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Problem Set 1 — Invariants and Induction

1. Problem 2.1-3

LINEAR-SEARCH(A,k)
  for $j =$ **to** A.length **do**
    if $A[i] = k$ **then**
    return

  return NULL

   Invariant: The start of each iteration A[1...i-1] are not equal to v.
   Initialization: i = 1 and is less than A.length. Maintence: During each step in the for loop we know that if we got to point A[i] then A[1...i] does not equal k. Otherwise the function would return. This perservers the invariant.
   Termination: The loop terminates if i ¿ A.length since i increases by one very time we know that A[1...i-1] does not equal k. And since i-1 == A.length this is the entire array and we return NULL since k does exist in the array.

2. Problem 2-2 in CLRS, 3rd edition.

   (a) We need to prove that the array is a permunation of the input array.
   (b) Initialization: It is the last element of A in the array
   Maintence: Switch A[j] and A[j-1] if it is smaller this means that the smallest item is always closer to the right.
   Termination: After the loop finishes j = i which implies that i is the smallest element from the right of the array.
   (c) Initialization: We start with the original array which holds.
   Maintence: A[i] is the smallest of the array for every loop iteration while the rest that are right of the array are shuffled, this implies that A[1..i] are sorted.
   Termination: After the loop finishes i = n. Which means that A[1..n] are sorted and this is the entire array.
   (d) The worst worst-case for bubble sort is $O(n^2)$. This happens in a reversely ordered list where each inner loop takes n accesses.

3. Prove by induction that for every non-negative integer n

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

   Proof. Base case: $n = 0$, then both $\sum_{i=0}^{0} i^2 = 0$ and $\frac{0(0+1)(0+1)}{6} = 0$.

Inductive step: Assume that $\sum_{i=0}^{k-1} i = \frac{k(k+1)(2k+1)}{6}$.

$$\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= \sum_{i=0}^{k} i^2 + (k+1)^2 \\
&= \frac{(k)(k+1)(2k+1)}{6} + (k+1)^2 && \text{I.H.} \\
&= \frac{(k+1)[(k)(2k+1)6(k+1)^2]}{6} \\
&= \frac{(k+1)(2k^2+7k-6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

$\square$

4. Prove that given an unlimited supply of 6-cent coins, 10-cent coins, and 15-cent coins, one can make any amount of change larger than 29 cents.

Proof. Let p be value of the coins we can make. We have two cases: $29 < p \leq 35$, and $p > 35$.

Case 1: We begin with $p > 35$. We inductively assume that we can make $p-6$ cents worth of postage (which is greater than 29). Then we add a 6-cent stamp to get $p = (p-6)+6$ cents of postage.

Case 2: We now show it is true for $29 < p \leq 35$, by manually showing that each value p is a sum of 6's 10's and 15's.

$p = 30 = 15 + 15$

$p = 31 = 6 + 15 + 10$

$p = 32 = 6 + 6 + 10 + 10$

$p = 33 = 6 + 6 + 6 + 15$

$p = 34 = 6 + 6 + 6 + 6 + 10$

$p = 35 = 15 + 10 + 10$

$\square$