Modeling a cooling cup of hot chocolate

Introduction

Cooling a hot drink is something everybody does at some point in their life. People often use methods such as blowing on the drink, stirring it, putting ice cubes in the shrink, or even putting it in the fridge. However, people are often unaware of how exactly a drink cools, and that is exactly what this exploration will be focused on.

I was personally inspired to do investigate the phenomenon of a cooling drink because of my time as a Boy Scout in Russia. Every year we went on a Winter Campout where I would drink the classic campout drink, hot chocolate. Over time, this drink would cool due to the sub-zero temperatures, which prompted me to think how about how much time I had before this drink would be too cold to drink. In this essay, the goal will be to develop an accurate model for calculating the temperature (T) of a cooling drink at any time (t).

The cooling of a liquid is typically calculated on Newton's law of cooling, which states that "the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature" (). This can be written as an equation:

$$\frac{dT}{dt} = k(T - T_a)$$

In this equation, T is the temperature of the liquid as a function of time (t), and T_a is the ambient temperature. k is a constant that is dependent a certain heat transfer coefficient and the surface area where heat can be transferred. This equation is a first-order differential equation and can be solves as follows.

Model 1: Liquid surrounded by air

In order to model the temperature, $T(t)$ must be defined in terms of t. The first step in doing this is rearranging Newton's law of Cooling:

$$\frac{dT}{T - T_a} = kdt$$

Both sides of the equation can be integrated to get:

$$\int \frac{1}{T - T_a} dT = \int k\, dt$$

$$\Rightarrow \int \frac{1}{T - T_a} dT = kt + c$$

Making the substitution $u = T - T_a$, the left hand side can be written as:

$$\int \frac{1}{u} du = kt + c$$

$$\Rightarrow ln|u| = kt + c$$

Resubstituting for u gives:

$$ln|T - T_a| = kt + c$$

$$\Rightarrow |T - T_a| = e^{kt+c}$$

$$\Rightarrow |T - T_a| = e^{kt}e^{c}$$

In the right-hand side, e^c is an arbitrary constant, so it can be written as C.

$$|T - T_a| = Ce^{kt}$$

Assuming that this equation is for a "cooling" drink, $T \geq T_a$, therefore, $|T - T_a| = T - T_a$.

$$T - T_a = Ce^{kt}$$

$$T(t) = T_a + Ce^{kt}$$

In this equation, C can be solved for when $t = 0$

$$T(0) = T_a + Ce^{k(0)}$$

$$T(0) = T_a + C(1)$$

$$C = T(0) - T_a$$

The initial temperature $T(0)$ will be written in the form T_0, since it is technically a constant. Therefore, the function for temperature can be written as:

$$T(t) = T_a + (T_0 - T_a)e^{kt}$$

One of the limitations of Model 1, is that is assumes that the water is only in contact with one surface, the air. In reality, the cooling liquid is often surrounded by an intermediate layer, the cup.

Model 2: Liquid in a container

To model the cooling of the liquid now, there are 2 temperature losses that must be considered. The first is the water losing heat to the air, the second is the water losing heat to the cup. The following notations will be used to model this relationship.

$$Temperature\ \ of\ \ Water = T_w(t)$$

$$Temperature\ \ of\ \ Cup = T_c$$

$$Initial\ \ Temperature\ \ of\ \ Water = T_{w,0}(t)$$

$$Initial\ \ Temperature\ \ of\ \ Cup = T_{c,0}$$

There are also two constants k, which are used to describe this transfer of heat. The transfer of heat between the water and the air will be called $k_{w,a}$ and the transfer of heat between the water and the cup will be called $k_{w,c}$.

Going back to Newton's law of cooling, the law was originally written as:

$$\frac{dT}{dt} = k(T - T_a)$$

However, now the liquid is losing heat to both the cup and the air in different proportions. The rate of temperature loss must then be written as:

$$\frac{dT}{dt} = k_{w,c}(T - T_c) + k_{w,a}(T - T_a)$$

It can be assumed that before the liquid is poured into the cup, the cup is in thermal equilibrium with the air. Therefore, the temperature of the cup would be the same as the temperature of the air.

$$\frac{dT}{dt} = k_{w,c}(T - T_a) + k_{w,a}(T - T_a)$$

$$\frac{dT}{dt} = (T - T_a)\left(k_{w,c} + k_{w,a}\right)$$

Rearranging this equation gives:

$$\frac{dT}{T - T_a} = \left(k_{w,c} + k_{w,a}\right)dt$$

Following the same steps as before gives the function for temperature:

$$T(t) = T_a + (T_0 - T_a)e^{\left(k_{w,c}+k_{w,a}\right)t}$$

The equation for temperature was then compared to a data set by Laura Lowe from the University of Georgia. The experiment was done by using a digital thermometer to measure the temperature of hot water in a ceramic cup every minute for 30 minutes. The starting temperature (T_0) of the water was 90.6℃ and the ambient temperature was 22.2℃. The Desmos graphing software was used to find the regression line for this data with the equation:

$$T(t) = 22.2 + (90.6 - 22.2)e^{(k_1+k_2)t}$$

The graph of this regression line compared to the data can be seen below.

