8. Fields in Anisotropic Regions

8.1 INTRODUCTION

The descriptive simplicity of wave excitation and propagation in uniaxial media no longer obtains in the case of general anisotropic media. As noted in Chapter 7, the reduced electromagnetic field equations in a uniaxial medium are characterized by a permittivity dyadic $\boldsymbol{\epsilon}$ and permeability dyadic $\boldsymbol{\mu}$ that are diagonal in appropriate real coordinate frames. In arbitrary anisotropic media, on the other hand, these dyadics cannot be so diagonalized and in a real coordinate basis, all matrix (tensor) elements of $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are in general non-vanishing. However, in gyrotropic regions, which possess a symmetry axis along, say, the $\mathbf{b}_0$ direction, the $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ dyadics assume the form

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_t + \epsilon_z \mathbf{b}_0 \mathbf{b}_0, \qquad \boldsymbol{\mu} = \boldsymbol{\mu}_t + \mu_z \mathbf{b}_0 \mathbf{b}_0, \tag{1}$$

where $\boldsymbol{\epsilon}_t$ and $\boldsymbol{\mu}_t$ are in general non-diagonal dyadics transverse to $\mathbf{b}_0$; for a uniaxial medium, $\boldsymbol{\epsilon}_t$ and $\boldsymbol{\mu}_t$ are of course diagonal.

In the present chapter we shall consider guided-wave descriptions in anisotropic media, with special emphasis on electromagnetic fields in gyrotropic regions. Descriptions of general linear fields in anisotropic media can be effected either in terms of first-order field equations or, on elimination of non-electromagnetic field variables, can be reduced to higher-order equations for just the electromagnetic variables. As noted in Sec. 1.5, this reduction process implies the introduction of equivalent permittivity and permeability dyadics that in general are spatially and temporally dispersive operators containing ∇ and $\partial/\partial t$. The reduced field formulation frequently leads to analytical complexities in identification of energy expressions, reciprocity properties, eigenmodes, etc., and may even omit non-electromagnetic types of wave phenomena. The first-order formulation, in which electromagnetic and non-electromagnetic field variables are given equal status, avoids many of these difficulties. Nevertheless, because of the widespread use of reduced descriptions in the literature, both first-order and reduced descriptions will be employed in the following. In certain cases, for example in the guided-wave descriptions of a cold magnetoplasma with non-spatially dispersive parameters, the reduced description is adequate and does not give rise to the complexities noted above.

In the guided-wave description of a homogeneous linear field developed in Sec. 1.4, the overall field is represented as a superposition of eigenmodes of the form $\Psi_\alpha \exp(i\kappa_\alpha z)$. The variable z distinguishes the guiding or symmetry axis, κ_α denotes the eigenvalue or mode wavenumber, and the field eigenvector Ψ_α depends on the spatial variable $\boldsymbol{\rho}$ transverse to z and on the time t. Explicitly, the overall field representation takes the form (at source-free points)

$$\Psi(\mathbf{r}, t) = \sum_\alpha a_\alpha(0)\Psi_\alpha(\boldsymbol{\rho}, t)e^{i\kappa_\alpha z} = \sum_\alpha a_\alpha(z)\Psi_\alpha(\boldsymbol{\rho}, t), \tag{2}$$

where $a_\alpha(z)$ distinguishes the amplitude of the αth mode at z. For time harmonicity and unbounded cross sections, $\Psi_\alpha(\boldsymbol{\rho}, t) = \Psi_\alpha \exp[i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)]$, while for bounded cross sections, $\Psi_\alpha(\boldsymbol{\rho}, t) = \Psi_\alpha(\boldsymbol{\rho}) \exp(-i\omega t)$; in either event, if the Ψ_α and their orthogonality properties are known, it is a simple matter to evaluate the amplitude coefficients $a_\alpha(z)$ from a knowledge of the total field Ψ at any reference plane z. In the following sections, the requisite information about the mode functions Ψ_α and wavenumbers κ_α will be ascertained for anisotropic media (Sec. 8.2) and gyrotropic media (Secs. 8.2 and 8.3).

Differences in the nature of wave propagation in isotropic and anisotropic regions are revealed by contrasting the transmission line behavior of mode amplitudes $a_\alpha(z)$ in such media. In all cases, assuming uniformity along the direction z, the z dependence of the a_α at source-free points is determined by the equations

$$\frac{d}{dz} a_\alpha = i\kappa_\alpha a_\alpha. \tag{3}$$

Only for regions reflection symmetric with respect to z, wherein there exists for each wave a_α a reflected wave with appropriate field symmetry and propagation wavenumber $\kappa_{-\alpha} = -\kappa_\alpha$, is it possible to develop a conventional transmission line description in terms of voltage and current amplitudes as we have done for isotropic and uniaxial regions. As noted in Sec. 7.1, in the guided-wave description of propagation in isotropic (unbounded) regions, one introduces a $\mathbf{k}_t, \omega$ basis of modes with an $\exp[i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)]$ dependence and thereby reduces the overall field problem to two independent E- and H-mode transmission-line problems with identical propagation wavenumbers $\pm\kappa'_\alpha = \pm\kappa''_\alpha = \pm\kappa_\alpha$ but unequal characteristic impedances $\pm Z'_\alpha \neq \pm Z''_\alpha$. In a uniaxial region with transmission direction chosen along the uniaxial axis (i.e., $\mathbf{b}_0$ parallel to $\mathbf{z}_0$), the field problem is still reducible (Sec. 7.2) to conventional E- and H-mode transmission-line problems, but with the parameters $\pm\kappa'_\alpha \neq \pm\kappa''_\alpha$ and $\pm Z'_\alpha \neq \pm Z''_\alpha$ characterizing the direct and reflected waves. However, nonconventional transmission-line descriptions (see Sec. 8.2.h) are possible for gyrotropic regions, or more generally for any region admitting $\pm\kappa_\alpha$ waves.

For a gyrotropic region with transmission along the gyrotropic axis, one finds direct and reflected waves with $\pm\kappa'_\alpha \neq \pm\kappa''_\alpha$. Moreover, the corresponding characteristic impedances are not $\pm$ of one another but rather one has Z'_α and $-Z'^*_\alpha$; also, one has Z''_α and $-Z''^*_\alpha$, which are not the same as the corresponding primed quantities. For a gyrotropic region with the direction of transmission chosen perpendicular to the gyrotropic axis (see Sec. 8.4b), one finds similar direct and reflected waves, but the characteristic impedances of these waves are not simply related. In the case of a general anisotropic region, one has $\kappa_\alpha \neq \kappa_\beta$ (i.e., it is not generally possible to find modes with equal but opposite wavenumbers). A schematization of these line descriptions in an unbounded electromagnetic region is displayed in Fig. 8.1.1.

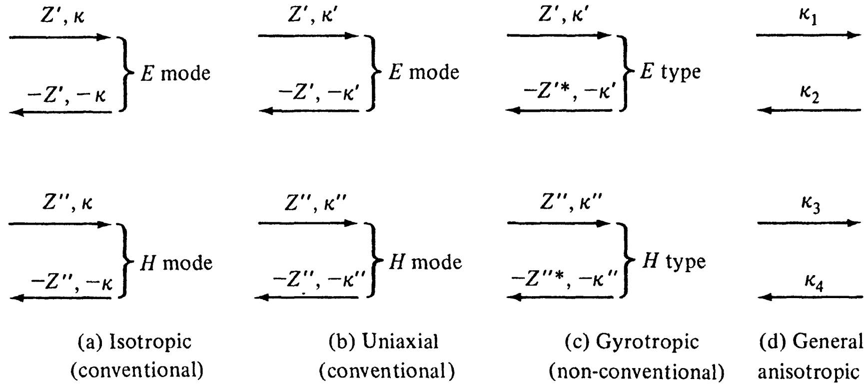

FIG. 8.1.1 Transmission-line descriptions in unbounded, homogeneous media (transmission direction along symmetry direction, if any).

8.2 GUIDED-WAVE REPRESENTATION IN ANISOTROPIC MEDIA (REDUCED FORMULATION)

8.2a Formulation for Arbitrary Media

As noted in Sec. 8.1, a reduced description of electromagnetic radiation and scattering in non-spatially dispersive media involves only the electromagnetic fields **E** and **H** and employs an equivalent permittivity dyadic $\boldsymbol{\epsilon}$ and permeability dyadic $\boldsymbol{\mu}$. For generality, both the permittivity $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\mathbf{r}, \omega)$ and permeability $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{r}, \omega)$ are assumed $\mathbf{r}$ and ω dependent, thereby making the discussion applicable to inhomogeneous anisotropic media under steady-state conditions. This reduced formulation will be seen as a generalization of procedures utilized previously in Chapters 5 and 7 for isotropic and uniaxial media.

In a reduced formulation, the steady-state Maxwell equations in a source-excited, inhomogeneous, anisotropic medium can be written in the operator form [omitting the time dependence $\exp(-i\omega t)$]

$$L\Psi(\mathbf{r}) = -\Phi(\mathbf{r}), \tag{1a}$$

where, as in Sec. 1.4,

$$L \to -i\begin{bmatrix} \omega\boldsymbol{\epsilon} & -i\nabla \times \mathbf{1} \\ i\nabla \times \mathbf{1} & \omega\boldsymbol{\mu} \end{bmatrix}, \quad \Psi \to \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}, \quad \Phi \to \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix}, \tag{1b}$$

J and **M** being the equivalent steady-state electric and magnetic current densities that appear in a reduced description on eliminating dynamical variables.

To effect a guided-wave representation with z as the transmission direction, one decomposes the field operator L into components that depend on either ∇_t or $\partial/\partial z$ $[\nabla = \nabla_t + \mathbf{z}_0(\partial/\partial z)]$,

$$L = -i\left(K - \frac{\Gamma}{i}\frac{\partial}{\partial z}\right), \tag{2a}$$

where

$$K \to \begin{bmatrix} \omega\boldsymbol{\epsilon} & -i\nabla_t \times \mathbf{1} \\ i\nabla_t \times \mathbf{1} & \omega\boldsymbol{\mu} \end{bmatrix} \quad \text{and} \quad \Gamma \to \begin{bmatrix} 0 & -\mathbf{z}_0 \times \mathbf{1} \\ \mathbf{z}_0 \times \mathbf{1} & 0 \end{bmatrix}. \tag{2b}$$

In a z-oriented vector basis the dyadics $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ may be represented for a general gyroelectric and gyromagnetic medium as

$$\begin{aligned} \boldsymbol{\epsilon} &= \boldsymbol{\epsilon}_t + \mathbf{z}_0\mathbf{z}_0\epsilon_z + \mathbf{z}_0\bar{\boldsymbol{\epsilon}}_{zt} + \bar{\boldsymbol{\epsilon}}_{tz}\mathbf{z}_0, \\ \boldsymbol{\mu} &= \boldsymbol{\mu}_t + \mathbf{z}_0\mathbf{z}_0\mu_z + \mathbf{z}_0\bar{\boldsymbol{\mu}}_{zt} + \bar{\boldsymbol{\mu}}_{tz}\mathbf{z}_0, \end{aligned} \tag{2c}$$

where $\mathbf{u}_t$ is a transverse dyadic such that $\mathbf{u}_t \cdot \mathbf{z}_0 = 0 = \mathbf{z}_0 \cdot \mathbf{u}_t$ and $\bar{\mathbf{u}}_{tz}$, $\bar{\mathbf{u}}_{zt}$ are vectors transverse to $\mathbf{z}_0$, $\mathbf{u}$ representing either $\boldsymbol{\epsilon}$ or $\boldsymbol{\mu}$. For a gyrotropic medium $\bar{\boldsymbol{\epsilon}}_{zt} = 0 = \bar{\boldsymbol{\epsilon}}_{tz}$ and $\bar{\boldsymbol{\mu}}_{zt} = 0 = \bar{\boldsymbol{\mu}}_{tz}$; this permits a simplification in the analysis, but for the time being we shall continue to treat the general case.

As in the isotropic case, the longitudinal components E_z and H_z of electric and magnetic fields are derivable from the transverse components $\mathbf{E}_t$ and $\mathbf{H}_t$. Thus from Eqs. (1) and (2) one derives

$$\begin{aligned} \begin{bmatrix} E_z \\ H_z \end{bmatrix} &= \begin{bmatrix} -\epsilon_z^{-1}\bar{\boldsymbol{\epsilon}}_{zt} & (i\omega\epsilon_z)^{-1}\nabla_t \times \mathbf{z}_0 \\ -(i\omega\mu_z)^{-1}\nabla_t \times \mathbf{z}_0 & -\mu_z^{-1}\bar{\boldsymbol{\mu}}_{zt} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{bmatrix} + \begin{bmatrix} \dfrac{J_z}{i\omega\epsilon_z} \\ \dfrac{M_z}{i\omega\mu_z} \end{bmatrix} \\ &\to \mathcal{N}\bar{\Psi} + \theta_z, \end{aligned} \tag{3a}$$

where it is recalled that $\nabla_t \times \mathbf{z}_0 \cdot \mathbf{A}_t \equiv \nabla_t \cdot (\mathbf{z}_0 \times \mathbf{A}_t)$ and

$$\bar{\Psi} \to \begin{bmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{bmatrix}, \qquad \theta_z \to \begin{bmatrix} \dfrac{J_z}{i\omega\epsilon_z} \\ \dfrac{M_z}{i\omega\mu_z} \end{bmatrix}. \tag{3b}$$

On elimination of the longitudinal field components from Eqs. (1) and (2) via

Eqs. (3a), one can obtain defining equations for the transverse wavevector $\bar{\Psi}(\mathbf{r})$. Because of the relative complexity of the resulting transverse equations in the general anisotropic case, we shall not exhibit them explicitly (however, see the source-free transverse modal equations (6) below).

To obtain the mode functions characteristic of guided waves propagating along z, we seek source-free ($\Phi = 0$) solutions of Eq. (1a) with the form

$$\Psi_\alpha(\mathbf{r}) = \Psi_\alpha(\boldsymbol{\rho})e^{i\kappa_\alpha z}, \tag{4a}$$

where the $\Psi_\alpha(\boldsymbol{\rho})$ and κ_α are eigenfunctions and eigenvalues, respectively, and $\mathbf{r} = \boldsymbol{\rho} + z\mathbf{z}_0$. The presence of the $\exp(i\kappa_\alpha z)$ factor implies that medium properties are independent of z, a restriction subsequently removed. As in Sec. 1.4, substitution of Eq. (4a) into the source-free equation (1a) yields the eigenvalue equation

$$K\Psi_\alpha = \kappa_\alpha \Gamma \Psi_\alpha \tag{4b}$$

subject on the transverse boundary s, if any, to the dyadic impedance condition

$$\boldsymbol{\nu} \times \mathbf{E}_\alpha = \mathscr{Z} \cdot \mathbf{H}_\alpha \qquad \text{on } s, \tag{4c}$$

or in a $\mathbf{s}_0, \mathbf{z}_0$ basis,

$$\begin{aligned} -\mathbf{s}_0 \cdot \mathbf{E}_\alpha &= \mathscr{Z}_{zs}\mathbf{H}_\alpha \cdot \mathbf{s}_0 + \mathscr{Z}_{zz}\mathbf{H}_\alpha \cdot \mathbf{z}_0, \\ \mathbf{z}_0 \cdot \mathbf{E}_\alpha &= \mathscr{Z}_{ss}\mathbf{H}_\alpha \cdot \mathbf{s}_0 + \mathscr{Z}_{sz}\mathbf{H}_\alpha \cdot \mathbf{z}_0, \end{aligned} \tag{4d}$$

where the tangential and normal unit vectors $\mathbf{s}_0$ and $\boldsymbol{\nu}$, respectively, are related by $\boldsymbol{\nu} \times \mathbf{z}_0 = \mathbf{s}_0$. The transverse nature of the electromagnetic field implies that the z components of the mode fields are derivable from the transverse components. If one decomposes the mode fields as

$$\mathbf{E}_\alpha = \mathbf{E}_{t\alpha} + E_{z\alpha}\mathbf{z}_0, \qquad \mathbf{H}_\alpha = \mathbf{H}_{t\alpha} + H_{z\alpha}\mathbf{z}_0, \tag{5}$$

and eliminates the longitudinal z components via [see Eq. (3a)],

$$\begin{aligned} -i\omega\mu_z H_{z\alpha} - i\omega\bar{\boldsymbol{\mu}}_{zt} \cdot \mathbf{H}_{t\alpha} &= \nabla_t \cdot (\mathbf{z}_0 \times \mathbf{E}_{t\alpha}), \\ -i\omega\epsilon_z E_{z\alpha} - i\omega\bar{\boldsymbol{\epsilon}}_{zt} \cdot \mathbf{E}_{t\alpha} &= \nabla_t \cdot (\mathbf{H}_{t\alpha} \times \mathbf{z}_0), \end{aligned} \tag{5a}$$

the eigenvalue equation (4) can be expressed solely in terms of transverse components as (see Sec. 2.2a)

$$\begin{aligned} &\omega\left[\boldsymbol{\epsilon}_t - \frac{\bar{\boldsymbol{\epsilon}}_{tz}\bar{\boldsymbol{\epsilon}}_{zt}}{\epsilon_z} - \frac{1}{\omega^2}\mathbf{z}_0 \times \nabla_t \frac{1}{\mu_z}\nabla_t \times \mathbf{z}_0\right] \cdot \mathbf{E}_{t\alpha} \\ &\qquad + i\left[\frac{\bar{\boldsymbol{\epsilon}}_{tz}}{\epsilon_z}\mathbf{z}_0 \times \nabla_t - \mathbf{z}_0 \times \nabla_t \frac{\bar{\boldsymbol{\mu}}_{zt}}{\mu_z}\right] \cdot \mathbf{H}_{t\alpha} = \kappa_\alpha \mathbf{H}_{t\alpha} \times \mathbf{z}_0, \\ &\omega\left[\boldsymbol{\mu}_t - \frac{\bar{\boldsymbol{\mu}}_{tz}\bar{\boldsymbol{\mu}}_{zt}}{\mu_z} - \frac{1}{\omega^2}\mathbf{z}_0 \times \nabla_t \frac{1}{\epsilon_z}\nabla_t \times \mathbf{z}_0\right] \cdot \mathbf{H}_{t\alpha} \\ &\qquad - i\left[\frac{\bar{\boldsymbol{\mu}}_{tz}}{\mu_z}\mathbf{z}_0 \times \nabla_t - \mathbf{z}_0 \times \nabla_t \frac{\bar{\boldsymbol{\epsilon}}_{zt}}{\epsilon_z}\right] \cdot \mathbf{E}_{t\alpha} = \kappa_\alpha \mathbf{z}_0 \times \mathbf{E}_{t\alpha}, \end{aligned} \tag{6}$$

which evidently assume a much simpler form in the gyrotropic case $\bar{\boldsymbol{\epsilon}}_{zt} = \bar{\boldsymbol{\epsilon}}_{tz} = \bar{\boldsymbol{\mu}}_{zt} = \bar{\boldsymbol{\mu}}_{tz} = 0$.

To ascertain orthogonality properties of the mode functions Ψ_α, one considers the eigenvalue problem adjoint to Eq. (4),

$$K^+ \Psi_\alpha^+ = \kappa_\alpha^* \Gamma^+ \Psi_\alpha^+, \tag{7a}$$

where the adjoint operators and eigenfunctions are

$$K^+ \rightarrow \begin{bmatrix} \omega \tilde{\boldsymbol{\epsilon}}^* & -i\nabla_t \times \mathbf{1} \\ i\nabla_t \times \mathbf{1} & \omega \tilde{\boldsymbol{\mu}}^* \end{bmatrix}, \quad \Gamma^+ = \Gamma, \quad \Psi_\alpha^+ \rightarrow \begin{bmatrix} \mathbf{E}_\alpha^+ \\ \mathbf{H}_\alpha^+ \end{bmatrix} \tag{7b}$$

subject to the condition on the boundary (if any)

$$\mathbf{v} \times \mathbf{E}_\alpha^+ = \tilde{\mathscr{Z}}^* \cdot \mathbf{H}_\alpha^+. \tag{7c}$$

The adjoint fields $\mathbf{E}_\alpha^+$ and $\mathbf{H}_\alpha^+$ satisfy the source-free equations in a transposed-conjugate (Hermitian adjoint) medium with parameters $\tilde{\boldsymbol{\epsilon}}^*$ and $\tilde{\boldsymbol{\mu}}^*$ obtained from the original medium by the replacements ($\sim$ denotes the transpose and * the complex-conjugate operation):

$$\boldsymbol{\epsilon}_t \rightarrow \tilde{\boldsymbol{\epsilon}}_t^*, \ \bar{\boldsymbol{\epsilon}}_{tz} \leftrightarrow \bar{\boldsymbol{\epsilon}}_{zt}^*, \ \boldsymbol{\mu}_t \rightarrow \tilde{\boldsymbol{\mu}}_t^*, \ \bar{\boldsymbol{\mu}}_{tz} \leftrightarrow \bar{\boldsymbol{\mu}}_{zt}^*, \ \mathscr{Z} \rightarrow \tilde{\mathscr{Z}}^*. \tag{7d}$$

In a conventional manner (see Sec. 1.4) one infers from the eigenvalue equations (4) and (7) the orthogonality properties

$$(\Psi_\alpha^+, \Gamma\Psi_\beta) = 2N_\alpha \delta_{\alpha\beta} = (\bar{\Psi}_\alpha^+, \Gamma\bar{\Psi}_\beta), \tag{8a}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol that equals 1 or 0 depending on whether or not κ_α is equal to κ_β; N_α is the αth-mode normalization constant, and the $\bar{\Psi}_\alpha$ and $\bar{\Psi}_\alpha^+$ are transverse eigenvectors with components $\mathbf{E}_{t\alpha}$, $\mathbf{H}_{t\alpha}$ and $\mathbf{E}_{t\alpha}^+$, $\mathbf{H}_{t\alpha}^+$, respectively. More specifically, the orthogonality properties (8a) are expressed as

$$\iint_S [\mathbf{E}_{t\alpha}^{+*} \cdot \mathbf{H}_{t\beta} \times \mathbf{z}_0 + \mathbf{H}_{t\alpha}^{+*} \cdot \mathbf{z}_0 \times \mathbf{E}_{t\beta}]\, dS = 2N_\alpha \delta_{\alpha\beta}, \tag{8b}$$

where the integral is extended over the cross section S transverse to z. As emphasized in Sec. 8.1, in the same waveguide there generally do not exist mode solutions with eigenvalues $+\kappa_\alpha$ and $-\kappa_\alpha$, thereby eliminating the possibility of a standing-wave transmission-line description in terms of interfering $+\kappa_\alpha$ and $-\kappa_\alpha$ waves. The latter obtains only for reflection symmetric regions.

The preceding description, valid for a general uniform electromagnetic waveguide, simplifies substantially in special situations.

8.2b Lossless Regions

If the anisotropic medium and waveguide boundary (if any) are lossless (i.e., reactive), then the constitutive medium parameters are Hermitian:

$$\boldsymbol{\epsilon} = \tilde{\boldsymbol{\epsilon}}^*, \qquad \boldsymbol{\mu} = \tilde{\boldsymbol{\mu}}^*, \qquad (i\mathscr{Z}) = (i\tilde{\mathscr{Z}})^*. \tag{9}$$

Under these conditions, it follows from a comparison of Eqs. (2b), (4), and (7) that

$$K = K^{+} \quad \text{and} \quad \Psi_\alpha^{+} = \Psi_{\alpha^*}, \tag{10a}$$

where Ψ_{α^*} is an eigenvector of the original waveguide with eigenvalue κ_α^*. It should be noted that eigenvalue problems of the type (4) do not in general possess real κ_α eigenvalues even in the Hermitian case, and furthermore, if κ_α is an eigenvalue, so also is its conjugate κ_α^*. For the special case of real eigenvalues κ_α one has $\Psi_\alpha^+ = \Psi_\alpha$, while for imaginary κ_α, one has $\Psi_\alpha^+ = -\Psi_\alpha$. In view of Eq. (10a), the orthogonality relation (8b) for a lossless region takes the form

$$\iint_S [\mathbf{E}_{t\alpha^*}^* \cdot \mathbf{H}_{t\beta} \times \mathbf{z}_0 + \mathbf{H}_{t\alpha^*}^* \cdot \mathbf{z}_0 \times \mathbf{E}_{t\beta}] dS = 0 \qquad \text{if } \kappa_\alpha \neq \kappa_\beta. \tag{10b}$$

The appearance of κ_α in the orthogonality statement is due to the fact that κ_α, rather than κ_α^* of Eq. 7(a), distinguishes the eigenvector $\Psi_\alpha^{+*} = \Psi_{\alpha^*}^*$ appearing in Eq. (10b).

8.2c Lossy (Symmetric) Regions

If the anisotropic medium and waveguide boundary (if any) are dissipative but symmetric, then the constitutive parameters are described by symmetric dyadics:

$$\boldsymbol{\epsilon} = \tilde{\boldsymbol{\epsilon}}, \qquad \boldsymbol{\mu} = \tilde{\boldsymbol{\mu}}, \qquad \mathscr{X} = \tilde{\mathscr{X}}. \tag{11}$$

For these conditions, comparison of Eqs. (2b), (4), and (7) reveals that

$$K^{+} = AK^{*}A \quad \text{and} \quad \Psi_\alpha^{+} = A\Psi_{-\alpha}^{*}, \tag{12a}$$

where

$$A = A^{-1} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A\Gamma = -\Gamma A.$$

$A\Psi_{-\alpha}^*$ is an eigenvector of the original waveguide with eigenvalue $-\kappa_\alpha$; it is derivable from Ψ_α on replacing κ_α by $-\kappa_\alpha$, taking the complex conjugate of the field components, and reversing the sign of H_α. The orthogonality relation (8b) now becomes

$$\iint_S [\mathbf{E}_{t,-\alpha} \cdot \mathbf{H}_{t\beta} \times \mathbf{z}_0 - H_{t,-\alpha} \cdot \mathbf{z}_0 \times \mathbf{E}_{t\beta}]\, dS = 0 \qquad \text{if } -\kappa_\alpha \neq \kappa_\beta. \tag{12b}$$

It should be observed in this case that if κ_α is an eigenvalue, so also is $-\kappa_\alpha$. However, a conventional transmission-line description is generally not possible since via Eq. (12a), Ψ_α and $\Psi_{-\alpha}$ are not related as in the conventional reflection symmetric case; differences arise because the characteristic wave impedances are complex quantities in the dissipative case.

