Optimizing Surfaces of Revolution

Extended Essay - Mathematics

How can the surface area of a function rotated over a horizontal line be minimized with respect to the location of the line, and which method of optimization is the most efficient?

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1. Introduction

The field of geometry has always been an integral part of mathematics and is one of the fields with visible real-world applications. The study of the three-dimensional shapes is especially relevant for things such as construction, architecture, imaging, and simulation technology. Although concepts such as prisms and spheres are taught at a young age, shapes in the real world are much more complex. Calculus offers many paths to dealing with complex shapes, one of them being solids of revolution.

Solids of revolution are solids that are formed when a curve on a plane is rotated around a straight line on the same plane. An example of this can be seen in *Figure 1*, where the region bounded by the x-axis, the graph of $y = f(\mathrm{x})$ and the lines $x = a$ and $x = b$, is rotated around the x-axis to produce a solid.

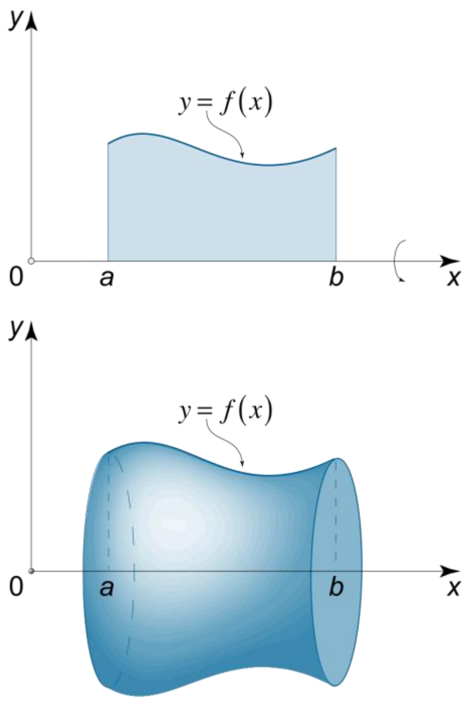

Figure 1: Region rotated around x-axis, *(Svirin)*

This solid can be described using its surface area, the calculation of which shall be explained in the following chapter. In order to link this to physical applications I involved another calculus topic, optimization - the surface area of the solid of revolution could be optimized with respect to a certain parameter.

Looking through the possible parameters to manipulate, I wanted to find one that could be applicable to a variety of functions, so I chose to look at the location of the axis of revolution, when performing the process of rotating the region bounded by the graph of the function. Although the axis of rotation could be any line, I decided to limit my investigation to the rotational axis being a horizontal line. Therefore, I decided to investigate the following research question:

How can the surface area of a function rotated over a horizontal line be minimized with respect to the location of the line, and which method of optimization is the most efficient for this type of function?

2. Calculating Surface Area

An approximation for the surface area of a solid of revolution can be found by subdividing the solid into adjacent shapes. The resulting divisions are a shape known as a frustum, which is essentially a section of a cone (grey region in *Figure 2*). To approximate the area of the surface of revolution, the lateral surfaces of each frustum must be calculated and then added together.

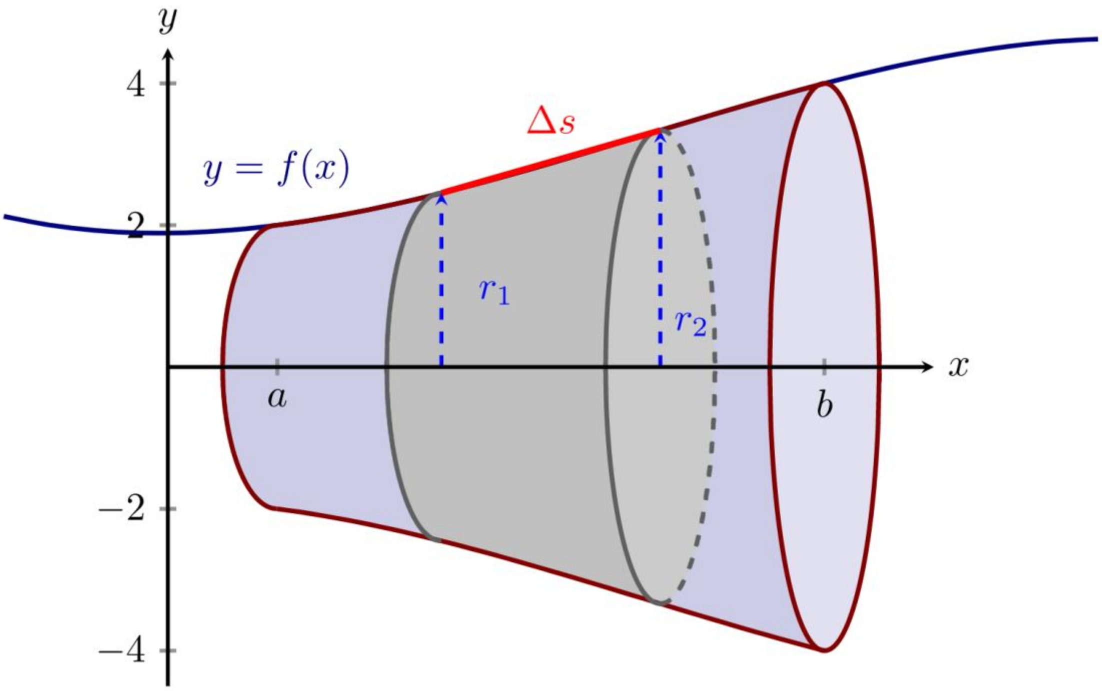

Figure 2: Function rotated over x- axis, *(Talamo)*

In order to find the surface area of a frustum, a sufficiently small part of the graph must be considered, such that the length of the arc denoted by Δs in *Figure 2,* can be approximated by s, the length of the line segment, as shown in *Figure 3,* which depicts the cross section of a frustum.

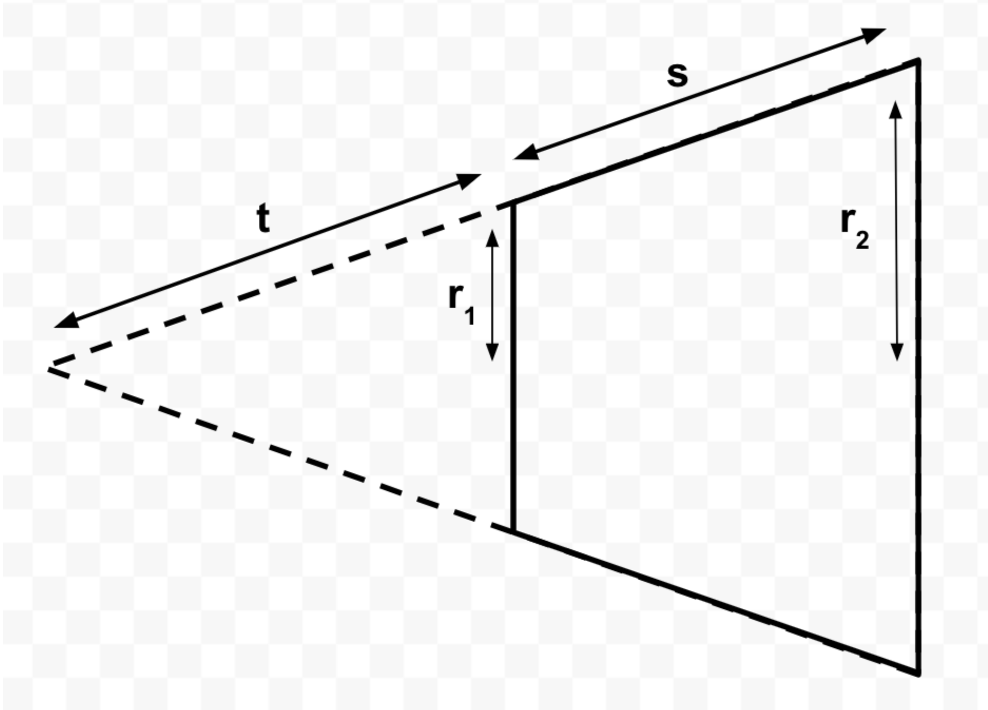

Figure 3: Cross section of a frustum, ***(Author's own)***

This image suggests how a frustum can be obtained from a large cone with a smaller cone removed from it. The lateral surface area of a cone of radius *r* is calculated by $A = \pi r l$, where l is the slant height. The surface area of the frustum SA can be derived by subtracting the lateral surface area of the smaller cone, $A_2 = \pi r_1 t$, from the lateral surface area of the larger cone, $A_1 = \pi r_2 (s + t)$:

