Why does sample mean has a prob distn ? $\bar{x}. \longrightarrow \bar{X}$

Recall: According to the CLT, when n is large and we wish to calculate a probability such as $P(a \leq \bar{X} \leq b)$, we need only to "pretend" that $\bar{X}$ is normal with mean μ and standard deviation $\sigma/\sqrt{n}$.

if $n \geq 30$

Let $T = X_1 + \cdots + X_n = n\bar{X}$ be the sample total. For large sample size, we can also "pretend" that $T = n\bar{X}$ is normal with mean $n\mu$ and standard deviation $\sqrt{n} \cdot \sigma$.

$X \sim N(\mu, \sigma^2)$

Then $aX + b \equiv Y \sim N(a\mu + b, a^2\sigma^2)$

Example. In a certain population of fish, the weight of the individual fish follow a distribution with mean 250g and standard deviation 50g. What is the probability that the total weight of a catch of 37 fish is at least 9 kg?

X be weight of a randomly selected fish

$E(X) = \mu = \underline{250g}$. $SD(X) = \sigma = 50g$.

37 fish regarded as a random sample. $n = 37$ large. $\bar{X}$ is approx normal with $\mu = 250$ and SD $50/\sqrt{37}$.

T normal with mean $= (37)(250)$ and $SD = \sqrt{37} \cdot 50$

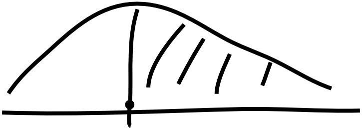
$$\text{Ans} = P(T > 9000)$$

$$= P(\bar{X} > 9000/37)$$

```
> 1-pnorm(9, 37*.25, sqrt(37)*.05)
[1] 0.7944601
> pnorm(9/37, .25, .05/sqrt(37), lower.tail=F)
[1] 0.7944601
```

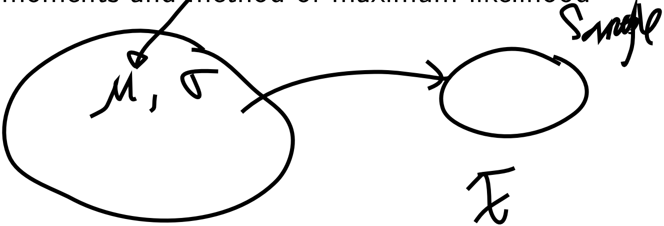
T

$$= P\left(Z > \frac{9000-(37)(500)}{\sqrt{37}\cdot 50}\right)$$



Chapter 7. Point Estimation

- Point estimator/estimate of a population parameter.
- Method of moments and method of maximum likelihood



Outline

Outline

- The objective of point estimation is to use a sample to compute a number that is a good guess for a parameter of interest.

- The parameter of interest about a population could be the population mean μ, population standard deviation σ, the population proportion p, and so on. We use generically θ to denote the parameter of interest.

$\Gamma(\alpha, \beta)$

Point estimate and point estimator

A **point estimate** of a parameter θ is a fixed number that can be regarded as a sensible value of θ. A point estimate is obtained by selecting a suitable statistic and computing its value from the given sample data.

The selected statistic is called the **point estimator** of θ.

statistic

The symbol $\hat{\theta}$ is customarily used to denote both the estimator of θ and the point estimate.

random variable

For example, $\hat{\mu} = \bar{X}$ is read as the point estimator of μ is the sample mean $\bar{X}$; $\hat{\mu} = \bar{x} = 5.77$: the point estimate of μ is 5.77.

Example 7.1 An automobile manufacturer has developed a new type of bumper, which is supposed to absorb impacts with less damage than previous bumpers. The manufacturer has used this bumper in a sequence of 25 controlled crashes against a wall, each at 10 mph, using one of its compact car models.

The conceptual population consists all such crashes.

The random variable X of interest is the indicator whether a crush results no visible damage.

$p = E(X)$

$$X = \begin{cases} 1, & \text{no visible damage} \\ 0, & \text{otherwise} \end{cases}$$

The random sample of size $n = 25$: $X_1, X_2, \ldots, X_{25}$.

