A BASIC MODEL OF BIFURCATIONS FOR A CLASS OF DEGREE ONE CIRCLE MAPS: THE ARNOL'D TONGUES REVISITED

F. BALIBREA AND A. FALCÓ

1. Introduction

The map $g_w(x) = x+w$ can be seen as the superposition of two simple sinusoidal oscillators where $x \in \mathbb{S}^1$ represents the value of the phase of one of the oscillators after the other has done one oscillation. The term $w \in [0,1)$ represents the ratio of the frequencies of the two oscillators.

When w is an irrational number the motion of the systems is called *quasiperiodic* and if w is a rational number then the motion is called *periodic.*

The more general map

$$H_{b,w}(x) = x + w + \frac{b}{2\pi}\sin(2\pi x) \tag{1}$$

where $x \in \mathbb{R}$ and $(b,w) \in \mathbb{R}^+ \times \mathbb{R}$. was introduced by Arnol'd [4] to study the behaviour of the motion of the system when a non–linear term is added. The resultant family of maps has been used to study some variety of forced systems where there are two competing frecuencies, for example, the case of a sinusoidally driven pendulum.

Depending on the range on b, the family of maps have different behaviours which has been consider in the literature (see [4], [12], [6], [16] and [10]).

Assume that $w \in \mathbb{R}$, when $b > 1$ we have bimodal degree one circle maps and in the case $b \in [0,1]$, $H_{b,w}$ is a orientation preserving circle homeomophism. We can have non–linear terms of different kinds, for example, piecewise–linear maps from which arise families of maps similar to the (1) one.

The aim of this paper is twofold. First we will define a generic class of families of maps like (1) one and then we will describe the basic structure of bifurcations. In this case similar phenomena appear: Arnol'd tongues, horns, phase–locking, etc.

2. Preliminary definitions an examples

We will start this section by defining what we understand by the class of bimodal degree one circle maps and orientation preserving circle homeomorphisms. As it is usual, instead of working with the circle maps themselves we will rather use their liftings to the universal covering space $\mathbb{R}$. In this spirit we define $\mathcal{L}$ to be the class of all continuous maps F from $\mathbb{R}$ into itself such that $F(x+1) = F(x)+1$ for all $x \in \mathbb{R}$. That is, $\mathcal{L}$ is the class of all liftings of degree one circle maps. We will denote

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by $\mathcal{A}$ the class of maps $F \in \mathcal{L}$ such that there exists $c_{_F} \in (0,1)$ with the property that F is strictly increasing in $[0, c_{_F}]$ and strictly decreasing in $[c_{_F}, 1]$.

We note that every map $F \in \mathcal{A}$ has a unique local maximum and a unique local minimum in $[0,1)$. To define the class $\mathcal{A}$ we restricted ourselves to the case in which F has the minimum at 0. Since each map from $\mathcal{L}$ is conjugate by a translation to a map from $\mathcal{L}$ having the minimum at 0, the fact that we fix that the maps from $\mathcal{A}$ have the minimum at 0 is not restrictive. Thus, class $\mathcal{A}$ models the bimodal degree one circle maps.

We will denote the class of all orientation preserving circle homeomorphisms by $\mathcal{H}$. More precisely, $F \in \mathcal{H}$ if and only if $F \in \mathcal{L}$ and it is strictly increasing. Next we will define the class of families of maps under consideration.

We will say that a parametriced family of maps $\{F_{b,w}\}_{(b,w)\in\mathbb{R}^+\times\mathbb{R}}$ is an *Arnol'd family of degree one circle maps* if for $(b,w) \in \mathbb{R}^+ \times \mathbb{R}$ the map $F_{b,w} : \mathbb{R} \to \mathbb{R}$ is given by $F_{b,w}(x) = G_b(x) + w$, where $G_b \in \mathcal{L}$ for all $b \in \mathbb{R}^+$, and the following conditions hold.

(A1) The map $b \longmapsto G_b$ from $\mathbb{R}^+$ to $\mathcal{L}$ is continuous.

(A2) For $b \in (1,\infty)$ the map $G_b \in \mathcal{A}$ and the map $b \longmapsto c_{_{G_b}}$ from $(1,\infty)$ to $\mathbb{R}$ is continuous and the following conditions hold.
  - (i) $\lim_{b\to 1} c_{_{G_b}} = 1$,
  - (ii) $\lim_{b\to\infty} c_{_{G_b}} = c \in (0,1)$,
  - (iii) $\lim_{b\to\infty} G_b(c_{_{G_b}}) = \infty$ and
  - (iv) $\lim_{b\to\infty} G_b(0) = -\infty$.

(A3) For $b \in [0,1]$ the map $G_b \in \mathcal{H}$ and $G_0(x) = x$ for all $x \in \mathbb{R}$.

We note that by definition of $F_{b,w}$ and G_b we have that for all $b > 1$ and $w \in \mathbb{R}$, $F_{b,w} \in \mathcal{A}$ and $c_{_{F_{b,w}}} = c_{_{G_b}}$. Thus, from now one we will denote $c_{_{G_b}}$ by c_b.

Now, we give two examples of Arnol'd families of degree one circle maps. The first one contains only piecewise–monote continuous maps.

Example 2.1. *Let $G_b \in \mathcal{L}$ be such that for $b > 1$ the map is a piecewise–monotone map $G_b \in \mathcal{A}$ with $c_b = 1 - \frac{1}{2}(1 - \exp -\{(b-1)\})$, $G_b(c_b) = \exp\{(b-1)\}$ and $G_b(0) = 1 - \exp\{(b-1)\}$ for all $b > 1$ and for $b \in [0,1]$, $G_b(x) = x$. It is not difficult to prove that G_b satifies (A1)–(A3) and in consequence $\{F_{b,w}\}_{(b,w)\in\mathbb{R}^+\times\mathbb{R}}$, is an Arnol'd family of degree one circle maps.*

In the next example we construct an Arnol'd family of maps for which there exists continuous family $\{h_b\}_{b\in\mathbb{R}^+}$ contained in $\mathcal{H}$ satisfying that $F_{b,w} \circ h_b = h_b \circ H_{b,w}$ for all $w \in \mathbb{R}$. In consequence the bifurcation diagrams of these two families are the same.

Example 2.2. *Let $\{H_{b,w}\}_{(b,w)\in\mathbb{R}^+\times\mathbb{R}}$, be the standard maps family given by (1). Assume that $b > 1$ and let $C(b)$ be the relative maximum of $H_{b,w}$ in $(1/4, 1/2)$ and let $M(b)$ be the relative minimum of $H_{b,w}$ in $(1/2, 3/8)$ that we can obtain from $\cos 2\pi x = -1/b$ for each $b > 1$ (see Figure 1). to see that*

$$\lim_{b\to 1} M(b) = \lim_{b\to 1} C(b) = 1/2,$$

$$\lim_{b\to\infty} C(b) = 1/4 \text{ and } \lim_{b\to\infty} M(b) = 3/8.$$

We may assume that $M(b) = C(b) = 1/2$ for all $b \in [0,1]$. Now, we define

$$G_b(x) = \begin{cases} x + \frac{b}{2\pi}\sin 2\pi(x + M(b)) & \text{if } b > 1, \\ x + \frac{b}{2\pi}\sin 2\pi(x + 1/2) & \text{otherwise.} \end{cases}$$

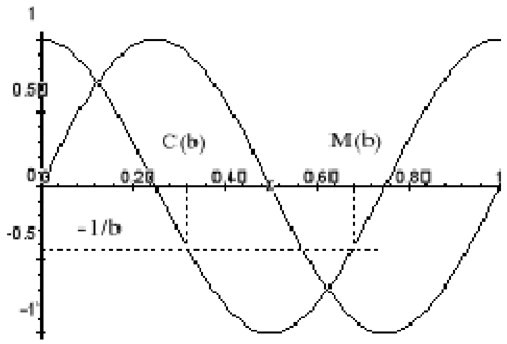

FIGURE 1. The maps $\sin 2\pi x$ and $\cos 2\pi x$.

