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ON THE STRUCTURE OF THE KNEADING SPACE OF BIMODAL DEGREE ONE CIRCLE MAPS

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Dedicated to the memory of Valery S. Melnik

In this paper, we introduce an index space and two $\star$-like operators that can be used to describe bifurcations for parametrized families of degree one circle maps. Using these topological tools, we give a description of the kneading space, that is, the set of all dynamical combinatorial types for the class of all bimodal degree one circle maps considered as dynamical systems.

Keywords: Kneading theory; degree one circle maps; bifurcation space.

1. Introduction and Statement of Main Theorem

For continuous maps on the interval with finitely many monotonicity intervals, the kneading theory developed by Milnor and Thurston [1988] gives a symbolic description of the dynamics of these maps. This description is given in terms of the kneading invariants which essentially consist of the symbolic orbits of the turning points of the map under consideration. Moreover, this theory also gives a classification of all such maps through these invariants. For continuous bimodal degree one circle maps similar invariants were introduced by Alsedà and Mañosas [1990]. In that paper, the first part of the program just described was carried through, and relations between the circle maps invariants and the rotation interval were elucidated. Later on, in [Alsedà & Falcó, 1997, Theorem A] the set of all these kneading invariants (the *kneading space*) was characterized. The main goal of this paper is to give a description of the kneading space of the bimodal degree one circle maps using some self-similarity operators which allow us to identify certain subsets with known structure. To state this description we need the appropriate notation. This paper is, in some sense, a continuation of [Alsedà & Falcó, 1997] and we use heavily the notation and results from that paper. Although we have tried to make this paper self-contained in the introduction we have repeated certain definitions in [Alsedà & Falcó, 1997] for readability.

As it is usual, instead of working with the circle maps themselves we will rather use their liftings to the universal covering space $\mathbb{R}$. To this end, we

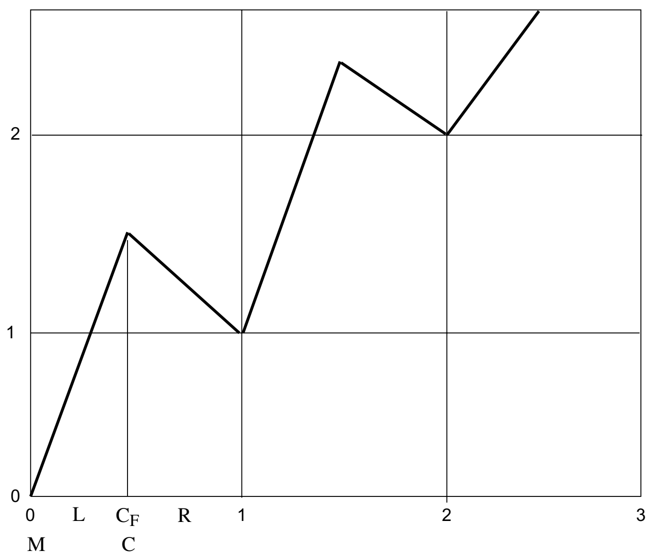

Fig. 1. An example of a map F in class $\mathcal{A}$.

introduce the following class $\mathcal{A}$ of maps. First, we define $\mathcal{L}$ to be the class of all continuous maps F from $\mathbb{R}$ into itself such that $F(x+1) = F(x) + 1$ for all $x \in \mathbb{R}$. That is, $\mathcal{L}$ is the class of all liftings of degree one circle maps. Then we will say that $F \in \mathcal{A}$ if (see Fig. 1):

(1) $F \in \mathcal{L}$.
(2) There exists $c_F \in (0,1)$ such that F is strictly increasing in $[0, c_F]$ and strictly decreasing in $[c_F, 1]$.

We note that every map $F \in \mathcal{A}$ has a unique local maximum and a unique local minimum in $[0,1)$. To define the class $\mathcal{A}$ we restricted ourselves to the case in which F has the minimum at 0. Since each map from $\mathcal{L}$ is conjugate by a translation to a map from $\mathcal{L}$ having the minimum at 0, the fact that in (2) we fix that F has a minimum in 0 is not restrictive.

For a map $F \in \mathcal{A}$ one can define the *kneading pair* denoted by $\mathcal{K}(F)$ (see Definition 2.4) which captures all dynamics of the map F (see [Alsedà & Mañosas, 1990, Proposition A]). The kneading space is a subset of the product space $\mathcal{E}_\epsilon \times \mathcal{E}_\delta$ where both $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ are totally ordered spaces equipped with the order topology (see Sec. 2.1). Also, the set of all kneading pairs will be called the *kneading space*. Now, we introduce the following index space. It will be used to characterize a class of basic subsets from each component of the kneading space.

Definition 1.1. Let $\mathcal{J}$ be the index space whose elements are sequences $\underline{x} = \{x_j\}_{j=1}^n$ with terms in $[0,1] \cup \{\delta, \epsilon\}$, where $n \in \mathbb{N} \cup \{\infty\}$ and either $n < \infty$, $\{x_j\}_{j=1}^{n-1} \subset \mathbb{Q} \cap (0,1)$ and $x_n \in \{0,1,\delta,\epsilon\} \cup \{(0,1) \cap (\mathbb{R}\backslash\mathbb{Q})\}$ or $n = \infty$ and $\{x_j\}_{j=1}^n \subset \mathbb{Q}\backslash\mathbb{Z}$.

Consider the set $\mathcal{J}$ endowed with the lexicographical ordering induced by the usual ordering of the real numbers and the following ordering:

$$\delta < 0 < 1 < \epsilon.$$

The ordering of $\mathcal{J}$ will denoted by $\prec$. Let $\mathcal{J}^\infty$ be the subset of $\mathcal{J}$ which contains all the infinite sequences in $\mathcal{J}$, and let $\mathcal{J}_\epsilon^*$ (resp., $\mathcal{J}_\delta^*$) be the set of all finite sequences $\{x_j\}_{j=1}^N$ in $\mathcal{J}$ such that the last term is either $x_N \in (0,1) \cap (\mathbb{R}\backslash\mathbb{Q})$ or $x_N = \epsilon$ (resp., $x_N = \delta$). Finally, set $\mathcal{J}_\epsilon = \ J^\infty \cup \mathcal{J}_\epsilon^*$ and $\mathcal{J}_\delta = \mathcal{J}^\infty \cup \mathcal{J}_\delta^*$. Note that $\mathcal{J}_\epsilon$ (resp., $\mathcal{J}_\delta$) has as maximum the finite sequence ϵ (resp., 1) and as minimum 0 (resp., δ). Also we denote by $\mathcal{I}$ the set of all finite sequences which do not end with $0, 1, \epsilon$ or δ, union the empty sequence.

Now, we are ready to state the main result of this paper. A crucial observation to the next theorem is that all maps appearing in it are defined in a constructive way using four symbolic operators to be defined in Sec. 3.

Main Theorem. *For $F \in \mathcal{A}$ there exist $a, b \in \mathbb{R}$, $a \le b$, and two closed intervals $Q_\epsilon(a)$ in $\mathcal{E}_\epsilon$ and $Q_\delta(b)$ in $\mathcal{E}_\delta$ such that $\mathcal{K}(F) \in Q_\epsilon(a) \times Q_\epsilon(b)$. Moreover, the numbers a and b are the endpoints of the rotation interval of F and the following statements hold.*

(a) *There exists $p_{\epsilon,a}\colon \mathcal{J}_\epsilon \to Q_\epsilon(a)$ which is nondecreasing, maps the endpoints of $\mathcal{J}_\epsilon$ into the endpoints of $Q_\epsilon(a)$ and if $a \in \mathbb{Q}\backslash\mathbb{Z}$ then $p_{\epsilon,a}$ is one-to-one. Moreover,*

$$\operatorname{Im}(p_{\epsilon,a}) = Q_\epsilon(a) \Big\backslash \bigcup_{\underline{x} \in \mathcal{I}} (p_{\epsilon,a}(\underline{x}1), p_{\epsilon,a}(\underline{x}\epsilon)) .$$

(b) *There exists*

$$\mathcal{P}_\epsilon(a) \subset \bigcup_{\underline{x} \in \mathcal{I}} [p_{\epsilon,a}(\underline{x}1), p_{\epsilon,a}(\underline{x}\epsilon)]$$

with the following property. For each $\underline{\alpha} \in \mathcal{P}_\epsilon(a)$ there is an $\underline{x} \in \mathcal{I}$ and a bijective strictly monotone map $u_{\underline{\alpha}}^\epsilon$ from the kneading space of all unimodal maps on the interval to a closed subinterval of $[p_{\epsilon,a}(\underline{x}1), p_{\epsilon,a}(\underline{x}\epsilon)]$ which contains $\underline{\alpha}$ as an endpoint. Moreover, for each $\underline{x} \in \mathcal{I}$ there exists $\underline{\alpha} \in \mathcal{P}_\epsilon(a)$ such that $\max(\operatorname{Im}(u_{\underline{\alpha}}^\epsilon)) = p_{\epsilon,a}(\underline{x}\epsilon)$.

(c) *There exists* $p_{\delta,b}\colon \mathcal{J}_\delta \to Q_\delta(b)$ *which is nondecreasing, maps the endpoints of* $\mathcal{J}_\delta$ *into the endpoints of* $Q_\delta(b)$ *and if* $b \in \mathbb{Q}\backslash\mathbb{Z}$ *then* $p_{\delta,b}$ *is one-to-one. Moreover,*

$$\mathrm{Im}(p_{\delta,b}) = Q_\delta(b) \Big\backslash \bigcup_{\underline{x}\in\mathcal{I}} (p_{\delta,b}(\underline{x}\delta), p_{\delta,b}(\underline{x}0)) .$$

(d) *There exists*

$$\mathcal{P}_\delta(b) \subset \bigcup_{\underline{x}\in\mathcal{I}} [p_{\delta,b}(\underline{x}\delta), p_{\delta,b}(\underline{x}0)]$$

with the following property. For each $\underline{\beta} \in \mathcal{P}_\delta(b)$ *there is an* $\underline{x} \in \mathcal{I}$ *and a bijective strictly monotone map* $u^\delta_{\underline{\beta}}$ *from the kneading space of all unimodal maps on the interval to a closed subinterval of* $[p_{\delta,b}(\underline{x}\delta), p_{\delta,b}(\underline{x}0)]$ *which contains* $\underline{\beta}$ *as an endpoint. Moreover, for each* $\underline{x} \in \mathcal{I}$ *there exists* $\underline{\beta} \in \mathcal{P}_\delta(b)$ *such that* $\min(\mathrm{Im}(u^\delta_{\underline{\beta}})) = p_{\delta,b}(\underline{x}\delta)$.

1.1. *Remarks to the Main Theorem*

The above theorem gives a characterization of a subclass of kneading pairs for maps from $\mathcal{A}$ which have a *noninteracting* renormalization structure in terms of the endpoints of the rotation interval. Here *noninteracting* means that the first (resp., second) element of the kneading pair has a renormalization structure depending only on the left (resp., right) endpoint of the rotation interval. This subclass is explicitly defined in terms of two symbolic operators that will be defined later.

To be more precise, the Main Theorem effectively gives a decomposition of the symbolic space $Q_\epsilon(a)$ (resp., $Q_\delta(b)$) into a set

$$K_{\epsilon,a} = \mathrm{Im}(p_{\epsilon,a}) \cup \left(\bigcup_{\underline{\alpha}\in\mathcal{P}_\epsilon(a)} \mathrm{Im}(u^\delta_{\underline{\alpha}}) \right)$$

and respectively,

$$K_{\delta,b} = \mathrm{Im}(p_{\delta,b}) \cup \left(\bigcup_{\underline{\beta}\in\mathcal{P}_\delta(b)} \mathrm{Im}(u^\delta_{\underline{\beta}}) \right),$$

whose points are completely characterized, and the open intervals in the complement of this set (which are gaps where we have not been able to characterize the sequences in their interior). The role of the above gaps in $Q_\epsilon(a)$ (resp., $Q_\delta(b)$) is to deal with the first (resp., second) component of the kneading pairs containing unbounded symbols (see Fig. 2 and compare with [Hockett & Holmes, 1988, Fig. 5]). These unbounded symbols appear in the kneading sequences due to the fact that b (resp., a) can be arbitrarily far from a (resp., b).

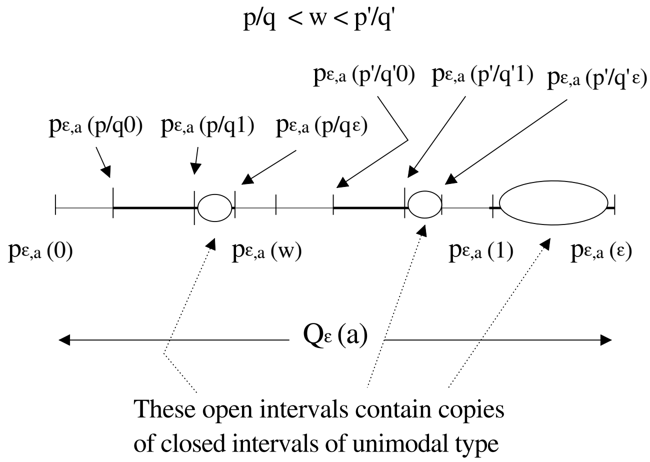

Fig. 2. The description of the closed interval $Q_\epsilon(a)$ using the map $p_{\epsilon,a}$. Note that the gaps also contain the sequences with unbounded symbols.

Another feature of the above decomposition theorem is the following one. Assume that we know the first (resp., second) component of the kneading pair of a given map from class $\mathcal{A}$ having a (resp., b) as a left (resp., right) endpoint of the rotation interval up to a given finite length n. From the above theorem we can obtain the minimal interval in $Q_\epsilon(a)$ (resp., $Q_\delta(b)$) with endpoints in $K_{\epsilon,a}$ (resp., $K_{\delta,b}$) containing all the first (resp., second) components of kneading pairs which coincide with the given one in the first n symbols (see Figs. 3 and 2).

We also remark that there exist pairs in $\mathcal{E}_\epsilon \times \mathcal{E}_\delta$ which cannot occur as kneading pairs of any map from $\mathcal{A}$. An explicit example of this fact that uses Theorem 2.6 and Proposition 2.7 is given after Proposition 2.7.

The study and binding of the dynamics (periodic points, topological entropy, ...) of maps associated to the endpoints of these intervals can be done with the help of techniques developed in [Alsedà & Mañosas, 1990]. In particular, this theorem gives some basic topological tools that can be used to describe bifurcations in parametrized families of bimodal degree one circle maps like the following well-known family

$$H_{a,w}(x) = x + w + \frac{a}{2\pi}\sin(2\pi x), \tag{1}$$

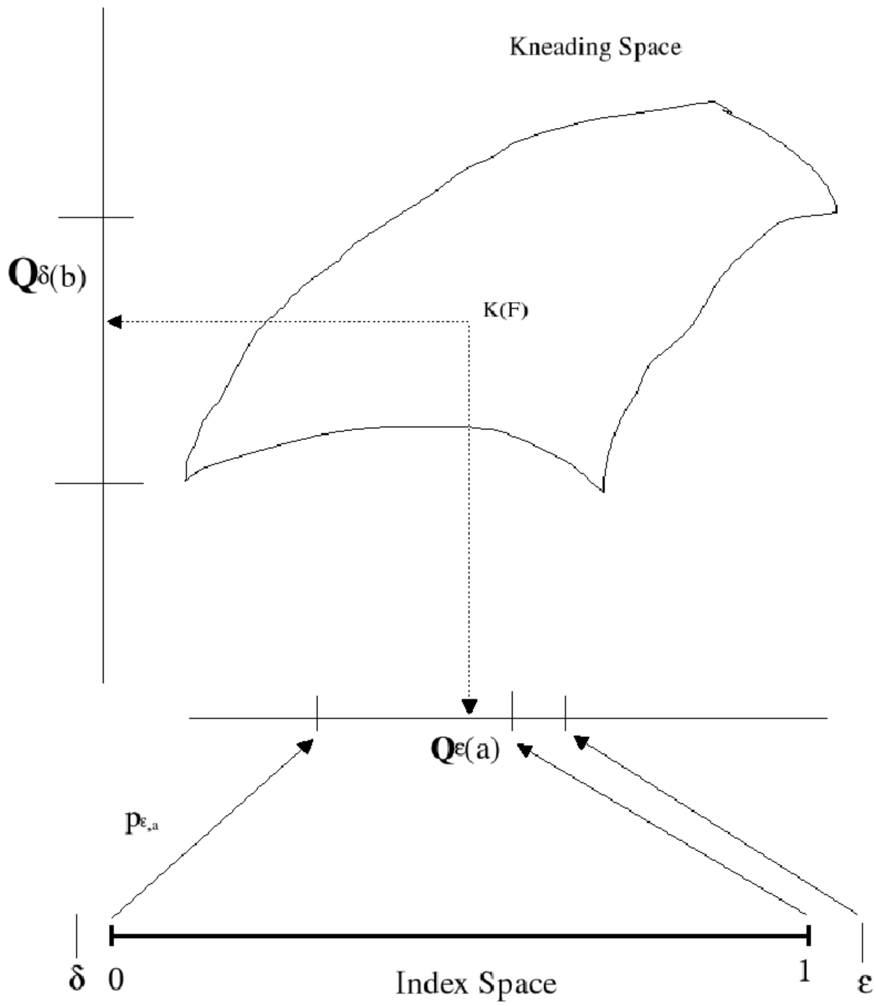

Fig. 3. The kneading space and its description using the index space and the map given in the Main Theorem.

where $x \in \mathbb{R}$ and $(a, w) \in \mathbb{R}^+ \times \mathbb{R}$ or the family considered by Hockett and Holmes in [1988]. We note that this study depends on the characterization of the endpoints of the rotation interval in terms of the parameter — usually a difficult task.

The fact that the Main Theorem characterizes the dynamics of the maps in terms of the rotation interval restricts ourselves to the class of circle maps of degree one; where the rotation theory holds. Thus, in principle the results and theory developed in this paper and predecessors cannot be extended to circle maps of degree different from one. However, the results and techniques of this paper can be straightforwardly extended to multimodal circle maps of degree one with considerably more effort and notational complexity. In any case, the preceding kneading theory must also be extended to this more complicate setting.

The paper is organized as follows. Unfortunately, the definitions of the $*$-like operators and the statements of the results used to proved the Main Theorem are rather technical and require a good deal of notation. In particular, before these definitions and statements, for completeness, we need to make a survey on Kneading Theory for maps in $\mathcal{A}$ and to fix the notation we will use when talking about unimodal interval maps (see Sec. 2). In Sec. 3 we introduce the four operators to state the results that we use in Sec. 6 to prove the Main Theorem. Finally, in Secs. 4 and 5 we shall prove the results stated in Sec. 3.

2. A Survey on Kneading Theory for Maps in $\mathcal{A}$ and Unimodal Maps

This section is divided into two subsections. The first one is essentially a survey of the papers by Alsedà and Mañosas [1990]; Alsedà and Falcó [1997]. The second one recalls the basic concepts of the kneading theory of unimodal maps.

2.1. *Kneading theory for maps in $\mathcal{A}$*

We start by recalling the notion of itinerary of a point. From now on, $E(\cdot)$ denotes the integer part function and $D(\cdot)$ denotes the decimal part function (that is, $D(z) = z - E(z)$). For $F \in \mathcal{A}$ and $x \in \mathbb{R}$ let

$$s(x) = \begin{cases} R & \text{if } D(x) \in (c_F, 1), \\ C & \text{if } D(x) = c_F, \\ L & \text{if } D(x) \in (0, c_F), \\ M & \text{if } D(x) = 0, \end{cases}$$

and $d(x) = E(F(x)) - E(x)$.

