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Vicsek Model for Collective Motions

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Abstract

Collective motion is a widespread phenomenon observed in various systems, from flocks of birds to traffic flow and even social dynamics. The Vicsek model, proposed by Tamás Vicsek and his colleagues in 1995, is a simple yet powerful mathematical framework that captures the emergence of collective motion from local alignment rules. This project reviews the physics behind the Vicsek model, its mathematical formulations, and its extensions and refinements to account for real-world complexities, such as noise, attraction and repulsion forces, heterogeneity, and three-dimensional environments. Additionally, we discuss how interdisciplinary approaches, including network theory, have enhanced our understanding of collective behavior. Through theoretical modeling and empirical observations, the Vicsek model continues to provide valuable insights into complex systems and offers potential solutions to real-world problems.

1. Introduction

Collective motion is a fascinating phenomenon observed in diverse systems, ranging from flocks of birds to traffic flow and even social dynamics. Understanding the underlying principles governing collective behavior is crucial for numerous fields, including biology, physics, and sociology. The Vicsek model, proposed by Tamás Vicsek et al. in 1995, has emerged as a powerful tool for studying and simulating collective motion. The model consists of self-propelled particles (SPPs) that move at constant speed and align their velocity with their neighbors within a given distance in the presence of noise. Such a simple model shows a phase transition from a disordered motion to large-scale ordered motion at low noise and high density of particles. The model is minimal and describes a kind of universality that applies to various systems exhibiting collective motion.

The motivation behind the Vicsek model was to capture the essence of flocking behavior observed in nature, such as birds (figure 1), fish or insects. Flocking is a form of collective motion where individuals move coherently in the same direction without any global coordination or leader. Flocking can provide several advantages for individuals, such as predator avoidance, resource exploitation, or information transfer. However, the mechanisms that enable flocking are not fully understood and pose many challenges for modeling and analysis.

The Vicsek model offers a simple and elegant way to understand how flocking can arise from local interactions and noise. The model has been extensively studied and extended by researchers to explore various aspects and implications of collective motion. In this project, we review the main features and results of the Vicsek model and its variations.

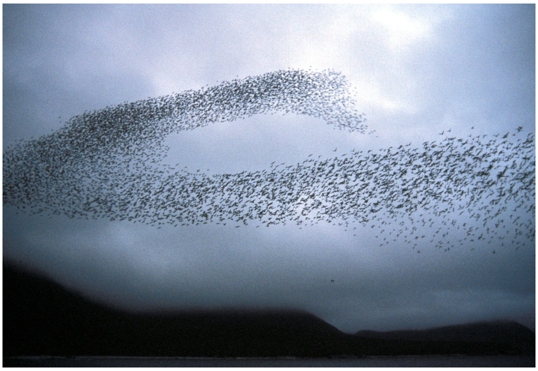

Figure 1. Starling flock

2. The Vicsek Model: Foundations and Mathematical Formulations

The Vicsek model consists of N point-like particles moving in a two-dimensional space with periodic boundary conditions. Each particle i has a position vector $r_i(t)$ and a velocity vector $v_i(t) = v_0 \hat{e}_i(t)$, where v_0 is the constant speed and $\hat{e}_i(t)$ is the unit vector defining the direction of motion at time t. The direction of each particle is given by an angle $\theta_i(t)$ with respect to a fixed axis; typically the positive x axis in the Cartesian coordinates.

The dynamics of each particle is governed by two rules: alignment and noise. At each discrete time step Δt, each particle aligns its direction with the average direction of its neighbors within a cut-off radius r_c around it. This alignment rule can be expressed as:

$$\theta_i(t + \Delta t) = \langle \theta_j \rangle_{|r_i - r_j| < r_c} + \eta_i(t)$$

where $\langle \theta_j \rangle_{|r_i - r_j| < r_c}$ is the average angle of all particles j (including i) within the interaction range r_c around particle i, and $\eta_i(t)$ is a random noise term uniformly distributed in the interval $\eta[-\frac{\pi}{2}, +\frac{\pi}{2}]$, where η is the noise amplitude.

