CHAPTER 1: LINEAR SYSTEM & GUASSIAN ELIMINATION

A linear system has either no solution, only one solution, or infinitely many solutions.

Elementary Row Operations:	
Multiply i^{th} row by constant k:	$A \xrightarrow{kR_i} B \xrightarrow{\frac{1}{k}R_i} A$ $EA = B$ and $E^{-1}B = A$
Interchange the i^{th} and the j^{th} row:	$A \xrightarrow{R_i \leftrightarrow R_j} B \xrightarrow{R_i \leftrightarrow R_j} A$ $EA = B$ and $EB = A$
Add k times of the i^{th} to the j^{th} row:	$A \xrightarrow{R_j + kR_i} B \xrightarrow{R_j - kR_i} A$ $EA = B$ and $E^{-1}B = A$

The linear system is - **inconsistent** if the last col of $(R|b')$ is a pivot col, **only one solution** if the last col of $(R|b')$ is not a pivot col and every col of R is a pivot col, has **infinitely many solutions** if last col of $(R|b')$ is not a pivot col and R has atleast one pivot col.

1. A homogeneous system of linear equations has either only the **trivial solution** or **infinitely many solutions** in addition to the trivial solution.
2. A homogenous system of linear equations with **more unknowns than** equations has infinitely many solutions.

CHAPTER 2: MATRICES

Let A, B, C be matrices of the same size and c, d are scalars.

$A + B = B + A$	$A + (B + C) = (A + B) + C$
$c(A + B) = cA + cB$	$(cd)A = c(dA) = d(cA)$
$A + 0 = 0 + A = A$	$A - A = 0$
$0A = 0$	$A(BC) = (AB)C$
$A(B_1 + B_2) = AB_1 + AB_2$	$(C_1 + C_2)A = C_1A + C_2A$
$c(AB) = (cA)B = A(cB)$	

Let A is an $m \times n$ matrix.

- $A0_{n\times q} = 0_{m\times q}$ • $0_{(p\times m)}A = 0_{p\times n}$
- $AI_n = I_mA = A$
- $A^n = I$ if $n = 0$, else $A^n = AA \dots A$.
- $A^mA^n = A^{m+n}$
- Generally, $AB \neq BA$. $AB = 0$, does not imply $A = 0$ *or* $B = 0$.
- and $(AB)^2 \neq A^2B^2$

Cancellation Laws *(May not hold if A is not invertible)*

Let A be an invertible $m \times m$ matrix.

1. If B_1 and B_2 are $m \times n$ matrices s.t. $AB_1 = AB_2 \rightarrow B_1 = B_2$
2. If C_1 and C_2 are $n \times m$ matrices s.t. $C_1A = C_2A \rightarrow C_1 = C_2$

Invertible matrices: Let A and B be square matrix of order n.
A is said to be **invertible** if there exists a square matrix B such that $\boldsymbol{AB = I}$ or $\boldsymbol{BA = I}$. The matrix B is the inverse of A. A square matrix is called singular of it has no inverse.

Basic Invertible Matrix Properties

- cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}$
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$
- A^{-1} is invertible and $(A^{-1})^{-1} = A$
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- If $A_1, A_2, \dots, A_k$ are invertible matrices, then $A_1A_2 \dots A_k$ is invertible and $(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$.
- $A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \dots A^{-1}$ (n times)
- A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$

Let A and B be a square matrix of the same size. If $\boldsymbol{AB = I}$:

- A is invertible B is invertible
- $A^{-1} = B$ $B^{-1} = A$ $BA = I$
- If A is singular, then both AB and BA are singular.

Method to find Inverses: Let A be an invertible matrix. Then $(A \mid I) \xrightarrow{GJE} (I \mid A^{-1})$
To check whether a square matrix is invertible, $A \xrightarrow{GE} R$. If R has no zero row, then A is invertible, otherwise A is singular.

