# Chapter 3

# Field equations

The essence of electromagnetics lies in the Maxwell equations, which were formed by James Clark Maxwell in 1864 and are still far from being solved. (Some people insist on calling them Maxwell's equations, but, following JACKSON (1975), we adopt similar usage as in 'the Bessel functions'.) The Maxwell equations form a highly symmetric set of partial differential equations, which can be cast in many forms using different mathematical formalisms, such as scalars, vectors, tensors, quaternions, differential forms, and Clifford algebras. Maxwell originally applied the quaternionic form of thinking and scalar form of writing and the present vector notation applied by electromagnetists is mainly due to HEAVISIDE from the 1880s. It is typical to all formalisms that, while a higher degree of abstraction allows simplicity in writing and interpreting equations, it is counterbalanced by the inconvenience of learning the use of new operations. This is why new formalisms have had a hard time penetrating into the electromagnetics literature. A hundred years ago this was true of the vector notation and it was mostly only overcome by the stubbornness of Oliver Heaviside. The same seems now to be the case in notations applying differential forms or Clifford algebras, which have certain advantages over the vector notation. This has provoked Georges A. Deschamps, the great promoter of differential forms, to exclame 'Too bad vector calculus was ever invented!'

The present exposition, however, lies firmly on the basis of vector analysis. Some logical connections, hard to see in plain vector notation but simple in differential form notation, will be emphasized using matrix operators.

## 3.1 The Maxwell equations

The Maxwell equations written in terms of real vector quantities, the electric and magnetic field vectors $\mathbf{E}$, $\mathbf{H}$ and the electric and magnetic flux densities $\mathbf{D}$, $\mathbf{B}$, as functions of the position vector $\mathbf{r}$ and time $t$, are

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t), \tag{3.1}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t), \tag{3.2}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0, \tag{3.3}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \varrho(\mathbf{r},t). \tag{3.4}$$

Here, the current vector $\mathbf{J}$ and the charge density $\varrho$ are the postulated sources of electromagnetic fields and they satisfy the continuity condition

$$\nabla \cdot \mathbf{J}(\mathbf{r},t) = -\frac{\partial}{\partial t}\varrho(\mathbf{r},t), \tag{3.5}$$

which is implicit in the equations (3.2) and (3.4) and can be obtained by divergence and $\partial/\partial t$ operations.

The electromagnetic force could be calculated without any reference to electromagnetic fields, but the introduction of vectors $\mathbf{E}$, $\mathbf{B}$, which are calculated first, facilitates the problem. The force density is then obtained from the Lorentz expression

$$\mathbf{F}(\mathbf{r},t) = \varrho(\mathbf{r},t)\mathbf{E}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t). \tag{3.6}$$

The electromagnetic fields $\mathbf{E}$, $\mathbf{B}$ have thus the physical significance in transmitting the force between currents and charges. Because they are not easily calculated from the sources, the problem is alleviated by introducing a second pair of auxiliary field vectors $\mathbf{H}$, $\mathbf{D}$.

Thus, we arrive at the Maxwell equations (3.1)–(3.4), which are a set of two vector and two scalar equations for four vector or twelve scalar unknowns. This is not enough for a unique solution of the fields. In fact, there should exist an additional set of two vector equations between the field vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$, $\mathbf{D}$. These equations are dependent on the medium and are called the constitutive equations or medium equations. For example, for the most general linear, local and non-dispersive media the additional equations can be written as follows:

$$\mathbf{D} = \overline{\overline{\epsilon}} \cdot \mathbf{E} + \overline{\overline{\xi}} \cdot \mathbf{H}, \tag{3.7}$$

$$\mathbf{B} = \overline{\overline{\zeta}} \cdot \mathbf{E} + \overline{\overline{\mu}} \cdot \mathbf{H}. \tag{3.8}$$

Here, the dyadic parameters $\overline{\overline{\epsilon}}$, $\overline{\overline{\xi}}$, $\overline{\overline{\zeta}}$, $\overline{\overline{\mu}}$ depend on properties of the medium. All physical phenomena within the medium are hidden behind these four dyadics. In fact, the macroscopic electromagnetic fields do not distinguish between two media if they have the same dyadic parameters, even if the physical processes behind those parameters are totally different. The problems considered here are concerned with effects of media on electromagnetic fields rather than the converse case.

### 3.1.1 Operator equations

The logic leading to the Maxwell equations can be seen more easily using a mathematical formalism higher than the vector algebra. For example, using differential forms the equations take a very simple form. This notation can be simulated using mixed matrix operators as follows.

Starting from the electromagnetic source defined as a current-charge four-vector $\mathsf{i}$:

$$\mathsf{i} = \begin{pmatrix} \mathbf{J} \\ \rho \end{pmatrix}, \tag{3.9}$$

the equation of charge conservation (3.5) can be written as

$$\mathsf{D}_1 \cdot \mathsf{i} = (\nabla \;\; \frac{\partial}{\partial t}) \cdot \begin{pmatrix} \mathbf{J} \\ \varrho \end{pmatrix} = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}\varrho = 0. \tag{3.10}$$

with a four-vector operator $\mathsf{D}_1$ defined by

$$\mathsf{D}_1 = (\nabla \;\; \frac{\partial}{\partial t}). \tag{3.11}$$

The physical fields $\mathbf{E}$, $\mathbf{B}$ satisfy the combination of two Maxwell equations (3.1), (3.3). Writing the vector pair as a single six vector $\mathsf{e}$:

$$\mathsf{e} = \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \tag{3.12}$$

and defining a second operator $\mathsf{D}_2$ by

$$\mathsf{D}_2 = \begin{pmatrix} \nabla \times \overline{\overline{I}} & \overline{\overline{I}}\frac{\partial}{\partial t} \\ 0 & \nabla \end{pmatrix}, \tag{3.13}$$

we can write

$$\mathsf{D}_2 \cdot \mathsf{e} = \begin{pmatrix} \nabla \times \overline{\overline{I}} & \overline{\overline{I}}\frac{\partial}{\partial t} \\ 0 & \nabla \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} \end{pmatrix} = 0. \tag{3.14}$$

Here, 0 on the right-hand side means a combination of the null vector and the scalar zero.

It is easy to check that the following operator product results in the null operator:

$$\mathsf{D}_1 \cdot \mathsf{K}\mathsf{D}_2 = 0, \tag{3.15}$$

where we denote

$$\mathsf{K} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.16}$$

Comparing (3.10) and (3.15) shows us that the former is satisfied identically if the source four-vector $\mathsf{i}$ is written in terms of another six-vector $\mathsf{h}$ in the form

$$\mathsf{i} = \mathsf{K}\mathsf{D}_2\mathsf{K}\cdot\mathsf{h}. \tag{3.17}$$

Writing $\mathsf{h} = (\mathbf{H},\ \mathbf{D})^T$ as a matrix of two vectors, this equation equals

$$\begin{pmatrix} \mathbf{J} \\ \varrho \end{pmatrix} = \begin{pmatrix} \nabla\times\overline{\overline{I}} & -\overline{\overline{I}}\frac{\partial}{\partial t} \\ 0 & \nabla \end{pmatrix}\cdot\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix}. \tag{3.18}$$

This together with (3.14) defines the Maxwell equations and can, thus, be interpreted as being just a definition of the source in terms of the fields $\mathbf{H},\ \mathbf{D}$.

Finally, the medium equation can be written in an operator form

$$\mathsf{e} = \mathsf{M}(\mathsf{h}), \tag{3.19}$$

where $\mathsf{M}$ contains the information of the medium. For a linear medium, the $\mathsf{M}$ operator is a matrix of four dyadics.

### 3.1.2 Medium equations

The most general linear medium can be described in terms of four dyadic parameters defined in (3.7), (3.8). These equations can also be written as a relation between the vector pairs $\mathsf{e} = (\mathbf{E},\ \mathbf{B})^T$ and $\mathsf{h} = (\mathbf{H},\ \mathbf{D})^T$:

$$\mathbf{H} = \overline{\overline{M}}\cdot\mathbf{E} + \overline{\overline{Q}}\cdot\mathbf{B}, \tag{3.20}$$

$$\mathbf{D} = \overline{\overline{P}}\cdot\mathbf{E} + \overline{\overline{L}}\cdot\mathbf{B}. \tag{3.21}$$

The relation between the two sets of medium dyadics can easily be derived by substitution:

$$\overline{\overline{\epsilon}} = \overline{\overline{P}} - \overline{\overline{L}}\cdot\overline{\overline{Q}}^{-1}\cdot\overline{\overline{M}}, \qquad \overline{\overline{\xi}} = \overline{\overline{L}}\cdot\overline{\overline{Q}}^{-1}, \tag{3.22}$$

$$\overline{\overline{\zeta}} = -\overline{\overline{Q}}^{-1}\cdot\overline{\overline{M}}, \qquad \overline{\overline{\mu}} = \overline{\overline{Q}}^{-1}, \tag{3.23}$$

or

$$\overline{\overline{P}} = \overline{\overline{\epsilon}} - \overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}}, \qquad \overline{\overline{L}} = \overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}, \tag{3.24}$$

$$\overline{\overline{M}} = -\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}}, \qquad \overline{\overline{Q}} = \overline{\overline{\mu}}^{-1}. \tag{3.25}$$

The general linear medium is also called *magnetoelectric* or *bianisotropic*. In the case of no special directions in the medium all medium dyadics $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$, $\overline{\overline{\xi}}$, $\overline{\overline{\zeta}}$ are multiples of the unit dyadic $\overline{\overline{I}}$ and the medium is called *bi-isotropic.* Special cases of bi-isotropic media are the *isotropic chiral medium* with $\overline{\overline{\zeta}} = -\overline{\overline{\xi}} = \zeta\overline{\overline{I}}$ and the *Tellegen medium* with $\overline{\overline{\zeta}} = \overline{\overline{\xi}} = \zeta\overline{\overline{I}}$. For $\overline{\overline{\xi}} = 0$, $\overline{\overline{\zeta}} = 0$, the medium is *anisotropic* and if the dyadics $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$ are multiples of $\overline{\overline{I}}$, it is simply *isotropic.*

The constitutive equations of the bi-isotropic medium in the frequency domain can be also written in the form

$$\mathbf{D} = \epsilon\mathbf{E} + (\chi - j\kappa)\sqrt{\epsilon_o\mu_o}\,\mathbf{H}, \tag{3.26}$$

$$\mathbf{B} = (\chi + j\kappa)\sqrt{\epsilon_o\mu_o}\,\mathbf{E} + \mu\mathbf{H}. \tag{3.27}$$

The dimensionless coefficients $\chi$ and $\kappa$ are called the Tellegen and the chirality parameters, respectively, and for a lossless medium they turn out to be real numbers. A chiral medium can be produced by inserting in a suitable base material particles with specific handedness, i.e. particles whose mirror image cannot be brought into coincidence with the original particle, like particles of helical form. The Tellegen medium can be produced by combining permanent electric and magnetic dipoles in similar parallel pairs and making a mixture with such particles. Such a medium was first suggested by TELLEGEN in 1948.

When the medium parameters are actually operators containing time differentiation $\partial/\partial t$, the medium is called *dispersive*, or, more exactly, *time dispersive.* If they contain the space differentiation $\nabla$, the medium is called *space dispersive* or *non-local.* If the linear relations do not hold, the medium is, of course, *non-linear.*

Every physical medium can be understood to present time-dispersive properties due to the inevitable inertia of its molecules. In particular, at frequencies high enough every physical medium should act as if unpolarizable, like the vacuum. However, for some frequency ranges medium parameters may depend very little on frequency and they may be considered non-dispersive as a first approximation.

### 3.1.3 Wave equations

Eliminating all but one of the field vectors from the Maxwell equations of first order, wave equations of the second order are produced. Let us consider the general bianisotropic medium with the parameter dyadics $\overline{\overline{\epsilon}}$, $\overline{\overline{\xi}}$, $\overline{\overline{\zeta}}$, $\overline{\overline{\mu}}$. Substituting (3.7), (3.8) in the Maxwell equations (3.1), (3.2), we obtain the following equations provided the medium is *time independent*:

$$\left(\nabla \times \overline{\overline{I}} + \overline{\overline{\zeta}}\frac{\partial}{\partial t}\right) \cdot \mathbf{E} = -\overline{\overline{\mu}} \cdot \frac{\partial}{\partial t}\mathbf{H}, \tag{3.28}$$

$$\left(\nabla \times \overline{\overline{I}} - \overline{\overline{\xi}}\frac{\partial}{\partial t}\right) \cdot \mathbf{H} = \overline{\overline{\epsilon}} \cdot \frac{\partial}{\partial t}\mathbf{E} + \mathbf{J}. \tag{3.29}$$

Differentiating the second of these equations by $\frac{\partial}{\partial t}$ and substituting $\frac{\partial}{\partial t}\mathbf{H}$ from the first one gives us the wave equation for the electric field

$$\overline{\overline{W}}_e(\nabla, \frac{\partial}{\partial t}) \cdot \mathbf{E} = \frac{\partial}{\partial t}\mathbf{J}, \tag{3.30}$$

Eliminating the $\mathbf{E}$ vector gives us the corresponding equation for the magnetic field:

$$\overline{\overline{W}}_m(\nabla, \frac{\partial}{\partial t}) \cdot \mathbf{H} = -\left(\nabla \times \overline{\overline{I}} + \overline{\overline{\zeta}}\frac{\partial}{\partial t}\right) \cdot \overline{\overline{\epsilon}}^{-1} \cdot \mathbf{J}. \tag{3.31}$$

The wave operators $\overline{\overline{W}}_e(\nabla, \frac{\partial}{\partial t})$ and $\overline{\overline{W}}_m(\nabla, \frac{\partial}{\partial t})$ are defined by

$$\overline{\overline{W}}_e(\nabla, \frac{\partial}{\partial t}) = -\left(\nabla \times \overline{\overline{I}} - \overline{\overline{\xi}}\frac{\partial}{\partial t}\right) \cdot \overline{\overline{\mu}}^{-1} \cdot \left(\nabla \times \overline{\overline{I}} + \overline{\overline{\zeta}}\frac{\partial}{\partial t}\right) - \overline{\overline{\epsilon}}\frac{\partial^2}{\partial t^2}, \tag{3.32}$$

$$\overline{\overline{W}}_m(\nabla, \frac{\partial}{\partial t}) = -\left(\nabla \times \overline{\overline{I}} + \overline{\overline{\zeta}}\frac{\partial}{\partial t}\right) \cdot \overline{\overline{\epsilon}}^{-1} \cdot \left(\nabla \times \overline{\overline{I}} - \overline{\overline{\xi}}\frac{\partial}{\partial t}\right) - \overline{\overline{\mu}}\frac{\partial^2}{\partial t^2}. \tag{3.33}$$

As special cases we have for the *homogeneous anisotropic medium*

$$\overline{\overline{W}}_e(\nabla, \frac{\partial}{\partial t}) = \overline{\overline{\mu}}^{-1} {}^{\times}_{\times} \nabla\nabla - \overline{\overline{\epsilon}}\frac{\partial^2}{\partial t^2}, \tag{3.34}$$

$$\overline{\overline{W}}_m(\nabla, \frac{\partial}{\partial t}) = \overline{\overline{\epsilon}}^{-1} {}^{\times}_{\times} \nabla\nabla - \overline{\overline{\mu}}\frac{\partial^2}{\partial t^2}, \tag{3.35}$$

and for the *homogeneous isotropic medium*

$$\overline{\overline{W}}_e(\nabla, \frac{\partial}{\partial t}) = \mu^{-1}\overline{\overline{I}} {}^{\times}_{\times} \nabla\nabla - \epsilon\overline{\overline{I}}\frac{\partial^2}{\partial t^2}, \tag{3.36}$$

$$\overline{\overline{W}}_m(\nabla, \frac{\partial}{\partial t}) = \epsilon^{-1}\overline{\overline{I}} {}^{\times}_{\times} \nabla\nabla - \mu\overline{\overline{I}}\frac{\partial^2}{\partial t^2}. \tag{3.37}$$

The other two Maxwell equations (3.3), (3.4) must be added as initial conditions to these wave equations and for anisotropic media they are

$$\nabla \cdot (\overline{\overline{\epsilon}} \cdot \mathbf{E}) + \nabla \cdot (\overline{\overline{\xi}} \cdot \mathbf{H}) = \varrho, \tag{3.38}$$

$$\nabla \cdot (\overline{\overline{\mu}} \cdot \mathbf{H}) + \nabla \cdot (\overline{\overline{\zeta}} \cdot \mathbf{E}) = 0. \tag{3.39}$$

**References**


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## 3.2 Fourier transformations

To simplify the solution of problems it is often useful not to consider fields in time and space but to make Fourier transformations from time and/or space to the corresponding Fourier variable domain. Thus, differential equations can be transformed to algebraic equations which are easier to solve. Difficulties are normally encountered when transformation back to physical time and/or space is performed. This step could in many cases be made numerically. However, Fourier transformed solutions give information in terms of sinusoidal signals or plane waves and are useful as such.

