ON THE STATIC HEDGING OF FIXED INCOME PORTFOLIOS UNDER STOCHASTIC INTEREST RATE MODELS

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Abstract. The aim of this paper is to compute optimal strategies for fixed income portfolio that guarantees a minimum return over a fixed planning period. We solve this problem in a context of Stochastic Interest Rate Models by using Linear Programming. Moreover, we provide some models that can be used in this framework.

1. Introduction and statement of main result

Suppose an investor in a fixed income market has certain obligations due to some specified future date, called the investor' s horizon planning period. A key problem facing such an investor is the problem of immunizing (protecting) his or her portfolio of bonds against interest rate risk. Bierwag and Khang [2] prove that the process of immunizing a bond portfolio can be described as a maxi-min strategy in a game against nature where the investor's target is to guarantee a minimum value at the end of his or her horizon planning period. Dantzig [7] shows that this maxi-min solution can be determined by solving an equivalent linear program that depends on the assumption about the term structure of interest rates.

One of the main results concerning the development of portfolio immunization strategies against interest rate risk is due to Khang [10] and is described by his dynamic global immunization theorem. Kang's strategy consists of a continuous portfolio rebalancing in order to keep portfolio duration equal to the length of the remaining planning period.

Specifically, consider an investor who has a horizon planning period of length t_0. Suppose the forward interest rates structureshifts up or down by a stochastic shift parameter at any time during the investor's planning period. If the investor follows Khang's strategy, then the

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investor's wealth at the end of his or her planning period will be no less than the amount anticipated on the basis of the foward interest rates structure observed initially (at time 0). Furthermore, the investor's wealth at time T will be greater than the amount anticipated initially if at least one interest shock takes place during the planning period.

The validity of Khang's sttategy rests on two key assumptions:

(i) If $g(t),\ t \geq 0$, denotes the forward interest rates structure, and the forward interest rates structure changes to $g^*(t)$, then

$$g * (t) = g(t) + \delta$$

where δ is a stochastic shift parameter; and (ii) there are no transaction costs.

The first assumption avoids the problem of the risk of misestimating the term structure behavior, which Fong and Vasicek [9] call the "immunization risk." Assumption (ii) avoids the high costs that a strategy of continuous portfolio rebalancing may incur.

In this framework dynamic immunization can be stated as the finite union of problems of static immunization. More precisely, this aproximation was used by Navarro and Nave [13] in order to avoid the initial restrictions imposed by Khang. Thus, it is important to understand the static immunization in order to extend this aproximation to a more complex problems that are close to the real life.

From all said above, our main goal is to investigate the construction of a fixed income portfolio that guarantees a minimum return over a fixed planning period.

We will consider that the short–term interest rate r follows an Itô process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the risk–neutral probability, and given by the following Stochastic Differential Equation

$$\begin{aligned} & dr(t,\omega) = \mu(r,\omega)dt + \sigma(r,\omega)dW_t \\ & r(0) = r > 0. \end{aligned} \tag{1}$$

From now one, we will consider that Ω represents the whole set of scenarios of the economy under consideration.

Let $p = p(t, r; T)$ denote the price of a zero coupon $T-$bond. Now, in order to obtain the Term Structure Equation, we assume that the bond market is arbitrage–free. Then, it follows (see Björk [3]) that p is the solution of the following Partial Differential Equation

$$p_t + \frac{1}{2}\sigma^2(r)p_{rr} + \mu(r)p_r - rp = 0, \tag{2}$$

with boundary condition

$$p(T, r; T) = 1.$$

From the Feynman–Kac Representation Theorem (see also Björk [3]) we have that

$$p(t,r;T) = E_{\mathbb{P}}^{(t,r)}\left[\exp\left(-\int_t^T r(s)ds\right)\right]. \tag{3}$$

