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SIE 321 Probabilistic Models in OR
Homework 2
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1. A fair coin is tossed consecutively 7 times. Find the conditional probability P(A | B), where the events A and B are defined as

$$A = \{\text{more heads than tails came up}\},\ B = \{\text{5th toss is a head}\}$$

   Bernoulli Trials:

   - Success and failure

   - Independennt events

   - Constant success rat

   Binomial Distribution:

   $P(x) = \begin{pmatrix} n \\ x \end{pmatrix} p^x(1-p)^{n-x}$

   x = # of successes, n = # of trials, p = success rate

   Out of seven consecutive fair coin tosses, the possible outcomes of getting more heads than tails are 4 heads, 5 heads, 6 heads, and 7 heads. Therefore, A = 4.

   $P(A) = P(4) = \begin{pmatrix} 7 \\ 4 \end{pmatrix} (0.5)^4(0.5)^3$

   $P(A) = 0.273$

   $P(A) = 27.3\%$

   $P(B) = ?$

   $P(A|B) = ?$

2. Consider rolling a pair of dice once. What is the probability of getting 8, given that the sum of the faces is an even number?d

   A = {values of both faces add up to 8}, B = {sum of both faces is an even number}

   $P(A = 8) = \frac{5}{36} = 0.139$

   $P(B) = \frac{18}{36} = 0.5$

   $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$

   $P(A|B) = \frac{0.139}{0.5}$

   $P(A|B) = 0.278 = 27.8\%$

3. A box contains four tickets labeled 112, 121, 211, and 222. Choose one ticket at random and consider events A_i= {1 occurs in i-th position}, i = 1, 2, 3.

   (a) Are events A_1, A_2, and A_3 pairwise independent?
   Pairwise Independece:
   $P(A \cap B) = P(A)P(B)$
   $P(A_1) = \frac{2}{4} = \frac{1}{2}$
   $P(A_2) = \frac{2}{4} = \frac{1}{2}$
   $P(A_3) = \frac{2}{4} = \frac{1}{2}$
   $P(A_1 \cap A_2) = \frac{1}{4}$
   $P(A_1)P(A_2) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$
   $\frac{1}{4} = \frac{1}{4}$

$P(A_2 \cap A_3) = \frac{1}{4}$
$P(A_2)P(A_3) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$
$\frac{1}{4} = \frac{1}{4}$
$P(A_1 \cap A_3) = \frac{1}{4}$
$P(A_1)P(A_3) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$
$\frac{1}{4} = \frac{1}{4}$
$\therefore$ Events A_1, A_2, and A_3 are pairwise independent

(b) Are they mutually independent?
Mutual Independence:
$P(A \cap B \cap C) = P(A)P(B)P(C)$
$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$
$P(A_1 \cap A_2 \cap A_3) = 0$
$P(A_1)P(A_2)P(A_3) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}$
$0 \neq \frac{1}{8}$
$\therefore$ Events A_1, A_2, and A_3 are NOT mutually independent

4. You enter a chess tournament where your probability of winning a game is 0.42 against 30% the players (call them type 1), 0.33 against other 10% of the players (call them type 2), and 0.25 against the remaining 60% players (call them type 3).

(a) You play a game against a randomly chosen opponent. What is the probability of winning?
type 1 = 0.3, type 2 = 0.1, type 3 = 0.6
Law of Total Probability:
$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$
$P(win|type1) = 0.42$
$P(win|type2) = 0.33$
$P(win|type3) = 0.25$
$P(win) = P(win|type1)P(type1) + P(win|type2)P(type2) + P(win|type3)P(type3)$
$P(win) = 0.42(0.3) + 0.33(0.1) + 0.25(0.6)$
$P(win) = 0.309$

(b) Suppose that you were the winner of a game. What is the probability that your opponent was of type 3?
Bayes' Rule (Reverse of the Law of Total Probability):
$P(A_k|B) = \frac{P(B|A_k)P(A_k}{P(B)}) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$
$P(type3|win) = \frac{P(win|type3)P(type3)}{P(win)}$
$P(type3|win) = \frac{0.25(0.6)}{0.309}$
$P(type3|win) = 0.485 = 48.5\%$

5. A customer has approached a bank for a loan. Without further information, the bank believes there is a 4% chance that the customer will default on the loan. The bank can run a credit check on the customer. The check will yield either a favorable or an unfavorable report. From past experience, the bank believes that

$$\text{P(favorable report being received} \mid \text{customer will default)} = \frac{4}{40},$$

and

$$\text{P(favorable report} \mid \text{customer will not default)} = \frac{96}{100}.$$

If a favorable report is received, what is the probability that the customer will default on the loan?

P(default) = 0.04

P(no default) = 0.96

Bayes' Rule (Reverse of the Law of Total Probability):

$P(A_k|B) = \frac{P(B|A_k)P(A_k}{P(B)}) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$

P(default | favorable report) = $\frac{P(favorable\,report|default)P(default)}{P(favorable\,report|default)P(default)+P(favorable\,report|no\,default)P(no\,default)}$

P(default | favorable report) = $\frac{\frac{4}{40}(0.04)}{\frac{4}{40}(0.04)+\frac{96}{100}(0.96)}$

P(default | favorable report) = $\frac{0.004}{0.004+0.9216}$

P(default | favorable report) = 0.0043 = 0.43 %