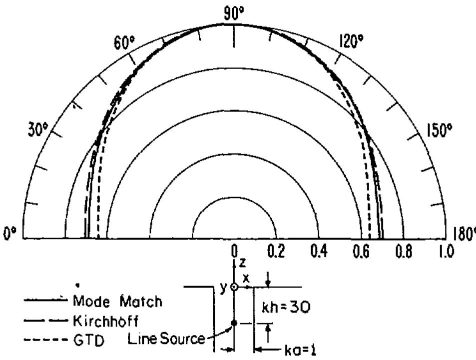

Fig. 1. Radiation comparison for $ka = 1$.

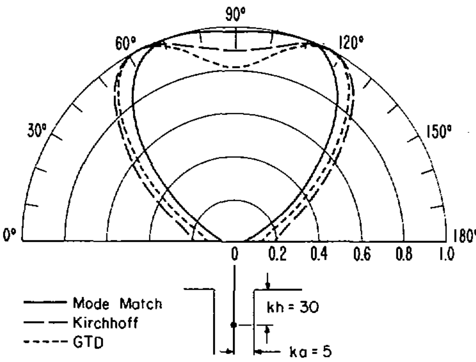

Fig. 2. Radiation comparison for $ka = 5$.

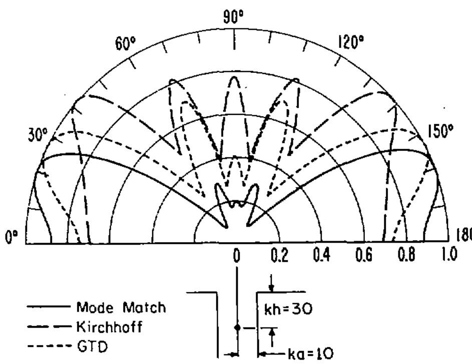

Fig. 3. Radiation comparison for $ka = 10$.

both theories would yield D_0^- a maximum. The narrow aperture results are thus typically as shown in Fig. 1. In Fig. 3, $ka \sim 3\pi$ which is near a resonance for the TM_3 mode incident from below. The figure demonstrates the inability of Kirchhoff theory to account for resonance. Actually, since Kirchhoff theory also improves as ka becomes large, the resonance corresponding to $ka \sim 3$ (given in [4]) could be expected to show greater discrepancy. In each figure the patterns show discrepancy in the low angle ($\psi \sim 0°$, $\psi \sim 180°$) regions. This is a consequence of the Kirchhoff theories inability to give the proper description of the edge diffracted components of the radiation. That is D_n^-, as given by (3), displays exponential behavior for large n, while the D_n^- obtained from the GTD technique and the mode match technique [4] shows algebraic dependence on n for large n. In fact it can be shown [4] that the edge condition [12] requires that

$$D_n^- = \frac{(-1)^n \alpha}{n^{5/3}} + 0(n^{-7/3}) \tag{10}$$

where α is a constant of proportionality. The aperture amplitudes, using the mode match method [4], converge to this proper dependence. Using condition (10) for a truncation criterion, the mode match patterns were generated with ten explicit terms of series (2) for Fig. 1, and thirty explicit terms for Figs. 2 and 3. These radiation conclusions are in agreement with the analysis of the unflanged case, which has been treated in an elegant fashion by Weinstein [13].

In conclusion we have shown that with the notable exceptions of near resonance and in the low angle observation regions, Kirchhoff theory gives a relatively good description of the power pattern in the forward direction. The GTD method improves upon the Kirchhoff technique and is much more versatile than the mode match method (it is not restricted to problems for which the wave equation separates). Finally, we have demonstrated that the GTD application to open ended waveguides, which has been shown to be highly accurate in predicting the modal reflection coefficients in the aperture [1], is also well suited in the same formalism to treat the radiated field.

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Current Distribution on a Cylindrical Antenna in a Plasma

PASCAL MEYER

***Abstract*—A general solution for the current distribution on a cylinder in a cold magnetoplasma has been obtained. It is complicated, but for practical applications several simplifications occur. We present an outline of this solution and a sample of results.**

As a step toward understanding the working of antennas in plasma, we have studied the following problem: a thin-walled open-ended cylinder of radius a, length h, in a gyrotropic plasma. A general solution for the time harmonic Maxwell equations in the anisotropic homogeneous medium is readily obtained by a space Fourier transform. Taking advantage of the geometry of the problem, one can perform some lengthy but straightforward calculations which give the electric field by its expansion

$$E_j(r,\Phi,z) = \sum_{\beta=-\infty}^{\beta=+\infty} \exp\,(i\beta\Phi) \int_{-\infty}^{+\infty} d\gamma \exp\,(i\gamma z) E_j(r,\beta,\gamma) \tag{1}$$

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where $j = r,z,\Phi$ and r,z,Φ are cylindrical coordinates with $0z$ along the antenna axis. $E_j(r,\beta,\gamma)$ is given in term of a similar expansion of the sources distribution S by

$$E_j(r,\beta,\gamma) = M_{ji}(r,\beta,n,\gamma) S_i(n,\gamma) \tag{2}$$

(summation on dummy indexes n and i).

