Chapter 5

Vector Analysis on Surface

5-1 Surface Symbolic Vector and Symbolic Expression for a Surface

Vector analysis on a surface has previously been treated by Weatherburn [10]. His works are summarized by Van Bladel [11]. Most books on vector analysis do not cover this subject. The approach taken by Weatherburn is to define a two-dimensional surface operator similar to the del operator in space. Some key differential functions analogous to gradient, divergence, and curl are then introduced.

In this work, the treatment is different. We approach the analysis based on a symbolic vector method similar to the one found in Chapter 4 for vector analysis in space. A symbolic expression for a surface is defined in terms of a surface symbolic vector. Afterwards, several essential functions in vector analysis for a surface are introduced. They are different from the ones defined by Weatherburn. The relationships between the set introduced in this book and Weatherburn's set will be discussed later. Finally, it will be shown that there is an intimate relationship between the symbolic expression for a surface and the symbolic expression in space. In fact, the former can be deduced from the latter without an independent formulation. However, it is more natural to treat the vector analysis on a surface as an independent discipline first, and then point out its relationship to the vector analysis in space.

Following the symbolic method discussed in Chapter 4, we will introduce a symbolic surface vector, denoted by ∇_s, and the corresponding symbolic vector expression $T(\nabla_s)$ for a surface that is defined by

$$T(\nabla_s) = \lim_{\Delta S \to 0} \frac{\sum_i T(\hat{m}_i)\, \Delta l_i}{\Delta S}, \tag{5.1}$$

where Δl_i denotes an elementary arc length of the contour enclosing ΔS, and $\hat{m}_i$ is the unit vector tangent to the surface and normal to its edge. The running index i covers the number of sides of ΔS. For a cell with four sides, i goes from 1 to 4. The symbolic expression is generated by replacing at least one vector in an algebraic vector expression with ∇_s. For example, $\nabla_s \times \mathbf{b}$ is created by replacing the vector $\mathbf{a}$ in $\mathbf{a} \times \mathbf{b}$ with ∇_s. The expression defined by (5.1) is invariant to, or independent of the choice of, the coordinates on the surface in the general Dupin system. It is recalled that the choice of (v_1, v_2) and the corresponding tangential unit vectors $(\hat{u}_1, \hat{u}_2)$ is quite arbitrary. To find the differential expression based on (5.1) in the general orthogonal Dupin system, let the sides of the surface cell be located at $v_1 \pm (\Delta v_1/2)$ and $v_2 \pm (\Delta v_2/2)$, with the corresponding unit normal vectors $\pm\hat{u}_1$ and $\pm\hat{u}_2$ located at these positions. The value of $\hat{u}_1$ evaluated at $v_1 + (\Delta v_1/2)$ is not equal to the value of the same unit vector evaluated at $v_1 - (\Delta v_1/2)$. The same is true for the metric coefficients h_1 and h_2. The area of the elementary surface ΔS is equal to $h_1 h_2\, \Delta v_1\, \Delta v_2$. Figure 5-1 shows the configuration of the cell. By substituting these quantities into (5.1) and taking the limit, one finds

$$T(\nabla_s) = \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial v_1} \left[h_2 T(\hat{u}_1) \right] + \frac{\partial}{\partial v_2} \left[h_1 T(\hat{u}_2) \right] \right\}. \tag{5.2}$$

For a plane surface located in the x–y plane in a rectangular system,

$$T(\nabla_s) = \frac{\partial}{\partial x} T(\hat{x}) + \frac{\partial}{\partial y} T(\hat{y}). \tag{5.3}$$

This is the only case where $T(\nabla_s)$ can be expressed conveniently in a rectangular system. In general, rectangular variables are not the proper ones to describe the

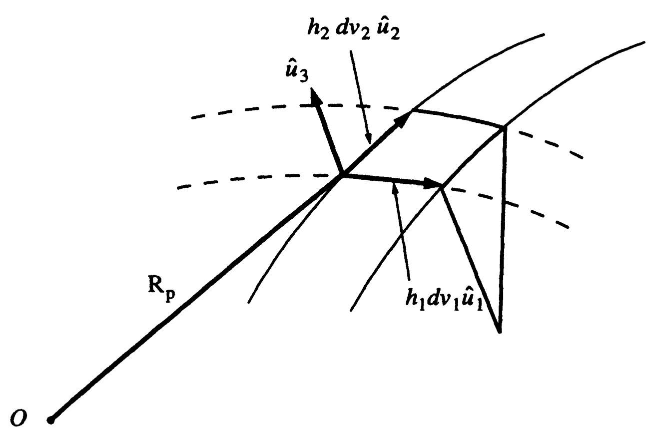

Figure 5-1 Cell on a surface in the general Dupin coordinate system.

function $T(\nabla_s)$ for a curved surface. From the definition of $T(\nabla_s)$ given by (5.1) and its differential form stated by (5.2), it is obvious that Lemma 4.1, introduced at the end of Chapter 4, Section 4-1, is also applicable to $T(\nabla_s)$ because $T(\hat{m}_i)$, $T(\hat{u}_1)$, and $T(\hat{u}_2)$ all have the proper form of a vector expression. By means of (5.2), it is now possible to derive the differential expression of some key functions in the vector analysis for a surface, analogous to the gradient, the divergence, and the curl in space.

5-2 Surface Gradient, Surface Divergence, and Surface Curl

For convenience, we repeat here the differential expression for $T(\nabla_s)$ expressed in the general Dupin system:

$$T(\nabla_s) = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial v_1}\left[h_2\, T(\hat{u}_1)\right] + \frac{\partial}{\partial v_2}\left[h_1\, T(\hat{u}_2)\right]\right\}. \tag{5.4}$$

5-2-1 Surface Gradient

If we let $T(\nabla_s) = \nabla_s f$, where f is a scalar function of position, then $T(\hat{u}_1) = f\hat{u}_1$, $T(\hat{u}_2) = f\hat{u}_2$. Upon substituting these quantities into (5.4), we obtain

$$\begin{aligned} T(\nabla_s) &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial v_1}(h_2 f\hat{u}_1) + \frac{\partial}{\partial v_2}(h_1 f\hat{u}_2)\right] \\ &= \frac{1}{h_1 h_2}\left[h_2 f\frac{\partial \hat{u}_1}{\partial v_1} + \left(f\frac{\partial h_2}{\partial v_1} + h_2\frac{\partial f}{\partial v_1}\right)\hat{u}_1\right. \\ &\qquad \left. + h_1 f\frac{\partial \hat{u}_2}{\partial v_2} + \left(f\frac{\partial h_1}{\partial v_2} + h_1\frac{\partial f}{\partial v_2}\right)\hat{u}_2\right]. \end{aligned}$$