Clearly, $G_b \in \mathcal{L}$ for all $b \in \mathbb{R}^+$. For $b > 1$ the map $G_b \in \mathcal{A}$ with $c_{G_b} = C(b) + 1 - M(b)$. Otherwise, $G_b \in \mathcal{H}$. Moreover, it is not difficult to see that G_b satisfies (A1)–(A3). Let $h_b(x) = x + M(b)$, then $h_b \in \mathcal{H}$ for all $b \in \mathbb{R}^+$ and

$$\begin{aligned} h_b(F_{b,w}(x)) &= h_b((G_b + w)(x)) \\ &= (G_b + w)(x) + M(b) \\ &= (H_{b,0} + w)(h_b(x)) \\ &= H_{b,w}(h_b(x)) \end{aligned}$$

for all $x \in \mathbb{R}$ and for all $b \in \mathbb{R}^+$. Thus, we have proved that for $\{H_{b,w}\}_{(b,w)\in\mathbb{R}^+\times\mathbb{R},}$ there exists $\{F_{b,w}\}_{(b,w)\in\mathbb{R}^+\times\mathbb{R},} \in \mathcal{T}$ and $\{h_b\}_{b\in\mathbb{R}^+} \in \mathcal{H}$ such that $h_b \circ F_{b,w} = H_{b,w} \circ h_b$ for all $w \in \mathbb{R}$ (i.e. the maps $F_{b,w}$ and $H_{b,w}$ are topologically conjugated).

3. A basic description of the bifurcation structure in $\mathbb{R}^+ \times \mathbb{R}$

The aim of the section is to give a two partition of the bifurcation plane $\mathbb{R}^+ \times \mathbb{R}$ associated to both extremes of the rotation interval. Next, we decribe the boundaries of such partitions and their relationship. This section is organiced as follows.

3.1. **Preliminary definitions and results.** In this section we will introduce the basic definitions to understand the dynamics of the degree one circle maps.

First, we recall that for $F \in \mathcal{L}$ *the rotation interval* R_F is defined to be the set

$$\{\rho_F(x) : x \in \mathbb{R}\},$$

where

$$\rho_F(x) = \rho(x) = \limsup_{n\to\infty} \frac{F^n(x) - x}{n}.$$

It is well known (see [14]) that the set R_F is a closed interval, perhaps degenerate to a single point. Also, if $F \in \mathcal{L}$ is a non–decreasing map then

$$R_F = \{\lim_{n\to\infty} \frac{F^n(x) - x}{n}\}.$$

Thus, to every non-decreasing map $F \in \mathcal{L}$ we can associate a real number

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n},$$

which is called *the rotation number of* F. Roughly speaking, $\rho(F)$ is the average angular speed of any point moving around the circle under iteration of the map. We note that $\rho(F)$ is a topological invariant of F. That is, if F and G are topologically conjugated (i.e. there exists $h \in \mathcal{H}$ such that $F \circ h = h \circ G$) then $\rho(F) = \rho(G)$.

The rotation interval is closely related with the existence of periodic orbits for degree one circle maps. To see this we will introduce the following definitions and notation.

Let $F \in \mathcal{L}$ and let $x \in \mathbb{R}$. Then the set $\{y \in \mathbb{R} : y = F^n(x)(\text{mod. } 1) \text{ for } n = 0, 1, \ldots\}$ will be called the *(mod. 1) orbit* of x by F. We stress the fact that if P is a (mod. 1) orbit and $x \in P$, then $x + k \in P$ for all $k \in \mathbb{Z}$.

It is not difficult to prove that each point from an orbit (mod. 1) P has the same rotation number. Thus, we can speak about the *rotation number* of P.

If $A \subset \mathbb{R}$ and $x \in \mathbb{R}$, we shall write $x+A$ or $A+x$ to denote the set $\{x+a : a \in A\}$. Also, if $B \subset \mathbb{R}$ we shall write $A + B$ to denote the set $\{a + b : a \in A,\ b \in B\}$.

If x is a periodic (mod. 1) point of F of period q with rotation number $\frac{p}{q}$ then its (mod. 1) orbit is called a *periodic (mod. 1) orbit of F of period q with rotation number* $\frac{p}{q}$. If P is a (mod. 1) orbit of F we denote by P_i the set $P \cap [i, i+1)$ for all $i \in \mathbb{Z}$. Obviously $P_i = i + P_0$. We note that if P is a periodic (mod. 1) orbit of F with period q, then $\text{Card}(P_i) = q$ for all $i \in \mathbb{Z}$.

For a map $F \in \mathcal{L}$ we define maps F_l and F_u by

$$F_u(x) = \sup\{F(y) : y \leq x\} \tag{2}$$

and

$$F_l(x) = \inf\{F(y) : y \geq x\} \tag{3}$$

(see [19], [2] and [9]). The maps F_u, F_l belong to $\mathcal{L}$ and are non-decreasing and $R_F = [\rho(F_l), \rho(F_u)]$.

3.2. **The bifurcation diagram for the Arnol'd family of degree one circle maps.** Let $\{F_{b,w}\}_{(b,w)\in\mathbb{R}^+\times\mathbb{R},}$ be an Arnol'd family of degree one circle maps. For each $b \in \mathbb{R}^+$ we define $\rho_b^-,\ \rho_b^+ : \mathbb{R} \to \mathbb{R}$ by

$$\rho_b^-(w) = \rho((F_{b,w})_l)$$

and

$$\rho_b^+(w) = \rho((F_{b,w})_u),$$

respectively. Thus, $R_{F_{b,w}} = [\rho_b^-(w), \rho_b^+(w)]$.

Remark 3.1. *Using (2) and (3) we have that $(F_{b,w})_l(x) = w + (G_b)_l(x)$ and $(F_{b,w})_u(x) = w + (G_b)_u(x)$. Moreover, from (3) and (2) it is not difficult to see that, for $b \in (1, \infty)$, $(F_{b,w})_l(0) = F_{b,w}(0)$ and $(F_{b,w})_u(c_b) = F_{b,w}(c_b)$ for all $w \in \mathbb{R}$.*

The next result follows from Remark 3.1, the definition of $F_{b,w}$ and [2, Lemma 3.7.12].

Proposition 3.1. *The maps ρ_b^- and ρ_b^+ are continuous, onto, ρ_b^- is non–increasing, ρ_b^+ is non–decreasing and satisfy that $\rho_b^-(w) \leq \rho_b^+(w)$ for all $(b, w) \in \mathbb{R}^+ \times \mathbb{R}$. Moreover, the maps $b \mapsto \rho_b^-$ and $b \mapsto \rho_b^+$ are continuous.*

The basic structure of the bifurcation diagram for a given Arnol'd family of degree one circle maps $\{F_{b,w}\}_{(b,w)\in\mathbb{R}^+\times\mathbb{R},}$ is given by the *upper (respectively, lower) a–Arnol'd tongue* as

$$A^-(a) = \{(b,w) \in \mathbb{R}^+ \times \mathbb{R} : \rho_b^-(w) = a\}$$

(respectively,

$$A^+(a) = \{(b,w) \in \mathbb{R}^+ \times \mathbb{R} : \rho_b^+(w) = a\}).$$

In order to describe the boundaries of $A^-(a)$ and $A^+(a)$, for each $a \in \mathbb{R}$, we define the following four maps. Let $a \in \mathbb{R}$, for each $b \in \mathbb{R}^+$ we take

$$\begin{aligned} \Phi_a^-(b) &= \sup\{w \in \mathbb{R} : \rho_b^-(w) = a\}, \\ \Psi_a^-(b) &= \inf\{w \in \mathbb{R} : \rho_b^-(w) = a\}, \\ \Phi_a^+(b) &= \inf\{w \in \mathbb{R} : \rho_b^+(w) = a\} \end{aligned}$$

and

$$\Psi_a^+(b) = \sup\{w \in \mathbb{R} : \rho_b^+(w) = a\}.$$

From Proposition 3.1 we have that the above maps, defined from $\mathbb{R}^+$ into $\mathbb{R}$, are well–defined. Note that since $F_{0,a}(x) = x + a$, then

$$\Phi_a^-(0) = \Psi_a^-(0) = \Psi_a^+(0) = \Phi_a^+(0) = a$$

for all $a \in \mathbb{R}$. The next lemma follows straithforward for the definitions (see Figure **??**).