Then the *reduced itinerary of* x, denoted by $\underline{\hat{I}}_F(x)$, is defined as follows. For $i \in \mathbb{N}$, set $s_i = s(F^i(x))$ and $d_i = d(F^{i-1}(x))$. Then $\underline{\hat{I}}_F(x)$ is defined by

$$\begin{cases} d_1^{s_1} d_2^{s_2} \cdots & \text{if } s_i \in \{L, R\} \quad \text{for all } i \geq 1, \\ d_1^{s_1} d_2^{s_2} \cdots d_n^{s_n} & \text{if } s_n \in \{M, C\} \quad \text{and} \quad s_i \in \{L, R\} \\ & \qquad\qquad \text{for all } i \in \{1, \ldots, n-1\}. \end{cases}$$

Note that since $F \in \mathcal{L}$ we have $\underline{\hat{I}}_F(x) = \underline{\hat{I}}_F(x + k)$ for all $k \in \mathbb{Z}$. Let $x, y \in \mathbb{R}$ be such that $D(x) \neq D(y)$. We say that x and y are *conjugate* if and only if $F(D(x)) = F(D(y))$. Note also that if x and y are conjugate then they have the same reduced itinerary.

Let $\mathcal{S} = \{M, L, C, R\}$ and let $\underline{\alpha} = \alpha_1\alpha_2 \cdots$ be a sequence of elements $\alpha_i = d_i^{s_i}$ of $\mathbb{Z} \times \mathcal{S}$.

Definition 2.1. We say that $\underline{\alpha}$ is *admissible* if one of the following two conditions is satisfied:

(1) $\underline{\alpha}$ is infinite, $s_i \in \{L, R\}$ for all $i \geq 1$ and there exists $k \in \mathbb{N}$ such that $|d_i| \leq k$ for all $i \geq 1$.
(2) $\underline{\alpha}$ is finite of length n, $s_n \in \{M, C\}$ and $s_i \in \{L, R\}$ for all $i \in \{1, \ldots, n-1\}$.

Notice that any reduced itinerary is an admissible sequence. Now we shall introduce some notation for admissible sequences (and hence for reduced itineraries).

The cardinality of an admissible sequence $\underline{\alpha}$ will be denoted by $|\underline{\alpha}|$ (if $\underline{\alpha}$ is infinite we write $|\underline{\alpha}| = \infty$).

We denote by S the shift operator which acts on the set of admissible sequences of length greater than one as follows: $S(\underline{\alpha}) = \alpha_2\alpha_3 \cdots$ if $\underline{\alpha} = \alpha_1\alpha_2\alpha_3 \cdots$. We will write S^k for the kth iterate of S. Obviously S^k is only defined for admissible sequences of length greater than k. Clearly, for each $x \in \mathbb{R}$ we have $S^n(\underline{\hat{I}}_F(x)) = \underline{\hat{I}}_F(F^n(x))$ if $|\underline{\hat{I}}_F(x)| > n$.

Let $\underline{\alpha} = \alpha_1\alpha_2 \cdots \alpha_n$ and $\underline{\beta} = \beta_1\beta_2 \cdots$ be two sequences of symbols in $\mathbb{Z} \times \mathcal{S}$. We shall write $\underline{\alpha}\ \underline{\beta}$ to denote the concatenation of $\underline{\alpha}$ and $\underline{\beta}$ (i.e. the sequence $\alpha_1\alpha_2 \cdots \alpha_n\beta_1\beta_2 \cdots$). We also shall use the symbols $\underline{\alpha}^n$ to denote $\overbrace{\underline{\alpha}\ \underline{\alpha} \cdots \ \underline{\alpha}}^{n \text{ times}}$ and $\underline{\alpha}^\infty$ to denote $\underline{\alpha}\ \underline{\alpha} \cdots$.

Let $\underline{\alpha} = \alpha_1\alpha_2 \cdots \alpha_n$ be a sequence of symbols in $\mathbb{Z} \times \mathcal{S}$. Set $\alpha_i = d_i^{s_i}$ for $i = 1, 2, \ldots, n$. We say that $\underline{\alpha}$ is *even* if $\mathrm{Card}\{i \in \{1, \ldots, n\} | s_i = R\}$ is even. Otherwise we say that $\underline{\alpha}$ is *odd*.

Now we endow the set of admissible sequences with a total ordering. First set $M < L < C < R$. Then we extend this ordering to $\mathbb{Z} \times \mathcal{S}$ lexicographically. That is, we write $d^s < t^m$ if and only if either $d < t$ or $d = t$ and $s < m$. Let now $\underline{\alpha} = \alpha_1\alpha_2 \ldots$ and $\underline{\beta} = \beta_1\beta_2 \ldots$ be two admissible sequences such that $\underline{\alpha} \neq \underline{\beta}$. Then there exists $n \in \mathbb{N}$ such that $\alpha_n \neq \beta_n$ and $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, n-1$. We write $\underline{\alpha} < \underline{\beta}$ if either $\alpha_1\alpha_2 \ldots \alpha_{n-1}$ is even and $\alpha_n < \beta_n$ or $\alpha_1\alpha_2 \ldots \alpha_{n-1}$ is odd and $\alpha_n > \beta_n$.

Let $\underline{\alpha} = \alpha_1\alpha_2\alpha_3 \cdots$ be an admissible sequence. We say that $\underline{\alpha}$ is *periodic of period* n if $S^n(\underline{\alpha}) = \underline{\alpha}$ and $S^i(\underline{\alpha}) \neq \underline{\alpha}$ for $i = 1, 2, \ldots, n-1$. We note that if $\underline{\alpha}$ is a periodic sequence of period n, then $|\underline{\alpha}| = \infty$ and there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{Z} \times \mathcal{S}$ such that $\underline{\alpha} = (\alpha_1 \cdots \alpha_n)^\infty$. We also note that if x is a periodic (mod 1) point of F such that $|\underline{\hat{I}}_F(x)| = \infty$, then $\underline{\hat{I}}_F(x)$ is periodic (recall that $S^n(\underline{\hat{I}}_F(x)) = \underline{\hat{I}}_F(F^n(x)))$ but their periods are not necessarily equal.

The following results (see [Alsedà & Mañosas, 1990]) show that the above ordering of reduced itineraries is, in fact, the ordering of points in $[0, c_F]$.

Proposition 2.2. *Let $F \in \mathcal{A}$. Then,*

(a) *If $x, y \in [0, c_F]$, and $x < y$ then $\underline{\hat{I}}_F(x) \leq \underline{\hat{I}}_F(y)$.*
(b) *If $x, y \in [c_F, 1)$, and $x < y$ then $\underline{\hat{I}}_F(x) \geq \underline{\hat{I}}_F(y)$.*

Corollary 2.3. *Let $F \in \mathcal{A}$. For all $x \in \mathbb{R}$ we have $\underline{\hat{I}}_F(0) \leq \underline{\hat{I}}_F(x) \leq \underline{\hat{I}}_F(c_F)$.*

To define the kneading pair of a map $F \in \mathcal{A}$ we introduce the following notation. For a point $x \in \mathbb{R}$ we define the sequences $\underline{\hat{I}}_F(x^+)$ and $\underline{\hat{I}}_F(x^-)$ as follows. For each $n \geq 0$ there exists $\delta(n) > 0$ such that $d(F^{n-1}(y))$ and $s(F^n(y))$ take constant values for each $y \in (x, x + \delta(n))$ (resp., $y \in (x - \delta(n), x)$). Denote these values by $d(F^{n-1}(x^+))$ and $s(F^n(x^+))$ (resp., $d(F^{n-1}(x^-))$ and $s(F^n(x^-))$). Then we set

$$\underline{\hat{I}}_F(x^+) = d(x^+)^{s(F(x^+))} d(F(x^+))^{s(F^2(x^+))} \ldots$$

and

$$\underline{\hat{I}}_F(x^-) = d(x^-)^{s(F(x^-))} d(F(x^-))^{s(F^2(x^-))} \cdots .$$

Clearly, $\underline{\hat{I}}_F(x^+)$ and $\underline{\hat{I}}_F(x^-)$ are infinite admissible sequences, and $\underline{\hat{I}}_F(x^+) = \underline{\hat{I}}_F((x+k)^+)$ and $\underline{\hat{I}}_F(x^-) = \underline{\hat{I}}_F((x+k)^-)$ for all $k \in \mathbb{Z}$. Moreover, if $x \notin \mathbb{Z}$ and $|\underline{\hat{I}}_F(x)| = \infty$ then $\underline{\hat{I}}_F(x^-) = \underline{\hat{I}}_F(x) = \underline{\hat{I}}_F(x^+)$.

Definition 2.4. Let $F \in \mathcal{A}$. The pair $(\underline{\hat{I}}_F(0^+), \underline{\hat{I}}_F(c_F^-))$ will be called *the kneading pair of* F and will be denoted by $\mathcal{K}(F)$.

From [Alsedà & Mañosas, 1990, Proposition A] it follows that $\mathcal{K}(F)$ characterizes the set of reduced itineraries (and hence the dynamics) of a map $F \in \mathcal{A}$.

Let $\mathcal{AD}$ denote the set of all infinite admissible sequences.

Note that for each $F \in \mathcal{A}$, we have $\mathcal{K}(F) \in \mathcal{AD} \times \mathcal{AD}$. To characterize the pairs in $\mathcal{AD} \times \mathcal{AD}$ that can occur as a kneading pair of a map from $\mathcal{A}$ we will define a subset $\mathcal{E}$ of $\mathcal{AD} \times \mathcal{AD}$ which appears to be the set of all kneading pairs of all maps from $\mathcal{A}$ (see [Alsedà & Falcó, 1997]). To this end we introduce the following notation.

Let $\underline{\alpha} = d_1^{s_1}\alpha_2, \ldots,$ be an admissible sequence. We will denote by $\underline{\alpha}'$ the sequence $(d_1+1)^{s_1}\alpha_2 \cdots$. Note that since for $F \in \mathcal{A}$ we have $d(F(0^+)) = d(F(0^-)) - 1$ we can write $(\underline{\hat{I}}_F(0^+))' = \underline{\hat{I}}_F(0^-)$.

We will denote by $\mathcal{E}^*$ the set of all pairs $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{AD} \times \mathcal{AD}$ such that the following conditions hold:

(1) $\underline{\nu}_1' < \underline{\nu}_2$.
(2) $\underline{\nu}_1 \le S^n(\underline{\nu}_i) \le \underline{\nu}_2$ for all $n > 0$ and $i \in \{1, 2\}$.
(3) If for some $n \ge 0$, $S^n(\underline{\nu}_i) = d^R, \ldots,$ then $S^{n+1}(\underline{\nu}_i) \ge \underline{\nu}_1'$ for $i \in \{1, 2\}$.

We note that Condition (2) says, in particular, that $\underline{\nu}_1$ and $\underline{\nu}_2$ are minimal and maximal, respectively, according to the following definition. Let $\underline{\alpha}$ be an admissible sequence. We say that $\underline{\alpha}$ is *minimal* (resp., *maximal*) if and only if $\underline{\alpha} \le S^n(\underline{\alpha})$ (resp., $\underline{\alpha} \ge S^n(\underline{\alpha})$) for all $n \in \{1, 2, \ldots |\underline{\alpha}| - 1\}$.

As we will see, the above set contains (among others) the kneading pairs of maps from $\mathcal{A}$ with nondegenerate rotation interval. To deal with some special kneading pairs associated to maps with degenerate rotation interval, we introduce the following definitions.

For $a \in \mathbb{R}$ we set $\epsilon_i(a) = E(ia) - E((i-1)a)$ and $\delta_i(a) = \tilde{E}(ia) - \tilde{E}((i-1)a)$, where $\tilde{E}\colon \mathbb{R} \to \mathbb{Z}$ is defined as follows

$$\tilde{E}(x) = \begin{cases} E(x) & \text{if } x \notin \mathbb{Z}, \\ x - 1 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Also, we set

$$\underline{\hat{I}}_\epsilon(a) = \epsilon_1(a)^L \epsilon_2(a)^L \cdots \epsilon_n(a)^L \cdots$$

and

$$\underline{\hat{I}}_\delta(a) = \delta_1(a)^L \delta_2(a)^L \cdots \delta_n(a)^L \cdots .$$

Let $\underline{\hat{I}}_\epsilon^*(a) = (\underline{\hat{I}}_\epsilon(a))'$ and let $\underline{\hat{I}}_\delta^*(a)$ denote the sequence that satisfies $(\underline{\hat{I}}_\delta^*(a))' = \underline{\hat{I}}_\delta(a)$. Let $a = p/q$ be such that $(p, q) = 1$. We denote by $\underline{\hat{I}}_R(a)$ the sequence

$$(\delta_1(a)^L \cdots \delta_{q-1}(a)^L \delta_q(a)^R)^\infty$$

and by $\underline{\hat{I}}_R^*(a)$ the sequence which satisfies $(\underline{\hat{I}}_R^*(a))' = \underline{\hat{I}}_R(a)$.

To simplify the use of the above sequences the following lemma will be helpful (see [Alsedà & Mañosas, 1990, (4.1–4.3)]).

Lemma 2.5. *Let $a \in \mathbb{R}$. Then the following statements hold.*

(a) *If $a \notin \mathbb{Z}$ then $\delta_1(a) = \epsilon_1(a) + 1$. Furthermore, if $a \notin \mathbb{Q}$ then $\delta_i(a) = \epsilon_i(a)$ for all $i > 1$. That is, $\underline{\hat{I}}_\delta^*(a) = \underline{\hat{I}}_\epsilon(a)$ and $\underline{\hat{I}}_\delta(a) = \underline{\hat{I}}_\epsilon^*(a)$. If $a = p/q$ with $(p, q) = 1$ and $q > 1$ then $\epsilon_i(a) = \delta_i(a)$ for $i = 2, \ldots, q-1$, $\delta_q(a) = \epsilon_q(a) - 1$ and, $\epsilon_{i+q}(a) = \epsilon_i(a)$ and $\delta_{i+q}(a) = \delta_i(a)$ for all $i \in \mathbb{N}$.*
(b) *If $a \in \mathbb{Z}$ then $\epsilon_i(a) = \delta_i(a) = a$ for all $i > 0$.*

Now, for each $a \in \mathbb{R}$, we set

$$\mathcal{E}_a = \begin{cases} \{(\underline{\hat{I}}_\epsilon(a), \underline{\hat{I}}_\epsilon^*(a)), (\underline{\hat{I}}_\delta^*(a), \underline{\hat{I}}_\delta(a)), \\ \qquad (\underline{\hat{I}}_R^*(a), \underline{\hat{I}}_R(a))\} \\ \qquad\qquad \text{if } a = \dfrac{p}{q} \in \mathbb{Q}, \quad \text{with } (p,q) = 1, \\ \{(\underline{\hat{I}}_\delta^*(a), \underline{\hat{I}}_\delta(a))\} \quad \text{if } a \notin \mathbb{Q}. \end{cases}$$

Finally we denote by $\mathcal{E}$ the set $\mathcal{E}^* \cup (\bigcup_{a \in \mathbb{R}} \mathcal{E}_a)$. The following two results, given by Alsedà and Falcó [1997, Theorem A], characterize the kneading pairs of the maps from class $\mathcal{A}$.

Theorem 2.6. *For $F \in \mathcal{A}$ we have $\mathcal{K}(F) \in \mathcal{E}$. Conversely, for each $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}$ there exists $F \in \mathcal{A}$ such that $\mathcal{K}(F) = (\underline{\nu}_1, \underline{\nu}_2)$.*

To define the ambient space of the set $\mathcal{E}$ we introduce the following two sets. Let

$$\mathcal{E}_\epsilon = \left\{\underline{\alpha} \in \mathcal{AD}\colon \exists \underline{\beta} \in \mathcal{AD} \text{ such that } (\underline{\alpha}, \underline{\beta}) \in \mathcal{E}\right\}$$

and

$$\mathcal{E}_\delta = \left\{\underline{\beta} \in \mathcal{AD}\colon \exists \underline{\alpha} \in \mathcal{AD} \text{ such that } (\underline{\alpha}, \underline{\beta}) \in \mathcal{E}\right\}.$$

The following result characterizes the sets $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ (see [Falcó, 1995, Theorem 3.1.1]).

Proposition 2.7. *The following statements hold.*

(a) $\underline{\alpha} \in \mathcal{E}_\epsilon$ *if and only if it is minimal and verifies that if for some* $n \geq 0$, $S^n(\underline{\alpha}) = d^R \cdots$ *then* $S^{n+1}(\underline{\alpha}) \geq \underline{\alpha}'$.
(b) $\underline{\beta} \in \mathcal{E}_\delta$ *if and only if it is maximal.*

We consider $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ endowed with the order topology and let $\mathcal{E}_\epsilon \times \mathcal{E}_\delta$ be with the product topology. We note that $\mathcal{E}$ is strictly contained in $\mathcal{E}_\epsilon \times \mathcal{E}_\delta$. To see this consider for example the set $A = \{(0^L)^\infty, (1^L)^\infty\}$ of admissible sequences. Since each of the three sequences $((-1^L)^\infty, (0^L)^\infty)$, $((0^L)^\infty, (1^L)^\infty)$ and $((1^L)^\infty, (2^L)^\infty)$ belongs to $\mathcal{E}$, it follows that $A \subset \mathcal{E}_\epsilon$ and $A \subset \mathcal{E}_\delta$. In consequence,

$$\{((0^L)^\infty, (1^L)^\infty), ((1^L)^\infty, (0^L)^\infty)\} \subset \mathcal{E}_\epsilon \times \mathcal{E}_\delta.$$

However, $((0^L)^\infty, (1^L)^\infty) \in \mathcal{E}$ but $((1^L)^\infty, (0^L)^\infty) \notin \mathcal{E}$.

For $a \in \mathbb{R}$ we define $\mathcal{Q}_\epsilon(a)$ as $[\underline{\hat{I}}^*_\delta(a), \underline{\hat{I}}_\epsilon(a)] \subset \mathcal{E}_\epsilon$ and $\mathcal{Q}_\delta(a) = [\underline{\hat{I}}_\delta(a), \underline{\hat{I}}^*_\epsilon(a)] \subset \mathcal{E}_\delta$ (recall that from Theorem 2.6, $\underline{\hat{I}}^*_\delta(a), \underline{\hat{I}}_\epsilon(a) \in \mathcal{E}_\epsilon$ and $\underline{\hat{I}}_\delta(a), \underline{\hat{I}}^*_\epsilon(a) \in \mathcal{E}_\delta$ for all $a \in \mathbb{R}$). From Lemma 2.5(a) we have that if $a \notin \mathbb{Q}$ then $\mathcal{Q}_\epsilon(a)$ and $\mathcal{Q}_\delta(a)$ are closed intervals degenerate to a point.

We recall that for $F \in \mathcal{L}$ the rotation interval R_F is defined to be the set

$$\{\rho_F(x)\colon x \in \mathbb{R}\},$$

where

$$\rho_F(x) = \rho(x) = \limsup_{n\to\infty} \frac{F^n(x) - x}{n}.$$

It is well known (see [Ito, 1981]) that the set R_F is a closed interval, perhaps degenerate to a single point. Also, if $F \in \mathcal{L}$ is a nondecreasing map then R_F is degenerate to a single point $\rho(F)$ and

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}$$

for every $x \in \mathbb{R}$. The next result gives a characterization of the rotation interval by using the kneading pair (see [Alsedà & Mañosas, 1990, Theorem B]).

Theorem 2.8. *Let* $F \in A$. *Then* $R_F = [a, b]$ *if and only if*

$$\mathcal{K}(F) \in Q_\epsilon(a) \times Q_\delta(b) = [\underline{\hat{I}}^*_\delta(a), \underline{\hat{I}}_\epsilon(a)] \times [\underline{\hat{I}}_\delta(b), \underline{\hat{I}}^*_\epsilon(b)].$$

2.2. *Kneading theory for unimodal maps*

In the last step of this survey we introduce the notation we shall use for the kneading theory of unimodal interval maps. Let I be a closed interval and let $f\colon I \to I$ be a continuous map. We say that f is *unimodal* if

(i) $f(\max I) = f(\min I) \in \partial I$, and
(ii) there exists $c \in \mathrm{Int}(I)$ such that the maps $f|_{[\min I, c]}$ and $f|_{[c, \max I]}$ are homeomorphisms.

The set of all unimodal maps from I to itself will be denoted by $U(I)$. A map $f \in U(I)$ will be called *positive* if $f|_{[\min I, c]}$ is increasing. Otherwise, f will be called *negative*.