The particle then moves at constant speed v_0 in the new direction:

$$\boldsymbol{r_i}(t + \Delta t) = \boldsymbol{r_i}(t) + v_0 \Delta t \hat{e}_i(t + \Delta t)$$

This model depends only on three control parameters: the total density of particles $\rho = N/L^2$, where L is the linear size of the system; the noise amplitude η, which controls the degree of randomness in the alignment; and the ratio $v_0 \Delta t / r$, which measures the relative importance of motion and interaction.

This model exhibits a phase transition from a disordered state to an ordered state as the noise amplitude η decreases or the density ρ increases. In the disordered state, the particles move in random directions and the average velocity of the system is zero. In the ordered state, the particles move coherently in the same direction and the average velocity of the system is nonzero. The order parameter that characterizes this transition is the normalized average velocity:

$$\varphi = \frac{1}{N v_0} \left| \sum_{i=1}^{N} \boldsymbol{v_i} \right|$$

which ranges from 0 in the disordered state to 1 in the ordered state.

The phase diagram of the Vicsek model shows a critical line that separates the disordered and ordered phases. The critical line can be approximated by a power law:

$$\eta_c \propto \rho^{-\alpha},$$

where η_c is the critical noise amplitude and α is a critical exponent. The value of α depends on the details of the model, such as the shape of the interaction region and the type of noise distribution. Numerical simulations have estimated α to be 0.45 for circular interaction regions and uniform noise distributions.

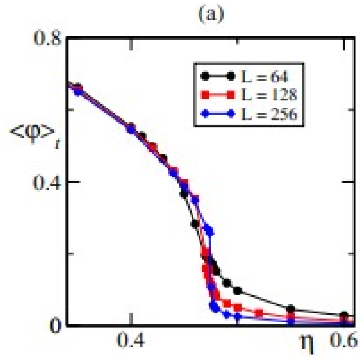

Figure 2. Characteristic order parameter curves vs. noise amplitude for fixed density and different system sizes. [Picture taken from F. Ginelli (2016)]

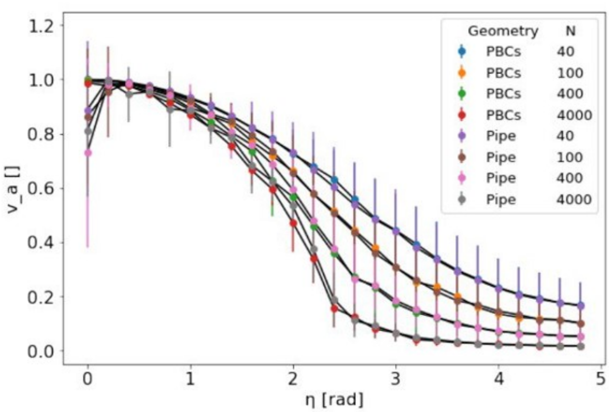

Figure 3. Order parameter versus noise strength η for systems with PBCs and a channel geometry, with increasing size and particle number but constant density $\rho = 1, v_0 = 5$ and $r_c = 18$. Data points are the mean values with standard deviations of 30 runs each. The black lines connect the data points for a better visual perception of the transitions. [Picture taken from Grégoire, et al. (2004).]

3. Incorporating Noise, Capturing Real-World Variability

Noise is an essential ingredient of the Vicsek model, as it introduces variability and uncertainty in individual behavior. Noise can be interpreted in different ways, depending on the context and system under study. For example, noise can represent individual preferences or opinions that deviate from the average direction of motion; random fluctuations or errors in sensing or processing information; external disturbances or perturbations from environmental factors or predators; or intrinsic variability or diversity in individual characteristics or abilities.