Lemma 3.2. *The following statements hold.*

(1) *For all $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$,*

$$\Phi_a^+(b) \le \Psi_a^+(b) \le \Phi_a^-(b)$$

and

$$\Phi_a^+(b) \le \Psi_a^-(b) \le \Phi_a^-(b).$$

(2) *For all $a \in \mathbb{R}$ and $b \in (0,1]$, $\Phi_a^-(b) = \Psi_a^+(b)$ and $\Psi_a^-(b) = \Phi_a^+(b)$.*

(3) *Let $a, a' \in \mathbb{R}$ with $a < a'$. Then*
   (a) *$\Phi_a^-(b) < \Phi_{a'}^+(b)$ for all $b \in (0,1]$ and*
   (b) *$\Phi_a^-(b) < \Psi_{a'}^-(b)$ and $\Psi_a^+(b) < \Phi_{a'}^+(b)$ for all $b \in (0,\infty)$.*

Clearly, all of these maps describe, for each $a \in \mathbb{R}$, the boundaries of $A^-(a)$ and $A^+(a)$. Note that we use the endpoints of the rotation interval in the definition of the boundary maps. Thus, when one or both of the endpoints of the rotation interval is a rational number it is possible to establish a equivalent definition of the boundary maps in the rational case (see [16]). To see this we will use the following definition.

Let $a = p/q \in \mathbb{Q}$ with $(p,q) = 1$ and let $i \in \{l,u\}$. We say that $F_{b,w}$ satifies the *a–upper property* (respectively, the *a–lower property*) if $\rho((F_{b,w})_i) = a$ and $((F_{b,w})_i^q - p)(x) \ge x$ (respectively, $((F_{b,w})_i^q - p)(x) \le x$) for all $x \in \mathbb{R}$.

The following theorem resumes most of classical results for the boundary maps.

Theorem A. *For each $a \in \mathbb{R}$ the maps Φ_a^-, Ψ_a^-, Φ_a^+ and Ψ_a^+ are continuous satisfying that*

$$\lim_{b\to\infty} \Phi_a^-(b) = \lim_{b\to\infty} \Psi_a^-(b) = \infty$$

and

$$\lim_{b\to\infty} \Phi_a^+(b) = \lim_{b\to\infty} \Psi_a^+(b) = -\infty.$$

Moreover, if $a = p/q \in \mathbb{Q}$*, with* $(p,q) = 1$*, then the following statements hold.*

(1) *For all* $b \in \mathbb{R}^+$ *we have that*

$$\begin{aligned}\Phi_a^-(b) &= \sup\{w \in \mathbb{R} : (F_{b,w})_l \text{ satisfies the } a\text{–upper condition}\},\\ \Phi_a^+(b) &= \inf\{w \in \mathbb{R} : (F_{b,w})_u \text{ satisfies the } a\text{–lower condition}\},\\ \Psi_a^-(b) &= \inf\{w \in \mathbb{R} : (F_{b,w})_l \text{ satisfies the } a\text{–lower condition}\}\end{aligned}$$

and

$$\Psi_a^+(b) = \sup\{w \in \mathbb{R} : (F_{b,w})_u \text{ satisfies the } a\text{–upper condition}\}.$$

(2) *Assume that the following conditions hold.*
   (a) *For all* $b \in \mathbb{R}^+$ *the map* $G_b(x) = x + b\gamma(x)$ *where* $\gamma \in \mathcal{C}^r(\mathbb{R},\mathbb{R})$ *with* $r \geq 1$
   (b) *There is a unique degenerate orbit for* $(F_{b,w})_u$ *(respectively,* $(F_{b,w})_l$*), that is, there is a unique (mod 1) periodic orbit* P *of period* q *and rotation number* a *holding* $D_x F_{b,w}^q(x) = 1$ *for all* $x \in P$*.*
   (c) *If* P *is the (mod 1) periodic orbit of period* q *and rotation number* a *for* $(F_{b,w})_u$ *(respectively,* $(F_{b,w})_l$*). Then* $D_{xx}F_{b,w}^q(x) \neq 0$ *for all* $x \in P$*.*

   Then the maps $\Phi_a^-, \Psi_a^-, \Phi_a^+$ *and* Ψ_a^+ *are uniformly Lipschitz.*

Let $a, a' \in \mathbb{R}$ with $a < a'$. Now, we will study the intersections of the boundaries of the upper a–Arnod'd tongue and the lower a'–Arnold'd tongue. A first result in this direction obtained from Lemma 3.2 and Theorem **??** is the following.

Corollary 3.3. *For each* $a, a' \in \mathbb{R}$ *with* $a < a'$ *there exist* $b_1, b_2, b_3, b_4 \in (1,\infty)$ *such that* $\Phi_a^-(b_1) = \Phi_a^+(b_1)$*,* $\Phi_a^-(b_2) = \Psi_{a'}^+(b_2)$*,* $\Phi_{a'}^+(b_3) = \Psi_a^-(b_3)$ *and* $\Psi_{a'}^+(b_4) = \Psi_a^-(b_4)$*. Moreover,* $b_1 < b_2$*, (resp.* $b_3 < b_4$*) if and only if* $a' \in \mathbb{Q}$*. (resp.* $a \in \mathbb{Q}$*)).*

The next result studies, when the image of the two critical points increases with respect to the parameter b, the unicity of the intersections of two different boundary maps.

Theorem B. *Assume that for each* $i \in \mathbb{N}$*, the maps from* $(1,\infty)$ *to* $\mathbb{R}$*,* $b \mapsto G_b^i(0)$ *is decreasing and* $b \mapsto G_b^i(c_b)$ *is increasing. Let* $a, a' \in \mathbb{R}$ *with* $a < a'$ *then the following statements hold.*

(1) *If* $a \notin \mathbb{Q}$ *(respectively,* $a' \notin \mathbb{Q}$*) then*

$$\mathrm{Card}\{b > 1 : \Phi_a^-(b) = \Phi_{a'}^+(b)\} = \mathrm{Card}\{b > 1 : \Phi_a^-(b) = \Psi_{a'}^+(b)\} = 1$$

(respectively,

$$\mathrm{Card}\{b > 1 : \Phi_{a'}^+(b) = \Phi_a^-(b)\} = \mathrm{Card}\{b > 1 : \Phi_{a'}^+(b) = \Psi_a^-(b)\} = 1).$$

(2) *If* $a \in \mathbb{Q}$ *and* $a' \in \mathbb{Q}$ *then the sets*

$$\begin{aligned}&\{w : w = \Phi_a^-(b) = \Phi_{a'}^+(b) \text{ for some } b > 1\},\\ &\{w : w = \Phi_a^-(b) = \Psi_{a'}^+(b) \text{ for some } b > 1\},\\ &\{w : w = \Phi_{a'}^+(b) = \Phi_a^-(b) \text{ for some } b > 1\},\end{aligned}$$

and

$$\{w : w = \Phi_{a'}^+(b) = \Psi_a^-(b) \text{ for some } b > 1\}$$

have cardinality 1.

It is not difficult to see that the family of maps given in Example 2.2 satisfies the assumptions of Theorem **??**. Thus, from Theorem **??**(a) follows Conjecture B of [10].

3.3. Introducing basic kneading theory to describe the bifurcation diagram.

3.3.1. *A basic survey on kneading theory.* Next, we will use *kneading theory* to describe the boundaries of $A^-(a)$ and $A^+(a)$ for all $a \in \mathbb{R}$. To see this we introduce some notation about the *kneading theory* developed by Alsedà and Mañosas in [3]. First we recall the notion of itinerary of a point. In what follows we shall denote the integer part function by $E(\cdot)$. For $x \in \mathbb{R}$ we set $D(x) = x - E(x)$.

For $F \in \mathcal{A}$ and $x \in \mathbb{R}$ let

$$s(x) = \begin{cases} R & \text{if } D(x) > c_F, \\ C & \text{if } D(x) = c_F, \\ L & \text{if } D(x) \in (0, c_F), \\ M & \text{if } D(x) = 0, \end{cases}$$

and $d(x) = E(F(x)) - E(x)$.