Let $f \in U(I)$ and let $x \in I$. We associate with x a finite or infinite sequence of the symbols L, C, R called its itinerary. To do it we introduce the following notation. Let $f\colon I \to I$ be continuous. We will say that f is *locally increasing* (resp. *decreasing*) at $x \in I$ if there exists an open (in I) neighborhood V of x such that $f|_V$ is increasing (resp. decreasing). Now, we define the ith address of a point x, that we denote by $\theta_i(x)$, as follows:

$$\theta_i(x) = \begin{cases} L & \text{if } f^i \text{ is locally increasing at } x, \\ C & \text{if } f^i(x) = c, \\ R & \text{if } f^i \text{ is locally decreasing at } x. \end{cases}$$

We define the itinerary of x denoted by $\underline{\theta}_f(x)$ as follows:

(i) $\underline{\theta}_f(x) = \theta_0(x)\theta_1(x)\cdots\theta_n(x)\cdots$ if $\theta_i(x) \in \{L, R\}$ for all $i \geq 0$.
(ii) $\underline{\theta}_f(x) = \theta_0(x)\theta_1(x)\cdots\theta_n(x)$ if $\theta_n(x) = C$, and $\theta_i(x) \in \{L, R\}$ for all $i \in \{0, 1, \ldots, n-1\}$.

Given $n \in \mathbb{N}$ and $x \in I$, there exists $\delta > 0$ such that $\theta_n(y)$ takes constant value L or R in the interval $(x, x+\delta)$. We denote this value by $\theta_n(x^+)$. In a similar way we can define $\theta_n(x^-)$. With this notation we set $\underline{\theta}_f(x^+) = \theta_1(x^+)\theta_2(x^+)\cdots$ and $\underline{\theta}_f(x^-) = \theta_1(x^-)\theta_2(x^-)\cdots$. We note that if $\underline{\theta}_f(x)$ is infinite then $\underline{\theta}_f(x) = \underline{\theta}_f(x^+) = \underline{\theta}_f(x^-)$.

The sequence $\underline{\theta}_f(f(c)^+)$ is called the *kneading sequence of* f. We will denote it by $k(f)$.

Let $\underline{A} = A_0A_1\cdots$ be a sequence of elements $A_i \in \{L, C, R\}$. We say that $\underline{A}$ is *admissible* if one of the following two conditions is satisfied:

(i) $\underline{A} = A_0A_1\cdots A_n\cdots$ if $A_i \in \{L, R\}$ for all $i \geq 0$.
(ii) $\underline{A} = A_0A_2\ldots A_n$ if $A_n = C$, and $A_i \in \{L, R\}$ for all $i \in \{0, 1, \ldots n-1\}$.

Now, we introduce an ordering in the set of all admissible sequences. We set $L < C < R$ and we extend this ordering lexicographically to the set of all admissible sequences as follows. Let $K_0K_1\cdots K_n$ be a finite (or empty) sequence of symbols L, R. We say that $K_0K_1\cdots K_n$ is even (resp. odd) if it has an even (resp. odd) number of R's. Assume that $\underline{K} = K_0K_1\cdots$ and $\underline{K}' = K_0'K_1'\cdots$ are admissible sequences such that $\underline{K} \neq \underline{K}'$. Let $n \in \mathbb{N}$ be such that $K_i = K_i'$ for $i < n$ and $K_n \neq K_n'$. Then we say that $\underline{K} < \underline{K}'$ if either

(i) $K_n < K_n'$ and $K_0K_1\cdots K_{n-1}$ is even.
(ii) $K_n > K_n'$ and $K_0K_1\cdots K_{n-1}$ is odd.

We note that if $x < y$ and $f \in U(I)$ then $\underline{\theta}_f(x) \leq \underline{\theta}_f(y)$ if f is positive and $\underline{\theta}_f(x) \geq \underline{\theta}_f(y)$ if f is negative.

The *shift operation* S on admissible sequences is defined as follows. If $\underline{K} = K_0K_1\cdots$ then we set $S(\underline{K}) = K_1K_2\cdots$ which is also an admissible sequence. If $K_0 = C$, then S is undefined. We write S^n to denote the nth iterate of S. Note that for each $x \in I$ and $f \in U(I)$ we have $S(\underline{\theta}(x)) = (\underline{\theta}(f(x)))$.

An admissible sequence $\underline{K}$ will be called *maximal* if and only if $S^n(\underline{K}) \leq \underline{K}$ for each $n < |\underline{K}|$, where $|\underline{K}|$ denotes the length of $\underline{K}$. We note that for each $f \in U(I)$ (independently of the fact that f is positive or negative), $k(f)$ is maximal and admissible with length infinite. Given $\underline{K} = K_0K_1\cdots$, an admissible sequence, we will write $\hat{\underline{K}}$ to denote $\hat{K}_0\hat{K}_1\cdots$ where $\hat{L} = R$, $\hat{R} = L$ and $\hat{C} = C$. We note that $\underline{K}$ is maximal if and only if $\hat{\underline{K}}$ is minimal; that is, $S^n(\hat{\underline{K}}) \geq \hat{\underline{K}}$ for each $n < |\underline{K}|$.

From [Collet & Eckmann, 1980], it follows that for each admissible infinite maximal sequence $\underline{K}$ there exist $f, g \in U(I)$, f positive and g negative, such that $k(f) = k(g) = \underline{K}$. We shall denote by $\mathcal{K}$ the set of all admissible infinite maximal sequences.

3. Self-Similarity Operators

In this section, first we state the results that we will use to prove the Main Theorem. In Sec. 3.1, we define the $\star$-operators and we state the main result about them (Theorem A). In Sec. 3.2 we define the $\odot$-operators and state Theorem B which studies them.

3.1. *The $\star$-operators*

The aim of this subsection is to characterize the sets of sequences which appear as the first (resp., second) component of the kneading pair of a map $F \in \mathcal{A}$ for which there exist $p \in \mathbb{N}$, $q \in \mathbb{Z}$ and a closed interval J containing c_F (resp. 0) such that $(F^q - p)|_J$ is a unimodal map. We make this study at a symbolic level by using a $\star$-operator which relates certain subsets of the symbolic spaces $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ with the space of kneading sequences of unimodal maps. Moreover, we will show how the "unimodal symbolic space" is embedded into $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$.

We start by introducing some notation. Let Ξ denote the set of all finite sequences with symbols in $\mathbb{Z} \times \{L, R\}$ (of course, we consider the empty sequence as an element of Ξ).

Now we consider the set of sequences which occur as reduced itineraries of periodic critical points. We will denote by $\mathcal{P}_\epsilon$ (resp. $\mathcal{P}_\delta$) the set of all minimal sequences of the form $\underline{\beta}d^M$ with $\underline{\beta} \in \Xi$ and $d \in \mathbb{Z}$ satisfying that if for some $n \in \{1, \ldots, |\underline{\beta}|\}$, $S^{n-1}(\underline{\beta}d^M) = t^R\cdots$ then $S^n(\underline{\beta}d^M) > \underline{\beta}'d^M$ (resp., the set of all maximal sequences of the form $\underline{\beta}d^C$) and such that if $\underline{\beta}$ is not empty then $\left\{(\underline{\beta}d^L)^\infty, \underline{\beta}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty\right\} \subset \mathcal{E}_\epsilon$ (resp. $\left\{(\underline{\beta}d^L)^\infty, (\underline{\beta}d^R)^\infty\right\} \subset \mathcal{E}_\delta$).

We are now ready to define the $\star$-operators. We start by defining the operator $\star_\delta\colon \mathcal{P}_\delta \times \mathcal{K} \to \mathcal{AD}$ as follows. Let $\underline{\gamma} = \underline{\beta}d^C \in \mathcal{P}_\delta$ and $\underline{K} = K_1K_2\cdots \in \mathcal{K}$. Then we define

$$\underline{\gamma} \star_\delta \underline{K} = \begin{cases} \underline{\beta}d^{K_1}\underline{\beta}d^{K_2}\underline{\beta}\cdots & \text{if } \underline{\beta} \text{ is even,} \\ \underline{\beta}d^{\hat{K}_1}\underline{\beta}d^{\hat{K}_2}\underline{\beta}\cdots & \text{if } \underline{\beta} \text{ is odd.} \end{cases}$$

Now we define $\star_\epsilon\colon \mathcal{P}_\epsilon \times \mathcal{K} \to \mathcal{AD}$. Let $\underline{\beta} \in \Xi$ and $s \in \{L, R\}$. We set

$$\chi(s, \underline{\beta}) = \begin{cases} \underline{\beta} & \text{if } s = L, \\ \underline{\beta}' & \text{if } s = R. \end{cases}$$

Also, for $d \in \mathbb{Z}$, we set

$$\varphi(s, d) = \begin{cases} d^L & \text{if } s = L, \\ (d-1)^R & \text{if } s = R. \end{cases}$$

Let $\underline{\gamma} = \underline{\beta}d^M \in \mathcal{P}_\epsilon$ and $\underline{K} = K_1K_2\cdots \in \mathcal{K}$. Then we define $\underline{\gamma} \star_\epsilon \underline{K}$ as follows. If $\underline{\beta}$ is not empty then

$$\underline{\gamma}\star_\epsilon \underline{K} = \begin{cases} \underline{\beta}\varphi(K_1,d)\chi(K_1,\underline{\beta})\varphi(K_2,d)\chi(K_2,\underline{\beta})\cdots \\ \qquad\qquad\qquad \text{if } \underline{\beta} \text{ is even,} \\ \underline{\beta}\varphi(\hat{K_1},d)\chi(K_1,\underline{\beta})\varphi(\hat{K_2},d)\chi(K_2,\underline{\beta})\cdots \\ \qquad\qquad\qquad \text{if } \underline{\beta} \text{ is odd.} \end{cases}$$

If $\underline{\beta}$ is empty then $\underline{\gamma} \star_\epsilon \underline{K} = d_1^{K_1} d_2^{K_2}\cdots$ where, if $K_1 = L$ then $d_i = d$ for all $i \geq 1$ and if $K_1 = R$ then $d_1 = d-1$ and

$$d_i = \begin{cases} d+1 & \text{if } K_{i-1}K_i = RL, \\ d & \text{if } K_{i-1}K_i \in \{LL, RR\}, \\ d-1 & \text{if } K_{i-1}K_i = LR, \end{cases}$$

for $i \geq 2$.

The main result of this subsection which studies the properties of the $\star$-operators is the following.

Theorem A. *Let $\underline{\gamma} = \underline{\beta} d^M \in \mathcal{P}_\epsilon$, $\underline{\alpha} = \underline{\beta} d^C \in \mathcal{P}_\delta$ and $\underline{K} \in \mathcal{K}$. Then $\underline{\gamma} \star_\epsilon \underline{K} \in \mathcal{E}_\epsilon$ and $\underline{\alpha} \star_\delta \underline{K} \in \mathcal{E}_\delta$. If $\underline{\beta}$ is even then $\underline{\gamma}\star_\epsilon$ is order reversing and $\underline{\alpha}\star_\delta$ is order preserving. Otherwise, $\underline{\gamma}\star_\epsilon$ is order preserving and $\underline{\alpha}\star_\delta$ is order reversing. Moreover $\underline{\gamma}\star_\epsilon \mathcal{K}$ and $\underline{\gamma}\star_\delta \mathcal{K}$ are connected in $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$, respectively.*

Theorem A will be proved in Sec. 4. It characterizes at a symbolic level the "unimodal boxes" in the spaces $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$. Indeed, if we consider the set $\mathcal{K}$ endowed with the order topology (that is, $\mathcal{K} = [L^\infty, RL^\infty]$) then, from Theorem A, we see that if $\underline{\gamma} = \underline{\beta} d^M \in \mathcal{P}_\epsilon$ (resp. $\underline{\gamma} = \underline{\beta} d^C \in \mathcal{P}_\delta$), then

$$\underline{\gamma}\star_\epsilon \mathcal{K} = \begin{cases} [\underline{\gamma}\star_\epsilon RL^\infty, \underline{\gamma}\star_\epsilon L^\infty] & \text{if } \underline{\beta} \text{ is even} \\ [\underline{\gamma}\star_\epsilon L^\infty, \underline{\gamma}\star_\epsilon RL^\infty] & \text{if } \underline{\beta} \text{ is odd} \end{cases}$$

$$\left(\text{resp.}\quad \underline{\gamma}\star_\delta \mathcal{K} = \begin{cases} [\underline{\gamma}\star_\delta RL^\infty, \underline{\gamma}\star_\delta L^\infty] \\ \qquad\qquad \text{if } \underline{\beta} \text{ is odd} \\ [\underline{\gamma}\star_\delta L^\infty, \underline{\gamma}\star_\delta RL^\infty] \\ \qquad\qquad \text{if } \underline{\beta} \text{ is even} \end{cases}\right)$$

where, given two sequences $\underline{\alpha}, \underline{\beta} \in \mathcal{AD}$ with $\underline{\alpha} \leq \underline{\beta}$, $[\underline{\alpha}, \underline{\beta}]$ denotes the set of all admissible sequences lying between $\underline{\alpha}$ and $\underline{\beta}$.

The set $\underline{\gamma}\star_\epsilon \mathcal{K}$ will be called the *ϵ-unimodal box of $\underline{\gamma}$* and the set $\underline{\gamma}\star_\delta \mathcal{K}$ will be called the *δ-unimodal box of $\underline{\gamma}$*.

3.2. *The $\odot$-operators*

Let $k \in \mathbb{Z}$. We denote by Σ_k the set of sequences in $\{k^L, (k+1)^L\}^{\mathbb{N}}$. Let $\underline{\alpha} = d_1^L d_2^L \cdots$ and $\underline{\beta} = t_1^L t_2^L \cdots$ be two sequences in Σ_k. We consider in Σ_k the topology defined by the metric $d(\underline{\alpha}, \underline{\beta}) = \sum_{i=0}^\infty 2^{-i}|d_i - t_i|$. With this topology, Σ_k is a compact metric space. Let $S_k \colon \Sigma_k \to \Sigma_k$ denote the usual shift transformation restricted to Σ_k. Clearly, S_k is continuous. Let $\pi_k \colon \Sigma_k \to \Sigma_0$ be the order preserving homeomorphism defined by $\pi_k(d_1^L d_2^L \cdots) = (d_1 - k)^L (d_2 - k)^L \cdots$. Clearly, $S_0 \circ \pi_k = \pi_k \circ S_k$.

For $k \in \mathbb{Z}$ we define the sets $\mathcal{B}_\epsilon(k) = \Sigma_k \cap \mathcal{E}_\epsilon$ and $\mathcal{B}_\delta(k) = \Sigma_k \cap \mathcal{E}_\delta$. We note that the sets $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ are invariant under "translations". That is, if $d_1^{s_1} d_2^{s_2}\cdots$ is a sequence in $\mathcal{E}_\epsilon$ (resp. in $\mathcal{E}_\delta$) then $(d_1 + k)^{s_1}(d_2+k)^{s_2}\cdots$ also belongs to $\mathcal{E}_\epsilon$ (resp. $\mathcal{E}_\delta$). Therefore, $\mathcal{B}_\epsilon(k) = \pi_k^{-1}(\mathcal{B}_\epsilon(0))$ and $\mathcal{B}_\delta(k) = \pi_k^{-1}(\mathcal{B}_\delta(0))$. From Proposition 2.7 we have $\mathcal{B}_\epsilon(k)$ (resp. $\mathcal{B}_\delta(k)$) are the minimal (resp. maximal) sequences in Σ_k.

For $a \in \mathbb{R}$ we will denote $a - \tilde{E}(a)$ by $\tilde{D}(a)$. Also, $\mathbb{Q}\backslash\mathbb{Z}$ will be denoted by $\mathbb{Q}^*$.

We note that from Lemma 2.5, if $a = p/q \in \mathbb{Q}^*$ with $(p,q) = 1$ and $q \neq 2$ then the finite sequences $\epsilon_2(a)^L \cdots \epsilon_{q-1}(a)^L$ and $\delta_2(a)^L \cdots \delta_{q-1}(a)^L$ are equal. We will denote this finite sequence by $\underline{r}(a)$ (we take the empty sequence as $r(1/2)$).

Now we are ready to define the $\odot$-operators.

For $\alpha = d^L$ with $d \in \{0,1\}$ we set $\hat{\alpha} = (1-d)^L$. Then for $a \in (0,1]$ and $\underline{\alpha} = \alpha_1\alpha_2\cdots \in \mathcal{B}_\epsilon(0)$ we define

$$a \odot_\epsilon \underline{\alpha} = \begin{cases} 0^L \underline{r}(a)\alpha_1\hat{\alpha}_1\underline{r}(a)\alpha_2\hat{\alpha}_2\cdots \\ \qquad\qquad \text{if } a \in \mathbb{Q}^*, \\ \underline{\hat{I}}_\epsilon(a) \qquad \text{if } a \notin \mathbb{Q}^* \text{ and } \underline{\alpha} = (1^L)^\infty, \\ \underline{\hat{I}}^*_\delta(a) \qquad \text{if } a \notin \mathbb{Q}^* \text{ and } \underline{\alpha} \neq (1^L)^\infty. \end{cases}$$

We extend the above definition to each $a \in \mathbb{R}$ by setting $a \odot_\epsilon \underline{\alpha} = \pi^{-1}_{\tilde{E}(a)}(\tilde{D}(a) \odot_\epsilon \underline{\alpha})$.

Now, we define the $\odot_\delta$ version of the $\odot$-operator as follows. Let $a \in [0,1)$ and $\underline{\alpha} = \alpha_1\alpha_2\cdots \in \mathcal{B}_\delta(0)$ be. Then we set

$$a \odot_\delta \underline{\alpha} = \begin{cases} 1^L \underline{r}(a)\alpha_1\hat{\alpha}_1\underline{r}(a)\alpha_2\hat{\alpha}_2\cdots \\ \qquad\qquad \text{if } a \in \mathbb{Q}^*, \\ \underline{\hat{I}}_\delta(a) \qquad \text{if } a \notin \mathbb{Q}^* \text{ and } \underline{\alpha} = (0^L)^\infty, \\ \underline{\hat{I}}^*_\epsilon(a) \qquad \text{if } a \notin \mathbb{Q}^* \text{ and } \underline{\alpha} \neq (0^L)^\infty, \end{cases}$$

and we extend the above definition to each $a \in \mathbb{R}$ by $a \odot_\delta \underline{\alpha} = \pi^{-1}_{E(a)}(D(a) \odot_\delta \underline{\alpha})$.

The next result which we will be proved in Sec. 5.1 gives a first motivation to the $\odot$-operators.

Proposition 3.1. *Let $a \in \mathbb{R}$. Then $a \odot_\epsilon (0^L)^\infty = \hat{\underline{I}}^*_\delta(a)$, $a \odot_\epsilon (1^L)^\infty = \hat{\underline{I}}_\epsilon(a)$, $a \odot_\delta (0^L)^\infty = \hat{\underline{I}}_\delta(a)$ and $a \odot_\delta (1^L)^\infty = \hat{\underline{I}}^*_\epsilon(a)$.*

From the above proposition we see that the fundamental boxes can be written as $\mathcal{Q}_\epsilon(a) = [a \odot_\epsilon (0^L)^\infty, a \odot_\epsilon (1^L)^\infty]$ and $\mathcal{Q}_\delta(a) = [a \odot_\delta (0^L)^\infty, a \odot_\delta (1^L)^\infty]$. The next theorem is the main result of this subsection.