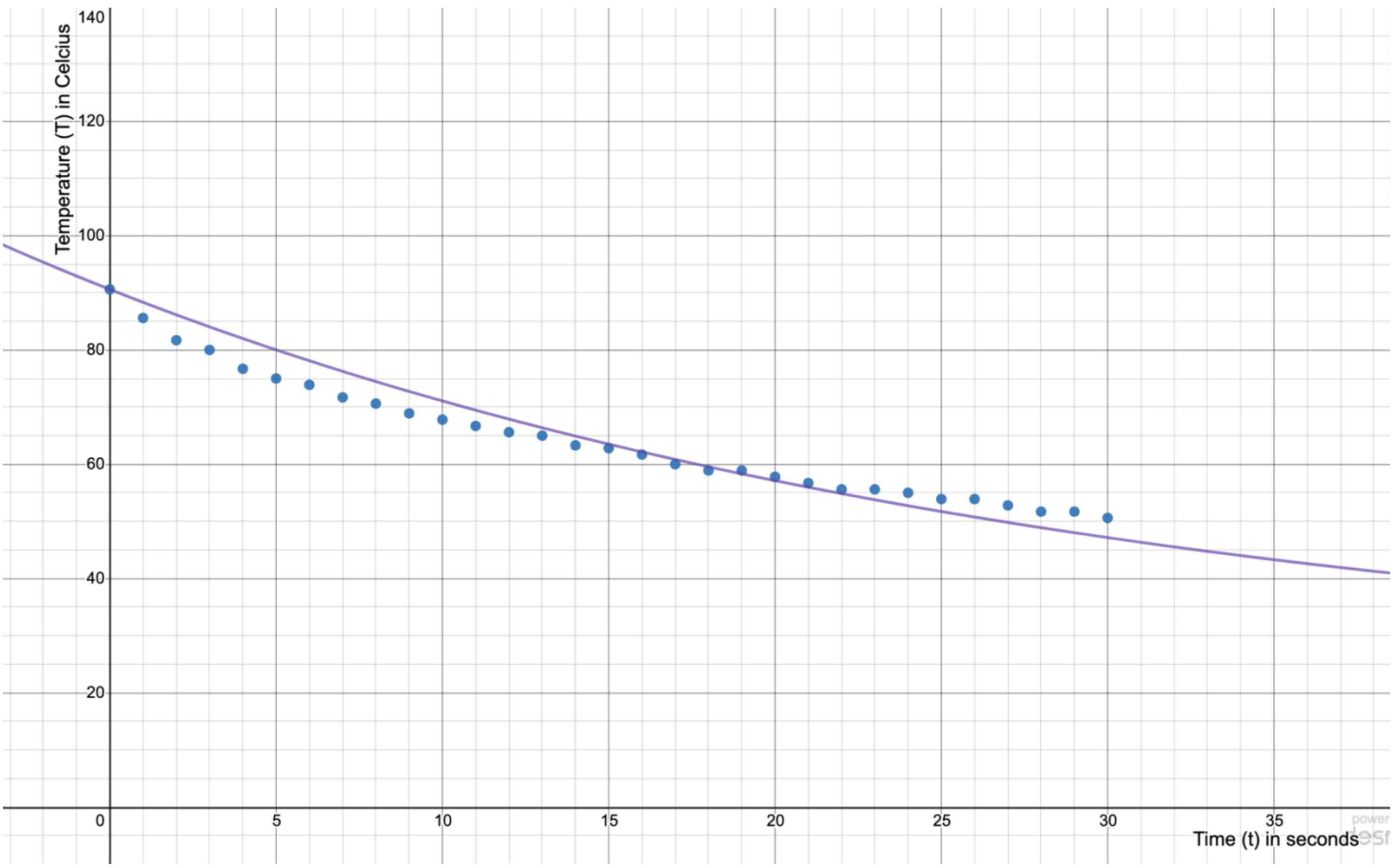

Figure 1: Graph showing Regression Line of Model 2 versus Data for a Cooling Cup of Hot Water

The regression line with the equation from Model 2 had an R^2 value of 0.9243, which is a relatively good correlation. However, the correlation can be improved by acknowledging on of the limitations of Model 2, The assumption that the temperature of the cup would be constant.