The investor's strategy consists of purchasing an allocation vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]' \in \mathbb{R}_+^n$ of bonds that satisfy the following budget constraint:

$$\sum_{i=1}^{n} x_i p_i = I, \tag{4}$$

where p_i denotes the current (at $t = 0$) price of one unit of i–th bond and I is the investor's initial whealth at $t = 0$. We will consider that $\tilde{T}$ denotes the upper bound of all bond maturities and let $0 < t_0 < \tilde{T}$ be denotes the horizon planning period. By using that are no arbitrage opportunies in the bond market, we can write

$$p_i = \sum_{s=1}^{n_i-1} C_s^{(i)} p(r,0;\tau_s^{(i)}) + (1 + C_{n_i}^{(i)}) p(r,0;T_i) \tag{5}$$

where n_i denotes the number of bond i coupon payments made after $t = 0$, $\tau_s^{(i)}$ is the time of i-bond s–th coupon payment after $t = 0$, with

$$0 < \tau_1^{(i)} < \cdots < \tau_{n_i-1}^{(i)} < \tau_{n_i}^{(i)} = T_i$$

for $i = 1, 2, \ldots, n$, and $C_i^{(s)}$ is the size of each coupon payment from bond i, at time $t = \tau_s^{(i)}$, for $s = 1, 2, \ldots, n_i$, and , finally, T_i is the time to maturity of bond i with

$$0 < T_1 \leq \cdots \leq T_n \leq T_{n+1} = \tilde{T}.$$

After the strategy $\mathbf{x}$ is choosen, interest rate changes from $r(0)$ to $r(t_0, \omega)$. Then the value at the end of the horizon planning period of a investement of p_i monetary units in bond i, is given by

$$V_i(\omega) = \sum_{s=1}^{n_i-1} \tilde{C}_s^{(i)} p(r(t_0,\omega), t_0; \tau_s^{(i)}) + (1 + \tilde{C}_{n_i}^{(i)}) p(r(t_0,\omega), t_0; T_i), \tag{6}$$

where

$$\tilde{C}_s^{(i)} = \begin{cases} C_s^{(i)} & \text{if} \quad t_0 \leq \tau_s^{(i)}, \\ 0 & \text{otherwise}, \end{cases}$$

for $s = 1, 2, \ldots, n_i$ and $i = 1, 2, \ldots, n$. Note that $V_i(\omega) = V_i(t_0, \omega)$. Thus, the portfolio value at the end of the horizon planning period is

$$\sum_{i=1}^{n} x_i V_i(\omega). \tag{7}$$

We remark that we can compute an aproximation of $r(t_0, \omega)$ using the Euler discretization of the associated Stochastic Differential Equation.

Let $V \geq 0$ denote a lower bound for the worst-case final portfolio value, that is,

$$\min_{\omega \in \mathcal{E}} \sum_{i=1\omega_1}^{n} x_i V_i(\omega) \geq V \geq 0.$$

where

$$\mathcal{E} = \{\omega_1, \ldots, \omega_m\} \subset \Omega$$

is a finite set of scenarios. Then, the investor strategy consists in to maximize the value of V. The porfolio selection process can be modelled by the following linear program.

$$\begin{aligned} & \max\nolimits_{x_1,\ldots,x_n,V} V \\ \text{subject to} \quad & \textstyle\sum_{i=1}^n x_i V_i(\omega_j) \geq V \quad j = 1, 2, \ldots, m \\ & \textstyle\sum_{i=1}^n x_i p_i = I \\ & V \geq 0, x_i \geq 0 \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{8}$$

This model was studied by Navarro y Nave [13] in 1997 using numerical procedures. The main result of this paper is the following

Theorem A. *The solution V^* of linear program (8) is given by the following equality*

$$V^* = \lambda^* I \tag{9}$$

where

$$\lambda^* = \min \left\{ \max \left\{ \frac{V_k(\omega_j)}{p_k} : \ k = 1, 2, \ldots, n \right\} : j = 1, 2, \ldots, m \right\} \tag{10}$$