By making use of (2.61), with $h_3 = 1$, we can express the derivatives of $\hat{u}_1$ and $\hat{u}_2$ in terms of $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$, which yields

$$\begin{aligned} T(\nabla_s) = \frac{1}{h_1 h_2}&\left[-h_2 f\left(\frac{1}{h_2}\frac{\partial h_1}{\partial v_2}\hat{u}_2 + \frac{\partial h_1}{\partial v_3}\hat{u}_3\right) + \left(f\frac{\partial h_2}{\partial v_1} + h_2\frac{\partial f}{\partial v_1}\right)\hat{u}_1\right. \\ &\left. - h_1 f\left(\frac{1}{h_1}\frac{\partial h_2}{\partial v_1}\hat{u}_1 + \frac{\partial h_2}{\partial v_3}\hat{u}_3\right) + \left(f\frac{\partial h_1}{\partial v_2} + h_1\frac{\partial f}{\partial v_2}\right)\hat{u}_2\right]. \end{aligned}$$

Some of the terms cancel each other. The net result is

$$\nabla_s f = \frac{1}{h_1}\frac{\partial f}{\partial v_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial f}{\partial v_2}\hat{u}_2 - \left(\frac{1}{h_1}\frac{\partial h_1}{\partial v_3} + \frac{1}{h_2}\frac{\partial h_2}{\partial v_3}\right) f\hat{u}_3. \tag{5.5}$$

The coefficient in front of $f\hat{u}_3$ can be written in several different forms. If we denote the product $h_1 h_2$ by H, which is equal to Ω $(= h_1 h_2 h_3)$ with $h_3 = 1$, then

$$\frac{1}{H}\frac{\partial H}{\partial v_3} = \frac{1}{h_1}\frac{\partial h_1}{\partial v_3} + \frac{1}{h_2}\frac{\partial h_2}{\partial v_3} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = -J, \tag{5.6}$$

where R_1 and R_2 denote the principal radii of curvature of the surface as stated by (2.73) and (2.76), and J is the surface curvature defined by (2.84). The differential function we have just derived is designated as the *surface gradient* of f, and it will be denoted by $\nabla_s f$, which can now be written in the form

$$\nabla_s f = \frac{\hat{u}_1}{h_1}\frac{\partial f}{\partial v_1} + \frac{\hat{u}_2}{h_2}\frac{\partial f}{\partial v_2} + J\hat{u}_3 f. \tag{5.7}$$

Equation (5.7) shows that the symbol ∇_s is indeed a differential-algebraic operator, defined by

$$\nabla_s = \frac{\hat{u}_1}{h_1}\frac{\partial}{\partial v_1} + \frac{\hat{u}_2}{h_2}\frac{\partial}{\partial v_2} + J\hat{u}_3. \tag{5.8}$$

The operation of $\hat{u}_3$ on f is a simple multiplication.

5-2-2 Surface Divergence

If we let $T(\nabla_s) = \nabla_s \cdot \mathbf{F}$, then $T(\hat{u}_i) = \hat{u}_i \cdot \mathbf{F}$; hence

$$\begin{aligned}\nabla_s \cdot \mathbf{F} &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial v_1}(h_2 F_1) + \frac{\partial}{\partial v_2}(h_1 F_2)\right] \\ &= \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}F_i\right).\end{aligned} \tag{5.9}$$

The function so obtained is designated as the *surface divergence* of $\mathbf{F}$, and it will be denoted by $\nabla_s \mathbf{F}$; thus,

$$\nabla_s \mathbf{F} = \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}F_i\right). \tag{5.10}$$

Equation (5.10) can be converted to an operational form as follows:

$$\begin{aligned}\nabla_s \mathbf{F} &= \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}\hat{u}_i \cdot \mathbf{F}\right) \\ &= \sum_{i=1}^{2}\left[\frac{\hat{u}_i}{h_i}\cdot\frac{\partial \mathbf{F}}{\partial v_i} + \frac{1}{H}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}\hat{u}_i\right)\cdot \mathbf{F}\right] \\ &= \sum_{i=1}^{2}\left[\frac{\hat{u}_i}{h_i}\cdot\frac{\partial \mathbf{F}}{\partial v_i} + J\hat{u}_3 \cdot \mathbf{F}\right].\end{aligned} \tag{5.11}$$

This expression shows that ∇_s is another differential-algebraic operator, defined by

$$\nabla_s = \left[\sum_{i=1}^{2}\frac{\hat{u}_i}{h_i}\cdot\frac{\partial}{\partial v_i} + J\hat{u}_3 \cdot\right]. \tag{5.12}$$

When this operator is applied to $\mathbf{F}$, it yields the surface divergence of $\mathbf{F}$.

5-2-3 Surface Curl

If we let $T(\nabla_s) = \nabla_s \times \mathbf{F}$, then $T(\hat{u}_i) = \hat{u}_i \times \mathbf{F}$; hence

$$\begin{aligned}\nabla_s \times \mathbf{F} &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial v_1}(h_2\hat{u}_1 \times \mathbf{F}) + \frac{\partial}{\partial v_2}(h_1\hat{u}_2 \times \mathbf{F})\right] \\ &= \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}\hat{u}_i \times \mathbf{F}\right) \\ &= \sum_{i=1}^{2}\left[\frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i} + \frac{1}{H}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}\hat{u}_i\right) \times \mathbf{F}\right] \\ &= \sum_{i=1}^{2}\left[\frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i} + J\hat{u}_3 \times \mathbf{F}\right].\end{aligned} \tag{5.13}$$

The function so created is called the *surface curl* of $\mathbf{F}$ and it will be denoted by $\nabla_s \mathbf{F}$; thus,

$$\nabla_s \mathbf{F} = \left[\sum_{i=1}^{2}\frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i} + J\hat{u}_3 \times\right]\mathbf{F}. \tag{5.14}$$

It is evident that ∇_s is another differential-algebraic operator, defined by

$$\nabla_s = \left(\sum_{i=1}^{2}\frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i} + J\hat{u}_3 \times\right). \tag{5.15}$$