Model 3: Liquid in a heating container

In order to calculate the temperature of the container, it is not so simple as to add the temperature lost from the water. The temperature lost from the water represents a certain amount of thermal energy lost. This thermal energy (Q) can be calculated using the equation:

$$Q = mc\Delta T,$$

where m is the mass of the water, c is the specific heat capacity of water, and ΔT is the change in temperature. Of course, m and c are constant, however, as the cup loses temperature its ΔT increases and naturally the thermal energy lost increases as well. The thermal energy lost by the water at the water-cup interface is equal to the thermal energy gained by the cup. The temperature change of the cup can therefore be calculated using the thermal energy equation above. The constant below will be used to calculate this temperature change,

$$Mass\ of\ water: 0.5kg$$

$$Mass\ of\ Cup: 0.32kg$$

$$Specific\ Heat\ Capacity\ of\ Water: 4200Jkg^{-1}C^{-1}$$

$$Specific\ Heat\ Capacity\ of\ Ceramic: 850Jkg^{-1}C^{-1}$$

Equating the energy loss and energy gain results in:

$$4200 * 0.5 * \left(T_{w,0} - Temperature\ of\ Water\right) = 850 * 0.32 * \left(T_c(t) - T_{c,0}\right)$$

Once again, the initial temperature of the cup can be considered the same as the ambient temperature. In this equation, it is important to notice that the temperature of the water should only represent the temperature change due to heat lost to the cup. This mean that the temperature of water is given by:

$$Temperature\ of\ water = T_a + (T_{w,0} - T_a)e^{k_{w,c}t}$$

Solving the equation for the temperature of the cup gives:

$$T_c(t) = \frac{4200 \times 0.5 \times \left(T_{w,0} - T_a - (T_{w,0} - T_a)e^{k_{w,c}t}\right)}{850\ \times\ 0.32} + T_a$$

$$T_c = 7.72 \times \left(T_{w,0} - T_a - (T_{w,0} - T_a)e^{k_{w,c}t}\right) + T_a \tag{1}$$

Now this can be used to find the function for the Temperature of the water. Model 2 was generated from the following equation:

$$\frac{dT}{dt} = k_{w,c}(T - T_c) + k_{w,a}(T - T_a)$$

Equation (1) can now be substituted into this equation:

$$\frac{dT}{dt} = k_{w,c}\left(T - \left[7.72 \times \left(T_{w,0} - T_a - (T_{w,0} - T_a)e^{k_{w,c}t}\right) + T_a\right]\right) + k_{w,a}(T - T_a)$$

This equation may seem very complex, however solving it becomes much simpler when grouping terms with T, t, and constants:

$$\frac{dT}{dt} = T(k_{w,c} - k_{w,a}) + 7.7.* (\mathrm{T_{w,0}} - \mathrm{T_a})\mathrm{e}^{\mathrm{k_{w,c}t}} + \left[7.72 \times \mathrm{k_{w,c}}(\mathrm{T_a} - \mathrm{T_{w,0}}) - \mathrm{k_{w,c}T_a} - \mathrm{k_{w,a}T_a}\right]$$

In order to solve this equation, it will be simplified further by replacing terms with constants a, b, c, and d such that:

$$\frac{dT}{dt} = aT + be^{ct} + d$$

It is now clear to see that this equation is a first-order differential equation. Now comes the process of solving the equation for T. Bringing aT to the left side gives:

$$\frac{dT}{dt} - aT = be^{ct} + d$$

A normal step in solving this type of differential equations is temporarily multiplying all terms by $e^{\int -a\,dt} = e^{-at}$, which results in:

$$e^{-at} \times \frac{dT}{dt} - ae^{-at} \times T = be^{-at} \times e^{ct} + de^{-at}$$

$$e^{-at} \times \frac{dT}{dt} - ae^{-at} \times T = be^{(c-a)t} + de^{-at}$$

Integrating both sides in terms of t, the following is obtained:

$$\int e^{-at} \times \frac{dT}{dt} - ae^{-at} \times T\,dt = \int be^{(c-a)t} + \int de^{-at}$$

The left-hand side will be focused on first. Looking closely at this side it can be observed that it is actually the product rule derivative of $e^{-at} \times T$.

