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1. Brewery market position problem

$$T = \begin{bmatrix} 0.7 & 0.15 & 0.15 \\ 0.35 & 0.2 & 0.45 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

a) Steady-state market shares for the 2 major breweries?

2 major breweries : A, B

$TL = L$ $\quad L = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$

$$\begin{bmatrix} 0.7 & 0.15 & 0.15 \\ 0.35 & 0.2 & 0.45 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\begin{aligned} 0.7A + 0.15B + 0.15C &= A \\ 0.35A + 0.2B + 0.45C &= B \\ 0.3A + 0.2B + 0.5C &= C \\ A + B + C &= 1 \end{aligned} \Rightarrow \begin{aligned} -0.3A + 0.15B + 0.15C &= 0 \\ 0.35A - 0.8B + 0.45C &= 0 \\ 0.3A + 0.2B - 0.5C &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} -0.3 & 0.15 & 0.15 & 0 \\ 0.35 & -0.8 & 0.45 & 0 \\ 0.3 & 0.2 & -0.5 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0.33 \\ 0 & 1 & 0 & 0 & 0.33 \\ 0 & 0 & 1 & 0 & 0.33 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$A = 0.33,\ B = 0.33,\ C = 0.33$
∴ Steady-state market shares for the top 2 breweries are both 33%.

b) What is the prob. that a current beer A customer will be purchasing beer B in 3 months?

$n = 3$ $\quad T = p$

$$p^3 = \begin{bmatrix} 0.7 & 0.15 & 0.15 \\ 0.35 & 0.2 & 0.45 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}^3 \Rightarrow \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.54 & 0.17 & 0.29 \\ 0.49 & 0.18 & 0.33 \\ 0.48 & 0.18 & 0.34 \end{bmatrix} \end{array}$$

$\boxed{\text{The probability a current beer A customer purchases beer B in 3 months is 17\%.}}$

c) What is the probability that a current beer B customer will be purchasing beer A in 3 months?

$$p^3 = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.54 & 0.17 & 0.29 \\ 0.49 & 0.18 & 0.33 \\ 0.48 & 0.18 & 0.34 \end{bmatrix} \end{array}$$

$\boxed{\text{The probability that a current beer B customer purchases beer A in 3 months is 49\%.}}$

d) Assume brewery A estimates it's monthly profit per customer at $14 and its potential customer base as 500,000 strong. What is the average monthly profit for brewery A?

expected cost per unit time:

$$\sum_{j=0}^{M} \pi_j C_j$$

Cost function $C(j)$:

$C_j = 500000(\$14) = 7\,000\,000$

$$\sum_{j=0}^{3} \pi_j C_j = (\$7M)(0.33) + (\$7M)(0.33) + (\$7M)(0.33)$$

$$E[D] = 0.35(1) + 0.1(2) + 0.15(3) = \boxed{1}$$

2 Blood inventory problem

D = demand of rare blood type in pints over any 3-day period

$P\{D=0\} = 0.4$, $P\{D=1\} = 0.35$, $P\{D=2\} = 0.1$, $P\{D=3\} = 0.15$

$E(D) = 0.35(1) + 0.1(2) + 0.15(3) = 1$

a) Construct the one-step transition matrix for this Markov Chain

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.35 & 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.1 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.35 & 0.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.1 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.35 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \end{bmatrix} \end{array}$$

b) Find the steady-state probabilities.

$pL = L$ don't forget $A+B+C+D+E+F+G+H$

$$\begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.35 & 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.1 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.35 & 0.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.1 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.35 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{bmatrix}$$

$0.4A + 0.6B = A$

$0.4A + 0.35B + 0.25C = B$

$0.4B + 0.1C + 0.5D = C$

$0.4C + 0.35D + 0.15E = D$

$0.4D + 0.1E + 0.5F = E$

$0.4E + 0.35F + 0.25G = F$

$0.4F + 0.6G = G$

$0.6G = H$

$A+B+C+D+E+F+G+H = 1$

$$\boxed{\pi_0 = 0.17,\ \pi_1 = 0.29,\ \pi_2 = 0.23,\ \pi_3 = 0.15,\ \pi_4 = 0.09,\ \pi_5 = 0.03,\ \pi_6 = 0.0097,\ \pi_7 = 0}$$

c) Use pt. (b) to find the steady-state prob. that a pint of blood will need to be discarded during a 3-day period.

$$P(\text{discarding a pint of blood}) = P(D=0 \mid \text{state}=7)\cdot\pi_7$$
$$= P_{7,0}\cdot\pi_7$$
$$= 0.05\,(0.003)$$

$$\boxed{P(\text{discarding a pint of blood}) = 0.00015}$$

d) Find the steady-state probability that an emergency delivery will be needed during the 3-day period between regular deliveries.

$$P(\text{needing emergency delivery}) = \pi_0 P_{0,0} + \pi_1(P_{1,0} + P_{1,1}) + \pi_2(P_{2,0} + P_{2,1} + P_{2,2})$$
$$+ \pi_3(P_{3,0} + P_{3,1} + P_{3,2} + P_{3,3})$$
$$+ \pi_4(P_{4,0} + P_{4,1} + P_{4,2} + P_{4,3} + P_{4,4})$$
$$= 0.266(4) + 0.312(0.35+0.4)$$
$$+ 0.205(0.2 + 0.35 + 0.4)$$
$$+ 0.098(0.05+0.2+0.35+0.4)$$
$$+ 0.047(0+0.05+0.2+0.35+0.4)$$

$$\boxed{P(\text{needing emergency delivery}) = 0.865}$$

3. Computer inspection problem

a) Construct the (one-step) transition matrix

$S = \{0, 1\}$ 1 = failed (down)