8.2d Transverse Anisotropy (Reflection Symmetry)

If the medium is gyrotropic, so that the anisotropy is confined to a plane transverse to z, then

$$\bar{\boldsymbol{\epsilon}}_{tz} = \bar{\boldsymbol{\epsilon}}_{zt} = \bar{\boldsymbol{\mu}}_{tz} = \bar{\boldsymbol{\mu}}_{zt} = \mathscr{X}_{zs} = \mathscr{X}_{sz} = 0, \tag{13}$$

and the region is said to be reflection symmetric with respect to a plane transverse to z. To examine the symmetry properties of Ψ vectors in such regions, it is convenient to introduce the reflection operator R defined by†

$$R \rightarrow \begin{bmatrix} \mathbf{1}_t - \mathbf{z}_0\mathbf{z}_0 & 0 \\ 0 & -\mathbf{1}_t + \mathbf{z}_0\mathbf{z}_0 \end{bmatrix} \quad \text{whence } R\Psi \rightarrow \begin{bmatrix} \mathbf{E}_t - E_z\mathbf{z}_0 \\ -\mathbf{H}_t + H_z\mathbf{z}_0 \end{bmatrix}. \tag{13a}$$

If Eq. (13) is satisfied, then $RK = KR$, $R\Gamma = -\Gamma R$, and hence if Ψ_α is a mode solution of Eq. (4) corresponding to κ_α, then $R\Psi_\alpha$ is a mode solution with eigenvalue $-\kappa_\alpha$. Similarly, if Ψ_α^+ has eigenvalue κ_α^*, then $R\Psi_\alpha^+$ has eigenvalue $-\kappa_\alpha^*$. Thus, one infers from Eq. (8a) the orthogonality statements

$$(\Psi_\alpha^+, \Gamma\Psi_\beta) = 0 \qquad \text{if } \kappa_\alpha \neq \kappa_\beta,$$

$$(\Psi_\alpha^+, \Gamma R\Psi_\beta) = 0 = (R\Psi_\alpha^+, \Gamma\Psi_\beta) \qquad \text{if } \kappa_\alpha \neq -\kappa_\beta,$$

and therefore, on addition,

$$(\Psi_\alpha^+, \Gamma(\Psi_\beta + R\Psi_\beta)) = 0 = (\Psi_\alpha^+ + R\Psi_\alpha^+, \Gamma\Psi_\beta) \qquad \text{if } \kappa_\alpha^2 \neq \kappa_\beta^2, \tag{14a}$$

which reads, in component form,

$$\iint_S \mathbf{H}_{t\alpha}^{+*} \cdot \mathbf{z}_0 \times \mathbf{E}_{t\beta}\, dS = 0 = \iint_S \mathbf{E}_{t\alpha}^{+*} \cdot \mathbf{H}_{t\beta} \times \mathbf{z}_0\, dS \qquad \text{if } \kappa_\alpha^2 \neq \kappa_\beta^2. \tag{14b}$$

It should be noted that the orthogonality statements (14) do not distinguish between $\pm\kappa$ modes.

For the lossless (Hermitian) case, in view of Eq. (10a), Eqs. (14b) become

$$\iint_S \mathbf{H}_{t\alpha^*}^* \cdot \mathbf{z}_0 \times \mathbf{E}_{t\beta}\, dS = 0 \qquad \text{if } \kappa_\alpha^2 \neq \kappa_\beta^2. \tag{15}$$

For the lossy symmetric case, in view of Eq. (12a), Eqs. (14b) take the form

$$\iint_S \mathbf{H}_{t\alpha} \cdot \mathbf{z}_0 \times \mathbf{E}_{t\beta}\, dS = 0 \qquad \text{if } \kappa_\alpha^2 \neq \kappa_\beta^2. \tag{16}$$

8.2e Isotropic Regions

In the case of an isotropic medium and waveguide boundary (if any),

$$\boldsymbol{\epsilon} = \mathbf{1}\epsilon, \qquad \boldsymbol{\mu} = \mathbf{1}\mu, \qquad \mathscr{X} = \mathscr{X}(\mathbf{z}_0\mathbf{z}_0 + \mathbf{s}_0\mathbf{s}_0), \tag{17}$$

where ϵ, μ, and $\mathscr{X}$ are scalars, and $\mathbf{s}_0$ is defined in Eq. (4d). In the lossless case, wherein ϵ, μ, and $i\mathscr{X}$ are real, the orthogonality statement (15) again obtains, whereas in the lossy case, wherein these parameters are complex, Eq. (16) is applicable.

†It is also possible to define a reflection operator that is the negative of the one in Eq. (13a).

8.2f Regions with *E*- and *H*-mode Decompositions

The total fields corresponding to the transverse eigenfunctions $\mathbf{E}_{t\alpha}$ and $\mathbf{H}_{t\alpha}$ generally have both longitudinal components $E_{z\alpha}$ and $H_{z\alpha}$. To explore the possibility of separating such eigenfields into E modes and H modes with respect to the z direction, one may investigate whether Eqs. (6) can be satisfied when either $H_{z\alpha} = 0$ or $E_{z\alpha} = 0$, respectively. The z components of the field are specified in terms of $\mathbf{E}_{t\alpha}$ and $\mathbf{H}_{t\alpha}$ by Eqs. (5a); these equations, together with Eqs. (6), hold little promise for achieving the indicated simplification in the general case. In an isotropically filled waveguide, however, Eqs. (6) become

$$\kappa_\alpha \mathbf{E}_{t\alpha} = \omega\left[\mathbf{1}_t\mu + \frac{1}{\omega^2}\nabla_t \frac{1}{\epsilon}\nabla_t\right] \cdot (\mathbf{H}_{t\alpha} \times \mathbf{z}_0), \tag{18a}$$

$$\kappa_\alpha \mathbf{H}_{t\alpha} = \omega\left[\mathbf{1}_t\epsilon + \frac{1}{\omega^2}\nabla_t \frac{1}{\mu}\nabla_t\right] \cdot (\mathbf{z}_0 \times \mathbf{E}_{t\alpha}), \tag{18b}$$

where on the isotropic guide wall s of impedance $\mathscr{Z}$ [see Eq. (17)] and unit normal $\mathbf{v}$,

$$\mathbf{v} \times \mathbf{E}_{t\alpha} = \mathscr{Z}\mathbf{H}_{t\alpha} \qquad \text{on } s. \tag{18c}$$

For an E mode, Eq. (5a) implies $\nabla_t \cdot (\mathbf{z}_0 \times \mathbf{E}_{t\alpha}) = 0$, and Eq. (18b) yields

$$\kappa_\alpha \mathbf{H}_{t\alpha} = \omega\epsilon\mathbf{z}_0 \times \mathbf{E}_{t\alpha}, \tag{19a}$$

whereas for an H mode, with $\nabla_t \cdot (\mathbf{H}_{t\alpha} \times \mathbf{z}_0) = 0$,

$$\kappa_\alpha \mathbf{E}_{t\alpha} = \omega\mu\mathbf{H}_{t\alpha} \times \mathbf{z}_0. \tag{19b}$$

Thus, the requirement $E_{z\alpha} = 0$ or $H_{z\alpha} = 0$ forces the vectors $\mathbf{E}_{t\alpha}$ and $\mathbf{H}_{t\alpha}$ to be mutually perpendicular.

It is convenient to consider how boundary impedance and medium inhomogeneity individually affect separability of the modal field into E and H modes. The impedance boundary condition (18c) implies that

$$-\mathbf{s}_0 \cdot \mathbf{E}_{t\alpha} = \mathscr{Z}H_{z\alpha}, \qquad \mathscr{Z}\mathbf{s}_0 \cdot \mathbf{H}_{t\alpha} = E_{z\alpha} \quad \text{on } s, \tag{20}$$

so a tangential component of $\mathbf{H}_{t\alpha}$ on the boundary generates a longitudinal component $E_{z\alpha}$, while a tangential component of $\mathbf{E}_{t\alpha}$ on the boundary generates a longitudinal component $H_{z\alpha}$. Thus, even if an E-mode field is capable of existing in the waveguide medium, its transverse tangential electric component on the boundary couples to the longitudinal magnetic field, thereby destroying the E-mode character unless $\mathscr{Z} = 0$ or ∞. Analogous remarks apply to the H modes. A waveguide bounded by a wall with finite surface impedance therefore does not generally admit separate E- and H-mode fields, although such a decomposition may be possible for special symmetries. For example, in the presence of E-mode excitation which does not generate transverse electric field components parallel to the boundary, no H-mode coupling is required.

A transversely inhomogeneous medium filling the waveguide cross section also prohibits in general the separate existence of E and H modes. This may be verified by examination of the boundary conditions across an interface $\tilde{s}$ which separates two cross-sectional regions characterized by constant constitutive parameters μ_1, ϵ_1, and μ_2, ϵ_2, respectively. If $\tilde{\mathbf{s}}_0$ and $\tilde{\mathbf{v}}_0$ denote transverse unit vectors parallel and perpendicular to $\tilde{s}$, respectively, with $\tilde{\mathbf{v}}_0 \times \tilde{\mathbf{z}}_0 = \tilde{\mathbf{s}}_0$, then the boundary conditions on $\tilde{s}$ require that

$$\tilde{\mathbf{v}}_0 \times \mathbf{E}_{\alpha 1} = \tilde{\mathbf{v}}_0 \times \mathbf{E}_{\alpha 2}, \qquad \epsilon_1 \tilde{\mathbf{v}}_0 \cdot \mathbf{E}_{\alpha 1} = \epsilon_2 \tilde{\mathbf{v}}_0 \cdot \mathbf{E}_{\alpha 2}, \tag{21a}$$

$$\tilde{\mathbf{v}}_0 \times \mathbf{H}_{\alpha 1} = \tilde{\mathbf{v}}_0 \times \mathbf{H}_{\alpha 2}, \qquad \mu_1 \tilde{\mathbf{v}}_0 \cdot \mathbf{H}_{\alpha 1} = \mu_2 \tilde{\mathbf{v}}_0 \cdot \mathbf{H}_{\alpha 2}. \tag{21b}$$

Imposition of the E-mode condition $\nabla_t \cdot \mathbf{z}_0 \times \mathbf{E}_{\alpha 1} = 0$, with $\kappa_\alpha \mathbf{H}_{t\alpha} = \omega\epsilon \mathbf{z}_0 \times \mathbf{E}_{t\alpha}$, in regions 1 and 2 permits the last of Eqs. (21b) to be written as

$$\mu_1 \epsilon_1 \tilde{\mathbf{s}}_0 \cdot \mathbf{E}_{t\alpha 1} = \mu_2 \epsilon_2 \tilde{\mathbf{s}}_0 \cdot \mathbf{E}_{t\alpha 2}, \tag{22}$$

which is incompatible with the first of Eqs. (21a) unless $\mu_1 \epsilon_1 = \mu_2 \epsilon_2$ or $\tilde{\mathbf{s}}_0 \cdot \mathbf{E}_{t\alpha} = 0$. An analogous result is obtained for the H modes. Thus, a decomposition into E and H modes is generally impossible when the local propagation speed $c = 1/\sqrt{\mu\epsilon}$ has a transverse variation, unless special symmetries eliminate the existence of tangential tansverse field components.

It is evident from the preceding discussion that E and H modes *may* exist individually in a homogeneously filled cross section bounded by perfectly conducting walls. Upon introducing the characteristic admittance by renormalizing $\mathbf{H}_{t\alpha}$ according to $\mathbf{H}_{t\alpha} = Y_\alpha \mathbf{h}_\alpha$ and setting $\mathbf{E}_{t\alpha} = \mathbf{e}_\alpha$, one obtains for the E modes [see Eqs. (18) and (19)],

$$\mathbf{h}_\alpha = \mathbf{z}_0 \times \mathbf{e}_\alpha, \quad Y_\alpha = \frac{\omega\epsilon}{\kappa_\alpha}, \quad \nabla_t \nabla_t \cdot \mathbf{e}_\alpha = -(k^2 - \kappa_\alpha^2)\mathbf{e}_\alpha, \quad k^2 = \omega^2 \mu\epsilon, \tag{23a}$$

and, for the H modes,

$$\mathbf{e}_\alpha = \mathbf{h}_\alpha \times \mathbf{z}_0, \quad Y_\alpha = \frac{\kappa_\alpha}{\omega\mu}, \quad \nabla_t \nabla_t \cdot \mathbf{h}_\alpha = -(k^2 - \kappa_\alpha^2)\mathbf{h}_\alpha. \tag{23b}$$

The normalized mode set $\mathbf{e}_\alpha, \mathbf{h}_\alpha$ is identical with that defined in Eqs. (2.2.10).

In a uniaxially anisotropic medium with

$$\boldsymbol{\epsilon} = \mathbf{1}_t \epsilon_t + \mathbf{z}_0 \mathbf{z}_0 \epsilon_z, \qquad \boldsymbol{\mu} = \mathbf{1}_t \mu_t + \mathbf{z}_0 \mathbf{z}_0 \mu_z, \tag{24}$$

the eigenvalue problem in Eqs. (6) reduces to

$$\kappa_\alpha Z_\alpha \mathbf{e}_\alpha = \omega \left[\mathbf{1}_t \mu_t + \frac{1}{\omega^2} \nabla_t \frac{1}{\epsilon_z} \nabla_t \right] \cdot \mathbf{h}_\alpha \times \mathbf{z}_0, \tag{25a}$$

$$\kappa_\alpha Y_\alpha \mathbf{h}_\alpha = \omega \left[\mathbf{1}_t \epsilon_t + \frac{1}{\omega^2} \nabla_t \frac{1}{\mu_z} \nabla_t \right] \cdot \mathbf{z}_0 \times \mathbf{e}_\alpha. \tag{25b}$$

These equations are almost identical with Eqs. (18a) and (18c) and the same considerations lead to the conclusion that a field decomposition into E and H modes is possible when the cross section is filled homogeneously and is bounded by a perfectly conducting wall. The resulting normalized mode set has been employed in Sec. 7.2.

8.2g Modal Representations for the Reduced Electromagnetic Field

The mode functions described in Secs. 8.2a through 8.2f may be employed to represent the electromagnetic field excited by arbitrary source configurations in various types of uniform waveguides. In abstract notation, the steady-state problem posed in Eqs. (1) and (2) may be solved for the transverse field $\bar{\Psi}(\mathbf{r})$ of Eq. (3b) by utilizing the complete set of transverse modes $\bar{\Psi}_\alpha(\boldsymbol{\rho})$ defined in Eq. (8a) [see also Eqs. (4b) and (7a)],

$$\bar{\Psi}(\mathbf{r}) = \sum_\alpha a_\alpha(z)\bar{\Psi}_\alpha(\boldsymbol{\rho}), \tag{26a}$$

where the summation† is to be extended over all α, which distinguish the eigenvalues κ_α of both the forward and backward travelling modes of amplitudes $a_\alpha(z)$. An $\exp(-i\omega t)$ dependence is implied. On adding to the transverse field representation (26a) the associated longitudinal field components, defined in terms of the transverse fields $\bar{\Psi}$ and $\bar{\Psi}_\alpha$ by Eqs. (3) and (5a), one obtains even at source points the total field representation

$$\Psi(\mathbf{r}) = \sum_\alpha a_\alpha(z)\Psi_\alpha(\boldsymbol{\rho}) - \theta_z(\mathbf{r}), \tag{26b}$$

where Ψ and Ψ_α are complete wavevectors of the excited and modal fields, respectively, and θ_z, defined in Eq. (3b), is related to the source wavevector. To determine the modal amplitudes $a_\alpha(z)$, we first observe from the orthogonality property (8b) and from (26b) that $a_\alpha = (2N_\alpha)^{-1}(\Psi_\alpha^+, \Gamma\Psi)$, a result that is independent of θ_z. Transformation of the field equations (1a) and (2a) by multiplication with the adjoint mode function $\Psi_\alpha^+(\boldsymbol{\rho})$ and integration over the transverse waveguide cross section then yields as the defining equation for the $a_\alpha(z)$ [we recall from Eq. (7a) that $(\Psi_\alpha^+, K\Psi) = (K^+\Psi_\alpha^+, \Psi) = (\kappa_\alpha^*\Gamma^+\Psi_\alpha^+, \Psi) = 2N_\alpha\kappa_\alpha a_\alpha$]:

$$\left(\frac{d}{dz} - i\kappa_\alpha\right)a_\alpha(z) = -b_\alpha(z), \tag{27}$$

where the source amplitude b_α is given by‡

$$b_\alpha = (2N_\alpha)^{-1}(\Psi_\alpha^+, \Phi), \tag{28a}$$

$$= (2N_\alpha)^{-1}\iint [\mathbf{J}_t \cdot \mathbf{E}_\alpha^{+*} + J_z E_{z\alpha}^{+*} + \mathbf{M}_t \cdot \mathbf{H}_\alpha^{+*} + M_z H_{z\alpha}^{+*}]\, dS \tag{28b}$$

On integration of the second and fourth terms in Eq. (28b) by parts (assuming that J_z vanishes on the guide walls), one can write

$$b_\alpha = (2N_\alpha)^{-1}\iint [\mathbf{J}_{te} \cdot \mathbf{E}_\alpha^{+*} + \mathbf{M}_{te} \cdot \mathbf{H}_\alpha^{+*}]\, dS, \tag{29a}$$

†In the present abstract notation, $\sum_\alpha$ implies both an integration over $\mathbf{k}_t$ and a sum over eigenvalues for each $\mathbf{k}_t, \omega$ [see the time-harmonic analogue of Eq. (31)].

‡Treatment of excitation by arbitrary sources may be simplified by use of Green's functions. The transmission equations then assume the form given in Eq. (1.4.15).

where

$$\mathbf{J}_{te} = \mathbf{J}_t - \frac{1}{i\omega}\nabla_t \times \frac{\mathbf{M}_z}{\mu_z} - \frac{\bar{\boldsymbol{\epsilon}}_{zt}J_z}{\epsilon_z}, \qquad \mathbf{M}_{te} = \mathbf{M}_t + \frac{1}{i\omega}\nabla_t \times \frac{\mathbf{J}_z}{\epsilon_z} - \frac{\bar{\boldsymbol{\mu}}_{zt}M_z}{\mu_z}; \tag{29b}$$

$\mathbf{J}_{te}$ and $\mathbf{M}_{te}$ are equivalent transverse current densities characteristic of a field description in terms of transverse fields only.

Alternative to the abstract procedure in Eqs. (26) and (27), one may obtain a field representation by first eliminating the longitudinal field components from Eqs. (1) and (2), and thereby derive explicit equations for the transverse field components (see Sec. 2.2a). The resulting transverse field equations resemble Eqs. (6), with $i\kappa_\alpha$ replaced by d/dz, and contain the equivalent transverse currents $\mathbf{J}_{te}$ and $\mathbf{M}_{te}$ defined in Eq. (29b) [see Eqs. (2.2.4) and (2.2.5) for the special case of homogeneous isotropic regions bounded by lossless walls]. The explicit modal representation of the transverse fields now becomes

$$\mathbf{E}_t(\mathbf{r}) = \sum_\alpha a_\alpha(z)\mathbf{E}_{t\alpha}(\boldsymbol{\rho}), \qquad \mathbf{H}_t(\mathbf{r}) = \sum_\alpha a_\alpha(z)\mathbf{H}_{t\alpha}(\boldsymbol{\rho}). \tag{30}$$

Substitution of these expansions into the transverse fields equations, recalling Eqs. (6) and the orthogonality property (10b), then yields the above transmission-line equation (27) for the modal amplitudes $a_\alpha(z)$. Since there are generally no solutions admitting both κ_α and $-\kappa_\alpha$ in the *same* region, the transmission lines representative of Eq. (27) are unilateral and propagate waves in one direction only. Because of this restriction, the conventional transmission-line procedure is of limited value in this general formulation.

As noted in Eq. (26b), to represent the *total* fields in a region containing sources, it is necessary to stipulate at each point $\mathbf{r}$ not only the total modal fields $\mathbf{E}_\alpha = \mathbf{E}_{t\alpha} + E_{z\alpha}\mathbf{z}_0$ and $\mathbf{H}_\alpha = \mathbf{H}_{t\alpha} + H_{z\alpha}\mathbf{z}_0$ but also the longitudinal source currents J_z and M_z in order to obtain a complete representation. This lack of vector completeness for the total fields is to be anticipated since the eigenvalue problem refers only to the transverse space.

8.2h Non-Conventional Transmission-Line Descriptions

The radiation of energy from a source in an unbounded, homogeneous, and stationary medium may be described in terms of a set of guided modes carrying energy away from the source. A complete set of such modes (see Sec. 1.4) comprises waves carrying energy both away from and toward the source, but only the former are excited in an unbounded medium. A boundary surface scatters the radiated waves, and the description of this scattering process requires in general the sophisticated analytical techniques of diffraction theory. However, if the boundary surface and the medium are appropriately simple, the scattering description may be effected by relatively simple transmission-line methods. These methods have been discussed in Sec. 2.4 for the simple case of an isotropic medium. In the present section, we develop similar but less conventional transmission-line methods for certain choices of transmission direction in appropriate gyrotropic media and thereby introduce a more general form of transmission-line theory.

Representation of a linear field in terms of a complete set of characteristic guided waves reduces an overall field description to a simple determination of wave amplitudes. In spatially homogeneous and stationary regions, each amplitude is determined by an ordinary differential equation and may be shown (see Sec. 1.4) to have a simple exponential dependence on distance along the guide direction. In unbounded regions, each wave amplitude is uncoupled from the other amplitudes and hence separately evaluable; the amplitude of a wave at any point is finite if the wave in question carries energy away from a source, or zero if the energy transport is toward a source. The presence of a boundary surface generally introduces coupling among the waves traveling toward or from sources and thereby makes any one amplitude dependent on the amplitudes of other waves. For boundary surfaces of arbitrary shape, all waves are coupled at the boundary; consequently, the evaluation of wave amplitudes may be prohibitively difficult, thus vitiating the utility of a guided-mode representation. However, for regions with media and boundaries of suitable symmetry, the wave coupling introduced by boundary surfaces may be simple and readily taken into account. In the present section we shall analyze mode coupling caused by plane-parallel boundaries in stratified gyrotropic media. This analysis can always be performed in terms of the traveling waves defined in Sec. 1.4. For guiding structures containing isotropic media and guide walls, not only traveling-wave (scattering) but also the standing-wave (impedance) descriptions of conventional transmission-line analysis can be employed, as has been indicated in Sec. 2.4. In more general anisotropic structures admitting $\pm\kappa$ waves, a simple but less conventional transmission-line analysis may still be possible.

For simplicity, we shall limit the discussion to reduced electromagnetic field descriptions in homogeneous media with non spatially dispersive $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ parameters. As shown in Sec. 1.4, a general field in a transversely unbounded region may be represented as

$$\Psi(\mathbf{r}, t) = \iiint \sum_{\alpha} a_\alpha(\mathbf{k}_t, \omega; z)\Psi_\alpha e^{i(\mathbf{k}_t\cdot\boldsymbol{\rho} - \omega t)} \frac{d\mathbf{k}_t\, d\omega}{(2\pi)^3}, \tag{31}$$

where for the special case of a reduced electromagnetic description

$$\Psi(\mathbf{r}, t) \rightarrow \begin{bmatrix} \mathbf{E}(\mathbf{r}, t) \\ \mathbf{H}(\mathbf{r}, t) \end{bmatrix}.$$

If the medium is Hermitian, i.e., the field operator $K = K^+$ (see Sec. 8.2b), then the eigenvectors Ψ_α and their adjoints Ψ_α^+ are representable as

$$\Psi_\alpha \rightarrow \begin{bmatrix} Z_\alpha \mathbf{e}_\alpha + \mathbf{e}_{z\alpha} \\ \mathbf{h}_\alpha + \mathbf{h}_{z\alpha} \end{bmatrix}, \qquad \Psi_\alpha^+ = \Psi_{\alpha^*} \rightarrow \begin{bmatrix} Z_{\alpha^*}\mathbf{e}_{\alpha^*} + \mathbf{e}_{z\alpha^*} \\ \mathbf{h}_{\alpha^*} + \mathbf{h}_{z\alpha^*} \end{bmatrix}, \tag{32}$$

where $\mathbf{e}_\alpha$, $\mathbf{h}_\alpha$ and $\mathbf{e}_{z\alpha}$, $\mathbf{h}_{z\alpha}$ are mode vectors transverse and parallel to z for the eigenvalue κ_α, while the adjoint eigenvectors $\mathbf{e}_{\alpha^*}$, $\mathbf{h}_{\alpha^*}$ and $\mathbf{e}_{z\alpha^*}$, $\mathbf{h}_{z\alpha^*}$ are the corresponding mode vectors with eigenvalue κ_α^*. The orthogonality property (1.4.4c) of the Ψ_α implies that in polarization space,

$$(\Gamma\Psi_\alpha^+, \Psi_\beta) = Z_\beta \mathbf{e}_\beta \cdot \mathbf{h}_{\alpha^*}^* \times \mathbf{z}_0 + Z_{\alpha^*}^* \mathbf{h}_\beta \cdot \mathbf{z}_0 \times e_{\alpha^*}^* = 2N_\alpha \delta_{\alpha\beta}. \tag{33}$$

Since the normalization of Ψ_α and Ψ_{α^*} is arbitrary, it is convenient to choose the mode parameters Z_α and Z_{α^*} so as to effect

$$\mathbf{e}_\alpha \cdot \mathbf{h}_{\alpha^*}^* \times \mathbf{z}_0 = 1 \qquad \text{and} \qquad \mathbf{h}_\alpha \cdot \mathbf{z}_0 \times \mathbf{e}_{\alpha^*}^* = 1 \tag{34}$$

whence the normalization constant is

$$2N_\alpha = Z_\alpha + Z_{\alpha^*}^* \tag{35}$$

In view of Eq. (35) and the orthogonality properties of the complete wave-vector $\Psi_\alpha \exp[i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)]$, one infers, just as in Eq. (1.4.6b), that the guided-wave amplitude $a_\alpha(\mathbf{k}_t, \omega; z)$ is related to the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$ on the plane z by

$$a_\alpha(\mathbf{k}_t, \omega; z) = \frac{1}{2N_\alpha} \iiint [\mathbf{E} \cdot \mathbf{h}_{\alpha^*}^* \times \mathbf{z}_0 + Z_{\alpha^*}^* \mathbf{H} \cdot \mathbf{z}_0 \times \mathbf{e}_{\alpha^*}^*] e^{-i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)} \, d\boldsymbol{\rho} \, dt, \tag{36}$$

where, for prescribed $\mathbf{k}_t, \omega$ and non-spatially dispersive medium parameters, the polarization index α distinguishes only four waves,† conforming to the four possible transverse (to z) components of the two vectors $\mathbf{E}$ and $\mathbf{H}$. In source-free regions, the z dependence of the a_α follows from Eq. (27) as

$$\left(\frac{d}{dz} - i\kappa_\alpha\right) a_\alpha(\mathbf{k}_t, \omega; z) = 0. \tag{37}$$

In a homogeneous isotropic or anisotropic Hermitian medium, the relation (36) between the total fields $\mathbf{E}$ and $\mathbf{H}$ and the wave amplitude $a_\alpha(\mathbf{k}_t, \omega; z)$ is valid at any plane z. Let us consider the coupling effects introduced by the presence of a planar boundary surface at z perpendicular to the guiding direction $\mathbf{z}_0$. If the planar boundary is spatially homogeneous and non-time-varying, only modes of the same $\mathbf{k}_t, \omega$ will be coupled at this boundary surface to the wave of amplitude $a_\alpha(\mathbf{k}_t, \omega; z)$. As noted above, there are four such modal waves with given $\mathbf{k}_t, \omega$. Hence, if $\Psi(z) \equiv \Psi(\mathbf{k}_t, \omega; z)$ denotes the $\mathbf{k}_t, \omega$ component of the total field $\Psi(\mathbf{r}, t)$ at the plane z, then on omission of the $\mathbf{k}_t, \omega$ arguments, Eq. (31) implies that

$$\Psi(z) = \sum_\alpha a_\alpha(z) \Psi_\alpha, \tag{38}$$

where for the reduced electromagnetic field description (in non spatially dispersive media), α distinguishes four possible waves. A basic analytical problem is that of ascertaining the interdependence among the $a_\alpha(z)$ caused by a planar boundary surface at some plane z_0.