### 3.2.1 Fourier transformation in time

The transformation pair in time can be written as

$$f(\omega) = \int_{-\infty}^{\infty} F(t)e^{-j\omega t}dt, \tag{3.40}$$

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{j\omega t}d\omega. \tag{3.41}$$

Since the fields in time domain are real, this sets a condition for the complex field dependence on frequency. In fact, the condition

$$f(-\omega) = f^*(\omega) \tag{3.42}$$

follows directly from the expressions defining the transform. This means that the complex fields in the frequency domain must in fact be real functions of the variable $j\omega$, a fact that can also be seen directly from (3.40). Thus, odd functions of $\omega$ must be pure imaginary and even functions of $\omega$ real and, conversely, the imaginary part of a field is an odd function of $\omega$ and the real part an even function of $\omega$.

The Maxwell equations in the frequency domain are

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}, \tag{3.43}$$

$$\nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J}, \tag{3.44}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.45}$$

$$\nabla \cdot \mathbf{D} = \varrho. \tag{3.46}$$

(3.45) and (3.46) are in fact unnecessary, because they follow from (3.43), (3.44) and the equation of continuity:

$$\nabla \cdot \mathbf{J} = -j\omega\varrho. \tag{3.47}$$

(3.43) and (3.44) are the time-harmonic Maxwell equations which can be applied directly to sources with sinusoidal time dependence.

*Operator equations*

To express the Maxwell equations (3.43), (3.44) concisely, the following operator formalism is of great use for time-harmonic fields. Similar operators could also have been introduced for time-dependent fields. To obtain more symmetry into the equations, the magnetic current density vector $\mathbf{J}_m$ is introduced in (3.43):

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B} - \mathbf{J}_m. \tag{3.48}$$

The magnetic source, first introduced by OLIVER HEAVISIDE in 1885, can be understood as being an equivalent source adopted for convenience, without making any speculations about the existence of magnetic monopoles, a subject of great controversy to physicists. The magnetic charge is automatically introduced through an equation of continuity similar to (3.47):

$$\nabla \cdot \mathbf{J}_m = -j\omega\varrho_m. \tag{3.49}$$

It is possible to write the equation pair (3.48), (3.44) as an operator equation in many different ways, for example:

$$\mathsf{L}_1 \cdot \mathsf{f} - j\omega \mathsf{L}_2 \cdot \mathsf{f} = \mathsf{g}, \tag{3.50}$$

where we define for the bianisotropic medium:

$$\mathsf{L}_1 = \begin{pmatrix} 0 & \nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}} \\ -\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\zeta}} & 0 \end{pmatrix}, \quad \mathsf{L}_2 = \begin{pmatrix} \overline{\overline{\epsilon}} & 0 \\ 0 & \overline{\overline{\mu}} \end{pmatrix}, \tag{3.51}$$

$$\mathsf{f} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad \mathsf{g} = \begin{pmatrix} \mathbf{J} \\ \mathbf{J}_m \end{pmatrix}. \tag{3.52}$$

For an anisotropic medium (3.50) can be written concisely as

$$\begin{pmatrix} -j\omega\overline{\overline{\epsilon}} & \nabla \times \overline{\overline{I}} \\ -\nabla \times \overline{\overline{I}} & -j\omega\overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{J} \\ \mathbf{J}_m \end{pmatrix}. \tag{3.53}$$

*Helmholtz equations*

The wave equations (3.30), (3.31) are reduced to Helmholtz equations after Fourier transformation in time when the dyadic wave operators are transformed to the dyadic Helmholtz operators through replacing $\partial/\partial t$ by $j\omega$:

$$\overline{\overline{H}}_e(\nabla) = \overline{\overline{W}}_e(\nabla, j\omega), \qquad \overline{\overline{H}}_m(\nabla) = \overline{\overline{W}}_m(\nabla, j\omega). \tag{3.54}$$

They can also be obtained from the operator form (3.50) through substitution, based on the fact that the dyadic matrix operator product $\mathsf{L}_1 \cdot \mathsf{L}_2^{-1} \cdot \mathsf{L}_1$ is diagonal. In fact, from (3.50) we can write

$$\mathsf{L}_1 \cdot \mathsf{L}_2^{-1} \cdot \mathsf{L}_1 \cdot \mathsf{f} - j\omega \mathsf{L}_1 \cdot \mathsf{f} = \mathsf{L}_1 \cdot \mathsf{L}_2^{-1} \cdot \mathsf{g}, \tag{3.55}$$

and substituting $\mathsf{L}_1 \cdot \mathsf{f}$ from (3.50), the equation

$$\mathsf{L}_1 \cdot \mathsf{L}_2^{-1} \cdot \mathsf{L}_1 \cdot \mathsf{f} + \omega^2 \mathsf{L}_2 \cdot \mathsf{f} = j\omega \mathsf{g} + \mathsf{L}_1 \cdot \mathsf{L}_2^{-1} \cdot \mathsf{g}. \tag{3.56}$$

The first matrix can be written in diagonal form after some algebra:

$$\mathsf{L}_1 \cdot \mathsf{L}_2^{-1} \cdot \mathsf{L}_1 = \begin{pmatrix} \overline{\overline{H}}_e(\nabla) & 0 \\ 0 & \overline{\overline{H}}_m(\nabla) \end{pmatrix} \tag{3.57}$$

with the Helmholtz dyadic operators defined as

$$\overline{\overline{H}}_e(\nabla) = -(\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot \overline{\overline{\mu}}^{-1} \cdot (\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) + \omega^2 \overline{\overline{\epsilon}}, \tag{3.58}$$

$$\overline{\overline{H}}_m(\nabla) = -(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \overline{\overline{\epsilon}}^{-1} \cdot (\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) + \omega^2 \overline{\overline{\mu}}. \tag{3.59}$$

Because of the diagonal property of $\mathsf{L}_1 \cdot \mathsf{L}_2^{-1} \cdot \mathsf{L}_1$ and $\mathsf{L}_2$ matrices, (3.56) is split into two Helmholtz equations for the two field vectors $\mathbf{E}$ and $\mathbf{H}$:

$$\overline{\overline{H}}_e(\nabla) \cdot \mathbf{E} = j\omega \mathbf{J} + (\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot (\overline{\overline{\mu}}^{-1} \cdot \mathbf{J}_m), \tag{3.60}$$

$$\overline{\overline{H}}_m(\nabla) \cdot \mathbf{H} = -(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot (\overline{\overline{\epsilon}}^{-1} \cdot \mathbf{J}) + j\omega \mathbf{J}_m. \tag{3.61}$$

For the homogeneous anisotropic medium the Helmholtz operators are reduced to

$$\overline{\overline{H}}_e(\nabla) = \overline{\overline{\mu}}^{-1} {}^{\times}_{\times} \nabla\nabla + \omega^2 \overline{\overline{\epsilon}}, \tag{3.62}$$

$$\overline{\overline{H}}_m(\nabla) = \overline{\overline{\epsilon}}^{-1} {}^{\times}_{\times} \nabla\nabla + \omega^2 \overline{\overline{\mu}}. \tag{3.63}$$

Finally, for the homogeneous bi-isotropic medium with the parameter dyadics $\overline{\overline{\xi}} = (\chi - j\kappa)\sqrt{\mu_o\epsilon_o}\,\overline{\overline{I}}$, $\overline{\overline{\zeta}} = (\chi + j\kappa)\sqrt{\mu_o\epsilon_o}\,\overline{\overline{I}}$, the operators can be written as

$$\overline{\overline{H}}_e(\nabla) = \mu^{-1}\overline{\overline{H}}(\nabla), \qquad \overline{\overline{H}}_m(\nabla) = \epsilon^{-1}\overline{\overline{H}}(\nabla), \tag{3.64}$$

with

$$\overline{\overline{H}}(\nabla) = -\overline{\overline{L}}_+(\nabla)\cdot\overline{\overline{L}}_-(\nabla) = -\overline{\overline{L}}_-(\nabla)\cdot\overline{\overline{L}}_+(\nabla), \tag{3.65}$$

$$\overline{\overline{L}}_\pm(\nabla) = \nabla\times\overline{\overline{I}} \mp k_\pm\overline{\overline{I}}, \qquad k_\pm = k_o(\sqrt{n^2-\chi^2}\pm\kappa), \qquad n = \sqrt{\mu_r\epsilon_r}. \tag{3.66}$$

For the isotropic medium $k_\pm = k$ and we have simply

$$\overline{\overline{H}}(\nabla) = \overline{\overline{I}}{}^{\times}_{\times}\nabla\nabla + k^2\overline{\overline{I}} = -[(\nabla\times\overline{\overline{I}})^2 - k^2\overline{\overline{I}}] = -(\nabla\times\overline{\overline{I}} - k\overline{\overline{I}})\cdot(\nabla\times\overline{\overline{I}} + k\overline{\overline{I}}). \tag{3.67}$$

It is worth noting that for the bi-isotropic medium, the Helmholtz operators can be factorized, i.e. written as a product of two operators of first order. This property is helpful in finding solutions for the Helmholtz equation, a subject to be discussed in Chapter 5.

### 3.2.2 Fourier transformation in space

When solving electromagnetic problems, Fourier transformation with respect to the space variable $\mathbf{r}$ often helps in finding the solution. This corresponds to expanding the fields and sources as a continuous sum of plane wave quantities in the Fourier variable $\mathbf{k}$ space. Denoting by $V_r$ and $V_k$ the respective physical and transformation spaces, the transformation to both directions is written as

$$f(\mathbf{k}) = \int_{V_r} F(\mathbf{r})e^{j\mathbf{k}\cdot\mathbf{r}}dV_r, \tag{3.68}$$

$$F(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{V_k} f(\mathbf{k})e^{-j\mathbf{k}\cdot\mathbf{r}}dV_k. \tag{3.69}$$

Thus, the Maxwell equations in the $\mathbf{k},\omega$ domain are obtained by replacing $\nabla$ through $-j\mathbf{k}$:

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B} - j\mathbf{J}_m, \tag{3.70}$$

$$\mathbf{k}\times\mathbf{H} = -\omega\mathbf{D} + j\mathbf{J}, \tag{3.71}$$

$$\mathbf{k}\cdot\mathbf{B} = j\varrho_m, \tag{3.72}$$

$$\mathbf{k}\cdot\mathbf{D} = j\varrho. \tag{3.73}$$

We make no attempt to introduce different notation for original and transformed quantities, because the argument space is always obvious from the context. The last two equations are unnecessary because they follow from the first two and

$$\mathbf{k}\cdot\mathbf{J} = \omega\varrho \qquad \mathbf{k}\cdot\mathbf{J}_m = \omega\varrho_m. \tag{3.74}$$

It is obvious that since $F(\mathbf{r})$ is a real function of $j\omega$, the doubly transformed quantity $f(\mathbf{k})$ must in fact be a real function of two variables, $j\mathbf{k}$ and $j\omega$.

Fourier transforming the Helmholtz equations for the bianisotropic medium results in

$$[(\mathbf{k}\times\overline{\overline{I}}+\omega\overline{\overline{\xi}})\cdot\overline{\overline{\mu}}^{-1}\cdot(\mathbf{k}\times\overline{\overline{I}}-\omega\overline{\overline{\zeta}})+\omega^2\overline{\overline{\epsilon}}]\cdot\mathbf{E} = j\omega\mathbf{J}-j(\mathbf{k}\times\overline{\overline{I}}+\omega\overline{\overline{\xi}})\cdot\overline{\overline{\mu}}^{-1}\cdot\mathbf{J}_m, \quad (3.75)$$

$$[(\mathbf{k}\times\overline{\overline{I}}-\omega\overline{\overline{\zeta}})\cdot\overline{\overline{\epsilon}}^{-1}\cdot(\mathbf{k}\times\overline{\overline{I}}+\omega\overline{\overline{\xi}})+\omega^2\overline{\overline{\mu}}]\cdot\mathbf{H} = j\omega\mathbf{J}_m+j(\mathbf{k}\times\overline{\overline{I}}-\omega\overline{\overline{\zeta}})\cdot\overline{\overline{\epsilon}}^{-1}\cdot\mathbf{J}. \quad (3.76)$$

Here, the linear medium equations (3.7), (3.8) have also been Fourier transformed. For non-dispersive media, medium parameters are the same constants for Fourier-transformed fields as for the original fields. For time-dispersive media, the parameters become functions of the frequency, whereas for space-dispersive media, they would be functions of the Fourier variable $\mathbf{k}$.

The equations in $\mathbf{k}$ space correspond to a representation of electromagnetic fields as a combination of plane waves, sometimes also called the plane-wave spectrum. These equations can be applied directly if the problem only deals with plane waves, for example, if only media with plane interfaces and boundaries are present and the excitation is a plane wave. Otherwise, the equations can be applied to solve a problem in Fourier space, after which the solution can be transformed back to the physical space. This, however, is hampered by computational difficulties because the inverse transformation can very seldom be made in analytic form and a numerical one is ill behaved. Fourier transformations in two space variables will be applied to planar structures in Chapter 7.

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## 3.3 Electromagnetic potentials

The electromagnetic problem can often be simplified by introducing potential functions in terms of which the electromagnetic fields can be expressed. The potential problem is simpler than the original electromagnetic field problem, if, for example, the original vector problem is reduced to a scalar problem, or if the dyadic operator of the original field problem is reduced to a scalar operator of the potential problem.

