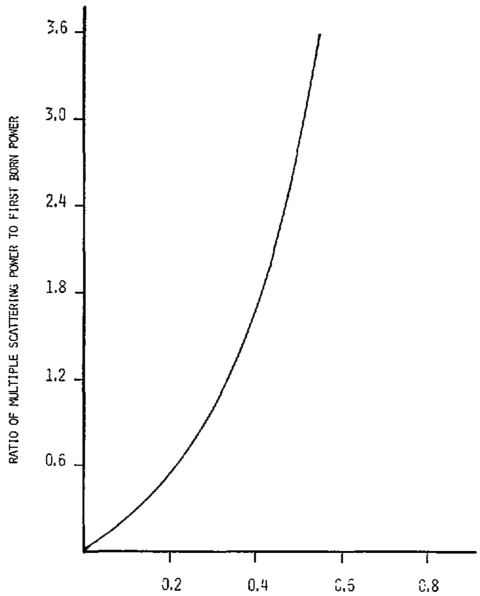

Fig. 1. Measure of multiple scattered power to first Born power versus B, sufficiency condition for first convergence of series.

Solution

To facilitate the analysis, (9) may be put in operational form as follows:

$$F(\bar{\kappa}) = \psi_0 + \frac{\epsilon}{(2\pi)^3}\int d\bar{p}K(\bar{\kappa},\bar{p})F(\bar{p}) \tag{10a}$$

or

$$F(\bar{\kappa}) = \psi_0 + \lambda\tilde{K}F \tag{10b}$$

where

ψ_0	first-order scattering term
$K(\bar{\kappa},\bar{p})$	kernel of the multiple scattering term
$\tilde{K}$	operator $[\epsilon/(2\pi)^3]\int d\bar{p}K(\bar{\kappa},\bar{p})$
λ	radius of convergence of the series, equal to 1 in our case.

To determine the convergence of the series solution, we note that the kernel $K(\bar{\kappa},\bar{p})$ has a weak or integrable singularity. It was proved in [4] that it is not necessary to assume continuity of the kernel for the method of successive approximation to apply: it is only sufficient that the double integral

$$B^2 \equiv \int_V\int d\bar{\kappa}\, d\bar{p}\left|\frac{\epsilon}{(2\pi)^3}K(\bar{\kappa},\bar{p})\right|^2 \tag{11a}$$

or

$$B^2 \equiv \int_V\int d\bar{r}\, d\bar{r}'\left|\epsilon\frac{\exp(-ik|\bar{r}-\bar{r}'|)}{-4\pi|\bar{r}-\bar{r}'|}\eta(r)\right|^2 \tag{11b}$$

should exist. Then the method of successive approximations converges for all values of λ lying within a radius of convergence $|\lambda| \leq 1/B$. The limit of successive approximations is the solution of (10) and this solution is unique. Thus the series-expansion method is extended to new classes of integral equations.

To determine the error due to the truncation of the series after the first term, as is the case for the first Born, let us find the ratio of the first term to the remaining terms. From (10b), the scalar product of the scattering amplitude is

$$(F,F) = (\psi_0,\psi_0) + (\psi_0,\tilde{K}F) + (\tilde{K}F,\psi_0) + (\tilde{K}F,\tilde{K}F). \tag{12}$$

Applying the Bunyakovski inequality, we obtain the norm of the function F as

$$\|F\|^2 \leq \|\psi_0\|^2 + 2B\|\psi_0\|\,\|F\| + B^2\|F\|^2 \tag{13}$$

which leads to the condition, after solution for $\|F\|$ in (13),

$$\|F\|^2 \leq (\|\psi_0\|^2)/(1-B)^2 \tag{14}$$

where $0 < B < 1$ as $\lambda = 1$. From (12) we note that

$$(F,F) = (\psi_0,\psi_0) + \text{multiple scattering terms.} \tag{15}$$

Combining (14) and (15), we obtain an analytical bound on the validity of the first Born

$$\frac{\text{multiple scattering power}}{\text{first Born power}} \leq \frac{B(2-B)}{(1-B)^2} \quad [\text{Fig. 1}]. \tag{16}$$

In conclusion, a simple quantitative criterion has been established to test the validity of the first Born and to ensure its convergence. While our method gives a "looser" bound, it appears well fitted, in radar practice, for rapid numerical estimates.

