Electromagnetic Waves
Lecture Notes

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Contents

1 TIME–VARYING FIELDS AND MAXWELL'S EQUATIONS

1.1 Introduction

Electrostatic Model:

$$\nabla \times \vec{\mathbf{E}} = 0 \tag{1.1}$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho \tag{1.2}$$

$\vec{\mathbf{E}}$: electric field intensity (V/m)
$\vec{\mathbf{D}}$: electric flux density (C/m^2)
ρ : charge density (C/m^3)
For linear (*doğrusal*) and isotropic (*yön bağımsız*) media we have

$$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}} \tag{1.3}$$

The electric field intensity $\vec{\mathbf{E}}$ for an electrostatic model is conservative ($\nabla \times \vec{\mathbf{E}} = 0$) and can be expressed as the gradient of a scalar potential.

$$\vec{\mathbf{E}} = -\nabla V \qquad \text{(Electrostatic model)} \tag{1.4}$$

Magnetostatic Model:

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{1.5}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} \tag{1.6}$$

$\vec{\mathbf{B}}$: magnetic flux density (T = Wb/m^2)
$\vec{\mathbf{H}}$: magnetic field intensity (A/m)
$\vec{\mathbf{J}}$: current density (A/m^2)
For linear and isotropic media we have

$$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}} \tag{1.7}$$

A changing magnetic field gives rise to an electric field, and a changing electric field gives rise to a magnetic field.

1.2 Faraday's Law of Electromagnetic Induction

Fundamental Postulate for Electromagnetic Induction

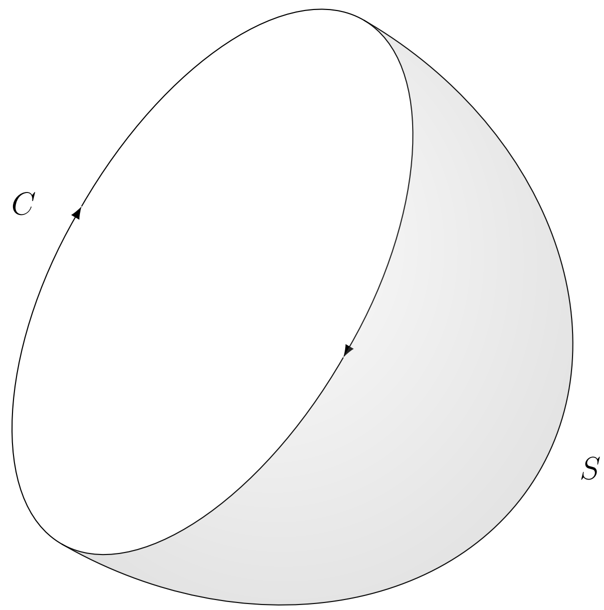
$$\boxed{\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}} \tag{1.8}$$

The electric field intensity $\vec{\mathbf{E}}$ in a region of time–varying magnetic flux density $\vec{\mathbf{B}}$ is non-conservative ($\nabla \times \vec{\mathbf{E}} \neq 0$) and cannot be expressed as the gradient of a scalar potential. We will see that

$$\vec{\mathbf{E}} = -\nabla V + ? \qquad \text{(Time–varying fields)} \tag{1.9}$$

Now, we will obtain the integral form of Faraday's law (Eq. 1.8). We will use Stokes' theorem.

$$\int_S (\nabla \times \vec{\mathbf{A}}) \cdot d\vec{\mathbf{s}} = \oint_C \vec{\mathbf{A}} \cdot d\vec{\ell} \tag{1.10}$$



Taking the surface integral of both sides of Eq. 1.8 over an open surface we have

$$\int_S (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{s}} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}} \tag{1.11}$$

Applying Stokes' theorem

$$\int_S (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{s}} = \oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} \tag{1.12}$$

$$\boxed{\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}}} \qquad \text{(Faraday's law)} \tag{1.13}$$

A Stationary Circuit in a Time–Varying Magnetic Field

For a stationary circuit with a contour C and surface S we have

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{1.14}$$

Let's define

$$V = \oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} \tag{1.15}$$

where V is the electromotive force (emf) induced in a circuit with contour C (V), and

$$\varPhi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{1.16}$$

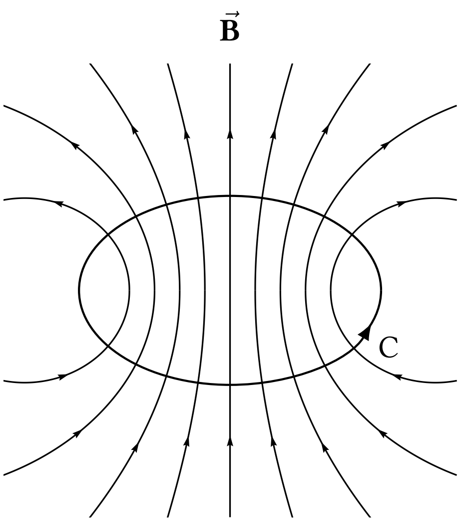
where $\varPhi$ is the magnetic flux crossing surface S (Wb). So we have

$$\boxed{V = -\frac{d\varPhi}{dt}} \qquad \text{(Faraday's law of electromagnetic induction)} \tag{1.17}$$

The electromotive force induced in a stationary closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.

The induced emf will cause a current to flow in the closed loop in a such a direction as to oppose the change in the linking magnetic flux.

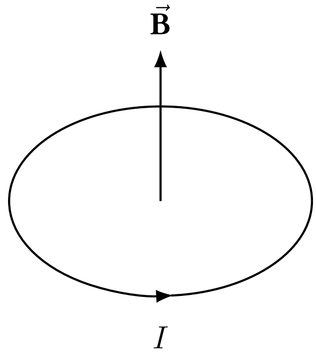
The current induced by the changing flux is called the induced current. The negative sign is known as **Lenz's law**.



Let's assume that the flux through a circuit C is decreasing.

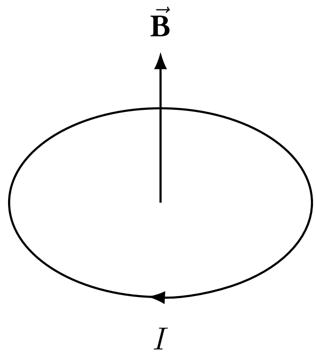
$$\frac{d\varPhi}{dt} < 0, \qquad V > 0 \tag{1.18}$$

Then by Lenz's law, the induced emf produces a current I in the circuit which gives rise to a magnetic field so directed as to increase the flux through C.



The reverse situation is shown in the following figure when the flux through C is increasing.

$$\frac{d\Phi}{dt} > 0, \qquad V < 0 \tag{1.19}$$

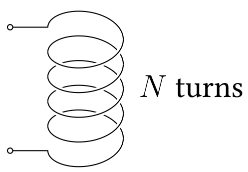


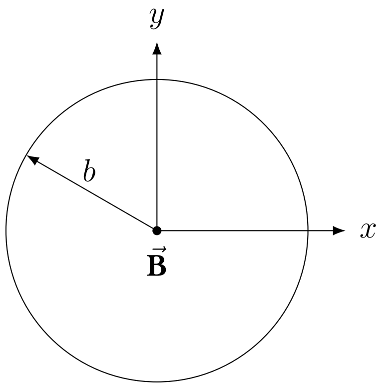
The change in flux can occur for a variety of reasons. For instance
i) the magnetic induction field $\vec{\mathbf{B}}$ may be changing in time,
ii) the size or shape of the circuit may be altering,
iii) the circuit may be moving in a manner which continually alters the flux passing through it.

Example 1.1 A circular loop (*çembersel halka*) of N turns (*sarım*) of conducting wire lies in the $x-y$ plane with its center at the origin of a magnetic field specified by

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z\, B_0\, \cos\left(\frac{\pi r}{2b}\right) \sin\omega t \tag{1.20}$$

where b is the radius of the loop and ω is the angular frequency. Find the emf induced in the loop.
<u>Solution</u>





$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z \, B_0 \, \cos\left(\frac{\pi r}{2b}\right) \sin \omega t$$

$$V = -\frac{d\Phi}{dt} \tag{1.21}$$

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{1.22}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z \, B_0 \, \cos\left(\frac{\pi r}{2b}\right) \sin \omega t \tag{1.23}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_z \, r \, d\phi \, dr \tag{1.24}$$

$$\hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_z = 1 \tag{1.25}$$

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \left[B_0 \, \cos\left(\frac{\pi r}{2b}\right) \sin \omega t\right] \, [r \, d\phi \, dr] \tag{1.26}$$

$$\Phi = \int_0^{2\pi} \int_0^b B_0 \, \cos\left(\frac{\pi r}{2b}\right) \sin \omega t \, r \, dr \, d\phi \tag{1.27}$$

$$\Phi = B_0 \, \sin \omega t \, \int_0^{2\pi} \, d\phi \int_0^b \cos\left(\frac{\pi r}{2b}\right) \, r \, dr \tag{1.28}$$

$$\int_0^{2\pi} \, d\phi = 2\pi \tag{1.29}$$

$$\Phi = 2\pi \, B_0 \, \sin \omega t \, \int_0^b \cos\left(\frac{\pi r}{2b}\right) \, r \, dr \tag{1.30}$$

$$I = \int_0^b r \, \cos\left(\frac{\pi r}{2b}\right) \, dr \tag{1.31}$$

$$u = \frac{\pi r}{2b} \Rightarrow r = \frac{2b}{\pi} u \tag{1.32}$$

$$du = \frac{\pi}{2b} dr \Rightarrow dr = \frac{2b}{\pi} du \tag{1.33}$$

$$I = \int \left[\frac{2b}{\pi} u\right] \, \cos(u) \, \left[\frac{2b}{\pi} du\right] \tag{1.34}$$

$$I = \left(\frac{2b}{\pi}\right)^2 \int \, u \, \cos u \, du \tag{1.35}$$

$$J = \int x \cos x \, dx \tag{1.36}$$

$$\int u \, dv = uv - \int v \, du \qquad \text{(integration by parts)} \tag{1.37}$$

$$u = x \qquad dv = \cos x \, dx \tag{1.38}$$

$$du = dx \qquad v = \sin x \tag{1.39}$$

$$J = x \sin x - \int \sin x \, dx = x \sin x + \cos x \tag{1.40}$$

$$I = \left(\frac{2b}{\pi}\right)^2 J = \left(\frac{2b}{\pi}\right)^2 (u \sin u + \cos u) \tag{1.41}$$

$$I = \left(\frac{2b}{\pi}\right)^2 \left[\left(\frac{\pi r}{2b}\right) \sin\left(\frac{\pi r}{2b}\right) + \cos\left(\frac{\pi r}{2b}\right)\right]_0^b \tag{1.42}$$

$$\begin{aligned} I = \left(\frac{2b}{\pi}\right)^2 & \left[\left(\frac{\pi b}{2b}\right) \sin\left(\frac{\pi b}{2b}\right) + \cos\left(\frac{\pi b}{2b}\right) \right. \\ & \left. - \left(\frac{\pi 0}{2b}\right) \sin\left(\frac{\pi 0}{2b}\right) - \cos\left(\frac{\pi 0}{2b}\right)\right] \end{aligned} \tag{1.43}$$

$$I = \left(\frac{2b}{\pi}\right)^2 \left[\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) - 0 - \cos(0)\right] \tag{1.44}$$

$$I = \left(\frac{2b}{\pi}\right)^2 \left[\frac{\pi}{2} - 1\right] \tag{1.45}$$

$$\Phi = 2\pi B_0 \sin \omega t \int_0^b \cos\left(\frac{\pi r}{2b}\right) r \, dr \tag{1.46}$$

$$\Phi = 2\pi B_0 \sin \omega t \left(\frac{2b}{\pi}\right)^2 \left(\frac{\pi}{2} - 1\right) \tag{1.47}$$

$$\Phi = \frac{8b^2}{\pi} \left(\frac{\pi}{2} - 1\right) B_0 \sin \omega t \tag{1.48}$$

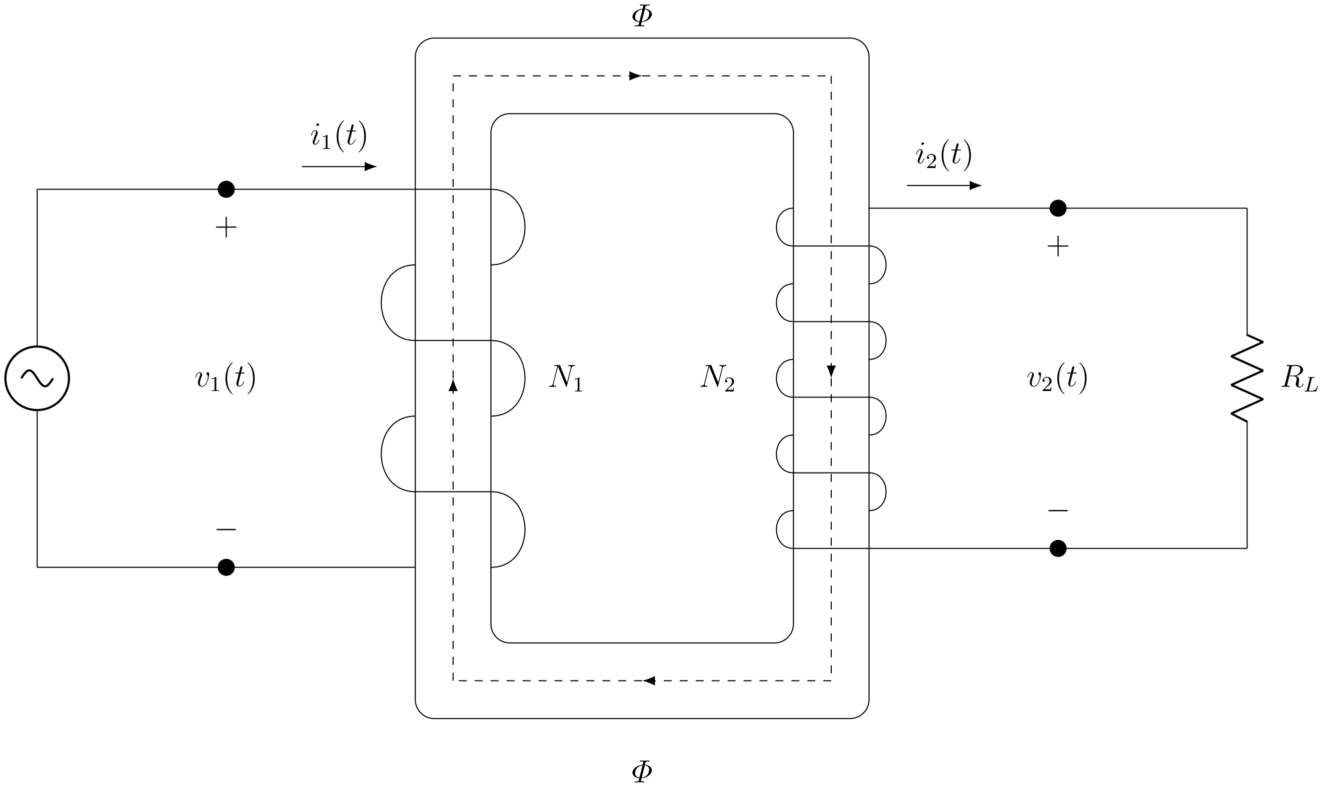
$$V = -\frac{d\Phi}{dt} \tag{1.49}$$

$$V = -\omega \frac{8b^2}{\pi} \left(\frac{\pi}{2} - 1\right) B_0 \cos \omega t \qquad \text{(for one turn)} \tag{1.50}$$

$$V = -N \omega \frac{8b^2}{\pi} \left(\frac{\pi}{2} - 1\right) B_0 \cos \omega t \quad \text{(V)} \qquad \text{(for } N \text{ turns)} \tag{1.51}$$

Transformers

A transformer is an alternating current (a-c) device that transforms voltages, currents and impedances. A transformer consists of two or more coils (sargı) coupled magnetically through a common ferromagnetic core (çekirdek).



A transformer

For any closed path in a magnetic circuit

$$\sum_j N_j\, I_j = \sum_k \mathcal{R}_k\, \Phi_k \tag{1.52}$$

The algebraic sum of ampere-turns (amper-sarım) around a closed path in a magnetic circuit is equal to the algebraic sum of the products of the reluctances and fluxes.

For the closed path in the magnetic circuit traced by magnetic flux Φ

$$N_1\, i_1 - N_2\, i_2 = \mathcal{R}\, \Phi \tag{1.53}$$

N_1 : the number of turns in the primary circuit
N_2 : the number of turns in the secondary circuit
i_1 : the current in the primary circuit (A)
i_2 : the current in the secondary circuit (A)
$\mathcal{R}$: reluctance of the magnetic circuit (manyetik direnç) (1/H)

The induced magnetomotive force (mmf) in the secondary circuit $N_2\, i_2$, opposes the flow of the magnetic flux Φ created by the mmf in the primary circuit $N_1\, i_1$. The reluctance of the ferromagnetic core is

$$\mathcal{R} = \frac{\ell}{\mu S} \tag{1.54}$$

ℓ : length of the core (m)
S : cross-sectional area of the core (m^2)
μ : permeability (manyetik geçirgenlik) of the core (H/m)

$$N_1\, i_1 - N_2\, i_2 = \mathcal{R}\,\Phi = \frac{\ell}{\mu S}\,\Phi \tag{1.55}$$

Ideal Transformer

For an ideal transformer we assume that $\mu \to \infty$. So

$$N_1\, i_1 - N_2\, i_2 = 0 \tag{1.56}$$

$$\boxed{\frac{i_1}{i_2} = \frac{N_2}{N_1}} \tag{1.57}$$

The ratio of the currents in the primary and secondary windings (sargı) of an ideal transformer is equal to the inverse ratio of the numbers of turns.

From Faraday's law

$$v_1 = -N_1\,\frac{d\Phi}{dt} \tag{1.58}$$

$$v_2 = -N_2\,\frac{d\Phi}{dt} \tag{1.59}$$

So, we have

$$\boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}} \tag{1.60}$$

The ratio of the voltages across the primary and secondary windings of an ideal transformer is equal to the turns ratio.

When the secondary winding is terminated in a load resistance R_L, the effective load seen by the source connected to primary winding is

$$(R_1)_{\text{eff}} = \frac{v_1}{i_1} = \frac{v_2\,\frac{N_1}{N_2}}{i_2\,\frac{N_2}{N_1}} = \frac{v_2}{i_2}\left(\frac{N_1}{N_2}\right)^2 = R_L\left(\frac{N_1}{N_2}\right)^2 \tag{1.61}$$

where

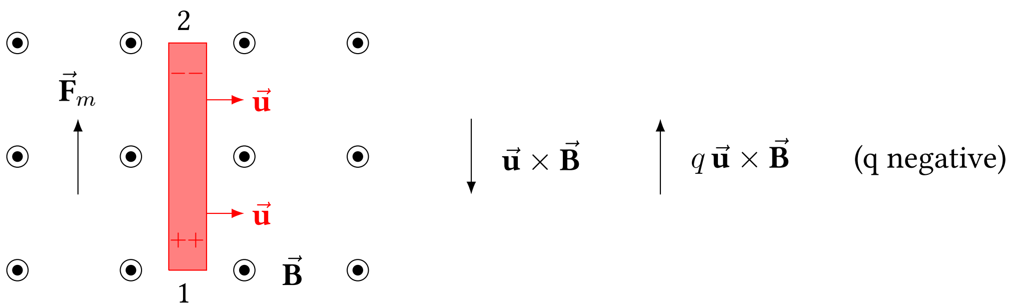
$$R_L = \frac{v_2}{i_2} \tag{1.62}$$

In a similar way, for a sinusoidal source $v_1(t)$ and a load impedance Z_L, the effective load seen by the source is

$$(Z_1)_{\text{eff}} = Z_L\left(\frac{N_1}{N_2}\right)^2 \tag{1.63}$$

A Moving Conductor in a Static Magnetic Field

When a conductor moves with a velocity $\vec{\mathbf{u}}$ in a static (non-time-varying) magnetic field $\vec{\mathbf{B}}$, a force $\vec{\mathbf{F}}_m = q\,\vec{\mathbf{u}} \times \vec{\mathbf{B}}$ will cause the free electrons in the conductor to drift towards one end of the conductor.



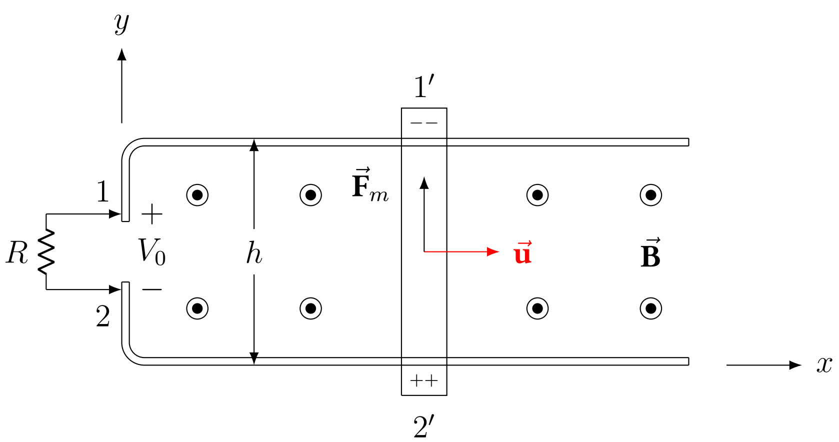
To an observer moving with the conductor there is no apparent motion, and the magnetic force per unit charge $\vec{\mathbf{F}}_m/q = \vec{\mathbf{u}} \times \vec{\mathbf{B}}$ can be interpreted as an induced electric field acting along the conductor and producing a voltage

$$V_{21} = \int_1^2 (\vec{\mathbf{u}} \times \vec{\mathbf{B}}) \cdot d\vec{\ell} \tag{1.64}$$

If the moving conductor is a part of closed circuit C, then the emf generated around the circuit is

$$V' = \oint_C (\vec{\mathbf{u}} \times \vec{\mathbf{B}}) \cdot d\vec{\ell} \quad \text{(V)} \tag{1.65}$$

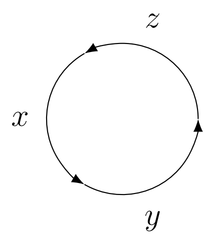
Example 1.2
A metal bar slides over a pair of conducting rails in a uniform magnetic field $\vec{\mathbf{B}} = \hat{\mathbf{a}}_z\, B_0$ with a constant velocity $\vec{\mathbf{u}}$, as shown in figure.
a) Determine the open circuit-voltage V_0 that appears across the terminals 1 and 2.
b) Assuming that a resistance R is connected between the terminals, find the electric power dissipated in R.
c) Show that this electric power is equal to the mechanical power required to move the sliding bar with a velocity $\vec{\mathbf{u}}$.



Solution

a) Solution I

$$V_0 = V_1 - V_2 = \oint_C (\vec{\mathbf{u}} \times \vec{\mathbf{B}}) \cdot d\vec{\ell} = \int_{2'}^{1'} (\vec{\mathbf{u}} \times \vec{\mathbf{B}}) \cdot d\vec{\ell} \tag{1.66}$$



$$\vec{\mathbf{u}} \times \vec{\mathbf{B}} = \hat{\mathbf{a}}_x\, u \times \hat{\mathbf{a}}_z\, B_0 = -\hat{\mathbf{a}}_y\, u\, B_0 \tag{1.67}$$

$$d\vec{\ell} = \hat{\mathbf{a}}_y\, dy \tag{1.68}$$

$$V_0 = \int_0^h (-)\, u\, B_0\, dy = -u\, B_0\, y\Big|_0^h = -u\, B_0\, h \qquad \text{(V)} \tag{1.69}$$

Solution II

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{1.70}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z\, B_0 \tag{1.71}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_z\, dx\, dy \tag{1.72}$$

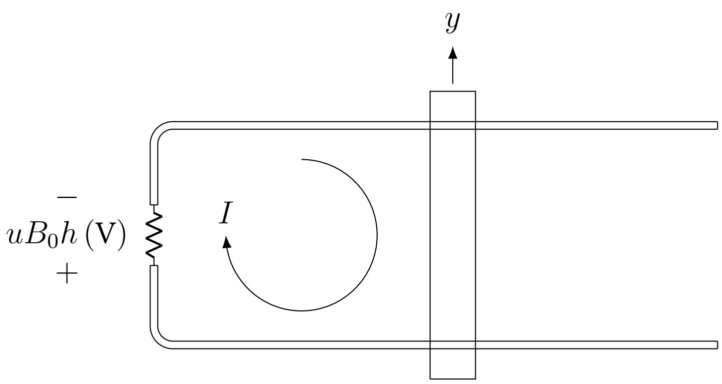
$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B_0\, dx\, dy \tag{1.73}$$

$$\Phi = \int_0^h \int_0^x B_0\, dx\, dy = B_0\, h\, x \tag{1.74}$$

$$x = u\, t \qquad (\text{distance} = \text{velocity} \times \text{time}) \tag{1.75}$$

$$\Phi = B_0\, h\, u\, t \tag{1.76}$$

$$V = -\frac{d\Phi}{dt} = -B_0\, h\, u \qquad \text{(V)} \tag{1.77}$$



$$I = \frac{V_0}{R} = \frac{B_0 h u}{R}$$

b)

$$P_e = I\, V_0 = \frac{V_0}{R}(V_0) = \frac{(V_0)^2}{R} = \frac{(u\, B_0\, h)^2}{R} \qquad \text{(Watt)} \tag{1.78}$$

c)

$$\text{Work} = \text{Force} \times \text{Distance} \tag{1.79}$$

$$\text{Power} = \frac{\text{Work}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \text{Force} \times \frac{\text{Distance}}{\text{Time}} \tag{1.80}$$

$$\text{Power} = \text{Force} \times \text{Velocity} \tag{1.81}$$

Mechanical power

$$P_{\text{mech}} = \vec{\mathbf{F}}_{\text{mech}} \cdot \vec{\mathbf{u}} \tag{1.82}$$

$\vec{\mathbf{F}}_{\text{mech}}$ is the mechanical force required to counteract the magnetic force $\vec{\mathbf{F}}_m$, which the magnetic field exerts on the current carrying metal bar.

$$\vec{\mathbf{F}}_m = I \oint_C d\vec{\ell} \times \vec{\mathbf{B}} \tag{1.83}$$

$d\vec{\ell}$ is in the direction of current flow.

$$d\vec{\ell} = \hat{\mathbf{a}}_y \, dy \tag{1.84}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z \, B_0 \tag{1.85}$$

$$d\vec{\ell} \times \vec{\mathbf{B}} = (\hat{\mathbf{a}}_y \times \hat{\mathbf{a}}_z) \, B_0 \, dy = \hat{\mathbf{a}}_x \, B_0 \, dy \tag{1.86}$$

I is in clockwise direction.

$$\vec{\mathbf{F}}_m = -I \int_{2'}^{1'} \hat{\mathbf{a}}_x \, B_0 \, dy = -\hat{\mathbf{a}}_x \, I \, B_0 \, y \Big|_0^h = -\hat{\mathbf{a}}_x \, I \, B_0 \, h \qquad \text{(magnetic force)} \tag{1.87}$$

$$\vec{\mathbf{F}}_{\text{mech}} = -\vec{\mathbf{F}}_m = \hat{\mathbf{a}}_x \, I \, B_0 \, h \tag{1.88}$$

$$P_{\text{mech}} = \vec{\mathbf{F}}_{\text{mech}} \cdot \vec{\mathbf{u}} = (\hat{\mathbf{a}}_x \, I \, B_0 \, h) \cdot (\hat{\mathbf{a}}_x \, u) = I \, B_0 \, h \, u \tag{1.89}$$

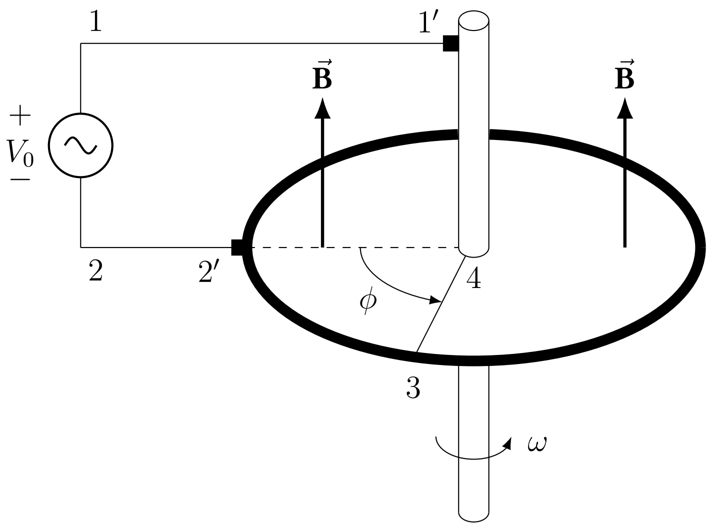
$$I = \frac{V_0}{R} = \frac{(B_0 \, h \, u)}{R} \tag{1.90}$$

$$P_{\text{mech}} = \left(\frac{B_0 \, h \, u}{R} \right) (B_0 \, h \, u) = \frac{(B_0 \, h \, u)^2}{R} \qquad \text{(Watt)} \tag{1.91}$$

$$P_{\text{mech}} = P_e \tag{1.92}$$

Example 1.3

The Faraday disk generator consists of a circular metal disk rotating with a constant angular velocity ω in a uniform and constant magnetic flux density $\vec{\mathbf{B}} = \hat{\mathbf{a}}_z \, B_0$ that is parallel to the axis of rotation. Brush contacts are provided at the axis and on the rim of the disk. Determine the open-circuit voltage of the generator if the radius of the disk is b.



Faraday disk generator

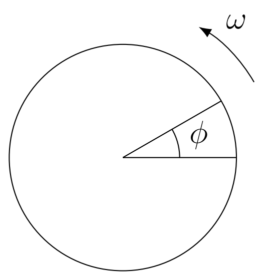
<u>Solution I</u>

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{1.93}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z \, B_0 \tag{1.94}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_z \, r \, d\phi \, dr \tag{1.95}$$

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B_0 \, d\phi \, r \, dr \tag{1.96}$$



$$\phi = \omega t, \quad \omega = \frac{d\phi}{dt}$$

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \int_0^b \int_0^{\omega t} B_0 \, d\phi \, r \, dr \tag{1.97}$$

$$\Phi = B_0 \int_0^{\omega t} d\phi \int_0^b r \, dr \tag{1.98}$$

$$\int_0^{\omega t} d\phi = \omega t \tag{1.99}$$

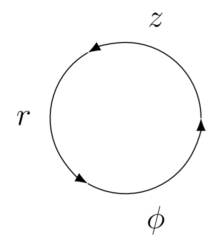
$$\int_0^b r \, dr = \frac{b^2}{2} \tag{1.100}$$

$$\Phi = \frac{1}{2} \, b^2 \, B_0 \, \omega \, t \tag{1.101}$$

$$V_0 = -\frac{d\Phi}{dt} = -\frac{1}{2} \, b^2 \, B_0 \, \omega \tag{1.102}$$

Solution II

$$V_0 = \oint_C (\vec{\mathbf{u}} \times \vec{\mathbf{B}}) \cdot d\vec{\ell} \tag{1.103}$$



$$\vec{\mathbf{u}} = \omega\, r\, \hat{\mathbf{a}}_\phi \tag{1.104}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z\, B_0 \tag{1.105}$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{B}} = \hat{\mathbf{a}}_r\, \omega\, r\, B_0 \tag{1.106}$$

$$d\vec{\ell} = d\vec{\mathbf{r}} = \hat{\mathbf{a}}_r\, dr \tag{1.107}$$

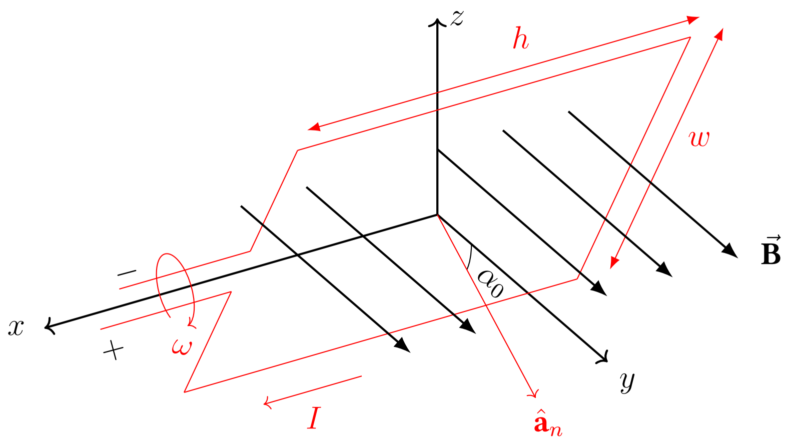
$$\vec{\mathbf{u}} \times \vec{\mathbf{B}} \cdot d\vec{\ell} = B_0\, \omega\, r\, dr \tag{1.108}$$

$$V_0 = \int_3^4 B_0\, \omega\, r\, dr = \omega\, B_0 \int_b^0 r\, dr = \omega\, B_0 \left[\frac{1}{2} r^2\right]_b^0 \tag{1.109}$$

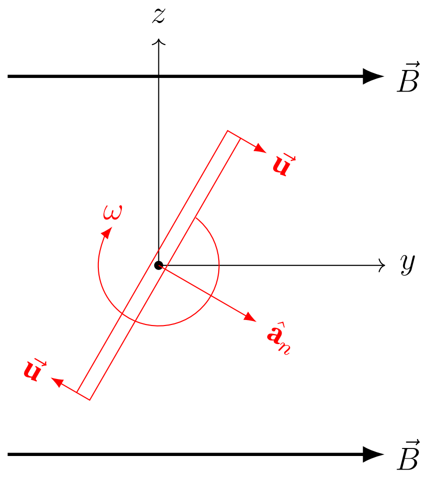
$$V_0 = -\frac{1}{2} b^2\, B_0\, \omega \tag{1.110}$$

Example 1.4

An h by w rectangular conducting loop is situated in a changing magnetic field $\vec{\mathbf{B}} = \hat{\mathbf{a}}_y\, B_0\, \sin \omega t$. The normal of the loop initially makes an angle α_0 with $\hat{\mathbf{a}}_y$. Find the induced emf in the loop a) when the loop is at rest, b) when the loop rotates with an angular velocity ω about the x-axis.



a) Perspective view



b) View from $+x$ direction

Solution

a)

$$V = -\frac{d\Phi}{dt} \tag{1.111}$$

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{1.112}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_y \, B_0 \, \sin \omega t \tag{1.113}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_n \, ds = \hat{\mathbf{a}}_n \, dx \, dz \tag{1.114}$$

$$\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_n = \cos \alpha_0 \tag{1.115}$$

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B_0 \, \cos \alpha_0 \sin \omega t \, dx \, dz \tag{1.116}$$

$$\Phi = \int_{-w/2}^{w/2} \int_{-h/2}^{h/2} B_0 \, \cos \alpha_0 \sin \omega t \, dx \, dz \tag{1.117}$$

$$\Phi = B_0 \, \cos \alpha_0 \sin \omega t \int_{-w/2}^{w/2} \int_{-h/2}^{h/2} dx \, dz \tag{1.118}$$

$$\int_{-w/2}^{w/2} \int_{-h/2}^{h/2} dx \, dz = w \, h = S \qquad \text{(area of the loop)} \tag{1.119}$$

$$\Phi = B_0 \, S \, \cos \alpha_0 \sin \omega t \tag{1.120}$$

$$V = -\frac{d\Phi}{dt} = -\omega \, B_0 \, S \, \cos \alpha_0 \cos \omega t \tag{1.121}$$

If the circuit is completed through an external load, V will produce a current that will oppose the change in Φ.

b)

$$V = -\frac{d\Phi}{dt} \tag{1.122}$$

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{1.123}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_y B_0 \sin \omega t \tag{1.124}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_n\, ds = \hat{\mathbf{a}}_n\, dx\, dz \tag{1.125}$$

$\hat{\mathbf{a}}_n$ rotates with angular velocity ω, so the angle between $\hat{\mathbf{a}}_n$ and $\hat{\mathbf{a}}_y$ changes with time.

$$\alpha = \alpha_0 + \omega t \tag{1.126}$$

$$\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_n = \cos \alpha = \cos(\alpha_0 + \omega t) \tag{1.127}$$

$$\Phi = B_0\, S\, \cos(\alpha_0 + \omega t) \sin \omega t \tag{1.128}$$

$$V = -\frac{d\Phi}{dt} = -B_0\, S\, \frac{d}{dt} [\cos(\alpha_0 + \omega t) \sin \omega t] \tag{1.129}$$

$$\frac{d}{dt} [\cos(\alpha_0 + \omega t) \sin \omega t] = -\omega\, \sin(\alpha_0 + \omega t)\, \sin \omega t + \omega\, \cos(\alpha_0 + \omega t) \cos \omega t \tag{1.130}$$

$$V = -B_0\, S\, [-\omega\, \sin(\alpha_0 + \omega t)\, \sin \omega t + \omega\, \cos(\alpha_0 + \omega t) \cos \omega t] \tag{1.131}$$

$$V = -\omega\, B_0\, S\, [\cos(\alpha_0 + \omega t) \cos \omega t - \sin(\alpha_0 + \omega t)\, \sin \omega t] \tag{1.132}$$

$$\cos(x + y) = \cos x\, \cos y - \sin x\, \sin y \tag{1.133}$$

$$V = -\omega\, B_0\, S\, \cos[(\alpha_0 + \omega t) + \omega t] \tag{1.134}$$

$$V = -\omega\, B_0\, S\, \cos(\alpha_0 + 2\omega t) \tag{1.135}$$

$$V = -\omega\, B_0\, S\, \cos(2\omega t + \alpha_0) \tag{1.136}$$

1.3 Maxwell's Equations

Differential form of Maxwell's equations

$$\boxed{\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}} \tag{1.137}$$

$$\boxed{\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t}} \tag{1.138}$$

$$\boxed{\nabla \cdot \vec{\mathbf{D}} = \rho} \tag{1.139}$$

$$\boxed{\nabla \cdot \vec{\mathbf{B}} = 0} \tag{1.140}$$

$$\nabla \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t} \qquad \text{(Equation of continuity; conservation of charge)} \tag{1.141}$$

$\vec{\mathbf{E}}$: electric field intensity (V/m)
$\vec{\mathbf{B}}$: magnetic flux density (T = Wb/m^2)
$\vec{\mathbf{H}}$: magnetic field intensity (A/m)
$\vec{\mathbf{D}}$: electric flux density (C/m^2)
$\vec{\mathbf{J}}$: current density (A/m^2)
ρ : volume charge density (C/m^3)

$$\vec{\mathbf{F}} = q\,(\vec{\mathbf{E}} + \vec{\mathbf{u}} \times \vec{\mathbf{B}}) \qquad \text{(Lorentz's force equation)} \tag{1.142}$$

These four Maxwell's equations, together with the equation of continuity and Lorentz's force equation, form the foundation of electromagnetic theory. These equations can be used to explain and predict all *macroscopic* electromagnetic phenomena.