In particular, we examine the coupling among the wave amplitudes a_α for Hermitian media that admit waves with wavenumbers (eigenvalues) $\pm\kappa_\alpha$ and transverse-mode vectors $\mathbf{e}_\alpha = \mathbf{e}_{-\alpha}$, $\mathbf{h}_\alpha = \mathbf{h}_{-\alpha}$. Such media are more general than

†The secular determinant for $\kappa_\alpha = \kappa_\alpha(\mathbf{k}_t, \omega)$ is of fourth degree for non spatially dispersive ϵ and μ.

the reflection symmetric regions described in Sec. 8.2d. For prescribed $\mathbf{k}_t$, ω, one then finds two distinctive pairs of waves with wavenumbers† $\pm\kappa'_\alpha$ and $\pm\kappa''_\alpha$. Accordingly, for each $\mathbf{k}_t, \omega$, one can represent the transverse fields $\mathbf{E}_t(z)$, $\mathbf{H}_t(z)$ at z by Eqs. (32) and (38) as

$$\begin{aligned} \mathbf{E}_t(z) &= V'_\alpha(z)\mathbf{e}'_\alpha + V''_\alpha(z)\mathbf{e}''_\alpha, \\ \mathbf{H}_t(z) &= I'_\alpha(z)\mathbf{h}'_\alpha + I''_\alpha(z)\mathbf{h}''_\alpha, \end{aligned} \tag{39a}$$

where for both the $'$ and $''$ waves, V_α and I_α are given by

$$\begin{aligned} V_\alpha(z) &= Z_\alpha a_\alpha(z) + Z_{-\alpha} a_{-\alpha}(z), \\ I_\alpha(z) &= a_\alpha(z) + a_{-\alpha}(z). \end{aligned} \tag{39b}$$

In general, the "characteristic impedance" parameters Z_α and $Z_{-\alpha}$ are not simply related (see Sec. 8.4b), whereas in conventional transmission-line descriptions, one recalls that $Z_{-\alpha} = -Z_\alpha$. Since it is assumed that $\mathbf{e}_\alpha = \mathbf{e}_{-\alpha}$, $\mathbf{h}_\alpha = \mathbf{h}_{-\alpha}$, one can derive from the orthogonality property (33) and the normalization in Eq. (34) the additional orthogonality properties (in polarization space)

$$\mathbf{e}_\beta \cdot \mathbf{h}^*_{\alpha^*} \times \mathbf{z}_0 = \delta_{\alpha\beta}, \qquad \mathbf{h}_\beta \cdot \mathbf{z}_0 \times e^*_{\alpha^*} = \delta_{\alpha\beta}, \tag{40}$$

for mode vectors $\mathbf{e}_\alpha$, $\mathbf{h}_\alpha$ distinguished by κ_α^2. It should be noted, as in Sec. 2.2, that the transverse-vector-mode functions $\mathbf{e}_\alpha \exp[i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)]$, $\mathbf{h}_\alpha \exp[i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)]$ determine a complete set of eigenvectors in the $\boldsymbol{\rho}, t$ and transverse vector (polarization) space. These vectors are distinguished by eigenvalues κ_α^2 and are to be contrasted with the eigenvectors $\Psi_\alpha \exp[i(\mathbf{k}_t \cdot \boldsymbol{\rho} - \omega t)]$ whose eigenvalues are κ_α. Accordingly, the mode summation for $\mathbf{e}_\alpha$, $\mathbf{h}_\alpha$ mode vectors, as employed in Chapters 2 and 3, ranges only over positive values of the index α, whereas for the Ψ_α-mode vectors the mode summation comprises $\pm\alpha$ values.

One can infer from Eqs. (36) and (40) the following relation between the "traveling-wave" amplitudes a_α and the "voltage" and "current" amplitudes V_α and I_α at z:

$$\begin{aligned} a_\alpha(z) &= \frac{1}{2N_\alpha}[V_\alpha(z) + Z^*_{\alpha^*} I_\alpha(z)], \\ a_{-\alpha}(z) &= \frac{1}{2N_{-\alpha}}[V_\alpha(z) + Z^*_{-\alpha^*} I_\alpha(z)], \end{aligned} \tag{41}$$

where, since the voltage and current amplitudes are associated with κ_α^2, there is no distinction between $\pm\alpha$ subscripts for these quantities. In transmission-line analyses one customarily introduces, looking in the direction of increasing z, a (voltage) "reflection coefficient" $\Gamma_\alpha(z)$ and a "terminal impedance" $Z_\alpha(z)$ defined by

$$\Gamma_\alpha(z) = \frac{a_{-\alpha}(z)N_{-\alpha}}{a_\alpha(z)N_\alpha}, \qquad Z_\alpha(z) = \frac{V_\alpha(z)}{I_\alpha(z)}, \tag{42}$$

†Note that the notation $\kappa'_\alpha = \kappa_\alpha$, $\kappa''_\alpha = \kappa_\beta$ could also be used to distinguish these modes.

to characterize coupling introduced between the $\pm\alpha$ waves. From Eqs. (41), one then derives the relation

$$\Gamma_\alpha(z) = \frac{Z_\alpha(z) + Z^*_{-\alpha^*}}{Z_\alpha(z) + Z^*_{\alpha^*}} \tag{43a}$$

and its converse,

$$Z_\alpha(z) = \frac{-Z^*_{-\alpha^*} + Z^*_{\alpha^*}\Gamma_\alpha(z)}{1 - \Gamma_\alpha(z)}. \tag{43b}$$

Equations (43) are to be contrasted with the conventional transmission-line relations

$$\Gamma_\alpha(z) = \frac{Z_\alpha(z) - Z_\alpha}{Z_\alpha(z) + Z_\alpha}, \qquad \frac{Z_\alpha(z)}{Z_\alpha} = \frac{1 + \Gamma_\alpha(z)}{1 - \Gamma_\alpha(z)}, \tag{44}$$

as derived in Eqs. (2.4.12) for a medium wherein $-N_{-\alpha} = N_\alpha$ and $-Z^*_{-\alpha^*} = Z^*_{\alpha^*} = Z_\alpha$.

If the impedance $Z_\alpha(z')$ is known at some terminal plane $z = z'$, as occurs if boundary conditions on this plane are given, then in order to calculate the field at some other plane z, it is necessary to ascertain the reflection coefficient $\Gamma_\alpha(z)$ or $Z_\alpha(z)$ thereon. To this end, one first evaluates $\Gamma_\alpha(z')$ from the known $Z_\alpha(z')$ by Eq. (43a) applied at $z = z'$. Since, in a source-free region, the traveling-wave amplitudes at z and z' are related in accordance with Eq. (37) by

$$a_\alpha(z) = a_\alpha(z')e^{i\kappa_\alpha(z-z')}$$

then, from Eq. (42), one finds that

$$\Gamma_\alpha(z) = \Gamma_\alpha(z')e^{-2i\kappa_\alpha(z-z')} \tag{45}$$

is the desired reflection coefficient at z. To find the corresponding relation between the impedances at z and z', one substitutes Eqs. (45) and (43a) into Eq. (43b) and obtains

$$Z_\alpha(z) = \frac{-2Z^*_{\alpha^*}Z^*_{-\alpha^*} - i[(Z^*_{\alpha^*} - Z^*_{-\alpha^*})\cot\kappa_\alpha(z-z') + i(Z^*_{\alpha^*} + Z^*_{-\alpha^*})]Z_\alpha(z')}{2Z_\alpha(z') + Z^*_{\alpha^*} + Z^*_{-\alpha^*} - i(Z^*_{\alpha^*} - Z^*_{-\alpha^*})\cot\kappa_\alpha(z-z')}. \tag{46}$$

For a conventional transmission line, wherein $-Z^*_{-\alpha^*} = Z^*_{\alpha^*} = Z_\alpha$, Eq. (46) reduces to [see Eq. (2.4.10) with $j \to -i$]

$$\frac{Z_\alpha(z)}{Z_\alpha} = \frac{Z_\alpha - iZ_\alpha(z')\cot\kappa_\alpha(z-z')}{Z_\alpha(z') - iZ_\alpha\cot\kappa_\alpha(z-z')}. \tag{47}$$

For the non-reflection symmetric media under consideration, Eqs. (45) and (46) provide the basis for a general transmission-line analysis of the coupling of waves by a planar boundary surface at z'.

For prescribed $\mathbf{k}_t, \omega$, the average electromagnetic power density in the positive z direction can be calculated from the real part of the complex Poynting vector,

$$P_z = \mathrm{Re}\,[\mathbf{E}(z) \cdot \mathbf{H}(z)^* \times \mathbf{z}_0]. \tag{48}$$

For this purpose one requires, in addition to the field representations (38) and (39a), a representation in terms of adjoint wavevectors (for Hermitian media)

$$\Psi(z) = \sum_\alpha a_\alpha^+(z)\Psi_\alpha^+ = \sum_\alpha a_{\alpha^*}(z)\Psi_{\alpha^*} \tag{49}$$

and correspondingly, by Eq. (32),

$$\begin{aligned} \mathbf{E}_t(z) &= V'_{\alpha^*}(z)\,\mathbf{e}'_{\alpha^*} + V''_{\alpha^*}(z)\,\mathbf{e}''_{\alpha^*}, \\ \mathbf{H}_t(z) &= I'_{\alpha^*}(z)\,\mathbf{h}'_{\alpha^*} + I''_{\alpha^*}(z)\,\mathbf{h}''_{\alpha^*}, \end{aligned} \tag{50a}$$

where, for both the $'$ and $''$ modes,

$$\begin{aligned} V_{\alpha^*}(z) &= Z_{\alpha^*}a_{\alpha^*}(z) + Z_{-\alpha^*}a_{-\alpha^*}(z), \\ I_{\alpha^*}(z) &= a_{\alpha^*}(z) + a_{-\alpha^*}(z). \end{aligned} \tag{50b}$$

In view of the orthonormality properties (40), one then finds on substitution of Eqs. (39a) and (50a) into (48) the average power relation

$$P_z = \mathrm{Re} \sum_{\alpha>0} V_\alpha(z)I_{\alpha^*}^*(z) = \mathrm{Re} \sum_{\alpha>0} V_{\alpha^*}^*(z)I_\alpha(z), \tag{51}$$

where, in the impedance description, the mode sum must be only over the $+\alpha$ modes. For the special case of propagating modes in isotropic media, wherein $V_{\alpha^*} = V_\alpha, I_{\alpha^*} = I_\alpha$, Eq. (51) reduces to the conventional relation

$$P_z = \mathrm{Re} \sum_{\alpha>0} V_\alpha(z)I_\alpha^*(z).$$

The transmission-line analysis outlined above is applicable in reflection symmetric Hermitian media (i.e., media admitting $\pm\kappa_\alpha$ waves for which $\mathbf{e}_\alpha = \mathbf{e}_{-\alpha}, \mathbf{h}_\alpha = \mathbf{h}_{-\alpha}, \mathbf{e}_{\alpha^*} = \mathbf{e}_{-\alpha^*}, \mathbf{h}_{\alpha^*} = \mathbf{h}_{-\alpha^*}$). Such conditions are fulfilled in non-dissipative gyrotropic media, but in general $Z_{\alpha^*}^* \neq Z_\alpha$ and $\mathbf{e}_\alpha \neq \mathbf{e}_{\alpha^*}$, $\mathbf{h}_\alpha \neq \mathbf{h}_{\alpha^*}$. In gyrotropic media with the gyrotropic axis parallel to the transmission direction, one finds that $Z_{\alpha^*}^* = Z_\alpha$, but $\mathbf{e}_\alpha \neq \mathbf{e}_{\alpha^*}, \mathbf{h}_\alpha \neq \mathbf{h}_{\alpha^*}$; thus the transmission-line relations (43) and (46) become conventional but the power relation (51) does not (see Sec. 8.3). In gyrotropic media with axis perpendicular to $\mathbf{z}_0$, one can find modes with $\mathbf{e}_\alpha = \mathbf{e}_{\alpha^*}, \mathbf{h}_\alpha = \mathbf{h}_{\alpha^*}$ but $Z_{\alpha^*}^* \neq Z_\alpha$, hence requiring the general transmission-line relations (46).

8.3 GUIDED WAVES IN A COLD MAGNETOPLASMA (GUIDE AXIS PARALLEL TO GYROTROPIC AXIS)

8.3a Evaluation of the Mode Functions

The guided-wave considerations of the preceding section on anisotropic media will now be specialized to the case of a gyrotropic medium wherein the guide and gyrotropic axes are parallel. The steady-state wavevectors Ψ_α descriptive of guided modes in such a medium will be exhibited explicitly, and their application to the theory of radiation from time-harmonic sources will be illustrated. As noted in Sec. 8.2, representation of radiated fields may be phrased either in terms of "first-order" Ψ vectors incorporating all electromagnetic and dynamical field variables, or in terms of "reduced" Ψ vectors displaying only electromagnetic variables.

Guided waves in a homogeneous gyrotropic medium, such as a collisionless cold magnetoplasma, are described by wavevectors of the form $\Psi_a(\mathbf{r}, t) = \Psi_\alpha \exp[j(\omega t - \mathbf{k} \cdot \mathbf{r})]$,† where the mode suffix a denotes both the polarization index α and the periodicities, ω, $\mathbf{k}_t$; $\mathbf{k}_t = \mathbf{k} - \mathbf{z}_0 \kappa_\alpha$ is the transverse wavenumber. In polarization space, for prescribed ω, $\mathbf{k}_t$, the Ψ_α satisfy the eigenvalue problem [see Eq. (8.2.2)]

$$L\Psi_\alpha = 0 = j(K - \kappa_\alpha \Gamma)\Psi_\alpha, \tag{1a}$$

where K and Γ are Hermitian operators and κ_α is the eigenvalue. The Ψ_α possess the orthogonality properties [see Eqs. (8.2.8a)]

$$(\Psi_\alpha^+, \Gamma\Psi_\beta) = 2N_\alpha\, \delta_{\alpha\beta} = (\bar{\Psi}_\alpha^+, \Gamma\bar{\Psi}_\beta); \tag{1b}$$

for the non-dissipative case, as observed in Sec. 8.2b, the adjoint wavevector $\Psi_\alpha^+ = \Psi_\alpha$ for $\kappa_\alpha = \kappa_\alpha^*$ and $\Psi_\alpha^+ = \Psi_{\alpha^*}$ for complex κ_α. Although these properties can be inferred a priori from the general form of the operator L indicated in Eq. (1a), one must know the specific structure of L to evaluate explicitly the various components of Ψ_α.

For a magnetoplasma fluid in which only electrons are mobile, the first-order form of L is displayed in Eq. (1.1.68) and the reduced form in Eqs. (8.2.1) and (8.2.2); in the first-order description, the components of Ψ are given by $\Psi \to (\mathbf{E}, \mathbf{H}, p, \mathbf{v})$, whereas in the reduced description, $\Psi \to (\mathbf{E}, \mathbf{H})$. The elimination of the dynamical variables p and $\mathbf{v}$, the electron pressure and average velocity, respectively, is characteristic of the reduced description. If, in addition, the magnetic field $\mathbf{H}$ is eliminated from the source-free plasma field equations, the resulting equation for the electric field $\mathbf{E}(\mathbf{r}, t)$ may be written in the form

$$\mathscr{Y}\left(\nabla, \frac{\partial}{\partial t}\right) \cdot \mathbf{E}(\mathbf{r}, t) = 0; \tag{2}$$

it may be noted that the dyadic "admittance" operator $\mathscr{Y}$ is the inverse of the electric component $\mathscr{G}_{11}$ of the plasma Green's function defined in Eq. (1.1.57).

Although solution of Eq. (2) is desired in a $\mathbf{k}_t, \omega$ basis, we shall find the solution in a $\mathbf{k}, \omega$ basis with $\mathbf{k} = \mathbf{k}_t + \kappa \mathbf{z}_0$. In a $\mathbf{k}, \omega$ basis (wherein $\nabla = -j\mathbf{k}$ and $\partial/\partial t = j\omega$, see Sec. 1.2), the admittance operator $\mathscr{Y}$ for a gyrotropic medium with permittivity $\boldsymbol{\epsilon}$ and permeability $\boldsymbol{\mu} = \mu_0 \mathbf{1}$ is given by

$$\mathscr{Y}(\mathbf{k}, \omega) = j\omega\epsilon_0\left[\boldsymbol{\epsilon} + \frac{\mathbf{k} \times (\mathbf{k} \times \mathbf{1})}{k_0^2}\right], \qquad k_0^2 = \omega^2 \mu_0 \epsilon_0, \tag{3a}$$

where, for the gyrotropic axis in the direction $\mathbf{b}_0$,

$$\frac{\boldsymbol{\epsilon}}{\epsilon_0} = \epsilon_1 \mathbf{1}_\perp - j\epsilon_2 \mathbf{b}_0 \times \mathbf{1}_\perp + \epsilon_3 \mathbf{1}_b. \tag{3b}$$

†Note that the time dependence in Sec. 8.3 is chosen to be $\exp(+j\omega t)$.

$\mathbf{1}_b = \mathbf{b}_0\mathbf{b}_0$ and $\mathbf{1}_\perp = \mathbf{1} - \mathbf{b}_0\mathbf{b}_0$ are unit dyadics longitudinal and transverse to $\mathbf{b}_0$, the unit vector defining the direction of the gyrotropic axis [in Eqs. (3), primes used elsewhere in this text to denote normalization with respect to ϵ_0 are omitted for simplicity]. For the special case of a cold collisionless magnetoplasma

$$\epsilon_1 = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}, \quad \epsilon_2 = -\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2 - \omega_c^2}, \quad \epsilon_3 = 1 - \frac{\omega_p^2}{\omega^2}, \tag{4}$$

where, as derived in Eqs. (1.5.20)–(1.5.22), the quantities ω_p and ω_c are the electron plasma and cyclotron angular frequencies, respectively.

To determine explicitly the eigenvalues κ_α and eigenvectors $\mathbf{\Psi}_\alpha$, it is generally more convenient to solve a reduced equation of the form (2) rather than the original eigenvalue problem (1). As was shown in Sec. 1.4, and noted in connection with Eq. (1a), in a homogeneous transversely unbounded region with guide axis along $\mathbf{z}_0$, there exists a set of guided modes of the form $\mathbf{\Psi}_a(\mathbf{r}, t) = \mathbf{\Psi}_\alpha \exp[j(\omega t - \mathbf{k}\cdot\mathbf{r})]$, with $\mathbf{k} = \mathbf{k}_t + \kappa_\alpha \mathbf{z}_0$. Accordingly, since $\mathbf{E}_a(\mathbf{r}, t) = \mathbf{E}_\alpha \exp[j(\omega t - \mathbf{k}\cdot\mathbf{r})]$, one infers from Eq. (2) that

$$\mathscr{Y}(\mathbf{k}, \omega)\cdot\mathbf{E}_\alpha = 0, \tag{5a}$$

and hence a non-vanishing mode field $\mathbf{E}_\alpha$ exists for those $\kappa = \kappa_\alpha$ that for prescribed $\omega, \mathbf{k}_t$ satisfy the secular equation

$$\det \mathscr{Y}(\mathbf{k}, \omega) = 0. \tag{5b}$$

To within a normalization constant, the mode fields $\mathbf{E}_\alpha$ corresponding to the κ_α are then derivable from Eq. (2). The remaining mode fields, for either the first-order or reduced descriptions, follow from $\mathbf{E}_\alpha$ and the source-free plasma field equations (1.1.54) as

$$\begin{aligned}
\mathbf{H}_\alpha &= \frac{\mathbf{k}\times\mathbf{E}_\alpha}{\omega\mu_0}, \\
\mathbf{v}_\alpha &= \frac{1}{j\omega\mu_0 n_0 q}[(k_0^2 - k^2)\mathbf{1} + \mathbf{k}\mathbf{k}]\cdot\mathbf{E}_\alpha, \\
p_\alpha &= \frac{\gamma p_0}{\omega}\mathbf{k}\cdot\mathbf{v}_\alpha.
\end{aligned} \tag{6}$$

For a cold plasma wherein $p_0 = 0$, it is evident that $p_\alpha = 0$. Thus, for prescribed $\omega, \mathbf{k}_t$, where $\mathbf{k} = \mathbf{k}_t + \kappa\mathbf{z}_0$, the orthogonality property derivable from Eq. (8.2.8b) in a cold plasma can be expected to have the form

$$\mathbf{E}_\alpha^{+*}\cdot\mathbf{H}_\beta\times\mathbf{z}_0 + \mathbf{H}_\alpha^{+*}\cdot\mathbf{z}_0\times\mathbf{E}_\beta = 2N_\alpha\,\delta_{\alpha\beta}, \tag{7}$$

with the adjoint components identifiable as $\mathbf{E}_\alpha^+ = \mathbf{E}_\alpha, \mathbf{H}_\alpha^+ = \mathbf{H}_\alpha$ for real κ_α, and as $\mathbf{E}_\alpha^+ = \mathbf{E}_{\alpha^*}, \mathbf{H}_\alpha^+ = \mathbf{H}_{\alpha^*}$ for complex κ_α; it should be recalled from Eq. (8.2.10a) that $\mathbf{E}_{\alpha^*}, \mathbf{H}_{\alpha^*}$ are components of the mode with eigenvalue κ_α^*. For modes with different $\omega, \mathbf{k}_t$, the orthogonality statement (7) also requires integration over both the cross section and time as in Eq. (1.4.2b).

The considerations in Eqs. (4)–(6), which are independent of the relative orientation of guide and gyrotropic axes, will now be specialized to the case wherein the guide axis is parallel to the static magnetic field impressed on the plasma. As a vector basis we choose the right-handed set of unit vectors $\mathbf{z}_0, \mathbf{k}_{t0}, \mathbf{z}_0 \times \mathbf{k}_{t0} = \hat{\mathbf{k}}_{t0}$ determined by the guide axis $\mathbf{z}_0$ and the transverse unit vector $\mathbf{k}_{t0}$, the latter defined by the decomposition $\mathbf{k} = \kappa \mathbf{z}_0 + k_t \mathbf{k}_{t0}$. In this basis, one represents the mode electric field as

$$\mathbf{E} = E_z \mathbf{z}_0 + E_{t'} \mathbf{k}_{t0} + E_{t''} \hat{\mathbf{k}}_{t0}, \qquad \hat{\mathbf{k}}_{t0} = \mathbf{z}_0 \times \mathbf{k}_{t0}, \tag{8}$$

and, furthermore,

$$\mathbf{1}_b = \mathbf{z}_0 \mathbf{z}_0,$$

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{1}) = -k_t^2 \mathbf{z}_0 \mathbf{z}_0 - \kappa^2 \mathbf{k}_{t0} \mathbf{k}_{t0} - k^2 \hat{\mathbf{k}}_{t0} \hat{\mathbf{k}}_{t0} + \kappa k_t (\mathbf{z}_0 \mathbf{k}_{t0} + \mathbf{k}_{t0} \mathbf{z}_0).$$

Thus, for the cold plasma defined by the permittivity parameters in Eq. (4), the defining equation (5a) for $\mathbf{E}$ may be represented as

$$j\omega\epsilon_0 \begin{pmatrix} \epsilon_3 - \dfrac{k_t^2}{k_0^2} & \dfrac{\kappa k_t}{k_0^2} & 0 \\ \dfrac{\kappa k_t}{k_0^2} & \epsilon_1 - \dfrac{\kappa^2}{k_0^2} & j\epsilon_2 \\ 0 & -j\epsilon_2 & \epsilon_1 - \dfrac{k_t^2 + \kappa^2}{k_0^2} \end{pmatrix} \begin{pmatrix} E_z \\ E_{t'} \\ E_{t''} \end{pmatrix} = 0. \tag{9}$$

The corresponding secular equation (5b) then yields for the eigenvalues κ the dispersion equation†[1]

$$\kappa'^4 + [(1 + \epsilon_1')k_t'^2 - 2\epsilon_1']\kappa'^2 + \epsilon_1'(k_t'^2 - 1)(k_t'^2 - \epsilon_\perp') = 0, \tag{10}$$

where we have employed the normalized quantities (for $\epsilon_3 > 0$)†

$$\kappa' = \frac{\kappa}{k_0\sqrt{\epsilon_3}}, \quad k_t' = \frac{k_t}{k_0\sqrt{\epsilon_3}}, \quad \epsilon_1' = \frac{\epsilon_1}{\epsilon_3}, \quad \epsilon_\perp' = \frac{\epsilon_1'^2 - \epsilon_2'^2}{\epsilon_1'}, \quad \epsilon_2' = \frac{\epsilon_2}{\epsilon_3}. \tag{10a}$$

κ' versus k_t' plots of the dispersion equation (10) in various ω ranges appear in Fig. 8.3.2.