Noise can also be modeled in different ways, depending on its source and nature. Classically, in the original Vicsek model, noise is modeled as a uniform random term added to the alignment angle of each particle (Vicsek, et al., 2012). This noise is isotropic, meaning that it does not favor any particular direction, and is uncorrelated, meaning that it does not depend on the previous or current state of the system. However, other forms of noise can be considered, such as:

- Anisotropic noise, which favors certain directions over others. For example, noise can be biased towards the current direction of motion, or towards a preferred direction given by an external cue or a global coordinate system.
- Correlated noise, which depends on the previous or current state of the system. For example, noise can have a memory effect, such that the noise term at time t depends on the noise term at time $t-1$, or a feedback effect, such that the noise term at time t depends on the order parameter v at time t .
- Heterogeneous noise, which varies across different particles or regions of the system. For example, noise can have a spatial gradient, such that the noise amplitude increases or decreases with distance from a certain point, or a distribution, such that the noise amplitude follows a certain probability distribution across particles.

Noise plays a crucial role in the Vicsek model, as it affects the emergence and stability of collective patterns and the phase transition. Noise can have different effects, depending on its strength and type. For example:

- Noise can induce collective motion, when it is weak and anisotropic. In this case, noise can break the symmetry of the system and create a net alignment among particles, leading to coherent motion.
- Noise can suppress collective motion, when it is strong and isotropic. In this case, noise can destroy the alignment among particles and create random motion, leading to disorder.
- Noise can enhance collective motion, when it is moderate and correlated. In this case, noise can increase the fluctuations and correlations among particles and create ordered domains, leading to coexistence of order and disorder.

The phase transition in the Vicsek model is also affected by noise, as it changes the critical line and the critical exponents. The critical line shifts to higher or lower values of noise amplitude or density depending on the type and strength of noise. The critical exponents change to different values depending on the type and nature of noise. For example, anisotropic noise can change the universality class of the phase transition from continuous to discontinuous.

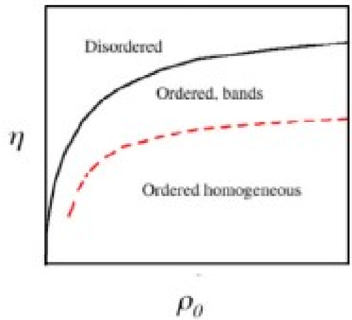

Figure 4. Qualitative VM phase diagram in the (ρ_0, η) plane. [Picture taken from F. Ginelli (2016)]

4. Simulation and Numerical Results

In order to simulate and construct the model in 2D, a functional approach was taken in the code. All the information about the SPPs were recorded in the (3, N) shaped matrix called *particles*. The first row of *particles* contains the x value of the positions of the SPPs, the second row contains the y value of the positions, and the third row contains the orientation of particles as an angle in the interval $[-\pi, \pi]$. The three following functions were then declared:

- `initialize_system()`:
  This simple function chooses random values for the position and orientation of each SPP in *particles*.

- `relative_distances()`:
  This function apllies Minimum Image Convention and computes the pairwise distances between particles. In order to make the function faster and avoid using for loops, a *numpy* broadcasting trick was employed. It involves using a 3D matrix where (i, j, k) shows the difference between the k'th coordinate of particle i and the k'th coordinate of particle j. Given periodic boundary conditions are used, care is taken to compute the minimum distance between one particle and the other particles, or the images of the other particles. Then, applying ternary conditions using *numpy.where*(), *NaN* is put inplace of distances larger than r_c. Finally, the 2D matrix of pairwise distances is returned.

- `vicsek_step()`:
  This function updates the positions and orientations after a time step Δt. It also applies boundary condition on the SPPs to make sure they are always within the desired limits L.

A number of animations of the system were made using *matplotlib.animation.FuncAnimation*() to better illustrate the dynamics of the Viscek model, for different values of η and N. Snapshots of these animations are given in figure 5. The full-length animations are also available. As evident from the snapshots, the low value of η means the system will demonstrate collective behavior very soon, as the effect of the noise is small and negligible.

Also, for a fixed density of $\rho \simeq 0.03$, phase transition was recorded as demonstrated in figure 6, for different system sizes. The continuous phase transition is evident from an ordered state in low noise to a disordered state in high noise values.