Transpose - Let A be a m x n matrix

$(A^T)^T = A$	$(A + B)^T = A^T + B^T$ **B is m x n*	$(cA)^T = cA^T$	$(AB)^T = B^TA^T$ **B is n x p*

Determinant

$A \rightarrow B_1$	kR_i	$\det(B_1) = k\det(A)$
$A \rightarrow B_2$	$R_i \leftrightarrow R_j$	$\det(B_2) = -\det(A)$
$A \rightarrow B_3$	$R_j \leftrightarrow kR_i$	$\det(B_3) = \det(A)$

- Determinant of a sq matrix with identical rows/columns is zero.
- $\det(AB) = \det(A)\det(B)$
- $\det(cA) = c^n\det(A)$ //A is order n
- $\det(A^{-1}) = \frac{1}{\det(A)}$ //A is an invertible matrix
- $\det(A^T) = \det(A)$ //A is a square matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi$$

If A is an $n \times n$ triangular matrix, $\det(A) = a_{11}a_{22} \dots a_{nn}$.
Let M_{ij} be an $(n-1)^2$ matrix obtained from A by deleting ith row and jth column.
$A_{ij} = (-1)^{i+j}\det(M_{ij})$ is called the (i,j)-**cofactor** of A. $A^{-1} = \frac{1}{\det(A)}adj(A)$.

CHAPTER 3: VECTOR SPACES

Let $S = \{u_1, u_2, \dots, u_k\}$ be a set of vectors in $\mathbb{R}^n$. $a_1u_1 + a_2u_2 + \dots + a_ku_k$ is called a linear combination of $u_1, u_2, \dots, u_k$. The set of all linear combinations of $\{a_1u_1 + a_2u_2 + \dots + a_ku_k | a_1, a_2, \dots, a_k \in \mathbb{R}\}$ is called a **linear span** of S and denoted by $span(S)$. If REF of $(u_1\ u_2 \dots u_k)$ has no zero row $\rightarrow span(\dots) = \mathbb{R}^n$.

Let V be a **subspace** of $\mathbb{R}^n$.

1. $0 \in V$
2. *For any* $v_1, v_2, \dots, v_r \in V$, *and* $c_1, c_2, \dots, c_r \in \mathbb{R}, c_1v_1 + c_2v_2 + \dots + c_rv_r \in V$

A subset V of $\mathbb{R}^n$ is a **subspace** of $\mathbb{R}^n \leftrightarrow$ **(i)** V is non-empty and **(ii)** for all $u, v \in V$ and $c, d \in \mathbb{R}$, $cu + dv \in V$

A set V is called a **vector space** if either $V = \mathbb{R}^n$ or V is a subspace of $\mathbb{R}^n$. A set W is called a subspace of a vector space V if W is a vector space and $W \subseteq V$.

Checking Subspaces: Let V be a subset of $\mathbb{R}^n$. To show that V is a subspace of $\mathbb{R}^n$, need to show that every vector $v \in V$ can be written in the form $a_1u_1 + a_2u_2 + \dots + a_ku_k$ where $a_1, \dots, a_k$ are arbitrary parameters and $u_1, \dots, u_k$ are constant vectors. To show that V is NOT a subspace, show that it violates some property of vector space.

Geometrical Intepretation

- Let u and v be a nonzero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. $span\{u\} = \{cu \mid c \in \mathbb{R}\}$ is a **line** through the origin. If u and v are not parallel, then $span\{u, v\} = \{su + tv \mid s, t \in \mathbb{R}\}$ is a **plane** containing the origin.
- The following are all subspaces of $\mathbb{R}^2$: zero space {(0,0)}, lines through origin, $\mathbb{R}^2$. The following are all subspaces of $\mathbb{R}^2$: zero space {(0,0)}, lines through origin, planes containing the origin, $\mathbb{R}^3$.

Linear Independence

Let $S = \{u_1, u_2, \dots, u_k\} \subseteq \mathbb{R}^n$. Consider $c_1u_1 + c_2u_2 + \dots + c_ku_k = \mathbf{0}$ $(k \geq 2)$
1. S is **Linearly Independent** if eqn only has **trivial soln** $(c_1 = 0, \dots, c_k = 0)$ OR no vector in S can be written as a linear combination of other vectors in S
2. S is **Linearly Dependent** if eqn has **non-trivial solns** OR iff there is at least 1 vector $u_i \in S$ can be written as a linear combination of other vectors in S.

Bases & Dimensions

Let V be a vector space and $S = \{u_1, u_2, \dots, u_k\}$ a subset of V. Then S is called a **basis** for V if **(i)** S is linearly independent, and **(ii)** S spans V.