Joseph C. Hassab
John Jarem
Dept. of Elec. Engrg.
Drexel Institute of Technology
Philadelphia, Pa.

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Receiving Antennas in a Compressible Plasma

***Abstract*—The receiving voltages and the receiving maximum available powers of a dipole antenna and a loop antenna immersed in an isotropic compressible plasma are obtained for both an electromagnetic and an electron-plasma wave incidence by making use of the reciprocity theorem.**

Let the receiving voltages induced across an opened terminal of an antenna immersed in an isotropic compressible plasma be denoted by V_e and V_p for an incident electromagnetic wave (e wave) and an electron-plasma wave (p wave), respectively. We define here the effective antenna lengths h_e and h_p for incident waves as

$$V_e = E_e h_e = [(\mu_0/\epsilon_0)^{1/2}P_e]^{1/2}(1-\omega_p^2/\omega^2)^{-1/4}h_e \tag{1}$$

$$V_p = E_p h_p = [(\mu_0/\epsilon_0)^{1/2}P_p(c/u)]^{1/2}(\omega_p/\omega)(1-\omega_p^2/\omega^2)^{-1/4}h_p \tag{2}$$

where E_e and E_p are, respectively, the incident transverse electric-field strength of an e wave and the longitudinal electric-field strength of a p wave at the origin of the coordinate; P_e and P_p are the incident power densities of the e and p waves; and c and u are, respectively, the light velocity and the root-mean-square electron thermal velocity.

Now we make use of the reciprocity theorem [1] in an isotropic compressible plasma to obtain the receiving voltages of antennas for e and p waves from the known results on transmitting antenna properties for the respective waves. In applying the reciprocity theorem it is assumed that the maximum receiving voltage of a Hertzian dipole is given by the product of the electric-field strength and the length of the dipole for both e and p waves. The radiated

Manuscript received September 25, 1969; revised January 5, 1970.

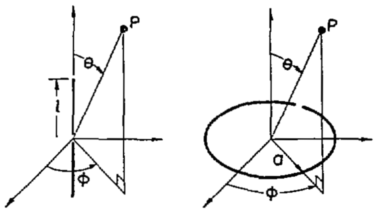

Fig. 1. Coordinate systems.

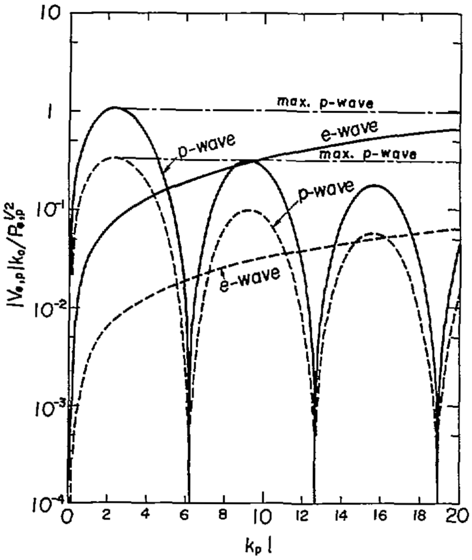

Fig. 2. Receiving voltages of dipole for e and p wave incidences ($\theta = \pi/2$ for e wave, $\theta = 0$ for p wave), $\omega_p^2/\omega^2 = 0.5$; solid curve: $c/u = 10^3$; dashed curve: $c/u = 10^4$.

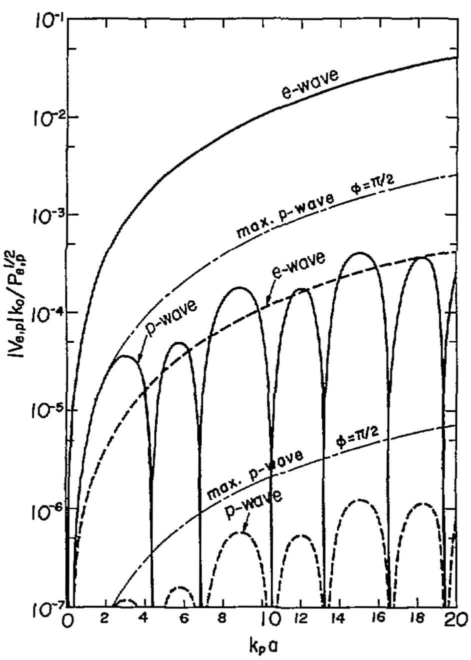

Fig. 3. Receiving voltages of loop antenna for e and p wave incidences ($\theta = \varphi = \pi/2$), $\omega_p^2/\omega^2 = 0.5$; solid curve: $c/u = 10^3$; dashed curve: $c/u = 10^4$.

