Convexe Optimization

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Homework 1:

Ex 1:

1) The rectangle set is convexe since we can write:

$$\{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i=1,\dots,n\} = \bigcap_{i=1}^{n}\left(\{x \in \mathbb{R}^n \mid \alpha_i \le x_i\} \cap \{x \in \mathbb{R}^n \mid x_i \le \beta_i\}\right)$$

but for all $i=1,\dots,n$ $\{x \in \mathbb{R}^n \mid \alpha_i \le x_i\}$ and $\{x \in \mathbb{R}^n \mid x_i \le \beta_i\}$ are half spaces of $\mathbb{R}^n$. Hence the rectangle is an intersection of half spaces and hence its convexe

2) Let $E = \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$

~~$(\lambda x_1 + (1-\lambda)$~~ Let (x_1, x_2) and $(y_1, y_2) \in E$ and $\lambda \in [0,1]$

$$(\lambda x_1 + (1-\lambda) y_1)(\lambda x_2 + (1-\lambda) y_2)$$

$$= \lambda^2 x_1 x_2 + \lambda(1-\lambda) x_1 y_2 + \lambda(1-\lambda) y_1 x_2 + (1-\lambda)^2 y_1 y_2$$

by def of E $\leftarrow$ $\geq \lambda^2 + (1-\lambda)^2 + \lambda(1-\lambda)[x_1 y_2 + x_2 y_1)$

$$= \lambda^2 + (1-\lambda)^2 + \lambda(1-\lambda)\left[\left(\sqrt{(x_1 y_2)}\right)^2 + \left(\sqrt{(x_2 y_1)}\right)^2\right)$$

$$\geq \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\sqrt{x_1 y_2 x_2 y_1}$$

$$\geq \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)$$

$$= 1$$

Hence, by definition of convexcity. E is a convexe set

3) Let $E = \{x \mid \|x - x_0\| \leq \|x - y\|_2, \forall y \in S\}$.

We can write:

$$E = \bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}.$$

Now, for every S, we can write:

$$\|x - x_0\|_2 \leq \|x - y\|_2 \iff (x - x_0)^T (x - x_0) \leq (x - y)^T (x - y)$$

$$\iff x^T x - 2x_0^T x + x_0^T x_0 \leq x^T x - 2y^T x + y^T y$$

$$\iff 2(y - x_0)^T x \leq y^T y - x_0^T x_0$$

Hence the set $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$ is convex (if $y \neq x_0$ its a halfspace, if $y = x_0$ its $\mathbb{R}^n$ which is convex)

Then E is convex a intersection of convex sets

4) This set is not convex in general. For example if $S = \{x_0, x_1\}$ and $T = \{y\}$. Then:

$$\{x \mid d(x,S) \leq d(x,T)\} = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ or } \|x - x_1\|_2 \leq \|x - y\|_2\}.$$

~~$\{x \mid \|x$~~

by the same calcul in previous question $= \{x \mid 2(y - x_0)^T x \leq y^T y - x_0^T x_0\} \cup \{x \mid 2(y - x_1)^T x \leq y^T y - x_1^T x_1\}$

So for example, if we take $y = 0$ and $x_1 = -x_0$ $(x_0 \neq 0)$

$$\{x \mid d(x,S) \leq d(x,T)\} = \{x \mid x_0^T x \geq \frac{x_0^T x_0}{2}\} \cup \{x \mid x_0^T x \leq \frac{-x_0^T x_0}{2}\}$$

This set is not convex, since the intersection of the two half spaces is empty, and we cannot go from a point in one to a point in the other using a segment.

