Chapter 8

A Historical Study of Vector Analysis

8-1 Introduction[1]

In a book on the history of vector analysis [21], Michael J. Crowe made a thorough investigation of the decline of quaternion analysis and the evolution of vector analysis during the nineteenth century until the beginning of this century. The topics covered are mostly vector algebra and quaternion analysis. He had few comments to offer on the technical aspects of the subject from the point of view of a mathematician or a theoretical physicist. For example, the difference between the presentations of Gibbs and Heaviside, considered to be two founders of modern vector analysis, is not discussed in Crowe's book, and less attention is paid to the history of vector differentiation and integration, and to the role played by the del operator, ∇.

Brief descriptions of the history of vector analysis from the technical point of view are found in a few books. For example, in a book by Burali-Forti and Marcolongo [22] published in 1920, there are four historical notes in the appendix entitled, respectively, "On the definition of abstraction," "On vectors," "On vector and scalar (interior) products," and "On grad, rot, div." In a book by Moon and Spencer [23] published in 1965, there is a brief but critical review of the history

[1] This chapter is based on C. T. Tai, "A historical study of vector analysis," Technical Report RL 915, The Radiation Laboratory, Department of Electrical Engineering and Computer Science, The University of Michigan, May 1995.

of vector analysis from a technical perspective; this book will receive further discussion later. In their introduction, Moon and Spencer firmly state an important reason why they present vector analysis by way of tensor analysis [23, p. 9]:

> The present book differs from the customary textbook on vectors in stressing the idea of invariance under groups of transformations. In other words, elementary tensor technique is introduced, and in this way, the subject is placed on the firm, logical foundation which vector textbooks have previously lacked.

Further, in Appendix C [23, p. 323], Moon and Spencer write:

> In reading the foregoing book [referring to their book], one may wonder why nothing has been said about the operator ∇, which is usually considered such an important part of vector analysis. The truth is that ∇, though providing the subject with fluency, is an unreliable device because it often gives incorrect results. For this reason—and because it is not necessary—we have omitted it in the body of the book. Here, however, we shall indicate briefly the use of the operator ∇....

These two quotations are sufficient to indicate that after decades of application of vector analysis, there seems to be no systematic treatment of the subject that could be considered satisfactory according to these two authors. This observation is also supported by the fact that we have so far no standard notations in vector analysis. Many books on electromagnetics, for example, use the linguistic notations for the gradient, divergence, and curl—grad u, div $\mathbf{f}$, and curl $\mathbf{f}$—while many others prefer Gibbs's notations for these functions—∇u, $\nabla \cdot \mathbf{f}$, and $\nabla \times \mathbf{f}$. Can we offer a better explanation to students as to why we do not yet have a universally accepted standard notation than to say it is merely a matter of personal choice? In regard to Moon and Spencer's comments about the lack of a firm, logical foundation in previous books on vector analysis, there has been no elaboration. They do give an example of an incorrect result from using ∇ to form a scalar product with $\mathbf{f}$ to find the expression for divergence in an orthogonal curvilinear coordinate system, but they do not explain *why* the result is incorrect. In fact, the views expressed by these two authors are also found in many books treating vector analysis. These will be reviewed and commented upon later.

In this chapter, we assume that the reader has already read the previous chapters of this book, particularly the method of symbolic vector in Chapter 4. The aim of this chapter is to point out the inadequacy or illogic in the treatments of some basic topics in vector analysis during a period of approximately one hundred years. The mistreatments are then rectified by the proper tools introduced in this book.

8-2 Notations and Operators

8-2-1 Past and Present Notations in Vector Analysis

In a book on advanced vector analysis published in 1924, Weatherburn [24] compiled a table of notations in vector analysis that had been used up to that time. The authors represented in that table are Gibbs, Wilson, Heaviside, Abraham, Ignatowsky, Lorentz, Burali-Forti, and Marcolongo. A table of notations is also given in Moon and Spencer's 1965 book *Vectors* (from which the quotations in the previous section were taken). The authors represented in that table are Maxwell, Gibbs, Wilson, Heaviside, Gans, Lagally, Burali-Forti, Marcolongo, Phillips, Moon, and Spencer. Among these authors, Gibbs, Wilson, Phillips, Moon, and Spencer are American; Maxwell and Heaviside belong to the English schools; while Abraham, Ignatowsky, Gans, and Lagally belong to the German schools. Lorentz was a Dutch physicist, Burali-Forti and Marcolongo were Italian, and Ignatowsky was a native of Russia but was trained in Germany. For our study, we have prepared another list representing several contemporary authors and some additional notations; this is given in Table 8.1. The dyadic notation is added because we need it to characterize the gradient of a vector, which is a dyadic function. In perusing this table, the reader will recognize the linguistic notations grad u, div **A**, curl **A**, or rot **A** for the three key functions. The reader is probably familiar with Gibbs's notations ∇u, $\nabla \cdot \alpha$, and $\nabla \times \alpha$, except that here the period in $\nabla.\alpha$ has been replaced by a raised dot ($\cdot$) as in Wilson's notations, and his Greek letters for vectors are now commonly replaced by boldface Clarendon (or equivalent fonts), while the linguistic notations are used by many authors in Europe and a few in the United States. Gibbs's notations have been adopted in many books published in the United States. We will later quote from two books on electromagnetic theory, one by Stratton and another by Jackson. Their treatises are well known to many electrical engineers, as well as physicists.

Historically, vector analysis was developed a few years after Maxwell formulated his monumental work in electromagnetic theory. When he wrote his treatise on electricity and magnetism [25] in 1873, vector analysis was not yet available. Its forerunner, quaternion analysis, developed by Hamilton (1805–1865) in 1843, was then advocated by many of Hamilton's followers. It is probably for this reason that Maxwell wrote an article in his book (Article 618) entitled "Quaternion Expressions for the Electromagnetic Equations." Maxwell's notations in our list are based on this document. Actually, he made little use of these notations in his book and in his papers published elsewhere.

The notation used by Heaviside is unconventional from the present point of view. His notation for the scalar product and the divergence does not have a dot and his notation for the curl is of the quaternion form, as is Maxwell's. The notations used by Burali-Forti and Marcolongo are obsolete now. Occasionally, we still see

Table 8.1: List of Notations

Author(s)	Vectors	Scalar Product	Vector Product	Dyadic in 3-space	Tensor in 3-space	Gradient of a Scalar	Gradient of a Vector	Divergence of a Vector	Curl or Rot of a Vector	Laplacian of a Scalar	Laplacian of a Vector
Maxwell [25]	σ, ρ $\mathcal{A}, \mathcal{B}$	$S\sigma\rho$	$V\sigma\rho$	—	—	∇u	—	$-S\nabla\rho$	$V\nabla\rho$	$\nabla^2 u$	—
Gibbs [4]	α, β	α, β	$\alpha \times \beta$	$\alpha\beta$		∇u	$\nabla\alpha$	$\nabla \cdot \alpha$	$\nabla \times \alpha$	$\nabla \cdot \nabla u$	$\nabla \cdot \nabla\alpha$
Wilson [30]	$\mathbf{A}, \mathbf{B}$	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{A} \times \mathbf{B}$	$\mathbf{AB}$	—	∇u	$\nabla\mathbf{A}$	$\nabla \cdot \mathbf{A}$	$\nabla \times \mathbf{A}$	$\nabla \cdot \nabla u$	$\nabla \cdot \nabla\mathbf{A}$
Heaviside [26]	$\mathbf{A}, \mathbf{B}$	$\mathbf{AB}$	$\mathbf{VAB}$	—	—	∇u	$\nabla \cdot \mathbf{A}$	$\nabla\mathbf{A}$; div $\mathbf{A}$	$\mathbf{V}\nabla\mathbf{A}$; curl $\mathbf{A}$	$\nabla^2 u$	$\nabla^2\mathbf{A}$
Gans [37]	$\mathcal{A}, \mathcal{B}$	$(\mathcal{A}, \mathcal{B})$	$[\mathcal{A}, \mathcal{B}]$	—	—	∇u; grad u	—	$\nabla \cdot \mathbf{A}$; div $\mathbf{A}$	$\nabla \times \mathbf{A}$; rot $\mathbf{A}$	Δu	$\Delta\mathbf{A}$
Burali-Forti/ Marcolongo [22]	$\mathbf{A}, \mathbf{B}$	$\mathbf{A} \times \mathbf{B}$	$\mathbf{A} \wedge \mathbf{B}$	—	—	grad u	—	div $\mathbf{A}$	rot $\mathbf{A}$	$\Delta_2 u$	$\Delta_2'\mathbf{A}$
Stratton [5]	$\mathbf{A}, \mathbf{B}$	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{A} \times \mathbf{B}$	—	T_{ij}	∇u	$\nabla\mathbf{A}$	$\nabla \cdot \mathbf{A}$	$\nabla \times \mathbf{A}$	$\nabla^2 u$	$\nabla^2\mathbf{A}$
Jackson [44]	$\mathbf{A}, \mathbf{B}$	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{A} \times \mathbf{B}$	$\overleftrightarrow{T}$	T_{ij}	∇u	$\nabla\mathbf{A}$	$\nabla \cdot \mathbf{A}$	$\nabla \times \mathbf{A}$	$\nabla^2 u$	$\nabla^2\mathbf{A}$
Moon/ Spencer [23]	$\mathbf{A}, \mathbf{B}$	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{A} \times \mathbf{B}$	—	T_{ij}	grad u	—	div$\mathbf{A}$	curl $\mathbf{A}$	$\nabla^2 u$	✡$\mathbf{A}$

Uppercase script symbols are used here in place of capital German letters originally used by Maxwell and Gans.

the notation $\mathbf{A} \wedge \mathbf{B}$ for the cross product in the works of European authors. On the whole, we now have basically two sets of notations in current use: the linguistic notation and Gibbs's notation. The names of Moon and Spencer are included in our list primarily because these two authors considered the use of ∇ to be unreliable and they frequently emphasize their view that the rigorous method of formulating vector analysis is to follow the route of tensor analysis. In addition, their new notation for the Laplacian of a vector function will receive detailed examination in the section on orthogonal curvilinear systems.