The term $\dfrac{\partial \vec{\mathbf{D}}}{\partial t}$ is called displacement current density (deplasman akımı; yer değiştirme akımı).

Derivation of equation of continuity

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \tag{1.143}$$

$$\nabla \cdot (\nabla \times \vec{\mathbf{A}}) = 0 \qquad \text{(Identity II)} \tag{1.144}$$

$$\nabla \cdot (\nabla \times \vec{\mathbf{H}}) = \nabla \cdot \left(\vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t}\right) \tag{1.145}$$

$$\nabla \cdot (\nabla \times \vec{\mathbf{H}}) = 0 \tag{1.146}$$

$$\nabla \cdot \left(\vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t}\right) = 0 \tag{1.147}$$

$$\nabla \cdot \vec{\mathbf{J}} + \nabla \cdot \frac{\partial \vec{\mathbf{D}}}{\partial t} = 0 \tag{1.148}$$

$$\nabla \cdot \vec{\mathbf{J}} + \frac{\partial}{\partial t}(\nabla \cdot \vec{\mathbf{D}}) = 0 \tag{1.149}$$

$$\nabla \cdot \vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{1.150}$$

$$\nabla \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t} \tag{1.151}$$

Integral form of Maxwell's equations

$$\int_S (\nabla \times \vec{\mathbf{A}}) \cdot d\vec{\mathbf{s}} = \oint_C \vec{\mathbf{A}} \cdot d\vec{\ell} \qquad \text{(Stokes' theorem)} \tag{1.152}$$

1)

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{1.153}$$

$$\int_S (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{s}} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}} \tag{1.154}$$

$$\boxed{\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}}} \qquad \text{(Faraday's law)} \tag{1.155}$$

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi}{dt} \tag{1.156}$$

2)

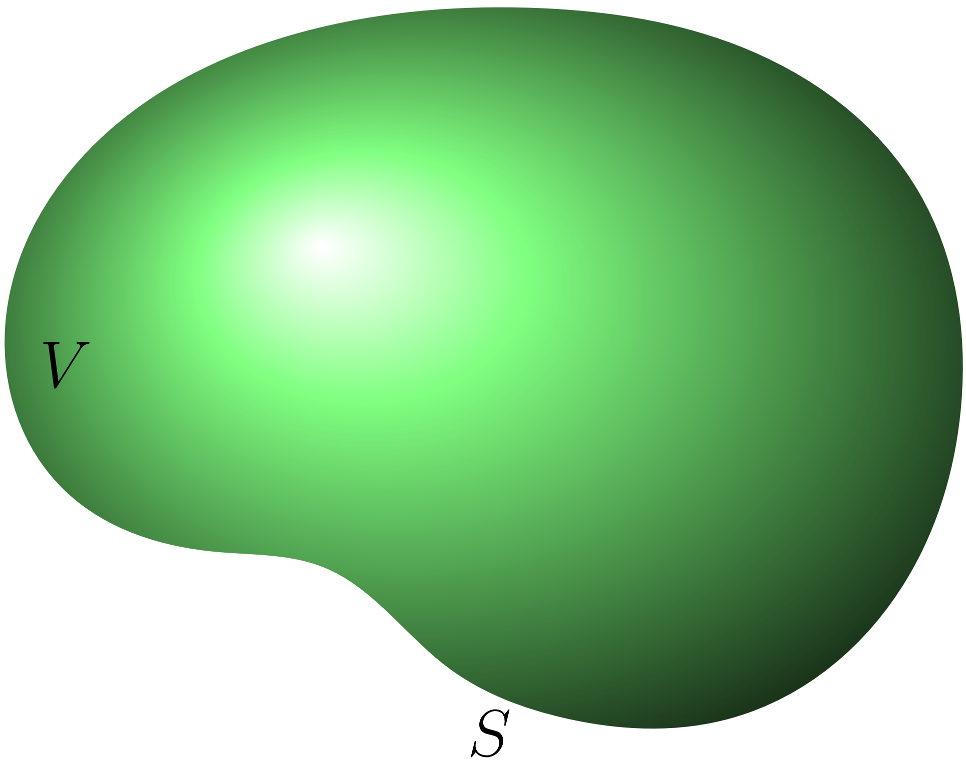
$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \tag{1.157}$$

$$\int_S (\nabla \times \vec{\mathbf{H}}) \cdot d\vec{\mathbf{s}} = \int_S \left(\vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right) \cdot d\vec{\mathbf{s}} \tag{1.158}$$

$$\boxed{\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_S \left(\vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right) \cdot d\vec{\mathbf{s}}} \qquad \text{(Ampere's circuital law)} \tag{1.159}$$

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} + \int_S \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} = I + \int_S \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} \tag{1.160}$$

$$\int_V (\nabla \cdot \vec{\mathbf{A}})\, dv = \oint_S \vec{\mathbf{A}} \cdot d\vec{\mathbf{s}} \qquad \text{(Divergence theorem)} \tag{1.161}$$



3)

$$\nabla \cdot \vec{\mathbf{D}} = \rho \tag{1.162}$$

$$\int_V (\nabla \cdot \vec{\mathbf{D}})\, dv = \int_V \rho\, dv \tag{1.163}$$

$$\boxed{\oint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{s}} = \int_V \rho\, dv} \qquad \text{(Gauss's law)} \tag{1.164}$$

$$\oint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{s}} = \int_V \rho\, dv = Q \tag{1.165}$$

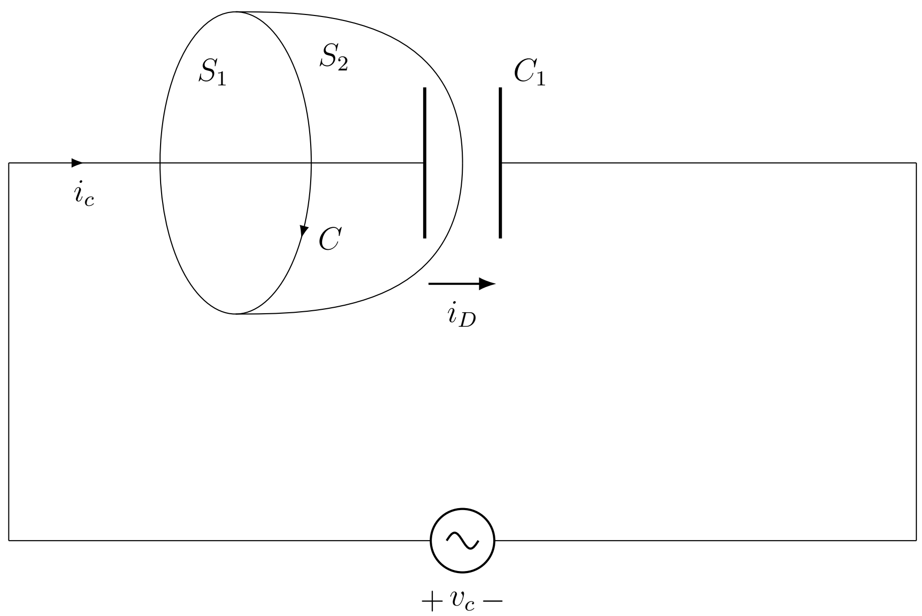
4)

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{1.166}$$

$$\int_V (\nabla \cdot \vec{\mathbf{B}})\, dv = \int_V 0\, dv \tag{1.167}$$

$$\boxed{\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 0} \qquad \text{(No isolated magnetic charge)} \tag{1.168}$$

Example 1.5
An ac voltage source of amplitude V_0 and angular frequency ω, $v_c = V_0 \sin \omega t$, is connected across a parallel-plate capacitor C_1, as shown in figure. a) Verify that the displacement current in the capacitor is the same as the conduction current in the wires. b) Determine the magnetic field intensity at a distance r from the wire.



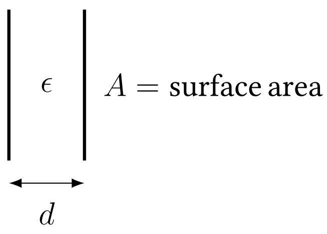
<u>Solution</u>

a)

$$i_c = C_1 \frac{dv_c}{dt} \qquad \text{(conduction current in the wires)} \tag{1.169}$$

$$i_c = C_1 \frac{d}{dt} (V_0 \ \sin \omega t) = \omega \, V_0 \, C_1 \ \cos \omega t \tag{1.170}$$

$$i_D = \int_S \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} \qquad \text{(displacement current in the capacitor)} \tag{1.171}$$

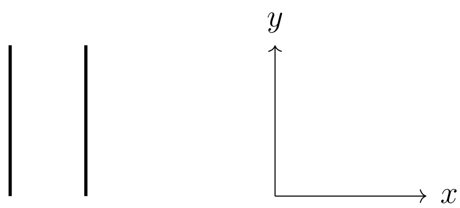


$$C_1 = \frac{\epsilon A}{d} \tag{1.172}$$

$$E = \frac{v_c}{d} \tag{1.173}$$

$$D = \epsilon \, E = \epsilon \, \frac{v_c}{d} = \frac{\epsilon}{d} \, V_0 \ \sin \omega t \tag{1.174}$$

$$\frac{\partial D}{\partial t} = \frac{\epsilon \, V_0}{d} \, \omega \cos \omega t \tag{1.175}$$



$$\vec{\mathbf{D}} = D \, \hat{\mathbf{a}}_x \tag{1.176}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_x \, dy \, dz \tag{1.177}$$

$$\frac{\partial \vec{\mathbf{D}}}{\partial t} = \hat{\mathbf{a}}_x \, \frac{\partial D}{\partial t} = \hat{\mathbf{a}}_x \, \frac{\epsilon \, V_0}{d} \, \omega \cos \omega t \tag{1.178}$$

$$\frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} = \frac{\epsilon \, V_0}{d} \, \omega \cos \omega t \, dy \, dz \tag{1.179}$$

$$i_D = \int \int \frac{\epsilon \, V_0}{d} \, \omega \cos \omega t \, dy \, dz = \frac{\epsilon \, V_0}{d} \, \omega \cos \omega t \int \int dy \, dz \tag{1.180}$$

$$\int \int dy \, dz = A \tag{1.181}$$

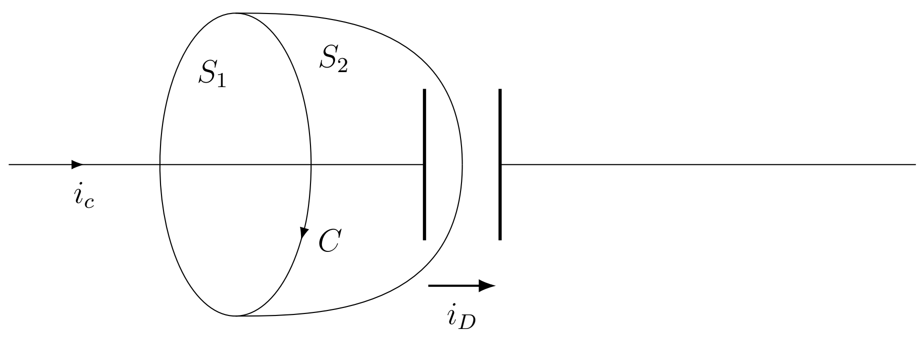
$$i_D = \frac{\epsilon \, A}{d} \, V_0 \, \omega \cos \omega t = C_1 \, V_0 \, \omega \cos \omega t \tag{1.182}$$

$$i_c = i_D = \omega \, V_0 \, C_1 \cos \omega t \tag{1.183}$$

The displacement current is equal to the conduction current.

b) The magnetic field intensity at a distance r from the conducting wire can be found by applying the generalized Ampere's circuital law:

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} + \int_S \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} \tag{1.184}$$



First let's choose planar disk surface S_1 for S. For this case $\vec{\mathbf{D}} = 0$, because no charges are deposited along the wire. So we have

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_{S_1} \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} \tag{1.185}$$

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_0^{2\pi} (H_\phi \, \hat{\mathbf{a}}_\phi) \cdot (r \, d\phi \, \hat{\mathbf{a}}_\phi) = \int_0^{2\pi} H_\phi \, r \, d\phi \; = \; H_\phi \, 2\pi r \tag{1.186}$$

$$\int_{S_1} \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} = i_c \; = \; \omega \, V_0 \, C_1 \, \cos \omega t \tag{1.187}$$

$$\Rightarrow H_\phi = \frac{C_1 \, V_0}{2\pi r} \omega \, \cos \omega t \tag{1.188}$$

Now let's choose curved surface S_2 passing through dielectric medium.

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_{S_2} \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} = i_D \tag{1.189}$$

$$\Rightarrow H_\phi \, 2\pi r = \omega \, V_0 \, C_1 \, \cos \omega t \tag{1.190}$$

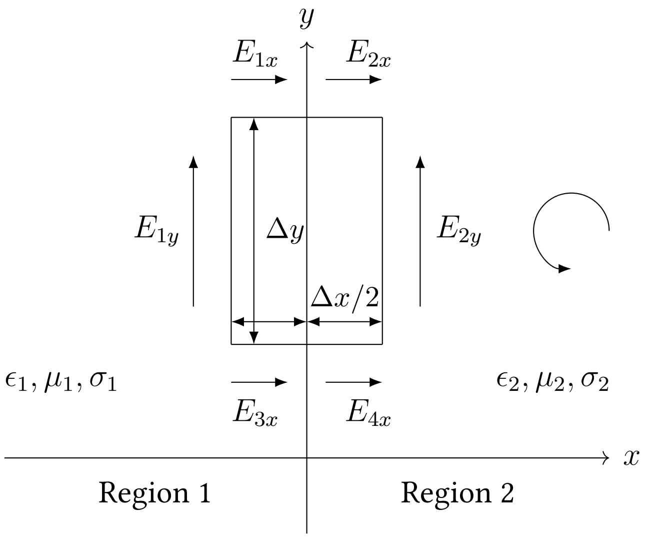
$$\Rightarrow H_\phi = \frac{C_1 \, V_0}{2\pi r} \omega \, \cos \omega t \tag{1.191}$$

1.4 Electromagnetic Boundary Conditions

Conditions on the Tangential Components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{1.192}$$

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}} \qquad \text{(Faraday's law)} \tag{1.193}$$



Suppose the surface of discontinuity to be the plane $x = 0$ as shown in figure. Consider the small rectangle of width Δx and length Δy enclosing a small portion of each media (1) and (2).

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}} \tag{1.194}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_z \, dx \, dy = \hat{\mathbf{a}}_z \, \Delta x \, \Delta y \tag{1.195}$$

$$E_{2y} \, \Delta y - E_{2x} \frac{\Delta x}{2} - E_{1x} \frac{\Delta x}{2} - E_{1y} \, \Delta y + E_{3x} \frac{\Delta x}{2} + E_{4x} \frac{\Delta x}{2} = -\frac{\partial B_z}{\partial t} \, \Delta x \, \Delta y \tag{1.196}$$

B_z is the average magnetic flux density through the rectangle $\Delta x \, \Delta y$. Now let $\Delta x \to 0$. So we obtain

$$E_{2y} \, \Delta_y - E_{1y} \, \Delta_y = 0 \tag{1.197}$$

$$\boxed{E_{1y} = E_{2y}} \tag{1.198}$$

The tangential components of an $\vec{\mathbf{E}}$ field is continuous across an interface.

$$E_{1y} - E_{2y} = 0 \tag{1.199}$$

$$\hat{\mathbf{a}}_n \times (\vec{\mathbf{E}}_1 - \vec{\mathbf{E}}_2) = 0 \tag{1.200}$$

$\hat{\mathbf{a}}_n = -\hat{\mathbf{a}}_x$: outward unit normal from medium 2 at the interface.

$$\vec{\mathbf{E}}_1 = \hat{\mathbf{a}}_x \, E_{1x} + \hat{\mathbf{a}}_y \, E_{1y} + \hat{\mathbf{a}}_z \, E_{1z} \tag{1.201}$$

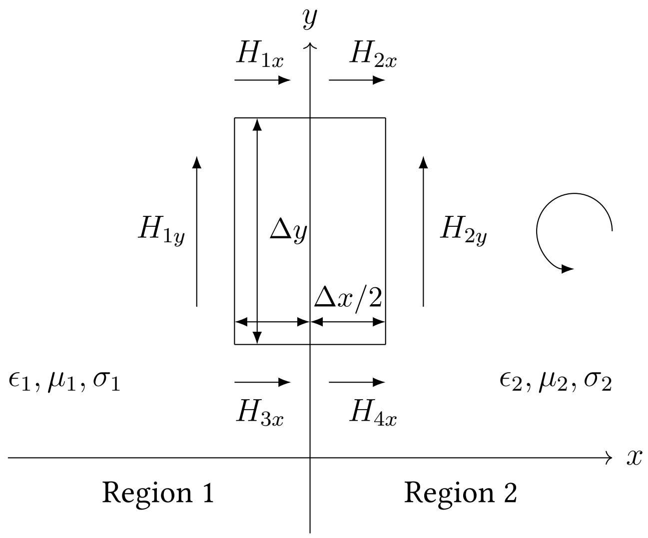
$$\vec{\mathbf{E}}_2 = \hat{\mathbf{a}}_x \, E_{2x} + \hat{\mathbf{a}}_y \, E_{2y} + \hat{\mathbf{a}}_z \, E_{2z} \tag{1.202}$$

$$\begin{aligned} \hat{\mathbf{a}}_n \times (\vec{\mathbf{E}}_1 \; - \; \vec{\mathbf{E}}_2) &= \\ &= -\hat{\mathbf{a}}_x \times (\hat{\mathbf{a}}_x \, E_{1x} + \hat{\mathbf{a}}_y \, E_{1y} + \hat{\mathbf{a}}_z \, E_{1z} - \hat{\mathbf{a}}_x \, E_{2x} - \hat{\mathbf{a}}_y \, E_{2y} - \hat{\mathbf{a}}_z \, E_{2z}) \\ &= -\hat{\mathbf{a}}_z \, E_{1y} + \hat{\mathbf{a}}_y \, E_{1z} + \hat{\mathbf{a}}_z \, E_{2y} - \hat{\mathbf{a}}_y \, E_{2z} \\ &= \hat{\mathbf{a}}_y (E_{1z} - E_{2z}) + \hat{\mathbf{a}}_z \, (E_{2y} - E_{1y}) = 0 \end{aligned} \tag{1.203}$$

$$\Rightarrow\ E_{1y} = E_{2y} \tag{1.204}$$

$$\Rightarrow\ E_{1z} = E_{2z} \tag{1.205}$$

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_S \left(\vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right) \cdot d\vec{\mathbf{s}} \qquad \text{(Ampere's circuital law)} \tag{1.206}$$



$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = H_{2y}\,\Delta y - H_{2x}\,\frac{\Delta x}{2} - H_{1x}\,\frac{\Delta x}{2} - H_{1y}\,\Delta_y + H_{3x}\,\frac{\Delta x}{2} + H_{4x}\,\frac{\Delta x}{2} \tag{1.207}$$

$$\lim_{\Delta x \to 0} \oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = H_{2y}\,\Delta y - H_{1y}\,\Delta y \tag{1.208}$$

$$\int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} = \vec{\mathbf{J}} \cdot \hat{\mathbf{a}}_z\,\Delta x\,\Delta y \qquad d\vec{\mathbf{s}} = \hat{\mathbf{a}}_z\,\Delta x\,\Delta y \tag{1.209}$$

$$\lim_{\Delta x \to 0} \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} = \lim_{\Delta x \to 0} [\vec{\mathbf{J}} \cdot \hat{\mathbf{a}}_z\,\Delta x\,\Delta y] = \lim_{\Delta x \to 0} [(\vec{\mathbf{J}}\,\Delta x) \cdot \hat{\mathbf{a}}_z\,\Delta y] \tag{1.210}$$

$$\lim_{\Delta x \to 0} \vec{\mathbf{J}}\,\Delta x = \vec{\mathbf{J}}_s \tag{1.211}$$

$\vec{\mathbf{J}}$: volume current density (A/m^2)
$\vec{\mathbf{J}}_s$: surface current density (A/m) (current sheet)

$$\lim_{\Delta x \to 0} \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} = \vec{\mathbf{J}}_s \cdot \hat{\mathbf{a}}_z\,\Delta y = J_{sz}\,\Delta y \tag{1.212}$$

$\vec{\mathbf{J}}_s \cdot \hat{\mathbf{a}}_z = J_{sz}$ (z component of $\vec{\mathbf{J}}_s$)

$$\lim_{\Delta x \to 0} \int_S \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} = \lim_{\Delta x \to 0} \frac{\partial D_z}{\partial t}\,\Delta x\,\Delta y = 0 \tag{1.213}$$

$$\Rightarrow\ H_{2y}\,\Delta y - H_{1y}\,\Delta y = J_{sz}\,\Delta y \tag{1.214}$$

$$\boxed{H_{2y} - H_{1y} = J_{sz}} \tag{1.215}$$

The tangential components of an $\vec{\mathbf{H}}$ field is discontinuous across an interface where a surface current exists.

$$H_{2y} - H_{1y} = J_{sz} \tag{1.216}$$

$$\hat{\mathbf{a}}_n \times (\vec{\mathbf{H}}_1 - \vec{\mathbf{H}}_2) = \vec{\mathbf{J}}_s \tag{1.217}$$

$\hat{\mathbf{a}}_n = -\hat{\mathbf{a}}_x$: outward unit normal from medium 2 at the interface.

$$\vec{\mathbf{H}}_1 = \hat{\mathbf{a}}_x\, H_{1x} + \hat{\mathbf{a}}_y\, H_{1y} + \hat{\mathbf{a}}_z\, H_{1z} \tag{1.218}$$

$$\vec{\mathbf{H}}_2 = \hat{\mathbf{a}}_x\, H_{2x} + \hat{\mathbf{a}}_y\, H_{2y} + \hat{\mathbf{a}}_z\, H_{2z} \tag{1.219}$$

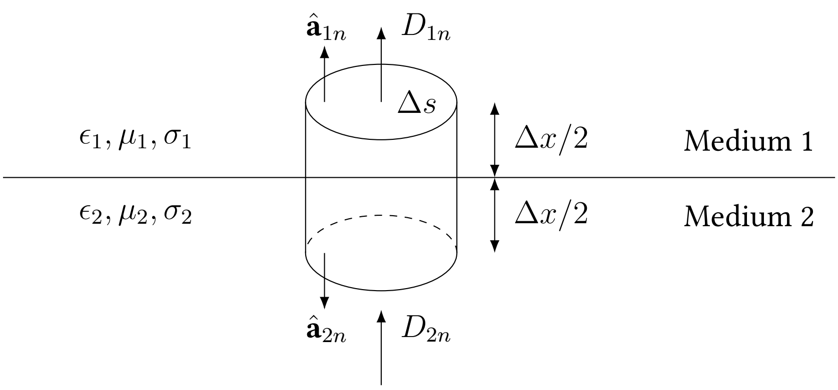
$$\vec{\mathbf{J}}_s = \hat{\mathbf{a}}_x\, J_{sx} + \hat{\mathbf{a}}_y\, J_{sy} + \hat{\mathbf{a}}_z\, J_{sz} \tag{1.220}$$

$$\begin{aligned} \hat{\mathbf{a}}_n \times (\vec{\mathbf{H}}_1 - \vec{\mathbf{H}}_2) &= \\ &= -\hat{\mathbf{a}}_x \times (\hat{\mathbf{a}}_x\, H_{1x} + \hat{\mathbf{a}}_y\, H_{1y} + \hat{\mathbf{a}}_z\, H_{1z} - \hat{\mathbf{a}}_x\, H_{2x} - \hat{\mathbf{a}}_y\, H_{2y} - \hat{\mathbf{a}}_z\, H_{2z}) \\ &= -\hat{\mathbf{a}}_z\, H_{1y} + \hat{\mathbf{a}}_y\, H_{1z} + \hat{\mathbf{a}}_z\, H_{2y} - \hat{\mathbf{a}}_y\, H_{2z} \\ &= \hat{\mathbf{a}}_y\, (H_{1z} - H_{2z}) + \hat{\mathbf{a}}_z\, (H_{2y} - H_{1y}) = \vec{\mathbf{J}}_s \end{aligned} \tag{1.221}$$

$$\Rightarrow H_{2y} - H_{1y} = J_{sz} \tag{1.222}$$

$$\Rightarrow H_{1z} - H_{2z} = J_{sy} \tag{1.223}$$

Conditions on the Normal Components of $\vec{\mathbf{B}}$ and $\vec{\mathbf{D}}$



$$\oint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{s}} = \int_V \rho\, dv \tag{1.224}$$

$$\vec{\mathbf{D}} \cdot \hat{\mathbf{a}}_{1n}\, \Delta s + \vec{\mathbf{D}} \cdot \hat{\mathbf{a}}_{2n}\, \Delta s + \Psi_{\text{edge}} = \rho\, \Delta x\, \Delta s \tag{1.225}$$

$$D_{1n}\, \Delta s - D_{2n}\, \Delta s + \Psi_{\text{edge}} = \rho\, \Delta x\, \Delta s \tag{1.226}$$

Ψ_{edge} : the outward electric flux through the curved edge surface of the pillbox.

$$\lim_{\Delta x \to 0} \Psi_{\text{edge}} = 0 \tag{1.227}$$

$$\lim_{\Delta x \to 0} \rho \, \Delta x = \rho_s \tag{1.228}$$

ρ : volume charge density (C/m^3)

ρ_s : surface charge density (C/m^2)

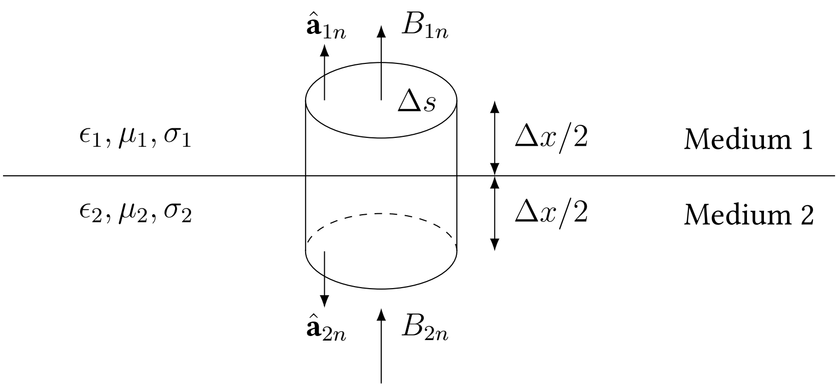
$$D_{1n} \, \Delta s - D_{2n} \, \Delta s = \rho_s \, \Delta s \tag{1.229}$$

$$\boxed{D_{1n} - D_{2n} = \rho_s} \tag{1.230}$$

The normal component of a $\vec{\mathbf{D}}$ field is discontinuous across an interface where a surface charge exists.

$$\hat{\mathbf{a}}_n \cdot (\vec{\mathbf{D}}_1 - \vec{\mathbf{D}}_2) = \rho_s \tag{1.231}$$

$\hat{\mathbf{a}}_n$: outward unit normal from medium 2 at the interface.



$$\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 0 \tag{1.232}$$

In a similar way

$$\vec{\mathbf{B}} \cdot \hat{\mathbf{a}}_{1n} \, \Delta s + \vec{\mathbf{B}} \cdot \hat{\mathbf{a}}_{2n} \, \Delta s + \Phi_{\text{edge}} = 0 \tag{1.233}$$

Φ_{edge} : the outward magnetic flux through the curved edge surface of the pillbox.

$$\lim_{\Delta x \to 0} \Phi_{\text{edge}} = 0 \tag{1.234}$$

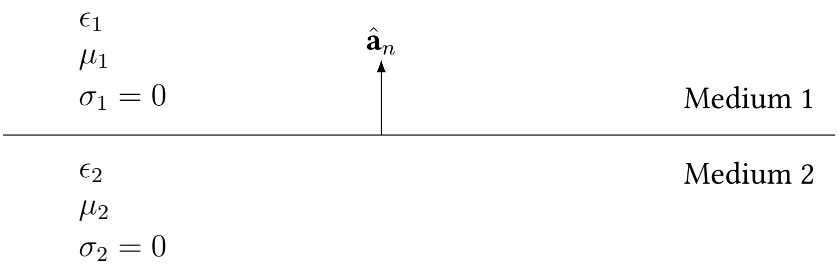
$$B_{1n} \, \Delta s - B_{2n} \, \Delta s = 0 \tag{1.235}$$

$$\boxed{B_{1n} = B_{2n}} \tag{1.236}$$

The normal component of a $\vec{\mathbf{B}}$ field is continuous across an interface.

Interface Between Two Lossless Linear Media

A lossless linear medium can be specified by a permittivity ϵ and permeability μ, with $\sigma = 0$. We set $\rho_s = 0$, $\vec{\mathbf{J}}_s = 0$.



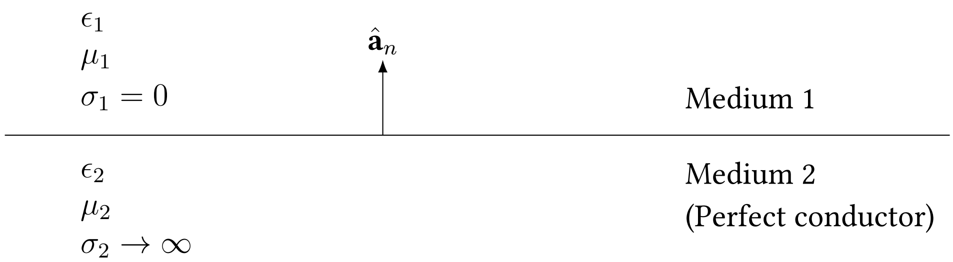
$$E_{1t} = E_{2t} \Rightarrow \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2} \Rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\epsilon_1}{\epsilon_2} \tag{1.237}$$

$$H_{1t} = H_{2t} \Rightarrow \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \Rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2} \tag{1.238}$$

$$D_{1n} = D_{2n} \Rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n} \tag{1.239}$$

$$B_{1n} = B_{2n} \Rightarrow \mu_1 H_{1n} = \mu_2 H_{2n} \tag{1.240}$$

Interface Between a Dielectric and a Perfect Conductor



In the interior of a perfect conductor the electric field is zero, and charges reside on the surface only.

$$\begin{aligned} \vec{\mathbf{E}}_2 &= 0 \\ \vec{\mathbf{H}}_2 &= 0 \\ \vec{\mathbf{D}}_2 &= 0 \\ \vec{\mathbf{B}}_2 &= 0 \end{aligned} \tag{1.241}$$

$$\begin{aligned} &E_{1t} = E_{2t} = 0 \\ &\hat{\mathbf{a}}_n \times (\vec{\mathbf{H}}_1 - \vec{\mathbf{H}}_2) = \vec{\mathbf{J}}_s \Rightarrow \hat{\mathbf{a}}_n \times \vec{\mathbf{H}}_1 = \vec{\mathbf{J}}_s \\ &\hat{\mathbf{a}}_n \cdot (\vec{\mathbf{D}}_1 - \vec{\mathbf{D}}_2) = \rho_s \Rightarrow \hat{\mathbf{a}}_n \cdot \vec{\mathbf{D}}_1 = \rho_s \\ &B_{1n} = B_{2n} = 0 \end{aligned} \tag{1.242}$$

$\hat{\mathbf{a}}_n$: outward unit normal from medium 2 at the interface.