5) Again, we write

$$\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{s_2 \in S_2} \{x \mid x + s_2 \in S_1\}$$
$$= \bigcap_{s_2 \in S_2} S_1 + s_2$$

Since S_1 is convex then $S_1 + s_2$ is convex for $s_2 \in S_2$ as the image of an affine application

Hence the set is convex

Ex2:

1) a. The function is not convex. For instance take $x = (1,2)$, $y = (3,1)$ and $\lambda = \frac{1}{2}$

$$f(\lambda x + (1-\lambda) y) = f(2, \tfrac{3}{2}) = 3$$

and $$\lambda f(x) + (1-\lambda) f(y) = \tfrac{1}{2} \times 2 + \tfrac{1}{2} \times 3 = \tfrac{5}{2}$$

The inequality of convexity is not verified

Its neither concave, for $x = (1,1)$, $y = (2,2)$, $\lambda = \cancel{0.5}\ \frac{1}{2}$:

$$f(\lambda x + (1-\lambda) y) = f(\tfrac{3}{2}, \tfrac{3}{2}) = \tfrac{9}{4}$$
$$\lambda f(x) + (1-\lambda) f(y) = \tfrac{5}{2} > \tfrac{9}{4}$$

1) a. suit $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\mathbb{R}^2_{++}$

Let $(x_1, x_2)\ (y_1, y_2) \in \mathbb{R}^2_{++}$ such as

$$y_1 y_2 \geq x_1 x_2$$

$$\nabla f(x_1, x_2)^T \left((y_1, y_2) - (x_1, x_2) \right) = x_2 [y_1 - x_1] + x_1 [y_2 - x_2]$$

$$= \sqrt{(x_2 y_1)^2} + \sqrt{(x_1 y_2)^2} - x_2 x_1 - x_1 x_2$$

$$\geq 2 \sqrt{x_1 x_2 y_1 y_2} - x_1 x_2 - x_1 x_2$$

$$\geq 2 \sqrt{(x_1 x_2)^2} - x_1 x_2 - x_1 x_2$$

$$= 0$$

By first order condition, f is quasi-concave

2) $$f(x_1, x_2) = \frac{1}{x_1 x_2}$$

Lets compute thes Hessian matrice

$$\frac{\partial f}{\partial x}(x_1, x_2) = -\frac{1}{x_2 x_1^2}, \quad \frac{\partial f}{\partial y}(x_1, x_2) = -\frac{1}{x_1 x_2^2}$$

$$\frac{\partial^2 f}{\partial x^2}(x_1, x_2) = \frac{2}{x_2 x_1^3}, \quad \frac{\partial^2 f}{\partial y^2}(x_1, x_2) = \frac{2}{x_1 x_2^3}, \quad \frac{\partial f}{\partial x \partial y}(x_1, x_2) = \frac{1}{x_1^2 x_2^2}$$

Hence $$\cancel{\nabla^2} H(f)(x_1, x_2) = \frac{1}{x_1 x_2} \begin{bmatrix} \frac{2}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} \end{bmatrix}$$

This is a positive definite matrice since $\forall (x_1, x_2) \in \mathbb{R}^2_{++}$

$$\begin{cases} \det(H(f)) = \frac{3}{x_1^2 x_2^2} > 0 \\ Tr(H(f)) = \frac{2}{x_1^2} + \frac{2}{x_2^2} > 0 \end{cases} \quad \cancel{\begin{cases} Tr(H(f)) = \frac{2}{x_1^2} + \frac{2}{x_2^2} > 0 \\ \det(H(f)) = \frac{3}{x_1^2 x_2^2} > 0 \end{cases}}$$

Hence f is convex (and also quasiconvex) on $\mathbb{R}^2_{++}$

h

- f is quasiconcave.

Let $\alpha \in \mathbb{R}$,

$$S_\alpha = \{(x_1, x_2) \in \mathbb{R}^2_{++} \mid -f(x_1, x_2) \leq \alpha\}$$

$$= \{(x_1, x_2) \in \mathbb{R}^2_{++} \mid \alpha x_2 - x_1 \geq 0\}.$$

S_α is a convex set (Half space $\cap \mathbb{R}^2_{++}$)

Hence f is quasi-linear on $\mathbb{R}^2_{++}$

4) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, $0 \leq \alpha \leq 1$

Lets compute the Hessian matrix.

$$\mathrm{Hess}(f)(x_1, x_2) = \alpha(1-\alpha) \begin{bmatrix} -x_1^{\alpha-2} x_2^{1-\alpha} & x_1^{\alpha-1} x_2^{-\alpha} \\ x_1^{\alpha-1} x_2^{-\alpha} & -x_1^{\alpha} x_2^{-\alpha-1} \end{bmatrix}$$

