2

The Green's Function Method

2.1 INTRODUCTION

In this chapter, we begin our study of linear ordinary differential equations of second order. Our goal is to develop a procedure whereby we can solve the differential equations using fundamental solutions called *Green's functions*.

We begin with a brief discussion of the delta function. We follow with a description of the Sturm–Liouville operator L and its properties. We define three types of Sturm–Liouville problems and investigate their properties. In all three types, we examine the role of the operator L and its *adjoint operator* L^*. These operators are used to define the Green's function and the adjoint Green's function, respectively. Our study culminates in a procedure for applying the Green's function and/or the adjoint Green's function in solving the differential equation $Lu = f$.

2.2 DELTA FUNCTION

The concept of the delta function arises when we wish to fix attention on the value of a function $f(x)$ at a given point x_0. Mathematically, we seek an operator T that transforms a function $f(x)$, continuous at x_0, into $f(x_0)$, the value of the function at x_0. In equation form, we require T such that

$$T[f(x)] = f(x_0) \tag{2.1}$$

We begin by considering the *pulse function* $p_\epsilon(x - x_0)$, defined by

$$p_\epsilon(x - x_0) = \begin{cases} \frac{1}{2\epsilon}, & x_0 - \epsilon < x < x_0 + \epsilon \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

Note that, regardless of the value of ϵ, the area under the pulse is unity. Indeed, if (a, b) is any interval containing $(x_0 - \epsilon, x_0 + \epsilon)$,

$$\int_a^b p_\epsilon(x - x_0)dx = \int_{x_0-\epsilon}^{x_0+\epsilon} \frac{1}{2\epsilon}\, dx = 1 \tag{2.3}$$

An important property of the pulse function is that it is even about x_0, viz.

$$p_\epsilon(x - x_0) = p_\epsilon(x_0 - x) \tag{2.4}$$

This property can be proved by interchanging x and x_0 in (2.2). The details are left for the problems. Multiplying the pulse function by $f(x)$ and integrating over any interval containing the pulse gives (Fig. 2-1)

$$\int_a^b f(x)p_\epsilon(x - x_0)dx = \frac{1}{2\epsilon}\int_{x_0-\epsilon}^{x_0+\epsilon} f(x)dx \tag{2.5}$$

By the mean value theorem for integrals [1], if $\hat{f}$ is the mean value of $f(x)$ on the interval $x \in (x_0 - \epsilon, x_0 + \epsilon)$,

$$\int_a^b f(x)p_\epsilon(x - x_0)dx = \hat{f} \tag{2.6}$$

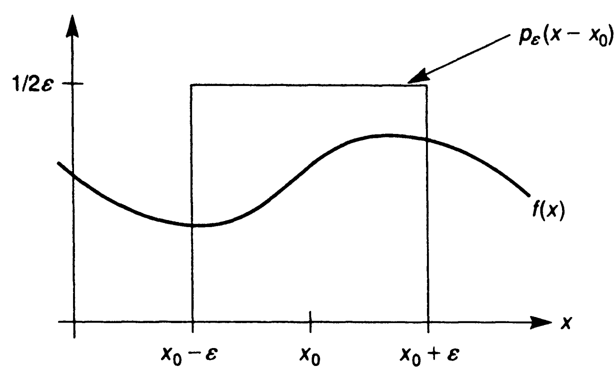

Fig. 2-1 Pulse function $p_\epsilon(x - x_0)$ and function $f(x)$.

Taking the limit as $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0}\int_a^b f(x)p_\epsilon(x-x_0)dx = f(x_0), \qquad x_0 \in (a,b) \tag{2.7}$$

The integration followed by the limiting operation in (2.7) transforms $f(x)$ to $f(x_0)$, the value of the function at x_0.

EXAMPLE 2.1 Let $f(x) = x^2$, $x_0 = 0$, and $x_0 \in (a,b)$. In this case, $f(x)$ is continuous at $x = 0$ and we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_a^b f(x)p_\epsilon(x-x_0)dx &= \lim_{\epsilon\to 0}\left[\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} x^2 dx\right]\\ &= \lim_{\epsilon\to 0}\left(\frac{\epsilon^2}{3}\right) = 0\end{aligned}$$

Since $f(0) = 0$, we have verified (2.7).

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EXAMPLE 2.2 Let $f(x) = \cos x$, $x_0 = \pi/3$, and $x_0 \in (a,b)$. In this case, $f(x)$ is continuous at $x = \pi/3$ and we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_a^b f(x)p_\epsilon(x-x_0)dx &= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\frac{\pi}{3}-\epsilon}^{\frac{\pi}{3}+\epsilon} \cos x \, dx\\ &= \lim_{\epsilon\to 0}\left\{\frac{1}{2\epsilon}\left[\sin\left(\frac{\pi}{3}+\epsilon\right) - \sin\left(\frac{\pi}{3}-\epsilon\right)\right]\right\} = \frac{1}{2}\end{aligned}$$

Since $f(\pi/3) = 1/2$, we have again verified (2.7).

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Expression (2.7) forms the cornerstone of our definition of the delta function, as follows:

$$\int_a^b f(x)\delta(x-x_0)dx = \lim_{\epsilon\to 0}\int_a^b f(x)p_\epsilon(x-x_0)dx \tag{2.8}$$

so that

$$\int_a^b f(x)\delta(x-x_0)dx = f(x_0) \tag{2.9}$$

for any x_0 in the interval (a,b). Note in (2.2) that as ϵ becomes smaller, the pulse function becomes narrower and higher while maintaining unit area. If the limit in (2.7) could be taken under the integral, we would have

$$\delta(x-x_0) \stackrel{s}{=} \lim_{\epsilon\to 0}[p_\epsilon(x-x_0)] \tag{2.10}$$

Since this limit does not exist, the interchange of limit and integration in (2.8) is not valid. We have therefore placed an "s" over the equality in (2.10) to indicate *symbolic equality* only.

The delta function $\delta(x - x_0)$ has two remarkable properties. Symbolically, it is a function that is zero everywhere except at $x = x_0$, where it is undefined. Second, when integrated against a function f that is continuous at x_0, it yields the value of the function at x_0. We note that (2.9) defines the operator T we were seeking in (2.1). Indeed, comparing (2.1) and (2.9) yields

$$T = \int_a^b (\bullet)\delta(x - x_0)dx \tag{2.11}$$

where $(\bullet)$ indicates the position of the function upon which T operates.

From the basic definition of the delta function in (2.9), we obtain some additional relations. If we set x_0 equal to zero, we find

$$\int_a^b f(x)\delta(x)dx = f(0) \tag{2.12}$$

Also, if in (2.9) we set $f(x) = 1$, we obtain

$$\int_a^b \delta(x - x_0)dx = 1 \tag{2.13}$$

Finally, from (2.4) and (2.8), we conclude symbolically that

$$\delta(x - x_0) \stackrel{s}{=} \delta(x_0 - x) \tag{2.14}$$

In concluding our development of the delta function and its properties, we remark that there are certain difficulties with the definitions. Indeed, any function that is zero everywhere except at one point must produce zero when Riemann integrated over any interval containing the point. The result in (2.13), for example, is therefore unacceptable in the Riemann sense. To interpret the integral, it seems that we must return to the basic definition in (2.8). The mathematical acceptability of integrals involving the delta function have, however, been formalized in the Theory of Distributions, introduced by Schwartz [2]. In the theory, the delta function is called a *generalized function*, and the integral in (2.9) is said to exist in the *distributional sense*. Although the theory is beyond the scope of this book, the interested reader can find introductory treatments in [3],[4].

The central role played by the delta function in the solution to certain differential equations becomes apparent in the following argument.

Suppose we wish to solve the equation

$$Lu = f \tag{2.15}$$

where L is a differential operator. Formally, the solution is given by multiplying both sides of (2.15) by the inverse operator, viz.

$$L^{-1}Lu = L^{-1}f$$

or

$$u = L^{-1}f \tag{2.16}$$

Since L is a differential operator, we shall assume that its inverse is an integral operator with kernel $g(x, \xi)$, so that

$$u(x) = \int g(x, \xi) f(\xi) d\xi \tag{2.17}$$

Substitution into (2.15) gives

$$\begin{aligned} f(x) &= L[u(x)] \\ &= L \int g(x, \xi) f(\xi) d\xi \\ &= \int Lg(x, \xi) f(\xi) d\xi \end{aligned} \tag{2.18}$$

where we have assumed, without proof, that we can move the operator L inside the integral. But, from the properties of the delta function, we have

$$f(x) = \int_a^b \delta(x - \xi) f(\xi) d\xi \tag{2.19}$$

for $x \in (a, b)$. Comparing (2.18) and (2.19), we identify

$$Lg(x, \xi) = \delta(x - \xi) \tag{2.20}$$

Presumably, if we can solve (2.20), then the solution to (2.15) is given explicitly by (2.17). The kernel $g(x, \xi)$ is called the *Green's function* for the problem.

It is the purpose of this chapter to formalize and structure the introductory ideas above. The result will be the solution to a class of linear ordinary differential equations of second order by the *Green's function method.*

2.3 STURM–LIOUVILLE OPERATOR THEORY

Consider the following linear, ordinary, differential equation of second order:

$$a_0(x)\frac{d^2u(x)}{dx^2}+a_1(x)\frac{du(x)}{dx}+a_2(x)u(x)-\lambda u(x)=f(x), \qquad a<x<b \tag{2.21}$$

where λ is, in general, a complex parameter independent of x. The functions a_0, a_1, and a_2 are real and assumed to have the following properties [5],[6]:

a. a_2, da_1/dx, and d^2a_0/dx^2 are continuous in $a \le x \le b$

b. $a_0 \neq 0$ in $a < x < b$

In (2.21), we also require that $u(x)$ be twice differentiable and that $f(x)$ be piecewise continuous. We may always recast this differential equation in *Sturm–Liouville form*, as follows:

$$-\frac{1}{w(x)}\frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right]+q(x)u(x)-\lambda u(x)=f(x) \tag{2.22}$$

The necessary coefficient transformations are given by [7]

$$q(x)=a_2(x) \tag{2.23}$$

$$p(x)=\exp\left[\int^x \frac{a_1(t)}{a_0(t)}dt\right] \tag{2.24}$$

$$w(x)=-\frac{p(x)}{a_0(x)} \tag{2.25}$$

We may verify these transformations by substituting (2.23)–(2.25) into (2.22) to produce (2.21). The details are left for the problems. We rewrite (2.22) in operator notation as follows:

$$(L-\lambda)u=f \tag{2.26}$$

where we identify the *Sturm–Liouville operator* L, viz.

$$L=-\frac{1}{w(x)}\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+q(x) \tag{2.27}$$

For the remainder of this chapter, without loss of generality, L will always mean the Sturm–Liouville operator in (2.27).

EXAMPLE 2.3 Consider Bessel's equation of order ν, given by

$$-u'' - \frac{1}{x}u' + \left(\frac{\nu^2}{x^2} - k^2\right)u = f \tag{2.28}$$

where "prime" indicates differentiation with respect to x. Comparing to (2.21), we identify

$$\begin{aligned}\lambda &= k^2\\ a_0 &= -1\\ a_1 &= -(1/x)\\ a_2 &= (\nu/x)^2\end{aligned}$$

To transform to Sturm–Liouville form, we use (2.23)–(2.25) and obtain

$$\begin{aligned}q(x) &= \frac{\nu^2}{x^2}\\ p(x) &= x\\ w(x) &= x\end{aligned}$$

so that

$$-\frac{1}{x}(xu')' + \left(\frac{\nu^2}{x^2} - k^2\right)u = f \tag{2.29}$$

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EXAMPLE 2.4 Consider Bessel's equation, given by (2.28), in a slightly different form, viz.

$$-x^2u'' - xu' - \left[(kx)^2 - \nu^2\right]u = \hat{f} \tag{2.30}$$

We note that (2.30) is obtained simply by multiplying both sides of (2.28) by x^2. In this case, we identify

$$\begin{aligned}\lambda &= -\nu^2\\ a_0 &= -x^2\\ a_1 &= -x\\ a_2 &= -(kx)^2\end{aligned}$$

Using (2.23)–(2.25), we obtain

$$\begin{aligned}q(x) &= -(kx)^2\\ p(x) &= x\\ w(x) &= \frac{1}{x}\end{aligned}$$

so that in Sturm–Liouville form,

$$-x(xu')' - (kx)^2u + \nu^2 u = \hat{f} \tag{2.31}$$

We note, in particular, that the weighting function $w(x)$ differs from that in Example 2.3.

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It might appear that the distinction between (2.29) and (2.31) is trivial since the latter can be obtained from the former by dividing by x^2. However, the difference in the weighting functions between (2.29) and (2.31) changes the Hilbert space, and makes a major difference in spectral representations associated with the radial portion of the Helmholtz equation in cylindrical coordinates [8], as we shall find in Chapter 4.

EXAMPLE 2.5 Consider Legendre's equation on the interval $x \in (-1, 1)$, as follows:

$$-(1-x^2)u'' + 2xu' - n(n+1)u = f \tag{2.32}$$

We identify

$$\begin{aligned} \lambda &= n(n+1) \\ a_0 &= -(1-x^2) \\ a_1 &= 2x \\ a_2 &= 0 \end{aligned}$$

Using (2.23)–(2.25), we obtain

$$\begin{aligned} q(x) &= 0 \\ p(x) &= 1-x^2 \\ w(x) &= 1 \end{aligned}$$

so that in Sturm–Liouville form,

$$-\left[(1-x^2)u'\right]' - n(n+1)u = f \tag{2.33}$$

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The Sturm–Liouville form of the second order differential equation, given by (2.22), plays a central role in the solution of electromagnetic boundary value problems. We distinguish three forms of the Sturm–Liouville problem, which we consider in the next three sections.