1.5 Potential Functions

Wave Equation for Vector Potential $\vec{\mathbf{A}}$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \qquad \text{(4th Maxwell's equation)} \tag{1.243}$$

$$\nabla \cdot (\nabla \times \vec{\mathbf{A}}) = 0 \qquad \text{(Identity II)} \tag{1.244}$$

$\vec{\mathbf{B}}$ is solenoidal ($\nabla \cdot \vec{\mathbf{B}} = 0$). So $\vec{\mathbf{B}}$ can be expressed as the curl of another vector field.

$$\boxed{\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}} \tag{1.245}$$

The vector field $\vec{\mathbf{A}}$ is called the vector magnetic potential (Wb/m).

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \text{(1st Maxwell's equation)} \tag{1.246}$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial}{\partial t}(\nabla \times \vec{\mathbf{A}}) \tag{1.247}$$

$$\nabla \times \left(\vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t}\right) = 0 \tag{1.248}$$

$$\nabla \times (\nabla V) = 0 \qquad \text{(Identity I)} \tag{1.249}$$

$$\vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t} = -\nabla V \tag{1.250}$$

$$\boxed{\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \qquad \text{(V/m)}} \tag{1.251}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \qquad \text{(2nd Maxwell's equation)} \tag{1.252}$$

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}} \tag{1.253}$$

$$\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \tag{1.254}$$

Let's substitute Eqs. 2.341 and 2.343 into Eq. 1.252 and make use of the constitutive relations.

$$\vec{\mathbf{H}} = \frac{\vec{\mathbf{B}}}{\mu} \tag{1.255}$$

$$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}} \tag{1.256}$$

For a homogenous medium we have

$$\nabla \times \left(\frac{\vec{\mathbf{B}}}{\mu}\right) = \vec{\mathbf{J}} + \frac{\partial}{\partial t}(\epsilon \vec{\mathbf{E}}) \tag{1.257}$$

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{\mathbf{A}} \right) = \vec{\mathbf{J}} + \frac{\partial}{\partial t} \left[\epsilon \left(-\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \right) \right] \tag{1.258}$$

$$\frac{1}{\mu} \nabla \times \nabla \times \vec{\mathbf{A}} = \vec{\mathbf{J}} + \epsilon \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \right) \tag{1.259}$$

$$\nabla \times \nabla \times \vec{\mathbf{A}} = \mu \vec{\mathbf{J}} + \mu\epsilon \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \right) \tag{1.260}$$

$$\nabla \times \nabla \times \vec{\mathbf{A}} = \nabla(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2 \vec{\mathbf{A}} \tag{1.261}$$

$$\nabla(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2 \vec{\mathbf{A}} = \mu \vec{\mathbf{J}} + \mu\epsilon \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \right) \tag{1.262}$$

$$\nabla(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2 \vec{\mathbf{A}} = \mu \vec{\mathbf{J}} + \mu\epsilon \frac{\partial}{\partial t} (-\nabla V) - \mu\epsilon \frac{\partial}{\partial t} \left(\frac{\partial \vec{\mathbf{A}}}{\partial t} \right) \tag{1.263}$$

$$\nabla(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2 \vec{\mathbf{A}} = \mu \vec{\mathbf{J}} - \nabla \left(\mu\epsilon \frac{\partial V}{\partial t} \right) - \mu\epsilon \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} \tag{1.264}$$

$$\nabla(\nabla \cdot \vec{\mathbf{A}}) + \nabla \left(\mu\epsilon \frac{\partial V}{\partial t} \right) - \mu \vec{\mathbf{J}} = \nabla^2 \vec{\mathbf{A}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} \tag{1.265}$$

$$\nabla^2 \vec{\mathbf{A}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = \nabla(\nabla \cdot \vec{\mathbf{A}}) + \nabla \left(\mu\epsilon \frac{\partial V}{\partial t} \right) - \mu \vec{\mathbf{J}} \tag{1.266}$$

$$\nabla^2 \vec{\mathbf{A}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = \nabla \left(\nabla \cdot \vec{\mathbf{A}} + \mu\epsilon \frac{\partial V}{\partial t} \right) - \mu \vec{\mathbf{J}} \tag{1.267}$$

The definiton of a vector requires the specification of both its curl and its divergence. The curl of $\vec{\mathbf{A}}$ is equal to $\vec{\mathbf{B}}$, i.e. ($\nabla \times \vec{\mathbf{A}} = \vec{\mathbf{B}}$). We can choose the divergence of $\vec{\mathbf{A}}$ as follows:

$$\boxed{\nabla \cdot \vec{\mathbf{A}} + \mu\epsilon \frac{\partial V}{\partial t} = 0} \qquad \text{(Lorentz condition for potentials)} \tag{1.268}$$

So we obtain

$$\boxed{\nabla^2 \vec{\mathbf{A}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu \vec{\mathbf{J}}} \tag{1.269}$$

This is the nonhomogenous wave equation for vector potential $\vec{\mathbf{A}}$

Wave Equation for Scalar Potential V

Now let's derive the wave equation for scalar potential V.

$$\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \tag{1.270}$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho \tag{1.271}$$

$$\vec{\mathbf{D}} = \epsilon\,\vec{\mathbf{E}} \tag{1.272}$$

$$\nabla\cdot(\epsilon\,\vec{\mathbf{E}}) = \rho \tag{1.273}$$

$$\nabla\cdot\left(\epsilon\left[-\nabla V - \frac{\partial\vec{\mathbf{A}}}{\partial t}\right]\right) = \rho \tag{1.274}$$

$$-\nabla\cdot\epsilon\left(\nabla V + \frac{\partial\vec{\mathbf{A}}}{\partial t}\right) = \rho \tag{1.275}$$

For a constant ϵ, we obtain

$$-\epsilon\,\nabla\cdot\left(\nabla V + \frac{\partial\vec{\mathbf{A}}}{\partial t}\right) = \rho \tag{1.276}$$

$$\nabla\cdot\left(\nabla V + \frac{\partial\vec{\mathbf{A}}}{\partial t}\right) = -\frac{\rho}{\epsilon} \tag{1.277}$$

$$\nabla\cdot(\nabla V) = \nabla^2 V \tag{1.278}$$

$$\nabla^2 V + \frac{\partial}{\partial t}\,(\nabla\cdot\vec{\mathbf{A}}) = -\frac{\rho}{\epsilon} \tag{1.279}$$

Using Lorentz condition

$$\nabla\cdot\vec{\mathbf{A}} + \mu\epsilon\,\frac{\partial V}{\partial t} = 0 \tag{1.280}$$

$$\nabla\cdot\vec{\mathbf{A}} = -\mu\epsilon\,\frac{\partial V}{\partial t} \tag{1.281}$$

$$\nabla^2 V + \frac{\partial}{\partial t}\left(-\mu\epsilon\,\frac{\partial V}{\partial t}\right) = -\frac{\rho}{\epsilon} \tag{1.282}$$

$$\boxed{\nabla^2 V - \mu\epsilon\,\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}} \tag{1.283}$$

This is the nonhomogenous wave equation for scalar potential V.

1.6 Wave Equations and Their Solutions

For given charge and current distributions, ρ and $\vec{\mathbf{J}}$, we first solve the following nonhomogeneous wave equations for potentials V and $\vec{\mathbf{A}}$.

$$\boxed{\nabla^2 V - \mu\epsilon\,\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}}$$

$$\boxed{\nabla^2\,\vec{\mathbf{A}} - \mu\epsilon\,\frac{\partial^2\vec{\mathbf{A}}}{\partial t^2} = -\mu\,\vec{\mathbf{J}}}$$

With V and $\vec{\mathbf{A}}$ determined, $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ can be found from the following equations by differentiation.

$$\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t}$$

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}$$

Solution of Wave Equations for Potentials

We now consider the solution of the nonhomogeneous wave equation for scalar electric potential V.

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{1.284}$$

First, let's find the solution for a point charge at time t, located at the origin of the coordinates. Then by summing the effects of all charge elements in a given region we can find the total solution. For a point charge at the origin it is convenient to use spherical coordinates. Because of spherical symmetry, V depends only on R and t (not on θ and ϕ). $V(R,t)$ satisfies the following homogenous equation:

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \tag{1.285}$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) \tag{1.286}$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = 0 \qquad \text{(Except at the origin)} \tag{1.287}$$

Let's introduce a new variable

$$V(R,t) = \frac{1}{R} U(R,t) \tag{1.288}$$

$$\frac{\partial V}{\partial R} = \frac{\partial}{\partial R}\left(\frac{U}{R}\right) = \frac{\left(\frac{\partial U}{\partial R}\right) R - U}{R^2} \tag{1.289}$$

$$R^2 \frac{\partial V}{\partial R} = R \frac{\partial U}{\partial R} - U \tag{1.290}$$

$$\frac{\partial V}{\partial t} = \frac{1}{R}\frac{\partial U}{\partial t} \tag{1.291}$$

$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{R}\frac{\partial^2 U}{\partial t^2} \tag{1.292}$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R \frac{\partial U}{\partial R} - U\right) - \mu\epsilon \frac{1}{R}\frac{\partial^2 U}{\partial t^2} = 0 \tag{1.293}$$

$$\frac{\partial}{\partial R}\left(R \frac{\partial U}{\partial R} - U\right) = \frac{\partial U}{\partial R} + R \frac{\partial^2 U}{\partial R^2} - \frac{\partial U}{\partial R} = R \frac{\partial^2 U}{\partial R^2} \tag{1.294}$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}-U\right)=\frac{1}{R}\frac{\partial^2 U}{\partial R^2} \tag{1.295}$$

$$\frac{1}{R}\frac{\partial^2 U}{\partial R^2}-\mu\epsilon\frac{1}{R}\frac{\partial^2 U}{\partial t^2}=0 \tag{1.296}$$

$$\boxed{\frac{\partial^2 U}{\partial R^2}-\mu\epsilon\frac{\partial^2 U}{\partial t^2}=0} \tag{1.297}$$

One-dimensional homogeneous wave equation.

$$U=f\left(t-\frac{R}{c}\right) \tag{1.298}$$

$U=f\left(t+\frac{R}{c}\right)$ does not corresspond to a physically useful solution. So we have

$$U(R,t)=f\left(t-\frac{R}{c}\right), \qquad c=\frac{1}{\sqrt{\mu\epsilon}} \tag{1.299}$$

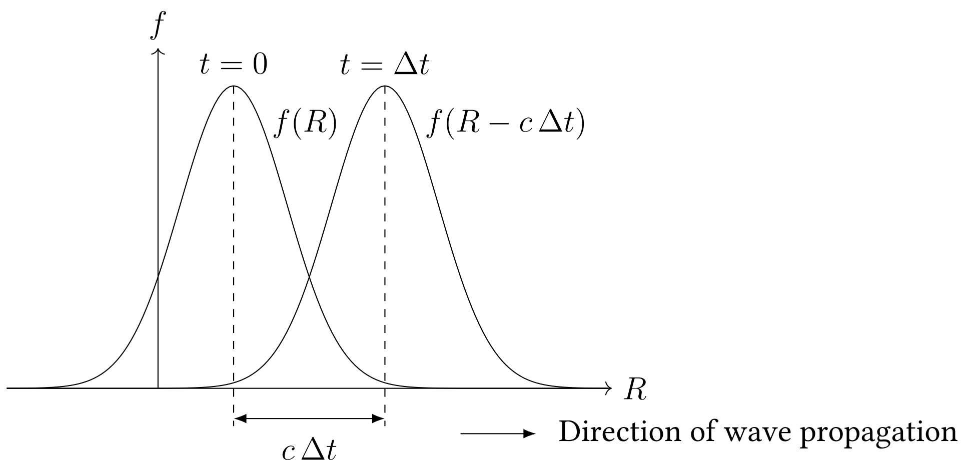
This represents a wave traveling in the positive R direction with a velocity $c=\frac{1}{\sqrt{\mu\epsilon}}$.

$$V(R,t)=\frac{1}{R}U(R,t) \tag{1.300}$$

$$V(R,t)=\frac{1}{R}f\left(t-\frac{R}{c}\right) \tag{1.301}$$

It can be also shown that

$$V(R,t)=\frac{1}{R}f(R-ct) \tag{1.302}$$



$$\underset{\text{Source}}{q \bullet} \xrightarrow{\quad R \quad} \underset{\text{Observation point}}{\bullet\, V(R,t)}$$

At an instant t, the potential at a distance R is a function of the charge that existed at the instant $\left(t - \frac{R}{c}\right)$. A time interval $\Delta t = \frac{R}{c}$ elapses before an observer at a distance R from the charge is able to notice any change occuring in the charge. This potential is therefore referred to as the retarded (gecikmeli) scalar potential.

To determine the function $f\left(t - \frac{R}{c}\right)$ more precisely, let us consider a point very close to the charge. In this case, the retardation may be ignored. If the charge varies according to the law $q(t)$, the potential is

$$V(R,t) = \frac{q(t)}{4\pi\epsilon R} \qquad \text{(Close to the charge)} \tag{1.303}$$

We have found the solution of wave equation as

$$V(R,t) = \frac{1}{R} f\left(t - \frac{R}{c}\right) \tag{1.304}$$

Comparing the last two equations we see that,

$$f\left(t - \frac{R}{c}\right) = \frac{q\left(t - \frac{R}{c}\right)}{4\pi\epsilon} \tag{1.305}$$

The resulting potential created by a varying point charge is

$$V(R,t) = \frac{q\left(t - \frac{R}{c}\right)}{4\pi\epsilon R}, \qquad c = \frac{1}{\sqrt{\mu\epsilon}} \tag{1.306}$$

The retarded potential at a point due to a cloud of charges of density $\rho(t)$ is given by

$$\boxed{V(R,t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho\left(t - \frac{R}{c}\right)}{R}\, dv' \qquad \text{(V)}} \tag{1.307}$$

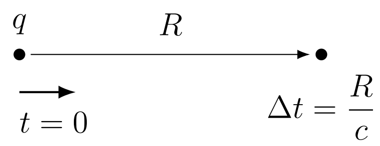
Retarded scalar potential

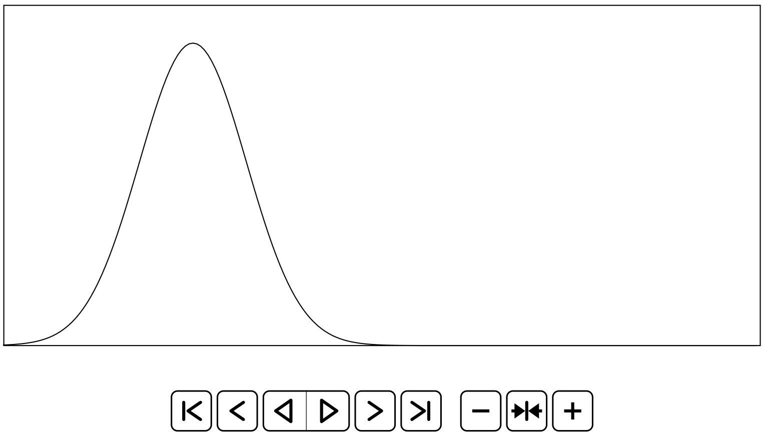
In a similar way

$$\boxed{\vec{\mathbf{A}}(R,t) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{\mathbf{J}}\left(t - \frac{R}{c}\right)}{R}\, dv' \qquad \text{(Wb/m)}} \tag{1.308}$$

Retarded vector potential

The electric and magnetic fields in the case of varying charges and currents need some time to change at points distant from the sources. In the quasi-static approximation we ignore this time-retardation effect and assume instant response. This assumption is implicit in dealing with circuit problems.





Source-Free Wave Equations

In problems of wave propagation we are interested in how an electromagnetic wave propagates in a source-free region where ρ and $\vec{\mathbf{J}}$ are both zero. In a simple (linear, isotropic,homogeneous) nonconducting medium ($\sigma = 0$) characterized by ϵ and μ, Maxwell's equations reduce to

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{1.309}$$

$$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}} \tag{1.310}$$

$$\boxed{\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t}} \tag{1.311}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \tag{1.312}$$

$$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}} \tag{1.313}$$

$$\vec{\mathbf{J}} = 0 \tag{1.314}$$

$$\boxed{\nabla \times \vec{\mathbf{H}} = \epsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}} \tag{1.315}$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho \tag{1.316}$$

$$\rho = 0 \tag{1.317}$$

$$\vec{\mathbf{D}} = \epsilon \, \vec{\mathbf{E}} \tag{1.318}$$

$$\boxed{\nabla \cdot \vec{\mathbf{E}} = 0} \tag{1.319}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{1.320}$$

$$\vec{\mathbf{B}} = \mu \, \vec{\mathbf{H}} \tag{1.321}$$

$$\boxed{\nabla \cdot \vec{\mathbf{H}} = 0} \tag{1.322}$$

$$\nabla \times \vec{\mathbf{E}} = -\mu \, \frac{\partial \vec{\mathbf{H}}}{\partial t} \tag{1.323}$$

$$\nabla \times (\nabla \times \vec{\mathbf{E}}) = -\mu \, \nabla \times \left(\frac{\partial \vec{\mathbf{H}}}{\partial t} \right) \tag{1.324}$$

$$\nabla \times (\nabla \times \vec{\mathbf{E}}) = -\mu \, \frac{\partial}{\partial t} (\nabla \times \vec{\mathbf{H}}) = -\mu \, \frac{\partial}{\partial t} \left(\epsilon \, \frac{\partial \vec{\mathbf{E}}}{\partial t} \right) = -\mu\epsilon \, \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{1.325}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} = \nabla(\nabla \cdot \vec{\mathbf{E}}) - \nabla^2 \, \vec{\mathbf{E}} \tag{1.326}$$

$$\nabla \cdot \vec{\mathbf{E}} = 0 \tag{1.327}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} = -\nabla^2 \, \vec{\mathbf{E}} \tag{1.328}$$

$$-\nabla^2 \, \vec{\mathbf{E}} = -\mu\epsilon \, \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{1.329}$$

$$\nabla^2 \, \vec{\mathbf{E}} - \mu\epsilon \, \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0 \tag{1.330}$$

$$c = \frac{1}{\sqrt{\mu\epsilon}} \tag{1.331}$$

$$\boxed{\nabla^2 \, \vec{\mathbf{E}} - \frac{1}{c^2} \, \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0} \tag{1.332}$$

Homogeneous vector wave equation

In a similar way

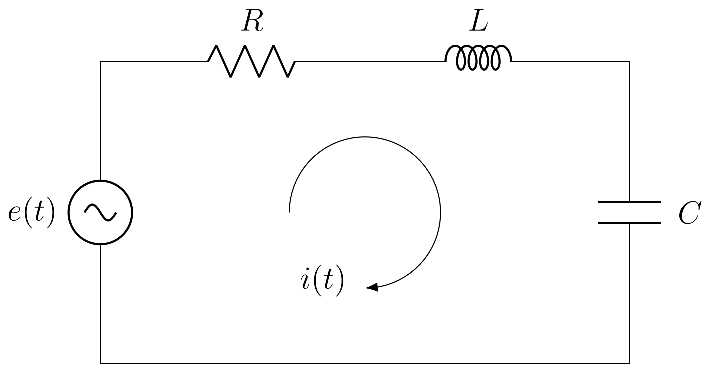
$$\boxed{\nabla^2 \, \vec{\mathbf{H}} - \frac{1}{c^2} \, \frac{\partial^2 \vec{\mathbf{H}}}{\partial t^2} = 0} \tag{1.333}$$

Homogeneous vector wave equation

1.7 Time-Harmonic Fields

The Use of Phasors

For time harmonic (steady-state sinusoidal) fields it is convenient to use a phasor notation.



$$R\,i + L\,\frac{di}{dt} + \frac{1}{C}\int i\,dt = e(t) \tag{1.334}$$

$$e(t) = E\cos\omega t = \text{Re}\left[E\,e^{j\omega t}\right] \qquad \text{(peak value)} \tag{1.335}$$

$$i(t) = I\cos(\omega t + \phi) = \text{Re}\left[I\,e^{j\phi}\,e^{j\omega t}\right] \qquad \text{(peak value)} \tag{1.336}$$

Re : the real part of

$$\frac{di(t)}{dt} = \text{Re}\left[I\,e^{j\phi}\,\frac{d}{dt}\,e^{j\omega t}\right] = \text{Re}\left[j\omega I\,e^{j\phi}\,e^{j\omega t}\right] \tag{1.337}$$

$$\int i(t)\,dt = \text{Re}\left[\int I\,e^{j\phi}\,e^{j\omega t}\,dt\right] = \text{Re}\left[\frac{1}{j\omega}\,I\,e^{j\phi}\,e^{j\omega t}\right] \tag{1.338}$$

$$R\,\text{Re}\left[I\,e^{j\phi}\,e^{j\omega t}\right] + L\,\text{Re}\left[j\omega I\,e^{j\phi}\,e^{j\omega t}\right] + \frac{1}{C}\,\text{Re}\left[\frac{1}{j\omega}\,I\,e^{j\phi}\,e^{j\omega t}\right] = \text{Re}\left[E\,e^{j\omega t}\right] \tag{1.339}$$

$$\left(R + j\omega L + \frac{1}{j\omega C}\right)\,I\,e^{j\phi} = E \tag{1.340}$$

Example 1.6
Express $3\cos\omega t - 4\sin\omega t$ as first a) $A_1\,\cos(\omega t + \theta_1)$; b) $A_2\,\sin(\omega t + \theta_2)$. Determine A_1, θ_1, A_2 and θ_2.

Solution
a)

$$3\cos\omega t = \text{Re}\left[3\,e^{j0^\circ}\,e^{j\omega t}\right] \tag{1.341}$$

$$\cos(x + y) = \cos x\,\cos y - \sin x\,\sin y \tag{1.342}$$

$$\cos(x-y) = \cos x \, \cos y + \sin x \, \sin y \tag{1.343}$$

$$\cos(x-90^\circ) = \cos x \, \cos(90^\circ) + \sin x \, \sin(90^\circ) \tag{1.344}$$

$$\cos(x-90^\circ) = (\cos x)\,(0) + (\sin x)\,(1) = \sin x \tag{1.345}$$

$$\sin \omega t = \cos(\omega t - 90^\circ) = \text{Re}\left[e^{-j90^\circ}\, e^{j\omega t}\right] \tag{1.346}$$

$$-4\, \sin \omega t = -4\, \cos(\omega t - 90^\circ) = \text{Re}\left[4\, e^{j180^\circ}\, e^{-j90^\circ}\, e^{j\omega t}\right] = \text{Re}\left[4\, e^{j90^\circ}\, e^{j\omega t}\right] \tag{1.347}$$

$$3\cos \omega t - 4\sin \omega t = \text{Re}\left[3\, e^{j0^\circ}\, e^{j\omega t}\right] + \text{Re}\left[4\, e^{j90^\circ}\, e^{j\omega t}\right] \tag{1.348}$$

$$3\cos \omega t - 4\sin \omega t = \text{Re}\left[\left(3\, e^{j0^\circ} + 4\, e^{j90^\circ}\right)\, e^{j\omega t}\right] \tag{1.349}$$

$$3\cos \omega t - 4\sin \omega t = \text{Re}\left[(3 + j4)\, e^{j\omega t}\right] \tag{1.350}$$

$$3\cos \omega t - 4\sin \omega t = \text{Re}\left[\left(5\, e^{j\tan^{-1}(4/3)}\right)\, e^{j\omega t}\right] \tag{1.351}$$

$$3\cos \omega t - 4\sin \omega t = \text{Re}\left[\left(5\, e^{j53.13^\circ}\right)\, e^{j\omega t}\right] \tag{1.352}$$

$$3\cos \omega t - 4\sin \omega t = 5\cos(\omega t + 53.13^\circ) \tag{1.353}$$

$A_1 = 5, \theta_1 = 53.13^\circ$

b)

$$3\cos \omega t - 4\sin \omega t = A_2\, \sin(\omega t + \theta_2) \tag{1.354}$$

$$\sin \omega t \rightarrow 1\angle 0^\circ \tag{1.355}$$

$$-4\sin \omega t \rightarrow 4\angle 180^\circ \tag{1.356}$$

$$\sin(x+y) = \sin x \, \cos y + \cos x \, \sin y \tag{1.357}$$

$$\sin(x+90^\circ) = \sin x \, \cos(90^\circ) + \cos x \, \sin(90^\circ) = \cos x \tag{1.358}$$

$$\cos \omega t = \sin(\omega t + 90^\circ) \rightarrow 1\angle 90^\circ \tag{1.359}$$

$$3\cos \omega t \rightarrow 3\angle 90^\circ \tag{1.360}$$

$$3\angle 90^\circ + 4\angle 180^\circ = j3 - 4 = -4 + j3 = 5\angle(-\tan^{-1} 3/4) = 5\angle 143.13^\circ \tag{1.361}$$

$$3\cos \omega t - 4\sin \omega t = 5\, \sin(\omega t + 143.13^\circ) \tag{1.362}$$

$A_2 = 5, \theta_2 = 143.13^\circ$

Time-Harmonic Electromagnetics

We can write a time-harmonic $\vec{\mathbf{E}}$-field as

$$\vec{\mathbf{E}}(x,y,z,t) = \mathrm{Re}\left[\vec{\mathbf{E}}(x,y,z)\, e^{j\omega t}\right] \qquad \text{(peak value)} \tag{1.363}$$

where $\vec{\mathbf{E}}(x,y,z)$ is a vector phasor that contains information on direction, magnitude and phase.

$$\frac{d\vec{\mathbf{E}}}{dt} = \mathrm{Re}\left[\vec{\mathbf{E}}(x,y,z)\, \frac{d}{dt}\, e^{j\omega t}\right] = \mathrm{Re}\left[j\omega\, \vec{\mathbf{E}}(x,y,z)\, e^{j\omega t}\right] \tag{1.364}$$

$$\frac{d\vec{\mathbf{E}}}{dt} \to j\omega\vec{\mathbf{E}} \tag{1.365}$$

$$\frac{d^2\vec{\mathbf{E}}}{dt^2} = \mathrm{Re}\left[\vec{\mathbf{E}}(x,y,z)\, \frac{d^2}{dt^2}\, e^{j\omega t}\right] = \mathrm{Re}\left[-\omega^2\, \vec{\mathbf{E}}(x,y,z)\, e^{j\omega t}\right] \tag{1.366}$$

$$\frac{d^2\vec{\mathbf{E}}}{dt^2} \to -\omega^2\vec{\mathbf{E}} \tag{1.367}$$

$$\int \vec{\mathbf{E}}\, dt = \int \mathrm{Re}\left[\vec{\mathbf{E}}(x,y,z)\, e^{j\omega t} dt\right] = \mathrm{Re}\left[\vec{\mathbf{E}}(x,y,z) \int e^{j\omega t}\, dt\right] = \mathrm{Re}\left[\frac{\vec{\mathbf{E}}(x,y,z)}{j\omega}\, e^{j\omega t}\right] \tag{1.368}$$

$$\int \vec{\mathbf{E}}\, dt \to \frac{\vec{\mathbf{E}}}{j\omega} \tag{1.369}$$

We can write time-harmonic Maxwell's equations in terms of vector field phasors $(\vec{\mathbf{E}}, \vec{\mathbf{H}})$ and source phasors $(\rho, \vec{\mathbf{J}})$ in a simple (linear, isotropic, and homogeneous) medium as follows:

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \quad \to \nabla \times \vec{\mathbf{E}} = -j\omega\, \vec{\mathbf{B}} \quad \Rightarrow \nabla \times \vec{\mathbf{E}} = -j\omega\mu\, \vec{\mathbf{H}} \tag{1.370}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \quad \to \nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\, \vec{\mathbf{D}} \quad \Rightarrow \nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\epsilon\, \vec{\mathbf{E}} \tag{1.371}$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho \quad \to \nabla \cdot \vec{\mathbf{D}} = \rho \quad \Rightarrow \nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon} \tag{1.372}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \quad \to \nabla \cdot \vec{\mathbf{B}} = 0 \quad \Rightarrow \nabla \cdot \vec{\mathbf{H}} = 0 \tag{1.373}$$

$$\boxed{\nabla \times \vec{\mathbf{E}} = -j\omega\mu\, \vec{\mathbf{H}}} \tag{1.374}$$

$$\boxed{\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\epsilon\, \vec{\mathbf{E}}} \tag{1.375}$$

$$\boxed{\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon}} \tag{1.376}$$

$$\boxed{\nabla \cdot \vec{\mathbf{H}} = 0} \tag{1.377}$$

$$\nabla^2 \vec{\mathbf{A}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu \vec{\mathbf{J}} \tag{1.378}$$

Nonhomogeneous wave equation for vector potential $\vec{\mathbf{A}}$.

$$\Rightarrow \nabla^2 \vec{\mathbf{A}} - \mu\epsilon\,(-\omega^2 \vec{\mathbf{A}}) = -\mu \vec{\mathbf{J}} \tag{1.379}$$

$$\Rightarrow \nabla^2 \vec{\mathbf{A}} + \mu\epsilon\,\omega^2 \vec{\mathbf{A}} = -\mu \vec{\mathbf{J}} \tag{1.380}$$

$$k^2 = \omega^2 \mu\epsilon \tag{1.381}$$

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c} \tag{1.382}$$

k : wavenumber

$$\boxed{\nabla^2 \vec{\mathbf{A}} + k^2 \vec{\mathbf{A}} = -\mu \vec{\mathbf{J}}} \tag{1.383}$$

Nonhomogeneous Helmholtz's equation.

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{1.384}$$

Nonhomogeneous wave equation for scalar potential V.

$$\Rightarrow \nabla^2 V - \mu\epsilon\,(-\omega^2 V) = -\frac{\rho}{\epsilon} \tag{1.385}$$

$$\Rightarrow \nabla^2 V + \mu\epsilon\,\omega^2 V = -\frac{\rho}{\epsilon} \tag{1.386}$$

$$\boxed{\nabla^2 V + k^2 V = -\frac{\rho}{\epsilon}} \tag{1.387}$$

Nonhomogeneous Helmholtz's equation.

Lorentz condition

$$\nabla \cdot \vec{\mathbf{A}} + \mu\epsilon \frac{\partial V}{\partial t} = 0 \qquad \rightarrow \nabla \cdot \vec{\mathbf{A}} + j\omega\mu\epsilon\, V = 0 \tag{1.388}$$

The phasor solution of nonhomogeneous Helmholtz's equation

$$\vec{\mathbf{A}}(R,t) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{\mathbf{J}}\left(t - \frac{R}{c}\right)}{R}\, dv' \tag{1.389}$$

$$\vec{\mathbf{J}}(x,y,z,t) = \mathrm{Re}\left[\vec{\mathbf{J}}(x,y,z)\, e^{j\omega t}\right] \tag{1.390}$$

$$\vec{\mathbf{J}}(x,y,z,t-R/c) = \text{Re}\left[\vec{\mathbf{J}}(x,y,z)\, e^{j\omega(t-R/c)}\right] \tag{1.391}$$

$$e^{j\omega(t-R/c)} = e^{j\omega t}\, e^{-j\omega R/c} = e^{j\omega t}\, e^{-jkR} \tag{1.392}$$

$$k = \omega/c \tag{1.393}$$

$$\vec{\mathbf{J}}(x,y,z,t-R/c) = \text{Re}\left[\vec{\mathbf{J}}(x,y,z)\, e^{-jkR}\, e^{j\omega t}\right] \tag{1.394}$$

$$\vec{\mathbf{J}}(t-R/c) \rightarrow \vec{\mathbf{J}}\, e^{-jkR} \tag{1.395}$$

$$\boxed{\vec{\mathbf{A}}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{\mathbf{J}}\, e^{-jkR}}{R}\, dv' \qquad (\text{Wb/m})} \tag{1.396}$$

In a similar way

$$\boxed{V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho\, e^{-jkR}}{R}\, dv' \qquad (\text{V})} \tag{1.397}$$

The procedure for determining the electric and magnetic fields due to time harmonic charge and current distributions is as follows:
1) Find phasors $V(R)$ and $\vec{\mathbf{A}}(R)$.

$$V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho\, e^{-jkR}}{R}\, dv' \tag{1.398}$$

$$\vec{\mathbf{A}}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{\mathbf{J}}\, e^{-jkR}}{R}\, dv' \tag{1.399}$$

2) Find $\vec{\mathbf{E}}(R)$ and $\vec{\mathbf{B}}(R)$.

$$\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \rightarrow \vec{\mathbf{E}}(R) = -\nabla V - j\omega\, \vec{\mathbf{A}} \tag{1.400}$$

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}} \rightarrow \vec{\mathbf{B}}(R) = \nabla \times \vec{\mathbf{A}} \tag{1.401}$$

3) Find instantaneous $\vec{\mathbf{E}}(R,t)$ and $\vec{\mathbf{B}}(R,t)$.

$$\vec{\mathbf{E}}(R,t) = \text{Re}\left[\vec{\mathbf{E}}(R)\, e^{j\omega t}\right] \tag{1.402}$$

$$\vec{\mathbf{B}}(R,t) = \text{Re}\left[\vec{\mathbf{B}}(R)\, e^{j\omega t}\right] \tag{1.403}$$

Source-Free Fields in Simple Media

General Maxwell's equations are given as follows:

$$\boxed{\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}}} \tag{1.404}$$

$$\boxed{\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\epsilon\,\vec{\mathbf{E}}} \tag{1.405}$$

$$\boxed{\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon}} \tag{1.406}$$

$$\boxed{\nabla \cdot \vec{\mathbf{H}} = 0} \tag{1.407}$$

In a simple, nonconducting source-free medium ($\rho = 0, \vec{\mathbf{J}} = 0, \sigma = 0$) the time-harmonic Maxwell's equations are

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}} \tag{1.408}$$

$$\nabla \times \vec{\mathbf{H}} = j\omega\epsilon\,\vec{\mathbf{E}} \tag{1.409}$$

$$\nabla \cdot \vec{\mathbf{E}} = 0 \tag{1.410}$$

$$\nabla \cdot \vec{\mathbf{H}} = 0 \tag{1.411}$$

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}} \tag{1.412}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\nabla \times \vec{\mathbf{H}} = -j\omega\mu\,(j\omega\epsilon\,\vec{\mathbf{E}}) = \omega^2\mu\epsilon\vec{\mathbf{E}} \tag{1.413}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} - \omega^2\mu\epsilon\vec{\mathbf{E}} = 0 \tag{1.414}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} = \nabla(\nabla \cdot \vec{\mathbf{E}}) - \nabla^2\,\vec{\mathbf{E}} \tag{1.415}$$

$$\nabla \cdot \vec{\mathbf{E}} = 0 \tag{1.416}$$

$$-\nabla^2\,\vec{\mathbf{E}} - \omega^2\mu\epsilon\vec{\mathbf{E}} = 0 \tag{1.417}$$

$$\nabla^2\,\vec{\mathbf{E}} + \omega^2\mu\epsilon\vec{\mathbf{E}} = 0 \tag{1.418}$$

$$\boxed{\nabla^2\,\vec{\mathbf{E}} + k^2\vec{\mathbf{E}} = 0} \tag{1.419}$$

Homogeneous vector Helmholtz's equation

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c} \tag{1.420}$$

In a similar way

$$\boxed{\nabla^2\,\vec{\mathbf{H}} + k^2\vec{\mathbf{H}} = 0} \tag{1.421}$$

Homogeneous vector Helmholtz's equation

Example 1.7

Show that if $(\vec{\mathbf{E}}, \vec{\mathbf{H}})$ are solutions of source–free Maxwell's equations in a simple medium characterized by ϵ and μ, then so also are $(\vec{\mathbf{E}}', \vec{\mathbf{H}}')$ where

$$\vec{\mathbf{E}}' = \eta\,\vec{\mathbf{H}} \tag{1.422}$$

$$\vec{\mathbf{H}}' = -\frac{\vec{\mathbf{E}}}{\eta} \tag{1.423}$$

In the above equation $\eta = \sqrt{\dfrac{\mu}{\epsilon}}$ is called the intrinsic impedance of the medium.