$$SA = \pi(\mathrm{r}_2(\mathrm{s} + \mathrm{t}) - \mathrm{r}_1\mathrm{t})$$

By simplifying, the following is obtained:

$$SA = \pi(\mathrm{r}_2\mathrm{s} + \mathrm{r}_2\mathrm{t} - \mathrm{r}_1\mathrm{t})$$

$$SA = \pi(r_2 s + (r_2 t - r_1 t)) \tag{1}$$

Using similar triangles, the ratio for their sides (and the cone's dimensions) is:

$$\frac{r_1}{t} = \frac{r_2}{s+t}$$

Rearranging this equation gives:

$$\Rightarrow r_1 t + r_1 s = r_2 t$$

$$\Rightarrow r_2 t - r_1 t = r_1 s \tag{2}$$

Substituting equation (2) into equation (1):

$$\Rightarrow SA = \pi(r_2 s + r_1 s)$$

$$\Rightarrow SA = \pi(r_2 + r_1) \cdot s \tag{3}$$

The interval $[a, b]$ can be partitioned into n intervals of arbitrary widths:

$$a \leq x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n \leq b$$

The i^{th} piece of the partition would therefore be $[x_i,\ x_{i+1}],\ i = \overline{0, n-1}$. Each piece of the partition now represents a single frustum. The radii of the frustum in the i^{th} piece of the partition would be $f(x_{i+1})$ and $f(x_i)$ respectively. The slant height s_i of this frustum would be the distance between the points $(x_{i+1}, f(x_{i+1}))$ and $(x_i, f(x_i))$, which can be calculated as:

$$s_i = \sqrt{\left(f(x_{i+1}) - f(x_i)\right)^2 + (x_{i+1} - x_i)^2}$$

It follows that the expression for the lateral surface area of a single frustum in the i^{th} partition, SA_i, is:

$$SA_i = \pi\big(f(x_{i+1}) + f(x_i)\big) \cdot \sqrt{\big(f(x_{i+1}) - f(x_i)\big)^2 + (x_{i+1} - x_i)^2} \tag{4}$$

The expression $\sqrt{\big(f(x_{i+1}) - f(x_i)\big)^2 + (x_{i+1} - x_i)^2}$, which is part of equation (4), can be simplified using the mean value theorem, which states that if a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point c in (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

By letting $c = w_i$, where $w_i \in (x_i, x_{i+1})$:

$$f'(w_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$\Rightarrow (x_{i+1} - x_i)f'(w_i) = f(x_{i+1}) - f(x_i) \tag{5}$$

Substituting equation (5) into equation (4), gives:

$$SA_i = \pi\big(f(x_{i+1}) + f(x_i)\big)\sqrt{\big((x_{i+1} - x_i)f'(w_i)\big)^2 + (x_{i+1} - x_1)^2}$$

Factoring leads to:

$$SA_i = \pi\big(f(x_{i+1}) + f(x_i)\big)\sqrt{(x_{i+1} - x_i)^2(f'(w_i)^2 + 1)}$$

$$\Rightarrow SA_i = \pi\big(f(x_{i+1}) + f(x_i)\big)(x_{i+1} - x_i)\sqrt{(f'(w_i)^2 + 1)}$$

Summing the surface areas of the individual frustums results into an approximation of the surface area of the solid of revolution:

$$SA \approx \sum_{i=0}^{n} \pi\big(f(x_{i+1}) + f(x_i)\big) \cdot (x_{i+1} - x_i)\sqrt{(f'(w_i)^2 + 1)}$$

To obtain the real value of SA , the number of frustums must be greatly increased, such that $n \to \infty$, and consequently $\Delta x_i = x_{i+1} - x_i \to 0$.

Recalling that $w_i \in (x_i, x_{i+1})$, as Δx_i approaches 0, the interval (x_i, x_{i+1}) becomes infinitesimally small, and x_i and x_{i+1} can be considered the same value as w_i, consequently $f(x_{i+1}) = f(x_i)$, leading also to $f(x_{i+1}) + f(x_i) = 2f(w_i)$.

Therefore, the summation equation becomes:

$$SA = \lim_{\Delta x \to 0} \sum_{i=0}^{n} 2\pi f(w_i) \cdot (x_{i+1} - x_i)\sqrt{(f'(w_i)^2 + 1)}$$

$$\Rightarrow SA = \lim_{\Delta x \to 0} 2\pi \sum_{i=0}^{n} f(w_i) \cdot (x_{i+1} - x_i)\sqrt{(f'(w_i)^2 + 1)}$$

This type of sum is known as a Riemann sum, and gives an approximation for the surface area. As the width of the frustums used to approximate the function grows smaller, the approximation becomes more accurate *(Patrick 38)*. Therefore, as Δx_i approaches 0, this sum approaches the value of the real area and can be found using the definite integral below:

$$SA = 2\pi \int_a^b f(x)\sqrt{(f'(x)^2 + 1)}\, dx$$

However, the surface area cannot often be calculated using one equation because of a certain restriction regarding the integral: when the graph of a function is below the x-axis, the value of the definite integral becomes negative, even though surface area is always positive. Therefore, applying this integral to a function where $f(x)$ is both positive and negative does not accurately represent the surface area. The function must be divided into intervals where $f(x)$ is either solely positive or negative, and the absolute value of these integrals should be summed. This can be done easier if the absolute value of the function is used instead of the original function, in which case the graph will always be above the x-axis, and the calculated area will be the same, consequently the required area is best evaluated by using:

$$SA = 2\pi \int_a^b |f(x)|\sqrt{(f'(x)^2 + 1)}\ dx$$

3. The Surface Area Function

3.1 Investigating the function

To begin, a simple example will be considered for finding the surface area of a solid of revolution: for the function $f(x) = \sin(x)$, the region bounded by its graph and the lines $x = 0$ and $x = \frac{5\pi}{3}$ is rotated around the x-axis. The interval $[0, \frac{5\pi}{3}]$ has been chosen such that global minimum and maximum points for the function are easily identified. Using the findings from the previous section, the surface area can be calculated as:

$$SA = 2\pi \int_0^{\frac{5\pi}{3}} |\sin x| \sqrt{(\cos^2 x + 1)}\ dx,$$

which must be split in two integrals to accommodate for the regions where the expression of the integrand is positive or negative:

$$SA = 2\pi \left(\int_0^{\pi} \sin x \sqrt{(\cos^2 x + 1)}\ dx - \int_{\pi}^{\frac{5\pi}{3}} \sin x \sqrt{(\cos^2 x + 1)}\ dx \right)$$

These integrals are very difficult to evaluate analytically, which is why their values will be calculated numerically instead, by using technology, namely the computational engine WolframAlpha. It is found that:

$$SA \approx 2\pi(2.2956 - (-1.6679))$$

$$\Rightarrow SA \approx 24.9034 \quad (4\text{ d.p.})$$

The same value is obtained if the following integral is calculated directly:

$$SA = 2\pi \int_0^{\frac{5\pi}{3}} |\sin x| \sqrt{(\cos^2 x + 1)}\ dx = 24.9034 \quad (4\text{ d.p.})$$