8-2-2 Quaternion Analysis

The rise of vector analysis as a distinct branch of applied mathematics has its origin in quaternion analysis. It is therefore necessary to review briefly the laws of quaternion analysis to show its influence on the development of vector analysis, and also explain the notations in the previous list. Quaternions are complex numbers of the form

$$q = w + ix + jy + kz, \tag{8.1}$$

where w, x, y, and z are real numbers, and i, j, and k are quaternion units, or quaternion unit vectors, associated with the x, y, and z axes, respectively. These units obey the following laws of multiplication:

$$\begin{gathered} ij = k, \qquad jk = i, \qquad ki = j, \\ ji = -k, \qquad kj = -i, \qquad ik = -j, \\ ii = jj = kk = -1. \end{gathered} \tag{8.2}$$

We must not at this stage associate these relations with our current laws of unit vectors in vector analysis. We consider the subject as a new algebra, which is indeed the case. The product of the multiplication of two quaternions σ and ρ in which the scalar parts w and w' are zero is obtained as follows:

We let

$$\sigma = iD_1 + jD_2 + kD_3,$$
$$\rho = iX + jY + kZ.$$

Then,

$$\begin{aligned} \sigma\rho = & -(D_1X + D_2Y + D_3Z) \\ & + i(D_2Z - D_3Y) + j(D_3X - D_1Z) + k(D_1Y - D_2X). \end{aligned} \tag{8.3}$$

The resultant quaternion, $\sigma\rho$, has two parts, one scalar and one vector. In Hamilton's original notation, they are

$$S.\sigma\rho = -(D_1X + D_2Y + D_3Z), \tag{8.4}$$

$$V.\sigma\rho = i(D_2Z - D_3Y) + j(D_3X - D_1Z) + k(D_1Y - D_2X). \tag{8.5}$$

The period between S or V and $\sigma\rho$ can be omitted without any resulting ambiguity. When one identifies σ as ∇, Hamilton's del operator defined with respect to the quaternion unit vectors, that is,

$$\sigma = \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}, \tag{8.6}$$

then,

$$S\nabla\rho = -\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right), \tag{8.7}$$

$$V\nabla\rho = i\left(\frac{\partial Z}{\partial z} - \frac{\partial Y}{\partial z}\right) + j\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + k\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right). \tag{8.8}$$

Maxwell used the quaternion notation $S\nabla\rho$ for the negative of the divergence of ρ, which he termed the *convergence*. He used the quaternion notation $V\nabla\rho$ for the curl of ρ. The term *curl*, now standard, was coined by Maxwell. According to Crowe [21, p. 142] the term *divergence* was originally due to William Kingdom Clifford (1845–1879), who was also the first person to define the modern notations for the scalar and vector products. However, his original definition of the scalar product is the negative of the modern scalar product. In the list of notations, we notice that Heaviside used the quaternion notation for the curl even though he was opposed to quaternion analysis. In one of his writings [26, p. 35], he concurred with Gibbs's treatment of vector analysis but criticized Gibbs's notations without offering a reason; we discuss this comment of Heaviside's later in Section 8-5. Before we discuss the works of these various authors, a review of the meaning of the algebraic and differential operators is necessary.

8-2-3 Operators

For our convenience, we would like to discuss in sufficient detail the classification and the characteristics of a number of operators appearing in this study. We will focus on unary and binary operators and consider such operators in cascade or compound arrangements as the complexity of the case at hand requires.

A unary operator involves only one operand. A binary operator needs two operands, one anterior and another posterior. A cascade operator could be unary or binary. As an example, we consider the derivative symbol $\partial/\partial x$ to be a unary operator. When it operates on an operand P, it produces the derivative, $\partial P/\partial x$. In some writings, the operator $\partial/\partial x$ is denoted by D_x. The operand under consideration can be a scalar function of x and other independent variables or a vector function or a dyadic function; that is,

$$\frac{\partial P}{\partial x}: \quad \frac{\partial a}{\partial x}, \frac{\partial \mathbf{A}}{\partial x}, \frac{\partial \bar{\bar{F}}}{\partial x}$$

are all valid applications of the unary differential operator. The partial derivative of a dyadic function in a rectangular system is defined by

$$\frac{\partial \bar{\bar{F}}}{\partial x} = \sum_j \frac{\partial \mathbf{F}_j}{\partial x}\hat{x}_j = \sum_i \sum_j \frac{\partial F_{ij}}{\partial x}\hat{x}_i\hat{x}_j. \tag{8.9}$$

We list in Table 8-2 several commonly used unary operators and their possible operands. The function a in the weighted differential operator $a(\partial/\partial x)$ is assumed to be a scalar function. A vector operator such as $\mathbf{A}(\partial/\partial x)$ can operate on a dyadic that would yield a "triadic"—a quantity that is not included in this study. The last operator in Table 8-2 is the del operator, or the gradient operator. It can be applied to an operand that is either a scalar or a vector.

Table 8-2: Valid Application of Some Unary Differential Operators

Operator	Type of Operand	Results
$\frac{\partial}{\partial x}$	$b,\ \mathbf{B},\ \bar{\bar{B}}$	$\frac{\partial b}{\partial x},\ \frac{\partial \mathbf{B}}{\partial x},\ \frac{\partial \bar{\bar{B}}}{\partial x}$
$a\frac{\partial}{\partial x}$	$b,\ \mathbf{B},\ \bar{\bar{B}}$	$a\frac{\partial b}{\partial x},\ a\frac{\partial \mathbf{B}}{\partial x},\ a\frac{\partial \bar{\bar{B}}}{\partial x}$
$\mathbf{A}\frac{\partial}{\partial x}$	$b,\ \mathbf{B}$	$\mathbf{A}\frac{\partial b}{\partial x},\ \mathbf{A}\frac{\partial \mathbf{B}}{\partial x}$
$\nabla = \sum_i \hat{x}_i \frac{\partial}{\partial x_i}$	$b,\ \mathbf{B}$	$\nabla b,\ \nabla \mathbf{B}$

A binary operator requires two operands. In arithmetic and algebra, we have four binary operators: $+$ (addition), $-$ (subtraction), $\times$ (multiplication), and $\div$ (division). In these cases, we need two operands, one anterior and another posterior, as in $2+3$, $4-3$, 5×3, and $6\div 3$. Note that the symbols $+$ and $-$ are also used to denote "plus" and "minus" signs. For example, $-a = |a|$ when a is negative. In this case, the minus sign is not considered to be a binary operator in our classification, but rather as a unary "sign change" operator. The two binary operators involved frequently in our work are the dot $(\cdot)$ and the cross $(\times)$. They appear in Gibbs's notations for the scalar and vector products, that is, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$. We consider the dot and the cross as two binary operators, and their operands, one anterior and one posterior, must be vectors; that is,

$$\mathbf{A}\cdot\mathbf{B} \quad \text{and} \quad \mathbf{A}\times\mathbf{B}.$$

The dot operator is not the same as the multiplication operator in arithmetic, nor is the cross operator the same as the multiplication operator, although we use the same symbol for both. According to the definitions of the scalar and vector products,

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} = |\mathbf{A}||\mathbf{B}|\cos\theta, \tag{8.10}$$

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} = |\mathbf{A}||\mathbf{B}|\sin\theta\hat{c}, \tag{8.11}$$

where θ is the angle measured from $\mathbf{A}$ to $\mathbf{B}$ in the plane containing these two vectors, and $\hat{c}$ is the unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$ and is pointed in the right-screw advancing direction when $\mathbf{A}$ turns into $\mathbf{B}$. The dot and the cross can also be applied to operands where one of them or both are dyadics. Thus, we have

$$\mathbf{A}\cdot\bar{\bar{B}},\quad \bar{\bar{B}}\cdot\mathbf{A},\quad \mathbf{A}\times\bar{\bar{B}},\quad \bar{\bar{B}}\times\mathbf{A},\quad \bar{\bar{A}}\cdot\bar{\bar{B}},\quad \bar{\bar{B}}\cdot\bar{\bar{A}}. \tag{8.12}$$

The first two entities are vectors and the remaining four are dyadics.

The last group of operators are called *cascade* or *compound operators*. Of particular concern in this study is the proper treatment of a pair of operators of different types, which are applied sequentially. When one of the operators is a scalar differential unary operator, and the other is a vector binary operator, there arise a number of hazards in their application which, if not properly treated, could lead to invalid results. Several commonly used cascade operators are of the forms

$$\cdot\frac{\partial}{\partial y},\qquad \cdot\nabla,\qquad \times\frac{\partial}{\partial y},\qquad \times\nabla. \tag{8.13}$$

These operators also require two operands; the anterior operand must be a vector or a dyadic and the posterior operand must be compatible with the part in front. Thus, we can have

$$\begin{array}{ll} \mathbf{A}\cdot\dfrac{\partial\mathbf{B}}{\partial y}, & \mathbf{A}\cdot\dfrac{\partial\bar{\bar{B}}}{\partial y}; \\ \mathbf{A}\cdot\nabla u, & \mathbf{A}\cdot\nabla\mathbf{B}; \\ \mathbf{A}\times\dfrac{\partial\mathbf{B}}{\partial y}, & \mathbf{A}\times\dfrac{\partial\bar{\bar{B}}}{\partial y}; \\ \mathbf{A}\times\nabla u, & \mathbf{A}\times\nabla\mathbf{B}. \end{array} \tag{8.14}$$

In (8.13), the unary operators, $\partial/\partial y$ and ∇, and the binary operators, $\cdot$ and $\times$, are not commutative; hence the following combinations or assemblies are not valid cascade operators:

$$\frac{\partial}{\partial y}\cdot,\qquad \nabla\cdot,\qquad \frac{\partial}{\partial y}\times,\qquad \nabla\times. \tag{8.15}$$

These assemblies are formed by interchanging the positions of the symbols in (8.13). They are not operators in the sense that we cannot find an operand to form a meaningful entity. For example,

$$\begin{array}{llll} \dfrac{\partial}{\partial y}\cdot\mathbf{A}, & \dfrac{\partial}{\partial y}\cdot\bar{\bar{B}}, & \nabla\cdot\mathbf{A}, & \nabla\cdot\bar{\bar{B}}, \\ \dfrac{\partial}{\partial y}\times\mathbf{A}, & \dfrac{\partial}{\partial y}\times\bar{\bar{B}}, & \nabla\times\mathbf{A}, & \nabla\times\bar{\bar{B}}, \end{array} \tag{8.16}$$

do not have any meaningful interpretation. The reader has probably noticed that there are two assemblies, $\nabla\cdot\mathbf{A}$ and $\nabla\times\mathbf{A}$, in (8.16) that correspond to *Gibbs's* *notation* for the divergence and curl. This is true, but that does not mean that $\nabla \cdot \mathbf{A}$ is a scalar product between ∇ and $\mathbf{A}$, nor is $\nabla \times \mathbf{A}$ a vector product between ∇ and $\mathbf{A}$. In fact, this is a central issue in this study to be examined critically in the following sections. We now have the necessary tools to investigate many of the past presentations of vector analysis.