2.4 STURM–LIOUVILLE PROBLEM OF THE FIRST KIND

For the first form of the Sturm–Liouville problem, we consider $(L-\lambda)u = f$ over a finite interval $x \in (a, b)$ and for real λ and real f. For $-\infty < a < b < \infty$, consider the Hilbert space $\mathcal{L}_2(a, b)$ with real inner product

$$\langle u, v\rangle = \int_a^b u(x)v(x)w(x)dx \tag{2.34}$$

for all $u, v \in \mathcal{L}_2(a, b)$. We define the *Sturm–Liouville Problem of the First Kind*, abbreviated SLP1, as follows:

$$L_\lambda u = f, \qquad a < x < b \tag{2.35}$$

where

$$L_\lambda = L - \lambda \tag{2.36}$$

and where

$$L = -\frac{1}{w(x)}\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x) \tag{2.37}$$

We impose the following restrictions [9]:

a. p, p', q, w are real and continuous for $a \le x \le b$
b. $p(x) > 0$, $w(x) > 0$ for $a \le x \le b$
c. λ is real and independent of x

In addition, we require $u(x) \in \mathcal{D}_L \subset \mathcal{L}_2(a, b)$, where $\mathcal{D}_L$ is the domain of the operator L. Because we are dealing with second-order differential operators, the domain is restricted to those functions that are twice differentiable. Finally, we require that $u(x)$ satisfy two boundary conditions as follows:

$$B_1(u) = \alpha = \alpha_{11}u(a) + \alpha_{12}u'(a) + \alpha_{13}u(b) + \alpha_{14}u'(b) \tag{2.38}$$

$$B_2(u) = \beta = \alpha_{21}u(a) + \alpha_{22}u'(a) + \alpha_{23}u(b) + \alpha_{24}u'(b) \tag{2.39}$$

where, for SLP1, α, β, and α_{ij} are real. Typically, in (2.38), if α is nonzero, the boundary condition is said to be *inhomogeneous*. If $\alpha = 0$, the boundary condition is *homogeneous*.

There are several important special cases contained in the boundary conditions in (2.38) and (2.39). A boundary condition is *unmixed* if it involves conditions on $u(x)$ at one boundary only. If SLP1 involves an unmixed condition at one end of the boundary and an unmixed condition at the other end, we refer to this case as SLP1 with *unmixed conditions.* The most general case of unmixed boundary conditions is $\alpha_{13} = \alpha_{14} = \alpha_{21} = \alpha_{22} = 0$, so that

$$B_1(u) = \alpha = \alpha_{11}u(a) + \alpha_{12}u'(a) \tag{2.40}$$

$$B_2(u) = \beta = \alpha_{23}u(b) + \alpha_{24}u'(b) \tag{2.41}$$

The relations in (2.38) and (2.39) are said to be *initial* conditions if $\alpha_{11} = \alpha_{22} = 1$ and all other α_{ij} coefficients are zero, so that

$$B_1(u) = \alpha = u(a) \tag{2.42}$$

$$B_2(u) = \beta = u'(a) \tag{2.43}$$

The two conditions in (2.38) and (2.39) are *periodic* if the value of the function $u(x)$ at one boundary is identical to the value at the other boundary, and if the value of the derivative $u'(x)$ at one boundary is identical to the value at the other boundary. To produce the periodic conditions, we require $\alpha_{11} = -\alpha_{13} = \alpha_{22} = -\alpha_{24} = 1$ and all other coefficients zero, so that

$$u(a) = u(b) \tag{2.44}$$

$$u'(a) = u'(b) \tag{2.45}$$

EXAMPLE 2.6 Consider the following differential equation on $x \in (0, 1)$:

$$-u'' - k^2u = f, \qquad k \in \mathbf{R}$$

with two homogeneous unmixed boundary conditions

$$u(0) = u(1) = 0$$

We identify $p(x) = w(x) = 1$, $q(x) = 0$, $\lambda = k^2$, $a = 0$, $b = 1$, $\alpha = \beta = 0$. In the boundary conditions in (2.38) and (2.39), all coefficients $\alpha_{ij} = 0$, except $\alpha_{11} = \alpha_{23} = 1$. We find that all requirements for SLP1 are satisfied.

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The operator L in SLP1 has a *formal adjoint*, which we construct by the following procedure. For $u, v \in \mathcal{L}_2(a, b)$, we form

$$\langle Lu, v\rangle = \int_a^b \left\{-\frac{1}{w(x)}\frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x)\right\} v(x)w(x)dx \tag{2.46}$$

Integrating by parts twice, we obtain

$$\langle Lu, v\rangle = \int_a^b u(x)\left\{-\frac{1}{w(x)}\frac{d}{dx}\left[p(x)\frac{dv(x)}{dx}\right] + q(x)v(x)\right\} w(x)dx$$
$$- \left\{p(x)\left[v(x)\frac{du(x)}{dx} - u(x)\frac{dv(x)}{dx}\right]\right\}\Bigg|_a^b \tag{2.47}$$

We write this result in inner product notation as

$$\langle Lu, v\rangle = \langle u, L^*v\rangle + J(u, v)\Bigg|_a^b \tag{2.48}$$

where $J(u, v)$ is called the *conjunct* and is given by

$$J(u, v) = -p(u'v - uv') \tag{2.49}$$

The operator L^*, produced in the integration by parts operation, is called the formal adjoint to L. We note that

$$L^* = L \tag{2.50}$$

When (2.50) is true, we say that L is *formally self-adjoint.* We conclude, in general, that the Sturm–Liouville operator for SLP1 is formally self-adjoint.

In our search for a solution, or solutions, to (2.35), we shall first assume that the boundary conditions in (2.38) and (2.39) are homogeneous. We then follow with the extension to the inhomogeneous case. Accordingly, if $u(x)$ is to be a solution to (2.35), we require the following restrictions:

a. $u \in \mathcal{L}_2(a, b)$
b. $u \in \mathcal{D}_L$
c. u satisfies two boundary conditions, $B_1(u) = 0$, $B_2(u) = 0$

These restrictions define a linear manifold $\mathcal{M}_L \subset \mathcal{L}_2(a, b)$. The proof is left for the problems. We next consider the function $v(x)$ in (2.48). We place the following restrictions on $v(x)$:

a. $v \in \mathcal{L}_2(a, b)$
b. $v \in \mathcal{D}_{L^*}$
c. v satisfies two *adjoint boundary conditions*, $B_1^*(v) = 0$, $B_2^*(v) = 0$

Since $v(x)$ is unspecified in the original problem statement in (2.35), we are free to choose the adjoint boundary conditions in any manner we wish, consistent with the integration by parts operation in (2.48). We define the adjoint boundary conditions to be those conditions $B_1^*(v) = 0$, $B_2^*(v) = 0$ that, when coupled with the boundary conditions on $u(x)$, result in the vanishing of the conjunct, viz.

$$J(u, v) \bigg|_a^b = 0 \tag{2.51}$$

These restrictions on $v(x)$ define a linear manifold $\mathcal{M}_{L^*} \subset \mathcal{L}_2(a, b)$. At present, it is not clear that it is possible to define the adjoint boundary conditions such that (2.51) is satisfied. We next show explicitly the adjoint boundary condition result for the unmixed, initial, and periodic cases.

We have defined the unmixed boundary case in (2.40) and (2.41). For the homogeneous case, they become

$$B_1(u) = \alpha_{11}u(a) + \alpha_{12}u'(a) = 0 \tag{2.52}$$

$$B_2(u) = \alpha_{23}u(b) + \alpha_{24}u'(b) = 0 \tag{2.53}$$

We use these expressions in the conjunct to eliminate $u'(a)$ and $u'(b)$, viz.

$$J(u, v) \bigg|_a^b = p(b)u(b) \left[\frac{\alpha_{23}}{\alpha_{24}}v(b) + v'(b)\right] - p(a)u(a) \left[\frac{\alpha_{11}}{\alpha_{12}}v(a) + v'(a)\right] \tag{2.54}$$

In this case, (2.51) is satisfied if we choose the following adjoint boundary conditions:

$$B_1^*(v) = \alpha_{11}v(a) + \alpha_{12}v'(a) = 0 \tag{2.55}$$

$$B_2^*(v) = \alpha_{23}v(b) + \alpha_{24}v'(b) = 0 \tag{2.56}$$

We note that in the unmixed boundary case, the boundary conditions on $v(x)$ in (2.55) and (2.56) are identical to those on $u(x)$ in (2.52) and (2.53). Therefore, for the case of unmixed boundary conditions, the linear manifold $\mathcal{M}_L$ is the same as the linear manifold $\mathcal{M}_{L^*}$. A formally self-adjoint operator with $\mathcal{M}_L = \mathcal{M}_{L^*}$ is said to be *self-adjoint*. We shall find subsequently that self-adjoint problems possess remarkable properties.

For homogeneous initial conditions, we have

$$u(a) = 0 \tag{2.57}$$

$$u'(a) = 0 \tag{2.58}$$

Substitution into the conjunct gives

$$J(u, v)\Big|_a^b = -p(b)\left[u'(b)v(b) - u(b)v'(b)\right] \tag{2.59}$$

In this case, (2.51) is satisfied if we choose adjoint boundary conditions

$$B_1^*(v) = v(b) = 0 \tag{2.60}$$

$$B_2^*(v) = v'(b) = 0 \tag{2.61}$$

We note that for initial conditions, the boundary conditions on $v(x)$ in (2.60) and (2.61) are not the same as those on $u(x)$ in (2.57) and (2.58). Therefore, $\mathcal{M}_L \neq \mathcal{M}_{L^*}$, and the initial condition case is never self-adjoint.

For periodic conditions, we substitute (2.44) and (2.45) into the conjunct to give

$$\begin{aligned} J(u, v)\Big|_a^b &= -p(b)\left[u'(a)v(b) - u(a)v'(b)\right] \\ &\quad + p(a)\left[u'(a)v(a) - u(a)v'(a)\right] \\ &= u'(a)\left[p(a)v(a) - p(b)v(b)\right] \\ &\quad - u(a)\left[p(a)v'(a) - p(b)v'(b)\right] \end{aligned} \tag{2.62}$$

In this case, (2.51) is satisfied if we choose adjoint boundary conditions

$$B_1^*(v) = p(a)v(a) - p(b)v(b) = 0 \tag{2.63}$$

$$B_2^*(v) = p(a)v'(a) - p(b)v'(b) = 0 \tag{2.64}$$

We note that for the general form of L in (2.37) and for periodic conditions, the boundary conditions on $v(x)$ in (2.63) and (2.64) are not the same as those on $u(x)$ in (2.44) and (2.45). However, if the operator L is such that $p(a) = p(b)$, the conditions are identical and the problem becomes self-adjoint.

To produce the solution to SLP1 by the Green's function method, we define two auxiliary problems: the *Green's function problem* and the *adjoint Green's function problem*. The Green's function problem is defined as follows:

$$L_\lambda g(x, \xi) = \frac{\delta(x - \xi)}{w(x)}, \qquad a < \xi < b \tag{2.65}$$

$$B_1(g) = 0 \tag{2.66}$$

$$B_2(g) = 0 \tag{2.67}$$

where $w(x)$ is the weight function defined in (2.25) and (2.27) and appearing in the inner product definition in (2.34). We note that, by definition, the boundary conditions on g are identical to the boundary conditions on u. The adjoint Green's function problem is defined as follows:

$$L_\lambda h(x,\xi) = \frac{\delta(x-\xi)}{w(x)}, \qquad a < \xi < b \tag{2.68}$$

$$B_1^*(h) = 0 \tag{2.69}$$

$$B_2^*(h) = 0 \tag{2.70}$$

We note that, by definition, the boundary conditions on h are identical to the boundary conditions on v. We also note that the boundary conditions associated with the Green's function and the adjoint Green's function are *always* homogeneous.

The solution to SLP1 by the Green's function method is obtained by taking the inner product of $L_\lambda u$ with h, viz.

$$\langle L_\lambda u, h\rangle = \langle u, L_\lambda h\rangle + J(u,h)\Big|_{x=a}^{x=b} \tag{2.71}$$

where the integrations indicated by the inner products are with respect to x. From (2.49), the conjunct $J(u,h)$ is given by

$$J(u,h) = -p(x)\left[\frac{du(x)}{dx}h(x,\xi) - u(x)\frac{dh(x,\xi)}{dx}\right] \tag{2.72}$$

Substitution of (2.35) and (2.68) into (2.71) gives

$$u(\xi) = \langle f, h\rangle - J(u,h)\Big|_a^b \tag{2.73}$$

or, explicitly,

$$u(\xi) = \int_a^b f(x)h(x,\xi)w(x)dx + \left\{p(x)\left[\frac{du(x)}{dx}h(x,\xi) - u(x)\frac{dh(x,\xi)}{dx}\right]\right\}\Bigg|_{x=a}^{x=b} \tag{2.74}$$

Equation (2.74) is the formal solution to SLP1, provided that we can solve the adjoint Green's function problem, given in (2.68)–(2.70). For homogeneous boundary conditions $B_1(u) = 0$, $B_2(u) = 0$, the selection

$B_1^*(h) = 0$, $B_2^*(h) = 0$ reduces the second term in (2.74) to zero. The extension to the inhomogeneous case, however, is now available. We simply apply the given boundary conditions $B_1(u) = \alpha$, $B_2(u) = \beta$ in conjunction with the adjoint boundary conditions $B_1^*(h) = 0$, $B_2^*(h) = 0$. We illustrate these results with an example.