If a vector space V has a basis S, then dimension of V is $\dim(V) = |S|$. $\dim(\mathbb{R}^n) = n$. Except {0} and $\mathbb{R}^2$, subspaces of $\mathbb{R}^2$ are lines through the origin of dim 1. Except {0} and $\mathbb{R}^3$, subspaces of $\mathbb{R}^3$ are either lines through the origin of dim 1, or planes containing the origin of dim 2.

Let V be a vector space which has basis and dim(V) = k. Any subset of V with **more than** k vectors is always **linearly dependent**. Any subset of V with **less than** k vectors **cannot span V**.

Let V be a vector space of dim k, and S a subset of V. **(1),(2),(3) are equivalent**:

1. S is a basis for V, (i.e. S is linearly independent and S spans V)
2. S is linearly independent and $|S| = k$
3. S spans V and $|S| = k$

To check that S is a basis for V, we only need to check any **two of the three conditions: (i):** S is linearly independent; **(ii):** S spans V; **(iii)** $|S| = k$.

Let U be a subspace of a vector space V. Then $\dim(U) \leq \dim(V)$. If $U \neq V, \dim(U) < \dim(V)$.

A basis for V is a set of the smallest size that can span V. The empty set $\emptyset$ is defined to be the basis for the zero space. Except the zero space, any vector space has **infinitely many** different bases.

Finding Bases: Suppose $W = span\{u_1, u_2, \dots u_k\}$.
Method 1 : Place the vectors as row vectors. Perform gaussian elimination. The row space of R forms the basis of W. **Method 2**: Place the vectors as column vectors. Find the pivot cols, the cols of vectors in W form the basis of W.

Implicit/Explicit Form: $\{ (x, y, z) \mid x + y + z = 0 \}, \{ (-s - t, s, t) \mid s, t \in \mathbb{R} \}$

CHAPTER 4: VECTOR SPACES ASSOCIATED W/ MATRICES

Let matrix A be m x n. Let R be the REF of A.
Row Space of A = span{$r_1, r_2, \dots, r_m$} $\subseteq \mathbb{R}^n$

- Row space of A = Row Space of R
- The nonzero rows of R form a basis for row space of R.
- The nonzero rows of R form a basis for the row space of A.

Column Space of A=span{$c_1, c_2, \dots, c_n$} $\subseteq \mathbb{R}^m$

- Column space of A ≠ Column space of R
- Pivot columns of R form a basis for the column space of R
- Corresponding columns of A form a basis for column space of A.

The solution set of the homogeneous linear system $Ax = 0$ is a subspace of $\mathbb{R}^n$ and is called the **nullspace** of A. Solve Ax=0 by GE/GJE, write general soln $t_1u_1 + \dots + t_ku_k$ where ti are arbitrary params and ui are fixed vectors. Then $\{u_1, \dots, u_k\}$ is a basis for the nullspace of A. The dimension of the nullspace of A is called the **nullity** of A. Since the nullspace is a subspace of $\mathbb{R}^n$, $nullity(A) = \dim(\textit{the nullspace of } A) \leq \dim(\mathbb{R}^n) = n$

Rank & Nullities

$\boldsymbol{rank(A)} = \dim(\textit{row space of } A) = \dim(\textit{col space of } A)$
$= \#\textit{ of nonzero rows of } R = \#\textit{ of pivot cols of } R$
$\boldsymbol{nullity(A)} = \dim(\textit{nullspace of } A) = \#\textit{ of non} - \textit{pivot cols of } R$

Dimension Theorem for Matrices: $rank(A) + nullity(A) = \#\textit{ of cols in } A$
= # of pivot-cols in R + # of non-pivot cols in R = # of cols of R = # of cols of A
A linear system $Ax = b$ is **consistent** iff b lies in the column space of A. Suppose x = v is a solution to Ax=b, then the solution set of the system Ax=b is given by $\{u + v \mid u \in \textit{nullspace of } A\}$. i.e. the system has a general solution x = (a general solution for Ax=0) + (one particular solution to Ax=b).
The **rank** of a matrix is the **dimension of its row space** (and its column space).

