathbf{k}\cdot\mathbf{r} = Nr = nr\cos\alpha, \tag{55}$$

where $\mathbf{r}$ is the vector coordinate along the ray, $n = |\bar{\mathbf{k}}|$ is the ordinary refractive index descriptive of the speed of the advancing phase front along $\bar{\mathbf{k}}$ (it is recalled that $\bar{\mathbf{k}}$ is normalized to k_0), and $N = n\cos\alpha$ is the "ray refractive index" pertaining to phase propagation along the *ray*. Since $\mathbf{k}$ and the ray are not codirectional but are displaced in general by an angle α (see Fig. 1.6.6), one has instead of the conventional $N = n$ in isotropic media the relation given in Eq. (55). It is to be emphasized that n and N, and hence α, are functions of the polar angles (θ, ϕ) which select a particular ray orientation. In terms of the ray velocity v_r defined in connection with Fig. 1.6.2, one has $v_r = c/N$, whereas for the phase velocity along $\mathbf{k}$, $v_p = c/n$. The phase function in Eq. (55) corresponds exactly to that given in Eq. (1.6.25).

When the field is generated by a point dipole (Sec. 7.3a), the rays diverge radially from the source point. As in the discussion relating to Fig. 1.7.1, energy is conserved in a ray tube, a fact that is exploited for the construction of the amplitude functions $|\mathbf{E}_0|$ and $|\mathbf{H}_0|$ in Eq. (51). One notes from Eq. (53a) that the ray intensity varies like $|\mathbf{E}_0|^2$, so the variation of $|\mathbf{E}_0|$ may be deduced from that of the intensity $\bar{S}$. From conservation of energy in a ray tube,

$$\bar{S}(\mathbf{r})\, dA(\mathbf{r}) = \text{constant}, \tag{55a}$$

where $dA(\mathbf{r})$ is the ray-tube cross section at $\mathbf{r}$, whence for a radially diverging ray system,

$$\bar{S}(\mathbf{r}) = \frac{\text{constant}}{r^2}. \tag{55b}$$

The same observation has been made in Sec. 1.6b, and it has been sharpened by relating the energy density to properties of the wavenumber surface (see Fig. 1.6.6): $\bar{S}(\mathbf{r}) \propto \bar{R}_1\bar{R}_2/r^2$, where $\bar{R}_1$ and $\bar{R}_2$ are the principal radii of curvature of the wavenumber surface at the point corresponding to the ray.

From these considerations, one constructs the following ray-optical approximation of the electric field generated by a point source in a homogeneous, unbounded, anisotropic dielectric:

$$\mathbf{E}(\mathbf{r}) = \sum_i \mathbf{E}_i(\mathbf{r}), \quad \mathbf{E}_i(\mathbf{r}) \sim \frac{e^{i\mathbf{k}_i\cdot\mathbf{r}}\sqrt{\bar{R}_{1i}\bar{R}_{2i}}}{r}\,\mathbf{A}_i, \qquad \mathbf{k}_i\cdot\mathbf{r} = k_0 r N_i, \tag{56}$$

where $\mathbf{r}$ is the vector distance from the source to the observation point, $\mathbf{k}_i$ locates a point on the wavenumber surface which has a normal along the ray direction $\mathbf{r}$ (with $\mathbf{k}_i\cdot\mathbf{r} \geq 0$; see Fig. 1.6.6), and the sum over i includes all $\mathbf{k}_i$ that satisfy this condition. $\bar{R}_{1i}$ and $\bar{R}_{2i}$ are the principal radii of curvature of the wavenumber surface at $\mathbf{k}_i$. The vector $\mathbf{A}_i$ contains the excitation amplitude and polarization of the field along the ray; the determination of the former requires the solution of the source problem (see Secs. 7.3, 8.3c, and 1.3d), whereas the polarization is found from the eigenvalue problem discussed in Sec. 1.3a (see also Reference 24). Multiple ray contributions may arise when the wavenumber surface has inflection points as in Fig. 1.6.6, or has several branches (see Fig. 1.6.10 for analogous phenomena in connection with space–time rays descriptive of transient fields). Since each ray has its own ray refractive index N_i, interference phenomena may be observed when two or more rays have comparable amplitudes. The ray-optical approximation fails when $\bar{R}_{1i}$ and (or) $\bar{R}_{2i} \to \infty$ (i.e., when a ray corresponds to a point of vanishing Gaussian curvature on the wavenumber surface). Calculations for this transition region are presented in Sec. 8.3c.

The form of Eq. (56) agrees with that in Eq. (1.6.25) derived by the modal procedure.

1.7d The Geometrical Theory of Diffraction

The simple ray solutions developed in Secs. 1.7b and 1.7c can be utilized for the study of more complicated high-frequency scattering problems via the geometrical theory of diffraction. The theory, developed initially by Keller for homogeneous isotropic media,[29] has subsequently been extended also to inhomogeneous media,[30] to anisotropic media,[31] and to media supporting several wave species (e.g., a warm plasma[32]). According to the theory, the scattered field at a given observation point consists of contributions from geometrically reflected or refracted, and from diffracted, rays whose individual properties depend on the local nature of the scattering object and the surrounding medium. The incident field is represented by a system of rays which, on striking an obstacle, are scattered; the nature of the scattered rays near the obstacle surface depends on the surface and medium properties at (more properly, in the vicinity of) the point of impact. The formulas developed in Secs. 1.7b and 1.7c can then be employed for tracking the scattered rays away from the obstacle boundary. A ray striking a flat surface element is reflected or refracted according to the relevant (plane-wave) reflection or refraction law, and is propagated with an amplitude specified by the plane-wave reflection coefficient. A ray striking a curved surface element (with principal radii of curvature much greater than the wavelength) is reflected as from a plane tangent to the surface at the point of impact, but the intensity along its path is governed by a "divergence coefficient," due to the surface curvature, which accounts for the spreading of the energy contained in a narrow ray bundle. The reflected and refracted rays are those encountered in the geometrical optics of isotropic and anisotropic media, and their properties are discussed in detail in this section.

In addition, the theory utilizes a class of diffracted rays whose characteristics in an isotropic medium are summarized briefly. A ray incident normally on an edge gives rise to "edge-diffracted" rays that emerge in all directions in the plane perpendicular to the edge at the point of impact; the intensity along an edge-diffracted ray decreases like $\rho^{-1/2}$, where ρ is the distance from the edge (see Sec. 6.4 for a detailed discussion and also the example at the end of this section). The edge-diffracted rays due to an obliquely incident ray lie on a cone having a semiangle at the apex equal to the angle between the incident ray and the edge; the incident and diffracted rays lie on opposite sides of the perpendic-

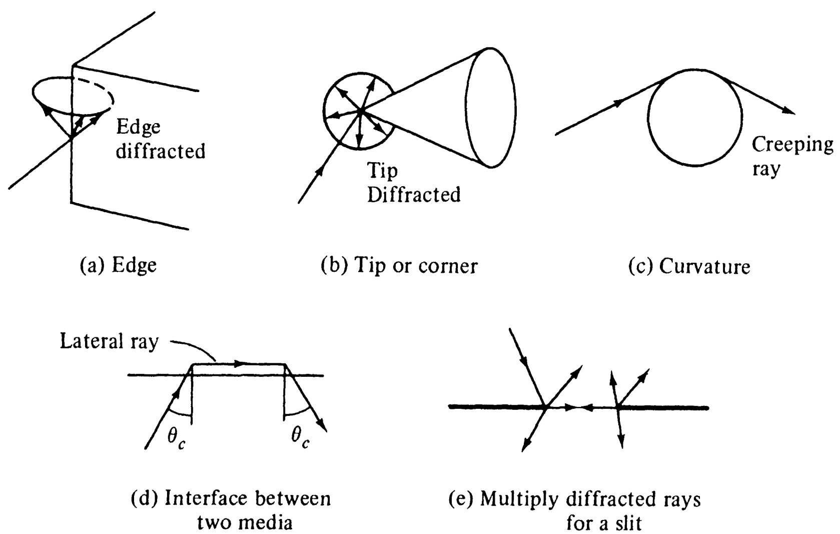


FIG. 1.7.2 Diffracted rays.

ular plane through the point of impact [see Fig. 1.7.2(a)]. A ray incident on a conical tip [Fig. 1.7.2(b)] or on a corner is scattered in all directions, and the intensity along a "tip-diffracted" ray decreases like r^{-1}, where r is the distance from the tip (see Sec. 6.8c). A ray incident tangentially on a curved object [Fig. 1.7.2(c)] creates a "creeping" ray which travels along the obstacle surface into the shadow region and sheds energy continually as it progresses; as a result, its amplitude along the surface is found to decay exponentially with distance (see Sec. 6.7). In the presence of an interface between two media, a ray incident at the critical angle θ_c from the optically denser medium launches a lateral ray that travels parallel to the interface in the optically thinner medium and sheds energy back to the denser medium by refraction [Fig. 1.7.2(d)]; the decay of the lateral ray amplitude due to energy leakage is found to be algebraic (see Sec. 5.5). The trajectories of the reflected and diffracted rays are in accord with Fermat's principle of least propagation time, and the manner of variation of the field, both in amplitude and phase, along a reflected or diffracted ray† can often be predicted from simple geometrical considerations, as illustrated in the example at the end of this section. The starting amplitude and phase along a given diffracted ray cannot in general be ascertained from the geometrical theory but must be taken from the rigorous asymptotic solution for a simple "canonical" scatterer whose contour at the point of diffraction is identical to that of the object under consideration (see the examples discussed elsewhere in this volume for a ray-optical interpretation of various rigorous asymptotic solutions).

Constructed in this manner, the scattered field for a canonical object, as predicted from the geometrical theory of diffraction, manifestly agrees with the asymptotic form of the rigorous solution in the range $k_0 \to \infty$. However, the geometrical theory, exploiting the local character of high-frequency diffraction effects, can also be applied to more complicated shapes for which exact solutions are not readily available. For example, by approximating the vicinity of a point on a variable-curvature cylindrical surface by a circular cylinder whose radius is identical with the local radius of curvature, one can predict the behavior of the diffracted (creeping) rays on such a surface; the validity of this prescription can be confirmed by comparing the result with the rigorous asymptotic solution for parabolic and elliptic cylinders.[33] Verification has also been obtained for the problem of diffraction by a circular aperture and by a slit, wherein one has to take into account multiply diffracted rays that travel from one edge to the other [Fig. 1.7.2(e)].[34,35] Corresponding results have been developed for scattering in anisotropic media, in media capable of supporting multiple wave species, and in wave guide regions.[36,37] An illustrative application of the theory to a non-elementary scattering problem is given later in this section.

†The category of diffracted rays following real trajectories in **r** space as shown in Fig. 1.7.2 can be enlarged by inclusion of rays defined on complex trajectories [e.g., rays descriptive of surface waves (see Sec. 5.6) and leaky waves, or of attenuated and evanescent fields].

An appealing feature of the geometrical theory of diffraction is its conceptual simplicity and its inherent broad range of applicability in homogeneous and inhomogeneous, isotropic or anisotropic regions. The theory is, however, based on a series of postulates whose validity has not as yet been proved in general, so it has been necessary to provide verification by recourse to the asymptotic expansions of rigorously solvable problems. In view of the substantial number of such successful comparisons, it appears likely that the geometrical theory of diffraction is capable of predicting at least the dominant effects of high-frequency scattering under relatively arbitrary conditions. The theory has also been extended to transient problems in dispersive media through use of space–time rays instead of time-harmonic rays.[19,24]

Ray reflection and refraction laws

When a ray strikes a surface across which the medium properties change discontinuously, it generates reflected and (in a penetrable medium) refracted rays which generally leave the interface along directions different from that of the incident ray. The new ray directions can be inferred from phase continuity of all relevant wave constituents generated at the boundary, a necessary requirement in order to satisfy field continuity and thence to maintain the local plane-wave character of the complex of incident, reflected and refracted waves. When applied to two ordinary dielectrics, the phase matching yields the conventional Snell's law, but a more general form results when the media are anisotropic and (or) when multiple wave species can be supported.[38] Thus, if m wave constituents are involved, each with a phase dependence $\exp[ik_0\psi_j(\mathbf{r})]$, $j = 1, \ldots, m$ [see Eq. (5)], then on the interface B,

$$\psi_1(\mathbf{r}) = \psi_2(\mathbf{r}) = \cdots = \psi_m(\mathbf{r}), \qquad \mathbf{r} \text{ on } B. \tag{57}$$

It follows from Eq. (57) that the derivatives of $\psi_j(\mathbf{r})$, $j = 1, \ldots, m$, in a direction ξ tangential to the interface are also continuous. Since

$$\frac{\partial \psi_j}{\partial \xi} = \xi_0 \cdot \nabla \psi_j = \bar{\mathbf{k}}_{tj}, \tag{58}$$

where ξ_0 is a unit vector and $\bar{\mathbf{k}}_{tj}$ is the projection of the jth wavevector onto the interface, the phase continuity condition can be written in terms of the normalized wavevectors $\bar{\mathbf{k}}_j = \nabla\psi_j$ as

$$\bar{\mathbf{k}}_{t1} = \bar{\mathbf{k}}_{t2} = \cdots = \bar{\mathbf{k}}_{tm} \qquad \text{on } B. \tag{59}$$

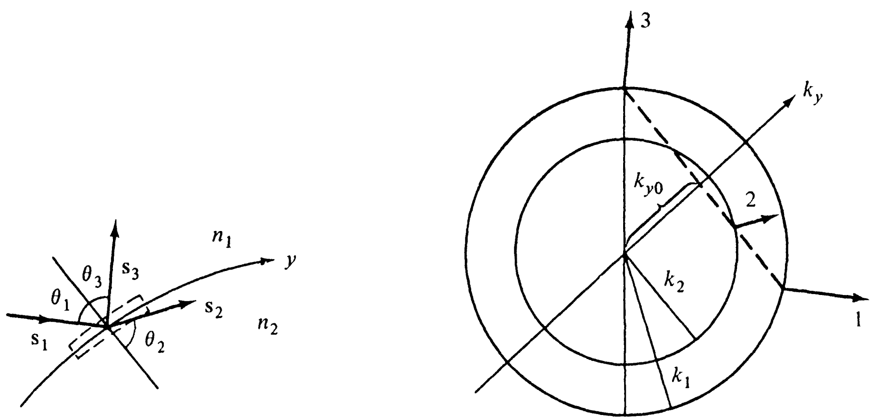
This relation will be used to derive reflected and refracted ray directions in isotropic, anisotropic, and compressible plasma media. The initial amplitude and polarization of the fields along reflected and refracted rays can be deduced either from additional continuity requirements applied to the asymptotic field expression or, equivalently, from the exact solution of plane-wave reflection and refraction at a plane boundary between two homogeneous media having constitutive parameters identical with the local medium parameters at the point of impact of the incident ray.

Isotropic media

At a plane interface between two ordinary dielectrics described by refractive indexes $n_1(\mathbf{r})$ and $n_2(\mathbf{r})$, a ray ($j = 1$) incident from medium 1 generates locally a single ray ($j = 3$) reflected into medium 1 and a single ray ($j = 2$) refracted into medium 2. Since $\bar{\mathbf{k}}_{1,3} = n_1 \mathbf{s}_{1,3}$, $\bar{\mathbf{k}}_2 = n_2 \mathbf{s}_2$ [see Eq. (6d)], and $\mathbf{s}_{tj} = \xi_0 \sin\theta_j$, where θ_j is the angle between the ray direction and the surface normal, Eq. (59) yields

$$\sin\theta_1 = \sin\theta_3, \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{60}$$

In solving for the angles, the choice of θ_j or $(\pi - \theta_j)$ is made subject to the radiation condition that both the reflected and refracted rays are directed away from the boundary. For the pair of incident and reflected rays $\mathbf{s}_1$ and $\mathbf{s}_3$ in Fig. 1.7.3(a), proceeding into the same medium, the first equation in Eq. (60) yields



(a) Physical configuration

(b) Wavenumber plot (section through the plane containing the rays)

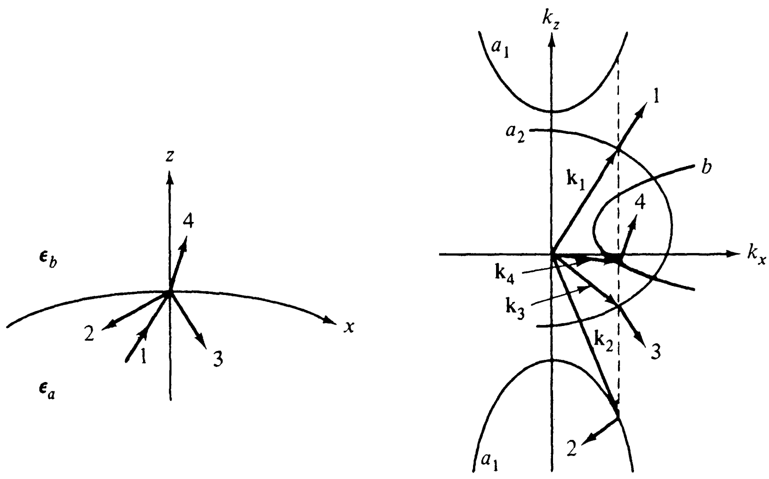
FIG. 1.7.3 Ray reflection and refraction law: $\theta_1 = \theta_3, n_1 \sin\theta_1 = n_2 \sin\theta_2$.

the specular reflection law, whereas for the incident and refracted rays $\mathbf{s}_1$ and $\mathbf{s}_2$, one has Snell's refraction law. In this instance Eq. (59) implies that the triplet of incident, reflected, and refracted rays lies in a plane perpendicular to the interface. For a curved interface as in Fig. 1.7.3(a), it must be kept in mind that ray-optical concepts apply only when the radius of curvature is large compared with the local wavelengths $\lambda_{1,2} = 2\pi/k_0 n_{1,2}$.

