Archiv für Elektrotechnik 55 (1973) 233—236

A new Polynomial Approach to the Numerical Solution of Thin Cylindrical Antenna Problem

D. M. Veličković, Niš, Jugoslavia

Contents: This paper is aimed at providing a new method of the numerical solution of the Hallén's equation for the thin cylindrical antenna. Essentially, the method is a combination of the well known polynomial approximation for the current distribution along the antenna and a semi-polynomial approximation for the kernel in the Hallén's equation which leads to integrals in closed form suitable for the rapid and accurate numerical calculations.

Übersicht: Dieser Aufsatz setzt sich das Ziel, eine neue Methode zur numerischen Lösung der Hallén'schen Gleichung für die dünne zylindrische Antenne zu geben. Im Wesentlichen ist die Methode eine Kombination der wohl bekannten Polynom-Approximation für die Stromverteilung entlang der Antenne und einer teilweisen Polynom-Approximation für den Kern der Hallén'schen Gleichung, welche auf Integrale in geschlossener Form und damit rasch zur genauen zahlenmäßigen Lösung führt.

Many methods treating the problem of a thin symmetrical dipole antenna (or a monopole antenna over the conducting ground) use as a starting point the well known Hallén's integral equation [1], or an equation which is a generalisation of the former (the antennas with arbitrary impedance loading). Although an exact method for solving such an equation does not exist, there are several methods for solving it approximately. A simple but very general method in obtaining approximate solution of the Hallén's equation is so called point-matching method. According to that method the unknown current distribution function is approximated by a finite series of known functions with unknown coefficients, which can be determined by satisfying the equation at a number of points along the antenna. Although the choice of the expansion functions is in principle arbitrary, a series of the trigonometric functions (or of the modified trigonometric functions) or polynomials [2] are commonly used.

Owing to its mathematical simplicity and satisfactory results it gives, the point-matching polynomial approach was successfully used in many problems in the field of linear antennas [3, 4].

In carrying out the determination of the unknown coefficients of the polynomial current distribution there are many integrals to be evaluated numerically. Both, the number of the integrals and the computation time for their evaluation increase rapidly with the order of the polynomial approximating the current distribution.

The present paper is aimed at proposing a conceptually new and powerfull approach to the numerical solution of the Hallén's equation based on the polynomial approximation for the current distribution. Essentially, the new method is based on a semi-polynomial approximation of the kernel in the Hallén's equation which, together with polynomial approximation for the current distribution, leads to the above mentioned integrals in closed form.

Although the proposed method can be adopted easily and used for analysing other types of linear antennas it will be demonstrated on a simple example of a thin, symmetricaly driven cylindrical dipole (Fig. 1). As usual, it is assumed that the dipole is driven by the delta-generator having a voltage U and frequency ω.

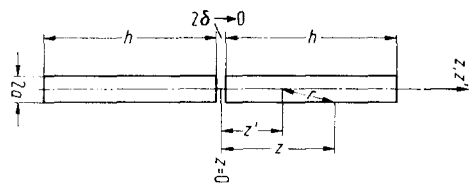

Fig 1. The symmetrically driven cylindrical dipole

Assuming an infinite conductivity of the antenna conductor, the Hallén's integral equation[1] governing the current distribution has the form

$$\int_{-h}^{h} I(z')\, K(z, z')\, \mathrm{d}z' = C \cos k z + \frac{U}{\mathrm{j}\, 60} \sin k\, |z|\,, \qquad (1)$$

where $I(z')$ is the unknown current distribution on the antenna and $K(z, z')$ is the kernel of the integral equation defined by $K(z, z') = \exp(-\mathrm{j}\, k\, r)/r$ with $r = [a^2 + (z - z')^2]^{1/2}$. h and a are the halflength and radius of the dipole, respectively; $k = 2\,\pi/\lambda$ and C is a constant to be determined.

Using the normalized coordinates $u = z/h$ and $u' = z'/h$ the following equation is obtained:

$$\int_{-1}^{1} G(u')\, K(u, u')\, \mathrm{d}u' = C_0 \cos H\, u + \frac{Z}{\mathrm{j}\, 60} \sin H\, |u| \quad (2)$$

where

$$G(u') = I\,(z' = h\, u')/I(0)\,,$$

$$K(u, u') = \exp\,(-\,\mathrm{j}\, H\, R)/R\,,$$

$$R = [A^2 + (u - u')^2]^{1/2}\,,$$

$$H = k\, h\,, \quad A = a/h\,, \quad Z = U/I(0)$$

and

$$C_0 = C/I(0)\,.$$

The kernel $K(u, u')$ in eq. (2) can be expressed as a sum of a static term $K_1 = 1/R$ and a remaining part $K_2 = [\exp\,(-\,\mathrm{j}\, H\, R) - 1]/R$.