Moreover, the effect of SPP velocities were studied. As shown in figure 7, the value of v_0 does not affect the phase transition at all. This means the emergent collective behavior will be seen in systems of different scales, from birds to bacteria, as the velocity of motion is irrelevent.

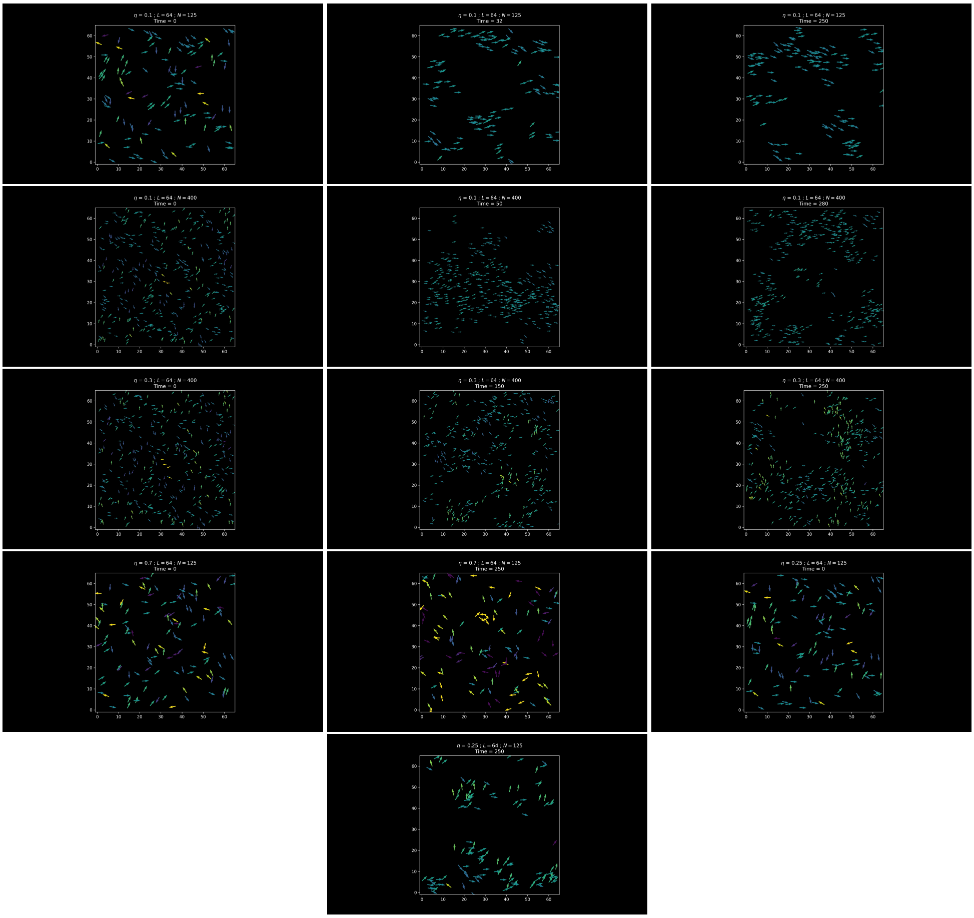

Figure 5. Snapshots of the animations. As evident, higher values of η increase the randomness and prevent the system from reaching an ordered phase.