- $rank(0) = 0$ and $rank(I_n) = n$
- $rank(A) \le \min\{m, n\}$ for an $m \times n$ matrix A
- If $rank(A) = \min\{m, n\}$, then A is said to have **full rank**
- A square matrix A is of full rank iff $\det(A) \neq 0$
- $rank(A) = rank(A^T)$

Let A and B be $m \times n$ and $m \times p$ matrices respectively.
Then $rank(AB) \le \min\{rank(A), rank(B)\}$

Extending bases
Step 1: Form a matrix A using the vectors in S as rows. **Step 2**: Reduce A to REF R. **Step 3**: Identify the non-pivot columns in R. **Step 4**: For each non-pivot column, get a vector such that the leading entry of the vector is at that column. **Step 5**: $S \cup$ (the set of vectors chosen in Step 4).

Let A be a $m \times n$ matrix. Column space of A $= \{Au \mid u \in \mathbb{R}^n\}$. A system of linear equations $Ax = b$ is consistent iff **b** lies in the column space of **A**. The system is also only consistent iff there exists $u \in \mathbb{R}^n$ s.t. $Au = b$, i.e. $b \in \{Au \mid u \in \mathbb{R}^n\}$.

CHAPTER 5: ORTHOGONALITY

Distance between u and v: $d(u,v) = ||u - v|| = \sqrt{(u-v)\cdot(u-v)}$

- $||u-v||^2 = ||u||^2 + ||v||^2 - 2||u||||v||\cos(\theta)$

Angle between u and v: $\theta = \cos^{-1}\left(\frac{||u||^2 + ||v||^2 - ||u-v||^2}{2||u||||v||}\right)$

Norm (or length) of u: $||u|| = \sqrt{u \cdot u} = \sqrt{u_1^2 + \cdots + u_n^2}$

Dot Product: $u \cdot v = uv^T = u_1v_1 + \cdots + u_nv_n$ $//u.v = v.u$

- $(u+v)\cdot w = u \cdot w + v \cdot w$ and $w \cdot (u+v) = w \cdot u + w \cdot v$
- $(cu)\cdot v = u \cdot (cv) = c(u \cdot v)$
- $||cu|| = |c|\,||u||$
- $u \cdot u \ge 0; and\ u \cdot u = 0\ iff\ u = 0$

Orthogonality
Two vectors u and v in $\mathbb{R}^n$ are called **orthogonal** if $u.v = 0$. A set S of vectors in $\mathbb{R}^n$ is called a **orthogonal set** if every pair of distinct vectors in S are orthogonal. Furthermore, S is an **orthonormal set** if every vector in S is a unit vector aswell. Given any two nonzero vectors in $\mathbb{R}^n$, angle between u and v $= \cos^{-1}(0) = \frac{\pi}{2}$.

A basis S for a vector space is called an **orthogonal basis** if S is orthogonal. Likewise for orthonormal. If we want to show that S is orthogonal or orthonormal basis for V, we only need to check:

1. S is orthogonal (or orthonormal)
2. $|S| = \dim(V)$ (if know the dim), or $span(S) = V$ (if dk the dim)

$V^\perp = \{u \mid u\ is\ orthogonal\ to\ V\} = \{u \mid v_i \cdot u = 0\ for\ i = 1,2,\ldots k\} \subseteq \mathbb{R}$^n.

Gram-Schmidt Process
Let $\{u_1, u_2, \ldots, u_k\}$ be a basis for a vector space V.

$$v_1 = u_1,$$
$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1}v_1$$
$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1}v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2}v_2$$
$$v_k = u_k - \frac{u_k \cdot v_1}{v_1 \cdot v_1}v_1 - \frac{u_k \cdot v_2}{v_2 \cdot v_2}v_2 - \cdots - \frac{u_k \cdot v_{k-1}}{v_{k-1} \cdot v_{k-1}}v_{k-1}$$

Then $\{v_1, v_2, \ldots, v_k\}$ is an **orthogonal basis** for V. Furthermore, $\left\{\frac{1}{||v_1||}v_1, \frac{1}{||v_2||}v_2, \ldots, \frac{1}{||v_k||}v_k\right\}$ is an **orthonormal basis** for V

Projections
Let V be a subspace of $\mathbb{R}^n$. Every $u \in \mathbb{R}^n$ can be written **uniquely** as $u = n + p$ where p is a vector in V, and n is a vector orthogonal to V. p is the **projection** of u onto V. Let $\{u_1, u_2, \ldots, u_k\}$ an orthogonal basis for V. Then for any $w \in \mathbb{R}^n$, $\frac{w \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2}u_2 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k}u_k$ is the projection of w onto V.
Let $\{v_1, v_2, \ldots, v_k\}$ be an orthonormal basis for V. Then for any $w \in \mathbb{R}^n$, $(w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \cdots + (w \cdot v_k)v_k$ is the projection of w onto V.

