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 SIE 321 Probabilistic Models in OR
 Homework 2
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1. A fair coin is tossed consecutively 7 times. Find the conditional probability $P(A | B)$, where the events A and B are defined as

$$A = \{\text{more heads than tails came up}\}, B = \{\text{5th toss is a head}\}$$

Bernoulli Trials:

- Success and failure
- Independent events
- Constant success rate

Binomial Distribution:

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$x = \#$ of successes, $n = \#$ of trials, $p =$ success rate

Out of seven consecutive fair coin tosses, the possible outcomes of getting more heads than tails are 4 heads, 5 heads, 6 heads, and 7 heads.

$$A = \{(4H, 3T), (5H, 2T), (6H, 1T), (7H, 0T)\}$$

$$B = \{(4H, 3T), (5H, 2T), (6H, 1T), (7H, 0T)\}$$

$$P(A) = P(4H, 3T) + P(5H, 2T) + P(6H, 1T) + P(7H, 0T)$$

$$P(A) = \binom{7}{4} (0.5)^7 + \binom{7}{5} (0.5)^7 + \binom{7}{6} (0.5)^7 + \binom{7}{7} (0.5)^7$$

$$P(A) = \left(\frac{1}{2}\right)^7 (35 + 21 + 7 + 1)$$

$$P(A) = \frac{64}{128} = \frac{1}{2}$$

$$P(B) = 1 - P(A)$$

$$P(B) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(A \cap B) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$

$$P(A|B) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$P(A|B) = \frac{1}{2}$$

2. Consider rolling a pair of dice once. What is the probability of getting 8, given that the sum of the faces is an even number?

$$A = \{\text{values of both faces add up to 8}\}, B = \{\text{sum of both faces is an even number}\}$$

$$P(A = 8) = \frac{5}{36} = 0.139$$

$$P(B) = \frac{18}{36} = 0.5$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

$$P(A|B) = \frac{0.139}{0.5}$$

$$P(A|B) = 0.278 = 27.8\%$$

3. A box contains four tickets labeled 112, 121, 211, and 222. Choose one ticket at random and consider events $A_i = \{1 \text{ occurs in } i\text{-th position}\}$, $i = 1, 2, 3$.

- (a) Are events A_1 , A_2 , and A_3 pairwise independent?

Pairwise Independence:

$$P(A \cap B) = P(A)P(B)$$

$$P(A_1) = \frac{2}{4} = \frac{1}{2}$$

$$P(A_2) = \frac{2}{4} = \frac{1}{2}$$

$$P(A_3) = \frac{2}{4} = \frac{1}{2}$$

$$P(A_1 \cap A_2) = \frac{1}{4}$$

$$P(A_1)P(A_2) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$

$$\frac{1}{4} = \frac{1}{4}$$

$$P(A_2 \cap A_3) = \frac{1}{4}$$

$$P(A_2)P(A_3) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$

$$\frac{1}{4} = \frac{1}{4}$$

$$P(A_1 \cap A_3) = \frac{1}{4}$$

$$P(A_1)P(A_3) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$

$$\frac{1}{4} = \frac{1}{4}$$

\therefore Events A_1 , A_2 , and A_3 are pairwise independent

- (b) Are they mutually independent?

Mutual Independence:

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = 0$$

$$P(A_1)P(A_2)P(A_3) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}$$

$$0 \neq \frac{1}{8}$$

\therefore Events A_1 , A_2 , and A_3 are NOT mutually independent

4. You enter a chess tournament where your probability of winning a game is 0.42 against 30% the players (call them type 1), 0.33 against other 10% of the players (call them type 2), and 0.25 against the remaining 60% players (call them type 3).

- (a) You play a game against a randomly chosen opponent. What is the probability of winning?

type 1 = 0.3, type 2 = 0.1, type 3 = 0.6

Law of Total Probability:

$$P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$$

$$P(\text{win}|\text{type1}) = 0.42$$

$$P(\text{win}|\text{type2}) = 0.33$$

$$P(\text{win}|\text{type3}) = 0.25$$

$$P(\text{win}) = P(\text{win}|\text{type1})P(\text{type1}) + P(\text{win}|\text{type2})P(\text{type2}) + P(\text{win}|\text{type3})P(\text{type3})$$

$$P(\text{win}) = 0.42(0.3) + 0.33(0.1) + 0.25(0.6)$$

$$P(\text{win}) = 0.309$$

- (b) Suppose that you were the winner of a game. What is the probability that your opponent was of type 3?

Bayes' Rule (Reverse of the Law of Total Probability):

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{P(B)} = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^n P(B|A_i)P(A_i)}$$

$$P(\text{type3}|\text{win}) = \frac{P(\text{win}|\text{type3})P(\text{type3})}{P(\text{win})}$$

$$P(\text{type3}|\text{win}) = \frac{0.25(0.6)}{0.309}$$

$$P(\text{type3}|\text{win}) = 0.485 = 48.5\%$$

5. A customer has approached a bank for a loan. Without further information, the bank believes there is a 4% chance that the customer will default on the loan. The bank can run a credit check on the customer. The check will yield either a favorable or an unfavorable report. From past experience, the bank believes that

$$P(\text{favorable report being received} \mid \text{customer will default}) = \frac{4}{40},$$

and

$$P(\text{favorable report} \mid \text{customer will not default}) = \frac{96}{100}.$$

If a favorable report is received, what is the probability that the customer will default on the loan?

$$P(\text{default}) = 0.04$$

$$P(\text{no default}) = 0.96$$

Bayes' Rule (Reverse of the Law of Total Probability):

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{P(B)} = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^n P(B|A_i)P(A_i)}$$

$$P(\text{default} \mid \text{favorable report}) = \frac{P(\text{favorable report} \mid \text{default})P(\text{default})}{P(\text{favorable report} \mid \text{default})P(\text{default}) + P(\text{favorable report} \mid \text{no default})P(\text{no default})}$$

$$P(\text{default} \mid \text{favorable report}) = \frac{\frac{4}{40}(0.04)}{\frac{4}{40}(0.04) + \frac{96}{100}(0.96)}$$

$$P(\text{default} \mid \text{favorable report}) = \frac{0.004}{0.004 + 0.9216}$$

$$P(\text{default} \mid \text{favorable report}) = 0.0043 = 0.43 \%$$