### 3.3.1 Vector and scalar potentials

Applying the operator formalism introduced in Section 3.1.1 it can easily be shown that, defining a differential operator $\mathsf{D}_3$ as follows,

$$\mathsf{D}_3 = \begin{pmatrix} -\overline{\overline{I}}\frac{\partial}{\partial t} & -\nabla \\ \nabla \times \overline{\overline{I}} & 0 \end{pmatrix}, \tag{3.77}$$

the identity

$$\mathsf{D}_2 \cdot \mathsf{D}_3 = \begin{pmatrix} \nabla \times \overline{\overline{I}} & \overline{\overline{I}}\frac{\partial}{\partial t} \\ 0 & \nabla \end{pmatrix} \cdot \begin{pmatrix} -\overline{\overline{I}}\frac{\partial}{\partial t} & -\nabla \\ \nabla \times \overline{\overline{I}} & 0 \end{pmatrix} = 0 \tag{3.78}$$

is valid. Here we have denoted the null matrix by '0'. Because from the Maxwell equations we have $\mathsf{D}_2 \cdot \mathsf{e} = 0$, we are tempted to express the electromagnetic fields in terms of a four-vector $\mathsf{a} = (\mathbf{A},\ \phi)^T$ in the form

$$\mathsf{e} = \mathsf{D}_3 \cdot \mathsf{a}, \tag{3.79}$$

or, equivalently,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{3.80}$$

which satisfies $\mathsf{D}_2 \cdot \mathsf{e} = 0$ automatically.

The potential $\mathsf{a}$ is not unique, because of the identity

$$\mathsf{D}_3 \cdot \mathsf{K}\mathsf{D}_1 = \begin{pmatrix} -\overline{\overline{I}}\frac{\partial}{\partial t} & -\nabla \\ \nabla \times \overline{\overline{I}} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nabla \\ \frac{\partial}{\partial t} \end{pmatrix} = 0, \tag{3.81}$$

which allows us to add to the potential $\mathsf{a}$ a term of the form $\mathsf{K}\mathsf{D}_1\psi$ where $\psi$ is any scalar function, without changing the resulting field $\mathsf{e} = \mathsf{D}_3 \cdot \mathsf{a} = \mathsf{D}_3 \cdot (\mathsf{a} + \mathsf{K}\mathsf{D}_1\psi)$. Thus, the potential function can be transformed into a form subject to a suitably chosen additional scalar condition to make the potential problem simpler to solve. This is called *the gauge transformation* of the potential and the additional condition the *gauge condition*.

The governing equation for the potential is obtained by combining the equation $\mathsf{K}\mathsf{D}_2\mathsf{K} \cdot \mathsf{h} = \mathsf{i}$ with the medium equation, which can be expressed as $\mathsf{h} = \mathsf{N}^{-1} \cdot \mathsf{e}$ and the equation for the potential reads

$$\mathsf{K}\mathsf{D}_2\mathsf{K} \cdot \mathsf{N}^{-1} \cdot \mathsf{D}_3 \cdot \mathsf{a} = \mathsf{i}. \tag{3.82}$$

Although the definition of the potential quantities is independent of the medium, it turns out that the equations governing the potentials are not essentially simpler than the wave equations for the electromagnetic fields unless the medium is bi-isotropic, in which case we have

$$\mathsf{N}^{-1} = \frac{1}{\mu}\begin{pmatrix} -\zeta & 1 \\ \mu\epsilon - \xi\zeta & \xi \end{pmatrix}, \tag{3.83}$$

in which case the equation (3.82) becomes

$$\nabla \times (\nabla \times \mathbf{A}) - (\xi - \zeta)\frac{\partial}{\partial t}\nabla \times \mathbf{A} + (\mu\epsilon - \xi\zeta)(\frac{\partial^2}{\partial t^2}\mathbf{A} + \frac{\partial}{\partial t}\nabla\phi) = \mu\mathbf{J}, \tag{3.84}$$

$$(\mu\epsilon - \xi\zeta)(\frac{\partial}{\partial t}\nabla \cdot \mathbf{A} + \nabla^2\phi) = -\mu\varrho. \tag{3.85}$$

These equations are decoupled if, following CHAMBERS (1956), we choose the gauge condition

$$\nabla \cdot \mathbf{A} = -(\mu\epsilon - \xi\zeta)\frac{\partial}{\partial t}\phi, \tag{3.86}$$

in which case we can write

$$[\nabla^2\overline{\overline{I}} + (\xi - \zeta)\frac{\partial}{\partial t}\nabla \times \overline{\overline{I}} - (\mu\epsilon - \xi\zeta)\frac{\partial^2}{\partial t^2}\overline{\overline{I}}] \cdot \mathbf{A} = -\mu\mathbf{J}, \tag{3.87}$$

$$[\nabla^2 - (\mu\epsilon - \xi\zeta)\frac{\partial^2}{\partial t^2}]\phi = -\frac{\mu}{\mu\epsilon - \xi\zeta}\varrho. \tag{3.88}$$

It is seen that the operator in the vector potential equation is dyadic unless we have $\xi = \zeta$, which is the condition of the Tellegen medium. Writing $\zeta = \xi = \chi\sqrt{\mu_o\epsilon_o}$, we have for the gauge condition in this case

$$\nabla \cdot \mathbf{A} = -(\mu\epsilon - \chi^2\mu_o\epsilon_o)\frac{\partial}{\partial t}\phi, \tag{3.89}$$

and the potential equations become

$$[\nabla^2 - (\mu\epsilon - \chi^2\mu_o\epsilon_o)\frac{\partial^2}{\partial t^2}]\mathbf{A} = -\mu\mathbf{J}, \tag{3.90}$$

$$[\nabla^2 - (\mu\epsilon - \chi^2\mu_o\epsilon_o)\frac{\partial^2}{\partial t^2}]\phi = -\frac{\mu}{\mu\epsilon - \chi^2\mu_o\epsilon_o}\varrho. \tag{3.91}$$

For isotropic, homogeneous media with parameters $\epsilon$, $\mu$, the gauge condition (3.86) is simplified to

$$\nabla \cdot \mathbf{A} = -\mu\epsilon\frac{\partial}{\partial t}\phi, \tag{3.92}$$

called the *Lorenz gauge* condition because it was first suggested by Ludwig Lorenz. The potential equations in this case become

$$[\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}]\mathbf{A} = -\mu\mathbf{J}, \qquad [\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}]\phi = -\frac{\varrho}{\epsilon}. \tag{3.93}$$

Because of the scalar operators in contrast to dyadic operators, vector and scalar potentials are most useful in problems involving homogeneous bi-isotropic media. Problems with inhomogeneities like scattering bodies can also be handled if the inhomogeneities are replaced by equivalent sources.

### 3.3.2 The Hertz vector

In the previous presentation the field was studied in terms of one vector and one scalar function totalling four scalar functions. Because of the gauge condition, the functions are not independent and there is some redundancy in their solution. In fact, from the knowledge of the vector potential, the scalar potential for bi-isotropic media can be solved by integrating the gauge condition (3.86), provided the value of the scalar potential is known at some time instant $t = t_o$:

$$\phi(\mathbf{r}, t) - \phi(\mathbf{r}, t_o) = -\frac{1}{\mu\epsilon - \xi\zeta}\int_{t_o}^{t} \nabla\cdot\mathbf{A}(\mathbf{r}, t')dt'. \tag{3.94}$$

A similar idea is to express the vector and scalar potentials in terms of another potential, the Hertz vector $\mathbf{\Pi}$. Writing the potential pair $\mathbf{A}$, $\phi$ in the form

$$\begin{pmatrix} \mathbf{A} \\ \phi \end{pmatrix} = \begin{pmatrix} (\mu\epsilon - \xi\zeta)\overline{\overline{I}}\frac{\partial}{\partial t} \\ -\nabla \end{pmatrix}\cdot\mathbf{\Pi}, \tag{3.95}$$

the gauge condition (3.86) is satisfied identically. The Hertz vector is seen to obey the wave equation of the form

$$[\nabla^2 - (\mu\epsilon - \xi\zeta)\frac{\partial^2}{\partial t^2}]\mathbf{\Pi} = -\frac{\mu}{\mu\epsilon - \xi\zeta}\mathbf{p}, \tag{3.96}$$

where $\mathbf{p}$ denotes the temporal integral of the current function

$$\mathbf{p}(\mathbf{r}, t) = \int_{t_o}^{t} \mathbf{J}(\mathbf{r}, t)dt. \tag{3.97}$$

The electromagnetic fields are obtained by substitution:

$$\mathbf{E} = \nabla\nabla\cdot\mathbf{\Pi} - (\mu\epsilon - \xi\zeta)\frac{\partial^2}{\partial t^2}\mathbf{\Pi} = \nabla\times(\nabla\times\mathbf{\Pi}) - \frac{\mu}{\mu\epsilon - \xi\zeta}\mathbf{p}. \tag{3.98}$$

$$\mathbf{B} = (\mu\epsilon - \xi\zeta)\frac{\partial}{\partial t}(\nabla \times \mathbf{\Pi}). \tag{3.99}$$

All the previous potential expressions can easily be written for time-harmonic fields by substituting $\partial/\partial t \to j\omega$.

### 3.3.3 Scalar Hertz potentials

The electromagnetic potential problem can be further reduced to solving two scalar functions. These functions can be defined in terms of components of the electric and magnetic fields along a certain direction in space, denoted by the unit vector $\mathbf{u}$. Let us only consider the special case of a bi-isotropic medium and time-harmonic fields arising from both electric and magnetic current sources. Writing the Maxwell equations in the form

$$\nabla \times \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = j\omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{J} \\ \mathbf{J}_m \end{pmatrix} \tag{3.100}$$

or, in shorthand, with respective symbols defined by

$$\nabla \times \mathsf{f} = j\omega \mathsf{J}\mathsf{N}\mathsf{f} + \mathsf{J}\mathsf{g}, \tag{3.101}$$

we can express the transverse fields $\mathsf{f}_t$ in terms of the longitudinal field components $\mathsf{f}_u$. In fact, taking the transverse component of (3.101) gives us

$$\mathbf{u} \times [(\mathbf{u} \cdot \nabla)\overline{\overline{I}}_t + j\omega \mathsf{J}\mathsf{N}\mathbf{u} \times \overline{\overline{I}}] \cdot \mathsf{f}_t = \mathbf{u} \times \nabla_t \mathsf{f}_u + \mathsf{J}\mathsf{g}_t. \tag{3.102}$$

Operating this by $-\mathbf{u} \times [(\mathbf{u} \cdot \nabla)\overline{\overline{I}}_t - j\omega \mathsf{J}\mathsf{N}\mathbf{u} \times \overline{\overline{I}}]\cdot$ leaves us with the equation

$$[(\mathbf{u} \cdot \nabla)^2 - \omega^2(\mathsf{J}\mathsf{N})^2]\mathsf{f}_t = [(\mathbf{u} \cdot \nabla)\overline{\overline{I}}_t - j\omega \mathsf{J}\mathsf{N}\mathbf{u} \times \overline{\overline{I}}] \cdot [\nabla_t \mathsf{f}_u - \mathsf{J}\mathbf{u} \times \mathsf{g}_t], \tag{3.103}$$

which states that if the longitudinal components $\mathsf{f}_u$ of the field are known, the transverse components $\mathsf{f}_t$ can be solved from a one-dimensional transmission-line type of an equation.

This prompts one to look for the solution, outside the sources, in terms of two scalar Hertz potentials

$$\Pi = \begin{pmatrix} \Pi_e \\ \Pi_m \end{pmatrix} \tag{3.104}$$

in the form

$$\mathsf{f}_u = [(\mathbf{u} \cdot \nabla)^2 - \omega^2(\mathsf{J}\mathsf{N})^2]\Pi, \tag{3.105}$$

$$\mathsf{f}_t = [(\mathbf{u} \cdot \nabla)\overline{\overline{I}}_t - j\omega \mathsf{J}\mathsf{N}\mathbf{u} \times \overline{\overline{I}}] \cdot \nabla\Pi, \tag{3.106}$$

because these satisfy (3.103) when $\mathbf{g} = 0$ and

$$[\nabla^2 \mathsf{I} - \omega^2 (\mathsf{JL})^2]\Pi = 0 \tag{3.107}$$

or

$$\begin{pmatrix} \nabla^2 + \omega^2(\mu\epsilon - \zeta^2) & \omega^2\mu(\xi - \zeta) \\ \omega^2\epsilon(\zeta - \xi) & \nabla^2 + \omega^2(\mu\epsilon - \xi^2) \end{pmatrix} \begin{pmatrix} \Pi_e \\ \Pi_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.108}$$

The total field can thus be expressed in the form

$$\mathsf{f} = [\nabla \times \overline{\overline{I}} \mathsf{I} + j\omega \mathsf{JN}] \cdot \nabla \times (\mathbf{u}\Pi), \tag{3.109}$$

or, more explicitly,

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \nabla \times \overline{\overline{I}} - j\omega\zeta\overline{\overline{I}} & -j\omega\mu\overline{\overline{I}} \\ j\omega\epsilon\overline{\overline{I}} & \nabla \times \overline{\overline{I}} + j\omega\xi\overline{\overline{I}} \end{pmatrix} \cdot \nabla \times \begin{pmatrix} \mathbf{u}\Pi_e \\ \mathbf{u}\Pi_m \end{pmatrix}. \tag{3.110}$$

Scalar Hertz potential expansions have also been developed for anisotropic media in the literature. They lead to fourth-order differential equations to be satisfied by the potentials, whereas for bi-isotropic media the equations are of the second order. This point is clearly seen from considerations of the Green dyadic in Chapter 5.

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## 3.4 Boundary, interface and sheet conditions

To define fields uniquely, in addition to the previous differential equations, more information on the solution is needed. Differential equations alone have an infinity of solutions, whence only by requiring suitable conditions to be satisfied, can the uniqueness of the solution be ensured. The additional conditions must not be too stringent, however, otherwise there might not exist a solution at all. The question of uniqueness will be considered in more detail in the subsequent section.

### 3.4.1 Discontinuities in fields, sources and media

The Maxwell equations in differential equation form are not valid everywhere in the classical sense if the medium parameters are discontinuous, because some of the field components may become discontinuous and their derivatives infinite. However, adopting generalized functions such as the delta function, differential operators in the Maxwell equations can be understood in a more general way, allowing us to include all discontinuities in the same equations.

Differential operators operating on functions with a step discontinuity at a surface $S$ may result in discontinuous functions involving the 'surface delta function' $\delta_s(\mathbf{r})$ defined by

$$\int_V f(\mathbf{r})\delta_s(\mathbf{r})dV = \int_S f(\mathbf{r})dS, \tag{3.111}$$

for functions $f(\mathbf{r})$ continuous at $S$. In fact, for the gradient operation we can write, following VAN BLADEL (1964), the result in two parts, the continuous part and the discontinuous part

$$\nabla f(\mathbf{r}) = \{\nabla f(\mathbf{r})\} + \nabla_s f(\mathbf{r})\delta_s(\mathbf{r}) \tag{3.112}$$

where the 'surface gradient' $\nabla_s$ is defined as[1]

$$\nabla_s f(\mathbf{r}) = f_1\mathbf{n}_1 + f_2\mathbf{n}_2. \tag{3.113}$$

Here $f_1$ and $f_2$ denote values of the field $f$ on each side 1 and 2 of the interface with unit normal vectors $\mathbf{n}_1$, $\mathbf{n}_2 = -\mathbf{n}_1$. $\{\}$ denotes the operation outside the surface $S$ of discontinuity.