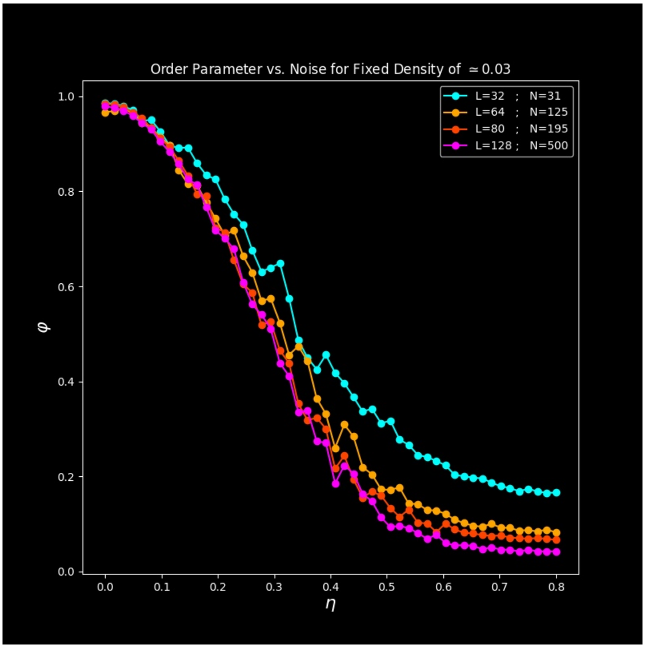

Figure 6. Phase transition for fixed ρ and v_0. For each value of ρ, 500 time steps were taken.

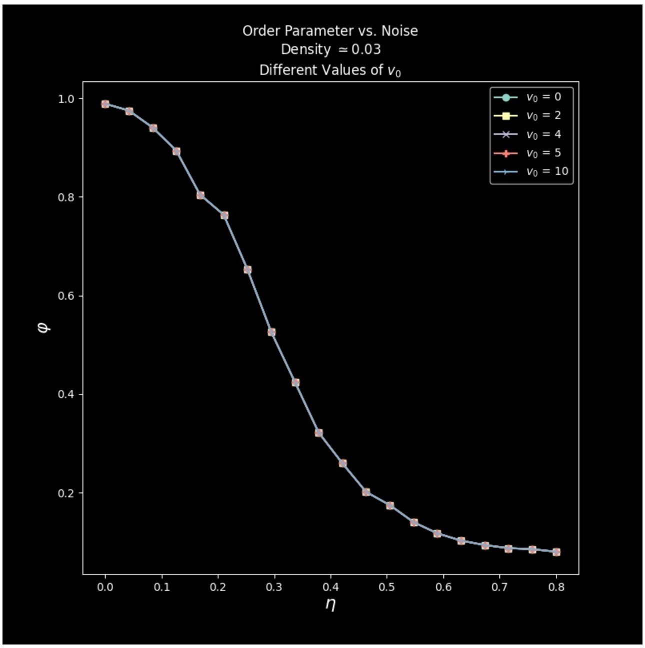

Figure 7. Effect of v_0 on phase transition.

Further on the Vicsek Model

5. Three-Dimensional Environments: Exploring Complex Spatial Patterns

The Vicsek model can be easily generalized to three-dimensional environments by adding a third component to the position and velocity vectors of each particle. The direction of each particle is now given by two angles: $\theta_i(t)$, which is the angle between the projection of $v_i(t)$ on the xy-plane and the x-axis; and $\phi_i(t)$, which is the angle between $v_i(t)$ and the z-axis. The dynamics of each particle is governed by the same rules as in two dimensions: alignment and noise. At each discrete time step Δt, each particle aligns its direction with the average direction of its neighbors within a sphere of radius r_c around it. This alignment rule can be expressed as:

$$\theta_i(t+\Delta t) = \langle \theta_j \rangle_{|r_i - r_j| < r_c} + \eta_i(t),$$

$$\phi_i(t+\Delta t) = \langle \phi_j \rangle_{|r_i - r_j| < r_c} + \eta_i(t)$$

As before, $\langle \theta_j \rangle_{|r_i - r_j| < r_c}$ is the average angle of all particles j (including i) within the interaction sphere around particle i, and $\eta_i(t)$ is a random noise term uniformly distributed in the interval $\eta[-\frac{\pi}{2}, +\frac{\pi}{2}]$, where η is the noise amplitude. The particle then moves at constant speed v_0 in the new direction:

$$\boldsymbol{r_i}(t+\Delta t) = \boldsymbol{r_i}(t) + v_0 \Delta t \hat{e}_i(t+\Delta t),$$

where $\hat{e}_i(t+\Delta t)$ is the unit vector defining the direction of motion at time $t+\Delta t$. These two rules are iterated for all particles until a steady state is reached.

