The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, SPRING SEMESTER 2022–2023

MATHEMATICS FOR COMPUTER SCIENTISTS 2

Time allowed: TWO Hours

This is a mock exam to help prepare for the take-home and open-book exam.

Answer ALL FOUR QUESTIONS

The real examination will be marked out of 100.

You may write/draw by hand your answers on paper and then scan them to a PDF file, or you may type/draw your answers into electronic form directly and generate a PDF file.

Your solutions should include complete explanations and should be based on the material covered in the module. Make sure your PDF file is easily readable and does not require magnification. Make sure that each page is in the correct orientation. Text/drawing which is not in focus or is not legible for any other reason will be ignored.

Submit your answers containing all the work you wish to have marked as a ***single PDF file****.*

Use the standard naming convention for your document: ***Student ID_COMP1043_202223****. Write your student ID number at the top of each page of your answers.*

Although you may use any notes or resources you wish to help you complete this open-book examination, the academic misconduct policies that apply to your coursework also apply here. You must be careful to avoid plagiarism, collusion or false authorship. Please familiarise yourself with the Guidance on Academic Integrity in Alternative Assessments, which is available on the Faculty of Science Moodle Page: Guidance for Remote Learning. The penalties for academic misconduct are severe.

Staff are not permitted to answer assessment or teaching queries during the period in which your examination is live. If you spot what you think may be an error on the exam paper, note this in your submission but answer the question as written.

1. Systems of linear equations

Consider the following system of linear equations

$$\begin{cases} y + w = -1 \\ -x + 2y - 2z + 3w = -3 \\ 2x + y + 4z - w = 1. \end{cases}$$

(a) Determine whether or not the system is consistent, and if it is, give the general solution in parametric form.

[15 Marks]

(b) Give the general solution to the corresponding homogeneous system in parametric form.

[10 Marks]

For all the points above, write full sentences explaining all the steps.

[End of Question 1: Total 25 marks]

Solution

(a) We reduce the augmented matrix to reduced row echelon form:

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 1 & -1 \\ -1 & 2 & -2 & 3 & -3 \\ 2 & 1 & 4 & -1 & 1 \end{array}\right) \xrightarrow{R_2=-R_2} \left(\begin{array}{cccc|c} 0 & 1 & 0 & 1 & -1 \\ 1 & -2 & 2 & -3 & 3 \\ 2 & 1 & 4 & -1 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -3 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 2 & 1 & 4 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3=R_3-2R_1} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -3 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 5 & 0 & 5 & -5 \end{array}\right) \xrightarrow{R_3=R_3-5R_2} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -3 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1=R_1+2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

There's no pivot in the augmentation column, so the system is consistent. The two other non-pivot columns are the ones corresponding to z and w, so we let $z = t$ and $w = u$, for parameters t, u, and solve for the general solution, getting:

$$w = u, \quad z = t, \quad y = -1 - u, \quad x = 1 - 2t + u.$$

The solution set is:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \middle| \, t, u \in \mathbf{R} \right\}$$

(b) We know that the solution set of the corresponding homogeneous equation is the pure parametric part of the solution set of the inhomogeneous equation. Hence we get:

$$\left\{ t \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \middle| \, t, u \in \mathbf{R} \right\} = \mathrm{Span} \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2. Linear maps

Consider the following three vectors in $\mathbf{R}^3$:

$$u_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Show that $\mathscr{B} = (u_1, u_2, u_3)$ is a basis for $\mathbf{R}^3$.

[4 Marks]

Consider the following four vectors in $\mathbf{R}^4$:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

(b) Show that $\mathscr{C} = (v_1, v_2, v_3, v_4)$ is a basis for $\mathbf{R}^4$.

[4 Marks]

A linear map $T : \mathbf{R}^3 \to \mathbf{R}^4$ is determined by:

$$T(u_1) = v_1, \quad T(u_2) = v_1 + v_2, \quad T(u_3) = 0,$$

(c) Give the standard matrix of T, i.e., in terms of the standard bases for $\mathbf{R}^3$ and $\mathbf{R}^4$.

[9 Marks]

(d) Give a basis for the image of T.

[4 Marks]

(e) Give a basis for the kernel of T.

[4 Marks]

For all the points above, write full sentences explaining all the steps.

[**End of Question 2: Total 25 marks**]

Solution

(a) Let B be the matrix whose columns are u_1, u_2, u_3. For part (c) we'll need the inverse of B (if B is indeed invertible as the question claims), so we row reduce:

$$(B \mid I_3) = \left(\begin{array}{ccc|ccc} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 7 & -2 & 1 & -3 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 7 & -2 & 1 & -3 & 0 \\ 0 & 3 & -1 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 3 & -1 & 0 & -2 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 - 3R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & -1 & -3 & -5 & 7 \end{array}\right)$$

$$\xrightarrow{R_3 = -R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 & 5 & -7 \end{array}\right) \xrightarrow{R_1 = R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -3 & -4 & 7 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 & 5 & -7 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 3 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 & 5 & -7 \end{array}\right), \quad \text{so } \mathscr{B} \text{ is a basis and } {}_{\mathscr{B}}[\mathrm{Id}]_{\mathscr{E}} = B^{-1} = \begin{pmatrix} -1 & -2 & 3 \\ 1 & 1 & -2 \\ 3 & 5 & -7 \end{pmatrix}$$

(b) We won't need the inverse change-of-basis matrix, so we just check the determinant of the matrix C whose columns are v_1, v_2, v_3, v_4, using expansion along the first column:

$$\det(C) = \det\begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \det\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1 - 1 = -2$$

This is nonzero, so $\mathscr{C}$ is a basis.