8-3 The Pioneer Works of J. Willard Gibbs (1839–1903)

8-3-1 Two Pamphlets Printed in 1881 and 1884

Gibbs's original works on vector analysis are found in two pamphlets entitled *Elements of Vector Analysis* [4], privately printed in New Haven. The first consists of 33 pages published in 1881 and the second of 40 pages published in 1884. These pamphlets were distributed to his students at Yale University and also to many scientists and mathematicians including Heaviside, Helmholtz, Kirchhoff, Lorentz, Lord Rayleigh, Stokes, Tait, and J. J. Thomson [27, Appendix IV]. The contents are divided into five chapters and a note on bivectors:

Chapter I.	Concerning the algebra of vectors
Chapter II.	Concerning the differential and integral calculus of vectors
Chapter III.	Concerning linear vector functions
Chapter IV.	Concerning the differential and integral calculus of vectors (Supplement to Chapter II)
Chapter V.	Concerning transcendental functions of dyadics
	A note on bivector analysis

The most important formulations for our immediate discussions are covered in Articles 50–54 and 68–71, which are reproduced here.

Functions of Positions in Space

50. Def.—If u is any scalar function of position in space (i.e., any scalar quantity having continuously varying values in space), ∇u is the vector function of position in space which has everywhere the direction of the most rapid increase of u, and a magnitude equal to the rate of that increase per unit of length. ∇u may be called the derivative of u, and u, the primitive of ∇u.

    We may also take any one of the Nos. 51, 52, 53 for the definition of ∇u.

51. If ρ is the vector defining the position of a point in space,

$$du = \nabla u \cdot d\rho.$$

52.

$$\nabla u = i\frac{du}{dx} + j\frac{du}{dy} + k\frac{du}{dz}. \tag{8.17}$$

53.

$$\frac{du}{dx} = i \cdot \nabla u, \qquad \frac{du}{dy} = j \cdot \nabla u, \qquad \frac{du}{dz} = k \cdot \nabla u.$$

54. Def.—If ω is a vector having continuously varying values in space,

$$\nabla \cdot \omega = i \cdot \frac{d\omega}{dx} + j \cdot \frac{d\omega}{dy} + k \cdot \frac{d\omega}{dz}, \tag{8.18}$$

$$\nabla \times \omega = i \times \frac{d\omega}{dx} + j \times \frac{d\omega}{dy} + k \times \frac{d\omega}{dz}. \tag{8.19}$$

$\nabla \cdot \omega$ is called the divergence of ω and $\nabla \times \omega$ its curl.

If we set

$$\omega = Xi + Yj + Zk,$$

we obtain by substitution the equation

$$\nabla \cdot \omega = \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} \tag{8.20}$$

and

$$\nabla \times \omega = i\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + j\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + k\left(\frac{dY}{dx} - \frac{dX}{dy}\right), \tag{8.21}$$

which may also be regarded as defining $\nabla \cdot \omega$ and $\nabla \times \omega$.

Combinations of the Operators ∇, $\nabla\cdot$, and $\nabla\times$

68. If ω is any vector function of space, $\nabla \cdot \nabla \times \omega = 0$. This may be deduced directly from the definition of No. 54.

The converse of this proposition will be proved hereafter.

69. If u is any scalar function of position in space, we have by Nos. 52 and 54

$$\nabla \cdot \nabla u = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dx^2}\right) u. \tag{8.22}$$

70. Def.—If ω is any vector function of position in space, we may define $\nabla \cdot \nabla\omega$ by the equation

$$\nabla \cdot \nabla\omega = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) \omega, \tag{8.23}$$

> the expression $\nabla\cdot\nabla$ being regarded, for the present at least, as a single operator when applied to a vector. (It will be remembered that no meaning has been attributed to ∇ before a vector.) Note that if
>
> $$\omega = iX + jY + kZ,$$
>
> then
>
> $$\nabla \cdot \nabla\omega = i\nabla \cdot \nabla X + j\nabla \cdot \nabla Y + k\nabla \cdot \nabla Z, \tag{8.24}$$
>
> that is, the operator $\nabla \cdot \nabla$ applied to a vector affects separately its scalar components.
>
> 71. From the above definition with those of Nos. 52 and 54, we may easily obtain
>
> $$\nabla \cdot \nabla\omega = \nabla\nabla \cdot \omega - \nabla \times \nabla \times \omega. \tag{8.25}$$
>
> The effect of the operator $\nabla \cdot \nabla$ is therefore independent of the direction of the axes used in its definition.

In quoting these sections, we have changed Gibbs's original notation for the divergence from $\nabla.\omega$ to $\nabla \cdot \omega$, that is, the period has been replaced by a dot. The equation numbers have been added for our reference later on.

After Gibbs revealed his new work on vector analysis, he was attacked fiercely by Tait, a chief advocate of the quaternion analysis, who stated [28, Preface]:

> Even Prof. Willard Gibbs must be ranked as one of the retarders of quaternion progress, in virtue of his pamphlet on vector analysis; a sort of hermaphrodite monster, compounded by the notations of Hamilton and Grassman.

This infamous statement has been quoted by many authors in the past. Gibbs's gentlemanly but firm response to Tait's attack [29]:

> The merit or demerits of a pamphlet printed for private distribution a good many years ago do not constitute a subject of any great importance, but the assumption implied in the sentence quoted are suggestive of certain reflections and inquiries which are of broad interest; and seem not untimely at a period when the methods and results of the various forms of multiple algebra are attracting so much attention. It seems to be assumed that a departure from quaternionic usage in the treatment of vectors is an enormity. If this assumption is true, it is an important truth; if not, it would be unfortunate if it should remain unchallenged, especially when supported by so high an authority. The criticism relates particularly to notations, but I believe that there is a deeper question of notions underlying that of notations. Indeed, if my offense had been solely in the matter of notation, it would have been less accurate to describe my production as a monstrosity, than to characterize its dress as uncouth.

Gibbs then went on to explain the advantage of his treatment of vector analysis compared with quaternion analysis. In the final part of that paper he stated:

> The particular form of signs we adopt is a matter of minor consequence. In order to keep within the resources of an ordinary printing office, I have used a dot and a cross, which are already associated with multiplication, which is best denoted by the simple juxtaposition of factors. I have no special predilection for these particular signs. The use of the dot is indeed liable to the objection that it interferes with its use as a separatrix, or instead of a parenthesis.

Although Gibbs considered his choice of the signs or notations a matter of minor importance, it was actually of great consequence, as will be shown in this study. Before we discuss his notations, a comment from Heaviside, generally considered by the scientific community as a cofounder with Gibbs of modern vector analysis, should be quoted. During the peak of the controversy between Tait and Gibbs, Heaviside made the following remark [26, p. 35]:

> Prof. W. Gibbs is well able to take care of himself. I may, however, remark that the modifications referred to are evidence of modifications felt to be needed, and that Prof. Gibbs' pamphlet (not published, New Haven, 1881–84, p. 83), is not a quaternionic treatise, but an able and in some respects original little treatise on vector analysis, though too condensed and also too advanced for learners' use, and that Prof. Gibbs, being no doubt a little touched by Prof. Tait's condemnation, has recently (in the pages of *Nature*) made a powerful defense of his position. He has by a long way the best of the argument, unless Prof. Tait's rejoinder has still to appear. Prof. Gibbs clearly separates the quaternionic question from the question of a suitable notation, and argues strongly against the quaternionic establishment of vector analysis. I am able (and am happy) to express a general concurrence of opinion with him about the quaternion and its comparative uselessness in practical vector analysis. As regards his notation, however, I do not like it. Mine is Tait's, but simplified, and made to harmonize with Cartesians.

There are two implications in Heaviside's remark that are of interest to us. When he considered Gibbs's pamphlet to be too condensed, it implies that some of the treatments may not have been obvious to him. Secondly, he stated his dislike for Gibbs's notations but without giving his reasons. The fact that Heaviside used some of Tait's quaternionic notations seems to indicate that he did not approve of Gibbs's notations at all. We now believe that many workers, including Heaviside, did not appreciate the most eloquent and complete theory of vector analysis formulated by Gibbs. For this reason, we would like to offer a digest of Gibbs's work so that we may have a clear understanding of his formulation.

8-3-2 Divergence and Curl Operators and Their New Notations

The basic definitions of the gradient, divergence, and the curl formulated by Gibbs are given by (8.18), (8.19), and (8.20). For convenience, we will make some changes in symbols to allow the convenience of using the summation sign. These changes are

$$x, y, z \quad \text{to} \quad x_1, x_2, x_3,$$
$$i, j, k \quad \text{to} \quad \hat{x}_1, \hat{x}_2, \hat{x}_3.$$

The old total derivative symbols will be replaced by partial derivatives and the Greek letters for vectors by boldface letters. Thus, Eqs. (8.17)–(8.19) become

$$\nabla u = \sum_i \hat{x}_i \frac{\partial u}{\partial x_i}, \tag{8.26}$$

$$\nabla \cdot \mathbf{F} = \sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i}, \tag{8.27}$$

$$\nabla \times \mathbf{F} = \sum_i \hat{x}_i \times \frac{\partial \mathbf{F}}{\partial x_i}. \tag{8.28}$$

It is understood that the summation goes from $i = 1$ to $i = 3$.

The most important information passed to us by Gibbs concerns the nomenclature for the notations in these expressions. In the title preceding Article 68 quoted previously, he designated ∇, $\nabla\cdot$, and $\nabla\times$ as operators. If we examine the expressions given by (8.26), (8.27), and (8.28) it is obvious that the gradient operator, or the del operator, is unmistakably given by

$$\nabla = \sum_i \hat{x}_i \frac{\partial}{\partial x_i}\,. \tag{8.29}$$

For the divergence, Gibbs used two symbols, a del followed by a dot, to denote his divergence operator. For the curl, he used a del followed by a cross to denote the curl operator. If we examine the expressions for the divergence and the curl defined by (8.27) and (8.28), it is clear that his two notations mean:

$$(\nabla\cdot)_{\mathrm{G}} \quad \rightarrow \quad \sum_i \hat{x}_i \cdot \frac{\partial}{\partial x_i}\,, \tag{8.30}$$

$$(\nabla\times)_{\mathrm{G}} \quad \rightarrow \quad \sum_i \hat{x}_i \times \frac{\partial}{\partial x_i}\,. \tag{8.31}$$

We emphasize this point by labeling his two notations with a subscript G, and we use an arrow instead of an equal sign to denote "a notation for."

