IMPEDANCE OF A THIN CYLINDRICAL ANTENNA IN AN ISOTROPIC PLASMA

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The problem of the input impedance of an antenna placed in an isotropic plasma is considered under the approximation of weak spatial dispersion and taking the radiation of both longitudinal and transverse waves into account. The case in which the antenna length approaches infinity is considered in detail.

This problem has been discussed in a number of papers (for example, [1–4]). However, the total input impedance of a thin antenna placed in an isotropic plasma taking spatial dispersion into account has only been investigated essentially in [2]. In that paper, an integral equation was obtained neglecting the transverse radiation field for the current distribution in the antenna, and the input impedance was calculated.

In this paper, we take the transverse radiation field into account. If the antenna length is fairly large this is very necessary (see below). In conclusion, we calculate the input impedance of an antenna whose length L tends to infinity.

§1. To solve this problem, it is convenient to start from the equation for the Fourier component of the longitudinal field $E_{\parallel k}$ and the transverse field $E_{\perp k}$ in the plasma (for example, [5]):

$$(k^2 - k_\perp^2)\, E_{\perp k} - \frac{\omega^2}{c^2}\, \varepsilon_\parallel E_{\parallel k} = 0, \tag{1}$$

and

$$E(r) = \int_{-\infty}^{\infty} E_k e^{ikr}\, dk = \int_{-\infty}^{\infty} E_{\perp k} e^{ikr}\, dk + \int_{-\infty}^{\infty} E_{\parallel k} e^{ikr}\, dk, \tag{2}$$

where $\varepsilon_\parallel = \varepsilon - D^2k^2$, $\varepsilon = 1 - \omega_0^2/\omega^2$, $D^2 = 3v_T^2/\omega^2$, $k_\perp^2 = \varepsilon\omega^2/c^2$, $\varepsilon \ll 1$, $v_T^2/c^2 \ll 1$, ω_0 is the Langmuir frequency, and v_T is the average thermal velocity of the plasma electrons.

The amplitudes of the fields $E_{\perp k}$ and $E_{\parallel k}$ are determined from the boundary conditions on the antenna surface. We assume that on the surface of the conductor the tangential component of the electric field is zero. We use as the second boundary condition the fact that the plasma current is zero on the antenna surface, which corresponds to the condition that the plasma electrons are specularly reflected from the boundary. As follows from [2], for a thin antenna this boundary condition is essentially the same as the natural requirement that the field in the plasma should be finite as $\varepsilon(\omega) \to 0$.

On the other hand, as shown in [6], the penetration of the field into the plasma is described by the same formulas for the two cases of diffuse and specular reflection of electrons from the plane boundary as $\varepsilon(\omega) \to 0$. In the case considered, we will use this result when $k_\parallel a \gg 1$, $|\varepsilon| \ll 1$. Therefore, when $|\varepsilon(\omega)| \ll 1$, if the antenna surface reflects the particles of the plasma very well, independently of the nature of this reflection the above condition gives a quite good approximation for the field in the plasma (see [2] for more detail).

Hence,

$$\begin{aligned} E_z + P(z) &= 0 \quad \text{for} \quad \rho = a \quad (-L < z < L), \\ j_{pe} &= 0 \quad \text{for} \quad \rho = a \quad (-L < z < L). \end{aligned} \tag{3}$$

here a is the radius ($a \gg D$); 2L is the length of the antenna, the coordinate axis Oz being directed along the axis of the cylinder; and P(z) is the external field, given on the surface of the antenna. It follows from the axial nature of the problem that it is sufficient to determine the field in the coordinate plane yz. The vectors $E_{\perp k}$ and $E_{\parallel k}$ can be defined in the form

$$E_{\perp k} = \{k_z \cos\Phi,\ k_z \sin\Phi,\ -\varkappa\} \frac{J_0(\varkappa a)\,\varkappa}{2\pi(\varkappa^2 - g_1^2)} A_1(k_z),$$

$$E_{\parallel k} = \{\varkappa \cos\Phi,\ \varkappa \sin\Phi,\ k_z\} \frac{J_0(\varkappa a)\,k_z}{2\pi(\varkappa^2 - g_2^2)} A_2(k_z);$$

$$k = \{\varkappa \cos\Phi,\ \varkappa \sin\Phi,\ k_z\},$$

$$g_1^2 = k_1^2 - k_z^2,\quad g_2^2 = k_\parallel^2 - k_z^2,\quad r = \{0,\ y,\ z\},\quad k_\parallel^2 = \frac{\varepsilon}{D^2}, \tag{4}$$

and the functions $A_1(k_z)$, $A_2(k_z)$ are determined from (3):

$$A_j(k_z) = \frac{1}{2\pi} \int_{-L}^{L} A_j(z) e^{-ik_z z}\, dz, \quad A_j(z) = 0 \text{ for } |z| > L \quad (j = 1,2).$$