Solution

The following equations are satisfied.

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}} \tag{1.424}$$

$$\nabla \times \vec{\mathbf{H}} = j\omega\epsilon\,\vec{\mathbf{E}} \tag{1.425}$$

$$\vec{\mathbf{H}} = \frac{\vec{\mathbf{E}}'}{\eta} \tag{1.426}$$

$$\vec{\mathbf{E}} = -\eta\,\vec{\mathbf{H}}' \tag{1.427}$$

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}} \tag{1.428}$$

$$\nabla \times (-\eta\,\vec{\mathbf{H}}') = -j\omega\mu\,\frac{\vec{\mathbf{E}}'}{\eta} \tag{1.429}$$

$$\eta\,\nabla \times \vec{\mathbf{H}}' = j\frac{\omega\mu}{\eta}\,\vec{\mathbf{E}}' \tag{1.430}$$

$$\nabla \times \vec{\mathbf{H}}' = j\frac{\omega\mu}{\eta^2}\,\vec{\mathbf{E}}' \tag{1.431}$$

$$\frac{\mu}{\eta^2} = \frac{\mu}{\mu/\epsilon} = \epsilon \tag{1.432}$$

$$\nabla \times \vec{\mathbf{H}}' = j\omega\epsilon\,\vec{\mathbf{E}}' \qquad (\text{2nd Maxwell's equation}) \tag{1.433}$$

$$\nabla \times \vec{\mathbf{H}} = j\omega\epsilon\,\vec{\mathbf{E}} \tag{1.434}$$

$$\nabla \times \frac{\vec{\mathbf{E}}'}{\eta} = j\omega\epsilon\,(-\eta\,\vec{\mathbf{H}}') \tag{1.435}$$

$$\nabla \times \vec{\mathbf{E}}' = -j\omega\epsilon\,\eta^2\,\vec{\mathbf{H}}' \tag{1.436}$$

$$\epsilon\,\eta^2 = \epsilon\,\frac{\mu}{\epsilon} = \mu \tag{1.437}$$

$$\nabla \times \vec{\mathbf{E}}' = -j\omega\mu\,\vec{\mathbf{H}}' \qquad (\text{1st Maxwell's equation}) \tag{1.438}$$

$$\nabla \cdot \vec{\mathbf{E}} = 0 \Rightarrow \nabla \cdot (-\eta\,\vec{\mathbf{H}}') = 0 \Rightarrow \nabla \cdot \vec{\mathbf{H}}' = 0 \tag{1.439}$$

$$\nabla \cdot \vec{\mathbf{H}} = 0 \Rightarrow \nabla \cdot \left(\frac{\vec{\mathbf{E}}'}{\eta}\right) = 0 \Rightarrow \nabla \cdot \vec{\mathbf{E}}' = 0 \tag{1.440}$$

Wave Equation in a Conducting Medium

If the simple medium is conducting ($\sigma \neq 0$), a current

$$\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}} \tag{1.441}$$

will flow and we obtain

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\epsilon\,\vec{\mathbf{E}} = \sigma\vec{\mathbf{E}} + j\omega\epsilon\,\vec{\mathbf{E}} \tag{1.442}$$

$$\nabla \times \vec{\mathbf{H}} = (\sigma + j\omega\epsilon)\,\vec{\mathbf{E}} \tag{1.443}$$

$$\nabla \times \vec{\mathbf{H}} = j\omega\left(\epsilon + \frac{\sigma}{j\omega}\right)\vec{\mathbf{E}} \tag{1.444}$$

$$\nabla \times \vec{\mathbf{H}} = j\omega\epsilon_c\,\vec{\mathbf{E}} \tag{1.445}$$

$$\epsilon_c = \epsilon + \frac{\sigma}{j\omega} = \epsilon - j\frac{\sigma}{\omega} \quad \text{(F/m)} \tag{1.446}$$

Maxwell's equations in a simple, conducting medium

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}} \tag{1.447}$$

$$\nabla \times \vec{\mathbf{H}} = j\omega\epsilon_c\,\vec{\mathbf{E}} \tag{1.448}$$

$$\nabla \cdot \vec{\mathbf{E}} = 0 \tag{1.449}$$

$$\nabla \cdot \vec{\mathbf{H}} = 0 \tag{1.450}$$

ϵ_c can be written as

$$\epsilon_c = \epsilon' - j\epsilon'' \quad \text{(F/m)} \tag{1.451}$$

where

$$\epsilon'' = \frac{\sigma}{\omega} \tag{1.452}$$

In a lossy dielectric medium, the real wave number k, changes to a complex wave number

$$k_c = \omega\sqrt{\mu\epsilon_c} = \omega\sqrt{\mu\left(\epsilon' - j\epsilon''\right)} \tag{1.453}$$

The ratio $\frac{\epsilon''}{\epsilon'}$ is called the **loss tangent** (kayıp tanjantı). It is a measure of the power loss in the medium.

$$\tan\delta_c = \frac{\epsilon''}{\epsilon'} \tag{1.454}$$

$$\epsilon' \simeq \epsilon \tag{1.455}$$

$$\tan\delta_c = \frac{\epsilon''}{\epsilon} = \frac{\frac{\sigma}{\omega}}{\epsilon} = \frac{\sigma}{\omega\epsilon} \tag{1.456}$$

The quantity δ_c is called the **loss angle** (kayıp açısı).

$$\epsilon_c = \epsilon - j\frac{\sigma}{\omega} \tag{1.457}$$

A medium is said to be a **good conductor** if

$$\frac{\sigma}{\omega} \gg \epsilon \tag{1.458}$$

In this case

$$\epsilon_c \simeq -j\frac{\sigma}{\omega} \tag{1.459}$$

A medium is said to be a **good insulator** if

$$\frac{\sigma}{\omega} \ll \epsilon \tag{1.460}$$

In this case

$$\epsilon_c \simeq \epsilon \tag{1.461}$$

For moist ground (nemli toprak)

$$\epsilon_r = 10 \tag{1.462}$$

$$\sigma = 10^{-2} \quad \text{(S/m)} \tag{1.463}$$

At $f = 1\,\text{KHz}$

$$\frac{\sigma}{\omega\epsilon} = \frac{\sigma}{\omega\epsilon_r\epsilon_0} = \frac{10^{-2}}{(2\pi \times 10^3)(10)(10^{-9}/36\pi)} \tag{1.464}$$

$$\frac{\sigma}{\omega\epsilon} = 18000 \qquad \text{Relatively good conductor} \tag{1.465}$$

At $f = 10\,\text{GHz}$

$$\frac{\sigma}{\omega\epsilon} = \frac{\sigma}{\omega\epsilon_r\epsilon_0} = \frac{10^{-2}}{(2\pi \times 10^{10})(10)(10^{-9}/36\pi)} \tag{1.466}$$

$$\frac{\sigma}{\omega\epsilon} = 1.8 \times 10^{-3} \qquad \text{Moist ground behaves like an insulator} \tag{1.467}$$

Example 1.8

A sinusoidal electric field intensity of amplitude 250 (V/m) and frequency 1 GHz exists in a lossy dielectric medium that has a relative permittivity of 2.5 and loss tangent of 0.001. Find the average power dissipated in the medium per cubic meter.

Solution

Average power dissipated per cubic meter

$$P = V_{rms}\,I_{rms} = \frac{V_p}{\sqrt{2}}\,\frac{I_p}{\sqrt{2}} = \frac{1}{2}\,V_p\,I_p \tag{1.468}$$

$$\frac{P}{\Delta V} = \frac{P}{\Delta x\,\Delta y\,\Delta z} = \frac{1}{2}\,\frac{V_p}{\Delta x}\,\frac{I_p}{\Delta y\,\Delta z} = \frac{1}{2}EJ \tag{1.469}$$

$$J = \sigma E \tag{1.470}$$

$$\frac{P}{\Delta V} = \frac{1}{2}\,\sigma E^2 \tag{1.471}$$

$$\epsilon_r = 2.5 \tag{1.472}$$

$$\frac{\epsilon''}{\epsilon'} = 0.001 \quad \text{loss tangent} \tag{1.473}$$

$$f = 10^9 \text{ Hz} \tag{1.474}$$

$$E = 250 \text{ (V/m)} \tag{1.475}$$

$$\epsilon'' = \frac{\sigma}{\omega} \Rightarrow \sigma = \omega\epsilon'' \tag{1.476}$$

$$\sigma = \omega\,(0.001\epsilon') \tag{1.477}$$

$$\sigma = (2\pi \times 10^9)\,(0.001\epsilon') \tag{1.478}$$

$$\epsilon' = \epsilon = \epsilon_r\epsilon_0 = (2.5)\,\frac{10^{-9}}{36\pi} \tag{1.479}$$

$$\sigma = (2\pi \times 10^9)\,(0.001)\,(2.5)\,\frac{10^{-9}}{36\pi} \tag{1.480}$$

$$\sigma = 1.39 \times 10^{-4} \text{ (S/m)} \tag{1.481}$$

$$\frac{P}{\Delta V} = \frac{1}{2}\,\sigma E^2 = \frac{1}{2}\,(1.39 \times 10^{-4})\,(250)^2 \tag{1.482}$$

$$\frac{P}{\Delta V} = 4.34 \text{ (W/m}^3\text{)} \tag{1.483}$$

For microwave oven $f = 2.45\,\text{GHz} = 2.45 \times 10^9\,\text{Hz}$.

$$\left.\begin{array}{l}\epsilon_r = 40 \\ \tan\delta_c = 0.35\end{array}\right\} \quad \text{for beefsteak} \tag{1.484}$$

$$\tan\delta_c = \frac{\sigma}{\omega\epsilon} = \frac{\sigma}{\omega\epsilon_r\epsilon_0} = 0.35 \tag{1.485}$$

$$\sigma = 0.35\,\omega\epsilon_r\epsilon_0 = 0.35\,(2\pi \times 2.45 \times 10^9)\,(40)\,\left(\frac{10^{-9}}{36\pi}\right) \tag{1.486}$$

$$\sigma = 1.91 \text{ (S/m)} \tag{1.487}$$

$$\frac{P}{\Delta V} = \frac{1}{2}\,\sigma E^2 = \frac{1}{2}\,(1.91)\,(250)^2 = 59600 \text{ (W/m}^3\text{)} = 59.6 \text{ (kW/m}^3\text{)} \tag{1.488}$$

2 PLANE ELECTROMAGNETIC WAVES

2.1 Introduction

A uniform plane wave (*düzlemsel dalga*) is a particular solution of Maxwell's equations with $\vec{\mathbf{E}}$ ($\vec{\mathbf{H}}$) assuming the same direction, same magnitude, and same phase in infinite planes perpendicular to the direction of propagation.

2.2 Plane Waves in Lossless Media

We focus our attention on wave behavior in the sinusoidal steady state. We will investigate the solutions of the homogeneous vector Helmholtz's equation in free space.

$$\nabla^2 \vec{\mathbf{E}} + k_0^2 \vec{\mathbf{E}} = 0 \tag{2.1}$$

k_0 : free-space wavenumber

$$k_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c} \qquad \left(\frac{\text{rad}}{\text{m}}\right) \tag{2.2}$$

$$\vec{\mathbf{E}} = \hat{\mathbf{a}}_x\, E_x + \hat{\mathbf{a}}_y\, E_y + \hat{\mathbf{a}}_z\, E_z \tag{2.3}$$

$$\nabla^2 E_x + k_0^2 E_x = 0 \tag{2.4}$$

$$\nabla^2 E_y + k_0^2 E_y = 0 \tag{2.5}$$

$$\nabla^2 E_z + k_0^2 E_z = 0 \tag{2.6}$$

For E_x component we have

$$\nabla^2 E_x + k_0^2 E_x = 0 \tag{2.7}$$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k_0^2 E_x = 0 \tag{2.8}$$

Consider a uniform plane wave characterized by a uniform E_x (uniform magnitude and constant phase) over plane surfaces perpendicular to z; that is

$$\frac{\partial^2 E_x}{\partial x^2} = 0 \tag{2.9}$$

$$\frac{\partial^2 E_x}{\partial y^2} = 0 \tag{2.10}$$

In this case we have

$$\frac{d^2 E_x}{dz^2} + k_0^2 E_x = 0 \tag{2.11}$$

$$E_x(z) = E_x^+(z) + E_x^-(z) \tag{2.12}$$

$$E_x(z) = E_0^+ \, e^{-jk_0 z} + E_0^- \, e^{jk_0 z} \tag{2.13}$$

E_0^+, E_0^- : arbitrary complex constants
E_0^+ and E_0^- are determined by boundary conditions.

$$E_x^+(z,t) = \text{Re}\left[E_x^+(z)\, e^{j\omega t}\right] \tag{2.14}$$

$$E_x^+(z,t) = \text{Re}\left[E_0^+ \, e^{-jk_0 z}\, e^{j\omega t}\right] \tag{2.15}$$

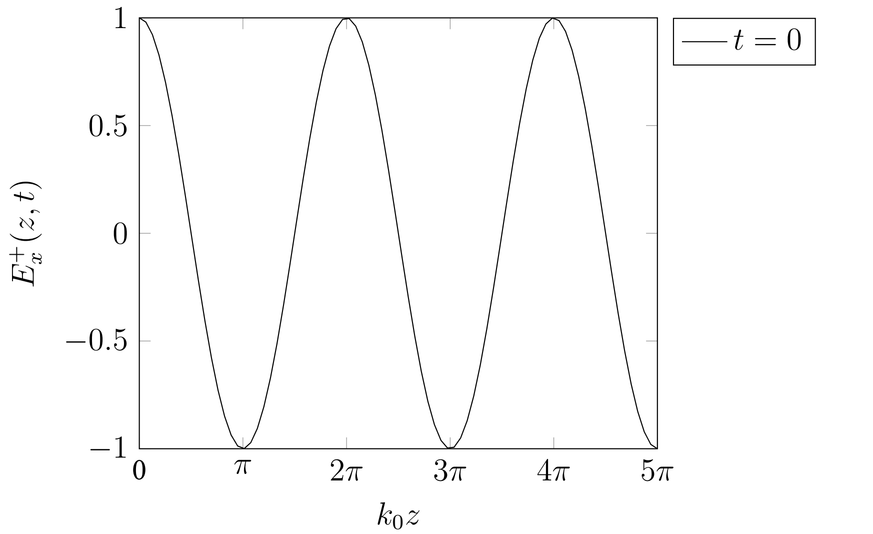
$$E_x^+(z,t) = E_0^+ \, \cos(\omega t - k_0 z) \qquad \text{(V/m)} \tag{2.16}$$

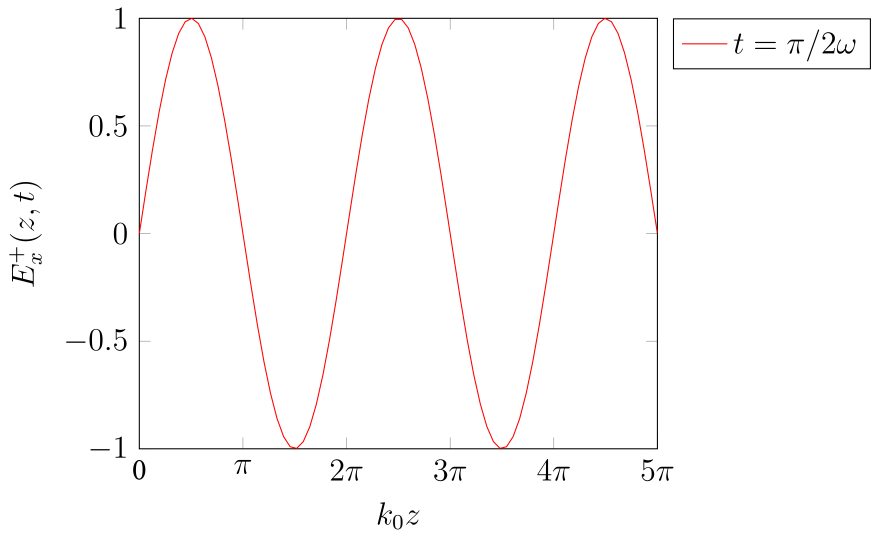
$$t = 0 \Rightarrow E_x^+(z,0) = E_0^+ \, \cos(-k_0 z) = E_0^+ \, \cos(k_0 z) \tag{2.17}$$

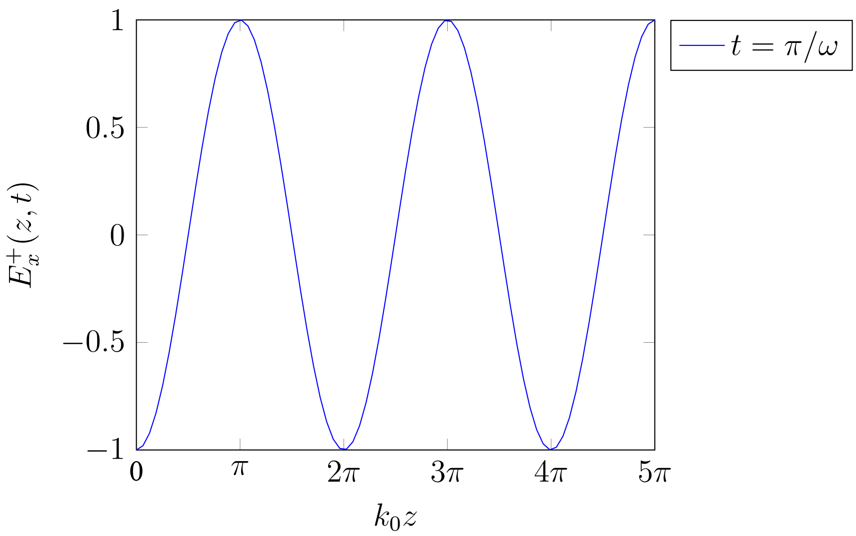
$$t = \frac{\pi}{2\,\omega} \Rightarrow E_x^+\left(z, \frac{\pi}{2\,\omega}\right) = E_0^+ \, \cos\left(\frac{\pi}{2} - k_0 z\right) = E_0^+ \, \sin(k_0 z) \tag{2.18}$$

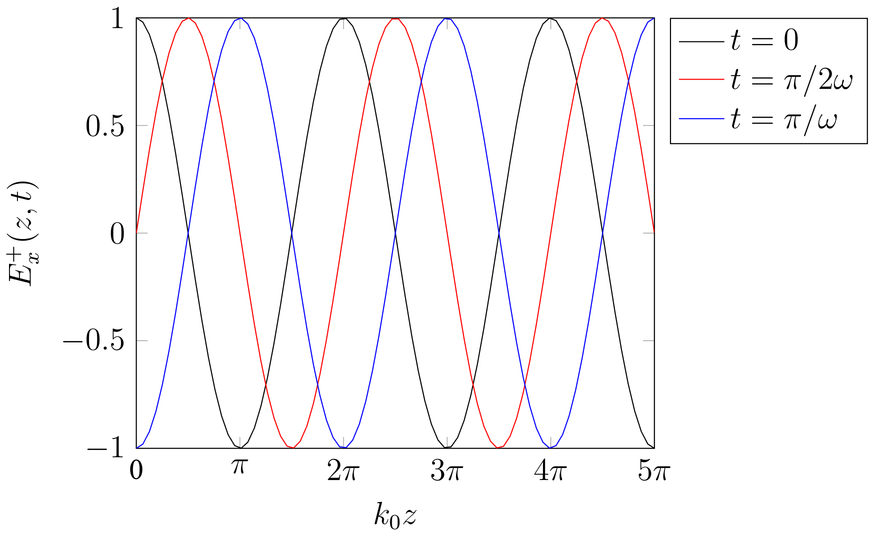
$$t = \frac{\pi}{\omega} \Rightarrow E_x^+\left(z, \frac{\pi}{\omega}\right) = E_0^+ \, \cos(\pi - k_0 z) = -E_0^+ \, \cos(k_0 z) \tag{2.19}$$

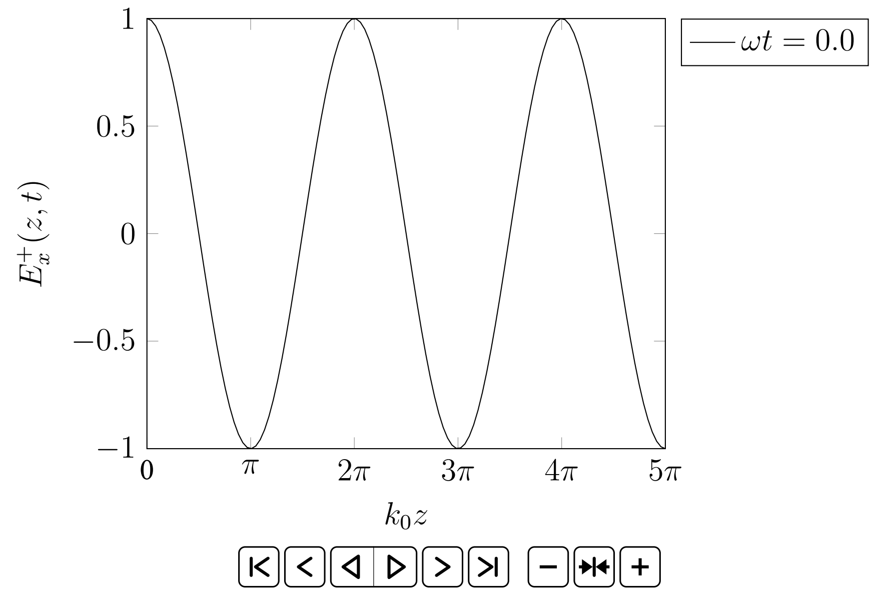
The curve travels in the $+z$ direction and we have a **traveling wave**.











$$E_x^+(z,t) = E_0^+ \cos(\omega t - k_0 z) \tag{2.20}$$

Let's fix our attention on a particular point on the wave:

$$\cos(\omega t - k_0 z) = \text{constant} \tag{2.21}$$

$$\Rightarrow \omega t - k_0 z = \text{constant phase} \tag{2.22}$$

$$u_p = \frac{dz}{dt} \quad \text{(phase velocity)} \tag{2.23}$$

$$\frac{d}{dt}(\omega t - k_0 z) = 0 \tag{2.24}$$

$$\omega - k_0 \frac{dz}{dt} = 0 \tag{2.25}$$

$$\frac{dz}{dt} = \frac{\omega}{k_0} = \frac{\omega}{\frac{w}{c}} = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{2.26}$$

$$u_p = c \tag{2.27}$$

The velocity of propagation of equiphase front (the phase velocity) in free space is equal to the velocity of light. The term $E_0^- \, e^{jk_0 z}$ represents a cosinusoidal wave traveling in $-z$ direction with velocity c.

$$E_x^+(z) = E_0^+ \, e^{-jk_0 z} \tag{2.28}$$

Now let's find $\vec{\mathbf{H}}$.

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu_0 \, \vec{\mathbf{H}} \tag{2.29}$$

$$\nabla \times \vec{\mathbf{E}} = \begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \tag{2.30}$$

$$\nabla \times \vec{\mathbf{E}} = \hat{\mathbf{a}}_x \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - \hat{\mathbf{a}}_y \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) + \hat{\mathbf{a}}_z \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \tag{2.31}$$

$$E_x = E_x^+(z) = E_0^+ \, e^{-jk_0 z} \tag{2.32}$$

$$E_y = 0 \tag{2.33}$$

$$E_z = 0 \tag{2.34}$$

$$\frac{\partial E_x}{\partial y} = 0 \tag{2.35}$$

$$\nabla \times \vec{\mathbf{E}} = \hat{\mathbf{a}}_y \frac{\partial E_x}{\partial z} = \hat{\mathbf{a}}_y \frac{\partial E_x^+(z)}{\partial z} = \hat{\mathbf{a}}_y \frac{\partial}{\partial z} \left[E_0^+ \, e^{-jk_0 z} \right] \tag{2.36}$$

$$\nabla \times \vec{\mathbf{E}} = \hat{\mathbf{a}}_y \, (-jk_0) \, E_0^+ \, e^{-jk_0 z} = \hat{\mathbf{a}}_y \, (-jk_0) \, E_x^+(z) \tag{2.37}$$

$$\vec{\mathbf{H}} = \frac{1}{-j\omega\mu_0} \nabla \times \vec{\mathbf{E}} \tag{2.38}$$

$$\vec{\mathbf{H}} = \frac{1}{-j\omega\mu_0}\,\hat{\mathbf{a}}_y\,(-jk_0)\,E_x^+(z) \tag{2.39}$$

$$\vec{\mathbf{H}} = \hat{\mathbf{a}}_y\,H_y^+ \tag{2.40}$$

$$H_y^+(z) = \frac{k_0}{\omega\mu_0}\,E_x^+(z) \tag{2.41}$$

$$\frac{k_0}{\omega\mu_0} = \frac{\frac{\omega}{c}}{\omega\mu_0} = \frac{1}{c\mu_0} = \frac{\sqrt{\mu_0\epsilon_0}}{\mu_0} = \sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{1}{\eta_0} \tag{2.42}$$

$$\eta_0 \triangleq \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \simeq\ 377\,(\Omega) \tag{2.43}$$

η_0 : the intrinsic impedance of the free space

$$H_y^+(z) = \frac{1}{\eta_0}\,E_x^+(z) \qquad \text{(A/m)} \tag{2.44}$$

$$H_y^+(z,t) = \mathrm{Re}\left[H_y^+(z)\,e^{j\omega t}\right] \tag{2.45}$$

$$H_y^+(z,t) = \mathrm{Re}\left[\frac{1}{\eta_0}\,E_0^+\,e^{-jk_0z}\,e^{j\omega t}\right] \tag{2.46}$$

$$H_y^+(z,t) = \frac{E_0^+}{\eta_0}\,\cos(\omega t - k_0z) \qquad \text{(A/m)} \tag{2.47}$$

$$\vec{\mathbf{E}}(z,t) = \hat{\mathbf{a}}_x\,E_x^+ = \hat{\mathbf{a}}_x\,E_0^+\,\cos(\omega t - k_0z) \qquad \text{(V/m)} \tag{2.48}$$

$$\vec{\mathbf{H}}(z,t) = \hat{\mathbf{a}}_y\,H_y^+ = \hat{\mathbf{a}}_y\,\frac{E_0^+}{\eta_0}\,\cos(\omega t - k_0z) \qquad \text{(A/m)} \tag{2.49}$$

For a uniform plane wave

$$\frac{|\vec{\mathbf{E}}|}{|\vec{\mathbf{H}}|} = \frac{|E_0^+\,\cos(\omega t - k_0z)|}{\frac{1}{\eta_0}\,|E_0^+\,\cos(\omega t - k_0z)|} = \eta_0 \tag{2.50}$$

Example 2.1

A uniform plane wave with $\vec{\mathbf{E}} = \hat{\mathbf{a}}_x\,E_x$ propagates in a lossless simple medium ($\epsilon_r = 4, \mu_r = 1, \sigma = 0$) in the $+z$ direction. Assume that E_x is sinuosidal with a frequency 100 MHz and has a maximum value 10^{-4} (V/m) at $t = 0$ and $z = \frac{1}{8}$ m.

a) Write the instantaneous expression for $\vec{\mathbf{E}}$ for any t and z.

b) Write the instantaneous expression for $\vec{\mathbf{H}}$.

c) Determine the locations where E_x is a positive maximum when $t = 10^{-8}$ s.

<u>Solution</u>

a)

$$f = 100\text{ MHz} = 100\times 10^6\text{ Hz} = 10^8\text{ Hz} \tag{2.51}$$

$$\omega = 2\pi f = 2\pi\times 10^8\ (\text{rad/s}) \tag{2.52}$$

$$\vec{\mathbf{E}}(z,t) = \hat{\mathbf{a}}_x\, E_x = \hat{\mathbf{a}}_x\, E_0\, \cos(\omega t - kz + \phi)\ (\mathrm{V/m}) \tag{2.53}$$

$$\vec{\mathbf{E}}(z = \frac{1}{8}, t = 0) = \hat{\mathbf{a}}_x\, E_0\, \cos\left(-k\,\frac{1}{8} + \phi\right) = \hat{\mathbf{a}}_x\, 10^{-4} \tag{2.54}$$

$$\Rightarrow \begin{cases} E_0 = 10^{-4}\ (\mathrm{V/m}) \\ \cos(-k\,\frac{1}{8} + \phi) = 1 \end{cases} \tag{2.55}$$

$$k = \omega\sqrt{\mu\epsilon} = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\mu_r\epsilon_r} \tag{2.56}$$

$$k = \frac{\omega}{c}\sqrt{\mu_r\epsilon_r} = \frac{2\pi f}{c}\sqrt{\mu_r\epsilon_r} \tag{2.57}$$

$$k = \frac{2\pi \times 10^8}{3 \times 10^8}\sqrt{(1)(4)} = \frac{4\pi}{3}\ (\mathrm{rad/m}) \tag{2.58}$$

$$-\frac{k}{8} + \phi = 0 \tag{2.59}$$

$$\phi = \frac{k}{8} = \frac{4\pi}{3}\,\frac{1}{8} = \frac{\pi}{6} \tag{2.60}$$

$$\vec{\mathbf{E}}(z,t) = \hat{\mathbf{a}}_x\, 10^{-4}\, \cos\left(2\pi \times 10^8 t - \frac{4\pi}{3}z + \frac{\pi}{6}\right) \tag{2.61}$$

$$\vec{\mathbf{E}}(z,t) = \hat{\mathbf{a}}_x\, 10^{-4}\, \cos\left[2\pi \times 10^8 t - \frac{4\pi}{3}\left(z - \frac{1}{8}\right)\right]\ (\mathrm{V/m}) \tag{2.62}$$

b)

$$H_y = \frac{E_x}{\eta} \tag{2.63}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0\mu_r}{\epsilon_0\epsilon_r}} = \sqrt{\frac{\mu_0}{\epsilon_0}}\,\sqrt{\frac{\mu_r}{\epsilon_r}} = \eta_0\,\sqrt{\frac{\mu_r}{\epsilon_r}} \tag{2.64}$$

$$\eta = \eta_0\,\sqrt{\frac{1}{4}} = \frac{\eta_0}{2} = \frac{120\pi}{2} = 60\pi \tag{2.65}$$

$$H_y = \frac{E_x}{60\pi} \tag{2.66}$$

$$\vec{\mathbf{H}}(z,t) = \hat{\mathbf{a}}_y\, \frac{10^{-4}}{60\pi}\, \cos\left[2\pi \times 10^8 t - \frac{4\pi}{3}\left(z - \frac{1}{8}\right)\right]\ (\mathrm{A/m}) \tag{2.67}$$

c)

$$t = 10^{-8}\ \mathrm{s} \tag{2.68}$$

$$\vec{\mathbf{E}}(z, t = 10^{-8}) = \hat{\mathbf{a}}_x\, 10^{-4}\, \cos\left[2\pi \times 10^8 \times 10^{-8} - \frac{4\pi}{3}\left(z - \frac{1}{8}\right)\right] \tag{2.69}$$

$$\vec{\mathbf{E}}(z, t = 10^{-8}) = \hat{\mathbf{a}}_x\, 10^{-4}\, \cos\left[2\pi - \frac{4\pi}{3}\left(z - \frac{1}{8}\right)\right] \tag{2.70}$$

$$\cos\left[2\pi - \frac{4\pi}{3}\left(z_m - \frac{1}{8}\right)\right]\Big|_{\max} = 1 \tag{2.71}$$

$$2\pi - \frac{4\pi}{3}\left(z_m - \frac{1}{8}\right) = 0, \mp 2\pi, \mp 4\pi, \ldots, \mp 2n\pi \tag{2.72}$$

$$2\pi - \frac{4\pi}{3}\left(z_m - \frac{1}{8}\right) = 2\pi - \frac{4\pi}{3}z_m + \frac{\pi}{6} = \frac{13\pi}{6} - \frac{4\pi}{3}z_m \tag{2.73}$$

$$\Rightarrow \frac{13\pi}{6} - \frac{4\pi}{3}z_m = 0, \mp 2\pi, \mp 4\pi, \ldots, \mp 2n\pi \tag{2.74}$$

$$\frac{13\pi}{6} - \frac{4\pi}{3}z_m = 2n\pi, \quad n \in \mathbb{Z},\ n = 0, \mp 1, \mp 2, \ldots \tag{2.75}$$

$$\frac{4\pi}{3}z_m = \frac{13\pi}{6} - 2n\pi \tag{2.76}$$

$$4\pi z_m = \frac{13\pi}{2} - 6n\pi = \frac{13\pi - 12n\pi}{2} = \frac{\pi}{2}(13 - 12n) \tag{2.77}$$

$$z_m = \frac{13 - 12n}{8}, \quad n \in \mathbb{Z},\ n = 0, \mp 1, \mp 2, \ldots \tag{2.78}$$

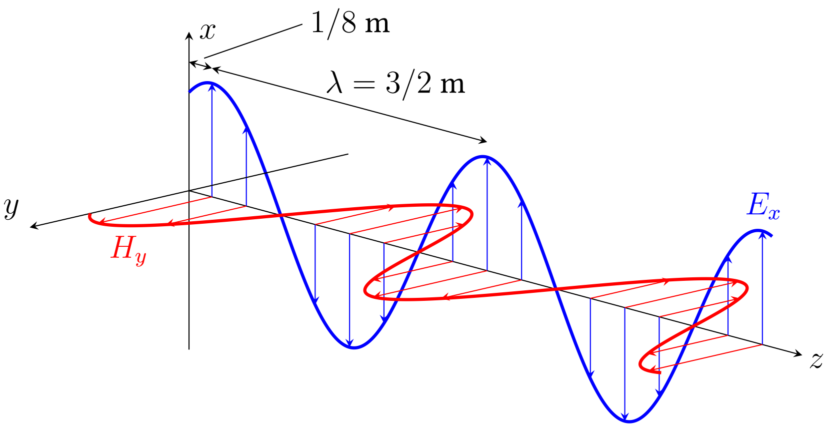
$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\frac{4\pi}{3}} = \frac{3}{2} = \frac{12}{8}\ \text{m} \tag{2.79}$$

$$z_m = \frac{13}{8} - n\lambda \quad \text{The locations where } E_x \text{ is a positive maximum} \tag{2.80}$$

For $t = 0$

$$\vec{\mathbf{E}}(z,0) = \hat{\mathbf{a}}_x\, 10^{-4} \cos\left[\frac{4\pi}{3}\left(z - \frac{1}{8}\right)\right] \ (\text{V/m}) \tag{2.81}$$

$$\vec{\mathbf{H}}(z,0) = \hat{\mathbf{a}}_y \frac{E_x(z,0)}{\eta} \tag{2.82}$$



2.3 Transverse Electromagnetic Waves (Enine Elektromanyetik Dalgalar)

$$\vec{\mathbf{E}} = \hat{\mathbf{a}}_x\, E_x \tag{2.83}$$

$$\vec{\mathbf{H}} = \hat{\mathbf{a}}_y\, H_y \tag{2.84}$$

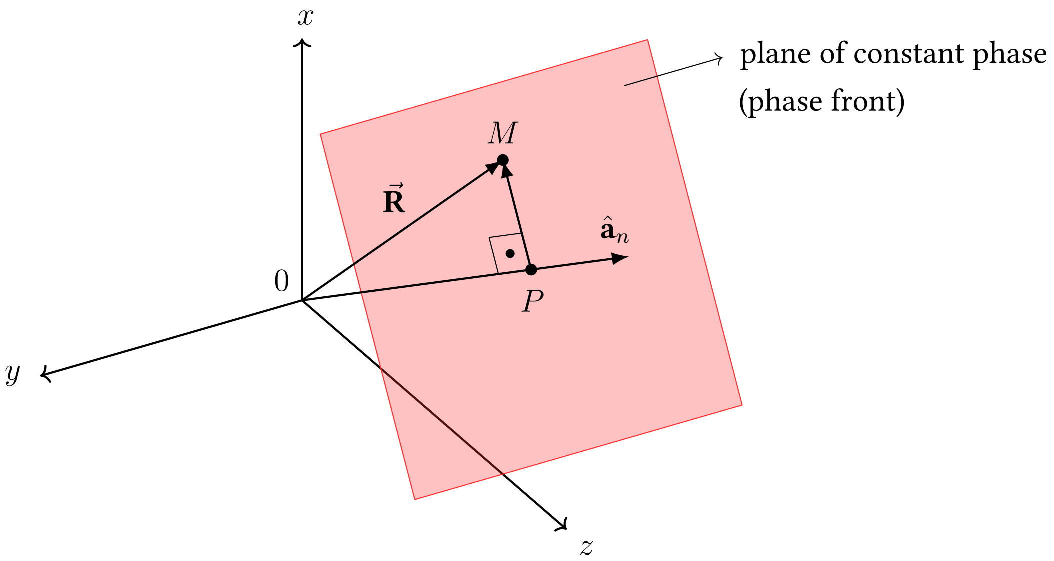
$\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ are perpendicular to each other, and both are transverse to the direction of propagation. It is a particular case of a transverse electromagnetic (TEM) wave. The phasor form E_x and H_y are functions of only the distance z.

