Agustin Espinoza SIE321 HW8 3/24/23

1. Gambler's ruin problem

$p(\text{winning}) = p$
$p(\text{losing}) = q = 1-p$

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w/prob } p \quad 0 < X_n < T \\ X_n - 1 & \text{w/prob } 1-p \end{cases}$$

$X_{n+1} = X_n$ for $X_n = 0$ or $X_n = T$

a) Construct (one-step) transition matrix

$$\boxed{P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1-p & 0 & p & \cdots & 0 & 0 \\ 0 & 1-p & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & p \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}$$

c) $T = 3$; $p = 0.4$

Req. prob $= \left(\frac{0.4}{0.7}\right)^2$

$= \boxed{\frac{16}{45}}$

b) $T = 3$; $p = 0.6$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Req. prob $= 1 - \left(\frac{1-p}{p}\right)^2$

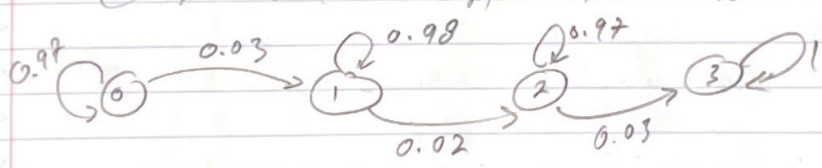
$= 1 - \left(\frac{0.3}{0.7}\right)^2$

$= \boxed{\frac{40}{45}}$

2. Video Cassette recorder manufacturer problem

$P(\text{fails during yr } 1) = 0.03$
$P(\text{fails during y}2 \mid \text{survived yr}1) = 0.02$

a) Formulate Markov Chain - 2 absorption states.
Construct (one-step) transition matrix

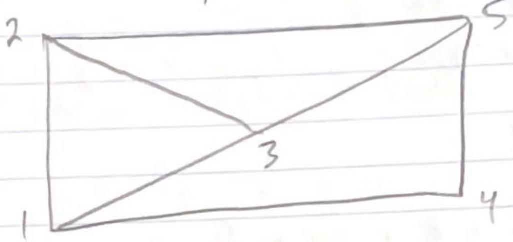


$$\boxed{P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.97 & 0.03 & 0 & 0 \\ 1 & 0 & 0.98 & 0.02 & 0 \\ 2 & 0 & 0 & 0.97 & 0.03 \\ 3 & 0 & 0 & 0 & 1 \end{array}}$$

b) $P(\text{honoring warranty}) = P[1,3] + P[1,4] = 0.03 + 0 = 0.03$

$\boxed{P(\text{honoring warranty}) = 3\%}$

3. Mouse Maze problem



a) Formulate (one-step) transition matrix.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

b) $\boxed{p(1 \text{ to } 5 \text{ in } 3 \text{ runs}) = 0}$

c) x_i = # of intersections from i

$$x_1 = \tfrac{1}{3}(1+x_2) + \tfrac{1}{3}(1+x_3) + \tfrac{1}{3}(1+x_4)$$
$$3x_1 = (1+x_2) + (1+x_3) + (1+x_4)$$
$$x_2 = \tfrac{1}{3}(1+x_1) + \tfrac{1}{3}(1+x_3) + \tfrac{1}{3}$$
$$3x_2 = (1+x_1) + (1+x_3) + 1$$
$$x_3 = \tfrac{1}{3}(1+x_1) + \tfrac{1}{3}(1+x_2) + \tfrac{1}{3}$$
$$3x_3 = (1+x_1) + (1+x_2) + 1$$
$$3x_3 = 1 + x_1 + x_2$$

$\boxed{x_3 = x_2} = k$

$3k = 1 + x_1 + k \qquad x_1 = 3k - 1 - k$

$\boxed{x_1 = 2k - 1}$

$x_4 = \frac{1}{2}(x_1 + 1) + \frac{1}{2}$

$2x_4 = (x_1 + 1) + 1, \quad x_4 = x_1 + \frac{1}{2} = \frac{x_1}{2} + \frac{1}{2}$

$x_4 = \frac{1}{2} + \frac{x_4}{2}$

$3x_1 = 3 + k + k + \frac{1}{2} + \frac{x_1}{2}$

$= 3x_1 - \frac{x_1}{2}$

$= 3 + k + k + \frac{1}{2}$

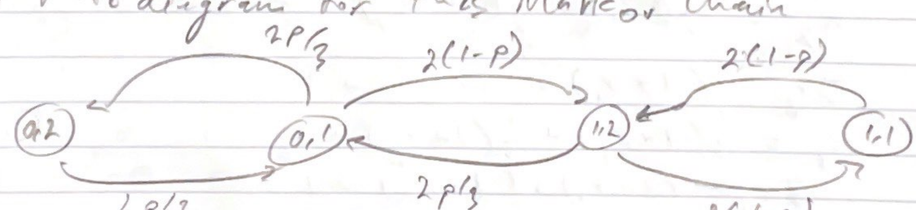
$\Rightarrow 2.5x_1 = 2k + 3.5$

$1.5x_1 = 4.5; \quad \boxed{x_1 = 3}$

The expected number of runs needed for the mouse to reach intersection 5 from intersection 1 is $\underline{3}$.

4. Shop machines problem

a) Rate diagram for this Markov Chain



b) down time-dependent diff. eqs.

$$\boxed{\frac{dp}{dt} = 3p + \frac{2}{3}(1-p)}$$

$$\boxed{\frac{dq}{dt} = 3p - 2q - (2/3)q}$$

c) construct steady-state equations:

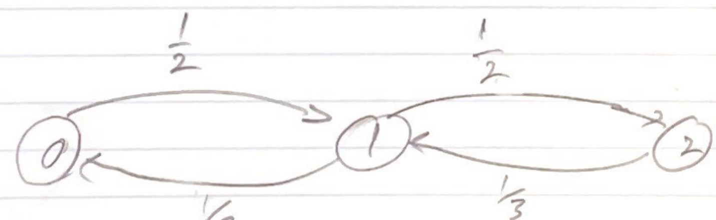
$$\boxed{\begin{aligned} -3p + 2/3\,(1-p) &= 0 \\ 3p - 2q - (2/3)q &= 0 \\ p + q + r &= 1 \end{aligned}}$$

d) determine the steady-state probabilities.

$$\boxed{\begin{aligned} p &= 2/11 \\ q &= 3/11 \\ r &= 6/11 \end{aligned}}$$

5. Work Center problem

a) develop rate diagram for this Markov chain



b) time-dependent ODEs for the Markov Chain

$$\boxed{\frac{d}{dt} x_0(t) = -0.5 x_0(t) + \frac{1}{3} x_1(t)}$$

$$\boxed{\frac{d}{dt} x_1(t) = 0.5 x_0(t) - \frac{1}{3} x_1(t) + \frac{1}{3} x_2(t)}$$

$$\boxed{\frac{d}{dt} x_2(t) = -\frac{1}{3} x_2(t)}$$

c) steady-state equations

$$-0.5 x_0 + \frac{1}{3} x_1 = 0$$

$$0.5 x_0 - \frac{1}{3} x_1 + \frac{1}{3} x_2 = 0$$

$$-\frac{1}{3} x_2 = 0$$

$$x_0 = \frac{1}{4} x_1$$

$$x_1 = \frac{1}{2} x_1$$

$$\frac{1}{4} x_1 + x_1 + \frac{1}{2} x_1 = 1$$

d) determine the steady-state probabilities

$$x_1 = 4/7$$

$$x_0 = \frac{1}{7}$$

$$x_2 = 2/7$$