Integrating over the variables Φ, $\varkappa$, B, in (2) in the same way as in [1] we obtain

$$E_z(\rho, z) = E_{\perp z} + E_{\parallel z} = \frac{i\pi}{2} \int_{-\infty}^{\infty} [A_1(k_z) J_0(ag_1) H_0^{(1)}(\rho g_1) g_1^2 +$$

$$+ A_2(k_z) k_z^2 J_0(ag_2) H_0^{(1)}(\rho g_2)] e^{ik_z z}\, dk_z,$$

$$E_\rho(\rho, z) = E_{\perp\rho} + E_{\parallel\rho} = \frac{\pi}{2} \int_{-\infty}^{+\infty} [A_1(k_z) J_0(ag_1) g_1 H_1^{(1)}(\rho g_1) -$$

$$- A_2(k_z) J_0(ag_2) g_2 H_1^{(1)}(\rho g_2)] k_z e^{ik_z z}\, dk_z, \tag{5}$$

where

$$y = \rho,\ g_j = \left|\sqrt{k_j^2 - k_z^2}\right| \text{for } k_z < k_j,\ g_j = i\left|\sqrt{k_z^2 - k_j^2}\right| \text{ for } k_z > k_j$$

$(j = 1,2)$; $k_1 = k_\perp$; $k_2 = k_\parallel$; $J_0(x)$, $H_0^{(1)}(x)$, $H_1^{(1)}(x)$ are Bessel and Hankel functions.

It is easy to verify by direct substitution that expressions (5) satisfy the field equations corresponding to (1). It should be noted that since we are considering a plasma in the approximation of weak spatial dispersion, generally speaking, the integration in (5) must be limited to $k_z \leqslant k_m \ll 1/D$. We can also obtain a relationship which defines the magnetic field in the plasma which, for the given symmetry, has only the H_φ-component, given by

$$H_\varphi(\rho, z) = \frac{\pi\omega\varepsilon}{2c} \int_{-\infty}^{\infty} A_1(k_z) J_0(ag_1) g_1 H_1^{(1)}(\rho g_1) e^{ik_z z}\, dk_z. \tag{6}$$

Inside the metal H = 0 when $\rho < a$, so that we can assume in the usual way that a current I(z) flows in the surface of the cylinder given by

$$I(z) = \frac{ac}{2} H_\varphi(\rho = a + 0) = 2\pi a j_0,$$

$$j = \{0,\ 0,\ j_z\},\ j_z = j_0 \delta(\rho - a). \tag{7}$$

Since I is not identically equal to zero as $\varepsilon \to 0$ (the current I depends on the external field P(z)), it is easy to see that A_1 has the form $A_1(k_z) = \widetilde{A}_1(k_z)/\varepsilon$, where $\widetilde{A}(k_z) \neq 0$ when $\varepsilon = 0$. To obtain integral equations for the required functions $A_1(k_z)$, and $A_2(k_z)$, we use the boundary conditions (3). Then, assuming that $j_{nn} = (\omega/4\pi i)[(\varepsilon - 1) \times E_\perp - E_\parallel]$, [see (5)], we have

$$\frac{i\pi}{2}\int\limits_{-\infty}^{\infty}[A_1(k_z)g_1^2J_{01}H_{01}+A_2(k_z)k_z^2J_{02}H_{02}]\,e^{ik_zz}\,dk_z+P(z)=0,$$

$$\int\limits_{-\infty}^{\infty}[(\varepsilon-1)A_1(k_z)g_1J_{01}H_{11}+A_2(k_z)g_2J_{02}H_{12}]\,k_ze^{ik_zz}\,dk_z=0,$$

$$-L<z<L, \tag{8}$$

where $H_{ij}=H_i^{(1)}(ag_j)$, $J_{0j}=J_0(ag_j)$ $(i, j=1,2)$.