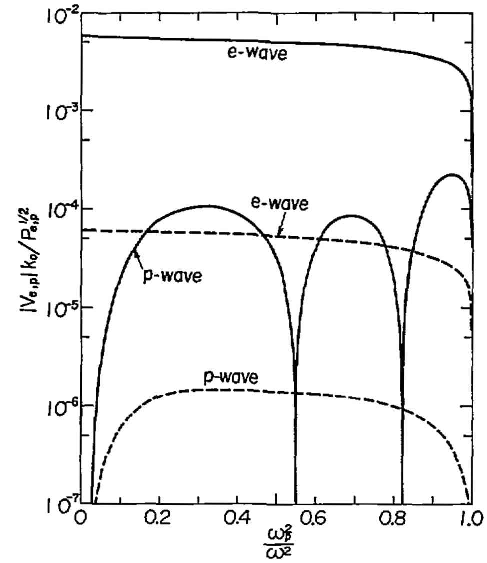

Fig. 4. Receiving voltages of loop antenna versus plasma density ($\theta = \varphi = \pi/2$), $c/u = 10^3$; solid curve: $k_0a = 10^{-2}$; dashed curve: $k_0a = 10^{-3}$.

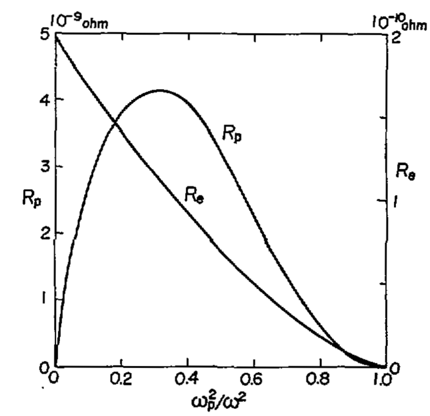

Fig. 5. Radiation resistances of loop versus plasma density, $k_0a = 10^{-3}$, $c/u = 10^3$.

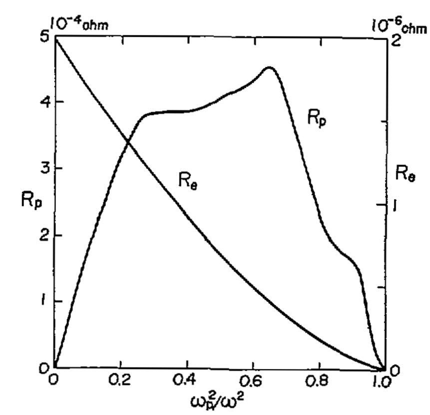

Fig. 6. Radiation resistances of loop versus plasma density, $k_0a = 10^{-2}$, $c/u = 10^3$.

fields of e and p waves from a dipole antenna having the current distribution whose propagation constant is set equal to k_e, the phase constant of an electromagnetic wave in a plasma, were calculated by Chen [2]. Those for a circular loop antenna were given by the present authors [3]. Thus the effective lengths h_e^d and h_p^d of the receiving dipole antenna for a θ-directed e wave and a p wave, respectively, are given by

$$h_e^d = \frac{2}{k_e \sin k_e l}\left\{\frac{\cos(k_e l \cos\theta) - \cos k_e l}{\sin\theta}\right\} \tag{3}$$

$$k_p^d = \frac{2}{k_e \sin k_e l}\left\{\frac{\cos(k_p l \cos\theta) - \cos k_e l}{1 - (c/u)^2 \cos^2\theta}\right\}\cos\theta \tag{4}$$

and the effective lengths $h_{e\theta}^l$, $h_{e\phi}^l$, and h_p^l of the receiving loop antenna for a θ-directed e wave, a ϕ-directed e wave, and a p wave, respectively, are expressed as

$$h_{e\theta}^l = 2k_e a^2 \tan(k_e a\pi) \sum_{n=0}^{\infty} \epsilon_n (-j)^n J_n(k_e a \sin\theta) \cdot\left[\frac{2n\cos\phi\sin n\phi + \{(k_e a)^2 - (n^2+1)\}\sin\phi\cos n\phi}{\{(k_e a)^2 - (n+1)^2\}\{(k_e a)^2 - (n-1)^2\}}\right]\cos\theta \tag{5}$$

