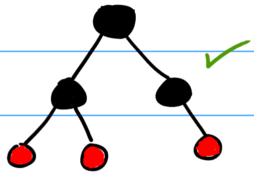
Red-Black Trees

```
data RBColour = Red | Black
data RBTree   = RBLeaf
              | RBNode RBColour Item RBTree RBTree
```

There is a judgment x Item for this



Equivalent inference rules:

$$\frac{}{\text{Red RBColour}} \qquad \frac{}{\text{Black RBColour}}$$

assuming all of this...

$$\frac{\overbrace{c \text{ RBColour} \quad x \text{ Item} \quad t_\ell \text{ RBTree} \quad t_r \text{ RBTree}}}{\text{RBLeaf RBTree}} \qquad \frac{c \text{ RBColour} \quad x \text{ Item} \quad t_\ell \text{ RBTree} \quad t_r \text{ RBTree}}{\underbrace{(\text{RBNode } c \ x \ t_\ell \ t_r)}_{\text{we can conclude this}} \text{ RBTree}}$$

Inference rules for proper red-black trees: Exercise!

(perhaps consider introducing another judgment alongside t OK)

Ambiguity and Syntax

Judgments not mutually defined.

$Bool ::= \top \mid \bot \mid (Bool) \mid \neg Bool \mid Bool \wedge Bool \mid Bool \vee Bool$

Has least # of rules needed to define the set.

Non-simultaneous inductive definition of Bool:

$$\frac{}{Bool} \qquad \frac{}{Bool} \qquad \frac{}{Bool} \qquad \frac{}{Bool}$$

Can be derived in multiple ways.

$$\frac{}{Bool} \qquad \frac{}{Bool}$$

Ambiguous expression?

Every expression has exactly 1 derivation.

Unambiguous inductive definition with precedence $\neg > \wedge > \vee$:

Simultaneous Induction

$$\frac{}{\varepsilon\ \mathbf{M}}M_E \qquad \frac{s\ \mathbf{M}}{(s)\ \mathbf{M}}M_N \qquad \frac{s_1\ \mathbf{M} \quad s_2\ \mathbf{M}}{s_1 s_2\ \mathbf{M}}M_J$$

$$\frac{}{\varepsilon\ \mathbf{L}}L_E \qquad \frac{s\ \mathbf{L}}{(s)\ \mathbf{N}}N_N \qquad \frac{s_1\ \mathbf{N} \quad s_2\ \mathbf{L}}{s_1 s_2\ \mathbf{L}}L_J$$

Stuck? 2 strategies:
① Prove a lemma
② Generalise your goal
(i.e what you're proving)

Prove that if s L or s N then s M

- Base case: From L_E, we get $s = \varepsilon$. Now, ε M by M_E.

- Inductive case: From L_J, we get $s = s_1 s_2$ where s_1 N and s_2 L. Assume that

$$\frac{}{s_1\ M}\text{I.H}_1 \qquad \frac{}{s_2\ M}\text{I.H}_2$$

Then

$$\frac{\dfrac{}{s_1\ M}\text{I.H}_1 \quad \dfrac{}{s_2\ M}\text{I.H}_2}{s_1 s_2\ M}M_J$$

Orange stuff not doable with original proof goal! We were being too specific, so couldn't assume anything about s_1.

- Inductive case: From N_N, we have $s = (s')$.
Remaining details: exercise! Should be straightforward.

Hence by rule induction, s L or s N $\Rightarrow$ s M. In particular our original goal follows by considering just the "s L" part. □

[Corollary: both sets of rules define the same set!]