We assume, as in [1], that $ak_0\ll 1$, where k_0 is the most significant value of k_z in the spectrum of $A_1(k_z)$, and $A_2(k_z)$ (see below). Then, using in the second equation the formulas for Bessel and Hankel functions with small values of the argument, we obtain $(a^2k_0^2\ln(ak_0/2)\ll 1$.

$$\int\limits_{-\infty}^{\infty}(\varepsilon-1)A_1(k_z)\,e^{ik_zz}\,k_z\,dk_z+\int\limits_{-\infty}^{\infty}A_2(k_z)\,e^{ik_zz}\,k_z\,dk_z=0.$$

Neglecting terms on the order of ε in comparison with unity and assuming that when $|z|>L$, $A_1(z)=A_2(z)=0$, we obtain $A_1(z)=A_2(z)$ $(|z|<L)$.

As stated earlier, the same result can be obtained from the condition that the field is finite as $\varepsilon\to 0$. In fact, if we let ε tend to zero in expression (5) for E_ρ, and take into account that E_ρ must in this case remain a finite quantity, we obtain immediately that $A_1(z)=A_2(z)=\widetilde{A}_1(z)/\varepsilon=\widetilde{A}_2(z)/\varepsilon$ when $|z|<L$, $A_1=A_2=0$ when $|z|>L$. Since $ak_0\ll 1$, substituting into (6) and (7) the values of the Bessel and Hankel functions for small arguments we have $A_1(z)=A_2(z)=2iI(z)/\omega\varepsilon$. If we substitute this expression into the first equation of (8), we obtain an integral equation for I(z) which agrees with the corresponding expression given in [1] (Eq. (8) of [1]). This means that when $ak_0\ll 1$ the field in the plasma, excited by the current I(z) flowing in the cylindrical conductor, is the same as the field excited by the current I(z) in an infinite medium, i.e., in a medium free of conductors (see also [2]).

Thus, we have for the current I(z) the integral equation

$$-\frac{1}{2\varepsilon\omega}\int\limits_{-L}^{L}\int\limits_{-\infty}^{\infty}(g_1^2J_{01}H_{01}+k_z^2J_{02}H_{02})\,e^{ik_z(z-\zeta)}I(\zeta)\,d\zeta\,dk_z+P(z)=0,$$

$$ak_0\ll 1. \tag{9}$$

Using the condition $ak_0\ll 1$ and the formula $\int\limits_{-\infty}^{\infty}e^{ik_zz}H_0^{(1)}\left(\rho\sqrt{k_2-k_z^2}\right)dk_z=-2ie^{ik\sqrt{\rho^2+z^2}}/\sqrt{\rho^2+z^2}$, Eq. (9) can be written in the form

$$-\frac{i}{\omega\varepsilon}\int\limits_{-L}^{L}\left[-k_\perp^2\frac{e^{ik_\perp R}}{R}-e^{ik_\perp R}F(k_\perp,\zeta_1,a)+e^{ik_\parallel R}F(k_\parallel,\zeta_1,a)\right]I(\zeta)\,d\zeta+$$

$$+P(z)=0, \tag{10}$$

where

$$\zeta_1=z-\zeta,\ R=\sqrt{a^2+\zeta_1^2},\ F(k,\zeta_1,a)=ik/R^2-(1+k^2\zeta_1^2)/R^3-$$
$$-3ik\zeta_1^2/R^4+3\zeta_1^2/R^5.$$

In formula (10) the first two terms in the brackets correspond to the transverse field $E_\perp$, and the last term, which is equal to $e^{ik_\parallel R}F(k_\parallel,\zeta_1,a)$, corresponds to the longitudinal field.

The integral equation (10), like that in [2], can be written in the form of a second-order Fredholm equation. In fact, the expression $\Phi(\zeta_1,a)=e^{ik_\parallel R}F(k_\parallel,\zeta_1,a)-e^{ik_\perp R}F(k_\perp,\zeta_1,a)$ as $\zeta\to 0$ and $a\to 0$ can be written as $\Phi(\zeta,a)\to-[(k_\parallel^2-k_\perp^2)\,a^2/2(a^2+\zeta_1^2)^{3/2}+i(k_\parallel^3-k_\perp^3)/3]\to-k_\parallel^2\delta(\zeta_1)-ik_\parallel^3/3$ $(k_\parallel^2\gg k_\perp^2)$. Hence, Eq. (8) can be represented in the form