Moreover, an optimal strategy $\mathbf{x}^* = [x_1^* \ x_2^* \ \ldots \ x_n^*]'$ is obtained from the solution

$$\mathbf{z}^* = [x_1^* \ x_2^* \ \ldots \ x_n^* \ y_1^*, \ \cdots \ y_m^*]'$$

of the following bounded least square problem

$$\min_{\mathbf{z} \geq \mathbf{0}} ||A\mathbf{z} - \mathbf{b}|| \tag{11}$$

where

$$A = \begin{bmatrix} V_1(\omega_1) & \cdots & V_n(\omega_1) & -1 & 0 & \cdots & 0 \\ V_1(\omega_2) & \cdots & V_n(\omega_2) & 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ V_1(\omega_m) & \cdots & V_n(\omega_m) & 0 & 0 & \cdots & -1 \\ p_1 & \cdots & p_n & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{12}$$

and

$$\mathbf{b} = [V^* \ V^* \ \cdots V^* \ I]. \tag{13}$$

We remark that the optimal strategy is not necessarily unique. However, we can use to compute it, for example, the `lsqnonneg` function included in the MATLAB Optimization Toolbox (its gives a numerical solution of (11)). On the other hand by using the Algorithm given in [6] (see also [5]) we can also compute the solution of (11) (see also Acedo, Benito Falcó, Rubia and Torres [1] for the use of this Algorithm to in order to compute either state price vector or arbitrage opportunities under the context of the Fundamental Theorem of Asset Pricing).

To end this section we introduce some models that can be used in this context and a numerical example. We include also the zero coupon bond price formula and the Euler Discretization.

1.1. **Merton's (1975) Nonlinear Mean Reversion Interest Rate Model.** This model was introduced by Merton [11] (see also [12], Chapter 17). It is defined by the following Stochastic Differential Equation

$$dr(t) \quad = \quad r(t)\,(a-br(t))\,dt+\sigma r(t)dW(t),\ \ r(0)=r, \tag{14}$$

where $a, b, \sigma \in \mathbb{R}^+$. From Itô's Lemma it can be shown that

$$r(t)=\frac{r\exp\left(\left(a-\frac{1}{2}\sigma^2\right)t+\sigma W(t)\right)}{1+rb\int_0^t \exp\left(\left(a-\frac{1}{2}\sigma^2\right)s+\sigma W(s)\right)ds}. \tag{15}$$

Thus if $r>0$, then $r(t)>0$, with probability one. Otherwise, if $r=0$, then $r(t)=0$, with probability one. Moreover, when $r>0$, we have that $r(t)$ has a Gamma distribution as $t\to\infty$ (see Merton [11]).

Now, in order to give the zero coupon bond pricing we need to introduce the Confluent Hypergeometric Functions.

A function $M(\alpha,\beta,z)$ is called a *Kummer function of the first order* if

$$M(\alpha,\beta,z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}\frac{z^n}{n!},$$

where

$$(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

and $(\alpha)_0=1$. These functions are well-defined for all values of α,β and z, if $1-\alpha, 1-\beta\notin\mathbb{Z}^+$.

The following result is due to Falcó and Nave [8].

Theorem 1.1. *Assume that $r(t)$ follows the Merton's Nonlinear Mean Reversion Interest Rate Model (14) and let*

$$\theta=\frac{a(T-t)-1}{\frac{1}{2}\sigma^2(T-t)}.$$

If $1-\theta\notin\mathbb{Z}^+$. Then the non-arbitrage price of a zero coupon $T-$bond is given by

$$p(t,r;T)=M\left(b^{-1},\theta,2br/\sigma^2\right).$$

Moreover, $p(t,r;T)$ is non-increasing and convex as a function of the initial rate r, and if $1-2a/\sigma^2 \notin \mathbb{Z}^+$, then

$$\lim_{(T-t)\to\infty} p(t,r;T) = M\left(b^{-1}, 2(a-\sigma\lambda)/\sigma^2, 2br/\sigma^2\right).$$

The aproximation $y_{t_0}(\omega)$ to $r(t_0,\omega)$ is calculated by:

Start: $\tau_0 = 0$, $y_0(\omega) = r$, $\Delta\tau = t_0/N$.
Loop: For $k = 0$ to $N-1$

$$\begin{aligned}\tau_{k+1} &= \tau_k + \Delta\tau;\\ \Delta W &= Z\sqrt{\Delta\tau}, \text{ with } Z \sim \mathcal{N}(0,1);\\ y_{k+1}(\omega) &= y_k(\omega) + y_k(\omega)(a - by_k(\omega))\Delta\tau + \sigma y_k(\omega)\Delta W;\end{aligned}$$

Return: $y_N(\omega)$.