For $\underline{\alpha} \in \Sigma_k$, $\underline{\alpha} = d_1^L d_2^L \cdots$ we define the *symbolic rotation number of* $\underline{\alpha}$ as

$$\rho(\underline{\alpha}) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d_i.$$

Theorem B. *Let $a, b \in \mathbb{R}$ with $a \le b$. Then the following statements hold*:

(a) *Let $\underline{\alpha}, \underline{\beta} \in \mathcal{B}_\epsilon(0)$ with $\underline{\alpha} < \underline{\beta}$. Then $a \odot_\epsilon \underline{\alpha} \le b \odot_\epsilon \underline{\beta}$. Moreover if $a \in \mathbb{Q}^*$ then $a \odot_\epsilon \underline{\alpha} < a \odot_\epsilon \underline{\beta}$.*
(b) *Let $\underline{\alpha}, \underline{\beta} \in \mathcal{B}_\delta(0)$ with $\underline{\alpha} < \underline{\beta}$. Then $a \odot_\delta \underline{\alpha} \le b \odot_\delta \underline{\beta}$. Moreover if $a \in \mathbb{Q}^*$ then $a \odot_\delta \underline{\alpha} < a \odot_\delta \underline{\beta}$.*
(c) *Let $\underline{\alpha} \in \mathcal{B}_\epsilon(0)$. Then $a \odot_\epsilon \underline{\alpha} \in \mathcal{B}_\epsilon(\tilde{E}(a)) \subset \mathcal{E}_\epsilon$ and $\rho(a \odot_\epsilon \underline{\alpha}) = a$.*
(d) *Let $\underline{\alpha} \in \mathcal{B}_\delta(0)$. Then $a \odot_\delta \underline{\alpha} \in \mathcal{B}_\delta(E(a)) \subset \mathcal{E}_\delta$ and $\rho(a \odot_\epsilon \underline{\alpha}) = a$.*
(e) *Let $a \in \mathbb{Q}^*$ and $(\underline{\alpha}, \underline{\beta}) \in \mathcal{B}_\epsilon(0) \times \mathcal{B}_\delta(0)$ be such that $\underline{\alpha} \ne (1^L)^\infty$ and $\underline{\beta} \ne (0^L)^\infty$. If $S^n(\underline{\alpha}) \le \underline{\beta}$ and $S^n(\underline{\beta}) \ge \underline{\alpha}$ for all $n \ge 0$, then $(a \odot_\epsilon \underline{\alpha}, a \odot_\delta \underline{\beta}) \in \mathcal{E}^* \subset \mathcal{E}$.*

We note that if $(\underline{\alpha}, \underline{\beta}) \in \mathcal{E}$, by Theorem 2.6 and Proposition 4.3 of [Alsedà & Falcó, 1997] we have $\underline{\alpha}' \le \underline{\beta}$, $S^n(\underline{\alpha}) \le \underline{\beta}$ and $S^n(\underline{\beta}) \ge \underline{\alpha}$ for all $n \ge 0$. Thus from Theorem B(e) we have the following.

Corollary 3.2. *Let $a \in \mathbb{Q}^*$ and let $(\underline{\alpha}, \underline{\beta}) \in (\mathcal{B}_\epsilon(0) \times \mathcal{B}_\delta(0)) \cap \mathcal{E}$ be such that $\underline{\alpha} \ne (1^L)^\infty$ and $\underline{\beta} \ne (0^L)^\infty$. Then $(a \odot_\epsilon \underline{\alpha}, a \odot_\delta \underline{\beta}) \in \mathcal{E}^* \subset \mathcal{E}$.*

We will prove Theorem B in Sec. 5.2.

We recall that in Sec. 3.1 we have defined the unimodal box of a periodic sequence $\underline{\gamma}$ from $\mathcal{P}_\epsilon$ (resp. $\mathcal{P}_\delta$) as $\underline{\gamma} \star_\epsilon \mathcal{K} = \underline{\gamma} \star_\epsilon [L^\infty, RL^\infty]$ (resp. $\underline{\gamma} \star_\delta \mathcal{K} = \underline{\gamma} \star_\delta [L^\infty, RL^\infty]$). Thus, in order that the unimodal boxes of $a \odot_\epsilon \underline{\alpha}$ and $a \odot_\delta \underline{\alpha}$ are defined, it is necessary that these sequences are periodic. The next result characterizes the periodic sequences of the form $a \odot_\epsilon \underline{\alpha}$ and $a \odot_\delta \underline{\alpha}$. It will be proved in Sec. 5.3.

Proposition 3.3. *Let $a \in \mathbb{R}$. The following statements hold.*

(a) *Let $\underline{\alpha} \in \mathcal{B}_\epsilon(0) \backslash \{(0^L)^\infty\}$ be periodic. If $a \notin \mathbb{Q}$ then $a \odot_\epsilon \underline{\alpha}$ is not periodic. If $a \in \mathbb{Z}$ then $a \odot_\epsilon \underline{\alpha}$ is periodic if and only if $\underline{\alpha} = (1^L)^\infty$. Moreover, $a^M \in P_\epsilon$ and $a \odot_\epsilon (1^L)^\infty = a^M \star_\epsilon L^\infty$. If $a \in \mathbb{Q}^*$ then $a \odot_\epsilon \underline{\alpha}$ is periodic. Moreover, there exists $\underline{\beta} d^M \in P_\epsilon$ such that $a \odot_\epsilon \underline{\alpha} = \underline{\beta} d^M \star_\epsilon L^\infty$.*
(b) *Let $\underline{\alpha} \in \mathcal{B}_\delta(0) \backslash \{(1^L)^\infty\}$ be periodic. If $a \notin \mathbb{Q}$ then $a \odot_\delta \underline{\alpha}$ is not periodic. If $a \in \mathbb{Z}$ then $a \odot_\delta \underline{\alpha}$ is periodic if and only if $\underline{\alpha} = (0^L)^\infty$. Moreover, $a^C \in P_\delta$ and $a \odot_\delta (0^L)^\infty = a^C \star_\delta L^\infty$. If $a \in \mathbb{Q}^*$ then $a \odot_\delta \underline{\alpha}$ is periodic. Moreover, there exists $\underline{\beta} d^C \in P_\delta$ such that $a \odot_\delta \underline{\alpha} = \underline{\beta} d^C \star_\delta L^\infty$.*

Now, we can define the *unimodal box* of a sequence of the form $a \odot_\epsilon \underline{\alpha}$ as follows. Let $a \in \mathbb{Q}$ and $\underline{\alpha} \in \mathcal{B}_\epsilon(0) \backslash \{(0^L)^\infty\}$ be periodic. Then, with the notation of Proposition 3.3(a), we set

$$\mathcal{U}_\epsilon(a \odot_\epsilon \underline{\alpha}) = \begin{cases} \underline{\beta} d^M \star_\epsilon \mathcal{K} & \text{if } a \in \mathbb{Q}^*, \\ a^M \star_\epsilon \mathcal{K} & \text{if } a \in \mathbb{Z} \text{ and } \underline{\alpha} = (1^L)^\infty. \end{cases}$$

Let now $\underline{\alpha} \in \mathcal{B}_\delta(0) \backslash \{(1^L)^\infty\}$ be periodic. With the notation of Proposition 3.3(b), we set

$$\mathcal{U}_\delta(a \odot_\delta \underline{\alpha}) = \begin{cases} \underline{\beta} d^C \star_\delta \mathcal{K} & \text{if } a \in \mathbb{Q}^*, \\ a^C \star_\delta \mathcal{K} & \text{if } a \in \mathbb{Z} \text{ and } \underline{\alpha} = (0^L)^\infty. \end{cases}$$

4. Proof of Theorem A

This section is organized as follows. In Sec. 4.1 we give some technical results and in Sec. 4.2 we prove Theorem A. Lastly, in Sec. 4.3, we give some remarks to Theorem A.

4.1. *Preliminary results*

In this subsection we study the itineraries of the critical points when they are periodic and some of the basic properties of the $\star$-operators. We start with the following technical lemmas and definitions.

Let $F \in \mathcal{L}$ and let $x \in \mathbb{R}$. Then the set $\{y \in \mathbb{R}\colon y = F^n(x) \pmod 1 \text{ for } n = 0, 1, \ldots\}$ will be called the (mod 1) *orbit* of x by F. We stress the fact that if P is a (mod 1) orbit and $x \in P$, then $x + k \in P$ for all $k \in \mathbb{Z}$. Let P be a (mod 1) orbit of a map $F \in \mathcal{L}$. We say that P is a *twist orbit* if F restricted to P is increasing. If a periodic (mod 1) orbit is twist then we say that P is a *twist periodic orbit.*

Lemma 4.1. *Let $F \in \mathcal{A}$. Then the following statements hold.*

(a) *Assume that 0 is a periodic* (mod 1) *point of period n. Then there exist $\underline{\beta} \in \Xi$ and $d \in \mathbb{Z}$,* *such that* $\underline{\hat{I}}_{F}(0^+)$ *is either* $(\underline{\beta}d^L)^\infty$ *with* $\underline{\beta}$ *even or* $\underline{\beta}d^R(\underline{\beta}'d^R)^\infty$ *with* $\underline{\beta}$ *odd. Moreover, if* $\underline{\hat{I}}_{F}(0^+) = (\underline{\beta}d^L)^\infty$ *then* $\underline{\beta}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty \in \mathcal{E}_\epsilon$ *and if* $\underline{\hat{I}}_{F}(0^+) = \underline{\beta}d^R(\underline{\beta}'d^R)^\infty$ *then* $(\underline{\beta}(d+1)^L)^\infty \in \mathcal{E}_\epsilon$.

(b) *Assume that* c_F *is a periodic* (mod 1) *point of period* n. *Then there exist* $\underline{\beta} \in \Xi$ *and* $d \in \mathbb{Z}$, *such that* $\underline{\hat{I}}_{F}(c_F^-)$ *is either* $(\underline{\beta}d^L)^\infty$ *with* $\underline{\beta}$ *even or* $(\underline{\beta}d^R)^\infty$ *with* $\underline{\beta}$ *odd. Moreover, if* $\underline{\hat{I}}_{F}(c_F^-) = (\underline{\beta}d^L)^\infty$ *then* $(\underline{\beta}d^R)^\infty \in \mathcal{E}_\delta$ *and if* $\underline{\hat{I}}_{F}(c_F^-) = (\underline{\beta}d^R)^\infty$ *then* $(\underline{\beta}d^L)^\infty \in \mathcal{E}_\delta$.

Proof. We start proving statement (a). Assume first that $\underline{\hat{I}}_{F}(0) = \underline{\beta}t^M$ for some $\underline{\beta} \in \Xi$ of length $n-1$ even. If $x > 0$ is sufficiently close to 0 we have $F^n|_{[0,x]}$ is increasing and $F^n(x)$ is also close to $F^n(0) = 0$. Therefore, $\underline{\hat{I}}_{F}(0^+) = \underline{\beta}t^L\underline{\hat{I}}_{F}(0^+)$. So $\underline{\hat{I}}_{F}(0^+) = (\underline{\beta}t^L)^\infty$. Now, assume that $\underline{\beta}$ is odd. Take $x < 0$ sufficiently close to 0. Then $F^n|_{[x,0]}$ is increasing and $F^n(x)$ is also close to $F^n(0)$. Thus $\underline{\hat{I}}_{F}(0^-) = \underline{\beta}'(t-1)^R\underline{\hat{I}}_{F}(0^-)$. Therefore $\underline{\hat{I}}_{F}(0^-) = (\underline{\beta}'(t-1)^R)^\infty$ and, in consequence, $\underline{\hat{I}}_{F}(0^+) = \underline{\beta}(t-1)^R(\underline{\beta}'(t-1)^R)^\infty$.

To prove the second statement of (a) in this case we only need to show that there exists $G \in \mathcal{A}$ such that $\underline{\hat{I}}_{G}(0^+) = \underline{\beta}(t-1)^R(\underline{\beta}'(t-1)^R)^\infty$ if $\underline{\beta}$ is even or $\underline{\hat{I}}_{G}(0^+) = (\underline{\beta}t^L)^\infty$ if $\underline{\beta}$ is odd. We note that the proof of Lemma 5.4 in [Alsedà & Falcó, 1997] does not depend on the fact that the orbit under consideration is twist. So, if $\underline{\hat{I}}_{F}(0) = \underline{\beta}t^M$ the statement follows from Lemma 5.4 in [Alsedà & Falcó, 1997] and the part of (a) already proved.

Now, assume that $\underline{\hat{I}}_{F}(0) = \underline{\gamma}k^C$ and $\underline{\hat{I}}_{F}(c_F) = \underline{\nu}t^M$ where $\underline{\gamma}, \underline{\nu} \in \Xi$, $\underline{\gamma}$ has length $n_1 - 1$, $\underline{\nu}$ has length $n_2 - 1$ and $n_1 + n_2 = n$. If $x > 0$ is sufficiently close to 0 then $F^{n_1}(x)$ is close to c_F. If $\underline{\gamma}$ is even then $F^{n_1}|_{[0,x]}$ is strictly increasing and, hence, $\underline{\hat{I}}_{F}(0^+) = \underline{\gamma}k^R\underline{\hat{I}}_{F}(c_F^+)$. Otherwise, if $\underline{\gamma}$ is odd, $F^{n_1}|_{[0,x]}$ is strictly decreasing and $\underline{\hat{I}}_{F}(0^+) = \underline{\gamma}k^L\underline{\hat{I}}_{F}(c_F^-)$. Now let $x > c_F$ be sufficiently close to c_F. If $\underline{\nu}$ is even, then $F^{n_2}|_{[c_F,x]}$ is strictly decreasing and $\underline{\hat{I}}_{F}(c_F^+) = \underline{\nu}(t-1)^R\underline{\hat{I}}_{F}(0^-)$. Otherwise, if $\underline{\nu}$ is odd, $F^{n_2}|_{[c_F,x]}$ is strictly increasing and $\underline{\hat{I}}_{F}(c_F^+) = \underline{\nu}t^L\underline{\hat{I}}_{F}(0^+)$. We recall that $\underline{\hat{I}}_{F}(c_F^+) = \underline{\hat{I}}_{F}(c_F^-)$ and that if $\underline{\hat{I}}_{F}(0^-) = (\underline{\hat{I}}_{F}(0^+))'$. Hence, if we set

$$\underline{\beta} = \begin{cases} \underline{\gamma}k^R\underline{\nu} & \text{if } \underline{\gamma} \text{ is even,} \\ \underline{\gamma}k^L\underline{\nu} & \text{if } \underline{\gamma} \text{ is odd,} \end{cases}$$

we get

$$\underline{\hat{I}}_{F}(0^+) = \begin{cases} \underline{\beta}(t-1)^R(\underline{\beta}'(t-1)^R)^\infty & \text{if } \underline{\nu} \text{ is even,} \\ (\underline{\beta}t^L)^\infty & \text{if } \underline{\nu} \text{ is odd.} \end{cases}$$

This ends the proof of the first part of statement (a).

Now, we prove the second part of statement (a) in this case. Let P be the (mod 1) orbit of 0 by F. Then $0, c_F \in P$. Let $x_0 = \min(P \cap (c_F, 1])$, $x_1 = \max(P \cap (0, c_F))$ and $J = (c_F, x_0)$ if $\underline{\gamma}$ is even and $J = (x_1, c_F)$ if $\underline{\gamma}$ is odd. Let $G \in \mathcal{A} \cap \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ be close enough to F such that $c_G \in J$, $G|_{[0,1]\setminus J} = F|_{[0,1]\setminus J}$ and $G(c_G) \in (F(c_F), \min(P \cap (F(c_F), \infty)))$. Thus, clearly, $\underline{\hat{I}}_{G}(0) = \underline{\beta}k^M$. From the proof of the previous case, since $\underline{\beta}$ has always different parity than $\underline{\nu}$, we get

$$\underline{\hat{I}}_{G}(0^+) = \begin{cases} (\underline{\beta}t^L)^\infty & \text{if } \underline{\nu} \text{ is odd } (\underline{\beta} \text{ even}) \\ \underline{\beta}(t-1)^R(\underline{\beta}'(t-1)^R)^\infty & \\ & \text{if } \underline{\nu} \text{ is even } (\underline{\beta} \text{ odd}) \end{cases}$$

and the proof of (a) follows by using G instead of F. Statement (b) follows in a similar way. ■

The next lemma gives some properties of the sequences in $\mathcal{P}_\epsilon$ an $\mathcal{P}_\delta$.

Lemma 4.2. *Let* $\underline{\beta} = \beta_1 \cdots \beta_{n-1} \in \Xi$. *The following statements hold.*

(a) *If* $\underline{\beta}d^M \in \mathcal{P}_\epsilon$, *then* $(\underline{\beta}d^L)^\infty$ *and* $(\underline{\beta}'(d-1)^R)^\infty$ *are periodic of period* n.

(b) *If* $\underline{\beta}d^C \in \mathcal{P}_\delta$, *then* $(\underline{\beta}d^L)^\infty$ *and* $(\underline{\beta}d^R)^\infty$ *are periodic of period* n.

Proof. By the minimality of $\underline{\beta}d^M$ we have $S^j(\underline{\beta}d^M) > \underline{\beta}d^M$ for $j = 1, 2, \ldots, n-1$. Assume that $(\underline{\beta}d^L)^\infty$ is periodic of period $k < n$ and set $m = n/k$. Then $\underline{\beta}d^L = (\beta_1 \cdots \beta_{k-1}d^L)^m$ and, hence,

$$\begin{aligned}(\beta_1 \cdots \beta_{k-1}d^L)^{m-1}\beta_1 \cdots \beta_{k-1}d^M \\ = \underline{\beta}d^M < S^{n-k}(\underline{\beta}d^M) = \beta_1 \cdots \beta_{k-1}d^M.\end{aligned}$$

In consequence $\beta_1 \cdots \beta_{k-1}$ is even and hence,

$$\beta_1 \cdots \beta_{k-1}d^L > \beta_1 \cdots \beta_{k-1}(d-1)^R.$$

Since $\underline{\beta}d^M \in \mathcal{P}_\epsilon$ then,

$$\beta_1 \cdots \beta_{k-1}d^L(\underline{\beta}(d-1)^R)(\underline{\beta}'(d-1)^R)^\infty \in \mathcal{E}_\epsilon.$$

Hence, by Proposition 2.7(a), we have

$$\begin{aligned}&(\beta_1 \cdots \beta_{k-1}d^L)^{m-1}\beta_1 \cdots \\ &\quad \beta_{k-1}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty \\ &\quad \le \beta_1 \cdots \beta_{k-1}(d-1)^R \cdots \\ &\quad = S^{n-k}(\underline{\beta}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty);\end{aligned}$$

a contradiction. The proof of statement (a) in the case $(\underline{\beta}'(d-1)^R)^\infty$, and statement (b) follow in a similar way. ■

The next lemma studies the relation between the periodic sequences in $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ and their shifts.

Lemma 4.3. *The following statements hold.*

(a) *Let* $\underline{\beta} = (\beta_1 \cdots \beta_n)^\infty \in \mathcal{E}_\epsilon$. *Then* $S^j(\underline{\beta}) > \underline{\beta}^*$ *for all* $j = 1, 2, \ldots, n-1$ *where* $\underline{\beta}^*$ *is either* $\underline{\beta}$ *if* $\beta_j = d^L \cdots$ *or* $\underline{\beta}'$ *if* $\beta_j = d^R \cdots$.

(b) *Let* $\underline{\beta} = (\beta_1 \cdots \beta_n)^\infty \in \mathcal{E}_\delta$. *Then* $S^j(\underline{\beta}) < \underline{\beta}$ *for all* $j = 1, 2, \ldots, n-1$.

Proof. We prove (a). Statement (b) follows in a similar way. Let $j \in \{2, \ldots, n\}$. If $\beta_{j-1} = d^L$ for some $d \in \mathbb{Z}$ then, by Proposition 2.7, since $S^{j-1}(\underline{\beta}) \geq \underline{\beta}$ and $S^{j-1}(\underline{\beta}) \neq \underline{\beta}$ the lemma follows in an obvious way. If $\beta_{j-1} = d^R$ for some $d \in \mathbb{Z}$, we have $S^{j-1}(\underline{\beta}) \geq \underline{\beta}'$. Assume that $S^{j-1}(\underline{\beta}) = \underline{\beta}'$. Then

$$\overbrace{\beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_{j-1}}^{n}\overbrace{\beta_j\cdots\beta_n\beta_1\cdots\beta_{j-1}}^{n}\cdots$$
$$= \beta_1'\beta_2\cdots\beta_n\beta_1\cdots\beta_n\cdots$$

and, hence, $\beta_1' = \beta_j = \beta_1$; a contradiction. This ends the proof of (a). ■

The proof of the following lemma follows by direct computation.