M_{ji} has complicated expressions (which can be found in the author's thesis). They are sums of terms like

$$F(\gamma) = \int_0^{2\pi} d\theta \int_1^{\infty} dk \exp[-i(\beta+n)\theta] J_\beta(kr) J_n(ka) \frac{Pji(k,\theta,\gamma)}{\Delta(k,\theta,\gamma)} \tag{3}$$

where J is Bessel functions and Pji/Δ is a rational fraction. This expression involves an integration which can be made for $\operatorname{Im}\omega > 0$ (time factor $\exp(-i\omega t)$) or $\operatorname{Im}\omega = 0^+$.

With some restrictions on S, the two useful components E_z and E_Φ are continuous at $r = a$, so that using the boundary conditions:

$$E_{z,\Phi}{}^{\text{total}} = 0 \qquad r = a \qquad |z| < h$$

we have the following relation, for $j = z$ or Φ:

$$-E_j{}^{\text{exc}}(\beta,\gamma) + E_j{}^{+}(\beta,\gamma) + E_j{}^{-}(\beta,\gamma) = M_{ji}(\beta,n,\gamma) S_i(n,\gamma) \tag{4}$$

where E^{exc} is the given excitation of the antenna and $E^{\pm}$ are the two unknown fields on the cylinder beyond the tips $z = \pm h$:

$$E^{\pm}(\gamma) = \int_{\pm h}^{\pm\infty} E(z) \exp(i\gamma z)\, dz. \tag{5}$$

From (4), using convolution, a Hállen type integral equation can be obtained; its solution would require heavy numerical methods, but a study of (3) shows analogy with the vacuum case. Hurd [1] has shown the usefulness of the Wiener–Hopf technique for its solution and we shall use it here.

A series of approximations is built in the following way. Let us approximate the infinite system (4) by the truncation $M \equiv 0$ for $\beta,n > N$. Using the standard Wiener–Hopf procedure, one can obtain a system of integral equations for the unknown field $E^{\pm}$. These equations are of Fredholm second kind and define one unique solution. It is the approximation of order N. Its convergence can be considered in the following way: for any given γ, M is a Hilbert–Schmidt operator in l^2 space; suppose that there is a small conductivity σ of the metal, then M is replaced by $M + \sigma$ which is an homeomorphism and can then be inverted by truncation; but σ, being small, can be neglected in low-order truncation; this inversion drives S from E, E being itself determined by an approximation of a Fredholm second kind equation. Then the l^2 convergence follows.

For practical purpose, the general method can be greatly simplified because we deal with thin and long antennas.

1) *Thin Antenna*: $a \ll$ all progressive wavelengths in the medium. For a cold plasma, two cases occur [2]: a) the refractive index is bounded, the condition is very well satisfied in most cases; and b) the index is not bounded, the condition is also satisfied, but not so well, if we take into account the limit of validity of the cold plasma approximation for large index values. One can then obtain an estimate of the rate of convergence with N. In (3), terms like $J_v(ka)$ are either 1 ($v = 0$) or 0 ($v \neq 1$) in the integration range; it remains then only a 3×3 matrix (a classical result, with a mixing here between TE and TM).

Furthermore, taking into account the various degrees of P_{ij}, an estimate of the size of the element shows that when $E_\Phi{}^{\text{exc}} \equiv 0$, only the element $i,j = z$ $\beta,n = 0$ of M_{ji} is useful. This element has properties quite similar to the ones found in the vacuum case. It can be written approximately for $\gamma < 1/a$ as

$$\frac{1}{\gamma_0{}^2}(\gamma^2 - \gamma_0{}^2)[\sum_s \ln a(\gamma - \gamma_s)] \tag{6}$$

where $\gamma_{0,s}$ are constants. This shows that there is a surface wave γ_0 on the cylinder. γ_0 has been found previously by several authors when studying more or less similar problems [3]–[5].