1.2. **The Vasicek (1977) Interest Rate Model.** In this model introduced in [14] the spot interest rate follows a diffusion process described by the following Stochastic Differential Equation

$$dr(t) = (\beta - \gamma r(t))\,dt + \rho dW(t),\ r(0) = r, \tag{16}$$

Then the non-arbitrage price of a zero coupon $T-$bond is given by

$$p(t,r;T) = \exp\left(F(T-t)(G-r) - (T-t)G - \frac{\rho^2}{4\beta}F(T-t)^2\right)$$

where

$$F(x) = \frac{1}{\beta}(1 - \exp(-\beta x))$$

and

$$G = \gamma - \frac{\rho^2}{2\beta^2}.$$

The aproximation $y_{t_0}(\omega)$ to $r(t_0,\omega)$ is calculated by:

Start: $\tau_0 = 0$, $y_0(\omega) = r$, $\Delta\tau = t_0/N$.
Loop: For $k = 0$ to $N-1$

$$\begin{aligned}\tau_{k+1} &= \tau_k + \Delta\tau;\\ \Delta W &= Z\sqrt{\Delta\tau}, \text{ with } Z \sim \mathcal{N}(0,1);\\ y_{k+1}(\omega) &= y_k(\omega) + (\beta - \gamma y_k(\omega))\Delta\tau + \rho\Delta W;\end{aligned}$$

Return: $y_N(\omega)$.

1.3. **The Cox–Ingersoll–Ross (1985) Model.** This model was introduced in [4]. Assume that interest rate $r(t)$ is given by the solution of the following Stochastic Differential Equation

$$dr(t) = \kappa(\mu - r(t))\,dt + \sigma r^{1/2}(r)dW(t),\ r(0) = r, \tag{17}$$

where the condition $2\kappa\mu > \sigma^2$ has to be imposed to ensure that the origin is innacesible to the process. Then the non-arbitrage price of a zero coupon $T-$bond is given by

$$p(t,r;T) = A(T-t)\exp(-rB(T-t))$$

where

$$A(x) = \left[\frac{2\lambda \exp((\kappa+\lambda)x/2)}{2\lambda + (\lambda+\kappa)(\exp(\lambda x)-1)} \right]^{2\kappa\mu/\sigma^2},$$

and

$$B(x) = \frac{2(\exp(\lambda x)-1)}{2\lambda + (\lambda+\kappa)(\exp(\lambda x)-1)},$$

where

$$\lambda = \sqrt{\kappa^2 + 2\sigma^2}.$$

The aproximation $y_{t_0}(\omega)$ to $r(t_0, \omega)$ is calculated by:

Start: $\tau_0 = 0$, $y_0(\omega) = r$, $\Delta\tau = t_0/N$.
Loop: For $k = 0$ to $N-1$

$$\begin{aligned} \tau_{k+1} &= \tau_k + \Delta\tau; \\ \Delta W &= Z\sqrt{\Delta\tau}, \text{ with } Z \sim \mathcal{N}(0,1); \\ y_{k+1}(\omega) &= y_k(\omega) + \kappa\left(\mu - y_k(\omega)\right)\Delta\tau + \sigma y_k^{1/2}(\omega)\Delta W; \end{aligned}$$

Return: $y_N(\omega)$.