Now we consider the propagation of a uniform plane wave along an arbitrary direction that does not necessarily coincide with a coordinate axis.

The phasor electric field intensity for a uniform plane wave propagating in the $+z$ direction is

$$\vec{\mathbf{E}}(z) = \vec{\mathbf{E}}_0\, e^{-jkz} \tag{2.85}$$

$\vec{\mathbf{E}}_0$: constant vector



The phasor electric field intensity for a uniform plane wave propagating in the $\hat{\mathbf{a}}_n$ direction is

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk|\overrightarrow{\mathbf{OP}}|} \tag{2.86}$$

$\hat{\mathbf{a}}_n$ is the unit vector in the direction of propagation. The position vector $\vec{\mathbf{R}}$ shows any point M on the phase front and it is given by

$$\vec{\mathbf{R}} = \hat{\mathbf{a}}_x\, x + \hat{\mathbf{a}}_y\, y + \hat{\mathbf{a}}_z\, z \tag{2.87}$$

The equation of the plane

$$\hat{\mathbf{a}}_n \cdot \overrightarrow{\mathbf{PM}} = 0 \tag{2.88}$$

$$\overrightarrow{\mathbf{PM}} - \vec{\mathbf{R}} + \overrightarrow{\mathbf{OP}} = 0 \tag{2.89}$$

$$\Rightarrow \overrightarrow{\mathbf{PM}} = \vec{\mathbf{R}} - \overrightarrow{\mathbf{OP}} = \vec{\mathbf{R}} - \hat{\mathbf{a}}_n\,|\overrightarrow{\mathbf{OP}}| \tag{2.90}$$

$$\hat{\mathbf{a}}_n \cdot \overrightarrow{\mathbf{PM}} = \hat{\mathbf{a}}_n \cdot (\vec{\mathbf{R}} - \hat{\mathbf{a}}_n\,|\overrightarrow{\mathbf{OP}}|) = 0 \tag{2.91}$$

$$\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}} - |\overrightarrow{\mathbf{OP}}| = 0 \tag{2.92}$$

$$|\overrightarrow{\mathbf{OP}}| = \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}} \tag{2.93}$$

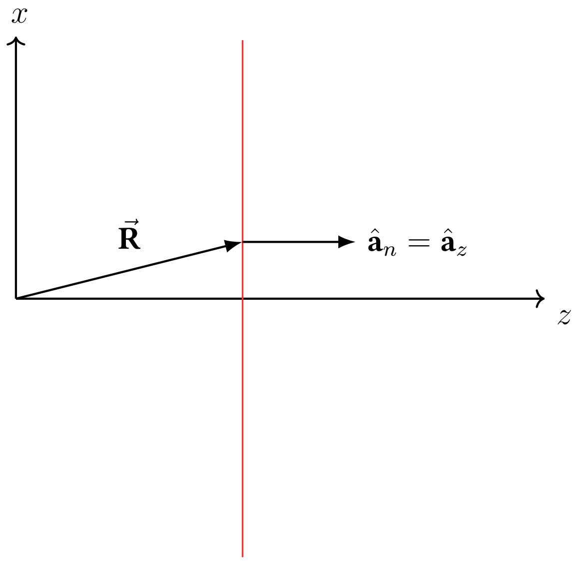
$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk|\overrightarrow{\mathbf{OP}}|} \tag{2.94}$$

$$\boxed{\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n\cdot\vec{\mathbf{R}}} \quad (\mathrm{V/m})} \tag{2.95}$$

$\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}} =$ constant is a plane of constant phase and uniform amplitude for the wave

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n\cdot\vec{\mathbf{R}}} \tag{2.96}$$

As a special case let $\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_z$.



$$\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_z \tag{2.97}$$

$$\vec{\mathbf{R}} = \hat{\mathbf{a}}_x\, x + \hat{\mathbf{a}}_y\, y + \hat{\mathbf{a}}_z\, z \tag{2.98}$$

$$\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}} = z \tag{2.99}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n\cdot\vec{\mathbf{R}}} = \vec{\mathbf{E}}_0\, e^{-jkz} \tag{2.100}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n\cdot\vec{\mathbf{R}}} \tag{2.101}$$

$$\vec{\mathbf{k}} = k\,\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_x k_x + \hat{\mathbf{a}}_y k_y + \hat{\mathbf{a}}_z k_z \tag{2.102}$$

$\vec{\mathbf{k}}$: wavenumber vector

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{R}} = (\hat{\mathbf{a}}_x k_x + \hat{\mathbf{a}}_y k_y + \hat{\mathbf{a}}_z k_z) \cdot (\hat{\mathbf{a}}_x\, x + \hat{\mathbf{a}}_y\, y + \hat{\mathbf{a}}_z\, z) \tag{2.103}$$

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{R}} = k_x x + k_y y + k_z z \tag{2.104}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-j\,\vec{\mathbf{k}}\cdot\vec{\mathbf{R}}} = \vec{\mathbf{E}}_0\, e^{-j(k_x x + k_y y + k_z z)} = \vec{\mathbf{E}}_0\, e^{-jk_x x}\, e^{-jk_y y}\, e^{-jk_z z} \tag{2.105}$$

Let's substitute this expression into the homogeneous Helmholtz's equation:

$$\nabla^2\, \vec{\mathbf{E}} + k^2\, \vec{\mathbf{E}} = 0 \tag{2.106}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk_x x}\, e^{-jk_y y}\, e^{-jk_z z} \tag{2.107}$$

$$\psi = e^{-jk_x x}\, e^{-jk_y y}\, e^{-jk_z z} \tag{2.108}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk_x x}\, e^{-jk_y y}\, e^{-jk_z z} = \vec{\mathbf{E}}_0\, \psi \tag{2.109}$$

$$\nabla^2\, \vec{\mathbf{E}} = \nabla^2\, (\vec{\mathbf{E}}_0\, \psi) = \vec{\mathbf{E}}_0\, \nabla^2 \psi \tag{2.110}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \tag{2.111}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial (e^{-jk_x x}\, e^{-jk_y y}\, e^{-jk_z z})}{\partial x} = -jk_x (e^{-jk_x x}\, e^{-jk_y y}\, e^{-jk_z z}) \tag{2.112}$$

$$\frac{\partial \psi}{\partial x} = -jk_x \psi \tag{2.113}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} \right) \tag{2.114}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x} \left(-jk_x \psi \right) \tag{2.115}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -jk_x \frac{\partial \psi}{\partial x} = -jk_x(-jk_x \psi) = -k_x^2 \psi \tag{2.116}$$

In a similar way

$$\frac{\partial^2 \psi}{\partial y^2} = -k_y^2 \psi \tag{2.117}$$

$$\frac{\partial^2 \psi}{\partial z^2} = -k_z^2 \psi \tag{2.118}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -k_x^2 \psi - k_y^2 \psi - k_z^2 \psi \tag{2.119}$$

$$\nabla^2 \psi = (-k_x^2 - k_y^2 - k_z^2)\, \psi \tag{2.120}$$

$$\nabla^2\, \vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, \nabla^2 \psi = \vec{\mathbf{E}}_0\, (-k_x^2 - k_y^2 - k_z^2)\, \psi \tag{2.121}$$

$$\nabla^2\, \vec{\mathbf{E}} + k^2\, \vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, (-k_x^2 - k_y^2 - k_z^2)\, \psi + k^2\, \vec{\mathbf{E}}_0\, \psi = 0 \tag{2.122}$$

$$\Rightarrow \boxed{k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu \epsilon} \tag{2.123}$$

In a charge free region

$$\nabla \cdot \vec{\mathbf{E}} = 0 \tag{2.124}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk\, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{2.125}$$

$$\nabla \cdot \vec{\mathbf{E}} = \nabla \cdot \left(\vec{\mathbf{E}}_0 \, e^{-jk \, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \right) \tag{2.126}$$

$$\nabla \cdot \left(\psi \vec{\mathbf{A}} \right) = \psi \, \nabla \cdot \vec{\mathbf{A}} + \vec{\mathbf{A}} \cdot \nabla \psi \tag{2.127}$$

$$\vec{\mathbf{A}} \to \vec{\mathbf{E}}_0 \tag{2.128}$$

$$\psi \to e^{-jk \, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{2.129}$$

$$\nabla \cdot \vec{\mathbf{E}} = \nabla \cdot \left(\vec{\mathbf{E}}_0 \, e^{-jk \, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \right) \tag{2.130}$$

$$\nabla \cdot \vec{\mathbf{E}} = e^{-jk \, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \, \nabla \cdot \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_0 \cdot \nabla e^{-jk \, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{2.131}$$

$$\nabla \cdot \vec{\mathbf{E}}_0 = 0 \quad (\vec{\mathbf{E}}_0 = \text{constant vector}) \tag{2.132}$$

$$\nabla \cdot \vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \cdot \nabla e^{-jk \, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} = \vec{\mathbf{E}}_0 \cdot \nabla \psi \tag{2.133}$$

$$\psi = e^{-jk \, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} = e^{-jk_x x} \, e^{-jk_y y} \, e^{-jk_z z} \tag{2.134}$$

$$\nabla \psi = \hat{\mathbf{a}}_x \frac{\partial \psi}{\partial x} + \hat{\mathbf{a}}_y \frac{\partial \psi}{\partial y} + \hat{\mathbf{a}}_z \frac{\partial \psi}{\partial z} \tag{2.135}$$

$$\frac{\partial \psi}{\partial x} = -jk_x \psi \tag{2.136}$$

$$\frac{\partial \psi}{\partial y} = -jk_y \psi \tag{2.137}$$

$$\frac{\partial \psi}{\partial z} = -jk_z \psi \tag{2.138}$$

$$\nabla \psi = \hat{\mathbf{a}}_x (-jk_x) \psi + \hat{\mathbf{a}}_y (-jk_y) \psi + \hat{\mathbf{a}}_z (-jk_z) \psi \tag{2.139}$$

$$\nabla \psi = -j (\hat{\mathbf{a}}_x k_x + \hat{\mathbf{a}}_y k_y + \hat{\mathbf{a}}_z k_z) \psi \tag{2.140}$$

$$\hat{\mathbf{a}}_x k_x + \hat{\mathbf{a}}_y k_y + \hat{\mathbf{a}}_z k_z = \vec{\mathbf{k}} = k \, \hat{\mathbf{a}}_n \tag{2.141}$$

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} \tag{2.142}$$

$$\nabla \psi = -jk \, \hat{\mathbf{a}}_n \psi \tag{2.143}$$

$$\nabla \cdot \vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \cdot \nabla \psi = \vec{\mathbf{E}}_0 \cdot (-jk \, \hat{\mathbf{a}}_n) \, \psi \tag{2.144}$$

$$\nabla \cdot \vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \cdot (-jk \, \hat{\mathbf{a}}_n) \, e^{-jk \, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} = 0 \tag{2.145}$$

$$\Rightarrow \boxed{\hat{\mathbf{a}}_n \cdot \vec{\mathbf{E}}_0 = 0} \tag{2.146}$$

$\vec{\mathbf{E}}_0$ is transverse to the direction of propagation. Now, let's find $\vec{\mathbf{H}}$.

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu \, \vec{\mathbf{H}} \tag{2.147}$$

$$\vec{\mathbf{H}} = -\frac{1}{j\omega\mu} \, \nabla \times \vec{\mathbf{E}} \tag{2.148}$$

$$\nabla \times \vec{\mathbf{E}} = \nabla \times \left(\vec{\mathbf{E}}_0 \, e^{-jk \, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \right) \tag{2.149}$$

$$\nabla \times \left(\psi \vec{\mathbf{A}}\right) = \psi \nabla \times \vec{\mathbf{A}} + \nabla\psi \times \vec{\mathbf{A}} \tag{2.150}$$

$$\vec{\mathbf{A}} \to \vec{\mathbf{E}}_0 \tag{2.151}$$

$$\psi \to e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{2.152}$$

$$\nabla \times \left(\vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) = e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\, \nabla \times \vec{\mathbf{E}}_0 + \left(\nabla e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) \times \vec{\mathbf{E}}_0 \tag{2.153}$$

$$\nabla \times \vec{\mathbf{E}}_0 = 0 \quad (\vec{\mathbf{E}}_0 = \text{constant vector}) \tag{2.154}$$

$$\nabla \times \vec{\mathbf{E}} = \left(\nabla e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) \times \vec{\mathbf{E}}_0 \tag{2.155}$$

$$\nabla e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} = \nabla\psi = -jk\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\, \hat{\mathbf{a}}_n \tag{2.156}$$

$$\nabla \times \vec{\mathbf{E}} = -jk\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\, \hat{\mathbf{a}}_n \times \vec{\mathbf{E}}_0 \tag{2.157}$$

$$\nabla \times \vec{\mathbf{E}} = -jk\, \hat{\mathbf{a}}_n \times \vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{2.158}$$

$$\nabla \times \vec{\mathbf{E}} = -jk\, \hat{\mathbf{a}}_n \times \vec{\mathbf{E}} \tag{2.159}$$

$$\vec{\mathbf{H}} = -\frac{1}{j\omega\mu} \nabla \times \vec{\mathbf{E}} = -\frac{1}{j\omega\mu} (-jk)\, \hat{\mathbf{a}}_n \times \vec{\mathbf{E}} \tag{2.160}$$

$$\vec{\mathbf{H}} = \frac{k}{\omega\mu}\, \hat{\mathbf{a}}_n \times \vec{\mathbf{E}} \tag{2.161}$$

$$\frac{k}{\omega\mu} = \frac{\frac{\omega}{c}}{\omega\mu} = \frac{1}{c\mu} = \frac{\sqrt{\mu\epsilon}}{\mu} = \sqrt{\frac{\epsilon}{\mu}} = \frac{1}{\eta} \tag{2.162}$$

$$\boxed{\vec{\mathbf{H}} = \frac{1}{\eta}\, \hat{\mathbf{a}}_n \times \vec{\mathbf{E}} \qquad (\mathrm{A/m})} \tag{2.163}$$

$$\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}} \quad (\Omega) \tag{2.164}$$

η : intrinsic impedance of the medium (wave impedance; karakteristik empedans)

A uniform plane wave propagating in an arbitrary direction, $\hat{\mathbf{a}}_n$, is a transverse electromagnetic (TEM) wave with $\vec{\mathbf{E}} \perp \vec{\mathbf{H}}$ and that both $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ are normal to $\hat{\mathbf{a}}_n$.

Example 2.2

If $\vec{\mathbf{H}}(R)$ of a TEM wave is given as

$$\vec{\mathbf{H}}(R) = \vec{\mathbf{H}}_0\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{2.165}$$

obtain $\vec{\mathbf{E}}(R)$.

Solution I

$$\nabla \times \vec{\mathbf{H}} = j\omega\epsilon\, \vec{\mathbf{E}} \tag{2.166}$$

$$\vec{\mathbf{E}} = \frac{1}{j\omega\epsilon} \nabla \times \vec{\mathbf{H}} \tag{2.167}$$

$$\nabla \times \vec{\mathbf{H}} = \nabla \times \left(\vec{\mathbf{H}}_0\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) \tag{2.168}$$

$$\nabla \times \left(\psi \vec{\mathbf{A}}\right) = \psi \, \nabla \times \vec{\mathbf{A}} + \nabla \psi \times \vec{\mathbf{A}} \tag{2.169}$$

$$\vec{\mathbf{A}} \to \vec{\mathbf{H}}_0 \tag{2.170}$$

$$\psi \to e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{2.171}$$

$$\nabla \times \vec{\mathbf{H}} = \nabla \times \left(\vec{\mathbf{H}}_0 \, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) = e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \, \nabla \times \vec{\mathbf{H}}_0 + \left(\nabla e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) \times \vec{\mathbf{H}}_0 \tag{2.172}$$

$$\nabla \times \vec{\mathbf{H}}_0 = 0 \quad \left(\vec{\mathbf{H}}_0 = \text{constant vector}\right) \tag{2.173}$$

$$\nabla \times \vec{\mathbf{H}} = \left(\nabla e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) \times \vec{\mathbf{H}}_0 \tag{2.174}$$

$$\nabla e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} = \nabla \psi = -jk \, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \, \hat{\mathbf{a}}_n \tag{2.175}$$

$$\nabla \times \vec{\mathbf{H}} = -jk \, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \, \hat{\mathbf{a}}_n \times \vec{\mathbf{H}}_0 \tag{2.176}$$

$$\nabla \times \vec{\mathbf{H}} = -jk \, \hat{\mathbf{a}}_n \times \vec{\mathbf{H}}_0 \, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{2.177}$$

$$\nabla \times \vec{\mathbf{H}} = -jk \, \hat{\mathbf{a}}_n \times \vec{\mathbf{H}} \tag{2.178}$$

$$\vec{\mathbf{E}} = \frac{1}{j\omega\epsilon} \nabla \times \vec{\mathbf{H}} = \frac{1}{j\omega\epsilon} (-jk) \, \hat{\mathbf{a}}_n \times \vec{\mathbf{H}} \tag{2.179}$$

$$\vec{\mathbf{E}} = -\frac{k}{\omega\epsilon} \, \hat{\mathbf{a}}_n \times \vec{\mathbf{H}} \tag{2.180}$$

$$\frac{k}{\omega\epsilon} = \frac{\frac{\omega}{c}}{\omega\epsilon} = \frac{1}{c\epsilon} = \frac{\sqrt{\mu\epsilon}}{\epsilon} = \sqrt{\frac{\mu}{\epsilon}} = \eta \tag{2.181}$$

$$\boxed{\vec{\mathbf{E}} = -\eta \, \hat{\mathbf{a}}_n \times \vec{\mathbf{H}}} \quad (\mathrm{V/m}) \tag{2.182}$$

<u>Solution II</u>

$$\vec{\mathbf{H}} = \frac{1}{\eta} \, \hat{\mathbf{a}}_n \times \vec{\mathbf{E}} \tag{2.183}$$

$$\hat{\mathbf{a}}_n \times \vec{\mathbf{H}} = \frac{1}{\eta} \, \hat{\mathbf{a}}_n \times \left(\hat{\mathbf{a}}_n \times \vec{\mathbf{E}}\right) \tag{2.184}$$

$$\vec{\mathbf{A}} \times \left(\vec{\mathbf{B}} \times \vec{\mathbf{C}}\right) = \vec{\mathbf{B}} \left(\vec{\mathbf{A}} \cdot \vec{\mathbf{C}}\right) - \vec{\mathbf{C}} \left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right) \quad (\text{"Back} - \text{cab" rule}) \tag{2.185}$$

$$\vec{\mathbf{A}} = \hat{\mathbf{a}}_n, \quad \vec{\mathbf{B}} = \hat{\mathbf{a}}_n, \quad \vec{\mathbf{C}} = \vec{\mathbf{E}} \tag{2.186}$$

$$\hat{\mathbf{a}}_n \times \left(\hat{\mathbf{a}}_n \times \vec{\mathbf{E}}\right) = \hat{\mathbf{a}}_n \left(\hat{\mathbf{a}}_n \cdot \vec{\mathbf{E}}\right) - \vec{\mathbf{E}} \left(\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_n\right) \tag{2.187}$$

$$\hat{\mathbf{a}}_n \cdot \vec{\mathbf{E}} = 0 \tag{2.188}$$

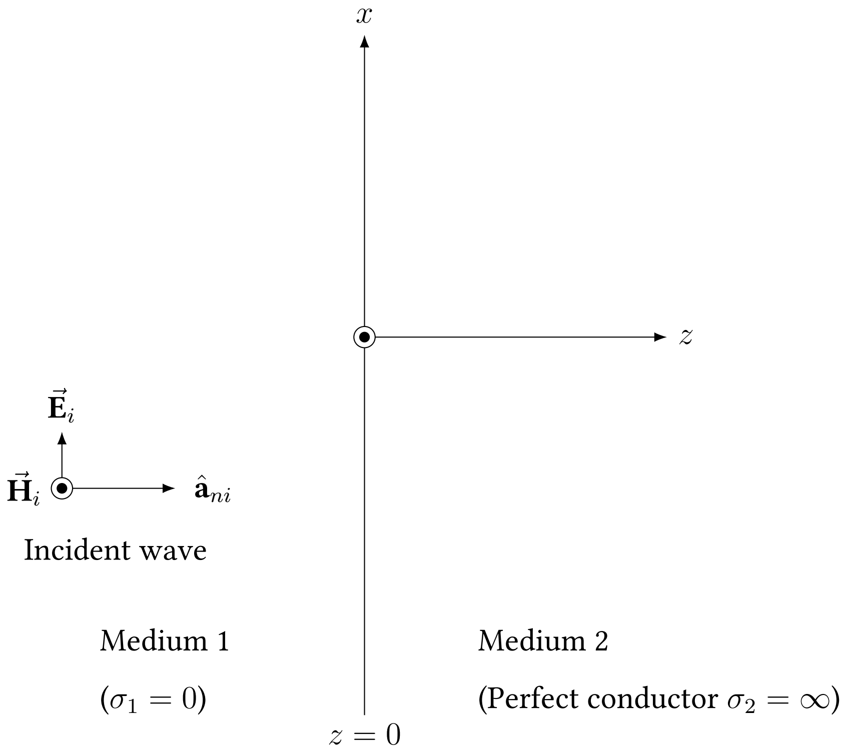
$$\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_n = 1 \tag{2.189}$$

$$\hat{\mathbf{a}}_n \times \left(\hat{\mathbf{a}}_n \times \vec{\mathbf{E}}\right) = -\vec{\mathbf{E}} \tag{2.190}$$

$$\hat{\mathbf{a}}_n \times \vec{\mathbf{H}} = -\frac{1}{\eta} \vec{\mathbf{E}} \tag{2.191}$$

$$\vec{\mathbf{E}} = -\eta \, \hat{\mathbf{a}}_n \times \vec{\mathbf{H}} \quad (\mathrm{V/m}) \tag{2.192}$$

2.4 Normal Incidence at a Plane Conducting Boundary



$$\vec{\mathbf{E}}_i(z) = \hat{\mathbf{a}}_x \, E_{i0} \, e^{-j\beta_1 z} \tag{2.193}$$

$$\vec{\mathbf{H}}_i(z) = \hat{\mathbf{a}}_y \, \frac{E_{i0}}{\eta_1} \, e^{-j\beta_1 z} \tag{2.194}$$

E_{i0} : the magnitude of $\vec{\mathbf{E}}_i$ at $z = 0$
β_1 : phase constant of medium 1
η_1 : intrinsic impedance of medium 1

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} \tag{2.195}$$

In a perfect conductor electric and magnetic fields are zero.

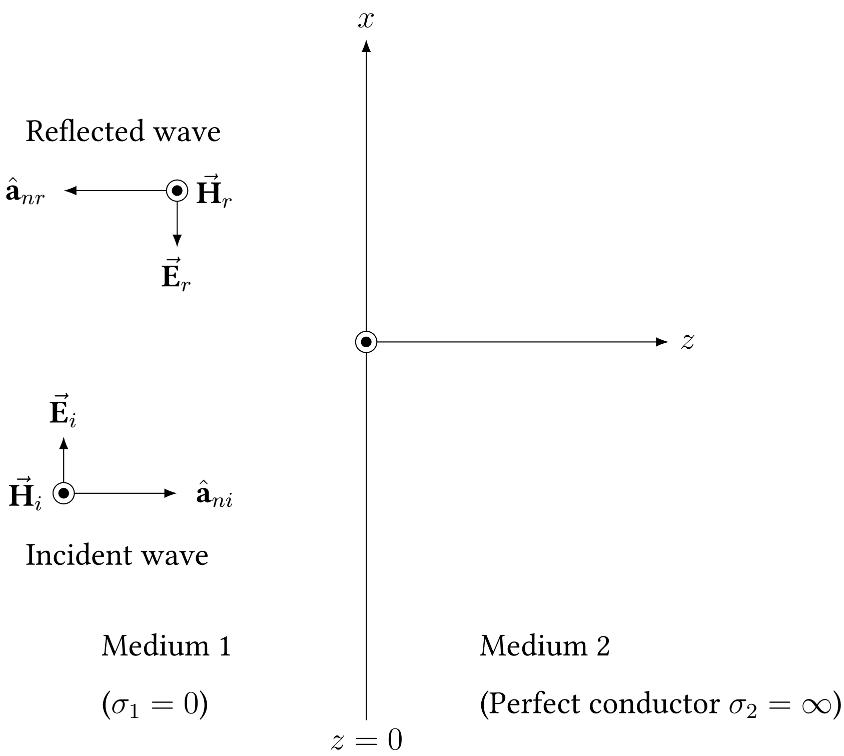
$$\vec{\mathbf{E}}_2 = 0 \tag{2.196}$$

$$\vec{\mathbf{H}}_2 = 0 \tag{2.197}$$

The incident wave is reflected giving rise to a reflected wave $(\vec{\mathbf{E}}_r, \vec{\mathbf{H}}_r)$.

$$\vec{\mathbf{E}}_r(z) = -\hat{\mathbf{a}}_x \, E_{r0} \, e^{j\beta_1 z} \tag{2.198}$$

Reflected wave travels in the $-z$ direction.



Total field in medium 1

$$\vec{\mathbf{E}}_1(z) = \vec{\mathbf{E}}_i(z) + \vec{\mathbf{E}}_r(z) \tag{2.199}$$

$$\vec{\mathbf{E}}_1(z) = \hat{\mathbf{a}}_x\, E_{i0}\, e^{-j\beta_1 z} - \hat{\mathbf{a}}_x\, E_{r0}\, e^{j\beta_1 z} \tag{2.200}$$

$$\vec{\mathbf{E}}_1(z) = \hat{\mathbf{a}}_x\, \left(E_{i0}\, e^{-j\beta_1 z} - E_{r0}\, e^{j\beta_1 z}\right) \tag{2.201}$$

$$\vec{\mathbf{E}}_2 = 0 \tag{2.202}$$

$$E_{1t} = E_{2t} = 0 \tag{2.203}$$

$$\vec{\mathbf{E}}_1(0) = 0 = \hat{\mathbf{a}}_x\, (E_{i0} - E_{r0}) \tag{2.204}$$

$$\Rightarrow \boxed{E_{r0} = E_{i0}} \tag{2.205}$$

$$\vec{\mathbf{E}}_r(z) = -\hat{\mathbf{a}}_x\, E_{i0}\, e^{j\beta_1 z} \tag{2.206}$$

$$\vec{\mathbf{E}}_1(z) = \hat{\mathbf{a}}_x\, E_{i0}\, \left(e^{-j\beta_1 z} - e^{j\beta_1 z}\right) \tag{2.207}$$

$$\vec{\mathbf{E}}_1(z) = \hat{\mathbf{a}}_x\, E_{i0}\, (-2j)\, \sin \beta_1 z \tag{2.208}$$

$$\vec{\mathbf{E}}_1(z) = -\hat{\mathbf{a}}_x\, 2j\, E_{i0}\, \sin \beta_1 z \tag{2.209}$$

$$\vec{\mathbf{H}}_r(z) = \frac{1}{\eta_1}\, \hat{\mathbf{a}}_{nr} \times \vec{\mathbf{E}}_r(z) \tag{2.210}$$

$$\hat{\mathbf{a}}_{nr} = -\hat{\mathbf{a}}_z \tag{2.211}$$

$$\vec{\mathbf{H}}_r(z) = \frac{1}{\eta_1}\, (-\hat{\mathbf{a}}_z) \times (-\hat{\mathbf{a}}_x)\, E_{i0}\, e^{j\beta_1 z} \tag{2.212}$$

$$\vec{\mathbf{H}}_r(z) = \hat{\mathbf{a}}_y\, \frac{E_{i0}}{\eta_1}\, e^{j\beta_1 z} \tag{2.213}$$

$$\vec{\mathbf{H}}_1(z) = \vec{\mathbf{H}}_i(z) + \vec{\mathbf{H}}_r(z) \tag{2.214}$$

$$\vec{\mathbf{H}}_1(z) = \hat{\mathbf{a}}_y \, \frac{E_{i0}}{\eta_1} \, e^{-j\beta_1 z} + \hat{\mathbf{a}}_y \, \frac{E_{i0}}{\eta_1} \, e^{j\beta_1 z} \tag{2.215}$$

$$\vec{\mathbf{H}}_1(z) = \hat{\mathbf{a}}_y \, \frac{E_{i0}}{\eta_1} \, \left(e^{-j\beta_1 z} + e^{j\beta_1 z}\right) \tag{2.216}$$

$$\vec{\mathbf{H}}_1(z) = \hat{\mathbf{a}}_y \, 2 \, \frac{E_{i0}}{\eta_1} \, \cos \beta_1 z \tag{2.217}$$

$$\vec{\mathbf{E}}_1(z,t) = \mathrm{Re}\left[\vec{\mathbf{E}}_1(z) \, e^{j\omega t}\right] \tag{2.218}$$

$$\vec{\mathbf{E}}_1(z,t) = \mathrm{Re}\left[-\hat{\mathbf{a}}_x \, 2j \, E_{i0} \, \sin \beta_1 z \, e^{j\omega t}\right] \tag{2.219}$$

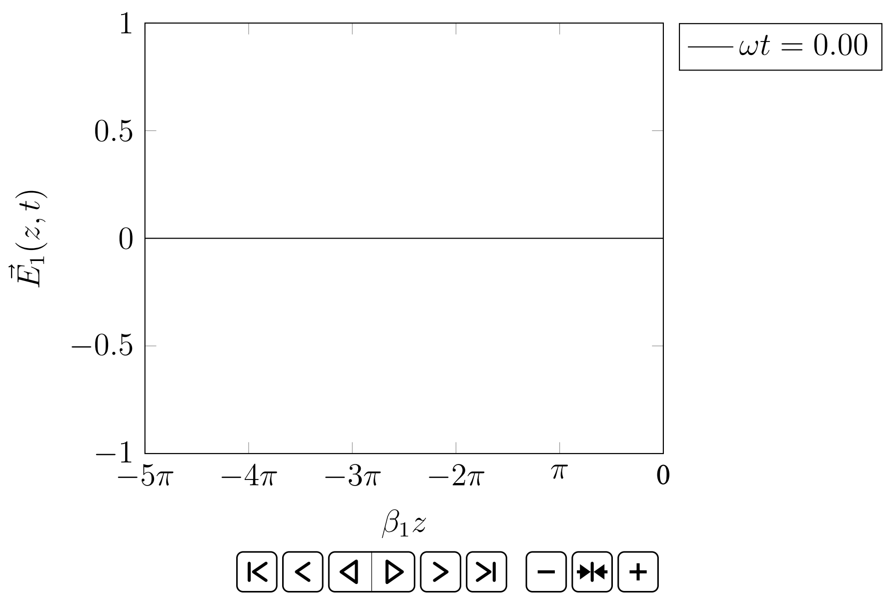
$$\vec{\mathbf{E}}_1(z,t) = \mathrm{Re}\left[-\hat{\mathbf{a}}_x \, 2j \, E_{i0} \, \sin \beta_1 z \, (\cos \omega t + j \sin \omega t)\right] \tag{2.220}$$

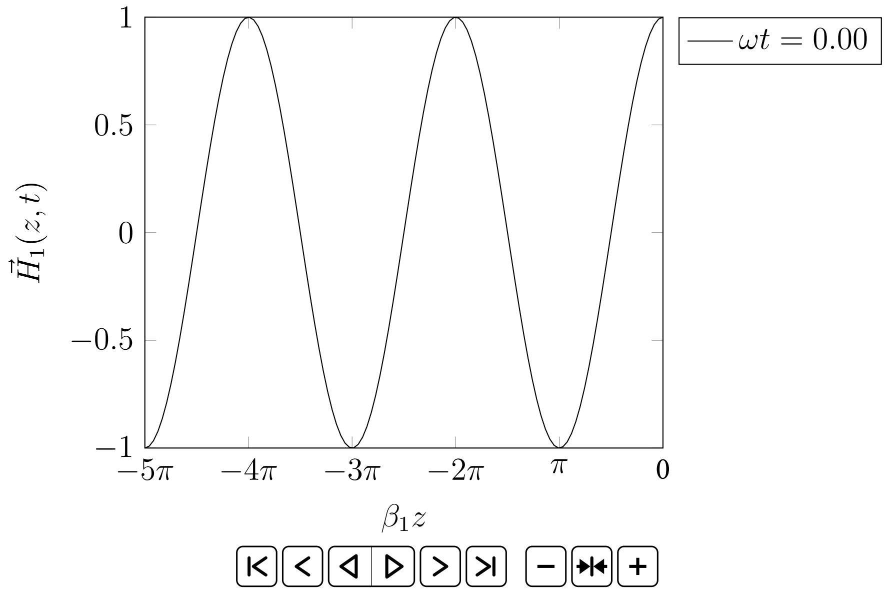
$$\vec{\mathbf{E}}_1(z,t) = \hat{\mathbf{a}}_x \, 2 \, E_{i0} \, \sin \beta_1 z \, \sin \omega t \tag{2.221}$$

$$\vec{\mathbf{H}}_1(z,t) = \mathrm{Re}\left[\vec{\mathbf{H}}_1(z) \, e^{j\omega t}\right] \tag{2.222}$$

$$\vec{\mathbf{H}}_1(z,t) = \mathrm{Re}\left[\hat{\mathbf{a}}_y \, 2 \, \frac{E_{i0}}{\eta_1} \, \cos \beta_1 z \, e^{j\omega t}\right] \tag{2.223}$$

$$\vec{\mathbf{H}}_1(z,t) = \hat{\mathbf{a}}_y \, 2 \, \frac{E_{i0}}{\eta_1} \, \cos \beta_1 z \, \cos \omega t \tag{2.224}$$



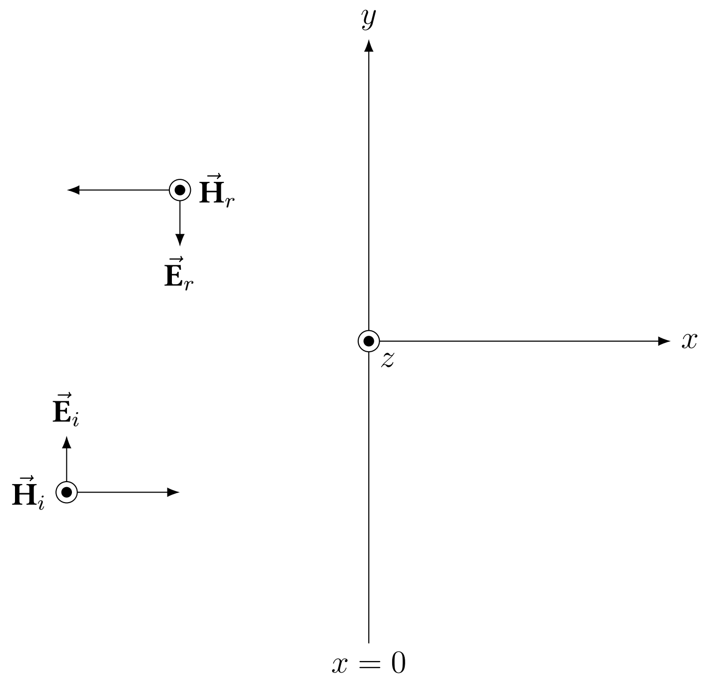


The total wave in medium 1 is not a traveling wave. It is a **standing wave**, resulting from the superpositon of two waves traveling in opposite directions.