According to our classification of the operators in Section 8.2, Gibbs's $(\nabla\cdot)_{\mathrm{G}}$ and $(\nabla\times)_{\mathrm{G}}$ are not compound operators; they are assemblies used by Gibbs as the notations for the divergence and curl. On the other hand, the terms at the right side of (8.30) and (8.31) are indeed compound operators, according to our classification. Because these operators are distinct from the gradient operator, we will introduce two notations for them. They are

$$ {\nabla\!\!\!\!\cdot}\; = \sum_i \hat{x}_i \cdot \frac{\partial}{\partial x_i}, \tag{8.32} $$

$$ {\nabla\!\!\!\!\times}\; = \sum_i \hat{x}_i \times \frac{\partial}{\partial x_i}. \tag{8.33} $$

They are called the *divergence operator* and the *curl operator*, respectively. Although these operators are so far defined in the rectangular coordinate system, we will demonstrate later that they are invariant to the choice of coordinate system. One important feature of ${\nabla\!\!\!\!\cdot}\;$ and ${\nabla\!\!\!\!\times}\;$ is that both these operators are independent of the gradient operator ∇. In other words, ∇ *is not a constituent of the divergence operator nor of the curl operator.* These two symbols are suggested by the appearance of the dot or the cross in between the unit vectors $\hat{x}_i$ and the partial derivatives $\partial/\partial x_i$ of the ∇ operator as defined by (8.29). In Gibbs's notations, $(\nabla\cdot)_G$ and $(\nabla\times)_G$, ∇ is a part of his notations for the divergence and the curl that leads to a serious misinterpretation by many later users and is a key issue in our study. With the introduction of these two new notations, Eqs. (8.18)–(8.26) become

$$ \nabla u = \sum_i \hat{x}_i \frac{\partial u}{\partial x_i}, \tag{8.34} $$

$$ {\nabla\!\!\!\!\cdot}\; \mathbf{F} = \sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i}, \tag{8.35} $$

$$ {\nabla\!\!\!\!\cdot}\; \mathbf{F} = \sum_i \frac{\partial F_i}{\partial x_i}, \tag{8.36} $$

$$ {\nabla\!\!\!\!\times}\; \mathbf{F} = \sum_i \hat{x}_i \times \frac{\partial \mathbf{F}}{\partial x_i}, \tag{8.37} $$

$$ {\nabla\!\!\!\!\times}\; \mathbf{F} = \sum_i \hat{x}_i \left(\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} \right) \tag{8.38} $$

with $(i, j, k) = (1, 2, 3)$ in cyclic order,

$$ {\nabla\!\!\!\!\cdot}\; \nabla u = \sum_i \frac{\partial^2 u}{\partial x_i^2}, \tag{8.39} $$

$$ {\nabla\!\!\!\!\cdot}\; \nabla \mathbf{F} = \sum_i \frac{\partial^2 \mathbf{F}}{\partial x_i^2}, \tag{8.40} $$

$$ {\nabla\!\!\!\!\cdot}\; \nabla \mathbf{F} = \sum_i \hat{x}_i {\nabla\!\!\!\!\cdot}\; \nabla F_i, \tag{8.41} $$

$$ {\nabla\!\!\!\!\cdot}\; \nabla \mathbf{F} = \nabla {\nabla\!\!\!\!\cdot}\; \mathbf{F} - {\nabla\!\!\!\!\times}\; {\nabla\!\!\!\!\times}\; \mathbf{F}. \tag{8.42} $$

In these formulas, the del operator only enters in the gradient of a scalar, (8.34), or of a vector, (8.40)–(8.42). Except for the notations for the divergence and the curl, we have not changed the content of Gibbs's work at all. These equations will be used later in our study of other people's presentations.

8-4 Book by Edwin Bidwell Wilson Founded upon the Lectures of J. Willard Gibbs

8-4-1 Gibbs's Lecture Notes

The first book on vector analysis by an American author was published in 1901. The author was Edwin Bidwell Wilson [30], then an instructor at Yale University. According to the general preface, the greater part of the material was taken from Prof. Gibbs's lectures on vector analysis delivered annually at Yale. A record of these lectures is preserved at Yale's Sterling Memorial Library: it is a clothbound book of notes, handwritten in ink on $8\frac{1}{2}$ in. × 11 in. ruled paper, about 290 pages long and consisting of 15 chapters [31]. The title page reads as follows:

Lectures Delivered upon Vector Analysis and its Applications to Geometry and Physics by Professor J. Willard Gibbs 1899–90 reported by Mr. E. B. Wilson

The table of contents is given in Table 8-3.

8-4-2 Wilson's Book

Presumably, Wilson's book (436 pages) is based principally on these notes. The preface states, however, that some use has been made of the chapters on vector analysis in Heaviside's *Electromagnetic Theory* (1893) and in Föppl's lectures

Table 8-3: Table of Contents of E. B. Wilson's *Lectures Delivered upon Vector Analysis*

on Maxwell's *Theory of Electricity* (1894). Apparently, Gibbs himself was not involved in the preparation of the body of the book, but he did contribute a preface, from which the following two paragraphs are taken:

> I was very glad to have one of the hearers of my course on Vector Analysis in the year 1899–1900 undertake the preparation of a text-book on the subject.
>
> I have not desired that Dr. Wilson should aim simply at the reproduction of my lectures, but rather that he should use his own judgment in all respects for the production of a text-book in which the subject should be so illustrated by an adequate number of examples as to meet the wants of students of geometry and physics.

In the general preface, Wilson stated:

> When I undertook to adapt the lectures of Professor Gibbs on Vector Analysis for publication in the Yale Bicentennial Series, Professor Gibbs himself was already so fully engaged in his work to appear in the same series, *Elementary Principles in Statistical Mechanics*, that it was understood no material assistance in the composition of this book could be expected from him. For this reason he wished me to feel entirely free to use my own discretion alike in the selection of the topics to be treated and in the mode

> of treatment. It has been my endeavor to use the freedom thus granted only in so far as was necessary for presenting his method in text-book form.

The following passage from Wilson's preface is particularly significant for the present discussion:

> It has been the aim here to give also an exposition of scalar and vector products of the operator ∇, of divergence and curl which have gained such universal recognition since the appearance of Maxwell's *Treatise on Electricity and Magnetism*, slope, potential, linear vector functions, etc. such as shall be adequate for the needs of students of physics at the present day and adapted to them.

We point out here that in Gibbs's pamphlets and in the lecture notes reported by Wilson, there is no mention of the scalar and vector products of the operator ∇. We believe this concept or interpretation was created by Wilson, and unfortunately it has had a detrimental effect upon the learning of vector analysis within the framework of Gibbs's original contributions.

In explaining the meaning of the divergence of a vector function, Wilson misinterpreted Gibbs's notation for this function, namely $\nabla \cdot \mathbf{F}$. After defining the ∇ operator for the gradient in a rectangular system as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\,, \tag{8.43}$$

he stated in Section 70, p. 150, of Wilson's book [30]:

> Although the operation $\nabla\mathbf{V}$ has not been defined and cannot be at present, two formal combinations of the vector operator ∇ and a vector function $\mathbf{V}$ may be treated. These are the (formal) scalar product and the (formal) vector product of ∇ into $\mathbf{V}$. They are:
>
> $$\nabla \cdot \mathbf{V} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \mathbf{V}, \tag{8.44}$$
>
> $$\nabla \times \mathbf{V} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \mathbf{V}. \tag{8.45}$$
>
> The differentiations $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, being scalar operators, pass by the dot and the cross, that is
>
> $$\nabla \cdot \mathbf{V} = \left(i \cdot \frac{\partial \mathbf{V}}{\partial x} + j \cdot \frac{\partial \mathbf{V}}{\partial y} + k \cdot \frac{\partial \mathbf{V}}{\partial z}\right), \tag{8.46}$$
>
> $$\nabla \times \mathbf{V} = \left(i \times \frac{\partial \mathbf{V}}{\partial x} + j \times \frac{\partial \mathbf{V}}{\partial y} + k \times \frac{\partial \mathbf{V}}{\partial z}\right). \tag{8.47}$$
>
> They may be expressed in terms of the components V_1, V_2, V_3 of $\mathbf{V}$.

We have identified the equations with our own numbers. In order to compare these expressions with Gibb's expressions now described by (8.34) to (8.38), we again will change the notations $\mathbf{V}, x, y, z, i, j, k$ to $\mathbf{F}, x_1, x_2, x_3, \hat{x}_1, \hat{x}_2, \hat{x}_3$; and $\nabla \cdot \mathbf{V}$ and $\nabla \times \mathbf{V}$ to $\overset{\cdot}{\nabla}\mathbf{F}$ and $\overset{\times}{\nabla}\mathbf{F}$. Equations (8.43) to (8.47) then become

$$\nabla = \sum_i \hat{x}_i \frac{\partial}{\partial x_i}\,, \tag{8.48}$$

$$\overset{\cdot}{\nabla}\mathbf{F} = \left(\sum_i \hat{x}_i \frac{\partial}{\partial x_i}\right) \cdot \mathbf{F}, \tag{8.49}$$

$$\overset{\times}{\nabla}\mathbf{F} = \left(\sum_i \hat{x}_i \frac{\partial}{\partial x_i}\right) \times \mathbf{F}, \tag{8.50}$$

$$\overset{\cdot}{\nabla}\mathbf{F} = \sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i}\,, \tag{8.51}$$

$$\overset{\times}{\nabla}\mathbf{F} = \sum_i \hat{x}_i \times \frac{\partial \mathbf{F}}{\partial x_i}\,. \tag{8.52}$$

Equations (8.48), (8.51), and (8.52) are identical to Gibbs's (8.29), (8.35), and (8.37). However, (8.49) and (8.50) are not found in Gibbs's works. Wilson obtained or derived (8.51) and (8.52) from (8.49) and (8.50). The derivation involves two crucial steps or assumptions. First, he considers Gibbs's notations $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ as "formal" scalar and vector products between ∇ and $\mathbf{F}$. In the following, we will refer to this model as the FSP (formal scalar product) and FVP (formal vector product). He did not explain the meaning of the word *formal*. Secondly, after he formed the FSP and FVP, he let the differentiation $\partial/\partial x_i$ "pass by" the dot and the cross with the argument that the differentiations $\partial/\partial x_i$ ($i = 1, 2, 3$) are scalar operators. Wilson's statement appears to be quite firm, but the standard books on mathematical analysis contain no proofs of any such theorem. Later on [30, p. 152], Wilson attempts to modify his position by saying:

> From some standpoints objections may be brought forward against treating ∇ as a symbolic vector and introducing $\nabla \cdot \mathbf{V}$ and $\nabla \times \mathbf{V}$ as the symbolic scalar and vector products of ∇ into $\mathbf{V}$, respectively. These objections may be avoided by simply laying down the definition that the symbol $\nabla\cdot$ and $\nabla\times$, which may be looked upon as entirely new operators quite distinct from ∇, shall be
>
> $$\nabla \cdot \mathbf{V} = i \cdot \frac{\partial \mathbf{V}}{\partial x} + j \cdot \frac{\partial \mathbf{V}}{\partial y} + k \cdot \frac{\partial \mathbf{V}}{\partial z} \tag{8.53}$$

and

$$\nabla \times \mathbf{V} = i \times \frac{\partial \mathbf{V}}{\partial x} + j \times \frac{\partial \mathbf{V}}{\partial y} + k \times \frac{\partial \mathbf{V}}{\partial z} . \tag{8.54}$$

But for practical purposes and for remembering formulas, it seems by all means advisable to regard

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

as a symbolic vector differentiator. This symbol obeys the same laws as a vector just in so far as the differentiations $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ obey the same laws as ordinary scalar quantities.