Then the *reduced itinerary of* x, denoted by $\underline{\widehat{I}}_F(x)$, is defined as follows. For $i \in \mathbb{N}$, set $s_i = s(F^i(x))$ and $d_i = d(F^{i-1}(x))$. Then $\underline{\widehat{I}}_F(x)$ is defined by

$$\begin{cases} d_1^{s_1} d_2^{s_2} \dots & \text{if } s_i \in \{L, R\} \text{ for all } i \geq 1, \\ d_1^{s_1} d_2^{s_2} \dots d_n^{s_n} & \text{if } s_n \in \{M, C\} \text{ and } s_i \in \{L, R\} \text{ for all } i \in \{1, \dots, n-1\}. \end{cases}$$

Note that since $F \in \mathcal{L}$ we have that $\underline{\widehat{I}}_F(x) = \underline{\widehat{I}}_F(x+k)$ for all $k \in \mathbb{Z}$.

Next, we define the kneading pair, (see [3]), it characterizes the set of reduced itineraries (and hence the dynamics) of a map $F \in \mathcal{A}$. Thus the study of the space of all kneading pairs, for maps from $\mathcal{A}$, provides a way to describe bifurcations for parametrized families of maps from $\mathcal{A}$. First, we introduce the following notation. For a point $x \in \mathbb{R}$ we define the sequences $\underline{\widehat{I}}_F(x^+)$ and $\underline{\widehat{I}}_F(x^-)$ as follows. For each $n \geq 0$ there exists $\delta(n) > 0$ such that $d(F^{n-1}(y))$ and $s(F^n(y))$ take constant values for each $y \in (x, x + \delta(n))$ (resp. $y \in (x - \delta(n), x)$). Denote these values by $d(F^{n-1}(x^+))$ and $s(F^n(x^+))$ (resp. $d(F^{n-1}(x^-))$ and $s(F^n(x^-))$). Then we set

$$\underline{\widehat{I}}_F(x^+) = d(x^+)^{s(F(x^+))} d(F(x^+))^{s(F^2(x^+))} \dots$$

and

$$\underline{\widehat{I}}_F(x^-) = d(x^-)^{s(F(x^-))} d(F(x^-))^{s(F^2(x^-))} \dots.$$

Let $F \in \mathcal{A}$. The pair $(\underline{\widehat{I}}_F(0^+), \underline{\widehat{I}}_F(c_F^-))$ will be called *the kneading pair of* F and will be denoted by $\mathcal{K}(F)$.

Now, we extend the definition of reduced itinerary to the orientation preserving circle homeomorphisms as follows. For $F \in \mathcal{H}$ and $x \in \mathbb{R}$ let

$$\widehat{s}(x) = \begin{cases} M & \text{if } D(x) = 0, \\ L & \text{if } D(x) \neq 0. \end{cases}$$

For $i \in \mathbb{N}$, set $s_i = \widehat{s}(F^i(x))$ and $d_i = d(F^{i-1}(x))$ (recall that $d(x) = E(F(x)) - E(x)$). Then $\underline{\widehat{I}}_F(x)$ is defined as

$$\begin{cases} d_1^{s_1} d_2^{s_2} \dots & \text{if } s_i = L \text{ for all } i \geq 1, \\ d_1^{s_1} d_2^{s_2} \dots d_n^{s_n} & \text{if } s_n = M \text{ and } s_i = L \text{ for all } i \in \{1, \dots, n-1\}. \end{cases}$$

In this context we define *the kneading pair* of a map $F \in \mathcal{H}$ as $(\underline{\widehat{I}}_F(0^+), \underline{\widehat{I}}_F(0^-))$. As above it will be denoted by $\mathcal{K}(F)$.

The following sequences are used to characterize the rotation interval by means the kneading pair (see [3, Proposition A]) and we will use its to characterize the boundaries of the Arnol'd tongues. For $a \in \mathbb{R}$ we set $\epsilon_i(a) = E(ia) - E((i-1)a)$ and $\delta_i(a) = \widetilde{E}(ia) - \widetilde{E}((i-1)a)$, where $\widetilde{E} : \mathbb{R} \longrightarrow \mathbb{Z}$ is defined as follows

$$\widetilde{E}(x) = \begin{cases} E(x) & \text{if } x \notin \mathbb{Z}, \\ x - 1 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Also, we set

$$\underline{\widehat{I}}_\epsilon(a) = \epsilon_1(a)^L \epsilon_2(a)^L \dots \epsilon_n(a)^L \dots$$

and

$$\underline{\widehat{I}}_\delta(a) = \delta_1(a)^L \delta_2(a)^L \dots \delta_n(a)^L \dots$$

Let $\underline{\widehat{I}}^*_\epsilon(a) = (\underline{\widehat{I}}_\epsilon(a))'$ and let $\underline{\widehat{I}}^*_\delta(a)$ denote the sequence that satisfies $(\underline{\widehat{I}}^*_\delta(a))' = \underline{\widehat{I}}_\delta(a)$.

From now one, $K_\epsilon(b,w)$ denotes $\underline{\widehat{I}}_{F_{b,w}}(0^+)$, the first component of the kneading pair of the map $F_{b,w}$ and $K_\delta(b,w)$ denotes $\underline{\widehat{I}}_{F_{b,w}}(c_b^-)$, the second one.

The following result, that will be useful to prove most of the results of the present paper, establishes the definition of the boundary maps using kneading invariants. It will be proved in Section .

Theorem 3.4. *For each $a \in \mathbb{R}$,*

$$\begin{aligned} \Phi_a^-(b) &= \sup\{w \in \mathbb{R} : K_\epsilon(b,w) = \underline{\widehat{I}}_\epsilon(a)\}, \\ \Psi_a^-(b) &= \inf\{w \in \mathbb{R} : K_\epsilon(b,w) = \underline{\widehat{I}}^*_\delta(a)\}, \\ \Phi_a^+(b) &= \inf\{w \in \mathbb{R} : K_\delta(b,w) = \underline{\widehat{I}}_\delta(a)\} \end{aligned}$$

and

$$\Psi_a^+(b) = \sup\{w \in \mathbb{R} : K_\delta(b,w) = \underline{\widehat{I}}^*_\epsilon(a)\}.$$

Using the above theorem we shall prove the next proposition. It gives a characterization of the bifurcation space in terms of the upper and lower a–Arnol'd Tongues.

Proposition 3.5. *The sets $\{A^-(a)\}_{a\in\mathbb{R}}$ and $\{A^+(a)\}_{a\in\mathbb{R}}$ give a two partition into closed disjoint sets of the bifurcation plane $\mathbb{R}^+ \times \mathbb{R}$. Moreover, for each $(b,w) \in \mathbb{R}^+ \times \mathbb{R}$. there exist $a, a' \in \mathbb{R}$, $a \le a'$, such that $(b,w) \in A^-(a) \cup A^+(a')$.*

In [6, Section 4] was introduced two maps in order to study the existence of superstable periodic orbits. In this sense and in a more general case we define the maps

$$\Phi_a^\epsilon(b) = \inf\{w \in \mathbb{R} : K_\epsilon(b,w) = \underline{\widehat{I}}_\epsilon(a)\}$$

and

$$\Phi_a^\delta(b) = \sup\{w \in \mathbb{R} : K_\delta(b,w) = \underline{\widehat{I}}_\delta(a)\}.$$

Then we have the following (see Figure **??** and compare with [6, Theorem 4.1]).

