Proofs of the paper: an entropy formula for a class of circle maps

by

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1 Preliminary results

For each $i \in \{1, 2, \ldots, 2p\}$, we define the *i-th kneading invariant of* F to be $\nu(c_i) = \kappa_F(c_i^+) - \kappa_F(c_i^-)$. Note that $\nu(c_i)$ is a power series with coefficients in $\mathbf{Z}[[\mathcal{V}_F]]$. Thus we can write $\nu(c_i) = \sum_{j=1}^{n} \nu_i^j(t) I_j$ with $\nu_i^j(t) \in \mathbf{Z}[[t]]$ for all i, j. The $(2p+1) \times (2p)$ matrix $\mathcal{K}_F(t) = (\nu_i^j(t))$ will be called the *kneading matrix of* F. Let $D_F^i(t)$ be the determinant which is obtained by deleting the i-th row of $\mathcal{K}_F(t)$.

The expression

$$\mathcal{D}_F(t) = \frac{(-1)^{i+1}}{(1-\epsilon(I_i)t)} D_F^i(t)$$

will be called the *kneading determinant of* F. It is well known [4] (see also [2]) that the above expression does not depend on i. Thus, the kneading determinant of F is well defined. Then, from [4] and [3] (see also [2]) we obtain the following result.

Theorem 1.1 *Let $F \in \mathcal{M}$. If $\mathcal{D}_F(t)$ does not vanish in $(0,1)$ then $h(F) = 0$. Otherwise, $h(F) = \log \frac{1}{\alpha}$ where α is the smallest zero of $\mathcal{D}_F(t)$ in $(0,1)$.*

Theorem 1.1 is the key point to prove Theorem 3.3. This is the main goal of the next section.

2 Proofs of the results

We start by proving Theorem 3.3. In view of Theorem 1.1 we only have to show that the zeroes of $K_F(t) \cdot P_F(t)$ and $\mathcal{D}_F(t)$ in $(0,1)$ coincide. Before starting the proof of Theorem 3.3 we shall compute the kneading invariants of the map under consideration. Since $c_{p+1} = c_F$ we have that $\nu(c_{p+1}) = \nu(c_F) = \kappa(c_F^+) - \kappa(c_F^-) = K_F(t)I_{p+1} + \widetilde{K}_F(t)I_{p+2}$. The next lemma takes care of the computation of the rest of kneading invariants.

Lemma 2.1 *For each $F \in \mathcal{M}$ we have $\nu(c_i) = I_{i+1} - I_i + R_F(t)$ for $i \neq p_F + 1$.*

Proof. First we compute $\nu(c_i)$ with $i \in \{1, 2, \ldots, p\}$. Since F is increasing in a neighborhood of c_i, $F(c_i) \in \mathbf{Z}$ and, $\kappa(x^+) = \kappa((x+m)^+)$ and $\kappa(x^-) = \kappa((x+m)^-)$ for all $x \in \mathbf{R}$ and $m \in \mathbf{Z}$ we have that $\kappa(c_i^+) = I_{i+1} + \epsilon(I_{i+1})t\kappa(0^+)$ and $\kappa(c_i^-) = I_i + \epsilon(I_i)t\kappa(0^-)$. Since $i \leq p$ we have $\epsilon(I_{i+1}) = \epsilon(I_i) = 1$ and, hence, $\nu(c_i) = I_{i+1} - I_i + t[\kappa(0^+) - \kappa(0^-)] = I_{i+1} - I_i + R_F(t)$. When $i \in \{p+2, \ldots, 2p\}$, since F is decreasing in a neighborhood of c_i, in a similar way we have

$\kappa(c_i^+) = I_{i+1} + \epsilon(I_{i+1})t\kappa(0^-)$ and $\kappa(c_i^-) = I_i + \epsilon(I_i)t\kappa(0^+)$. Now we have $\epsilon(I_{i+1}) = \epsilon(I_i) = -1$ and, hence, $\nu(c_i) = I_{i+1} - I_i - t[\kappa(0^-) - \kappa(0^+)] = I_{i+1} - I_i + R_F(t)$. ■