The Vicsek model in three dimensions has the same main parameters as in two dimensions: the density $\rho = N/L^3$, the noise amplitude η, and the ratio $v_0 \Delta t / t$.

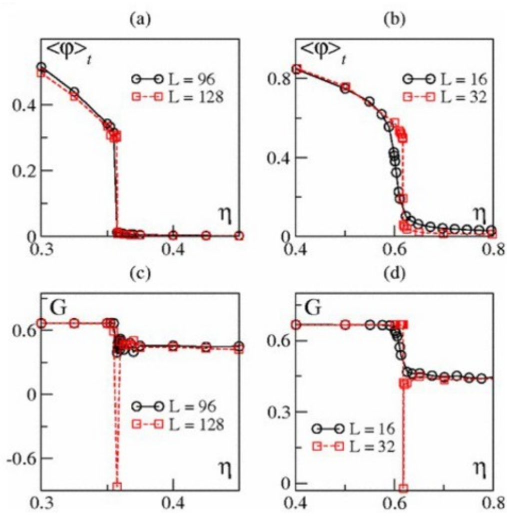

Figure 8. (Color online) Transition to collective motion in three spatial dimensions. Left panels: angular noise. Right panels: vectorial noise. (a),(b) Time-averaged order parameter vs noise amplitude at different system sizes. (c),(d) Binder cumulant G as a function of noise amplitude. ($\rho = 0.5$, $v_0 = 0.5$, time averages carried over 10^5 time steps.) [Picture taken from F. Ginelli (2016)]

The model in three dimensions exhibits a similar phase transition from a disordered state to an ordered state as in two dimensions. However, the ordered state in three dimensions is more complex and diverse than in two dimensions, as it can display various spatial patterns and structures. Some examples of these patterns are:

- Vortex formation, which is the appearance of rotating structures in the system. Vortices can have different shapes, sizes, and orientations, depending on the parameters and initial conditions. Vortices can also interact with each other, forming clusters or lattices.
- Helical motion, which is the appearance of spiral trajectories in the system. Helices can have different pitches, radii, and directions, depending on the parameters and initial conditions. Helices can also interact with each other, forming bundles or braids
- Complex spatial patterns, which are combinations or variations of vortices and helices. These patterns can have different symmetries, topologies, and dynamics, depending on the parameters and initial conditions. These patterns can also evolve over time, forming transitions or bifurcations.

6. Heterogeneity

Heterogeneity is a common feature of real systems, as individuals can have different characteristics or abilities that influence their behavior. Heterogeneity can be intrinsic, meaning that it is inherent to the individuals, or extrinsic, meaning that it is imposed by external factors or conditions. Heterogeneity can also be static, meaning that it is fixed and constant, or dynamic, meaning that it is variable and adaptive. Heterogeneity can be easily incorporated into the Vicsek model by allowing different particles to have different values of certain parameters. For example:

- Perception range heterogeneity, which means that different particles have different interaction radii r_i. This can represent intrinsic differences in individual sensing or processing abilities, or extrinsic differences in environmental cues or signals. This type of heterogeneity can induce cohesion, when it is small and dynamic. So, perception range heterogeneity can create adaptive interactions among particles, leading to increased alignment and compactness.
- Alignment strength heterogeneity, which means that different particles have different alignment weights w_i. This can represent intrinsic differences in individual social or behavioral tendencies, or extrinsic differences in social or environmental influences. Alignment strength heterogeneity can create different groups of particles with different average directions, leading to directional separation and reduced coherence
- Speed heterogeneity, which means that different particles have different speeds v0i. This can represent intrinsic differences in individual capabilities or preferences, or extrinsic differences in environmental conditions or constraints. In this case, heterogeneity can induce segregation, when it is large and static. So, speed heterogeneity can create different groups of particles with different average speeds, leading to spatial separation and reduced mixing

The phase transition in the Vicsek model is also affected by heterogeneity, as it changes the critical line and the critical exponents. The critical line shifts to higher or lower values of noise amplitude or density depending on the type and degree of heterogeneity. The critical exponents change to different values depending on the type and nature of heterogeneity. For example , speed heterogeneity can change the universality class of the phase transition from continuous to discontinuous .