Theorem C. *Let $a \in \mathbb{R}$. Then the maps Φ_a^ϵ and Φ_a^δ are continuous and the following statements hold.*

(1) *$w = \Phi_a^\epsilon(b)$ if and only if there exists a periodic (mod. 1) orbit $P_{b,w}$ of period q and rotation number a such that $0 \in P_{b,w}$ and $F_{b,w}|_{P_{b,w}} = (F_{b,w})_l|_{P_{b,w}}$.*

(2) *$w = \Phi_a^\delta(b)$ if and only if there exists a periodic (mod. 1) orbit $P_{b,w}$ of period q and rotation number a such that $c_{F_{b,w}} \in P_{b,w}$ and $F_{b,w}|_{P_{b,w}} = (F_{b,w})_u|_{P_{b,w}}$.*

(3) *Assume that $a = p/q \in \mathbb{Q}$ with $(p,q) = 1$. Then $\Psi_a^-(b) < \Phi_a^\epsilon(b) \leq \Phi_a^-(b)$ and $\Phi_a^+(b) \leq \Phi_a^\delta(b) < \Psi_a^+(b)$ for all $b > 1$. Moreover, $\Phi_a^\epsilon(b) = \Phi_a^\delta(b)$ for all $b \in [0,1]$.*

(4) *Assume that $a \notin \mathbb{Q}$. Then $\Phi_a^\epsilon(b) = \Phi_a^-(b)$ and $\Phi_a^\delta(b) = \Phi_a^+(b)$ for all $b > 1$. Moreover, $\Phi_a^\epsilon(b) = \Phi_a^\delta(b)$ for all $b \in [0,1]$.*

Corollary 3.6. *For each $a \in \mathbb{R}$, $\lim_{b\to\infty} \Phi_a^\epsilon(b) = \infty$ and $\lim_{b\to\infty} \Phi_a^\delta(b) = -\infty$. Moreover, let $a = p/q \in \mathbb{Q}$ with $(p,q) = 1$. If we assume that $\Phi_a^\epsilon(b) < \Phi_a^-(b)$ (respectively, $\Phi_a^+(b) < \Phi_a^\delta(b)$) for all $b > 1$, then there exists $b^* > 1$ such that $\Phi_a^\epsilon(b^*) = \Psi_a^+(b^*)$ (respectively, $\Phi_a^\delta(b^*) = \Psi_a^-(b^*)$).*

The next proposition and theorem are our version of Proposition 3.5 and Proposition 5.5 of [6] respectively.

Proposition 3.7. *The following statements hold.*

(1) *Assume that $G_b \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$ for all $b \in \mathbb{R}^+$. If $a \notin \mathbb{Q}$ then $\Phi_a^-(b) > \Phi_a^+(b)$ for all $b > 1$.*

(2) *Assume that $G_b \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ for all $b \in \mathbb{R}^+$. If $a = p/q \in \mathbb{Q}$, with $(p,q) = 1$, then $\Phi_a^\epsilon(b) < \Phi_a^-(b)$ and $\Phi_a^+(b) < \Phi_a^\delta(b)$ for all $b > 1$. Moreover, there exist $b', b'' \in (1,\infty)$ such that $\Phi_a^\epsilon(b') = \Psi_a^+(b')$ and $\Phi_a^\delta(b'') = \Psi_a^-(b'')$.*

4. Proof of Theorem A

We will use the definition of the boundaries of the Arnol'd tongues given by Theorem 3.4 in order to prove all results. Thus, we start this section introducing the basic definitions and results of kneading theory for maps in $\mathcal{A} \cup \mathcal{H}$.

In the case of the standard maps family, given by (1), (C1) of the above proposition holds. From the fact that $F_{b,w}$ has negative Schwarzian derivative, it is not difficult to see that (C3) holds (see [6], for instance). Finally, to see that (C2) condition is held we have to consider the complexification of the family given by

$$\widetilde{H}_{b,w}(z) = z + w + \frac{b}{2\pi} \sin 2\pi z$$

and then apply the fact that a neutral periodic point z_0 contains in its immediate basin of attraction at least one critical point (see for instance Theorem 2.3 [7]).

4.1. **Proof of Theorem A.** Clearly, the maps Φ_a^-, Ψ_a^-, Φ_a^+ and Ψ_a^+ are continuous. It follows the first part of the theorem. Now, from Remark 3.1, $(F_{b,w})_l(0) = w + G_b(0)$ and $(F_{b,w})_u(c_b) = w + G_b(c_b)$. By using (A2)(iii)–(iv) and the definitions of ρ_b^+ and ρ_b^- we give that $\lim_{b\to\infty} \rho_b^+(w) = \infty$ and $\lim_{b\to\infty} \rho_b^-(w) = -\infty$. Since $\rho_b^-(\Psi_a^-(b)) = \rho_b^+(\Psi_a^+(b)) = a$, then ,by using the fact that with respect to b the map $\rho_b^-(w)$ is non–increasing and $\rho_b^+(w)$ is non–decreasing, $\lim_{b\to\infty} \Psi_a^-(b) = \infty$ and $\lim_{b\to\infty} \Psi_a^+(b) = -\infty$. The rest of equalities are obtained using Lemma 3.2(a). This follows the first statement of the theorem.

Theorem A(a) follows from [16] and as consequence of the above statement we follow the next corollary that will be use to prove statement (b).

Corollary 4.1. *Let $a = p/q \in \mathbb{Q}$, with $(p,q) = 1$, and assume that for all $b \in \mathbb{R}^+$ the map $G_b \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$. Let $w \in \{\Phi_a^-(b), \Phi_a^+(b), \Psi_a^-(b), \Psi_a^+(b)\}$ for some $b \in \mathbb{R}^+$. If P is a periodic (mod. 1) orbit of period q and rotation number a then $D_xF_{b,w}^q(x) = 1$ for all $x \in P$.*

Now, we shall prove Theorem A(b). Assume that $w_0 = \Phi_a^-(b_0)$ for some $b_0 > 1$. Let x_0 be the (mod. 1) periodic point of period q and rotation number a. From Proposition 4.1, $D_xF_{b_0,w_0}^q(x_0) = 1$. Now, we define the map $G : \mathbb{R}^2 \times (0,\infty) \longrightarrow \mathbb{R}^2$ as $G(x,w,b) = (F_{b,w}^q(x) - x - p, D_xF_{b,w}^q(x) - 1)$. Clearly, $G(x_0,b_0,w_0) = (0,0)$. Then

$$\begin{vmatrix} D_xF_{b_0,w_0}^q(x_0) - 1 & D_wF_{b_0,w_0}^q(x_0) \\ D_{xx}F_{b_0,w_0}^q(x_0) & D_{wx}F_{b_0,w_0}^q(x_0) \end{vmatrix} = D_{xx}F_{b_0,w_0}^q(x_0)D_wF_{b_0,w_0}^q(x_0).$$

Since, $D_wF_{b_0,w_0}^q(x_0) = 1 + \sum_{j=1}^{q-1} \prod D_xF_{b_0,w_0}\left(F_{b_0,w_0}^i(x_0)\right) > 1$ and (C2) then by the Implicit Function Theorem there exists a neighboorhod U_{b_0} of b_0, $V_{(x_0,w_0)}$ a neighboorhod of (x_0,w_0) and a $\mathcal{C}^1$–map $f : U_{b_0} \to V_{(x_0,w_0)}$ given by $f(b) = (x(b),w(b))$ such that $G(f(b),b) = (0,0)$ for all $b \in U_{b_0}$. Moreover

$$\frac{dw(b)}{db} = -\frac{D_bF_{b,w}^q(x(b))}{D_wF_{b,w}^q(x(b))}. \tag{4}$$

From (ii)–(iii) we have that $\Phi_a^-(b) = w(b)$ for all $b \in U_{b_0}$ and the first part of theorem follows.

Since $F_{b,w(b)}^q(x(b)) - x(b) - p = 0$, then we can write (4) for $b = b_0$, when $F_{b,w}$ is a diffeomorphism (i.e. $b < 1$), as

$$\left.\frac{dw(b)}{db}\right|_{b=b_0} = -\left.\frac{\sum_{i=0}^{q-1} \frac{1}{D_xF_{b,w(b)}^i(x(b))}\gamma\left(F_{b,w(b)}^i(x(b))\right)}{\sum_{i=0}^{q-1} \frac{1}{D_xF_{b,w(b)}^i(x(b))}}\right|_{b=b_0} \tag{5}$$

using some calculus on Fréchet manifolds (see [13]). Slammert [22] has obtained (5) in a more efficient way. To prove (5) for any $b > 0$, it is necesary to see that the two following formulas hold for any (b,w,x)

$$D_wF_{b,w}^q(x) = \sum_{i=1}^{q} F_{b,w}^{q-i}(F_{b,w}^i(x)), \tag{6}$$

$$D_bF_{b,w}^q(x) = \sum_{i=1}^{q} DF_{b,w}^{q-i}(F_{b,w}^i(x))p(F_{b,w}^i(x)) \tag{7}$$

We remark that (6) and (7) follow after some tedious calculus by induction over q. Finally, (5) is obtained from (4) multiplying by

$$\frac{D_xF_{b,w}^i(x)}{D_xF_{b,w}(x)}$$

and using the chain rule taking into consideration that if $x \in P$ then $D_xF_{b,w}^i(x) = 1$ for $i = 1,2,\ldots,q-1$ (see [5]). Therefore,

$$\left|\frac{dw(b)}{db}\right| \le \sup_x |\gamma(x)| \tag{8}$$

and Φ_a^- is an uniformly Lipchipz curve. This ends the proof of the statement (b).