EXAMPLE 2.7 Consider the following differential equation on $x \in (a, b)$:

$$-u'' - \lambda u = f$$

with boundary conditions

$$u(a) = \alpha$$

$$u'(a) = \beta$$

In this case, $p(x) = w(x) = 1$, and (2.74) yields

$$u(\xi) = \int_a^b f(x)h(x,\xi)dx + \alpha\frac{dh(a,\xi)}{dx} - \beta h(a,\xi)$$

where we have applied the boundary conditions $u(a) = \alpha$, $u'(a) = \beta$ and the adjoint boundary conditions

$$h(b,\xi) = \frac{dh(b,\xi)}{dx} = 0$$

Note that for homogeneous boundary conditions, $\alpha = \beta = 0$, the conjunct vanishes and

$$u(\xi) = \int_a^b f(x)h(x,\xi)dx$$

■

We should note that in (2.74) and in Example 2.7, the solution is given in terms of the variable ξ, with x as the dummy variable of integration. This notation causes no difficulty since ξ simply refers to a point of evaluation of u on the interval (a, b). We shall subsequently obtain the solution u in terms of x with ξ as the dummy integration variable by a simple interchange of x and ξ. The reader is cautioned, however, to withhold performing this step until after explicit evaluation of the adjoint Green's function. This evaluation is the next subject for discussion.

We now show that it is never necessary to find the adjoint Green's function $h(x, \xi)$ directly from (2.68)–(2.70). Indeed, if we determine the Green's function $g(x, \xi)$, defined by (2.65)–(2.67), the adjoint Green's function follows immediately. The details follow. We form

$$\langle L_\lambda g(x,\xi), h(x,\xi')\rangle = \langle g(x,\xi), L_\lambda h(x,\xi')\rangle + J(g,h)\Big|_{x=a}^{x=b} \tag{2.75}$$

where integrations in the inner products are with respect to x. Application of (2.66), (2.67), (2.69), and (2.70) reduces the conjunct to zero, viz.

$$J(g,h)\Big|_{x=a}^{x=b}=0$$

Using this result and (2.65) and (2.68) in (2.75) gives

$$h(\xi,\xi')=g(\xi',\xi)$$

A simple variable change gives

$$h(x,\xi)=g(\xi,x) \tag{2.76}$$

We conclude that, if we can find the Green's function $g(x,\xi)$, the adjoint Green's function $h(x,\xi)$ follows immediately by an interchange of x and ξ. Substitution of $h(x,\xi)$ into (2.74) completes the solution to SLP1.

A further simplification occurs if the Green's function problem is self-adjoint. In this case, the operator and boundary conditions for the Green's function and the adjoint Green's function are identical, and we must have $h=g$. Therefore,

$$h(x,\xi)=g(x,\xi)=g(\xi,x) \qquad \text{(self-adjoint case)} \tag{2.77}$$

When $g(x,\xi)=g(\xi,x)$, the Green's function g is said to be *symmetric.* Substituting (2.77) into (2.74) gives, for the self-adjoint case,

$$\begin{aligned} u(\xi) &= \int_a^b f(x)g(x,\xi)w(x)dx \\ &\quad + \left\{p(x)\left[\frac{du(x)}{dx}g(x,\xi)-u(x)\frac{dg(x,\xi)}{dx}\right]\right\}\Bigg|_{x=a}^{x=b} \end{aligned} \tag{2.78}$$

We note that in the self-adjoint case, we have produced the useful result that it is unnecessary to consider any aspect of the adjoint problem. Indeed, (2.78) involves the Green's function $g(x,\xi)$ rather than the adjoint Green's function $h(x,\xi)$.

The only remaining step in the solution to SLP1 is the specific determination of $g(x,\xi)$. The differential equation that describes the Green's function is given by (2.65). We write this equation for $x\neq\xi$ as follows:

$$L_\lambda g(x,\xi)=0, \qquad x\neq\xi \tag{2.79}$$

This homogeneous second-order equation can be solved in the following two regions:

a. Region 1: $a < x < \xi$

b. Region 2: $\xi < x < b$

Since the equation is of second order, the solution in Region 1 will contain two as yet undetermined coefficients. In Region 2, the solution will contain two additional undetermined coefficients. These four coefficients require four constraints on the Green's function for their determination. The conditions $B_1(g) = 0$ and $B_2(g) = 0$ provide two constraints. The remaining two are provided by conditions joining together the two regions at $x = \xi$. Recall that in seeking a solution $u(x)$ to $(L - \lambda)u = f$, we have required that $u(x)$ be twice differentiable. In the solution to (2.65), however, we relax this requirement so that the Green's function is required to be simply differentiable on the interval $a < x < b$. This relaxation is logical since the second differentiation of the Green's function produces a singularity function $\delta(x - \xi)$ at $x = \xi$. Since a differentiable function is continuous, the third constraint on the Green's function is that it must be continuous at $x = \xi$. For the fourth constraint, we write explicitly the Sturm–Liouville operator in (2.65), viz.

$$\left\{-\frac{1}{w(x)}\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x) - \lambda\right\} g(x, \xi) = \frac{\delta(x - \xi)}{w(x)} \tag{2.80}$$

We multiply by $w(x)$ and integrate over the region $(\xi - \epsilon, \xi + \epsilon)$ to give

$$-\int_{\xi-\epsilon}^{\xi+\epsilon} \frac{d}{dx}\left(p\frac{dg}{dx}\right) dx + \int_{\xi-\epsilon}^{\xi+\epsilon} (q - \lambda) g w dx = 1 \tag{2.81}$$

In the second integral, since q, g, and w are continuous, $(q - \lambda)gw$ is continuous over the interval $[\xi - \epsilon, \xi + \epsilon]$. Since the interval is closed and bounded, the continuous function $(q - \lambda)gw$ is bounded on the interval. Let M be the upper bound on $|(q - \lambda)gw|$. Then,

$$\left|\int_{\xi-\epsilon}^{\xi+\epsilon} (q - \lambda) g w dx\right| \leq 2\epsilon M$$

and therefore,

$$\lim_{\epsilon \to 0}\left|\int_{\xi-\epsilon}^{\xi+\epsilon} (q - \lambda) g w dx\right| = 0$$

Performing the integration in the first integral in (2.81) and taking the limit as $\epsilon \to 0$ gives the fourth constraint, viz.

$$\left.\frac{dg}{dx}\right|_{\xi^+} - \left.\frac{dg}{dx}\right|_{\xi^-} = -\frac{1}{p(\xi)} \tag{2.82}$$

where we have used the continuity of $p(x)$ at $x = \xi$. Typically, our notation ξ^- indicates the limit as $x \to \xi$ from below. We collect the characteristics of the Green's function $g(x, \xi)$ that allow for its determination as follows:

$$L_\lambda g = 0, \qquad x \neq \xi \tag{2.83}$$

$$B_1(g) = 0 \tag{2.84}$$

$$B_2(g) = 0 \tag{2.85}$$

$$g|_{\xi^-} = g|_{\xi^+} \tag{2.86}$$

$$\left.\frac{dg}{dx}\right|_{\xi^+} - \left.\frac{dg}{dx}\right|_{\xi^-} = -\frac{1}{p(\xi)} \tag{2.87}$$

We summarize the procedure for determining the Green's function as follows:

1. Solve (2.83) for $x < \xi$ and for $x > \xi$. The result will contain four as yet undetermined coefficients.
2. Apply the boundary conditions indicated in (2.84) and (2.85). These two conditions will result in determination of two of the four coefficients.
3. Apply the *continuity condition* (2.86) and the *jump condition* (2.87). These two conditions will result in the determination of the final two coefficients.

We demonstrate the procedure in several examples.

EXAMPLE 2.8 Consider the following Green's function problem on $x \in (0, b)$:

$$-\frac{d^2g}{dx^2} - k^2 g = \delta(x - \xi) \tag{2.88}$$

$$g(0, \xi) = \frac{dg(0, \xi)}{dx} = 0 \tag{2.89}$$

We shall solve for the Green's function $g(x, \xi)$ by using (2.83)–(2.87). For $x \neq \xi$, (2.88) becomes

$$-\frac{d^2g}{dx^2} - k^2 g = 0, \qquad x \neq \xi \tag{2.90}$$

A solution to (2.90) can be written for the two regions bisected by $x = \xi$ as follows:

$$g(x, \xi) = \begin{cases} A \sin kx + B \cos kx, & x < \xi \\ C \sin kx + D \cos kx, & x > \xi \end{cases} \tag{2.91}$$

This result can be verified by substituting (2.91) into (2.90) to show that the differential equation is satisfied. We apply the two boundary conditions, given explicitly by (2.89). The result is $A = B = 0$, so that

$$g(x,\xi) = \begin{cases} 0, & x < \xi \\ C \sin kx + D \cos kx, & x > \xi \end{cases} \tag{2.92}$$

The remaining two coefficients C, D are evaluated by applying the continuity and jump conditions. Continuity gives

$$C \sin k\xi + D \cos k\xi = 0$$

Jump gives

$$k(C \cos k\xi - D \sin k\xi) = -1$$

Solving simultaneously gives

$$C = -\frac{\cos k\xi}{k}$$

$$D = \frac{\sin k\xi}{k}$$

Substitution into (2.92) and application of a trigonometric identity yields

$$g(x,\xi) = \begin{cases} 0, & x < \xi \\ \dfrac{\sin k(\xi - x)}{k}, & x > \xi \end{cases} \tag{2.93}$$

It is instructive to verify that (2.93) satisfies the requirements for the Green's function given in (2.83)–(2.87). The details are left for the problems.

■

EXAMPLE 2.9 Consider the following Green's function problem on $x \in (0, a)$:

$$-\frac{d^2 g}{dx^2} - k^2 g = \delta(x - \xi) \tag{2.94}$$

$$g(0,\xi) = g(a,\xi) = 0 \tag{2.95}$$

Proceeding as in Example 2.8, we obtain

$$g(x,\xi) = \begin{cases} A \sin kx + B \cos kx, & x < \xi \\ C \sin k(a - x) + D \cos k(a - x), & x > \xi \end{cases} \tag{2.96}$$

Note that the form of solution for $x > \xi$ is chosen so that the arguments for the sine and cosine are equal to zero at the boundary $x = a$. This form satisfies (2.90), while making the evaluation of the undetermined coefficients C and D easier. (The reader should verify that selection of the form $C \sin kx + D \cos kx$ would yield the same result; but the process of coefficient evaluation would be more complicated.) Applying the boundary conditions given in (2.95), we obtain $B = D = 0$, and

$$g(x,\xi) = \begin{cases} A \sin kx, & x < \xi \\ C \sin k(a-x), & x > \xi \end{cases} \tag{2.97}$$

The remaining two coefficients C, D are evaluated by applying the continuity and jump conditions. The results are

$$C = \frac{\sin k\xi}{k \sin ka} \tag{2.98}$$

$$A = \frac{\sin k(a-\xi)}{k \sin ka} \tag{2.99}$$

Substitution into (2.97) yields

$$g(x,\xi) = \frac{1}{k \sin ka} \begin{cases} \sin k(a-\xi) \sin kx, & x < \xi \\ \\ \sin k(a-x) \sin k\xi, & x > \xi \end{cases} \tag{2.100}$$

Note that the Green's function derived in (2.100) is symmetric, $g(x,\xi) = g(\xi,x)$, a result that we anticipate from the unmixed boundary conditions in (2.95) and the self-adjoint property in (2.77).

■

We next summarize the steps for solving the differential equation $L_\lambda u = f$ by the Green's function method. We distinguish two cases.

Nonself-Adjoint Green's Function Problem

1. Write the solution in the form given by (2.74).
2. Substitute the boundary conditions $B_1(u) = \alpha$, $B_2(u) = \beta$ into (2.74).
3. Substitute the adjoint boundary conditions $B_1^*(h) = 0$, $B_2^*(h) = 0$ into (2.74).
4. Solve the Green's function problem given by (2.65)–(2.67).
5. Obtain the adjoint Green's function through (2.76) and substitute into (2.74).
6. Interchange the variables x and ξ in (2.74).

Self-Adjoint Green's Function Problem

1. Write the solution in the form given by (2.78).
2. Substitute the boundary conditions $B_1(u) = \alpha$, $B_2(u) = \beta$ into (2.78).
3. Substitute the boundary conditions $B_1(g) = 0$, $B_2(g) = 0$ into (2.78).
4. Solve the Green's function problem given by (2.65)–(2.67) and substitute into (2.78).
5. Interchange the variables x and ξ in (2.78).

We remark again that, in the self-adjoint case, there is no necessity for considering any aspect of the adjoint problem. We next illustrate these procedures with some examples.