To investigate what happens to the value of the surface area of the solid of revolution when the graph of $f(x) = \sin(x)$ is rotated about a horizontal line $y = k$, other that the x- axis, a new function, $S(k)$, must be used, this new function being the result of the original function undergoing a vertical translation of k units up or down, so that the new function $S(k) = f(x) - k$ will be rotated around the x-axis.The surface area function in this case can be written as:

$$S(k) = 2\pi \int_0^{\frac{5\pi}{3}} |\sin(x) - k|\sqrt{(\cos^2(x) + 1)}\ dx.$$

In order to observe the behavior of $S(k)$, the surface area for various values of k will be calculated. As the minimum and maximum values of $f(x) = \sin(x)$ are -1 and 1 respectively, the values for k will be taken from the interval $[-3,3]$, to include them. The results of the calculations can be seen in *Table 1*:

Value of k	Value of $S(k)$ $(3\ \mathrm{d.p.})$
-3	122.239
-2	82.806
-1	43.373
-0.8	37.652
-0.6	33.300
-0.4	29.765
-0.2	26.974
0	24.903
0.2	23.901
0.4	24.337
0.6	26.267
0.8	29.823
1	35.488
2	74.921
3	114.354

Table 1: Surface area $S(k)$, calculated for various values of k

To observe the trend of the graph of $S(k)$, the points $(k, S(k))$ are plotted, as seen *Figure 4* below, using the graphing package Desmos Software:

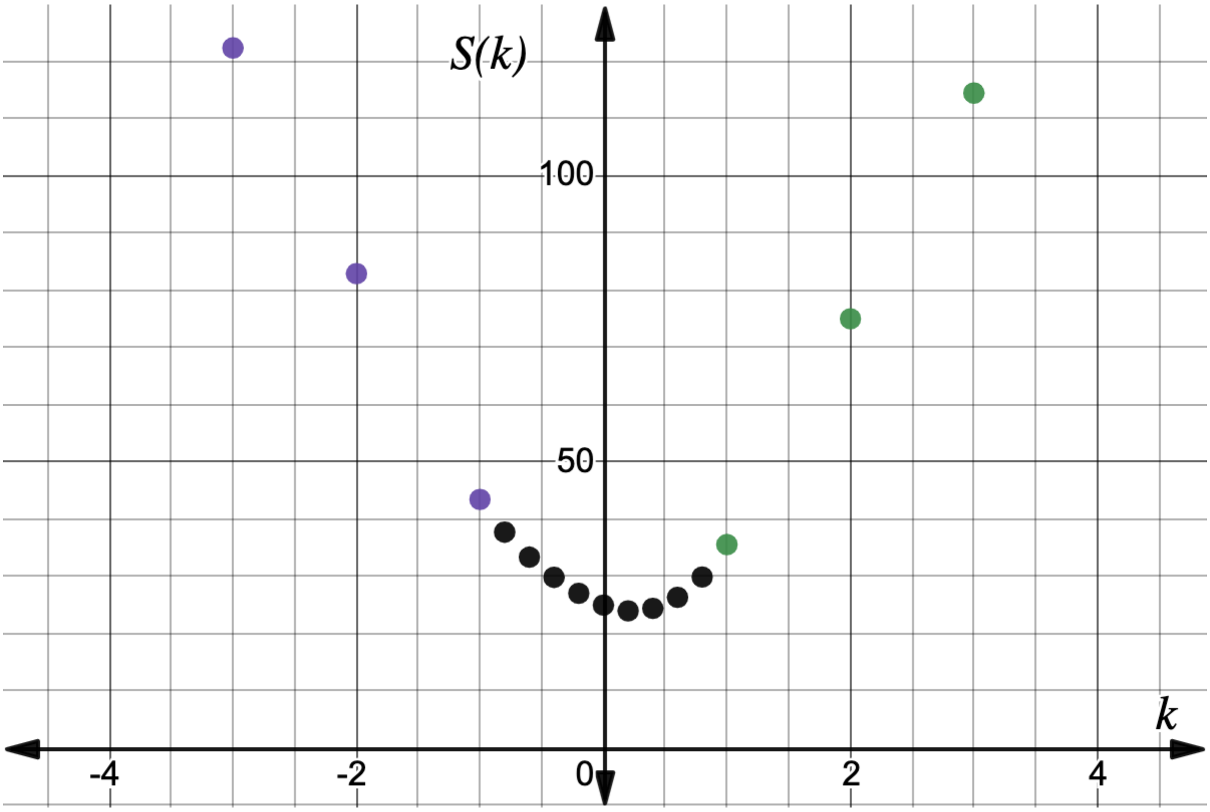

Figure 4: Surface Area Function for $f(x) = sin(x)$, *(Author's own)*

In this graph, when $k \in \{-3, -2, -1\}$ and $k \in \{1,2,3\}$ the trend of the scatter plot seems to be linear. This can be confirmed by finding the line of best fit in each case by using the Desmos software - the correlation coefficient, which shows the strength of a relationship, is 1, implying a perfect fit, as shown in *Table 2* below (all values are rounded to one decimal place):

Values of k	Equation of regression line	Correlation coefficient
$k \in \{-3,-2,-1\}$	$y = -39.4x + 3.9$	$r = 1$
$k \in \{-3,-2,-1\}$	$y = 39.4x - 3.9$	$r = 1$

Table 2: Regression line for $\boldsymbol{S(k)}$ when $\boldsymbol{k} \in \{-3,-2,-1\}$ and $\boldsymbol{k} \in \{1,2,3\}$

The non-linear behavior occurs only when $k \in [-1,1]$, i.e. when k takes values between the minimum and the maximum value of the function $f(x) = \sin(x)$. To confirm these findings, a different approach will now be used.

Case 1. When $k \leq -1,\ S(k) = \sin x - k$ is completely above the x-axis, so the general surface area equation to calculate the area of the surface of revolution is:

$$\mathrm{S}(k) \ = 2\pi \int_{0}^{\frac{5\pi}{3}} (\sin x - k)\sqrt{(\cos x)^2 + 1}\, dx$$

Expanding the integral gives:

$$\mathrm{S}(k) = 2\pi \int_{0}^{\frac{5\pi}{3}} (\sin x)\sqrt{\cos^2 x + 1}\, dx - 2\pi \int_{0}^{\frac{5\pi}{3}} k\sqrt{\cos^2 x + 1}\, dx$$

In the second integral, the integration is relative to x, so k (which has no dependence on x) can be moved out of the integral:

$$\mathrm{S}(k) = 2\pi \int_{0}^{\frac{5\pi}{3}} \sin x\sqrt{\cos^2 x + 1}\, dx - 2\pi k \int_{0}^{\frac{5\pi}{3}} \sqrt{\cos^2 x + 1}\, dx$$

Solving the integrals using WolframAlpha gives:

$$\mathrm{S}(k) = 2\pi(0.6277) - 2\pi k(6.27575)$$

$S(k)$ can therefore be written as a linear function:

$$\mathrm{S}(k) = 3.943955 - 39.4317k$$

When all coefficients are rounded to one decimal place, the same equation as the of the regression line found previously for $k \in \{-3, -2, -1\}$ is obtained:

$$\mathrm{S}(k) = 3.9 - 39.4k$$

The minimum value of $\mathrm{S}(k)$, which is a decreasing function, is $\mathrm{S}(-1) = 43.7$ (1 d.p.)

Case 2. When $k \geq 1$, $S(k) = \sin x - k$ is completely below the x-axis, the opposite of the general surface area equation must then be used to calculate the area of the surface of revolution:

$$\mathrm{S}(k) = -2\pi \int_{0}^{\frac{5\pi}{3}} (\sin x - k)\sqrt{(\cos x)^2 + 1}\, dx$$

This consequently leads to the following expression for $S(k)$:

$$\mathrm{S}(k) = -3.9 + 39.4k$$

The minimum value in this case (when $\mathrm{S}(k)$ is an increasing function) is $S(1) = 35.5$ (1 d.p.), which is less than the minimum value found for $k \leq -1$.