For prescribed ω and $\mathbf{k}_t$, there are four eigenvalues κ_α, given by the solution of the quartic equation (10) as

$$\kappa_\alpha' = \pm\sqrt{\epsilon_1' - (\epsilon_1' + 1)(k_t'^2/2) \pm \Delta[(\epsilon_1' - 1)/2]} \tag{11}$$

with

$$\Delta = \sqrt{k_t'^4 + (1 - k_t'^2)[2\epsilon_2'/(\epsilon_1' - 1)]^2}\ \mathrm{sgn}\,(\epsilon_1' - 1). \tag{11a}$$

For spatially dispersive $\boldsymbol{\epsilon}$, additional roots are possible. The sign of the discriminant Δ has been chosen to facilitate subsequent discussion of the analytic properties of the mode functions in the $\mathbf{k}_t$ plane. For each eigenvalue κ_α in Eq. (11), one finds on substitution into Eq. (9) that the corresponding mode electric-field components are related by

†When $\epsilon_3 < 0$, the normalization of κ and k_t is with respect to $k_0\sqrt{|\epsilon_3|}$. In Eq. (10), one then replaces κ' by $j\kappa'$ and k_t' by jk_t'.

$$\frac{E_z}{E_{t'}} = \frac{k_t' \kappa_\alpha'}{k_t'^2 - 1}, \qquad \frac{E_{t''}}{E_{t'}} = -j \frac{\epsilon_2'}{k_t'^2 + \kappa_\alpha'^2 - \epsilon_1'}, \tag{12a}$$

whence, from Eq. (6), the corresponding mode magnetic field is given by

$$\mathbf{H} = \frac{k_t}{\omega\mu_0} E_{t''} \mathbf{z}_0 - \frac{E_{t''}}{Z_\alpha''} \mathbf{k}_{t0} + \frac{E_{t'}}{Z_\alpha'} \hat{\mathbf{k}}_{t0}; \tag{12b}$$

the parameters

$$Z_\alpha' = \frac{\omega\mu_0}{\kappa_\alpha}(1 - k_t'^2), \qquad Z_\alpha'' = \frac{\omega\mu_0}{\kappa_\alpha} \tag{12c}$$

reduce to the characteristic E- and H-mode impedances in the isotropic case $\epsilon_2' = 0$, $\epsilon_1' = 1$ [see Eqs. (2.2.15c) and (2.2.15d)]. Disregarding the velocity components of the κ_α mode, which are readily evaluated from Eqs. (6) and (12a), we shall exhibit only reduced wavevectors Ψ_α defined by

$$\Psi_\alpha \rightarrow \begin{bmatrix} \mathbf{E}_\alpha \\ \mathbf{H}_\alpha \end{bmatrix} = \begin{bmatrix} Z_\alpha \mathbf{e}_\alpha + \mathbf{e}_{z\alpha} \\ \mathbf{h}_\alpha + \mathbf{h}_{z\alpha} \end{bmatrix}. \tag{13a}$$

From Eqs. (12) and (8), on setting $E_{t'}/(\sqrt{2} Z_\alpha')$ equal to unity, one can identify[2]

$$\begin{aligned} \sqrt{2}\,\mathbf{e}_\alpha &= \frac{\Delta - k_t'^2}{\Delta} \mathbf{k}_{t0} - j \frac{\delta}{\Delta} \hat{\mathbf{k}}_{t0}, & \sqrt{2}\,\mathbf{e}_{z\alpha} &= -\frac{k_t}{\omega\epsilon_0\epsilon_3} \mathbf{z}_0, \\ \sqrt{2}\,\mathbf{h}_\alpha &= j \frac{\Delta + k_t'^2}{\delta} \mathbf{k}_{t0} + \hat{\mathbf{k}}_{t0}, & \sqrt{2}\,\mathbf{h}_{z\alpha} &= -j \frac{k_t}{\kappa_\alpha} \frac{\Delta + k_t'^2}{\delta} \mathbf{z}_0, \end{aligned} \tag{13b}$$

where $\delta = 2\epsilon_2'/(\epsilon_1' - 1)$ and the mode impedance in Eq. (13a), denoted by

$$Z_\alpha = \frac{\omega\mu_0}{\kappa_\alpha} \frac{\Delta(1 - k_t'^2)}{\Delta - k_t'^2} = \frac{\omega\mu_0}{\kappa_\alpha} \frac{\Delta^2 + k_t'^2 \Delta}{\delta^2}, \tag{13c}$$

is chosen to secure the normalization in Eq. (15). It should be noted that $\mathbf{e}_\alpha$ and $\mathbf{h}_\alpha$ are fixed for given ω and $\mathbf{k}_t$, while Z_α may assume four distinct values, depending on the choice of sign for the eigenvalue κ_α in Eq. (11) that distinguishes the various Ψ_α wavevectors. The adjoint mode vectors $\mathbf{e}_{\alpha^*}$ and $\mathbf{h}_{\alpha^*}$, and adjoint mode impedance Z_{α^*}, follow from Eqs. (13b) and (13c) on the replacements

$$\Delta \rightarrow \Delta^* \qquad \text{and} \qquad \kappa_\alpha \rightarrow \kappa_\alpha^*; \tag{13d}$$

note that $Z_{\alpha^*}^* = Z_\alpha$ but $\mathbf{e}_\alpha \neq \mathbf{e}_{\alpha^*}$ and $\mathbf{h}_\alpha \neq \mathbf{h}_{\alpha^*}$, so there exist conventional transmission-line relations but the power expression involves the amplitudes of both the orginal and adjoint modes (see Sec. 8.2h).

The derivation of the transverse mode functions in Eq. (13a) can equally well be effected by use of the transverse field equations (8.2.25), which may be written as ($\epsilon_z = \epsilon_3$)

$$\begin{aligned} \kappa_\alpha Z_\alpha \mathbf{e}_a &= \omega\mu_0 \left(\mathbf{1}_t + \frac{\nabla_t \nabla_t}{k_0^2 \epsilon_z} \right) \cdot \mathbf{h}_a \times \mathbf{z}_0, \\ \kappa_\alpha Y_\alpha \mathbf{h}_a &= \omega\epsilon_0 \left(\boldsymbol{\epsilon}_t + \frac{\nabla_t \nabla_t}{k_0^2} \right) \cdot \mathbf{z}_0 \times \mathbf{e}_a, \end{aligned} \tag{14}$$

where $Y_\alpha = 1/Z_\alpha$, $\boldsymbol{\epsilon}_t = \epsilon_1 \mathbf{1}_t - j\epsilon_2 \mathbf{z}_0 \times \mathbf{1}_t$, and the suffix a on $\mathbf{e}_a, \mathbf{h}_a$ implies that they include the exp $[j(\omega t - \mathbf{k}_t \cdot \boldsymbol{\rho})]$ dependence. The indicated derivation via the reduced equation (2) or (5a) for the electric field $\mathbf{E}$ constitutes a more general procedure, applicable as well to spatially dispersive systems.

As anticipated from the reflection symmetry with respect to the gyrotropic axis, $\pm\kappa_\alpha$ eigenvalues are possible. Consequently, the orthogonality properties (1b) or (7) can be simplified by distinguishing modes by κ_α^2 rather than κ_α. Thus, from Eq. (13a), on adding the relations (7) for $+\kappa_\alpha$ and $-\kappa_\alpha$ and choosing $N_\alpha = Z_\alpha$, one infers for prescribed ω, $\mathbf{k}_t$ the orthonormality properties

$$\begin{aligned} \mathbf{e}_\alpha \cdot \mathbf{h}_\beta^* \times \mathbf{z}_0 &= \delta_{\alpha\beta} \qquad \text{for real } \kappa_\alpha, \\ \mathbf{e}_{\alpha^*} \cdot \mathbf{h}_\beta^* \times \mathbf{z}_0 &= \delta_{\alpha\beta} \qquad \text{for complex } \kappa_\alpha, \end{aligned} \tag{15}$$

where now $\delta_{\alpha\beta} = 1$ if $\kappa_\alpha^2 = \kappa_\beta^2$ and $\delta_{\alpha\beta} = 0$ if $\kappa_\alpha^2 \neq \kappa_\beta^2$. One notes that mode functions distinguished by suffices α and α^*, respectively, correspond to eigenvalues κ_α and κ_α^* and thus to opposite choices of sign before Δ in Eq. (11).

The above mode functions $\mathbf{e}_\alpha$ and $\mathbf{h}_\alpha$, distinguished by eigenvalues κ_α^2, permit a transmission-line representation of the steady-state transverse electric and magnetic fields and their sources in the cold magnetoplasma as

$$\begin{aligned} \mathbf{E}_t(\mathbf{r}) &= \sum_a V_\alpha(z)\mathbf{e}_\alpha e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}}, \\ \mathbf{H}_t(\mathbf{r}) &= \sum_a I_\alpha(z)\mathbf{h}_\alpha e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}}, \\ \mathbf{M}_{te} \times \mathbf{z}_0 &= \sum_a v_\alpha(z)\mathbf{e}_\alpha e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}}, \\ \mathbf{z}_0 \times \mathbf{J}_{te} &= \sum_a i_\alpha(z)\mathbf{h}_\alpha e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}}, \end{aligned} \tag{16a}$$

where the summation $\sum_a \equiv \sum_\alpha \iint d\mathbf{k}_t/(2\pi)^2$, the α subscript distinguishes "ordinary" and "extraordinary" waves in polarization space as will be explained in Sec. 8.3b, and the voltage (V_α) and current (I_α) amplitudes are determined by transmission-line equations with propagation constant κ_α and characteristic impedance $Z_\alpha = 1/Y_\alpha$ given by Eqs. (11) and (13c). The transmission-line equations are derived by following a procedure similar to that in Sec. 2.2 [however, subject to the orthogonality conditions in Eqs. (8.2.15) or (8.2.16)], and take the form

$$-\frac{dV_\alpha}{dz} = j\kappa_\alpha Z_\alpha I_\alpha + v_\alpha, \qquad -\frac{dI_\alpha}{dz} = j\kappa_\alpha Y_\alpha V_\alpha + i_\alpha, \tag{16b}$$

with source voltages and currents

$$v_\alpha(z) = \int_S \mathbf{M}_{te} \cdot \mathbf{H}_a^{+*}\, dS, \qquad i_\alpha(z) = \int_S \mathbf{J}_{te} \cdot \mathbf{E}_a^{+*}\, dS, \tag{16c}$$

where S denotes the waveguide cross section and the equivalent source currents $\mathbf{M}_{te}$ and $\mathbf{J}_{te}$ are defined in Eq. (8.2.29b). The details are left as an exercise for the reader.

Correspondingly, knowledge of the complete set of wavevectors $\boldsymbol{\Psi}_\alpha$, each distinguished by the eigenvalue κ, permits a representation of an arbitrary steady-state field $\Psi(\mathbf{r})$ as shown in Eq. (8.2.26b). In a cold magnetoplasma, using wavevectors defined in Eqs. (13), one thereby obtains in component form the field representation

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \sum_\alpha \int a_\alpha(z)(Z_\alpha \mathbf{e}_\alpha + \mathbf{e}_{z\alpha})e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}} \frac{d\mathbf{k}_t}{(2\pi)^2} + \frac{\mathbf{J}_z}{j\omega\epsilon_0\epsilon_z}, \\ \mathbf{H}(\mathbf{r}) &= \sum_\alpha \int a_\alpha(z)(\mathbf{h}_\alpha + \mathbf{h}_{z\alpha})e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}} \frac{d\mathbf{k}_t}{(2\pi)^2} + \frac{\mathbf{M}_z}{j\omega\mu_0\mu_z}. \end{aligned} \tag{17}$$

Solution of the defining equation (8.2.27) for a_α yields

$$a_\alpha(z) = -\int_{-\infty}^{z} b_\alpha(z')e^{-j\kappa_\alpha(z-z')}\,dz', \qquad \alpha = \alpha_>, \tag{18a}$$

for modes with $\kappa_\alpha = \kappa_{\alpha_>}$ carrying power in the increasing z direction, and

$$a_\alpha(z) = \int_{z}^{\infty} b_\alpha(z')e^{-j\kappa_\alpha(z-z')}\,dz', \qquad \alpha = \alpha_<, \tag{18b}$$

for modes with $\kappa_\alpha = \kappa_{\alpha_<}$ carrying power in the descreasing z direction; the source term b_α follows from Eq. (8.2.28) as (note $N_\alpha = Z_\alpha$)

$$b_\alpha(z) = \frac{1}{2Z_\alpha}\iint [(Z_{\alpha^*}^*\mathbf{e}_{\alpha^*}^* + \mathbf{e}_{z\alpha^*}^*)\cdot \mathbf{J}(\mathbf{r}) + (\mathbf{h}_{\alpha^*}^* + \mathbf{h}_{z\alpha^*}^*)\cdot \mathbf{M}(\mathbf{r})]e^{+j\mathbf{k}_t\cdot\boldsymbol{\rho}}\,dS. \tag{18c}$$

The results in Eqs. (17) and (18) follow equally well from the general Green's function representation in Eq. (1.4.17), the field representation (1.1.73a), and Eqs. (13).

8.3b Wavenumber Surfaces

The real solutions of the dispersion equation (10) define wavenumber (or refractive index) surfaces in (k_t', κ') space whose shape depends strongly on $\epsilon_1, \epsilon_2, \epsilon_3$ via the wave frequency ω, the plasma frequency ω_p, and the cyclotron frequency ω_c [see Eq. (4)]. For various ranges of $X = (\omega_p/\omega)^2$ and $Y = \omega_c/\omega$ bounded by the solid curves in Fig. 8.3.1, the surface shapes may be grouped into the eight categories shown in Fig. 8.3.2; the X and Y notation is customary in ionospheric propagation theory.[3-5] The branches labeled "o" and "e" represent, respectively, the real "ordinary" and "extraordinary" mode solutions; they correspond to the plus and minus signs, respectively, under the radical in Eq. (11), and the corresponding κ_α' values are denoted by $\pm\kappa_o'$ and $\pm\kappa_e'$ [the $\pm$ here refer to the signs outside the radical in Eq. (11)]. In k_t' ranges for which no branch is shown, κ_o' and (or) κ_e' is complex, with $\operatorname{Im}\kappa_{o,e}' < 0$ in order to satisfy the radiation condition for traveling modal fields [see Eq. (18a) requiring decay of $\exp(-j\kappa_{o,e}z)$ as $z \to \infty$]. This definition, based on the analytic properties of the multivalued functions $\kappa_{o,e}'$, does not always coincide with an alternative one used in the literature wherein "extraordinary" and "ordinary" distinguishes those wave types whose propagation in the direction transverse to z is, or is not, affected by the presence of the external magnetic field B_0.

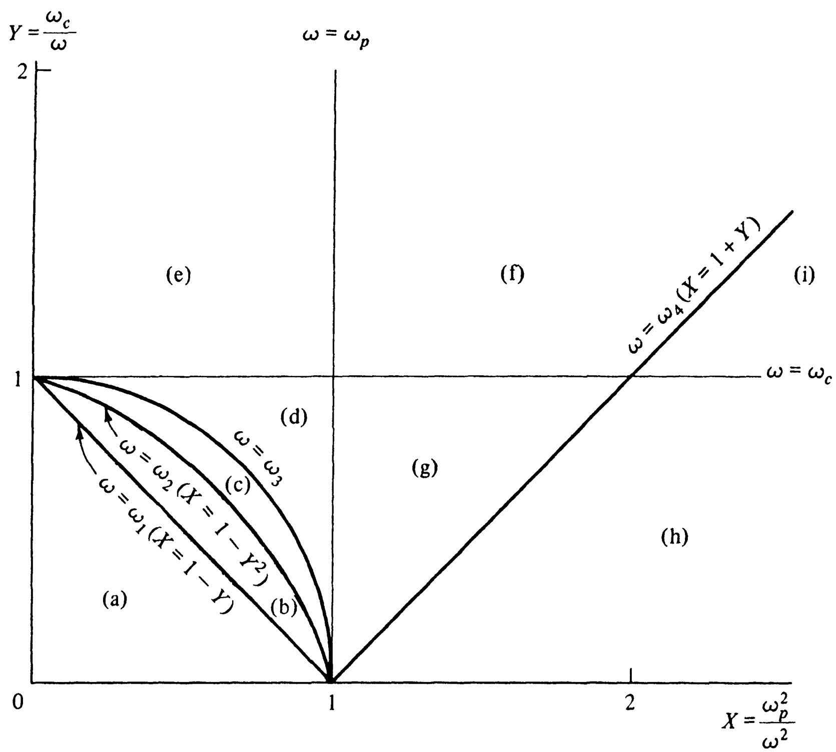

FIG. 8.3.1 Subdivision of the XY plane (the portion shown excludes the low-frequency domain $\omega \to 0$). The frequencies delineating various regions correspond to the following parameter values: $\omega_{1,4}$: $\epsilon_1^2 = \epsilon_2^2$; ω_2: $\epsilon_1 = 0$; ω_3: $2\epsilon_1'^2 = (\epsilon_1' + 1)(\epsilon_1'^2 - \epsilon_2'^2)$.

Transverse propagation corresponds to $\kappa' = 0$, and since $B_0 = 0$ (i.e., $\omega_c = 0$) implies an isotropic medium with wavenumber $k = k_0\sqrt{\epsilon_3}$ [see Eqs. (3b) and (4)], the corresponding k_t' values are $k_t'^2 = 1$ for the ordinary modes and $k_t'^2 \neq 1$ for the extraordinary modes.

The longitudinal wavenumbers $\kappa'_{o,e}$ in Eq. (11) evidently possess branch-point singularities in the complex k_t' plane.[2,6] Most relevant are those occurring for real values of k_t' and signifying the k_t' terminus of the interval of propagation of a non-evanescent mode with real κ'. Such branch points $\bar{k}_{ti}'$ may be located on Fig. 8.3.2 by the condition $d\kappa'/dk_t' = \infty$. $\bar{k}_{t1}'^2 = 1$ and $\bar{k}_{t2}'^2 = (\epsilon_1'^2 - \epsilon_2'^2)/\epsilon_1'$ distinguish k_t' values at which $\kappa'_{o,e} = 0$ while branch points at $\bar{k}_{t4}' = \pm\delta[(\delta/2) - \sqrt{(\delta^2/4) - 1}]$ arise from $\Delta = 0$ in Eq. (11a), and it is recalled that $\delta = 2\epsilon_2'/(\epsilon_1' - 1)$. The algebraic sign of $\kappa'_{o,e}$, when real, must be determined in accord with the radiation condition requiring that the waves with dependence $\exp(-jk\kappa'_{o,e}z)$ carry *energy* in the $+z$ direction. From the discussion in Sec. 1.7c, the average power flow density vector $\bar{\mathbf{S}} = \mathrm{Re}\,(\mathbf{E} \times \mathbf{H}^*)$ is normal to the wavenumber surface and drawn so that $\mathbf{k}' \cdot \bar{\mathbf{S}} > 0$, where $\mathbf{k}' = \mathbf{k}_t' + \mathbf{z}_0\kappa'$ is

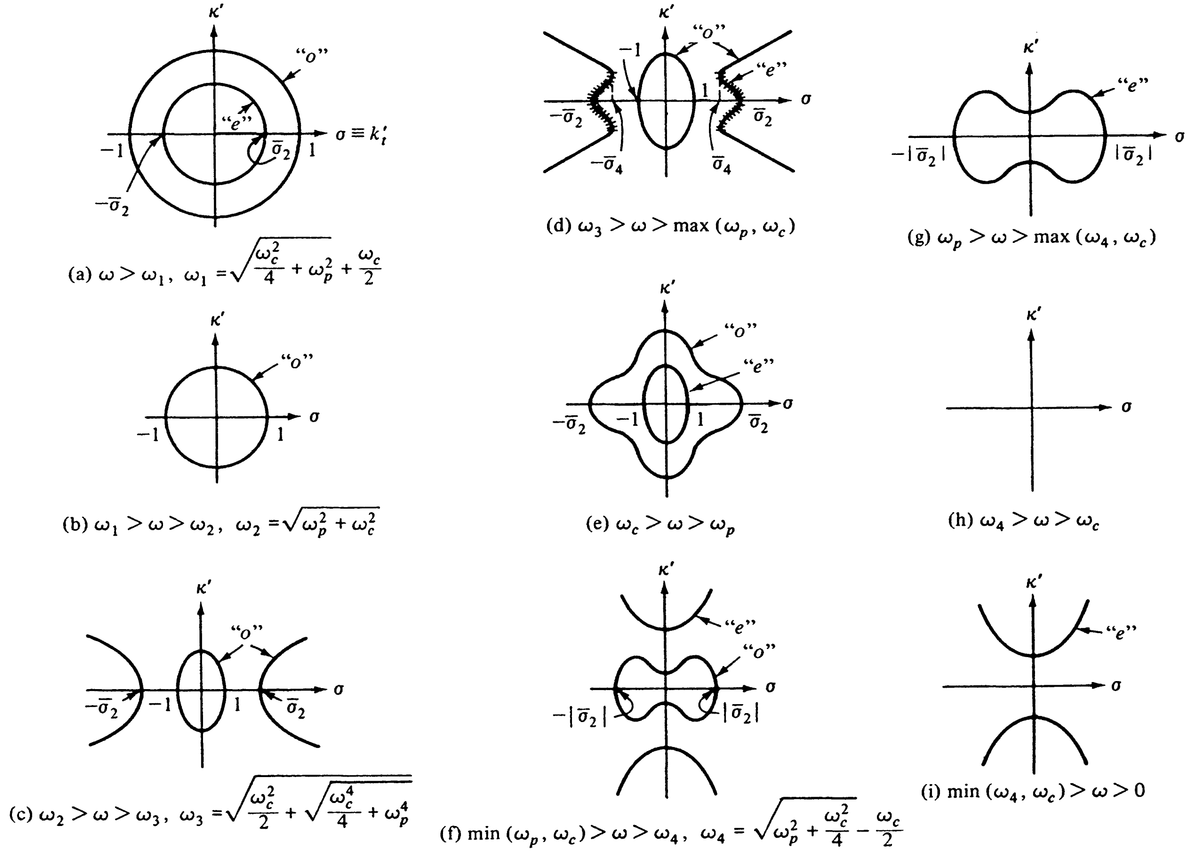

FIG. 8.3.2 Wavenumber surfaces for various ranges of plasma parameters shown in Fig. 8.3.1. $\sigma \equiv k_t' = k_t(k_0\sqrt{\epsilon_3})^{-1}$ and $\kappa' = \kappa(k_0\sqrt{\epsilon_3})^{-1}$ denote the normalized transverse and longitudinal wavenumbers, respectively, while $\bar{\sigma}_i \equiv \bar{k}_{ti}'$ locate branch points of κ'.

the normalized wavevector. Since we are interested in waves with $\bar{\mathbf{S}} \cdot \mathbf{z}_0 > 0$, this implies, for example, that $\kappa'_{o,e} > 0$ for the propagating ranges in Figs. 8.3.2(a) and 8.3.2(e); that $\kappa'_0 > 0$ for $k_t'^2 < 1$ and $\kappa'_0 < 0$ for $\bar{k}_{t2}'^2 < k_t'^2 < \infty$ in Fig 8.3.2(c); and that $\kappa'_e > 0$ for $-\infty < k_t'^2 < \infty$ in Fig. 8.3.2(i). When passing from the propagating to the non-propagating range, the analytic continuation of $\kappa'_{o,e}$ must be performed so that $\operatorname{Im} \kappa'_{o,e} < 0$ and thereby specifies the corresponding sign of $\operatorname{Re} \kappa'_{o,e}$.

The explicit form of the mode functions $\mathbf{e}_\alpha = \mathbf{e}_{o,e}$, $\mathbf{h}_\alpha = \mathbf{h}_{o,e}$ and their adjoints for the ordinary and extraordinary modes follows from Eqs. (13) for $\epsilon_3 > 0$ as

$$\sqrt{2}\, \mathbf{e}_{o,e} = \frac{\pm\Delta - k_t'^2}{\pm\Delta} \mathbf{k}_{to} \mp j \frac{\delta}{\Delta} \hat{\mathbf{k}}_{to}, \quad \sqrt{2}\, \mathbf{e}_{o^*,e^*} = \frac{\pm\Delta^* - k_t'^2}{\pm\Delta^*} \mathbf{k}_{to} \mp j \frac{\delta}{\Delta^*} \hat{\mathbf{k}}_{to}, \tag{19a}$$

$$\sqrt{2}\, \mathbf{h}_{o,e} = j \frac{k_t'^2 \pm \Delta}{\delta} \mathbf{k}_{t0} + \hat{\mathbf{k}}_{to}, \quad \sqrt{2}\, \mathbf{h}_{o^*,e^*} = j \frac{k_t'^2 \pm \Delta^*}{\delta} \mathbf{k}_{to} + \hat{\mathbf{k}}_{to}, \tag{19b}$$

with the characteristic impedances $Z_{o,e}$ in Eq. (13c) given correspondingly by

$$Z_{o,e} = \frac{\omega\mu_0}{k\kappa'_{o,e}} \frac{\Delta^2 \pm k_t'^2 \Delta}{\delta^2}, \qquad k = k_0\sqrt{\epsilon_3}, \quad \kappa'_{o,e} = \frac{1}{k} \kappa_{o,e}. \tag{20}$$

According to Eq. (15), the mode functions in Eqs. (19a, b) satisfy the orthonormality conditions

$$\mathbf{e}_{e,o} \cdot \mathbf{h}^*_{o,e} \times \mathbf{z}_0 = 0, \qquad \mathbf{e}_{e,o} \cdot \mathbf{h}^*_{e,o} \times \mathbf{z}_0 = 1, \qquad \Delta \text{ real}, \tag{21a}$$

$$\mathbf{e}_{e^*,o^*} \cdot \mathbf{h}^*_{o,e} \times \mathbf{z}_0 = 0, \qquad \mathbf{e}_{e^*,o^*} \cdot \mathbf{h}^*_{e,o} \times \mathbf{z}_0 = 1, \qquad \Delta \text{ imaginary}, \tag{21b}$$

with the recognition that $\kappa'_o = -\kappa_e'^*$ when Δ is imaginary (the minus sign being chosen to assure $\operatorname{Im} \kappa'_{o,e} < 0$). It then follows from Eq. (20) that

$$Z_0 = -Z_e^* \qquad \text{when } \Delta \text{ is imaginary.} \tag{21c}$$

When the medium is lossy, the matrix elements in Eq. (4) are complex and the mode orthogonality condition (8.2.8b), or its simplified form (8.2.14b) for the present reflection symmetric case, involves the eigenfunctions for the adjoint medium. It can be shown that results for this case can be obtained by analytic continuation from those for real Δ.

8.3c Green's Functions for Unbounded Regions

Modal representation

Application of the preceding results is now illustrated for the problem of radiation from a time-harmonic electric current element in an unbounded, cold homogeneous, and lossless magnetoplasma. While radiation in an unbounded medium can also be described in terms of the plane-wave modes of Sec. 1.2,[7,8] we employ the guided-wave formulation[1,2] in anticipation of the plane-stratified medium to be discussed in Sec. 8.3d. Since the gyrotropic axis is parallel to the symmetry axis z, one may employ Eqs. (16) or (17), which represent the electromagnetic fields and their source excitations in terms of the eigenfunctions in Eqs. (19).