When it is operated on $\mathbf{F}$, it yields the surface curl of $\mathbf{F}$. As with the vector analysis in space, we have three independent surface operators; they are partly differential and partly algebraic, a special feature of the surface operators. By evaluating the derivatives of $\mathbf{F}$ with respect to v_i in (5.14) and simplifying the result with the aid of (2.59) and (2.61), with $h_3 = 1$, one finds

$$\begin{aligned}\nabla_s \mathbf{F} = \frac{1}{H}\Bigg\{&\left[h_1\frac{\partial F_3}{\partial v_2} + h_1 F_2 \frac{\partial h_1}{\partial v_3}\right]\hat{u}_1 \\ &- \left[h_2\frac{\partial F_3}{\partial v_1} + h_1 F_1 \frac{\partial h_2}{\partial v_3}\right]\hat{u}_2 \\ &+ \left[\frac{\partial(h_2 F_2)}{\partial v_1} - \frac{\partial(h_1 F_1)}{\partial v_2}\right]\hat{u}_3\Bigg\}.\end{aligned} \tag{5.16}$$

The $\hat{u}_3$ component of $\nabla_s \mathbf{F}$ is the same as the corresponding component of the three-dimensional $\nabla \mathbf{F}$ in a Dupin coordinate system, but the two transversal components are different.

5-3 Relationship Between the Volume and Surface Symbolic Expressions

Although the surface symbolic vector ∇_s and the symbolic expression $T(\nabla_s)$ involving ∇_s as defined by (5.1) and its differential form by (5.2) appear to be independent of ∇ and $T(\nabla)$, actually they are intimately related. If we express $T(\nabla)$ in the general Dupin system ($h_3 = 1$), then (4.80) becomes

$$T(\nabla) = \frac{1}{H}\sum_{i=1}^{3}\frac{\partial}{\partial v_i}\left[\frac{H}{h_i}T(\hat{u}_i)\right], \tag{5.17}$$

where $H = h_1 h_2$. The differential expression of $T(\nabla_s)$ as given by (5.2) can be written in the form

$$T(\nabla_s) = \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left[\frac{H}{h_i}T(\hat{u}_i)\right]. \tag{5.18}$$

It is obvious that the first two terms of (5.17) are exactly the same as $T(\nabla_s)$; hence

$$T(\nabla) = T(\nabla_s) + \frac{1}{H}\frac{\partial}{\partial v_3}\left[H\,T(\hat{u}_3)\right]. \tag{5.19}$$

Equation (5.19), therefore, can be used to find $T(\nabla_s)$ once $T(\nabla)$ is known or $T(\nabla_s)$ can be defined as the sum of the first two terms of $T(\nabla)$. From this point of view, $T(\nabla_s)$ is not an independent function, and ∇_s is not an independent symbolic vector.

The last term of (5.19) can be written in the form

$$\begin{aligned}\frac{1}{H}\frac{\partial}{\partial v_3}\left[HT(\hat{u}_3)\right] &= \frac{\partial T(\hat{u}_3)}{\partial v_3} + \frac{1}{H}\frac{\partial H}{\partial v_3}T(\hat{u}_3)\\ &= \frac{\partial T(\hat{u}_3)}{\partial v_3} - J\,T(\hat{u}_3).\end{aligned} \tag{5.20}$$

Equation (5.19) is therefore equivalent to

$$T(\nabla) = T(\nabla_s) + \frac{\partial T(\hat{u}_3)}{\partial v_3} - J\,T(\hat{u}_3). \tag{5.21}$$

By using the expression of ∇f, $\nabla \mathbf{F}$, and $\nabla \mathbf{F}$ in the Dupin system and the expressions of $\nabla_s f$, $\nabla_s \mathbf{F}$, and $\nabla_s \mathbf{F}$ given by (5.7), (5.10), (5.14), it can be easily verified that (5.21) is indeed satisfied when we let $T(\nabla)$ equal ∇f, $\nabla \mathbf{F}$, $\nabla \times \mathbf{F}$, respectively.

5-4 Relationship Between Weatherburn's Surface Functions and the Functions Defined in the Method of Symbolic Vector

In the classic work of Weatherburn [10], he defines the surface gradient, the surface divergence, and the surface curl by retaining the two transversal parts of the three-dimensional functions. Our notations for his functions are $\nabla_t f$, $\nabla_t \mathbf{F}$, and $\nabla_t \mathbf{F}$.

Weatherburn originally used the same notations as the three-dimensional functions ∇f, $\nabla \cdot \mathbf{F}$, and $\nabla \times \mathbf{F}$. The surface functions defined by Brand [1] in an orthogonal Dupin system are the same as Weatherburn's. Van Bladel [11] uses three linguistic notations for these functions, namely, $\text{grad}_s f$, $\text{div}_s \mathbf{F}$, and $\text{curl}_s \mathbf{F}$. They are defined by

$$\text{grad}_s f = \nabla_t f = \sum_{i=1}^{2} \frac{\hat{u}_i}{h_i} \frac{\partial f}{\partial v_i}, \tag{5.22}$$

$$\text{div}_s \mathbf{F} = \boldsymbol{\nabla}_t \mathbf{F} = \sum_{i=1}^{2} \frac{\hat{u}_i}{h_i} \cdot \frac{\partial \mathbf{F}}{\partial v_i}, \tag{5.23}$$

$$\text{curl}_s \mathbf{F} = \boldsymbol{\bar{\nabla}}_t \mathbf{F} = \sum_{i=1}^{2} \frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i}. \tag{5.24}$$

These three terms have appeared before in our surface functions defined by (5.7), (5.11), and (5.14). Thus, the relations between the two sets are

$$\nabla_s f = \nabla_t f + J\hat{u}_3 f, \tag{5.25}$$

$$\boldsymbol{\nabla}_s \mathbf{F} = \boldsymbol{\nabla}_t \mathbf{F} + J\hat{u}_3 \cdot \mathbf{F}, \tag{5.26}$$

$$\boldsymbol{\bar{\nabla}}_s \mathbf{F} = \boldsymbol{\bar{\nabla}}_t \mathbf{F} + J\hat{u}_3 \times \mathbf{F}. \tag{5.27}$$

We must emphasize that $\boldsymbol{\nabla}_t \mathbf{F}$ and $\boldsymbol{\bar{\nabla}}_t \mathbf{F}$ are not scalar and vector products between ∇_t and $\mathbf{F}$. As with our ∇_s, $\boldsymbol{\nabla}_s$, and $\boldsymbol{\bar{\nabla}}_s$, they are three independent operators.

