Parsing Relation

Why do we even need this? No ambiguity!

Abstract syntax for $3 + \text{let } x = 5 \text{ in } x+2 \text{ end}$:

[Write N = Num, V=Var]

$$
\dfrac{
\dfrac{\dfrac{3 \in \mathbb{Z}}{3\ \text{Atom} \leftrightarrow (N\ 3)}}{3\ \text{PExp} \leftrightarrow (N\ 3)}
\quad
\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{5 \in \mathbb{Z}}{5\ \text{Atom} \leftrightarrow (N\ 5)}}{5\ \text{PExp} \leftrightarrow (N\ 5)}}{5\ \text{SExp} \leftrightarrow (N\ 5)}
\quad
\dfrac{
\dfrac{\dfrac{}{x\ \text{Atom} \leftrightarrow (V\ x)}}{x\ \text{PExp} \leftrightarrow (V\ x)}
\quad
\dfrac{\dfrac{\dfrac{2 \in \mathbb{Z}}{2\ \text{Atom} \leftrightarrow (N\ 2)}}{2\ \text{PExp} \leftrightarrow (N\ 2)}}{2\ \text{SExp} \leftrightarrow (N\ 2)}
}{x+2\ \text{SExp} \leftrightarrow \text{Plus}\ (V\ x)\ (N\ 2)}
}{\text{let } x=5 \text{ in } x+2 \text{ end Atom} \leftrightarrow \text{Let } x\ (N\ 5)\ (\text{Plus}\ (V\ x)\ (N\ 2))}
}{\text{let } x=5 \text{ in } x+2 \text{ end PExp} \leftrightarrow \text{Let } x\ (N\ 5)\ (\text{Plus}\ (V\ x)\ (N\ 2))}
}{\text{let } x=5 \text{ in } x+2 \text{ end SExp} \leftrightarrow \text{Let } x\ (N\ 5)\ (\text{Plus}\ (V\ x)\ (N\ 2))}
}{3 + \text{let } x=5 \text{ in } x+2 \text{ end}\ \ \text{SExp} \leftrightarrow \text{Plus}\ (N\ 3)\ (\text{Let } x\ (N\ 5)\ (\text{Plus}\ (V\ x)\ (N\ 2)))}
$$

$$
\dfrac{i \in \mathbb{Z}}{i\ \mathsf{A} \longleftrightarrow (\texttt{Num}\ i)}
\qquad
\dfrac{a\ \mathsf{S} \longleftrightarrow a'}{(a)\ \mathsf{A} \longleftrightarrow a'}
\qquad
\dfrac{e\ \mathsf{A} \longleftrightarrow a}{e\ \mathsf{P} \longleftrightarrow a}
\qquad
\dfrac{e\ \mathsf{P} \longleftrightarrow a}{e\ \mathsf{S} \longleftrightarrow a}
$$

$$
\dfrac{a\ \mathsf{A} \longleftrightarrow a' \qquad b\ \mathsf{P} \longleftrightarrow b'}{a \times b\ \mathsf{P} \longleftrightarrow (\texttt{Times}\ a'\ b')}
\qquad
\dfrac{a\ \mathsf{P} \longleftrightarrow a' \qquad b\ \mathsf{S} \longleftrightarrow b'}{a + b\ \mathsf{S} \longleftrightarrow (\texttt{Plus}\ a'\ b')}
$$

$$
\dfrac{x\ \mathsf{Ident}}{x\ \mathsf{Atom} \longleftrightarrow (\texttt{Var}\ x)\ \mathsf{AST}}
$$

$$
\dfrac{x\ \mathsf{Ident} \qquad e_1\ \mathsf{SExp} \longleftrightarrow a_1\ \mathsf{AST} \qquad e_2\ \mathsf{SExp} \longleftrightarrow a_2\ \mathsf{AST}}{\texttt{let}\ x = e_1\ \texttt{in}\ e_2\ \texttt{end}\ \mathsf{Atom} \longleftrightarrow (\texttt{Let}\ x\ a_1\ a_2)\ \mathsf{AST}}
$$

Substitution

Potential capture!

$$(\text{Let } x\ (y.\ (\text{Plus } (\text{N } 1)\ x)))[x := y]$$

$$\equiv_\alpha (\text{Let } x\ (z.\ (\text{Plus } (\text{N } 1)\ x)))[x := y] \qquad [\alpha\text{-Renaming}]$$

$$\equiv (\text{Let } y\ (z.\ (\text{Plus } (\text{N } 1)\ y)))$$

$$(\text{Let } y\ (z.\ (\text{Plus } (\text{N } 1)\ z)))[x := y]$$

$$\equiv (\text{Let } y\ (z.\ (\text{Plus } (\text{N } 1)\ z)))$$

$$(\text{Let } x\ (z.\ (\text{Plus } x\ z)))[x := y]$$

$$\equiv (\text{Let } y\ (z.\ (\text{Plus } y\ z)))$$

Not free!

$$(\text{Let } x\ (x.\ (\text{Plus } (\text{N } 1)\ x)))[x := y]$$

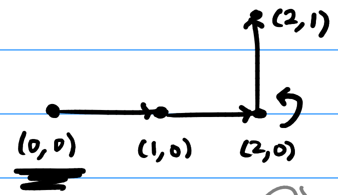
[Substitution only applies to free occurrences!]

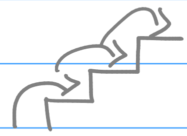
$$\equiv (\text{Let } y\ (x.\ (\text{Plus } (\text{N } 1)\ x)))$$

Semantics

$$R ::= \text{move}; R \mid \text{turn}; R \mid \text{stop}$$

Example: move; move; turn; move; stop





Shows evaluation step-by-step.

Small-step semantics

- **States:** $\left\{ (\underset{\sim}{p}, \underset{\sim}{d}, r) : \underset{\sim}{p}, \underset{\sim}{d} \in \mathbb{Z}^2, r \in R \right\}$

  current pos — direction — remaining instructions

- **Initial States:** (Things we start in) $\left\{ \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, r \right) : r \in R \right\}$

  start — initial dir — whatever instructions

- **Final States:** (Things we end in) $\left\{ (\underline{p}, \underline{d}, \text{stop}) : \underline{p}, \underline{d} \in \mathbb{Z}^2 \right\}$

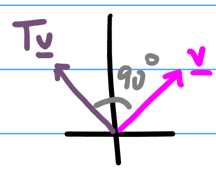
- **Transitions:** (Relationship between states)

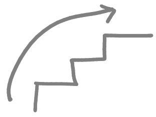
$$\frac{}{(\underline{p}, \underline{d}, \text{move}; r) \mapsto (\underline{p} + \underline{d}, \underline{d}, r)} \qquad \frac{}{(\underline{p}, \underline{d}, \text{turn}; r) \mapsto (\underline{p}, T\underline{d}, r)}$$

No relation for stop – they're final states.

A fact from linear algebra:

$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ encodes a rotation by 90° anticlockwise in $\mathbb{R}^2$



How does this compare to small-step? 

Focuses on final values, not step-by-step execution.

Big-step semantics

- **Evaluable expressions:**

- **Values:**

- **Evaluation rules:**

$\Downarrow$ $\Downarrow$

$\Downarrow$

⚠ Denotational semantics $[\![\cdot]\!] : R \to \mathbb{Z}^2$

Associates a mathematical object with a syntactic object.

$[\![\text{stop}]\!] =$

$[\![\text{turn}; r]\!] =$

$[\![\text{move}; r]\!] =$