Case 3. When $k \in (-1,1)$, a non-linear behavior is observed, as was shown in *Figure 4* . Given that this is a general equation and the domains where when $f(x) - k$ is positive or negative are unknown, it shall be left in the form:

$$\mathrm{S}(k) = 2\pi \int_{a}^{b} |\sin x - k|\sqrt{\cos^2 x + 1}\, dx.$$

The trend of the graph when $k \in (-1,1)$, and the values shown in *Table 1*, suggest that there is a value of $S(k)$ which is less than 35.5. However, it needs to be rigorously proven that a minimum to the function S can be found in this domain. This will be done in the next section, by providing a general proof which will show that there will always be a single minimum when $k \in (f_{min}, f_{max})$.

3.2 Proof of Unimodality

When searching for a minimum, any optimization algorithm will search for a location where $S'(k) = 0$. However, the minimum that is found may be a local minimum instead of a global minimum. Therefore, to ensure that only the global minimum is ever found, the function must have only one minimum. This type of function is known as a unimodal function, and has the feature that the second derivative is greater than 0 on its entire domain. Therefore, for any function f it must be shown that the function $S(k) = f(x) - k$ is unimodal, which can be done by proving that $S''(k) > 0$ for all $k \in \mathbb{R}$.

Theorem: *Let $S(k)$ be the area of the surface produced by rotating the region created by the graph of a continuous function $f(x)$, defined over the interval $[a, b]$, where the axis of rotation is $y = k$. For all $f(x)$, $S''(k) > 0$ for all values of k.*

Proof:

It can be assumed that the line $y = k$ intersect $f(x)$ finitely many times. These points of intersection shall be denoted (c_i, k) were $i = 1, 2, \dots, n - 1$ and $c_{i+1} > c_i$. Any points of intersection where $f'(c_i) = 0$ can be ignored as the points of intersection are there to represent a change in sign for $f(x) - k$, so that the domain of $S(k)$ can be subdivided into domains where S is positive and negative. Let $c_0 = a$ and $c_n = b$.

Assuming that $f(x) - k$ begins by being positive and then alternates sign for each successive interval, the expression for surface area can be written using sigma notation:

$$S(k) = 2\pi \sum_{i=0}^{n} (-1)^i \int_{c_i}^{c_{i+1}} (f(x) - k)\sqrt{(f'(x)^2 + 1)}\ dx$$

To find the second derivative of this $S(k)$, consider the integrals defining the surface area before and after a random point of intersection of the line $y = k$ and the graph of f, denoted c_i $(c_i = f^{-1}(k))$. Assuming that $S(k) = f(x) - k$ is positive for $x \in [c_{i-1}, c_i]$ and negative in $x \in [c_i, c_{i+1}]$, the surface area given by these two integrals would be:

$$S(k) = 2\pi \int_{c_i - 1}^{c_i} (f(x) - k)\sqrt{(f'(x)^2 + 1)}\ dx - 2\pi \int_{c_i}^{c_{i+1}} (f(x) - k)\sqrt{(f'(x)^2 + 1)}\ dx$$

This equation can then be expanded:

$$\begin{aligned} S(k) = 2\pi \int_{c_{i-1}}^{c_i} f(x)\sqrt{(f'(x)^2 + 1)}\ dx - 2\pi \int_{c_{i-1}}^{c_i} k\sqrt{(f'(x)^2 + 1)}\ dx \\ -2\pi \int_{c_i}^{c_{i+1}} f(x)\sqrt{(f'(x)^2 + 1)}\ dx + 2\pi \int_{c_i}^{c_{i+1}} k\sqrt{(f'(x)^2 + 1)}\ dx \end{aligned}$$

In the second and fourth integrals, k can be taken outside the integral as the integral is in terms of x:

$$\begin{aligned} S(k) = 2\pi \int_{c_{i-1}}^{c_i} f(x)\sqrt{(f'(x)^2 + 1)}\ dx - 2\pi k \int_{c_{i-1}}^{c_i} \sqrt{(f'(x)^2 + 1)}\ dx \\ -2\pi \int_{c_i}^{c_{i+1}} f(x)\sqrt{(f'(x)^2 + 1)}\ dx + 2\pi k \int_{c_i}^{c_{i+1}} \sqrt{(f'(x)^2 + 1)}\ dx \end{aligned}$$

In this equation, when differentiating both sides with respect to k, any terms without k can be considered constants:

$$S'(k) = 2\pi k \int_{c_i}^{c_{i+1}} \sqrt{(f'(x)^2 + 1)}\ dx - 2\pi k \int_{c_{i-1}}^{c_i} \sqrt{(f'(x)^2 + 1)}\ dx$$

In order to momentarily simplify this equation, the following substitution is made: $q'(x) = \sqrt{f'(x)^2 + 1}$. It follows that:

$$S'(k) = 2\pi\left(\int_{c_i}^{c_{i+1}} q'(x)\ dx - \int_{c_{i-1}}^{c_i} q'(x)\ dx\right)$$

$$S'(k) = 2\pi\big(q(c_{i+1}) - q(c_i) - q(c_i) + q(c_{i-1})\big)$$

Differentiating again with respect to k leads to:

$$S''(k) = \frac{d}{dk}\left[2\pi\big(q(c_{i+1}) - 2q(c_i) + q(c_{i-1})\big)\right]$$

This equation is focused on $c_i = f^{-1}(k)$, so c_{i+1} and c_{i-1} can simply be considered bounds to the interval without any dependence on k. For that reason, when differentiating with respect to k, $q(c_{i+1})$ and $q(c_{i-1})$ can be considered constants, while $q(c_i)$ is not, since $c_i = f^{-1}(k)$. Hence $\frac{d}{dk}\big(q(c_{i+1})\big) = 0$ and $\frac{d}{dk}\big(q(c_{i-1})\big) = 0$, so it follows that:

$$S''(k) = 2\pi\left(-2\frac{d}{dk}\big(q(c_i)\big)\right)$$

Using the chain rule to expand:

$$\begin{aligned} S''(k) &= -4\pi\frac{d}{dk}\Big(q\big(f^{-1}(k)\big)\Big) \\ &= -4\pi \cdot q'\big(f^{-1}(k)\big) \cdot \big((f^{-1})'(k)\big) \end{aligned}$$

The expression $(f^{-1})'(k)$ can be simplified to (see Appendix 1):

$$(f^{-1})'(k) = \frac{1}{f'\big(f^{-1}(k)\big)}$$

Substituting this back into the equation for surface area yields:

$$S''(k) \;=\; -4\pi\frac{q'\big(f^{-1}(k)\big)}{f'\big(f^{-1}(k)\big)}$$

Now c_i can be substituted into the equation for $f^{-1}(k)$:

$$S''(k) \;=\; -4\pi\frac{q'(c_i)}{f'(c_i)}$$

At this point, $\sqrt{f'(x)^2+1}$ can be substituted back for $q'(x)$:

$$S''(k) = \frac{-4\pi\sqrt{(f'(c_i)^2+1)}}{f'(c_i)}$$

Given that $S(k) = f(x) - k$ is positive in $x \in [c_{i-1}, c_i]$, it can be assumed that S(k) is decreasing at c_i. Therefore $f(c_i) < 0$, which implies that $S''(k) > 0$ around this random c_i. The same argument with opposite signs could be made with $f'(c_i) > 0$ showing that $S''(k) > 0$. This same logic can be applied to every term in the sum:

$$\frac{S''(k)}{2\pi} = \sum_{i=1}^{n/2} \frac{-2\sqrt{(f'(c_{2i})^2+1)}}{f'(c_{2i})}$$

This means that the entire sum is positive and that $S''(k) > 0$ (Thompson 83).

4. Methods of Optimization

Now that it has been shown that $S(k)$ is unimodal, and the existence of a minimum point is ensured, the minimum value of $S(k)$ must be calculated. This has to be done numerically, as the integrals and derivatives that must be calculated are too complex to solve analytically. There are many numerical methods of searching for a minimum for a function. Regardless of the method, a program must run a certain amount of times to find the minimum, and it must calculate a certain number of values each time. These are known as the number of iterations and evaluations. A program is most "efficient" when it has the lowest number of iterations and evaluations. This section will be comparing methods for finding the minimum of the $S(k)$ function by considering which is the most efficient.