We have so far presented the three key surface functions in orthogonal Dupin coordinate systems. By following the same procedure in the three-dimensional GCS, it is not difficult to extend the formulation to nonorthogonal Dupin systems ($\hat{u}_1 \cdot \hat{u}_2 \neq 0$, but $\hat{u}_1 \cdot \hat{u}_3 = \hat{u}_2 \cdot \hat{u}_3 = 0$). By introducing the primary and reciprocal vectors on a curved surface, we can show the invariance of these surface functions in the nonorthogonal Dupin coordinate systems.

As far as the surface functions are concerned, once the relationships between the two sets are known, it is a matter of personal preference as to which set should be considered as the standard surface functions. In a subsequent section dealing with integral theorems, it will be evident that the set derived from the present method, that is, $\nabla_s f$, $\boldsymbol{\nabla}_s \mathbf{F}$, and $\boldsymbol{\bar{\nabla}}_s \mathbf{F}$, or in general, $T(\nabla_s)$, is much more convenient to formulate the generalized Gauss theorem for a surface. We may also inject a remark that in electromagnetic theory, the equation of continuity (the law of conservation of charge) relating the surface current density $\mathbf{J}_s$ and the time rate of change of the surface charge density ρ_s is described by

$$\boldsymbol{\nabla}_s \mathbf{J}_s = -\frac{\partial \rho_s}{\partial t} \tag{5.28}$$

when the surrounding medium has no loss. Here, it is $\boldsymbol{\nabla}_s \mathbf{J}$, not $\nabla_t \cdot \mathbf{J}$ or $\text{div}_s \mathbf{f}$, that enters the formulation. On the other hand, for the rate of change of a scalar function on a surface in a direction tangent to the surface, both $\nabla_s f$ and $\nabla_t f$ produce the same result:

$$\frac{\partial f}{\partial s} = \hat{u}_s \cdot \nabla_s f = \hat{u}_s \cdot \nabla_t f. \tag{5.29}$$

The vector component $(\partial f/\partial v_3)\hat{u}_3$ in $\nabla_s f$ does not affect the value of $\partial f/\partial s$. For the curl function, one finds

$$\hat{u}_3 \cdot \nabla_s \mathbf{f} = \hat{u}_3 \cdot \nabla_t \times \mathbf{f} = \hat{u}_3 \cdot \nabla \mathbf{f}, \tag{5.30}$$

an identity to be used later.

5-5 Generalized Gauss Theorem for a Surface

By integrating the differential expression for $T(\nabla_s)$, (5.2), on an open surface S with contour L, we have

$$\iint_S T(\nabla_s)\, dS = \iint_S \sum_{i=1}^{2} \frac{\partial}{\partial v_i}\left[\frac{H}{h_i} T(\hat{u}_i)\right] dv_1\, dv_2, \tag{5.31}$$

where $dS = h_1 h_2\, dv_1\, dv_2 = H\, dv_1\, dv_2$. We assume that $T(\hat{u}_i)$ is continuous throughout S. The integrals in (5.31) can be carried out as follows:

$$\begin{aligned}
\iint_S \frac{\partial}{\partial v_1}[h_2\, T(\hat{u}_1)]\, dv_1\, dv_2 &= \int_{v_{2\,\min}}^{v_{2\,\max}} [h_2\, T(\hat{u}_1)]_{P_1}^{P_2}\, dv_2 \\
&= \int_{L_2} h_2\, T(\hat{u}_1)\, dv_2 - \int_{-L_1} h_2\, T(\hat{u}_1)\, dv_2 \qquad (5.32)\\
&= \oint_L h_2\, T(\hat{u}_1)\, dv_2,
\end{aligned}$$

where the locations (P_1, P_2), the segments L_1, L_2, and the two extreme values $v_{2\,\min}$, $v_{2\,\max}$ are shown in Fig. 5-2a. Similarly,

$$\begin{aligned}
\iint_S \frac{\partial}{\partial v_2}[h_1\, T(\hat{u}_2)]\, dv_1\, dv_2 &= \int_{v_{1\,\min}}^{v_{1\,\max}} [h_1\, T(\hat{u}_2)]_{P_3}^{P_4}\, dv_1 \\
&= \int_{-L_4} h_1\, T(\hat{u}_2)\, dv_1 - \int_{L_3} h_1\, T(\hat{u}_2)\, dv_1 \qquad (5.33)\\
&= -\oint_L h_1\, T(\hat{u}_2)\, dv_1,
\end{aligned}$$

where the locations P_3, P_4, the segments L_3, L_4, and the two extreme values $v_{1\,\min}$, $v_{1\,\max}$ are shown in Fig. 5-2b. Hence

$$\iint_S T(\nabla_s)\, dS = \oint_L \left[h_2\, T(\hat{u}_1)\, dv_2 - h_1\, T(\hat{u}_2)\, dv_1\right]. \tag{5.34}$$

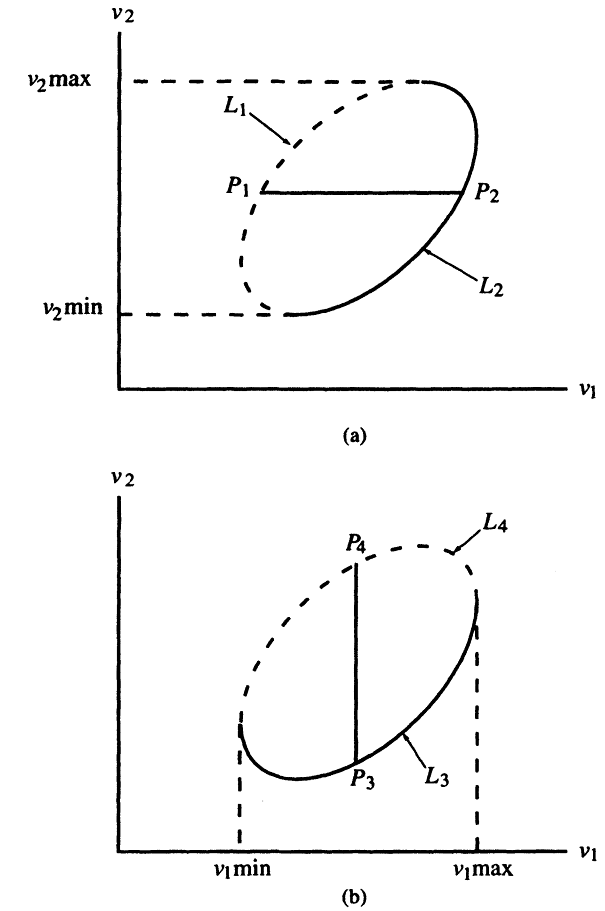

Figure 5-2 Domain of integration in the (v_1, v_2) plane of a simple region.