The refraction and reflection laws contained in Eqs. (59) and (60) may be visualized conveniently with the aid of refractive index or wavenumber surfaces. As noted in Sec. 1.6, for a given medium, such surfaces are obtained by plotting the refractive index n or the wavenumber $k = k_0 n = k_0 \bar{k}$ as a function of direction. The vector from the origin to a point on the surface specifies the direction of phase propagation in the corresponding plane-wave $\exp(i\mathbf{k}\cdot\mathbf{r})$; as shown in Sec. 1.6a, the normal to the surface at this point yields the direction of energy propagation (ray direction) in the wave. In an isotropic medium, n is independent of direction, so the plots of n and k are spherical, whence the phase and ray propagation vectors are codirectional. At an interface between two media, the condition $\bar{\mathbf{k}}_{t1} = \bar{\mathbf{k}}_{t2} = \bar{\mathbf{k}}_{t3}$ leads to the graphical construction in Fig. 1.7.3(b), where the k_y axis is drawn parallel to the local direction of the interface coordinate y in Fig. 1.7.3(a) and k_{y0} specifies the tangential wavenumber of the incident ray 1; the reflected ray 3 and the refracted ray 2 are then determined from those points on the wavenumber surfaces corresponding to the same value of k_{y0}. This graphical procedure has also been employed in Fig. 1.6.7 to chart the progress of a ray in a continuously varying plane-stratified medium.

Anisotropic media

When a ray strikes an interface between two different anisotropic media ϵ_a and ϵ_b, the initial directions of the reflected and refracted rays may be ascertained by the phase-matching procedure given in Eq. (59): all wave constituents must have the same wavevector component $\mathbf{k}_t$ tangential to the interface. An analytical form of the ray reflection and refraction laws is generally quite involved [see Eq. (7.5.9b) for the special case of a uniaxially anisotropic medium], but the ray trajectories may be visualized with the aid of the wavenumber surfaces for the two media. Medium ϵ_a is assumed to have a wavenumber surface with two branches a_1 and a_2; medium ϵ_b has a single branch b, only relevant portions of which are shown in Fig. 1.7.4. Consider an incident ray 1 corresponding to point $\mathbf{k}_1$ on the a_2 branch of the lower medium. When the xy



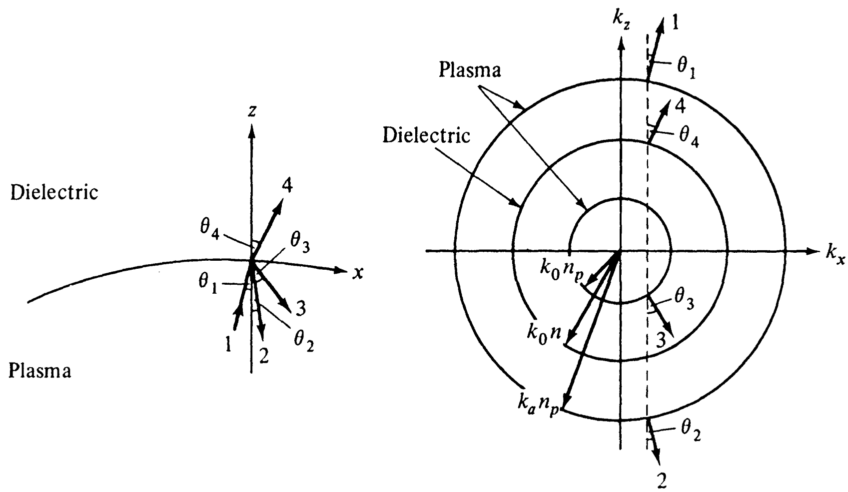
(a) Physical configuration (b) Wavenumber plot

FIG. 1.7.4 Ray reflection and refraction for anisotropic media (projection on the xz plane).

coordinate system in the tangent plane is oriented appropriately, the incident wavevector tangential to the interface is $\mathbf{k}_t \equiv \mathbf{k}_x$, and this value of $\mathbf{k}_x$ determines other points $\mathbf{k}_i$ on the composite wavenumber diagram. The corresponding rays $\bar{\mathbf{S}}_i$ perpendicular to the surfaces are drawn so as to ensure $\mathbf{k}_i \cdot \bar{\mathbf{S}}_i \geq 0$ [see Eq. (53a)], thereby providing the trajectories in Fig. 1.7.4. Only the projections on the xz plane are shown; although all wavevectors $\mathbf{k}_i$ lie in this plane, the rays are generally not coplanar and possess a y component, unless the wavenumber surfaces are symmetrical about the $k_y = 0$ plane. It is to be noted that the incident ray excites two reflected rays, one of which (ray 2) is turned back toward the direction of incidence. Such phenomena of "backward" reflection or refraction of *rays* may occur in an anisotropic medium because of the general character of the wavenumber surfaces, but no backward characteristics are exhibited by the wavevectors, and hence the directions of advance of the phase fronts, in view of the phase matching condition $\mathbf{k}_t =$ constant. One observes furthermore that the refracted ray 4 carries energy away from the interface, whereas the phase fronts in this ray propagate toward the interface (see direction of $\mathbf{k}_4$), which feature identifies a "backward wave" with respect to the direction z normal to the interface. Of all points $\mathbf{k}_i$ having the same value $\mathbf{k}_t$ on the composite diagram, only those are permissible which satisfy the radiation condition; the latter stipulates that each reflected and refracted ray must carry energy away from the boundary. These constraints eliminate rays corresponding to the upper branch of a_1 and to the upper intersection on branch b.

Warm isotropic plasma

In a homogeneous warm plasma medium, electromagnetic (transverse) and electroacoustic (longitudinal) waves propagate independently with the phase speeds $v = c/n_p$ and $v = a/n_p$, respectively, where c is the speed of light in vacuum, a is the electron-acoustic speed in the plasma, and $n_p = [1 - (\omega_p^2/\omega^2)]^{1/2}$ is the plasma refractive index (see Sec. 1.2c). To a first approximation, these waves also propagate independently in a slowly varying medium, but they are coupled at boundaries across which medium properties change abruptly. Imposition of the phase-matching procedure is illustrated for an electronacoustic ray incident on an interface separating the plasma from an exterior dielectric medium with refractive index n (Fig. 1.7.5). Since the wave processes are isotropic, they are described by spherical wavenumber surfaces having radii $k_0 n_p$, $k_a n_p$, and $k_0 n$ for the electromagnetic and electronacoustic waves in the plasma, and the electromagnetic wave in the dielectric, respectively, with $k_0 = \omega/c$ and $k_a = \omega/a$. The incident electronacoustic ray 1 has a wavevector component $k_{t1} = k_a \sin\theta_1$ parallel to the interface and via Eq. (59) and Fig. 1.7.5(b) establishes the directions of the reflected acoustic ray 2, the reflected electromagnetic ray 3, and the transmitted electromagnetic ray 4, respectively. Since the ray and wavevector directions coincide in the present example, a simple analytical statement of the ray reflection and refraction laws follows at once from Eq. (59):



(a) Physical configuration (b) Wavenumber plot

FIG. 1.7.5 Ray reflection and refraction for warm plasma.

$$\sin\theta_1 = \sin\theta_2 = \frac{a}{c}\sin\theta_3 = \frac{a}{c}\frac{n}{n_p}\sin\theta_4. \tag{61}$$

Analogous considerations apply when an electromagnetic ray is incident either from the plasma or from the dielectric side.

Diffracted rays

Although it is necessary, in general, to solve "canonical" scattering problems for specification of initial field values on diffracted rays, certain general observations can be made concerning the multiplicity and orientation of diffracted rays launched by a prescribed incident ray. For example, to a lowest order of approximation, a ray incident on a slowly curved edge in a slowly varying medium excites the same scattered field as a plane wave in a homogeneous medium incident on a straight edge extended indefinitely along the tangent at the point of impact of the ray. In this canonical problem, the plane-wave and the medium parameters are chosen so as to represent local conditions near the point of impact; because of translational invariance along the direction x parallel to the edge in a homogeneous medium, an incident plane wave with wavenumber k_{xi} along x excites scattered fields characterized by the same value $k_x = k_{xi}$. This condition can be utilized to chart the initial direction of edge-diffracted rays in a manner analogous to that used for reflected and refracted rays. For example, consider an anisotropic medium characterized by a wavenumber surface having three branches, a, b_1, and b_2, symmetrical about an axis w, as in a cold plasma subjected to an external magnetic field along the direction w [Fig. 1.7.6(b)]. An edge is embedded in this medium as shown and is excited by an incident ray $1a$ with $k_x = k_{xi}$ [Fig. 1.7.6(a) and (b)]. Diffracted rays correspond

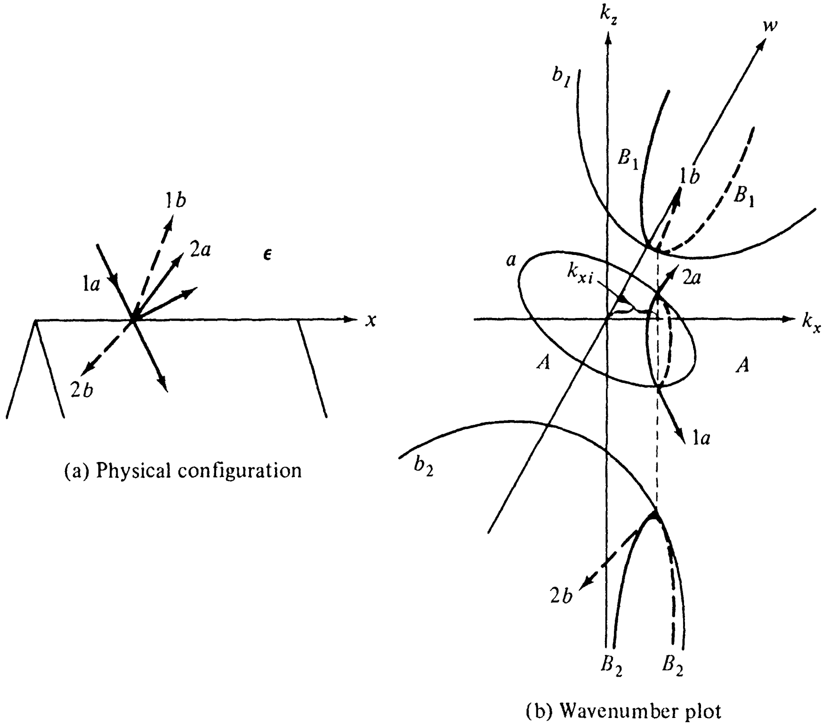


FIG. 1.7.6 Trajectories of edge-diffracted rays.

to **k** values lying on the intersection curves A, B_1, and B_2 of the plane $k_x = k_{xi}$ and the multibranched wavenumber surface; the rays proceed in the direction perpendicular to the wavenumber surface [see Eq. (53a) for constraints on their orientation] to points exterior to the wedge region in Fig. 1.7.6(a). As noted in Fig. 1.7.6, the diffracted rays corresponding to various branches of the wavenumber surface lie along distinct cones (or sections thereof when the relevant branch is open) which are generally non-symmetric about the edge. In the special case of an isotropic medium, the wavenumber surface or surfaces are spherical (see Figs. 1.7.3 and 1.7.5), with consequent x symmetry of the resulting diffracted ray cones. When the medium is an ordinary isotropic dielectric, existence of only a single spherical wavenumber surface implies that the angle of the diffracted ray cone is the same as the angle between the incident ray and the edge, as noted in connection with Fig. 1.7.2(a). It may be noted that the determination of the edge-diffracted ray trajectories is the same as for a line source with phase progression $\exp(ik_{xi}x)$ located on the x axis. This aspect is explored in Secs. 5.4d, 7.3c and 7.3d.

The trajectories of lateral rays excited on the interface between two media may be discussed in a similar manner.[39] Lateral rays arise whenever an incident ray of a given species can give rise to a ray of another species directed parallel to the interface; the latter ray may proceed either in the original medium (if several wave types can be supported therein) or in the exterior medium. For an interface between two ordinary dielectrics, one observes from Fig. 1.7.3(b) that $\theta_2 = 90°$ when $k_{y0} = k_2$ [i.e., $\theta_1 = \theta_c = \sin^{-1}(n_2/n_1)$, where θ_c is the critical angle (for $\theta_1 > \theta_c$, the incident rays are totally reflected and no transmission into medium 2 takes place)]. The critically refracted ray 2 travels along the interface in medium 2 and refracts energy continuously back into region 1 along ray 3, as shown in Fig. 1.7.2(d). The triplet of rays 1, 2, 3 is subject to the phase-matching condition in Eq. (59). When anisotropic media are involved as in Fig. 1.7.4, a multiplicity of lateral rays can be excited. For example, when the incident ray 1 on branch a_2 in Fig. 1.7.4 has a direction such that the refracted ray 4 is parallel to the interface in region b, one obtains the lateral-ray configuration schematized in the first sketch in Fig. 1.7.7(a); refraction into medium a occurs in ray species 2 and 3, corresponding to different branches of the wavenumber surface for region a. Alternatively, when the incident ray 1′ in region a originates on branch a_1 as shown, it excites a lateral ray 2′ on branch a_2 which travels along the interface in medium a and sheds energy continuously into media a and b along rays 3′ and 4′, respectively. These aspects are treated further in Secs. 5.5a, 5.5b, and 7.5d. One observes from Fig. 1.7.7(b) that lateral ray 4 can also be excited by a critically incident ray 1″ on branch a_1 and that

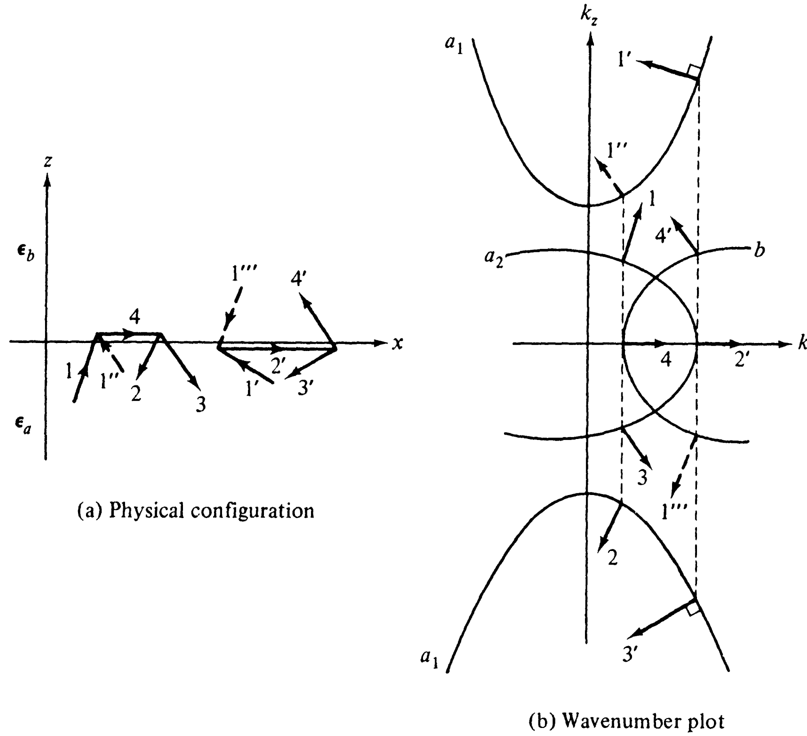


FIG. 1.7.7 Lateral ray trajectories on interface between anisotropic media (projection on xz plane).

lateral ray $2'$ can also be excited by a critically incident ray $1'''$ from region b. Lateral rays occur also on the dielectric–warm plasma interface in Fig. 1.7.5. For example, when the incidence angle for the electron-acoustic ray 1 is $\theta_1 = \sin^{-1}(a/c)$, ray 3 becomes an electromagnetic lateral ray which travels parallel to the interface in the plasma and refracts as the electronacoustic ray 2 in the plasma and the electromagnetic ray 4 in the dielectric. When the incidence angle of ray 1 is $\theta_1 = \sin^{-1}(an/cn_p)$, the electromagnetic ray 4 in the dielectric becomes a lateral ray that refracts as the electronacoustic ray 2 in the plasma. It may be noted that when the dispersion relation is written $k_z = k_z(k_x, k_y)$, appropriate for analysis of propagation in regions with planar stratification along z (see Sec. 1.6b), the lateral ray corresponds to a double root on the dispersion surface and is thus characterized by a branch-point singularity in the $\mathbf{k}_t$ integral representation [Eq. (1.6.21)].

Example: Diffraction by a conducting half-plane on the interface between two isotropic dielectrics

The asymptotic representation of the high-frequency field for radiation and diffraction problems comprises in general several constituents, each of which has a dominant behavior of the form exhibited in Eqs. (37), (40), or their equivalents. When reflection and (or) refraction at a boundary is relevant, the associated phenomena are accounted for by the geometric-optical field u_g, whereas shadowing effects, structural singularities, etc., give rise to an additional diffracted field u_d. Thus, the total field at an observation point $\mathbf{r}$ is given in general by

$$u(\mathbf{r}) \sim u_g(\mathbf{r}) + u_d(\mathbf{r}), \tag{62}$$

where $u_g(\mathbf{r}) = \sum_i u_g^{(i)}(\mathbf{R}_i)$, with $u_g^{(i)}(\mathbf{R}_i)$ representing the direct, reflected, or refracted field that reaches $\mathbf{r}$ along a ray $\mathbf{R}_i$. Similarly, the diffracted field may also have several distinct species, $u_d(\mathbf{r}) = \sum_j u_d^{(j)}(\mathbf{R}_j)$.