We assume the current element to be directed parallel to the z axis:

$$\mathbf{J}(\mathbf{r}) = \mathbf{z}_0 J^0 \delta(\mathbf{r}), \qquad \mathbf{M}(\mathbf{r}) \equiv 0. \tag{22}$$

Employing the field formulation in Eqs. (16), one finds, in view of Eqs. (8.2.29b) that $\mathbf{J}_{te} = 0, \mathbf{M}_{te} = (j/\omega\epsilon_3\epsilon_0)(\nabla_t \times \mathbf{J}_z)$, so the current generator $i_\alpha(z) = 0$ while $v_\alpha(z) = v_\alpha \delta(z)$ is given by (note: $\mathbf{H}_\alpha^+ = \mathbf{h}_{\alpha^*} + \mathbf{h}_{z\alpha^*}$)

$$v_\alpha = \frac{j}{\omega\epsilon_3\epsilon_0} \iint_{-\infty}^{\infty} J^0 \delta(\boldsymbol{\rho}) \nabla_t \cdot \mathbf{h}_{a^*}^*(\boldsymbol{\rho}) \times \mathbf{z}_0 \, dx\, dy = -\frac{k_t J^0}{\sqrt{2}\, \omega\epsilon_3\epsilon_0} = e_{z\alpha} J^0, \tag{23}$$

where $\mathbf{h}_a(\boldsymbol{\rho}) = \mathbf{h}_\alpha \exp(-j\mathbf{k}_t \cdot \boldsymbol{\rho})$ from Eq. (13b), a denotes both α and $\mathbf{k}_t$, and α stands for o or e. Also recall that $\epsilon_z = \epsilon_3\epsilon_0$. The solution of the relevant transmission-line equations (2.2.15a) and (2.2.15b) in the unbounded medium is given for $\epsilon_3 > 0$ by Eqs. (2.4.19):

$$V_\alpha(z) = -(\operatorname{sgn} z) \frac{v_\alpha}{2} e^{-j\kappa_\alpha |z|} = (\operatorname{sgn} z) Z_\alpha I_\alpha(z) \tag{23a}$$

with $Z_\alpha = 1/Y_\alpha$ given in Eq. (20). The total electromagnetic fields may now be written down from Eqs. (16) and (8.2.3).

Alternatively, proceeding from Eqs. (17) and (22), one finds in a source-free region,

$$\mathbf{E}(\mathbf{r}) = \sum_{\alpha \gtrless} \int [b_\alpha (Z_\alpha \mathbf{e}_\alpha + \mathbf{e}_{z\alpha}) e^{-j(\mathbf{k}_t \cdot \boldsymbol{\rho} + \kappa_\alpha z)}] \frac{d\mathbf{k}_t}{(2\pi)^2}, \qquad z \gtrless 0, \tag{24a}$$

$$\mathbf{H}(\mathbf{r}) = \sum_{\alpha \gtrless} \int [b_\alpha (\mathbf{h}_\alpha + \mathbf{h}_{z\alpha}) e^{-j(\mathbf{k}_t \cdot \boldsymbol{\rho} + \kappa_\alpha z)}] \frac{d\mathbf{k}_t}{(2\pi)^2}, \qquad z \gtrless 0. \tag{24b}$$

The suffixes $\alpha_>$ and $\alpha_<$ denote the modes carrying power in the increasing $(z > 0)$ and decreasing $(z < 0)$ directions respectively, and, as indicated, the summation is over one type of mode or the other depending on whether $z \gtrless 0$; furthermore, from Eq. (18c), with $b_\alpha(z) = b_\alpha \delta(z)$,

$$b_\alpha = -(\operatorname{sgn} z) \frac{e_{z\alpha^*}^*}{2Z_\alpha} J^0 = \frac{v_\alpha}{2Z_\alpha}, \qquad e_{z\alpha^*}^* = e_{z\alpha}. \tag{24c}$$

Because of reflection symmetry, to each value κ_α there corresponds an eigenvalue $-\kappa_\alpha$ (see Sec. 8.2d), so the two ranges $z \gtrless 0$ can be accommodated by the $\alpha_>$ expression provided that in the range $z < 0$, one replaces κ_α by $-\kappa_\alpha$. Equations (24) then yield the same result as that obtained from Eqs. (16) and (8.2.3).

Since the integrands in Eqs. (24a) and (24b) depend only on $\mathbf{k}_t$ and ρ, it is convenient to exploit the azimuthal symmetry and change to a Fourier–Bessel (cylindrical-wave) representation. On noting that $\mathbf{k}_{to} = \boldsymbol{\rho}_0 \cos\phi - \boldsymbol{\phi}_0 \sin\phi$, $\hat{\mathbf{k}}_{to} = \boldsymbol{\rho}_0 \sin\phi + \boldsymbol{\phi}_0 \cos\phi$, one finds from Eqs. (5.2.7) with $f(k_t) = 1, \boldsymbol{\rho}' = 0$, that

$$\int e^{-j\mathbf{k}_t \cdot \boldsymbol{\rho}} \, d\mathbf{k}_t = 2\pi \int_0^\infty J_0(k_t \rho) k_t \, dk_t, \qquad d\mathbf{k}_t = k_t \, dk_t \, d\phi, \tag{24d}$$

while from ρ differentiation of the same formula with $f(k_t) = 1/k_t$, $\boldsymbol{\rho}' = 0$,

$$\int_0^\infty dk_t \int_0^{2\pi} d\phi\, k_t \cos\phi\, e^{-jk_t\rho\cos\phi} = -j2\pi \int_0^\infty J_1(k_t\rho)k_t\, dk_t. \tag{24e}$$

Moreover, by direct integration of the ϕ integral,

$$\int_0^\infty dk_t \int_0^{2\pi} d\phi\, k_t \sin\phi\, e^{-jk_t\rho\cos\phi} = 0. \tag{24f}$$

When these results are utilized in Eqs. (24a), (24b), and (13), with $\boldsymbol{\rho}_0$ and $\boldsymbol{\phi}_0$ fixed, one finds on employing the transformation leading from Eq. (3.2.62) to (3.2.69) that for $z \neq 0$ and $\epsilon_3 > 0$,[2]†

$$\mathbf{E}_{t_{o,e}} = jA(\operatorname{sgn} z) \int_{-\infty}^\infty \sigma^2\left(\boldsymbol{\rho}_0 \frac{\sigma^2 \mp \Delta}{\pm\Delta} + \boldsymbol{\phi}_0 \frac{j\delta}{\pm\Delta}\right) H_1^{(2)}(k\sigma\rho)e^{-jk\kappa'_{o,e}|z|}\, d\sigma, \tag{25a}$$

$$E_{z_{o,e}} = A\int_{-\infty}^\infty \frac{\sigma^2 \mp \Delta}{\pm\Delta} \frac{\sigma^3\kappa'_{o,e}}{1-\sigma^2} H_0^{(2)}(k\sigma\rho)e^{-jk\kappa'_{o,e}|z|}\, d\sigma, \tag{25b}$$

$$\mathbf{H}_{t_{o,e}} = \frac{A}{\zeta}\int_{-\infty}^\infty \sigma^2\left(\boldsymbol{\rho}_0 \frac{\delta\kappa'_{o,e}}{\pm\Delta} + \boldsymbol{\phi}_0 j \frac{\sigma^2 \mp \Delta}{\pm\Delta} \frac{\kappa'_{o,e}}{1-\sigma^2}\right) H_1^{(2)}(k\sigma\rho)e^{-jk\kappa'_{o,e}|z|}\, d\sigma, \tag{25c}$$

$$H_{z_{o,e}} = -j(\operatorname{sgn} z)\frac{A\delta}{\zeta}\int_{-\infty}^\infty \frac{\sigma^3}{\pm\Delta} H_0^{(2)}(k\sigma\rho)e^{-jk\kappa'_{o,e}|z|}\, d\sigma, \qquad \sigma \equiv k'_t, \tag{25d}$$

where the upper and lower signs refer to the ordinary and extraordinary mode contributions, respectively, $A = k^2\zeta J^0/16\pi$ with $\zeta = (\mu_0/\epsilon_0\epsilon_3)^{1/2}$, and $\boldsymbol{\rho}_0$, $\boldsymbol{\phi}_0$, $\mathbf{z}_0$ are unit vectors in a cylindrical coordinate system. For brevity, the notation $\sigma \equiv k'_t$ has been introduced, where k'_t is defined in Eq. (10a). The total fields are then obtained from

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_o, \qquad \mathbf{H} = \mathbf{H}_e + \mathbf{H}_o. \tag{26}$$

The integration path is indented into the lower half of the σ plane to avoid the branch point at $\sigma = 0$ arising from the presence of the Hankel functions. The branch-point singularities due to Δ and $\kappa'_{o,e}$ are displaced from the real axis when losses are assumed, and their disposition in the lossless limit is thus made unambiguous; alternatively, one may follow the considerations in Sec. 8.3b for directly determining the path deformation around significant real branch points. These considerations are linked with the radiation condition requiring $\operatorname{Im}\kappa'_{o,e} \leq 0$ on the integration path. In the lossless case, where $\kappa'_{o,e}$ may be real over certain ranges of σ, its algebraic sign can be determined as noted in Sec. 8.3b, or by treating the problem in the limit of vanishing dissipation.

The radiation condition for propagating waves may alternatively be imposed by requiring the mode admittances $Y_{o,e} = 1/Z_{o,e}$ in Eq. (20) to be positive when $\kappa'_{o,e}$ is real. For proof, we begin with the requirement that the total average power flow through a plane $z > 0$ must be non-negative and given by

$$\bar{S}_z = \operatorname{Re}\iint_{-\infty}^{\infty} \mathbf{E}_t^* \cdot \mathbf{H}_t \times \mathbf{z}_0\, dx\, dy \geq 0. \tag{27}$$

†For $\epsilon_3 < 0$, replace $\sqrt{\epsilon} \to -j\sqrt{|\epsilon|}$, $\sigma \to j\sigma$, $\kappa' \to j\kappa'$.

Upon substituting the modal representations for $\mathbf{E}_t^*$ and $\mathbf{H}_t$ from Eq. (16a), interchanging the orders of the xy and $\mathbf{k}_t$ integrations, and recalling the orthogonality relation (8.2.15),† one finds

$$\bar{S}_z = \mathrm{Re} \iint [V_o^* I_o + V_e^* I_e] \frac{d\mathbf{k}_t'}{(2\pi)^2} \geq 0, \tag{28}$$

with the integrals extending over those portions of the k_t' plane for which $\kappa_{o,e}'^2$ is both real (Δ real) and complex (Δ imaginary), respectively. The relation in Eq. (28) is valid for arbitrary source distributions confined to or below the plane $z = 0$, and the form of $V_{o,e}$ and $I_{o,e}$ for $z > 0$ is then as in Eq. (23a); for the special case of the longitudinal electric dipole, $v_{o,e}$ is specified in Eq. (23). Equation (28) may therefore be written as

$$\bar{S}_z = \mathrm{Re} \iint [Y_0 |V_0|^2 + Y_e |V_e|^2] \frac{d\mathbf{k}_t'}{(2\pi)^2} \geq 0, \tag{29}$$

and since $Y_e = -Y_o^*$ when κ'^2 is complex [see Eq. (21c)], the corresponding integral representative of energy coupling in the non-propagating ordinary and extraordinary modes does not contribute to the real power. When κ' is imaginary with Δ real, $Y_{o,e} = 1/Z_{o,e}$ in Eq. (20) is also imaginary and there is again no contribution to $\bar{S}_z$. The only remaining modes are the propagating modes along z for which $\kappa_{o,e}'$ is real:

$$\bar{S}_z = \iint_{\kappa_o' \text{ real}} Y_0 |V_0|^2 \frac{d\mathbf{k}_t'}{(2\pi)^2} + \iint_{\kappa_e' \text{ real}} Y_e |V_e|^2 \frac{d\mathbf{k}_t'}{(2\pi)^2} \geq 0. \tag{30}$$

Since this inequality must be satisfied for arbitrary source distributions (i.e., arbitrary V_o and V_e) it follows that

$$Y_{o,e} > 0 \qquad \text{when } \kappa'_{o,e} \text{ is real,} \tag{31}$$

which statement implies that each propagating ordinary and extraordinary mode carries energy away from the source region (see also Reference 6).

Asymptotic evaluation of far fields

Since it does not seem possible to express the integrals in Eqs. (25) exactly in terms of known functions (a reduction is possible, however, for the special case of uniaxial anisotropy, $\epsilon_2' = 0$ [see Eq. (7.3.8) et seq.]), it is necessary to resort to asymptotic techniques in order to infer specific information about the behavior of the radiated fields. Generic double Fourier integrals encountered in the theory of radiation by arbitrary sources in anisotropic dispersive media have been discussed in Sec. 1.6b, and an asymptotic evaluation at large distances from the source has been performed by the stationary-phase method. Because of the complexity exhibited by multiple integral representations, the asymptotic

†On multiplying $\mathbf{e}_{o,e}$ and $\mathbf{h}_{o,e}$ in Eqs. (19) by $\exp(-j\mathbf{k}_t \cdot \boldsymbol{\rho})$ to synthesize the mode fields $\mathbf{e}_a = \mathbf{E}_{ta}/Z_\alpha$ and $\mathbf{h}_a = \mathbf{H}_{ta}$, one finds from Eq. (8.2.15),

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{e}_a(\boldsymbol{\rho}, \mathbf{k}_{t1}) \cdot \mathbf{h}_{a'}^*(\boldsymbol{\rho}, \mathbf{k}_{t2}) \times \mathbf{z}_0 \, dS = \mathbf{e}_\alpha \cdot \mathbf{h}_{\alpha'}^* \times \mathbf{z}_0 \, \delta(\mathbf{k}_{t1} - \mathbf{k}_{t2})(2\pi)^2.$$

calculation in Sec. 1.6b accounts only for saddle-point contributions without including effects of singularities in the integrands. For the special case under discussion here, the double integrals have been reduced to the single integrals in Eqs. (25), which can be treated more thoroughly by the methods of Chapter 5. For this reason, the asymptotic evaluation is presented again, due regard now being given to the steepest-descent paths through saddle points, and to the manner of dealing with singularities and transition phenomena. Graphical methods for determining saddle points are the same as in Sec. 1.6b and are summarized briefly.

As in the investigations of similar integrals encountered in problems of radiation in isotropic regions (see Sec. 5.3), it is assumed that $|k\sigma\rho| \gg 1$, so the Hankel functions may be replaced by their asymptotic approximation in Eq. (5.3.13). If $k\rho \gg 1$, this condition may be satisfied over the entire integration interval provided that the path is distorted away from the vicinity of $\sigma = 0$. Each of the integrals is then of the form [see Eq. (1.6.22), with $A \equiv L$, $\psi \equiv krM$, $i \rightarrow -j$, $k_t \rightarrow k\sigma$]

$$J = \int L(\sigma)e^{-jkrM(\sigma)}\, d\sigma, \tag{32}$$

where

$$M(\sigma) = \kappa'(\sigma)|\cos\theta| + \sigma\sin\theta, \tag{32a}$$

and (r, θ) are the spherical coordinates with respect to the source point $\mathbf{r} = 0$,

$$\rho = r\sin\theta, \qquad z = r\cos\theta. \tag{32b}$$

For $kr \gg 1$, the major contribution to the integral arises from the vicinity of the saddle points σ_i at which

$$\frac{d}{d\sigma}M(\sigma) = 0 \qquad \text{or} \qquad \left.\frac{d\kappa'(\sigma)}{d\sigma}\right|_{\sigma_i} = -|\tan\theta|. \tag{33}$$

The real solutions $\kappa'(\sigma_i) \equiv \kappa_i'$, σ_i, yield propagating waves in the far field while complex saddle points give rise to exponentially attenuated solutions.The geometrical significance of the saddle-point condition (33) has been discussed in Sec. 1.6b and also in connection with radiation in the uniaxial medium [see Eq. (7.3.11) et seq]: the real saddle points locate those points κ_i', σ_i on the wavenumber surface for the medium [i.e., the plot of the real κ', $k_t' \equiv \sigma$ solutions of Eq. (11)] at which the normal makes an angle θ with the positive κ'axis. The normal to the wavenumber surface points in the direction of power flow (ray direction), so the contribution from each saddle point represents a wave that, at least in the far field, carries energy radially away from the source region. This radial power flow property does not necessarily apply to the total field contributed by several saddle points, in view of interference that generally occurs between the various field constituents. These remarks are illustrated in general in Fig. 8.3.3 (see Fig. 1.6.5 for further elaboration on the use of the diagram), with Fig. 8.3.2 showing the actual curve shapes corresponding to various parameters in a plasma described by the permittivity tensor in Eq. (3b).

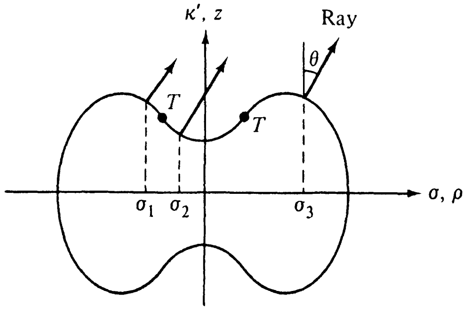

FIG. 8.3.3 Wavenumber surface and saddle points.

To carry out the asymptotic evaluation of the integral in Eq. (32), it remains to be shown that the integration path can be deformed into steepest-descent paths (SDP) through the various saddle points. The steepest-descent contours through the saddle point σ_i are defined by the equation $\operatorname{Re} M(\sigma) = \operatorname{Re} M(\sigma_i)$, but a solution is not simply obtained in view of the complicated structure of $\kappa'(\sigma)$. It suffices, however, to follow the SDP only in the vicinity of σ_i, where its progress is determined readily, provided that the remaining portions C' of the path lie in "valley regions" below the level of the various contributing saddle points. In the valleys of the complex σ plane, one has $\operatorname{Im} M(\sigma) < 0$, whence the contribution from any path segment C' is

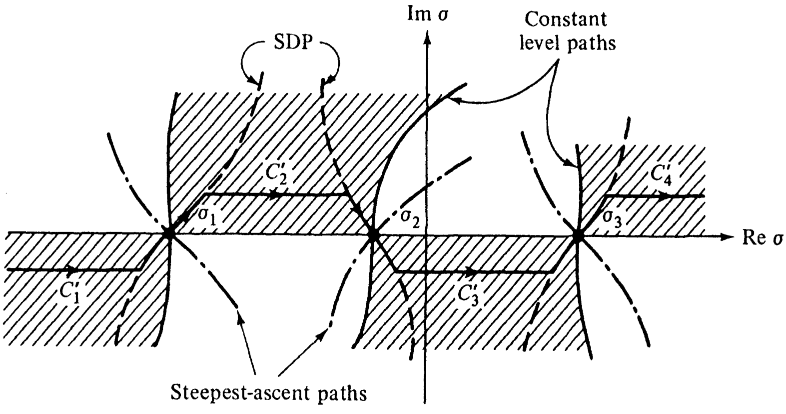

FIG. 8.3.4 Path deformation in the complex σ plane. Valley region [Im $M(\sigma) < 0$] is shaded. Note that $[d^2M(\sigma)/d\sigma^2]_{\sigma_{1,3}} < 0$, $[d^2M(\sigma)/d\sigma^2]_{\sigma_2} > 0$.

$O[\exp(-akr)]$, where $a > 0$ is the smallest value of $|\mathrm{Im}\, M(\sigma)|$ along C'. This exponentially small term can be neglected in comparison with the saddle-point approximation, for which $\mathrm{Im}\, M(\sigma_i) = 0$.† For the configuration in Fig. 8.3.3, the disposition of the complex σ plane in the vicinity of the saddle points is shown in Fig. 8.3.4, and one observes that the path may be deformed from the real axis so as to proceed entirely in valley regions when $\sigma \neq \sigma_i$. To achieve this path distortion it is necessary that the algebraic sign of $d^2M(\sigma_i)/d\sigma_i^2$ alternate from one saddle point to the other since the argument of the vector element $\boldsymbol{\sigma} - \boldsymbol{\sigma}_i$ leading away from the saddle point along the SDP is $\mp\pi/4$ when $d^2M(\sigma_i)/d\sigma_i^2 \gtrless 0$ [see Eqs. (4.2.1) and the similar analysis in Sec. 7.5e]. This behavior is met as long as $\kappa'(\sigma)$ is a regular function of σ over the relevant interval on the real axis. One may also verify that none of the branch points in the integrand is crossed during the path deformation so that the asymptotic approximation of the integral in Eq. (32) arises from the saddle points only, with the lowest-order contribution from each taking the form given in Eq. (4.2.1b) (see Sec. 8.3b for the analytic properties of $\kappa'_{o,e}$).

After these preliminaries, the asymptotic solution for the field components in Eqs. (25) at the distant observation point (r, θ) may be written as follows[2]:

$$\boldsymbol{\rho}_0 E_\rho + \mathbf{z}_0 E_z \sim 2jA \sum_i \beta_i \sqrt{R_i}\, G_i \frac{e^{-jkrN_i(\theta)}}{kr} (\mathbf{q}'_i \times \boldsymbol{\phi}_0), \tag{34a}$$

$$E_\phi \sim 2A \sum_i \beta_i \sqrt{R_i}\, F_i \frac{e^{-jkrN_i(\theta)}}{kr}, \qquad A = \frac{k^2\zeta J^0}{16\pi}, \tag{34b}$$

$$\boldsymbol{\rho}_0 H_\rho + \mathbf{z}_0 H_z \sim -\frac{2A}{\zeta} \sum_i \beta_i \sqrt{R_i}\, F_i \frac{e^{-jkrN_i(\theta)}}{kr} (\mathbf{p}'_i \times \boldsymbol{\phi}_0), \tag{34c}$$

$$H_\phi \sim -\frac{2jA}{\zeta} \sum_i \beta_i \sqrt{R_i}\, G_i \frac{e^{-jkrN_i(\theta)}}{kr}, \qquad \zeta = \sqrt{\frac{\mu_0}{\epsilon_0 \epsilon_3}}, \tag{34d}$$

where the summation extends over all contributing saddle points and $N_i(\theta) \equiv M(\sigma_i)$ is the ray refractive index. R_i is the radius of curvature of the wavenumber surface at the point κ'_i, σ_i,‡

$$\frac{1}{R_i} \equiv \frac{|d^2\kappa'_i/d\sigma_i^2|}{[1 + (d\kappa'_i/d\sigma_i)^2]^{3/2}} = \left|\frac{d^2\kappa'_i}{d\sigma_i^2}\right| |\cos^3\theta| = \frac{d^2M(\sigma_i)}{d\sigma_i^2} \cos^2\theta, \tag{35a}$$

and F_i, G_i are amplitude functions defined as follows:

$$F_i(\theta) = \sqrt{\frac{\sigma_i}{\sin\theta}} \frac{\sigma_i \delta}{\gamma_i \Delta_i} \cos\theta, \qquad G_i(\theta) = \sqrt{\frac{\sigma_i}{\sin\theta}} \frac{\kappa'_i \sigma_i}{1 - \sigma_i^2} \frac{\sigma_i^2 - \gamma_i \Delta_i}{\gamma_i \Delta_i} \cos\theta, \tag{35b}$$

with $\Delta_i \equiv \Delta(\sigma_i)$, and $\gamma_i = +1$ or -1 if the saddle point arises in the ordinary and extraordinary integral, respectively. Finally, $\beta_i = +1$ if $d^2M(\sigma_i)/d\sigma_i^2 < 0$, $\beta_i = -j$ if $d^2M(\sigma_i)/d\sigma_i^2 > 0$, and $\mathbf{p}'_i$, $\mathbf{q}'_i$ are the polarization vectors

†This estimate of the exponentially small error made by neglecting the contribution from C' is used in addition to the error estimate of the saddle-point evaluation in Sec. 4.2b based on the distance from the saddle point to the nearest singularity in the integrand.

‡The symbol $dA(\sigma_i)/d\sigma_i$ denotes evaluation of $dA(\sigma)/d\sigma$ at σ_i, and similarly for higher derivatives.

$$\mathbf{p}_i' = \boldsymbol{\rho}_0\sigma_i + \mathbf{z}_0\kappa_i', \qquad \mathbf{q}_i' = \boldsymbol{\rho}_0\sigma_i + \mathbf{z}_0\frac{1-\sigma_i^2}{\kappa_i'^2}, \tag{35c}$$

where $k\mathbf{p}_i'\cdot\mathbf{r} \equiv krN_i(\theta)$, so $N_i(\theta) = \mathbf{p}_i'\cdot\mathbf{r}_0$. These results, valid when $\epsilon_3 > 0$, also apply to $\epsilon_3 < 0$ with the substitutions listed in the footnote to Eqs. (25). Further investigation of the formulas in Eqs. (34) is required when the radius of curvature $R_i \to \infty$, and the pertinent considerations are given below. The form of Eqs. (34) is the same as in Eq. (1.6.25) if one observes that in view of the rotational symmetry of the wavenumber surface about the κ' axis, one of the principal radii of curvature is constant.

The structure of the asymptotic formulas substantiates the remarks made earlier about the use of the refractive index surfaces in predicting the number of propagating ray contributions in the far field. As noted in Sec. 1.6b, certain information regarding the ray amplitudes is also obtainable in view of the presence of the radius of curvature R_i in the result; field enhancement is evidently associated with rays corresponding to segments having small curvature. If the surface has points of inflection or open branches, certain rays will contribute only over a limited range of observation angles; in the vicinity of the boundaries of the domains of existence of these rays, $R_i \to \infty$ and a more careful evaluation of the original integral is required. The restricted domain of existence of certain rays associated with a wavenumber plot having points of inflection may be visualized from Fig. 8.3.3, which shows that the saddle points σ_1 and σ_2 yield propagating waves only when $0 \leq \theta < \theta_c$, where θ_c is the angle between the positive κ' (or z) axis and the normal at the inflection point T. For $\theta_c < \theta < \pi/2$, σ_1 and σ_2 are complex and the associated ray contributions are evanescent; this range of observation angles therefore defines the "shadow region" for these rays. These considerations do not affect the ray corresponding to σ_3 which contributes at all observation points. Analogous remarks have been made in Sec. 7.3 to explain phenomena associated with an open-branched surface.

Transition region: coalescence of two saddle points

As noted earlier, the asymptotic formulas in Eqs. (34) become invalid when $R_i \to \infty$, as, for example, at a point of inflection on the dispersion curve (see points T in Fig. 8.3.3). As the observation angle θ in the far field approaches the value θ_c defining the direction of the normal at T, the two first-order saddle points at σ_1 and σ_2 approach one another and coalesce into a second-order saddle point on the shadow boundary $\theta = \theta_c$. The asymptotic evaluation of the integral must then be carried out in terms of Airy functions (see Sec. 4.5a), and the result is directly analogous to that in Sec. 7.5e [see the portion of Eq. (7.5.59) relating to the saddle points β_1 and β_2]. On the dark side ($\theta > \theta_c$ for the diagram in Fig. 8.3.3), the contributions from the two rays corresponding to $\sigma_{1,2}$ are exponentially small and therefore negligible [the ray refractive index $N(\theta)$ is no longer real], whereas on the shadow boundary, the variation is $\sim (kr)^{-5/6} \exp[-jkrN(\theta_c)]$. Thus, the fields on the boundary are stronger than at other observation angles where the distance dependence is according to the conventional $(kr)^{-1}$. Larger field concentrations may therefore be expected in the transition region where the two ray contributions from σ_1 and σ_2 have almost identical properties and interact strongly.