The expression (3.113) can be readily checked by integrating (3.112) over $S$. If $f$ is continuous at $S$, we have $f_1 = f_2$, whence $\nabla_s f = 0$,

[1] The operator $\nabla_s$ must not be mistaken as the gradient *along* the surface $S$, sometimes also denoted by $\nabla_S$.

otherwise the step discontinuity is recovered in integration. Defining in the same way the surface divergence and curl operators:

$$\nabla_s \cdot \mathbf{F} = \mathbf{n}_1 \cdot \mathbf{F}_1 + \mathbf{n}_2 \cdot \mathbf{F}_2 = \mathbf{n}_1 \cdot (\mathbf{F}_1 - \mathbf{F}_2), \tag{3.114}$$

$$\nabla_s \times \mathbf{F} = \mathbf{n}_1 \times \mathbf{F}_1 + \mathbf{n}_2 \times \mathbf{F}_2 = \mathbf{n}_1 \times (\mathbf{F}_1 - \mathbf{F}_2), \tag{3.115}$$

the Maxwell equations in the frequency domain can be written explicitly as

$$\{\nabla \times \mathbf{E}\} + (\nabla_s \times \mathbf{E})\delta_s + j\omega\mathbf{B} = -\{\mathbf{J}_m\} - \mathbf{J}_{ms}\delta_s, \tag{3.116}$$

$$\{\nabla \times \mathbf{H}\} + (\nabla_s \times \mathbf{H})\delta_s - j\omega\mathbf{D} = \{\mathbf{J}\} + \mathbf{J}_s\delta_s, \tag{3.117}$$

$$\{\nabla \cdot \mathbf{D}\} + (\nabla_s \cdot \mathbf{D})\delta_s = \{\rho\} + \rho_s\delta_s, \tag{3.118}$$

$$\{\nabla \cdot \mathbf{B}\} + (\nabla_s \cdot \mathbf{B})\delta_s = \{\rho_m\} + \rho_{ms}\delta_s. \tag{3.119}$$

Here we have assumed that there are no delta discontinuities in the fields themselves, only in their derivatives.

Obviously, the delta discontinuous parts of the equations above must be equal on both sides. This gives us the following equations relating the discontinuous fields to the discontinuous sources:

$$\nabla_s \times \mathbf{E} = \mathbf{n}_1 \times \mathbf{E}_1 + \mathbf{n}_2 \times \mathbf{E}_2 = -\mathbf{J}_{ms}, \tag{3.120}$$

$$\nabla_s \times \mathbf{H} = \mathbf{n}_1 \times \mathbf{H}_1 + \mathbf{n}_2 \times \mathbf{H}_2 = \mathbf{J}_s, \tag{3.121}$$

$$\nabla_s \cdot \mathbf{D} = \mathbf{n}_1 \cdot \mathbf{D}_1 + \mathbf{n}_2 \cdot \mathbf{D}_2 = \rho_s, \tag{3.122}$$

$$\nabla_s \cdot \mathbf{B} = \mathbf{n}_1 \cdot \mathbf{B}_1 + \mathbf{n}_2 \cdot \mathbf{B}_2 = \rho_{ms}. \tag{3.123}$$

These equations also express the relations of delta-discontinuous fields to the step discontinuities of the media, as is seen by inserting medium equations of both sides of the discontinuity.

*Vector circuits*

The expressions (3.120), (3.121) are easily remembered when interpreted as generalized circuit equations. The quantity $\mathbf{n} \times \mathbf{H}$ can be interpreted as a vector current coming to the surface $S$ from the direction of $\mathbf{n}$, i.e. flowing in the direction of $-\mathbf{n}$. In this way, (3.121) tells us that the currents $\mathbf{n}_1 \times \mathbf{H}_1$ and $\mathbf{n}_2 \times \mathbf{H}_2$ both flow towards the surface $S$ and add up to the current $\mathbf{J}_s$. On the other hand, the tangential electric field $\mathbf{E}_t$ can be interpreted as a vector voltage. If (3.120) is written otherwise:

$$\mathbf{E}_{1t} - \mathbf{E}_{2t} = \mathbf{n}_1 \times \mathbf{J}_{ms}, \tag{3.124}$$

it can be interpreted as the difference of vector voltages on each side of the surface $S$, equating the voltage over the surface, which from side 1 to side 2 equals $\mathbf{n}_1 \times \mathbf{J}_{ms}$.

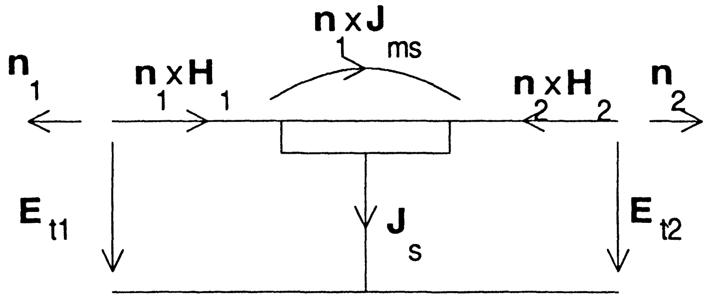


**Fig. 3.1** Vector circuit corresponding to a discontinuity in an electromagnetic field.

We could, alternatively, have chosen the magnetic circuit interpretation in terms of magnetic currents and magnetic voltages.

### 3.4.2 Boundary conditions

A closed surface $S$ containing the region $V$ is called a *boundary* for electromagnetic fields if the fields are zero outside $V$ (behind $S$) for any sources in $V$. From Huygens's principle to be discussed in Chapter 6, it follows that there must be secondary or induced surface sources on $S$ which outside $V$ exactly cancel the fields produced by the original sources in $V$. The required surface sources can also be obtained from the surface conditions on $S$ by requiring the fields $\mathbf{E}_2$ and $\mathbf{H}_2$ to vanish behind $S$. Suppressing the indices $()_1$ in (3.120), (3.121), we can write

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H}, \tag{3.125}$$

$$\mathbf{J}_{ms} = -\mathbf{n} \times \mathbf{E} \tag{3.126}$$

on the boundary. The boundary terminates the fields at the surface $S$ just like a terminating impedance at the end of a transmission line and it can be represented in terms of surface impedance. Henceforth, $\mathbf{n}$ points into the region $V$ where the fields are non-zero.

A boundary with a surface impedance independent of the fields is a mathematical idealization like many other concepts in electromagnetic theory. Sometimes an *interface* between media 1 and 2 can be approximately replaced by such a boundary when only fields in medium 1 are of interest. In doing so, the fields in medium 2 are set equal to zero and a suitable boundary impedance on the interface surface is introduced. While this method is only approximate, because a field-independent surface impedance does not correspond to the true interface conditions, it usually leads to a great simplification of the problem, and thus is useful if the error made is not significant. In practice this is done when the refraction factor of medium 2 is much smaller than that of medium 1. In exact form, the field dependence of the surface impedance must be taken care of through an impedance operator.

Applying the vector circuit concept to the boundary condition, we write a linear relation between the surface current $\mathbf{J}_s$ and the tangential electric field $\mathbf{E}_t$ in the form

$$\mathbf{E}_t = \overline{\overline{Z}}_s \cdot \mathbf{J}_s, \qquad \mathbf{J}_s = \overline{\overline{Y}}_s \cdot \mathbf{E} = \overline{\overline{Z}}_s^{-1} \cdot \mathbf{E}. \tag{3.127}$$

Here, $\overline{\overline{Z}}_s$ is a two-dimensional surface impedance dyadic satisfying $\mathbf{n} \cdot \overline{\overline{Z}}_s = \overline{\overline{Z}}_s \cdot \mathbf{n} = 0$ and $\overline{\overline{Y}}_s$ is its two-dimensional inverse. Similarly, we can write

$$\mathbf{H}_t = \overline{\overline{Z}}_{ms} \cdot \mathbf{J}_{ms}, \qquad \mathbf{J}_{ms} = \overline{\overline{Y}}_{ms} \cdot \mathbf{H} = \overline{\overline{Z}}_{ms}^{-1} \cdot \mathbf{H}, \tag{3.128}$$

where $\overline{\overline{Z}}_{ms}$ and $\overline{\overline{Y}}_{ms}$ are the corresponding two-dimensional magnetic impedance and admittance dyadics.

In terms of fields, the boundary conditions take the following equivalent forms:

$$\mathbf{n} \times \mathbf{H} = \overline{\overline{Y}}_s \cdot \mathbf{E}, \qquad \mathbf{E}_t = \overline{\overline{Z}}_s \cdot (\mathbf{n} \times \mathbf{H}), \tag{3.129}$$

$$\mathbf{n} \times \mathbf{E} = -\overline{\overline{Y}}_{ms} \cdot \mathbf{H}, \qquad \mathbf{H}_t = -\overline{\overline{Z}}_{ms} \cdot (\mathbf{n} \times \mathbf{E}), \tag{3.130}$$

implying the equivalence of electric and magnetic dyadics:

$$\overline{\overline{Y}}_s = \overline{\overline{Z}}_{ms} {}^{\times}_{\times} \mathbf{nn}, \qquad \overline{\overline{Z}}_s = \overline{\overline{Y}}_{ms} {}^{\times}_{\times} \mathbf{nn}. \tag{3.131}$$

Applying the definition of the inverse of a two-dimensional dyadic $\overline{\overline{A}}$:

$$\overline{\overline{A}} \cdot (\overline{\overline{A}}^T {}^{\times}_{\times} \mathbf{nn}) = (\mathrm{spm}\overline{\overline{A}})\overline{\overline{I}}_t, \qquad \mathrm{spm}\overline{\overline{A}} = \frac{1}{2}\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{A}} : \overline{\overline{I}}, \tag{3.132}$$

we can write relations between impedance and admittance dyadics

$$\overline{\overline{Y}}_s = \overline{\overline{Z}}_s^{-1} = \frac{\overline{\overline{Z}}_s^T {}^{\times}_{\times} \mathbf{nn}}{\mathrm{spm}\overline{\overline{Z}}_s}, \qquad \overline{\overline{Y}}_{ms} = \overline{\overline{Z}}_{ms}^{-1} = \frac{\overline{\overline{Z}}_{ms}^T {}^{\times}_{\times} \mathbf{nn}}{\mathrm{spm}\overline{\overline{Z}}_{ms}}. \tag{3.133}$$

*Special boundary conditions*

The *isotropic boundary* is defined to have the impedance dyadic of the form $\overline{\overline{Z}}_s = Z_s \overline{\overline{I}}_t$, whence $Z_{ms} = Y_s$ and $Y_{ms} = Z_s$. A perfect electric conductor (PEC) surface is characterized by $\overline{\overline{Z}}_s = 0$ and a perfect magnetic conductor (PMC) by $\overline{\overline{Z}}_{ms} = 0$ or, what is equivalent, by $\overline{\overline{Y}}_s = 0$. A 'good conductor' corresponds to a surface impedance which is small with respect to the impedance of the medium $\eta = \sqrt{\mu/\epsilon}$. Similarly, a 'good magnetic conductor' surface corresponds to a surface admittance dyadic whose components are small with respect to $1/\eta$.

The *radiation condition*, also known as the *Silver–Müller condition*, is an asymptotic relation between the radiation fields far from the source. It can be written as an isotropic impedance boundary condition on the sphere at infinity with $\mathbf{n} = \mathbf{u}_r$ and the impedance dyadic

$$\overline{\overline{Z}}_s = \eta_o \overline{\overline{I}}, \qquad \eta_o = \sqrt{\mu_o/\epsilon_o}. \tag{3.134}$$

The isotropic boundary is a special case of the more general *bi-isotropic boundary*, neither of which has any preferred direction in the tangent plane normal to $\mathbf{n}$. Because the only two-dimensional dyadics with this property are linear combinations of $\overline{\overline{I}}_t$ and $\overline{\overline{J}} = \mathbf{n} \times \overline{\overline{I}}$, the most general bi-isotropic boundary impedance dyadic must be of the form

$$\overline{\overline{Z}}_s = Z_a \overline{\overline{I}}_t + Z_b \mathbf{n} \times \overline{\overline{I}} \tag{3.135}$$

with two impedance parameters $Z_a$ and $Z_b$. From Section 2.9 we know that this kind of dyadic satisfies $\overline{\overline{Z}}_s {}^\times_\times \mathbf{nn} = \overline{\overline{Z}}_s$, whence for the bi-isotropic surface we have the simple rules

$$\overline{\overline{Z}}_{ms} = \overline{\overline{Y}}_s, \qquad \overline{\overline{Y}}_{ms} = \overline{\overline{Z}}_s. \tag{3.136}$$

The most general surface impedance dyadic contains four scalar parameters. If the impedance depends on the direction of the fields, it is called anisotropic. A reciprocal anisotropic surface has a symmetric dyadic

$$\overline{\overline{Z}}_s = \mathbf{uu}Z_u + \mathbf{vv}Z_v, \qquad \overline{\overline{Y}}_s = \frac{\mathbf{uu}}{Z_u} + \frac{\mathbf{vv}}{Z_v}, \tag{3.137}$$

with orthogonal two-dimensional unit vectors $\mathbf{u}$, $\mathbf{v}$. This means that the impedance is $Z_u$ for the tangential electric field polarized along $\mathbf{u}$ and, correspondingly, $Z_v$ for the field along $\mathbf{v}$.

As an example we may consider a corrugated surface, which is composed of shorted thin slots parallel to a unit vector $\mathbf{u}$. For a tangential field $\mathbf{E}_t$ parallel to $\mathbf{u}$ it behaves very like a PEC surface, whereas for $\mathbf{E}_t$ parallel to the orthogonal direction $\mathbf{v}$, the impedance is some scalar $Z_v$ depending on the nature of the slots. Thus, the surface impedance dyadic has the form $\overline{\overline{Z}}_s = \mathbf{uu}0 + \mathbf{vv}Z_v$. Being a linear dyadic it does not have a planar inverse in a strict mathematical sense, although we may write one in the form $\overline{\overline{Y}}_s = \mathbf{uu}\ \infty + \mathbf{vv}/Z_v$. Still worse from the mathematical point of view is a balanced, or tuned, corrugated surface, with $Z_v = \infty$.

