Regula Falsi Method

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Introduction

- The Regula Falsi method (False Position) is a bracketing root-finding algorithm.
- It is used to solve nonlinear equations of the form:

$$f(x) = 0$$

- Requires an initial interval $[a, b]$ where $f(a) \cdot f(b) < 0$ (the root is bracketed).
- Combines reliability of bisection with faster convergence using linear interpolation.

Regula Falsi Method

The Regula Falsi (False Position) method is used to find the root of a function $f(x)$ by iteratively updating an approximation based on a secant line.

Given:

Two points (x_{n-1}) and (x_n) such that:

$$f(x_{n-1}) \cdot f(x_n) < 0$$

there exists a root in $[x_{n-1}, x_n]$.

Step 1: Choose an Initial Interval

Objective:

Select an interval $[x_{n-1}, x_n]$ where the function $f(x)$ changes sign.

Condition

$$f(x_{n-1}) \cdot f(x_n) < 0$$

This ensures that a root exists in the interval $[x_{n-1}, x_n]$ (by the Intermediate Value Theorem).

Why is this important?

- Ensures that a real root exists within the interval.
- Prevents unnecessary computations on intervals without a root.
- The method will refine this interval in each iteration.

Derivation of the Formula

The equation of a straight line passing through a point (x_1, y_1) with slope m is:

$$y - y_1 = m(x - x_1)$$

The slope m between two points $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$ is:

$$m = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Using point $(x_n, f(x_n))$, the secant line equation is:

$$y - f(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_n)$$

Derivation of the Formula

The equation of the secant line passing through $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$ is:

$$y - f(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_n)$$

Setting $y = 0$ (to find the root approximation):

$$0 - f(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_n)$$

Derivation of the Formula

Solving for x:

$$x = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Hence, the iterative formula is:

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Algorithm Steps

Algorithm

1. Choose initial points x_{n-1} and x_n such that $f(x_{n-1}) \cdot f(x_n) < 0$.
2. Compute the next approximation:

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

3. Update the interval: Replace either x_n or x_{n-1} so that the sign change condition holds.
4. Repeat until $|x_{n+1} - x_n| < \epsilon$.

False Position Method: Algorithm

Goal

Find a solution to $f(x) = 0$ given two initial points x_0, x_1 such that:

$$f(x_0) \cdot f(x_1) < 0$$

Algorithm

1. Set $i = 2$, compute $f(x_0)$, $f(x_1)$.
2. While $i \leq N_0$, do steps 2.1-2.5:
   1. Compute:
   $$x_2 = x_1 - f(x_1)\frac{(x_1 - x_0)}{f(x_1) - f(x_0)}$$
   2. If $|x_2 - x_1| < \textbf{tol}$, output x_2 (root found), stop.
   3. Set $i = i + 1$, compute $f(x_2)$.
   4. If $f(x_2) \cdot f(x_1) < 0$, update $x_0 = x_2$, $f(x_0) = f(x_2)$.
   5. Set $x_1 = x_2$, $f(x_1) = f(x_2)$.
3. If maximum iterations reached, output failure.

Step 1: Bracketing of the Root

Bracketing

- We select x_0 and x_1 such that:

$$f(x_0) \cdot f(x_1) < 0$$

- The root lies in the interval $[x_0, x_1]$ by the Intermediate Value Theorem (IVT)

Step 2: Compute New Approximation x_2

Computation

- Compute the new approximation p using:

$$x_2 = x_1 - f(x_1)\frac{(x_1 - x_0)}{f(x_1) - f(x_0)}$$

- If $|x_2 - x_1| < \textbf{tol}$, the procedure is successful.

Step 3: Updating the Interval

Interval Update

- Compute $f(x_2)$.
- If $f(x_2) \cdot f(x_1) < 0$, update $x_0 = x_2$, $f(x_0) = f(x_2)$.
- Otherwise, update $x_1 = x_2$, $f(x_1) = f(x_2)$.

Numerical Example

Example

Find the root of:

$$f(x) = x^3 - x - 2$$

in the interval $[1, 2]$.

- $f(1) = -2$, $f(2) = 4 \rightarrow$ root is bracketed.
- Compute first approximation:

$$x_1 = 2 - \frac{4(2-1)}{4-(-2)} = 1.6667$$

- Check $f(1.6667)$, update interval accordingly.
- Continue until $|x_{n+1} - x_n| < 10^{-6}$.

Advantages and Disadvantages

Advantages

- Always brackets the root, ensuring convergence.
- Faster than the bisection method due to linear interpolation.

Disadvantages

- Can converge slowly if one endpoint remains fixed.
- Slower than methods like Newton-Raphson for well-behaved functions.