The contradictions between Wilson's statement above and his assertion concerning FSP and FVP are evident. Equations (8.53) and (8.54), of course, are the same as Gibbs's (8.27) and (8.28) with $\mathbf{V}$ replaced by $\mathbf{F}$ and x, y, z, i, j, k replaced by $x_1, x_2, x_3, \hat{x}_1, \hat{x}_2, \hat{x}_3$. The difference is that Gibbs never spoke of an FSP and FVP; Wilson introduced these concepts to derive the expressions for div $\mathbf{F}$ and curl $\mathbf{F}$ by imposing some nonvalid manipulations. What is the consequence? Many later authors followed his practice and encountered difficulties when the same treatment was applied to orthogonal curvilinear coordinate systems. Before we discuss this topic, we must review Heaviside's treatment of vector analysis, particularly his handling of ∇.

We have pointed out that Gibbs's pamphlets were communicated to Heaviside. On the other hand, Wilson also mentioned some use of Heaviside's treatment of vector analysis in the preface to his book, *Electromagnetic Theory* (1893). The exchange between Heaviside and Wilson was therefore reciprocal. However, Heaviside goes his own way in presenting the same topics. Before we turn to the next section, Wilson's FSP and FVP model will be analytically examined. If we start with one of Gibbs's definitions of divergence, without using his notation but rather by using the linguistic notation, that is,

$$\operatorname{div} \mathbf{F} = \sum_i \frac{\partial F_i}{\partial x_i} , \tag{8.55}$$

then, by substituting $F_1 = \hat{x}_1 \cdot \mathbf{F}$ into (8.55), we find

$$\operatorname{div} \mathbf{F} = \sum_i \frac{\partial(\hat{x}_i \cdot \mathbf{F})}{\partial x_i} = \sum_i \left[\hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} + \frac{\partial \hat{x}_i}{\partial x_i} \cdot \mathbf{F}\right] . \tag{8.56}$$

Because $\partial \hat{x}_i / \partial x_i = 0$, (8.56) reduces to

$$\operatorname{div} \mathbf{F} = \sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} , \tag{8.57}$$

which is obviously not equal to

$$\left(\sum \hat{x}_i \frac{\partial}{\partial x_i}\right) \cdot \mathbf{F},$$

or $\nabla \cdot \mathbf{F}$. This is a proof of the lack of validity of the FSP. A similar proof can be executed with respect to the FVP. Another demonstration of the fallacy of an FSP is to consider a "twisted" differential operator of the form

$$\nabla_t = \hat{x}_2 \frac{\partial}{\partial x_1} + \hat{x}_3 \frac{\partial}{\partial x_2} + \hat{x}_1 \frac{\partial}{\partial x_3} \tag{8.58}$$

and a "twisted" vector function defined by

$$\mathbf{F}_t = \hat{x}_2 F_1 + \hat{x}_3 F_2 + \hat{x}_1 F_3. \tag{8.59}$$

If the FSP were a valid product, then, by following Wilson's pass-by procedure, we obtain

$$\nabla_t \cdot \mathbf{F}_t = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}\,. \tag{8.60}$$

In other words, div $\mathbf{F}$ is now treated as the formal scalar product between ∇_t and $\mathbf{F}_t$. The result is the same as Wilson's FSP between ∇ and $\mathbf{F}$. Such a manipulation is not, of course, a valid mathematical procedure. We have now refuted Wilson's treatment of div $\mathbf{F}$ and curl $\mathbf{F}$ based on the FSP and FVP. The legitimate compound differential operators for the divergence and the curl are, respectively, $\odot\!\!\!\!\nabla$ and $\times\!\!\!\!\nabla$, defined by (8.32) and (8.33). $(\nabla\cdot)_G$ and $(\nabla\times)_G$ are merely Gibbs's notations suggested for the divergence and the curl. They are not operators.

8-4-3 The Spread of the Formal Scalar Product (FSP) and Formal Vector Product (FVP)

Being the first book on vector analysis published in the United States, Wilson's book became quite popular. It received its eighth printing in 1943, and a paperback reprint by Dover Publications appeared in 1960. Many later authors freely adopted Wilson's presentation using the FSP and FVP to derive the expressions for divergence and curl in the Cartesian coordinate system. We have found over 50 books [32] containing such a treatment. We now quote a few examples to show Wilson's influence.

1. In the book by Weatherburn [24] published in 1924, we find the following statement:

> To justify the notation, we have only to expand the formal products according to the distributive law, then
>
> $$\nabla \cdot \mathbf{f} = \left[\sum_i \left(\hat{x}_i \frac{\partial}{\partial x_i}\right)\right] \cdot \mathbf{f} = \sum_i \frac{\partial f_i}{\partial x_i} = \text{div}\,\mathbf{f}.$$

We remark here that any distributive law in mathematics must be proved. In this case, there is no distributive law to speak of because the author is dealing with an assembly of mathematical symbols and not a compound operator. Incidentally, Weatherburn's book appears to be the first book published in England wherein Gibbs's notations, but not Heaviside's, have been used in addition to the linguistic notations, namely, grad u, div $\mathbf{f}$, and curl $\mathbf{f}$.

2. A book by Lagally [33] published in 1928 contains the following statement on p. 123 (the original text is in German):

   The rotation (curl) of $\mathbf{f}$ is denoted by the vector product between ∇ with field function $\mathbf{f}$ ... and

   $$\text{div grad} f = \nabla \cdot \nabla f = \left(\sum_i \hat{x}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_i \hat{x}_j \frac{\partial f}{\partial x_j}\right) = \nabla^2 f.$$

   It is seen that a term like $\hat{x}_1(\partial/\partial x_1) \cdot \hat{x}_1(\partial/\partial x_1)$ is an assembly of symbols. It is not a compound operator.

3. In a book by Mason and Weaver [34, p. 336] we find the following statement:

   The differential operator ∇ can be considered formally as a vector of components $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, so that its scalar and vector products with another vector may be taken.

   In comparison with Wilson's treatment, Mason and Weaver have used the word *formally* to be associated with ∇ and then speak of scalar and vector products with vector functions.

4. In his book *Applied Mathematics*, Schelkunoff first derived the differential expression for the divergence based on the flux model [35, p. 126]; then he added:

   In Section 6 the vector operator del was introduced. If we treat it as a vector and multiply it by a vector $\mathbf{F}$, we find

   $$\nabla \cdot \mathbf{F} = \left(\sum_i \hat{x}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_j \hat{x}_j F_j\right) = \sum_i \frac{\partial F_i}{\partial x_i} = \text{div } \mathbf{F}.$$

   For this reason, $\nabla\cdot$ may be used as an alternative for div; however, the notation is tied too specifically to Cartesian coordinates.

   There are two messages in this statement: the first one is his acceptance of the FSP as a valid entity. The second one is his implication that FSP only applies to the Cartesian system. Actually, the divergence operator, ∇, is invariant with respect to the choice of the coordinate system, a property shown in Chapter 4, but $\nabla\cdot$ is an assembly, not an operator. Only by means of an illegitimate manipulation does it yield the differential expression for the divergence in the Cartesian coordinate system.

5. In a well-known book by Feynman, Leighton, and Sands [36, pp. 2–7] we find the following statement:

Let us try the dot product between ∇ with a vector field that we know, say $\mathbf{f}$; we write

$$\nabla \cdot \mathbf{f} = \nabla_x f_x + \nabla_y f_y + \nabla_z f_z$$

or

$$\nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} .$$

The authors remarked on the same page before this statement:

> With operators we must always keep the sequence right, so that the operations make the proper sense ....

This remark is important. Our discussion and use of the operators in Section 8-2, particularly that related to the compound operators, closely adheres to this principle. In the case of Gibbs's notation, $\nabla \cdot \mathbf{f}$, we are faced with a dot symbol after ∇, so that the differentiation cannot be applied to $\mathbf{f}$; it is blocked by a dot in the assembly. Thus, the authors seem to have violated their own rule by trying to form a dot product or FSP.

6. In the English translation of a Russian book by Borisenko and Tarapov [2, p. 157], we find the following statement:

> The expression $\nabla = \sum i_k(\partial/\partial x_k)$ for the operator ∇ implies the following representation for the divergence of $\mathbf{A}$:
>
> $$\text{div } \mathbf{A} = \frac{\partial A_k}{\partial x_k} = i_k \frac{\partial}{\partial x_k} \cdot \mathbf{A} = \nabla \cdot \mathbf{A}.$$
>
> A coordinate-free symbolic representation of the operator ∇ is
>
> $$\nabla(\ldots) = \lim_{V \to 0} \frac{1}{V} \oint_s \mathbf{N}(\ldots)\, dS \qquad (8.61)$$
>
> where $(\ldots)$ is some expression (possibly preceded by a dot or a cross) on which the given operator acts. In fact, according to (4.31) and (4.29) [of their book],
>
> $$\text{grad } \varphi = \lim_{V \to 0} \frac{1}{V} \oint_s \varphi \mathbf{N}\, dS, \qquad (8.62)$$
>
> $$\text{div } \mathbf{A} = \lim_{V \to 0} \frac{1}{V} \oint_s \mathbf{A} \cdot \mathbf{N}\, dS. \qquad (8.63)$$

From this passage, we see that the two authors believe the FSP is valid. Equation (8.61) also implies that they consider ∇ to be a constituent of the divergence and the curl in addition to comprising the gradient operator. The formula described by (8.61) appeared earlier in the book by Gans [37,

p. 49], who used both Gibbs's notations and the linguistic notations in this edition.

There are several authors presenting ∇ as defined as $\sum(\partial/\partial x_i)\hat{x}_i$ instead of $\sum \hat{x}_i(\partial/\partial x_i)$; and the Laplacian, defined as div grad, is often treated as the scalar product between two nablas, presumably because Gibbs used $\nabla \cdot \nabla$ as the notation for this compound operator. These practices, including the use of an FSP and FVP, are not confined within the boundaries of the United States and continental Europe; some Chinese and Japanese books, for example, commit the same errors.