The method of constructing the ray-optical field in this manner is illustrated by the example in Fig. 1.7.8, which shows a plane interface separating two different homogeneous isotropic media having refractive indexes n_1 and $n_2 < n_1$. A semiinfinite, perfectly conducting plane sheet is placed on the interface, and the entire configuration is illuminated by a line source oriented parallel to the edge (this renders the problem two-dimensional). The source emits rays in all directions, and some of the rays reach a given observation point either directly, by reflection, by refraction, or by diffraction. An observation point $\mathbf{r}_1$ in the upper medium is reached by the direct ray 1 and the reflected ray 3 which constitute the geometric-optical field. Contributions arise also from diffraction effects that are associated with the onset of total reflection and with the presence of an edge singularity, respectively. The former gives rise to the lateral ray 5 which travels along the interface in the low-refractive-index region and sheds energy back into the high-refractive-index region along ray 6 (for details, see Sec. 5.5). When the lateral ray strikes the edge, a radially spreading diffracted

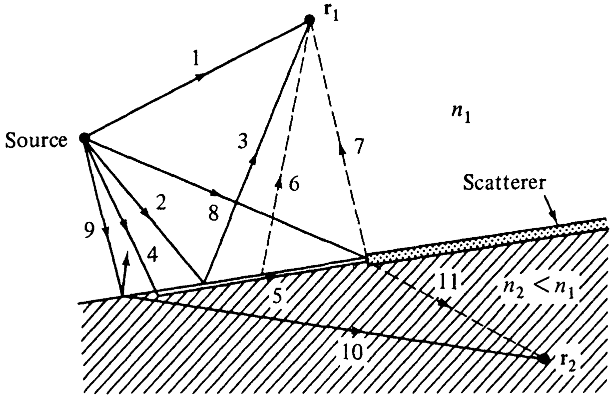


FIG. 1.7.8 Ray-optical construction of the field when a source illuminates a plane interface on which is placed a semiinfinite scatterer (dashed lines denote diffracted rays).

field is generated which reaches point $\mathbf{r}_1$ along ray 7. The edge is also excited directly by the incident ray 8 which sends its own edge-diffracted field along ray 7 (for details of scattering by a half-plane, see Sec. 6.5). Thus

$$u(\mathbf{r}_1) \sim u_g^{(1)} + u_{g2}^{(3)} + u_{d5}^{(6)} + u_{d5}^{(7)} + u_{d8}^{(7)}, \tag{63}$$

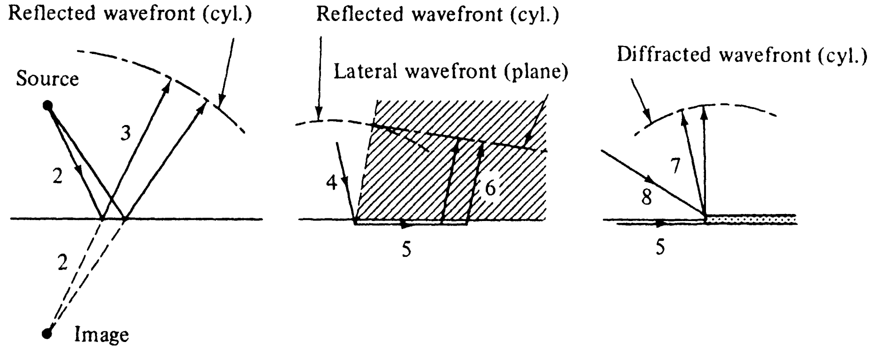
where the superscripts identify the rays leading to point $\mathbf{r}_1$, the letter subscripts distinguish geometric-optical and diffracted constituents, and the number subscripts specify the appropriate exciting ray.

The form of the individual field contributions in Eq. (63) can be determined from Eq. (37) or (40). The line-source field is assumed to be so normalized as to generate in an unbounded medium a unit amplitude field at unit radial distance, with the phase taken as zero at the source. Since the rays diverge radially from the source in the plane of the figure, but are parallel at successive cross sections along the line source (see Sec. 5.4d), the ray tubes are wedge shaped and $dA(\mathbf{R}) \sim 1/R$, where R is the distance from the source. Thus, the amplitude along an incident ray varies like $R^{-1/2}$, representative of a cylindrically spreading wave. From Eq. (37), with the above-mentioned normalization, one obtains for the direct field contribution,

$$u_g^{(1)}(\mathbf{R}_1) = \frac{e^{ik_0 n_1 R_1}}{\sqrt{R_1}}, \tag{64a}$$

where it has been recognized that $\psi(R) = k_0 n_1 R$ for the homogeneous medium in the present example. In this equation and the ones to follow, R_i is the length along the ith ray in Fig. 1.7.8.

The reflected field may also be calculated from Eq. (37), with the reference point on ray 3 chosen at the interface. Since the interface is planar and the media are isotropic, the tube of reflected rays appears to originate at the image point of the source [Fig. 1.7.9(a)], whence the ratio of the area at the interface



(a) Reflected rays (b) Lateral ray (c) Edge-diffracted rays

FIG. 1.7.9 Various ray-tube constructions.

to that at $\mathbf{r}_1$ is given by $R_2/(R_2 + R_3)$ (for a general treatment of reflection from a curved surface, see Reference 40). The reflected field at the interface, equal to the incident field multiplied by a reflection coefficient Γ, is given from Eq. (64a) by $\Gamma R_2^{-1/2} \exp(ik_0 n_1 R_2)$. The phase accumulation along ray 3 equals $\exp(ik_0 n_1 R_3)$, whence, from Eq. (37),

$$u_{g2}^{(3)} = \Gamma \frac{e^{ik_0 n_1 (R_2 + R_3)}}{\sqrt{R_2 + R_3}}. \tag{64b}$$

The Fresnel reflection coefficient Γ is calculated by field matching between the incident, reflected, and refracted rays at the interface and, in view of the previously noted plane-wave character of the local field along a ray, is the same as for a *plane-wave* field incident along the direction of ray 2.

The diffracted field along ray 6 is launched by continuous refraction of the field carried along the lateral ray 5, which, in turn, is excited by ray 4 incident at the angle of total reflection $\theta_c = \sin^{-1}(n_2/n_1)$ measured from the normal to the interface. Since the refraction angle equals θ_c and is therefore constant, the emerging tube is bounded by parallel rays [Fig. 1.7.9(b)], thereby furnishing a constant amplitude along ray 6. The phase accumulation along ray 6 is given by $\exp(ik_0 n_1 R_6)$, so $u_{d5}^{(6)} = u_{d5} \exp(ik_0 n_1 R_6)$, where u_{d5}, the reference value of the field at the interface, must again be determined by solving the boundary-value problem (alternatively, see Reference 41). It is found (Sec. 5.5) that the phase of u_{d5} is the one predicted from the trajectory along rays 4 and 5 (note that ray 5 travels in the medium with refractive index n_2) and that the amplitude is proportional to $1/k_0 R_5^{3/2}$. Thus,

$$u_{d5}^{(6)} \propto \frac{e^{ik_0(n_1 R_4 + n_2 R_5 + n_1 R_6)}}{k_0 R_5^{3/2}} U(R_5), \tag{64c}$$

where the Heaviside unit function $U(R_5)$ signifies that this contribution exists only when the ray path has a finite lateral segment R_5 [i.e., for observation points in the shaded region of Fig. 1.7.9(b), which contains the totally reflected geometric-optical rays]. Evidently, the formula fails near the total reflection boundary $R_5 = 0$ and must be replaced there by a transition function that cannot be deduced by ray-optical means (see Sec. 5.5).

When an incident ray strikes an edge, it generates a cylindrically spreading diffraction field that can be described in terms of a set of radial rays originating at the edge [Fig. 1.7.9(c)]. Thus, the field along an edge-diffracted ray 7 decays like $R_7^{-1/2}$ and its phase varies like $\exp(ik_0 n_1 R_7)$. The reference field at unit distance from the edge must now be determined from the solution of the relevant diffraction problem. For the diffracted field excited by ray 8 in Fig. 1.7.8 it suffices, because of the local plane-wave character of the incident field, to treat the simpler plane-wave scattering problem. Its solution furnishes a diffraction coefficient D_{87} which specifies the diffracted reference field generated by a plane wave of unit strength incident along the direction of ray 8. Since the actual incident field has a strength given by Eq. (64a) with R_1 replaced by R_8, one finds that

$$u_{d8}^{(7)} = D_{87} \frac{e^{ik_0 n_1 (R_8 + R_7)}}{\sqrt{R_8 R_7}}. \tag{64d}$$

The diffraction coefficient D_{87} depends on the angle of the incident and diffracted rays with respect to the half-plane; for scattering in a homogeneous medium, it has been calculated in Sec. 6.5.

An analogous calculation must be performed for the incident lateral wave to yield a diffraction coefficient D_{57}. Then, by considerations similar to the above, one may write

$$u_{d5}^{(7)} \propto D_{57} \frac{e^{ik_0(n_1 R_4 + n_2 R_5 + n_1 R_7)}}{k_0 R_5^{3/2} R_7^{1/2}} U(R_5), \tag{64e}$$

with R_5 representing the entire lateral distance between the point of impact of ray 4 and the edge. The edge also excites a lateral wave traveling to the left along the interface; its contribution has not been shown but should be included in general. Moreover, different wave constituents emerge when the observation point is far to the right of the edge, in which instance the reflected field is modified and the lateral wave field in Eq. (64c) is absent.

The same procedure may be employed for the construction of the ray-optical field in the lower medium of Fig. 1.7.8. Instead of the geometric-optical contributions in Eqs. (64a) and (64b), one now has the refracted field carried along ray 10. The diffracted fields are similar to the above, except for the omission of the lateral-wave constituent in Eq. (64c), which is absent in the optically thinner medium (see Fig. 1.7.3). When the edge of the obstacle is located to the left of the point of impact of the critically refracted ray 4, there exists a geometric-optical shadow region from which the refracted rays 10 are excluded and which is penetrated only by the edge-diffracted rays 11.

Although the dependence on k_0 has not been given explicitly in the reflection and diffraction coefficients in Eqs. (64), the distance dependence has been shown. One observes that for large distances R_i, the diffraction fields are considerably weaker than the geometric optical fields, thereby complicating the detection of the former in the time-harmonic regime (an exception occurs in shadow regions where the diffraction fields are dominant). This disadvantage is not present under transient conditions since in view of the discussion in Sec. 1.6c, the distinctive phases of the various field constituents in Eq. (63) imply different times of arrival of the transient response at an observation point (see Fig. 1.7.9 for a sketch of the wavefronts, and also Figs. 5.5.8 and 6.4.2). Thus, the time resolution under pulse conditions provides an important means of identifying field contributions, which may be obscured in the time-harmonic steady state.

Finally, it should be emphasized that retention of only the dominant term in Eq. (37) for the asymptotic representation of each of the constituent fields does not assure that the composite solution in Eq. (63) is consistent to a certain order in the small parameter $1/k_0$. The k_0 dependence of a diffraction field amplitude (e.g., that of the lateral wave) may correspond to that of a second term in the asymptotic expansion of the geometric-optical field [see Eq. (16)], so it may be necessary to include such terms if a consistent result to a given order in $1/k_0$ is desired.

1.7e Transient Fields

Non-time-harmonic fields in a spatially and temporally inhomogeneous medium satisfy the system of equations given abstractly in Eq. (2). To a lowest order of approximation as in Eq. (1), in a parameter regime validating a wave-packet description, the general field equations reduce to the simple geometric-optic form in Eq. (3), which are integrable along space–time ray trajectories defined in Eq. (10). The phase of the assumed field (1) follows from solution of the dispersion equation (4a) as in Eq. (11), while the amplitude and polarization of the field vector Ψ_0 requires evaluation of the eigenvectors $\Psi_{0\alpha}$ from Eq. (3). Because of the complicating effects of dispersion, performance of these calculations is generally quite involved. To illustrate salient features, it is instructive to treat the simple example of an isotropic cold plasma for which the lowest-order, as well as higher-order, approximations can be developed without undue difficulty.[19]

As in the simple time-harmonic field problems in Sec. 1.7b and 1.7c, it is convenient to employ the second-order form of the field equations to derive the dispersion and energy-conservation equations for the lowest-order space–time ray amplitudes. Thus, we consider the scalar wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_p^2(\mathbf{r})}{c^2}\right]u(\mathbf{r}, t) = 0, \tag{65}$$

satisfied approximately by the scalar electromagnetic potential function u in a spatially inhomogeneous cold plasma characterized by the variable plasma frequency $\omega_p(\mathbf{r})$, with c denoting the speed of light in vacuum. To introduce the parameter ν with respect to which the asymptotic development is to be carried out, one scales the space and time coordinates,

$$\mathbf{R} = \frac{\mathbf{r}}{\nu}, \qquad \tau = \frac{t}{\nu}, \tag{66}$$

and obtains

$$\left[\nabla_R^2 - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2} - \nu^2\frac{\bar{\omega}_p^2(\mathbf{R})}{c^2}\right]u(\mathbf{R}, \tau) = 0, \tag{67}$$

where ∇_R is the gradient operator in the $\mathbf{R}$ coordinate system, $\bar{\omega}_p(\mathbf{R}) = \omega_p(\mathbf{r}) = \omega_p(\mathbf{R}\nu)$ and $u(\mathbf{R}, \tau)$ is $u(\mathbf{r}, t)$ in the scaled coordinates. ν is now taken to be large, so non-vanishing values of $\mathbf{R}$ and τ imply large values of $\mathbf{r}$ and t. As noted previously, this is the range where wavepackets are fully developed; also, $\nabla\omega_p(\mathbf{r}) = (1/\nu)\nabla_R\bar{\omega}_p(\mathbf{R}) = O(1/\nu)$, so $\omega_p(\mathbf{r})$ is slowly varying. A solution for u is assumed in the form of an asymptotic series,

$$u(\mathbf{R}, \tau) = e^{i\nu\psi(\mathbf{R},\tau)}\sum_{m=0}^{\infty}\frac{u_m(\mathbf{R}, \tau)}{(i\nu)^m}, \tag{68}$$

with ψ and u_m independent of ν; on substitution into Eq. (67), assuming that the orders of summation and differentiation can be exchanged, one finds

$$(i\nu)^2\left[(\nabla_R\psi)^2 - \frac{1}{c^2}\left(\frac{\partial\psi}{\partial\tau}\right)^2 + \frac{\bar{\omega}_p^2}{c^2}\right]F + (i\nu)\left[\nabla_R^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial\tau^2} + 2\nabla_R\psi\cdot\nabla_R - \frac{1}{c^2}2\frac{\partial\psi}{\partial\tau}\frac{\partial}{\partial\tau}\right]F + \left(\nabla_R^2 - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right)F = 0, \tag{69}$$

where F stands for the sum in Eq. (68). Since this equation must hold for arbitrary (though large) values of ν, the coefficient of each power of ν must vanish independently. From the ν^2 term, one finds

$$(\nabla_R\psi)^2 - \frac{1}{c^2}\left(\frac{\partial\psi}{\partial\tau}\right)^2 + \frac{\bar{\omega}_p^2(\mathbf{R})}{c^2} = 0; \tag{70}$$

from the ν^1 term,

$$\left(\nabla_R^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial\tau^2} + 2\nabla_R\psi\cdot\nabla_R - \frac{2}{c^2}\frac{\partial\psi}{\partial\tau}\frac{\partial}{\partial\tau}\right)u_0 = 0; \tag{71}$$

and from the ν^{-m} term, $m \geq 0$,

$$\left(\nabla_R^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial\tau^2} + 2\nabla_R\psi\cdot\nabla_R - \frac{2}{c^2}\frac{\partial\psi}{\partial\tau}\frac{\partial}{\partial\tau}\right)u_{m+1} + \left(\nabla_R^2 - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right)u_m = 0. \tag{72}$$

Equation (70) is the dispersion equation for the medium which, on introduction of a temporal frequency $\bar{\omega}$ and a wavevector $\bar{\mathbf{k}}$ as in Eq. (4b),

$$\bar{\omega} = -\frac{\partial\psi}{\partial\tau}, \qquad \bar{\mathbf{k}} = \nabla_R\psi, \tag{73}$$

takes on the more familiar form

$$\bar{k}^2 - \frac{\bar{\omega}^2}{c^2} + \frac{\bar{\omega}_p^2(\mathbf{R})}{c^2} = 0, \qquad \text{i.e., } \bar{\omega} = \bar{\omega}(\bar{\mathbf{k}}, \mathbf{R}) = \pm\,[\bar{k}^2c^2 + \bar{\omega}_p^2(\mathbf{R})]^{1/2}. \tag{74}$$

Equation (71) is a transport equation for the leading amplitude term u_0, and Eq. (72) is a transport equation for the higher-order amplitudes. The higher-order amplitudes can be evaluated recursively in terms of the lower-order ones and ψ, but this calculation will not be performed here.

Solution of the dispersion equation

By integrating along space–time ray trajectories defined in Eq. (10), one may solve for the phase function ψ as in Eq. (11). Since the dispersion relation (74) does not contain the time variable explicitly, $d\bar{\omega}/dt = 0$ in Eq. (10), so $\bar{\omega} =$ constant along a ray. Thus, Eq. (11) reduces to

$$\psi(\mathbf{R}, \tau) - \psi(\mathbf{R}_1, \tau_1) = \int_{\mathbf{R}_1}^{\mathbf{R}} \bar{\mathbf{k}}\cdot d\mathbf{R} - \bar{\omega}(\tau - \tau_1), \tag{75}$$

where $\bar{\mathbf{k}}$ is obtained from Eq. (74). For the special case of planar stratification, this phase function is the same as the one obtained in Eq. (1.6.30) by saddle-point evaluation of the integral representation of the field $u(\mathbf{R}, \tau)$. A graphical procedure for tracking the rays has been discussed in connection with Fig. 1.6.9. For a homogeneous medium, $\bar{\omega}$ and $\bar{\mathbf{k}}$ are constant along a ray and one obtains the phase function in Eq. (1.6.5), with $v\psi$ defined in Eq. (1.6.1).