1.4. **A numerical example.** To end this section we give the following numerical example. We consider the model the Merton's (1975) model

$$\begin{aligned} dr(t) &= r(t)\left(0.5 - 0.3r(t)\right)dt + 0.29r(t)dW(t), \\ r(0) &= 0.045, \end{aligned} \tag{18}$$

and a bond market with four zero coupon bonds with maturities $T_1 = 0.5$, $T_2 = 1.0$, $T_3 = 1.5$ and $T_4 = 2.0$. Then, by using Theorem 1.1, we obtain $p_1 = 0.9777$, $p_2 = 0.9542$, $p_3 = 0.9296$ and $p_4 = 0.9036$. We generate, from the Euler Discretization of (18), $m = 25$ scenarios taking $t_0 = 0.2$, (see the matrix Simulations below). Then, for $I = 1$, we obtain that $V^* = \lambda^* = 1.0085$. Finally, the bounded least square problem (11) gives as solution the vector $\mathbf{z}^*$. This implies that an optimal strategy is given by $\mathbf{x} = [1.0228,\ 0,\ 0,\ 0]'$.

$$\text{Simulations} = \begin{bmatrix} 0.9851 & 0.9636 & 0.9401 & 0.9223 \\ 0.9887 & 0.9620 & 0.9328 & 0.9300 \\ 0.9876 & 0.9711 & 0.9466 & 0.9308 \\ 0.9868 & 0.9639 & 0.9340 & 0.9101 \\ 0.9828 & 0.9630 & 0.9401 & 0.9061 \\ 0.9886 & 0.9634 & 0.9398 & 0.9097 \\ 0.9826 & 0.9590 & 0.9320 & 0.9061 \\ 0.9835 & 0.9645 & 0.9458 & 0.8980 \\ 0.9866 & 0.9648 & 0.9300 & 0.9137 \\ 0.9869 & 0.9634 & 0.9379 & 0.9315 \\ 0.9870 & 0.9622 & 0.9413 & 0.8941 \\ 0.9830 & 0.9596 & 0.9279 & 0.9166 \\ 0.9869 & 0.9620 & 0.9375 & 0.9246 \\ 0.9835 & 0.9713 & 0.9440 & 0.9142 \\ 0.9849 & 0.9610 & 0.9390 & 0.9140 \\ 0.9865 & 0.9675 & 0.9408 & 0.9110 \\ 0.9852 & 0.9652 & 0.9404 & 0.9043 \\ 0.9879 & 0.9715 & 0.9266 & 0.8909 \\ 0.9889 & 0.9585 & 0.9341 & 0.9185 \\ 0.9885 & 0.9619 & 0.9379 & 0.9010 \\ 0.9861 & 0.9637 & 0.9386 & 0.9231 \\ 0.9879 & 0.9690 & 0.9419 & 0.9103 \\ 0.9879 & 0.9682 & 0.9236 & 0.9120 \\ 0.9851 & 0.9702 & 0.9418 & 0.9225 \\ 0.9843 & 0.9650 & 0.9454 & 0.9026 \end{bmatrix}, \quad \mathbf{z}^* = \begin{bmatrix} 1.0228 \\ 0 \\ 0 \\ 0 \\ 0.0025 \\ 0.0062 \\ 0.0051 \\ 0.0043 \\ 0.0002 \\ 0.0061 \\ 0 \\ 0.0009 \\ 0.0041 \\ 0.0043 \\ 0.0044 \\ 0.0004 \\ 0.0043 \\ 0.0009 \\ 0.0023 \\ 0.0039 \\ 0.0026 \\ 0.0054 \\ 0.0064 \\ 0.0060 \\ 0.0036 \\ 0.0054 \\ 0.0054 \\ 0.0025 \\ 0.0017 \end{bmatrix}$$