As $|u| \leqq 1$ and $|u'| \leqq 1$, then $0 \leqq |u - u'| \leqq 2$. The remaining part of the kernel K_2 can be approximated on the segment $[0, 2]$ with a polynomial series of the N-th order with the unknown complex coefficients B_n:

$$K_2 = [\exp\,(-\,\mathrm{j}\, H\, R) - 1]\Big/R \cong \sum_{n=0}^{N} B_n\, |u - u'|^n\,. \quad (3)$$

The $N + 1$ unknown coefficients B_n can be determined easily by satisfying the equation (3) in $N + 1$ points $|u - u'|_p$, $p = 0, 1, \ldots, N$.

If the matching points $|u - u'|_p$ are selected to be equidistant on the segment $[0, 2]$, i. e. $|u - u'|_p = 2\, p/N$, $p = 0, 1, \ldots, N$, the system of linear equations thus obtained can be easily solved, because the determinant of the system can be reduced to the Vandermonde's determinant.

Once the coefficients B_n have been determined, the Hallén's integral equation (2) can be put in the approximate form

$$\int_{-1}^{1} G(u')\, [A^2 + (u - u')^2]^{-1/2}\, \mathrm{d}u' + \sum_{n=0}^{N} B_n \int_{-1}^{1} G(u')\, |u - u'|^n\, \mathrm{d}u' \cong C_0 \cos H\, u + \frac{Z}{\mathrm{j}\, 60} \sin H\, |u|\,. \quad (4)$$

After obtaining eq. (4) in the above form the usual pointmatching method can be used for its solution. However if the normalized current distribution function is approximated by a polynomial of the form

$$G(u') \cong \sum_{m=0}^{M} A_m\, |u'|^m\,, \quad (5)$$

all the integrals appearing in (4) can be put in the closed form. In all other respects the point-matching polynomial procedure is to be performed in a conventional manner. The integral equation (4), with the polynomial approximation (5) included, should be satisfied in the $M + 1$ points equidistantly selected along the halflength of the dipole, i.e. in the points $u = u_i = i/M$, $i = 0, 1, \ldots, M$. In addition the two following conditions for the normalized current distribution along the dipole must be satisfied: $G(0) = A_0 = 1$ and $G(+1) = G(-1) = 0$.

In this way the system of $M + 2$ linear equations is obtained:

$$\sum_{m=0}^{M} A_m = 0$$

and

$$\sum_{m=0}^{M} A_m\, g_{m\,i} = C_0 \cos H\, u_i + \frac{Z}{\mathrm{j}\, 60} \sin H\, u_i\,,$$

$$i = 0, 1, \ldots, M\,,$$

where

$$g_{m\,i} = y_m(u_i) + \sum_{n=0}^{N} B_n\, Y_{m\,n}(u_i)\,, \quad (6)$$

$$Y_{m\,n}(u_i) = \int_{-1}^{1} |u'|^m\, |u_i - u'|^n\, \mathrm{d}u' = \sum_{s=0}^{n} \binom{n}{s} \frac{[1 + (-1)^{n-s}]\, u_i^{n-s} + (-1)^s\, [1 - (-1)^n]\, u_i^{m+n+1}}{m + s + 1}\,, \quad (7)$$

$$y_m(u_i) = \int_{-1}^{1} |u'|^m\, [A^2 + (u_i - u')^2]^{-1/2}\, \mathrm{d}u'\,. \quad (8)$$

By means of the reccurent formula, integrals $y_m(u_i)$ defined by (8), are reduced to the elementary integrals: $y_m(u_i) = \sum_{q=1}^{2} T_{mq}(u_i)$ where:

$$T_{mq}(u_i) = \int_{0}^{1} u'^m\, [A^2 + (u' + w\, u_i)^2]^{-1/2}\, \mathrm{d}u'$$

$$= \begin{cases} \ln \dfrac{1 + w\, u_i + [A^2 + (1 + w\, u_i)^2]^{1/2}}{w\, u_i + (A^2 + u_i^2)^{1/2}}\,, & \text{for } m = 0\,, \\[2ex] [A^2 + (1 + w\, u_i)^2]^{1/2} - (A^2 + u_i^2)^{1/2} - w\, u_i\, T_{0\,q}(u_i)\,, & \text{for } m = 1\,, \\[2ex] \dfrac{[A^2 + (1 + w\, u_i)^2]^{1/2} - (2\, m - 1)\, w\, u_i\, T_{m-1,\,q}(u_i)}{m} - \dfrac{(m - 1)\, (u_i^2 + A^2)\, T_{m-2,\,q}(u_i)}{m}\,, & \text{for } m \geqq 2 \end{cases}$$

and $w = (-1)^q$.