Example 2.3

A y-polarized uniform plane wave ($\vec{\mathbf{E}}_i, \vec{\mathbf{H}}_i$) with a frequency 100 MHz propagates in air in the $+x$-direction and impinges normally on a perfectly conducting plane at $x = 0$. Assuming the amplitude of $\vec{\mathbf{E}}_i$ to be 6 mV/m, write the phasor and instantaneous expressions for (a) $\vec{\mathbf{E}}_i$ and $\vec{\mathbf{H}}_i$ of the incident wave; (b) $\vec{\mathbf{E}}_r$ and $\vec{\mathbf{H}}_r$ of the reflected wave; and (c) $\vec{\mathbf{E}}_1$ and $\vec{\mathbf{H}}_1$ of the total wave in air. (d) Determine the location nearest to the conducting plane where $\vec{\mathbf{E}}_1$ is zero.

Solution



a)

$$\vec{\mathbf{E}}_i(x) = \hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, e^{-jkx} \quad (\mathrm{V/m}) \tag{2.225}$$

$$\eta_1 = \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \tag{2.226}$$

$$\vec{\mathbf{H}}_i(x) = \hat{\mathbf{a}}_z\, \frac{6 \times 10^{-3}}{120\pi}\, e^{-jkx} \quad (\mathrm{A/m}) \tag{2.227}$$

$$k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi \times 10^8}{3 \times 10^8} = \frac{2\pi}{3} \tag{2.228}$$

$$\vec{\mathbf{E}}_i(x,t) = \mathrm{Re}\left[\vec{\mathbf{E}}_i(x)\, e^{j\omega t}\right] \tag{2.229}$$

$$\vec{\mathbf{E}}_i(x,t) = \mathrm{Re}\left[\hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, e^{-jkx}\, e^{j\omega t}\right] \tag{2.230}$$

$$\vec{\mathbf{E}}_i(x,t) = \mathrm{Re}\left[\hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, e^{j(\omega t - kx)}\right] \tag{2.231}$$

$$\vec{\mathbf{E}}_i(x,t) = \hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, \cos(\omega t - kx) \quad (\mathrm{V/m}) \tag{2.232}$$

In a similar way

$$\vec{\mathbf{H}}_i(x,t) = \mathrm{Re}\left[\hat{\mathbf{a}}_z\, \frac{6 \times 10^{-3}}{120\pi}\, e^{-jkx}\, e^{j\omega t}\right] \tag{2.233}$$

$$\vec{\mathbf{H}}_i(x,t) = \hat{\mathbf{a}}_z\, \frac{6 \times 10^{-3}}{120\pi}\, \cos(\omega t - kx) \quad (\mathrm{A/m}) \tag{2.234}$$

b)

$$\vec{\mathbf{E}}_r(x) = -\hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, e^{jkx} \quad (\mathrm{V/m}) \tag{2.235}$$

$$\vec{\mathbf{H}}_r(x) = \frac{1}{\eta_1}\, \hat{\mathbf{a}}_n \times \vec{\mathbf{E}}_r(x) \tag{2.236}$$

$$\hat{\mathbf{a}}_n = -\hat{\mathbf{a}}_x \tag{2.237}$$

$$\vec{\mathbf{H}}_r(x) = \hat{\mathbf{a}}_z\, \frac{6 \times 10^{-3}}{120\pi}\, e^{jkx} \quad (\mathrm{A/m}) \tag{2.238}$$

$$\vec{\mathbf{E}}_r(x,t) = \mathrm{Re}\left[\vec{\mathbf{E}}_r(x)\, e^{j\omega t}\right] \tag{2.239}$$

$$\vec{\mathbf{E}}_r(x,t) = \mathrm{Re}\left[-\hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, e^{jkx}\, e^{j\omega t}\right] \tag{2.240}$$

$$\vec{\mathbf{E}}_r(x,t) = \mathrm{Re}\left[-\hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, e^{j(\omega t + kx)}\right] \tag{2.241}$$

$$\vec{\mathbf{E}}_r(x,t) = -\hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, \cos(\omega t + kx) \quad (\mathrm{V/m}) \tag{2.242}$$

In a similar way

$$\vec{\mathbf{H}}_r(x,t) = \mathrm{Re}\left[\hat{\mathbf{a}}_z\, \frac{6 \times 10^{-3}}{120\pi}\, e^{jkx}\, e^{j\omega t}\right] \tag{2.243}$$

$$\vec{\mathbf{H}}_r(x,t) = \hat{\mathbf{a}}_z\, \frac{6 \times 10^{-3}}{120\pi}\, \cos(\omega t + kx) \quad (\mathrm{A/m}) \tag{2.244}$$

c)

$$\vec{\mathbf{E}}_1(x) = \vec{\mathbf{E}}_i(x) + \vec{\mathbf{E}}_r(x) \tag{2.245}$$

$$\vec{\mathbf{E}}_1(x) = \hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, e^{-jkx} - \hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, e^{jkx} \tag{2.246}$$

$$\vec{\mathbf{E}}_1(x) = \hat{\mathbf{a}}_y\, 6 \times 10^{-3}\, \left(e^{-jkx} - e^{jkx}\right) \tag{2.247}$$

$$e^{-jkx} - e^{jkx} = (\cos kx - j \sin kx) - (\cos kx + j \sin kx) = -2j \sin kx \tag{2.248}$$

$$\vec{\mathbf{E}}_1(x) = -\hat{\mathbf{a}}_y\, j\, 12 \times 10^{-3}\, \sin kx \quad (\mathrm{V/m}) \tag{2.249}$$

$$\vec{\mathbf{E}}_1(x,t) = \mathrm{Re}\left[\vec{\mathbf{E}}_1(x)\, e^{j\omega t}\right] \tag{2.250}$$

$$\vec{\mathbf{E}}_1(x,t) = \mathrm{Re}\left[-\hat{\mathbf{a}}_y\, j\, 12 \times 10^{-3}\, \sin kx\, e^{j\omega t}\right] \tag{2.251}$$

$$\vec{\mathbf{E}}_1(x,t) = \mathrm{Re}\left[-\hat{\mathbf{a}}_y\, j\, 12 \times 10^{-3}\, \sin kx\, (\cos \omega t + j \sin \omega t)\right] \tag{2.252}$$

$$\vec{\mathbf{E}}_1(x,t) = \mathrm{Re}\left[-\hat{\mathbf{a}}_y\, j\, 12 \times 10^{-3}\, \sin kx\, \cos \omega t + \hat{\mathbf{a}}_y\, 12 \times 10^{-3}\, \sin kx\, \sin \omega t\right] \tag{2.253}$$

$$\vec{\mathbf{E}}_1(x,t) = \hat{\mathbf{a}}_y\, 12 \times 10^{-3}\, \sin kx\, \sin \omega t \quad (\mathrm{V/m}) \tag{2.254}$$

$$\vec{\mathbf{H}}_1(x) = \vec{\mathbf{H}}_i(x) + \vec{\mathbf{H}}_r(x) \tag{2.255}$$

$$\vec{\mathbf{H}}_1(x) = \hat{\mathbf{a}}_z\, \frac{6 \times 10^{-3}}{120\pi}\, e^{-jkx} + \hat{\mathbf{a}}_z\, \frac{6 \times 10^{-3}}{120\pi}\, e^{jkx} \tag{2.256}$$

$$\vec{\mathbf{H}}_1(x) = \hat{\mathbf{a}}_z\, \frac{6 \times 10^{-3}}{120\pi}\, \left(e^{-jkx} + e^{jkx}\right) \tag{2.257}$$

$$e^{-jkx} + e^{jkx} = (\cos kx - j \sin kx) + (\cos kx + j \sin kx) = 2 \cos kx \tag{2.258}$$

$$\vec{\mathbf{H}}_1(x) = \hat{\mathbf{a}}_z\, \frac{12 \times 10^{-3}}{120\pi}\, \cos kx \quad (\mathrm{A/m}) \tag{2.259}$$

$$\vec{\mathbf{H}}_1(x,t) = \mathrm{Re}\left[\vec{\mathbf{H}}_1(x)\, e^{j\omega t}\right] \tag{2.260}$$

$$\vec{\mathbf{H}}_1(x,t) = \mathrm{Re}\left[\hat{\mathbf{a}}_z\, \frac{12 \times 10^{-3}}{120\pi}\, \cos kx\, e^{j\omega t}\right] \tag{2.261}$$

$$\vec{\mathbf{H}}_1(x,t) = \mathrm{Re}\left[\hat{\mathbf{a}}_z\, \frac{12 \times 10^{-3}}{120\pi}\, \cos kx\, (\cos \omega t + j \sin \omega t)\right] \tag{2.262}$$

$$\vec{\mathbf{H}}_1(x,t) = \hat{\mathbf{a}}_z\, \frac{12 \times 10^{-3}}{120\pi}\, \cos kx\, \cos \omega t \quad (\mathrm{A/m}) \tag{2.263}$$

d)

$$\vec{\mathbf{E}}_1(x,t) = \hat{\mathbf{a}}_y\, 12 \times 10^{-3}\, \sin kx\, \sin \omega t = 0 \tag{2.264}$$

$$\Rightarrow \sin kx = 0 \tag{2.265}$$

$$\Rightarrow kx = n\pi, \quad n = 0, -1, -2, -3, \cdots \tag{2.266}$$

$$\Rightarrow \frac{2\pi}{\lambda}x = n\pi \tag{2.267}$$

$$\Rightarrow x = n\,\frac{\lambda}{2}, \quad n = 0, -1, -2, -3, \cdots \tag{2.268}$$

$$f = 100 \text{ MHz} = 10^8 \tag{2.269}$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^8} = 3 \text{ m} \tag{2.270}$$

$$\Rightarrow x = n\,\frac{3}{2}, \quad n = 0, -1, -2, -3, \cdots \tag{2.271}$$

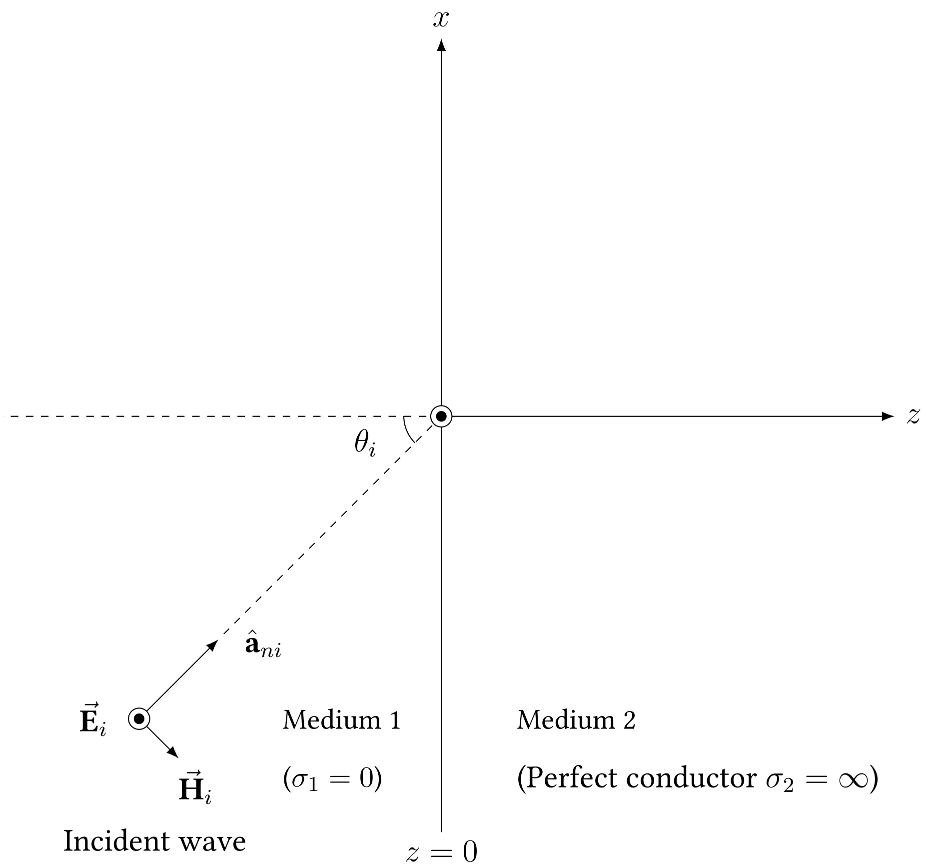
$$n = 0 \text{ gives the boundary} \tag{2.272}$$

$$n = -1 \text{ gives the nearest location} \tag{2.273}$$

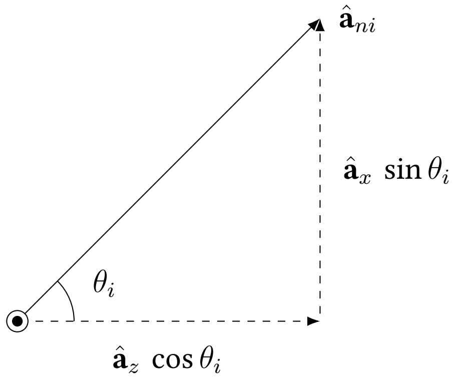
$$\Rightarrow x = -\frac{3}{2} \text{ m} \tag{2.274}$$

2.5 Oblique Incidence at a Plane Conducting Boundary

Horizontal Polarization (Yatay Kutuplanma)



θ_i: angle of incidence



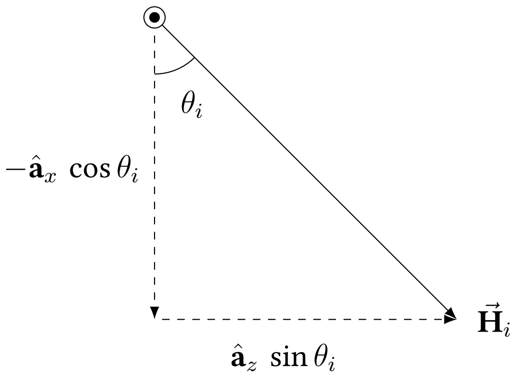
$$\hat{\mathbf{a}}_{ni} = \hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i \tag{2.275}$$

$$\vec{\mathbf{E}}_i(x,z) = \hat{\mathbf{a}}_y\, E_{i0}\, e^{-j\beta_1\, \hat{\mathbf{a}}_{ni}\cdot\vec{\mathbf{R}}} \tag{2.276}$$

$$\hat{\mathbf{a}}_{ni}\cdot\vec{\mathbf{R}} = (\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i)\cdot(\hat{\mathbf{a}}_x\, x + \hat{\mathbf{a}}_y\, y + \hat{\mathbf{a}}_z\, z) \tag{2.277}$$

$$\hat{\mathbf{a}}_{ni}\cdot\vec{\mathbf{R}} = (x\sin\theta_i + z\cos\theta_i) \tag{2.278}$$

$$\vec{\mathbf{E}}_i(x,z) = \hat{\mathbf{a}}_y\, E_{i0}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{2.279}$$

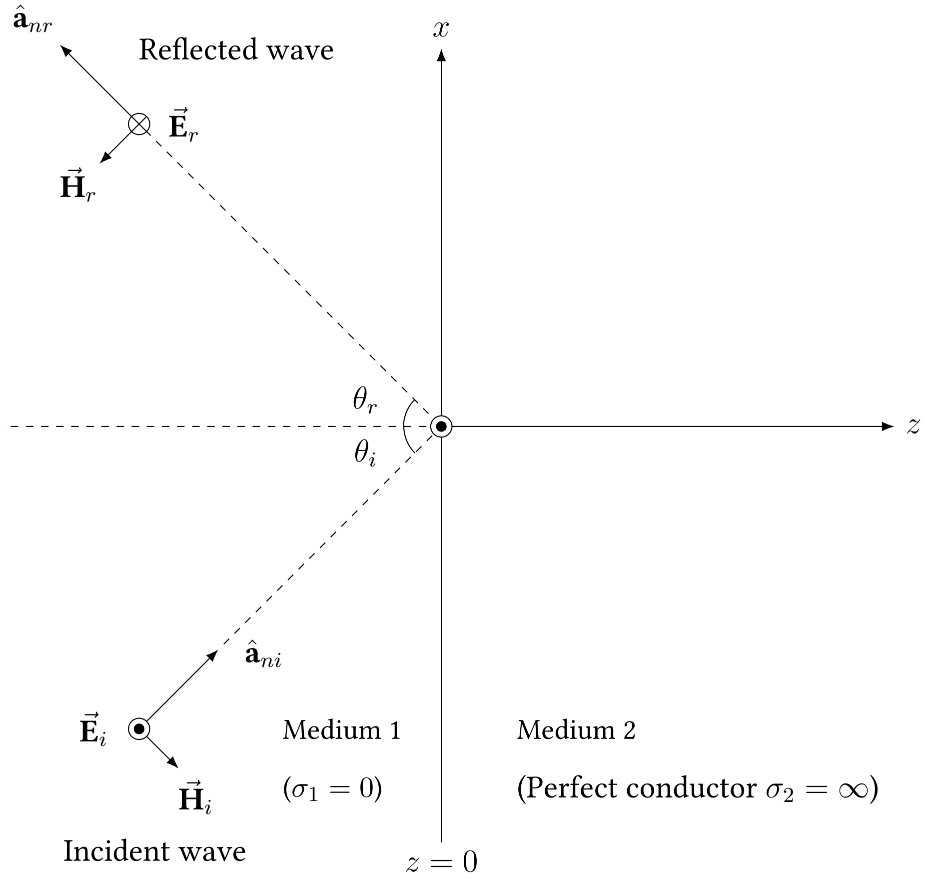


$$\vec{\mathbf{H}}_i(x,z) = \frac{1}{\eta_1}\left[\hat{\mathbf{a}}_{ni}\times\vec{\mathbf{E}}_i(x,z)\right] \tag{2.280}$$

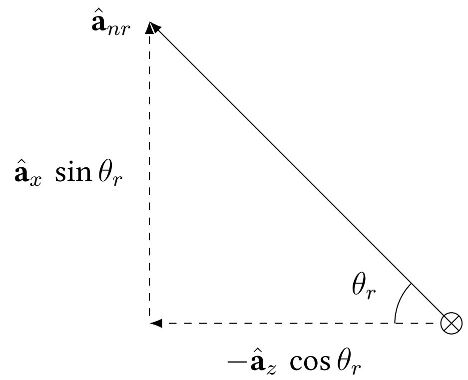
$$\vec{\mathbf{H}}_i(x,z) = \frac{1}{\eta_1}\left[(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i)\times\hat{\mathbf{a}}_y\, E_{i0}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\right] \tag{2.281}$$

$$\vec{\mathbf{H}}_i(x,z) = (\hat{\mathbf{a}}_z \sin\theta_i - \hat{\mathbf{a}}_x \cos\theta_i)\, \frac{E_{i0}}{\eta_1}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{2.282}$$

$$\vec{\mathbf{H}}_i(x,z) = (-\hat{\mathbf{a}}_x \cos\theta_i + \hat{\mathbf{a}}_z \sin\theta_i)\, \frac{E_{i0}}{\eta_1}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{2.283}$$



θ_r: angle of reflection



$$\vec{\mathbf{E}}_r(x,z) = -\hat{\mathbf{a}}_y\, E_{r0}\, e^{-j\beta_1\, \hat{\mathbf{a}}_{nr}\cdot\vec{\mathbf{R}}} \tag{2.284}$$

$$\hat{\mathbf{a}}_{nr} = \hat{\mathbf{a}}_x \sin\theta_r - \hat{\mathbf{a}}_z \cos\theta_r \tag{2.285}$$

$$\hat{\mathbf{a}}_{nr}\cdot\vec{\mathbf{R}} = (\hat{\mathbf{a}}_x \sin\theta_r - \hat{\mathbf{a}}_z \cos\theta_r)\cdot(\hat{\mathbf{a}}_x\, x + \hat{\mathbf{a}}_y\, y + \hat{\mathbf{a}}_z\, z) \tag{2.286}$$

$$\hat{\mathbf{a}}_{nr}\cdot\vec{\mathbf{R}} = (x\sin\theta_r - z\cos\theta_r) \tag{2.287}$$

$$\vec{\mathbf{E}}_r(x,z) = -\hat{\mathbf{a}}_y\, E_{r0}\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{2.288}$$

At the boundary surface, $z = 0$, the tangential components of $\vec{\mathbf{E}}_1$ is zero ($E_{1t} = E_{2t} = 0$).

$$\vec{\mathbf{E}}_1(x,z) = \vec{\mathbf{E}}_i(x,z) + \vec{\mathbf{E}}_r(x,z) \tag{2.289}$$

$$\vec{\mathbf{E}}_1(x, z = 0) = \vec{\mathbf{E}}_i(x, z = 0) + \vec{\mathbf{E}}_r(x, z = 0) = 0 \tag{2.290}$$

$$\vec{\mathbf{E}}_1(x, z = 0) = \hat{\mathbf{a}}_y\, E_{i0}\, e^{-j\beta_1 x \sin\theta_i} - \hat{\mathbf{a}}_y\, E_{r0}\, e^{-j\beta_1 x \sin\theta_r} = 0 \tag{2.291}$$

$$E_{i0}\, e^{-j\beta_1 x \sin\theta_i} = E_{r0}\, e^{-j\beta_1 x \sin\theta_r} \qquad \text{valid for all } x \tag{2.292}$$

For $x = 0$,

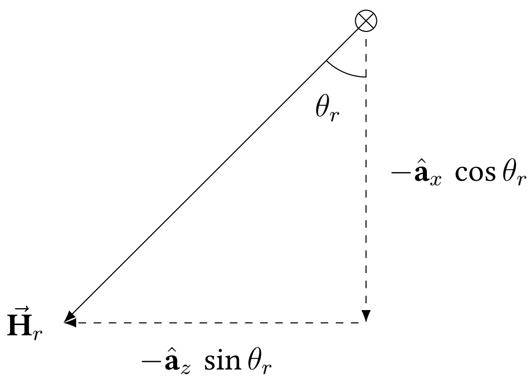
$$\boxed{E_{r0} = E_{i0}} \tag{2.293}$$

$$\Rightarrow \boxed{\theta_r = \theta_i} \qquad (\text{Snell's law of reflection}) \tag{2.294}$$

The angle of reflection is equal to the angle of incidence. So we have

$$\vec{\mathbf{E}}_r(x, z) = -\hat{\mathbf{a}}_y\, E_{i0}\, e^{-j\beta_1(x \sin\theta_i - z\cos\theta_i)} \tag{2.295}$$

$$\vec{\mathbf{H}}_r(x, z) = \frac{1}{\eta_1}\left[\hat{\mathbf{a}}_{nr} \times \vec{\mathbf{E}}_r(x, z)\right] \tag{2.296}$$



$$\vec{\mathbf{H}}_r(x, z) = \frac{1}{\eta_1}\left[(\hat{\mathbf{a}}_x\ \sin\theta_r - \hat{\mathbf{a}}_z\ \cos\theta_r) \times (-)\,\hat{\mathbf{a}}_y\, E_{i0}\, e^{-j\beta_1(x \sin\theta_i - z\cos\theta_i)}\right] \tag{2.297}$$

$$\vec{\mathbf{H}}_r(x, z) = (-\hat{\mathbf{a}}_z\ \sin\theta_r - \hat{\mathbf{a}}_x\ \cos\theta_r)\ \frac{E_{i0}}{\eta_1}\, e^{-j\beta_1(x \sin\theta_i - z\cos\theta_i)} \tag{2.298}$$

$$\vec{\mathbf{H}}_r(x, z) = (-\hat{\mathbf{a}}_x\ \cos\theta_i - \hat{\mathbf{a}}_z\ \sin\theta_i)\ \frac{E_{i0}}{\eta_1}\, e^{-j\beta_1(x \sin\theta_i - z\cos\theta_i)} \tag{2.299}$$

The total electric field

$$\vec{\mathbf{E}}_1(x, z) = \vec{\mathbf{E}}_i(x, z) + \vec{\mathbf{E}}_r(x, z) \tag{2.300}$$

$$\vec{\mathbf{E}}_1(x, z) = \hat{\mathbf{a}}_y\, E_{i0}\, e^{-j\beta_1(x \sin\theta_i + z\cos\theta_i)} - \hat{\mathbf{a}}_y\, E_{i0}\, e^{-j\beta_1(x \sin\theta_i - z\cos\theta_i)} \tag{2.301}$$

$$\vec{\mathbf{E}}_1(x, z) = \hat{\mathbf{a}}_y\, E_{i0}\left(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i}\right)\, e^{-j\beta_1 x \sin\theta_i} \tag{2.302}$$

$$e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i} = -2j\,\sin(\beta_1 z\cos\theta_i) \tag{2.303}$$

$$\vec{\mathbf{E}}_1(x, z) = -\hat{\mathbf{a}}_y\, j2\, E_{i0}\,\sin(\beta_1 z\cos\theta_i)\, e^{-j\beta_1 x \sin\theta_i} \tag{2.304}$$

The total magnetic field

$$\vec{\mathbf{H}}_1(x,z) = \vec{\mathbf{H}}_i(x,z) + \vec{\mathbf{H}}_r(x,z) \tag{2.305}$$

$$\begin{aligned}\vec{\mathbf{H}}_1(x,z) = &(-\hat{\mathbf{a}}_x\ \cos\theta_i + \hat{\mathbf{a}}_z\ \sin\theta_i)\ \frac{E_{i0}}{\eta_1}\ e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \\ &+ (-\hat{\mathbf{a}}_x\ \cos\theta_i - \hat{\mathbf{a}}_z\ \sin\theta_i)\ \frac{E_{i0}}{\eta_1}\ e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}\end{aligned} \tag{2.306}$$

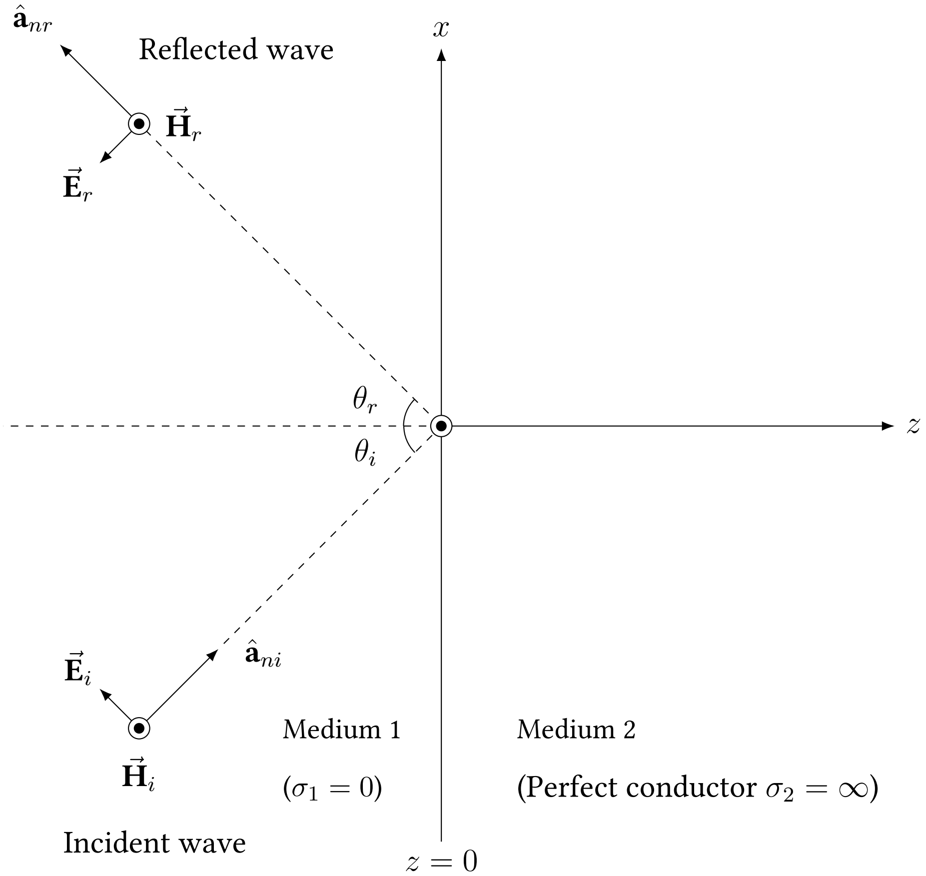
$$\begin{aligned}\vec{\mathbf{H}}_1(x,z) = &-\hat{\mathbf{a}}_x\ \cos\theta_i\ \frac{E_{i0}}{\eta_1}\ \left(e^{-j\beta_1 z\cos\theta_i} + e^{j\beta_1 z\cos\theta_i}\right) e^{-j\beta_1 x\sin\theta_i} \\ &+\hat{\mathbf{a}}_z\ \sin\theta_i\ \frac{E_{i0}}{\eta_1}\ \left(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i}\right) e^{-j\beta_1 x\sin\theta_i}\end{aligned} \tag{2.307}$$

$$e^{-j\beta_1 z\cos\theta_i} + e^{j\beta_1 z\cos\theta_i} = 2\cos(\beta_1 z\cos\theta_i) \tag{2.308}$$

$$e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i} = -2j\sin(\beta_1 z\cos\theta_i) \tag{2.309}$$

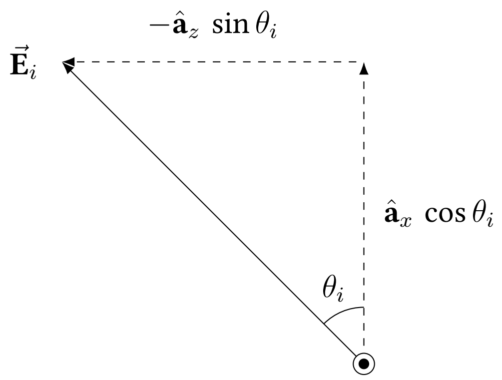
$$\begin{aligned}\vec{\mathbf{H}}_1(x,z) = &-\hat{\mathbf{a}}_x\ \frac{2E_{i0}}{\eta_1}\ \cos\theta_i\ \cos(\beta_1 z\cos\theta_i)\, e^{-j\beta_1 x\sin\theta_i} \\ &-\hat{\mathbf{a}}_z\ \frac{2jE_{i0}}{\eta_1}\ \sin\theta_i\ \sin(\beta_1 z\cos\theta_i)\, e^{-j\beta_1 x\sin\theta_i}\end{aligned} \tag{2.310}$$

Vertical Polarization (Dikey Kutuplanma)



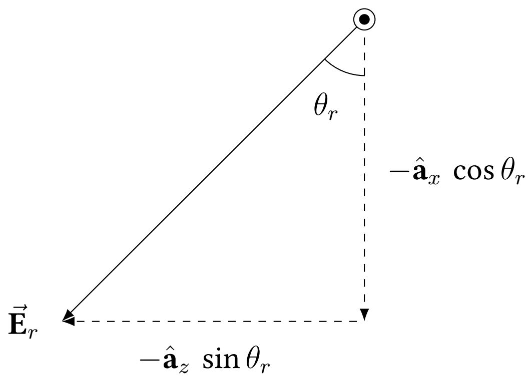
$$\hat{\mathbf{a}}_{ni} = \hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i \tag{2.311}$$

$$\hat{\mathbf{a}}_{nr} = \hat{\mathbf{a}}_x \sin\theta_r - \hat{\mathbf{a}}_z \cos\theta_r \tag{2.312}$$



$$\vec{\mathbf{E}}_i(x,z) = (\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\ E_{i0}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{2.313}$$

$$\vec{\mathbf{H}}_i(x,z) = \hat{\mathbf{a}}_y \frac{E_{i0}}{\eta_1}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{2.314}$$



$$\vec{\mathbf{E}}_r(x,z) = (-\hat{\mathbf{a}}_x \cos\theta_r - \hat{\mathbf{a}}_z \sin\theta_r)\ E_{r0}\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{2.315}$$

$$\vec{\mathbf{H}}_r(x,z) = \hat{\mathbf{a}}_y \frac{E_{r0}}{\eta_1}\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{2.316}$$

At $z = 0$, the tangential component of the total electric field intensity must vanish.

$$E_{ix}(x, z = 0) + E_{rx}(x, z = 0) = 0 \tag{2.317}$$

$$E_{i0} \cos\theta_i\, e^{-j\beta_1 x \sin\theta_i} - E_{r0} \cos\theta_r\, e^{-j\beta_1 x \sin\theta_r} = 0 \tag{2.318}$$

$$\Rightarrow \boxed{E_{r0} = E_{i0}} \tag{2.319}$$

$$\boxed{\theta_r = \theta_i} \qquad \text{(Snell's law of reflection)} \tag{2.320}$$