~~If $\alpha = 0$ or $\alpha = 1$:~~

~~f is convexe and concave on $\mathbb{R}^2_{++}$~~

~~If $0 < \alpha \leq 1$.~~

~~$\det(\mathrm{Hess}(f))$~~

$$\mathrm{Hess}(f)(x_1, x_2) = \alpha(1-\alpha) x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} -\frac{1}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & -\frac{1}{x_2^2} \end{bmatrix}$$

Thus, $\mathrm{Hess}(f)(x_1, x_2) \not\succeq 0$ if $0 < \alpha \leq 1$ (since $\det(\mathrm{Hess}(f)) = 0$ and $-\frac{1}{x_1^2} < 0$ So one of the eigenvalue is negative)

So f is neither convex, nor concave or quasiconvex.

and if $\alpha = 0$ or $\alpha = 1$, f is convexe and concave

Ex 3:

1) To prove f is convex, we will prove that:

$$\forall X \in S_m^{++}, \forall V \in S_m, \ g: t \mapsto f(X+tV) \text{ is convex.} \quad (P)$$

It suffices to take $V \in S_m$. In fact, if we prove this, take $X, Y \in S_m^{++}$, and $\lambda \in [0,1]$., let $s, t \in [0,1]$ $s \neq t$

Let
$$\begin{cases} V = \dfrac{X-Y}{s-t} \in S_m \\ A = X - tV. \end{cases}$$

Then
$$\begin{cases} X = A + tV = g(t) \\ Y = A + sV = g(s) \end{cases}$$

and by the convexity of $g: t \mapsto A+tV$.

$$g(\lambda t + (1-\lambda)s) \leq \lambda g(t) + (1-\lambda) g(s)$$

which gives
$$f(\lambda X + (1-\lambda) Y) \leq \lambda f(X) + (1-\lambda) f(Y)$$

Lets prove (P). Let $X \in S_m^{++}$, $V \in S_m$

$$f(X+tV) = \text{Tr}((X+tV)^{-1})$$

$$= \text{Tr}\left(\left(X^{\frac{1}{2}}[I + tX^{-\frac{1}{2}}VX^{-\frac{1}{2}}]X^{\frac{1}{2}}\right)^{-1}\right)$$

$$= \text{Tr}\left(X^{-1}(I + tX^{-\frac{1}{2}}VX^{-\frac{1}{2}})^{-1}\right)$$

Since $X^{-\frac{1}{2}}VX^{-\frac{1}{2}} \in S_m$, we have the decomposition $\cancel{X^{-\frac{1}{2}}VX^{-\frac{1}{2}}}$

$$X^{-\frac{1}{2}}VX^{-\frac{1}{2}} = Q\Lambda Q^T$$

Then
$$f(X+tV) = \text{Tr}(X^{-1}[I + tQ\Lambda Q^T]^{-1})$$
$$= \text{Tr}(X^{-1}Q[I + t\Lambda]^{-1}Q^T)$$

Hence $$f(X + tV) = \mathrm{Tr}\left(Q^T X^{-1} Q \underbrace{[I + t\Lambda]^{-1}}_{\text{diagonal}}\right)$$

$$= \sum_{i=1}^{n} (Q^T X^{-1} Q)_{ii} \, (1 + t\lambda_i)^{-1}$$

where $Sp(X^{-\frac{1}{2}} V X^{-\frac{1}{2}}) = \{\lambda_1, \dots, \lambda_n\}$.

Hence g is a positive weighted sum of convex functions

(since $(Q^T X^{-1} Q)_{ii} > 0$, $X \in S_n^{++}$)

$\left\{ \begin{array}{l} \forall t, \ (1 + t\lambda_i)^{-1} > 0, \\ \text{since } I + t\Lambda \in S_n^{++} \end{array} \right.$

And ~~Avg~~ g is convex. hence f is convex

2) For $X \in S_n^{++}$, we can write

$$X = Q \Lambda Q^T.$$

Let
$$g(X, y, x) = 2y^T B x - x^T C x.$$

$$\underbrace{d_x g}_{\substack{\downarrow \\ \text{driving} \\ \text{over } x}} (X, y, x)(h) = h^T [2By - Cx]$$