*Scalar potential fields*

For scalar potential fields, the impedance boundary condition is defined as a linear relation between values of the potential and its normal derivative at the boundary:

$$\alpha\phi + \beta\mathbf{n}\cdot\nabla\phi = 0. \tag{3.138}$$

This condition can be interpreted physically in terms of acoustic pressure $\phi$ and velocity, which is proportional to $\nabla\phi$. For $\alpha = 0$ one has $\mathbf{n}\cdot\nabla\phi = 0$, which is called the ***hard boundary condition***, because the velocity of air is zero in the normal direction at a rigid wall. For $\beta = 0$ one has $\phi = 0$, which is called the ***soft boundary condition.***

The condition for radiated potentials at infinity is called the *Sommerfeld radiation condition* with $\mathbf{n} = \mathbf{u}_r$:

$$\mathbf{u}_r\cdot\nabla\phi + jk\phi = 0. \tag{3.139}$$

### 3.4.3 Interface conditions

Let us consider discontinuities of the fields across an interface $S$ between two media 1 and 2. If there are no surface sources, from (3.120)–(3.123) we have the following conditions at $S$:

$$\mathbf{n}_1\times\mathbf{E}_1 + \mathbf{n}_2\times\mathbf{E}_2 = 0, \tag{3.140}$$

$$\mathbf{n}_1\times\mathbf{H}_1 + \mathbf{n}_2\times\mathbf{H}_2 = 0, \tag{3.141}$$

$$\mathbf{n}_1\cdot\mathbf{D}_1 + \mathbf{n}_2\cdot\mathbf{D}_2 = 0, \tag{3.142}$$

$$\mathbf{n}_1\cdot\mathbf{B}_1 + \mathbf{n}_2\cdot\mathbf{B}_2 = 0. \tag{3.143}$$

These equations present the dependence between fields on one and the other side of the interface. This dependence can be written in dyadic form depending on the parameters of the two media. The counterpart of an interface in transmission-line theory is a junction between two transmission lines.

*Isotropic media*

For an interface between two isotropic media, the fields in medium 2 can be expressed in terms of fields in medium 1. Since the normal vector appears twice, $\mathbf{n}$ in the following equations stands for either $\mathbf{n}_1$ or $\mathbf{n}_2$:

$$\mathbf{E}_2 = (\overline{\overline{I}}_t + \mathbf{nn}) \cdot \mathbf{E}_2 = \left(\overline{\overline{I}}_t + \frac{\epsilon_1}{\epsilon_2}\mathbf{nn}\right) \cdot \mathbf{E}_1 = \overline{\overline{D}}(\frac{\epsilon_1}{\epsilon_2}) \cdot \mathbf{E}_1, \tag{3.144}$$

$$\mathbf{H}_2 = \overline{\overline{D}}(\frac{\mu_1}{\mu_2}) \cdot \mathbf{H}_1. \tag{3.145}$$

Here, $\overline{\overline{D}}(\alpha)$ denotes a symmetric uniaxial dyadic, which can also be defined in the notation of Section 2.8 as

$$\overline{\overline{D}}(\alpha) = \overline{\overline{I}} + (\alpha - 1)\mathbf{nn} = \overline{\overline{D}}(1|\alpha - 1). \tag{3.146}$$

It is easy to see that $\overline{\overline{D}}(1) = \overline{\overline{I}}$ and also that this kind of dyadic satisfies the simple condition

$$\overline{\overline{D}}(\alpha) \cdot \overline{\overline{D}}(\beta) = \overline{\overline{D}}(\beta) \cdot \overline{\overline{D}}(\alpha) = \overline{\overline{D}}(\alpha\beta), \tag{3.147}$$

whence the inverse dyadic is

$$\overline{\overline{D}}(\alpha)^{-1} = \overline{\overline{D}}(1/\alpha). \tag{3.148}$$

The isotropic interface relations for electric and magnetic fields can be written symmetrically in terms of relative permittivities and permeabilities

$$\overline{\overline{D}}(\epsilon_{1r}) \cdot \mathbf{E}_1 = \overline{\overline{D}}(\epsilon_{2r}) \cdot \mathbf{E}_2, \tag{3.149}$$

$$\overline{\overline{D}}(\mu_{1r}) \cdot \mathbf{H}_1 = \overline{\overline{D}}(\mu_{2r}) \cdot \mathbf{H}_2, \tag{3.150}$$

from which it is also easy to see that the inverse dyadic of (3.144) is obtained by an interchange of indices 1 and 2.

*Anisotropic media*

For the case of two anisotropic media, the interface condition is just a little more involved. We define the more general uniaxial dyadic of vector variable

$$\overline{\overline{D}}(\mathbf{a}) = \overline{\overline{I}} + \mathbf{na}, \tag{3.151}$$

and $\mathbf{n}$ is defined as either $\mathbf{n}_1$ or $\mathbf{n}_2$. The special case $\mathbf{a} = \alpha\mathbf{n}$ can be expressed in terms of the previous symmetric dyadic (3.146) as $\overline{\overline{D}}(\alpha\mathbf{n}) = \overline{\overline{D}}(\alpha + 1)$. Other properties are

$$\overline{\overline{D}}(\mathbf{a}) \cdot \overline{\overline{D}}(\mathbf{b}) = \overline{\overline{D}}(\mathbf{a} + (1 + \mathbf{n} \cdot \mathbf{a})\mathbf{b}), \tag{3.152}$$

$$\overline{\overline{D}}^{-1}(\mathbf{a}) = \overline{\overline{D}}(\frac{-\mathbf{a}}{1+\mathbf{n}\cdot\mathbf{a}}). \tag{3.153}$$

$$\overline{\overline{D}}^{-1}(\mathbf{a})\cdot\overline{\overline{D}}(\mathbf{b}) = \overline{\overline{D}}(-\frac{\mathbf{a}-\mathbf{b}}{1+\mathbf{n}\cdot\mathbf{a}}). \tag{3.154}$$

The two uniaxial dyadics $\overline{\overline{D}}(\mathbf{a})$, $\overline{\overline{D}}(\mathbf{b})$ do not commute in general. They do so only when $\mathbf{a}$ is a multiple of $\mathbf{b}$.

In terms of the defined uniaxial dyadic, the interface conditions with relative medium parameters can be expressed in symmetric form

$$\overline{\overline{D}}(\mathbf{n}\cdot\overline{\overline{\epsilon}}_{2r}-\mathbf{n})\cdot\mathbf{E}_2 = \overline{\overline{D}}(\mathbf{n}\cdot\overline{\overline{\epsilon}}_{1r}-\mathbf{n})\cdot\mathbf{E}_1, \tag{3.155}$$

$$\overline{\overline{D}}(\mathbf{n}\cdot\overline{\overline{\mu}}_{2r}-\mathbf{n})\cdot\mathbf{H}_2 = \overline{\overline{D}}(\mathbf{n}\cdot\overline{\overline{\mu}}_{1r}-\mathbf{n})\cdot\mathbf{H}_1, \tag{3.156}$$

and, hence, in the non-symmetric form

$$\mathbf{E}_2 = \overline{\overline{D}}(\frac{\mathbf{n}\cdot(\overline{\overline{\epsilon}}_{1r}-\overline{\overline{\epsilon}}_{2r})}{\overline{\overline{\epsilon}}_{2r}:\mathbf{nn}})\cdot\mathbf{E}_1, \tag{3.157}$$

$$\mathbf{H}_2 = \overline{\overline{D}}(\frac{\mathbf{n}\cdot(\overline{\overline{\mu}}_{1r}-\overline{\overline{\mu}}_{2r})}{\overline{\overline{\mu}}_{2r}:\mathbf{nn}})\cdot\mathbf{H}_1. \tag{3.158}$$

It is easy to show that, again, the inverse of the dyadics in (3.157), (3.158) is of the same form with indices 1 and 2 interchanged. If $\mathbf{n}$ is an eigenvector of all medium dyadics, the uniaxial $\overline{\overline{D}}$ dyadics become symmetric. This condition is satisfied when there is an optical axis in the direction of $\mathbf{n}$ in both media. The dyadics in (3.157), (3.158) obviously reduce to those of isotropic media for $\overline{\overline{\epsilon}}_i = \epsilon_i\overline{\overline{I}}$, $\overline{\overline{\mu}}_i = \mu_i\overline{\overline{I}}$.

*Bianisotropic media*

The previous relations can finally be generalized to bianisotropic media. In fact, instead of (3.157), (3.158) we have for the interface condition of two bianisotropic media

$$\begin{pmatrix} \mathbf{E}_2 \\ \mathbf{H}_2 \end{pmatrix} = \begin{pmatrix} \overline{\overline{D}}(\mathbf{a}_{21}) & \mathbf{nb}_{21} \\ \mathbf{nc}_{21} & \overline{\overline{D}}(\mathbf{d}_{21}) \end{pmatrix}\cdot\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{H}_1 \end{pmatrix}, \tag{3.159}$$

with

$$\mathbf{a}_{21} = \frac{1}{D_{22}}\mathbf{n}\cdot[(\overline{\overline{\mu}}_2:\mathbf{nn})(\overline{\overline{\epsilon}}_1-\overline{\overline{\epsilon}}_2)-(\overline{\overline{\xi}}_2:\mathbf{nn})(\overline{\overline{\zeta}}_1-\overline{\overline{\zeta}}_2)], \tag{3.160}$$

$$\mathbf{b}_{21} = \frac{1}{D_{22}}\mathbf{n}\cdot[(\overline{\overline{\mu}}_2:\mathbf{nn})(\overline{\overline{\xi}}_1-\overline{\overline{\xi}}_2)-(\overline{\overline{\xi}}_2:\mathbf{nn})(\overline{\overline{\mu}}_1-\overline{\overline{\mu}}_2)], \tag{3.161}$$

$$\mathbf{c}_{21} = \frac{1}{D_{22}}\mathbf{n}\cdot[(\overline{\overline{\epsilon}}_2 : \mathbf{nn})(\overline{\overline{\zeta}}_1 - \overline{\overline{\zeta}}_2) - (\overline{\overline{\zeta}}_2 : \mathbf{nn})(\overline{\overline{\epsilon}}_1 - \overline{\overline{\epsilon}}_2)], \tag{3.162}$$

$$\mathbf{d}_{21} = \frac{1}{D_{22}}\mathbf{n}\cdot[(\overline{\overline{\epsilon}}_2 : \mathbf{nn})(\overline{\overline{\mu}}_1 - \overline{\overline{\mu}}_2) - (\overline{\overline{\zeta}}_2 : \mathbf{nn})(\overline{\overline{\xi}}_1 - \overline{\overline{\xi}}_2)], \tag{3.163}$$

$$D_{22} = (\overline{\overline{\epsilon}}_2 : \mathbf{nn})(\overline{\overline{\mu}}_2 : \mathbf{nn}) - (\overline{\overline{\zeta}}_2 : \mathbf{nn})(\overline{\overline{\xi}}_2 : \mathbf{nn}). \tag{3.164}$$

All dyadics in (3.159) are symmetric only if $\mathbf{n}$ is an eigenvector of all difference dyadics. Also, the anisotropic conditions (3.157), (3.158) are special cases of the bianisotropic condition (3.159) when $\overline{\overline{\xi}}_i = 0$ and $\overline{\overline{\zeta}}_i = 0$.

For bi-isotropic media, the condition (3.159) is simplified to

$$\begin{pmatrix} \mathbf{E}_2 \\ \mathbf{H}_2 \end{pmatrix} = \begin{pmatrix} \overline{\overline{D}}(\alpha_{21}) & \mathbf{nn}\beta_{21} \\ \mathbf{nn}\gamma_{21} & \overline{\overline{D}}(\delta_{21}) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{H}_1 \end{pmatrix}, \tag{3.165}$$

with

$$\alpha_{21} = \frac{\mu_2\epsilon_1 - \xi_2\zeta_1}{\mu_2\epsilon_2 - \xi_2\zeta_2}, \qquad \beta_{21} = \frac{\mu_2\xi_1 - \xi_2\mu_1}{\mu_2\epsilon_2 - \xi_2\zeta_2}, \tag{3.166}$$

$$\gamma_{21} = \frac{\epsilon_2\zeta_1 - \zeta_2\epsilon_1}{\epsilon_2\mu_2 - \zeta_2\xi_2}, \qquad \delta_{21} = \frac{\epsilon_2\mu_1 - \zeta_2\xi_1}{\epsilon_2\mu_2 - \zeta_2\xi_2}, \tag{3.167}$$

which, again, has the isotropic conditions (3.144), (3.145) as the limiting case $\xi_i \to \zeta_i \to 0$.

### 3.4.4 Sheet conditions

Thin material slabs or shells can be approximately handled through sheet conditions. This amounts to approximating shells or slabs of finite thickness and finite medium parameters by infinitely thin sheets with infinite medium parameters. Relations between fields on each side of the sheet are specified in terms of sheet conditions. Some special structures like dense metallic grid or mesh surfaces can also be approximated by suitable sheet conditions.

The sheet conditions give linear relations between the tangential electric and magnetic fields across the sheet. A counterpart of an impedance sheet in circuit theory is a shunt or series impedance, or, more generally, a two port. The relation with circuit quantities is obtained again by considering $\mathbf{E}_t$ as the vector voltage and $\mathbf{n}\times\mathbf{H}$ as the vector current flowing in the $-\mathbf{n}$ direction.

*Impedance sheet*

The impedance sheet condition is similar to the terminating boundary impedance condition but with non-zero fields on both sides of the surface.

Assuming the magnetic surface current to be zero on the sheet we have

$$\nabla_s \times \mathbf{E} = \mathbf{n}_1 \times (\mathbf{E}_1 - \mathbf{E}_2) = -\mathbf{J}_{ms} = 0. \qquad (3.168)$$

The corresponding vector circuit is a shunt impedance since the vector voltage is continuous at the sheet. Writing from the Maxwell equations

$$\mathbf{J}_s = \nabla_s \times \mathbf{H} = \mathbf{n}_1 \times \mathbf{H}_1 + \mathbf{n}_2 \times \mathbf{H}_2 = \mathbf{n}_1 \times (\mathbf{H}_1 - \mathbf{H}_2), \qquad (3.169)$$

we have by denoting the surface admittance of the sheet by $\overline{\overline{Y}}_s$

$$\mathbf{n}_1 \times \mathbf{H}_1 + \mathbf{n}_2 \times \mathbf{H}_2 = \mathbf{J}_s = \overline{\overline{Y}}_s \cdot \mathbf{E}_t. \qquad (3.170)$$

Thus, the impedance sheet condition in this case can be expressed in terms of a dyadic shunt admittance $\overline{\overline{Y}}_s$.

As an example we may consider a thin slab of dielectric material with parameters $\epsilon_r \epsilon_o$, $\mu_o$ in air. The polarization current in the slab is

$$\mathbf{J}_p = j\omega \mathbf{D} - j\omega\epsilon_o \mathbf{E} = j\omega\epsilon_o(\epsilon_r - 1)\mathbf{E}_t, \qquad (3.171)$$

and if the thickness $t$ of the slab approaches zero so that $(\epsilon_r - 1)t$ remains finite, we have as the limit the surface current

$$\mathbf{J}_s = \mathbf{J}_p t = j\omega\epsilon_o(\epsilon_r - 1)t\mathbf{E}_t = Y_s \mathbf{E}_t. \qquad (3.172)$$

Thus, the shunt admittance of the dielectric sheet is

$$Y_s = j\omega\epsilon_o(\epsilon_r - 1)t = \frac{j}{\eta_o}(\epsilon_r - 1)k_o t. \qquad (3.173)$$

In the lossless case with $\epsilon_r$ real this is a capacitive reactance.