$$I_0(z) = i\omega D^2 P(z) + \frac{1}{k_\parallel^2}\int_{-L}^{L} G_1(\zeta_1)\, I(\zeta)\, d\zeta,$$

where

$$G_1(\zeta_1) = -k_\perp^2 e^{ik_\perp|\zeta_1|}/R + \Phi(\zeta_1,\ a = 0),$$

$$\Phi(\zeta_1,\ 0) = e^{ik_\parallel|\zeta_1|}F(k_\parallel,\ \zeta_1,\ 0) - e^{ik_\perp|\zeta_1|}F(k_\perp,\ \zeta_1\ 0),$$

$$F(k,\ \zeta_1,\ 0) = \frac{2}{|\zeta_1|^3} - \frac{2ik}{\zeta_1^2} - \frac{k^2}{|\zeta_1|}.$$

Hence, in the kernel $G_1(\zeta_1)$ the dependence on a as $a \to 0$ remains only in the first term, which is equal to $k_\perp^2 e^{ik_\perp|\zeta_1|}/R$.

In the case of an elementary dipole when $k_\parallel L \gg 1$, the kernel of the integral equation (10) can be expanded in a series, assuming $k_\parallel \zeta_1$ to be small. If also $v_i^2/c^2 \ln(4L^2/a^2) \ll 1$, we can neglect the transverse wave field, i.e., we can set $k_\perp = 0$ in (10). In this case, the integral equation (10) becomes trivial, and we obtain for the current and the input impedance the expressions (see [2] for more details)

$$I(z) = \omega D^2 P_0\left[\Pi(d,\ z)i + \frac{2k_\parallel d}{3}\right],\quad Z_{\text{in}} = \frac{2dP_0}{I(0)} = \frac{2d}{\omega D^2}\left(-i + \frac{2k_\parallel d}{3}\right), \tag{11}$$

where the external electromotive force is given in the form

$$P(z) = P_0\Pi(d,\ z)\quad (\Pi(d,\ z) = 1 \text{ for } |z| < d,\ \Pi(d,\ z) = 0 \text{ for } |z| > d).$$

It is easy to see that in the case of an elementary dipole, $(k_\parallel L \gg 1)$ $k_0 \simeq 1/d$, and the condition under which formula (11) holds can be written in the form $a \ll d$.

§2. Let us use the relations obtained above to calculate the impedance of an infinite antenna ($L \to \infty$). We note that this case has already been considered in [3] (see also [4]). However, in the papers just mentioned, only the energy radiated by the antenna is calculated. To obtain an expression for the total input impedance of an infinite antenna, we can start from the system of equations (8), whence as $L \to \infty$ we obtain the following expression for the Fourier components of $A_1(k_z)$ and $A_2(k_z)$:

$$A_1(k_z) = \frac{2i}{\pi\Delta} P_k g_2 J_{02} H_{12},\quad A_2(k_z) = -\frac{2i}{\pi\Delta} P_k(\varepsilon_z' - 1) g_1 J_{01} H_{11},$$

$$\Delta = g_1 J_{01} J_{02}[g_1 g_2 H_{01} H_{12} - (\varepsilon_z' - 1) k_z^2 H_{11} H_{02}],$$

$$P_k = \frac{1}{2\pi}\int_{-L}^{L} P(z)\, e^{-ik_z z}\, dz. \tag{12}$$

Substituting these values of A_1 and A_2 into formulas (5)–(7), we obtain expressions for the electric and magnetic fields in the plasma, and also expressions for the current I(z) flowing in the surface of the conductor. If the external electromotive force is given in the form $P(z) = P_0\Pi(d,\ z)$, $dk_\parallel \ll 1$, we obtain the following expression for the total input impedance:

$$Z_{\text{in}} = \frac{2P_0 d}{I(0)} = \frac{8P_0 d}{\pi a\omega\varepsilon \int_{-\infty}^{\infty} A_1(k_z)\, g_1 H_{11}\, dk_z}. \tag{13}$$