Proof of Theorem 3.3. We recall that $R_F(t) = \sum_{i=1}^{2p+1} \phi_i(t) I_i$ (in this proof p_F will be denoted by p for simplicity). Then, by Lemma 2.1, we have that $\mathcal{K}_F(t)$ is (in the following matrices, again for simplicity, $\phi_i(t)$ will be denoted by ϕ_i)

$$\begin{pmatrix}
\phi_1 - 1 & \phi_1 & \cdots & \phi_1 & 0 & \phi_1 & \phi_1 & \cdots & \phi_1 & \phi_1 \\
\phi_2 + 1 & \phi_2 - 1 & \cdots & \phi_2 & 0 & \phi_2 & \phi_2 & \cdots & \phi_2 & \phi_2 \\
\phi_3 & \phi_3 + 1 & \cdots & \phi_3 & 0 & \phi_3 & \phi_3 & \cdots & \phi_3 & \phi_3 \\
\phi_4 & \phi_4 & \cdots & \phi_4 & 0 & \phi_4 & \phi_4 & \cdots & \phi_4 & \phi_4 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_p & \phi_p & \cdots & \phi_p - 1 & 0 & \phi_p & \phi_p & \cdots & \phi_p & \phi_p \\
\phi_{p+1} & \phi_{p+1} & \cdots & \phi_{p+1} + 1 & K_F(t) & \phi_{p+1} & \phi_{p+1} & \cdots & \phi_{p+1} & \phi_{p+1} \\
\phi_{p+2} & \phi_{p+2} & \cdots & \phi_{p+2} & \widetilde{K}_F(t) & \phi_{p+2} - 1 & \phi_{p+2} & \cdots & \phi_{p+2} & \phi_{p+2} \\
\phi_{p+3} & \phi_{p+3} & \cdots & \phi_{p+3} & 0 & \phi_{p+3} + 1 & \phi_{p+3} - 1 & \cdots & \phi_{p+3} & \phi_{p+3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{2p} & \phi_{2p} & \cdots & \phi_{2p} & 0 & \phi_{2p} & \phi_{2p} & \cdots & \phi_{2p} + 1 & \phi_{2p} - 1 \\
\phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & 0 & \phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & \phi_{2p+1} + 1
\end{pmatrix}.$$

Now, $D_F^{p+2}(t)$ is equal to

$$\begin{vmatrix}
\phi_1 - 1 & \phi_1 & \cdots & \phi_1 & 0 & \phi_1 & \phi_1 & \cdots & \phi_1 & \phi_1 \\
\phi_2 + 1 & \phi_2 - 1 & \cdots & \phi_2 & 0 & \phi_2 & \phi_2 & \cdots & \phi_2 & \phi_2 \\
\phi_3 & \phi_3 + 1 & \cdots & \phi_3 & 0 & \phi_3 & \phi_3 & \cdots & \phi_3 & \phi_3 \\
\phi_4 & \phi_4 & \cdots & \phi_4 & 0 & \phi_4 & \phi_4 & \cdots & \phi_4 & \phi_4 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_p & \phi_p & \cdots & \phi_p - 1 & 0 & \phi_p & \phi_p & \cdots & \phi_p & \phi_p \\
\phi_{p+1} & \phi_{p+1} & \cdots & \phi_{p+1} + 1 & K_F(t) & \phi_{p+1} & \phi_{p+1} & \cdots & \phi_{p+1} & \phi_{p+1} \\
\phi_{p+3} & \phi_{p+3} & \cdots & \phi_{p+3} & 0 & \phi_{p+3} + 1 & \phi_{p+3} - 1 & \cdots & \phi_{p+3} & \phi_{p+3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{2p} & \phi_{2p} & \cdots & \phi_{2p} & 0 & \phi_{2p} & \phi_{2p} & \cdots & \phi_{2p} + 1 & \phi_{2p} - 1 \\
\phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & 0 & \phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & \phi_{2p+1} + 1
\end{vmatrix}_{2p} =$$