Best Approximations
Take any $u \in \mathbb{R}^n$ and let p be the projection of u onto V. Then $d(u,p) \le d(u,v)\ \forall v \in V$. i.e. p is the best approximation of u in V. u is called a **least square solution** to the linear system $Ax = b$ if $||b - Au|| \le ||b - Av||\ \forall v \in \mathbb{R}^n$.
u is a least square solution to $Ax = b \leftrightarrow p = Au$ is the projection of b onto the column space of A $\leftrightarrow u$ is a solution to $A^TAx = A^Tb$.

Coordinate Systems
Let $S = \{u_1, \ldots, u_k\}$ be the basis for vector space V. $v \in V, v = c_1u_1 + \cdots + c_ku_k$. The vector $(v)_S = (c_1, c_2, \ldots, c_k)$ (as row vector) is called the **coordinate vector** of v relative to S. $E = \{e_1, e_2, \ldots, e_n\}$ is a **standard basis** for $\mathbb{R}^n$.
$For\ any\ v \in \mathbb{R}^n, (v)_E = v.$ Let A be a $m \times n$ matrix. Then $Ae_j^T = i^{th} col.\ of\ A.$
$(w)_S$ is essentially the projection of w onto S. (See *PROJECTIONS*)

Transition Matrices
Let $S = \{u_1, \ldots u_k\}$ and $T = \{v_1, \ldots, v_k\}$ be two bases for vector space V.

Transition Matrix from S to T : $P = [[u_1]_T\ [u_2]_T \ldots [u_k]_T] = \begin{bmatrix} a_{11} & \ldots & a_{1k} \\ \vdots & \ldots & \vdots \\ a_{k1} & \ldots & a_{kk} \end{bmatrix}$

where $u_1 = a_{11}v_1 + a_{21}{+}v_2 + a_{k1}v_k, \ldots,\ u_k = a_{1k}v_1 + a_{2k} + v_2 + a_{kk}v_k$.

- For any vector $w \in V, [w]_T = P[w]_S$
- P is **invertible** and P[-1] is the transition matrix from T to S
- If S and T are **orthonormal bases**, then P is an **orthogonal matrix**
- An Invertible matrix A is **orthogonal** if A[-1]=A[T].
- A is orthogonal matrix of order n ↔ Rows of A form orthogonal basis for $\mathbb{R}^n$ ↔ Cols of A form orthonormal basis for $\mathbb{R}^n$.

CHAPTER 6: DIAGONALIZATION

Let A be a square matrix of order n.

- A nonzero column vector $u \in \mathbb{R}^n$ is called an eigenvector of A if $Au = \lambda u$ for some scalar λ
- $\det(\lambda I - A) = 0$ is the characteristic equation of A
- $\det(\lambda I - A)$ is the characteristic polynomial of A

λ is an eigenvalue of A
↔ $Au = \lambda u$ for some nonzero column vector $u \in \mathbb{R}^n$
↔ $\lambda u - Au = 0$ for some nonzero column vector $u \in \mathbb{R}^n$
↔ $(\lambda I - A)u = 0$ for some nonzero column vector $u \in \mathbb{R}^n$
↔ the linear system $(\lambda I - A)x = 0$ (***eigenspace***) has non-trivial solutions
↔ $\det(\lambda I - A) = 0$.

If expanded, $\det(\lambda I - A)$ is a polynomial in λ degree n.