Because $T(\hat{u}_i)$ is linear with respect to $\hat{u}_i$ with $i = 1, 2$, the integrand in the line integral is proportional to

$$\hat{u}_1 h_2 \, dv_2 - \hat{u}_2 h_1 \, dv_1, \tag{5.35}$$

which can be simplified. Let us consider a segment of the contour L_1, which is the edge of S. In the tangential plane containing $\hat{u}_1$ and $\hat{u}_2$ at a typical point P, the four key unit vectors are shown in Fig. 5-3. All of them are tangential to the surface at P. A three-dimensional display of these vectors and the normal vector $\hat{u}_3$ is shown in Fig. 5-4. In these figures, $\hat{u}_\ell$ is tangential to the edge of the surface, and $\hat{u}_\mathrm{m}$ is normal to the edge, but tangential to the surface. The relations between these unit vectors are

$$\hat{u}_1 \times \hat{u}_2 = \hat{u}_3 = \hat{u}_\mathrm{m} \times \hat{u}_\ell. \tag{5.36}$$

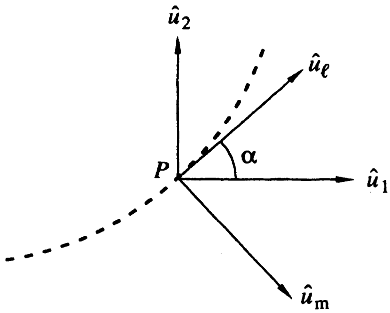

Figure 5-3 Four tangential vectors in the plane containing $\hat{u}_1$ and $\hat{u}_2$.

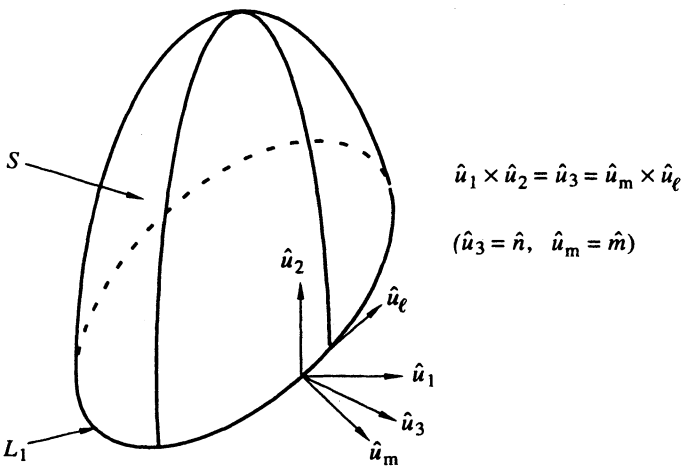

Figure 5-4 Three-dimensional view of the unit vectors at the edge of an open surface.

The algebraic relations between them are

$$\hat{u}_1 = \cos\alpha\,\hat{u}_\ell + \sin\alpha\,\hat{u}_m, \tag{5.37}$$

$$\hat{u}_2 = \sin\alpha\,\hat{u}_\ell - \cos\alpha\,\hat{u}_m, \tag{5.38}$$

where α is the angle between $\hat{u}_1$ and $\hat{u}_\ell$. If we denote the total differential arc length of the contour at P by dl, then

$$h_1\,dv_1 = \cos\alpha\,dl, \tag{5.39}$$

$$h_2\,dv_2 = \sin\alpha\,dl. \tag{5.40}$$

Upon substituting (5.39) and (5.40) into (5.35), we find

$$\hat{u}_1 h_2\,dv_2 - \hat{u}_2 h_1\,dv_1 = \hat{u}_m\,dl; \tag{5.41}$$

hence

$$T(\hat{u}_1)\,h_2\,dv_2 - T(\hat{u}_2)\,h_1\,dv_1 = T(\hat{u}_m)\,dl. \tag{5.42}$$

Equation (5.31) therefore reduces to

$$\iint_S T(\nabla_s)\, dS = \oint_L T(\hat{u}_m)\, dl. \tag{5.43}$$

The unit vector $\hat{u}_m$ is commonly denoted by $\hat{m}$, in contrast to the notation $\hat{n}$ used for $\hat{u}_3$; (5.43), therefore, will be written as

$$\iint_S T(\nabla_s)\, dS = \oint_L T(\hat{m})\, dl. \tag{5.44}$$

Equation (5.44) is designated as the *generalized Gauss theorem for a surface* or the *generalized surface Gauss theorem.* It converts an open surface integral into a closed line integral, and it has the same significance as the generalized Gauss theorem in space, which converts a volume integral into a closed surface integral. Various integral theorems can be derived by choosing the proper form for $T(\nabla_s)$.

1. **Surface gradient theorem**
   Let $T(\nabla_s) = \nabla_s f = \nabla_s f$; then $T(\hat{m}) = \hat{m} f$. Hence

$$\iint_S \nabla_s f\, dS = \oint_L f\hat{m}\, dl. \tag{5.45}$$

2. **Surface divergence theorem**
   Let $T(\nabla_s) = \nabla_s \cdot \mathbf{F} = \nabla \cdot \mathbf{F}$; then $T(\hat{m}) = \hat{m} \cdot \mathbf{F}$. Hence

$$\iint_S \nabla \cdot \mathbf{F}\, dS = \oint_L \hat{m} \cdot \mathbf{F}\, dl. \tag{5.46}$$

3. **Surface curl theorem**
   Let $T(\nabla_s) = \nabla_s \times \mathbf{F} = \nabla_s \mathbf{F}$; then $T(\hat{m}) = \hat{m} \times \mathbf{F}$. Hence

$$\iint_S \nabla_s \mathbf{F}\, dS = \oint_L \hat{m} \times \mathbf{F}\, dl. \tag{5.47}$$

In view of the relationship between $T(\nabla)$ and $T(\nabla_s)$ as described by (5.19), the generalized Gauss theorem for a surface can be written in the form

$$\iint_S \left\{ T(\nabla) - \frac{1}{H}\frac{\partial}{\partial v_3}\left[H\, T(\hat{u}_3)\right] \right\} dS = \oint_L T(\hat{m})\, dl. \tag{5.48}$$

If $T(\nabla)$ is proportional to $\hat{u}_3 \times \nabla$ or $\hat{n} \times \nabla$, then $T(\hat{u}_3) = 0$, and (5.48) becomes

$$\iint_S T(\nabla)\, dS = \oint_L T(\hat{m})\, d\ell, \qquad T(\hat{n}) = 0. \tag{5.49}$$

Three cases are considered now.