$$(-1)^{2(p+1)} K_F(t) \begin{vmatrix}
\phi_1 - 1 & \phi_1 & \cdots & \phi_1 & \phi_1 & \phi_1 & \cdots & \phi_1 & \phi_1 \\
\phi_2 + 1 & \phi_2 - 1 & \cdots & \phi_2 & \phi_2 & \phi_2 & \cdots & \phi_2 & \phi_2 \\
\phi_3 & \phi_3 + 1 & \cdots & \phi_3 & \phi_3 & \phi_3 & \cdots & \phi_3 & \phi_3 \\
\phi_4 & \phi_4 & \cdots & \phi_4 & \phi_4 & \phi_4 & \cdots & \phi_4 & \phi_4 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_p & \phi_p & \cdots & \phi_p - 1 & \phi_p & \phi_p & \cdots & \phi_p & \phi_p \\
\phi_{p+3} & \phi_{p+3} & \cdots & \phi_{p+3} & \phi_{p+3} + 1 & \phi_{p+3} - 1 & \cdots & \phi_{p+3} & \phi_{p+3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{2p} & \phi_{2p} & \cdots & \phi_{2p} & \phi_{2p} & \phi_{2p} & \cdots & \phi_{2p} + 1 & \phi_{2p} - 1 \\
\phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & \phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & \phi_{2p+1} + 1
\end{vmatrix}_{2p-1} =$$

$$
K_F(t)\left|\begin{array}{cccccc|cccccc}
\phi_1-1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\
\phi_2+1 & -2 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1\\
\phi_3 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_4 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\phi_{p-1} & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_p & 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0\\
\cline{2-12}
\phi_{p+3} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0\\
\phi_{p+4} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\phi_{2p-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0\\
\phi_{2p} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1\\
\phi_{2p+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right|_{2p-1}.
$$

If $p = 1$ then it follows that

$$
D_F^{p+2}(t) = K_F(t)(\phi_1(t) - 1) = K_F(t) \cdot P_F(t).
$$

Now, suppose that $p \geq 2$. Then by adding the first row of the determinant to the second one we get

$$
D_F^{p+2}(t) = K_F(t)\left|\begin{array}{cccccc|cccccc}
\phi_1-1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\
\phi_2+\phi_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_3 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_4 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\phi_{p-1} & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_p & 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0\\
\cline{2-12}
\phi_{p+3} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0\\
\phi_{p+4} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\phi_{2p-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0\\
\phi_{2p} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1\\
\phi_{2p+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right|_{2p-1}.
$$

Let $u_k = \sum_{i=k}^{2p+1} \phi_i$ for $k = p+3, p+4, \ldots, 2p+1$. Then we have that

$$
D_F^{p+2}(t) = K_F(t)\left|\begin{array}{ccccc|cccccc}
\phi_1-1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\
\phi_2+\phi_1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_3 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_4 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\phi_{p-1} & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_p & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0\\
\cline{2-11}
u_{p+3} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0\\
u_{p+4} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
u_{2p} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0\\
u_{2p+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right|_{2p-1}.
$$

Now we add the p-th column of the above determinant to the $(p-1)$-th one. Then we add the $(p-1)$-th column of the new determinant to the $(p-2)$-th one and, by iterating this process $p-2$ times we get

$$
D_F^{p+2}(t) = K_F(t) \left| \begin{array}{cccccc|cccccc}
\phi_1 - 1 & p-1 & p-2 & \cdots & 2 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
\phi_2 + \phi_1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_3 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_{p-1} & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_p & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cline{2-11}
u_{p+3} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
u_{p+4} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
u_{2p} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
u_{2p+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array} \right|_{2p-1}.
$$

