Problem 3

a) $\mu_1 = [2 \ \ 0]$; $\mu_2 = [3, 1]$ with $\Sigma_1 = \Sigma_2 = 3 I_{2\times 2}$

With equal priors and equal covariance matrices:

$$g_i(x) = x^T \Sigma^{-1} \mu_i - \frac{1}{2} \mu_i^T \Sigma^{-1} \mu_i$$

$$g_1(x) = g_2(x) \Leftrightarrow ① \Leftrightarrow x^T \Sigma^{-1} \mu_1 - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 = x^T \Sigma^{-1} \mu_2 - \frac{1}{2} \mu_2^T \Sigma^{-1} \mu_2$$

Weight vector w:

$$w = \Sigma^{-1}(\mu_1 - \mu_2) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1/3 \end{bmatrix} \times \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$$

$$\boxed{w = \begin{bmatrix} -1/3 \\ -1/3 \end{bmatrix}}.$$

Bias term b:

$$b = 0{,}5 \times \left((\mu_2^T \Sigma^{-1} \mu_2) - (\mu_1 \Sigma^{-1} \mu_1) \right) = 0{,}5 \times \left([3 \ \ 1] \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} - [2 \ \ 0] \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right)$$

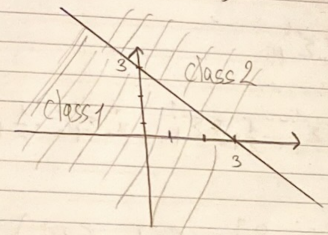
$$\boxed{b = 1}$$

Decision boundary:

$$w[0]\,x + w[1]\,y + b = 0 \; : (2)$$

$$(2) \iff y = \frac{-w[0]x - b}{w[1]} = \frac{1/3 \times x - 1}{-1/3} \boxed{= -(x-3)}$$

Plot:



b) $\mu_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$; $\mu_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$; $\Sigma_1 = \Sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

$$g_i(x) = x^T \Sigma^{-1} \mu_i - \frac{1}{2} \mu_i^T \Sigma^{-1} \mu_i + \ln(P(C_i))$$

$$g(x) = g_1(x) - g_2(x) = x^T \Sigma^{-1} \mu_1 - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \ln(P(C_1)) - x^T \Sigma^{-1} \mu_2 - \frac{1}{2} \mu_2^T \Sigma^{-1} \mu_2 + \ln(P(C_2))$$

weight vector w:

$$w = \Sigma^{-1}(\mu_1 - \mu_2) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \times \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Bias term b:

$$b = \frac{1}{2} \times \left(\mu_2^T \Sigma^{-1} \mu_2 - \mu_1^T \Sigma^{-1} \mu_1 \right)$$

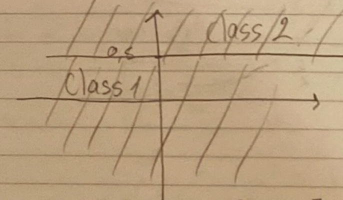
$$b = \frac{1}{2} \left(\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right)$$

$$b = \frac{1}{2}$$

Decision boundary:

$$w[0]x + w[1]y + b = 0$$

$$0 + -1y + \frac{1}{2} = 0 \iff y = 1/2$$



c) $\mu_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mu_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$; $\Sigma_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$; $\Sigma_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

$$g_i(x) = -\frac{1}{2}(x - \mu_i)^T \Sigma_i^{-1} (x - \mu_i) + \ln(P(C_i)) - \frac{1}{2} \ln(|\Sigma_i|)$$

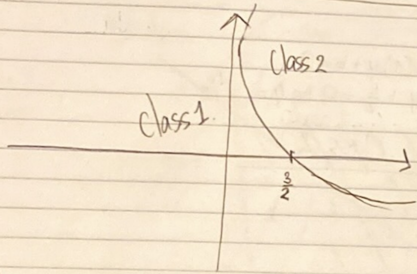
Can ignore $\ln(P(C_i))$ term since it's same for both.

$$g(x) = 0 \iff g_1(x) - g_2(x) = 0$$

$$g(x) = 0 \iff \frac{1}{2}\left(x^T \Sigma_2^{-1} x - x^T \Sigma_1^{-1} x \right) + \mu_1^T \Sigma_1^{-1} x - \mu_2^T \Sigma_2^{-1} x$$

$$- \frac{1}{2} \mu_1^T \Sigma_1^{-1} \mu_1 + \frac{1}{2} \mu_2^T \Sigma_2^{-1} \mu_2 + \frac{1}{2} \ln\left(\frac{|\Sigma_2|}{|\Sigma_1|} \right)$$

$$\boxed{g(x)=0 \iff \underbrace{\frac{x_1^2+4x_1x_2-x_2^2}{2}}_{\text{quadraterm}} + \underbrace{\frac{2x_1+3x_2}{2}}_{\text{linear term}} - \underbrace{\frac{3}{2}}_{\text{constant term}}}$$