Table I. Admittances (in mA/V) of the cylindrical dipoles having radius $a = 0.007022\,\lambda$ and different lengths $2\,h \cdot M$ and N are orders of the polynomials approximating the current distribution and kernel, respectively

		M		
		2	3	4
$2\,h = 0.50\,\lambda$	$N = 4$	9.16 − j 3.57	9.16 − j 3.55	8.81 − j 3.30
	$N = 20$	9.16 − j 3.57	9.16 − j 3.55	8.81 − j 3.31
	Popović	9.16 − j 3.57	9.16 − j 3.55	8.81 − j 3.31
	Measured (Mack)		8.90 − j 3.46	
$2\,h = 0.75\,\lambda$	$N = 5$	1.53 − j 0.36	1.54 − j 0.26	1.53 − j 0.10
	$N = 20$	1.52 − j 0.36	1.54 − j 0.26	1.53 − j 0.10
	Popović	1.52 − j 0.36	1.54 − j 0.27	1.53 − j 0.10
	Measured (Mack)		1.58 − j 0.17	
$2\,h = \lambda$	$N = 5$	0.98 + j 1.55	0.96 + j 1.58	0.97 + j 1.68
	$N = 20$	0.98 + j 1.54	0.96 + j 1.58	0.96 + j 1.67
	Popović	0.98 + j 1.54	0.96 + j 1.58	0.97 + j 1.67
	Measured (Mack)		1.02 + j 1.68	

Table II ($M = 3$). Current Distribution (in mA/V) along the cylindrical dipole having radius $a = 0.007022\,\lambda$ and length $2\,h = \lambda \cdot M$ and N are the orders of the polynomials approximating the current distribution and kernel, respectively

$u' = z'/h$	$N = 3$	$N = 5$	$N = 10$	$N = 20$	Popović
0.0	0.93 + j 1.48	0.96 + j 1.58	0.96 + j 1.58	0.96 + j 1.58	0.96 + j 1.58
0.1	0.93 + j 0.15	0.95 + j 0.24	0.95 + j 0.23	0.95 + j 0.24	0.95 + j 0.24
0.2	0.89 − j 0.89	0.89 − j 0.82	0.89 − j 0.83	0.89 − j 0.82	0.89 − j 0.82
0.3	0.81 − j 1.66	0.80 − j 1.60	0.80 − j 1.60	0.80 − j 1.60	0.80 − j 1.60
0.4	0.71 − j 2.16	0.69 − j 2.11	0.69 − j 2.11	0.69 − j 2.11	0.69 − j 2.11
0.5	0.59 − j 2.40	0.56 − j 2.36	0.56 − j 2.36	0.56 − j 2.36	0.56 − j 2.36
0.6	0.46 − j 2.39	0.43 − j 2.36	0.42 − j 2.35	0.43 − j 2.37	0.43 − j 2.37
0.7	0.33 − j 2.14	0.30 − j 2.12	0.29 − j 2.11	0.30 − j 2.12	0.30 − j 2.12
0.8	0.20 − j 1.65	0.18 − j 1.63	0.17 − j 1.63	0.18 − j 1.64	0.18 − j 1.64
0.9	0.09 − j 0.93	0.07 − j 0.93	0.07 − j 0.92	0.08 − j 0.93	0.08 − j 0.93
1.0	0.00 + j 0.00	0.00 + j 0.00	0.00 + j 0.00	0.00 + j 0.00	0.00 + j 0.00

The integrals (7) and (8) are given in closed form and can be evaluated accurately and very quickly.

The proposed method was checked in the case of a thin cylindrical dipole of a radius $a = 0.007022\,\lambda$ and for the different dipole lengths $2\,h/\lambda = 0.50$, 0.75, 1.00.

First, the convergence of the semi-polynomial approximation for the kernel was examined by comparing the approximate with the exact values. For the dipole lengths $2\,h \leqq 1.5\,\lambda$ the order $N \geqq 5$ of the polynomial approximating the kernel was found to be very satisfactory.

Afterwards, the impedances and current distributions were calculated and compared with the Mack's experimental data[5] and with Popović's theoretical results[2]. The latter were calculated by the usual point-matching polynomial method. In the Popović's and present theoretical procedures the same order M of the polynomial for the current approximation was used. The results for the admittances are presented in the Table I and those for the current distribution in the Table II.

The comparison of the data presented in the Tables I and II shows results which convergence quickly. The main advantage of the present method is in its ecconomy with respect to the necessary computer time. This is of a particular interest in the case when the high order of the polynomial approximation for the current distribution is used and when the calculations have to be performed by a smallsize computer.

The author is indebted to Prof. J. V. Surutka for his interest, encouragement and for illuminating discussions.

References

1. King, R. W. P.: 'The theory of linear antennas' (Harvard University Press, 1956).
2. Popović, B. D.: Polynomial approximation of current along thin cylindrical dipoles', Proc. IEE, Vol. 117, No. 5, 1970.
3. Popović, B. D.: 'Analysis of two identical parallel arbitralily located thin asymmetrical antennas', Proc. IEE, Vol. 117, No. 9, september 1970.
4. Popović, B. D.: 'Theory of cylindrical antennas with arbitrary impedance loading', Proc. IEE, Vol. 118, No. 10, 1971.
5. Mack, R. B.: 'A study of circular arrays', Cruft Laboratory, Harvard University, technical reports 382 and 383, 1963.

Received August 18, 1972

Dragutin M. Veličković, M. Sc.
Dept. of Electronic Engineering
University of Niš
Stanka, Puanovića 43/45-I
Niš, Yugoslavia