Alternatively: Use row reduction to show that the rank of C is 4.

(c) From the given information we can read off the $(\mathscr{B}, \mathscr{C})$-matrix for T:

$${}_{\mathscr{C}}[T]_{\mathscr{B}} = \begin{pmatrix} | & | & | \\ {}_{\mathscr{C}}[T(u_1)] & {}_{\mathscr{C}}[T(u_2)] & {}_{\mathscr{C}}[T(u_3)] \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ {}_{\mathscr{C}}[v_1] & {}_{\mathscr{C}}[v_1 + v_2] & {}_{\mathscr{C}}[0] \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We thus get:

$$[T] = {}_{\mathscr{E}_4}[\mathrm{Id}_{\mathbf{R}^4}]_{\mathscr{C}}\; {}_{\mathscr{C}}[T]_{\mathscr{B}}\; {}_{\mathscr{B}}[\mathrm{Id}_{\mathbf{R}^3}]_{\mathscr{E}_3} = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & -2 & 3 \\ 1 & 1 & -2 \\ 3 & 5 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(d) Since the $(\mathscr{B}, \mathscr{C})$-matrix is in row echelon form with pivots in the first two columns, the first two vectors of $\mathscr{C}$ form a basis for the image of T:

$$(v_1, v_2)$$

Alternatively: The standard matrix for T has (after row reduction) pivots in the first two columns, so the first two columns of the standard matrix gives a basis for the image of T:

$$\left(\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 0 \end{pmatrix}\right)$$

It's also easy to see that another basis is given by the first two standard basis vectors: (e_1, e_2).

(e) Since the $(\mathscr{B}, \mathscr{C})$-matrix is in row echelon with the only non-pivot in last column, the last vector of $\mathscr{B}$ forms a basis for the kernel of T:

$$(u_3)$$

Alternatively: From the RREF of the standard matrix of T,

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we again see that (u_3) is a basis for null space, which is the kernel of T.

3. Subspaces

Consider the subspaces U, V of $\mathbf{R}^4$:

$$U = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbf{R}^4 \,\middle|\, x + y - z - w = 0 \right\}, \quad V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

(a) Give bases for U and V.

[8 Marks]

(b) Give a basis for the subspace sum $U+V$.

[8 Marks]

(c) Give a basis for the intersection $U \cap V$.

[9 Marks]

For all the points above, write full sentences explaining all the steps.

[**End of Question 3: Total 25 marks**]

Solution

(a) Note that U is the null space of the matrix in RREF,

$$\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}.$$

The last three columns are non-pivot columns, giving a basis for U consisting of the three vectors:

$$(u_1, u_2, u_3) = \left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

The two vectors defining V are linearly independent, as the matrix they form has rank two,

$$(v_1 \; v_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{yielding the basis} \quad (v_1, v_2) = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right).$$

(b) Putting the bases for V and U together in a matrix and row reducing, we get:

$$\begin{pmatrix} | & | & | & | & | \\ u_1 & u_2 & u_3 & v_1 & v_2 \\ | & | & | & | & | \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{R_2 = R_2 + R_1} \begin{pmatrix} -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{R_4 = R_4 + R_3} \begin{pmatrix} -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

This is in echelon form with a pivot in the rows 1, 2, 3, and 5. By the pivotal theorem, this shows that $U+V$ is 4-dimensional with a basis (u_1, u_2, u_3, v_2).

(c) The row echelon form from (b) shows that $v_1 \in U \cap V$, and Grassmann's formula says $\dim(U \cap V) + \dim(U+V) = \dim(U) + \dim(V)$, so $\dim(U \cap V) = 1$. Hence (v_1) is a basis for $U \cap V$.

4. Diagonalisation

Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

(a) Diagonalise A, if possible; that is, find matrices C and D such that $A = CDC^{-1}$, or show that A is not diagonalisable.

[20 Marks]

(b) Compute $A^{1\,000\,000}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

[5 Marks]

For all the points above, write full sentences explaining all the steps.

[**End of Question 4: Total 25 marks**]

Solution

(a) We compute the characteristic polynomial by expansion along the first row. (The second column would also work nicely.)

$$f(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 0 & 0 \\ 2 & 2-\lambda & -1 \\ 2 & 0 & 1-\lambda \end{pmatrix} = (2-\lambda)^2(1-\lambda)$$

The eigenvalues are 1 and 2. We check whether we can find a basis of eigenvectors. (This happens if and only if the geometric multiplicity of 2 is 2.)

For $\lambda = 1$, we get

$$A - I = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 2 & 0 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{so pick } v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$, we get

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -1 \\ 2 & 0 & -1 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{so we have a basis } \mathscr{C}_2 = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right)$$

Then we have that A is diagonalisable with

$$A = CDC^{-1}, \quad \text{with} \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(b) We note that the vector $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = v_1$ is in the eigenspace for $\lambda_1 = 1$, so $Ax = x$, $A^2x = x$, and generally, $A^n x = x$ for positive integers n. Thus:

$$A^{1\,000\,000}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$