EXAMPLE 2.10 Consider the following differential equation on $x \in (0, b)$:

$$-u'' - k^2 u = f$$

with boundary conditions

$$u(0) = \alpha$$

$$u'(0) = \beta$$

These boundary conditions define an initial value problem, which we know is never self-adjoint. We therefore use (2.74) and obtain

$$u(\xi) = \int_0^b f(x)h(x,\xi)dx + \alpha\frac{dh(0,\xi)}{dx} - \beta h(0,\xi) \tag{2.101}$$

where the adjoint Green's function equation is given by

$$-\frac{d^2h}{dx^2} - k^2 h = \delta(x - \xi) \tag{2.102}$$

and where we have used the adjoint boundary conditions

$$h(b,\xi) = \frac{dh(b,\xi)}{dx} = 0 \tag{2.103}$$

As shown in (2.76), we can obtain the adjoint Green's function $h(x,\xi)$ directly from the Green's function problem, given in this case by

$$-\frac{d^2g}{dx^2} - k^2 g = \delta(x - \xi) \tag{2.104}$$

$$g(0,\xi) = \frac{dg(0,\xi)}{dx} = 0 \tag{2.105}$$

But, we have obtained this Green's function in (2.93), Example 2.8, as follows:

$$g(x,\xi) = \begin{cases} 0, & x < \xi \\ \dfrac{\sin k(\xi - x)}{k}, & x > \xi \end{cases} \tag{2.106}$$

Application of (2.76) yields the adjoint Green's function, viz.

$$h(x,\xi) = \begin{cases} 0, & x > \xi \\ \dfrac{\sin k(x - \xi)}{k}, & x < \xi \end{cases} \tag{2.107}$$

From (2.107), we also obtain

$$h(0,\xi) = -\frac{\sin k\xi}{k} \tag{2.108}$$

$$\frac{dh(0,\xi)}{dx} = \cos k\xi \tag{2.109}$$

Substitution of (2.107)–(2.109) into (2.101) gives

$$u(\xi) = \int_0^{\xi} f(x)\frac{\sin k(x-\xi)}{k}dx + \beta\frac{\sin k\xi}{k} + \alpha\cos k\xi$$

An interchange of x and ξ yields the final solution, viz.

$$u(x) = \int_0^{x} f(\xi)\frac{\sin k(\xi-x)}{k}d\xi + \beta\frac{\sin kx}{k} + \alpha\cos kx \tag{2.110}$$

It is important to assure that our solution in (2.110) satisfies the differential equation and the boundary conditions. To do so, it is necessary to twice differentiate (2.110). This differentiation requires some care. We note that the integral in (2.110) involves a variable upper limit. To differentiate, we make use of a theorem [10], as follows: "The derivative of the definite integral of a continuous function with respect to the upper limit of integration is equal to the value of the integrand function at this upper limit." In (2.110), however, the variable x occurs not only in the upper limit, but also under the integral sign. To remedy this problem, we write (2.110) as follows:

$$u(x) = \frac{1}{k}\mathrm{Im}\left[e^{-ikx}\int_0^{x} f(\xi)e^{ik\xi}d\xi\right] + \beta\frac{\sin kx}{k} + \alpha\cos kx \tag{2.111}$$

Since real differentiation and the imaginary part operator can be interchanged, it is now straightforward to show that this solution satisfies the original differential equation and the boundary conditions. The details are left for Problem 2.11.

■

EXAMPLE 2.11 Consider the following differential equation on $x \in (0, 1)$:

$$-u'' = f$$

with boundary conditions

$$u(0) = \alpha$$

$$u(1) = 0$$

The associated Green's function problem is

$$-\frac{d^2 g}{dx^2} = \delta(x - \xi)$$

$$g(0, \xi) = g(1, \xi) = 0$$

Since the boundary conditions are unmixed, the Green's function problem is self-adjoint. We therefore use (2.78). After application of the boundary conditions on $u(x)$ and $g(x, \xi)$, we have

$$u(\xi) = \int_0^1 f(x) g(x, \xi) dx + \alpha \frac{dg(0, \xi)}{dx} \tag{2.112}$$

Using the procedure for Green's function evaluation, we find that

$$g(x, \xi) = \begin{cases} (1 - \xi)x, & x < \xi \\ (1 - x)\xi, & x > \xi \end{cases}$$

and

$$\frac{dg(0, \xi)}{dx} = 1 - \xi$$

Substitution into (2.112) followed by an interchange of variables yields

$$u(x) = (1 - x) \int_0^x \xi f(\xi) d\xi + x \int_x^1 (1 - \xi) f(\xi) d\xi + \alpha(1 - x)$$

We leave it to the reader to show that this solution satisfies the differential equation and the boundary conditions.

■

In this section, we have defined the requirements for SLP1 and have given a procedure for its solution by the Green's function method. Note that the parameter λ and the forcing function $f(x)$ were constrained to be real. In many of the interesting problems of electromagnetic theory, λ and $f(x)$ are complex. We consider this case in the next section.

2.5 STURM–LIOUVILLE PROBLEM OF THE SECOND KIND

For the second form of the Sturm–Liouville problem, we consider $(L - \lambda)u = f$ over a finite interval $x \in (a, b)$ and for complex λ and complex f. Since we are now dealing with complex quantities, the Hilbert space $\mathcal{L}_2(a, b)$ is now defined with complex inner product

$$\langle u, v \rangle = \int_a^b u(x)\bar{v}(x)w(x)dx \tag{2.113}$$

for all $u, v \in \mathcal{L}_2(a, b)$. We define the *Sturm–Liouville Problem of the Second Kind*, abbreviated SLP2, as follows:

$$L_\lambda u = f, \qquad -\infty < a < x < b < \infty \tag{2.114}$$

where

$$L_\lambda = L - \lambda \tag{2.115}$$

and where

$$L = -\frac{1}{w(x)}\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x) \tag{2.116}$$

We impose the following restrictions

a. p, p', q, w are real and continuous for $a \leq x \leq b$
b. $p(x) > 0, w(x) > 0$ for $a \leq x \leq b$
c. λ is complex and independent of x

In addition, we require $u(x) \in \mathcal{D}_{L_\lambda} \subset \mathcal{L}_2(a, b)$, where $\mathcal{D}_{L_\lambda}$ is the domain of the operator L_λ. Because we are dealing with second-order differential operators, the domain is restricted to those functions that are twice differentiable. Finally, we require that $u(x)$ satisfy two boundary conditions as follows:

$$B_1(u) = \alpha = \alpha_{11}u(a) + \alpha_{12}u'(a) + \alpha_{13}u(b) + \alpha_{14}u'(b) \tag{2.117}$$

$$B_2(u) = \beta = \alpha_{21}u(a) + \alpha_{22}u'(a) + \alpha_{23}u(b) + \alpha_{24}u'(b) \tag{2.118}$$

where the α_{ij} are real. Because of the generalization of λ and $f(x)$ to include complex values, we anticipate that the solution $u(x)$ to (2.114) will be a complex function. Since $u(x)$ is complex, (2.117) and (2.118) will, in general, generate complex values of α and β. We note, however, that the operator L is real; that is,

$$\overline{Lu} = L\bar{u} \tag{2.119}$$

In a similar manner to the SLP1 development, we may show that the operator L in SLP2 is *formally self-adjoint*. Indeed, for $u, v \in \mathcal{L}_2(a, b)$, we find that

$$\begin{aligned}
\langle Lu, v\rangle &= \int_a^b \left\{-\frac{1}{w(x)}\frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x)\right\}\bar{v}(x)w(x)dx \\
&= \int_a^b u(x)\left\{-\frac{1}{w(x)}\frac{d}{dx}\left[p(x)\frac{d\bar{v}(x)}{dx}\right] + q(x)\bar{v}(x)\right\}w(x)dx \\
&\quad - \left\{p(x)\left[\bar{v}(x)\frac{du(x)}{dx} - u(x)\frac{d\bar{v}(x)}{dx}\right]\right\}\Bigg|_a^b \\
&= \langle u, Lv\rangle + J(u, v)\Bigg|_a^b
\end{aligned} \tag{2.120}$$

where we have used $L\bar{v} = \overline{Lv}$ and where, for SLP2,

$$J(u, v) = -p(u'\bar{v} - u\bar{v}') \tag{2.121}$$

We shall require the same restrictions on u and v as those developed for SLP1. For homogeneous boundary conditions, we place the following restrictions on $u(x)$:

a. $u \in \mathcal{L}_2(a, b)$

b. $u \in \mathcal{D}_L$

c. u satisfies two boundary conditions, $B_1(u) = 0$, $B_2(u) = 0$

We place the following restrictions on $v(x)$:

a. $v \in \mathcal{L}_2(a, b)$

b. $v \in \mathcal{D}_{L^*}$

c. v satisfies two *adjoint boundary conditions*, $B_1^*(v) = 0$, $B_2^*(v) = 0$

We define the adjoint boundary conditions to be those conditions $B_1^*(v) = 0$ $B_2^*(v) = 0$ that, when coupled with the boundary conditions on $u(x)$, result in the vanishing of the conjunct, viz.

$$J(u, v)\Bigg|_a^b = 0 \tag{2.122}$$

We note in (2.121) that the conjunct involves $\bar{v}$, rather than v. However, the conditions on v are easily produced from the conditions on $\bar{v}$. We define the *conjugate adjoint boundary conditions* by

$$\alpha_{11}^*\bar{v}(a) + \alpha_{12}^*\bar{v}'(a) + \alpha_{13}^*\bar{v}(b) + \alpha_{14}^*\bar{v}'(b) = 0 \tag{2.123}$$

$$\alpha_{21}^*\bar{v}(a) + \alpha_{22}^*\bar{v}'(a) + \alpha_{23}^*\bar{v}(b) + \alpha_{24}^*\bar{v}'(b) = 0 \tag{2.124}$$

where the α_{ij}^*'s are chosen so that (2.122) is satisfied. Taking the complex conjugate of (2.123) and (2.124) and noting that the α_{ij}^*'s are real, we obtain

$$\alpha_{11}^*v(a) + \alpha_{12}^*v'(a) + \alpha_{13}^*v(b) + \alpha_{14}^*v'(b) = 0 \tag{2.125}$$

$$\alpha_{21}^*v(a) + \alpha_{22}^*v'(a) + \alpha_{23}^*v(b) + \alpha_{24}^*v'(b) = 0 \tag{2.126}$$

We conclude that, because the α_{ij}'s are real in SLP2, the conjugate adjoint boundary conditions and the adjoint boundary conditions are identical.

We next consider specific conditions that result in L being self-adjoint in SLP2. For the unmixed boundary case, we again have

$$\alpha_{11}u(a) + \alpha_{12}u'(a) = 0 \tag{2.127}$$

$$\alpha_{23}u(b) + \alpha_{24}u'(b) = 0 \tag{2.128}$$

We use these expressions in the conjunct to eliminate $u'(a)$ and $u'(b)$, viz.

$$J(u, v)\Big|_a^b = p(b)u(b)\left[\frac{\alpha_{23}}{\alpha_{24}}\bar{v}(b) + \bar{v}'(b)\right] - p(a)u(a)\left[\frac{\alpha_{11}}{\alpha_{12}}\bar{v}(a) + \bar{v}'(a)\right] \tag{2.129}$$

The conjunct in (2.129) vanishes provided

$$\alpha_{11}\bar{v}(a) + \alpha_{12}\bar{v}'(a) = 0 \tag{2.130}$$

$$\alpha_{23}\bar{v}(b) + \alpha_{24}\bar{v}'(b) = 0 \tag{2.131}$$

which, in SLP2, always implies that

$$\alpha_{11}v(a) + \alpha_{12}v'(a) = 0 \tag{2.132}$$

$$\alpha_{23}v(b) + \alpha_{24}v'(b) = 0 \tag{2.133}$$

We note that, in SLP2, the unmixed boundary case yields boundary conditions on $v(x)$ identical to those on $u(x)$. Therefore, the linear manifold $\mathcal{M}_L$ is the same as the linear manifold $\mathcal{M}_{L^*}$. We conclude that unmixed boundary conditions in SLP2 yield a self-adjoint operator just as in SLP1. We remark that this result depends on the restriction to real α_{ij}. We shall find in the next chapter that this restriction has a dramatic effect on the eigenvalues of the operator L.

EXAMPLE 2.12 We consider characteristics of the following differential equation on $x \in (0, 1)$:

$$(L - \lambda)u = f$$

with $L = -d^2/dx^2$ and with boundary conditions

$$u'(0) - \alpha u(0) = 0$$
$$u(1) = 0$$

where λ and f are complex and α is real. We note that the problem meets all of the requirements for SLP2. In addition, the boundary conditions are unmixed. We therefore conclude that the operator L is self-adjoint. We stress that a different result would have been obtained if $\alpha \in$ C. Indeed, for $u, v \in \mathcal{L}_2(a, b)$, we have

$$\langle Lu, v\rangle = \langle u, Lv\rangle - u'(1)\bar{v}(1) - u(0)\left[\bar{v}'(0) - \alpha\bar{v}(0)\right]$$

where we have applied the boundary conditions on u. The conjugate adjoint boundary conditions that reduce the conjunct to zero are

$$\bar{v}'(0) - \alpha\bar{v}(0) = 0$$
$$\bar{v}(1) = 0$$

Taking the complex conjugate, we have

$$v'(0) - \bar{\alpha}v(0) = 0$$
$$v(1) = 0$$

We conclude that the conditions on v are not the same as the conditions on u, and therefore the operator L is no longer self-adjoint. We shall investigate the distinction between real and complex α in this example again in the next chapter.

