Day 13: Shuttle Search, Part 2

This document is a work in progress. It currently contains a worked example that reflects the way I solved the puzzle, but I'm planning to add a general solution later.

Worked example

First, we solve the puzzle for the input `17,x,13,19` provided in the description.

Let $t \geq 0$ be the earliest timestamp such that all of the listed bus IDs depart at offsets matching their positions in the list.

Directly from the description of the puzzle we get the following equations for t:

$$\begin{aligned} t &= 17k_1, \\ t+2 &= 13k_2, \\ t+3 &= 19k_3 \end{aligned}$$

for some integers k_1, k_2, k_3. We can rearrange these equations to get

$$\begin{aligned} t &= 17k_1 \\ &= 13k_2 - 2 \\ &= 19k_3 - 3. \end{aligned}$$

We now consider the equation $17k_1 = 13k_2 - 2$. Using a brute-force approach we find that the solution with the lowest values of k_1 and k_2 is $k_1 = 6$ and $k_2 = 8$, that is

$$17 \times 6 = 13 \times 8 - 2 = 102.$$

Hence the first timestamp at which the buses 17 and 13 depart at the given offsets is 102. This repeats every $17 \times 13 = 221$ minutes. (Note that all bus IDs are prime numbers.) Thus we have

$$\begin{aligned} t &= 17 \times 13l_1 + 17 \times 6 \\ &= 17(13l_1 + 6) \end{aligned}$$

for some integer l_1.

We apply the same approach to the equation $17k_1 = 19k_3 - 3$. This time we find that

$$17 \times 11 = 19 \times 10 - 3,$$

which gives

$$t = 17(19l_2 + 11)$$

for some integer l_2.

This way we obtained two new equations for t:

$$\begin{aligned} t &= 17(13l_1 + 6), \\ t &= 17(19l_2 + 11). \end{aligned}$$

We can rearrange these equations to get

$$t = 17(t_1 + 6),$$

where

$$\begin{aligned} t_1 &= 13l_1 \\ &= 19l_2 + 5. \end{aligned}$$

and $t_1 \geq 0$.

Now we can apply the same process we have used to get the new equations for t to the equation $13l_1 = 19l_2 + 5$. We get

$$13 \times 15 = 19 \times 10 + 5,$$

and thus

$$t_1 = 13(19m_1 + 15)$$

for some integer m_1.

This time, however, there is only one unknown, and thus we know that the lowest possible value of t_1 occurs when $m_1 = 0$.

Hence

$$t_1 = 13 \times 15 = 195$$

and

$$t = 17(195 + 6) = 3417,$$

which is the correct solution.