$$h_{e\phi}^l = -2k_e a^2 \tan(k_e a\pi) \sum_{n=0}^{\infty} \epsilon_n (-j)^n J_n(k_e a \sin\theta) \cdot\left[\frac{2n\sin\phi\sin n\phi - \{(k_e a)^2 - (n^2+1)\}\cos\phi\cos n\phi}{\{(k_e a)^2 - (n+1)^2\}\{(k_e a)^2 - (n-1)^2\}}\right] \tag{6}$$

$$h_p^l = \frac{4k_e a}{k_p}\tan(k_e a\pi)\sum_{n=1}^{\infty}(-j)^{n+1}J_n(k_p a\sin\theta)\frac{n}{(k_e a)^2 - n^2}\sin n\phi \tag{7}$$

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n = 1,2,3,\cdots \end{cases}$$

where (θ,ϕ) is the spherical coordinate as shown in Fig. 1, l is a half-length of a dipole, a a radius of a loop, and k_p the propagation constant of an electron-plasma wave given by

$$k_p^2 = (\omega^2/u^2)(1 - \omega_p^2/\omega^2) = k_e^2(c/u)^2. \tag{8}$$

Fig. 2 shows the receiving voltages calculated for a short dipole versus the dimension of the antenna for e and p waves incident from the particular directions ($\theta = \pi/2$ for e wave, $\theta = 0$ for p wave). The receiving voltage due to p wave incidence is maximum in the direction $\theta = 0$ for a very short dipole ($k_p l \to 0$ as a limit), and the direction of the maximum radiation deviates from $\theta = 0$ towards $\theta = \pi/2$ as $k_p l$ increases. The dot–dash lines in Fig. 2 show the maximum values for the p wave given by

$$h_{pm}^d \simeq (4/k_e\pi)(c/u)^{-1} \tag{9}$$

in the direction of

$$\theta \simeq \cos^{-1}(\pi/k_p l). \tag{10}$$

Fig. 3 shows the receiving voltages calculated for a small loop versus the antenna dimension. The receiving voltage due to p wave incidence is maximum in the direction $\theta = \phi = \pi/2$ for a very small loop ($k_p a \to 0$ as a limit), and the direction of the maximum radiation in this case deviates from $\theta = \pi/2$ towards $\theta = 0$ as $k_p a$ increases. The maximum effective length of the loop is calculated as

$$h_{pm}^l \simeq 2.48\pi(c/u)^{-1}k_e a^2 \tag{11}$$

in the direction of

$$\theta \simeq \sin^{-1}(2.0/k_p a) \tag{12}$$

for the conditions $k_e a \ll 1$, $k_p a \gg 1$, and $\phi = \pi/2$. The dot–dash lines in Fig. 3 show the maximum values for p waves calculated from (11).

Fig. 4 shows the variation of the respective voltages versus the plasma density for a loop of constant radius. It is of interest to find that the receiving voltage due to a p wave is fairly small compared to that due to an e wave of the same power density except in the case of a very small dipole. At $\omega_p^2/\omega^2 = 0.5$, the calculation for a dipole shows that the receiving voltages for e and p waves are of the same order in magnitude at about $k_p l = 10$ for $c/u = 10^3$, and at about $k_p l = 17$ for $c/u = 10^4$ (Fig. 2).

The receiving maximum available power can be obtained by using the radiation resistances for e and p waves R_e and R_p. Namely,

$$W_{em} = |V_e|^2/[4(R_e + R_p)] \tag{13}$$

$$W_{pm} = |V_p|^2/[4(R_e + R_p)]. \tag{14}$$

The accurate values of these radiation resistances for a dipole antenna were given by Gupta [4] and those for a loop antenna were recently given by the present authors [3]. The radiation resistances R_e and R_p versus ω_p^2/ω^2 for $k_0 a = 10^{-3}$ and 10^{-2}, both for $c/u = 10^3$, are plotted in Figs. 5 and 6, respectively, and were not given numerically in the previous paper [3].

S. ADACHI
T. KASAHARA
Y. MUSHIAKE
Dept. Commun. Engrg.
Tohoku University
Sendai, Japan

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