Solution of the transport equation

In view of Eq. (73), Eq. (71) can be written as

$$\left(\nabla_R\cdot\bar{\mathbf{k}} + \frac{1}{c^2}\frac{\partial\bar{\omega}}{\partial\tau} + 2\bar{\mathbf{k}}\cdot\nabla_R + \frac{2}{c^2}\bar{\omega}\frac{\partial}{\partial\tau}\right)u_0 = 0. \tag{76}$$

By changing variables from τ to $c\tau$, introducing a four-dimensional group velocity vector $\bar{\mathbf{V}}_g = \bar{\mathbf{v}}_g + \boldsymbol{\tau}_0 c$ as in Fig. 1.6.1(b), and noting from Eq. (74) that $\bar{\mathbf{v}}_g = \nabla_k\bar{\omega} = \bar{\mathbf{k}}c^2/\bar{\omega}$, one may verify that Eq. (76) can be written in the alternative form:

$$\bar{\Box}\cdot\left(\bar{u}_0^2\frac{\bar{\omega}}{c^2}\bar{\mathbf{V}}_g\right) = 0, \qquad \bar{\Box} \equiv \nabla_R + \boldsymbol{\tau}_0\frac{\partial}{\partial(c\tau)}, \tag{77}$$

where $\boldsymbol{\tau}_0$ is a unit vector along the $c\tau$ axis. Equation (77) implies conservation of the energy flux, $\bar{u}_0^2\bar{V}_g\bar{\omega}/c^2$, in $(\mathbf{R}, c\tau)$ space. $\bar{\mathbf{V}}_g$ is tangent to the space–time rays defined in Eq. (10), which fact has been utilized in Sec. 1.6b for the special case of plane-stratified media. If the divergence theorem in four-dimensional $(\mathbf{R}, c\tau)$ space is applied to a volume generated by a small tube of rays and terminated by hyperplanes $\tau =$ constant (Fig. 1.7.10), the only contribution to the hypersurface integral arises from the cross-section "planes" $\Delta\mathbf{R}_1$ and $\Delta\mathbf{R}$, since on the walls of the tube, $\mathbf{b}\cdot\bar{\mathbf{V}}_g = 0$, $\mathbf{b}$ being a vector normal to the hypersurface. Since the outward normals at $\Delta\mathbf{R}_1$ and $\Delta\mathbf{R}$ have opposite directions along the τ axis and $\boldsymbol{\tau}_0\cdot\mathbf{V}_g = c$, one finds

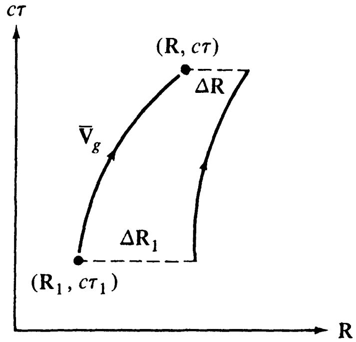


FIG. 1.7.10 Energy conservation in a space-time ray tube.

$$\frac{\bar{\omega}_1}{c^2}\, u_0^2(\mathbf{R}_1, \tau_1)\Delta\mathbf{R}_1 = \frac{\bar{\omega}}{c^2}\, u_0^2(\mathbf{R}, \tau)\Delta\mathbf{R}, \tag{78a}$$

whence, in view of the constancy of $\bar{\omega}$ along a ray, one has for the amplitude u_0 at the point $(\mathbf{R}, \tau)$ along a ray in terms of its value at a reference point $(\mathbf{R}_1, \tau_1)$:

$$u_0(\mathbf{R}, \tau) = u_0(\mathbf{R}_1, \tau_1)\sqrt{\frac{\Delta\mathbf{R}_1}{\Delta\mathbf{R}}}. \tag{78b}$$

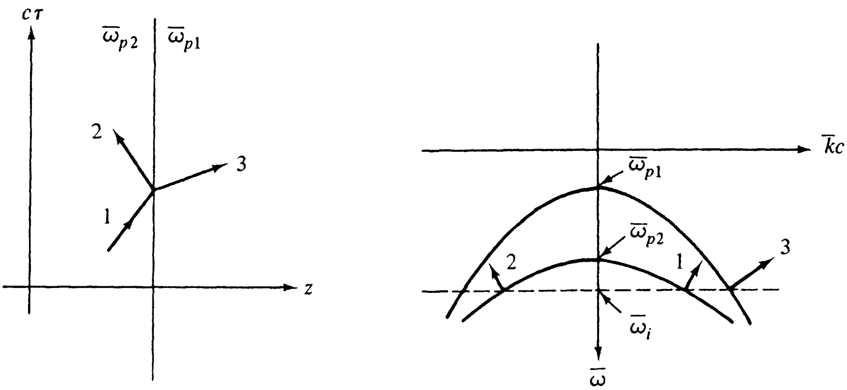
Since the hypersurface areas $\Delta\mathbf{R}_1$ and $\Delta\mathbf{R}$ in planes τ = constant in the four-dimensional $(\mathbf{R}, c\tau)$ space actually represent volumes in the three-dimensional $\mathbf{R}$-space, one recognizes that the energy-conservation statement in Eq. (78a) is equivalent to the statement $\Delta W = |I|^2\Delta\mathbf{R}$ = constant used in connection with Eq. (1.6.9) [see also Eq. (1.6.15)]. Via the present analysis, the validity of this earlier energy-conservation theorem for wavepackets has been extended to propagation in inhomogeneous media. Note that in the inhomogeneous medium, only the frequency spread $\Delta\bar{\omega}$ of waves in the wavepacket is preserved, whereas in a homogeneous medium, both $\Delta\bar{\omega}$ and $\Delta\bar{\mathbf{k}}$ remain invariant. One may also use the considerations of Eqs. (1.6.8)–(1.6.10) for the inhomogeneous case provided that the mapping is carried out locally in such a manner as to conserve energy in the wavepacket; the initial amplitude $u_0(\mathbf{R}_1, \tau_1)$ and phase $\nu\psi(\mathbf{R}_1, \tau_1)$ near the source region can be calculated from Eq. (1.6.5) by assuming the medium to be locally homogeneous, and Eqs. (75) and (78b) are used thereafter, due allowance being made for the variability of $\bar{\mathbf{k}}$ and the constancy of $\bar{\omega}$ along a ray.

Reflection and refraction of space-time rays

As in the time-harmonic case, a space–time ray generates reflected and refracted rays on striking a surface B across which the medium properties vary discontinuously. By considerations analogous to those leading to Eq. (57), one requires continuity of the phase function $\psi(\mathbf{R}, \tau)$ on the surface B for all relevant wave constituents.[38] It then follows that partial derivatives of ψ in the direction tangential to B, and $\partial\psi/\partial\tau$, are also continuous whence from Eq. (73), if m wave constituents are involved,

$$\bar{\mathbf{k}}_{t1} = \bar{\mathbf{k}}_{t2} = \cdots = \bar{\mathbf{k}}_{tm}, \qquad \bar{\omega}_1 = \bar{\omega}_2 = \cdots = \bar{\omega}_m. \tag{79}$$

Because of constancy of the frequency $\bar{\omega}$ in the complex of incident, reflected, and refracted rays, the matching condition (79) on the tangential wave-vector components $\bar{\mathbf{k}}_t$ is imposed as in the time-harmonic problem, and thus the graphical procedures depicted in Figs. 1.7.3–1.7.5 remain relevant. The initial directions of the time-harmonic rays in $\mathbf{r}$ space determined in this manner represent the projections of the corresponding space-time rays onto hyperplanes perpendicular to the τ axis. The ray directions in $(\mathbf{R}, c\tau)$ space are obtained by constructing the normal vectors to relevant branches of the $(c\bar{\mathbf{k}}, \bar{\omega})$ dispersion surfaces at $(c\bar{\mathbf{k}}_t, \bar{\omega})$ points descriptive of the ray complex (see Sec. 1.6a for details on the graphical construction). For two cold isotropic plasma media characterized by Eq. (74), with local plasma frequencies $\bar{\omega}_{p1}$ and $\bar{\omega}_{p2}$ on opposite sides of a boundary separating them, the matching technique is depicted in Fig. 1.7.11.



(a) Configuration space (b) Dispersion surface

FIG 1.7.11 Ray reflection and refraction at interface between two media.

As in the analogous situation in Fig. 1.6.9, to simplify the graphical construction, only one-dimensional propagation (with $\bar{\mathbf{k}}_t = 0$) is considered. The matching parameter $\bar{\omega}_i$ is determined from the incident wavepacket described by ray 1 and is used as in Fig. 1.7.11 to yield the initial directions of the reflected and refracted rays.

Fields near the wavefront

The asymptotic expansion (68) for the transient field $u(\mathbf{R}, \tau) = u(\mathbf{r}, t)$, valid for the large $(\mathbf{r}, t)$ (wavepacket) regime, is inadequate to describe phenomena at times $t \approx r/c$ corresponding to arrival of the wavefront. As noted in Sec.

1.6c, the wavefront fields are synthesized by the high-frequency time-harmonic wave constituents. The asymptotic expansions of the time-harmonic fields in Secs. 1.7b and 1.7c can be employed to develop corresponding expansions of the transient fields for $t \approx r/c$, thereby improving on the leading term expressions in Eqs. (1.6.45) and (1.6.47).

By the considerations of Sec. 1.6c, it follows that if the time-harmonic field is given by a generalized version of Eq. (16),

$$u(\mathbf{r}) \sim \frac{1}{2\pi} e^{ik_0\psi(\mathbf{r})}\left[A'(\mathbf{r}) + \frac{1}{(-i\omega)^{\beta+1}} \sum_{m=0}^{\infty} \frac{A_m(\mathbf{r})}{(-i\omega)^m} \Gamma(\beta + m + 1)\right], \qquad k_0 = \frac{\omega}{c}, \tag{80}$$

then, from Eq. (1.6.20), the transient field near $t = \psi(\mathbf{r})/c$ is

$$\hat{u}(\mathbf{r}, t) \sim A'(\mathbf{r})\delta(t') + t'^{\beta} \sum_{m=0}^{\infty} A_m(\mathbf{r}) t'^m, \qquad \text{where } t' = t - \frac{\psi(\mathbf{r})}{c} \geq 0. \tag{81}$$

The appearance of the phase $\psi(\mathbf{r})$ given in Eq. (28), instead of r, generalizes applicability of corresponding results in Sec. 1.6c to media possessing spatial inhomogeneities at the highest propagation speed c. Equation (81) implies that the wavefronts advance along directions of the wavenormal, which, in the present case of an isotropic medium, are parallel to the rays defined by Eq. (23). (For an asymptotic expansion of the fields covering the range between the wavefront and wavepacket regimes, see reference 22.)

Problems

1. Derive second-order wave equations for the acoustic pressure p and velocity $\mathbf{v}$ as in Eqs. (1.1.3):
   (a) with inclusion of the source terms given in the first-order equations [Eqs. (1.1.1)];
   (b) allowing for either spatial or temporal variability, or both, of the background density n_0 and pressure p_0.
   (c) Find the second-order equations satisfied by the Green's functions G_{11} and $\mathscr{G}_{22}$ in Eqs. (1.1.5) when the background density and pressure are spatially and (or) temporally variable.

2. In a volume V bounded by a surface S whereon the "impedance" boundary conditions in Eq. (1.1.5a) apply, use Green's functions satisfying Eqs. (1.1.5) and simpler boundary conditions $G_{11} = 0 = \mathbf{G}_{12}$ on S to derive within V representations for $p(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$ excited by prescribed sources $s(\mathbf{r}, t)$ and $\mathbf{f}(\mathbf{r}, t)$ in V. *Hint:* In addition to integration over the source region as in Eq. (1.1.4), find and integrate over equivalent source distributions on S to assure satisfaction of Eq. (1.1.5a).

3. While the adjoint problem preserves the original boundary surface S, the boundary conditions on S, as well as the medium filling the volume V enclosed by S, may differ in general. By treating an as yet unspecified adjoint medium

with parameters n_0^+, p_0^+, T_0^+ and boundary parameter α^+, *show* that for derivation of the reciprocity relation in Eq. (1.1.9), the adjoint medium and boundary parameters are identical with the original ones.

4. Show that the reciprocity relations in Eqs. (1.1.11) and (1.1.12) remain valid when the background density and pressure are spatially inhomogeneous.

5. In a volume bounded by a surface S, use electromagnetic dyadic Green's functions satisfying perfect conductor boundary conditions in Eq. (1.1.20c) and derive field representations corresponding to Eqs. (1.1.19), satisfying the impedance boundary conditions in Eq. (1.1.16d). Show that an additional surface integral contribution is required and explain in terms of equivalent electric and magnetic induced currents $\mathbf{H} \times \mathbf{n}$ and $\mathbf{n} \times \mathbf{E}$ on the boundary surface S, where $\mathbf{n}$ is the normal to the boundary. Repeat with use of the free-space dyadic Green's function.

6. (a) Derive the second-order form of the Maxwell field equations (1.1.18) with inclusion of the $\mathbf{J}$ and $\mathbf{M}$ source terms in Eq. (1.1.16). Interpret the source terms in the second-order equations as *equivalent* electric or magnetic currents.
(b) Use the dyadic Green's functions defined by the second-order equations (1.1.21) to represent the electric and magnetic fields excited by the source distributions in (a). Compare the result with Eqs. (1.1.19), which are based on the first-order form of the field equations and the *actual* source distributions. Explain the difference in the two representations and convert one into the other (*Hint:* Employ the formula $\mathbf{A}\cdot\nabla\times\mathbf{B}=\mathbf{B}\cdot\nabla\times\mathbf{A}-\nabla\cdot(\mathbf{A}\times\mathbf{B})$ and use the divergence theorem). Which formulation is more direct?
(c) Repeat for an inhomogeneous medium with permittivity $\epsilon(\mathbf{r})$ and permeability $\mu(\mathbf{r})$.

7. Use the time-dependent free space Green's functions in Eqs. (1.1.34) to calculate via Eqs. (1.1.19) the electromagnetic fields radiated by a pulsed circular ring electric current: $\mathbf{J}(\mathbf{r}, t) = \boldsymbol{\phi}_0 A\delta(\rho - a)\delta(z)\delta(t)$. The ring has a radius $\rho = a$ and lies in the $z = 0$ plane; the current direction is along the ϕ coordinate in a (ρ, ϕ, z) circular cylindrical coordinate system. When the ring radius a tends to zero and the source strength A is increased so that (aA) remains finite, show that the fields radiated by the electric ring current are equivalent to those radiated by a magnetic current dipole oriented perpendicularly to the plane of the ring. Relate to $\mathscr{G}_{i2}(i = 1, 2)$ of Eq. (1.1.19).

8. A pulsed, arbitrarily oriented electric current dipole with $\mathbf{J}(\mathbf{r}, t) = \mathbf{p}\delta(\mathbf{r})\delta(t)$ is located in the presence of a perfectly conducting infinite plane. Calculate the "half-space" Green's functions for this region and therefrom the radiated electric and magnetic fields. Interpret the result in terms of the free space field and the field reflected by the plane. Show that the reflected field can be interpreted as arising from a properly oriented image dipole. (*Hint:* Decompose the dipole into components parallel and normal to the boundary and treat each separately.) Repeat for a pulsed magnetic dipole current $\mathbf{M}(\mathbf{r}, t) = \mathbf{m}\delta(\mathbf{r})\delta(t)$.

9. (a) Find the time-dependent fields radiated by an arbitrarily oriented electric current dipole located at the point (x', y', z') in the quarter space region $x > 0$, $y > 0$ formed by two perfectly conducting planes intersecting at right angles. Show that the solution can be obtained by replacing the boundary by three ap-

propriately oriented image dipole sources located at $(-x', y', z')$, $(x', -y', z')$ and $(-x', -y', z')$. Interpret the solution in terms of free space fields and reflected contributions from the bounding surface. Draw relevant wave fronts. Repeat for excitation by a magnetic dipole.

(b) Replace the quarter space region by a trough (sector) of angle $\alpha = \pi/n$, where n is a positive integer. (Problem 8 and part (a) are special cases of this configuration, with $n = 1$ and $n = 2$.) Show that the image concept can be applied to replace the trough by $(n - 1)$ appropriately placed and oriented image sources.

10. By integrating over z' the time-dependent three-dimensional scalar Green's function in Eq. (1.1.31), show that the causal two-dimensional scalar Green's function satisfying the equation

$$\left(\nabla_t^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)g(\boldsymbol{\rho}, \boldsymbol{\rho}'; t, t') = -\delta(\boldsymbol{\rho} - \boldsymbol{\rho}')\delta(t - t'), \qquad \nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}, \tag{1a}$$

is given by:

$$g(\boldsymbol{\rho}, \boldsymbol{\rho}'; t, t') = \frac{1}{2\pi}\frac{1}{\sqrt{(t - t')^2 - (|\boldsymbol{\rho} - \boldsymbol{\rho}'|^2/c^2)}}\,U\left(t - t' - \frac{|\boldsymbol{\rho} - \boldsymbol{\rho}'|}{c}\right), \tag{1b}$$

where $\boldsymbol{\rho} = (x, y)$ and the Heaviside unit function $U(\alpha)$ equals unity for $\alpha > 0$ and vanishes for $\alpha < 0$. (*Hint:* Note that $\delta[f(z')] = \sum_i \delta(z' - z_i)/|f'(z_i)|$, where z_i are the zeros of $f(z')$ and $f' = df/dz$.)

11. A flat, perfectly conducting obstacle of area S lying in the $z = 0$ plane is illuminated by a prescribed electromagnetic field $\mathbf{E}_i(\mathbf{r}, t)$, $\mathbf{H}_i(\mathbf{r}, t)$. The electric currents $\mathbf{J}(\mathbf{r}, t)$ induced on the obstacle by the incident field give rise to the scattered field $\mathbf{E}_s(\mathbf{r}, t)$, $\mathbf{H}_s(\mathbf{r}, t)$.

(a) Use Eqs. (1.1.19) to represent the scattered field in terms of the induced currents. (*Note:* The currents flow on both sides of the obstacle.) Invoke the boundary condition $\mathbf{n} \times \mathbf{E} = 0$ on S, where $\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s$ is the total electric field and $\mathbf{n}$ is the outward unit normal on S, to derive an integral equation for the induced currents [include the "edge condition"; see Eq. (1.5.37)].