Remark 4.1. *If $\gamma(x) = \frac{b}{2\pi}\sin(2\pi x)$ (the case of the standard maps family), the Lipschitz constant given by (8) is equal to $\frac{1}{2\pi}$ (see* [6] *and* [10]*). Moreover, if we assume that $x(b)$ converges to $x(0)$ when b converges to 0. Then*

$$\lim_{b\to 0}\frac{dw(b)}{db} = \frac{\sum_{i=0}^{q}\gamma\left(x(0)+\dfrac{ip}{q}\right)}{q},$$

which coincides with the result proved in [12].

5. Proof of Theorem 3.4 Proposition 3.5 and Theorem B

5.1. **A characterization of the rotation interval using kneading pairs and the proof of Theorem 3.4.** Let $\mathcal{S} = \{M, L, C, R\}$ and let $\underline{\alpha} = \alpha_0\alpha_1\dots$ be a sequence of elements $\alpha_i = d_i^{s_i}$ of $\mathbb{Z}\times\mathcal{S}$. We say that $\underline{\alpha}$ is *admissible* if one of the following two conditions is satisfied:

(1) $\underline{\alpha}$ is infinite, $s_i \in \{L, R\}$ for all $i \geq 1$ and there exists $k \in \mathbb{N}$ such that $|d_i| \leq k$ for all $i \geq 1$.
(2) $\underline{\alpha}$ is finite of length n, $s_n \in \{M, C\}$ and $s_i \in \{L, R\}$ for all $i \in \{1, \dots, n-1\}$.

Notice that any reduced itinerary is an admissible sequence. Now we shall introduce some notation for admissible sequences (and hence for reduced itineraries).

The cardinality of an admissible sequence $\underline{\alpha}$ will be denoted by $|\underline{\alpha}|$ (if $\underline{\alpha}$ is infinite we write $|\underline{\alpha}| = \infty$).

We denote by S the shift operator which acts on the set of admissible sequences of length greater than one as follows : $S(\underline{\alpha}) = \alpha_2\alpha_3\dots$ if $\underline{\alpha} = \alpha_1\alpha_2\alpha_3\dots$. We will write S^k for the k-th iterate of S. Obviously S^k is only defined for admissible sequences of length greater than k. Clearly, for each $x \in \mathbb{R}$ we have $S^n(\widehat{\underline{I}}_F(x)) = \widehat{\underline{I}}_F(F^n(x))$ if $|\widehat{\underline{I}}_F(x)| > n$.

Let $\underline{\alpha} = \alpha_1\alpha_2\dots\alpha_n$ and $\underline{\beta} = \beta_1\beta_2\dots$ be two sequences of symbols in $\mathbb{Z}\times\mathcal{S}$. We shall write $\underline{\alpha}\ \underline{\beta}$ to denote the concatenation of $\underline{\alpha}$ and $\underline{\beta}$ (i. e. the sequence $\alpha_1\alpha_2\dots\alpha_n\beta_1\beta_2\dots$). We also shall use the symbols $\underline{\alpha}^n$ to denote $\overbrace{\underline{\alpha}\ \underline{\alpha}\dots\ \underline{\alpha}}^{n\ times}$ and $\underline{\alpha}^\infty$ to denote $\underline{\alpha}\ \underline{\alpha}\dots$.

Let $\underline{\alpha} = \alpha_1\alpha_2\dots\alpha_n$, be a sequence of symbols in $\mathbb{Z}\times\mathcal{S}$. Set $\alpha_i = d_i^{s_i}$ for $i = 1, 2, \dots, n$. We say that $\underline{\alpha}$ is *even* if $\mathrm{Card}\{i \in \{1, \dots, n\} | s_i = R\}$ is even. Otherwise we say that $\underline{\alpha}$ is *odd*.

Now we endow the set of admissible sequences with a total ordering. First set $M < L < C < R$. Then we extend this ordering to $\mathbb{Z}\times\mathcal{S}$ lexicographically. That is, we write $d^s < t^m$ if and only if either $d < t$ or $d = t$ and $s < m$. Let now $\underline{\alpha} = \alpha_1\alpha_2\dots$ and $\underline{\beta} = \beta_1\beta_2\dots$ be two admissible sequences such that $\underline{\alpha} \neq \underline{\beta}$. Then there exists n such that $\alpha_n \neq \beta_n$ and $\alpha_i = \beta_i$ for $i = 1, 2, \dots, n-1$. We say that $\underline{\alpha} < \underline{\beta}$ if either $\alpha_1\alpha_2\dots\alpha_{n-1}$ is even and $\alpha_n < \beta_n$ or $\alpha_1\alpha_2\dots\alpha_{n-1}$ is odd and $\alpha_n > \beta_n$.

The following theorem, proved by Alsedà and Mañosas [3], establishes the characterization of the rotation interval using the kneading pair.

Theorem 5.1. *Let $F \in \mathcal{A}\cup\mathcal{H}$ and $a, a' \in \mathbb{R}$. Then $R_F = [a, a']$ if and only if*

$$\widehat{\underline{I}}_\delta^*(a) \leq \widehat{\underline{I}}_F(0^+) \leq \widehat{\underline{I}}_\epsilon(a)$$

and

$$\widehat{\underline{I}}_\delta^*(a') \leq \widehat{\underline{I}}_{F}(u^-) \leq \widehat{\underline{I}}_\epsilon(a'),$$

where either $u = c_F$ *if* $F \in \mathcal{A}$ *or* $u = 0$ *if* $F \in \mathcal{H}$.

Now, we obtain Theorem 3.4 as a consequence of the above theorem.

Proof of Theorem 3.4. Fix $b \in \mathbb{R}^+$, from Theorem 5.1, for $w \in \mathbb{R}$. we have that $\rho_b^-(w) = a$ and $\rho_b^+(w) = a'$ if and only if

$$\widehat{\underline{I}}_\delta^*(a) \leq K_\epsilon(b, w) \leq \widehat{\underline{I}}_\epsilon(a)$$

and

$$\widehat{\underline{I}}_\delta(a') \leq K_\delta(b, w) \leq \widehat{\underline{I}}_\epsilon^*(a').$$

Then, by using the definition of $\Phi_{a'}^+$, Φ_a^-, $\Psi_{a'}^+$ and Ψ_a^- and Lemma 5.3 the theorem follows.

5.2. **The set of all kneading pairs for maps from $\mathcal{A}$ and $\mathcal{H}$..** To describe the set of all kneading pairs for maps from $\mathcal{A}$ and $\mathcal{H}$ we introduce the following notation. Let $\underline{\alpha} = d_1^{s_1}\alpha_2\ldots$, be an admissible sequence. We will denote by $\underline{\alpha}'$ the sequence $(d_1+1)^{s_1}\alpha_2\ldots$.

Let $\underline{\alpha} \in \mathcal{AD}$. We say that $\underline{\alpha}$ is *minimal* (respectively *maximal*) if and only if $\underline{\alpha} \leq S^n(\underline{\alpha})$ (respectively $\underline{\alpha} \geq S^n(\underline{\alpha})$) for all $n \in \{1, 2, \ldots \mid \underline{\alpha} \mid -1\}$.

We will denote by $\mathcal{E}^*$ the set of all pairs $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{AD} \times \mathcal{AD}$ such that the following conditions hold:

(1) $\underline{\nu}_1' < \underline{\nu}_2$.
(2) $\underline{\nu}_1 \leq S^n(\underline{\nu}_i) \leq \underline{\nu}_2$ for all $n > 0$ and $i \in \{1, 2\}$.
(3) If for some $n \geq 0$, $S^n(\underline{\nu}_i) = d^R\ldots$, then $S^{n+1}(\underline{\nu}_i) \geq \underline{\nu}_1'$ for $i \in \{1, 2\}$.

We note that condition (2) says, in particular, that $\underline{\nu}_1$ and $\underline{\nu}_2$ are minimal and maximal, respectively, according the following definition. Let $\underline{\alpha} \in \mathcal{AD}$, we say that $\underline{\alpha}$ is *minimal* (respectively *maximal*) if and only if $\underline{\alpha} \leq S^n(\underline{\alpha})$ (respectively $\underline{\alpha} \geq S^n(\underline{\alpha})$) for all $n \in \{1, 2, \ldots \mid \underline{\alpha} \mid -1\}$.