If the dielectric has conductive loss, $\epsilon = \epsilon_r\epsilon_o - j\sigma/\omega$, we can write for the shunt admittance

$$Y_s = \sigma t + \frac{j}{\eta_o}(\epsilon_r - 1)k_o t, \qquad (3.174)$$

which can be interpreted as being duc to two parallel sheets: a resistive sheet with the admittance $\sigma t$ and a pure reactive sheet.

*Magnetic impedance sheet*

The dual case of the previous example is a sheet with magnetic current and with no electric current: $\mathbf{J}_s = 0$. In this case the tangential magnetic field is continuous and the tangential electric field discontinuous. The condition

$$\mathbf{n}_1 \times \mathbf{H}_1 + \mathbf{n}_2 \times \mathbf{H}_2 = \mathbf{J}_s = 0 \qquad (3.175)$$

means that the vector currents at both sides of the sheet are the same, whence the equivalent circuit must be a series impedance. The sheet condition can be derived from (3.120) in the form

$$\mathbf{n}_1 \times \mathbf{J}_{ms} = \mathbf{E}_{1t} - \mathbf{E}_{2t} = \mathbf{n}_1 \times \overline{\overline{Y}}_{ms} \cdot \mathbf{H}_t = (\overline{\overline{Y}}_{ms} {}^{\times}_{\times} \mathbf{n}_1 \mathbf{n}_1) \cdot (\mathbf{n}_1 \times \mathbf{H}_t). \quad (3.176)$$

This corresponds to the series surface impedance dyadic $\overline{\overline{Z}}_s = \overline{\overline{Y}}_{ms} {}^{\times}_{\times} \mathbf{n}_1 \mathbf{n}_1$.

As an example let us consider a thin slab of magnetic material, $\epsilon_o$, $\mu_r \mu_o$ with magnetic polarization current approximated through the magnetic surface current

$$\mathbf{J}_{ms} = j\omega\mu_o(\mu_r - 1)t\mathbf{H}_t = j\eta_o(\mu_r - 1)k_o t\mathbf{H}_t. \quad (3.177)$$

Writing

$$\mathbf{n}_1 \times \mathbf{J}_{ms} = j\eta_o(\mu_r - 1)k_o t \mathbf{n}_1 \times \mathbf{H} = \mathbf{E}_{1t} - \mathbf{E}_{2t}, \quad (3.178)$$

we see that the vector voltage over the series impedance is $\mathbf{n}_1 \times \mathbf{J}_{ms}$ and the series impedance equals

$$\overline{\overline{Z}}_s = j\eta_o(\mu_r - 1)k_o t \overline{\overline{I}}. \quad (3.179)$$

*Combined sheet*

A generalization of the electric and magnetic impedance sheet is a sheet, where both electric and magnetic surface currents can flow. Because both tangential fields are now discontinuous across the sheet, the right-hand sides of (3.127) and (3.128) are taken as average values of the fields at the two sides. Thus, for the sheet conditions we can write the following pair of equations:

$$\mathbf{n}_1 \times \mathbf{H}_1 + \mathbf{n}_2 \times \mathbf{H}_2 = \overline{\overline{Y}}_s \cdot \frac{1}{2}(\mathbf{E}_{1t} + \mathbf{E}_{2t}), \quad (3.180)$$

$$\mathbf{E}_{1t} - \mathbf{E}_{2t} = (\mathbf{n}_1 \mathbf{n}_1 {}^{\times}_{\times} \overline{\overline{Y}}_{ms}) \cdot \frac{1}{2}(\mathbf{n}_1 \times \mathbf{H}_1 - \mathbf{n}_2 \times \mathbf{H}_2). \quad (3.181)$$

For the general sheet, $\overline{\overline{Y}}_s$ and $\overline{\overline{Y}}_{ms}$ are independent dyadics. The above equations can be expressed in a form relating vector voltages and currents to each other:

$$\begin{pmatrix} \mathbf{E}_{t1} \\ \mathbf{E}_{2t} \end{pmatrix} = \begin{pmatrix} \overline{\overline{Z}}_{11} & \overline{\overline{Z}}_{12} \\ \overline{\overline{Z}}_{12} & \overline{\overline{Z}}_{11} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{n}_1 \times \mathbf{H}_1 \\ \mathbf{n}_2 \times \mathbf{H}_2 \end{pmatrix}, \quad (3.182)$$

where we have assumed $Z_{11} = Z_{22}$ due to symmetry and $Z_{12} = Z_{21}$ due to reciprocity of the isotropic slab. From a simple comparison, we can write expressions for the dyadic impedance parameters

$$\overline{\overline{Z}}_{11} = \overline{\overline{Z}}_s + \frac{1}{4}\mathbf{n}_1\mathbf{n}_1{}^{\times}_{\times}\overline{\overline{Y}}_{ms}, \tag{3.183}$$

$$\overline{\overline{Z}}_{12} = \overline{\overline{Z}}_s - \frac{1}{4}\mathbf{n}_1\mathbf{n}_1{}^{\times}_{\times}\overline{\overline{Y}}_{ms}. \tag{3.184}$$

The slab can be simply described by an equivalent circuit of T type by noting that two series impedance dyadics are each $2(\overline{\overline{Z}}_{11} - \overline{\overline{Z}}_{12}) = \mathbf{n}_1\mathbf{n}_1{}^{\times}_{\times}\overline{\overline{Y}}_{ms}$, which is the same as the series impedance of a magnetic impedance slab. The shunt impedance in this case equals $\overline{\overline{Z}}_{12} = \overline{\overline{Z}}_s - \frac{1}{4}\mathbf{n}_1\mathbf{n}_1{}^{\times}_{\times}\overline{\overline{Y}}_{ms}$.

The two ports of the T circuit are disconnected in the case $\overline{\overline{Z}}_{12} = 0$, which corresponds to the condition

$$\overline{\overline{Z}}_s = \frac{1}{4}\mathbf{n}_1\mathbf{n}_1{}^{\times}_{\times}\overline{\overline{Y}}_{ms}. \tag{3.185}$$

This actually means that the two ports are terminated by a boundary surface impedance dyadic $\overline{\overline{Z}}_{11} - \overline{\overline{Z}}_{12} = \frac{1}{2}\mathbf{n}_1\mathbf{n}_1{}^{\times}_{\times}\overline{\overline{Y}}_{ms}$, which corresponds to a PEC plane positioned in the symmetry plane of the sheet short circuiting the surface admittance $\overline{\overline{Y}}_s$ and halving the magnetic surface admittance.

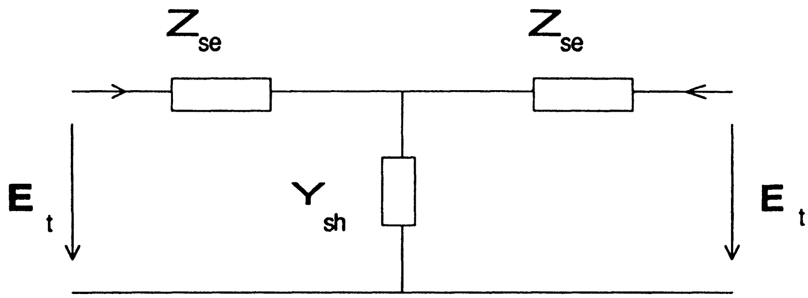


**Fig. 3.2** T circuit representation of a sheet of dielectric and magnetic material with series impedances and a shunt admittance.

As an example of a combined sheet we may take a slab with both relative permittivity and permeability different from unity. Comparing expressions we can write $\overline{\overline{Y}}_s = j\omega(\epsilon - \epsilon_o)t\overline{\overline{I}}$ and $\overline{\overline{Y}}_{ms} = j\omega(\mu - \mu_o)t\overline{\overline{I}}$, whence the T circuit series impedances are $Z_{se} = j\eta_o(\mu_r - 1)k_o t$ and the shunt admittance $Y_{sh} = j4k_o t(\epsilon_r - 1)/\eta_o$, Fig. 3.2. The disconnection condition becomes $(\epsilon_r - 1)(\mu_r - 1)(k_o t)^2 + 4 = 0$, which requires either $\mu_r < 1$ or $\epsilon_r < 1$.

Anisotropic sheets are sometimes used when different transmission and reflection properties are needed for different field polarizations. A simple example is a planar sheet of equidistant parallel metallic wires, which reflects well the polarization parallel to the wires and transmits the orthogonal polarization. The analysis is, however, not touched here.

### 3.4.5 Boundary and sheet impedance operators

The surface impedance concept is an approximation when replacing the physical interface of two media, because it assumes a constant ratio of tangential fields on the other side of the interface for all incident fields. At best this is approximately valid when the refraction factor of the second medium is much lower than that of the first medium, as demonstrated by USLENGHI (1991), because then the transmitted field is almost transverse electromagnetic (TEM) to the normal direction and the ratio of the tangential fields equals the wave impedance of the second medium. In more exact form the surface impedance can be written as an operator taking the field dependence into account. Likewise, the sheet impedance parameters can also be generalized to operators.

Let us consider the electromagnetic field vectors in a homogeneous isotropic medium. In particular, we are interested in a relation between the field components tangential to a plane with normal unit vector $\mathbf{n}$. This relation is carried over to another medium across a planar interface because of continuity. Starting from the Maxwell equations, we obtain the following second-order equations through elimination of the normal field components $\mathbf{n}\cdot\mathbf{E}$, $\mathbf{n}\cdot\mathbf{H}$:

$$(\nabla_t\nabla_t + k^2\overline{\overline{I}}_t)\cdot(\mathbf{n}\times\mathbf{H}) = -j\omega\epsilon(\mathbf{n}\cdot\nabla)\mathbf{E}_t, \tag{3.186}$$

$$(\mathbf{n}\mathbf{n}{}^{\times}_{\times}\nabla\nabla + k^2\overline{\overline{I}}_t)\cdot\mathbf{E}_t = -j\omega\mu(\mathbf{n}\cdot\nabla)\mathbf{n}\times\mathbf{H}. \tag{3.187}$$

These do not yet present conditions for the tangential field vectors at the plane because of the normal differentiation operator $\mathbf{n}\cdot\nabla$ on the right-hand sides. However, the normal differentiations of source-free fields can be obtained from the scalar Helmholtz equations

$$(\nabla^2 + k^2)\mathbf{E}_t = 0 \quad\Rightarrow\quad (\mathbf{n}\cdot\nabla)^2\mathbf{E}_t = -(\nabla_t^2 + k^2)\mathbf{E}_t, \tag{3.188}$$

$$(\nabla^2 + k^2)(\mathbf{n}\times\mathbf{H}) = 0 \quad\Rightarrow\quad (\mathbf{n}\cdot\nabla)^2(\mathbf{n}\times\mathbf{H}) = -(\nabla_t^2 + k^2)(\mathbf{n}\times\mathbf{H}). \tag{3.189}$$

Accepting a pseudo-differential operator involving a square root of differential operators, we may further write

$$(\mathbf{n}\cdot\nabla)\mathbf{E}_t = \pm j\sqrt{\nabla_t^2 + k^2}\mathbf{E}_t, \tag{3.190}$$

$$(\mathbf{n}\cdot\nabla)(\mathbf{n}\times\mathbf{H}) = \pm j\sqrt{\nabla_t^2 + k^2}(\mathbf{n}\times\mathbf{H}). \tag{3.191}$$

The sign is chosen so that for the case $\nabla_t = 0$ we obtain the boundary impedance expressions, which is denoted by taking $\pm \to +$. Inserting (3.190), (3.191) in (3.186), (3.187) results in relations similar to those of the surface impedance conditions (3.129):

$$\mathbf{E}_t = \overline{\overline{Z}}_s(\nabla_t)\cdot(\mathbf{n}\times\mathbf{H}), \tag{3.192}$$

$$\mathbf{n}\times\mathbf{H} = \overline{\overline{Y}}_s(\nabla_t)\cdot\mathbf{E}_t, \tag{3.193}$$

with the definition of the following impedance and admittance operators,

$$\overline{\overline{Z}}_s(\nabla_t) = \frac{\eta}{k}\frac{\nabla_t\nabla_t + k^2\overline{\overline{I}}_t}{\sqrt{\nabla_t^2 + k^2}}, \tag{3.194}$$

$$\overline{\overline{Y}}_s(\nabla_t) = \frac{\mathbf{nn}{}^\times_\times\nabla\nabla + k^2\overline{\overline{I}}_t}{k\eta\sqrt{\nabla_t^2 + k^2}}. \tag{3.195}$$

These two dyadics can be easily shown to be two-dimensional inverses of each other so that (3.192) and (3.193) actually represent the same equation. In fact, after showing that $\mathrm{spm}\overline{\overline{Z}}_s = \eta^2$, we can write

$$\overline{\overline{Z}}_s^{-1}(\nabla_t) = \frac{\mathbf{nn}{}^\times_\times\overline{\overline{Z}}_s^T}{\mathrm{spm}\overline{\overline{Z}}_s} = \overline{\overline{Y}}_s(\nabla_t). \tag{3.196}$$

It is also immediately seen that if the fields do not vary tangentially, i.e. for $\nabla_t = 0$, we simply have $\overline{\overline{Z}}_s = \eta\overline{\overline{I}}_t$ and $\overline{\overline{Y}}_s = \overline{\overline{I}}_t/\eta$. If the variance is slight, we may treat the operator $\nabla_t$ like a small quantity and use Taylor expansions for the square roots to arrive at the following two-term series expansions:

$$\overline{\overline{Z}}_s(\nabla_t) = \frac{\eta}{k^2}(\nabla_t\nabla_t + k^2\overline{\overline{I}}_t)\left(1 - \frac{\nabla_t^2}{2k^2} + \cdots\right) \approx$$

$$\eta\left(\overline{\overline{I}}_t + \frac{\nabla_t\nabla_t}{k^2} - \frac{\nabla_t^2}{2k^2}\overline{\overline{I}}_t\right) = \eta\left(\overline{\overline{I}}_t + \frac{\nabla_t\nabla_t}{2k^2} - \frac{\mathbf{nn}{}^\times_\times\nabla\nabla}{2k^2}\right), \tag{3.197}$$

$$\overline{\overline{Y}}_s(\nabla_t) = \frac{1}{k^2\eta}(\mathbf{nn}{}^\times_\times\nabla_t\nabla_t + k^2\overline{\overline{I}}_t)\left(1 - \frac{\nabla_t^2}{2k^2} + \cdots\right) \approx$$

$$\frac{1}{\eta}\left(\overline{\overline{I}}_t + \frac{\mathbf{nn}{}^\times_\times\nabla\nabla}{k^2} - \frac{\nabla_t^2}{2k^2}\overline{\overline{I}}_t\right) = \frac{1}{\eta}\left(\overline{\overline{I}}_t + \frac{\mathbf{nn}{}^\times_\times\nabla\nabla}{2k^2} - \frac{\nabla_t\nabla_t}{2k^2}\right). \tag{3.198}$$

The truncated operators $\overline{\overline{Y}}_s(\nabla)$ and $\overline{\overline{Z}}_s(\nabla)$ are not exact inverses of each other, but the error is a fourth order operator, only. These impedance and admittance operator expressions can be applied for problems with fields varying sufficiently slowly in the transverse direction.