Now, we will construct a continuous path of optimal strategies. To see this we write

$$A = [\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_{28}\ \mathbf{a}_{29}],$$

where $\mathbf{a}_i \in \mathbb{R}^{26}$ for $i = 1, \ldots, 29$. Then, it is easy to see that the three linear systems (note that if $\mathbf{z}^* = [z_1^*, \ldots, z_{29}^*]'$, then $z_2^* = z_3^* = z_4^* = z_{11}^* = 0$)

$$[\mathbf{a}_1 \mathbf{a}_5 \cdots \mathbf{a}_{10} \mathbf{a}_{12} \cdots\ \mathbf{a}_{29}] \mathbf{u}_i = \mathbf{a}_i$$

have the following three solutions ,

$$\mathbf{u}_2 = \begin{bmatrix} 0.9772 \\ -0.0006 \\ 0.0045 \\ -0.0057 \\ 0.0004 \\ -0.0026 \\ 0.0027 \\ -0.0034 \\ -0.0007 \\ 0.0010 \\ 0.0023 \\ 0.0010 \\ 0.0024 \\ -0.0102 \\ 0.0015 \\ -0.0035 \\ -0.0024 \\ -0.0061 \\ 0.0079 \\ 0.0041 \\ -0.0001 \\ -0.0036 \\ -0.0028 \\ -0.0075 \\ -0.0031 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 0.9511 \\ -0.0026 \\ 0.0081 \\ -0.0067 \\ 0.0045 \\ -0.0054 \\ 0.0004 \\ -0.0104 \\ 0.0083 \\ 0.0007 \\ -0.0026 \\ 0.0070 \\ 0.0011 \\ -0.0086 \\ -0.0023 \\ -0.0026 \\ -0.0034 \\ 0.0130 \\ 0.0064 \\ 0.0022 \\ -0.0007 \\ -0.0023 \\ 0.0160 \\ -0.0049 \\ -0.0093 \end{bmatrix}, \ \mathbf{u}_4 = \begin{bmatrix} 0.9330 \\ -0.0006 \\ -0.0050 \\ -0.0069 \\ 0.0106 \\ 0.0108 \\ 0.0126 \\ 0.0196 \\ 0.0068 \\ -0.0108 \\ 0.0267 \\ 0.0005 \\ -0.0039 \\ 0.0034 \\ 0.0049 \\ 0.0094 \\ 0.0149 \\ 0.0308 \\ 0.0041 \\ 0.0212 \\ -0.0031 \\ 0.0114 \\ 0.0097 \\ -0.0034 \\ 0.0157 \end{bmatrix}.$$

Thus, if $\mathbf{u}_i = [u_i^{(1)}, \ldots, u_i^{(25)}]'$ for $i = 2, 3, 4$, we have that

$$\mathbf{a}_i = u_i^{(1)}\mathbf{a}_1 + \sum_{j=5, j\neq 11}^{29} u_i^{(j)}\mathbf{a}_j$$

for $i = 2, 3, 4$. Since,

$$z_1^*\mathbf{a}_1 + \sum_{j=5, j\neq 11}^{29} z_j^*\mathbf{a}_j = \mathbf{b}.$$

We can obtain other solutions writting

$$\begin{aligned}
\mathbf{b} &= \sum_{k=1,k\neq 11}^{29} \hat{z}_k \mathbf{a}_k \\
&= \hat{z}_1\mathbf{a}_1 + \sum_{j=2}^{4} \hat{z}_j \mathbf{a}_j + \sum_{k=5,k\neq 11}^{29} \hat{z}_k \mathbf{a}_k \\
&= \hat{z}_1\mathbf{a}_1 + \sum_{j=2}^{4} \hat{z}_j \left(u_j^{(1)} \mathbf{a}_1 + \sum_{t=5,t\neq 11}^{29} u_j^{(t)} \mathbf{a}_t \right) + \sum_{k=5,k\neq 11}^{29} \hat{z}_k \mathbf{a}_k \\
&= \left(\hat{z}_1 + \sum_{j=2}^{4} u_j^{(1)} \hat{z}_j \right) \mathbf{a}_1 + \sum_{k=5,k\neq 11}^{29} \left(\hat{z}_k + \sum_{j=2}^{4} u_j^{(k)} \hat{z}_j \right) \mathbf{a}_k.
\end{aligned}$$