The total electric field intensity in medium 1

$$\vec{\mathbf{E}}_1(x,z) = \vec{\mathbf{E}}_i(x,z) + \vec{\mathbf{E}}_r(x,z) \tag{2.321}$$

$$\begin{aligned}\vec{\mathbf{E}}_1(x,z) =& \left(\hat{\mathbf{a}}_x\ \cos\theta_i - \hat{\mathbf{a}}_z\ \sin\theta_i\right)\ E_{i0}\ e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\\ &- \left(\hat{\mathbf{a}}_x\ \cos\theta_i + \hat{\mathbf{a}}_z\ \sin\theta_i\right)\ E_{i0}\ e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}\end{aligned} \tag{2.322}$$

$$\begin{aligned}\vec{\mathbf{E}}_1(x,z) =& \hat{\mathbf{a}}_x\ \cos\theta_i\, E_{i0}\ \left(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i}\right) e^{-j\beta_1 x\sin\theta_i}\\ &- \hat{\mathbf{a}}_z\ \sin\theta_i\, E_{i0}\ \left(e^{-j\beta_1 z\cos\theta_i} + e^{j\beta_1 z\cos\theta_i}\right) e^{-j\beta_1 x\sin\theta_i}\end{aligned} \tag{2.323}$$

$$e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i} = -2j\ \sin(\beta_1 z\cos\theta_i) \tag{2.324}$$

$$e^{-j\beta_1 z\cos\theta_i} + e^{j\beta_1 z\cos\theta_i} = 2\cos(\beta_1 z\cos\theta_i) \tag{2.325}$$

$$\begin{aligned}\vec{\mathbf{E}}_1(x,z) =& -\hat{\mathbf{a}}_x\, 2jE_{i0}\ \cos\theta_i\ \sin(\beta_1 z\cos\theta_i)\, e^{-j\beta_1 x\sin\theta_i}\\ &- \hat{\mathbf{a}}_z\, 2E_{i0}\ \sin\theta_i\ \cos(\beta_1 z\cos\theta_i)\, e^{-j\beta_1 x\sin\theta_i}\end{aligned} \tag{2.326}$$

Total magnetic field intensity in medium 1

$$\vec{\mathbf{H}}_1(x,z) = \vec{\mathbf{H}}_i(x,z) + \vec{\mathbf{H}}_r(x,z) \tag{2.327}$$

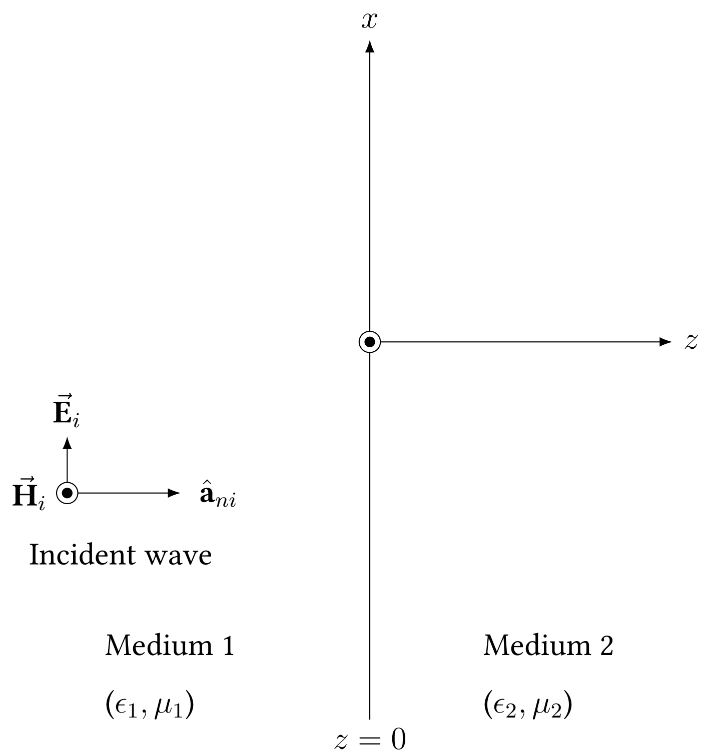
$$\vec{\mathbf{H}}_1(x,z) = \hat{\mathbf{a}}_y\ \frac{E_{i0}}{\eta_1}\ e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} + \hat{\mathbf{a}}_y\ \frac{E_{i0}}{\eta_1}\ e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)} \tag{2.328}$$

$$\vec{\mathbf{H}}_1(x,z) = \hat{\mathbf{a}}_y\ \frac{E_{i0}}{\eta_1}\ \left(e^{-j\beta_1 z\cos\theta_i} + e^{j\beta_1 z\cos\theta_i}\right) e^{-j\beta_1 x\sin\theta_i} \tag{2.329}$$

$$e^{-j\beta_1 z\cos\theta_i} + e^{j\beta_1 z\cos\theta_i} = 2\cos(\beta_1 z\cos\theta_i) \tag{2.330}$$

$$\vec{\mathbf{H}}_1(x,z) = \hat{\mathbf{a}}_y\ \frac{2E_{i0}}{\eta_1}\ \cos(\beta_1 z\cos\theta_i)\, e^{-j\beta_1 x\sin\theta_i} \tag{2.331}$$

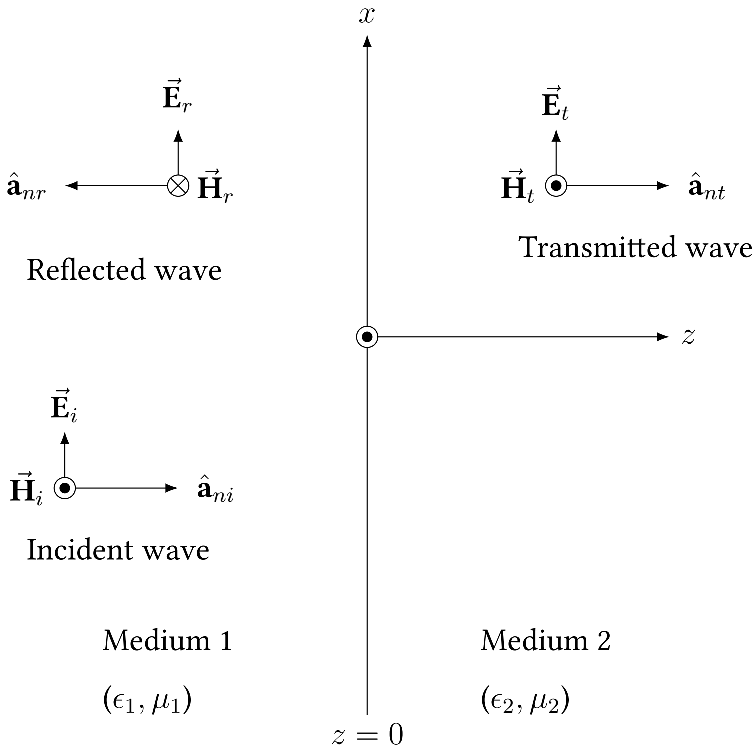
2.6 Normal Incidence at a Plane Dielectric Boundary



$$\vec{\mathbf{E}}_i(z) = \hat{\mathbf{a}}_x\, E_{i0}\, e^{-j\beta_1 z} \tag{2.332}$$

$$\vec{\mathbf{H}}_i(z) = \hat{\mathbf{a}}_y\, \frac{E_{i0}}{\eta_1}\, e^{-j\beta_1 z} \tag{2.333}$$

When an electromagnetic wave is incident on the surface of a different dielectric medium, part of the incident power is reflected and part is transmitted.



$$\vec{\mathbf{E}}_r(z) = \hat{\mathbf{a}}_x\, E_{r0}\, e^{j\beta_1 z} \tag{2.334}$$

$$\vec{\mathbf{H}}_r(z) = \frac{1}{\eta_1}\,(-\hat{\mathbf{a}}_z) \times \vec{\mathbf{E}}_r(z) = -\hat{\mathbf{a}}_y\, \frac{E_{r0}}{\eta_1}\, e^{j\beta_1 z} \tag{2.335}$$

$$\vec{\mathbf{E}}_t(z) = \hat{\mathbf{a}}_x\, E_{t0}\, e^{-j\beta_2 z} \tag{2.336}$$

$$\vec{\mathbf{H}}_t(z) = \frac{1}{\eta_2}\,\hat{\mathbf{a}}_z \times \vec{\mathbf{E}}_t(z) = \hat{\mathbf{a}}_y\, \frac{E_{t0}}{\eta_2}\, e^{-j\beta_2 z} \tag{2.337}$$

E_{t0} : the magnitude of $\vec{\mathbf{E}}_t$ at $z = 0$.
β_2 : phase constant of medium 2.
η_2 : intrinsic impedance of medium 2.

$$\beta_2 = \omega\,\sqrt{\mu_2\,\epsilon_2} \tag{2.338}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} \tag{2.339}$$

E_{r0} and E_{t0} may be positive or negative, depending on $\epsilon_1, \mu_1, \epsilon_2, \mu_2$. E_{r0} and E_{t0} must be determined.

The tangential components (the $x-$ and $y-$ components) of the electric and magnetic field intensities must be continous at $z = 0$.

$$\vec{\mathbf{E}}_i(0) + \vec{\mathbf{E}}_r(0) = \vec{\mathbf{E}}_t(0) \tag{2.340}$$

$$\Rightarrow E_{i0} + E_{r0} = E_{t0} \tag{2.341}$$

$$\vec{\mathbf{H}}_i(0) + \vec{\mathbf{H}}_r(0) = \vec{\mathbf{H}}_t(0) \tag{2.342}$$

$$\Rightarrow \frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2} \tag{2.343}$$

$$\begin{aligned} E_{i0} + E_{r0} &= E_{t0} \\ \frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} &= \frac{E_{t0}}{\eta_2} \end{aligned} \tag{2.344}$$

$$\begin{aligned} E_{r0} - E_{t0} &= -E_{i0} \\ -\frac{E_{r0}}{\eta_1} - \frac{E_{t0}}{\eta_2} &= -\frac{E_{i0}}{\eta_1} \qquad (//\,\eta_1) \end{aligned} \tag{2.345}$$

$$\begin{aligned} E_{r0} - E_{t0} &= -E_{i0} \\ -E_{r0} - \frac{\eta_1}{\eta_2} E_{t0} &= -E_{i0} \end{aligned} \tag{2.346}$$

$$-E_{t0}\left(1 + \frac{\eta_1}{\eta_2}\right) = -2\, E_{i0} \tag{2.347}$$

$$E_{t0}\left(1 + \frac{\eta_1}{\eta_2}\right) = 2\, E_{i0} \tag{2.348}$$

$$E_{t0} = \frac{2}{1 + \frac{\eta_1}{\eta_2}} E_{i0} = \frac{2}{\frac{\eta_2 + \eta_1}{\eta_2}} E_{i0} \tag{2.349}$$

$$\boxed{E_{t0} = \frac{2\,\eta_2}{\eta_1 + \eta_2} E_{i0}} \tag{2.350}$$

$$E_{r0} = E_{t0} - E_{i0} = \frac{2\,\eta_2}{\eta_1 + \eta_2} E_{i0} - E_{i0} \tag{2.351}$$

$$E_{r0} = \left(\frac{2\,\eta_2}{\eta_1 + \eta_2} - 1\right) E_{i0} = \frac{2\,\eta_2 - (\eta_1 + \eta_2)}{\eta_1 + \eta_2} E_{i0} = \frac{2\,\eta_2 - \eta_1 - \eta_2}{\eta_1 + \eta_2} E_{i0} \tag{2.352}$$

$$\boxed{E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0}} \tag{2.353}$$

$$\boxed{\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}} \quad \text{(dimensionless)} \quad \text{(Reflection coefficient)} \tag{2.354}$$

$$\boxed{\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\,\eta_2}{\eta_2 + \eta_1}} \quad \text{(dimensionless)} \quad \text{(Transmission coefficient)} \tag{2.355}$$

The reflection coefficient Γ can be positive or negative. The tranmission coefficient τ is always positive.

$$E_{r0} = E_{t0} - E_{i0} \tag{2.356}$$

$$\frac{E_{r0}}{E_{i0}} = \frac{E_{t0}}{E_{i0}} - 1 \tag{2.357}$$

$$\Gamma = \tau - 1 \tag{2.358}$$

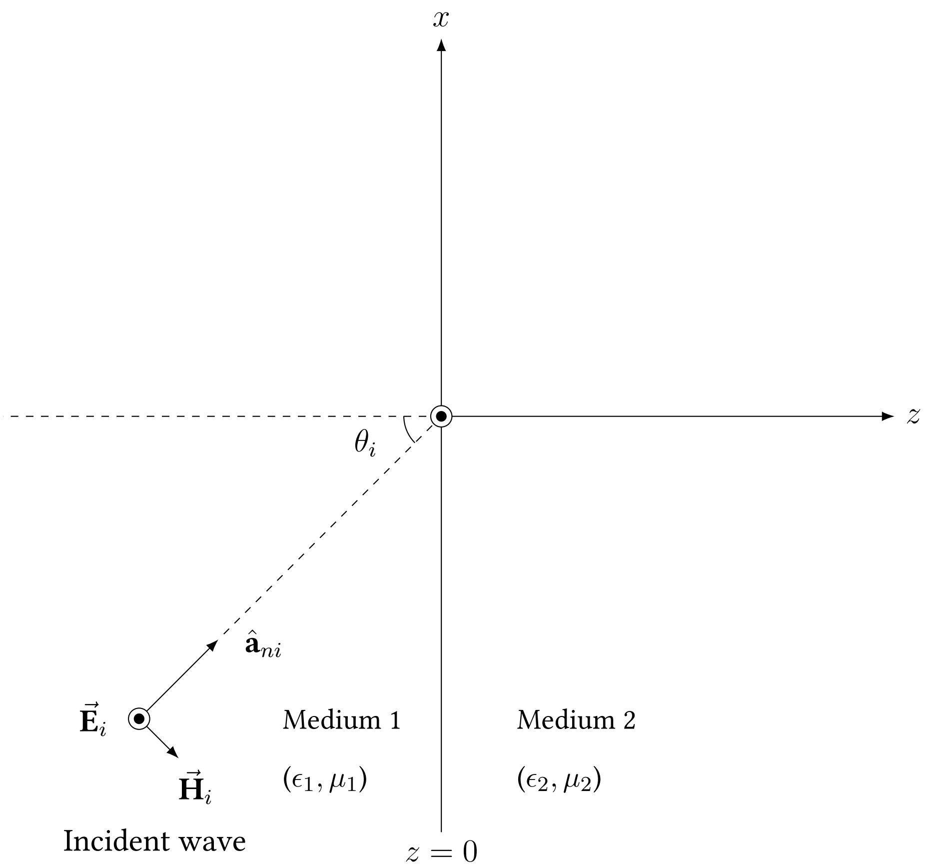
$$\boxed{\tau = \Gamma + 1} \tag{2.359}$$

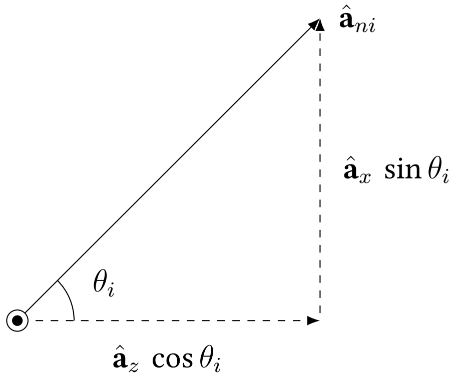
The total electric field in medium 1

$$\vec{\mathbf{E}}_1(z) = \vec{\mathbf{E}}_i(z) + \vec{\mathbf{E}}_r(z) = \hat{\mathbf{a}}_x\, E_{i0}\, \left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right) \tag{2.360}$$

2.7 Oblique Incidence at a Plane Dielectric Boundary

Horizontal Polarization (Yatay Kutuplanma)





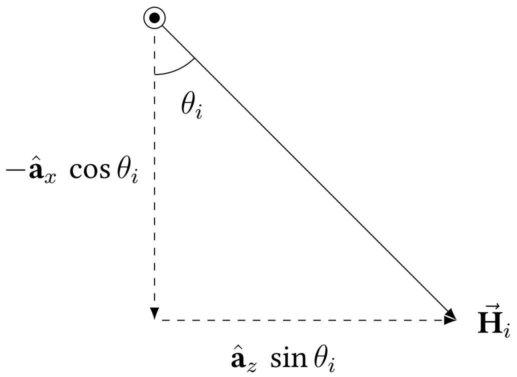
$$\vec{\mathbf{E}}_i(x,z) = \hat{\mathbf{a}}_y \, E_{i0} \, e^{-j\beta_1 \, \hat{\mathbf{a}}_{ni} \cdot \vec{\mathbf{R}}} \tag{2.361}$$

$$\hat{\mathbf{a}}_{ni} = \hat{\mathbf{a}}_x \ \sin\theta_i + \hat{\mathbf{a}}_z \ \cos\theta_i \tag{2.362}$$

$$\hat{\mathbf{a}}_{ni} \cdot \vec{\mathbf{R}} = (\hat{\mathbf{a}}_x \ \sin\theta_i + \hat{\mathbf{a}}_z \ \cos\theta_i) \cdot (\hat{\mathbf{a}}_x \, x + \hat{\mathbf{a}}_y \, y + \hat{\mathbf{a}}_z \, z) \tag{2.363}$$

$$\hat{\mathbf{a}}_{ni} \cdot \vec{\mathbf{R}} = (x \sin\theta_i + z \cos\theta_i) \tag{2.364}$$

$$\vec{\mathbf{E}}_i(x,z) = \hat{\mathbf{a}}_y \, E_{i0} \, e^{-j\beta_1 (x \sin\theta_i + z \cos\theta_i)} \tag{2.365}$$

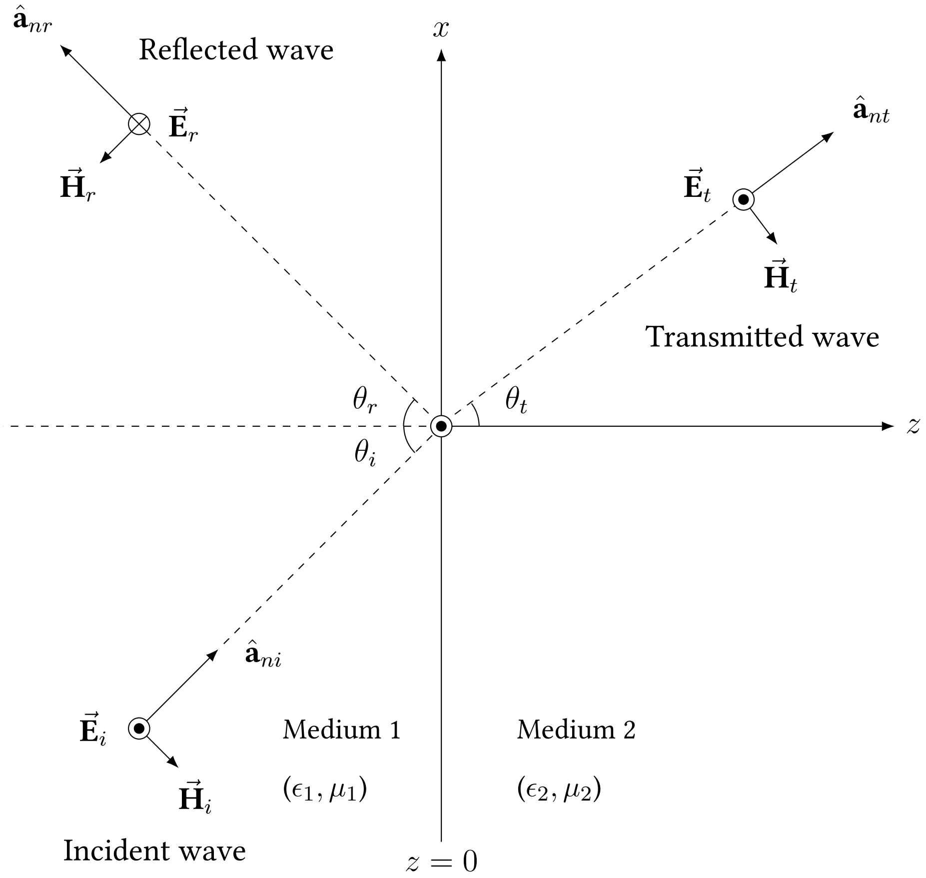


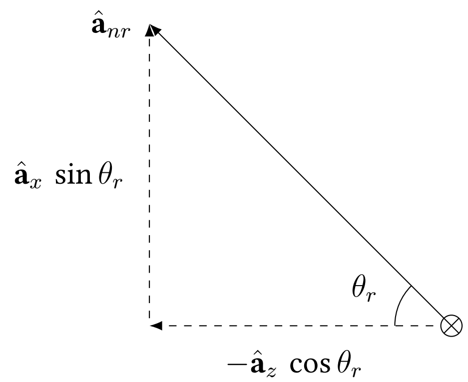
$$\vec{\mathbf{H}}_i(x,z) = \frac{1}{\eta_1} \left[\hat{\mathbf{a}}_{ni} \times \vec{\mathbf{E}}_i(x,z) \right] \tag{2.366}$$

$$\vec{\mathbf{H}}_i(x,z) = \frac{1}{\eta_1} \left[(\hat{\mathbf{a}}_x \ \sin\theta_i + \hat{\mathbf{a}}_z \ \cos\theta_i) \times \hat{\mathbf{a}}_y \, E_{i0} \, e^{-j\beta_1 (x \sin\theta_i + z \cos\theta_i)} \right] \tag{2.367}$$

$$\vec{\mathbf{H}}_i(x,z) = (\hat{\mathbf{a}}_z \ \sin\theta_i - \hat{\mathbf{a}}_x \ \cos\theta_i) \ \frac{E_{i0}}{\eta_1} \ e^{-j\beta_1 (x \sin\theta_i + z \cos\theta_i)} \tag{2.368}$$

$$\vec{\mathbf{H}}_i(x,z) = (-\hat{\mathbf{a}}_x \ \cos\theta_i + \hat{\mathbf{a}}_z \ \sin\theta_i) \ \frac{E_{i0}}{\eta_1} \ e^{-j\beta_1 (x \sin\theta_i + z \cos\theta_i)} \tag{2.369}$$





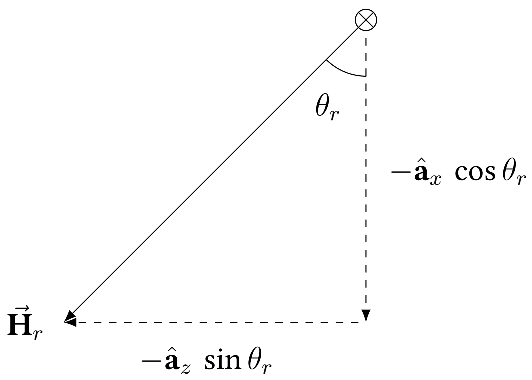
$$\vec{\mathbf{E}}_r(x,z) = -\hat{\mathbf{a}}_y\, E_{r0}\, e^{-j\beta_1\, \hat{\mathbf{a}}_{nr}\cdot\vec{\mathbf{R}}} \tag{2.370}$$

$$\hat{\mathbf{a}}_{nr} = \hat{\mathbf{a}}_x \ \sin\theta_r - \hat{\mathbf{a}}_z \ \cos\theta_r \tag{2.371}$$

$$\hat{\mathbf{a}}_{nr}\cdot\vec{\mathbf{R}} = (\hat{\mathbf{a}}_x \ \sin\theta_r - \hat{\mathbf{a}}_z \ \cos\theta_r)\cdot(\hat{\mathbf{a}}_x\, x + \hat{\mathbf{a}}_y\, y + \hat{\mathbf{a}}_z\, z) \tag{2.372}$$

$$\hat{\mathbf{a}}_{nr}\cdot\vec{\mathbf{R}} = (x\sin\theta_r - z\cos\theta_r) \tag{2.373}$$

$$\vec{\mathbf{E}}_r(x,z) = -\hat{\mathbf{a}}_y\, E_{r0}\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{2.374}$$



$$\vec{\mathbf{H}}_r(x,z) = \frac{1}{\eta_1}\left[\hat{\mathbf{a}}_{nr}\times\vec{\mathbf{E}}_r(x,z)\right] \tag{2.375}$$

$$\vec{\mathbf{H}}_r(x,z) = \frac{1}{\eta_1}\left[(\hat{\mathbf{a}}_x\,\sin\theta_r - \hat{\mathbf{a}}_z\,\cos\theta_r)\times(-)\,\hat{\mathbf{a}}_y\,E_{r0}\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\right] \tag{2.376}$$

$$\vec{\mathbf{H}}_r(x,z) = (-\hat{\mathbf{a}}_z\,\sin\theta_r - \hat{\mathbf{a}}_x\,\cos\theta_r)\,\frac{E_{r0}}{\eta_1}\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{2.377}$$

$$\vec{\mathbf{H}}_r(x,z) = (-\hat{\mathbf{a}}_x\,\cos\theta_r - \hat{\mathbf{a}}_z\,\sin\theta_r)\,\frac{E_{r0}}{\eta_1}\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{2.378}$$

$$\vec{\mathbf{E}}_t(x,z) = \hat{\mathbf{a}}_y\,E_{t0}\,e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \tag{2.379}$$

$$\vec{\mathbf{H}}_t(x,z) = (-\hat{\mathbf{a}}_x\,\cos\theta_t + \hat{\mathbf{a}}_z\,\sin\theta_t)\,\frac{E_{t0}}{\eta_2}\,e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \tag{2.380}$$

E_{r0}, E_{t0}, θ_r and θ_t are unknown quantities.
Tangential components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ are continuous at $z = 0$.

$$E_{iy}(x, z=0) + E_{ry}(x, z=0) = E_{ty}(x, z=0) \tag{2.381}$$

$$H_{ix}(x, z=0) + H_{rx}(x, z=0) = H_{tx}(x, z=0) \tag{2.382}$$

$$E_{i0}\,e^{-j\beta_1\,x\sin\theta_i} - E_{r0}\,e^{-j\beta_1\,x\sin\theta_r} = E_{t0}\,e^{-j\beta_2\,x\sin\theta_t} \tag{2.383}$$

$$-\frac{E_{i0}}{\eta_1}\cos\theta_i\,e^{-j\beta_1\,x\sin\theta_i} - \frac{E_{r0}}{\eta_1}\cos\theta_r\,e^{-j\beta_1\,x\sin\theta_r} = -\frac{E_{t0}}{\eta_2}\cos\theta_t\,e^{-j\beta_2\,x\sin\theta_t} \tag{2.384}$$

These equations must be satisfied for all x. So

$$\beta_1\,x\sin\theta_i = \beta_1\,x\sin\theta_r = \beta_2\,x\sin\theta_t \qquad \text{(phase matching)} \tag{2.385}$$

$$\beta_1\,x\sin\theta_i = \beta_1\,x\sin\theta_r \tag{2.386}$$

$$\sin\theta_i = \sin\theta_r \tag{2.387}$$

$$\boxed{\theta_r = \theta_i} \qquad \text{(Snell's law of reflection)} \tag{2.388}$$

$$\beta_1 \sin\theta_i = \beta_2 \sin\theta_t \tag{2.389}$$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} = \frac{\omega\sqrt{\mu_1\,\epsilon_1}}{\omega\sqrt{\mu_2\,\epsilon_2}} = \frac{\sqrt{\mu_1\,\epsilon_1}}{\sqrt{\mu_2\,\epsilon_2}} = \frac{n_1}{n_2}} \qquad \text{(Snell's law of refraction)} \tag{2.390}$$

n_1 : the index of refraction for medium 1.
n_2 : the index of refraction for medium 2.

For $x = 0$

$$E_{i0} - E_{r0} = E_{t0} \tag{2.391}$$

$$-\frac{E_{i0}}{\eta_1}\cos\theta_i - \frac{E_{r0}}{\eta_1}\cos\theta_r = -\frac{E_{t0}}{\eta_2}\cos\theta_t \tag{2.392}$$

$$E_{i0} - E_{r0} = E_{t0} \tag{2.393}$$

$$-\frac{E_{i0}}{\eta_1}\cos\theta_i - \frac{E_{r0}}{\eta_1}\cos\theta_i = -\frac{E_{t0}}{\eta_2}\cos\theta_t \tag{2.394}$$

$$E_{r0} + E_{t0} = E_{i0} \qquad //\,(\frac{1}{\eta_1}\cos\theta_i) \tag{2.395}$$

$$-\frac{E_{r0}}{\eta_1}\cos\theta_i + \frac{E_{t0}}{\eta_2}\cos\theta_t = \frac{E_{i0}}{\eta_1}\cos\theta_i \tag{2.396}$$

$$\frac{E_{r0}}{\eta_1}\cos\theta_i + \frac{E_{t0}}{\eta_1}\cos\theta_i = \frac{E_{i0}}{\eta_1}\cos\theta_i \tag{2.397}$$

$$-\frac{E_{r0}}{\eta_1}\cos\theta_i + \frac{E_{t0}}{\eta_2}\cos\theta_t = \frac{E_{i0}}{\eta_1}\cos\theta_i \tag{2.398}$$

$$\frac{E_{t0}}{\eta_1}\cos\theta_i + \frac{E_{t0}}{\eta_2}\cos\theta_t = 2\,\frac{E_{i0}}{\eta_1}\cos\theta_i \tag{2.399}$$

$$E_{t0}\left(\frac{1}{\eta_1}\cos\theta_i + \frac{1}{\eta_2}\cos\theta_t\right) = 2\,\frac{E_{i0}}{\eta_1}\cos\theta_i \tag{2.400}$$

$$E_{t0}\left(\frac{\eta_2\,\cos\theta_i + \eta_1\,\cos\theta_t}{\eta_1\,\eta_2}\right) = 2\,\frac{E_{i0}}{\eta_1}\cos\theta_i \tag{2.401}$$

$$E_{t0}\,(\eta_2\,\cos\theta_i + \eta_1\,\cos\theta_t) = 2\,\eta_2\,E_{i0}\,\cos\theta_i \tag{2.402}$$

$$E_{t0} = \frac{2\,\eta_2\,E_{i0}\,\cos\theta_i}{\eta_2\,\cos\theta_i + \eta_1\,\cos\theta_t} \tag{2.403}$$

$$\boxed{\tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\,\eta_2\,\cos\theta_i}{\eta_2\,\cos\theta_i + \eta_1\,\cos\theta_t}} \tag{2.404}$$

$\tau_\perp$: transmission coefficient for horizontal polarization

$$E_{r0} = E_{i0} - E_{t0} \tag{2.405}$$

$$E_{r0} = E_{i0}\left(1 - \frac{E_{t0}}{E_{i0}}\right) \tag{2.406}$$

$$E_{r0} = E_{i0}\,(1 - \tau_\perp) \tag{2.407}$$

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = 1 - \tau_\perp \tag{2.408}$$

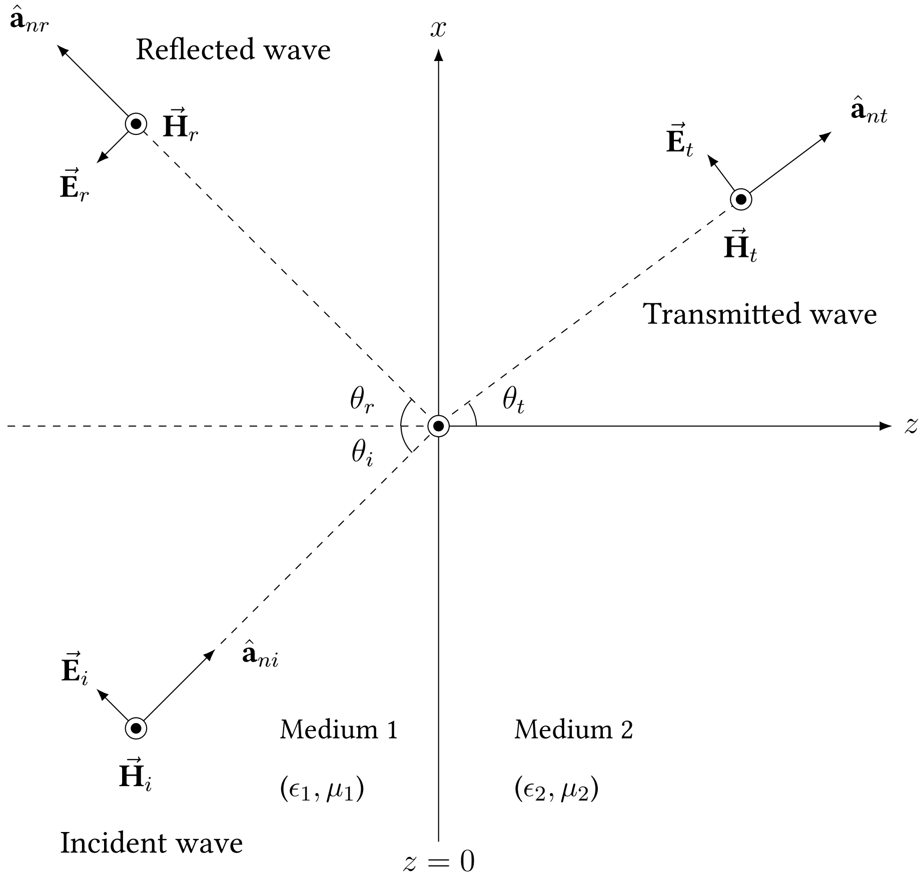
$$\Gamma_\perp = 1 - \frac{2\,\eta_2\,\cos\theta_i}{\eta_2\,\cos\theta_i + \eta_1\,\cos\theta_t} \tag{2.409}$$

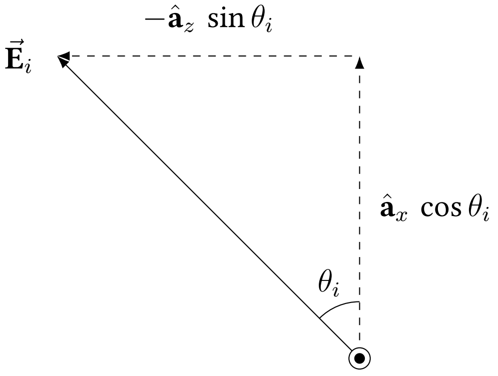
$$\Gamma_\perp = \frac{\eta_2\,\cos\theta_i + \eta_1\,\cos\theta_t - 2\,\eta_2\,\cos\theta_i}{\eta_2\,\cos\theta_i + \eta_1\,\cos\theta_t} \tag{2.410}$$

$$\boxed{\Gamma_\perp = \frac{\eta_1\,\cos\theta_t - \eta_2\,\cos\theta_i}{\eta_1\,\cos\theta_t + \eta_2\,\cos\theta_i}} \tag{2.411}$$

$\Gamma_\perp$: reflection coefficient for horizontal polarization

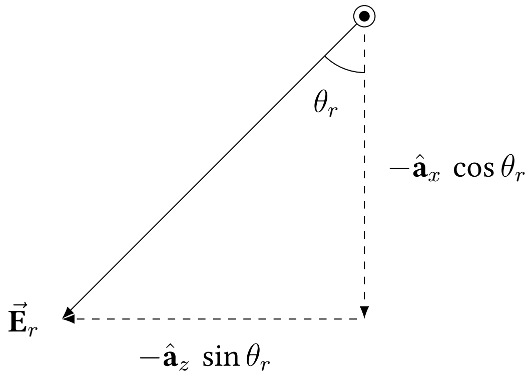
Vertical Polarization (Dikey Kutuplanma)





$$\vec{\mathbf{E}}_i(x,z) = (\hat{\mathbf{a}}_x \ \cos\theta_i - \hat{\mathbf{a}}_z \ \sin\theta_i) \ E_{i0} \, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{2.412}$$

$$\vec{\mathbf{H}}_i(x,z) = \hat{\mathbf{a}}_y \, \frac{E_{i0}}{\eta_1} \, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{2.413}$$



$$\vec{\mathbf{E}}_r(x,z) = (-\hat{\mathbf{a}}_x \ \cos\theta_r - \hat{\mathbf{a}}_z \ \sin\theta_r) \ E_{r0} \, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{2.414}$$

$$\vec{\mathbf{H}}_r(x,z) = \hat{\mathbf{a}}_y \, \frac{E_{r0}}{\eta_1} \, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{2.415}$$

$$\vec{\mathbf{E}}_t(x,z) = (\hat{\mathbf{a}}_x \ \cos\theta_t - \hat{\mathbf{a}}_z \ \sin\theta_t) \ E_{t0} \, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \tag{2.416}$$

$$\vec{\mathbf{H}}_t(x,z) = \hat{\mathbf{a}}_y \, \frac{E_{t0}}{\eta_2} \, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \tag{2.417}$$

E_{r0}, E_{t0}, θ_r and θ_t are unknown quantities.

