Topics Covered

Page

Jeffreys Prior

The likelihood function is given by

$$f(\{n_i\}|\eta, \{s_i\}, \{b_i\}) = \prod_{i=1}^{N_{bins}} \frac{\lambda_i^{n_i} e^{-\lambda_i}}{n_i!} \tag{1}$$

Where the mean λ_i of each Poisson distribution is given by

$$\lambda_i = \eta s_i + b_i \tag{2}$$

Calculation of the Jeffreys Prior for the signal strength parameter η gives

$$p(\eta) \propto \sqrt{\mathrm{E}\left[\left(\frac{\partial}{\partial\eta}\log f(\{n_i\}|\eta)\right)^2\right]} \tag{3}$$

$$\propto \sqrt{\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_{N_{bins}}=0}^{\infty}\left\{\left[\frac{\partial}{\partial\eta}\log\prod_{i=1}^{N_{bins}}\frac{\lambda_i^{n_i}e^{-\lambda_i}}{n_i!}\right]^2\prod_{k=1}^{N_{bins}}\frac{\lambda_k^{n_k}e^{-\lambda_k}}{n_k!}\right\}} \tag{4}$$

$$\propto \sqrt{\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_{N_{bins}}=0}^{\infty}\left\{\left[\frac{\partial}{\partial\eta}\sum_{i=1}^{N_{bins}}\left(n_i\log\lambda_i-\lambda_i-\log(n_i!)\right)\right]^2\prod_{k=1}^{N_{bins}}\frac{\lambda_k^{n_k}e^{-\lambda_k}}{n_k!}\right\}} \tag{5}$$

$$\propto \sqrt{\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_{N_{bins}}=0}^{\infty}\left\{\left[\sum_{i=1}^{N_{bins}}\frac{n_i s_i}{\lambda_i}-s_i\right]^2\prod_{k=1}^{N_{bins}}\frac{\lambda_k^{n_k}e^{-\lambda_k}}{n_k!}\right\}}, \tag{6}$$

and expanding the quadratic gives

$$p(\eta) \propto \sqrt{\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_{N_{bins}}=0}^{\infty}\left\{\left[\sum_{i=1}^{N_{bins}}\sum_{j=1}^{N_{bins}}\left(\frac{n_i}{\lambda_i}-1\right)\left(\frac{n_j}{\lambda_j}-1\right)s_is_j\right]\prod_{k=1}^{N_{bins}}\frac{\lambda_k^{n_k}e^{-\lambda_k}}{n_k!}\right\}} \tag{7}$$

$$\propto \sqrt{\sum_{i=1}^{N_{bins}}\sum_{j=1}^{N_{bins}}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_{N_{bins}}=0}^{\infty}\left(\frac{n_i}{\lambda_i}-1\right)\left(\frac{n_j}{\lambda_j}-1\right)s_is_j\prod_{k=1}^{N_{bins}}\frac{\lambda_k^{n_k}e^{-\lambda_k}}{n_k!}}. \tag{8}$$

Jeffreys Prior continued on next page...

We then separate out the diagonal terms in the sum:

$$p(\eta) \propto \left\{ \sum_{i=1}^{N_{bins}} \sum_{\substack{j=1 \\ j\neq i}}^{N_{bins}} \left[\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{N_{bins}=0}}^{\infty} \left(\frac{n_i}{\lambda_i} - 1\right)\left(\frac{n_j}{\lambda_j} - 1\right) s_i s_j \prod_{k=1}^{N_{bins}} \frac{\lambda_k^{n_k} e^{-\lambda_k}}{n_k!} \right] \right. \tag{9}$$

$$\left. + \sum_{i=1}^{N_{bins}} \left[\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{N_{bins}=0}}^{\infty} \left(\frac{n_i}{\lambda_i} - 1\right)^2 s_i^2 \prod_{k=1}^{N_{bins}} \frac{\lambda_k^{n_k} e^{-\lambda_k}}{n_k!} \right] \right\}^{1/2}$$

$$\propto \left\{ \sum_{i=1}^{N_{bins}} \sum_{\substack{j=1 \\ j\neq i}}^{N_{bins}} \left[\underbrace{\sum_{n_l=0}^{\infty} \cdots}_{l\neq i,j} \left(\frac{1}{\lambda_i} \mathrm{E}_i[n_i] - \mathrm{E}_i[1]\right)\left(\frac{1}{\lambda_j} \mathrm{E}_j[n_j] - \mathrm{E}_j[1]\right) s_i s_j \prod_{\substack{k=1 \\ k\neq i,j}}^{N_{bins}} \frac{\lambda_k^{n_k} e^{-\lambda_k}}{n_k!} \right] \right. \tag{10}$$

$$\left. + \sum_{i=1}^{N_{bins}} \frac{s_i^2}{\lambda_i^2} \left[\underbrace{\sum_{n_l=0}^{\infty} \cdots}_{l\neq i} \mathrm{Var}(n_i) \prod_{\substack{k=1 \\ k\neq i}}^{N_{bins}} \frac{\lambda_k^{n_k} e^{-\lambda_k}}{n_k!} \right] \right\}^{1/2},$$

where $\mathrm{E}_i[f] \equiv \sum_{n_i=0}^{\infty} f \, \frac{\lambda_i^{n_i} e^{-\lambda_i}}{n_i!}$, implying $\mathrm{E}_i[n_i] = \lambda_i$ and $\mathrm{E}_i[1] = 1$. Furthermore, $\mathrm{Var}(n_i) = \mathrm{E}_i[(n_i - \lambda_i)^2] = \sum_{n_i=0}^{\infty} (n_i - \lambda_i)^2 \frac{\lambda_i^{n_i} e^{-\lambda_i}}{n_i!} = \lambda_i$. This greatly simplifies our expression to

$$p(\eta) \propto \sqrt{\sum_{i=1}^{N_{bins}} \frac{s_i^2}{\lambda_i} \underbrace{\sum_{n_l=0}^{\infty} \cdots}_{l\neq i} \prod_{\substack{k=1 \\ k\neq i}}^{N_{bins}} \frac{\lambda_k^{n_k} e^{-\lambda_k}}{n_k!}} \tag{11}$$

$$\propto \sqrt{\sum_{i=1}^{N_{bins}} \frac{s_i^2}{\lambda_i} \prod_{\substack{k=1 \\ k\neq i}}^{N_{bins}} \sum_{n_k=0}^{\infty} \frac{\lambda_k^{n_k} e^{-\lambda_k}}{n_k!}} \tag{12}$$

$$\propto \sqrt{\sum_{i=1}^{N_{bins}} \frac{s_i^2}{\lambda_i} \prod_{\substack{k=1 \\ k\neq i}}^{N_{bins}} 1} \tag{13}$$

$$\propto \sqrt{\sum_{i=1}^{N_{bins}} \frac{s_i^2}{\lambda_i}}\,. \tag{14}$$

We are left with the astoundingly simple expression

$$\boxed{p(\eta) \propto \sqrt{\sum_{i=1}^{N_{bins}} \frac{s_i^2}{\lambda_i}}}\,. \tag{15}$$