$$\frac{d}{dt}[e^{-at} \times \mathrm{T}] = \frac{d}{dt}e^{-at} \times \mathrm{T} + e^{-at} \times \frac{d}{dt}\mathrm{T}$$

$$= e^{-at} \times \frac{dT}{dt} - ae^{-at} \times T$$

Therefore, the equation can be simplified to:

$$e^{-at} \times \mathrm{T} = \int be^{(c-a)t} + \int de^{-at}$$

Now focusing on the right-hand side, the integral are variations of the parent function e^x which can be solved for easily:

$$\mathrm{e}^{-\mathrm{at}} \times \mathrm{T} = \frac{\mathrm{be}^{(\mathrm{c}-\mathrm{a})\mathrm{t}}}{(\mathrm{c}-\mathrm{a})} - \frac{\mathrm{de}^{-\mathrm{at}}}{\mathrm{a}} + \mathrm{C}$$

Now T can be made the subject by dividing both sides by $\mathrm{e}^{-\mathrm{at}}$:

$$\mathrm{T} = \frac{\mathrm{be}^{\mathrm{ct}}}{\mathrm{c}-\mathrm{a}} - \frac{\mathrm{d}}{\mathrm{a}} + \mathrm{Ce}^{\mathrm{at}}$$

Of course, the constant C can be solved for by setting $t = 0$:

$$T_{w,0} = \frac{be^{c(0)}}{c-a} - \frac{d}{a} + Ce^{-a(0)}$$

$$T_{w,0} = \frac{b}{c-a} - \frac{d}{a} + C$$

$$C = T_{w,0} - \frac{b}{c-a} + \frac{d}{a}$$

This can be substituted into the equation to get:

$$\mathrm{T} = \frac{be^{ct}}{c-a} - \frac{d}{a} + (T_{w,0} - \frac{b}{c-a} + \frac{d}{a})e^{at}$$

Replacing all the coefficients and constant but leaving the k coefficients as parameters gives:

$$T(t) = \frac{\big(7.7 \cdot (90.6 - 22.2)\big)e^{(k_1)t}}{(k_1) - (k_1 - k_2)} - \frac{(7.72 \times k_1 \cdot (22.2 - 90.6) - k_1 \cdot 22.2 - 26 \cdot k_2)}{(k_1 - k_2)}$$
$$+ \left(90.6 - \frac{\big(7.7 \cdot (90.6 - 22.2)\big)}{k_1 - (k_1 - k_2)}\right.$$
$$\left. + \frac{(7.72 \times k_1 \cdot (22.2 - 90.6) - k_1 \cdot 22.2 - 26 \cdot k_2)}{(k_1 - k_2)}\right) e^{(k_1 - k_2)t}$$

Similar to Model 2, this equation was also plotted as a regression line against the data from before to give the graph below.