Lemma 4.4. *The following statements hold.*

(a) *Let* $f \in U(I)$ *be negative. If* $f(c) \geq c$, *then* $k(f) = L^\infty$. *Otherwise* $k(f) = RS(k(f))$ *and there exists* $c_- < c < c_+$ *with* $f(c_-) = f(c_+) = c$. *Then the following statements hold.*
  - (a.1) $\underline{\theta}(x) = RL\cdots$ *if and only if* $x \in [\inf I, c_-)$.
  - (a.2) $\underline{\theta}(x) = RR\cdots$ *if and only if* $x \in (c_-, c)$.
  - (a.3) $\underline{\theta}(x) = LR\cdots$ *if and only if* $x \in (c, c_+)$.
  - (a.4) $\underline{\theta}(x) = LL\cdots$ *if and only if* $x \in (c_+, \sup I]$.

(b) *Let* $f \in U(I)$ *be positive. If* $f(c) \leq c$, *then* $k(f) = L^\infty$. *Otherwise* $k(f) = RS(k(f))$ *and there exists* $c_- < c < c_+$ *with* $f(c_-) = f(c_+) = c$. *Then the following statements hold.*
  - (b.1) $\underline{\theta}(x) = LR\cdots$ *if and only if* $x \in (c_+, \sup I]$.
  - (b.2) $\underline{\theta}(x) = RR\cdots$ *if and only if* $x \in (c, c_+)$.
  - (b.3) $\underline{\theta}(x) = LR\cdots$ *if and only if* $x \in (c_-, c)$.
  - (b.4) $\underline{\theta}(x) = LL\cdots$ *if and only if* $x \in [\inf I, c_-)$.

Let $I, J \subset \mathbb{R}$ be two closed intervals. Let $f\colon I \to I$ and $g\colon J \to J$ be two continuous maps. We say that f is *topologically conjugate* to g if there exists a homeomorphism $H\colon I \to J$ such that $h \circ f = g \circ h$. From [Collet & Eckmann, 1980] (see also [De Melo & Van Strien, 1993]) we have that if $f \in U(I)$ and $g \in U(J)$ are topologically conjugate then $k(f) = k(g)$.

The next proposition justifies the definition of the $\star$-operators in the case $\underline{\beta}$ empty.

Proposition 4.5. *Let* $\underline{K} \in \mathcal{K}$ *and* $d \in \mathbb{Z}$. *Then the following statements hold.*

(a) *There exist* $F \in \mathcal{A}$ *and* $J \subset \mathbb{R}$, *a closed interval containing* 0, *such that* $(F-d)|_J$ *is unimodal with* $k((F-d)|_J) = \underline{K}$ *and* $\underline{\hat{I}}_F(0^+) = d^M \star_\epsilon \underline{K}$.

(b) *There exists* $F \in \mathcal{A}$ *and* $J \subset \mathbb{R}$, *a closed interval containing* c_F, *such that* $(F-d)|_J$ *is unimodal with* $k((F-d)|_J) = \underline{K}$ *and* $\underline{\hat{I}}_F(c_F^-) = d^C \star_\delta \underline{K}$.

Proof. Let $f \in U(I)$ be negative such that $k(f) = \underline{K}$. Take $\epsilon > 0$ and $J = [-\epsilon, \epsilon]$, and let $h\colon I \to J$ be the unique increasing map such that $h(c) = 0$ and h is affine in $[\min I, c]$, $[c, \max I]$. Let $F \in \mathcal{A}$ be such that $F(x) = h \circ f \circ h^{-1}(x) + d$ for each $x \in J$. Clearly, $(F-d)|_J$ is topologically conjugate to f. Then $k((F-d)|_J) = k(f) = K_1K_2\cdots$. We observe that since $(F-d)$ maps J into itself we have $F(J) \subset J + d$. Since $F \in \mathcal{L}$ we have that for all $j \geq 1$, $F^j(J) \subset J + jd$. On the other hand, since $s((F-d)^j(0^+)) = s(F^j(0^+))$ we get that for all $j \geq 1$, $s(F^j(0^+)) = K_j$. Assume that $(F-d)(0) \geq 0$, then $f(c) \geq c$ and, from Lemma 4.4, we have $k(f) = L^\infty$. Since $F(0) \geq d$ we have $F^j(0) \in [0, \epsilon] + jd$ for all $i \geq 0$. Then for all $i \geq 1$ we have $d(F^j(0^+)) = jd - (j-1)d = d$ and $\underline{\hat{I}}_F(0^+) = d^M \star_\epsilon \underline{K}$.

Now, assume that $(F-d)(0) < 0$. Then $f(c) < c$ and, from Lemma 4.4(a), we have $K_1 = R$. Since $F(0) < d$ we obtain that $F(0) \in [-\epsilon, 0] + d$. Then $d(0^+) = d - 1$ and so $\underline{\hat{I}}_F(0^+) = (d-1)^R \cdots$. Let $j \geq 2$. Assume that $K_{j-1}K_j = RL$. Then $S^{j-2}(k(f)) = \underline{\theta}(f^{j-2}(x)) = RL\cdots$ for $x > f(c)$, close enough to $f(c)$. From Lemma 4.4(a.1) we have $f^{j-1}(c) \in [\min I, c_-)$ and, hence, $F^{j-1}(0) \in [-\epsilon, h(c_-)) + (j-1)d$. Moreover $F^j(0) \in (0, \epsilon] + jd$. Then $d(F^{j-1}(0^+)) = jd - ((j-1)d - 1) = d + 1$. If $K_{j-1}K_j = LL$, then, $F^{j-1}(0) \in (h(c_+), \epsilon] + (j-1)d$

and $F^j(0) \in (0,\epsilon] + id$. So $d(F^{j-1}(0^+)) = jd - (j-1)d = d$. If $K_{j-1}K_j = RR$, then $F^{j-1}(0^+) \in (h(c_-),0)+(j-1)d$ and $F^j(0^+) \in [-\epsilon,0)+jd$. Thus, $d(F^{j-1}(0^+)) = (jd-1)-((j-1)d-1) = d$. Finally, if $K_{j-1}K_j = LR$ then $F^{j-1}(0) \in (0,h(c_+))+(j-1)d$, $F^j(0) \in [-\epsilon,0)+jd$. Therefore, $d(F^{j-1}(0^+)) = (jd-1)-(j-1)d = d-1$. From the definition of $\star_\epsilon$ we have $\underline{\hat{I}}_F(0^+) = d^M \star_\epsilon \underline{K}$.

Statement (b) follows in a similar way. ■

4.2. *Proof of Theorem A*

We will only prove Theorem A for $\star_\epsilon$. The proof for $\star_\delta$ follows in a similar way. Let $\underline{\gamma} = \underline{\beta}d^M \in \mathcal{P}_\epsilon$ and $\underline{K} \in \mathcal{K}$. We will only prove the statement in the case $\underline{\beta}$ even. The case $\underline{\beta}$ odd follows analogously. First we are going to prove that $\underline{\gamma} \star_\epsilon \underline{K} \in \mathcal{E}_\epsilon$. If $\underline{\beta}$ is empty then this follows from Proposition 4.5(a), the definition of $\mathcal{E}_\epsilon$ and Theorem 2.6. Assume now that $\underline{\beta}$ is not empty. We note that $\underline{\gamma} \star_\epsilon L^\infty = (\underline{\beta}d^L)^\infty$ and $\underline{\gamma} \star_\epsilon R^\infty = \underline{\beta}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty$. Since $\underline{\beta}d^M \in \mathcal{P}_\epsilon$ these two sequences belong to $\mathcal{E}_\epsilon$ and we are done. Thus, we can assume that $\underline{K} \notin \{L^\infty, R^\infty\}$. From [Collet & Eckmann, 1980] we have $\underline{K} = RL\cdots$. Let $\underline{\beta} = \beta_1\beta_2\cdots\beta_{n-1}$, $\underline{K} = K_1K_2\cdots$ and $j = nm$ with $m \geq 0$. Then we have $\underline{\gamma}\star_\epsilon \underline{K} = \underline{\beta}\varphi(K_1,d)\chi(K_1,\underline{\beta})\varphi(K_2,d)\chi(K_2,\underline{\beta})\cdots$. It is not difficult to see that, since $\underline{K}$ is maximal, then $\varphi(K_1,d)\varphi(K_2,d)\cdots \in \mathcal{AD}$ is minimal. Therefore, if $K_{m-1} = L$ then

$$\begin{aligned} S^j(\underline{\gamma} \star_\epsilon \underline{K}) &= \underline{\beta}\varphi(K_m,d)\chi(K_m,\underline{\beta})\varphi(K_{m+1},d)\cdots \\ &\geq \underline{\gamma} \star_\epsilon \underline{K}. \end{aligned}$$

Otherwise,

$$\begin{aligned} S^j(\underline{\gamma} \star_\epsilon \underline{K}) &= \underline{\beta}'\varphi(K_m,d)\chi(K_m,\underline{\beta})\varphi(K_{m+1},d)\cdots \\ &\geq (\underline{\gamma} \star_\epsilon \underline{K})' \end{aligned}$$

and, by Proposition 2.7(a), we are done. So, take now $j = nm+p$ with $m \geq 0$, $1 \leq p < n$. Then we have to compare

$$\begin{aligned} &S^j(\underline{\gamma} \star_\epsilon \underline{K}) \\ &\quad = \beta_{p+1}\cdots\beta_{n-1}\varphi(K_m,d)\chi(K_m,\underline{\beta})\varphi(K_{m+1},d)\cdots \\ &\quad = \underline{\upsilon}\varphi(K_m,d)\chi(K_m,\underline{\beta})\varphi(K_{m+1},d)\cdots, \end{aligned} \tag{2}$$

with

$$\begin{aligned} &\underline{\gamma} \star_\epsilon \underline{K} \\ &\quad = \beta_1\cdots\beta_{n-p-1}\beta_{n-p}\cdots\beta_{n-1}\varphi(K_1,d)\cdots \\ &\quad = \overline{\underline{\upsilon}}\beta_{n-p}\cdots\beta_{n-1}\varphi(K_1,d)\cdots. \end{aligned} \tag{3}$$

Set

$$\underline{\upsilon}^* = \begin{cases} \overline{\underline{\upsilon}} & \text{if } \beta_p = t^L, \\ \overline{\underline{\upsilon}}' & \text{if } \beta_p = t^R, \end{cases}$$

where $t \in \mathbb{Z}$ and

$$(\underline{\gamma} \star_\epsilon \underline{K})^* = \underline{\upsilon}^*\beta_{n-p}\cdots\beta_{n-1}\varphi(K_1,d)\cdots.$$

By Proposition 2.7(a) we have to show that

$$S^j(\underline{\gamma} \star_\epsilon \underline{K}) \geq (\underline{\gamma} \star_\epsilon \underline{K})^*.$$

Since $\underline{\beta}d^M \in \mathcal{P}_\epsilon$, $\underline{\beta}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty$, $(\underline{\beta}d^L)^\infty \in \mathcal{E}_\epsilon$. Therefore, by Proposition 2.7(a) and Lemma 4.3(a), for all $1 \leq p < n$, we have

$$\begin{aligned} &\underline{\upsilon}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty \\ &\qquad \geq \underline{\upsilon}^*\beta_{n-p}\cdots\beta_{n-1}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty \end{aligned} \tag{4}$$

and

$$\underline{\upsilon}d^L(\underline{\beta}d^L)^\infty > \underline{\upsilon}^*\beta_{n-p}\cdots\beta_{n-1}d^L(\underline{\beta}d^L)^\infty. \tag{5}$$

Clearly if $\underline{\upsilon} \neq \underline{\upsilon}^*$ then $S^j(\underline{\gamma} \star_\epsilon \underline{K}) > (\underline{\gamma} \star_\epsilon \underline{K})^*$ and we are done. So assume that $\underline{\upsilon} = \underline{\upsilon}^*$. First, we consider the case $\underline{\upsilon}$ even. If $\varphi(K_m,d) = d^L$ then either $d^L > \beta_{n-p}$ and, from (2) and (3), we see that $S^j(\underline{\gamma} \star_\epsilon \underline{K}) > (\underline{\gamma} \star_\epsilon \underline{K})^*$ or $d^L = \beta_{n-p}$. In the latter, since $\underline{\upsilon}d^L$ is even, from (5) we have that

$$(\underline{\beta}d^L)^\infty > \beta_{n-p+1}\cdots\beta_{n-1}d^L(\underline{\beta}d^L)^\infty;$$

a contradiction with Lemma 4.3(a). Now, let $\varphi(K_m,d) = (d-1)^R$. From (4) we have

$$\beta_{n-p} \leq (d-1)^R.$$

If $\beta_{n-p} < (d-1)^R$, then $S^j(\underline{\gamma} \star_\epsilon \underline{K}) > (\underline{\gamma} \star_\epsilon \underline{K})^*$ by (2) and (3). So, assume that $\beta_{n-p} = (d-1)^R$. Then $\underline{\upsilon}(d-1)^R = \underline{\upsilon}^*(d-1)^R$ is odd and, from (4), we have

$$\begin{aligned} &(\underline{\beta}'(d-1)^R)^\infty \\ &\qquad \leq \beta_{n-p+1}\cdots\beta_{n-1}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty. \end{aligned}$$

We note that

$$\begin{aligned} &S^{n-p}((\underline{\beta}'(d-1)^R)^\infty) \\ &\qquad = (\beta_{n-p+1}\cdots\beta_{n-1}(d-1)^R\beta_1'\cdots\beta_{n-p})^\infty. \end{aligned}$$

Therefore, if

$$\begin{aligned} &\beta_1'\beta_2\cdots\beta_{n-1}(d-1)^R \\ &\qquad = \beta_{n-p+1}\cdots\beta_{n-1}(d-1)^R\beta_1'\cdots\beta_{n-p} \end{aligned}$$

then, $S^{n-p}((\underline{\beta}'(d-1)^R)^\infty) = (\underline{\beta}'(d-1)^R)^\infty$ which is a contradiction by Lemma 4.2(a). In consequence,

$$\begin{aligned} &\beta_1'\beta_2\cdots\beta_{n-1}(d-1)^R \\ &\qquad < \beta_{n-p+1}\cdots\beta_{n-1}(d-1)^R\beta_1'\cdots\beta_{n-p} \end{aligned} \tag{6}$$

and, by (2) and (3), $S^j(\underline{\gamma} \star_\epsilon \underline{K}) > (\underline{\gamma} \star_\epsilon \underline{K})^*$ if $\varphi(K_{m+1}, d) = (d-1)^R$ (recall that $\varphi(K_1, d) = (d-1)^R$). Now, assume that $\varphi(K_{m+1}, d) = d^L$. If

$$\beta_1' \beta_2 \cdots \beta_{n-1} < \beta_{n-p+1} \cdots \beta_{n-1}(d-1)^R \beta_1' \cdots \beta_{n-p-1}$$

then we also have $S^j(\underline{\gamma} \star_\epsilon \underline{K}) > (\underline{\gamma} \star_\epsilon \underline{K})^*$. Otherwise, since $\underline{\beta}'$ is even, from (6), we have

$$\beta_1' \beta_2 \cdots \beta_{n-1} = \beta_{n-p+1} \cdots \beta_{n-1}(d-1)^R \beta_1' \cdots \beta_{n-p-1}$$

and $\beta_{n-p} \geq d^L$. If $\beta_{n-p} > d^L$ then the statement follows as above. Hence, $\beta_{n-p} = d^L$ and so

$$\beta_1' \beta_2 \cdots \beta_{n-1} d^L = \beta_{n-p+1} \cdots \beta_{n-1}(d-1)^R \beta_1' \cdots \beta_{n-p-1} \beta_{n-p}.$$

This is a contradiction because the left-hand side of the above equation has different parity than the right-hand side. The case $\underline{\upsilon}$ odd is handled by analogy. This ends the proof of the first statement of the theorem.

Now, we are going to prove that $\underline{\gamma}\star_\epsilon$ is order reversing. Let $\underline{K}, \underline{K}' \in \mathcal{K}$ be such that $\underline{K} < \underline{K}'$. Set $\underline{K} = K_1 K_2 \cdots$ and $\underline{K}' = K_1' K_2' \cdots$. Then there exists $n \geq 1$ such that $K_1 \cdots K_{n-1} = K_1' \cdots K_{n-1}'$ and $K_n < K_n'$ if $K_1 \cdots K_{n-1}$ is even and $K_n > K_n'$ if $K_1 \cdots K_{n-1}$ is odd. We will only consider the case $K_1 \cdots K_{n-1}$ even. The proof in the case odd follows similarly. Then we have $K_n = L < R = K_n'$. Assume that $\underline{\beta}$ is not the empty sequence. Then

$$\underline{\gamma} \star_\epsilon \underline{K} = \underline{\beta} d_1^{K_1} \chi(K_1, \underline{\beta}) d_2^{K_2} \cdots \chi(K_{n-1}, \underline{\beta}) d_n^{K_n} \cdots$$

and

$$\underline{\gamma} \star_\epsilon \underline{K}' = \underline{\beta} t_1^{K_1'} \chi(K_1, \underline{\beta}) t_2^{K_2'} \cdots \chi(K_{n-1}, \underline{\beta}) t_n^{K_n'} \cdots .$$

Hence,

$$\underline{\beta} d_1^{K_1} \chi(K_1, \underline{\beta}) d_2^{K_2} \cdots \chi(K_{n-1}, \underline{\beta}) = \underline{\beta} t_1^{K_1'} \chi(K_1, \underline{\beta}) t_2^{K_2'} \cdots \chi(K_{n-1}, \underline{\beta}),$$

$d_n^{K_n} = d^L$, $t_n^{K_n'} = (d-1)^R$ and the sequence $\underline{\beta} d_1^{s_1} \chi(K_1, \underline{\beta}) d_2^{s_2} \cdots \chi(K_{n-1}, \underline{\beta})$ is even. Then, clearly, $\underline{\gamma}\star_\epsilon \underline{K}' < \underline{\gamma} \star_\epsilon \underline{K}$. Now, assume that $\underline{\beta}$ is the empty sequence. Then

$$\underline{\gamma} \star_\epsilon \underline{K} = d_1^{K_1} \cdots d_{n-1}^{K_{n-1}} d_n^{K_n} \cdots$$

and

$$\underline{\gamma} \star_\epsilon \underline{K}' = t_1^{K_1'} \cdots t_{n-1}^{K_{n-1}'} t_n^{K_n'} \cdots = d_1^{K_1} \cdots d_{n-1}^{K_{n-1}} t_n^{K_n'} \cdots$$

and the result follows as in the case $\underline{\beta}$ not empty. From the assumptions only one of the following two possibilities can occur: either $K_{n-1}K_n = RL$ and $K_{n-1}'K_n' = RR$, or $K_{n-1}K_n = LL$ and $K_{n-1}'K_n' = LR$. Assume that $K_{n-1}K_n = RL$ and $K_{n-1}'K_n' = RR$. Then $d_n^{K_n} = (d+1)^L$ and $t_n^{K_n'} = d^R$ and $\underline{\gamma} \star_\epsilon \underline{K}' < \underline{\gamma} \star_\epsilon \underline{K}$. Now, let $K_{n-1}K_n = LL$ and $K_{n-1}'K_n' = LR$. Then $d_n^{K_n} = d^L$ and $t_n^{K_n'} = (d-1)^R$ and also, $\underline{\gamma} \star_\epsilon \underline{K}' < \underline{\gamma} \star_\epsilon \underline{K}$. This concludes the proof of the second statement.

The third statement of the theorem follows from [Collet & Eckmann, 1980, Theorem II.2.7]. This ends the proof of Theorem A.

4.3. *Remarks to Theorem A*

In the preceding subsection we have shown that the unimodal boxes $\underline{\gamma} \star_\epsilon \mathcal{K}$ and $\underline{\gamma} \star_\delta \mathcal{K}$ are connected. However, the topological structure of the spaces

$$\mathcal{E}_\epsilon(\underline{\gamma}) = (\underline{\gamma} \star_\epsilon \mathcal{K}) \times \mathcal{E}_\delta$$

(resp.

$$\mathcal{E}_\delta(\underline{\gamma}) = \mathcal{E}_\epsilon \times (\underline{\gamma} \star_\delta \mathcal{K}))$$

is much more complicated. We illustrate this fact by the following examples. Let $\underline{\gamma} = 0^L 1^M$. Then $\underline{\gamma} \star_\epsilon L^\infty = (0^L 1^L)^\infty$ and $\underline{\gamma} \star_\epsilon RL^\infty = 0^L 0^R 1^L 1^L (0^L 1^L)^\infty$. Therefore, $\underline{\gamma} \star_\epsilon \mathcal{K} = [(0^L 1^L)^\infty, 0^L 0^R 1^L 1^L (0^L 1^L)^\infty]$.

