Orthogonality and its relation to the Fourier series

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1 Introduction

A Fourier series is the expansion of an arbitrary periodic function $f(x)$ as a linear combination of sines and cosines.

2 Fourier sine and cosine series

A function $f(x)$ that is periodic with period 2L it is claimed can be written as a Fourier series with has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{1}$$

The constants a_n and b_n are known as Fourier coefficients. In the expansion it is not necessary to include a b_0 term as $\sin(0) = 0$. Equation (1) happens to be a special case of a formalism as we are expanding $f(x)$ as a linear combination of basis functions. Thus (1) can be re-written as

$$f(x) = a_0\psi_0(x) + \sum_{n=1}^{\infty} \left[a_n\psi_n(x) + b_n\phi_n(x) \right] \tag{2}$$

where we define

$$\begin{aligned} \psi_0(x) &= \frac{1}{2} \\ \psi_n(x) &= \cos\left(\frac{n\pi x}{L}\right) \\ \phi_n(x) &= \sin\left(\frac{n\pi x}{L}\right) \end{aligned} \tag{3}$$

We can omit the ϕ_0 term for the same reason that b_0 could be omitted.

3 The inner product

We now define the inner product for a pair of periodic functions to be

$$\langle f, g \rangle = \int_{-L}^{L} f(x)\, g(x)\, dx,$$

where L is half the period of the function. By using the slider on the left and selecting which inner product you would like to calculate you should be able to verify the following relations

$$\begin{aligned}
\langle \psi_0, \psi_n \rangle &= \frac{L}{2}\delta_{0n} \\
\langle \psi_n, \psi_m \rangle &= L\delta_{0n} \\
\langle \phi_n, \phi_m \rangle &= L\delta_{0n} \\
\langle \psi_0, \psi_n \rangle &= 0,
\end{aligned}$$

where $n = 0, 1, 2, \ldots$.

4 Finding the Fourier coefficients

Using the same technique we can find the Fourier coefficients of a given function $f(x)$ by taking the inner product of f with the corresponding basis function.

$$\begin{aligned}
a_n &= \frac{\langle \psi_n, f \rangle}{||\psi_n||^2} = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx, \\
b_n &= \frac{\langle \phi_n, f \rangle}{||\phi_n||^2} = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L} f(x)\right) dx
\end{aligned} \tag{4}$$