- A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix
- A is diagonalizable ↔ A has n linearly independent eigenvectors
- A has n distinct eigenvalues ⟶ A is diagonalizable
- A is called **orthogonally diagonalizable** if there exists an orthogonal matrix P ($P^{-1} = P^T$) s.t. P^TAP is a diagonal matrix
- A is orthogonally diagonalizable ↔ A is symmetric ($A^T = A$)

Diagonalize Matrix: Let A be a square matrix of order n. **Step 1**: Find all distinct eigenvalues $\lambda_1 \ldots \lambda_k$ (by solving characteristic eqn). **Step 2:** For each eigenvalue, find a basis S_{λ_i} for the eigenspace E_{λ_I}. **Step 3:** Let $S = S_{\lambda 1,\ldots} \cup S_{\lambda k}$. If $|S| < n$, then A is not diagonalizable. If $|S| = n, P = [u_1\ u_2 \ldots u_n]$ is an invertible matrix that diagonalizes A.

Orthogonally Diagonalize: Let A be a symmetric matrix of order n. Find eigenvalue and eigenspace as (1),(2). **Step 2b**: Use Gram-Schmidt Process to transform $S_{\lambda i}$ to an orthonormal basis $T_{\lambda i}$. **Step 3**: Let $T = T_{\lambda 1} \cup \ldots \cup T_{\lambda k}$. Then $P = [v_1\ v_2 \ldots v_k]$ is an orthogonal matrix that orthogonally diagonalizes A.

Power of matrices
For any integer m (if A is singular, then m must be non-negative).

$$A^m = P\begin{bmatrix} \lambda_1^m & 0 & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^m \end{bmatrix}P^{-1}, \quad P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Theorem 6.1.8: Invertible Matrices: Let A be an $n \times n$ matrix. The following statements are **equivalent**.

1. A is invertible.
2. The linear system $Ax = 0$ has only the trivial solution.
3. The RREF of A is an identity matrix.
4. A can be expressed as a product of elementary matrices.
5. $\det(A) \neq 0$
6. The rows of A form a basis for $\mathbb{R}^n$.
7. The columns of A form a basis for $\mathbb{R}^n$.
8. $rank(A) = n$
9. 0 is not an eigenvalue of A.

CHAPTER 7: LINEAR TRANSFORMATIONS

A **linear transformation** is a mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ of the form

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} \text{ for } (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n.$$

If $n = m$, then T is a **linear operator** on $\mathbb{R}^n$. Let $A = (a_{ij})_{n \times n}$. $T(u) = Au$ for all $u \in \mathbb{R}^n$. The matrix A is called the standard matrix for T. If $\{e_1, e_2, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$, then $\boldsymbol{A} = [\boldsymbol{T}(\boldsymbol{e_1})\ \boldsymbol{T}(\boldsymbol{e_2}) \ldots \boldsymbol{T}(\boldsymbol{e_n})]$.

Compositions of mappings: Let $S: \mathbb{R}^n \to \mathbb{R}^m$ and $T: \mathbb{R}^m \to \mathbb{R}^k$ be mappings. The composition of T with S denoted by $T \circ S$, is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$ defined by: $(T \circ S)(u) = T(S(u))$ for $u \in \mathbb{R}^n$. Suppose S and T are linear transformations. Then $T \circ S$ is also a linear transformation. Furthermore, if A and B are standard matrices for S and T respectively, then BA is the standard matrix for $T \circ S$.

Range/Kernel: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
The **range** of T denoted by $R(T)$ is the set of images of T, i.e.,
$R(T) = \{\, T(u) \mid u \in \mathbb{R}^n \,\} \subseteq \mathbb{R}^m$.
The **kernel** of T denoted by $Ker(T)$ is the set of images of vectors in $\mathbb{R}^n$ whose image is the zero vector in $\mathbb{R}^m$, i.e. $Ker(T) = \{\, u \mid T(u) = 0 \,\} \subseteq \mathbb{R}^n$.

If A is the standard matrix for T, then $R(T) = the\ column\ space\ of\ A$, and $Ker(T) = the\ nullspace\ of\ A.$
The **rank** of T, denoted rank(T), is the dimension of R(T). The **nullity** of T, denoted nullity(T), is the dimension of Ker(T). If A is the standard matrix for T, then $rank(T) = rank(A)$ and $nullity(T) = nullity(A)$.

Dimension Theorem for Linear Transformations: $rank(T) + nullity(T) = n$