Let $u = \phi_1 - 1 - \sum_{k=p+3}^{2p+1} u_k$. Then

$$
D_F^{p+2}(t) = K_F(t) \left| \begin{array}{cccccc|cccccc}
u & p-1 & p-2 & \cdots & 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_2 + \phi_1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_3 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_{p-1} & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_p & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cline{2-11}
u_{p+3} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
u_{p+4} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
u_{2p-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
u_{2p} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array} \right|_{2p-1}.
$$

We note that $\sum_{k=p+3}^{2p+1} u_k = \sum_{k=p+3}^{2p+1} \sum_{i=k}^{2p+1} \phi_i(t) = \sum_{i=p+3}^{2p+1} (i-p-2)\phi_i(t)$. Therefore,

$$
\begin{aligned}
P_F(t) &= -1 + \sum_{i=1}^{p} (p-i+1)\phi_i(t) - \sum_{i=p+3}^{2p+1} (i-p-2)\phi_i(t) \\
&= -1 + \phi_1(t) + (p-1)(\phi_1(t) + \phi_2(t)) + \sum_{i=3}^{p} (p-i+1)\phi_i(t) - \sum_{k=p+3}^{2p+1} u_k \\
&= u + (p-1)(\phi_1(t) + \phi_2(t)) + \sum_{i=3}^{p} (p-i+1)\phi_i(t).
\end{aligned}
$$

Thus

$$
D_F^{p+2}(t) = K_F(t) \left| \begin{array}{cccccc|cccccc}
P_F(t) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_2 + \phi_1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_3 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_{p-1} & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_p & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cline{2-11}
u_{p+3} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
u_{p+4} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
u_{2p-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
u_{2p} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array} \right|_{2p-1}.
$$

Hence, $D_F^{p+2}(t)$ is equal to $(-1)^{p-1}K_F(t)\cdot P_F(t)$.

Since

$$\mathcal{D}_F(t)=\frac{(-1)^{p+3}}{(1-\epsilon(I_{p+2})t)}D_F^{p+2}(t)=\frac{(-1)^{p+3}}{(1-t)}D_F^{p+2}(t)$$

we have that the zeroes of $\mathcal{D}_F(t)$ and $D_F^{p+2}(t)$ in $(0,1)$ coincide. Therefore, the zeroes of $\mathcal{D}_F(t)$ and $K_F(t)\cdot P_F(t)$ in $(0,1)$ are the same. This ends the proof of Theorem 3.3. ■

Proof of Corollary 3.4. The rotation interval of each map $F\in\mathcal{M}$ is of the form $[c,d_F]$ with $d_F=E(F(c_F))$. By Theorem B of [1] we get that $h(F)\geq\log\beta_{d_F-c}$ where β_{d_F-c} is the largest root of the equation

$$z+1-2\sum_{n=0}^{\infty}z^{-E(n/(d_F-c))}=0.$$

In view of Theorem C.(c) and Lemma 22 of [1] we obtain that β_{d_F-c} is larger than or equal to the largest root of the equation $x^3-x^2-3x+1=0$. This root is $2.1700864866\ldots$. This ends the proof of the corollary. ■

Proof of Corollary 3.6. For such a family of maps one has $w_\mu<b$ and $E(F_\mu(c_F))=E(F_\mu(w_\mu))$ for each $\mu\in[\mu_0,\mu_1]$. Therefore, $A_{F_\mu}(F_\mu^n(c_{F_\mu}))=I_{p+2}$ for all $n\geq 1$ and, hence, $\kappa(c_{F_\mu}^-)=I_{p+1}+\sum_{i=1}^{\infty}(-1)^{i-1}t^iI_{p+2}$. Thus, by Remark 3.2, $K_{F\mu}(t)=-1$. From Theorem 3.3, we obtain the desired result. ■

References

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