(b) Assume that the induced currents on S are the same as if S were contained in an infinite perfectly conducting plane; i.e., $\mathbf{J} = \mathbf{n} \times \mathbf{H}_i$ on S^+, $\mathbf{J} = 0$ on S^-, where S^+ and S^- denote the "illuminated" and "dark" sides of S facing toward and away from the incident field, respectively, and $\mathbf{n}$ is the outward unit normal on S. Write explicit expressions for $\mathbf{E}_s$ and $\mathbf{H}_s$. This method of approximation is known as the (Kirchhoff) physical optics approximation and is valid when the obstacle is large compared to the wavelengths contained in the temporal spectral decomposition of the incident field.

12. Consider an aperture of area S in a perfectly conducting screen lying in the plane $z = 0$. A known electromagnetic field $\mathbf{E}_i(\mathbf{r}, t)$, $\mathbf{H}_i(\mathbf{r}, t)$ is incident from the half space $z < 0$.

(a) Show that the required continuity of the total tangential electric and magnetic fields in the aperture can be secured on replacing the aperture by a perfect conductor and placing thereon (on each side) an equivalent magnetic current $\mathbf{M} = \mathbf{E} \times \mathbf{n}$, where $\mathbf{n}$ is the outward unit normal. By this device, the problem of determining the overall fields in the complicated region containing the *perforated* plane is reduced to the problem of first finding the induced magnetic currents **M** flowing on an *infinite* perfectly conducting plane. These induced currents give rise to the scattered field $\mathbf{E}_s(\mathbf{r}, t)$, $\mathbf{H}_s(\mathbf{r}, t)$.

(b) Use the half-space dyadic Green's functions of Problem 8 to represent the fields $\mathbf{E}_s$, $\mathbf{H}_s$ radiated by the induced currents **M** into the regions $z < 0$ and $z > 0$. Then invoke the required continuity of the total tangential magnetic field $\mathbf{n} \times (\mathbf{H}_i + \mathbf{H}_s)$ in the aperture to derive an integral equation for **M** (note the "edge condition," Eq. (1.5.37); see also Problem 6 of Chapter 2). $\mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{H}_i$ in the aperture, i.e., the tangential component of the incident magnetic field is not modified by the presence of the aperture.

(c) Assume that the tangential electric field in the aperture is the same as the incident electric field, i.e., $\mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{E}_i$ in S, and give explicit expressions for $\mathbf{E}_s$ and $\mathbf{H}_s$. The resulting "physical optics" approximation is valid under conditions analogous to those stated in Problem 11b.

13. Using the free-space dyadic Green's function representations in Eqs. (1.1.38a) and (1.1.41), derive the response to an electric current excitation $\mathbf{J}(\mathbf{r}, t) = \mathbf{J}\delta(\mathbf{r})e^{-i\omega_0 t}U(t)$, where $U(\alpha) = 1$ for $\alpha > 0$ and $U(\alpha) = 0$ for $\alpha < 0$. Separate the transient contributions from those descriptive of the time-harmonic steady-state ($t \to \infty$). Identify the resulting steady-state dyadic Green's functions $\mathscr{G}_{21}(\mathbf{r}, \mathbf{r}')$ and compare with those listed in Sec. 5.2. Using Eqs. (1.1.38b) and (1.1.39a,b), find the equations satisfied by the steady-state form of the potential function $\mathscr{S}(\mathbf{r}, \mathbf{r}')$, as well as the steady-state form of the expressions for $\nabla_t^2 \mathscr{S}(\mathbf{r}, \mathbf{r}')$ and $(\partial/\partial\rho)\mathscr{S}(\mathbf{r}, \mathbf{r}')$, and compare with results in Sec. 2.4b. Discuss the relation between causality and the steady-state "radiation condition."

14. Calculate the adjoint (time-reversed, inward-radiating) fields $\mathbf{E}^+(\mathbf{r}, t)$ and $\mathbf{H}^+(\mathbf{r}, t)$ radiated by the analogue of the electric dipole current in Problem 13: $\mathbf{J}^+(\mathbf{r}, t) = \mathbf{J}^+ \delta(\mathbf{r})e^{i\omega_0 t}U(-t)$. Separate the steady-state and transient constituents and interpret the behavior of each. What is the steady-state analogue of the "time reversed radiation condition"? Using complex conjugate notation and referring to the results of Problem 13, find a corresponding phrasing of the time-dependent reciprocity conditions (1.1.28) for the steady-state time-harmonic field.

15. Using a field representation in terms of the potential functions Π' and Π'' analogous to that in Eqs. (1.1.42), derive the E- and H-mode constituents of the fields generated by a pulsed magnetic dipole $\mathbf{M}(\mathbf{r}, t) = \mathbf{y}_0\delta(\mathbf{r})\delta(t)$, oriented perpendicularly to the symmetry direction $\mathbf{a} \equiv \mathbf{z}_0$. Deduce the result alternatively by duality considerations ($\mathbf{E} \to \mathbf{H}$, $\mathbf{H} \to -\mathbf{E}$, $\mathbf{J} \to \mathbf{M}$, $\mu \longleftrightarrow \epsilon$) applied to Eqs. (1.1.42)-(1.1.47).

16. Recognizing that an electric current element directed along the symmetry axis $\mathbf{a} \equiv \mathbf{z}_0$ does not generate H-mode fields [$\Pi'' \equiv 0$ in Eq. (1.1.42b)], compute the (E-mode) fields radiated by a so-directed element with an impulsive time dependence. Referring to Eq. (1.1.38a), show that the result does not exhibit the singularity difficulty [as in Eq. (1.1.46a)] across the transverse plane containing the source. If the source currents are spatially distributed in a volume V, does Eq. (1.1.42a) yield a valid field description within V?

17. In a medium which is inhomogeneous but non-dispersive (all wave frequencies propagate at the same speed), ϵ_0 and μ_0 in Eqs. (1.1.16) et seq. are replaced by $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$, respectively. Show that reciprocity as in Eq. (1.1.25) remains valid

and derive relations corresponding to Eqs. (1.1.28) and (1.1.29). Repeat for the case of an anisotropic, inhomogeneous, non-dispersive medium wherein $\epsilon_0 \partial \mathbf{E}/\partial t$ and $\mu_0 \partial \mathbf{H}/\partial t$ are replaced by $\boldsymbol{\epsilon}(\mathbf{r}) \cdot \partial \mathbf{E}/\partial t$ and $\boldsymbol{\mu}(\mathbf{r}) \cdot \partial \mathbf{H}/\partial t$, respectively, with $\boldsymbol{\epsilon}(\mathbf{r})$ and $\boldsymbol{\mu}(\mathbf{r})$ representing permittivity and permeability dyadics.

18. When defining an adjoint electromagnetic problem, the spatial boundaries S of the region V are preserved but the medium properties in V and the boundary conditions on S must be suitably determined. *Show* that the constitutive parameters ϵ_0^+, μ_0^+, and $\mathscr{Z}^+$ of the adjoint problem required to achieve reciprocity as in Eq. (1.1.24) are $\epsilon_0^+ = \epsilon_0$, $\mu_0^+ = \mu_0$, $\mathscr{Z}^+ = -\tilde{\mathscr{Z}}$, where $\tilde{\mathscr{A}}$ denotes the transpose of the dyadic $\mathscr{A}$. Repeat for an anisotropic medium (see Problem 17) and show that this requires an adjoint medium with $\boldsymbol{\epsilon}^+ = \tilde{\boldsymbol{\epsilon}}$, $\boldsymbol{\mu}^+ = \tilde{\boldsymbol{\mu}}$, $\mathscr{Z}^+ = -\tilde{\mathscr{Z}}$.

19. An impulsive source acting at time $t = t'$ is distributed uniformly over the plane $z = z'$ in an infinite, homogeneous, non-dispersive medium characterized by the propagation speed c_0. At time $t_0 > t'$, the medium propagation speed changes abruptly (and uniformly throughout all space) from c_0 to c_1. The scalar Green's function $g(z, z'; t, t')$ descriptive of the space-time dependent field is defined as follows:

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2(t)}\frac{\partial^2}{\partial t^2}\right] g(z, z'; t, t') = -\delta(t - t')\delta(z - z'), \tag{2}$$

with $c(t) = c_0$ for $t < t_0$, $c(t) = c_1$ for $t > t_0$, and subject to the causality condition $g \equiv 0$ for $t < t'$. To ensure continuity of the electric and magnetic flux densities at $t = t_0$,† the boundary condition

$$g, \frac{\partial g}{\partial t} \text{ continuous at } t = t_0 \tag{2a}$$

is imposed.

Show that the solution is given for $t < t_0$ by

$$g = \frac{c_0}{2} U(c_0 T - |Z|), \qquad T = t - t', \qquad Z = z - z', \tag{2b}$$

and for $t > t_0$ by

$$\begin{aligned} g = \frac{c_0}{4}\left(1 + \frac{c_0}{c_1}\right) U(c_1\tau + c_0 T_0 - |Z|) \\ + \frac{c_0}{4}\left(1 - \frac{c_0}{c_1}\right)[U(c_0 T_0 - c_1\tau - |Z|) - U(c_1\tau - c_0 T_0 - |Z|)], \end{aligned} \tag{2c}$$

with $T_0 = t_0 - t'$, $\tau = t - t_0$. The Heaviside unit function $U(x)$ equals unity for $x > 0$ and vanishes for $x < 0$. Explain the various terms in Eqs. (2b) and (2c) as traveling wavefronts and follow their progress in various time intervals. Note that a sudden change in the temporal properties of a medium introduces a reflected as well as a transmitted wave.

20. Repeat the calculation of Problem 19 for an impulsive point source located at $r = 0$, whence the Green's function is defined by:

$$\left[\nabla^2 - \frac{1}{c^2(t)}\frac{\partial^2}{\partial t^2}\right] g(\mathbf{r}; t, t') = \delta(t - t')\delta(\mathbf{r}), \tag{3}$$

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†M. Kline and I. Kay, *Electromagnetic Theory and Geometrical Optics*, New York: Interscience (1965), Sec. 1.3.

subject to $g \equiv 0$ for $t < t'$, and to the continuity condition in Eq. (2a). Show that the result for $t < t_0$ is

$$g = \frac{c_0 \delta(c_0 T - r)}{4\pi r}, \tag{4a}$$

and for $t > t_0$,

$$g = \frac{c_0}{8\pi r}\left\{\left(1 + \frac{c_0}{c_1}\right)\delta(c_0 T_0 + c_1\tau - r) + \left(1 - \frac{c_0}{c_1}\right)[\delta(c_0 T_0 - c_1 \tau - r) - \delta(c_1\tau - c_0 T_0 - r)]\right\}, \tag{4b}$$

where the notation is the same as in Problem 19. Interpret the result in terms of spherical wavefronts and follow their progress through various time intervals. Show that the reflected field constituent in Eq. (4b) remains finite at $r = 0$. Examine the field at $r = 0$ in detail by using the Green's functions in Eqs. (4a) and (4b) to synthesize response to a source function $f(t)$ that vanishes for $t < 0$ and acts throughout a time interval $0 < t < \bar{t}$. Distinguish between the cases $\bar{t} < t_0$ and $\bar{t} > t_0$.

21. An electromagnetic current source is distributed over a plane $z = z'$ in a homogeneous, non-dispersive medium, and has the impulsive temporal behavior $\delta'(t - t')$. At time $t = t_0 > t'$, a cold homogeneous plasma is created suddenly (for example, by subjecting a neutral gas to ionizing radiation). The relevant Green's function problem is as follows:

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_p^2(t)}{c^2}\right]\hat{V}(z, z'; t, t') = -\delta'(t - t')\delta(z - z'), \tag{5}$$

where $\delta'(t) = d\delta(t)/dt$ and

$$\omega_p(t) = bU(t - t_0), \qquad b = \text{constant}. \tag{5a}$$

A causality condition $\hat{V} \equiv 0$ for $t < t'$ and a continuity condition as in Eq. (2a) are imposed. Note that $\hat{V}$ represents the time derivative of the Green's function g for $t \neq t'$.

(a) Show that for $t_0 = t'$, i.e., for radiation into a homogeneous, stationary plasma,

$$\hat{V} = \frac{c^2}{2}\delta(cT - |Z|) - \frac{cbT}{2}\frac{J_1(b\sqrt{T^2 - (Z/c)^2})}{\sqrt{T^2 - (Z/c)^2}}U(cT - |Z|). \tag{6}$$

This is the one-dimensional version of the results in Eqs. (1.1.61).

(b) Show that for $t < t_0$, with $t_0 > t'$,

$$\hat{V} = \frac{c^2}{2}\delta(cT - |Z|), \qquad T = t - t', \qquad Z = z - z', \tag{7a}$$

and for $t > t_0$,

$$\hat{V} = \frac{c^2}{2}\delta(cT - |Z|) - \frac{cb}{4}\left\{\frac{J_1(b\zeta_+^{1/2})}{\zeta_+^{1/2}}\left(\tau - T_0 - \frac{Z}{c}\right)U(\zeta_+) + \frac{J_1(b\zeta_-^{1/2})}{\zeta_-^{1/2}}\left(\tau - T_0 + \frac{Z}{c}\right)U(\zeta_-)\right\}, \qquad T_0 = t_0 - t', \tau = t - t_0, \tag{7b}$$

where $\zeta_\pm = \tau^2 - [(Z \pm cT_0)/c]^2$. Explain the results in physical terms, and compare with those for the non-dispersive media in Problem 19 (see also Problem 38).

22. Repeat Problem 21 for excitation by a point source with impulsive behavior $\delta(t-t')$. For $t_0 = t'$, the solution for all $t > t'$ is given in Eq. (1.1.61b). When $t_0 > t'$, the solution for $t' < t < t_0$ is that in Eq. (4a), with $c_0 = c$. Show that for $t > t_0$, one obtains

$$g = \frac{c\delta(cT - r)}{4\pi r} + \frac{b}{8\pi r}\left\{\frac{J_1(b\zeta_+^{1/2})}{\zeta_+^{1/2}}\left(\tau - T_0 - \frac{r}{c}\right)U(\zeta_+) - \frac{J_1(b\zeta_-^{1/2})}{\zeta_-^{1/2}}\left(\tau - T_0 + \frac{r}{c}\right)U(\zeta_-)\right\}, \tag{8}$$

with $\zeta_\pm$ defined as in Problem 21 provided that Z is replaced by r. Show that for $\tau > T_0$, the Green's function remains finite at $r = 0$ and has the value

$$g = \frac{b^2}{4\pi c}\left[-\frac{J_1(b\sqrt{\tau^2 - T_0^2})}{b\sqrt{\tau^2 - T_0^2}} + \frac{T_0}{T}J_2(b\sqrt{\tau^2 - T_0^2})\right], \qquad r = 0. \tag{9}$$

Interpret the results in physical terms and compare with those for the non-dispersive media in Problem 20.

23. Show that in a linear medium the constitutive relations can be expressed in derivative operator form as

$$\mathbf{D}(\mathbf{r}, t) = \boldsymbol{\epsilon}\left(\nabla, \frac{\partial}{\partial t}; \mathbf{r}, t\right) \cdot \mathbf{E}(\mathbf{r}, t), \qquad \mathbf{B}(\mathbf{r}, t) = \boldsymbol{\mu}\left(\nabla, \frac{\partial}{\partial t}; \mathbf{r}, t\right) \cdot \mathbf{H}(\mathbf{r}, t) \tag{10a}$$

or in integral operator form as

$$\mathbf{D}(\mathbf{r}, t) = \int \boldsymbol{\epsilon}(\mathbf{r}, t; \mathbf{r}', t') \cdot \mathbf{E}(\mathbf{r}', t')\, d\mathbf{r}'\, dt',$$

$$\mathbf{B}(\mathbf{r}, t) = \int \boldsymbol{\mu}(\mathbf{r}, t; \mathbf{r}', t') \cdot \mathbf{H}(\mathbf{r}', t')\, d\mathbf{r}'\, dt' \tag{10b}$$

if the polarization currents $\mathbf{J}$ and $\mathbf{M}$ in the medium are determined by linear differential equations of the form:

$$\mathscr{L}_e\left(\nabla, \frac{\partial}{\partial t}; \mathbf{r}, t\right) \cdot \mathbf{J}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t), \qquad \mathscr{L}_m\left(\nabla, \frac{\partial}{\partial t}; \mathbf{r}, t\right) \cdot \mathbf{M}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r}, t). \tag{11}$$

For homogeneous and stationary media, show that the dyadic operators have the form

$$\boldsymbol{\epsilon}(\mathbf{r}, t; \mathbf{r}', t') = \boldsymbol{\epsilon}(\mathbf{r} - \mathbf{r}'; t - t'), \qquad \boldsymbol{\mu}(\mathbf{r}, t; \mathbf{r}', t') = \boldsymbol{\mu}(\mathbf{r} - \mathbf{r}'; t - t'). \tag{12}$$

24. By rotating coordinates in the four-dimensional $(\mathbf{r}, t)$ space so that the time axis passes through the observation point, show that the asymptotic approximation (for large r) to the integral in Eq. (1.6.1) can be given in the invariant form [alternative to that in Eq. (1.6.5)]:

$$I(\mathbf{r}, t) \sim A(\mathbf{k}_s)e^{i\psi(\mathbf{r}, t; \mathbf{k}_s)}(2\pi)^{3/2}e^{-i(\pi/4)\sigma}(R_1R_2R_3)^{1/2}(r^2 + t^2)^{-3/4}, \tag{13}$$

where $\sigma = \sum_{j=1}^{3} \operatorname{sgn} R_j$ and R_j, $j = 1, 2, 3$ are the principal radii of curvature of the four-dimensional $(\mathbf{k}, \omega)$ dispersion surface at the saddle point $\mathbf{k}_s$. [*Hint:* See Eqs. (1.6.24) and (1.6.25).]

25. The time-dependent, two-dimensional scalar Green's function $g(\boldsymbol{\rho}, \boldsymbol{\rho}'; t, t')$ for an isotropic, homogeneous, cold plasma is defined by:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_p^2}{c^2}\right) g(\boldsymbol{\rho}, \boldsymbol{\rho}'; t, t') = -\delta(\boldsymbol{\rho} - \boldsymbol{\rho}')\delta(t - t'), \qquad \boldsymbol{\rho} = (y, z), \quad (14)$$

subject to the causality condition $g \equiv 0$ for $t < t'$.