The above method of determining the input impedance can be checked by direct calculation of the energy flow radiated by the antenna followed by a comparison of the resulting expression with the value of the radiated energy W, determined by means of formulas (6) and (7) introduced above. Thus,

$$W = \frac{1}{2} u \operatorname{Re} I(0),\quad u = 2P_0 d. \tag{14}$$

The energy flux radiated by the antenna is the sum of the energy flux of the transverse waves, characterized by the Poynting vector, and the energy flux of the longitudinal waves. The total flux through an infinitely remote cylindrical surface with axis Oz is easily seen to be

$$W = W_{\perp} + W_{\parallel} = 2\pi\rho\left(\frac{c}{8\pi}\,\mathrm{Re}\int\limits_{-\infty}^{\infty} E_{\perp z} H_{\varphi}^{*} dz - \omega\frac{\partial\varepsilon_{\parallel}}{\partial k}\int\limits_{-\infty}^{\infty}\frac{\overline{E_{\parallel}^{2}}}{8\pi}\,dz\right),$$

$$\frac{\partial\varepsilon_{\parallel}}{\partial k} = -2D^{2}k_{\parallel}, \quad \rho \to \infty. \tag{15}$$

Using formulas (12) we obtain, after simple reduction,

$$W = \frac{2\omega\varepsilon(P_0 d)^2}{\pi^2}\left\{\int\limits_{0}^{k_{\perp}} \frac{g_1^2 J_{01}^2 J_{02}^2}{|\Delta|^2}\left[k_z^2|H_{11}|^2 + g_2^2|H_{12}|^2\right]dk_z + \right.$$

$$\left. + \int\limits_{k_{\perp}}^{k_{\parallel}} \frac{k_z^2|g_1|^2}{|\Delta|^2} J_{01}^2 J_{02}^2 |H_{11}|^2\, dk_z\right\}, \tag{16}$$

where we have set $P_k = P_0 d/\pi$ when $k_{\parallel} d \ll 1$ and we have neglected terms on the order of ε in comparison with unity. If we now calculate W using formula (14), it can be shown that the expression obtained in this way is the same as (16).

If the condition $ak_0 \ll 1$ is observed in the case considered when $L \to \infty$, we can obtain simple estimates of the value of the input impedance of the antenna. In this case, it is more convenient to start from expression (7) whence, for $P(z) = P_0\Pi(d, z)$, $k_{\parallel} d \ll 1$ we have

$$I(z) = \frac{\omega\varepsilon P_0}{2\pi^2 i}\int\limits_{-\infty}^{\infty}\frac{\sin(kd)}{k}\,\frac{e^{ikz}dk}{G(k)} \qquad (L \to \infty), \tag{17}$$

where, as in all the formulas given above, integration must be extended only over the region $k \leqslant k_m \ll 1/D$, and when $ka \ll 1$ $(k_0 \simeq 1/d,\ a \ll d)$

$$G(k) = \frac{1}{2\pi}\left\{-k_{\perp}^2 \ln\frac{4}{a^2|k_{\perp}^2 - k^2|} + k^2\ln\frac{|k_{\parallel}^2 - k^2|}{|k_{\perp}^2 - k^2|} - i\pi k_{\perp}^2\Pi(k_{\perp}, k) - \right.$$

$$\left. - i\pi k^2\left[\Pi(k_{\parallel}, k) - \Pi(k_{\perp}, k)\right]\right\}. \tag{18}$$

The need to take into account the transverse radiation field in the case considered arises from formulas (17) and (18). If this field is ignored, i.e., if we set $k_{\perp} = 0$ in (17) and (18), the integral in (17) becomes infinite due to the behavior of G(k) as $k \to 0$.

If $\varepsilon = 0$, then from (17) and (18) and from the obvious relationship $\lim\limits_{x\to\infty} x^2\ln[(x^2-a^2)/(x^2-b^2)] \simeq -a^2$ $(a^2 \gg b^2)$, provided the condition of the type $\beta^2\ln(4d^2/a^2) \ll 1$, $\beta^2 \sim k_{\perp}^2 k_{\parallel}^2$ $(k_{\max} \simeq k_0 \sim 1/d)$ still holds, as previously, it follows that

