Predicate Logic

(1) When does $\varphi := \neg(A \Rightarrow B) \vee \neg B$ hold?

Can use a truth table:

A	B	$A \Rightarrow B$	$\neg(A \Rightarrow B)$	$\neg B$	φ
$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\top$

So φ holds when $B = \bot$ and regardless of A.
Could also use laws of logic to show that $\varphi \equiv \neg B$: exercise!

(2) Simplify $(A \Rightarrow B) \vee (B \Rightarrow A)$

A proof using logic laws:

$$
\begin{aligned}
(A \Rightarrow B) \vee (B \Rightarrow A) &\equiv (\neg A \vee B) \vee (\neg B \vee A) && \text{(material implication)} \\
&\equiv (\neg A \vee A) \vee (\neg B \vee B) && (\vee \text{ commutes}) \\
&\equiv \top \vee \top && \text{(law of excluded middle)} \\
&\equiv \top.
\end{aligned}
$$

Could use a truth table again, though.

[Note: law of excluded middle is not constructively valid, so this proof is not <u>constructive</u>. More on this later!]

③

A	B	$A \bullet B$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\top$

$A \vee B \equiv A \bullet (\neg B)$

since $A \vee \neg B = \bot$ when $\begin{cases} A = \bot \\ B = \top \end{cases}$

just like $\bullet$

What is $\bullet$? Flipped implication, $\Leftarrow$

since $A \bullet B = B \Rightarrow A = A \Leftarrow B$

④

$F(x)$ = "person x is my friend"
$P(x)$ = "person x is perfect"

predicates? functions of spec.
p: Objects $\to \{\top, \bot\}$

"none of my friends is perfect" = Any of these will do:

$$\neg(\exists x.\, F(x) \wedge P(x)) \equiv \forall x.\, F(x) \Rightarrow \neg P(x) \equiv \forall x.\, P(x) \Rightarrow \neg F(x)$$

"There does not exist a person x who is perfect and my friend"

"For all people x, if they are my friend, then they aren't perfect"

"For all people x, if they are perfect, then they aren't my friend"

⑤ Prove $\forall n \in \mathbb{N}$ that $f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2n-1+f(n-1) & \text{if } n > 0 \end{cases} = n^2$

- Base case: By definition, $f(0) = 0 = 0^2$.

- Inductive case: Assume $f(n) = n^2$ for some $n \in \mathbb{N}$. Now,

$$\begin{aligned} f(n+1) &= 2(n+1) - 1 + f(n) && \text{(definition)} \\ &= 2n+1+n^2 && \text{(inductive hypothesis)} \\ &= (n+1)^2. && \text{(algebra)} \end{aligned}$$

Thus $f(n) = n^2$ $\forall n \in \mathbb{N}$ by induction. $\square$