■

We next consider homogeneous initial conditions. Following a similar procedure to that in (2.127)–(2.133), we find that the initial condition case is not self-adjoint in SLP2. The details are left for the problems. For periodic conditions in SLP2, a similar procedure shows that the operator L is self-adjoint, provided that $p(a) = p(b)$. Again, the details are left for the problems.

The solution procedure for SLP1 has been given in (2.71)–(2.74). We now show that the solution to SLP2 follows along similar lines, with modification to accommodate complex λ and $f(x)$. We take the inner product of $L_\lambda u$ with the adjoint Green's function h and integrate by parts twice to give

$$\langle L_\lambda u, h\rangle = \langle u, L_\lambda^* h\rangle + J(u,h)\Big|_{x=a}^{x=b} \tag{2.134}$$

where

$$L_\lambda^* = L - \bar{\lambda} \tag{2.135}$$

and

$$J(u,h) = -p(x)\left[\frac{du(x)}{dx}\bar{h}(x,\xi) - u(x)\frac{d\bar{h}(x,\xi)}{dx}\right] \tag{2.136}$$

The adjoint Green's function problem is given by

$$L_\lambda^* h = \frac{\delta(x-\xi)}{w(x)} \tag{2.137}$$

Substitution of (2.137) into (2.134) gives

$$u(\xi) = \langle f, h\rangle - J(u,h)\Big|_{x=a}^{x=b} \tag{2.138}$$

or, explicitly,

$$u(\xi) = \int_a^b f(x)\bar{h}(x,\xi)w(x)dx + \left\{p(x)\left[\frac{du(x)}{dx}\bar{h}(x,\xi) - u(x)\frac{d\bar{h}(x,\xi)}{dx}\right]\right\}\Big|_{x=a}^{x=b} \tag{2.139}$$

We note that (2.139) is the solution to SLP2, provided that we can determine the *conjugate adjoint Green's function* $\bar{h}(x,\xi)$. Taking the complex conjugate of both sides of (2.137), we obtain

$$\overline{L_\lambda^* h(x,\xi)} = L_\lambda \bar{h}(x,\xi) = \frac{\delta(x-\xi)}{w(x)} \tag{2.140}$$

$$B_1^*(\bar{h}) = 0 \tag{2.141}$$

$$B_2^*(\bar{h}) = 0 \tag{2.142}$$

Similar to the SLP1 case, the selection of the boundary conditions proceeds as follows. We first assume that the boundary conditions on u are homogeneous. Then, $B_1^*(\bar{h})$ and $B_2^*(\bar{h})$ are chosen such that the conjunct in (2.138) vanishes. The extension to the inhomogeneous case, however, is now available. We simply apply the given boundary conditions $B_1(u) = \alpha$,

$B_2(u) = \beta$ in conjunction with the conjugate adjoint boundary conditions $B_1^*(\bar{h}) = 0$, $B_2^*(\bar{h}) = 0$.

As in the case of SLP1, we can show that it is never necessary to find the conjugate adjoint Green's function directly. Indeed, we form

$$\langle L_\lambda g(x,\xi), h(x,\xi')\rangle = \langle g(x,\xi), L_\lambda^* h(x,\xi')\rangle + J(g,h)\Big|_{x=a}^{x=b} \tag{2.143}$$

from which

$$\bar{h}(\xi,\xi') = g(\xi',\xi)$$

or, with a variable change,

$$\bar{h}(x,\xi) = g(\xi,x) \tag{2.144}$$

We shall always proceed to find the Green's function $g(x,\xi)$, and then produce the conjugate adjoint Green's function $\bar{h}(x,\xi)$ from (2.144). Substitution of $\bar{h}(x,\xi)$ into (2.139) completes the solution to SLP2.

A further simplification occurs when the operator L is self-adjoint. We have

$$L_\lambda g(x,\xi) = \frac{\delta(x-\xi)}{w(x)} \tag{2.145}$$

$$B_1(g) = 0 \tag{2.146}$$

$$B_2(g) = 0 \tag{2.147}$$

and

$$L_\lambda \bar{h}(x,\xi) = \frac{\delta(x-\xi)}{w(x)} \tag{2.148}$$

$$B_1(\bar{h}) = 0 \tag{2.149}$$

$$B_2(\bar{h}) = 0 \tag{2.150}$$

The fact that the boundary conditions on $\bar{h}$ are identical to the boundary conditions on g is deduced as follows. First, the conditions on g and h are identical from the self-adjoint property of L; second, the conditions on h and $\bar{h}$ are always identical in SLP2. We conclude from (2.145)–(2.150) that $g(x,\xi) = \bar{h}(x,\xi)$, and therefore, using (2.144), we find that

$$g(x,\xi) = \bar{h}(x,\xi) = g(\xi,x) \qquad \text{(self-adjoint case)} \tag{2.151}$$

Substitution into (2.139) gives, for the self-adjoint case,

$$u(\xi) = \int_a^b f(x)g(x,\xi)w(x)dx + \left\{ p(x)\left[\frac{du(x)}{dx} g(x,\xi) - u(x)\frac{dg(x,\xi)}{dx} \right] \right\} \Bigg|_{x=a}^{x=b} \tag{2.152}$$

We shall summarize the steps for solving SLP2 by the Green's function method. We distinguish two cases.

Nonself-Adjoint Green's Function Problem

1. Write the solution in the form given by (2.139).
2. Substitute the boundary conditions $B_1(u) = \alpha$, $B_2(u) = \beta$ into (2.139).
3. Substitute the adjoint boundary conditions $B_1^*(\bar{h}) = 0$, $B_2^*(\bar{h}) = 0$ into (2.139).
4. Solve the Green's function problem given by (2.65)–(2.67).
5. Obtain the conjugate adjoint Green's function $\bar{h}$ through (2.144) and substitute into (2.139).
6. Interchange the variables x and ξ in (2.139).

Self-Adjoint Green's Function Problem

1. Write the solution in the form given by (2.152).
2. Substitute the boundary conditions $B_1(u) = \alpha$, $B_2(u) = \beta$ into (2.152).
3. Substitute the boundary conditions $B_1(g) = 0$, $B_2(g) = 0$ into (2.152).
4. Solve the Green's function problem given by (2.65)–(2.67) and substitute into (2.152).
5. Interchange the variables x and ξ in (2.152).

It is interesting and extremely useful to note that for L self-adjoint, the procedure for obtaining the solution to SLP1 and SLP2 is identical. Indeed, we have proved that in both of these cases, we may obtain the solution in terms of the Green's function $g(x,\xi)$ rather than the adjoint Green's function (SLP1) or the conjugate adjoint Green's function (SLP2). Specifically, (2.78) and (2.151) are identical. It is only in the cases of nonself-adjoint operators where we use the adjoint Green's function (SLP1) or the conjugate adjoint Green's function (SLP2), respectively. We illustrate these ideas in the following examples.

EXAMPLE 2.13 Consider the following differential equation on $x \in (0, a)$:

$$(L - \lambda)u = f$$

$$L = -\frac{d^2}{dx^2}$$

with boundary conditions

$$u'(0) = 0$$

$$u'(a) = 0$$

where f and λ are complex. The problem is of class SLP2. Since the boundary conditions are unmixed, the operator L is self-adjoint with respect to the complex inner product

$$\langle u, v \rangle = \int_0^a u(x)\bar{v}(x)dx$$

Because of the self-adjoint property of L, the form of the solution is given by (2.152). In this case,

$$u(\xi) = \int_0^a f(x)g(x, \xi)dx$$

The self-adjoint property produces a symmetric Green's function, so that

$$u(x) = \int_0^a f(\xi)g(x, \xi)d\xi$$

where we require the solution to the Green's function problem

$$-\frac{d^2g}{dx^2} - \lambda g = \delta(x - \xi)$$

with boundary conditions

$$\frac{dg(0, \xi)}{dx} = \frac{dg(a, \xi)}{dx} = 0$$

We form

$$g = \begin{cases} A\cos\sqrt{\lambda}x, & x < \xi \\ B\cos\sqrt{\lambda}(a - x), & x > \xi \end{cases}$$

where we have applied the boundary conditions at $x = 0$ and $x = a$ to eliminate two coefficients. Application of the continuity and jump conditions at $x = \xi$ yields

$$A = -\frac{\cos\sqrt{\lambda}(a - \xi)}{\sqrt{\lambda}\sin\sqrt{\lambda}a}$$

$$B = -\frac{\cos\sqrt{\lambda}\xi}{\sqrt{\lambda}\sin\sqrt{\lambda}a}$$

The Green's function therefore is given by

$$g(x,\xi) = -\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}a}\begin{cases}\cos\sqrt{\lambda}x\cos\sqrt{\lambda}(a-\xi), & x<\xi\\ \cos\sqrt{\lambda}\xi\cos\sqrt{\lambda}(a-x), & x>\xi\end{cases}$$

As expected from the self-adjoint property, the Green's function is symmetric.

■

EXAMPLE 2.14 Consider the following differential equation on $x \in (0, b)$:

$$-u'' - k^2 u = f$$

with boundary conditions

$$u(0) = \alpha$$

$$u'(0) = \beta$$

where k, α, β, f are complex. The problem is of class SLP2. It is identical to Example 2.10 except for the extension to complex k, α, β, f. Since the boundary conditions define an initial value problem, it is not self-adjoint. We therefore use (2.139) and find that

$$u(\xi) = \int_0^b f(x)\bar{h}(x,\xi)dx + \alpha\frac{d\bar{h}(0,\xi)}{dx} - \beta\bar{h}(0,\xi)$$

where the conjugate adjoint Green's function equation is given by

$$-\frac{d^2\bar{h}}{dx^2} - k^2\bar{h} = \delta(x-\xi)$$

and where we have used the conjugate adjoint boundary conditions

$$\bar{h}(b,\xi) = \frac{d\bar{h}(b,\xi)}{dx} = 0$$

We note that the conjugate adjoint Green's function problem is identical to the adjoint Green's function problem in Example 2.10. The solution therefore proceeds identically, and we produce the following result:

$$u(x) = \int_0^x f(\xi)\frac{\sin k(\xi - x)}{k}d\xi + \beta\frac{\sin kx}{k} + \alpha\cos kx$$

Although the form of solution is the same as in Example 2.10, complex k, α, β, f produces a complex solution $u(x)$.

■

In our study of SLP1 and SLP2 problems, SLP1 could properly be considered as SLP2, with the specialization that all quantities are real. We now take that point of view and classify all problems so far studied in this chapter as SLP2.

Green's function problems classified as SLP2 do not nearly exhaust all of the cases of practical interest. There are many problems of interest in electromagnetics that do not satisfy the requirements of SLP2. Such problems are classified SLP3 and are considered in the next section.

2.6 STURM–LIOUVILLE PROBLEM OF THE THIRD KIND

In defining the *Sturm–Liouville Problem of the Third Kind*, abbreviated SLP3, we again consider the following differential equation:

$$L_\lambda u = f, \qquad a < x < b \tag{2.153}$$

where

$$L_\lambda = L - \lambda, \qquad \lambda \in \mathrm{C} \tag{2.154}$$

and where

$$L = -\frac{1}{w(x)}\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x) \tag{2.155}$$

In SLP2, we demanded that the interval (a, b) be finite and that the coefficients in (2.155) satisfy the following conditions:

a. p, p', q, w are real and continuous for $a \leq x \leq b$
b. $p(x) > 0$, $w(x) > 0$ for $a \leq x \leq b$

If the interval (a, b) is not finite, or if any of the above conditions on the coefficients is violated, the problem is SLP3. In the mathematical literature, SLP2 problems are termed *regular* Sturm–Liouville problems, while SLP3 problems are termed *singular* [11]. We consider the SLP3 problem in Hilbert space $\mathcal{L}_2(a, b)$ with inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g}(x)w(x)dx \tag{2.156}$$

There are several classes of SLP3 problems that are important in electromagnetic applications, defined by the following situations:

1. The interval is semi-infinite. In this problem, there is a *singular point* as $x \to \infty$.

2. The interval is finite, but $p(x) = 0$ at an endpoint. In this problem, there is a singular point at the endpoint where $p(x)$ vanishes.
3. The interval is $(-\infty, \infty)$. In this problem, there are singular points as $x \to \pm\infty$.
4. The interval is semi-infinite, and $p(x)$ vanishes at the finite endpoint. In this problem, there are singular points as $x \to \infty$ and at the finite endpoint.

We shall classify singular problems by considering the homogeneous equation associated with (2.153), viz.

$$L_\lambda u = 0 \tag{2.157}$$

According to Weyl's theorem [12]:

1. If for a particular value of λ, every u that is a solution to (2.157) is in $\mathcal{L}_2(a, b)$, then for all λ, every u is in $\mathcal{L}_2(a, b)$.
2. For every λ with $\mathrm{Im}(\lambda) \neq 0$, there exists at least one $u \in \mathcal{L}_2(a, b)$.

We omit the proof of this theorem and refer the reader to [12]. The theorem effectively divides singular problems into two mutually exclusive cases [13]:

1. The *limit circle* case: All solutions u are in $\mathcal{L}_2(a, b)$ for all λ.
2. The *limit point* case: There is either one solution or no solutions in $\mathcal{L}_2(a, b)$, according to the following:
   a. If $\mathrm{Im}(\lambda) \neq 0$, there exists exactly one solution u in $\mathcal{L}_2(a, b)$.
   b. If $\mathrm{Im}(\lambda) = 0$, there is either one solution or no solutions in $\mathcal{L}_2(a, b)$.