4. **Cross-gradient theorem**

Let $T\,(\nabla) = \left(\hat{n} \times \nabla\right) f$. As a result of Lemmas 4.1 and 4.2, we have

$$\begin{aligned}\left(\hat{n} \times \nabla\right) f &= \left(\hat{n} \times \nabla_n\right) f + \left(\hat{n} \times \nabla_f\right) f \\ &= -\left(\nabla\, \hat{n}\right) f + \hat{n} \times \nabla f = \hat{n} \times \nabla f,\end{aligned}$$

because $\nabla\, \hat{n} = 0$, and

$$T\left(\hat{m}\right) = \left(\hat{n} \times \hat{m}\right) f = \hat{u}_\ell f.$$

Hence

$$\iint_S \hat{n} \times \nabla f\, dS = \oint_L f\, d\boldsymbol{\ell}. \tag{5.50}$$

For identification purposes, we designate it as the cross-gradient theorem.

5. **Stokes's theorem**

Let

$$\begin{aligned}T\,(\nabla) &= \left(\hat{n} \times \nabla\right) \cdot \mathbf{F} \\ &= \left(\hat{n} \times \nabla_n\right) \cdot \mathbf{F} + \left(\hat{n} \times \nabla_F\right) \cdot \mathbf{F} \\ &= -\left(\nabla\, \hat{n}\right) \cdot \mathbf{F} + \hat{n} \cdot \left(\nabla_F \times \mathbf{F}\right) = \hat{n} \cdot \nabla\, \mathbf{F}.\end{aligned}$$

Then

$$T\left(\hat{m}\right) = \left(\hat{n} \times \hat{m}\right) \cdot \mathbf{F} = \hat{u}_\ell \cdot \mathbf{F}.$$

Hence

$$\iint_S \hat{n} \cdot \nabla\, \mathbf{F}\, dS = \oint_L \mathbf{F} \cdot d\boldsymbol{\ell}, \tag{5.51}$$

which is the famous theorem named after George Gabriel Stokes (1819–1903).

6. **Cross–∇–cross theorem**

Let $T(\nabla) = (\hat{n} \times \nabla) \times \mathbf{F}$. By means of Lemma 4.2, we have

$$\begin{aligned}(\hat{n} \times \nabla) \times \mathbf{F} &= (\hat{n} \times \nabla_n) \times \mathbf{F} + (\hat{n} \times \nabla_F) \times \mathbf{F} \\ &= -(\nabla\, \hat{n}) \times \mathbf{F} + (\hat{n} \times \nabla_F) \times \mathbf{F}.\end{aligned}$$

Now, $\nabla\, \hat{n} = 0$ because $\hat{n}$ is a linear vector, not curvilinear, and $T(\hat{m}) = (\hat{n} \times \hat{m}) \times \mathbf{F} = \hat{\ell} \times \mathbf{F}$; hence

$$\iint (\hat{n} \times \nabla_F) \times \mathbf{F}\, dS = \oint (\hat{\ell} \times \mathbf{F})\, d\ell. \tag{5.52}$$

The function $(\hat{n} \times \nabla_F) \times \mathbf{F}$ is given by (4.162) with $\mathbf{a}$ and $\mathbf{b}$ therein replaced by $\hat{n}$ and $\mathbf{F}$, respectively.

5-6 Surface Symbolic Expressions with a Single Symbolic Vector and Two Functions

A complete line of formulas and theorems can be derived covering these two topics. However, we will present only the essential formulation without actually going into detail. A third lemma dealing with symbolic expressions with a single-surface S vector and two functions is one of the main subjects to be covered. A scalar Green's theorem on a surface involving the surface Laplacian will also be presented.

For a symbolic expression with two functions, its differential form in the general Dupin system according to (5.2) is defined by

$$T(\nabla_s, \widetilde{a}, \widetilde{b}) = \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left[\frac{H}{h_i}T(\hat{u}_i, \widetilde{a}, \widetilde{b})\right], \tag{5.53}$$

where $H = h_1 h_2$. We now introduce two expressions with two *partial surface S vectors*, denoted by ∇_{sa} and ∇_{sb}, as follows:

$$T(\nabla_{sa}, \widetilde{a}, \widetilde{b}) = \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left[\frac{H}{h_i}T(\hat{u}_i, \widetilde{a}, \widetilde{b})\right]_{\widetilde{b}=c}, \tag{5.54}$$

$$T(\nabla_{sb}, \widetilde{a}, \widetilde{b}) = \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left[\frac{H}{h_i}T(\hat{u}_i, \widetilde{a}, \widetilde{b})\right]_{\widetilde{a}=c}. \tag{5.55}$$

It is obvious that Lemma 4.1 also applies to (5.54) and (5.55). Equation (5.53) can now be decomposed into three parts:

$$T(\nabla_s, \widetilde{a}, \widetilde{b}) = T(\nabla_{sa}, \widetilde{a}, \widetilde{b}) + T(\nabla_{sb}, \widetilde{a}, \widetilde{b}) - \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left[\frac{H}{h_i}T(\hat{u}_i, \widetilde{a}, \widetilde{b})\right]_{\widetilde{a},\widetilde{b}=c}. \tag{5.56}$$

Because $T(\hat{u}_i)$ is linear with respect to $\hat{u}_i$, we can combine $\hat{u}_i$ with H/h_i to form one function, and examine its derivatives. According to (2.62), with $h_3 = 1$,

$$\frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}\hat{u}_i\right) + \frac{1}{H}\frac{\partial}{\partial v_3}(H\hat{u}_3) = 0.$$

Thus,

$$\frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}\hat{u}_i\right) = -\frac{1}{H}\frac{\partial H}{\partial v_3}(H\hat{u}_3) = J\hat{u}_3. \tag{5.57}$$

The last term in (5.56), therefore, can be written as

$$-\frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left[\frac{H}{h_i}T(\hat{u}_i, \widetilde{a}, \widetilde{b})\right]_{\widetilde{a},\widetilde{b}=c} = -J\,T(\hat{u}_3, \widetilde{a}, \widetilde{b}). \tag{5.58}$$

Lemma 5.1. For a symbolic expression defined with respect to a single-surface S vector and two functions, the following relation holds true:

$$T(\nabla_s, \widetilde{a}, \widetilde{b}) = T(\nabla_{sa}, f_1, f_2) + T(\nabla_{sb}, \widetilde{a}, \widetilde{b}) - J\,T(\hat{u}_3, \widetilde{a}, \widetilde{b}). \tag{5.59}$$

The proof of this lemma follows directly from (5.56) and (5.58). By means of this lemma, we can derive all the possible surface vector identities similar to the identities described by (4.140)–(4.151) and (4.156)–(4.160). We merely write down these relations without detailed explanation.