8-5 ∇ in the Hands of Oliver Heaviside (1850–1925)

Although we have traced the concept of the FSP and FVP as due to Wilson, the same practice is found in the works of Heaviside. In Volume I of his book *Electromagnetic Theory* [26, §127] published in 1893, Heaviside stated:

> When the operand of ∇ is a vector, say $\mathbf{D}$, we have both the scalar product and the vector product to consider. Taking the formula along first, we have
>
> $$\text{div } \mathbf{D} = \nabla_1 D_1 + \nabla_2 D_2 + \nabla_3 D_3.$$
>
> This function of $\mathbf{D}$ is called the divergence and is an important function in physical mathematics.

He then considered the curl of a vector function as the vector product between ∇ and that vector. At the time of his writing, he was already aware of Gibbs's pamphlets on vector analysis but Wilson's book was not yet published. It seems, therefore, that Heaviside and Wilson independently introduced the misleading concept for the scalar and vector products between ∇ and a vector function. Both were, perhaps, induced by Gibbs's notations for the divergence and the curl. Heaviside did not even include the word *formal* in his description of the products. We should mention that Heaviside's notations for these two products and the gradient are not the same as Gibbs's (see the table of notations in Section 2.1). His notation for the divergence of $\mathbf{f}$ is $\nabla\mathbf{f}$ and his notation for the curl of $\mathbf{f}$ is $V\nabla\mathbf{f}$ (a quaternion notation), while his notation for the gradient of a scalar function f is $\nabla.f$. Having treated $\nabla\cdot\mathbf{f}$ and $\nabla\times\mathbf{f}$ (Gibbs's notations for the divergence and the curl) as two "products," Heaviside simply considered ∇ as a vector in deriving various differential identities. One of them was presented as follows [26, §132]:

> The examples relate principally to the modification introduced by the differentiating functions of ∇.
>
> (a) We have the parallelopiped property
>
> $$\mathbf{N}V\nabla\mathbf{E} = \nabla V\mathbf{EN} = \mathbf{E}V\mathbf{N}\nabla \tag{176}$$
>
> where ∇ is a common vector. The equations remain true when ∇ is vex, provided we consistently employ the differentiating power in the three forms.

Thus, the first form, expressing $\mathbf{N}$ component of curl $\mathbf{E}$, is not open to misconception. But in the second form, expressing the divergence of $V\mathbf{EN}$, since $\mathbf{N}$ follows ∇, we must understand that $\mathbf{N}$ is supposed to remain constant. In the third form, again, the operand $\mathbf{E}$ precedes the differentiator; we must either, then, assume that ∇ acts backwards, or else, which is preferable, change the third form to $V\mathbf{N}\nabla.\mathbf{E}$, the scalar product of $V\mathbf{N}\nabla$ and $\mathbf{E}$, or $(V\mathbf{N}\nabla)\mathbf{E}$ if that is plainer.

(b) Suppose, however, that both vectors in the vector product are variable. Thus, required the divergence of $V\bar{E}\bar{H}$, expanded vectorially. We have,

$$\nabla V\mathbf{EH} = \mathbf{E}V\mathbf{H}\nabla = \mathbf{H}V\nabla\mathbf{E}, \tag{177}$$

where the first form alone is entirely unambiguous. But we may use either of the others, provided that the differentiating power of ∇ is made to act on both $\mathbf{E}$ and $\mathbf{H}$. But if we keep to the plainer and more usual convention that the operand is to follow the operator, then the third term, in which $\mathbf{E}$ alone is differentiated, gives one part of the result, whilst the second form, or rather its equivalent, $-\mathbf{E}\nabla\mathbf{H}$, wherein $\mathbf{H}$ alone is differentiated, gives the rest. So we have, complete, and without ambiguity

$$\operatorname{div} V\mathbf{EH} = \mathbf{H} \operatorname{curl} \mathbf{E} - \mathbf{E} \operatorname{curl} \mathbf{H}, \tag{178}$$

an important transformation.

First of all, in terms of Gibbs's notations, Heaviside's Eqs. (176), (177), and (178) would be written in the form

$$\mathbf{N}\cdot\nabla\times\mathbf{E} = \nabla\cdot(\mathbf{E}\times\mathbf{N}) = \mathbf{E}\cdot(\mathbf{N}\times\nabla), \tag{8.64}$$

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{E}\cdot(\mathbf{H}\times\nabla) = \mathbf{H}\cdot(\nabla\times\mathbf{E}), \tag{8.65}$$

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H}. \tag{8.66}$$

According to the established mathematical rules, Heaviside's logic in arriving at his (178) or our (8.66) is entirely unacceptable; in particular, present-day students would never write an equation (177) or (8.65) with ∇ being the ∇ operator. The second term in (8.65) is a weighted operator, while the first and the third are functions and they are not equal to each other. His Eq. (178) or (8.66) in Gibbs's notation is a valid vector identity but his derivation of this identity is not based on established mathematical rules. It is obtained by a manipulation of mathematical symbols and selecting the desired forms. The most important message passed on to us is his practice of considering $\nabla\cdot\mathbf{f}$ and $\nabla\times\mathbf{f}$ as two legitimate products, the same as Wilson's FSP and FVP. Heaviside's "equations" will be examined again in a later section and will be cast in proper form in terms of the symbolic vector and/or a partial symbolic vector.

Many authors in the past have considered Heaviside to be a cofounder with Gibbs of modern vector analysis. We do not share this view. In Heaviside's treatment of vector analysis, he spoke freely of the scalar product and the vector product between ∇ and a vector function **F**, and he used ∇ as a vector in deriving algebraic vector identities that incorporate differential entities. In view of these mathematically insupportable treatments, Heaviside's status as a pioneer in vector analysis is not of the same level as Gibbs's. In the historical introduction of a 1950 edition of Heaviside's book on *Electromagnetic Theory* [26], Ernst Weber stated:

> Chap. III of the *Electromagnetic Theory* dealing with 'The Elements of Vectorial Algebra and Analysis' is practically the model of modern treatises on vector analysis. Considerable moral assistance came from a pamphlet by J. W. Gibbs who independently developed vector analysis during 1881–84 in Heaviside sense—but using the less attractive notation of Tait; however, Gibbs deferred publication until 1901.

This statement unfortunately contains several misleading messages. In the first place, in view of our detailed study of Heaviside's works, his treatment would be a poor model if it were used to teach vector calculus. Secondly, if Heaviside truly received moral assistance from Gibbs's pamphlet, he would not have committed himself to the improper use of ∇, and would have restricted his use of it to the expression for the gradient. Most important of all, Gibbs did not develop his theory in the Heaviside sense. His development is completely different from that of Heaviside. Finally, the book published in 1901 was written by Wilson, not by Gibbs himself. Even though it was founded upon the lectures of Gibbs, it contained some of Wilson's own interpretations, which are not found in Gibbs's original pamphlets nor in his lecture notes reported by Wilson. The two prefaces, one by Gibbs and another by Wilson, which we quote in Section 8-4-2, are proofs of our assertion. We were reluctant to criticize a scientist of Heaviside's status and the opinion expressed by Prof. Weber. After all, Heaviside had contributed much to electromagnetic theory and had been recognized as a rare genius. However, in the field of vector analysis, we must set the record straight and call attention to the outstanding contribution of Gibbs, who stood above all his contemporaries in the last century. For the sake of future generations of students, we have the obligation to remove unsound arguments and arbitrary manipulations in an otherwise precise branch of mathematical science.

8-6 Shilov's Formulation of Vector Analysis

A book in Russian on vector analysis was written by Shilov [38] in 1954, who advocated a new formulation with the intent of providing a rather broader treatment of vector analysis. Shilov's work was adopted by Fang [39], who studied in the U.S.S.R. We were informed of Shilov's work through Fang. After a careful examination of the English translation of the two key chapters in that book, we found the contradictions as described below:

Shilov defined an "expression" for ∇ denoted by $T(\nabla)$ as

$$T(\nabla) = \frac{\partial}{\partial x}T(i) + \frac{\partial}{\partial y}T(j) + \frac{\partial}{\partial z}T(k), \tag{8.67}$$

where i, j, k denote the Cartesian unit vectors and ∇ (nabla) is identified as the Hamilton differential operator, that is,

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

Equation (8.67) is the same as Shilov's Eq. (18) on p. 18 of [35]. We want to emphatically call attention to the fact that the only meaningful expression for $T(\nabla)$ involving ∇ are ∇f and $\nabla\mathbf{f}$, the gradient of f and $\mathbf{f}$. In the case of ∇f, (8.67) is an identity, because the right side of (8.67) yields

$$\frac{\partial}{\partial x}(if) + \frac{\partial}{\partial y}(jf) + \frac{\partial}{\partial z}(kf) = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z},$$

which is ∇f.

The most serious contradiction in Shilov's work is his derivation of the expression for the divergence and the curl by letting $T(\nabla)$ equal to $\nabla \cdot \mathbf{f}$ and $\nabla \times \mathbf{f}$, respectively. We have pointed out before that these two products do not exist. Shilov is defining a meaningless assembly to make it meaningful. It is like defining $2 + \times 3$ to be equal to $2 \times +3$ $(= +6)$.

8-7 Formulations in Orthogonal Curvilinear Systems

After having revealed a number of "historical" confusions and contradictions in vector analysis so far presented in the rectangular system, we now examine several presentations in curvilinear coordinate systems. We will show even more clearly the sources of the various misrepresentations.

8-7-1 Two Examples from the Book by Moon and Spencer

In their book, Moon and Spencer write [23, p. 325]:

> Let me apply the definition, Eq. (1.4) [of ∇ in the orthogonal curvilinear system, our (7.20)], to divergence. By the usual definition of a scalar product,
>
> $$\nabla \cdot \mathbf{V} = \frac{1}{(g_{11})^{1/2}}\frac{\partial(V)_1}{\partial x^1} + \frac{1}{(g_{22})^{1/2}}\frac{\partial(V)_2}{\partial x^2} + \frac{1}{(g_{33})^{1/2}}\frac{\partial(V)_3}{\partial x^3}. \tag{8.68}$$
>
> But this is *not* divergence, which is found to be ....
>
> Similar inconsistencies are obtained with other applications of Eq. (1.4).

In (8.68), their $(g_{ii})^{1/2}$ correspond to our metric coefficients h_i and their x^i to our variables v_i.

In the first place, they have now applied the FSP to ∇ and $\mathbf{V}$ in an orthogonal curvilinear coordinate system without realizing that the FSP is not a valid entity in any coordinate system including the rectangular system. After obtaining a wrong formula for the divergence, (8.68), they did not offer an explanation of the reason for the failure.

