Appendix D

Relationships Between Integral Theorems

The integral theorems stated by (1)–(3) and (7)–(8) in Appendix C are closely related. By means of the gradient theorem, we can derive both the divergence theorem and the curl theorem. From this point of view, we must first prove the gradient theorem, leaving aside its derivation by the symbolic method. The theorem states that

$$\iiint \nabla f \, dV = \oiint_S f \, d\mathbf{S}. \tag{D.1}$$

In a rectangular system with coordinate variables (x_1, x_2, x_3), the x_1 component of (D.1) corresponds to

$$\iiint \frac{\partial f}{\partial x_1} dx_1 \, dx_2 \, dx_3 = \iint_{S_2} f \, dx_2 \, dx_3 - \iint_{S_1} f \, dx_2 \, dx_3, \tag{D.2}$$

where S_1 and S_2 denote the two sides of an enclosed surface S viewed in the x_1 direction. The negative sign associated with the surface integral evaluated on S_1 is due to the fact that the vector component of $d\mathbf{S}_1$ is equal to $-dx_2 \, dx_3 \, \hat{x}_1$. Equation (D.2) is a valid identity because the volume integral is given by

$$\begin{aligned} \iiint \frac{\partial f}{\partial x_1} dx_1 \, dx_2 \, dx_3 &= \iint [f(P_2) - f(P_1)] \, dx_2 \, dx_3 \\ &= \iint_{S_2} f \, dx_2 \, dx_3 - \iint_{S_1} f \, dx_2 \, dx_3, \end{aligned} \tag{D.3}$$

where P_2 and P_1 denote two stations located at opposite sides of the surface along the x_1 direction. The same procedure can be used to prove the remaining two components of (D.1). Having proved the validity of (D.1), we can use it to deduce the divergence theorem (Gauss theorem) and the curl theorem.

We now consider three distinct sets of (D.1) in the form

$$\iiint \nabla F_i \, dV = \oiint_S F_i \, d\mathbf{S}, \qquad i = 1, 2, 3. \tag{D.4}$$

By taking the scalar product of (D.4) with $\hat{x}_i$ and summing the resultant equations, we obtain

$$\sum_i \hat{x}_i \cdot \iiint \nabla F_i \, dV = \sum_i \hat{x}_i \cdot \oiint_S F_i \, d\mathbf{S}. \tag{D.5}$$

Let

$$\mathbf{F} = \sum F_i \hat{x}_i, \tag{D.6}$$

and because

$$\hat{x}_i \cdot \nabla F_i = \nabla \cdot (F_i \hat{x}_i), \tag{D.7}$$

we obtain

$$\iiint \nabla \mathbf{F} \, dV = \oiint_S \mathbf{F} \cdot d\mathbf{S}, \tag{D.8}$$

which is the divergence theorem. Similarly, by taking the cross product of $\hat{x}_i$ with (D.4), we obtain

$$\sum_i \hat{x}_i \times \iiint \nabla F_i \, dV = \sum_i \hat{x}_i \times \oiint F_i \, d\mathbf{S}. \tag{D.9}$$

Because

$$\hat{x}_i \times \nabla F_i = -\nabla (F_i \hat{x}_i), \tag{D.10}$$

(D.9) is equivalent to

$$\iiint \nabla \mathbf{F} \, dV = -\oiint \mathbf{F} \times d\mathbf{S}, \tag{D.11}$$

which is the curl theorem. The approach that we took can be applied to the other two theorems listed as (7)–(8) in Appendix C. In this case, we consider the cross-gradient theorem as the key theorem that must be proved first. The theorem states that

$$\iint \hat{n} \times \nabla f \, dS = \oint f \, d\ell. \tag{D.12}$$

In a rectangular system, we can write

$$\hat{n} = \sum_i n_i \hat{x}_i,$$

$$d\boldsymbol{\ell} = \sum_i dx_i\, \hat{x}_i.$$

Then the x_1 component of (D.12) reads

$$\iint \left(n_2 \frac{\partial f}{\partial x_3} - n_3 \frac{\partial f}{\partial x_2} \right) dS = \oint f\, dx_1. \tag{D.13}$$

The surface integral in (D.13) can be written in the form

$$\begin{aligned}\iint \left(\frac{\partial f}{\partial x_3}\, dx_1\, dx_3 + \frac{\partial f}{\partial x_2}\, dx_1\, dx_2 \right) &= \iint \left(\frac{\partial f}{\partial x_2}\, dx_2 + \frac{\partial f}{\partial x_3}\, dx_3 \right) dx_1 \\ &= \iint df\, dx_1 \\ &= \int [f(p_2) - f(p_1)]\, dx_1 \\ &= \int_{c_2} f(p_2)\, dx_1 - \int_{-c_1} f(p_1)\, dx_1 \\ &= \oint_c f\, dx_1,\end{aligned}$$

where p_1 and p_2 denote two stations on the closed contour, which consists of two segments $c_1 + c_2$. We have thus proved the validity of (D.13). The same procedure applies to the x_2 and x_3 components of (D.12). Once we have proved the cross-gradient theorem, it can be used to deduce the Stokes theorem.

We consider three distinct sets of (D.12) in the form

$$\iint \hat{n} \times \nabla F_i\, dS = \oint F_i\, d\boldsymbol{\ell}, \qquad i = 1, 2, 3. \tag{D.14}$$

By taking the scalar product of (D.14) with $\hat{x}_i$ and summing the resultant equations, we obtain

$$\sum_i \hat{x}_i \cdot \iint (\hat{n} \times \nabla F_i)\, dS = \sum_i \hat{x}_i \cdot \oint F_i\, d\boldsymbol{\ell}. \tag{D.15}$$

Because

$$\hat{x}_i \cdot (\hat{n} \times \nabla F_i) = -\hat{n} \cdot (\hat{x}_i \times \nabla F_i) = \hat{n} \cdot \nabla (F_i \hat{x}_i), \tag{D.16}$$

and we let

$$\mathbf{F} = \sum_i F_i \hat{x}_i,$$

(D.15) can be written in the form

$$\iint \hat{n} \cdot \nabla \mathbf{F} \, dS = \oint \mathbf{F} \cdot d\boldsymbol{\ell}, \tag{D.17}$$

which is the Stokes theorem. It is seen that in this analysis, the gradient theorem is considered the key theorem based on which the other three theorems can be readily derived. The approach taken here has its own merit without considering the derivation of these theorems, independently, by the symbolic method.

The relationships among the gradient theorem, the divergence theorem, and the curl theorem have previously been pointed out by Van Bladel [11, Appendix I]. Alternatively, we can use the divergence theorem and the Stokes theorem as the key theorems to derive the other three theorems. The manipulations, however, are more complicated.