So minimizing over x yeilds

$$\min_x g(X, y, x) = x^T B^T C^{-1} B^T x$$

We chose $B = Q^T$ and $C = \Lambda$.

$$g(X, y, x) = 2y^T Q x - x^T \Lambda x$$

and
$$f(X, y) = \min_x g(X, y, x) = x^T Q \Lambda^{-1} Q^T x$$
$$= x^T X^{-1} x$$

$$\cancel{\Rightarrow f(X, y) = \min_{x \in \mathbb{R}^n} g(X,}$$

Hence f is a convesc function since g is convesc (slide 21)

and we optimize over $\mathbb{R}^n$ (convesc)

$$f(X, y) = \min_{x \in \mathbb{R}^n} g(X, y, x) = \min_{x \in \mathbb{R}^n} (2y^T Q x - x^T \Lambda x)$$

Ex 3:

3. $f : X \longmapsto \sum_{i=1}^{m} \sigma_i(X)$ on $\mathrm{dom}\, f = S_m$

Lets prove that, $f(X) = \sup_{\|B\|_2 \leq 1} |\langle X, B\rangle|$

Let $X \in S_m$, we write the SVD of X. $X = U \Sigma V^T$

Let $B \in \mathcal{M}_n(\mathbb{R})$, $\|B\|_2 \leq 1$.

$$\begin{aligned} \langle X, B\rangle &= \mathrm{Tr}(X^T B) \\ &= \mathrm{Tr}(U\Sigma V^T B) \\ &= \mathrm{Tr}(U^T B V \Sigma) \\ &= \sum_{i=1}^{m} \sigma_i(X) \left|(U^T B^T V)_{ii}\right| \end{aligned}$$

Since $\|B\|_2 = \sigma_{max}(B)$, then we also have ~~$\|VB^TU\|_2\ \|B$~~

$\|VB^TU\|_2 = \|B\|_2 \leq 1$ (Since U and V are unitary)

Hence $\langle X, B\rangle = \sum_{i=1}^{m} \sigma_i(X) \left|(U^T B V)_{ii}\right|$

$\leq \sum_{i=1}^{m} \sigma_i(X)$ since $\|VB^TU\|_2 = \sup_{\|x\|_2 = 1} \|VB^TUx\|_2 \leq 1$

This upper bound is achieved for $B = UV^T$.

since $\langle X, B\rangle = \sum_{i=1}^{m} \sigma_i(X) \left|(U^T U V V^T)_{ii}\right|$

$= \sum_{i=1}^{m} \sigma_i(X) = f(X)$

Hence $f(X) = \sup_{\|B\|_2 \leq 1} |\langle X, B\rangle|$

we $X \mapsto \langle X, B\rangle$ is convex in X for each $B \in \mathcal{M}_n(\mathbb{R})$ and we optimize over the unit ball for spectral norm which is convex

Hence f is convex

Ex 4: $K_{m+} = \{ x \in \mathbb{R}^m \mid x_1 \geq x_2 \geq \cdots \geq x_m \geq 0 \}$.

1. First, K_{m+} is a cone.

If $x, y \in K_{m+}$, $\theta_1, \theta_2 \in \mathbb{R}^+$.

$$\theta_1 x + \theta_2 y = ((\theta_1 x_i + \theta_2 y_i))_{1 \leq i \leq m}$$

and $\theta_1 x_1 + \cancel{\theta_2} y_1 \geq \theta_1 x_2 + \theta_1 y_2 \geq \cdots \geq \theta_1 x_m + \theta_1 y_m \geq 0$

. K_{m+} is closed.

let p_i the projection, $p_i(x) = \cancel{p_i}\, x_i$, p_i is continuous.

Then, $$K_{m+} = \bigcap_{i=1}^{m-1} \left((p_i - p_{i+1})^{-1} (\mathbb{R}^+) \right) \cap p_m^{-1}(\mathbb{R}^+) \quad (\cdot)$$

Hence K_m^+ is an intersection of closed sets and hence

K_{m+} is closed

K_{m+} is solid

$\cancel{\overset{\circ}{K}_{m+} = \{ x \in \mathbb{R}^m \}}$

by $(\cdot)$. $$\overset{\circ}{K}{}_m^+ = \bigcap_{i=1}^{m-1} \left((p_i - p_{i+1})^{-1} (\mathbb{R}^{+*}) \right) \cap p_m^{-1}(\mathbb{R}^{+*})$$

$$= \{ x \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_m > 0 \}$$

which is non empty.