In discussing the Laplacian of a vector function, Moon and Spencer state [23, p. 235]:

> Section (7.08) showed that there are three meaningful combinations of differential operators: div grad, grad div, and curl curl. Of these, the first is the scalar Laplacian, ∇^2. It is convenient to combine the other two operators to form the vector Laplacian, ✡ :
>
> $$\text{✡} = \text{grad div} - \text{curl curl}. \tag{8.69}$$
>
> Evidently the vector Laplacian can operate only on a vector, so
>
> $$\text{✡}\,\mathbf{E} = \text{grad div}\,\mathbf{E} - \text{curl curl}\,\mathbf{E}. \tag{8.70}$$
>
> Since the quantities on the right are vectors, ✡ **E** transforms as a univalent tensor or vector.
>
> As noted in Table 1.01 [their table of notations on p. 10], the scalar and vector Laplacians are often represented by the same symbol. This is poor practice, however, since the two are basically quite different:
>
> $$\nabla^2 = \text{div grad}, \tag{8.71}$$
>
> $$\text{✡} \equiv \text{grad div} - \text{curl curl}. \tag{8.72}$$
>
> This difference is evident also when the expression for the vector Laplacian is expanded....

Analytically, we have proved that in any general curvilinear system,

$$\nabla\cdot\nabla\mathbf{f} = \nabla\nabla\cdot\mathbf{f} - \nabla\times\nabla\times\mathbf{f}, \tag{8.73}$$

where $\nabla\mathbf{f}$ denotes the gradient of a vector function that is a dyadic function. The divergence of a dyadic function is a vector function. The use of ∇^2 to denote the Laplacian is an old practice, but the use of $\nabla\cdot\nabla$ is preferred because it shows the structure of the Laplacian when it is applied to either a scalar function or a vector function. By treating (8.69) as the definition for the Laplacian applied to a vector function, the two authors have probably been influenced by a remark made by Stratton [5, p. 50]:

> The vector $\nabla\cdot\nabla\mathbf{F}$ may now be obtained by subtraction of (85) [an expansion of $\nabla\times\nabla\times\mathbf{F}$ in an orthogonal curvilinear system] from the expansion of $\nabla\nabla\cdot\mathbf{F}$, and the result differs from that which follows a direct application of the Laplacian to the curvilinear components of **F**.

As shown in our proof, $\nabla \mathbf{F}$ is a dyadic, where the gradient operator must apply to the entire vector function containing both the components and the unit vectors. When this is done, we find that (8.73) is indeed an identity. In view of our analysis, it is clear that a special symbol for the Laplacian is not necessary when it is operating on a vector function. The same remark holds true for the two different notations for the Laplacian introduced by Burali-Forti and Marcolongo, as shown in Table 8.1.

These two examples also show why Moon and Spencer thought that ∇ is an unreliable device. The past history of vector analysis seems to have led them to such a conclusion. ∇ is a reliable device when it is used in the gradient of a scalar or vector function, but not in any other application. We emphasize once more that for the divergence and the curl, the divergence operator, $\nabla\!\!\!\!\cdot\;$, and the curl operator, $\nabla\!\!\!\!\times$, are the proper operators. They are distinctly different from ∇.

8-7-2 A Search for the Divergence Operator in Orthogonal Curvilinear Coordinate Systems

In a well-known book on the methods of theoretical physics [40, p. 44], the authors, Morse and Feshbach, try to find the differential operators for the three key functions in an orthogonal curvilinear coordinate system. They state:

The vector operator must have different forms for its different uses:

$$\nabla = \sum_i \frac{\hat{u}_i}{h_i} \frac{\partial}{\partial v_i} \quad \text{for the gradient}$$

$$= \frac{1}{\Omega} \sum_i \hat{u}_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \right) \quad \text{for the divergence}$$

and no form which can be written for the curl.

We have used Ω to represent $h_1 h_2 h_3$ and have changed their coordinate variables ξ_i to v_i and their symbols a_i to $\hat{u}_i$. It is obvious that the "operator" introduced by these two authors for the divergence can produce the correct expression for the divergence only if the operation is interpreted as

$$\left[\frac{1}{\Omega} \sum_i \hat{u}_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \right) \right] \cdot \mathbf{f} \quad \rightarrow \quad \frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \hat{u}_i \cdot \mathbf{f} \right). \tag{8.74}$$

Such an interpretation is quite arbitrary, and it does not follow the accepted rule of a differential operator because the first term within the bracket is a function, so the entire expression represents the scalar product of $[\cdots]$ and $\mathbf{f}$. One is not supposed to move the unit vector $\hat{u}_i$ to the right side of Ω/h and then combine $\hat{u}_i$ with $\cdot\mathbf{f}$, as shown in the right term of (8.74). It is a matter of creating a desired expression by arbitrarily rearranging the terms in a function and the position of the dot operator. A reader must recognize now that ∇ can never be a part of the divergence operator nor the curl operator. The proper operators for the divergence and the curl are $\nabla\!\!\!\!\cdot\;$

and $\nabla\!\!\!\!\!\times$, respectively. We could have used any two symbols for that matter, such as **D** and **C**.

8-8 The Use of ∇ to Derive Vector Identities

There are many authors who have tried to apply identities in vector algebra to "derive" vector identities involving the differential functions ∇f, $\nabla\!\!\!\!\cdot\; f$, and $\nabla\!\!\!\!\!\times \mathbf{f}$. We quote here two examples. The first example is from the book by Borisenko and Tarapov [2, p. 180], where a problem is posed and "solved":

> Prob. 7. Find $\nabla(\mathbf{A}\cdot\mathbf{B})$.
>
> Solution. Clearly
>
> $$\nabla(\mathbf{A}\cdot\mathbf{B}) = \nabla(\mathbf{A}_c\cdot\mathbf{B}) + \nabla(\mathbf{A}\cdot\mathbf{B}_c) \tag{8.75}$$
>
> where the subscript 'c' has the same meaning as on p. 170 [the subscript 'c' denotes that the quantity to which it is attached is momentarily being held fixed]. According to formula (1.30)
>
> $$\mathbf{c}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\mathbf{c})\mathbf{B} - \mathbf{A}\times(\mathbf{B}\times\mathbf{c}). \tag{8.76}$$
>
> Hence setting
>
> $$\mathbf{A} = \mathbf{A}_c, \qquad \mathbf{B} = \mathbf{B}, \qquad \mathbf{c} = \nabla,$$
>
> we have
>
> $$\nabla(\mathbf{A}_c\cdot\mathbf{B}) = (\mathbf{A}_c\cdot\nabla)\mathbf{B} + \mathbf{A}_c\times(\nabla\times\mathbf{B}), \tag{8.77}$$
>
> and similarly,
>
> $$\nabla(\mathbf{A}\cdot\mathbf{B}_c) = \nabla(\mathbf{B}_c\cdot\mathbf{A}) = (\mathbf{B}_c\cdot\nabla)\mathbf{A} + \mathbf{B}_c\times(\nabla\times\mathbf{A}). \tag{8.78}$$
>
> Thus, finally,
>
> $$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times\ \text{curl}\,\mathbf{B} + \mathbf{B}\times\ \text{curl}\,\mathbf{A}. \tag{8.79}$$

As far as the final result, (8.79), is concerned, they have indeed obtained a correct answer. But there is no justification for applying (8.76) with **c** replaced by ∇.

The second example is found in the book by Panofsky and Phillips [41, p. 470]. They wrote:

> $$\begin{aligned}\nabla\times(\mathbf{A}\times\mathbf{B}) &= (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B}\\ &= (\nabla\cdot\mathbf{B}_c)\mathbf{A} - (\nabla\cdot\mathbf{B})\mathbf{A}_c\\ &\quad - (\nabla\cdot\mathbf{A}_c)\mathbf{B} - (\nabla\cdot\mathbf{A})\mathbf{B}_c\end{aligned} \tag{8.80}$$
>
> where the subscript 'c' indicates that the function is constant and may be permuted with the vector operator, with due regard to sign changes if such changes are indicated by the ordinary vector relations.

It is seen that their $(\nabla \cdot \mathbf{B})\mathbf{A}$ in the first line is not (div $\mathbf{B}$)$\mathbf{A}$. Rather, it is equal to $(\nabla \cdot \mathbf{B}_c)\mathbf{A} + (\nabla \cdot \mathbf{B})\mathbf{A}_c$. Secondly, if $\mathbf{B}_c$ is constant, the established rule in differential calculus would consider their $\nabla \cdot \mathbf{B}_c$, (i.e., div $\mathbf{B}_c$) $= 0$. The use of algebraic identities to derive differential identities by replacing a vector by ∇ has no foundation—the first line of (8.80). For the exercise in consideration, one way to find the identity is to prove first that

$$\overset{\times}{\nabla} (\mathbf{A} \times \mathbf{B}) = \dot{\nabla} (\mathbf{BA} - \mathbf{AB}), \tag{8.81}$$

where $\mathbf{AB}$ is a dyadic and $\mathbf{BA}$ its transpose. Then, by means of dyadic analysis, one finds

$$\dot{\nabla} (\mathbf{BA}) = (\dot{\nabla} \mathbf{B})\mathbf{A} + \mathbf{B} \cdot \nabla \mathbf{A}, \tag{8.82}$$

$$\dot{\nabla} (\mathbf{AB}) = (\dot{\nabla} \mathbf{A})\mathbf{B} + \mathbf{A} \cdot \nabla \mathbf{B}. \tag{8.83}$$

Hence

$$\overset{\times}{\nabla} (\mathbf{A} \times \mathbf{B}) = (\dot{\nabla} \mathbf{B})\mathbf{A} + \mathbf{B} \cdot \nabla \mathbf{A} - (\dot{\nabla} \mathbf{A})\mathbf{B} - \mathbf{A} \cdot \nabla \mathbf{B}, \tag{8.84}$$

where $\nabla \mathbf{A}$ and $\nabla \mathbf{B}$ are two dyadic functions. A simpler method of deriving (8.84) will be shown in a later section. It should be emphasized that one cannot legitimately write

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B}$$

as the two authors did and then change $(\nabla \cdot \mathbf{B})\mathbf{A}$ to $\nabla \cdot (\mathbf{BA})$, and similarly for $(\nabla \cdot \mathbf{A})\mathbf{B}$, in order to create a desired identity.