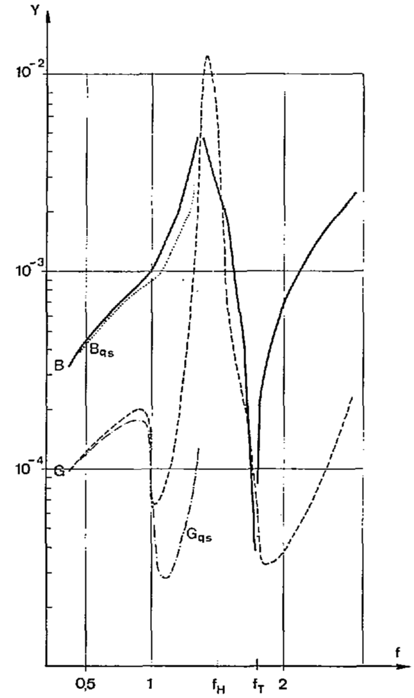

Fig. 1. Y (mhos) versus f (MHz), for $\alpha = 10°$.

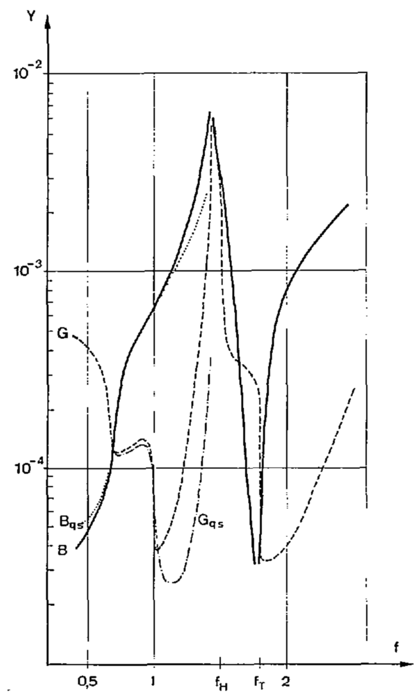

Fig. 2. Y (mhos) versus f (MHz), for $\alpha = 50°$.

2) *Long Antenna*: $h/a \gg 1$. When calculating the current distribution from (4), $E^{\pm}$ acts as secondary excitations from (5). One can see that they behave as $\Phi(\gamma)\exp(\pm i\gamma h)$, where $\Phi(\gamma)$ is algebraic at $|\gamma| \to \infty$. In this distribution, a term such as:

$$\int \exp(-i\gamma z) \frac{E^+(\gamma)}{M(\gamma)} = \int \exp[-i\gamma(z-h)] \frac{\Phi(\gamma)}{M(\gamma)} \tag{7}$$

can be viewed as the current "reflected" from the tip $z = +h$. From (2), the major part of it is calculated with $z - h > a$ and we need only to know $\Phi(\gamma)$ in the limited range $\operatorname{Im}\gamma < 1/|z-h|$,

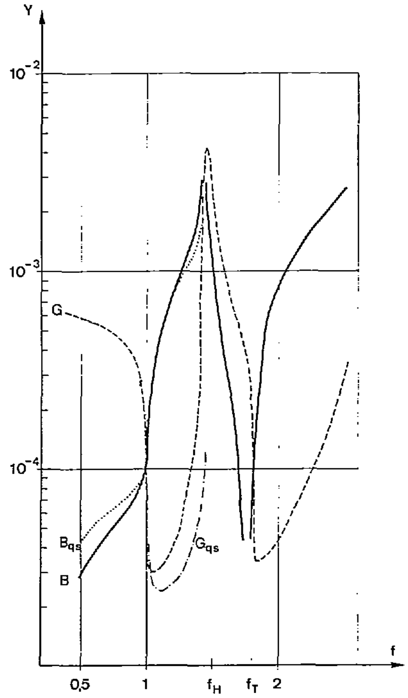

Fig. 3. Y (mhos) versus f (MHz), for $\alpha = 90°$.

for an approximate integration. From (1), the integral equation using (7) gives $E \simeq C^{te}$ in the range $\gamma < 1/a$, so that,

$$I^{\text{ref}}(z) \simeq C^{te} I^{\infty}(z + h)$$

when I^{∞} is the current on an infinite ($h = \infty$) antenna. Although this is not valid near the tip, one can see that I^{ref} does not deviate much from I^{∞} so that C^{te} can be obtained by the condition $I^{\text{total}}(\pm h) = 0$. One can obtain then a very simple approximation for the current and the admittance follows. The validity of this approximation is supported by the vacuum case where it works very well, and also by [1, figs. 4–7]. Note too that the basic idea is the same as in Keller's theory of diffraction.