(a) Show that for $\boldsymbol{\rho}' = t' = 0$, g can be represented in the following alternative forms [cf. Eqs. (1.5.51)-(1.5.53)]:

$$g = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{e^{i(\eta y + \zeta z - \omega t)}}{\eta^2 + \zeta^2 + (\omega_p^2/c^2) - (\omega^2/c^2)}\, d\eta\, d\zeta\, d\omega \quad (15)$$

$$g = \left(\frac{c}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\omega \int_0^{\infty} dk \frac{kJ_0(kR)e^{-i\omega t}}{c^2k^2 + \omega_p^2 - \omega^2} \quad (16a)$$

$$= \frac{1}{2}\left(\frac{c}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk \frac{kH_0^{(1)}(kR)e^{-i\omega t}}{c^2k^2 + \omega_p^2 - \omega^2}, \qquad R = \sqrt{y^2 + z^2} \quad (16b)$$

$$g = \frac{c^2}{4\pi} \int_{-\infty}^{\infty} \frac{kH_0^{(1)}(kR) \sin \omega_1 t}{\omega_1}\, dk, \qquad \omega_1 = \sqrt{\omega_p^2 + k^2c^2} \quad (17)$$

$$g = \frac{i}{8\pi} \int_{-\infty}^{\infty} H_0^{(1)}(k_1 R)e^{-i\omega t}\, d\omega, \qquad k_1 = \frac{\sqrt{\omega^2 - \omega_p^2}}{c}. \quad (18)$$

The transition from Eq. (15) to Eq. (16a) follows on introduction of polar coordinates in (y, z) and (η, ζ) space, and use of the integral representation $J_0(kR) = (1/2\pi)\int_0^{2\pi} e^{ikR\cos\psi}\, d\psi$. Discuss the disposition of the various integration paths with respect to singularities of the integrands. From a table of Laplace transforms, one may show that g is given in closed form by:

$$g = \frac{\cos[\omega_p\sqrt{t^2 - (R/c)^2}]}{2\pi\sqrt{t^2 - (R/c)^2}}\, U\!\left(t - \frac{R}{c}\right), \quad (19)$$

where $U(x) = 0$ or 1 for $x < 0$, or $x > 0$, respectively.

(b) Substitute into Eqs. (17) and (18) the asymptotic form

$$H_0^{(1)}(w) \sim \sqrt{\frac{2}{\pi w}}\, e^{i[w - (\pi/4)]}, \qquad |w| \gg 1, \quad (20)$$

valid for large R and non-vanishing k or k_1. Decomposing the integrand in Eq. (17) into two parts comprising exponential functions, show that each part gives rise to a single relevant saddle point k_{s1} or k_{s2} as follows:

$$k_{s1,2} = \frac{\pm R\omega_p}{c^2\sqrt{t^2 - (R/c)^2}}, \qquad t > \frac{R}{c}. \quad (21)$$

Show that the sum of the asymptotic approximations of each integral, as obtained from the one-dimensional (single k integral) version of Eq. (1.6.5), yields Eq. (19).

Similarly, proceeding from Eq. (18) with Eq. (20), show that the integrand has two relevant saddle points located at

$$\omega_{s1,2} = \pm\frac{\omega_p}{\sqrt{1 - (R/ct)^2}}, \qquad t > \frac{R}{c}. \quad (22)$$

Use Eq. (1.6.43) for each saddle point to show that the asymptotic approximation of Eq. (18) also yields the result in Eq. (19). Discuss the restrictions on the asymptotic results although they happen to agree with the exact solution.
(c) Use the procedure leading to Eq. (1.6.49), in conjunction with Eqs. (18) and (20), to find the transition function linking the asymptotic approximations with the wavefront regime. Compare with the exact solution in Eq. (19).

26. A line source along the z axis is embedded in a cold, isotropic, homogeneous plasma as in Problem 25. The time dependence of the source is that of a suddenly switched-on harmonic signal

$$f(t) = e^{-i\omega_0 t} U(t), \tag{23a}$$

whence its Fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt = \frac{i}{\omega - \omega_0}. \tag{23b}$$

(a) Show that the response $\bar{g}$ to this excitation is given by Eq. (18) provided that the integrand is multiplied by $F(\omega)$ (discuss the disposition of the integration path). Perform an asymptotic evaluation to show that $\bar{g}$ behaves for $t > R/c$ as follows:

$$\bar{g} \sim \frac{i}{4\pi\tau}\left[\frac{e^{-i\omega_p\tau}}{(\omega_p t/\tau) - \omega_0} - \frac{e^{i\omega_p\tau}}{(\omega_p t/\tau) + \omega_0}\right] + \frac{i}{4} H_0^{(1)}\left(\frac{R}{c}\sqrt{\omega_0^2 - \omega_p^2}\right) e^{-i\omega_0 t} U(t - t_0), \tag{24}$$

where

$$\tau = \sqrt{t^2 - \left(\frac{R}{c}\right)^2}, \qquad t_0 = \frac{R}{c}\,\omega_0(\omega_0^2 - \omega_p^2)^{-1/2} = \frac{R}{v_{g0}},$$

and with $v_{g0} = [dk_1(\omega_0)/d\omega_0]^{-1}$ denoting the group speed corresponding to the signal frequency ω_0. Interpret this result by referring to Fig. 1.6.12. Note that as $t \to \infty$ there remains only the second term in Eq. (24), the *exact* time-harmonic response [see Eq. (5.4.25)]. Discuss the conditions under which Eq. (24) is valid and note the different behavior of the solution for $\omega_0 > \omega_p$ and $\omega_0 < \omega_p$.
(b) From $(\operatorname{Re}\bar{g})$ and $(\operatorname{Im}\bar{g})$, obtain the response to a suddenly switched-on signal $f_1 = U(t)\cos\omega_0 t$ and $f_2 = U(t)\sin\omega_0 t$, respectively. Referring to Eq. (1.6.49), find the transition functions for $(\operatorname{Re}\bar{g})$ and $(\operatorname{Im}\bar{g})$ near the wavefront $t = R/c$. Discuss the difference in behavior of the two solutions and relate it to the initial time dependence of the source functions f_1 and f_2. Compare with the transition function for the $\delta(t)$ excitation in Problem 25.

27. Consider the one-dimensional form of the wave equation (1.7.15), with $\nabla^2 \to \partial^2/\partial z^2$ and $n(\mathbf{r}) = n(z)$. Assume an asymptotic solution of the form (1.7.16), with $\mathbf{r} \to z$. Show that the second term in the asymptotic expansion is given by [see also Problem 40(a)]:

$$u_1(z) = \frac{-1}{2\sqrt{n(z)}}\int^z \frac{d^2u_0(\zeta)/d\zeta^2}{\sqrt{n(\zeta)}}\, d\zeta, \tag{25}$$

with $u_0(z) = A/\sqrt{n(z)}$, $A =$ constant. To justify use of only the first term in the asymptotic expansion (1.7.16), it is necessary that $|u_1| \ll |k_0 u_0|$. Show from Eq.

(25) that this requirement is equivalent to condition (1.7.21), viz., $|dn/dz| \ll k_0 n^2$.

28. A source distributed over the entire plane $z = z'$ in vacuum is comprised of electric currents flowing parallel to the y direction, uniformly along y but with a linearly progressing phase $\exp(ik_0\alpha x)$ along x, where $k_0 = \omega/c$, c is the speed of light in vacuum, and α is real. A time dependence $\exp(-i\omega t)$ is implied.

Show that the electromagnetic field has non-vanishing components E_y, H_x, H_z; that H_x and H_z are derivable from E_y; and that E_y is proportional to the Green's function $g(x, z\,; z')$ defined by:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_0^2\right) g(x, z\,; z') = -\delta(z - z')e^{ik_0\alpha x}, \tag{26}$$

subject to an outward radiation condition at $|z| \to \infty$. Show that the solution is

$$g(x, z\,; z') = \frac{1}{-2ik_0\sqrt{1-\alpha^2}} \exp\left[ik_0(\alpha x + \sqrt{1-\alpha^2}\,|z - z'|)\right]. \tag{27}$$

Discuss the properties of the radiated field for $|\alpha| < 1$ and $|\alpha| > 1$.

Assuming knowledge of the field at $z = z'$, i.e.,

$$g(x, z'; z') \equiv u(x, z') = \frac{e^{ik_0\alpha x}}{-2ik_0\sqrt{1-\alpha^2}}, \tag{28}$$

derive the solution in Eq. (27) by the ray-optical procedure that leads to Eq. (1.7.40). Why is the ray-optical result exact in this case? Show that the ray configuration for $0 < \alpha < 1$ is as shown in Fig. P1.1.

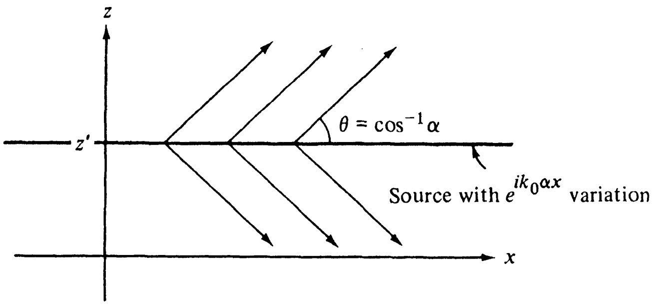


FIG. P1.1 Ray configuration for phased plane sheet source.

29. In vacuum a y-directed electric current source is distributed over a cylindrical surface $\bar{\rho} = a$, uniformly along y and with an azimuthal phase progression $\exp(im\phi)$, where $\bar{\rho}$ and ϕ are polar coordinates in the xz plane and m is a positive integer.

(a) Write $\exp(im\phi) = \exp(ik_0\alpha s)$, where $s = a\phi$ measures linear distance along the source distribution and the wavenumber $k_0\alpha = (m/a)$ describes the source phase. For $k_0 a \gg 1$, a source element at A may be assumed to radiate as though it were contained in an infinite planar source distribution tangent to the cylinder at A. Show that for $\alpha = (m/k_0 a) < 1$, the geometric-optical ray family appears as in Fig. P1.2, with a caustic formed at $\bar{\rho} = b = a\cos\theta = \alpha a$ (see also Fig.

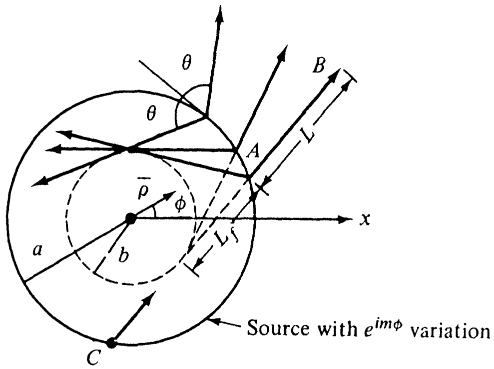


FIG. P1.2 Ray configuration for phased cylindrical sheet source.

5.4.14). Show also that the ray tube cross section ratio $\Delta s_B/\Delta s_A$ at points B and A along a ray is given for $\bar{\rho} > a$ by:

$$\frac{\Delta s_B}{\Delta s_A} = \frac{L_f + L}{L_f}, \qquad L = \sqrt{\bar{\rho}^2 - b^2} - \sqrt{a^2 - b^2}, \qquad L_f = \sqrt{a^2 - b^2}. \tag{29}$$

(b) Use in Eq. (1.7.40) the initial field value from Eq. (28) and the cross section ratio in Eq. (29) to construct the geometric optical field at any point $(\bar{\rho}, \phi)$, $\bar{\rho} > a$, as follows:

$$u_1(\bar{\rho}, \phi) \sim M \exp\,[im(\phi - \gamma + \theta) + ik_0(\bar{\rho}\sin\gamma - a\sin\theta)] \left(\frac{a^2 - b^2}{\bar{\rho}^2 - b^2}\right)^{1/4}, \tag{30}$$

where $\gamma = \cos^{-1}(b/\bar{\rho})$ and M is a constant. The contribution in Eq. (30) is due to a ray reaching an observation point B from a source point A. The ray originating at C and passing through the interior of the source also contributes at B. Calculate the field due to the ray from C, noting that a phase change $\exp(-i\pi/2)$ must be included due to the contact of this ray with the caustic [see Eq. (5.8.55) for the more general problem of a caustic in an inhomogeneous medium].

(c) In the region $b < \bar{\rho} < a$, each observation point is also reached by two rays. Using the ray optical formula (1.7.40), show that the resulting field is given by:

$$u(\bar{\rho}, \phi) \sim 2M \exp\left[im(\phi - \theta) + ik_0 a \sin\theta - \frac{i\pi}{4}\right]$$
$$\times \left(\frac{a^2 - b^2}{\bar{\rho}^2 - b^2}\right)^{1/4} \cos\left(k_0\bar{\rho}\sin\gamma - m\gamma - \frac{\pi}{4}\right). \tag{31}$$

30. A line source at Q is parallel to a curved interface separating two different homogeneous isotropic media; a time dependence $\exp(-i\omega t)$ is implied.

(a) Referring to Fig. P1.3 for construction of the reflected ray-tube configuration, show that since $\theta_r = \theta_i$ and $\alpha = \beta$,

$$f = r\cos\theta_r \frac{\Delta\varphi}{\Delta\psi}, \qquad \Delta\psi = 2\Delta\varphi + \Delta\theta_0, \qquad \frac{\Delta\theta_0}{\Delta\varphi} = \frac{r\cos\theta_i}{l_1}, \tag{32a}$$

$$\frac{f}{f + l} = \frac{r l_1 \cos\theta_i}{2 l l_1 + (l + l_1) r \cos\theta_i}, \tag{32b}$$

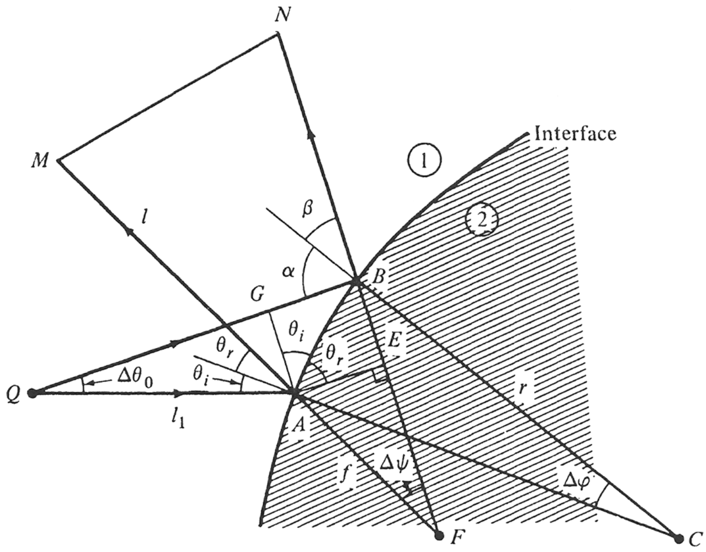


FIG. P1.3 Reflection from a curved boundary.

where r is the radius of curvature of the surface at the point of impact A of the incident ray, M is the observation point, and F is the virtual focus of the reflected ray tube. If the incident field at A is taken as $l_1^{-1/2}\exp(ik_1 l_1)$, show that the reflected field at M is given by:

$$u_{\text{refl}} \sim \Gamma \frac{e^{ik_1(l_1+l)}}{\sqrt{l_1+l}} D, \qquad D = \left[\frac{(l_1+l)r\cos\theta_i}{2l_1 l + (l_1+l)r\cos\theta_i}\right]^{1/2}, \tag{33}$$

where Γ is the reflection coefficient appropriate to the incidence angle θ_i and k_1 is the wavenumber in medium 1. The geometrical divergence coefficient D incorporates the effect of interface curvature; for a plane interface, one has $r = \infty$ and $D = 1$, whence Eq. (33) reduces to Eq. (1.7.64b). For a circular cylinder, $r =$ constant. For calculation of fields in the far zone, one lets $l \to \infty$. By letting $l_1 \to \infty$ and renormalizing the source strength so that the incident field is the plane-wave $\exp(ik_0 l_1)$, one obtains the reflected field

$$\bar{u}_{\text{refl}} \sim \Gamma e^{ik_1(l_1+l)} \sqrt{\frac{r\cos\theta_i}{2l + r\cos\theta_i}}. \tag{34}$$

(b) Referring to Fig. P1.4 for construction of the refracted ray-tube configuration, show that

$$f' = \frac{r\cos\theta_R}{\Delta\tau/\Delta\varphi} = \frac{rl_1\cos\theta_R}{(m-1)l_1 + mr\cos\theta_i}, \tag{35}$$

where $m = d\theta_R/d\theta_i = (n_1\cos\theta_i)(n_2\cos\theta_R)^{-1}$, and Snell's law $n_2\sin\theta_R = n_1\sin\theta_i$ has been used. With $k_{1,2} = k_0 n_{1,2}$, show that for an incident field as in (a), the transmitted field at M' is given by [cf. Eqs. (5.5.9)]:

$$E_{\text{trans}} \sim T \frac{e^{i(k_1 l_1 + k_2 l_2)}}{[(ml_2\cos\theta_i + l_1\cos\theta_R)/\cos\theta_R]^{1/2}} D', \tag{36}$$