Example 1. *The space $\mathcal{E}_\epsilon(\underline{\gamma})$ contains "accumulating" holes in $\mathcal{E}$ consisting of "horizontal lines".*

Let $\underline{\alpha} = (3^L)^\infty \in \mathcal{E}_\delta$. Clearly $[\underline{\gamma} \star_\epsilon RL^\infty, \underline{\gamma} \star_\epsilon L^\infty] \times \{\underline{\alpha}\} \subset \mathcal{E}^* \subset \mathcal{E}$. Let now $\underline{\alpha}_n = (3^L)^n (-1^L)^\infty \in \mathcal{E}_\delta$. Then $\underline{\alpha}_n < \underline{\alpha}_{n+1} < \underline{\alpha}$ for all $n \in \mathbb{N}$. Since $S^{n-1}(\underline{\alpha}_n) = (-1^L)^\infty < \underline{\omega}$ for all $\underline{\omega} \in \underline{\gamma} \star_\epsilon \mathcal{K}$ we have that for all $n \in \mathbb{N}$, $[\underline{\gamma} \star_\epsilon RL^\infty, \underline{\gamma} \star_\epsilon L^\infty] \times \{\underline{\alpha}_n\} \notin \mathcal{E}$. We also note that $d(\underline{\alpha}_n, \underline{\alpha})$ tends to 0 as $n \to \infty$.

Example 2. *The "accumulating" holes in $\mathcal{E}$ consisting of "horizontal lines" are intertwined with "horizontal lines" inside $\mathcal{E}$.*

Let $\underline{\beta}_n = (3^L)^n (2^L)^\infty \in \mathcal{E}_\delta$. Then for all $n \in \mathbb{N}$, $[\underline{\gamma} \star_\epsilon RL^\infty, \underline{\gamma} \star_\epsilon L^\infty] \times \{\underline{\beta}_n\} \subset \mathcal{E}$ but $d(\underline{\alpha}_n, \underline{\beta}_n) = \sum_{i=n+1}^\infty 1/2^i = 1/2^n$ which tends to 0 when $n \to \infty$.

Example 3. *There exist "rectangles" in*

$$\mathcal{E} \cap (\underline{\gamma} \star_\epsilon \mathcal{K} \times \underline{\beta} \star_\delta \mathcal{K}).$$

Let $\underline{\beta} = 3^M$. Then $\underline{\beta} \star_\delta L^\infty = (3^L)^\infty$ and $\underline{\beta} \star_\delta RL^\infty = 3^R (3^L)^\infty$. It is not difficult to see that $[\underline{\gamma} \star_\epsilon RL^\infty, \underline{\gamma} \star_\epsilon L^\infty] \times [\underline{\beta} \star_\delta L^\infty, \underline{\beta} \star_\delta RL^\infty] \subset \mathcal{E}$.

5. Proof of Theorem B

In Sec. 5.1 we give some preliminary results and prove Proposition 3.1 and in Sec. 5.2 we prove Theorem B.

5.1. *Preliminary results*

We start by introducing some technical results about the sequences $\hat{\underline{I}}^*_\delta(a), \hat{\underline{I}}_\epsilon(a), \hat{\underline{I}}_\delta(a)$ and $\hat{\underline{I}}^*_\epsilon(a)$. The following lemma is due to Alsedà and Mañosas [1990].

Lemma 5.1. *The following statements hold*:

(a) *If* $a = p/q$ *with* $(p,q) = 1$, *then* $\hat{\underline{I}}_\epsilon(a)$ *and* $\hat{\underline{I}}_\delta(a)$ *are periodic with period* q (*i.e.* $S^q(\hat{\underline{I}}_\epsilon(a)) = \hat{\underline{I}}_\epsilon(a)$ *and* $S^q(\hat{\underline{I}}_\delta(a)) = \hat{\underline{I}}_\delta(a)$).

(b) *Let* $a, b \in \mathbb{R}$ *with* $a < b$. *Then* $\hat{\underline{I}}_\epsilon(a) < \hat{\underline{I}}_\epsilon(b)$, $\hat{\underline{I}}_\delta(a) < \hat{\underline{I}}_\delta(b)$, $\hat{\underline{I}}^*_\epsilon(a) < \hat{\underline{I}}^*_\epsilon(b)$ *and* $\hat{\underline{I}}^*_\delta(a) < \hat{\underline{I}}^*_\delta(b)$.

From Theorem 2.6 and Proposition 2.7 we have the following.

Lemma 5.2. *Let* $a \in \mathbb{R}$. *Then* $\hat{\underline{I}}^*_\delta(a), \hat{\underline{I}}_\epsilon(a) \in \mathcal{E}_\epsilon$ *are minimal and* $\hat{\underline{I}}_\delta(a), \hat{\underline{I}}^*_\epsilon(a) \in \mathcal{E}_\delta$ *are maximal.*

Lemma 5.3. *Let* $a \in \mathbb{R}$. *Then* $\epsilon_1(a) \le \epsilon_i(a) \le \epsilon_1(a)+1$ *and* $\delta_1(a)-1 \le \delta_i(a) \le \delta_1(a)$ *for all* $i \ge 1$.

Proof. We recall that $\epsilon_i(a) = E(ia) - E((i-1)a) = E(a + (i-1)a) - E((i-1)a)$. Then, from the fact that $E(x) + E(y) \le E(x+y) \le E(x) + E(y) + 1$ for all $x, y \in \mathbb{R}$, we have $\epsilon_1(a) \le \epsilon_i(a) \le \epsilon_1(a) + 1$ for all $i \ge 1$. In a similar way we can prove that $\delta_1(a) - 1 \le \delta_i(a) \le \delta_1(a)$ for all $i \ge 1$. ∎

The next lemma follows by direct computation.

Lemma 5.4. *Let* $a \in \mathbb{Z}$ *then* $\epsilon_i(a) = \delta_i(a) = a$ *for all* $i > 0$.

Lemma 5.5. *Let* $a \in \mathbb{R}$. *Then* $\hat{\underline{I}}_\epsilon(a), \hat{\underline{I}}^*_\delta(a) \in \Sigma_{\tilde{E}(a)}$ *and* $\hat{\underline{I}}_\delta(a), \hat{\underline{I}}^*_\epsilon(a) \in \Sigma_{E(a)}$.

Proof. From Lemmas 5.1(a) and 2.5, the fact that $\epsilon_1(a) = \delta_1(a) - 1 = E(a) = \tilde{E}(a)$ if $a \notin \mathbb{Z}$ and Lemma 5.3 the statement follows when $a \notin \mathbb{Z}$. If $a \in \mathbb{Z}$, then from Lemma 5.4 we have $\hat{\underline{I}}_\epsilon(a) = \hat{\underline{I}}_\delta(a) = (a^L)^\infty$, $\hat{\underline{I}}^*_\epsilon(a) = (a+1)^L(a^L)^\infty$ and $\hat{\underline{I}}^*_\delta(a) = (a-1)^L(a^L)^\infty$. Since $E(a) = a$ and $\tilde{E}(a) = a - 1$ the statement follows also in this case. ∎

We now have the following corollaries which will be useful in the next section.

Corollary 5.6. *Let* $a \in \mathbb{R}$. *Then* $\hat{\underline{I}}_\epsilon(a), \hat{\underline{I}}^*_\delta(a) \in \mathcal{B}_\epsilon(\tilde{E}(a))$ *and* $\hat{\underline{I}}_\delta(a), \hat{\underline{I}}^*_\epsilon(a) \in \mathcal{B}_\delta(E(a)))$.

Proof. It follows from Lemmas 5.5 and 5.2. ∎

Corollary 5.7. *Let* $a \in \mathbb{R}$. *Then*

$$\hat{\underline{I}}_\epsilon(a) = \pi^{-1}_{\tilde{E}(a)}(\hat{\underline{I}}_\epsilon(\tilde{D}(a))),$$

$$\hat{\underline{I}}^*_\delta(a) = \pi^{-1}_{\tilde{E}(a)}(\hat{\underline{I}}^*_\delta(\tilde{D}(a))),$$

$$\hat{\underline{I}}_\delta(a) = \pi^{-1}_{E(a)}(\hat{\underline{I}}_\delta(D(a))) \quad \textit{and}$$

$$\hat{\underline{I}}^*_\epsilon(a) = \pi^{-1}_{E(a)}(\hat{\underline{I}}^*_\epsilon(D(a))).$$

Proof. Let $a \in \mathbb{R}$. Then

$$\begin{aligned}
\epsilon_i(a) &= E(ia) - E((i-1)a) \\
&= E(i(D(a) + E(a))) \\
&\quad - E((i-1)(D(a) + E(a)))) \\
&= E(iD(a)) + iE(a) \\
&\quad - E((i-1)D(a)) - (i-1)E(a) \\
&= E(iD(a)) - E((i-1)D(a)) + E(a) \\
&= \epsilon_i(D(a)) + E(a).
\end{aligned}$$

If $a \notin \mathbb{Z}$, since $\tilde{E}(a) = E(a)$ and $\tilde{D}(a) = D(a)$ we have $\hat{\underline{I}}_\epsilon(a) = \pi^{-1}_{\tilde{E}(a)}(\hat{\underline{I}}_\epsilon(\tilde{D}(a)))$. Otherwise, by Lemma 5.4, $\hat{\underline{I}}_\epsilon(a) = (E(a)^L)^\infty$ and since $\tilde{D}(a) = 1$ and $\tilde{E}(a) = E(a) - 1$ we get $\hat{\underline{I}}_\epsilon(a) = \pi^{-1}_{\tilde{E}(a)}(\hat{\underline{I}}_\epsilon(\tilde{D}(a)))$. Also, $\hat{\underline{I}}^*_\epsilon(a) = \pi^{-1}_{E(a)}(\hat{\underline{I}}^*_\epsilon(D(a)))$ if $a \notin \mathbb{Z}$. Otherwise, $\hat{\underline{I}}^*_\epsilon(a) = (E(a)+1)^L(E(a)^L)^\infty = \pi^{-1}_{E(a)}(\hat{\underline{I}}^*_\epsilon(D(a)))$. The other two cases follow in a similar way. ∎

Lemma 5.8. *Let* $a = p/q \in \mathbb{Q}^*$ *be with* $(p,q) = 1$. *Then* $\epsilon_q(a) = \epsilon_1(a) + 1$.

Proof. If $\epsilon_q(a) \ne \epsilon_1(a) + 1$ then, by Lemma 5.3, we can assume that $\epsilon_q(a) = \epsilon_1(a)$. Then, by Lemma 5.1(a), $\hat{\underline{I}}_\epsilon(a) = (\epsilon_1(a)^L \underline{r}(a) \epsilon_1(a)^L)^\infty$. By Lemma 5.2,

$$S^{q-1}(\hat{\underline{I}}_\epsilon(a)) = (\epsilon_1(a)^L \epsilon_1(a)^L \underline{r}(a))^\infty \ge \hat{\underline{I}}_\epsilon(a).$$

Thus, by Lemma 5.3, $\epsilon_2(a) = \epsilon_1(a)$ and, proceeding inductively, we obtain that $\hat{\underline{I}}_\epsilon(a) = (\epsilon_1(a)^L)^\infty$; a contradiction by Lemma 5.1(a). ∎

Remark 5.9. In view of Lemmas 2.5 and 5.8, for $a \in \mathbb{Q}^*$, we can write

$$\hat{\underline{I}}^*_\delta(a) = \epsilon_1(a)^L \underline{r}(a)(\epsilon_1(a)^L(\epsilon_1(a)+1)^L \underline{r}(a))^\infty,$$

$$\hat{\underline{I}}_\epsilon(a) = (\epsilon_1(a)^L \underline{r}(a)(\epsilon_1(a)+1)^L)^\infty,$$

$\underline{\hat{I}}_\delta(a) = ((\epsilon_1(a)+1))^L \underline{r}(a)\epsilon_1(a)^L)^\infty$, and

$\underline{\hat{I}}^*_\epsilon(a) = (\epsilon_1(a)+1)^L \underline{r}(a)((\epsilon_1(a)+1)^L \epsilon_1(a)^L \underline{r}(a))^\infty$.

The above observation already allows us to prove Proposition 3.1.

Proof of Proposition 3.1. We will only prove that $a \odot_\epsilon (1^L)^\infty = \underline{\hat{I}}_\epsilon(a)$. The proof of the other three statements follows similarly. From Corollary 5.7 and the definition of $\odot_\epsilon$ we can assume that $a \in (0,1]$. Now, the statement follows directly from the definitions if $a \notin \mathbb{Q}^*$. If $a \in \mathbb{Q}^*$ the statement follows from Remark 5.9 and the fact that $\epsilon_1(a) = 0$. ■

5.2. *Proof of Theorem B*

We start with a technical lemma.

Lemma 5.10. *Let $a = p/q \in \mathbb{Q}^*$ be with $(p,q) = 1$. Then*

(a) $\epsilon_1(a)^L(\epsilon_1(a)+1)^L\underline{r}(a) > \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L$.

(b) *For $1 < j \le q-1$ we have*

$$\begin{aligned}&\epsilon_j(a)^L \cdots \epsilon_{q-1}(a)^L \\ &\qquad \times \epsilon_1(a)^L(\epsilon_1(a)+1)^L\epsilon_2(a)^L \cdots \epsilon_{j-1}(a)^L \\ &\quad > \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L,\end{aligned}$$

and

$$\begin{aligned}&\epsilon_j(a)^L \cdots \epsilon_{q-1}(a)^L \\ &\qquad \times (\epsilon_1(a)+1)^L\epsilon_1(a)^L\epsilon_2(a)^L \cdots \epsilon_{j-1}(a)^L \\ &\quad > \epsilon_1(a)^L\underline{r}(a)(\epsilon_1(a)+1)^L.\end{aligned}$$

(c)

$$\begin{aligned}&(\epsilon_1(a)+1)^L\epsilon_1(a)^L\underline{r}(a) \\ &\qquad < (\epsilon_1(a)+1)^L\underline{r}(a)(\epsilon_1(a)+1)^L.\end{aligned}$$

(d) *For $1 < j \le q-1$ we have*

$$\begin{aligned}&\epsilon_j(a)^L \cdots \epsilon_{q-1}(a)^L \\ &\qquad \times (\epsilon_1(a)+1)^L\epsilon_1(a)^L\epsilon_2(a)^L \cdots \epsilon_{j-1}(a)^L \\ &\quad < (\epsilon_1(a)+1)^L\underline{r}(a)(\epsilon_1(a)+1)^L,\end{aligned}$$

and

$$\begin{aligned}&\epsilon_j(a)^L \cdots \epsilon_{q-1}(a)^L \\ &\qquad \times \epsilon_1(a)^L(\epsilon_1(a)+1)^L\epsilon_2(a)^L \cdots \epsilon_{j-1}(a)^L \\ &\quad < (\epsilon_1(a)+1)^L\underline{r}(a)\epsilon_1(a)^L.\end{aligned}$$

Proof. Since, by Remark 5.9 and Lemma 5.2,

$$\underline{\hat{I}}^*_\delta(a) = \epsilon_1(a)^L\underline{r}(a)(\epsilon_1(a)^L(\epsilon_1(a)+1)^L\underline{r}(a))^\infty$$

and is a minimal sequence we have

$$\epsilon_1(a)^L(\epsilon_1(a)+1)^L\underline{r}(a) \ge \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L.$$

If

$$\epsilon_1(a)^L(\epsilon_1(a)+1)^L\underline{r}(a) = \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L,$$

then

$$\begin{aligned}\underline{\hat{I}}^*_\delta(a) &= \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L(\epsilon_1(a)+1)^L \cdots \\ &> \epsilon_1(a)^L(\epsilon_1(a)+1)^L\underline{r}(a)\epsilon_1(a)^L \cdots \\ &= S^{q-1}(\underline{\hat{I}}^*_\delta(a));\end{aligned}$$

a contradiction with the minimality of $\underline{\hat{I}}^*_\delta(a)$. This ends the proof of (a). Now, we prove (b). Again by the minimality of $\underline{\hat{I}}^*_\delta(a)$, for $1 < j \le q-1$ we have

$$\begin{aligned}&\epsilon_j(a)^L \cdots \epsilon_{q-1}(a)^L\epsilon_1(a)^L \\ &\qquad \times (\epsilon_1(a)+1)^L\epsilon_2(a)^L \cdots \epsilon_{j-1}(a)^L \\ &\quad \ge \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L.\end{aligned}$$

If the above inequality holds, we have

$$\begin{aligned}S^{j-1}(\underline{\hat{I}}^*_\delta(a)) &= \epsilon_j(a)^L \cdots \epsilon_{q-1}(a)^L\epsilon_1(a)^L \\ &\qquad \times (\epsilon_1(a)+1)^L\epsilon_2(a)^L \cdots \\ &\qquad \times \epsilon_{j-1}(a)^L\epsilon_j(a)^L \cdots \\ &= \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L\epsilon_1(a)^L \cdots \\ &< \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L(\epsilon_1(a)+1)^L\underline{r}(a) \cdots \\ &= \underline{\hat{I}}^*_\delta(a);\end{aligned}$$

a contradiction. Hence,

$$\begin{aligned}&\epsilon_j(a)^L \cdots \epsilon_{q-1}(a)^L\epsilon_1(a)^L \\ &\qquad \times (\epsilon_1(a)+1)^L\epsilon_2(a)^L \cdots \epsilon_{j-1}(a)^L \\ &\quad > \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L.\end{aligned}$$

Now, we prove the second part of statement (b). Since by Remark 5.9 and Lemma 5.2

$$\underline{\hat{I}}_\epsilon(a) = (\epsilon_1(a)^L\underline{r}(a)(\epsilon_1(a)+1)^L)^\infty$$

is a periodic minimal sequence of period q then, for $1 < j \le q-1$, we have that $S^{j-1}(\underline{\hat{I}}_\epsilon(a)) > \underline{\hat{I}}_\epsilon(a)$. Thus

$$\begin{aligned}&\epsilon_j(a)^L \cdots \epsilon_{q-1}(a)^L \\ &\qquad \times (\epsilon_1(a)+1)^L\epsilon_1(a)^L\epsilon_2(a)^L \cdots \epsilon_{j-1}(a)^L \\ &\quad > \epsilon_1(a)^L\underline{r}(a)(\epsilon_1(a)+1)^L.\end{aligned}$$

Otherwise, the equality holds and so $S^{j-1}(\underline{\hat{I}}_\epsilon(a)) = \underline{\hat{I}}_\epsilon(a)$ with $j < q$; a contradiction. This concludes the proof of statement (b). By using the sequences $\underline{\hat{I}}^*_\epsilon(a)$ and $\underline{\hat{I}}_\delta(a)$ instead of $\underline{\hat{I}}^*_\delta(a)$ and $\underline{\hat{I}}_\epsilon(a)$, statements (c) and (d) follow in a similar way. ■

Proof of Theorem B. We start by proving (a). Assume that $\tilde{E}(a) = k < \tilde{E}(b)$. From the definition of $\odot_\epsilon$ it follows that $a \odot_\epsilon \underline{\alpha} \in \Sigma_k$ and $a \odot_\epsilon \underline{\beta} \in \Sigma_{\tilde{E}(b)}$. Then, if $a \odot_\epsilon \underline{\alpha} = k^L, \ldots$, clearly, $a \odot_\epsilon \underline{\alpha} < a \odot_\epsilon \underline{\beta}$. If $a \odot_\epsilon \underline{\alpha} = (k+1)^L \cdots$ then, from the definition of $\odot_\epsilon$ it follows that $a \notin \mathbb{Q}^*$. Moreover, from the definition of $\underline{\hat{I}}_\epsilon(a)$ and $\underline{\hat{I}}^*_\delta(a)$ (see also Lemma 5.4) it follows that $a = k+1$ and $a \odot_\epsilon \underline{\alpha} = \underline{\hat{I}}_\epsilon(k+1) = ((k+1)^L)^\infty$. Clearly, $((k+1)^L)^\infty < \underline{\gamma}$ for each $\underline{\gamma} \in \Sigma_m$ with $m > k$. This proves statement (a) in this case. So, assume that $\tilde{E}(a) = \tilde{E}(b)$. By the definition of $\odot_\epsilon$, Corollary 5.7 and the fact that $\pi_{\tilde{E}(a)}$ is order preserving we may assume that $\tilde{E}(a) = \tilde{E}(b) = 0$ (that is, $a, b \in (0,1]$). We consider first the case $a = b$. If $a \notin \mathbb{Q}^*$ then, from Theorem 2.8, we have $\underline{\hat{I}}^*_\delta(a) \leq \underline{\hat{I}}_\epsilon(a)$. Hence, for each $\underline{\alpha} \in \mathcal{B}_\epsilon(0)\backslash\{(1^L)^\infty\}$, $a \odot_\epsilon \underline{\alpha} = \underline{\hat{I}}^*_\delta(a) \leq \underline{\hat{I}}_\epsilon(a) = a \odot_\epsilon (1^L)^\infty$. Therefore, $a \odot_\epsilon \underline{\alpha} \leq a \odot_\epsilon \underline{\beta}$ for each $\underline{\alpha}, \underline{\beta} \in \mathcal{B}_\epsilon(0)$. Take now $a \in \mathbb{Q}^*$ and set $\underline{\alpha} = \alpha_1\alpha_2\cdots$ and $\underline{\beta} = \beta_1\beta_2\cdots$. Since $\underline{\alpha} < \underline{\beta}$, there exists $k \geq 1$ such that $\alpha_1\cdots\alpha_{k-1} = \beta_1\cdots\beta_{k-1}$ and $\alpha_k < \beta_k$. Then $a \odot_\epsilon \underline{\alpha} < a \odot_\epsilon \underline{\beta}$ directly from the definition. This ends the proof of statement (a) in the case $a = b$. We note that in particular, from Proposition 3.1, we have proved that

$$\begin{aligned}\underline{\hat{I}}^*_\delta(a) = a \odot_\epsilon (0^L)^\infty \leq a \odot_\epsilon \underline{\alpha} \leq a \odot_\epsilon (1^L)^\infty \\ = \underline{\hat{I}}_\epsilon(a)\end{aligned}$$

for each $\underline{\alpha} \in \mathcal{B}_\epsilon(0)$. Now we assume that $a \neq b$. Take $c \in (a, b)$ irrational. Then since $\underline{\hat{I}}_\epsilon(c) = \underline{\hat{I}}^*_\delta(c)$ (see Lemma 2.5), from Lemma 5.1(b) we get that $\underline{\hat{I}}_\epsilon(a) < \underline{\hat{I}}_\epsilon(c) = \underline{\hat{I}}^*_\delta(c) < \underline{\hat{I}}^*_\delta(b)$. So, from above we have

$$a \odot_\epsilon \underline{\alpha} \leq \underline{\hat{I}}_\epsilon(a) < \underline{\hat{I}}^*_\delta(b) \leq b \odot_\epsilon \underline{\beta}.$$

This concludes the proof of statement (a). Statement (b) follows in a similar way.