As we will see, the above set contains (among others) the kneading pairs of maps from $\mathcal{A}$ with non–degenerate rotation interval. To deal with some special kneading pairs associated to maps with degenerate rotation interval we introduce the following sets.

When $a = p/q$ with $(p, q) = 1$ we denote by $\widehat{\underline{I}}_R(a)$ the sequence

$$(\delta_1(a)^L \ldots \delta_{q-1}(a)^L \delta_q(a)^R)^\infty$$

and by $\widehat{\underline{I}}_R^*(a)$ the sequence which satisfies $(\widehat{\underline{I}}_R^*(a))' = \widehat{\underline{I}}_R(a)$.

Now we set

$$\mathcal{E}_a = \begin{cases} \{(\widehat{\underline{I}}_\epsilon(a), \widehat{\underline{I}}_\epsilon^*(a)), (\widehat{\underline{I}}_\delta^*(a), \widehat{\underline{I}}_\delta(a)), (\widehat{\underline{I}}_R^*(a), \widehat{\underline{I}}_R(a))\} & \text{if } a = p/q \in \mathbb{Q}, \text{ with } (p,q) = 1, \\ \left\{(\widehat{\underline{I}}_\delta^*(a), \widehat{\underline{I}}_\delta(a))\right\} & \text{if } a \notin \mathbb{Q}. \end{cases}$$

and

$$\widehat{\mathcal{E}}_a = \begin{cases} \{(\widehat{\underline{I}}_\epsilon(a), \widehat{\underline{I}}_\epsilon^*(a)), (\widehat{\underline{I}}_\delta^*(a), \widehat{\underline{I}}_\delta(a)), (\widehat{\underline{I}}_\epsilon(a), \widehat{\underline{I}}_\delta(a))\} & \text{if } a = p/q \in \mathbb{Q}, \text{ with } (p,q) = 1, \\ \left\{(\widehat{\underline{I}}_\delta^*(a), \widehat{\underline{I}}_\delta(a))\right\} & \text{if } a \notin \mathbb{Q}, \end{cases} .$$

Finally we denote by $\mathcal{E}$ the set $\mathcal{E}^* \cup (\cup_{a\in\mathbb{R}}\mathcal{E}_a)$ and by $\widehat{\mathcal{E}}$ the set $\cup_{a\in\mathbb{R}}\widehat{\mathcal{E}}_a$. From [1, Theorem A] we have that $\mathcal{E}$ (respectively, $\widehat{\mathcal{E}}$) is the set of all kneading pairs for maps from $\mathcal{A}$ (respectively, $\mathcal{H}$).

5.3. **Proof of Proposition 3.5.** The sequences defined above satifisfy the following lemma.

Lemma 5.2. *Let $a \in \mathbb{R}$. If $a \notin \mathbb{Q}$ then $\widehat{\underline{I}}^*_\delta(a) = \widehat{\underline{I}}_\epsilon(a)$ and $\widehat{\underline{I}}_\delta(a) = \widehat{\underline{I}}^*_\epsilon(a)$. Otherwise, if $a \in \mathbb{Q}$ then $\widehat{\underline{I}}^*_\delta(a) < \widehat{\underline{I}}_\epsilon(a)$ and $\widehat{\underline{I}}_\delta(a) < \widehat{\underline{I}}^*_\epsilon(a)$.*

Let $\underline{\alpha} \in \mathcal{AD}$. We will say that $\underline{\alpha} \in \mathcal{E}_\epsilon$ if and only if it is minimal and satisfies that if for some $n \geq 0$, $S^n(\underline{\alpha}) = d^R \ldots$ then $S^{n+1}(\underline{\alpha}) \geq \underline{\alpha}'$. In a similar way, $\underline{\alpha} \in \mathcal{E}_\delta$ if and only if it is maximal. From [11, Theorem 3.1.1] it follows that

$$\mathcal{E}_\epsilon = \{\underline{\alpha} \in \mathcal{AD} : \exists \underline{\beta} \in \mathcal{AD} \text{ such that } (\underline{\alpha}, \underline{\beta}) \in \mathcal{E}\}$$

and

$$\mathcal{E}_\delta = \{\underline{\beta} \in \mathcal{AD} : \exists \underline{\alpha} \in \mathcal{AD} \text{ such that } (\underline{\alpha}, \underline{\beta}) \in \mathcal{E}\}$$

We consider $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ endowed with the order topology and let $\mathcal{E}_\epsilon \times \mathcal{E}_\delta$ be with the product topology. Note that $\mathcal{E}$ and $\widehat{\mathcal{E}}$ are strictly contained in $\mathcal{E}_\epsilon \times \mathcal{E}_\delta$ Now, we consider the maps $K_\epsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathcal{E}_\epsilon$ and $K_\delta : \mathbb{R}^+ \times \mathbb{R} \to \mathcal{E}_\delta$. Then we have the following.

Lemma 5.3. *The following statements hold.*

(1) *The maps K_ϵ and K_δ are continuous.*
(2) *Let $a \in \mathbb{R}$. For each $b \in \mathbb{R}^+$ there exist $w_1^\epsilon, w_2^\epsilon \in \mathbb{R}$ (respectively, $w_1^\delta, w_2^\delta \in \mathbb{R}$) such that $K_\epsilon(b, w_1^\epsilon) = \widehat{\underline{I}}_\epsilon(a)$ and $K_\epsilon(b, w_2^\epsilon) = \widehat{\underline{I}}^*_\delta(a)$ (respectively, $K_\delta(b, w_1^\delta) = \widehat{\underline{I}}_\delta(a)$ and $K_\epsilon(b, w_2^\delta) = \widehat{\underline{I}}^*_\epsilon(a)$).*

Proof. The first assertion follows using the continuous dependence of $F_{b,w}$ from the parameter values b and w. Now, from Theorem 5.1, we have that

$$\mathcal{E}_\epsilon = \cup_{a\in\mathbb{R}} \left[\widehat{\underline{I}}^*_\delta(a), \widehat{\underline{I}}_\epsilon(a)\right]. \tag{9}$$

Fixed $b \in \mathbb{R}^+$. Assume first that $a \notin \mathbb{Q}$,. From Proposition 3.1, there exists $w_1^\epsilon \in \mathbb{R}$ such that $\rho_b^-(w_1^\epsilon) = a$. Then, by using Lemma 5.2 and [3, Theorem B], $K_\epsilon(b, w_1^\epsilon) = \widehat{\underline{I}}_\epsilon(a) = \widehat{\underline{I}}^*_\delta(a)$. Now, let $a \in \mathbb{Q}$. Since ρ_b^- is a continuous non–increasing onto map, from Theorem 5.1, there exists $w \in \mathbb{R}$ such that $K_\epsilon(b, w) \in \left[\widehat{\underline{I}}^*_\delta(a), \widehat{\underline{I}}_\epsilon(a)\right]$. Let $a', a'' \notin \mathbb{Q}$, satisfying that $a' < a < a''$, we have from the above case that there exist $w_1^\epsilon(a'), w_1^\epsilon(a'') \in R$ such that $K_\epsilon(b, w_1^\epsilon(a')) = \widehat{\underline{I}}_\epsilon(a') = \widehat{\underline{I}}^*_\delta(a')$ and $K_\epsilon(b, w_1^\epsilon(a'')) = \widehat{\underline{I}}_\epsilon(a'') = \widehat{\underline{I}}^*_\delta(a'')$. Using that $\left[\widehat{\underline{I}}^*_\delta(a), \widehat{\underline{I}}_\epsilon(a)\right]$ is strictly contained in $\left[\widehat{\underline{I}}_\epsilon(a'), \widehat{\underline{I}}_\epsilon(a'')\right]$ and the continuity of $K_\epsilon(b, \cdot)$, statement (2) follows. ∎

Proof of Proposition 3.5. From Theorem 5.1 we have that

$$A_\epsilon(a) = K_\epsilon^{-1}\left(\left[\widehat{\underline{I}}^*_\delta(a), \widehat{\underline{I}}^*_\epsilon(a)\right]\right)$$

and

$$A_\delta(a) = K_\delta^{-1}\left(\left[\widehat{\underline{I}}_\delta(a), \widehat{\underline{I}}^*_\epsilon(a)\right]\right).$$