Tangential components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ are continuous at $z = 0$.

$$E_{ix}(x,z=0) + E_{rx}(x,z=0) = E_{tx}(x,z=0) \tag{2.418}$$

$$H_{iy}(x,z=0) + H_{ry}(x,z=0) = H_{ty}(x,z=0) \tag{2.419}$$

$$E_{i0} \ \cos\theta_i \, e^{-j\beta_1 \, x\sin\theta_i} - E_{r0} \ \cos\theta_r \, e^{-j\beta_1 \, x\sin\theta_r} = E_{t0} \ \cos\theta_t \, e^{-j\beta_2 \, x\sin\theta_t} \tag{2.420}$$

$$\frac{E_{i0}}{\eta_1} \, e^{-j\beta_1 \, x\sin\theta_i} + \frac{E_{r0}}{\eta_1} \, e^{-j\beta_1 \, x\sin\theta_r} = \frac{E_{t0}}{\eta_2} \, e^{-j\beta_2 \, x\sin\theta_t} \tag{2.421}$$

These equations must be satisfied for all x. So

$$\beta_1\, x \sin\theta_i = \beta_1\, x \sin\theta_r = \beta_2\, x \sin\theta_t \qquad \text{(phase matching)} \tag{2.422}$$

$$\beta_1\, x \sin\theta_i = \beta_1\, x \sin\theta_r \tag{2.423}$$

$$\sin\theta_i = \sin\theta_r \tag{2.424}$$

$$\boxed{\theta_r = \theta_i} \qquad \text{(Snell's law of reflection)} \tag{2.425}$$

$$\beta_1\, \sin\theta_i = \beta_2\, \sin\theta_t \tag{2.426}$$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} = \frac{\omega\sqrt{\mu_1\,\epsilon_1}}{\omega\sqrt{\mu_2\,\epsilon_2}} = \frac{\sqrt{\mu_1\,\epsilon_1}}{\sqrt{\mu_2\,\epsilon_2}} = \frac{n_1}{n_2}} \qquad \text{(Snell's law of refraction)} \tag{2.427}$$

n_1 : the index of refraction for medium 1.
n_2 : the index of refraction for medium 2.

For $x = 0$

$$E_{i0}\, \cos\theta_i - E_{r0}\, \cos\theta_r = E_{t0}\, \cos\theta_t \tag{2.428}$$

$$\frac{E_{i0}}{\eta_1} + \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2} \tag{2.429}$$

$$E_{i0}\, \cos\theta_i - E_{r0}\, \cos\theta_i = E_{t0}\, \cos\theta_t \tag{2.430}$$

$$\frac{E_{i0}}{\eta_1} + \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2} \tag{2.431}$$

$$E_{r0}\, \cos\theta_i + E_{t0}\, \cos\theta_t = E_{i0}\, \cos\theta_i \tag{2.432}$$

$$E_{i0} + E_{r0} = \frac{\eta_1}{\eta_2}\, E_{t0} \tag{2.433}$$

$$E_{r0}\, \cos\theta_i + E_{t0}\, \cos\theta_t = E_{i0}\, \cos\theta_i \tag{2.434}$$

$$E_{r0} - \frac{\eta_1}{\eta_2}\, E_{t0} = -E_{i0} \qquad //\,(-\cos\theta_i) \tag{2.435}$$

$$E_{r0}\, \cos\theta_i + E_{t0}\, \cos\theta_t = E_{i0}\, \cos\theta_i \tag{2.436}$$

$$-E_{r0}\, \cos\theta_i + \frac{\eta_1}{\eta_2}\, E_{t0}\, \cos\theta_i = E_{i0}\, \cos\theta_i \tag{2.437}$$

$$E_{t0}\, \cos\theta_t + \frac{\eta_1}{\eta_2}\, E_{t0}\, \cos\theta_i = 2\, E_{i0}\, \cos\theta_i \qquad //\,(\eta_2) \tag{2.438}$$

$$\eta_2\, E_{t0}\, \cos\theta_t + \eta_1\, E_{t0}\, \cos\theta_i = 2\, \eta_2\, E_{i0}\, \cos\theta_i \tag{2.439}$$

$$E_{t0}\ (\eta_2\ \cos\theta_t + \eta_1\ \cos\theta_i) = 2\,\eta_2\, E_{i0}\ \cos\theta_i \tag{2.440}$$

$$\frac{E_{t0}}{E_{i0}} = \frac{2\,\eta_2\ \cos\theta_i}{\eta_2\ \cos\theta_t + \eta_1\ \cos\theta_i} \tag{2.441}$$

$$\boxed{\tau_{\|} = \frac{E_{t0}}{E_{i0}} = \frac{2\,\eta_2\ \cos\theta_i}{\eta_2\ \cos\theta_t + \eta_1\ \cos\theta_i}} \tag{2.442}$$

$\tau_{\|}$: transmission coefficient for vertical polarization

$$E_{r0} = \frac{\eta_1}{\eta_2}\, E_{t0} - E_{i0} \tag{2.443}$$

$$\frac{E_{r0}}{E_{i0}} = \frac{\eta_1}{\eta_2}\,\frac{E_{t0}}{E_{i0}} - 1 \tag{2.444}$$

$$\frac{E_{r0}}{E_{i0}} = \frac{\eta_1}{\eta_2}\,\tau_{\|} - 1 \tag{2.445}$$

$$\frac{E_{r0}}{E_{i0}} = \frac{\eta_1}{\eta_2}\,\frac{2\,\eta_2\ \cos\theta_i}{\eta_2\ \cos\theta_t + \eta_1\ \cos\theta_i} - 1 \tag{2.446}$$

$$\frac{E_{r0}}{E_{i0}} = \frac{2\,\eta_1\ \cos\theta_i}{\eta_2\ \cos\theta_t + \eta_1\ \cos\theta_i} - 1 \tag{2.447}$$

$$\frac{E_{r0}}{E_{i0}} = \frac{2\,\eta_1\ \cos\theta_i - \eta_2\ \cos\theta_t - \eta_1\ \cos\theta_i}{\eta_2\ \cos\theta_t + \eta_1\ \cos\theta_i} \tag{2.448}$$

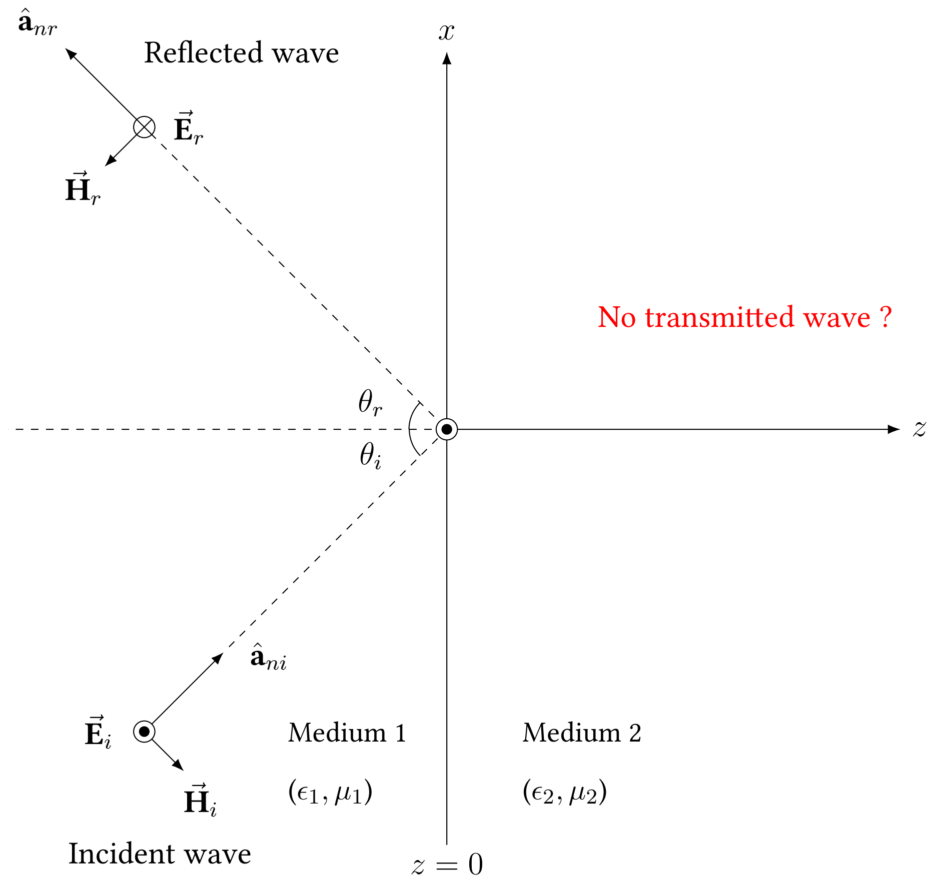
$$\frac{E_{r0}}{E_{i0}} = \frac{\eta_1\ \cos\theta_i - \eta_2\ \cos\theta_t}{\eta_2\ \cos\theta_t + \eta_1\ \cos\theta_i} \tag{2.449}$$

$$\boxed{\Gamma_{\|} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_1\ \cos\theta_i - \eta_2\ \cos\theta_t}{\eta_2\ \cos\theta_t + \eta_1\ \cos\theta_i}} \tag{2.450}$$

$\Gamma_{\|}$: reflection coefficient for vertical polarization

2.8 Total Reflection

Horizontal Polarization (Yatay Kutuplanma)



For total reflection we need $|\Gamma_\perp| = 1$ or $\Gamma_\perp = \mp 1$.
Let's first try $\Gamma_\perp = 1$.

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_1 \cos\theta_t - \eta_2 \cos\theta_i}{\eta_1 \cos\theta_t + \eta_2 \cos\theta_i} = 1 \tag{2.451}$$

$$\eta_1 \cos\theta_t - \eta_2 \cos\theta_i = \eta_1 \cos\theta_t + \eta_2 \cos\theta_i \tag{2.452}$$

$$\Rightarrow 2\,\eta_2 \cos\theta_i = 0 \tag{2.453}$$

$$\cos\theta_i = 0 \Rightarrow \theta_i = 90^\circ \tag{2.454}$$

This does not give us an interesting solution.
Now let's try $\Gamma_\perp = -1$.

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_1 \cos\theta_t - \eta_2 \cos\theta_i}{\eta_1 \cos\theta_t + \eta_2 \cos\theta_i} = -1 \tag{2.455}$$

$$\eta_1 \cos\theta_t - \eta_2 \cos\theta_i = -\eta_1 \cos\theta_t - \eta_2 \cos\theta_i \tag{2.456}$$

$$2\,\eta_1 \cos\theta_t = 0 \tag{2.457}$$

$$\cos\theta_t = 0 \Rightarrow \theta_t = 90^\circ \tag{2.458}$$

From Snell's law

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\sqrt{\mu_1\,\epsilon_1}}{\sqrt{\mu_2\,\epsilon_2}} \tag{2.459}$$

For nonmagnetic media $\mu_1 = \mu_2 = \mu_0$.

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \tag{2.460}$$

$$\theta_t = 90^\circ \Rightarrow \sin\theta_t = 1 \tag{2.461}$$

$$\frac{1}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \tag{2.462}$$

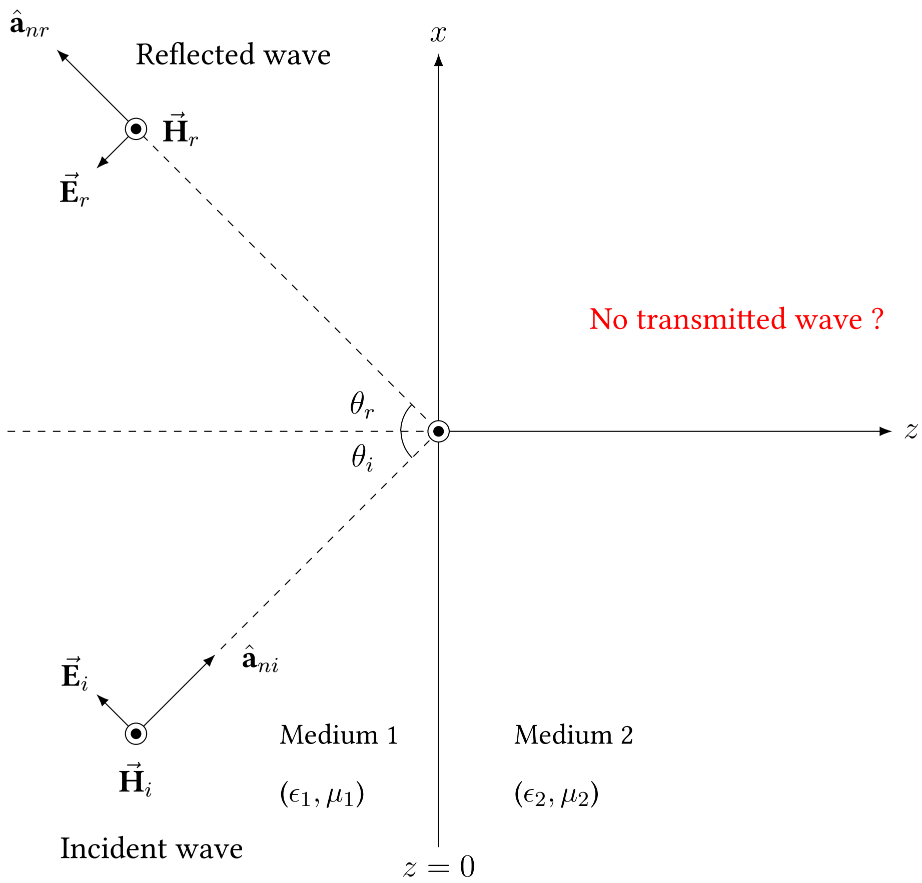
$$\sin\theta_i = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \tag{2.463}$$

$$\sin\theta_c = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \tag{2.464}$$

$$\boxed{\theta_c = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}} \tag{2.465}$$

The angle of incidence θ_c is called the **critical angle**.

Vertical Polarization (Dikey Kutuplanma)



For total reflection we need $|\Gamma_{\|}| = 1$ or $\Gamma_{\|} = \mp 1$.
Let's first try $\Gamma_{\|} = -1$.

$$\Gamma_{\|} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} = -1 \tag{2.466}$$

$$\eta_1 \cos\theta_i - \eta_2 \cos\theta_t = -\eta_2 \cos\theta_t - \eta_1 \cos\theta_i \tag{2.467}$$

$$\Rightarrow 2\,\eta_1 \cos\theta_i = 0 \tag{2.468}$$

$$\cos\theta_i = 0 \Rightarrow \theta_i = 90^\circ \tag{2.469}$$

This does not give us an interesting solution.
Now let's try $\Gamma_{\|} = 1$.

$$\Gamma_{\|} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} = 1 \tag{2.470}$$

$$\eta_1 \cos\theta_i - \eta_2 \cos\theta_t = \eta_2 \cos\theta_t + \eta_1 \cos\theta_i \tag{2.471}$$

$$2\,\eta_2 \cos\theta_t = 0 \tag{2.472}$$

$$\cos\theta_t = 0 \Rightarrow \theta_t = 90^\circ \tag{2.473}$$

From Snell's law

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\sqrt{\mu_1\,\epsilon_1}}{\sqrt{\mu_2\,\epsilon_2}} \tag{2.474}$$

For nonmagnetic media $\mu_1 = \mu_2 = \mu_0$.

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \tag{2.475}$$

$$\theta_t = 90^\circ \Rightarrow \sin\theta_t = 1 \tag{2.476}$$

$$\frac{1}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \tag{2.477}$$

$$\sin\theta_i = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \tag{2.478}$$

$$\sin\theta_c = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \tag{2.479}$$

$$\boxed{\theta_c = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}} \tag{2.480}$$

The angle of incidence θ_c is called the **critical angle**. Note that the critical angle θ_c does not depend on polarization, i.e. it is same for both horizontal and vertical polarizations.

If $\theta_i = \theta_c$ then $\theta_t = 90°$.
If $\theta_i > \theta_c$ then

$$\sin\theta_i > \sin\theta_c \tag{2.481}$$

$$\sin\theta_i > \sqrt{\frac{\epsilon_2}{\epsilon_1}} \tag{2.482}$$

$$\sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i > 1 \tag{2.483}$$

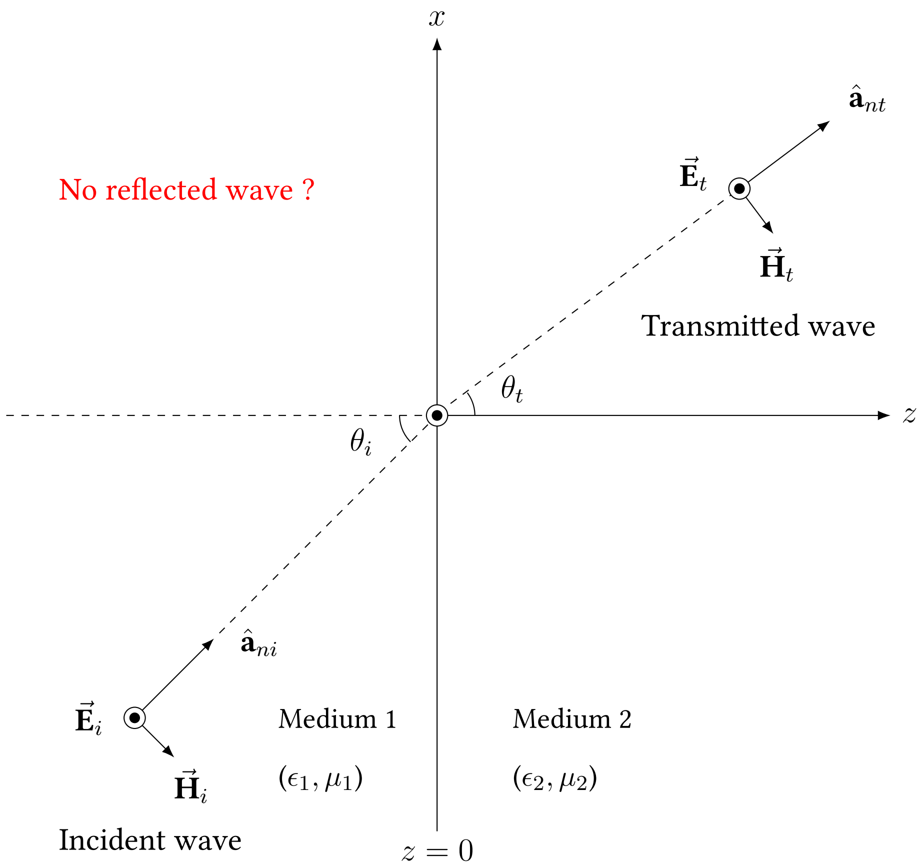
$$\sin\theta_t = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i \tag{2.484}$$

$$\Rightarrow \sin\theta_t > 1 \tag{2.485}$$

There is no real solution for θ_t and total reflection occurs.

2.9 Total Transmission

Horizontal Polarization (Yatay Kutuplanma)



For total transmission we need $\Gamma_\perp = 0$.

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_1\,\cos\theta_t - \eta_2\,\cos\theta_i}{\eta_1\,\cos\theta_t + \eta_2\,\cos\theta_i} = 0 \tag{2.486}$$

$$\Rightarrow \eta_1 \cos\theta_t - \eta_2 \cos\theta_i = 0 \tag{2.487}$$

$$\Rightarrow \eta_1 \cos\theta_t = \eta_2 \cos\theta_i \tag{2.488}$$

From Snell's law

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\sqrt{\mu_1\,\epsilon_1}}{\sqrt{\mu_2\,\epsilon_2}} \tag{2.489}$$

$$\sin\theta_t = \frac{\sqrt{\mu_1\,\epsilon_1}}{\sqrt{\mu_2\,\epsilon_2}}\,\sin\theta_i \tag{2.490}$$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{\mu_1\,\epsilon_1}{\mu_2\,\epsilon_2}\,\sin^2\theta_i} \tag{2.491}$$

$$\cos\theta_i = \sqrt{1-\sin^2\theta_i} \tag{2.492}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} \tag{2.493}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} \tag{2.494}$$

$$\eta_1 \cos\theta_t = \eta_2 \cos\theta_i \tag{2.495}$$

$$\sqrt{\frac{\mu_1}{\epsilon_1}}\,\sqrt{1-\frac{\mu_1\,\epsilon_1}{\mu_2\,\epsilon_2}\,\sin^2\theta_i} = \sqrt{\frac{\mu_2}{\epsilon_2}}\,\sqrt{1-\sin^2\theta_i} \tag{2.496}$$

$$\frac{\mu_1}{\epsilon_1}\left(1-\frac{\mu_1\,\epsilon_1}{\mu_2\,\epsilon_2}\,\sin^2\theta_i\right) = \frac{\mu_2}{\epsilon_2}\,\left(1-\sin^2\theta_i\right) \tag{2.497}$$

$$\frac{\mu_1}{\epsilon_1} - \frac{\mu_1^2}{\mu_2\,\epsilon_2}\,\sin^2\theta_i = \frac{\mu_2}{\epsilon_2} - \frac{\mu_2}{\epsilon_2}\,\sin^2\theta_i \tag{2.498}$$

$$\frac{\mu_2}{\epsilon_2}\,\sin^2\theta_i - \frac{\mu_1^2}{\mu_2\,\epsilon_2}\,\sin^2\theta_i = \frac{\mu_2}{\epsilon_2} - \frac{\mu_1}{\epsilon_1} \tag{2.499}$$

$$\left(\frac{\mu_2}{\epsilon_2} - \frac{\mu_1^2}{\mu_2\,\epsilon_2}\right)\,\sin^2\theta_i = \frac{\mu_2}{\epsilon_2} - \frac{\mu_1}{\epsilon_1} \tag{2.500}$$

$$\sin^2\theta_i = \frac{\frac{\mu_2}{\epsilon_2} - \frac{\mu_1}{\epsilon_1}}{\frac{\mu_2}{\epsilon_2} - \frac{\mu_1^2}{\mu_2\,\epsilon_2}} \tag{2.501}$$

$$\sin^2\theta_i = \frac{\epsilon_1\,\mu_2 - \epsilon_2\,\mu_1}{\epsilon_1\,\epsilon_2}\,\frac{\mu_2\,\epsilon_2}{\mu_2^2-\mu_1^2} \tag{2.502}$$

$$\sin^2\theta_i = \frac{\epsilon_1\,\mu_2 - \epsilon_2\,\mu_1}{\epsilon_1}\,\frac{\mu_2}{\mu_2^2-\mu_1^2} \tag{2.503}$$

$$\sin^2\theta_i = \frac{\epsilon_1\,\mu_2 - \epsilon_2\,\mu_1}{\epsilon_1}\,\frac{\mu_2}{\mu_2^2\,[1-(\mu_1/\mu_2)^2]} \tag{2.504}$$

$$\sin^2\theta_i = \frac{\epsilon_1\,\mu_2 - \epsilon_2\,\mu_1}{\epsilon_1\,\mu_2}\,\frac{1}{[1-(\mu_1/\mu_2)^2]} \tag{2.505}$$

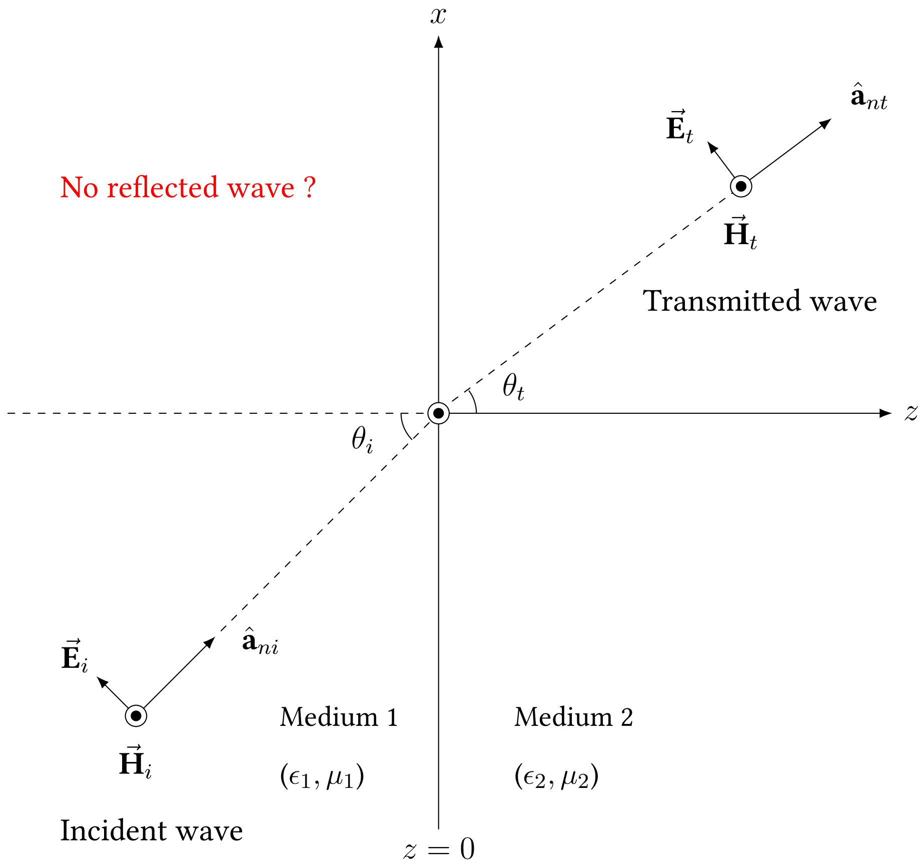
$$\sin^2\theta_i = \frac{1-\frac{\epsilon_2\,\mu_1}{\epsilon_1\,\mu_2}}{1-\frac{\mu_1^2}{\mu_2^2}} \tag{2.506}$$

For nonmagnetic media $\mu_1 = \mu_2 = \mu_0$ and we have

$$\sin^2\theta_i = \infty \tag{2.507}$$

So for horizontal polarization there is no incidence angle that makes the reflection coefficient $\Gamma_\perp = 0$.

Vertical Polarization (Dikey Kutuplanma)



For total transmission we need $\Gamma_\parallel = 0$.

$$\Gamma_\parallel = \frac{E_{r0}}{E_{i0}} = \frac{\eta_1\,\cos\theta_i - \eta_2\,\cos\theta_t}{\eta_2\,\cos\theta_t + \eta_1\,\cos\theta_i} = 0 \tag{2.508}$$

$$\Rightarrow \eta_1\,\cos\theta_i - \eta_2\,\cos\theta_t = 0 \tag{2.509}$$

$$\Rightarrow \eta_1\,\cos\theta_i = \eta_2\,\cos\theta_t \tag{2.510}$$

From Snell's law

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\sqrt{\mu_1\,\epsilon_1}}{\sqrt{\mu_2\,\epsilon_2}} \tag{2.511}$$

$$\sin\theta_t = \frac{\sqrt{\mu_1\,\epsilon_1}}{\sqrt{\mu_2\,\epsilon_2}}\,\sin\theta_i \tag{2.512}$$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1 - \frac{\mu_1\,\epsilon_1}{\mu_2\,\epsilon_2}\,\sin^2\theta_i} \tag{2.513}$$

$$\cos\theta_i = \sqrt{1-\sin^2\theta_i} \tag{2.514}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} \tag{2.515}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} \tag{2.516}$$

$$\eta_1\,\cos\theta_i = \eta_2\,\cos\theta_t \tag{2.517}$$

$$\sqrt{\frac{\mu_1}{\epsilon_1}}\,\sqrt{1-\sin^2\theta_i} = \sqrt{\frac{\mu_2}{\epsilon_2}}\,\sqrt{1-\frac{\mu_1\,\epsilon_1}{\mu_2\,\epsilon_2}\,\sin^2\theta_i} \tag{2.518}$$

$$\frac{\mu_1}{\epsilon_1}\,\left(1-\sin^2\theta_i\right) = \frac{\mu_2}{\epsilon_2}\,\left(1-\frac{\mu_1\,\epsilon_1}{\mu_2\,\epsilon_2}\,\sin^2\theta_i\right) \tag{2.519}$$

$$\frac{\mu_1}{\epsilon_1}-\frac{\mu_1}{\epsilon_1}\,\sin^2\theta_i = \frac{\mu_2}{\epsilon_2}-\frac{\mu_2}{\epsilon_2}\,\frac{\mu_1\,\epsilon_1}{\mu_2\,\epsilon_2}\,\sin^2\theta_i \tag{2.520}$$

$$\frac{\mu_1}{\epsilon_1}-\frac{\mu_1}{\epsilon_1}\,\sin^2\theta_i = \frac{\mu_2}{\epsilon_2}-\frac{\mu_1\,\epsilon_1}{\epsilon_2\,\epsilon_2}\,\sin^2\theta_i \tag{2.521}$$

$$-\frac{\mu_1}{\epsilon_1}\,\sin^2\theta_i+\frac{\mu_1\,\epsilon_1}{\epsilon_2\,\epsilon_2}\,\sin^2\theta_i = \frac{\mu_2}{\epsilon_2}-\frac{\mu_1}{\epsilon_1} \tag{2.522}$$

$$-\frac{\mu_1}{\epsilon_1}\,\sin^2\theta_i+\frac{\mu_1\,\epsilon_1}{\epsilon_2^2}\,\sin^2\theta_i = \frac{\mu_2}{\epsilon_2}-\frac{\mu_1}{\epsilon_1} \tag{2.523}$$

$$\left(-\frac{\mu_1}{\epsilon_1}+\frac{\mu_1\,\epsilon_1}{\epsilon_2^2}\right)\,\sin^2\theta_i = \frac{\mu_2}{\epsilon_2}-\frac{\mu_1}{\epsilon_1} \tag{2.524}$$

$$\left(\frac{\mu_1\,\epsilon_1}{\epsilon_2^2}-\frac{\mu_1}{\epsilon_1}\right)\,\sin^2\theta_i = \frac{\mu_2}{\epsilon_2}-\frac{\mu_1}{\epsilon_1} \tag{2.525}$$

$$\sin^2\theta_i = \frac{\frac{\mu_2}{\epsilon_2}-\frac{\mu_1}{\epsilon_1}}{\frac{\mu_1\,\epsilon_1}{\epsilon_2^2}-\frac{\mu_1}{\epsilon_1}} \tag{2.526}$$

$$\sin^2\theta_i = \frac{\frac{\epsilon_1\,\mu_2-\epsilon_2\,\mu_1}{\epsilon_1\,\epsilon_2}}{\frac{\epsilon_1^2\,\mu_1-\epsilon_2^2\,\mu_1}{\epsilon_1\,\epsilon_2^2}} \tag{2.527}$$

$$\sin^2\theta_i = \frac{\epsilon_1\,\mu_2-\epsilon_2\,\mu_1}{\epsilon_1\,\epsilon_2}\,\frac{\epsilon_1\,\epsilon_2^2}{\epsilon_1^2\,\mu_1-\epsilon_2^2\,\mu_1} \tag{2.528}$$

$$\sin^2\theta_i = \left[\epsilon_1\,\mu_2-\epsilon_2\,\mu_1\right]\,\frac{\epsilon_2}{\epsilon_1^2\,\mu_1-\epsilon_2^2\,\mu_1} \tag{2.529}$$

$$\sin^2\theta_i = \left[\epsilon_1\,\mu_2-\epsilon_2\,\mu_1\right]\,\frac{\epsilon_2}{\left[\epsilon_1^2-\epsilon_2^2\right]\,\mu_1} \tag{2.530}$$

$$\sin^2\theta_i = \epsilon_2\,\mu_1\,\left[\frac{\epsilon_1\,\mu_2}{\epsilon_2\,\mu_1}-1\right]\,\frac{\epsilon_2}{\epsilon_2^2\,\left[\frac{\epsilon_1^2}{\epsilon_2^2}-1\right]\,\mu_1} \tag{2.531}$$

$$\sin^2\theta_i = \frac{\left[\frac{\epsilon_1\,\mu_2}{\epsilon_2\,\mu_1}-1\right]}{\left[\frac{\epsilon_1^2}{\epsilon_2^2}-1\right]} \tag{2.532}$$

$$\sin^2\theta_i = \frac{1-\frac{\epsilon_1\,\mu_2}{\epsilon_2\,\mu_1}}{1-\frac{\epsilon_1^2}{\epsilon_2^2}} \tag{2.533}$$

For nonmagnetic media $\mu_1 = \mu_2 = \mu_0$ and we have

$$\sin^2\theta_i = \frac{1 - \frac{\epsilon_1}{\epsilon_2}}{1 - \frac{\epsilon_1^2}{\epsilon_2^2}} = \frac{1 - \frac{\epsilon_1}{\epsilon_2}}{\left(1 - \frac{\epsilon_1}{\epsilon_2}\right)\left(1 + \frac{\epsilon_1}{\epsilon_2}\right)} \tag{2.534}$$

$$\sin^2\theta_i = \frac{1}{1 + \frac{\epsilon_1}{\epsilon_2}} \tag{2.535}$$

$$\boxed{\sin\theta_i = \frac{1}{\sqrt{1 + \frac{\epsilon_1}{\epsilon_2}}}} \tag{2.536}$$

The angle satisfying this equation is called as the Brewster angle θ_B. If $\theta_i = \theta_B$ then $\Gamma_{\parallel} = 0$ and there is no reflection from the interface, the wave is totally transmitted to the second medium.

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