Now, we prove the first statement of (c). Without loss of generality we may assume that $a \in (0,1]$. If $a \notin \mathbb{Q}^*$ then the statement follows from the definition of $\odot_\epsilon$ and Lemma 5.2. Now, assume that $a \in \mathbb{Q}^*$. From Theorem 2.8 and Lemma 5.2 we also have that $a \odot_\epsilon (0^L)^\infty, a \odot_\epsilon (1^L)^\infty \in \mathcal{B}_\epsilon(0) \subset \mathcal{E}_\epsilon$. Therefore, we may assume that $\underline{\alpha} \notin \{(0^L)^\infty, (1^L)^\infty\}$. Since $\underline{\alpha}$ is minimal, we have $\underline{\alpha} = 0^L\cdots$. Indeed, otherwise we have $S^n(\underline{\alpha}) \geq \underline{\alpha} = 1^L\cdots$ for each $n \geq 0$. Hence $\underline{\alpha} = (1^L)^\infty$; a contradiction. Consequently, $a \odot_\epsilon \underline{\alpha} = 0^L\underline{r}(a)0^L1^L\cdots$. To end the proof of the first statement of (c) we have to prove that $S^j(a \odot_\epsilon \underline{\alpha}) \geq a \odot_\epsilon \underline{\alpha}$ for each $j \geq 1$. Let $\underline{\alpha} = \alpha_1\alpha_2\cdots$ and $a = p/q$ with $(p,q) = 1$ and $m \geq 1$. Then

$$S^{qm}(a \odot_\epsilon \underline{\alpha}) = \hat{\alpha}_m\underline{r}(a)\alpha_{m+1}\hat{\alpha}_{m+1}\cdots.$$

If $\alpha_m = 1^L$, then $\hat{\alpha}_m = 0^L$ and, since $\underline{\alpha}$ is minimal, we have $S^{qm}(\underline{\alpha} \odot_\epsilon a) \geq a \odot_\epsilon \underline{\alpha}$. If $\alpha_m = 0^L$ and $\hat{\alpha}_m = 1^L$ then clearly, we are done. Now we look at

$$S^{mq-1}(a \odot_\epsilon \underline{\alpha}) = \alpha_m\hat{\alpha}_m\underline{r}(a)\alpha_{m+1}\hat{\alpha}_{m+1}\cdots.$$

If $\alpha_m = 1^L$, obviously $S^{mq-1}(\underline{\alpha} \odot_\epsilon a) \geq a \odot_\epsilon \underline{\alpha}$. Assume that $\alpha_m = 0^L$. Then $\alpha_m\hat{\alpha}_m = 0^L1^L$ and the desired inequality follows from Lemma 5.10(a) (recall that we are assuming that $a \in (0,1]$ and $a \in \mathbb{Q}^*$; that is $\epsilon_1(a) = 0$). Now, assume that $1 < j \leq q-1$. Then

$$S^{(m-1)q+j-1}(a \odot_\epsilon \underline{\alpha}) = \epsilon_j(a)^L\cdots\epsilon_{q-1}(a)^L\alpha_m\hat{\alpha}_m\cdots$$

and, from Lemma 5.10(b), we get $S^{(m-1)q+j-1}(\underline{\alpha} \odot_\epsilon a) \geq a \odot_\epsilon \underline{\alpha}$. This ends the proof of the first statement of (c). The fact that $\rho(\underline{\alpha} \odot_\epsilon a) = a$ follows straightforwardly from the definition of $\odot_\epsilon$ and the fact that $\rho(\underline{\hat{I}}_\epsilon(a)) = \rho(\underline{\hat{I}}^*_\delta(a)) = a$. This ends the proof of (c). Statement (d) follows in a similar way.

Now, we prove (e). Assume that $a = p/q$ with $(p,q) = 1$ and set $\underline{\alpha} = \alpha_1\alpha_2\cdots$ and $\underline{\beta} = \beta_1\beta_2\cdots$. Since $a \in \mathbb{Q}^*$ we have $E(a) = \tilde{E}(a) = \epsilon_1(a)$. Hence,

$$a \odot_\epsilon \underline{\alpha} = \epsilon_1(a)^L\underline{r}(a)\alpha_1\hat{\alpha}_1\underline{r}(a)\alpha_2\hat{\alpha}_2\cdots$$

and

$$a \odot_\delta \underline{\beta} = (\epsilon_1(a)+1)^L\underline{r}(a)\beta_1\hat{\beta}_1\underline{r}(a)\beta_2\hat{\beta}_2\cdots.$$

Since $\underline{\alpha} \neq (1^L)^\infty$ is minimal and $\underline{\beta} \neq (0^L)^\infty$ is maximal, in a similar way as before we obtain that $\underline{\alpha} = 0^L\cdots$ and $\underline{\beta} = 1^L\cdots$. Therefore $\underline{\alpha} < \underline{\beta}$ and $(a \odot_\epsilon \underline{\alpha})' < a \odot_\delta \underline{\beta}$. Moreover, since $S^n(\underline{\alpha}) \leq \underline{\beta}$, we obtain $S^n(a \odot_\epsilon \underline{\alpha}) \leq a \odot_\delta \underline{\beta}$ in a similar way as above by using Lemma 5.10(c) instead of Lemmas 5.10(a) and 5.10(d) instead of Lemma 5.10(b). On the other hand, from $S^n(\underline{\beta}) \geq \underline{\alpha}$ and Lemma 5.10(a) and (b) we obtain $S^n(a \odot_\delta \underline{\beta}) \geq a \odot_\epsilon \underline{\alpha}$. Then statement (d) follows from the definition of $\mathcal{E}^* \subset \mathcal{E}$. ■

5.3. *Proof of Proposition 3.3*

We need three preliminary results. The next lemma follows easily.

Lemma 5.11. *Let $\underline{\alpha} = \alpha_1\alpha_2\cdots, \underline{\beta} = \alpha_1\beta_2\cdots \in \mathcal{AD}$ be such that $\underline{\alpha} < \underline{\beta}$. Then the following statements hold.*

(a) *If $\alpha_1 = d^L$ then $S(\underline{\alpha}) < S(\underline{\beta})$.*
(b) *If $\alpha_1 = d^R$ then $S(\underline{\alpha}) > S(\underline{\beta})$.*

The following proposition characterizes the sequences in $\mathcal{P}_\epsilon$ and $\mathcal{P}_\delta$.

Proposition 5.12. *The following statements hold.*

(a) *Let $\underline{\beta} \in \Xi$ be such that $\underline{\gamma} = \underline{\beta} d^M$ is minimal satisfying that if $S^{j-1}(\underline{\gamma}) = d^R \cdots$ for some $j = 0, 1, \ldots, |\underline{\gamma}| - 1$, $S^j(\underline{\gamma}) \geq \underline{\gamma}'$. Then there exists $F \in \mathcal{A}$ such that $\hat{\underline{I}}_F(0) = \underline{\gamma}$. Moreover $\underline{\gamma} \in \mathcal{P}_\epsilon$.*

(b) *Let $\underline{\beta} \in \Xi$ be such that $\underline{\gamma} = \underline{\beta} d^C$ is maximal. Then there exists $F \in \mathcal{A}$ such that $\hat{\underline{I}}_F(c_F) = \underline{\gamma}$. Moreover $\underline{\gamma} \in \mathcal{P}_\delta$.*

Proof. We will prove statement (a). Statement (b) follows similarly. The strategy of the proof will be to construct effectively a map $F \in \mathcal{A}$ such that $\hat{\underline{I}}_F(0) = \underline{\gamma}$. We proceed as follows. Set $\underline{\gamma} = d_1^{s_1} d_2^{s_2} \cdots d_{n-1}^{s_{n-1}} d_n^{s_n}$ with $s_n = M$. Let $k \in \mathbb{Z}$ be such that $\max\{|d_i|: i = 1, \ldots, n\} < k$ and let $c \in (0,1)$. Now, for $j = 0, 1, \ldots n-1$, we choose points $x(S^j(\underline{\gamma})) \in [0,1)$ such that

(i) $x(\underline{\gamma}) = 0$,

(ii) if for $j = 1, \ldots, n-1$ we have $S^{j-1}(\underline{\gamma}) = d_j^L \cdots$ (resp. $S^{j-1}(\underline{\gamma}) = d_j^R \cdots$) then $x(S^j(\underline{\gamma})) \in (0, c)$ (resp. $x(S^j(\underline{\gamma})) \in (c, 1)$),

(iii) if for $i \neq j$, $i, j \in \{1, 2, \ldots, n-1\}$ we have $x(S^i(\underline{\gamma})), x(S^j(\underline{\gamma})) \in [0, c)$ (resp. $x(S^i(\underline{\gamma})), x(S^j(\underline{\gamma})) \in (c, 1)$), then $x(S^i(\underline{\gamma})) < x(S^j(\underline{\gamma}))$ if and only if $S^i(\underline{\gamma}) < S^j(\underline{\gamma})$ (resp. $S^i(\underline{\gamma}) > S^j(\underline{\gamma})$).

We note that, by the minimality of $\underline{\gamma}$, we have $x(\underline{\gamma}) < x(S^j(\underline{\gamma}))$ for $j = 1, 2, \ldots n-1$. Therefore, we can write

$$\begin{aligned} x(\underline{\gamma}) &< x(S^{j_1}(\underline{\gamma})) \\ &< \cdots < x(S^{j_k}(\underline{\gamma})) < c < x(S^{j_{k+1}}(\underline{\gamma})) \\ &< \cdots < x(S^{j_{n-1}}(\underline{\gamma})). \end{aligned}$$

Then we set $j_0 = 0$ and we take $F \in \mathcal{L}$ such that $F(c) = k$, $F(x(S^{j_t}(\underline{\gamma})) = x(S^{j_{t+1}}(\underline{\gamma})) + d_{j_t+1}$ if $j_t \neq n-1$, $F(x(S^{n-1}(\underline{\gamma})) = d_n$ and F is affine in $[x(S^{j_t}(\underline{\gamma})), x(S^{j_{t+1}}(\underline{\gamma}))]$ for $t \in \{0, 1, \ldots, n-1\} \backslash \{k\}$ and in $[x(S^{j_k}(\underline{\gamma})), c]$ and $[c, x(S^{j_{k+1}}(\underline{\gamma}))]$. Now, we claim that $F \in \mathcal{A}$. To prove it note that $F(c) = k > F(x(S^j(\underline{\gamma}))$ for $j = 0, \ldots, n-1$. Then $F|_{[x(S^{j_k}(\underline{\gamma})), c]}$ is strictly increasing and $F|_{[c, x(S^{j_{k+1}}(\underline{\gamma}))]}$ is strictly decreasing. Let t be such that $[x(S^{j_t}(\underline{\gamma})), x(S^{j_{t+1}}(\underline{\gamma}))] \subset [0, c)$. We have $S^{j_t}(\underline{\gamma}) = d_{j_t+1}^{s_{j_t+1}} \cdots < d_{j_{t+1}+1}^{s_{j_{t+1}+1}} \cdots = S^{j_{t+1}}(\underline{\gamma})$. If either $d_{j_t+1} < d_{j_{t+1}+1}$ or $d_{j_t+1} = d_{j_{t+1}+1}$ and $s_{j_t+1} < s_{j_{t+1}+1}$, then clearly $F(x(S^{j_t}(\underline{\gamma}))) < F(x(S^{j_{t+1}}(\underline{\gamma})))$. Now, assume $d_{j_t+1}^{s_{j_t+1}} = d_{j_{t+1}+1}^{s_{j_{t+1}+1}}$. From Lemma 5.11 we have that either $S^{j_t+1}(\underline{\gamma}) < S^{j_{t+1}+1}(\underline{\gamma})$ if $s_{j_t+1} = L$ or $S^{j_t+1}(\underline{\gamma}) > S^{j_{t+1}+1}(\underline{\gamma})$ if $s_{j_t+1} = R$. In both cases $x(S^{j_t+1}(\underline{\gamma})) < x(S^{j_{t+1}+1}(\underline{\gamma}))$ and, as a consequence, $F(x(S^{j_t}(\underline{\gamma}))) < F(x(S^{j_{t+1}}(\underline{\gamma})))$. Thus $F|_{[x(S^{j_t}(\underline{\gamma})), x(S^{j_{t+1}}(\underline{\gamma}))]}$ is strictly increasing. In a similar way we can prove that if $[x(S^{j_t}(\underline{\gamma})), x(S^{j_{t+1}}(\underline{\gamma}))] \subset (c, 1)$ then $F|_{[x(S^{j_t}(\underline{\gamma})), x(S^{j_{t+1}}(\underline{\gamma}))]}$ is strictly decreasing. To end the proof of the claim we have to prove that $F(x(S^{j_{n-1}}(\underline{\gamma}))) > F(1)$. Since $x(S^{j_{n-1}}(\underline{\gamma})) \in (c, 1)$ we have $S^{j_{n-1}-1}(\underline{\gamma}) = d_{j_{n-1}}^R \cdots$. Then $S^{j_{n-1}}(\underline{\gamma}) > \underline{\gamma}'$. If either $d_{j_{n-1}+1} > (d_1+1)$ or $d_{j_{n-1}+1} = (d_1+1)$ and $s_{j_{n-1}+1} = R > L = s_1$ then, since $F(1) = F(0) + 1 = x(S(\underline{\gamma})) + d_1 + 1$ and $F(x(S^{j_{n-1}}(\underline{\gamma}))) = x(S^{j_{n-1}+1}(\underline{\gamma})) + d_{j_{n-1}+1}$, we have $F(x(S^{j_{n-1}}(\underline{\gamma}))) > F(1)$. On the other hand, assume that $d_{j_{n-1}+1}^{s_{j_{n-1}+1}} = (d_1+1)^{s_1}$. We obtain $F(x(S^{j_{n-1}}(\underline{\gamma}))) > F(1)$ as above by using Lemma 5.11. This ends the proof of the claim. Lastly, we have $\hat{\underline{I}}_F(0) = \underline{\gamma}$ by construction. Also, from Lemma 4.1(a) we have $\underline{\gamma} \in \mathcal{P}_\epsilon$. This ends the proof of the proposition. ■

The next lemma characterizes the periodic sequences in $\mathcal{B}_\epsilon(0)$ and $\mathcal{B}_\delta(0)$.

Lemma 5.13. *The following statements hold.*

(a) *Let $\underline{\alpha} \in \mathcal{B}_\epsilon(0) \backslash \{(0^L)^\infty, (1^L)^\infty\}$ be periodic. Then $\underline{\alpha} = (0^L \underline{\beta} 1^L)^\infty$ for some $\underline{\beta} \in \Xi$.*

(b) *Let $\underline{\alpha} \in \mathcal{B}_\delta(0) \backslash \{(0^L)^\infty, (1^L)^\infty\}$ be periodic. Then $\underline{\alpha} = (1^L \underline{\beta} 0^L)^\infty$ for some $\underline{\beta} \in \Xi$.*

Proof. Clearly $\underline{\alpha}$ is of the form $(d_1^L \underline{\beta} d_n^L)^\infty$ with $\underline{\beta} \in \Xi$. Assume that $d_1 = 1$. Since $\underline{\alpha}$ is minimal we have $\underline{\alpha} = 1^L \cdots \leq S^j(\underline{\alpha})$ for all j. Then $S^j(\underline{\alpha}) = 1^L \cdots$ for all j and, as a consequence, $\underline{\alpha} = (1^L)^\infty$; a contradiction. Hence $d_1 = 0$. Now, assume that $d_n = 0$. Then $\underline{\alpha} = (0^L \underline{\beta} 0^L)^\infty$. If $\underline{\beta}$ is the empty sequence then $\underline{\alpha} = (0^L)^\infty$; a contradiction. Now assume that $\underline{\beta}$ is not the empty sequence and set $\underline{\beta} = \beta_2 \cdots \beta_{n-1}$. Since $\underline{\alpha}$ is minimal $\underline{\alpha} = 0^L \beta_2 \cdots \leq 0^L 0^L \beta_2 \cdots = S^{n-1}(\underline{\alpha})$. Thus

$\beta_2 = 0^L$. Proceeding inductively we obtain that $\beta_i = 0^L$ for $i = 2, \ldots, n-1$. Thus $\underline{\alpha} = (0^L)^\infty$; a contradiction. This ends the proof of (a). Statement (b) follows in a similar way. ■

Proof of Proposition 3.3. We will only prove statement (a). Statement (b) follows in a similar way. The fact that $a \odot_\delta \underline{\alpha}$ is not periodic when $a \notin \mathbb{Q}$ and when $a \in \mathbb{Z}$ is periodic if and only if $\underline{\alpha} = (1^L)^\infty$ follows from the definitions of $\odot_\epsilon$ and of the sequences $\hat{\underline{I}}^*_\delta(a)$ and $\hat{\underline{I}}_\epsilon(a)$. The third statement follows directly from the definitions. Now we prove the last two statements. Assume that $a \in \mathbb{Q}^*$. If $\underline{\alpha} = (1^L)^\infty$ then $a \odot_\epsilon \underline{\alpha}$ is periodic by Proposition 3.1 and Lemma 5.1(a). Moreover if $a = p/q$ with $(p, q) = 1$ then $a \odot_\delta \underline{\alpha} = \left(\epsilon_1(a)^L \epsilon_2(a)^L \cdots \epsilon_q(a)^L\right)^\infty$. Let $\underline{\alpha} \in \mathcal{B}_\epsilon(0) \backslash \{(1^L)^\infty\}$. By Lemma 5.13(a) we get $\underline{\alpha} = (0^L \alpha_2 \cdots \alpha_{n-1} 1^L)^\infty$. Without loss of generality assume that $\tilde{E}(a) = 0$. Then

$$a \odot_\epsilon \underline{\alpha} = (0^L \underline{r}(a) 0^L 1^L \underline{r}(a) \alpha_2 \cdots \underline{r}(a) \alpha_{n-1} \hat{\alpha}_{n-1} \underline{r}(a) 1^L)^\infty$$

is periodic. Now, let

$$\underline{\gamma} = 0^L \underline{r}(a) 0^L 1^L \underline{r}(a) \alpha_2 \cdots \underline{r}(a) \alpha_{n-1} \hat{\alpha}_{n-1} \underline{r}(a) 1^M.$$

Clearly, $a \odot_\epsilon \underline{\alpha} = \underline{\gamma} \star_\epsilon L^\infty$. Since, from Proposition 4.6(b) of [Alsedà & Falcó, 1997], $0^L \underline{r}(a) 1^M$ is a minimal sequence, by using Lemma 5.10(a) and (b), we have $\underline{\gamma}$ is a minimal sequence (note that $\epsilon_1(a) = 0$). Then by Proposition 5.12(a) we have $\underline{\gamma} \in \mathcal{P}_\epsilon$. ■

6. Proof of Main Theorem

Let $F \in \mathcal{A}$ be such that $R_F = [a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$. By using Theorem 2.8 it follows the first statement of the Main Theorem.

