Discrete Optimization - Assignment 1

In this assignment you will be asked questions related to the traveling salesman problem (TSP) and the use of branch-and-bound methods for solving the TSP.

The assignment is due 23:55 on 06.10.2014 and should be completed in groups of 2 or 3. Please upload your written report and source code to absalon. You are permitted to discuss the assignment with students from other groups but should not copy from each other.

Theoretical part - formulation and lower bounds

Given a complete directed graph $G = (V, E, c)$ with $|V| = n$ where c_{ij} is the cost of the edge from vertex i to vertex j, the (asymmetric) travelling salesman problem is to find a minimum cost Hamiltonian tour of G.

1.1

We will now provide an integer programming formulation, known as the subtour formulation, of the TSP. Here

$$x_{ij} = \begin{cases} 1, & \text{if edge (i,j) is included} \\ 0, & \text{otherwise} \end{cases}$$

and the problem is

$$\begin{aligned} \min \; & \sum_{i,j \in V} c_{ij} x_{ij} \\ \text{s.t.} \; & \sum_{i \in V, i \neq j} x_{ij} = 1 \text{ for all } j \in V \\ & \sum_{j \in V, j \neq i} x_{ij} = 1 \text{ for all } i \in V \\ & \sum_{i,j \in S} x_{ij} \leq |S| - 1 \\ & S \subset V, 2 \leq |S| \leq n - 2 \\ & x_{ij} \in \{0, 1\} \end{aligned}$$

Prove that the formulation is correct. You must prove that any feasible solution will satisfy the above constraints, and that any solution satisfying the above constraints is feasible.

1.2

We have constraints $\sum_{i,j \in S} x_{ij} \leq |S| - 1$ for every $S \subset V$, such that $2 \leq |S| \leq n - 2$. How many of these constraints are there (in terms of n)?

1.3

We now present the compact formulation from the lectures. Let G and x_{ij} and n be as in the subtour formulation. Let t_i be integers (note that we can choose to use real numbers for t_i as well) representing the order in which the vertices are visited. Thus if $t_i = 1$, vertex i is the first vertex visited.

$$\begin{aligned}
\min \quad & \sum_{i,j \in V} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i \in V, i \neq j} x_{ij} = 1 \text{ for all } j \in V \\
& \sum_{j \in V, j \neq i} x_{ij} = 1 \text{ for all } i \in V \\
& t_j \geq t_i + 1 - n(1 - x_{ij}) \\
& i \in V, j \in V \setminus \{1\} \\
& x_{ij} \in \{0, 1\} \\
& t_i \in \mathbb{R}_+
\end{aligned}$$

How many constraints of the form $t_j \geq t_i + 1 - n(1 - x_{ij})$ are there (in terms of n)?

1.4

You are not required to give a proof, but give a possible reason for why the subtour formulation from exercise 1.1 might be preferred over the compact formulation from exercise 1.3, despite having a much greater number of constraints.

1.5

Let G be as before and denote one of the vertices as vertex 1. Consider now the subset of edges of G consisting of a tree together with an additional edge incident to vertex 1 such that vertex 1 is in a cycle. Prove that the minimum cost such subset is a lower bound for the minimum cost Hamiltonian tour.

Implementation part - branch-and-bound

You will now attempt to solve the three instances included in the source code uploaded to absalon.

2.1

Provide an upper bound for the first problem instance and describe a simple heuristic for finding upper bounds for the TSP.

2.2

Attempt to solve the three instances using branch and bound and in CPLEX. If you cannot solve one of the instances, simply report this. You may choose to use the LP relaxation, the relaxation from exercise 1.5, or any other relaxation of your choosing as a lower bound in your branch and bound implementation.

2.3

Report the number of branch and bound nodes and the running time for each instance. There will be a competition for who can obtain the best running times and the fewest branch and bound nodes. The winning group will be given a bottle of wine for each group member.