Transition region: saddle point moves to infinity

The radius of curvature of the dispersion surface also grows without limit on the remote segments of open branches. A simple but typical example has already been encountered in the uniaxially anisotropic medium in Chapter 7 (see Fig. 7.3.2); the difficulty in the present asymptotic evaluation arises when $\theta \to \theta_c(\sigma_i \to \infty)$, where θ_c relates to the normal to the asymptote of the surface. Open branches of the dispersion curve may arise when $\epsilon_1' < 0$, as found from a study of the real solutions of Eq. (10) when $\kappa', \sigma \to \infty$; the equation for the asymptotes is easily shown to be $\theta = \hat{\theta} = \pm\tan^{-1}\sqrt{-1/\epsilon_1'}$, it being recalled that $\sigma \equiv k_t'$.

To assess the modification of the asymptotic procedure required when $\sigma_i \to \infty$, it suffices to consider the integrands in Eqs. (25) in the range of large σ. From Eq. (11) and its counterpart when $\epsilon_3 < 0$, one finds, for $|\sigma| \gg 1$,

$$\kappa_0' \approx -\sqrt{\epsilon_1' + \frac{\epsilon_2'^2}{\epsilon_1' - 1} - \epsilon_1'\sigma^2}\,, \qquad \epsilon_3 > 0\,, \quad \epsilon_1' < 0\,, \tag{36a}$$

$$\kappa_e' \approx \sqrt{-\epsilon_1' + \frac{\epsilon_2'^2}{1 - \epsilon_1'} - \epsilon_1'\sigma^2}\,, \qquad \epsilon_3 < 0\,, \quad \epsilon_1' < 0\,, \tag{36b}$$

where it has been recognized that $\kappa_0' < 0$ and $\kappa_e' > 0$ (see Fig. 8.3.2), and that the open branch is associated with the ordinary and extraordinary modes when $\epsilon_3 > 0$ and $\epsilon_3 < 0$, respectively. To approximate the factors multiplying the exponential and Hankel function in the integrands of Eqs. (25), we may put $\kappa \propto \sigma$, $\Delta \approx -\sigma^2$, thereby obtaining simple powers of σ which may be reexpressed as appropriate spatial derivatives of the remaining integrand. The resulting fields may thus be formulated as spatial derivatives of an integral that has the same form as the one in Eq. (7.3.8), and the latter may be evaluated in the closed form listed in Eq. (7.3.3b).† Since the potential function in Eq. (7.3.3b) behaves like $[rN(\theta)]^{-1} \exp[-jkrN(\theta)]$, one observes that a true field singularity exists as $\theta \to \theta_c$ [i.e., $N(\theta) \to 0$] and that the field behavior near this type of shadow boundary in the gyrotropic case is the same as in the simpler uniaxially anisotropic medium.[2] The reader is therefore referred to the detailed discussion of the latter problem in Secs. 7.3a and 7.3e. It is perhaps worth emphasizing again that the strong field and power flow singularities at $\theta = \theta_c$ disappear when losses are present, and also when a smooth extended source distribution is employed in place of the point dipole current.

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†Allowance must be made for the different time dependence $\exp(j\omega t)$ employed in this chapter. While Eq. (7.3.8) represents directly the form corresponding to Eq. (36b), one may show that the integral corresponding to Eq. (36a) leads to the same type of result.

8.3d Green's Functions for Plane-Stratified Regions

Representation in terms of ordinary and extraordinary modes

In a medium consisting of piecewise constant layers whose interfaces are perpendicular to the rectilinear coordinate z, the electromagnetic boundary conditions across each interface may be satisfied in a systematic manner in a representation comprised of the guided-wave eigenfunctions.[2,9] If the gyrotropic axis is inclined arbitrarily with respect to the z axis, the general mode functions in Sec. 8. 2 must be employed in each region; since there exists in general no mode pair $\exp(-j\kappa_\alpha z)$ and $\exp(+j\kappa_\alpha z)$ descriptive of waves propagating along the $+z$ and $-z$ directions with the same propagation constant κ_α, the analysis in layered media must be effected in a traveling-wave, rather than a standing-wave, description. However, for the special case of longitudinal orientation of the gyrotropic axis as in this section, the region becomes reflection symmetric and a standing-wave approach may be used to advantage (see Sec. 8.2d). A number of pertinent considerations are presented below.

If the region in question consists of layers, each of which is characterized by a dielectric tensor of the form shown in Eq. (3b) with $\mathbf{b}_0 \equiv \mathbf{z}_0$, the transverse electric and magnetic fields in the νth layer are represented in the form [see Eq. (16)]

$$\mathbf{E}_{t\nu}(\mathbf{r}) = \iint_{-\infty}^{\infty} [V_{o\nu}(z, \mathbf{k}_t)\mathbf{e}_{o\nu}(\mathbf{k}_t) + V_{e\nu}(z, \mathbf{k}_t)\mathbf{e}_{e\nu}(\mathbf{k}_t)] \exp(-j\mathbf{k}_t\cdot\boldsymbol{\rho})\frac{d\mathbf{k}_t}{(2\pi)^2}, \tag{37a}$$

$$\mathbf{H}_{t\nu}(\mathbf{r}) = \iint_{-\infty}^{\infty} [I_{o\nu}(z, \mathbf{k}_t)\mathbf{h}_{o\nu}(\mathbf{k}_t) + I_{e\nu}(z, \mathbf{k}_t)\mathbf{h}_{e\nu}(\mathbf{k}_t)] \exp(-j\mathbf{k}_t\cdot\boldsymbol{\rho})\frac{d\mathbf{k}_t}{(2\pi)^2}, \tag{37b}$$

where $\mathbf{e}_{\alpha\nu}$ and $\mathbf{h}_{\alpha\nu}$, ($\alpha = o, e$), are the eigenfunctions in Eqs. (19) and $V_{\alpha\nu}$, $I_{\alpha\nu}$ are the modal voltage and current which satisfy the transmission-line equations (16a). Let us suppose that the νth layer is separated from the μth layer by a plane interface at $z = z_\nu$; then the required continuity of $\mathbf{E}_t$ and $\mathbf{H}_t$ across z_ν may be satisfied if

$$V_{o\nu}\mathbf{e}_{o\nu} + V_{e\nu}\mathbf{e}_{e\nu} = V_{o\mu}\mathbf{e}_{o\mu} + V_{e\mu}\mathbf{e}_{e\mu}\,, \tag{38a}$$

$$I_{o\nu}\mathbf{h}_{o\nu} + I_{e\nu}\mathbf{h}_{e\nu} = I_{o\mu}\mathbf{h}_{o\mu} + I_{e\mu}\mathbf{h}_{e\mu}\,. \tag{38b}$$

In view of the orthogonality relation in Eqs. (21), Eqs. (38) may be reduced to the matrix form

$$\begin{pmatrix} V_\nu \\ I_\nu \end{pmatrix} = T_z\begin{pmatrix} V_\mu \\ I_\mu \end{pmatrix}, \quad V_\nu \to \begin{pmatrix} V_{o\nu} \\ V_{e\nu} \end{pmatrix}, \quad \text{etc.}, \tag{39}$$

where V_i, I_i, ($i = \nu, \mu$), are column vectors while T_z is the impedance transfer matrix

$$T_z \to \begin{pmatrix} t_{\nu\mu} & 0 \\ 0 & \hat{t}_{\nu\mu} \end{pmatrix}, \tag{39a}$$

whose elements $t_{\nu\mu}$, $\hat{t}_{\nu\mu}$ are themselves 2×2 matrices:

$$t_{\nu\mu} \rightarrow \begin{pmatrix} \mathbf{h}_{o^*\nu}^* \times \mathbf{z}_0 \cdot \mathbf{e}_{o\mu} & \mathbf{h}_{o^*\mu}^* \times \mathbf{z}_0 \cdot \mathbf{e}_{e\mu} \\ \mathbf{h}_{e^*\nu}^* \times \mathbf{z}_0 \cdot \mathbf{e}_{o\mu} & \mathbf{h}_{e^*\mu}^* \times \mathbf{z}_0 \cdot \mathbf{e}_{e\mu} \end{pmatrix}, \tag{39b}$$

$$\hat{t}_{\nu\mu} \rightarrow \begin{pmatrix} \mathbf{e}_{o^*\nu}^* \cdot \mathbf{h}_{o\mu} \times \mathbf{z}_0 & \mathbf{e}_{o^*\nu}^* \cdot \mathbf{h}_{e\mu} \times \mathbf{z}_0 \\ \mathbf{e}_{e^*\nu}^* \cdot \mathbf{h}_{o\mu} \times \mathbf{z}_0 & \mathbf{e}_{e^*\nu}^* \cdot \mathbf{h}_{e\mu} \times \mathbf{z}_0 \end{pmatrix}. \tag{39c}$$

Since the submatrices $t_{\nu\mu}$ and $\hat{t}_{\nu\mu}$ are generally not diagonal, the ordinary and extraordinary modes are coupled at the interface.

In the interior of each slab region, the modal network comprises two uncoupled transmission lines representative of the o and e modes, respectively (see Fig. 8.3.5). The impedance transfer matrix $\hat{T}_z$ for a slab of length d is readily constructed from the transmission-line solution in Sec. 2.4:

$$\hat{T}_z \rightarrow \begin{pmatrix} \cos(k\kappa'd) & jZ\sin(k\kappa'd) \\ jY\sin(k\kappa'd) & \cos(k\kappa'd) \end{pmatrix}, \tag{40}$$

where κ' and $Z = Y^{-1}$ represent the diagonal 2×2 matrices

$$\kappa' \rightarrow \begin{pmatrix} \kappa_o' & 0 \\ 0 & \kappa_e' \end{pmatrix}, \qquad Z \rightarrow \begin{pmatrix} Z_o & 0 \\ 0 & Z_e \end{pmatrix}. \tag{40a}$$

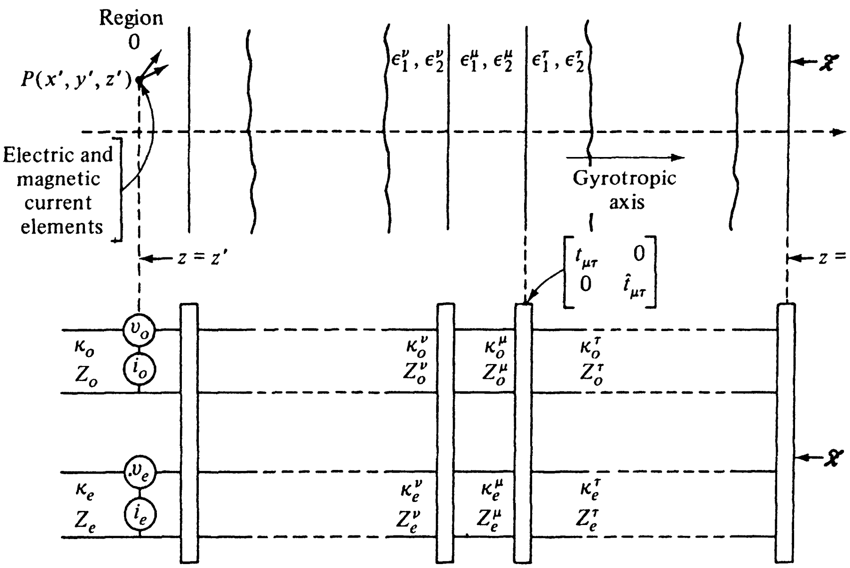


FIG. 8.3.5. Physical structure and associated network problem.

$\cos(k\kappa'd)$ is a diagonal matrix having as elements $\cos(k\kappa_o'd)$ and $\cos(k\kappa_e'd)$, respectively, with a similar interpretation applicable to $\sin(k\kappa'd)$. The voltage and current vectors V_1, I_1 at the slab face $z = z_1$ are then related to the analogous quantities at $z = z_2$ in the *same* slab via

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \hat{T}_z \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}, \qquad z_2 - z_1 = d\,, \tag{41}$$

and the transition across the interface is accomplished through use of Eq. (39). Repeated application of Eqs. (39) and (41) allows the systematic calculation of the voltage and current at any point in a plane-stratified region and leads to an overall transfer matrix given by the ordered product of the individual T_z and $\hat{T}_z$.

If the region is terminated at $z = a$ by a plane with constant anisotropic surface impedance $\mathscr{Z}$ (i. e., $\mathbf{E}_t = \mathscr{Z}\cdot\mathbf{H}_t \times \mathbf{z}_0$ at $z = a$), then by considerations analogous to those leading to Eqs. (38),

$$V = \hat{\mathscr{Z}} I\,, \tag{42}$$

where V and I are column vectors as in Eq. (39). and $\hat{\mathscr{Z}}$ is the 2×2 impedance matrix,

$$\hat{\mathscr{Z}} \rightarrow \begin{pmatrix} \mathbf{h}_{o^*}^* \times \mathbf{z}_0\cdot\mathscr{Z}\cdot\mathbf{h}_o \times \mathbf{z}_0 & \mathbf{h}_{o^*}^* \times \mathbf{z}_0\cdot\mathscr{Z}\cdot\mathbf{h}_e \times \mathbf{z}_0 \\ \mathbf{h}_{e^*}^* \times \mathbf{z}_0\cdot\mathscr{Z}\cdot\mathbf{h}_o \times \mathbf{z}_0 & \mathbf{h}_{e^*}^* \times \mathbf{z}_0\cdot\mathscr{Z}\cdot\mathbf{h}_e \times \mathbf{z}_0 \end{pmatrix}, \tag{42a}$$

the mode functions being those for the medium adjoining the boundary. One observes that even a scalar surface impedance $\mathscr{Z}_s$, for which $\mathscr{Z} = \mathbf{1}_t\mathscr{Z}_s$, generally couples the ordinary modes unless $\mathscr{Z} = 0$ (perfect conductor) or $k_t = 0$ (normally incident mode).

Since an abrupt change in the medium properties leads to a coupling between the o and e modes as in Eq. (39), it follows that these modes are coupled continuously in a gyrotropic medium with continuous variation along z. The modal representations in Eqs. (37) may then be employed only locally and the mode functions become z dependent (the discrete index ν goes over into a continuous function of z). This complication disappears in the special case of uniaxial anisotropy (Sec. 7. 2) and, of course, for an isotropic dielectric (Sec. 2.3e). For a treatment of the coupled equations, see Reference 10.

Rapid variations in the medium parameters lead to strong coupling of the o and e modes, thereby making calculation difficult in this representation. It may then be convenient to employ an alternative method wherein the modal equations are uncoupled when $\epsilon_2 = 0$, for an arbitrary z dependence of ϵ_1 and ϵ_3; mode coupling is now introduced by the off-diagonal term in the dielectric tensor in Eq. (3b). The "unperturbed" solution in this instance is evidently either the uniaxial one discussed in Sec. 7.2 ($\epsilon_1 \neq \epsilon_3$) or the isotropic one in Sec. 2. 3e ($\epsilon_1 = \epsilon_3$). In a gyrotropic plasma described by Eqs. (4), $\epsilon_2 = 0$ obtains for $\omega_c = 0$ or ∞, so a perturbation scheme about the isotropic and uniaxial cases is possible for small values of ω_c and $1/\omega_c$, respectively, for relatively arbitrary electron density variations $X(z) = \omega_p^2(z)/\omega^2$.[11]

Asymptotic evaluation of the fields

The reduction of the integral representations for the fields when the observation point is located many wavelengths from the source region may be accomplished by the techniques described in Secs. 5.5 and 7.5, and will be discussed only briefly. For example, the solution of the problem of radiation from a dipole source in a homogeneous, gyrotropic plasma half-space $z < 0$ is given in the form of integrals that, for observation points inside the plasma, include those in Eqs. (25) representing the primary field, as well as a similar set for the perturbation introduced by the interface; the integrands of the latter contain in addition the appropriate modal reflection coefficients derivable from the network considerations earlier in this section. The exponential dependence in the integrands is of the form

$$\exp\{-jk[\kappa'_{o,e}(\sigma)\,|z| + \kappa'_{o,e}(\sigma)\,|z'| + \sigma\rho]\}, \tag{43}$$

where the subscripts o and e may occur in any combination. An exponent of the same type has been investigated in connection with Eq. (7.5.12), where $q_{1,2}(\beta)$ takes the place of $\kappa'_{o,e}$; the subsequent derivation and interpretation of the saddle-point condition may be utilized also in the present case after it is recalled that the rays have a direction perpendicular to the wavenumber surface. Again, the wavenumber surfaces play an important role in providing a physical picture, as noted in Secs. 1.6 and 1.7d. If the plasma has a wavenumber surface as in diagram (f) of Fig. 8.3.2, for example, then any of the rays A through D in Fig. 8.3.6 correspond to a typical saddle point σ_i as shown. Upon striking the interface, the incident ordinary ray A in Fig 8.3.6(a) excites reflected rays C and D in the ordinary and extraordinary modes, respectively; the latter arise from saddle points in integrands with exponential dependence

$$\exp[-j\,|k|\,(\kappa'_o\,|z'| + \kappa'_o\,|z| + \sigma\rho)]$$

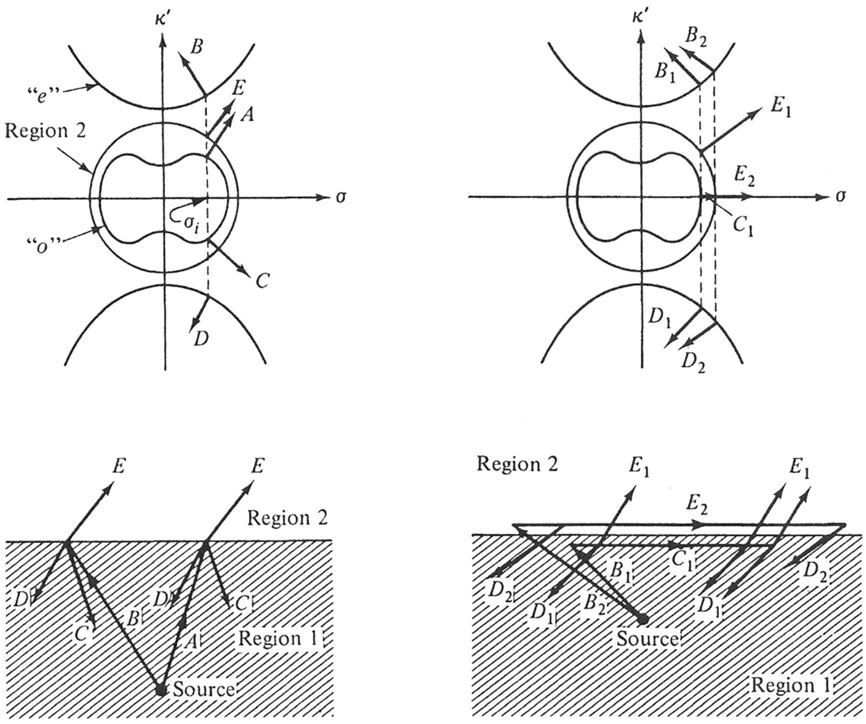
and

$$\exp[-j\,|k|\,(\kappa'_o\,|z'| + \kappa'_e\,|z| + \sigma\rho)],$$

respectively. Analogous considerations apply to the incident extraordinary ray B. When the observation point is located in the exterior isotropic half-space $z > 0$, then $k\kappa'_{o,e}(\sigma)$ multiplying $|z|$ in Eq. (43) is replaced by $\kappa = k\kappa' = \sqrt{k_0^2 - k^2\sigma^2}$ for both the E and H modes, and the refracted ray E is perpendicular to the corresponding circular diagram (see Sec. 7.5e, also Fig. 1.7.4). These simple ray pictures, which may also be constructed when the interface is inclined with respect to the z axis, provide a direct insight into the reflection and refraction mechanism pertaining to confined source distributions; the reflected and transmitted ray amplitudes emerge from the amplitude coefficients in the asymptotic approximation. The diagram may also be employed to predict the occurrence of lateral rays that are associated intimately with the phenomenon of total reflection (see Fig. 1.7.7). It is especially interesting to note that in addition to the lateral rays traveling on the vacuum side of the interface [ray path $B_2E_2D_2$ in Fig. 8.3.6(b)], such rays may also occur on the plasma side in view of the mode coupling at the boundary. When σ_i in Fig. 8.3.6(a) moves to the right until it coincides with the σ intercept of the o branch, the incident extraordinary ray B_1 excites an ordinary ray C_1 traveling parallel to the boundary which, in turn, sheds extraordinary rays D_1 back into the medium and refracted rays E_1 into the vacuum region [Fig. 8.3. 6(b)]. It should also be mentioned that the diversity of refractive index profiles in Fig. 8.3.2 admits of focusing phenomena more varied than those described in Sec. 7.5e. In addition to the focusing arising from the *e* branch in Fig. 8.3.6, a similar effect occurs due to those portions of the *o* branch having a curvature opposite to that of the circular contour representing the vacuum half-space.

These procedures, applying to an abrupt change in the medium properties, may also be utilized to chart the progress of a ray in a slowly varying medium, as noted in Sec. 1.6. In this instance, one approximates the medium by piecewise constant layers, draws the appropriate wavenumber or refractive index diagrams, and determines the trajectory of the continuously refracted rays as above (internal reflection is neglected in a lowest-order approximation).

Although these simple considerations provide an interpretation of the saddle-point contributions to the integrals, and also of the lateral-wave effects ascribed to branch-point singularities, they do not account for wave types arising from pole singularities that may be crossed during the deformation of the integration path into the steepest-descent contour. Because of the complicated dependence of $\kappa_{o,e}$ on σ, the determination of pole singularities in the modal reflection coefficients poses a formidable task and must generally be accomplished by numerical means (see Secs. 8.4b and 8.4c for a simple example).



(a) Reflected rays (b) Lateral rays

FIG. 8.3.6 Ray configurations.

8.4 GUIDED WAVES IN A COLD MAGNETOPLASMA (GUIDE AXIS PERPENDICULAR TO GYROTROPIC AXIS)

As discussed in Sec. 8.2h, the analysis of boundary-value problems in a gyrotropic medium is in general quite involved. However, for special geometries, medium parameters, and excitation, the analysis becomes quite tractable and may be effected by transmission-line methods. For example, it was found in Secs. 7.2b and 7.4b that a certain class of boundary value problems in a uniaxially anisotropic medium can be reduced to equivalent problems in an isotropic environment. Similar reductions occur in a gyrotropic medium if the fields are independent of the coordinate parallel to the gyrotropic axis (i.e., if wave propagation is perpendicular to this axis). To explore this latter class of problems, we shall first ascertain the possible guided waves in a gyrotropic medium with the gyrotropic axis $\mathbf{b}_0$ perpendicular to the guiding direction $\mathbf{z}_0$. In this guide structure the problems in question then correspond to the special case of waves with vector wavenumbers $\mathbf{k}$ in the plane containing $\mathbf{z}_0$ and perpendicular to $\mathbf{b}_0$; however, other waves also exist in this structure, with wavenumbers $\mathbf{k}$ in the plane formed by $\mathbf{b}_0$ and $\mathbf{z}_0$, for instance.

8.4a Eigenfunctions and Eigenvalues for $\mathbf{b}_0$ Perpendicular to $\mathbf{z}_0$

In a waveguide with $\mathbf{b}_0$ perpendicular to $\mathbf{z}_0$, the guided wavevectors Ψ_α and eigenvalues κ_α, defined in Secs. 1.4, and 8.2. and 8.3, are strongly dependent on the orientation φ of the gyrotropic axis $\mathbf{b}_0 = \mathbf{k}_{t0} \cos\varphi + \hat{\mathbf{k}}_{t0} \sin\varphi$ in the plane transverse to the guide axis $\mathbf{z}_0$. The notation is the same as that in the reduced electromagnetic description of a gyrotropic medium with $\mathbf{b}_0$ parallel to $\mathbf{z}_0$ and discussed in Sec. 8.3a. The dependence of Ψ_α and κ_α on φ can be ascertained by solution of the eigenvalue problem (8.3.1a) via the method indicated in Eqs. (8.3.2)–(8.3.6). For $\mathbf{b}_0$ perpendicular to $\mathbf{z}_0$, the $\mathbf{E}_\alpha$ component of Ψ_α can be derived from Eq. (8.3.5a), evaluated in the $\mathbf{z}_0$, $\mathbf{k}_{t0}$, $\hat{\mathbf{k}}_{t0}$ basis defined by Eq. (8.3.8) (assuming $\omega \neq 0$):

$$\begin{bmatrix} \epsilon_1 - \dfrac{k_t^2}{k_0^2} & \dfrac{\kappa k_t}{k_0} + j\epsilon_2 \sin\varphi & -j\epsilon_2 \cos\varphi \\ \dfrac{\kappa k_t}{k_0^2} - j\epsilon_2 \sin\varphi & \epsilon_1 \sin^2\varphi + \epsilon_3 \cos^2\varphi - \dfrac{\kappa^2}{k_0^2} & (\epsilon_1 - \epsilon_3)\sin\varphi\cos\varphi \\ j\epsilon_2 \cos\varphi & (\epsilon_1 - \epsilon_3)\sin\varphi\cos\varphi & \epsilon_1 \cos^2\varphi + \epsilon_3 \sin^2\varphi - \dfrac{k_t^2 + \kappa^2}{k_0^2} \end{bmatrix} \cdot \begin{bmatrix} E_z \\ E_{t'} \\ E_{t''} \end{bmatrix} = 0 \qquad (1)$$

with the notation and procedure the same as that employed in Eq. (8.3.9) et seq., for the case $\mathbf{b}_0$ parallel to $\mathbf{z}_0$.

Non-vanishing solutions of Eq. (1) exist for those values $\kappa = \kappa_\alpha$ that satisfy the determinental equation [note that the present normalization $\kappa' = \kappa/k_0$, $k_t' = k_t/k_0$ differs from that in Eq. (8.3.10)]:

$$\kappa'^4 - \left[\epsilon_\perp + \epsilon_3 - 2k_t'^2 + \left(1 - \frac{\epsilon_3}{\epsilon_1}\right) k_t'^2 \cos^2\varphi\right]\kappa'^2 + (\epsilon_\perp - k_t'^2)(\epsilon_3 - k_t'^2)$$
$$- \cos^2\varphi\left[4(\epsilon_1 - \epsilon_3)\sin^2\varphi - (\epsilon_\perp - \epsilon_3)k_t'^2 + \left(1 - \frac{\epsilon_3}{\epsilon_1}\right)k_t'^4\right] = 0, \tag{2a}$$

which for $\varphi = \pi/2$ becomes

$$[\kappa'^2 - (\epsilon_\perp - k_t'^2)]\,[\kappa'^2 - (\epsilon_3 - k_t'^2)] = 0, \tag{2b}$$

and for $\varphi = 0$ becomes

$$\kappa'^4 - \left[\epsilon_\perp + \epsilon_3 - \left(1 + \frac{\epsilon_3}{\epsilon_1}\right)k_t'^2\right]\kappa'^2 + \frac{\epsilon_3}{\epsilon_1}\left[\left(\epsilon_1 - k_t'^2\right)^2 - \epsilon_2^2\right] = 0 \tag{2c}$$

with $\epsilon_\perp = (\epsilon_1^2 - \epsilon_2^2)/\epsilon_1$, and $\epsilon_{1,2,3}$ defined in Eq. (8.3.4).