Then,

$$z_k^* = \hat{z}_k + \sum_{j=2}^{4} u_j^{(k)} \hat{z}_j.$$

for $k = 1, 5, \ldots, 10, 12, \ldots, 29$. We only need to choose $\hat{z}_j = \eta > 0$ for $j = 2, 3, 4$ and

$$\hat{z}_k = z_k^* - \eta \left(\sum_{j=2}^{4} u_j^{(k)} \right) > 0,$$

for $k = 1, 5, \ldots, 10, 12, \ldots, 29$. Let be the set

$$\mathcal{K} = \left\{ k \in \{1, 5, \ldots, 10, 12, \ldots, 29\} : \sum_{j=2}^{4} u_j^{(k)} > 0 \right\}$$

and we take

$$\delta = \min \left\{ \frac{z_k^*}{\sum_{j=2}^{4} u_j^{(k)}} : k \in \mathcal{K} \right\}.$$

Then

$$\mathbf{z}_\eta^* = [\hat{z}_1, \eta, \eta, \eta, \hat{z}_5, \ldots, \hat{z}_{10}, 0, \hat{z}_{12}, \ldots, \hat{z}_{29}]$$

is also a solution of (11) for each $\eta \in [0, \delta)$. In consequence, we have that $\mathbf{x}_\eta = [1.0228 - 2.8613\eta, \eta, \eta, \eta]'$ is an optimal strategy for each $\eta \in [0, 0.3574599)$. The rest of the paper is devoted to the proof of Theorem A.

2. Proof of Theorem A

Proof. First at all we write the linear system (8) in the standard form as follows

$$
\begin{aligned}
&\min{}_{x_1,\dots,x_n,V,y_1,\dots,y_m} -V \\
&\text{subject to} \quad \textstyle\sum_{i=1}^n x_iV_i(\omega_j) - V - y_j = 0 \qquad j=1,2,\dots,m \\
&\textstyle\sum_{i=1}^n x_ip_i = I \\
&V \geq 0, x_i \geq 0, \qquad i=1,2,\dots,n \\
&y_j \geq 0, \qquad j=1,2,\dots,m.
\end{aligned} \tag{19}
$$

Then the dual program is given by

$$
\begin{aligned}
&\max{}_{\lambda_1,\dots,\lambda_m,\lambda_{m+1}} \lambda_{m+1}I \\
&\text{subject to} \quad \textstyle\sum_{j=1}^m \lambda_jV_i(\omega_j) + \lambda_{m+1}p_i \leq 0 \qquad i=1,2,\dots,n \\
&\textstyle\sum_{j=1}^m \lambda_j \geq 1 \\
&\lambda_j \geq 0, \qquad j=1,2,\dots,m.
\end{aligned} \tag{20}
$$

Remark 2.1. We note that since $\sum_{j=1}^m \lambda_jV_i(\omega_j) \geq 0$ and $p_i > 0$, for $i=1,2,\dots,n$, then $\lambda_{m+1} \leq 0$. In consequence, the maximum for the objective function in the linear program (20) is obtained for values of $\lambda_{m+1}I$ close to zero.

Let $a > 1$ be a real number. Assume that we have one finite sequence of feasible values $\lambda_1^{(a)},\dots,\lambda_m^{(a)},\lambda_{m+1}^{(a)}$ for which

$$\sum_{j=1}^m \lambda_j^{(a)} = a \geq 1.$$

Then,

$$\sum_{j=1}^m \lambda_j^{(a)}V_i(\omega_j) + \lambda_{m+1}^{(a)}p_i \leq 0.$$

for $i=1,2,\dots,n$. This inequality is equivalent to

$$\sum_{j=1}^m \frac{\lambda_j^{(a)}}{a}V_i(\omega_j) + \frac{\lambda_{m+1}^{(a)}}{a}p_i \leq 0.$$

for $i=1,2,\dots,n$. Thus, we obtain that for each real number $a \geq 1$, the sequence $\lambda_1^{(a)}/a,\dots,\lambda_m^{(a)}/a,\lambda_{m+1}^{(a)}/a$ is also feasible with

$$\sum_{j=1}^m \frac{\lambda_{m+1}^{(a)}}{a} = 1.$$

By using the fact that $\lambda_{m+1}^{(a)}$ is negative, we conclude that

$$\lambda_{m+1}^{(a)} I < \frac{\lambda_{m+1}^{(a)}}{a} I.$$

In consequence, we can state that the optimal feasible solution is obtained for values satisfying the condition