4.1 Golden Section Search

The goal of any search method is to reduce the size of the interval where the minimum could lie, or the "uncertainty" interval. The golden section search places value in reducing the number of times a function must be evaluated using previously evaluated points, thereby increasing the efficiency of the algorithm.

This application of this search method for a function over the interval $[a_0, b_0]$ can be seen in *Figure 5*. This interval is divided into three sections by evaluating two new points, a_1 and b_1.

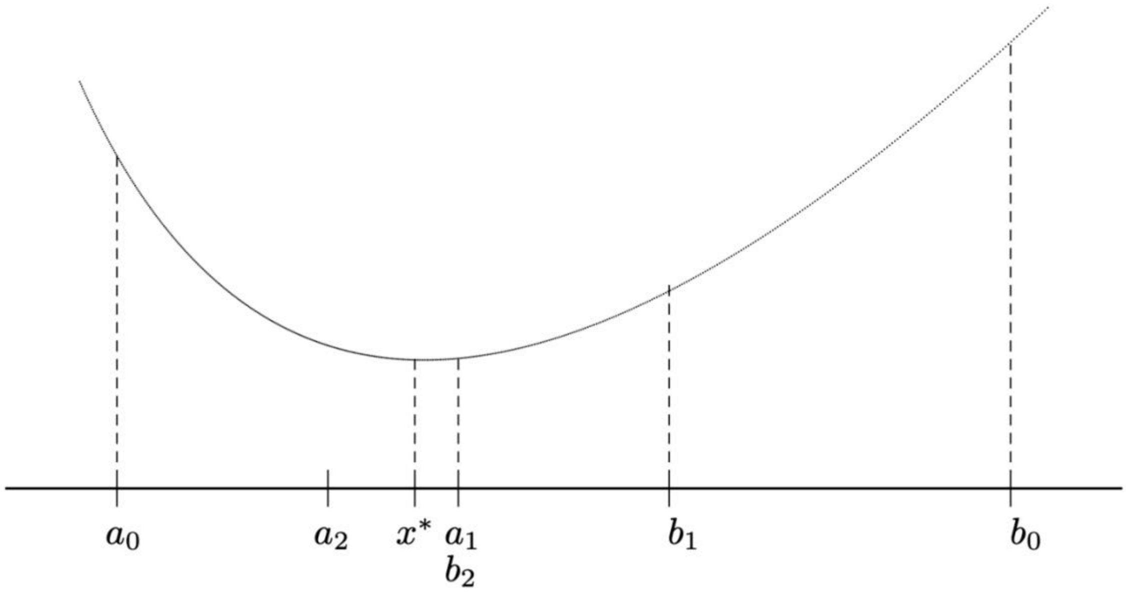

Figure 5: Golden Search Method, *(Butenko and Pardalos 323)*

These points are chosen such that:

$a_1 - a_0 = b_0 - b_1 = \rho(b_0 - a_0)$, where $\rho < \frac{1}{2}$.

If $a_1 < b_1$ then it can be said that the minimum is on the interval $[a_0, b_1]$. The reverse argument can be applied to $a_1 > b_1$. Now in the new interval $[a_0, b_1]$ the interval must again be subdivided into three parts by evaluating two points in the new interval. Given that a_1 is already in the interval $[a_0, b_1]$, it can be reused, renaming it b_2, as in *Figure 5*. Now only one new point must be evaluated, which will be called a_2.

The points in the new interval $[a_0, b_1]$ must be in the same ratio as the points in the original interval. Using this knowledge, a value for ρ can be calculated. First, a few terms must be defined. Let $l = b_0 - a_0$ be the original length of the interval and let $d = b_1 - a_0 = b_0 - a_1$ be the length of the new interval. The ratio of the new interval to the original interval can be described as:

$$d = (1 - \rho)l \quad \text{or} \quad l = \frac{d}{1-\rho} \tag{1}$$

The interval $[a_0, a_1]$ has a length of ρl. When considering the new interval, the same interval now known as $[a_0, b_2]$ has a length of $(1 - \rho)d$, giving the equation:

$$\rho l = (1 - \rho)d \tag{2}$$

Substituting equation (1) into (2), the following is obtained:

$$\rho \frac{d}{1-\rho} = (1 - \rho)d$$

$$\Rightarrow 0 = \rho^2 - 3\rho + 1 \tag{3}$$

The solution to equation (3) which is less than 1 is: $\rho \approx 0.382$ This method of eliminating ρ of the interval is repeated over and over again, which is how the method operates. If the method is repeated n times, then the resulting size of the uncertainty interval would be step $l(1 - \rho)^n$. If a certain uncertainty (Δx) is necessary, the number of steps can be solved for by using the inequality $l(1 - \rho)^n \leq \Delta x$ (Butenko and Pardalos 323-325).

4.2 Bisection Method

The bisection method is a method for finding the root of a function. The surface area function $S(k)$ is always positive, thus has no roots, however, $S'(k)$ must have a root as Section 3 proved that $S(k)$ has a minimum, so $S'(k) = 0$. In essence, the method involves repeatedly dividing the interval where the root could lie (the uncertainty interval) in half, and eliminating one of those halves. Consider *Figure 6*, where a minimum for $F(x)$ exists on the interval $[a_1, b_1]$.

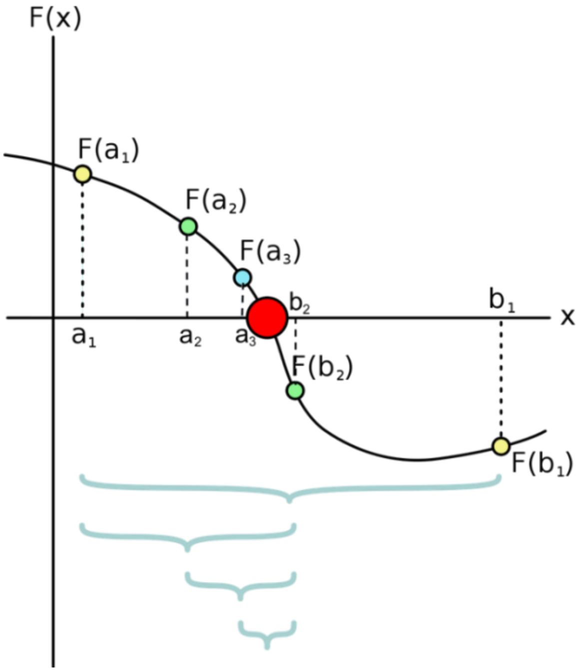

Figure 6: Bisection Method applied to $F(x)$, *(Tokuchan)*

Firstly, the midpoint of the interval is calculated to be b_2. If $F(b_2)$ has the same sign as $F(b_2)$, then it can be concluded that the minimum is on the interval $[a_1, b_2]$. This scenario can be seen in *Figure 6*, with the blue lines showing how the uncertainty interval has been reduced to $[a_1, b_2]$.

Similarly, if $F(b_2)$ were to have the same sign as $F(a_1)$ then the uncertainty interval would be $[a_1, b_2]$ instead. Subsequently, in *Figure 6* it can be seen that the midpoint of the new interval is found to be a_2. The steps taken to eliminate half of this interval are repeated, and this process continues until the value of the function at the calculated root is sufficiently close to zero. Similar to the golden section search, since the interval is cut down by half each time, the number of steps required can be represented by the inequality $l(0.5)^n \leq \Delta x$, where Δx is the required uncertainty, l length of the interval, and n is the number of steps.