Now, we will define the map $p_{\epsilon,a} \colon \mathcal{J}_\epsilon \to Q_\epsilon(a)$ (resp., $p_{\delta,a} \colon \mathcal{J}_\delta \to Q_\delta(a)$ in two steps. First, we will introduce the notation that will allow us to speak about iterated $\odot$-operators.

Let $x_1, x_2 \in (0, 1]$ and $\underline{\alpha} \in \mathcal{B}_\epsilon(0)$. We note that if $\underline{\beta} \in \mathcal{B}_\epsilon(0)$ then, by Theorem B(c) and the definition of $\odot_\epsilon$, $x_i \odot_\epsilon \underline{\beta}$ also lies in $\mathcal{B}_\epsilon(0)$. Therefore, the sequence

$$x_1 \odot_\epsilon (x_2 \odot_\epsilon \underline{\alpha})$$

is well defined. Now we take $x_1, x_2 \in (k, k+1]$ with $k \in \mathbb{Z}$ and we extend the notation to this case as follows. Let

$$x_1 \odot_\epsilon (x_2 \odot_\epsilon \underline{\alpha}) = \pi_k^{-1}(\tilde{D}(x_1) \odot_\epsilon (\tilde{D}(x_2) \odot_\epsilon \underline{\alpha})).$$

In a similar way let $x_1, x_2 \in [0, 1)$ and $\underline{\alpha} \in \mathcal{B}_\delta(0)$. Then, by using Theorem B(d), the sequence

$$x_1 \odot_\delta (x_2 \odot_\delta \underline{\alpha})$$

is well defined. If $x_1, x_2 \in [k, k+1)$ with $k \in \mathbb{Z}$ then we set

$$x_1 \odot_\delta (x_2 \odot_\delta \underline{\alpha}) = \pi_k^{-1}(D(x_1) \odot_\delta (D(x_2) \odot_\delta \underline{\alpha})).$$

In the first step we define $p_{\epsilon,a}$ from $\mathcal{J}^*_\epsilon$ (resp., $\mathcal{J}^*_\delta$) into $Q_\epsilon(a)$ (resp., $Q_\delta(a)$). Let $a \in \mathbb{R}$ and $\underline{x}\omega \in \mathcal{J}^*_\epsilon$ (resp., $\underline{x}\omega \in \mathcal{J}^*_\delta$). Assume that $\underline{x} = \{x_i\}_{i=1}^{n-1} \in \mathcal{I}$, if $n = 1$ (i.e. $\underline{x}$ is the empty sequence) we set

$$p_{\epsilon,a}(\omega) = \begin{cases} a \odot_\epsilon \hat{\underline{I}}_\epsilon(0) & \text{if } \omega = 0, \\ a \odot_\epsilon \hat{\underline{I}}_\epsilon(\omega) & \text{if } \omega \notin (0,1) \cap \mathbb{Q}, \\ a \odot_\epsilon \hat{\underline{I}}^*_\delta(1) & \text{if } \omega = 1, \\ a \odot_\epsilon \hat{\underline{I}}_\epsilon(1) & \text{if } \omega = \epsilon, \end{cases}$$

respectively,

$$p_{\delta,a}(\omega) = \begin{cases} a \odot_\epsilon \hat{\underline{I}}_\delta(0) & \text{if } \omega = \delta, \\ a \odot_\epsilon \hat{\underline{I}}^*_\epsilon(0) & \text{if } \omega = 0, \\ a \odot_\epsilon \hat{\underline{I}}_\delta(\omega) & \text{if } \omega \notin (0,1) \cap \mathbb{Q}, \\ a \odot_\epsilon \hat{\underline{I}}_\delta(1) & \text{if } \omega = 1. \end{cases}$$

Otherwise, if $n \geq 2$ we set $p_{\epsilon,a}(\underline{x}\omega) = a \odot_\epsilon p_{\epsilon,x_1}(x_2 x_3 \cdots x_{n-1}\omega)$ (resp., $p_{\epsilon,a}(\underline{x}\omega) = a \odot_\delta p_{\delta,x_1}(x_2 x_3 \cdots x_{n-1}\omega)$).

Recall that $\hat{\underline{I}}_\epsilon(1) = \hat{\underline{I}}_\delta(1) = (1^L)^\infty$ and $\hat{\underline{I}}_\epsilon(0) = \hat{\underline{I}}_\delta(0) = (0^L)^\infty$. From Proposition 3.1 and Theorem B(a) and (b) we get:

Lemma 6.1. *Let $a \in \mathbb{R}$. Then the maps $p_{\epsilon,a} \colon \mathcal{J}^*_\epsilon \to Q_\epsilon(a)$ and $p_{\delta,a} \colon \mathcal{J}^*_\delta \to Q_\delta(a)$ are nondecreasing. Moreover, if $a \in \mathbb{Q}^*$ then $p_{\epsilon,a}$ and $p_{\delta,a}$ are strictly increasing.*

To extend the definition of $p_{\epsilon,a}$ and $p_{\delta,a}$ to $\mathcal{J}^\infty$, respectively, we introduce the following notation. For $\underline{x} = \{x_i\}_{i=1}^\infty \in \mathcal{J}$ we set $\lambda^n_0(\underline{x}) = x_1 x_2 \cdots x_n 0$ and $\lambda^n_1(\underline{x}) = x_1 x_2 \cdots x_n 1$ for each $n \in \mathbb{N}$.

Proposition 6.2. *Let $a \in \mathbb{R}$. Then for each $\underline{x} = \{x_i\}_{i=1}^\infty \in \mathcal{J}^\infty$ we have*

$$\bigcap_{n=1}^\infty [p_{\epsilon,a}(\lambda^n_0(\underline{x})), p_{\epsilon,a}(\lambda^n_1(\underline{x}))] \in B_\epsilon(\tilde{E}(a)) \quad \textit{and}$$

$$\bigcap_{n=1}^\infty [p_{\delta,a}(\lambda^n_0(\underline{x})), p_{\delta,a}(\lambda^n_1(\underline{x}))] \in B_\delta(E(a)).$$

To prove this proposition we shall use the following.

Lemma 6.3. *Let $a \in \mathbb{Q}^*$. Then*

$$\lim_{n\to\infty} d(p_{\epsilon,a}(\lambda_0^n(\underline{x})), p_{\epsilon,a}(\lambda_1^n(\underline{x}))) = 0 \quad \textit{and}$$
$$\lim_{n\to\infty} d(p_{\delta,a}(\lambda_0^n(\underline{x})), p_{\delta,a}(\lambda_1^n(\underline{x}))) = 0.$$

Proof. Let $a = p/q \in \mathbb{Q}^*$, with $(p,q) = 1$. Without loss of generality, we may assume that $\tilde{E}(a) = 0$. If $p_{\epsilon,x_1}(0) = d_{1,1}^L d_{1,2}^L \cdots$ and $p_{\epsilon,x_1}(1) = t_{1,1}^L t_{1,2}^L \cdots$ are two sequences in $B_\epsilon(0)$ then, since

$$\begin{aligned} p_{\epsilon,a}(\lambda_0^1(\underline{x})) &= a \odot_\epsilon p_{\epsilon,x_1}(0) \\ &= 0^L \underline{r}(a) d_{1,1}^L \hat{d}_{1,1}^L \underline{r}(a) d_{1,2}^L \hat{d}_{1,2}^L \underline{r}(a) \cdots \end{aligned}$$

and

$$\begin{aligned} p_{\epsilon,a}(\lambda_1^1(\underline{x})) &= a \odot_\epsilon p_{\epsilon,x_1}(1) \\ &= 0^L \underline{r}(a) t_{1,1}^L \hat{t}_{1,1}^L \underline{r}(a) t_{1,2}^L \hat{t}_{1,2}^L \underline{r}(a) \cdots, \end{aligned}$$

we have

$$\begin{aligned} &d(p_{\epsilon,a}(\lambda_0^1(\underline{x})), p_{\epsilon,a}(\lambda_1^1(\underline{x}))) \\ &\quad = \sum_{i=1}^{\infty} (2^{-qi}|d_{1,i} - t_{1,i}| + 2^{-(q+1)i}|\hat{d}_{1,i} - \hat{t}_{1,i}|), \end{aligned}$$

because the differences between these two sequences occur at the positions $q, q+1, 2q, 2q+1, \ldots$. Finally, by using the fact that $|d_{1,i} - t_{1,i}| = |\hat{d}_{1,i} - \hat{t}_{1,i}|$, we obtain

$$\begin{aligned} &d(p_{\epsilon,a}(\lambda_0^1(\underline{x})), p_{\epsilon,a}(\lambda_1^1(\underline{x}))) \\ &\quad = \sum_{i=1}^{\infty} (2^{-qi} + 2^{-(qi+1)})|d_{1,i} - t_{1,i}| \\ &\quad = \frac{3}{2} \sum_{i=1}^{\infty} 2^{-qi}|d_{1,i} - t_{1,i}| \\ &\quad < \frac{3}{2}\left(\frac{1}{1-2^{-q}} - 1\right). \end{aligned}$$

Now, assume that $x_1 = p_1/q_1$, with $(p_1, q_1) = 1$,

$$p_{\epsilon,x_2}(0) = d_{2,1}^L d_{2,2}^L \cdots \quad \text{and}$$
$$p_{\epsilon,x_2}(1) = t_{2,1}^L t_{2,2}^L \cdots,$$

then

$$\begin{aligned} p_{\epsilon,x_1}(x_2 0) &= x_1 \odot_\epsilon p_{\epsilon,x_2}(0) \\ &= 0^L \underline{r}(x_1) d_{2,1}^L \hat{d}_{2,1}^L \underline{r}(x_1) d_{2,2}^L \hat{d}_{2,2}^L \underline{r}(x_1) \cdots \end{aligned}$$

and

$$\begin{aligned} p_{\epsilon,x_1}(x_2 1) &= x_1 \odot_\epsilon p_{\epsilon,x_1}(1) \\ &= 0^L \underline{r}(x_1) t_{2,1}^L \hat{t}_{2,1}^L \underline{r}(x_1) t_{2,2}^L \hat{t}_{2,2}^L \underline{r}(x_1) \cdots . \end{aligned}$$

Note that in this case the differences between these two sequences appear at the positions $q_1 q_1 + 1, 2q_1, 2q_1+1, \ldots$. Since $p_{\epsilon,a}(\lambda_i^2(\underline{x})) = a \odot_\epsilon p_{\epsilon,x_1}(x_2 i)$ for $i = 0$ and 1 and by using some similar arguments as above, it is not difficult to prove that the differences between the two sequences will take place at the $qq_1, qq_1+1, 2qq_1, 2qq_1+1, \ldots$ positions. Thus,

$$\begin{aligned} &d\left(p_{\epsilon,a}(\lambda_0^2(\underline{x})), p_{\epsilon,a}(\lambda_2^2(\underline{x}))\right) \\ &\quad = \sum_{i=1}^{\infty} (2^{-qq_1 i} + 2^{-(qq_1 i+1)})|d_{2,i} - t_{2,i}| \\ &\quad < \frac{3}{2}\left(\frac{1}{1-2^{-qq_1}} - 1\right). \end{aligned}$$

Proceeding inductively, set $x_i = p_i/q_i$, with $(p_i, q_i) = 1$ for $i = 1, 2, \ldots, n-1$, $p_{\epsilon,x_n}(0) = d_{n,1}^L d_{n,2}^L \cdots$ and $p_{\epsilon,x_n}(1) = t_{n,1}^L t_{n,2}^L \cdots$ then

$$\begin{aligned} &d\left(p_{\epsilon,a}(\lambda_0^n(\underline{x})), p_{\epsilon,a}(\lambda_2^n(\underline{x}))\right) \\ &\quad = \sum_{i=1}^{\infty} (2^{-qq_1\cdots q_{n-1} i} + 2^{-(qq_1\cdots q_{n-1} i+1)})|d_{n,i} - t_{n,i}| \\ &\quad < \frac{3}{2}\left(\frac{1}{1-2^{-qq_1\cdots q_{n-1}}} - 1\right). \end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(p_{\epsilon,a}(\lambda_0^n(\underline{x})), p_{\epsilon,a}(\lambda_1^n(\underline{x}))) = 0,$$

because $q_i, q \geq 2$, and then the first equality follows. The second one can be computed in a similar way. ■

Proof of Proposition 6.2. We only prove the first statement, the second one is given in a similar way. First of all, we remark that $\mathcal{B}_\epsilon(\tilde{E}(a))$, the minimal sequences in two symbols, is a closed invariant set of $\Sigma_{\tilde{E}(a)}$, because the shift map is continuous. Let $\underline{x} = \{x_i\}_{i=1}^\infty \in \mathcal{J}^\infty$, if $a \notin \mathbb{Q}^*$ then by the definition of $\odot_\epsilon$ we have $p_{\epsilon,a}(\lambda_0^n(\underline{x})) = p_{\epsilon,a}(\lambda_1^n(\underline{x})) = \underline{\hat{I}}_\delta^*(a) \in \mathcal{B}_\epsilon(\tilde{E}(a))$ for all $n \in \mathbb{N}$ and the proposition follows. Now, assume that $a \in \mathbb{Q}^*$. By using the fact that

$$\lambda_0^n(\underline{x}) \prec \lambda_0^{n+1}(\underline{x}) \prec \lambda_1^{n+1}(\underline{x}) \prec \lambda_1^n(\underline{x}),$$

from Lemma 6.1 it follows that $[p_{\epsilon,a}(\lambda_0^{n+1}(\underline{x})), p_{\epsilon,a}(\lambda_1^{n+1}(\underline{x}))]$ is strictly contained in $[p_{\epsilon,a}(\lambda_0^n(\underline{x})), p_{\epsilon,a}(\lambda_1^n(\underline{x}))] \subset \mathcal{B}_\epsilon(\tilde{E}(a)) \subset \Sigma_{\tilde{E}(a)}$. Then, $\bigcap_{n=1}^\infty [p_{\epsilon,a}(\lambda_0^n(\underline{x})), p_{\epsilon,a}(\lambda_1^n(\underline{x}))] \neq \emptyset$, because $\Sigma_{\tilde{E}(a)}$ is a compact set. Moreover, by using Lemma 6.3 and the fact that $B_\epsilon(\tilde{E}(a))$ is a closed set in $\Sigma_{\tilde{E}(a)}$, we have $\bigcap_{n=1}^{\infty}[p_{\epsilon,a}(\lambda_0^n(\underline{x})), p_{\epsilon,a}(\lambda_1^n(\underline{x}))] \in B_\epsilon(\tilde{E}(a))$ and the proposition follows. ■

Now, let $a \in \mathbb{R}$ then, by using Proposition 6.2, we can define $p_{\epsilon,a}\colon \mathcal{J}^\infty \to Q_\epsilon(a)$ by

$$p_{\epsilon,a}(\underline{x}) = \bigcap_{n=1}^{\infty} [p_{\epsilon,a}(\lambda_0^n(\underline{x})), p_{\epsilon,a}(\lambda_1^n(\underline{x}))]$$

and $p_{\delta,a}\colon \mathcal{J}^\infty \to Q_\delta(a)$ by

$$p_{\delta,a}(\underline{x}) = \bigcap_{n=1}^{\infty} [p_{\delta,a}(\lambda_0^n(\underline{x})), p_{\delta,a}(\lambda_1^n(\underline{x}))]\,.$$

We note that for $\underline{x} = \{x_i\}_{i=1}^\infty \in \mathcal{J}^\infty$ we have

$$p_{\epsilon,a}(\underline{x}) \notin (p_{\epsilon,a}(x_1x_2\cdots x_n1), p_{\epsilon,a}(x_1x_2\cdots x_n\epsilon))$$

and

$$p_{\epsilon,a}(\underline{x}) \notin (p_{\delta,a}(x_1x_2\cdots x_n\delta), p_{\delta,a}(x_1x_2\cdots x_n0))$$

for all $n \in \mathbb{N}$.

By the construction of $p_{\epsilon,a}$ and $p_{\delta,a}$ we obtain the following corollary, it reiterates all said above and gives the proof of statements (a) and (c) of our Main Theorem.

Corollary 6.4. *Let $a \in \mathbb{R}$. Then the following statements hold.*

(a) *The map $p_{\epsilon,a}\colon \mathcal{J}_\epsilon \to Q_\epsilon(a)$ is nondecreasing, maps the endpoints of $\mathcal{J}_\epsilon$ into the endpoints of $Q_\epsilon(a)$ and if $a \in \mathbb{Q}\backslash\mathbb{Z}$ then $p_{\epsilon,a}$ is one-to-one. Moreover,*

$$\mathrm{Im}(p_{\epsilon,a}) = Q_\epsilon(a) \Big\backslash \bigcup_{\underline{x}\in\mathcal{I}} (p_{\epsilon,a}(\underline{x}1), p_{\epsilon,a}(\underline{x}\epsilon)).$$

(b) *The map $p_{\delta,b}\colon \mathcal{J}_\delta \to Q_\delta(a)$ is nondecreasing, maps the endpoints of $\mathcal{J}_\delta$ into the endpoints of $Q_\delta(b)$ and if $a \in \mathbb{Q}\backslash\mathbb{Z}$ then $p_{\delta,b}$ is one-to-one. Moreover,*

$$\mathrm{Im}(p_{\delta,b}) = Q_\delta(a) \Big\backslash \bigcup_{\underline{x}\in\mathcal{I}} (p_{\delta,b}(\underline{x}\delta), p_{\delta,b}(\underline{x}0))\,.$$

Remark 6.5. It is not difficult to see that in the case that $a \in \mathbb{Q}^*$ and $\underline{x} = \{x_i\}_{i=1}^\infty \in \mathcal{J}^\infty$ is such that $x_i = a$ for all $i \geq 1$. Then we have $a \odot_\epsilon p_{\epsilon,a}(\underline{x}) = p_{\epsilon,a}(\underline{x})$ and $a \odot_\delta p_{\delta,a}(\underline{x}) = p_{\delta,a}(\underline{x})$. That is, the sequences $p_{\epsilon,a}(\underline{x}) \in Q_\epsilon(a)$ and $p_{\delta,a}(\underline{x}) \in Q_\delta(a)$ are, respectively, fixed points of the operators $a\odot_\epsilon\colon \mathcal{B}_\epsilon(0) \to Q_\epsilon(a)$ and $a\odot_\delta\colon \mathcal{B}_\delta(0) \to Q_\delta(a)$, respectively.

Finally, statements (b) and (d) follow from the definition of $\mathcal{P}_\epsilon$, $\mathcal{P}_\delta$ and the $\star$-operators given in Sec. 3.1 and Proposition 5.3. This ends the proof of the Main Theorem. ■

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