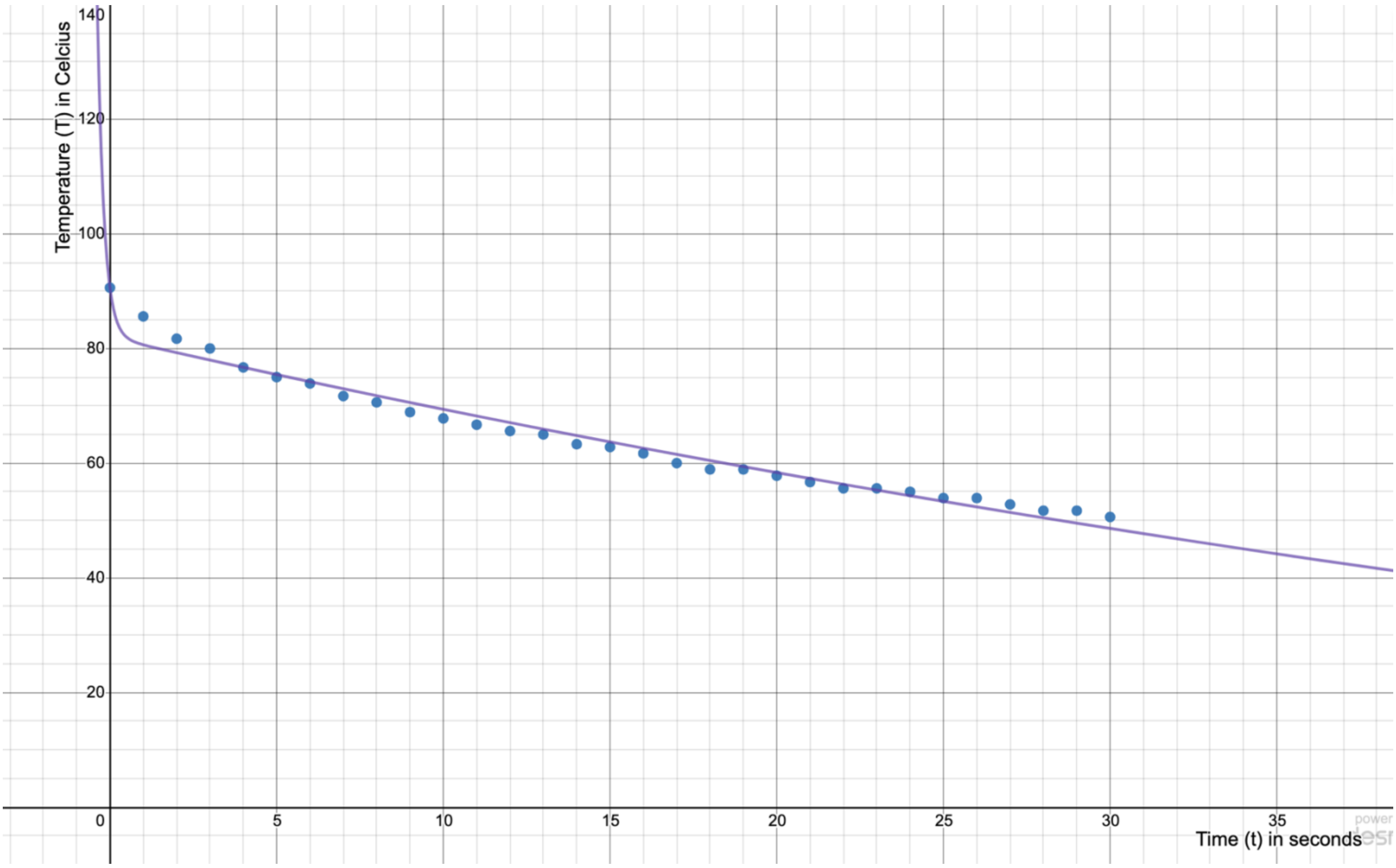

Figure 2: Graph showing Regression Line of Model 2 versus Data for a Cooling Cup of Hot Water

The regression line with the equation from Model 3 had an R^2 value of 0.9787, which is a better correlation than for the regression line of Model 2. This shows that by accounting for the increase in temperature of the container, the model can actually be made more accurate.

Limitations of the Models

Although the models have been shown a relatively accurate representation of real-life, they are not completely accurate for a few reasons. Firstly, the models assume a perfect distribution of thermal energy or temperature throughout the liquid. However, as can be seen in Figure 3, the surface of the liquid is not at a single temperature, and a gradation of temperature occurs across the surface. Another assumption of the model is that the heat transfer coefficient of water to the air is constant, yet it is a well-known fact that blowing on a cup of tea cools it faster. Any form of wind or could potentially increase the heat transfer coefficient and cause the temperature to decrease faster. A third limitation is on that applies to the third model. This model assumes the temperature of the container only increases, however, just as the water loses heat to the air, the cup will also lose heat to the air. As the temperature of the cup decreases, the temperature difference between the water and the cup would increase, causing more temperature to be lost through the cup. Accounting for this additional loss could be a feasible extension to this investigation.

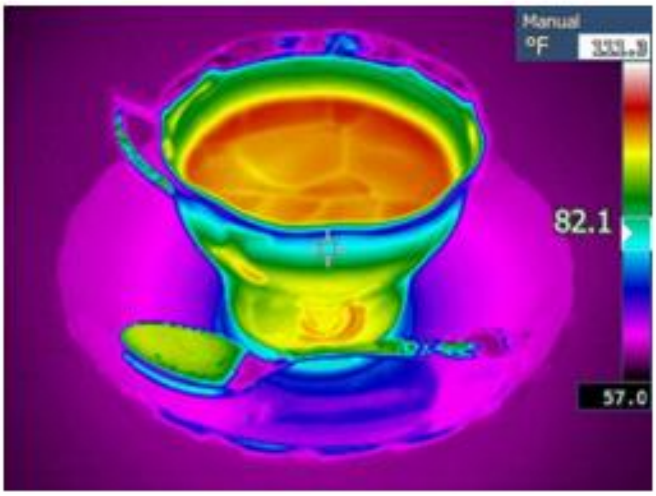

Figure 3: Thermal image of Cup of Hot Water (Conductive Heat)

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