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1. Small two-bay car-wash operation problem

$\lambda = \frac{1}{5}$ customers/min, $\mu = \frac{1}{4}$ customers/min, $s = 2$ servers

Multi-server finite-queue model (M/M/s/K)

$$\rho = \frac{\lambda}{s\mu}$$

$$\rho = \frac{(1/5)}{(1/4)} = 0.8$$

$$\rho = 0.8$$

$$P_n = \begin{cases} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} P_0, & n = 1, 2, \ldots, s \\ \frac{\left(\frac{\lambda}{\mu}\right)^n}{s!\, s^{n-s}} P_0, & n = s, s+1, \ldots, K \\ 0, & n > K \end{cases}$$

a) Expected fraction of potential customers that will be lost because of inadequate waiting space: 2 spaces.

$$P_3 = \frac{(1-\rho)}{(1-\rho^{3+1})}\rho^3 = \frac{1-0.8}{(1-0.8)^4}(0.8)^3 = 0.173$$

$$\boxed{P_3 = 0.173}$$

b) $$P_5 = \frac{(1-\rho)}{(1-\rho^{5+1})}(\rho^5) = \left(\frac{1-\rho}{1-\rho^6}\right)(\rho^5) = \frac{(1-0.8)}{(1-0.8^6)}(0.8)^5$$

$$\boxed{P_5 = 0.089}$$

2. Forrester Manufacturing Company problem

M/M/1 queueing model

$\lambda = 4$ machines/hr

$\mu = 8$ machines/hr

$\rho = \frac{\lambda}{\mu} = \frac{4}{8} = \frac{1}{2}$

a) Find the probability distribution of the number of machines not running, and the mean of this distribution.

Single-server queueing systems (M/M/1)

$P_n = 1 - P_0$

$P_0 = 1 - \rho = \frac{1}{2}$

$P_n = 1 - \frac{1}{2} = \frac{1}{2}$

$E[n] = \frac{1}{(1-\rho)\mu} = \frac{1}{(1-\frac{1}{2})8} = \frac{1}{4}$

$\boxed{E[n] = \frac{1}{4} = 0.25 = 25\%}$

b) The expected fraction of time that the repair technician will be busy:

$W_q = \frac{\rho}{(1-\rho)\mu} = \frac{\frac{1}{2}}{(1-\frac{1}{2})8} = \frac{\frac{1}{2}}{4} = \frac{1}{8}$

$\boxed{W_q = 0.125}$

3. 3 frisky puppies problem

$S = \{0, 1, 2, 3\}$

Birth-and-death process

$\lambda_n = 1/15$ for $n = 0, 1, 2$ $\qquad \rho = \dfrac{\lambda_n}{\mu_n} = \dfrac{\frac{1}{15}}{1/10} = 1.5$

$\mu_n = 1/10$ for $n = 1, 2, 3$

$$\begin{tikzcd}
0 \arrow[r, bend left, "\lambda_0"] & 1 \arrow[l, bend left, "\mu_0"] \arrow[r, bend left, "\lambda_1"] & 2 \arrow[l, bend left, "\mu_1"] \arrow[r, bend left, "\lambda_2"] & 3 \arrow[l, bend left, "\mu_2"]
\end{tikzcd}$$

$$T = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 \\ \mu_1 & -\lambda_1+\mu_1 & \lambda_1 & 0 \\ 0 & \mu_2 & -\lambda_2+\mu_2 & \lambda_2 \\ 0 & 0 & \mu_3 & -\mu_3 \end{bmatrix} \end{array} = \begin{bmatrix} -\frac{1}{15} & \frac{1}{15} & 0 & 0 \\ \frac{1}{10} & \frac{1}{30} & \frac{1}{15} & 0 \\ 0 & \frac{1}{10} & \frac{1}{30} & \frac{1}{15} \\ 0 & 0 & \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$$

$\pi T = 0$ and $\Sigma \pi_i = 1$

$$\pi_0 = \frac{1}{1+3\rho+9\rho^2+27\rho^3}$$

$$\pi_1 = \frac{3\rho}{1+3\rho+9\rho^2+27\rho^3}$$

$$\pi_2 = \frac{9\rho^2}{1+3\rho+9\rho^2+27\rho^3}$$

$$\pi_3 = \frac{27\rho^3}{1+3\rho+9\rho^2+27\rho^3}$$

a) At any given time what is the prob. that more puppies will be out of the box than in?

$P = \pi_0 + \pi_1$

$$P = \frac{1}{1+2\rho+3\rho^2}$$

$$\boxed{P = 0.391 = 39.1\%}$$

$\therefore$ At any given time, there is a 39% that more puppies will be out of the box than in the box.

b) On average, how many puppies will be in the box?

$E[n] = \sum_{i=0}^{3} i \pi_i$

$E[n] = 0\pi_0 + 1\pi_1 + 2\pi_2 + 3\pi_3$

$\rho = 1.5$

$\boxed{E[n] = 1.75}$

$\therefore$ On average, there will be 1.75 puppies outside the box.

4. Espresso stand problem.

$\lambda = 40$ customers/hr $= \frac{2}{3}$ customers/min

$\mu = \frac{1}{\frac{75}{60}} = 0.8$ customers/min

a) Find L, L_q, W, W_q

$\boxed{L = \frac{\lambda}{\mu - \lambda} = \frac{\frac{2}{3}}{0.8 - \frac{2}{3}} = 5 \text{ customers.}}$

$\boxed{L_q = L - \frac{\lambda}{\mu} = 5 - \frac{\frac{2}{3}}{0.8} = 4.167 \text{ customers}}$

$\boxed{W = \frac{L}{\lambda} = \frac{5}{\frac{2}{3}} = 7.5 \text{ min}}$

$\boxed{W_q = \frac{L_q}{\lambda} = \frac{4.167}{\frac{2}{3}} = 6.25 \text{ min}}$

b) Suppose Marsha is replaced by an expresso vending machine that requires exactly 75 s for each customer to operate.
Find L, L_q, W, W_q

M/D/1 model

$\lambda = 40$ customers/hr
$\mu = 48$ customers/hr

$\rho = \frac{\lambda}{\mu} = \frac{40}{48} = 0.83$

$$L_q = \frac{(\lambda^2 \sigma^2)}{2(1-\rho)} + \frac{\rho^2}{2(1-\rho)}$$

$\sigma = 0$

$$\boxed{L_q = \frac{(\rho^2)}{2(1-\rho)} = \frac{(0.83)^2}{2(1-0.83)} = 2.026 \text{ customers}}$$

$$\boxed{W_q = \frac{L_q}{\lambda} = \frac{2.03}{40} = 0.05 \text{ hrs/customer}}$$

$$\boxed{L = L_q + \rho = 2.03 + 0.83 = 2.86 \text{ customers}}$$

$$\boxed{W = W_q + \frac{1}{\mu} = 0.05 + \frac{1}{48} = 0.071 \text{ hrs/customer}}$$

5. Shoe repair store problem

$\lambda = 2$ customers/hr

$\mu = \frac{1}{12/60} = 5$ customers/hr

a) calculate the expected # of pairs of shoes in shop

$$L = \frac{\lambda}{\mu - \lambda} = \frac{2}{5-2} = \frac{2}{3} = 0.67$$

$\boxed{L = 0.67 \text{ pairs of shoes}}$

b) Calc. the expected amount of time from shoes getting dropped off until they are repaired and picked up.

$$W = \frac{L}{\lambda}$$

$$W = \frac{0.67}{2} = 0.335$$

$\boxed{W = 0.335 \text{ hr} = 20.1 \text{ mins}}$