Let Y = the (random) number of crashes that result in no visible damage to the automobile:

$$Y = X_1 + X_2 + \cdots + X_{25}$$

The parameter to be estimated is p = the proportion of all such crashes that result in no damage [alternatively, p = P(no damage in a single crash)].

If Y is observed to be $y = x_1 + \cdots + x_{25} = 15$, the most reasonable estimator and estimate are

estimator $\hat{p} = \dfrac{Y}{n} = \dfrac{X_1 + \cdots + X_n}{n}$ estimate $\hat{p} = \dfrac{y}{n} = \dfrac{15}{25} = .60.$

The sample proportion $\hat{p}$ is a special case of sample mean.

Example 7.3
Studies have shown that a calorie-restricted diet can prolong life. Of course, controlled studies are much easier to do with lab animals. Here is a random sample of eight lifetimes (days) taken from a population of 106 rats that were fed a restricted diet.

716 1144 1017 1138 389 1221 530 958

Let $X_1, \ldots, X_8$ denote the lifetimes as random variables, before the observed values are available. For the population mean μ, a natural estimator is the sample mean:

estimator: $\hat{\mu} = \bar{X}$ estimate: $\hat{\mu} = \bar{x} = 889.1$

For the population variance σ^2, we may use the sample variance:

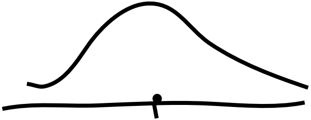
$$\text{estimator: } \hat{\sigma}^2 = S^2 \quad \text{estimate: } \hat{\sigma}^2 = s^2 = 95315$$

For the population standard deviation σ,

$$\text{estimator: } \hat{\sigma} = S \quad \text{estimate: } = s = 308.7$$

Outline

Mean squared error



- There may be more than one reasonable estimator for a parameter. For example, for a normal population, μ is the population mean and population median, we could use $\hat{\mu} = \bar{X}$ or $\hat{\mu} = \tilde{X}$.

$X_1, \cdots, X_{100}$

- A sensible way to quantify the idea of $\hat{\theta}$ being close to θ is to consider the squared error $\rightarrow$ random

$$(\hat{\theta} - \theta)^2$$

- A better estimator has a smaller **mean squared error (MSE)**:

$$\text{MSE} = E[(\hat{\theta} - \theta)^2]$$

Mean squared error

$$
\begin{aligned}
\mathsf{MSE} &= E[(\hat{\theta}-\theta)^2] \\
&= E\{[(\hat{\theta}-E(\hat{\theta}))+(E(\hat{\theta})-\theta)]^2\} \\
&= E[(\hat{\theta}-E(\hat{\theta}))^2]+[E(\hat{\theta})-\theta]^2 \\
&= \text{variance of estimator} + (\text{bias})^2
\end{aligned}
$$

$$+E[(\hat{\theta}-E(\hat{\theta}))(E(\hat{\theta})-\theta)] = 0$$

Outline

May restrict our attention to only those unbiased estimators.

A point estimator $\hat{\theta}$ is said to be an **unbiased** estimator of θ if $E(\hat{\theta}) = \theta$ for **every possible value** of θ.

If $\hat{\theta}$ is not unbiased, the difference $E(\hat{\theta}) - \theta$ is called the **bias** of $\hat{\theta}$.

Principle of Unbiased Estimation
When choosing among several different estimator of θ, select one that is unbiased.

Let $X_1, \ldots, X_n$ be a random sample from a distribution with mena μ and variance σ^2. Then

1. $\bar{X}$ is an unbiased estimator of μ.
2. The estimator

$$\hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

is unbiased for estimating σ^2.