A general comment on the analogy and no analogy between algebraic vector identities and differential vector identities was made by Milne [42]. He states on p. 77:

> The above examples [referring to nine differential vector identities expressed in linguistic notations such as grad $(\hat{x} \cdot \mathbf{Y}) =$ (grad $\hat{x}$) $\cdot\, \mathbf{Y} +$ (grad $\mathbf{Y}$) $\cdot\, \hat{x}$, etc.] whilst exhibiting the relations between the symbols in vector or tensor form, conceal the nature of the identities. A little gain of insight is obtained occasionally if the symbol is employed. E.g., Example (9) [curl curl $\hat{x} =$ grad div $\hat{x} - \nabla^2 \hat{x}$] may be written
>
> $$\nabla \times (\nabla \times \hat{x}) = \nabla(\nabla \cdot \hat{x}) - \nabla^2 \hat{x}, \tag{8.85}$$
>
> which bears an obvious analogy to
>
> $$\mathbf{Q} \times (\mathbf{Q} \times \hat{x}) = \mathbf{Q}(\mathbf{Q} \cdot \hat{x}) - \mathbf{Q}^2 \hat{x} \tag{8.86}$$
>
> where $\mathbf{Q}$ denotes a vector function.
>
> On the other hand Example (5)
>
> $$\text{Curl } (\hat{x} \times \mathbf{Y}) = \mathbf{Y} \cdot \text{grad } \hat{x} - \hat{x} \cdot \text{grad } \mathbf{Y} + \hat{x} \text{ div } \mathbf{Y} - \mathbf{Y} \text{ div } \hat{x}$$
>
> may be written
>
> $$\nabla \times (\hat{x} \times \mathbf{Y}) = \mathbf{Y} \cdot \nabla \hat{x} - \hat{x} \cdot \nabla \mathbf{Y} + \hat{x}(\nabla \cdot \mathbf{Y}) - \mathbf{Y}(\nabla \cdot \hat{x}) \tag{8.87}$$

which bears no obvious analogy to

$$\mathbf{Q} \times (\hat{x} \times \mathbf{Y}) = \hat{x}(\mathbf{Q} \cdot \mathbf{Y}) - \mathbf{Y}(\mathbf{Q} \cdot \hat{x}) \tag{8.88}$$

To obtain a better analogy, one would have to write

$$\mathbf{Q} \times (\hat{x} \times \mathbf{Y}) = \mathbf{Q} \cdot (\mathbf{YX} - \mathbf{XY}) \tag{8.89}$$

and replace $\mathbf{Q}$ by ∇.

We do not understand why (8.89) is a better analogy than (8.88) because, as algebraic vector identities, they are equivalent. There is only one interpretation of (8.89), namely,

$$\mathbf{Q} \times (\hat{x} \times \mathbf{Y}) = (\mathbf{Q} \cdot \mathbf{Y})\hat{x} - (\mathbf{Q} \cdot \hat{x})\mathbf{Y}, \tag{8.90}$$

which is the same as (8.88).

By replacing $\mathbf{Q}$ by ∇ in (8.89), and treating the resultant expression as the divergence of the dyadic $\mathbf{XY} - \mathbf{YX}$, the manipulation is identical to the one used by Panofsky and Phillips. This short paragraph on the role played by del in an authoritative book on vectorial mechanics shows the consequence of treating Gibbs's notations for the divergence and the curl as two products, one scalar and one vector.

We have now shown the failures by several authors in trying to invoke ∇ as an operator, not only for the gradient but also for the divergence and the curl. The role is now filled in by the symbolic vector, to be discussed in the next section as introduced in this book. Many of the ambiguities that have occurred in the past presentations covered in this paper will be recast correctly and unambiguously by our new method utilizing the symbolic vector.

8-9 A Recasting of the Past Failures by the Method of Symbolic Vector

If we replace Heaviside's "equations" (8.64)–(8.66) with

$$\mathbf{N} \cdot \boldsymbol{\nabla} \times \mathbf{E} = \boldsymbol{\nabla}(\mathbf{E} \times \mathbf{N}) = \mathbf{E} \cdot (\mathbf{N} \times \boldsymbol{\nabla}), \tag{8.91}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot (\mathbf{H} \times \boldsymbol{\nabla}) = \mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E}), \tag{8.92}$$

$$\begin{aligned}\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) &= \boldsymbol{\nabla}_{\mathrm{E}} \cdot (\mathbf{E} \times \mathbf{H}) - \boldsymbol{\nabla}_{\mathrm{H}} \cdot (\mathbf{H} \times \mathbf{E}) \\ &= \mathbf{H} \cdot (\boldsymbol{\nabla}_{\mathrm{E}} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla}_{\mathrm{H}} \times \mathbf{H}),\end{aligned} \tag{8.93}$$

then (8.93) yields

$$\boldsymbol{\nabla}(\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \boldsymbol{\nabla} \mathbf{E} - \mathbf{E} \cdot \boldsymbol{\nabla} \mathbf{H}. \tag{8.94}$$

Although (8.91) and (8.92) have the same form as Heaviside's except that his ∇ has been replaced by the symbolic vector $\boldsymbol{\nabla}$, yet there is a vast difference in meaning between the two sets. For example, his $\mathbf{H} \cdot \nabla \times \mathbf{E}$ in (8.65) is interpreted as $\mathbf{H} \cdot \text{curl } \mathbf{E}$, but our $\mathbf{H} \cdot \triangledown \times \mathbf{E}$ is the same as $\triangledown \cdot (\mathbf{E} \times \mathbf{H})$ because of Lemma 4.2 and it is equal to $\triangledown(\mathbf{E} \times \mathbf{H})$.

Every term in (8.91) to (8.94) is well defined. Both Lemma 4.1 and Lemma 4.2 are used to obtain the vector identity stated by (8.94).

Returning now to the problems posed by Borisenko and Tarapov, we start with the symbolic expression $\triangledown(\mathbf{A} \cdot \mathbf{B})$ for $\nabla(\mathbf{A} \cdot \mathbf{B})$; then, by applying Lemma 4.2, we have

$$\triangledown(\mathbf{A} \cdot \mathbf{B}) = \triangledown_A(\mathbf{A} \cdot \mathbf{B}) + \triangledown_B(\mathbf{A} \cdot \mathbf{B}). \tag{8.95}$$

Applying Lemma 4.1, we have

$$\triangledown_A(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \triangledown_A)\mathbf{A} - \mathbf{B} \times (\mathbf{A} \times \triangledown_A) \tag{8.96}$$

and

$$\triangledown_B(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \triangledown_B)\mathbf{B} - \mathbf{A} \times (\mathbf{B} \times \triangledown_B). \tag{8.97}$$

Hence

$$\triangledown_A(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B} \cdot \nabla\mathbf{A} + \mathbf{B} \times \triangledown\, \mathbf{A} \tag{8.98}$$

and

$$\triangledown_B(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla\mathbf{B} + \mathbf{A} \times \triangledown\, \mathbf{B}. \tag{8.99}$$

Thus,

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla\mathbf{B} + \mathbf{B} \cdot \nabla\mathbf{A} + \mathbf{A} \times \triangledown\, \mathbf{B} + \mathbf{B} \times \triangledown\, \mathbf{A}. \tag{8.100}$$

Our derivation of (8.100) appears to be similar to the derivation by Borisenko and Tarapov in form, but the use of the FSP and FVP in their formulation and the treatment of (8.77) as an algebraic identity is entirely unacceptable, while each of our steps are supported by the basic principle in the method of symbolic vector, particularly the two lemmas therein.

The exercise posed by Panofsky and Phillips can be formulated correctly by our new method. The steps are as follows:

We start with $\triangledown \times (\mathbf{A} \times \mathbf{B})$, which is the symbolic expression of $\triangledown\, (\mathbf{A} \times \mathbf{B})$; then by means of Lemma 4.2,

$$\triangledown \times (\mathbf{A} \times \mathbf{B}) = \triangledown_A \times (\mathbf{A} \times \mathbf{B}) + \triangledown_B \times (\mathbf{A} \times \mathbf{B}). \tag{8.101}$$

By means of Lemma 4.1, we have

$$\begin{aligned} \triangledown_A \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \triangledown_A)\mathbf{A} - (\triangledown_A \cdot \mathbf{A})\mathbf{B} \\ &= \mathbf{B} \cdot \nabla\mathbf{A} - \mathbf{B}\triangledown\, \mathbf{A}. \end{aligned}$$

Similarly,

$$\begin{aligned} \triangledown_B \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \triangledown_B)\mathbf{A} - (\mathbf{A} \cdot \triangledown_B)\mathbf{B} \\ &= \mathbf{A}\triangledown\, \mathbf{B} - \mathbf{A} \cdot \nabla\mathbf{B}. \end{aligned}$$

Hence

$$\not\nabla\,(\mathbf{B} \times \mathbf{B}) = \mathbf{A}\not\nabla\mathbf{B} - \mathbf{A} \cdot \nabla\mathbf{B} - \mathbf{B}\not\nabla\mathbf{A} + \mathbf{B} \cdot \nabla\mathbf{A}, \tag{8.102}$$

which is the same as (8.84) obtained previously in Section 8-8 by a more complicated analysis. The convenience and the simplicity of the method of symbolic vector to derive vector identities has been clearly demonstrated in the last two examples. All commonly used vector identities have been derived in this way, as shown in Chapter 4.

8-9-1 In Retrospect

In this work, we have examined critically some practices of presenting vector analysis in several early works and in a few contemporary writings. We state with emphasis that the whole subject of vector analysis was formulated by the great American scientist J. Willard Gibbs in a precise and elegant fashion. Although his original works are confined to formulations in a Cartesian coordinate system, they can be extended to curvilinear systems as a result of the invariance of the differential operators, as reviewed in this book, without the necessity of resorting to the aid of tensor analysis.

In spite of the richness of Gibbs's theory of vector analysis, his notations for the divergence and the curl, in the opinion of this author, have induced several later workers, including one of his students, Wilson, to make some inappropriate interpretations. The adoption of these interpretations has been worldwide. We have selected a few examples from the works of several seasoned scientists and engineers to illustrate the prevalence of the improper use of ∇.

As a result of this study, we have justified our adoption of the new operational symbols as the notations for the divergence and the curl to replace Gibbs's old notations. It seems that our move is reasonable from the logistical point of view.

We have examined a history covering a period of over one hundred years. It represents a most interesting period in the development of the mathematical foundations of electromagnetic theory. However, in view of the long-entrenched and widespread misuse of the gradient operator ∇ as a constituent of the divergence and curl operators, the obligation of sharing the insight presented here with many of our colleagues in this field has been a labor fraught with frustration.

We hope that this historical study has been sufficiently clear to enable the serious workers in this subject to understand the issues, and that future students will not have to ponder over contradictions and misrepresentations to learn this subject.

It may be proper to conclude this chapter and the book by quoting a remark made by E. B. Wilson 87 years ago [43]. In reviewing two Italian books on vector analysis by Burali-Forti and Marcolongo, Wilson concluded his article with the following remark:

> What the resulting residual system may be we will not venture to predict, but that there will be such a system fifty years hence we fully believe. And

whatever that system may be it should and probably will conform to two requirements:

1. correct ideas relative to vector fields;
2. analytical suggestions of notations.

We sincerely believe that the method of symbolic vector together with the new operational notations for the divergence and the curl have fulfilled Dr. Wilson's wish. Whether the new notations will be adopted by potential users we leave to future generations of students to decide. Our goal is to put a logical approach on record.