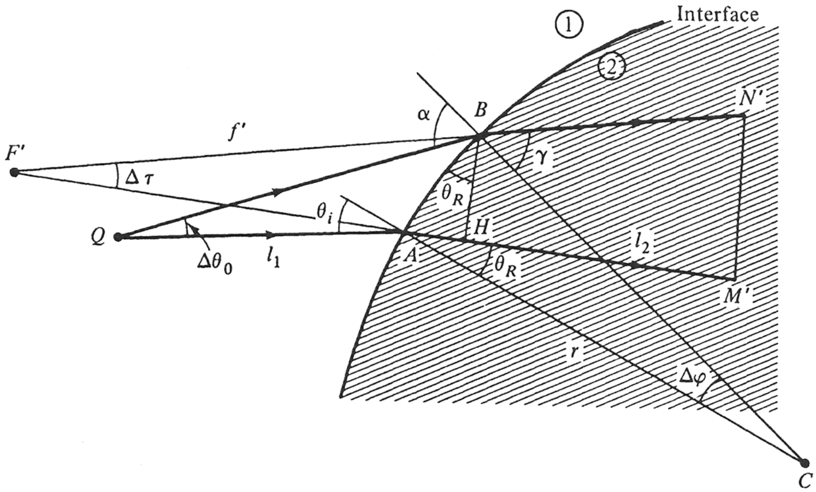


FIG. P1.4 Refraction at a curved boundary.

where T is the plane-wave transmission coefficient. The geometrical divergence coefficient D' incorporates the effect of interface curvature,

$$D' = \left[\frac{(ml_2 \cos\theta_i + l_1 \cos\theta_R)r}{(m-1)l_1 l_2 + (ml_2 \cos\theta_i + l_1 \cos\theta_R)r}\right]^{1/2}, \tag{36a}$$

whence $D' = 1$ for a plane interface. For an incident plane-wave $\exp(ik_0 l_1)$,

$$\bar{u}_{\text{trans}} \sim T e^{i(k_1 l_1 + k_2 l_2)} \sqrt{\frac{r \cos\theta_R}{(m-1)l_2 + r\cos\theta_R}}. \tag{37}$$

31. A slowly curved interface B separates two isotropic homogeneous or weakly inhomogeneous media 1 and 2 that support, respectively, M and N wave species. Sufficiently close to a point P on the interface, the media may be regarded as locally homogeneous, and the various wave species u_j satisfy there the time-harmonic scalar wave equation $(\nabla^2 + k_j^2)u_j = 0$, $k_j = kn_j$, with k representing a reference wavenumber and n_j the refractive index. Assume all field and geometrical quantities to be independent of the rectilinear coordinate z so that the problem is two-dimensional in the variable $\boldsymbol{\rho} = (x, y)$. Assume that wave type 1 in region 1 is incident on B at P, and that this incident wave u_i has the high-frequency asymptotic approximation

$$u_i(\boldsymbol{\rho}) \sim A_i(\boldsymbol{\rho}) e^{ik_1\psi_i'(\boldsymbol{\rho})}\left[1 + O\left(\frac{1}{k}\right)\right], \tag{38a}$$

with an implied $\exp(-i\omega t)$ dependence. The interface couples u_i to M reflected waves in region 1,

$$u_r(\boldsymbol{\rho}) \sim \sum_{m=1}^{M} B_{rm}(\boldsymbol{\rho}) e^{ik_m\psi_{rm}'(\boldsymbol{\rho})}\left[1 + O\left(\frac{1}{k}\right)\right], \tag{38b}$$

and to N transmitted waves in region 2,

$$u_t(\boldsymbol{\rho}) \sim \sum_{n=1}^{N} D_{tn}(\boldsymbol{\rho}) e^{ik_n\psi_{tn}'(\boldsymbol{\rho})}\left[1 + O\left(\frac{1}{k}\right)\right]. \tag{38c}$$

Each of the phases satisfies near P the eiconal equation (1.7.18a), $(\nabla\psi'_j)^2 = 1$, and the amplitudes satisfy the transport equation (1.7.18b). Note that $n_j\psi'_j$ corresponds to ψ_j in Eq. (1.7.57).

(a) Use the interface matching condition for the tangential wavevector components in Eq. (1.7.59) and the eiconal equation to show that the normal components of the various wavevectors satisfy the relation:

$$k_i\frac{\partial\psi'_i}{\partial\nu} = -k_m\left[\left(\frac{\partial\psi'_{rm}}{\partial\nu}\right)^2 + \frac{k_i^2}{k_m^2} - 1\right]^{1/2} = k_n\left[\left(\frac{\partial\psi'_{tn}}{\partial\nu}\right)^2 + \frac{k_i^2}{k_n^2} - 1\right]^{1/2}, \tag{39}$$

where ν is the normal to the boundary and $m = 1 \cdots M$, $n = 1 \cdots N$.

(b) Assume that region 1 is a warm plasma that can support two wave types (electromagnetic and electron-acoustic), and that B is a perfect conductor, thereby eliminating region 2 (see the plasma portion of Fig. 1.7.5). The relevant scalar wave equations in region 1 are

$$(\nabla^2 + k_1^2)H(\boldsymbol{\rho}) = 0, \qquad k_1 = kn_p, \qquad k = k_0 \tag{40a}$$

$$(\nabla^2 + k_2^2)p(\boldsymbol{\rho}) = 0, \qquad k_2 = k_a n_p \tag{40b}$$

where H is the y component of magnetic field and p the acoustic pressure deviation from the mean. The following boundary conditions are assumed to be satisfied on B [expressive of the vanishing of the tangential component of electric field, and of an "impedance" condition for the normal component of the velocity; cf. Eq. (1.1.1b)]:

$$\begin{gathered} 0 = q\frac{\partial p}{\partial s} - \frac{1}{i\omega\epsilon_0 n_p^2}\frac{\partial H_1}{\partial\nu}, \\ p\beta = \frac{1}{i\omega m N_0 n_p^2}\frac{\partial p}{\partial\nu} - q\frac{\partial H_1}{\partial s}, \qquad n_p^2 = 1 - \left(\frac{N_0 e^2}{\epsilon_0 m\omega^2}\right), \end{gathered} \tag{41}$$

where q, m, N_0 and β are real parameters and s is the coordinate tangential to B. Derive asymptotic forms of the boundary conditions in Eq. (41) by assuming that H and p can be represented as in Eqs. (38a) and (38b), with $M = 2$, and retaining only dominant terms. Show that if an acoustic ray is incident ($m = 2$), the reflected acoustic ray has an initial amplitude $\Gamma_{22}(\boldsymbol{\rho})$ given by

$$\begin{aligned} \Gamma_{22}(\boldsymbol{\rho}) &= \frac{B_{r2}(\boldsymbol{\rho})}{A_{i2}(\boldsymbol{\rho})} \\ &= \frac{q[k_1(\partial\psi'_1/\partial\nu)k_2(\partial\psi'_2/\partial\nu) - [k_2(\partial\psi'_2/\partial s)]^2(1-n_p^2)] + \beta k_1(\partial\psi'_1/\partial\nu)(N_0e/\omega\epsilon_0)}{q[k_1(\partial\psi'_1/\partial\nu)k_2(\partial\psi'_2/\partial\nu) + [k_2(\partial\psi'_2/\partial s)]^2(1-n_p^2)] - \beta k_1(\partial\psi'_1/\partial\nu)(N_0e/\omega\epsilon_0)}. \end{aligned} \tag{42}$$

Derive analogous expressions for the coupling coefficient $\Gamma_{21} = B_{r1}/A_{i2}$ to the reflected magnetic field. Repeat for an incident electromagnetic field and derive the corresponding coefficients Γ_{11} and Γ_{12}.

(c) Assuming that the boundary is plane and the plasma homogeneous, solve Eqs. (40) and (41) for the exact reflection coefficients when a plane acoustic or electromagnetic wave is incident. Compare the result with Eq. (42), etc.

32. Assume that excitation for the warm plasma in (b) of Problem 31 is provided by a line source and that the medium is homogeneous throughout. Tracking an incident acoustic ray, one finds that the relation between angles of incidence

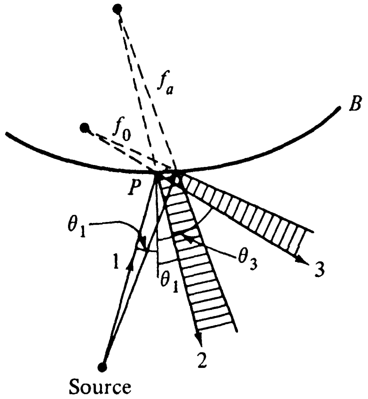


FIG. P1.5 Incident acoustic ray (1) and reflected acoustic (2) and electromagnetic (3) rays.

and reflection is given by Eq. (1.7.61) (see also Fig. 1.7.5). Use these reflection laws and the known radius of curvature r_c of the boundary at P to determine the focal lengths f_a and f_0 for the reflected acoustic and electromagnetic ray tubes, respectively (see Fig. P1.5). If L_2 and L_3 are the distances from P to a point on the reflected acoustic and electromagnetic rays, respectively, show that the ratio of the ray-tube cross sections at P and at the observation point is given for the acoustic ray by (see Problem 30)

$$\frac{f_a}{f_a + L_2} = \frac{r_c L_1 \cos\theta_1}{2L_1L_2 + (L_1 + L_2)r_c \cos\theta_1}, \tag{43a}$$

and for the electromagnetic ray by

$$\frac{f_0}{f_0 + L_3} = \frac{r_c L_1 \cos^2\theta_3}{L_1L_3(\cos\theta_3 + n_a\cos\theta_1) + r_c(L_1\cos^2\theta_3 + n_a L_3 \cos^2\theta_1)}, \tag{43b}$$

where L_1 is the distance from the source to P, and $n_a = k_a/k_0$. Repeat the calculation for an incident electromagnetic ray.

Use the results in (b) of Problem 31 together with the above to construct asymptotic approximations of the fields radiated by a line source in a homogeneous warm plasma bounded by a smoothly curved conducting surface. Assume that the line source generates known initial amplitudes of acoustic and electromagnetic fields.

33. In a gyrotropic medium, with a steady external magnetic field applied along the z axis, the surface described by the wavevector $\mathbf{k} = \mathbf{k}(\bar{\theta}, \omega)$ is rotationally symmetric and depends on the frequency ω and the polar angle $\bar{\theta}$ between $\mathbf{k}$ and the z axis. Instead of this k surface, it is often convenient to use the refractive index surface, defined in terms of k as follows:

$$n(\bar{\theta}, \omega) = \frac{c}{\omega} k(\bar{\theta}, \omega). \tag{44}$$

By implicit differentiation of the function $f = kc - \omega n = 0$, show that the components of the group velocity vector $\mathbf{v}_g = \nabla_k \omega$ are given by (rotate coordinates so that $k_y = 0$):

$$v_{gx} = c\frac{\sin\bar{\theta} - (\cos\bar{\theta}/n)(\partial n/\partial\bar{\theta})}{\partial(n\omega)/\partial\omega} \tag{45a}$$

$$v_{gz} = c\frac{\cos\bar{\theta} + (\sin\bar{\theta}/n)(\partial n/\partial\bar{\theta})}{\partial(n\omega)/\partial\omega} \tag{45b}$$

[Note that $\partial k/\partial k_x = \sin\bar{\theta}$, $\partial n/\partial k_x|_{\omega=\text{const.}} = k^{-1}\cos\bar{\theta}(\partial n/\partial\bar{\theta})$, etc.] Show also that

$$v_g = \sqrt{v_{gx}^2 + v_{gz}^2} = \frac{c}{\cos\alpha\,\partial(n\omega)/\partial\omega}, \tag{46}$$

where

$$\alpha = \tan^{-1}\left(\frac{1}{n}\frac{\partial n}{\partial\bar{\theta}}\right) \tag{46a}$$

and that the components of $\mathbf{v}_g$ along and perpendicular to $\mathbf{k}$ are:

$$v_{gk} = \frac{c}{\partial(n\omega)/\partial\omega} = v_g\cos\alpha, \qquad v_{g\bar{\theta}} = -\frac{c\tan\alpha}{\partial(n\omega)/\partial\omega}. \tag{47}$$

Thus, α in Eq. (46a) is the angle between $\mathbf{k}$ and $\mathbf{v}_g$. Show by a graphical construction utilizing Eq. (46a) and the refractive index surface that the vector $\mathbf{v}_g$ is normal to the surface.

34. A source distributed uniformly over the plane $z = 0$ and having the impulsive behavior $\delta(t)$ is embedded in a cold, isotropic, homogeneous plasma described by the constant plasma frequency ω_p. The fields are derivable from a scalar Green's function g that satisfies the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_p^2}{c^2}\right)g(z,t) = -\delta(z)\delta(t) \tag{48}$$

subject to the causality requirement $g \equiv 0$ for $t < 0$. Show that the solution for g is given by [cf. the time-integrated form of Eq. (6)]:

$$g = \frac{c}{2}J_0\left(\omega_p\sqrt{t^2 - \left(\frac{z}{c}\right)^2}\right)U\left(t - \frac{|z|}{c}\right) \tag{49}$$

(a) The source in Eq. (48) is localizde at the space-time point $(z, t) = (0, 0)$ so that the space-time rays descriptive of the radiation process converge at the origin. Use the space-time ray method of Secs. 1.7a and 1.7e to show that the solution of the ray equation is [cf. notation in Eq. (1.7.67)],

$$Z = v_g(\bar{\omega})\tau, \qquad v_g = \frac{\bar{k}c^2}{\bar{\omega}}, \qquad \bar{\omega} = \pm\sqrt{\bar{k}c^2 + \bar{\omega}_p^2}, \tag{50a}$$

and that the field amplitude along a ray varies like $\tau^{-1/2}$ [cf. Eq. (1.7.78b)]. Show also from Eq. (50a) that the ray parameter $\bar{\omega}$ can be expressed explicitly in the form

$$\bar{\omega} = \frac{\pm\bar{\omega}_p}{\sqrt{1 - (Z/c\tau)^2}}, \qquad \text{whence } \bar{k}c = \frac{\bar{\omega}_p}{\sqrt{(c\tau/Z)^2 - 1}}, \tag{50b}$$

and that the phase at (Z, τ) on a ray correiponding to $\bar{\omega} \gtrless 0$ is given via Eqs. (1.7.75) and (50b) by:

$$\psi_\pm(Z,\tau) - \psi_\pm(0,0) = \mp\omega_p\tau\sqrt{1 - (Z/c\tau)^2}, \qquad \bar{\omega} \gtrless 0. \tag{50c}$$

Obtain the field by superposition of the ray contributions corresponding to $\bar{\omega} \gtrless 0$ as [$m = 0$ term in Eq. (1.7.68)]:

$$u(Z, \tau) \sim \frac{|A(\bar{\omega})|}{\sqrt{\tau}} \cos\left[\nu\bar{\omega}_p\sqrt{\tau^2 - \left(\frac{Z}{c}\right)^2} - \nu\psi_+(0, 0) + \alpha\right], \tag{50d}$$

where $A(\bar{\omega}) = |A(\bar{\omega})| \exp(i\alpha)$ is a quantity descriptive of the initial field on a ray, and it has been recognized that $\psi_-(0, 0) = -\psi_+(0, 0)$ in order to render $u(Z, \tau)$ real.

(b) To determine the initial values A and $\psi_+(0, 0)$ along a ray, it is necessary to solve the space-time source problem in the vicinity of the source region. Since this cannot be done by the ray method (note that $u \to \infty$ as $\tau \to 0$ along a ray), the initial values must be found from an exact solution valid near the source point, as provided by Eq. (49). For sufficiently large $(ct - z)$ values, the Bessel function in Eq. (49) may be replaced by its large argument asymptotic form, and the quantities A and $\psi_+(0, 0)$ in Eq. (50d) are then determined by the requirement that $u(Z, \tau)$ in Eq. (50d) agree with the asymptotic form of Eq. (49) when Z and τ approach the source region. Perform this evaluation.

Comment: It may be noted that since Eq. (49) provides the exact solution at all space-time points, the ray method offers no advantage for the problem of radiation into a homogeneous plasma. However, if the medium is inhomogeneous with $\omega_p = \omega_p(z)$, closed-form solutions can generally not be obtained. The present results then provide the initial field values in the vicinity of $z = 0$, and the ray method can be employed to yield asymptotic solutions at other space-time points (see Problem 35). The ray solution becomes invalid near the wavefronts $|z| = ct$ where a transition function as in Eq. (1.6.49) must be utilized. For the present problem, Eq. (49) yields the transition function if $\sqrt{t^2 - (z/c)^2}$ is replaced by $\sqrt{2\xi(t - \xi)}$, $\xi = |z|/c$. In view of the dispersion equation $ck = (\omega^2 - \omega_p^2)^{1/2}$, one identifies ω_α^2 in Eq. (1.6.48) as $\omega_\alpha^2 = \omega_p^2/2$, and $\nu = -1$ in Eq. (1.6.49).

35. Assume that the source configuration in Problem 34 is embedded in an inhomogeneous plasma described by a plasma frequency $\omega_p = \omega_p(z)$. In the following, we employ the notation of Sec. 1.7e.

(a) Show that the space-time rays (up to the turning point, if any) are described by the curves

$$c\tau = \int_0^Z \frac{\bar{\omega}}{\sqrt{\bar{\omega}^2 - \bar{\omega}_p^2(\zeta)}}\, d\zeta, \qquad \bar{\omega} = \text{constant on a ray.} \tag{51a}$$

(b) Show that for a fixed increment $d\bar{\omega}$ defining two neighboring rays, the ray-tube cross section dZ in planes $\tau =$ constant is given by:

$$\frac{dZ}{d\bar{\omega}} = \frac{\sqrt{\bar{\omega}^2 - \bar{\omega}_p^2(Z)}}{\bar{\omega}} \int_0^Z \frac{\bar{\omega}_p^2(\zeta)\, d\zeta}{[\bar{\omega}^2 - \bar{\omega}_p^2(\zeta)]^{3/2}}. \tag{51b}$$

[Note that on writing Eq. (51a) as $F(\tau, Z, \bar{\omega}) = 0$, one has $dZ/d\bar{\omega} = -(\partial F/\partial\bar{\omega})(\partial F/\partial Z)^{-1}$.]