7. The Vicsek Model and Network Theory: Bridging Disciplines

Network theory is a branch of mathematics and physics that studies the structure and function of complex systems by modeling them as networks or graphs. A network is a collection of nodes or vertices that represent the elements of the system, and links or edges that represent the interactions or relations between the elements. Networks can have different properties, such as size, density, degree distribution, clustering coefficient, path length, modularity, assortativity, etc., that characterize their topology and complexity.

Network theory can be used to study collective motion by constructing networks from the Vicsek model. The nodes of the network can represent the particles of the system, and the links of the network can represent the interactions or influences among the particles. The network can be static or dynamic, depending on whether the links are fixed or variable over time. The network can also be weighted or unweighted, depending on whether the links have different strengths or not.Some examples of networks constructed from the Vicsek model are:

- Interaction network, which is a dynamic and unweighted network where two nodes are linked if they are within the interaction range r of each other. This network captures the local alignment rule of the Vicsek model and reflects the spatial structure of the system.
- Influence network, which is a static and weighted network where two nodes are linked with a weight proportional to their velocity correlation. This network captures the global order parameter of the Vicsek model and reflects the directional structure of the system.
- Communication network, which is a dynamic and weighted network where two nodes are linked with a weight proportional to their information transfer. This network captures the noise term of the Vicsek model and reflects the variability structure of the system.
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Network theory can provide useful tools and methods to analyze and predict collective motion patterns from the Vicsek model such as:

- Centrality measures, which are numerical values that quantify the importance or influence of nodes in a network. Centrality measures can be used to identify leaders or followers in collective motion patterns, or to assess the robustness or vulnerability of collective motion patterns to perturbations or attacks.
- Community detection algorithms, which are techniques that partition a network into groups of nodes that are more densely connected within than between. Community detection algorithms can be used to identify clusters or subgroups in collective motion patterns, or to reveal the hierarchical or modular structure of collective motion patterns
- Network dynamics, which are models that describe the evolution of networks over time. Network dynamics can be used to simulate or predict the formation or transition of collective motion patterns, or to explore the effects of feedback or adaptation on collective motion patterns.

8. conclusion

The Vicsek model is a valuable tool for studying emergent phenomena and understanding the principles governing complex systems. The model is simple enough to be tractable and universal, but also rich enough to be realistic and diverse. The model is versatile enough to be adaptable and extendable, but also robust enough to be consistent and reliable. It is powerful enough to be descriptive and predictive, but also inspiring enough to be creative and innovative.

The Vicsek model has been extensively studied and extended by researchers to explore various aspects and implications of collective motion. The model has been generalized to three-dimensional environments, where it displays complex spatial patterns and structures. The model has also incorporated attraction and repulsion forces, which influence the formation and stability of collective patterns. The model has also introduced heterogeneity, which impacts the emergence and robustness of collective patterns.

This model has also been combined with network theory and machine learning algorithms to enhance our understanding of collective behavior. Network theory also links the microscopic level of individual behavior with the macroscopic level of collective behavior, and reveals the mechanisms and principles that govern the emergence and self-organization of complex systems. Machine learning provides novel techniques to study collective motion from data generated from the Vicsek model. Machine learning also integrates different disciplines and domains, and creates novel solutions and applications for real-world problems.

However, there are still many open questions and challenges for research in this field. What are the effects of higher dimensions or different geometries on collective motion patterns? What are the roles of different types or sources of noise on collective motion patterns? What are the mechanisms or criteria for leader-follower or consensus formation in collective motion patterns?What are the best ways to measure or quantify collective motion patterns? What are the best ways to control or manipulate collective motion patterns? What are the best ways to generalize or transfer collective motion models across different systems or domains? And so on. These questions pose interesting challenges and opportunities for future researches in this field.

9. Acknowledgement

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