The wavevectors Ψ_α corresponding to the roots $\kappa = \kappa_\alpha$ of Eq. (2) are somewhat complicated for arbitrary orientation φ of the gyrotropic axis in the transverse plane. For simplicity we consider first the case $\varphi = \pi/2$, wherein the gyrotropic axis is perpendicular to both the guide axis $\mathbf{z}_0$ and the wave direction $\mathbf{k}_0$. For prescribed $\mathbf{k}_t$, ω, one obtains from Eq. (1) and Eq. (8.3.6) wavevectors Ψ_α of the same form as in Eq. (8.3.13a). For one type of mode one finds

$$\begin{aligned} \mathbf{e}_\alpha &= \hat{\mathbf{k}}_{t0}, & \mathbf{e}_{z\alpha} &= 0, \\ \mathbf{h}_\alpha &= -\mathbf{k}_{t0}, & \mathbf{h}_{z\alpha} &= \frac{k_t}{\kappa_\alpha}\mathbf{z}_0, \end{aligned} \tag{3a}$$

with the characteristic mode impedance Z_α and wavenumber κ_α given by

$$Z_\alpha = \frac{\omega\mu_0}{\kappa_\alpha}, \qquad \kappa_\alpha = \pm\sqrt{k_0^2\epsilon_3 - k_t^2}. \tag{3b}$$

This is an ordinary H-type mode [see Eq. (2.2.15d)], with electric field parallel to $\mathbf{b}_0$, appropriate to propagation in a medium with relative dielectric constant ϵ_3, the latter being given in Eq. (8.3.4). Since $\kappa_{\pm\alpha} = \pm\kappa_\alpha$, $\mathbf{e}_\alpha = \mathbf{e}_{-\alpha} = \mathbf{e}_{\alpha^\bullet}$, $\mathbf{h}_\alpha = \mathbf{h}_{-\alpha} = \mathbf{h}_{\alpha^\bullet}$, and $Z_{\alpha^\bullet}^* = Z_\alpha$ for these modes, the transmission-line description is conventional.

There also exists another type of mode for $\varphi = \pi/2$, again obtained from Eqs. (1) and (8.3.6), with

$$\begin{aligned} \mathbf{e}_\alpha &= \mathbf{k}_{t0}, & \mathbf{e}_{z\alpha} &= \frac{\kappa_\alpha k_t + jk_0^2\epsilon_2}{k_t^2 - k_0^2\epsilon_1}\mathbf{z}_0, \\ \mathbf{h}_\alpha &= \mathbf{k}_{t0}, & \mathbf{h}_{z\alpha} &= 0, \end{aligned} \tag{4a}$$

and with characteristic impedance and wavenumber given by

$$Z_\alpha = \frac{\kappa_\alpha - jk_t\epsilon_2/\epsilon_1}{\omega\epsilon_0\epsilon_\perp}, \qquad \kappa_\alpha = \pm\sqrt{k_0^2\epsilon_\perp - k_t^2}. \tag{4b}$$

In this second type of polarization, the "extraordinary" mode, the gyrotropic axis $\mathbf{b}_0$ is parallel to the a.c. magnetic field. Although $\kappa_{\pm\alpha} = \pm\kappa_\alpha$, $\mathbf{e}_\alpha = \mathbf{e}_{-\alpha} = \mathbf{e}_{\alpha^*}$, and $\mathbf{h}_\alpha = \mathbf{h}_{-\alpha} = \mathbf{h}_{\alpha^*}$ for this mode type, the impedance $Z^*_{\alpha^*} \neq Z_\alpha$ and hence the general transmission-line formalism of Eqs. (8.2.43) and (8.2.46) must be employed; in connection with the latter one notes that

$$\begin{aligned} Z^*_{\alpha^*} &= Z^*_\alpha, \quad & Z^*_{-\alpha^*} &= -Z_\alpha \quad & &\text{for } \kappa_\alpha \text{ real,} \\ Z^*_{\alpha^*} &= -Z_{-\alpha}, \quad & Z^*_{-\alpha^*} &= -Z_\alpha \quad & &\text{for } \kappa_\alpha \text{ imaginary.} \end{aligned} \tag{4c}$$

For given $\mathbf{k}_t$, ω, the four Ψ_α wavevectors defined in Eqs. (3a) and (4a) possess the orthonormality properties (8.2.40) and also $\mathbf{e}_\alpha = \mathbf{h}_\alpha \times \mathbf{z}_0$.

For $\mathbf{b}_0 = \mathbf{k}_{t0}$ (i.e., $\varphi = 0$) the mode vectors $\mathbf{e}_\alpha$ and $\mathbf{h}_\alpha$ can again be determined via the general procedure outlined in Sec. 8.3a. However, in this case, $\mathbf{e}_\alpha \neq \mathbf{e}_{-\alpha}$ and $\mathbf{h}_\alpha \neq \mathbf{h}_{-\alpha}$, whence no simple transmission-line (impedance) formalism is apparent under this circumstance. We shall relegate explicit evaluation of these mode vectors to the Problems section.

8.4b Two-dimensional Boundary-value Problems in Gyrotropic Media

Simplifications in the solution of boundary-value problems in a gyrotropic medium occur for certain choices of excitation and geometry. Such is the case if the fields are independent of the coordinate parallel to the gyrotropic direction $\mathbf{b}_0$ and the medium contains planar or cylindrical scatterers with planes or cylindrical axes parallel to $\mathbf{b}_0$. It is then convenient to regard the system as a waveguide with axis $\mathbf{z}_0$ perpendicular to $\mathbf{b}_0$, whence, as noted in Sec. 8.4a, only the $\varphi = \pi/2$ modes are necessary for the field description. These waves have two distinctive polarizations and can be separately excited, depending on the source orientation.

The ordinary waves, whose wavevectors are indicated in Eq. (3a), have their electric field polarized along the gyrotropic direction $\mathbf{b}_0 \equiv \hat{\mathbf{k}}_{t0}$. The propagation characteristics of these waves are dependent only on ϵ_3 and are independent of the anisotropy transverse to $\mathbf{b}_0$. In the presence of sources, boundaries, or obstacles that do not generate other electric field components, radiation and scattering problems reduce to conventional ones in an isotropic medium with wavenumber $k_0\sqrt{\epsilon_3}$. From Eqs. (3a) and (8.2.28b), it is seen that electric line currents parallel to $\mathbf{b}_0$ excite only these ordinary waves, as do scatterer configurations in the form of arbitrarily shaped perfectly conducting cylinders whose axes are aligned along $\mathbf{b}_0$.

The extraordinary waves described by Eq. (4a) have their a.c. magnetic field polarized along the gyrotropic axis $\mathbf{b}_0 \equiv \hat{\mathbf{k}}_{t0}$. In consequence, wave propagation is now affected by the anisotropy transverse to $\mathbf{b}_0$, but in a sufficiently simple manner to make certain propagation characteristics analogous to isotropic ones, while others exhibit strikingly the influence of anisotropy. Because of the relative simplicity of this class of radiation and scattering problems, having only a single magnetic field component, let us consider scattering of waves from a magnetic line current oriented parallel to $\mathbf{b}_0 \equiv \hat{\mathbf{k}}_{t0}$.

From Eqs. (4a) and (8.2.28b) one verifies that only extraordinary waves are excited in this case; there is no electric field along $\mathbf{b}_0$ and hence only the transverse-to-$\mathbf{b}_0$ part of the dielectric dyadic is effective.

It is instructive to contrast the reduced Maxwell field and transmission-line descriptions for the case of excitation from a magnetic line current.[12] Let us choose a rectangular x, y, z coordinate system oriented such that $\mathbf{x}_0 = \mathbf{k}_{t0}$, $\mathbf{y}_0 = \mathbf{b}_0 = \hat{\mathbf{k}}_{t0}$, with $\mathbf{z}_0$ along the guide axis. Also, let $\mathbf{M} = M\mathbf{y}_0$ be the magnetic line current density, whence $\partial/\partial y \equiv 0$, and denote the non-vanishing field components by $H_{t''} = H_y = H$, by $E_{t'} = E_x$ and by E_z. On elimination of E_z, the reduced Maxwell field equations (8.2.1), with $\boldsymbol{\epsilon}$ given as in Eq. (8.3.3b), can then be written in the form

$$\begin{aligned}\frac{\partial E_x}{\partial z} &= -j\omega\mu_0\Big(1 + \frac{1}{k_0^2\epsilon_1}\frac{\partial^2}{\partial x^2}\Big)H - j\frac{\epsilon_2}{\epsilon_1}\frac{\partial}{\partial x}E_x - M,\\ \frac{\partial H}{\partial z} &= j\frac{\epsilon_2}{\epsilon_1}\frac{\partial}{\partial x}H - j\omega\epsilon_0\epsilon_\perp E_x,\end{aligned} \tag{5a}$$

and on elimination of E_x as

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_0^2\epsilon_\perp\Big)H = j\omega\epsilon_0\epsilon_\perp M, \tag{5b}$$

where $\epsilon_\perp = (\epsilon_1^2 - \epsilon_2^2)/\epsilon_1$ is the effective relative dielectric constant for the wave propagation. While Eq. (5b) is the same as for an isotropic medium with relative permittivity $\epsilon_\perp$, the effect of anisotropy is evident from the presence of the ϵ_2 dependent terms in Eqs. (5a). It is not difficult to verify from Eq. (8.3.3b) that for a cold electron plasma, $\epsilon_\perp$ has the following behavior as a function of frequency:

$$\begin{aligned}&\epsilon_\perp < 0 \qquad \text{when } \omega < \omega_4 \text{ and } \omega_2 < \omega < \omega_1,\\ &\epsilon_\perp > 0 \qquad \text{where } \omega_4 < \omega < \omega_2 \text{ and } \omega > \omega_1,\end{aligned} \tag{6a}$$

where (see Fig. 8.3.2)

$$\omega_{4,1} = \frac{\sqrt{\omega_c^2 + 4\omega_p^2} \mp \omega_c}{2}, \qquad \omega_2 = \sqrt{\omega_c^2 + \omega_p^2}. \tag{6b}$$

For the above magnetic line current excitation, there also exists a transmission-line description of the z dependence of the extraordinary wave fields. This description is particularly appropriate for wave fields with an x dependence $\exp(-jk_t x)$ and for plane-stratified scatterers with axis of stratification along z. As indicated in the general transmission-line formalism of Sec. 8.2h, one describes the αth-mode behavior in terms of a (voltage) reflection coefficient $\Gamma_\alpha(z)$ which is related to the impedance $Z_\alpha(z)$ at some scattering plane at z by Eq. (8.2.43). For given k_t, ω, the extraordinary modes have characteristic impedances given by Eqs. (4b) and (4c), whence by Eq. (8.2.43) the reflection coefficient, looking in the direction of increasing z, becomes for propagating modes with real κ_α,

$$\Gamma_\alpha(z) = \frac{Z_\alpha(z) - Z_\alpha}{Z_\alpha(z) + Z_\alpha^*}. \tag{7}$$

For example, if there exists a perfectly conducting plane at $z = 0$ so that $Z_\alpha(0) = 0$, then by Eqs. (4b) and (7), the reflection coefficients $\overrightarrow{\Gamma}_\alpha$, $\overleftarrow{\Gamma}_\alpha$ at $z = 0$, looking in the positive and negative direction, respectively, are

$$\overrightarrow{\Gamma}_\alpha(0) = -\frac{Z_\alpha}{Z_\alpha^*} = -\frac{1 + j(k_t\epsilon_2/\kappa_\alpha\epsilon_1)}{1 - j(k_t\epsilon_2/\kappa_\alpha\epsilon_1)}, \quad \overleftarrow{\Gamma}_\alpha(0) = -\frac{Z_{-\alpha}}{Z_{-\alpha}^*} = -\frac{1 - j(k_t\epsilon_2/\kappa_\alpha\epsilon_1)}{1 + j(k_t\epsilon_2/\kappa_\alpha\epsilon_1)}, \tag{8}$$

whence by an equation of the form (8.2.45), the reflection coefficient at any plane z can be readily ascertained. The non-symmetric results in Eqs. (8) for the reflection coefficient of a perfectly conducting plane in a gyrotropic medium are to be contrasted for an isotropic medium with the single symmetric reflection coefficient of value -1. Furthermore, the requirement $E_x = 0$ on a perfectly conducting plane at $z = 0$ is seen from Eq. (5a) to imply a relation between $\partial H/\partial z$ and $\partial H/\partial x$ that contrasts with the $\partial H/\partial z = 0$ requirement in an isotropic medium. In this case the effect of medium anisotropy produces non–reciprocal behavior since $\Gamma_\alpha(0)$ is not an even function of k_t (i.e., waves with transverse periodicity described by $-k_t$ are reflected differently from those with $+ k_t$).

The above transmission analysis in terms of z-guided waves is no longer useful when the scatterer is a perfectly conducting cylinder of arbitrary cross section, with axis oriented parallel to $\mathbf{b}_0$ and to the magnetic line current direction. In this case, an incident wave with prescribed k_t is then coupled to an infinity of scattered waves with $-\infty < k_t < \infty$. Even if the scattering surface S lies on a coordinate surface $u =$ constant in an orthogonal (u, v) coordinate system that renders the wave equation (5b) separable, and if the incident field is represented in terms of v-dependent eigenfunctions, the boundary condition $E_v = 0$ on S will generally introduce coupling between the modes. An exception occurs for a circular cylinder if the azimuthal eigenfunctions are chosen as $\exp(-jn\phi)$, $n = 0, \pm 1, \pm 2, \ldots.$

8.4c Radiation from a Magnetic Line Source in the Presence of a Perfectly Conducting Plane

To exhibit the effects of anisotropy alluded to in Sec. 8.4b, we shall consider in more detail the problem of radiation from a y-directed magnetic line source in the presence of a perfectly conducting plane boundary at $z = 0$ (Fig. 8.4.1). The entire configuration is embedded in a cold plasma, with the external magnetic field axis $\mathbf{b}_0$ parallel to y. The transverse electric-field component E_x is obtainable from the single magnetic-field component $H_y \equiv H$ from the second of Eqs. (5a), while the longitudinal component E_z follows in a similar manner from $E_z = (j\omega\epsilon_0\epsilon_\perp)^{-1}[j(\epsilon_2/\epsilon_1)(\partial H/\partial z) + (\partial H/\partial x)]$. H satisfies Eq. (5b) with $M = \delta(x)\delta(z - z')$, subject on $z = 0$ to the boundary condition

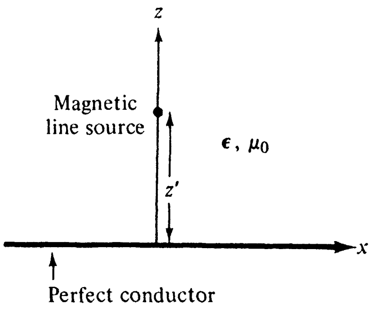


FIG. 8.4.1 Line source and perfectly conducting plane.

$E_x = 0$ [seeEq. (5a)], whence

$$\frac{\partial H}{\partial z} - j\frac{\epsilon_2}{\epsilon_1}\frac{\partial H}{\partial x} = 0 \qquad \text{at } z = 0. \tag{9}$$

In an unbounded medium, the magnetic Green's function solution H_0 is given by [see Eqs. (5.4.26a) and (5.4.40)]

$$H_0(\mathbf{r}, \mathbf{r}') = -\frac{\omega\epsilon_\perp}{4} H_0^{(2)}(k\sqrt{x^2 + (z - z')^2}), \qquad \mathbf{r} = (x, z), \tag{10}$$

with $k = k_0\sqrt{\epsilon_\perp}$. When $\epsilon_\perp < 0$, no radiation occurs. Since propagation is transverse to the direction of the gyrotropic axis (corresponding to $\kappa' = 0$ in Fig. 8.3.2), the wavevector and ray directions coincide in this class of problems. One may also verify that the ray direction is radially outward from the source point $\mathbf{r}'$. It is not difficult to ascertain that the boundary conditions for a perfectly conducting plane are not satisfied by the insertion of an image source at $(-z', 0)$, so it is necessary to employ a modal representation. In view of the translational invariance along x, the transverse dependence of a modal field is $\exp(-j\xi x)$, $-\infty < \xi < \infty$, whence $H(\mathbf{r}, \mathbf{r}')$ can be represented as

$$H(\mathbf{r}, \mathbf{r}') = \int_{-\infty}^{\infty} I(z, z'; \xi)\frac{e^{-j\xi x}}{2\pi}\, d\xi, \tag{11}$$

where $\xi \equiv k_t$ and the 2π factor has been included for normalization. Insertion into Eq. (5b) and use of the Fourier integral representation for the delta function $\delta(x) = (1/2\pi)\int_{-\infty}^{\infty} \exp(-j\xi x)\, d\xi$ yields

$$\frac{d^2 I}{dz^2} + (k^2 - \xi^2)I = j\omega\epsilon_\perp\delta(z - z'), \qquad k = k_0\sqrt{\epsilon_\perp}, \tag{12a}$$

subject to the boundary condition

$$\epsilon_2\xi I - \epsilon_1\frac{dI}{dz} = 0 \qquad \text{at } z = 0, \tag{12b}$$

and a radiation condition at $z \to \infty$. With $\kappa = \sqrt{k^2 - \xi^2}$ defined as the modal propagation constant, these equations are in the standard transmission line form. By the traveling-wave techniques described in Chapter 2 [see Eq. (2.4.23b)], the solution for I may be written as

$$I = -j\omega\epsilon_\perp \frac{e^{-j\kappa|z-z'|} - \Gamma e^{-j\kappa(z+z')}}{2j\kappa}, \qquad \kappa = \sqrt{\kappa^2 - \xi^2}, \quad \operatorname{Im}\kappa \leq 0, \tag{13}$$

where use has been made of the one-dimensional Green's function for the unbounded z-domain in Eq. (5.4.7). The modal reflection coefficient Γ is given by Eq. (8),

$$\Gamma = -\frac{1 - (\xi\epsilon_2/j\kappa\epsilon_1)}{1 + (\xi\epsilon_2/j\kappa\epsilon_1)}. \tag{13a}$$

The present problem is similar to that in Sec. 5.7, which deals with the radiation from sources in an isotropic medium in the presence of a plane surface having a constant surface impedance, the difference between the two problems arising from the functional form of Γ in Eq. (13a). Thus, the asymptotic analysis required in the derivation of explicit far-field formulas is essentially the same as in Sec. 5.7b. As noted previously, an important feature is introduced by the behavior $\Gamma(\xi) \neq \Gamma(-\xi)$, implying that waves with identical spatial periodicity, but incident on opposite sides of the surface normal, are affected unequally by the boundary. This non-reciprocal characteristic is ascribable to the gyrotropic properties of the medium.

Unidirectional surface wave

Since $\Gamma(\xi)$ in Eq. (13a) has the form corresponding to an inductive surface impedance in an isotropic medium, one expects from Sec. 5.7 to find a surface wave that decays exponentially with z and propagates undamped in the x direction. Indeed, the reflection coefficient Γ has poles ξ_p when

$$\sqrt{k^2 - \xi_p^2}\,\epsilon_1 = j\xi_p\epsilon_2, \tag{14}$$

which equation may easily be solved by squaring to yield $\xi_p = \pm k_0\sqrt{\epsilon_1}$. To obtain a surface-wave solution with real ξ_p, ϵ_1 must be positive and in that case, $\xi_p^2 > k^2$, so $\kappa_p = -j|\kappa_p|$ is imaginary. The algebraic sign of ξ_p is therefore identical with that of $-\epsilon_2/\epsilon_1$ and, since $\epsilon_1 > 0$, $\operatorname{sgn}\xi_p = -\operatorname{sgn}\epsilon_2$; the sign of ϵ_2 may be changed by reversing the direction of the d.c. magnetic field impressed on the plasma [see Eq. (8.3.4)]. Thus,

$$\xi_p = \pm k_0\sqrt{\epsilon_1} \qquad \text{when } \epsilon_2 \lessgtr 0, \quad \epsilon_1 > 0. \tag{15}$$

If a small amount of dissipation is present, then $\operatorname{Im}\sqrt{\epsilon_1} < 0$, thereby fixing the pole position with respect to the integration path. In contrast to the situation in isotropic configurations, only a single pole exists instead of the customary symmetrical pair at $\pm\xi_p$ as in Sec. 5.7. This characteristic leads to unidirectional wave propagation, as noted below.[12]

An examination of the elements of the dielectric tensor shows that $\epsilon_1 = (\omega^2 - \omega_p^2 - \omega_c^2)(\omega^2 - \omega_c^2)^{-1}$ is positive when $\omega > \sqrt{\omega_p^2 + \omega_c^2}$ or $\omega < \omega_c$, where ω_c and ω_p denote the electron cyclotron and plasma frequencies, respectively, and that, in particular,

$$\begin{aligned} &\text{(a)} \quad 0 < \epsilon_1 < 1 \quad \text{when } \omega > \sqrt{\omega_p^2 + \omega_c^2}; \\ &\text{(b)} \quad \epsilon_1 > 1 \qquad \text{when } \omega < \omega_c. \end{aligned} \tag{16}$$

The surface-wave phase velocity $c/\sqrt{\epsilon_1}$ along the y direction parallel to the plane is therefore greater than the velocity c of light in vacuum (fast wave) in case (a), but less than c (slow wave) in case (b). These observations are of interest in connection with the problem of radiation from a line source embedded in a magneto-plasma sheath of finite thickness, in which instance the perturbed slow wave of case (b) remains a surface wave while the perturbed fast wave corresponding to case (a) becomes a leaky wave that may strongly influence the radiation pattern; a fast wave distribution radiates energy away from the boundary (see Sec. 5.5g). One observes also from Eqs. (5), (14), and (15) that the field components in the surface wave are

$$H \equiv H_y = C \exp\left(\pm jk_0\sqrt{\epsilon_1}\, x - k_0 |\epsilon_2| \epsilon_1^{-1/2} z\right), \tag{17a}$$

$$E_z = \pm \frac{k_0}{\omega\epsilon_0\sqrt{\epsilon_1}} H, \qquad E_x = 0, \tag{17b}$$

where C is a constant-amplitude factor and the upper and lower signs correspond to $\epsilon_2 > 0$ and $\epsilon_2 < 0$, respectively, with $\epsilon_1 > 0$. Since $E_x = 0$, the surface wave is TEM (transverse electromagnetic) with respect to the x direction. This property could have been deduced directly from the requirement that the surface wave is a solution of the source-free Maxwell field equations, satisfies the boundary conditions at $z = 0$, and has field components that behave according to $\exp[-j\xi_p x - |\kappa_p| z]$. Because E_x must vanish at $z = 0$, it must therefore vanish everywhere. It is noted that the surface wave propagates (and carries energy) only in the $-x$ direction when ϵ_2 is positive, and only in the $+x$ direction when ϵ_2 is negative, thereby exhibiting the unidirectional characteristic mentioned earlier.

A comparison of Eqs. (6) and (16) shows that the surface waves may propagate even when the medium itself cannot sustain propagation; this condition applies when $\epsilon_\perp < 0$, $\epsilon_1 > 0$ [i.e., $\omega_2 < \omega < \omega_3$ or $\omega < \min(\omega_c, \omega_1)$]. One may then conceive of a physical situation that leads to a "thermodynamic paradox" since energy is apparently absorbed by a lossless termination. Consider the configuration in Fig. 8.4.1, without the line source and with a non-absorbing boundary inserted along the $x = 0$ plane; the medium parameters are chosen so that $\epsilon_\perp < 0$ and a surface wave may propagate along the $+x$ direction. Since no propagation is possible in the medium itself nor along the plane toward $x = -\infty$, there is no mechanism to account for the energy transported toward the boundary by an incident surface wave (the lossless $x = 0$ plane may be chosen so as to be incapable of supporting a surface wave). This difficulty exists only in the absence of dissipation and disappears when slight losses are assumed. The boundary value problem is, in fact, poorly posed in the lossless case but may be treated correctly when the non-dissipative problem is derived from the lossy one in the limit of vanishing conductivity.[13,14]

The far field

For large values of $kR = k_0\sqrt{\epsilon_\perp}R$, where $R = \sqrt{x^2 + (z + z')^2}$ is the distance from the image point $(-z', 0)$ to the observation point (x, z), the scattered portion $H_s(\mathbf{r}, \mathbf{r}')$ of the integral in Eq. (11) may be evaluated by the asymptotic techniques employed elsewhere for analogous problems (see Secs. 5.3d and 7.5d). The unperturbed field H_0 in the unbounded medium is given in closed form in Eq. (10). To evaluate H_s we introduce the polar coordinates (R, θ) via $z + z' = R\cos\theta$, $x = R\sin\theta$, and also transform into the w plane according to $\xi = k\sin w$. When k is real and positive, the integration path maps into the contour $\hat{P}$ shown in Fig. 5.3.5(b), whereas for imaginary $k = -j|k|$, the contour follows the imaginary axis in the w plane (see Fig. 8.4.2). The steepest-descent

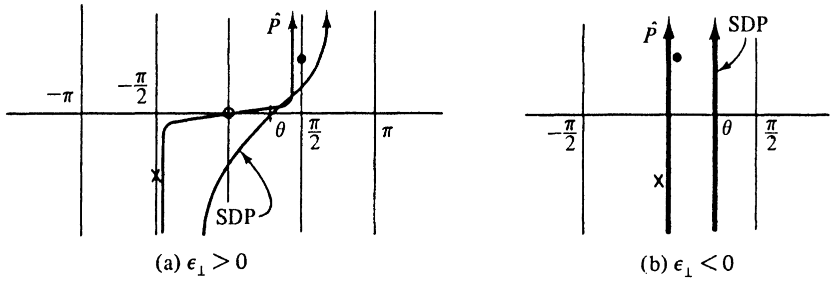


FIG. 8.4.2 Integration paths and pertinent singularities ($\epsilon_1 > 0$)
•-Pole location when $\epsilon_2 < 0$
×-Pole location when $\epsilon_2 > 0$

paths through the saddle point at $w_s = \theta$ are also shown, and one notes that when $\epsilon_2 < 0$, a surface-wave pole may be captured for sufficiently large, positive observation angles θ, but not for $\theta < 0$; the converse is true when $\epsilon_2 > 0$. Since the surface waves have been shown to possess unidirectional characteristics, with the wave going in the $+x$ direction ($\theta > 0$) when $\epsilon_2 < 0$ and in the $-x$ direction ($\theta < 0$) when $\epsilon_2 > 0$, these domains of contribution to the line-source field satisfy the radiation condition requiring the transport of energy away from the source. The asymptotic evaluation of the integral along the steepest-descent path furnishes the reflected field, which appears in this case to originate from the mirror-image point (it is worth emphasizing that the image construction is valid only in the asymptotic sense),

$$H_{SDP} \sim \frac{\omega\epsilon_\perp}{2\sqrt{2\pi}}\Gamma(k\sin\theta)\frac{e^{-j(kR-\pi/4)}}{\sqrt{kR}} + O\left(\frac{1}{(kR)^{3/2}}\right), \tag{18}$$

where

$$\Gamma(k\sin\theta) = \frac{\epsilon_2\sin\theta - j\epsilon_1\cos\theta}{\epsilon_2\sin\theta + j\epsilon_1\cos\theta}. \tag{18a}$$

The total far-field approximation is then given by

$$H \sim H_0 + H_{SDP} + U(\theta - \theta_p)\text{ (surface wave)}, \quad |kR| \gg 1,\ \theta \not\approx \theta_p, \tag{19}$$

where θ_p is the observation angle at which the steepest-descent path crosses the pole and $U(\theta - \theta_p)$ is the Heaviside unit function. This simple formula must be modified in accord with the procedure in Sec. 4.4 if one wishes to include as well observation angles $\theta \approx \theta_p$. When k is imaginary, both the primary and reflected fields are evanescent, but real energy may be extracted from the source by the surface waves.

