SHARIF UNIVERSITY OF TECHNOLOGY
PHYSICS DEPARTMENT

Problem Set 2

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Exercise 2.2

Matrix Formalism

Let A be the $N \times N$ adjacency matrix of an undirected unweighted network, without self-loops. Let 1 be a column vector of N elements, all equal to 1. In other words $\mathbf{1} = (1, 1, ..., 1)^T$, where the superscript T indicates the transpose operation. Use the matrix formalism (multiplicative constants, multiplication row by column, matrix operations like transpose and trace, etc, but avoid the sum symbol $\sum$) to write expressions for:

(a) The vector $\mathbf{k}$ whose elements are the degrees k_i of all nodes $i = 1, 2, ..., N$.

(b) The total number of links, L, in the network.

(c) The number of triangles T present in the network, where a triangle means three nodes, each connected by links to the other two (Hint: you can use the trace of a matrix).

(d) The vector $\mathbf{k}_{nn}$ whose element i is the sum of the degrees of node i's neighbors.

(e) The vector $\mathbf{k}_{nnn}$ whose element i is the sum of the degrees of node i's second neighbors.

Answer.

(a)

$$\mathbf{k} = \mathbf{1}^T \mathbf{A}$$

(b)

$$L = \frac{1}{2}\mathbf{1}\mathbf{k}^T = \frac{1}{2}\mathbf{1}\mathbf{A}^T\mathbf{1}$$

(c) Theorem: Raising adjacency matrix to the n-th power gives n-length walks between two vertices (Link to the Proof: Here). Using this theorem, we have:

$$T = \frac{tr(\mathbf{A}^3)}{3!}$$

(d) Let e_i be an $N \times 1$ vector of zeros with a one in i-th position. Then the i-th column of the adjacency matrix A is:

$$\mathbf{A}_i = \mathbf{A}e_i$$

These show the neighbors of node i. Multiplying this vector with the vector $\mathbf{k}$ from part of (a) will give:

$$\text{the sum of the degrees of node i's neighbors} = \mathbf{A}_i\mathbf{k} = (\mathbf{A}e_i)^T\mathbf{k} = (\mathbf{A}e_i)^T\mathbf{1}^T\mathbf{A}$$

$$\Longrightarrow \mathbf{k}_{nn} = \mathbf{A}^T\mathbf{k}$$

(e) Following the argument from part (d), we have:

$$\mathbf{k}_{nnn} = \mathbf{A}^T\mathbf{k}_{nn} = \mathbf{A}^T(\mathbf{A}^T\mathbf{k})$$

Exercise 2.4

Degree, Clustering Coefficient and Components

(a) Consider an undirected network of size N in which each node has degree k = 1. Which condition does N have to satisfy? What is the degree distribution of this network? How many components does the network have?

(b) Consider now a network in which each node has degree k = 2 and clustering coefficient C = 1. How does the network look like? What condition does N satisfy in this case?

Answer.

(a) N has to be even. The degree distribution is:

$$p_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

The network has $\frac{N}{2}$ components because every two nodes are linked together.

(b) This network is made up of many triangles. These triangles are not connected to one another. As a result, N has to be divisible by 3.