class09

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Outline

§7.2 Methods of point estimation
Maximum likelihood estimation

§8.1 Basic properties of confidence intervals

Let $X_1, \ldots, X_n$ have joint pmf or pdf

$$f(x_1, \ldots, x_n; \theta_1, \ldots, \theta_m) \quad (7.6)$$

where the parameters $\theta_1, \ldots, \theta_m$ have unknown values. When $x_1, \ldots, x_n$ are the observed sample values and (7.6) is regarded as a function of $\theta_1, \ldots, \theta_m$, it is called the **likelihood function**.

The **maximum likelihood estimates** $\hat{\theta}_1, \ldots, \hat{\theta}_m$ are those values of the θ_i's that maximize the likelihood function, so that

$$f(x_1, \ldots, x_n; \hat{\theta}_1, \ldots, \hat{\theta}_m) \geq f(x_1, \ldots, x_n; \theta_1, \ldots, \theta_m)$$

for all $\theta_1, \ldots, \theta_m$.

When the X_i's are substituted in place of the x_i's, the **maximum likelihood estimators** (mle's) result.

$f(x;\theta)$

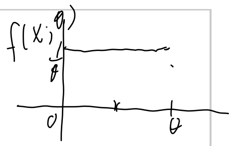
$L(\theta) = f(x_1;\theta) f(x_2;\theta) \cdots f(x_n;\theta)$

$\hat{\theta}$

$\ell(\theta) = \ln(L(\theta))$

$= \ln f(x_1;\theta) + \ln f(x_2;\theta) + \cdots + \ln f(x_n;\theta)$

Example (Uniform).
X is uniform on $(0, \theta)$.



$$f(x) = \begin{cases} \frac{1}{\theta}, 0 < x < \theta. \\ 0, \text{ elsewhere} \end{cases}$$

Method of moments: $E(X) = \frac{\theta}{2} = \bar{X} \Rightarrow \hat{\theta} = 2\bar{X}$.

$\mu = \frac{\theta}{2} = \bar{X}$

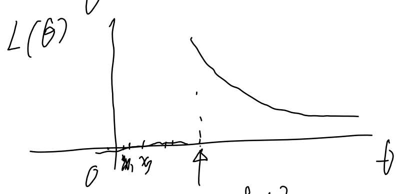
Method of maximum likelihood:
The likelihood function

$$L(\theta) = \frac{1}{\theta^n}, 0 < x_1, \ldots, x_n < \theta$$

$$\hat{\theta} = \max(X_i)$$

$\ell(\theta) = -n \ln(\theta)$, $0 < x_1, \cdots, x_n < \theta$

$\frac{d\ell(\theta)}{d\theta} = -\frac{n}{\theta} = 0$?



$\hat{\theta} = \max\{x_i\}$

θ $\hat{\theta} = \max\{x_i\}$

The invariance principle

Let $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$ be the mle's of the parameters $\theta_1, \theta_2, \ldots, \theta_m$.

Then the mle of any function $h(\theta_1, \theta_2, \ldots, \theta_m)$ is $h(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m)$.

Example 7.21

In the normal case, the mle's of μ and σ^2 are $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$. biased

Sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ unbiased

To obtain the mle of the function $h(\mu, \sigma^2) = \sqrt{\sigma^2} = \sigma$, substitute the mle's into the function:

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \left[\frac{1}{n}\sum(X_i - \bar{X})^2\right]^{1/2}$$

$X \sim N(\mu, \sigma^2)$

of size $n=10$

Based on a sample, $\bar{x} = 2.5$, $s = 1.5$. Using the invariance principle, find the maximum likelihood estimate of

$\hat{\mu} = \bar{x} = 2.5$, $\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{\frac{1}{n}\sum(x_i - \bar{x})^2} = 1.42$

$S = 1.5$

(a) the population coefficient of variation $100\sigma/\mu\%$

$\widehat{h(\mu,\sigma)} = \frac{100\hat{\sigma}}{\hat{\mu}} = 56.9(\%)$ $\hat{\sigma}^2 = \frac{(n-1)S^2}{n} = \frac{1}{n}\sum(x_i - \bar{x})^2$

$\hat{\sigma} = \sqrt{\frac{(n-1)S^2}{n}} = \sqrt{\frac{(10-1)}{10}(1.5)^2} = 1.42$

(b) $P(X < 3.0) = \int_{-\infty}^{3.0} f(x;\mu,\sigma)\,dx = F(3;\mu,\sigma)$

$F(3, \hat{\mu}, \hat{\sigma})$

(c) the 95th percentile for the distribution of X.

$F^{-1}(.95, \hat{\mu}, \hat{\sigma})$

qnorm(.95, 2.5, 1.42)

$= 4.84$.

95th percentile of $Z = 1.645$

$\hat{\mu} + (1.645)(\hat{\sigma}) = 4.84$.

$\widehat{P(X<3.0)} = P\left(\frac{X-\mu}{\sigma} < \frac{3.0-\mu}{\sigma}\right)$

$= P\left(Z < \frac{3.0-\hat{\mu}}{\hat{\sigma}}\right) = P\left(Z < \frac{3.0-2.5}{1.42}\right)$

= pnorm(3, 2.5, 1.42) = .64

Large-sample behavior of the mle

Under very general conditions on the joint distribution of the sample, when the **sample size is large**,

the mle $\hat{\theta}$ is close to θ (consistency),

is approximately unbiased $[E(\hat{\theta}) \approx \theta]$, and

has variance that is nearly as small as can be achieved by any unbiased estimator, i.e., $\hat{\theta}$ is approximately the MVUE of θ.