0 = working (up)

$$\boxed{P = \begin{array}{c|cc} & \text{up} & \text{down} \\ \hline \text{up} & 0.9 & 0.1 \\ \text{down} & 0.25 & 0.75 \end{array}}$$

b) What is the long-term fraction downtime of the computer?

$$pL = L, \quad \pi_0 + \pi_1 = 1$$

$$\begin{bmatrix} 0 & 0.25 \\ 0.1 & \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix}$$

$$\begin{aligned} 0.25\pi_0 + 0.75\pi_1 &= \pi_0 \\ 0.9\pi_0 + 0.1\pi_1 &= \pi_1 \\ \pi_0 + \pi_1 &= 1 \end{aligned} \Rightarrow \begin{aligned} -0.75\pi_0 + 0.25\pi_1 &= 0 \\ 0.1\pi_0 - 0.9\pi_1 &= 0 \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

$$\left[\begin{array}{cc|c} -0.75 & 0.25 & 0 \\ 0.1 & -0.9 & 0 \\ 1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0.71 \\ 0 & 1 & 0.29 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\pi_{up} = 0.71, \ \pi_{down} = 0.286}$$

$\boxed{\text{The long-term fraction downtime of the computer is .29\%.}}$

c) If the computer is working (up) right now, what is the probability that it will be down 8 hrs from now?

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.25 & 0.75 \end{bmatrix}$$

$\boxed{\text{The probability of the computer, currently working (1), failing in 8 hrs is 4.8\%}}$

$$P^{8}_{up/down} = P^{7}_{up/up} \cdot P^{1}_{up/down}$$
$$= 0.9^7 (0.1) = 0.048$$

d) Hourly operational cost = \$0.2, when the computer is up, and \$6.5 when the computer is down.

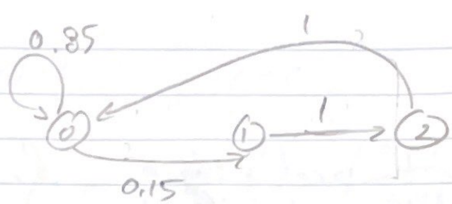
$$E[\text{cost per hour}] = 0.2\,\pi_{up} + 0.5\,\pi_{down}$$
$$= 0.2(0.71) + 0.5(0.29)$$
$$= 0.287$$
$$= \$0.29$$

$\boxed{\text{The hourly operational cost is \$0.29.}}$

4. Manufacturing machine problem.
$P(\text{break down}) = 0.15$

a) Formulate Markov Chain, and then the (one-step) transition matrix.

$S = \{0, 1, 2\}$ 0: machine not broken
1: machine broken
2: machine under repair.



$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline & & & \end{array}$$

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \end{array}\begin{bmatrix} 0.85 & 0.15 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

b) Find the expected first passage time u_{ij}, from state i to j.

$\mu_{01} = 1 + 0.85\mu_{01}$

$0.15\mu_{01} = 1$

$\boxed{\mu_{01} = 6.67}$

$\mu_{10} = 1 + \mu_{20} \Rightarrow \boxed{\mu_{10} = 2}$

$\boxed{\mu_{20} = 1}$

c) Given the machine has gone without a breakdown for 15 days, how does the expected # of full days hereafter that the machine will remain operational compare with the results in part b?

Markovian property:

μ_{10} | working already for 15 days $= 20 + 6.67$
$= \boxed{26.67}$

5. Gambler's ruin problem

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } p \\ X_n - 1 & \text{with probability } 1-p \end{cases} \quad \text{for } 0 \le X_n < T$$

$$X_{n+1} = X_n \quad \text{for } X_n = 0 \text{ or } X_n = T$$

gambler $\boxed{\text{bets } \$1.00}$

$P(\text{winning}) = p$

$P(\text{losing}) = q = 1-p$

X_n is a Markov Chain. The gambler starts with X_0 dollars where $0 \le X_0 < T$

a) Construct (one-step) transition matrix

$$\boxed{P = \begin{array}{c|cccccc} & 0 & 1 & 2 & \cdots & T-1 & T \\ \hline 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1-p & 0 & p & \cdots & 0 & 0 \\ 2 & 0 & 1-p & 0 & p & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ T-1 & 0 & 0 & 0 & 1-p & 0 & p \\ T & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}}$$

b) $T = 3 \quad p = 0.55$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.45 & 0 & 0.55 & 0 \\ 0 & 0.45 & 0 & 0.55 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$T = 3, \therefore T-1 = 2$

$$p(\text{bankrupt}) = 1 - \left(\frac{1-p}{p}\right)^2 = 1 - \left(\frac{0.45}{0.55}\right)^2 = \boxed{\frac{40}{121}}$$

c) $T = 3,\ p = 0.45$

$$p(\text{bankrupt}) = \left(\frac{0.45}{1-0.45}\right)^2 = \boxed{\frac{81}{121}}$$