1.

$$\nabla_s(ab) = \nabla_{sa}(ab) + \nabla_{sb}(ab) - J\hat{u}_3 ab;$$

hence

$$\nabla_s(ab) = b\nabla_s a + a\nabla_s b - Jab\hat{u}_3. \tag{5.60}$$

2.

$$\nabla_s(a\mathbf{b}) = \nabla_{sa} \cdot (a\mathbf{b}) + \nabla_{sb} \cdot (a\mathbf{b}) - J\hat{u}_3 \cdot a\mathbf{b};$$

hence

$$\nabla_s(a\mathbf{b}) = a\nabla_s\mathbf{b} + \mathbf{b} \cdot \nabla_s a - Ja\mathbf{b} \cdot \hat{u}_3. \tag{5.61}$$

3.

$$\nabla_s \times (a\mathbf{b}) = \nabla_{sa} \times (a\mathbf{b}) + \nabla_{sb} \times (a\mathbf{b}) - J\hat{u}_3 \times a\mathbf{b};$$

hence

$$\nabla_s(a\mathbf{b}) = -b \times \nabla_s a + a\nabla_s\mathbf{b} + Ja\mathbf{b} \times \hat{u}_3. \tag{5.62}$$

4.

$$\begin{aligned}\nabla_s(\mathbf{a} \cdot \mathbf{b}) &= \nabla_{sa}(\mathbf{a} \cdot \mathbf{b}) + \nabla_{sb}(\mathbf{a} \cdot \mathbf{b}) - J\hat{u}_3(\mathbf{a} \cdot \mathbf{b}) \\ &= \mathbf{b} \times (\nabla_{sa} \times \mathbf{a}) + \mathbf{b} \cdot \nabla_{sa}\mathbf{a} \\ &\qquad + \mathbf{a} \times (\nabla_{sb} \times \mathbf{b}) + \mathbf{a} \cdot \nabla_{sb}\mathbf{b} - J\hat{u}_3(\mathbf{a} \cdot \mathbf{b});\end{aligned}$$

hence

$$\nabla_s(\mathbf{a} \cdot \mathbf{b}) = \mathbf{b} \times \nabla_s\mathbf{a} + \mathbf{b} \cdot \nabla_s\mathbf{a} + \mathbf{a} \times \nabla_s\mathbf{b} + \mathbf{a} \cdot \nabla_{sb}\mathbf{b} - J(\mathbf{a} \cdot \mathbf{b})\hat{u}_3. \tag{5.63}$$

We are making effective use of Lemma 4.1 in these exercises.

5.

$$\begin{aligned}\nabla_s \cdot (\mathbf{a} \times \mathbf{b}) &= \nabla_{sa} \cdot (\mathbf{a} \times \mathbf{b}) + \nabla_{sb} \cdot (\mathbf{a} \times \mathbf{b}) - J\hat{u}_3 \cdot (\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{b} \cdot \nabla_{sa} \times \mathbf{a} + \mathbf{a} \cdot (\mathbf{b} \times \nabla_{sb}) - J\hat{u}_3 \cdot (\mathbf{a} \times \mathbf{b});\end{aligned}$$

hence

$$\nabla_s(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla_s\mathbf{a} - \mathbf{a} \cdot \nabla_s\mathbf{b} - J(\mathbf{a} \times \mathbf{b}) \cdot \hat{u}_3. \tag{5.64}$$

6.

$$(\nabla_s \cdot \mathbf{a})\mathbf{b} = (\nabla_{sa} \cdot \mathbf{a})\mathbf{b} + (\nabla_{sb} \cdot \mathbf{a})\mathbf{b} - J(\hat{u}_3 \cdot \mathbf{a})\mathbf{b};$$

hence

$$\nabla_s \mathbf{ab} = \mathbf{b}\nabla_s \mathbf{a} + \mathbf{a} \cdot \nabla_s \mathbf{b} - J\mathbf{ba} \cdot \hat{u}_3. \tag{5.65}$$

7.

$$\begin{aligned}\nabla_s \times (\mathbf{a} \times \mathbf{b}) &= \nabla_{sa} \times (\mathbf{a} \times \mathbf{b}) + \nabla_{sb} \times (\mathbf{a} \times \mathbf{b}) - J\hat{u}_3 \times (\mathbf{a} \times \mathbf{b}) \\ &= (\nabla_{sa} \cdot \mathbf{b})\mathbf{a} - (\nabla_{sa} \cdot \mathbf{a})\mathbf{b} + (\nabla_{sb} \cdot \mathbf{b})\mathbf{a} - (\nabla_{sb} \cdot \mathbf{a})\mathbf{b} \\ &\quad - j\hat{u}_3 \times (\mathbf{a} \times \mathbf{b});\end{aligned}$$

hence

$$\nabla_s(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla_s \mathbf{a} - \mathbf{b}\nabla_s \mathbf{a} + \mathbf{a}\nabla_s \mathbf{b} - \mathbf{a} \cdot \nabla_s \mathbf{b} + J(\mathbf{a} \times \mathbf{b}) \times \hat{u}_3. \tag{5.66}$$

8.

$$\begin{aligned}(\nabla_s \times \mathbf{a}) \times \mathbf{b} &= (\nabla_{sa} \times \mathbf{a}) \times \mathbf{b} + (\nabla_{sb} \times \mathbf{a}) \times \mathbf{b} - J(\hat{u}_3 \times \mathbf{a}) \times \mathbf{b} \\ &= (\nabla_{sa} \times \mathbf{a}) \times \mathbf{b} + \mathbf{a}(\nabla_{sb} \cdot \mathbf{b}) - \nabla_{sb}(\mathbf{a} \cdot \mathbf{b}) - J(\hat{u}_3 \times \mathbf{a}) \times \mathbf{b} \\ &= (\nabla_{sa} \times \mathbf{a}) \times \mathbf{b} + \mathbf{a}(\nabla_{sb} \cdot \mathbf{b}) - \mathbf{a} \times (\nabla_{sb} \times \mathbf{b}) - (\mathbf{a} \cdot \nabla_{sb})\mathbf{b} \\ &\quad - J(\hat{u}_3 \times \mathbf{a}) \times \mathbf{b};\end{aligned}$$

hence

$$(\nabla_s \times \mathbf{a}) \times \mathbf{b} = -\mathbf{b} \times \nabla_s \mathbf{a} + \mathbf{a}\nabla_s \mathbf{b} - \mathbf{a} \times \nabla_s \mathbf{b} - \mathbf{a} \cdot \nabla_s \mathbf{b} - J\mathbf{b} \times (\mathbf{a} \times \hat{u}_3). \tag{5.67}$$

As with (4.159), the surface symbolic expression involves the vector product between $\mathbf{b}$ and $\nabla_s \mathbf{a}$ and the products of $\mathbf{a}$ with the surface gradient, the surface divergence, and the surface curl of $\mathbf{b}$.