Using this, a relatively simple algorithm has been programmed. It gives a value of admittance Y in about 2 min on IBM 360/65; the overall accuracy cannot of course be controlled but comparison with simpler case ($\alpha = 0$ and short antenna) show a good check. We present here a sample of values of $Y = G + iB$ versus the frequency f in the case $a = 10^{-2}$, $h = 24$, f_p (plasma frequency) $= 10^6$, f_h (gyrofrequency) $= 1.5 \times 10^6$, and α (angle between the antenna and the magnetostatic field) $= 10°, 50°, 90°$, (see Figs. 1–3); MKSA units are used and $|B|$ is drawn when negative, here between f_h and $f_t = SQRT\ (f_p^2 + f_h^2)$. For comparison, the result of the quasi-static approximation [6] has been drawn in the range $f = (0.5, 1.3)$ where this approximation is roughly valid, apart from some discrepancies, the general agreement is satisfactory. The index QS identifies these curves.

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Radiation from a Paraboloid with an Axially Defocused Feed

PAUL G. INGERSON AND WILLARD V. T. RUSCH

Abstract—**The radiation characteristics of a paraboloid with an axially defocused idealized point-source feed are derived using numerical integration of the physical-optics currents. Results from the standard "linearized" analysis are compared with results obtained without approximating various factors in the integrand. In particular, the more complete analysis reveals that the defocusing curves are not symmetrical about the focus. At values of defocusing which cause deep nulls in the defocusing curves of less-tapered feeds, the angular beam exhibits bifurcation. Consequently, these less-tapered feeds are not generally suited to beam broadening applications.**

The experimental results of Landry and Chassé [1] for an axially defocused dipole-type feed illuminating a paraboloidal reflector confirm our analysis of the same geometry reported in [2]. Calculations made for $f/D = 0.35$ and $D/\lambda = 69.67$ yield results which are virtually identical to the data in [1, fig. 3], particularly with respect to the position of the displaced minima and the nonsymmetric maxima of the defocusing peaks. Because we feel that [2] may not have come to the general attention of the readers, the principal results are summarized in this communication.

A perfectly conducting paraboloidal reflector is illuminated by an idealized point source on the reflector axis (Fig. 1). The signed distance beyond the focus to the source is d. The vector from the focus to a point on the reflector is $\bar{\rho}$. The vector from the source to the same point is $\bar{\rho}'$. The difference in the magnitudes of these two vectors is primarily (but not exclusively) responsible for the antenna's defocusing characteristics. The physical-optics far-zone field is

$$\bar{E}_S = -\frac{j\omega\mu_0}{2\pi}\frac{\exp(-jkR)}{R}\int_{\text{front}}[\hat{n} \times \bar{H}_F(\rho,\theta',\phi')]_{\text{trans}} \cdot \exp(k\bar{\rho}\cdot\bar{a}_R)\,dS(\theta',\phi'). \quad (1)$$

The field of the feed is an idealized vector spherical wave

$$\bar{E}_F = \frac{\exp(-jk\rho')}{\rho'}\,\bar{e}(\theta'',\phi'')$$

$$\bar{H}_F = \left(\frac{\epsilon_0}{\mu_0}\right)^{1/2}(\bar{a}_{\rho'} \times \bar{E}_F). \quad (2)$$

The E- and H-fields are orthogonal to each other and to the direction of propagation. The constant-phase surfaces are spheres centered at the source. The directivity pattern is a function of the source angles θ'' and ϕ''. The distance relationship is

$$\rho' = \rho\left\{1 + \left[\frac{d}{\rho}\left(\frac{d}{\rho} - 2\cos\theta'\right)\right]\right\}^{1/2}. \quad (3)$$

The simplest line of analysis employs these approximations

$$\rho' \approx \rho - d\cos\theta' \quad \text{(in phase expressions)} \quad (4a)$$

$$\rho' \approx \rho \quad \text{(in amplitude expressions)} \quad (4b)$$

$$\bar{e}(\theta'',\phi'') \approx \bar{e}(\theta',\phi') \quad (4c)$$

$$\bar{a}_{\rho'} \approx \bar{a}_{\rho}. \quad (4d)$$

This set of approximations may be designated the "linear" analysis because the phase expression has been linearized. This linear analysis, or its equivalent, is used most commonly in [3], [4].

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