Problem 4

a) $\prod \frac{1}{\sqrt{2\pi\theta^2}} \exp\left(-\frac{(x_i-\mu_1)^2}{2\sigma}\right)$ 3

take $-\log($3$)$: 4

4 $\iff -\log\left[\prod \frac{1}{\sqrt{2\pi\sigma^2}}\right] - \log\left(\prod \exp\left(\frac{-(x_i-\mu_1)^2}{2\sigma}\right)\right)$

4 $\iff -\sum \cancel{\log(1)}^{\,0} - \log\left(\sqrt{2\pi\sigma^2}\right) + \sum \frac{(x_i-\mu_1)^2}{2\sigma}$

4 $\iff \sum \log\left(\sqrt{2\pi\sigma^2}\right) + \sum \frac{(x_i-\mu_1)^2}{2\sigma}$

4 $\iff N\frac{\log(2\pi\sigma^2}{2} + \sum \frac{(x_i-\mu_1)^2}{2\sigma}$

Take derivative of 4 with resped to μ_1: 5

5 $\iff \sum \frac{1}{2\sigma} \times 2(x_i-\mu_1)$; set 5 $=0$

5 $=0 \iff \frac{2}{2\sigma}\sum x_i - \mu_1 = 0 \iff \sum x_i - \sum \mu_1 = 0 \iff \sum x_i = \sum \mu_1$

5 $=0 \iff \sum x_i = N\mu_1 \iff \boxed{\mu_1 = \frac{1}{N}\sum x_i}$

Similarly for μ_2 we can safely say:

$$\boxed{\mu_2 = \frac{1}{N}\sum x_i}$$

b) $\prod^{N_1} \frac{1}{2\pi^{n/2}\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(x_i-\mu_1)^T\Sigma^{-1}(x_i-\mu_1)\right) +$

$\prod^{N_2} \frac{1}{2\pi^{n/2}\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(y_i-\mu_2)^T\Sigma^{-1}(y_i-\mu_2)\right)$: 1

Take $-\log($1$)$: 2

2 $\iff \sum^{N_1} -\log\left(\frac{1}{2\pi^{n/2}\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(x_i-\mu_1)^T\Sigma^{-1}(x_i-\mu_1)\right)\right) +$

$\sum^{N_2} -\log\left(\frac{1}{2\pi^{n/2}\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(y_i-\mu_2)^T\Sigma^{-1}(y_i-\mu_2)\right)\right)$

2 $\iff N_1\log(2\pi^{n/2}) - \frac{N_1}{2}\log(|\Sigma|) + \frac{1}{2}\sum(x_i-\mu_1)^T\Sigma^{-1}(x_i-\mu_1) +$

$N_2\log(2\pi^{n/2}) - \frac{N_2}{2}\log(|\Sigma|) - \frac{1}{2}\sum(y_i-\mu_2)^T\Sigma^{-1}(y_i-\mu_2)$

$$(2) \Leftrightarrow \frac{-N_1}{2}|\Sigma| + \frac{1}{2}\sum (x_i-\mu_1)^T(x_i-\mu_1) - \frac{N_2}{2}|\Sigma| + \frac{1}{2}\sum (y_i-\mu_2)^T(y_i-\mu_2)$$

Set (2) to 0: (3)

$$(3) \Leftrightarrow \overbrace{\frac{1}{2}\left(\sum^{N_1}(x_i-\mu_1)^T(x_i-\mu_1) + \sum^{N_2}(y_i-\mu_2)^T(y_i-\mu_2)\right)}^{(a)} - \frac{1}{2}(N_1+N_2)|\Sigma| = 0$$

$$(3) \Leftrightarrow \frac{N_1+N_2}{2}|\Sigma| = (a)$$

$$\boxed{(3) \Leftrightarrow \Sigma_{ML} = \frac{1}{N_1+N_2} \times \left[\sum^{N_1}(x_i-\mu_1)(x_i-\mu_1)^T + \sum^{N_2}(y_i-\mu_2)(y_i-\mu_2)^T\right]}$$