While attention has already been called to the non-reciprocal behavior exhibited by the surface-wave contribution, the same phenomenon may be observed in connection with the reflected field H_{SDP} since Γ is not an even function of θ. $H(\mathbf{r}, \mathbf{r}')$ is therefore not equal to $H(\mathbf{r}', \mathbf{r})$. An examination of Γ reveals, however, that the simultaneous changes $\theta \to -\theta$, $\epsilon_2 \to -\epsilon_2$, leave the reflection coefficient unchanged; the pole configuration in Fig. 8.4.2 is likewise symmetrized when a reversal in the sign of θ is made jointly with a corresponding modification of ϵ_2. From these considerations, the fields at P in the two configurations depicted in Fig. 8.4.3 are seen to be equivalent, and since a sign change in ϵ_2 produces the transposed tensor $\boldsymbol{\epsilon}$ in Eq. (1.5.20) [see Eq. (8.3.3b)], reciprocity is found to be satisfied if an interchange of source and observation points is accompanied by insertion of the transposed medium. The general conclusions derived in Sec. 1.5b are therefore verified. In the plasma medium under consideration, the transposed medium corresponds to a reversal in the direction of the externally impressed magnetic field.

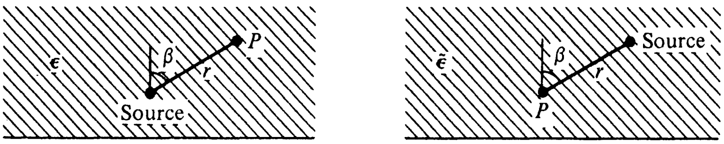


FIG. 8.4.3 Equivalent problems.

The preceding result is relevant for antenna pattern measurements in an anisotropic region. A customary technique in an isotropic environment is to determine the radiation pattern by measuring the energy received by the antenna from plane waves incident at various angles. Use of this procedure in a gyrotropic region requires as well the transposition of the medium parameters.[15]

8.4d Diffraction by a Half-plane

If the perfectly conducting boundary in Fig. 8.4.1(a) extends only from $x = 0$ to $x = \infty$, with the remainder of space filled with the same anisotropic medium, the resulting boundary-value problem may not be solved by separation of variables but is amenable to treatment by the Wiener–Hopf technique.[16] The details are omitted here and only some physical characteristics of the solution are mentioned. If conditions are adjusted so that a surface wave may propagate along the top face toward $x = -\infty$, then it follows that the bottom face of the screen supports a wave traveling toward $x = +\infty$ (rotate the configuration in Fig. 8.4.1 by 180° about the y axis). These are the only possible wave types and no propagation occurs along the top and bottom faces toward $x = \infty$ and $x = -\infty$, respectively. An incident surface wave on the top face therefore excites a reflected surface wave on the bottom face and a radiation field when $\epsilon_\perp > 0$; for negative $\epsilon_\perp$, no radiation occurs from the edge and the entire incident energy is returned along the bottom side. The process is schematized in Fig. 8.4.4.

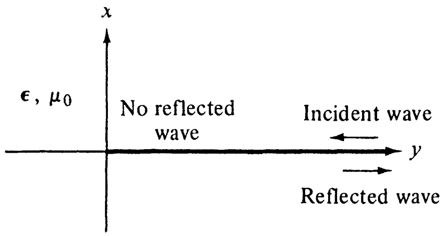


FIG. 8.4.4 Half-plane.

PROBLEMS

1. Show that the transverse field equations for a medium with arbitrary dyadic permittivity $\boldsymbol{\epsilon}(\mathbf{r})$ and permeability $\boldsymbol{\mu}(\mathbf{r})$ are given by (time dependence $\exp(j\omega t)$):

$$-\frac{\partial \mathbf{E}_t}{\partial z} = j\omega\left[-\mathbf{z}_0 \times \boldsymbol{\mu}_t \times \mathbf{z}_0 + \frac{1}{\omega^2}\nabla_t \frac{1}{\epsilon_z}\nabla_t + \frac{\mathbf{z}_0 \times \bar{\boldsymbol{\mu}}_{tz}\bar{\boldsymbol{\mu}}_{zt} \times \mathbf{z}_0}{\mu_z}\right]\cdot \mathbf{H}_t \times \mathbf{z}_0$$
$$-\left[\nabla_t \frac{1}{\epsilon_z}\bar{\boldsymbol{\epsilon}}_{zt} \times \mathbf{z}_0 + \frac{\mathbf{z}_0 \times \bar{\boldsymbol{\mu}}_{tz}\nabla_t}{\mu_z}\right]\cdot \mathbf{z}_0 \times \mathbf{E}_t + \mathbf{M}_{te} \times \mathbf{z}_0, \tag{1a}$$

$$-\frac{\partial \mathbf{H}_t}{\partial z} = j\omega\left[-\mathbf{z}_0 \times \boldsymbol{\epsilon}_t \times \mathbf{z}_0 + \frac{1}{\omega^2}\nabla_t \frac{1}{\mu_z}\nabla_t + \frac{\mathbf{z}_0 \times \bar{\boldsymbol{\epsilon}}_{tz}\bar{\boldsymbol{\epsilon}}_{zt} \times \mathbf{z}_0}{\epsilon_z}\right]\cdot \mathbf{z}_0 \times \mathbf{E}_t$$
$$-\left[\nabla_t \frac{1}{\mu_z}\bar{\boldsymbol{\mu}}_{zt} \times \mathbf{z}_0 + \frac{\mathbf{z}_0 \times \bar{\boldsymbol{\epsilon}}_{tz}\nabla_t}{\epsilon_z}\right]\cdot \mathbf{H}_t \times \mathbf{z}_0 + \mathbf{z}_0 \times \mathbf{J}_{te}, \tag{1b}$$

where z is the longitudinal direction, with respect to which $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ have the representations in Eq. (8.2.2c), and where the equivalent electric and magnetic source currents $\mathbf{J}_{te}$ and $\mathbf{M}_{te}$ are given in Eq. (8.2.29b) (with $i \to -j$). Show also that the longitudinal field components are obtained from the transverse components as follows:

$$H_z = \frac{1}{j\omega\mu_z}[\nabla_t\cdot(\mathbf{z}_0 \times \mathbf{E}_t) - j\omega\bar{\boldsymbol{\mu}}_{zt}\cdot\mathbf{H}_t - M_z], \tag{2a}$$

$$E_z = \frac{1}{j\omega\epsilon_z}[\nabla_t\cdot(\mathbf{H}_t \times \mathbf{z}_0) - j\omega\bar{\boldsymbol{\epsilon}}_{zt}\cdot\mathbf{E}_t - J_z]. \tag{2b}$$

2. While ordinary and extraordinary modes propagate independently in a homogeneous gyrotropic medium, mode coupling occurs in the presence of spatial inhomogeneities. Coupling is weak, and computation thereby facilitated, when

the medium is slowly varying but the anisotropy is arbitrary. Alternatively, in a uniaxially anisotropic medium, inhomogeneities along the gyrotropic axis do not introduce mode coupling (see Sec. 7.2). Weakly coupled modes may therefore be found for a strongly inhomogeneous medium whose anisotropy deviates only slightly from the uniaxial (or from the isotropic, as a special case of the uniaxial medium).[11]

(a) To derive the coupled mode equation, note from Sec. 7.2a that the transverse eigenfunctions in a transversely homogeneous, unbounded, isotropic or uniaxially anisotropic dielectric medium are identical and may be decomposed into E (extraordinary) and H (ordinary) modes with respect to the optic (z) axis. Use this mode set for representation of fields in the gyrotropic plasma described by Eq. (8.3.3b), with $\epsilon_2 \neq 0$. Substitute Eqs. (7.2.6) into the transverse field equations (1), with $\bar{\epsilon}_{zt} = \bar{\epsilon}_{tz} = 0$, $\boldsymbol{\mu} = \mu_0 \mathbf{1}$, and follow the procedure in Sec. 7.2a to derive transmission-line equations for the modal voltages and currents. Derive the relations

$$\int_S \mathbf{h}_j'^* \cdot \boldsymbol{\epsilon}_t \cdot \mathbf{h}_i' \, dS = \epsilon_1 \int_S \mathbf{h}_j'^* \cdot \mathbf{h}_i' \, dS - j\epsilon_2 \int_S \mathbf{h}_j'^* \cdot \mathbf{e}_i' \, dS = \epsilon_1 \delta_{ij}$$
$$= \int_S \mathbf{h}_j''^* \cdot \boldsymbol{\epsilon}_t \cdot \mathbf{h}_i'' dS, \tag{3a}$$

$$\int_S \mathbf{h}_j'^* \cdot \boldsymbol{\epsilon}_t \cdot \mathbf{h}_i'' \, dS = \epsilon_1 \int_S \mathbf{h}_j'^* \cdot \mathbf{h}_i'' \, dS + j\epsilon_2 \int_S \mathbf{h}_j'^* \cdot \mathbf{e}_i'' \, dS = -j\epsilon_2 \delta_{ij}$$
$$= \int_S \mathbf{h}_j''^* \cdot \boldsymbol{\epsilon}_t \cdot \mathbf{h}_i' \, dS, \tag{3b}$$

and show therefrom that the E-mode amplitudes satisfy the equations

$$-\frac{dV_i'}{dz} = j\kappa_i' Z_i' I_i' + v_i', \qquad -\frac{dI_i'}{dz} = j\kappa_i' Y_i' V_i' + i_i' + \kappa_i' Y_i' \frac{\epsilon_2}{\epsilon_1} V_i'', \tag{4a}$$

and the H modes satisfy

$$-\frac{dV_i''}{dz} = j\kappa_i'' Z_i'' I_i'' + v_i'', \qquad -\frac{dI_i''}{dz} = j\kappa_i'' Y_i'' V_i'' + i_i'' - \kappa_i' Y_i' \frac{\epsilon_2}{\epsilon_1} V_i', \tag{4b}$$

where κ_i, $Z_i = 1/Y_i$, v_i and i_i are the uniaxial medium quantities in Eqs. (7.2.8) and (7.2.9). Show that the uniaxial equations (7.2.7) (with $\epsilon_t \equiv \epsilon_1 \to \epsilon_0$, $\mu_t = \mu_z = \mu_0$) are recovered when $\epsilon_2 = 0$, whence the presence of a non-vanishing ϵ_2 introduces coupling between E and H modes.

(b) A source of inhomogeneity in an anisotropic plasma [see Eqs. (8.3.3b) and (8.3.4)] is a variation in the plasma density or, equivalently, in the parameter $X \equiv (\omega_p/\omega)^2$, with the external magnetic field H_0, and therefore $Y \equiv \omega_c/\omega$, taken to be constant. In this instance, a gyrotropic deviation from the uniaxial case $Y = \infty$ may be analyzed by representing the voltages and currents as series involving integral powers of the small, constant parameter $(1/Y)$:†

$$V_i = \sum_{n=0} \frac{1}{Y^n} V_i^{(n)}, \qquad I_i = \sum_{n=0} \frac{1}{Y^n} I_i^{(n)}. \tag{5}$$

†When Y is small, analogous considerations may be applied to develop an expansion in terms of Y^n. The customary ionospheric notation Y should not be confused with the same symbol used elsewhere for characteristic admittance.

Other Y-dependent quantities are expanded in a similar manner, and like coefficients of $(1/Y)^n$ are equated. Show that the zeroth order approximations $V_i^{(0)}$ and $I_i^{(0)}$ are the solutions of the uniaxial medium equations (7.2.7) (with $\epsilon_1 \to \epsilon_0$), while the higher-order terms $n \geq 1$ are obtained from:

$$-\frac{dV_i'^{(n)}}{dz} = j\kappa_i'^{(0)} Z_i'^{(0)} I_i'^{(n)}, \tag{6a}$$

$$-\frac{dI_i'^{(n)}}{dz} = j\kappa_i'^{(0)} Y_i'^{(0)} V_i'^{(n)} + \omega\epsilon_0 X[V_i''^{(n-1)} + V_i''^{(n-3)} + \ldots$$
$$+ j(V_i'^{(n-2)} + V_i'^{(n-4)} + \ldots)], \tag{6b}$$

$$-\frac{dV_i''^{(n)}}{dz} = j\kappa_i''^{(0)} Z_i''^{(0)} I_i''^{(n)}, \tag{6c}$$

$$-\frac{dI_i''^{(n)}}{dz} = j\kappa_i''^{(0)} Y_i''^{(0)} V_i''^{(n)} - \omega\epsilon_0 X[V_i'^{(n-1)} + V_i'^{(n-3)} + \ldots$$
$$-j(V_i''^{(n-2)} + V_i''^{(n-4)} + \ldots)], \tag{6d}$$

where the quantities with index$^{(0)}$ pertain to $Y = \infty$, and quantities with negative index vanish identically. Observe that the voltages and currents of any order $n \geq 1$ satisfy the same basic (zeroth order) transmission-line equations, with the continuously distributed source terms provided by the solutions of lower order. For example, when $n = 2$, show that the lower-order solutions $V_i''^{(1)}$ and $V_i'^{(0)}$ act as current sources for the E modes while $V_i'^{(1)}$ and $V_i''^{(0)}$ play a similar role with respect to the H modes. Equations (6) therefore represent a recursive system wherein the true sources v_i, i_i excite the uniaxial quantities $V_i^{(0)}, I_i^{(0)}$, which in turn generate $V_i^{(1)}, I_i^{(1)}$, etc.

(c) Obtain a typical solution by finding first the Green's functions $Z_i'^{(0)}(z, z')$, $Z_i''^{(0)}(z, z')$ and $[T_i'^I(z, z')]^{(0)}$, $[T_i''^I(z, z')]^{(0)}$ [see Eq. (7.2.15b) and Fig. 7.2.2], and then performing an integration over the source region distributed continuously along z:

$$V_i'^{(1)}(z) = -\omega\epsilon_0 \int Z_i'^{(0)}(z, z')X(z')V_i''^{(0)}(z')dz', \tag{7a}$$

$$V_i''^{(1)}(z) = \omega\epsilon_0 \int Z_i''^{(0)}(z, z')X(z')V_i'^{(0)}(z')dz', \tag{7b}$$

$$\begin{aligned} V_i'^{(2)}(z) &= -\omega\epsilon_0 \int Z_i'^{(0)}(z, z')X(z')[V_i''^{(1)}(z') + jV_i'^{(0)}(z')]dz' \\ &= -j\omega\epsilon_0 \int Z_i'^{(0)}(z, z')X(z')V_i'^{(0)}(z')dz' \\ &\quad -(\omega\epsilon_0)^2 \int dz' Z_i'^{(0)}(z, z')X(z') \int dz'' Z_i''^{(0)}(z', z'')X(z'')V_i'^{(0)}(z''), \end{aligned} \tag{7c}$$

etc. The integration interval in Eqs. (7) spans the entire longitudinal extent of the source region. Show that the current Green's function T^I appears in the analogous expressions for $I_i^{(n)}$, and that the nth-order approximation involves an n-fold integration of the zeroth-order solutions. Note that since each integral contains the density function $X(z)$, the nth-order term in Eq. (5) involves the factor $[X/Y]^n$. Discuss the convergence properties of the solution.

3. Starting from the eigenvalue equations (8.2.6), when applied to a gyrotropic plasma described by the dielectric tensor in Eq. (8.3.3b), show that the polarization vectors $\mathbf{e}_\alpha$ and $\mathbf{h}_\alpha$ for modes with transverse spatial and time dependence $\exp[-j\mathbf{k}_t \cdot \boldsymbol{\rho} + j\omega t]$ satisfy the equations

$$\mathbf{P}\cdot\mathbf{e}_\alpha = \kappa_\alpha'^2 \mathbf{e}_\alpha, \qquad \tilde{\mathbf{P}}^* \cdot (\mathbf{h}_\alpha \times \mathbf{z}_0) = \kappa_\alpha'^2 (\mathbf{h}_\alpha \times \mathbf{z}_0) \tag{8}$$

where $\mathbf{P} = \mathbf{R}\cdot\mathbf{S}$, and

$$\mathbf{R} = \mathbf{1}_t - \mathbf{k}_t'\mathbf{k}_t', \qquad \mathbf{S} = \boldsymbol{\epsilon}_t' - (\mathbf{z}_0 \times \mathbf{k}_t')(\mathbf{z}_0 \times \mathbf{k}_t'). \tag{8a}$$

The notation is that of Eqs. (8.3.10a) and (8.3.13a). Representing $\mathbf{e}_\alpha$ and $\mathbf{h}_\alpha$ in a $\mathbf{k}_{t0}$, $\hat{\mathbf{k}}_{t0}$ vector basis [see Eq. (8.3.8)], show that Eqs. (8) can be solved to yield the result in Eqs. (8.3.13b).

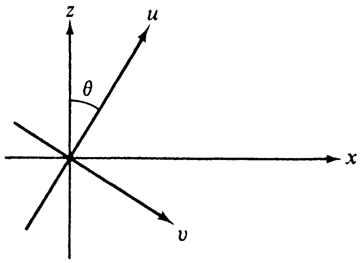


FIG. P8.1 Coordinate systems (y coordinate not shown).

4. In the (v, y, u) coordinate system of Fig. P8.1, a uniaxial medium (with optic axis along u) has the dielectric tensor (normalized to ϵ_0)

$$\boldsymbol{\epsilon} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}. \tag{9}$$

(a) Show that in the (x, y, z) coordinate frame rotated through an angle θ about the y axis, the $\boldsymbol{\epsilon}$ tensor has the representation:

$$\boldsymbol{\epsilon} \rightarrow \begin{pmatrix} \epsilon_{11} & 0 & \epsilon_{13} \\ 0 & 1 & 0 \\ \epsilon_{13} & 0 & \epsilon_{33} \end{pmatrix} \tag{10}$$

where

$$\begin{aligned} \epsilon_{11} &= \cos^2\theta + \epsilon\sin^2\theta, \qquad \epsilon_{13} = (\epsilon - 1)\sin\theta\cos\theta \\ \epsilon_{33} &= \epsilon\cos^2\theta + \sin^2\theta. \end{aligned} \tag{10a}$$

(b) Show that for a time dependence $\exp(j\omega t)$, x-independent plane-wave solutions of the form

$$\begin{aligned} \mathbf{E}_a(y, z) &= \mathbf{E}_\alpha e^{-jk_0\bar{k}_t y - jk_0\bar{\kappa}_\alpha z}, \qquad \bar{k}_t = k_t/k_0, \bar{\kappa}_\alpha = \kappa_\alpha/k_0 \\ \mathbf{H}_a(y, z) &= \mathbf{H}_\alpha e^{-jk_0\bar{k}_t y - jk_0\bar{\kappa}_\alpha z} \end{aligned} \tag{11}$$

can exist in this medium provided that $\bar{\kappa}_\alpha$ takes on one of the four values:

$$\begin{aligned}\bar{\kappa}_\alpha(\bar{k}_t) &= \pm\kappa_o, \qquad \kappa_o = \sqrt{1-\bar{k}_t^2}\\ \bar{\kappa}_\alpha(\bar{k}_t) &= \pm\kappa_e, \qquad \kappa_e = \frac{1}{\sqrt{\epsilon_{33}}}\sqrt{\epsilon-\bar{k}_t^2}.\end{aligned} \tag{11a}$$

κ_o describes ordinary modes which propagate as in vacuum while κ_e describes extraordinary modes. Interpret these $\bar{\kappa}$ versus $\bar{k}_t$ solutions by referring to the wavenumber surfaces as in Fig. 7.1.1, noting that the (x, y, z) coordinate system does not coincide with the principal axes of the medium, and that for the waves considered, one has $k_x \equiv 0$. Show that $\kappa_{o,e} > 0$ and $\kappa_{o,e} < 0$ describe waves carrying energy in the $+z$ and $-z$ directions, respectively.

(c) Show that with $\kappa_{o,e}$ defined as in Eq. (11a), the polarization vectors $\mathbf{E}_\alpha$ and $\mathbf{H}_\alpha$ in Eq. (11) have the form:

$$\mathbf{E}_\alpha \to \mathbf{E}_{o,e} = \mathbf{x}_o \pm g_{o,e}\mathbf{y}_o - h_{o,e}\mathbf{z}_0 \tag{12}$$

where

$$g_{o,e} = \frac{\bar{k}_t\kappa_{o,e}(\epsilon_{11} - \bar{k}_t^2 - \kappa_{o,e}^2)}{\epsilon_{13}(1-\kappa_{o,e}^2)}. \tag{12a}$$

and

$$h_{o,e} = \frac{\epsilon_{11} - \bar{k}_t^2 - \kappa_{o,e}^2}{\epsilon_{13}}. \tag{12b}$$

Also,

$$\mathbf{H}_\alpha \to \mathbf{H}_{o,e} = \frac{k_0}{\omega\mu_0}[-(\kappa_{o,e}g_{o,e} + \bar{k}_t h_{o,e})\mathbf{x}_0 \pm \kappa_{o,e}\mathbf{y}_0 - \bar{k}_t\mathbf{z}_0]. \tag{13}$$

Upper and lower signs correspond to subscripts o and e, respectively. Note that although the eigenvalues $\bar{\kappa}_\alpha$ occur in pairs $\pm\kappa_o$, $\pm\kappa_e$, the transverse (to z) eigenvectors corresponding to $+\kappa_{o,e}$ differ from those corresponding to $-\kappa_{o,e}$.
(d) Show that the mode fields in Eq. (11), with Eqs. (12)-(13), reduce to E modes and H modes with respect to u (i.e., $H_{eu} = 0$ and $E_{ou} = 0$, respectively).
(e) Verify that the modes in Eqs. (12)-(13) satisfy the orthogonality condition in Eq. (8.2.10b).

5. A perfectly conducting cylindrical obstacle embedded in a homogeneous cold magnetoplasma is excited by a line source of magnetic currents of strength M, the arrangement being such that the cylinder axis, gyrotropic axis, and line-source direction are parallel to the z axis. Assuming that in an orthogonal, separable, curvilinear (u, v) coordinate system, the cylinder boundary s in a plane transverse to z is described by the equation u = constant, show that for an $\exp(j\omega t)$ dependence, the electromagnetic fields can be derived from the single component $H_z \equiv H$ satisfying the equation

$$(\nabla_t^2 + k_0^2\epsilon_\perp)H = j\omega\epsilon_0\epsilon_\perp M, \tag{14}$$

subject to a radiation condition at infinity and the boundary condition

$$j\frac{\epsilon_2}{\epsilon_1}\frac{1}{h_v}\frac{\partial H}{\partial v} - \frac{1}{h_u}\frac{\partial H}{\partial u} = 0 \text{ on } s, \tag{14a}$$

where $\epsilon_\perp = (\epsilon_1^2 - \epsilon_2^2)/\epsilon_1$ [see Eqs. (8.4.5b) and (8.3.4)], $\nabla_t = \mathbf{u}_0(\partial/h_u\partial u) + \mathbf{v}_0(\partial/h_v\partial_v)$, and h_u, h_v are the metric coefficients in the (u, v) coordinate system. Show that while the method of separation of variables can be employed to solve Eq. (14), separability does not obtain in general for the boundary condition in Eq. (14a) although the boundary coincides with a curve $u =$ constant. However, if the v-dependent orthogonal eigenfunctions $\psi_i(v)$ are such that

$$\frac{1}{h_v}\frac{\partial\psi_i}{\partial v} = \alpha_i\psi_i, \tag{15}$$

where α_i does not depend on v, and if h_u is also independent of v, show that the representation

$$H(u, v) = \sum_i I_i(u)\psi_i(v) \tag{16}$$

leads to a separable form. Note that included among the separable configurations are the plane boundary $u \equiv x =$ constant, whence $h_u \equiv 1$ and $\psi_i(v) \equiv \psi_i(y) \propto \exp(-j\eta y)$, $-\infty < \eta < \infty$ (cf. Sec. 8.4c), and also the circular cylindrical boundary $u \equiv \rho =$ constant, whence $h_u \equiv 1$ and $\psi_i(v) \equiv \psi_i(\phi) \propto \exp(-jn\phi)$, $n = 0, \pm 1, \pm 2, \ldots$. Construct the solution for the electromagnetic fields in the presence of the circular cylindrical scatterer and compare its form with that for the isotropic medium ($\epsilon_2 = 0$) in Sec. 6.7.

6. Calculate the electromagnetic fields and the power flow properties for the line source in Fig. 8.4.1 in an infinite medium (in the absence of the conducting screen at $z = 0$). Place a source at the image point $(-z', 0)$ and calculate the fields at $z = 0$. Referring to the boundary condition (8.4.9), show that the image source is incapable of accounting properly for the effect of a perfect conductor at $z = 0$. Explain this conclusion in reference to Fig. 8.4.3 and propose an alternative procedure.

7. The sectoral region between two intersecting, perfectly conducting half-planes inclined at an angle $\varphi < \pi$ is filled with a gyrotropic dielectric as in Eq. (8.3.3b), the gyrotropic axis being parallel to the apex. Referring to Eq. (6.5.13) or (6.5.16b) for the formulation in the isotropic case, with $J_\mu(k\rho)$ replaced by the leading terms in its expansion as $\rho \to 0$, construct expressions for the electromagnetic fields near the apex when the medium is gyrotropic and the polarization is such that $\mathbf{H} = \mathbf{z}_0 H$. Calculate the energy stored in the vicinity of the apex (cf. Sec. 1.5b) for the cases of vanishing and non-vanishing dissipation in the medium. To impose a "boundary condition" near the apex, show that the lossless case must be approached as the limiting case of a dissipative medium, and that direct consideration of a purely lossless medium poses an ill-defined problem.[13,14]

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