K_{m+} is pointed.

Let $x \in K_{m+}$ such as $-x \in K_{m+}$

Then $\forall i = 1, \ldots, m$. $x_i \geq 0$ and $x_i \leq 0$

hence $x_i = 0$

and $x \cancel{\neq} 0$.

C/C K_{m+} is a proper cone

2. Lets prove that

$$K^*_{m+} = \{ y \in \mathbb{R}^n \mid 1^T y \geq 0 \}.$$

Let $y \in K^*_{m+}$, ~~and~~ since $1 \in K_{m+}$

then $1^T y \geq 0$

Conversly, if $1^T y \geq 0$. Let $x \in K_{m+}$.

$$x^T y = \sum_{i=1}^{n} x_i y_i \geq x_n \sum_{i=1}^{n} y_i = x_n 1^T y \geq 0$$

then $x \in K^*_{m+}$

Ex 5:

1) $f(x) = \max_{i=1,\dots,n} x_i$

Let $y \in \mathbb{R}^n$, and

$$h(x) = y^T x - f(x)$$

~~h(x~~ Suppose that $1^T y = \sum_{i=1}^{n} y_i = 1$, then

$$h(x) = \sum_{i=1}^{n} y_i - \max x_i = \sum_{i=1}^{n} y_i [x_i - \max_{i=1,\dots,n} x_i]$$

if $\exists j$, $y_j < 0$, Consider the sequence $x_p = (p, \dots, p, \underset{j \text{ position}}{0}, p, \dots, p)$ $p \in \mathbb{N}$

$$h(x_p) = -y_j p \xrightarrow[p \to +\infty]{} +\infty$$

then $f^*(y) = +\infty$

if $y \geq 0$, $h(x) \leq 0$

and $h(1, \dots, 1) = 0$

Hence $f^*(y) = 0$

Now if $\sum_{i=1}^{n} y_i \neq 1$, let $\varepsilon = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} y_i \geq 1 \\ -1 & \text{otherwise} \end{cases}$

and $x_p = \varepsilon(p, p, \dots, p)$, $p \in \mathbb{N}$

then $$h(x_p) = \varepsilon \sum_{i=1}^{n} p y_i - \varepsilon p$$
$$= \varepsilon p \left[\sum_{i=1}^{n} y_i - 1 \right] \xrightarrow[p \to +\infty]{} +\infty$$

Hence $$f^*(y) = +\infty$$

C/C
$$f^*(y) = \begin{cases} 0 & \text{if } y \succeq 0 \text{ and } 1^T y \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

2
$$f(x) = \sum_{i=1}^{r} x_{[i]}$$
Let $y \in \mathbb{R}^n$
$$h(x) = y^T x - f(x)$$

~~Let If we take $x \in \mathbb{R}^n$ such as $x_1 \geq \dots \geq x_n$.~~

$\cancel{h(x) = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{r} x_i}$

$\cancel{= \sum_{i=1}^{r} x_i (y_i - 1) + \sum_{i=r+1}^{n} x_i y_i}$

If we take $x_p = (p, \dots, p)$, $p \in \mathbb{N}$.
$$h(x_p) = \sum_{i=1}^{r} p(y_i - 1) + \sum_{i=r+1}^{n} p y_i$$
$$= p\left(\sum_{i=1}^{n} y_i - r\right)$$

Then, if $\sum_{i=1}^{n} y_i \neq r$, and $\varepsilon = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} y_i > r \\ -1 & \text{otherwise} \end{cases}$

$$h(\varepsilon x_p) \xrightarrow[p \to +\infty]{} +\infty$$
and $$f^*(y) = +\infty$$

We now suppose that $\sum_{i=1}^{n} y_i = r$

- Suppose that $\exists j,\ y_j < 0$.