4.3 Newton's Method

Any point on a function $f(x)$ can be approximated almost perfectly using a Taylor series at a point $x = x_0$ (the more terms are considered, the better the approximation):

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2} + \frac{f'''(x_0)(x - x_0)^3}{6} + \cdots$$

However, for simplicity, the function can be approximated by using only a second order Taylor polynomial:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2}$$

Differentiating this with respect to x gives:

$$f'(x) = f'(x_0) + f''(x_0)(x - x_0)$$

The goal of this equation is to find an approximation for the root of the function, which will be denoted x_1. Since this x_1 is a minimum, $f'(x_1) = 0$. Therefore, the following equation is derived to calculate the approximation x_1.

$$f'(x_1) = f'(x_0) + f''(x_0)(x_1 - x_0)$$

$$\Rightarrow 0 = f'(x_0) + f''(x_0)(x_1 - x_0)$$

Rearranging this to solve for x_1 gives:

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)}$$

This equation has resulted in an approximation of the *x*-value of the minimum. By applying this equation with x_1 as the starting value, an approximation x_2 can be derived, which would be even closer to the minimum. Repeating this step over and over, will result in a series of x-values, x_n, that can be defined by the formula:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

These iterations can also be demonstrated graphically. Consider *Figure 7*, with the blue line representing $f(x)$.

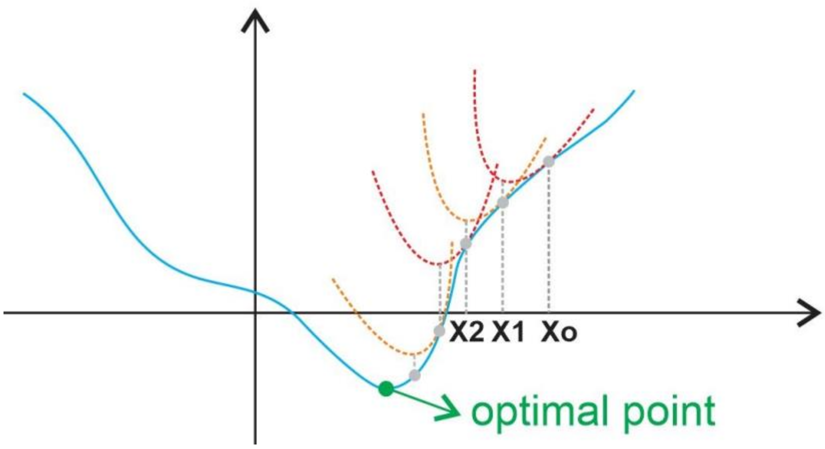

Figure 7: Newton's method showing quadratic approximations for graph of f(x), ***(Umam)***

The first point that is evaluated is x_0. At this point, the quadratic approximation (the dotted red line) is defined by the second-order Taylor polynomial. The minimumof the approximation, the point where $f'(x) = 0$, is recorded as x_1. The quadratic approximation at x_1 is then taken, giving a new x-value. This process is repeated until the value of $f'(x_n)$ is sufficiently close to zero. In the image it can be seen how this series of x-values approaches the real minimum of the function.

4.4 An Example

This section will give a short example of how the number of iterations for these methods can be calculated for a specific function $f(x) = \sin(x)$ where $x \in \left[0, \frac{5\pi}{3}\right]$.

The function representing the area of the surface of revolution is:

$$S(k) = 2\pi \int_0^c (\sin(x) - k)\sqrt{(cos^2(x) + 1)}\ dx - \int_c^{\frac{5\pi}{3}} (\sin(x) - k)\sqrt{(cos^2(x) + 1)}\ dx$$

The goal of the methods will be to achieve an uncertainty of 1×10^{-3}, and the number of necessary iterations will be calculated.

Using the Golden Section Search, the number of iterations is calculated by $l(1-\rho)^n \leq \Delta x$, as mentioned before. For this specific function and interval, $l = f_{max} - f_{min} = 2$, $\rho = 0.382$, and $\Delta x = 1 \times 10^{-3}$. The inequality is therefore:

$$2(0.618)^n \leq 10^{-3}$$

$$\Rightarrow (0.618)^n \leq 5 \times 10^{-4}$$

Taking the natural logarithm of both sides:

$$n\ln(0.618) \leq \ln(5 \times 10^{-4})$$

$$\Rightarrow n \geq \frac{\ln(5 \times 10^{-4})}{ln(0.618)}$$

$$\Rightarrow n \geq 15.794 \text{ (3 d.p.)}$$

Therefore, the minimum number of iterations required using the Golden Section Search is 16.

Using the bisection method, the number of iterations is calculating using a similar inequality:

$$l(0.5)^n \leq \Delta x$$

$$\Rightarrow 2(0.5)^n \leq 10^{-3}$$

$$\Rightarrow (0.5)^n \leq 5 \times 10^{-4}$$

Taking the natural logarithm of both sides:

$$n\ln(0.5) \leq \ln((5 \times 10^{-4})$$

$$\Rightarrow n \geq \frac{\ln(5 \times 10^{-4})}{\ln(0.5)}$$

$$\Rightarrow n \geq 10.966 \text{ (3 d.p.)}$$

Therefore, the number of iterations required using the bisection method is 11.

Using the Newton-Raphson method, a starting point must be considered. Given that

$k \in [f_{min}, f_{max}]$, this point will be arbitrarily chosen as $k = 0.5$. The values for the first derivative at this point are extremely difficult to calculate analytically, so they will be calculated numerically using SciPy modules for integration and differentiation.

At $k = 0.5$, $S'(0.5) = 9.623$, and $S''(0.5) = 38.444$.

Therefore, the new approximation for the minimum is $0.5 - \frac{-9.623}{38.444} = 0.250$.

Now evaluating at $k = 0.250$, $S'(0.250) = 0.348$, and $S''(0.250) = 36.150$.

The new approximation is $0.250 - \frac{0.348}{36.150} = 0.240$. When the first derivative is evaluated at this point is gives $S'(0.240) = 0.0006$. This number is smaller than the required uncertainty, so the approximation of the root is at $k = 0.240$, and $S(k) = 23.872$ (3 d.p.). With this function, the number of iterations was 2, a relatively small number compared to the other two methods.

Using the methods from above, I wrote a program to find the minimum surface area $S(k)$ and the line over which it is rotated $y = k$, using Python. The program recorded the number of iterations to get an uncertainty of 10^{-3}. I then wrote a program to graph these minimized surfaces, an example of which can be seen below. Further examples can be seen in Appendix 2.

Example: $f(x) = \sin(\mathrm{x}), x \in \left[0, \frac{5\pi}{3}\right]$

Method	Equation of rotation axis	$\boldsymbol{S(k)}$	Number of Iterations
Bisection	$\boldsymbol{y = 0.240}$	23.872	10
Golden Search	$\boldsymbol{y = 0.240}$	23.872	16
Newton-Raphson	$\boldsymbol{y = 0.240}$	23.872	2-5