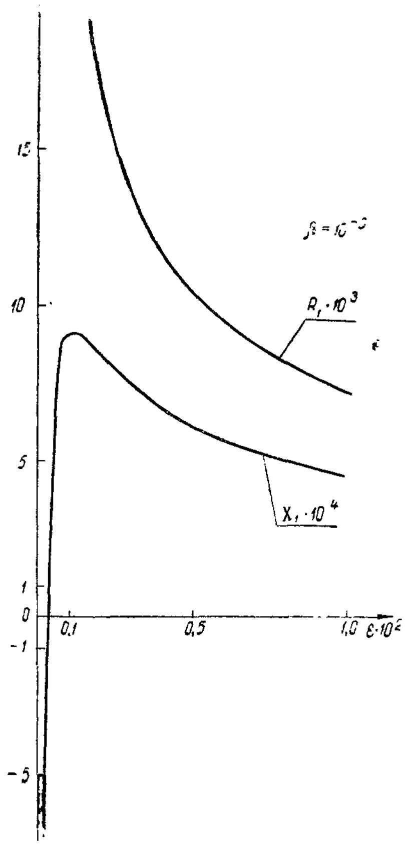
$$I(z) \simeq \omega D^2 P_0 \Pi(d, z)\, i, \qquad \varepsilon = 0 \quad (L \to \infty).$$

By integrating (17) numerically we can obtain the impedance Z_{in} of an infinite conductor. In this case, it is convenient to express Z_{in} in the form

$$Z_{\text{вх}} = X + iR = \frac{2d}{\omega D^2}(R_1 + iX_1).$$

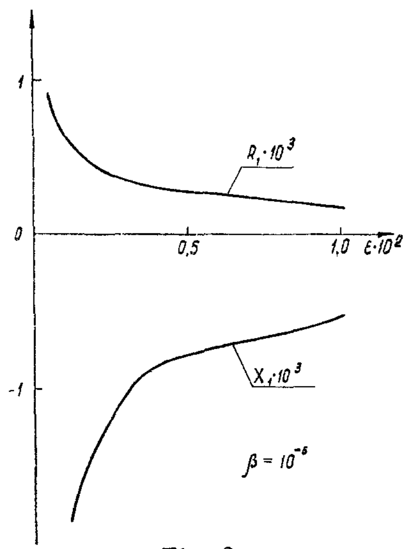
Figures 1 and 2 show X_1 and R_1 as functions of ε for values of $\beta^2 = k_{\perp}^2/k_{\parallel}^2 = 10^{-5}$ and 10^{-6}, and $dk_{\parallel} = 30\sqrt{\varepsilon}$, $a^2k_{\parallel}^2 = 25\varepsilon$, $\varepsilon > 0$. If follows from the figures that the reactance of an infinite antenna behaves in the following way. For very small values of ε, the reactance is inductive, and as ε increases it becomes capacitive. If $\varepsilon = 0$, then $X \simeq -2d/\omega_0 D^2$ (see above).

In conclusion, let us consider the possible existence of surface waves in the considered system ($L \to \infty$). The



15
10
5
1
0
-1
-5
0,1
0,5
1,0
$\varepsilon \cdot 10^2$
$R_1 \cdot 10^3$
$X_1 \cdot 10^4$

Fig. 1.



1
0
-1
0,5
1,0
$\varepsilon \cdot 10^2$
$R_1 \cdot 10^3$
$X_1 \cdot 10^3$
$\beta = 10^{-6}$

Fig. 2

dispersion relation for these waves is

$$\Delta(\omega, k_z) = 0, \tag{19}$$

where k_z is the real root of Eq. (19) and $k_z > k_\parallel$, which is necessary so that both fields (the longitudinal $E_\parallel$ and the transverse $E_\perp$) should fall off exponentially with distance from the surface of the antenna as $\rho \to \infty$ (5). The quantity Δ can be written in the form

$$\Delta = I_{01} I_{02} |g_1|^2 |g_2| \{- K_{01} K_{12} |g_1| |g_2| + (1 - \varepsilon) k_z^2 K_{02} K_{11}\}. \tag{20}$$

Here, $I_{0j} = I_0(a|g_j|)$; $K_{ij} = K_i(a|g_j|)$ $(i, j = 1, 2)$: $I_0(x)$, $K_0(x)$, $K_1(x)$ are Bessel functions of imaginary argument. It can be shown, for example, graphically, that as $\varepsilon \to 0$ the equation $\Delta(k_z^2) = 0$ has one root $k_{0z}^2 \simeq 1/2D^2$ ($\lambda_0 = 2\pi/k_{z0}$) . However, as already stated, integration in all the formulas which give the field in the plasma must be limited to $k_z < k_m \ll 1/D$ the weak spatial dispersion approximation). This means, strictly speaking, that an unattenuated surface wave cannot exist in the system considered. A more detailed investigation of this problem is outside the framework of the hydrodynamic treatment used above by virtue of the fact that here wavelength λ_0 is of the order of the Debye radius.

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