$\hat{\mu} + (1.645)(\hat{\sigma}) = 4.84.$

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Maximum likelihood estimation is the most widely used estimation technique among statisticians.

Note that there is no similar result for method of moments estimators. In general, if there is a choice between maximum likelihood and moment estimators, the mle is preferable.

$\hat{\theta} = 2\overline{X}$

$\hat{\theta} = \max\{X_i\}$

Chapter 8. Statistical Intervals Based on a Single Sample

- A point estimate by itself provides no info about the precision and reliability of estimation.
- Standard error is a measure of the precision and reliability.
- Alternatively, we may calculate and report an entire **interval of plausible estimates** – an **interval estimate or confidence interval (CI)**.

μ, σ

$\overline{x} = 7.5$

$SE = \frac{s}{\sqrt{n}}$

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§8.1 Basic properties of confidence intervals

Normal population with known σ

Consider a simple situation –

Suppose that the parameter of interest is a population mean μ and that

1. The population distribution is normal
2. the value of population standard deviation σ is known.

$= 1.5$

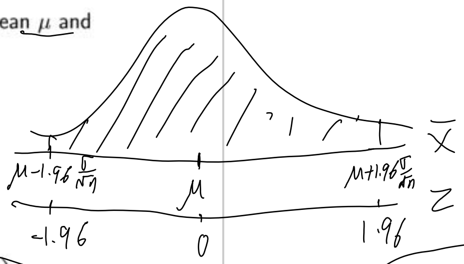
Random interval

- The sample mean $\bar{X}$ is normal with mean μ and standard deviation $\sigma/\sqrt{n}$.
- Standardizing $\bar{X}$:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

  is standard normal.
- It then follows that

$$P\left(-1.96 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.96\right) = .95$$



$\mu > \bar{X} - 1.96\frac{\sigma}{\sqrt{n}}$

$P\left(-1.96\frac{\sigma}{\sqrt{n}} < \bar{X} - \mu < 1.96\frac{\sigma}{\sqrt{n}}\right) = .95$

$\Updownarrow$

$\mu < \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}$

Random interval

- Rearranging terms of the inequalities,

$$P(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96\frac{\sigma_0}{\sqrt{n}}) = .95$$

- The **random interval**

$$(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}) \qquad (8.4)$$

  includes or covers the true value of μ with a probability of 0.95.

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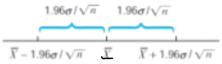

Figure 8.2 The random interval (8.4) centered at $\bar{X}$

95% Confidence interval

If, after observing $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, we compute the observed sample mean $\bar{x}$ and then substitute $\bar{x}$ into the random interval in place of $\bar{X}$, the resulting **fixed** interval is called a **95% confidence interval for** μ. This CI can be expressed as

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right) \text{ is a 95\% CI for } \mu$$

or as

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96\frac{\sigma}{\sqrt{n}} \text{ with 95\% confidence}$$

A concise expression for the interval is $\bar{x} \pm 1.96\sigma/\sqrt{n}$, where $-$ gives the left endpoint (lower limit) and $+$ gives the right endpoint (upper limit).

$\bar{x} = 2.6$ $n = 36$

Example. The **average zinc concentration** recovered from a sample of zinc measurements in 36 different locations is found to be 2.6 grams per milliliter. Find a 95% confidence interval for the **mean zinc concentration in the river**. (μ) Assume that the population distribution is normal and that the population standard deviation is 0.3. $= \sigma$

95% confidence

$\bar{x} = 2.6$, $n = 36$, $\sigma = 0.3$. The resulting interval is

$$\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}} = 2.6 \pm (1.96)\frac{0.3}{\sqrt{36}} = 2.6 \pm 0.10 = (2.50, 2.70)$$

We are 95% confident that $2.50 < \mu < 2.70$.

Interpreting a confidence interval

The statement

$$P(2.50 < \mu < 2.70) = 0.95$$

is not quite right: (2.50, 2.70) is a fixed interval and μ is a constant; either μ is within the interval or it does not!

Interpretation of the 95% confidence level:
If we could take samples (of size n) over and over again, 95% of these samples would give an interval that includes μ, and 5% of all samples would yield an erroneous interval.

Interpreting a confidence interval

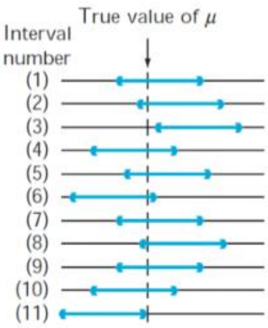

Figure 8.3 Repeated construction of 95% CIs

An R simulation

```
# M: number of samples.
# n: sample size.
M<-20
n<-10
mu<-0; sigma=1

for (m in 1:M){
# Generate a sample of size n from standard normal.
x<-rnorm(n, mean=mu, sd=sigma)
xbar<-mean(x); ME<-1.96*sigma/sqrt(n)
LowerLimit<-xbar-ME; UpperLimit<-xbar+ME
Outside<-ifelse(mu<LowerLimit | mu>UpperLimit, 1, 0)
# 95% CI.
cat("Sample ", m, ": ",
round(c(xbar-ME, xbar+ME), 2), "\t ", Outside, "\n")
}
```