5-7 Surface Symbolic Expressions with Two Surface Symbolic Vectors and a Single Function

In presenting the symbolic expressions with two symbolic expressions in a three-dimensional space, we define these functions in the rectangular system and then their relations. The rectangular system is not a convenient coordinate system for a curved surface. For this reason, the subject will be presented using the basic definition of the surface symbolic expression as stated by (5.1), namely,

$$T(\nabla_s) = \lim_{\Delta S \to 0} \frac{\sum_i T(\hat{m})\,\Delta \ell_i}{\Delta S}. \tag{5.68}$$

For an expression consisting of two surface symbolic vectors and a single function, we can apply the same formula twice, that is,

$$T(\nabla_s, \nabla_s', \widetilde{f}) = \lim_{\Delta S' \to 0} \lim_{\Delta S \to 0} \frac{\sum_i \sum_j T(\hat{m}, \hat{m}', \widetilde{f})\,\Delta \ell_\ell\,\Delta \ell_j'}{\Delta S\,\Delta S'}. \tag{5.69}$$

Several cases will be considered.

1. **Surface Laplacian of a scalar function**
   Let $T(\nabla_s, \nabla'_s, \tilde{f}) = \nabla_s \cdot \nabla'_s f$; then $T(\hat{m}, \hat{m}', \tilde{f}) = \hat{m} \cdot \hat{m}' f$. Thus,

$$\begin{aligned} \nabla_s \cdot \nabla'_s f &= \lim_{\Delta S \to 0} \lim_{\Delta S' \to 0} \frac{\sum_i \sum_j \hat{m} \cdot \hat{m}' f \, \Delta \ell \, \Delta \ell'}{\Delta S \, \Delta S'} \\ &= \lim_{\Delta S \to 0} \frac{\sum_i \hat{m} \cdot \nabla_s f \, \Delta \ell}{\Delta S} = \nabla_s \nabla_s f. \end{aligned} \tag{5.70}$$

   The double operator $\nabla_s \nabla_s$ is designated as the surface Laplacian. No special notation will be introduced for this operator.

2. **Surface Laplacian of a vector function**
   Let $T(\nabla_s, \nabla'_s, \tilde{f}) = (\nabla_s \cdot \nabla'_s)\mathbf{F}$; then $T(\hat{m}, \hat{m}', \tilde{f}) = (\hat{m} \cdot \hat{m}')\mathbf{F}$. Following the same procedure as the previous case, we obtain

$$\nabla_s \cdot \nabla'_s \mathbf{F} = \nabla_s \nabla_s \mathbf{F}. \tag{5.71}$$

3. **The surface gradient of the surface divergence of a vector function**
   Let $T(\nabla_s, \nabla'_s, \tilde{f}) = \nabla_s \nabla'_s \cdot \mathbf{F}$; then $T(\hat{m}, \hat{m}', \tilde{f}) = \hat{m}(\hat{m}' \cdot \mathbf{F})$. The double limit of this function yields

$$\nabla_s \cdot \nabla'_s \mathbf{F} = \nabla_s \nabla_s \mathbf{F}. \tag{5.72}$$

4. **The double surface curl of a vector function**
   Let $T(\nabla_s, \nabla'_s, \tilde{f}) = \nabla_s \times (\nabla'_s \times \mathbf{F})$; then $T(\hat{m}, \hat{m}', \mathbf{F}) = \hat{m} \times (\hat{m}' \times \mathbf{F})$. We find

$$\nabla_s \times (\nabla'_s \times \mathbf{F}) = \boldsymbol{\nabla}_s \boldsymbol{\nabla}_s \mathbf{F}. \tag{5.73}$$

   As a result of Lemma 4.1,

$$\begin{aligned} \nabla_s \times (\nabla'_s \times \mathbf{F}) &= (\nabla_s \cdot \mathbf{F})\nabla'_s - (\nabla_s \cdot \nabla'_s)\mathbf{F} \\ &= \nabla'_s(\nabla_s \cdot \mathbf{F}) - \nabla_s \cdot \nabla'_s \mathbf{F}. \end{aligned}$$

   We obtain the identity

$$\begin{aligned} \boldsymbol{\nabla}_s \boldsymbol{\nabla}_s \mathbf{F} &= \nabla_s \nabla_s \mathbf{F} - \nabla_s \nabla_s \mathbf{F} \\ &= (\nabla_s \nabla_s - \nabla_s \nabla_s)\mathbf{F}. \end{aligned} \tag{5.74}$$

   This is analogous to the three-dimensional identity

$$\begin{aligned} \boldsymbol{\nabla} \boldsymbol{\nabla} \mathbf{F} &= \nabla \nabla \mathbf{F} - \nabla \nabla \mathbf{F} \\ &= (\nabla \nabla - \nabla \nabla)\mathbf{F}. \end{aligned}$$

Finally, we would like to present a scalar Green's theorem for a curved surface. This theorem can be obtained by applying the generalized surface Gauss theorem, (5.44), with

$$T(\nabla_s) = \nabla_s \cdot (a \nabla_s b - b \nabla_s a). \tag{5.75}$$

As a result of Lemmas 4.1 and 5.1 or by means of (5.61), we find

$$T(\nabla_s) = a\nabla_s\nabla_s b - b\nabla_s\nabla_s a \tag{5.76}$$

and

$$\begin{aligned} T(\hat{m}) &= \hat{m} \cdot (a\nabla_s b - b\nabla_s a) \\ &= a\frac{\partial b}{\partial m} - b\frac{\partial a}{\partial m} . \end{aligned} \tag{5.77}$$

Substituting (5.76) and (5.77) into (5.44), we obtain the scalar surface Green's theorem of the second kind:

$$\iint (a\nabla_s\nabla_s b - b\nabla_s\nabla_s a)\, ds = \oint \left(a\frac{\partial b}{\partial m} - b\frac{\partial a}{\partial m} \right) d\ell. \tag{5.78}$$

Other theorems can be derived following similar procedures.