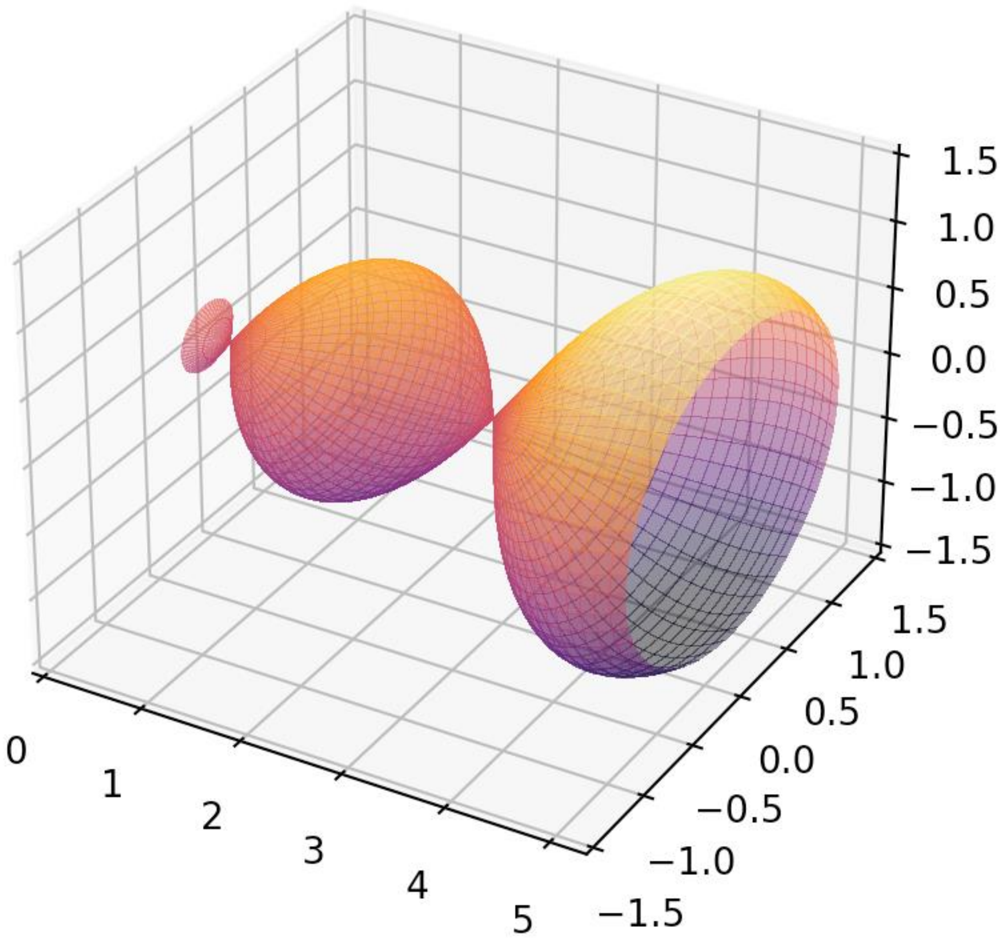

Figure 8: A Graphical Representation of Rotated Region of $\boldsymbol{f(x) = sin(x) - 0.240}$ (Author's Own)

4.5 Comparing Methods

The error (e) in each of the iterations of the optimization program can be written as $e = |k_n - k^*|$, where k^* is the minimum value for $S(k)$ and k_n is the approximation given by the n^{th} iteration. With each iteration, the goal is to decrease the error. This relationship between successive errors can be written as $e_{n+1} = \mu {e_n}^{\alpha}$, where $\mu \in (0,1]$ since the error is decreasing. In this equation, μ is known as the rate of convergence and α is known as the order of convergence *("What is Order")*. With this in mind a few things can be concluded. If α is equal to 1 (linear):

$$\lim_{n\to+\infty} \frac{e_{n+1}}{e_n} = \lim_{n\to+\infty} \frac{\mu {e_n}^{\alpha}}{e_n} = \mu$$

Similarly, if α is greater than 1 (superlinear):

$$\lim_{n\to+\infty} \frac{e_{n+1}}{e_n} = \lim_{n\to+\infty} \frac{\mu {e_n}^{\alpha}}{e_n} = \lim_{n\to+\infty} {e_n}^{\alpha-1} = 0$$

With the bisection method and golden section search method the error is the size of the interval. For bisection this size is halved for each iteration, therefore:

$$\lim_{n\to+\infty} \frac{e_{n+1}}{e_n} = \lim_{n\to+\infty} \frac{0.5e_n}{e_n} = 0.5$$

This means that the rate of convergence is 0.5. Similarly for the golden search, the rate of convergence is 0.618. This shows that the rate of convergence is faster using the bisection method. The one disadvantage of the bisection method is that it requires the evaluation of the derivative of $S(k)$, while the golden search only requires $S(k)$, but this factor is negligible due to the low number of iterations.

The Newton-Raphson method, on the other hand, has an order of convergence of 2. It is clear that with $e_{n+1} = \mu {e_n}^2$, the error would decrease much faster if $e_n < 1$. However, consider what would happen if the error began incredibly large. The error would not converge quickly. When an interval was increased to over 1000 units large, and the starting point k_0 was the midpoint of the large interval $[f_{min}, f_{max}]$, the program takes over 1000 iterations to converge on the minimum. For this reason, Newton's method is unreliable for situations when k_0 is far away from the root.

5. Conclusion

In this essay, it has been shown that for any function f that reaches a minimum value, f_{min}, and a maximum one, f_{max}, on an interval $[a, b]$, three different cases were considered:

Case 1. When $k \leq f_{min}$, $f(x) - k$ is completely above the x-axis, so the general surface area equation to calculate the area of the surface of revolution.

$$S(k) = 2\pi \int_{a}^{b} (f(x) - k)\sqrt{(f'(x)^2 + 1)}\, dx$$

Case 2. When $k \geq f_{max}$, the graph of $S(k) = f(x) - k$ is completely below the x-axis, so the opposite of the general surface area equation must be considered to calculate the area of the surface of revolution.

Case 3. When $k \in (f_{min}, f_{max})$, the graph of $S(k) = f(x) - k$ will be both above and below the x-axis.

$$\mathrm{S}(k) = 2\pi \int_{a}^{b} |f(x) - k|\sqrt{(f'(x)^2 + 1)}\, dx$$

By considering $f(x) = \sin(x)$ and looking at the behaviour of the surface area function, the trend of a global minimum for $k \in (f_{min}, f_{max})$ was observed and its existence was subsequently proved by showing that $S(k)$ is unimodal for all functions f.

To numerically find this global minimum, several methods were discussed: Bisection, Golden Search, and Newton-Raphson. When applying these methods, it became clear Newton's method was the fastest and most efficient of the methods due to it being quadratically convergent, however, when k_0 is far from the minimum the method converges slower than the bisection method. For this reason, a mixed method could be used. Bisection could be utilized until the interval has shrunk to a certain size. After this point, the faster-converging Newton's method could be used. This option allows for fast convergence while accounting for the limitations of Newton's method.

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7. Appendices

Appendix 1: Derivative of an Inverse Function

The expression $\left(f^{-1}(k)\right)'$ can be simplified through a short proof.

$$f\left(f^{-1}(k)\right) = k$$

Differentiating both sides with respect to k:

$$\frac{df\left(f^{-1}(k)\right)}{dk} = 1$$

Applying chain rule to this equation gives:

$$f'\left(f^{-1}(k)\right) \cdot (f^{-1})'^{(k)} = 1$$

Rearranging this for $(f^{-1})'(k)$ results in:

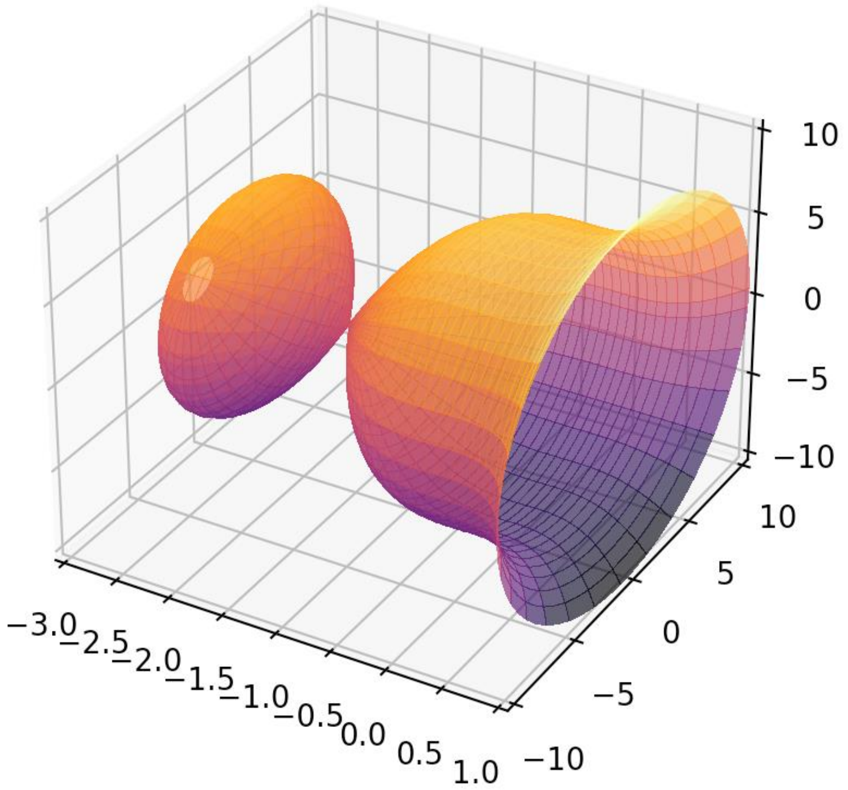
$$(f^{-1})'^{(k)} = \frac{1}{f'\left(f^{-1}(k)\right)}$$

Appendix 2: Examples of Minimized Surfaces

Example 1: $f(x) = x^4 + 3x^3 - x^2 + 4$, $x \in [-3,1]$.