3. But S is biased for σ.

$E(\bar{X}) = \mu$ ✓

$$E(S^2) = E\left\{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right\}$$
$$= \frac{1}{n-1}E\left\{\sum_{i=1}^{n}[(X_i - \mu) - (\bar{X} - \mu)]^2\right\}$$
$$= \frac{1}{n-1}E\left\{\left[\sum_{i=1}^{n}(X_i - \mu)^2\right] - n(\bar{X} - \mu)^2\right\}$$
$$= \frac{1}{n-1}E\{n\sigma^2 - \sigma^2\} = \sigma^2$$

$= \sigma^2$ (unbiased)

If S *were* unbiased for σ, then

$$V(S) = E(S^2) - [E(S)]^2 = \sigma^2 - \sigma^2 = 0$$

Then S would be a constant! Thus S is biased for σ.

Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli population with probability of success p. Then the sample total $Y = X_1 + \cdots + X_n$ is a binomial rv with parameters n and p, and the sample proportion $\hat{p} = Y/n$ is an unbiased estimator for p.

$$E(X_i) = p \Rightarrow$$ (The population proportion is the population mean)

$$\hat{p} = \frac{X_1 + \cdots + X_n}{n} = \bar{X} \Rightarrow$$ (The sample proportion is the sample mean)

Outline

Among all estimators of θ that are unbiased, choose the one that has minimum variance. The resulting $\hat{\theta}$ is called the **minimum variance unbiased estimator (MVUE)** of θ.

Since MSE$=$variance$+$(bias)^2, seeking an unbiased estimator with minimum variance is the same as seeking an unbiased estimator that has minimum mean squared error.

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with parameters μ and σ. Then the estimator $\hat{\mu} = \bar{X}$ is the MVUE for μ, and S^2 is the MVUE for σ^2.

Outline

The **standard error** of an estimator $\hat{\theta}$ is its standard deviation $\sigma_{\hat{\theta}} = \sqrt{V(\hat{\theta})}$. It is the magnitude of a typical or representative deviation between an estimate and the value of θ.

The standard error of an estimator typically involves unknown parameters and thus unknown. Substitution of the estimates of these parameters into $\sigma_{\hat{\theta}}$ yield the **estimated standard error** (of the estimator). The estimated standard error can be denoted either by $\hat{\sigma}_{\hat{\theta}}$ (the $\hat{}$ over σ emphasizes that $\sigma_{\hat{\theta}}$ is being estimated) or by $s_{\hat{\theta}}$.

Let $\bar{X}$ denote the sample mean of a random sample from a population distribution with mean μ and variance σ^2.

The standard deviation of sample mean $\bar{X}$ is given by $\sigma_{\bar{X}} = \sigma/\sqrt{n}$, which is the standard error (of the sample mean as an estimator of μ).

As σ is unknown, we may estimate it with $s_{\bar{X}} = s/\sqrt{n}$.

$\sigma/\sqrt{n}$ is standard error of $\bar{X}$

$S/\sqrt{n}$ is estimated standard error.

E.g., from a population with mean μ, we take a sample of size $n = 25$, $\bar{x} = 1.2$, $s = 0.6$. An estimate of μ is $\hat{\mu} = 1.2$. The (estimated) standard error (of the estimator $\bar{X}$) is $0.6/\sqrt{25} = 0.12$.

(**Note**. Here we have omitted the word "estimated": We estimate μ as 1.2 with a standard error of 0.12.)

In the population: (population) standard deviation $= \sigma$ is a parameter; as is a (true) standard error $\sigma_{\hat{\theta}}$.

From data: The (sample) standard deviation s is a fixed number, as an estimate of the population/true mean μ; the (estimated) standard error (of the mean) $s/\sqrt{n}$ is an estimate (much smaller number than s).

Example 7.11 (Example 7.1 continued)
The standard error (of $\hat{p} = Y/n$) is

$$\sigma_{\hat{p}} = \sqrt{V(X/n)} = \sqrt{\frac{V(X)}{n^2}} = \sqrt{\frac{np(1-p)}{n^2}} = \sqrt{\frac{p(1-p)}{n}}$$

A point estimate of p is

$$\hat{p} = x/n = 15/25 \stackrel{\bullet}{=} .6$$

The (estimated) standard error (of $\hat{p}$) is

$$\hat{\sigma}_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{(.6)(1-.6)}{25}} = .098.$$