We note that we can determine the limit point or limit circle case by examination of the solutions to (2.157) at a single value of λ. If all solutions u are in $\mathcal{L}_2(a, b)$, the limit circle case applies. If not, by exclusion, the limit point case applies.

EXAMPLE 2.15 Consider the following differential equation in $\mathcal{L}_2(0, \infty)$:

$$\left(-\frac{d^2}{dx^2} - \lambda\right) u = 0 \tag{2.158}$$

The endpoint $x = 0$ is a regular point. The endpoint $x \to \infty$ is a singular point. To determine the limit point or limit circle case, we examine solutions to (2.158) for $\lambda = 0$. Two linearly independent solutions are $u_1 = 1$ and $u_2 = x$, neither of which is absolutely square integrable over $(0, \infty)$. Therefore, the solutions are not in $\mathcal{L}_2(0, \infty)$ and we have the limit point case.

■

EXAMPLE 2.16 Consider Bessel's equation of order zero in $\mathcal{L}_2(0, a)$:

$$\left[-\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right) - \lambda\right]u = 0, \qquad 0 < a < \infty \tag{2.159}$$

The endpoint $x = a$ is a regular point. Since $p(x) = x = 0$ at $x = 0$, the endpoint $x = 0$ is a singular point. To determine the limit point or limit circle case, we examine two linearly independent solutions to (2.159) for $\lambda = 0$, namely, $u_1 = 1$ and $u_2 = \log x$. Although u_2 is logarithmically singular at $x = 0$, both u_1 and u_2 are absolutely square integrable over $(0, a)$. Therefore, both solutions are in $\mathcal{L}_2(0, a)$, and we have the limit circle case.

■

EXAMPLE 2.17 Consider Bessel's equation of order zero in $\mathcal{L}_2(0, \infty)$:

$$\left[-\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right) - \lambda\right]u = 0 \tag{2.160}$$

Both endpoints are singular points. In such cases, we pick an interior point $x = \xi$ and examine limit point or limit circle conditions on two intervals: $\xi < x < \infty$ and $0 < x < \xi$. From Example 2.16, we have the limit circle case on $0 < x < \xi$. For $\xi < x < \infty$, the endpoint $x = \xi$ is regular and the endpoint $x \to \infty$ is singular. Further, neither $u_1 = 1$ nor $u_2 = \log x$ is absolutely square integrable over (ξ, ∞), and therefore neither is in $\mathcal{L}_2(\xi, \infty)$. We conclude that we have the limit point case. We say that Bessel's equation of order zero in $\mathcal{L}_2(0, \infty)$ is in the limit circle case at $x = 0$ and the limit point case as $x \to \infty$.

■

The method of construction of the Green's function for SLP3 problems is directly related to the limit point and limit circle classifications. We shall proceed by considering a few examples, and follow with some conclusions and generalizations.

EXAMPLE 2.18 Consider the following Green's function problem on the interval $x \in (0, \infty)$:

$$-\frac{d^2g}{dx^2} - \lambda g = \delta(x - \xi), \qquad \lambda \in \mathbf{C} \tag{2.161}$$

with the boundary condition

$$g(0, \xi) = 0$$

In the beginning, we shall not assign a boundary condition as $x \to \infty$. However, the method for dealing with this deficiency will emerge as we proceed. From Example 2.15, we have the limit point case. We begin the construction of the Green's function by considering solutions to the homogeneous equation for $x \neq \xi$, viz.

$$-\frac{d^2 g}{dx^2} - \lambda g = 0, \qquad x \neq \xi \tag{2.162}$$

Possible forms of solution to this equation are $\sin\sqrt{\lambda}x$, $\cos\sqrt{\lambda}x$, $\exp(i\sqrt{\lambda}x)$, and $\exp(-i\sqrt{\lambda}x)$. Since the problem is a limit point problem, we know from Weyl's Theorem that, if $\mathrm{Im}(\lambda) \neq 0$, there is exactly one solution in $\mathcal{L}_2(0, \infty)$. Our task is to find it. (We shall have no need to consider the case where $\mathrm{Im}(\lambda) = 0$ since we can always approach this case by taking a limit as $\mathrm{Im}(\lambda) \to 0$.) Since neither $\sin\sqrt{\lambda}x$ nor $\cos\sqrt{\lambda}x$ is absolutely square integrable over $(0, \infty)$, neither is in $\mathcal{L}_2(0, \infty)$. Consider the two exponential solution forms. We have

$$\int_0^\infty |e^{-i\sqrt{\lambda}x}|^2 dx = \int_0^\infty e^{2(\mathrm{Im}\sqrt{\lambda})x} dx \tag{2.163}$$

and

$$\int_0^\infty |e^{i\sqrt{\lambda}x}|^2 dx = \int_0^\infty e^{-2(\mathrm{Im}\sqrt{\lambda})x} dx \tag{2.164}$$

Which of these two exponential solution forms is in $\mathcal{L}_2(0, \infty)$ depends on whether $\mathrm{Im}(\sqrt{\lambda})$ is negative or positive. Since λ is a parameter specified in the problem statement, we shall choose for definiteness

$$\mathrm{Im}(\sqrt{\lambda}) < 0 \tag{2.165}$$

With this choice

$$\int_0^\infty |e^{-i\sqrt{\lambda}x}|^2 dx < \infty \tag{2.166}$$

and we conclude that $\exp(-i\sqrt{\lambda}x)$ is the one solution to (2.162) in $\mathcal{L}_2(0, \infty)$. We now proceed with the construction of the Green's function in the usual manner. We write

$$g(x, \xi) = \begin{cases} A \sin\sqrt{\lambda}x + C\cos\sqrt{\lambda}x, & x < \xi \\ Be^{-i\sqrt{\lambda}x} + De^{i\sqrt{\lambda}x}, & x > \xi \end{cases} \tag{2.167}$$

Application of the boundary condition at $x = 0$ results in $C = 0$, with the result

$$g(x, \xi) = \begin{cases} A \sin\sqrt{\lambda}x, & x < \xi \\ Be^{-i\sqrt{\lambda}x} + De^{i\sqrt{\lambda}x}, & x > \xi \end{cases} \tag{2.168}$$

In SLP1 or SLP2 problems, we would next apply a second boundary condition to eliminate another coefficient. In the limit point case in SLP3, however, we replace the second boundary condition with the requirement that the solution be in $\mathcal{L}_2(0, \infty)$. Since $\sin\sqrt{\lambda}x$, as used in (2.168), has support only on $(0, \xi)$, the only part of the solution in (2.168) that is not in $\mathcal{L}_2(0, \infty)$ is $\exp(i\sqrt{\lambda}x)$. We therefore choose $D = 0$ and obtain

$$g(x, \xi) = \begin{cases} A \sin\sqrt{\lambda}x, & x < \xi \\ Be^{-i\sqrt{\lambda}x}, & x > \xi \end{cases} \tag{2.169}$$

We next apply the continuity and jump conditions at $x = \xi$ in the usual manner and obtain

$$A = \frac{e^{-i\sqrt{\lambda}\xi}}{\sqrt{\lambda}} \tag{2.170}$$

$$B = \frac{\sin\sqrt{\lambda}\xi}{\sqrt{\lambda}} \tag{2.171}$$

Substitution of these constants into (2.170) gives

$$g(x, \xi) = \frac{1}{\sqrt{\lambda}} \begin{cases} e^{-i\sqrt{\lambda}\xi} \sin\sqrt{\lambda}x, & x < \xi \\ e^{-i\sqrt{\lambda}x} \sin\sqrt{\lambda}\xi, & x > \xi \end{cases} \tag{2.172}$$

where $\sqrt{\lambda}$ is constrained by (2.165). We note that the Green's function derived in (2.172) is symmetric, $g(x, \xi) = g(\xi, x)$.

■

Example 2.18 suggests the following procedure for dealing with Green's functions associated with problems in the limit point case at one boundary, say $x = b$, and regular at the other boundary. First, we write the solution to the Green's function problem in the usual manner, in terms of four undetermined coefficients, as in (2.167). To determine one of the four coefficients, we apply the boundary condition at the regular endpoint. Next, to determine a second coefficient, we apply the requirement that the solution on the interval $\xi < x < b$ must be in $\mathcal{L}_2(a, b)$. The remaining two coefficients are determined in the usual manner by the continuity and jump conditions at $x = \xi$.

Mathematically, for the limit point case, we may show that for $\text{Im}(\lambda) \neq 0$, the single solution to $L_\lambda u = 0$ in $\mathcal{L}_2(a, b)$ is always obtained simply by invoking the $\mathcal{L}_2$ requirement. In addition, if an unmixed boundary condition is applied at the regular endpoint, this condition, together with the $\mathcal{L}_2$ requirement, renders the problem self-adjoint. *No boundary condition* *is required at the limit point boundary*. The proof of this crucial result is contained in a review paper by Hajmirzaahmad and Krall [14]. As we have shown, the self-adjoint property results in a symmetric Green's function.

There is an alternate method leading to the determination of the Green's function in Example 2.18 [15]. Indeed, if we invoke in (2.168) the physically reasonable condition that the Green's function vanishes as $x \to \infty$, we produce the same result as we do by invoking the $\mathcal{L}_2(0, \infty)$ requirement. That is, we can invoke a *limit condition*

$$\lim_{x \to \infty} g(x, \xi) = 0$$

in place of the second "boundary" condition in the statement of the problem. This is an appealing procedure since such an unmixed limit condition can be viewed as an extension of the regular boundary condition

$$g(b, \xi) = 0$$

Indeed, consider the Green's function problem

$$-\frac{d^2 g}{dx^2} - \lambda g = \delta(x - \xi)$$

with boundary conditions

$$g(0, \xi) = g(b, \xi) = 0$$

The result in (2.172) can be obtained by solving this problem and then taking the limit as $b \to \infty$. The details are left for the problems.

In summary, for the case of a regular unmixed boundary condition at $x = a$ and the limit point case at $x = b$, the $\mathcal{L}_2$ requirement takes the place of a boundary condition at $x = b$. Furthermore, the problem is self-adjoint. In the case where $b \to \infty$, we may use the alternate procedure of applying a limit condition in place of the $\mathcal{L}_2$ requirement. We remark that it is sufficient to have a procedure that picks out the one $\mathcal{L}_2$ solution required in the mathematical proofs, such as those in [14]. Invoking the limit condition is such a procedure. We consider these ideas further in the following example.

EXAMPLE 2.19 Consider the following differential equation on $x \in (0, \infty)$:

$$-u'' - \lambda u = f \tag{2.173}$$

with boundary condition

$$u(0) = 0$$

where u, f, λ are complex. We choose the inner product

$$\langle u, v\rangle = \int_0^\infty u(x)\bar{v}(x)dx$$

From Example 2.18, we know that this problem is singular in the limit point case as $x \to \infty$. We therefore invoke the limit condition

$$\lim_{x\to\infty} u(x) = 0$$

The problem is self-adjoint and the Green's function is symmetric. We therefore use (2.152) which, specialized to this case, yields

$$u(\xi) = \int_0^\infty f(x)g(x,\xi)dx + \left[\frac{du(x)}{dx}g(x,\xi) - u(x)\frac{dg(x,\xi)}{dx}\right]\Bigg|_{x=0}^{x=\infty} \tag{2.174}$$

We apply the boundary condition and limit condition on $u(x)$ and choose

$$g(0,\xi) = 0$$

$$\lim_{x\to\infty} g(x,\xi) = 0$$

and find that

$$u(\xi) = \int_0^\infty f(x)g(x,\xi)dx$$

Finally, after interchanging x and ξ, we obtain

$$u(x) = \int_0^\infty f(\xi)g(x,\xi)d\xi$$

where the Green's function $g(x,\xi)$ is given by (2.172).

■

We note that the result in (2.174) is an extension to the result for self-adjoint operators in SLP2. Specifically, the arguments for the SLP2 unmixed boundary case given in (2.122)–(2.133) carry over to the SLP3 limit point case at infinity, provided again that the α_{ij}'s are constrained to be real. We may establish this result simply by observing that the arguments in (2.122)–(2.133) are not altered by taking the limit as $a \to -\infty$ or $b \to \infty$, or both. Since the problem is self-adjoint, the Green's function is symmetric, and the result in (2.174) is assured before solving for the specific Green's function. We shall illustrate this important point in an additional example.