*Wire grid*

A useful approximate impedance condition for a grid or mesh of thin metallic wires, with cells much smaller than the wavelength, was formulated by KONTOROVICH (1963). In practice, this condition has been observed to give good results with grid sizes as large as a quarter of a wavelength. Only the simple theory for a planar grid with square cells and bonded junctions is discussed here.

The grid plane is approximated by a continuous surface with the sheet current condition (3.169). In the Kontorovich average boundary condition theory, the tangential electric field is given by

$$\mathbf{E}_t = j\eta\gamma\left(\mathbf{J}_s + \frac{1}{2k^2}\nabla_t\nabla_t\cdot\mathbf{J}_s\right) = \overline{\overline{Z}}_s(\nabla_t)\cdot\mathbf{J}_s. \tag{3.199}$$

This is of a form similar to the operator surface impedance condition above with the operator

$$\overline{\overline{Z}}(\nabla_t) = j\eta\gamma\left(\overline{\overline{I}}_t + \frac{\nabla_t\nabla_t}{2k^2}\right). \tag{3.200}$$

The geometry of the grid is taken into account through the parameter $\gamma$:

$$\gamma = \frac{a}{\lambda}\ln\left(\frac{a}{2\pi r_o}\right), \tag{3.201}$$

where $a$ is the spacing between the wires, $r_o$ is the wire radius and $\lambda$ the wavelength. For $\gamma \to 0$ the grid becomes a conducting plane with infinitesimal cells and, in fact, we have $\overline{\overline{Z}}_s \to 0$ corresponding to a PEC plane. The approximation assumes the inequalities $r_o \ll a$ and $a < 0.25\lambda$ for reasonable accuracy.

Because there are no magnetic currents on the grid, we have $\mathbf{n}\times(\mathbf{E}_1 - \mathbf{E}_2) = 0$, and the circuit equivalent is a shunt impedance operator. The grid theory will be applied in Chapter 7.

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## 3.5 Uniqueness

In solving an electromagnetic problem numerically, it is essential to make sure that the problem has a unique solution. Non-uniqueness may lead to ill-behaved procedures, like singular or almost singular matrices, which either means a halt in running the computer program or a high error level. The uniqueness of a problem defined through a differential equation is dependent on the type of associated boundary conditions.

If a linear deterministic differential equation of the general form $Lf = g$ possesses two solutions $f_1$ and $f_2$ which are different $f_1 - f_2 = f_o \neq 0$, there exists a non-null solution to the corresponding homogeneous equation $Lf_o = 0$. Thus, the question whether or not the solution is unique equals the question whether or not the homogeneous equation possesses a non-null solution $f_o$.

The homogeneous equation can be understood as a generalized eigenvalue problem when one of the parameters of the problem, say $p_n$ is defined as the eigenvalue parameter $\lambda$:

$$L(p_1, p_2, ..p_n, ...p_N)f_o = 0, \qquad p_n = \lambda. \tag{3.202}$$

By a 'parameter' we mean any geometrical or physical parameter like a measure, material parameter, operating frequency or propagation factor. Normally the problem does not have a non-null solution $f_o$ for arbitrary value of $\lambda$, but only if the value of $\lambda$ is among the set of eigenvalues $\lambda_i$, in which case the corresponding $f_o$ can be understood as an eigenfunction of the problem.

The eigenvalue problem is not necessarily of the standard form $L_1 f = \lambda L_2 f$, because the parameter $\lambda$ may, for example, appear in an exponent or under a square-root sign. If the assigned value of the parameter $p_n$ does not coincide with any of the eigenvalues $\lambda_i$ of the homogeneous problem, the solution of the deteministic problem, if it exists, is unique. On the other hand, if the parameter value coincides with one of the eigenvalues, then we may add any multiple of the corresponding eigensolution to a solution of the deteministic problem and the result is again a solution, which thus is non-unique. For example, for a problem of a source inside a metallic enclosure, the field is unique if the operating frequency does not equal one of the resonance frequencies of the enclosure; otherwise we may add to the solution the field of the resonance mode with any amplitude.

The idea of uniqueness is easily understood but to check any specific problem for uniqueness is a difficult task in general, because it requires solving the homogeneous (sourceless) problem for eigenvalues. For some cases the uniqueness can, however, be simply established through a suitable integral theorem which ascertains that for some range of parameter values the homogeneous problem only has the null solution. Although these theorems do not give all possible values of parameters for uniqueness, they usually cover a range of practical values. A few such theorems are considered in the following.

### 3.5.1 Electrostatic problem

Let us consider the sourceless electrostatic problem, the scalar potential obeying the Laplace equation $\nabla^2\phi_o = 0$. For a closed problem with a surface $S$ bounding the volume $V$ of interest, the following integral expression can be written:

$$\int_V |\nabla\phi_o|^2 dV = \int_V \nabla\cdot(\phi_o\nabla\phi_o)dV - \int_V \phi_o\nabla^2\phi_o dV = \oint_S \phi_o(\mathbf{n}\cdot\nabla\phi_o)dS. \tag{3.203}$$

If the surface integral vanishes because of suitable boundary conditions, we have $\nabla\phi_o = 0$ in $V$, which makes the original problem unique since the corresponding sourceless electric field $\mathbf{E}_o = -\nabla\phi_o$ vanishes in $V$. We can list a number of boundary conditions giving unique solutions.

- $\phi_o = 0$ on $S$, whence $\phi_1 = \phi_2$ on $S$. This corresponds to the Dirichlet boundary condition $\phi = F(\mathbf{r})$ on $S$, meaning that the potential is assigned a given function on the boundary.

- $\mathbf{n} \cdot \nabla\phi_o = 0$ on $S$. This corresponds to the Neumann boundary condition $\mathbf{n} \cdot \nabla\phi = F(\mathbf{r})$ on $S$.

- For the combined condition which is Dirichlet on part of $S$ and Neumann on the rest of $S$ the uniqueness is still valid, because, again, the right-hand side of (3.203) vanishes.

- Finally also a mixed condition of the form $\alpha\phi + \beta\mathbf{n} \cdot \nabla\phi = F(\mathbf{r})$ may lead to a unique solution. To see this, the integral expression (3.203) is written in the following form:

$$\int_V |\nabla\phi_o|^2 dV =$$

$$\oint_S \frac{|\alpha\phi_o + \beta\mathbf{n} \cdot \nabla\phi_o|^2}{2\alpha\beta} dS - \oint_S \frac{|\alpha\phi_o|^2}{2\alpha\beta} dS - \oint_S \frac{|\beta\mathbf{n} \cdot \nabla\phi_o|^2}{2\alpha\beta} dS. \tag{3.204}$$

Uniqueness is obtained because the first surface integral vanishes owing to the boundary conditions and provided the condition $\alpha\beta > 0$ is satisfied. In fact, in this case the right-hand side is non-positive while the left-hand side is non-negative, whence both sides must vanish. Considering $\alpha$ an eigenvalue parameter, we might ask about its values giving non-null solutions $\phi_o$. The previous reasoning shows us only that they are *not* of the form $A/\beta$ where $A$ is any positive real number.

The previous is relevant to interior problems. If the volume $V$ is bounded by a surface $S$ and the sphere at infinity, we have the exterior problem. For the surface integral in infinity to vanish it is sufficient to have the far-field condition $\phi$ vanishing as $1/r$.

### 3.5.2 Scalar electromagnetic problem

The electromagnetic time-harmonic scalar potential $\phi$ is assumed to satisfy the Helmholtz equation with a suitable source term $g$:

$$\nabla^2\phi + k^2\phi = g. \tag{3.205}$$

Such a potential might be, for example, a component of the Hertz vector, from which the electromagnetic field can be derived. Uniqueness for $\phi$ presumes no non-null solution for the homogeneous equation $(\nabla^2 + k^2)\phi_o = 0$. This is a typical eigenvalue problem $L\phi_o = \lambda\phi_o$ and possesses a non-null solution if the quantity $k$ satisfies the condition $k^2 = -\lambda_i$, where $\lambda_i$ is one of the eigenvalues. The spectrum of eigenvalues depends on the boundary values of the problem. In some cases we know certainly that the solution is unique, for example when $k^2 = \omega^2\mu\epsilon$ is real and the eigenvalues are all non-real, or conversely. These two cases can be considered separately.

**$k^2$ not real:**

From the integral identity

$$(k^2 - k^{*2}) \int_V |\phi_o|^2 dS = \oint_S (\phi_o \mathbf{n} \cdot \nabla\phi_o^* - \phi_o^* \mathbf{n} \cdot \nabla\phi_o) dS. \tag{3.206}$$

we see that the surface integral vanishes for the Dirichlet and Neumann boundary conditions, $\phi = F(\mathbf{r})$ and $\mathbf{n} \cdot \nabla\phi = F(\mathbf{r})$. The volume integral vanishes for sure only if $k^2 = \omega^2\mu\epsilon$ is non-real, or if the medium is lossy and the frequency is real. Also, for impedance conditions $\alpha\phi + \beta\mathbf{n} \cdot \nabla\phi = F(\mathbf{r})$ on $S$ the surface integral can be shown to vanish provided $\alpha/\beta$ is real.

**$k^2$ real:**

In this case we apply the identity

$$\alpha \oint_S |\phi_o|^2 dS = \oint_S \phi_o^*(\alpha\phi_o + \beta\mathbf{n}\cdot\nabla\phi_o) dS - \beta \int_V (|\nabla\phi_o|^2 - k^2|\phi_o|^2) dV. \tag{3.207}$$

Assuming $\Im(\alpha/\beta) \neq 0$ (and hence $\alpha \neq 0$), we see that for the lossy impedance boundary condition $\alpha\phi + \beta\mathbf{n} \cdot \nabla\phi = F(\mathbf{r})$ on $S$, the second surface integral vanishes for the homogeneous solution $\phi_o$. Because the quantities on each side of the equation have different phase angles, the integrals must be zero. Thus, $\phi_o$ must vanish on $S$, and, from the boundary condition, so also must $\mathbf{n} \cdot \nabla\phi_o$. Vanishing of these two on $S$ implies vanishing of $\phi_o$ in the whole $V$, which can be shown by Green's formula, to be discussed in Chapter 5. Hence, if $k^2$ is real and $\alpha/\beta$ not real, the solution for the electromagnetic potential problem is unique.

This is also valid for exterior problems, if we take the sphere in infinity as the outer surface. The Sommerfeld conditions

$$\lim_{r\to\infty} \left[ r \left( \frac{\partial\phi_o}{\partial r} + jk\phi_o \right) \right] = 0, \qquad \lim_{r\to\infty} |r\phi_o| < K, \tag{3.208}$$

are in the form of impedance conditions with, $\alpha = jk$ and $\beta = 1$.

The reason for the uniqueness is that the eigenvalues in this case are complex and cannot coincide with the real $k^2$ values. The eigenvalues can also be real and the problem still unique if the frequency does not coincide with any of the resonant frequencies. Because these depend not only on the boundary conditions but also on the shape of the boundary surface, it does not seem possible to find a condition without actually solving the eigenvalue problem.

### 3.5.3 Vector electromagnetic problem

For the electromagnetic problem enclosed by a surface $S$, the solution $\mathbf{E}_o$, $\mathbf{H}_o$, again, is zero except when $\omega$ is one of the resonance frequencies of the cavity resonator defined by the surface $S$. This can be seen from the identity

$$\oint_S \mathbf{n} \cdot \mathbf{E}_o \times \mathbf{H}_o^* dS = j\omega\epsilon^* \int_V |\mathbf{E}_o|^2 dV - j\omega\mu \int_V |\mathbf{H}_o|^2 dV. \tag{3.209}$$

For the PEC or PMC boundary conditions the surface integral vanishes. If $arg(\mu) \neq -arg(\epsilon)$, or the equivalent, $\Im\{k^2\} \neq 0$, the medium is lossy and both volume integrals must vanish separately, whence $\mathbf{E}_o = 0$ and $\mathbf{H}_o = 0$. For real $k^2$ this conclusion does not result from the identity. In fact, at resonance, the volume integrals are proportional to magnetic and electric energies which are known to be equal. In this case, the uniqueness cannot be deduced simply because it depends on the shape of the resonator.

For lossy impedance boundary conditions the surface integral does not vanish, but the integrand is $\mathbf{H}_o^* \cdot \overline{\overline{Z}}_s \cdot \mathbf{H}_o$, which has a positive real part at every point of the boundary surface. Assuming now a lossless medium, (3.209) has a real part which is non-zero only on the left-hand side. This leads to the vanishing of $\mathbf{H}_o$, and hence $\mathbf{E}_o$, on the boundary and, from the Huygens' theorem (to be discussed in Chapter 6), at every point in $V$. Thus, lossy boundary, lossless medium and real frequency imply uniqueness for electromagnetic problems.

In other words, a cavity resonator with a lossy boundary has non-real eigenfrequencies, and a real frequency never coincides with them. If the boundary is a sphere receding to infinity we have basically a similar problem even if the discrete spectrum of eigenvalues becomes continuous at the limit. Taking the lossy surface impedance as $\overline{\overline{Z}}_s = \eta_o \overline{\overline{I}}_t$, the impedance condition becomes the Silver–Müller radiation condition and thus we have obtained uniqueness for the radiation field problem. This, of course, relies more on physical intuition than mathematics, those wishing a more saticfactory proof can consult JONES (1979) or COLTON and KRESS (1983).

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## 3.6 Conditions for medium parameters

In macroscopic electromagnetics all physics of media are hidden behind the medium parameters. The four dyadics of the linear medium have algebraic properties reflecting their different physical properties like passivity, reciprocity or losslessness, which are studied in this Section. The discussion is restricted to time-harmonic fields only.

### 3.6.1 Energy conditions

The real part of the complex Poynting vector

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* \tag{3.210}$$

is classically accepted to represent the directed average power flow density of the electromagnetic field. The origin of this flow is obtained from the divergence of the Poynting vector. From the Maxwell equations we can write the divergence in terms of the electromagnetic fields in the form

$$\nabla \cdot \mathbf{S} = -\frac{1}{2}(\mathbf{E} \cdot \mathbf{J}^* + \mathbf{H}^* \cdot \mathbf{J}_m) + 2j\omega\frac{1}{4}(\mathbf{E} \cdot \mathbf{D}^* - \mathbf{H}^* \cdot \mathbf{B}), \tag{3.211}$$

whose real part has the following termwise interpretation, each at the point of the argument $\mathbf{r}$:

- $\Re\{\nabla \cdot \mathbf{S}\}$ rate of electromagnetic energy generated at a point;
- $\frac{1}{2}\Re\{\mathbf{E} \cdot \mathbf{J}^* + \mathbf{H}^* \cdot \mathbf{J}_m\}$ amount of power needed to sustain the electric and magnetic currents at that point;
- $W_e = \frac{1}{4}\Re\{\mathbf{E} \cdot \mathbf{D}^*\}$ electric energy density;
- $W_m = \frac{1}{4}\Re\{\mathbf{H}^* \cdot \mathbf{B}\}$ magnetic energy density.