Since $\left[\widehat{\underline{I}}_\delta^*(a), \widehat{\underline{I}}_\epsilon(a)\right]$ and $\left[\widehat{\underline{I}}_\delta(a), \widehat{\underline{I}}_\epsilon^*(a)\right]$ are closed sets in $\mathcal{E}_\epsilon$ and $\mathcal{E}_\epsilon$, respectively, then, by Lemma 5.3, $A_\epsilon(a)$ and $A_\delta(a)$ are closed sets in $\mathbb{R}^+ \times \mathbb{R}$. This ends the first statement of proposition. To prove the sencond one we observe that

$$\bigcup_{-\infty < a \leq a' < \infty} (A_\epsilon(a) \cup A_\delta(a')) \subset \mathbb{R}^+ \times \mathbb{R}..$$

Now, let $(b, w) \in \mathbb{R}^+ \times \mathbb{R}$. be such that $R_{F_{b,w}} = [\rho_b^-(w), \rho_b^+(w)]$, then $(b, w) \in A_\epsilon(\rho_b^-(w)) \cup A_\delta(\rho_b^+(w))$ and proposition folows.

5.4. **Proof of Theorem B.** From the assumptions of Theorem B we have the following.

Lemma 5.4. *For each $i \in \mathbb{N}$ the map $b \to E(G_b^i(0))$ from $(1, \infty)$ to $\mathbb{Z}$ is non–increasing and the map $b \to E(G_b^i(c_b))$ from $(1, \infty)$ to $\mathbb{Z}$ is non–decreasing.*

Lemma 5.5. *Let $a \notin \mathbb{Q}$. Assume that $\rho((G_b)_l) = a$ (respectively, $\rho((G_b)_u) = a$) for some $b \in (1, \infty)$. If for some $b_0 > b$ we have that $\rho((G_{b_0})_l) \neq a$ (respectively, $\rho((G_{b_0})_u) \neq a$), then $\rho((G_{b'})_l) < a$ (respectively, $\rho((G_{b'})_u) > a$) for all $b' > b_0$.*

Proof. First at all, we note that $\widehat{\underline{I}}_{G_b}(0) = \widehat{\underline{I}}_\epsilon(a) = \widehat{\underline{I}}_\delta^*(a)$. Assume that for some $b' > b_0$ we have that $\rho((G_{b'})_l) = a$, then $\widehat{\underline{I}}_{G_{b'}}(0) = \widehat{\underline{I}}_\epsilon(a) = \widehat{\underline{I}}_\delta^*(a)$. Thus, $E(G_b^i(0)) = E(G_{b'}^i(0))$ for all $i \in \mathbb{N}$. By using Lemma 5.5 we have that $E(G_b^i(0)) = E(G_{b''}^i(0))$ for all $b'' \in [b, b']$. In consequence, $\widehat{\underline{I}}_{G_{b''}}(0) = \widehat{\underline{I}}_\epsilon(a)$ for all $b'' \in [b, b']$, a contradiction. ∎

Proof of Theorem **??**. Assume first that $a \notin \mathbb{Q}$, the case $a' \notin \mathbb{Q}$ follows in a similar way. Note that $b \mapsto (F_{b,w})_l(0) = w + (G_b)_l(0)$, recall Remark 3.1, is strictly decreasing. Thus, by usig the persistence of the endpoinds of the rotation interval (see [20]), the map $b \mapsto \rho((F_{b,w})_l)$ satisfies that if for some $b \in (0, \infty)$ we have that $\rho((F_{b,w})_l) \notin \mathbb{Q}$, then

$$\rho((F_{b',w})_l) < \rho((F_{b,w})_l) < \rho((F_{b'',w})_l)$$

for all $0 < b' < b < b''$. Let $w = \Phi_a^-(b) = \Phi_{a'}^+(b)$ for some $b \in (1, \infty)$. Then $\rho_b^-(w) = a < a' = \rho_b^+(w)$. From Proposition **??**(a) and (b) we have that $\Psi_{\epsilon,a}(b) = \Psi_{\delta,a}(b)$. Thus, $K_\epsilon(b, w) = \widehat{\underline{I}}_\epsilon(a) = \widehat{\underline{I}}_\delta^*(a)$ and $K_\delta(b, w) = \widehat{\underline{I}}_\delta(a) = \widehat{\underline{I}}_\epsilon^*(a)$, a contradiction. Let $\Phi_{\epsilon,a}(b) < \Phi_{\delta,a}(b)$. Then $\rho_b^+(\Phi_{\epsilon,a}(b)) < \rho_b^+(\Phi_{\delta,a}(b)) = a$. Since $a = \rho_b^-(\Phi_{\epsilon,a}(b)) \leq \rho_b^+(\Phi_{\epsilon,a}(b))$, we have a contradiction and statement (a) follows. Now, we prove statement (b), (c) follows in a similar way. We start by proving that (b.1) is equivalent to (b.2). The proof that (b.1) is equivalent to (b.3) is quite similar. Let $\Psi_{\epsilon,a}(b) = w_0$ and $\Psi_{\delta,a}(b) = w$. Assume that $w < w_0$. Then ,from Proposition we have $\rho_b^-(w) < a = \rho_b^-(w_0)$. Now, assume that $\rho_b^-(w) < \rho_b^-(w_0)$, then, from Lemma 3.1, $w < w_0$ and statement (b) follows.

5.5. **Proof of Theorem C.** To prove Theorem C we shall use the following proposition of [3].

Proposition 5.6. *Let $F \in A$ and let P be a TPO with rotation number a and assume that $a = p/q \in \mathbb{Q}$, with $(p, q) = 1$. Then the following statements hold.*

1. $\{0, c_F\} \nsubseteq P_0$.
2. *Assume that $\nu_P \neq c_F$. If $\mu_P \neq 0$ then $\widehat{\underline{I}}_F(\mu_P) = \widehat{\underline{I}}_\epsilon(a)$. Otherwise, $\widehat{\underline{I}}_F(\mu_P) = \widehat{\underline{I}}_F(0) = \epsilon_1(a)^L \epsilon_2(a)^L \ldots \epsilon_q(a)^M$ and $\widehat{\underline{I}}_F(0^+) = \widehat{\underline{I}}_\epsilon(a)$.*

(3) *Assume that* $\mu_P \neq 0$. *If* $\nu_P \neq c_F$ *then* $\widehat{\underline{I}}_F(\nu_P) = \widehat{\underline{I}}_\delta(a)$. *Otherwise,* $\widehat{\underline{I}}_F(\nu_P) = \widehat{\underline{I}}_F(c_F) = \delta_1(a)^L \delta_2(a)^L \dots \delta_q(a)^M$ *and* $\widehat{\underline{I}}_F(c_F^-) = \widehat{\underline{I}}_\delta(a)$.

6. Algunas sobras

Theorem 6.1. *The following statements hold.*

(1) *For all* $b > 1$ *we have that* $\Psi_a^-(b) < \Phi_a^-(b)$ *and* $\Phi_a^+(b) < \Psi_a^+(b)$.

(2) *If for a given* $b \in (0,1]$ *we have that for all* $w \in \mathbb{R}$ *where* $\rho(F_{b,w}) = a$, $F_{b,w}(x) \neq x + a$ *(respectively,* $F_{b,w}(x) = x + a$*) for all* $x \in \mathbb{R}$. *Then*

$$\Phi_a^+(b) = \Psi_a^+(b) < \Psi_a^-(b) = \Phi_a^-(b)$$

(respectively,

$$\Psi_a^-(b) = \Phi_a^-(b) = \Phi_a^+(b) = \Psi_a^+(b).$$

(3) *Let* $a \notin \mathbb{Q}$. *Then* $\Psi_a^-(b) = \Phi_a^-(b)$ *and* $\Phi_a^+(b) = \Psi_a^+(b)$ *for all* $b > 0$.

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Departamento de Matemáticas, Campus de Espinardo, Murcia, SPAIN
E-mail address: balibrea@fcm.um.es

Fundación Universitaria San Pablo, Carrer Comissari 1, 03202 Elche, SPAIN
E-mail address: falco@mat.uab.es