EXAMPLE 2.20 Consider the following differential equation on $x \in (-\infty, \infty)$:

$$(L - \lambda)u = f, \qquad \text{Im}(\sqrt{\lambda}) < 0$$

where

$$L = -\frac{d^2}{dx^2}$$

This problem is in the limit point case as $x \to \infty$ and as $x \to -\infty$. Our procedure in dealing with limit points at both ends of the interval along the real line is to pick an interior point $x = \xi$. Since $\text{Im}(\lambda) \neq 0$, there is exactly one solution to $L_\lambda u = 0$ in $\mathcal{L}_2(-\infty, \xi)$ and exactly one solution in $\mathcal{L}_2(\xi, \infty)$. These two solutions to the homogeneous equation form the building blocks for the construction of the Green's function. Hajmirzaahmad and Krall [14] prove the following: For the Sturm–Liouville operator L with the limit point case at both ends of the interval,

1. No boundary conditions need be invoked.
2. L is self-adjoint.

Again, in lieu of the $\mathcal{L}_2$ requirement, we shall invoke limiting conditions, one at each end of the interval, viz.

$$\lim_{x \to -\infty} u(x) = 0$$
$$\lim_{x \to \infty} u(x) = 0$$

We assume that u, λ, f are complex. Since L is self-adjoint, the Green's function is symmetric. The solution to the differential equation is therefore given by

$$u(x) = \int_{-\infty}^{\infty} f(\xi) g(x, \xi) d\xi$$

where the Green's function must satisfy

$$(L - \lambda) g(x, \xi) = \delta(x - \xi)$$
$$\lim_{x \to -\infty} g(x, \xi) = \lim_{x \to \infty} g(x, \xi) = 0$$

We write the solution for the Green's function as

$$g(x, \xi) = \begin{cases} Ae^{-i\sqrt{\lambda}x}, & x > \xi \\ Be^{i\sqrt{\lambda}x}, & x < \xi \end{cases}$$

where

$$\text{Im}(\sqrt{\lambda}) < 0$$

and where we have invoked the two limit conditions. We note that, consistent with the limit point case and $\mathrm{Im}(\lambda) \neq 0$, our two limit conditions have produced exactly one solution in $\mathcal{L}_2(-\infty, \xi)$ and exactly one solution in $\mathcal{L}_2(\xi, \infty)$. Applying the continuity and jump conditions at $x = \xi$, we obtain

$$A = \frac{e^{i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}}$$

$$B = \frac{e^{-i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}}$$

Therefore,

$$g(x,\xi) = \frac{1}{2i\sqrt{\lambda}} \begin{cases} e^{-i\sqrt{\lambda}(x-\xi)}, & x > \xi \\ e^{-i\sqrt{\lambda}(\xi-x)}, & x < \xi \end{cases}$$

or, more compactly,

$$g(x,\xi) = \frac{e^{-i\sqrt{\lambda}|x-\xi|}}{2i\sqrt{\lambda}} \tag{2.175}$$

■

We have established in the above paragraphs a procedure for deriving the Green's function in limit point problems. We now turn to a consideration of the limit circle case. We begin with two examples.

EXAMPLE 2.21 Consider the following Green's function problem on $x \in (0, \infty)$:

$$-\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{dg}{dx}\right)\right] - \lambda g = \frac{\delta(x-\xi)}{x}, \qquad \lambda \in \mathbf{C} \tag{2.176}$$

From the results in Example 2.17, we have the limit circle case at $x = 0$ and the limit point case as $x \to \infty$. We begin our construction of the Green's function in the usual manner by considering the homogeneous equation

$$-\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{dg}{dx}\right)\right] - \lambda g = 0, \qquad x \neq \xi$$

which is Bessel's equation of order zero. Solutions can be constructed from linear combinations of the Bessel function $J_0(\sqrt{\lambda}x)$, the Neumann function $Y_0(\sqrt{\lambda}x)$, and the two Hankel functions $H_0^{(1)}(\sqrt{\lambda}x)$ and $H_0^{(2)}(\sqrt{\lambda}x)$. We write

$$g = \begin{cases} AJ_0(\sqrt{\lambda}x) + CY_0(\sqrt{\lambda}x), & x < \xi \\ BH_0^{(2)}(\sqrt{\lambda}x) + DH_0^{(1)}(\sqrt{\lambda}x), & x > \xi \end{cases} \tag{2.177}$$

We may evaluate one of the coefficients in (2.177) by following our procedure for dealing with limit points. We therefore invoke the following limiting condition as $x \to \infty$:

$$\lim_{x\to\infty} g(x,\xi) = 0 \tag{2.178}$$

The asymptotic forms of the two Hankel functions are given by [16]

$$H_0^{(1)}(t) \sim \sqrt{\frac{2}{i\pi t}}e^{it}$$

$$H_0^{(2)}(t) \sim \sqrt{\frac{2i}{\pi t}}e^{-it}$$

These asymptotic forms show that if we constrain $\mathrm{Im}\sqrt{\lambda} < 0$, $H_0^{(1)}(\sqrt{\lambda}x)$ diverges as $x \to \infty$. We therefore set $D = 0$ and obtain

$$g = \begin{cases} AJ_0(\sqrt{\lambda}x) + CY_0(\sqrt{\lambda}x), & x < \xi \\ BH_0^{(2)}(\sqrt{\lambda}x), & x > \xi \end{cases} \tag{2.179}$$

Determining the remaining three coefficients requires three conditions. The continuity and jump conditions at $x = \xi$ will provide two conditions. To produce the third, we consider the limit circle case at $x = 0$. The leading terms in the expansion of the Bessel and Neumann functions are given by [16]

$$J_0(t) = 1 + \cdots$$

$$Y_0(t) = \frac{2}{\pi}\ln\frac{\gamma t}{2} - \cdots$$

where $\ln\gamma$ is Euler's constant. Since we have the limit circle case, we know *a priori* that both of these functions are square integrable over $(0, \xi)$. Therefore, invoking the requirement that the solution be in $\mathcal{L}_2(0,\xi)$ does not evaluate a coefficient, as was the case for limit points. We do have, however, a condition that we can invoke from physical principles. Bessel's equation with forcing function at $x = \xi$ normally results from considerations of the radial dependence in problems in cylindrical coordinates. In such problems, we shall find in Chapter 4 that, based on physical grounds, the solution must remain finite as $x \to 0$. The Neumann function does not meet such a condition at $x = 0$, and we therefore set $C = 0$ and obtain

$$g = \begin{cases} AJ_0(\sqrt{\lambda}x), & x < \xi \\ BH_0^{(2)}(\sqrt{\lambda}x), & x > \xi \end{cases} \tag{2.180}$$

The continuity condition at $x = \xi$ yields

$$AJ_0(\sqrt{\lambda}\xi) = BH_0^{(2)}(\sqrt{\lambda}\xi) \tag{2.181}$$

The jump condition gives

$$\left[B\frac{dH_0^{(2)}(\sqrt{\lambda}x)}{dx} - A\frac{dJ_0(\sqrt{\lambda}x)}{dx}\right]_{x=\xi} = -\frac{1}{\xi} \tag{2.182}$$

Performing the indicated derivatives and solving (2.181) and (2.182) simultaneously for A gives

$$A\left[J_0(\sqrt{\lambda}\xi)H_1^{(2)}(\sqrt{\lambda}\xi) - J_1(\sqrt{\lambda}\xi)H_0^{(2)}(\sqrt{\lambda}\xi)\right] = \frac{H_0^{(2)}(\sqrt{\lambda}\xi)}{\sqrt{\lambda}\xi} \tag{2.183}$$

By a well-known Wronskian relationship [17], we have

$$J_1(t)H_0^{(2)}(t) - J_0(t)H_1^{(2)}(t) = \frac{2}{i\pi t}$$

Using this relation in (2.183), we obtain

$$A = \frac{\pi}{2i}H_0^{(2)}(\sqrt{\lambda}\xi)$$

Substitution of this result into (2.181) gives

$$B = \frac{\pi}{2i}J_0(\sqrt{\lambda}\xi)$$

Therefore,

$$g(x,\xi) = \frac{\pi}{2i}\begin{cases} H_0^{(2)}(\sqrt{\lambda}\xi)J_0(\sqrt{\lambda}x), & x < \xi \\ H_0^{(2)}(\sqrt{\lambda}x)J_0(\sqrt{\lambda}\xi), & x > \xi \end{cases} \tag{2.184}$$

We note that the Green's function is symmetric, $g(x,\xi) = g(\xi,x)$. A useful specialization of the result in (2.184) can be obtained by taking the limit as $\xi \to 0$, with the result

$$g(x,0) = \frac{\pi}{2i}H_0^{(2)}(\sqrt{\lambda}x) \tag{2.185}$$

■

We have noted in the above example that the Green's function is symmetric. We are led to inquire if the operator that produced the Green's function is self-adjoint. Consider first the case where we have the Sturm–Liouville operator L with the limit circle case at $x = a$ and a regular unmixed boundary condition at $x = b$. This case has been clarified by Kaper, Kwong, and Zettl [18], who have proved the following: The operator L is self-adjoint if $u \in \mathcal{D}_L$, and if:

1. u satisfies an unmixed condition at the regular boundary $x = b$.
2. u exists and is finite as $u \to a$, the limit circle boundary.

In addition, they show that the finiteness condition is mathematically equivalent to the condition

$$\lim_{x \to a} [p(x)u'(x)] = 0$$

(This equivalence is important in making the connection between the physically appealing finiteness condition and the classical Weyl theory.) This important result has been extended [14] to show that the self-adjoint property is retained when the regular point at $x = b$ is replaced by a limit point, as in the previous example. We consider another example.

EXAMPLE 2.22 Consider the following Green's function problem on $x \in (0, \infty)$:

$$-\frac{1}{x^2}\left[\frac{d}{dx}\left(x^2\frac{dg}{dx}\right)\right] - k^2 g = \frac{\delta(x-\xi)}{x^2}, \qquad k \in \mathbf{C} \tag{2.186}$$

This problem is in the limit circle case at $x = 0$ and the limit point case as $x \to \infty$. We therefore invoke a finiteness condition at $x = 0$ and the following limit condition as $x \to \infty$:

$$\lim_{x \to \infty} g(x, \xi) = 0$$

Equation (2.186) is the *spherical Bessel equation* of order zero [19]. Its solution is given by linear combinations of spherical Bessel, spherical Neumann, and spherical Hankel functions. We therefore write

$$g = \begin{cases} Aj_0(kx) + Ch_0^{(2)}(kx), & x < \xi \\ Bh_0^{(2)}(kx) + Dh_0^{(1)}(kx), & x > \xi \end{cases} \tag{2.187}$$

where

$$j_0(kx) = \frac{\sin kx}{kx} \tag{2.188}$$

$$h_0^{(1)}(kx) = \frac{e^{ikx}}{ikx} \tag{2.189}$$

$$h_0^{(2)}(kx) = -\frac{e^{-ikx}}{ikx} \tag{2.190}$$

To preserve finiteness as $x \to 0$, we set $C = 0$. To satisfy the limit condition at infinity, we adopt the constraint

$$\mathrm{Im}(k) < 0 \tag{2.191}$$

and therefore set $D = 0$. With these conditions, we find that

$$g = \begin{cases} Aj_0(kx), & x < \xi \\ Bh_0^{(2)}(kx), & x > \xi \end{cases} \tag{2.192}$$

The continuity condition at $x = \xi$ yields

$$Aj_0(k\xi) = Bh_0^{(2)}(k\xi) \tag{2.193}$$

The jump condition gives

$$\left[B\frac{dh_0^{(2)}(kx)}{dx} - A\frac{dj_0(kx)}{dx} \right]_{x=\xi} = -\frac{1}{\xi^2} \tag{2.194}$$

Solving for A in (2.193) and substituting into (2.194), we have

$$B\left[h_0^{(2)}(k\xi)\frac{dj_0(k\xi)}{dx} - \frac{dh_0^{(2)}(k\xi)}{dx} j_0(k\xi) \right] = \frac{j_0(k\xi)}{\xi^2} \tag{2.195}$$

The expression in square brackets in (2.195) is one of many Wronskian expressions involving the spherical Bessel functions. In this case, we find [20] that

$$h_0^{(2)} j_0' - h_0^{(2)\prime} j_0 = \frac{i}{z^2} \tag{2.196}$$

where all arguments are with respect to z and differentiation is with respect to argument. Using (2.196) in (2.195) gives

$$B = -ikj_0(k\xi) \tag{2.197}$$

Substitution into (2.193) yields

$$A = -ikh_0^{(2)}(k\xi) \tag{2.198}$$

Substitution of (2.197) and (2.198) into (2.192) yields the Green's function

$$g = -ik \begin{cases} j_0(kx)h_0^{(2)}(k\xi), & x < \xi \\ j_0(k\xi)h_0^{(2)}(kx), & x > \xi \end{cases} \tag{2.199}$$

An alternate form can be obtained by using (2.188) and (2.190). We have

$$g = \frac{1}{kx\xi} \begin{cases} e^{-ik\xi}\sin kx, & x < \xi \\ e^{-ikx}\sin k\xi, & x > \xi \end{cases} \tag{2.200}$$

We note that in the limit as $\xi \to 0$, we produce the result

$$\lim_{\xi \to 0} g(x, \xi) = \frac{e^{-ikx}}{x} \tag{2.201}$$

We shall use this important result in Chapter 4.

■

In dealing with the singular point in limit circle cases, we have not been able to evaluate a coefficient by invoking the requirement that the solution be in $\mathcal{L}_2$. Instead, we have invoked a condition based on physical grounds. The self-adjoint property of L results in a Green's function that is symmetric.

The above examples all concern operators that are self-adjoint. We next present an example where the operator manifold contains two unmixed conditions, but the operator is not self-adjoint.