(c) Construct the asymptotic solution given by the $m = 0$ term in Eq. (1.7.68) on use of Eq. (51b) and Eqs. (1.7.75) and (1.7.78b). Obtain the initial field value on a ray by letting the observation point approach the source region and requiring that the asymptotic solution agrees with that for a homogeneous medium having a plasma frequency $\omega_p(0)$; the latter result has been found in Problem 34 (note that in this limiting process, Eq. (51a) becomes $c\tau \to Z\bar{\omega}[\bar{\omega}^2 - \bar{\omega}_p^2(0)]^{-1/2}$; other integrals are reduced in a similar manner). Show that for (Z, τ) values on ray segments between the source and the turning point (if any), the asymptotic form of the Green's function in Eq. (48) with $\omega_p = \omega_p(z)$ is then given by:†

$$g(Z, \tau) \sim \frac{c^{3/2}}{\sqrt{2\pi\nu}} \frac{\cos [\nu/c \int_0^Z \sqrt{\bar{\omega}^2 - \bar{\omega}_p^2(\zeta)}\, d\zeta - \nu\bar{\omega}\tau + (\pi/4)]}{[\bar{\omega}^2 - \bar{\omega}_p^2(Z)]^{1/4}[\bar{\omega}^2 - \bar{\omega}_p^2(0)]^{1/4}\{\int_0^Z (\bar{\omega}_p^2(\zeta)\, d\zeta/[\bar{\omega}^2 - \bar{\omega}_p^2(\zeta)]^{3/2})\}^{1/2}} \tag{51c}$$

The turning point is defined by $d\tau/dZ = \infty$ in Eq. (51a). The ray parameter $\bar{\omega}$ depends on (Z, τ) implicitly through Eq. (51a). When Eq. (51a) can be solved for $\bar{\omega} = \bar{\omega}(Z, \tau)$, one may eliminate $\bar{\omega}$ from Eq. (51c).

(d) If a ray has a turning point at (Z_t, τ_t) (see Fig. 1.6.9), show that beyond the turning point, the ray is defined by the equation:

$$c\tau = \int_0^{Z_t} \frac{\bar{\omega}\, d\zeta}{\sqrt{\bar{\omega}^2 - \bar{\omega}_p^2(\zeta)}} + \int_Z^{Z_t} \frac{\bar{\omega}\, d\zeta}{\sqrt{\bar{\omega}^2 - \bar{\omega}_p^2(\zeta)}}. \tag{52}$$

Calculate the field [note the analogous calculation leading to Eq. (5.8.55)].

36. Consider the scalar wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2} - \nu^2 \frac{\bar{\omega}_p^2(\tau)}{c^2}\right] u(\mathbf{R}, \tau) = 0, \tag{53}$$

descriptive of electromagnetic wave processes in a cold, isotropic plasma whose properties are constant in space but vary (sufficiently slowly) with time.

(a) Assuming a solution of the form given in Eq. (1.7.68), show that the disper-

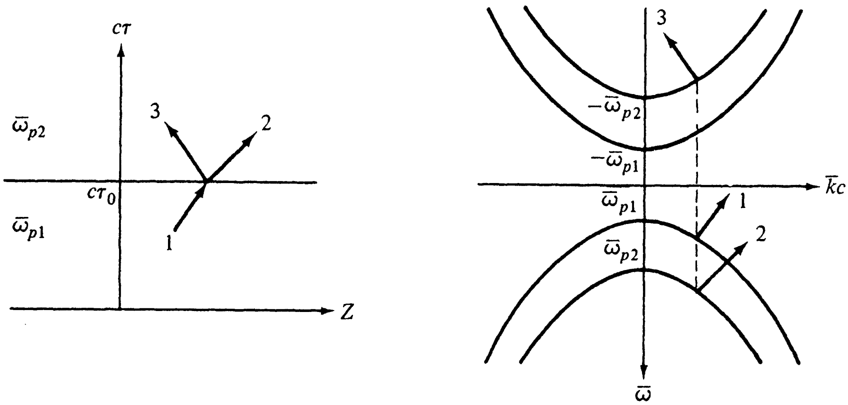


FIG. P1.6 Space-time ray refraction at a time discontinuity in the medium properties.

†N. Bleistein and R. M. Lewis, "Space-Time Diffraction for Dispersive Hyperbolic Equations," *SIAM J. of Appl. Math.*, Vol. 14, No. 6, November 1966, pp. 1454-1470.

sion equation and the transport equation for $u_0(\mathbf{R}, \tau)$ are given, respectively, by Eqs. (1.7.70) and (1.7.71), with $\bar{\omega}_p(\mathbf{R})$ replaced by $\bar{\omega}_p(\tau)$. Show also that Eq. (1.7.77) remains valid, but that Eq. (1.7.78b) must be modified since $\bar{\omega} \neq$ constant (whereas $\bar{\mathbf{k}} =$ constant) on a ray [see Eqs. (1.7.10) et seq.]. Discuss the applicability (or non-applicability) of energy conservation in a ray tube for time-varying media.

(b) If the medium properties change abruptly (and homogeneously throughout all space) at time $\tau = \tau_0$, show that instead of Eq. (1.7.79), the phase-matching condition requires

$$\bar{\mathbf{k}}_1 = \bar{\mathbf{k}}_2 = \cdots = \bar{\mathbf{k}}_m \qquad \text{at } \tau = \tau_0 \tag{54}$$

where m denotes the number of wave constituents. This matching condition can be schematized as in Fig. P1.6, with $\bar{\omega}_{p1}$ and $\bar{\omega}_{p2}$ denoting the values of plasma frequency before and after τ_0. Compare analogies and differences between the configuration in Fig. P1.6 and the spatial discontinuity in Fig. 1.7.11. If the medium changes continuously with time, find a graphical construction for the ray trajectories analogous to that in Fig. 1.6.9.

37. Plane-wave propagation in media with temporal inhomogeneity may be characterized by the dispersion equation $\omega = \omega(\mathbf{k}, t)$.

(a) Show that since the eigenfunctions $\exp(i\mathbf{k} \cdot \mathbf{r})$ account properly for the spatial dependence of the fields, the oscillatory representation in Eqs. (1.3.2b) provides the proper basis for synthesis of source-excited fields in such media. Show also that when such oscillatory representations are substituted into the field equations, the reduced equation for the time-dependent modal amplitudes takes the form of Eq. (1.3.13), with the operator W being time-dependent. This equation cannot generally be solved in closed form. However, show that for weak inhomogeneities, one may employ WKB procedures (see Sec. 3.5c) to effect an asymptotic solution of the time-dependent modal amplitudes, and that this leads to consideration of integrals of the form:

$$I(\mathbf{r}, t) \sim \int A(\mathbf{k}, t) e^{i\psi(\mathbf{r},t;\mathbf{k})}\, d\mathbf{k}, \tag{55}$$

where

$$\psi(\mathbf{r}, t; \mathbf{k}) = \mathbf{k} \cdot \mathbf{r} \pm \int_{t'}^{t} \omega(\mathbf{k}, \eta)\, d\eta \tag{55a}$$

with $A(\mathbf{k}, t)$ and $\omega(\mathbf{k}, t)$ representing weakly time-dependent functions.

(b) Use the asymptotic procedure described in Sec. 1.6a to show that

$$I(\mathbf{r}, t) \sim A(\mathbf{k}_s, t) e^{i\psi(\mathbf{r},t;\mathbf{k}_s)} \frac{(2\pi)^{3/2} e^{\pm i(\pi/4)\sigma}}{|\det Q(\mathbf{k}_s, t)|^{1/2}}, \tag{56}$$

where the saddle points $\mathbf{k}_s$ are defined by

$$\frac{d\mathbf{r}}{dt} \pm \nabla_k \omega(\mathbf{k}, t) = 0 \qquad \text{at } \mathbf{k}_s(\mathbf{r}, t), \tag{56a}$$

and where Q is a matrix whose elements are $(\partial^2\Omega/\partial k_i \partial k_j)$, with $i, j = x, y, z$; $\Omega(\mathbf{k}, t)$ denotes the integral on the right-hand side of Eq. (55a). As in Eq. (1.6.5), $\sigma = \sum_{j=1}^{3} \operatorname{sgn} \bar{R}_i$ where $\bar{R}_i$ are the eigenvalues of the matrix Q.

(c) Relate the saddle-point condition (56a) to the space-time ray equation for a time-varying medium and use the graphical procedure of Problem 36b to locate the saddle points $\mathbf{k}_s$. Show that the magnitude of $I(\mathbf{r}, t)$ can be expressed in the form

$$|I(\mathbf{r}, t)| \sim |I(\mathbf{r}, t_1)|\left|\frac{A^2(\mathbf{k}_s, t)\Delta\mathbf{r}_1}{A^2(\mathbf{k}_s, t_1)\Delta\mathbf{r}}\right|, \tag{57}$$

where $\Delta\mathbf{r} = |\det Q|\Delta\mathbf{k}$ [cf. Eq. (1.6.10)]. Interpret this result in terms of space-time rays and compare with Eq. (1.6.15).

38. To illustrate the use of oscillatory representations for a time-varying medium,† consider the configuration of Problem 21, with $\omega_p(t)$ defined by

$$\begin{aligned} \omega_p(t) &= a, \qquad t < t_0 \\ &= b, \qquad t > t_0 \end{aligned} \tag{58}$$

thereby allowing for a homogeneous plasma medium before and after t_0.
(*a*) Writing

$$\hat{V} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikz}\bar{g}(k, t, t')\,dk, \qquad z' = 0, \tag{59}$$

substituting into Eq. (5) and simplifying, show that

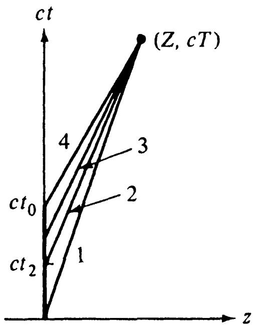
$$\bar{g} = c^2 \cos\omega_1 T, \qquad t < t_0 \tag{60a}$$

$$= c^2[\cos\omega_1 T_0 \cos\omega_2\tau - (\omega_1/\omega_2)\sin\omega_1 T_0 \sin\omega_2\tau], \qquad t > t_0, \tag{60b}$$

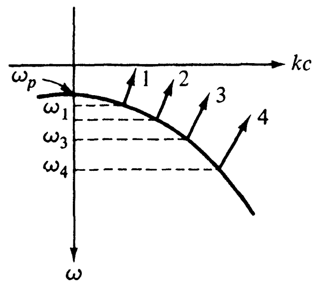
where $T = t - t'$, $T_0 = t_0 - t'$, $\tau = t - t_0$, with $\omega_1 = \sqrt{k^2c^2 + a^2}$ and $\omega_2 = \sqrt{k^2c^2 + b^2}$. For $a = 0$, show that Eqs. (59) and (60) yield the result in Eqs. (7) [cf. Eqs. (1.1.61)].
(b) Decompose the trigonometric functions in Eqs. (60) into exponentials and apply to each of the resulting integrals the asymptotic procedure of Problem 37 when modified for a single k integration. Derive the relevant saddle-point equations and interpret them ray-optically. Show that for $a = 0$, the saddle-point (or ray) equation can be solved explicitly for $k_s = k_s(z, t)$; derive the asymptotic solution for $\hat{V}$ and compare it with the asymptotic form of the exact solution in Eq. (7). For $a \neq 0$, show from the space-time ray configuration that the family of reflected rays corresponding to ray 3 in Fig. P1.6 may form a caustic, thereby producing wavepacket contraction (focusing).

39. While dispersion generally acts to spread out an originally narrow pulse, frequency modulated (FM) pulses can be designed so as to interact constructively with a dispersive medium. In particular, a source-pulse frequency spectrum can besought such that pulse compression takes place at a specified space-time point.
(a) To illustrate these aspects, consider FM pulse propagation in a homogeneous, cold, isotropic plasma, with excitation occurring in the plane $z = 0$ for a time interval $0 < t < t_0$; an enhanced and compressed signal is desired at $z = Z$ at time $t = T$, where $T > t_0 + (Z/c)$. For sufficiently large (Z, T), employ asymptotic considerations and schematize the process by the focusing at (Z, T)

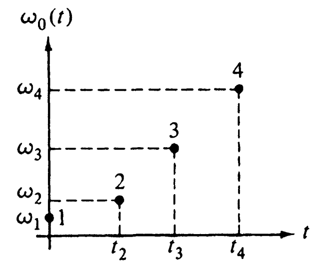
†L. B. Felsen and G.M. Whitman, "Wave Propagation in Time-Varying Media," *IEEE Transactions on Antennas and Propagation AP-18* (1970), pp. 242–253.



(a) Ray configuration for focusing at (Z, T)



(b) Dispersion curve and frequencies corresponding to different rays



(c) Source frequency profile

FIG. P1.7 Synthesis of source-frequency profile for FM pulse compression.

of the space-time rays emanating from the source region [Fig. P1.7(a)]. Determine the required source frequency spectrum $\omega_0(t)$ by locating each ray α in Fig. P1.7(a) on the dispersion curve in Fig. P1.7(b) and noting the corresponding ray frequency ω_α; the source-frequency profile $\omega_0 = \omega_0(t)$ in Fig. P1.7(c) then follows from the intersections of the various rays with the t axis.

(b) To determine the source profile $\omega_0(t')$ analytically, use the space-time ray equation

$$v_g(\omega)(T - t') = Z \tag{61}$$

where t' denotes the departure time of the ray from the source plane $z = 0$, and $v_g(\omega)$ is the group velocity. Show for the cold isotropic plasma, whose dispersion relation is $\omega^2 - k^2c^2 - \omega_p^2 = 0$, that

$$\omega_0(t') = \frac{(T - t')\omega_p}{\sqrt{(T - t')^2 - (Z/c)^2}}. \tag{62}$$

When the plasma is inhomogeneous, show that Eq. (61) is replaced by

$$t' = T - \int_0^z \frac{d\zeta}{v_g(\omega, \zeta)}, \tag{63}$$

where $v_g(\omega, z)$ is the variable group velocity along a ray. Develop a graphical procedure for determination of $\omega_0(t')$, analogous to that of Fig. P1.7. (Note that $\omega =$ constant on a ray in a spatially inhomogeneous medium.)

40. Show that the first-order differential equation

$$\frac{dw(s)}{ds} + \alpha(s)w(s) = \beta(s) \tag{64}$$

has the solution

$$w(s) = w(s_1)\exp\left(-\int_{s_1}^{s}\alpha(\eta)\,d\eta\right) + \int_{s_1}^{s}\beta(\xi)\exp\left(-\int_{\xi}^{s}\alpha(\eta)d\eta\right)d\xi. \tag{65}$$

(a) Referring to the transport equations (1.7.18c) satisfied by the amplitude coefficients in the high-frequency asymptotic expansion (1.7.16) of the time-harmonic field in an inhomogeneous isotropic medium, and using Eq. (65), show that the solution for the mth order coefficient $u_m(\mathbf{r})$ is given by:

$$u_m(\mathbf{r}) = u_m(\mathbf{r}_1)\left[\frac{n(\mathbf{r}_1)\,dA(\mathbf{r}_1)}{n(\mathbf{r})\,dA(\mathbf{r})}\right]^{1/2} - \int_{\mathbf{r}_1}^{\mathbf{r}}\frac{1}{2n(s)}\left[\frac{n(s)\,dA(s)}{n(\mathbf{r})\,dA(\mathbf{r})}\right]^{1/2}\nabla^2 u_{m-1}(s)\,ds, \tag{66}$$

where $m = 0, 1, 2\cdots$ (with $u_{-1} \equiv 0$); the integration variable s runs along a ray between points $\mathbf{r}_1$ and $\mathbf{r}$, and dA denotes the ray tube cross section. In obtaining the result in Eq. (66), note the equivalent forms (1.7.34) and (1.7.35b) for the lowest-order solution $u_0(\mathbf{r})$. Show that for one-dimensional propagation in a plane-stratified medium, the result for u_1 reduces to that in Problem 27. Show also that for validity of the lowest-order (geometric optical) approximation $u(\mathbf{r}) \sim u_0(\mathbf{r})\exp[ik_0\psi(\mathbf{r})]$ in Eq. (1.7.16), it is sufficient to require

$$[k_0 u_0(\mathbf{r})n(\mathbf{r})]^{-1}\,\nabla^2 u_0(\mathbf{r}) \ll 1.$$

(b) Transforming into the form of Eq. (65) the transport equations (1.7.72) satisfied by the amplitude coefficients in the asymptotic expansion (1.7.68) of the time-dependent field in an inhomogeneous, cold, isotropic plasma medium, show that the solution for the mth order coefficient $u_m(\mathbf{R}, \tau)$ is given by:

$$u_m(\mathbf{R}, \tau) = u_m(\mathbf{R}_1, \tau_1)\left[\frac{\Delta\mathbf{R}_1}{\Delta\mathbf{R}}\right]^{1/2} - \frac{c^2}{2\bar{\omega}}\int_{(\mathbf{R}_1,\tau_1)}^{(\mathbf{R},\tau)}\frac{1}{V_g(s)}\left[\frac{\Delta\mathbf{R}(s)}{\Delta\mathbf{R}}\right]^{1/2}\bar{\nabla}^2 u_{m-1}(s)\,ds, \tag{67}$$

where $m = 0, 1, 2\ldots$ (with $u_{-1} \equiv 0$), the integration variable s runs along a space-time ray between points $(\mathbf{R}_1, \tau_1)$ and $(\mathbf{R}, \tau)$, and $\Delta\mathbf{R}$ denotes (in hyperplanes $\tau =$ constant) the ray-tube cross section in the $(\mathbf{R}, \tau)$ four space (i.e., $\Delta\mathbf{R}$ is a volume element in three space). The operator $\bar{\nabla}^2$ is defined as $\bar{\nabla}^2 = \nabla_R^2 - \partial^2/c^2\partial\tau^2$, and $\bar{\omega} = -\partial\psi/\partial\tau$, while V_g is the four-dimensional group speed. In deriving Eq. (67), observe that the lowest-order solution $u_0(\mathbf{R}, \tau)$ may be expressed either as in the first term on the right-hand side of Eq. (67) [see Eq. (1.7.78b)] or in a manner analogous to that of Eq. (1.7.35b). Noting that $ds/V_g = d\tau/c$ (see Fig. 1.7.10), simplify the form of the integral in Eq. (67) by changing the integration variable to τ.

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