Consider $x_p = (p, \dots, p, \underset{\substack{\uparrow \\ j \text{ position}}}{0}, p, \dots, p)$

$$h(x_p) = -y_i\, p \xrightarrow[p \mapsto +\infty]{} +\infty$$

and $f^*(y) = \cancel{+}\infty$

- suppose then that $y \geq 0$, suppose that $\exists j.\ y_j > 1$.

and take $x_p = (0, \dots, \underset{\substack{\uparrow \\ j \text{ position}}}{p}, 0, \dots, 0)$, $p \in \mathbb{N}$

$$h(x_p) = p(y_i - 1) \xrightarrow[p \mapsto +\infty]{} +\infty$$

then $f^*(y) = +\infty$

- We suppose then that $0 \leq y \leq 1$, and $\sum_{i=1}^{n} y_i = 1$.

Let $x \in \mathbb{R}^n$, and write $x_{i_1} \geq x_{i_2} \geq \dots \geq x_{i_n}$.

then $h(x) = \sum_{j=1}^{n} x_{i_j} y_{i_j} - \sum_{j=1}^{r} x_{i_j}$

$$= \sum_{j=1}^{r} x_{i_j}(y_{i_j} - 1) + \sum_{j=r+1}^{n} x_{i_j} y_{i_j}$$

$$\leq \sum_{j=1}^{r} x_{i_r}(y_{i_j} - 1) + \sum_{j=r+1}^{n} x_{i_r} y_{i_j}$$

$$= \sum_{j=1}^{n} y_{i_j} - r = 0$$

then $h(x) \leq 0$

and $h(1, \dots, 1) = 0$

c/c

$$f^*(y) = \begin{cases} 0 & \text{if } 0 \preceq y \preceq 1,\ 1^T y = n \\ +\infty & \text{otherwise} \end{cases}$$

3. $$f(x) = \max_{i=1,\dots,m} (a_i x + b_i)$$

Let $y \in \mathbb{R}$, and $h(x) = xy - f(x)$

remark that, $a_i x + b_i \geq a_j x + b_j$ with $i > j$

if and only if $x \geq \dfrac{b_j - b_i}{a_i - a_j}$

since $a_1 \leq \dots \leq a_m$ and no term is redundant, the function is piecewise linear and the dominating term changes from $a_i x + b_i$ to $a_{i+1} \cancel{x} + b_{i+1}$ at the breaking point $\dfrac{b_i - b_{i+1}}{a_{i+1} - a_i}$

If $y > a_m$, then for sufficiently large x,

$$\cancel{f(x)} \ h(x) \underset{x \to +\infty}{\sim} xy - a_m x - b \xrightarrow[x \to +\infty]{} +\infty \text{ as } y > a_m$$

also if $a_1 > y$, for $x \to -\infty$

$$h(x) \underset{x \to +\infty}{\sim} xy - a_1 x - b \xrightarrow[x \to -\infty]{} +\infty$$

Now if $y \in [a_1, a_m]$. let i such as $a_i \leq y \leq a_{i+1}$.

note $c_i = \dfrac{b_i - b_{i+1}}{a_{i+1} - a_i}$ for $i = 1, \dots, m-1$.

~~but~~ on $[c_i, c_{i+1}]$, $f(x) = a_i x + b_i$ and so

$$\sup_{x \in [c_i, c_{i+1}]} h(x) = (y - a_i) \times \frac{b_{i} - b_i}{a_{i+1} - a_i} - b_i$$

Now, on $[-\infty, c_i]$, f is linear with ~~increasing~~ positive slope so its maximum is attained on c_i

and on $[c_i, +\infty)$ its a linear function with negative slope and the maximum is in c_i

Hence

$$\sup_{\mathbb{R}} h(x) = \sup_{x \in [c_i, c_{i+1}]} h(x) = (y - a_i) \times \frac{b_{i} - b_{i+1}}{a_{i+1} - a_i}$$

And

$$f^*(y) = \begin{cases} (y - a_i) \times \dfrac{b_i - b_{i+1}}{a_{i+1} - a_i} & \text{if } a_i \leq y \leq a_{i+1} \\ +\infty & \text{if } y \notin [a_1, a_m] \end{cases}$$