The interpretations for $W_e$ and $W_m$ above are, however, only valid for non-dispersive media. In macroscopic form we can write by integration

$$-\oint_S \mathbf{S} \cdot \mathbf{n}dS = \frac{1}{2}\int_V (\mathbf{E} \cdot \mathbf{J}^* + \mathbf{H}^* \cdot \mathbf{J}_m)dV - 2j\omega \int_V (W_e - W_m)dV. \tag{3.212}$$

The real part of this equation expresses the power balance in the volume $V$. The left-hand side gives the power flow through the surface $S$ into the volume $V$ (direction $-\mathbf{n}$), the real part of the second term corresponds to the power needed to sustain the currents in $V$ and the real part of the third term gives the power dissipation in the material medium, zero if the medium is lossless.

*Active and passive media*

Electromagnetic media can be classified as being active or passive corresponding to their properties of increasing or reducing the energy of an electromagnetic field. The property is inherent in the medium parameters. Although there may exist non-isotropic media active for some fields and passive for some other fields, let us only consider media which are either active or passive for all fields. This property can be seen directly from the sign of the divergence of the real part of the Poynting vector. The following characterizations are valid if the energy conditions are satisfied for all possible electromagnetic fields:

- $\Re\{\nabla\cdot\mathbf{S}\} > 0$ active medium (medium gives energy to the field);
- $\Re\{\nabla\cdot\mathbf{S}\} \le 0$ passive medium (medium does not give energy to the field);
- $\Re\{\nabla\cdot\mathbf{S}\} < 0$ lossy medium (field gives energy to the medium);
- $\Re\{\nabla\cdot\mathbf{S}\} = 0$ lossless medium (energy is not transferred between the medium and the field).

These energy conditions are transferred to conditions for the medium parameters of a linear medium if we write from (3.211) for a point $\mathbf{r}$ without sources $\mathbf{J}$, $\mathbf{J}_m$:

$$\Re\{\nabla\cdot\mathbf{S}\} = \Re\{\frac{j\omega}{2}(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{H}^*\cdot\mathbf{B})\} =$$

$$\frac{j\omega}{4}(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{E}^*\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}^* - \mathbf{H}^*\cdot\mathbf{B}) =$$

$$\frac{j\omega}{4}(\mathbf{E}\ \mathbf{H})\cdot\begin{pmatrix} \overline{\overline{\epsilon}}^* - \overline{\overline{\epsilon}}^T & \overline{\overline{\xi}}^* - \overline{\overline{\zeta}}^T \\ \overline{\overline{\zeta}}^* - \overline{\overline{\xi}}^T & \overline{\overline{\mu}}^* - \overline{\overline{\mu}}^T \end{pmatrix}\cdot\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}^*, \tag{3.213}$$

and require that this quantity be positive, negative or zero, for all fields $\mathbf{E}$, $\mathbf{H}$, corresponding to the different cases above.

*Lossless media*

For lossless media, $\Re\{\nabla \cdot \mathbf{S}\} = 0$ must be valid for all vectors $\mathbf{E}$, $\mathbf{H}$. For example, when $\mathbf{H} = 0$, this should be valid for all vectors $\mathbf{E}$. Invoking the theorem $\overline{\overline{A}} : \mathbf{aa}^* = 0$ for all $\mathbf{a} \Rightarrow \overline{\overline{A}} = 0$ we see that the dyadic $j(\overline{\overline{\epsilon}}^* - \overline{\overline{\epsilon}}^T)$ must vanish. Thus, $\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}^{T*}$ is hermitian, because its antihermitian part vanishes. In a similar way, other conditions are obtained and the conditions for lossless media can be written as

$$\overline{\overline{\epsilon}}^T = \overline{\overline{\epsilon}}^*, \quad \overline{\overline{\xi}}^T = \overline{\overline{\zeta}}^*, \quad \overline{\overline{\mu}}^T = \overline{\overline{\mu}}^*. \tag{3.214}$$

For bi-isotropic media, the conditions simplify to

$$\epsilon^* = \epsilon, \quad \xi^* = \zeta, \quad \mu^* = \mu. \tag{3.215}$$

Writing $\xi = (\chi + j\kappa)\sqrt{\mu_o\epsilon_o}$ and $\zeta = (\chi - j\kappa)\sqrt{\mu_o\epsilon_o}$, we see that, for a lossless bi-isotropic medium all four parameters $\epsilon$, $\mu$, $\chi$ and $\kappa$ must be real.

*Lossy media*

For lossy bianisotropic media, the antihermitian part of the $6 \times 6$ matrix defined by the medium parameter dyadics $\overline{\overline{\epsilon}}$, $\overline{\overline{\xi}}$, $\overline{\overline{\zeta}}$, $\overline{\overline{\mu}}$, null in the lossless case, must be positive definite. For example, for a conductive anisotropic medium, the conduction dyadic $\overline{\overline{\sigma}} = j\omega(\overline{\overline{\epsilon}} - \overline{\overline{\epsilon}}^{T*})$ must be positive definite for lossy media.

For a lossy bi-isotropic medium we can write an inequality limiting the magnitudes of $\chi$ and $\kappa$ parameters. In fact, because the matrix

$$\begin{pmatrix} -j(\epsilon - \epsilon^*) & -j(\xi - \zeta^*) \\ -j(\zeta - \xi^*) & -j(\mu - \mu^*) \end{pmatrix} \tag{3.216}$$

must be negative definite, its eigenvalues must be negative numbers. Writing the parameters in terms of real and imaginary parts,

$$\epsilon = \epsilon_{\text{re}} + j\epsilon_{\text{im}}, \quad \xi = \xi_{\text{re}} + j\xi_{\text{im}}, \quad \zeta = \zeta_{\text{re}} + j\zeta_{\text{im}}, \quad \mu = \mu_{\text{re}} + j\mu_{\text{im}}, \tag{3.217}$$

we actually arrive at the three conditions

$$\epsilon_{\text{im}} < 0, \quad \mu_{\text{im}} < 0, \quad (\xi_{\text{im}} + \zeta_{\text{im}})^2 + (\xi_{\text{re}} - \zeta_{\text{re}})^2 < 4\epsilon_{\text{im}}\mu_{\text{im}}, \tag{3.218}$$

of which the last one can be written

$$\chi_{\text{im}}^2 + \kappa_{\text{im}}^2 < \frac{\mu_{\text{im}}\epsilon_{\text{im}}}{\mu_o\epsilon_o}. \tag{3.219}$$

Thus there is a physical limit for the magnitudes of the imaginary parts of the parameters $\chi$ and $\kappa$ due to the passivity of the medium. The medium cannot be lossy through the $\chi$ and $\kappa$ parameters alone, because $\mu_{\text{im}} = 0$ and $\epsilon_{\text{im}} = 0$ are seen to imply $\chi_{\text{im}} = 0$ and $\kappa_{\text{im}} = 0$.

*Active media*

For active media, all the conditions given for lossy media are valid with inequality signs turned around. In particular, the conduction dyadic $\overline{\overline{\sigma}} = j\omega(\overline{\overline{\epsilon}} - \overline{\overline{\epsilon}}^{T*})$ must be negative definite for active media, which corresponds to negative resistance in circuit theory.

*Positive energy density function*

Yet another condition for the medium parameters is obtained by requiring that the energy function be positive. For a dispersive and lossless bi-isotropic medium, the following is an extension of the energy density expression in a dispersive anisotropic medium given by YEH and LIU (1972), and CHEN (1983):

$$W = \frac{1}{8}(\mathbf{E}\ \mathbf{H}) \cdot \frac{d}{d\omega} \begin{pmatrix} \omega(\epsilon + \epsilon^*) & \omega(\zeta + \xi^*) \\ \omega(\xi + \zeta^*) & \omega(\mu + \mu^*) \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}^* =$$

$$\frac{1}{4}(\mathbf{E}\ \mathbf{H}) \cdot \begin{pmatrix} \epsilon & (\chi + j2\kappa)\sqrt{\mu_o \epsilon_o} \\ (\chi - j2\kappa)\sqrt{\mu_o\epsilon_o} & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}^* . \qquad (3.220)$$

In the last expression we have assumed that the real parameters $\epsilon$, $\mu$ and $\chi$ are independent and the imaginary parameter $j\kappa$ is linearly dependent on $\omega$, which appears a fair assumption for low enough frequencies.

To have a positive energy function, the matrix should be positive definite, which leads to the following three conditions for the real parameters,

$$\epsilon > 0, \quad \mu > 0, \quad \chi^2 + (2\kappa)^2 < n^2 = \frac{\mu\epsilon}{\mu_o\epsilon_o}, \qquad (3.221)$$

which in particular gives a limiting condition for the magnitudes of the parameters $\chi$ and $\kappa$. It must be emphasized, however, that these are valid for low frequencies only and the $\omega$ dependence of the parameters according to any particular model of the medium affects the inequalities at higher frequencies.

*Boundary conditions*

The power absorbed by a boundary surface with the impedance dyadic $\overline{\overline{Z}}_s$ can be obtained from

$$\Re\{\mathbf{n} \cdot \mathbf{S}\} = \frac{1}{4}\mathbf{n} \cdot (\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}) = \frac{1}{4}\mathbf{H} \cdot (\overline{\overline{Z}}_s{}^* + \overline{\overline{Z}}_s{}^T) \cdot \mathbf{H}^* \cdot \qquad (3.222)$$

Here the unit vector $\mathbf{n}$ points towards the impedance surface. If (3.222) is positive for all fields $\mathbf{H}$, the power is flowing from the fields into the absorbing boundary and not the other way. Thus, the condition for a lossy boundary is positive definiteness of the hermitian part of the surface impedance dyadic. This means a positive real symmetric part and a negative imaginary antisymmetric part.

For a lossless boundary the hermitian part of the surface impedance dyadic is zero. In other words, the impedance satisfies

$$\overline{\overline{Z}}^T = -\overline{\overline{Z}}^*, \tag{3.223}$$

or $\overline{\overline{Z}}_s$ must be an antihermitian dyadic. For a lossless bi-isotropic impedance, $\overline{\overline{Z}}_s = Z_a\overline{\overline{I}}_t + Z_b\mathbf{n} \times \overline{\overline{I}}$, we thus see that $Z_a$ must be imaginary and $Z_b$ real. In the isotropic case, the boundary impedance must be imaginary to be lossless, as is well understood from circuit theory.

### 3.6.2 Reciprocity conditions

Besides energy quantites, reaction is another quadratic function of electromagnetic fields and sources. It was introduced by RUMSEY in 1954 to account for the interaction of fields and antennas. Let us consider two sources $\mathsf{g}_a$ and $\mathsf{g}_b$ and their fields $\mathsf{f}_a$, $\mathsf{f}_b$. The reaction of the field $a$ on the source $b$ is a complex number defined by

$$< \mathsf{f}_a, \mathsf{g}_b >= \int\limits_{V_b} (\mathbf{E}_a \cdot \mathbf{J}_b - \mathbf{H}_a \cdot \mathbf{J}_{mb})dV, \tag{3.224}$$

where the volume $V_b$ contains all of the sources $b$. (The original notation $< a, b >$ does not distinguish between the source and the field.) The reaction is a measurable quantity. For example, if the source $b$ is a current element $\mathbf{J} = \mathbf{u}I_bL\delta(\mathbf{r} - \mathbf{r}_b)$, the reaction of the unknown field $\mathbf{E}_a(\mathbf{r})$ is

$$< \mathsf{f}_a, \mathsf{g}_b >= I_bL\mathbf{u} \cdot \mathbf{E}_a(\mathbf{r}_b), \tag{3.225}$$

which is $I_b$ times the voltage induced by the field $\mathbf{E}_a$ in the dipole $b$. In general, setting a test source $b$ in the field due to a source $a$ and measuring the reaction gives a method to determine the field $a$.

Reaction is symmetric under conditions called reciprocal. The difference of two reactions can be expanded from the Maxwell equations

$$< \mathsf{f}_a, \mathsf{g}_b > - < \mathsf{f}_b, \mathsf{g}_a >=$$

$$\int\limits_V (\mathbf{E}_a \cdot \mathbf{J}_b - \mathbf{H}_a \cdot \mathbf{J}_{mb} - \mathbf{E}_b \cdot \mathbf{J}_a + \mathbf{H}_b \cdot \mathbf{J}_{ma})dV =$$

$$\int_S \mathbf{n}\cdot(\mathbf{E}_a\times\mathbf{H}_b-\mathbf{E}_b\times\mathbf{H}_a)dS+$$
$$j\omega\int_V\Big(\mathbf{E}_b\cdot(\overline{\overline{\epsilon}}^T-\overline{\overline{\epsilon}})\cdot\mathbf{E}_a-\mathbf{E}_b\cdot(\overline{\overline{\zeta}}^T+\overline{\overline{\xi}})\cdot\mathbf{H}_a+$$
$$\mathbf{H}_b\cdot(\overline{\overline{\xi}}^T+\overline{\overline{\zeta}})\cdot\mathbf{E}_a-\mathbf{H}_b\cdot(\overline{\overline{\mu}}^T-\overline{\overline{\mu}})\cdot\mathbf{H}_a\Big)dV, \tag{3.226}$$

where $S$ is the surface of the volume $V$ containing all sources and $\mathbf{n}$ is the outward normal unit vector of $S$.

Reciprocity requires (3.226) to vanish for all fields. Being a property of the medium, we can see the conditions for the medium to be reciprocal:

$$\overline{\overline{\epsilon}}=\overline{\overline{\epsilon}}^T,\quad \overline{\overline{\mu}}=\overline{\overline{\mu}}^T,\quad \overline{\overline{\xi}}=-\overline{\overline{\zeta}}^T,\quad \overline{\overline{\zeta}}=-\overline{\overline{\xi}}^T, \tag{3.227}$$

because the volume integral in (3.226) vanishes. For the bi-isotropic medium, this reduces to $\xi=-\zeta$, or $\chi=0$. Thus, the parameter $\chi$ can be called the non-reciprocity parameter. Because a medium with $\chi\neq 0$ was introduced by TELLEGEN in 1948, it is also called the Tellegen parameter.

Requiring the surface integral of (3.226) to vanish, we can obtain conditions for a reciprocal boundary. Inserting the impedance boundary condition

$$\mathbf{E}_t=\overline{\overline{Z}}_s\cdot(\mathbf{n}\times\mathbf{H}), \tag{3.228}$$

the boundary term in (3.226) can be written as

$$\int_S \mathbf{n}\cdot(\mathbf{E}_a\times\mathbf{H}_b-\mathbf{E}_b\times\mathbf{H}_a)dS=\int_S \mathbf{H}_b\cdot(\overline{\overline{Z}}_s-\overline{\overline{Z}}_s{}^T)\cdot\mathbf{H}_a dS, \tag{3.229}$$

which is seen to vanish for any fields $\mathbf{H}_a$, $\mathbf{H}_b$ with the condition

$$\overline{\overline{Z}}_s=\overline{\overline{Z}}_s{}^T. \tag{3.230}$$

Thus, a symmetric impedance dyadic is reciprocal. The bi-isotropic impedance dyadic $\overline{\overline{Z}}_s=Z_a\overline{\overline{I}}_t+Z_b\mathbf{n}\times\overline{\overline{I}}$ is reciprocal only for $Z_b=0$, or when the impedance is actually isotropic.

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