Method	Equation of rotation axis	$S(k)$	Number of Iterations
Bisection	y =-3.786	653.142	24
Golden Search	y =-3.786	653.142	35
Newton-Raphson	y =-3.786	653.142	3-12

A graphical representation of the minimized surface is seen below:

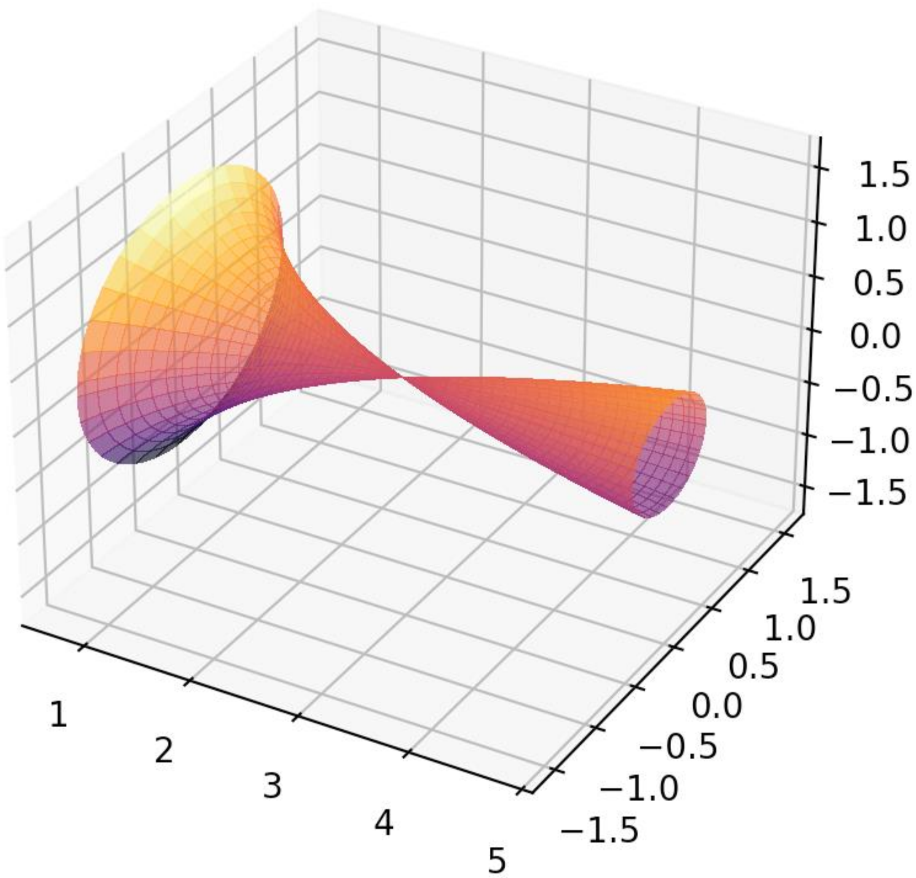


(Author's Own)

Example 2: $f(x) = \log_4(x)$, $x \in [0.5,5]$

Method	**Equation of rotation axis**	$\boldsymbol{S(k)}$	**Number of Iterations**
Bisection	$\boldsymbol{y = 0.687}$	11.429	26
Golden Search	$\boldsymbol{y = 0.687}$	11.429	31
Newton-Raphson	$\boldsymbol{y = 0.687}$	11.429	2-12

A graphical representation of the minimized surface is seen below:

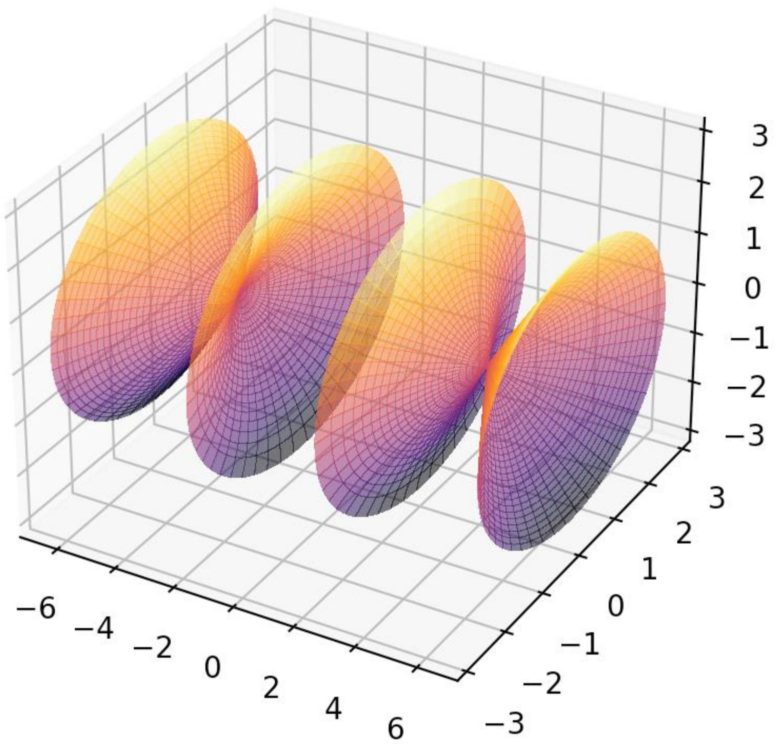


(Author’s Own)

Example 3: $f(x) = \sqrt{-3 + \frac{7x^2}{10}}$, $x \in [-4.7, -2.07] \cup [2.07, 4.7]$, the results found are displayed in the table below.

Method	**Equation of rotation axis**	$\boldsymbol{S(k)}$	**Number of Iterations**
Bisection	$\boldsymbol{y = 2.585}$	44.610	12
Golden Search	$\boldsymbol{y = 2.585}$	44.610	32
Newton-Raphson	$\boldsymbol{y = 2.585}$	44.610	2-4

A graphical representation of the minimized surface is seen below:



(Author's Own)

Appendix 3: 3-Dimensional Graphing Program

```
import numpy as np
import matplotlib.pyplot as plt
import mpl_toolkits.mplot3d.axes3d as axes3d
import math
from matplotlib import cm
from matplotlib.ticker import LinearLocator, FormatStrFormatter
fig = plt.figure()
ax = fig.add_subplot(1, 1, 1, projection='3d')

u = np.linspace(-7, -2.1, 1000) #replace first two terms with the interval

def f(x):
	return np.sqrt(-3+(7*(x**2))/10) - 3.048 #replace this line with appropriate function

v = np.linspace(0, 2*np.pi, 1000)
U, V = np.meshgrid(u, v)
R, V = np.meshgrid(r, v)

X1 = U
Y1 = f(X1)*np.cos(V)
Z1 = f(X1)*np.sin(V)

ax.plot_surface(X1, Y1, Z1, alpha=0.6, cmap=cm.inferno,linewidth=0, antialiased=False)

# set the bounds of the axes
ax.set_xlim3d(-7, 7)
ax.set_ylim3d(-3.1, 3.1)
ax.set_zlim3d(-3.1, 3.1)

plt.show()
```