$$\sum_{j=1}^{m} \lambda_j = 1,$$

with $0 \leq \lambda_j \leq 1$ for $j = 1, 2, \ldots, m$. We remark that the first restriction of linear program (20) implies that

$$\lambda_{m+1} \leq -\frac{1}{p_i} \sum_{j=1}^{m} \lambda_j V_i(\omega_j)$$

for $i = 1, 2, \ldots, n$ and $[\lambda_1, \ldots, \lambda_m]' \in K$, where

$$K = \left\{ [z_1 \; z_2 \; \cdots z_m]' \in [0,1]^m : z_1 + z_2 + \cdots + z_m = 1 \right\}$$

is a compact and convex set.

Now, for each $i = 1, 2, \ldots, m$, consider the linear map

$$z_{i,m+1} = z_{i,m+1}(\lambda_1, \ldots, \lambda_m) = -\frac{1}{p_i} \sum_{j=1}^{m} \lambda_j V_i(\omega_j).$$

defined in K. Since the graph of $z_{i,m+1}$ is a hyperplane in $\mathbb{R}^{m+1}$. Then the extremal points of $z_{i,m+1}$ in K are attained in the convex boundary of K which is given as the convex hull generated by the points

$$\left\{ \mathbf{e}_j = \left[0, \ldots, 0, \underbrace{1}_{j-th \ position}, 0 \ldots, 0 \right]' \in \mathbb{R}^m : j = 1, 2, \ldots, m \right\}$$

This fact implies that

$$\min \left\{ -\frac{V_k(\omega_j)}{p_k} : \; k = 1, 2, \ldots, n \right\} \leq z_{i,m+1}(\mathbf{e}_j) \leq \max \left\{ -\frac{V_k(\omega_j)}{p_k} : \; k = 1, 2, \ldots, n \right\}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Let

$$\lambda_{m+1}^{(j)} = \min \left\{ -\frac{V_k(\omega_j)}{p_k} : \; k = 1, 2, \ldots, n \right\},$$

then the set of possible solutions for the dual program is given by,

$$\left\{ \begin{bmatrix} \mathbf{e}_j \\ \\ \lambda_{m+1}^{(j)} \end{bmatrix} \in \mathbb{R}^{m+1} : j = 1, 2, \ldots, m \right\}.$$

This implies that the maximum is attained at

$$\begin{bmatrix} \mathbf{e}_{j^*} \\ \\ \lambda^*_{m+1} \end{bmatrix}$$

where

$$\lambda^*_{m+1} = \lambda^{(j^*)}_{m+1} = \max\left\{\lambda^{(j)}_{m+1} : j = 1, 2, \ldots, m\right\}.$$

Thus, the maximum of the dual program is attained at

$$\lambda^*_{m+1} I.$$

Note that since $-V_i(\omega_j)/p_j < 0$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$; we can write

$$-\lambda^* = \lambda^*_{m+1} = -\min\left\{\max\left\{\frac{V_k(\omega_j)}{p_k} :\ k = 1, 2, \ldots, n\right\} : j = 1, 2, \ldots, m\right\}$$

From the Duality Theorem of Linear Programming we obtain that the primal program has a finite optimal solution $V^* = \lambda^* I$ and (9)-(10) hold. This ends the proof of the first statement of Theorem A.

Finally, from (19), we have that the optimal portfolio satisfy that

$$\begin{aligned} &\textstyle\sum_{i=1}^n x_i V_i(\omega_j) - V^* - y_j = 0 \quad && j = 1, 2, \ldots, m \\ &\textstyle\sum_{i=1}^n x_i p_i = I && \\ &x_i \geq 0, && i = 1, 2, \ldots, n \\ &y_j \geq 0, && j = 1, 2, \ldots, m. \end{aligned} \tag{21}$$

In order to solve (21) we use the bounded least squares version of this problem given by (11)–(13). □

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