EXAMPLE 2.23 Consider the following differential equation on $x \in (0, \infty)$:

$$-u'' - \lambda u = f \tag{2.202}$$

where

$$u'(0) = \alpha u(0), \qquad \alpha \in \mathrm{C} \tag{2.203}$$

and where we assume that u, λ, f are complex. The boundary $x = 0$ is a regular point. As $x \to \infty$, we have the limit point case. We therefore apply the limit condition

$$\lim_{x \to \infty} u(x) = 0 \tag{2.204}$$

In addition, since α is complex, the problem is not an extension to a finite interval self-adjoint problem. (See Example 2.12 for a discussion.) We therefore proceed using (2.139). Applying the boundary and limit conditions given in (2.203) and (2.204), we obtain

$$u(\xi) = \int_0^\infty f(x)\bar{h}(x, \xi)dx + u(0)\left[\alpha\bar{h}(0, \xi) - \frac{d\bar{h}(0, \xi)}{dx}\right] \tag{2.205}$$

where we have chosen

$$\lim_{x \to \infty} \bar{h}(x, \xi) = 0 \tag{2.206}$$

If we now choose

$$\frac{d\bar{h}(0, \xi)}{dx} = \alpha\bar{h}(0, \xi) \tag{2.207}$$

we obtain

$$u(\xi) = \int_0^\infty f(x)\bar{h}(x, \xi)dx \tag{2.208}$$

We note that the boundary conditions on $\bar{h}(x, \xi)$ are identical to the boundary conditions on $u(x)$. If we recall that the boundary conditions on the Green's function $g(x, \xi)$ are always identical to the boundary conditions on $u(x)$, we find that

$$\bar{h}(x, \xi) = g(x, \xi) \tag{2.209}$$

We therefore have

$$u(\xi) = \int_0^\infty f(x)g(x, \xi)dx \tag{2.210}$$

where we must solve

$$-\frac{dg^2}{dx^2} - \lambda g = \delta(x - \xi) \tag{2.211}$$

$$\frac{dg(0, \xi)}{dx} = \alpha g(0, \xi) \tag{2.212}$$

$$\lim_{x \to \infty} g(x, \xi) = 0 \tag{2.213}$$

We write

$$g = \begin{cases} A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x, & x < \xi \\ Ce^{-i\sqrt{\lambda}x}, & x > \xi \end{cases} \tag{2.214}$$

where we have applied the limit condition as $x \to \infty$ and have chosen

$$\mathrm{Im}(\sqrt{\lambda}) < 0$$

Applying the boundary condition at $x = 0$, we obtain

$$g = \begin{cases} A \left(\cos \sqrt{\lambda} x + \dfrac{\alpha}{\sqrt{\lambda}} \sin \sqrt{\lambda} x \right), & x < \xi \\ Ce^{-i\sqrt{\lambda}x}, & x > \xi \end{cases} \tag{2.215}$$

Invoking the continuity and jump conditions at $x = \xi$ results in

$$A = \frac{e^{-i\sqrt{\lambda}\xi}}{i\sqrt{\lambda} + \alpha}$$

$$C = \frac{\cos \sqrt{\lambda}\xi + \frac{\alpha}{\sqrt{\lambda}} \sin \sqrt{\lambda}\xi}{i\sqrt{\lambda} + \alpha}$$

Therefore, the Green's function is given by

$$g(x,\xi) = \frac{1}{i\sqrt{\lambda}+\alpha}\begin{cases} e^{-i\sqrt{\lambda}\xi}\left(\cos\sqrt{\lambda}x + \dfrac{\alpha}{\sqrt{\lambda}}\sin\sqrt{\lambda}x\right), & x<\xi \\ e^{-i\sqrt{\lambda}x}\left(\cos\sqrt{\lambda}\xi + \dfrac{\alpha}{\sqrt{\lambda}}\sin\sqrt{\lambda}\xi\right), & x>\xi \end{cases} \tag{2.216}$$

We note that, although the operator in this problem is not self-adjoint, we still have

$$g(x,\xi) = g(\xi,x)$$

and, therefore, interchanging x and ξ in (2.210), we have

$$u(x) = \int_0^\infty f(\xi)g(x,\xi)d\xi$$

We remark that if an operator is self-adjoint, the Green's function is symmetric. However, if the operator is not self-adjoint, it does not necessarily follow that the Green's function is not symmetric. This seemingly small distinction has a marked effect on characteristics of eigenvalues, as will be discussed in the next chapter.

■

In certain cases, determination of the limit point or limit circle is dependent on parameters in the differential equation. We illustrate this fact with the following example.

EXAMPLE 2.24 Consider the following Green's function problem associated with Bessel's equation of order ν in $\mathcal{L}_2(0,\infty)$:

$$(L-\lambda)g = \frac{\delta(x-\xi)}{x} \tag{2.217}$$

where

$$L = -\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{d}{dx}\right)\right] + \frac{\nu^2}{x^2} \tag{2.218}$$

and where ν and λ are complex parameters. We assume that

$$\mathrm{Re}(\nu) > 0 \tag{2.219}$$

We define the inner product for the space to be

$$\langle u,v\rangle = \int_0^\infty u\bar{v}x dx \tag{2.220}$$

We note that both endpoints are singular points. Proceeding as in Example 2.17, we pick an interior point $x=\xi$ and examine limit point or limit circle conditions on two intervals $\xi < x < \infty$ and $0 < x < \xi$. For $\lambda = 0$, the homogeneous equation $(L - \lambda)u = 0$ has the two independent solutions

$$u_1 = x^{\nu} \tag{2.221}$$

and

$$u_2 = x^{-\nu} \tag{2.222}$$

Consider the interval $\xi < x < \infty$. We first examine whether u_1 is in $\mathcal{L}_2(\xi, \infty)$. We have

$$\int_{\xi}^{\infty} x^{\nu} x^{\bar{\nu}} x dx = \int_{\xi}^{\infty} x^{2\mathrm{Re}(\nu)+1} dx$$

By (2.219), this integral diverges and u_1 is not in $\mathcal{L}_2(\xi, \infty)$. We therefore have the limit point case as $x \to \infty$ and assign the limit condition

$$\lim_{x\to\infty} g(x, \xi) = 0 \tag{2.223}$$

The situation on the interval $0 < x < \xi$ is more delicate. We first examine whether u_1 is in $\mathcal{L}_2(0, \xi)$. We have

$$\int_{0}^{\xi} x^{\nu} x^{\bar{\nu}} x dx = \int_{0}^{\xi} x^{2\mathrm{Re}(\nu)+1} dx$$

This integral exists when $2\mathrm{Re}(\nu) + 1 > -1$ or $\mathrm{Re}(\nu) > -1$. Replacing ν by $-\nu$, we find that u_2 is in $\mathcal{L}_2(0, \xi)$, provided that $\mathrm{Re}(-\nu) > -1$ or $\mathrm{Re}(\nu) < 1$. Combining these two results, we find that both solutions are in $\mathcal{L}_2(0, \xi)$, provided that $-1 < \mathrm{Re}(\nu) < 1$. We therefore have the limit circle case as $x \to 0$ for $-1 < \mathrm{Re}(\nu) < 1$ and the limit point case otherwise. In either case, we shall demand satisfaction of the physically motivated limit condition

$$\lim_{x\to 0} g(x, \xi) \text{ finite} \tag{2.224}$$

The Green's function problem defined by (2.217) and (2.218) with limiting conditions given by (2.223) and (2.224) yields the solution

$$g(x, \xi) = \frac{\pi}{2i} \begin{cases} H_{\nu}^{(2)}(\sqrt{\lambda}\xi) J_{\nu}(\sqrt{\lambda}x), & x < \xi \\ H_{\nu}^{(2)}(\sqrt{\lambda}x) J_{\nu}(\sqrt{\lambda}\xi), & x > \xi \end{cases} \tag{2.225}$$

where

$$\mathrm{Im}(\sqrt{\lambda}) < 0$$

That this is the solution for $g(x, \xi)$ can be determined by construction in the usual manner. We defer the details until Example 3.6 in the next chapter.

■

The limit point and limit circle cases in SLP3 problems are a subject of continuing interest to mathematicians. We have only considered the portion of the theory of interest to us in our application to electromagnetic boundary value problems. For an in-depth discussion of limit point and limit circle cases, as well as a well-compiled bibliography, the reader is referred to [14] and [21].

We have now completed our discussion of the solution to linear, ordinary, second-order differential equations by the Green's function method. In the next chapter, we shall discuss an alternate method where we shall determine the solution to $(L-\lambda)u = f$ by finding the spectral representation associated with the differential operator L.

PROBLEMS

2.1. For the pulse function $p_\epsilon(x - x_0)$, defined in (2.2), show that

$$p_\epsilon(x - x_0) = p_\epsilon(x_0 - x)$$

2.2. By substituting (2.23)–(2.25) into (2.22), verify that the Sturm–Liouville differential equation is transformed into the general form in (2.21).

2.3. The Chebyshev differential equation is defined on the interval $x \in (-1, 1)$, as follows:

$$-(1 - x^2)u'' + xu' - n^2 u = f$$

Transform to Sturm–Liouville form.

2.4. Transform the Laguerre differential equation

$$-xu'' - (1 - x)u' - nu = f$$

to Sturm–Liouville form.

2.5. In SLP1, the following are restrictions on $u(x)$:

(a) $u \in \mathcal{L}_2(a, b)$;
(b) $u \in \mathcal{D}_L$;
(c) u satisfies two boundary conditions, $B_1(u) = 0$, $B_2(u) = 0$.

Show that these restrictions define a linear manifold $\mathcal{M}_L \subset \mathcal{L}_2(a, b)$.

2.6. Solve the following differential equation:

$$-\frac{d^2 g(x, \xi)}{dx^2} = \delta(x - \xi)$$

$$g(0, \xi) = \frac{dg(1, \xi)}{dx} = 0$$

2.7. Verify that (2.93) satisfies the requirements for the Green's function given in (2.83)–(2.87).

2.8. Solve the following differential equation:

$$-\frac{d^2g(x,\xi)}{dx^2} = \delta(x-\xi)$$

$$\frac{dg(0,\xi)}{dx} = \frac{dg(L,\xi)}{dx} = 0$$

2.9. Consider the following differential equation:

$$-u'' = f(x), \qquad x \in (0, L)$$

with $f(x)$ real and with boundary conditions

$$u(0) = \alpha, \qquad \alpha \in \mathbf{R}$$

$$u'(L) = 0$$

(a) Show that, if a solution exists, it is unique.
(b) Construct the solution by the Green's function method.

2.10. Given the differential equation $Lu = f$ on the interval $x \in (0, L)$ with $f(x)$ real and with the following boundary conditions:

$$u(0) = \alpha u'(0), \qquad \alpha > 0$$

$$u'(L) = 0$$

Show that this problem is self-adjoint.

2.11. Show that the solution given in (2.111) satisfies the differential equation and the boundary conditions in Example 2.10.

2.12. Solve the following SLP1 differential equation on the interval $x \in (0, L)$:

$$-u'' - k^2u = f$$

$$u'(0) = 0$$

$$u'(L) = \beta$$

2.13. Consider the following differential equation:

$$-\frac{d^2u}{dx^2} = f$$

$$u(0) = \frac{du(1)}{dx}$$

$$\frac{du(0)}{dx} = \alpha, \qquad \alpha \in \mathbf{R}$$

where $x \in (0, 1)$ and where f is a real-valued function of x. Solve the differential equation by the Green's function method.

2.14. For SLP2, show that the operator L is not self-adjoint for the case of initial conditions.

2.15. For SLP2, show that the operator L is self-adjoint for periodic conditions, provided $p(a) = p(b)$.

2.16. Repeat Problem 2.12 for the case where $k \in \mathbf{C}$ and $f(x)$ is complex so that the problem becomes SLP2.

2.17. Repeat Problem 2.9 for the case where $f(x)$ is complex so that the problem becomes SLP2.

2.18. Consider the following SLP2 Green's function problem:

$$-\frac{d^2 g}{dx^2} - \lambda g = \delta(x - \xi)$$

with periodic boundary conditions

$$g(0, \xi) = g(2\pi, \xi)$$

$$\frac{dg(0, \xi)}{dx} = \frac{dg(2\pi, \xi)}{dx}$$

Show that the solution is

$$g(x, \xi) = -\frac{\cos\left[\sqrt{\lambda}(|x - \xi| - \pi)\right]}{2\sqrt{\lambda}\sin\sqrt{\lambda}\pi}$$

2.19. Consider the following SLP3 boundary-value problem, defined on the interval $x \in (0, 1)$:

$$(L - \lambda)u = f$$

$$L = -\frac{d^2}{dx^2}$$

where $\lambda \in \mathbf{C}$ and where the following boundary conditions apply:

$$u'(0) - \alpha u(0) = 0, \qquad \alpha \in \mathbf{C}$$

$$u(1) = 0$$

(a) Show that the operator L is not self-adjoint.

(b) Find the Green's function $g(x, \xi)$, the adjoint Green's function $h(x, \xi)$, and the conjugate adjoint Green's function $\bar{h}(x, \xi)$.

(c) Solve the differential equation.

2.20. Consider the following SLP3 boundary-value problem, defined on the interval $x \in (0, \infty)$:

$$-u'' - k^2 u = f$$

$$u'(0) = 0$$

where $k \in C$ and $\mathrm{Im}(k) < 0$. Obtain the solution in terms of the appropriate Green's function $g(x, \xi)$ by demanding that

$$\lim_{x \to \infty} [u(x)] = 0$$

2.21. Consider the Green's function problem

$$-\frac{d^2 g}{dx^2} - \lambda g = \delta(x - \xi)$$

with boundary conditions

$$g(0, \xi) = g(b, \xi) = 0$$

Obtain the result in (2.172) by solving this problem and then taking the limit as $b \to \infty$.

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