Suresh Chandra
Mohit Kumar Sharma

A Textbook of Optics

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ISBN 978-3-031-55675-3 ISBN 978-3-031-55676-0 (eBook)
https://doi.org/10.1007/978-3-031-55676-0

Jointly published with Ane Books Pvt. Ltd.
The print edition is not for sale in South Asia (India, Pakistan, Sri Lanka, Bangladesh, Nepal and Bhutan) and Africa. Customers from South Asia and Africa can please order the print book from: ANE Books Pvt. Ltd.
ISBN of the Co-Publisher's edition: 978-9-386-76156-9

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Dedicated to my Maternal Grand Parents
Shri Naubat Ram Sharma & Smt. Ganga Devi

—Suresh Chandra

Preface

Light (radiation) is the means by which we obtain knowledge about various places in the world. Therefore, the nature of light has been a point of discussion for most of the time in the history of science. In the early seventeenth century, Newton considered light to be a stream of tiny, perfectly elastic particles, called corpuscles. This treatment could explain the reflection and refraction phenomena but failed to explain the phenomena such as interference, diffraction and polarization, known at that time. Later on, in the year 1679, Christian Huygens proposed that the light behaves like a wave. The wave nature of light explained successfully the reflection, refraction, interference, diffraction, polarization phenomena.

The wave nature of light was unique till the end of the nineteenth century when other phenomena such as photoelectric effect, Compton effect, atomic spectra, black-body spectrum were discovered. The wave nature of light was found unable to explain these phenomena. These phenomena were explained by considering the light as packets of energy, called quanta. The idea of particle (quantum) theory of light was formulated by Planck in 1900. Interesting to note that the wave theory of light had monopoly for 220 years. Nowadays, for explaining some optical, light is considered as packets of energy whereas for explaining some other phenomena, the light is considered as wave. These two distinct behaviours of light together are known as the dual nature of light.

Optics is a part of electromagnetic spectrum, which can be seen by the eyes. It ranges from $\sim$3900 Å to $\sim$7500 Å in wavelengths. The electromagnetic spectrum, having wavelengths from $\sim$0.1 Å to more than 10 m, consists of gamma rays, X-rays, ultra-violet radiation, visible radiation, infrared radiation, microwaves, radio waves. Thus, the visible part (also known as the optical region) is a very small portion of electromagnetic spectrum. Luckily, the earth's atmosphere is transparent in this range also and we can see the light coming from outside the earth's atmosphere. Due to transparency of the earth's atmosphere, we enjoy numerous natural phenomena, such as rising and setting of sun, phases of moon, solar and lunar eclipses, comets, meteors, star constellations, and so on.

The earth's atmosphere is opaque for most of the parts of electromagnetic spectrum and is transparent for some wavelength bands. Two of these bands, known as the optical window and radio window, play important roles in the study of cosmic objects. The radio window having range from few mm to few decametres becomes very important when one wants to study very far distant objects, as the scattering of radiation is inversely proportional to the fourth power of its wavelength. The optical window that covers the visible range is very important as it can be realized directly by human being.

After creation of human being probably he/she would have noticed the rising and setting of sun each day, and when the sun was not there (nighttime), a large number of stars were visible. So, the astronomy (which may be considered as the spectroscopy of cosmic objects) may be considered as the inception of science. Optics is a science which can be visualized by eyes.

A number of books by Indian as well as foreign authors are written on optics for science and engineering students of graduate class. Many of them claimed about their books to be a complete course. We also decided to write a book on optics where student-friendly style in a simple language is adopted. The material presented in this book is based on the long and extensive teaching and research experience of Prof. Suresh Chandra in various universities in India and abroad. The text is supplemented by the exercises, which help for better understanding of the subject. For further improvement, scope is always there. We shall appreciate receiving comments/suggestions from the readers of this book, which can be sent to the following emails also:

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While preparing the manuscript, we have been helped, advised, and encouraged by our seniors, friends, and colleagues working in various institutions in India as well as abroad, and by our friends in personal life. We are heartily thankful to all of them.

We are grateful to Hon'ble Dr. Ashok K. Chauhan, Founder President, Amity University, Hon'ble Dr. Atul Chauhan, Chancellor Amity University Uttar Pradesh, Prof. Balvinder Shukla, Vice-Chancellor Amity University Uttar Pradesh, Prof. Sunita Rattan, Director, Amity Institute for Applied Sciences, Amity University, Noida for their moral support and encouragement.

We are thankful to Prof. A. L. Verma, Prof. S. N. Shukla, Prof. C. K. Goel, Prof. Sangeeta Tiwari, Prof. R. S. Pandey, Dr. Prakriti Rai, Dr. Jitender Kumar (all from Amity Institute for Applied Sciences, Amity University, Noida) for their valuable cooperation and support.

Suresh Chandra is especially thankful to his wife Mrs. Purnima Sharma for her valuable cooperation in his life and for sharing a major part of responsibility of family affairs. We are deeply grateful to our family members who always have been a source of inspiration and happiness for us. The last but not least, we are highly thankful to the Publisher of this book for inviting us to write this book.

Noida, India

Suresh Chandra
Mohit Kumar Sharma

About This Book

- Subject is introduced in a very simple manner so that a common student could understand it.
- Each topic is explained with the help of simple exercises.
- Details of mathematical expressions are given up to the extent so that the students could understand independently.
- Simple language, understandable to a common student, is used.

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About the Authors

Prof. Dr. Suresh Chandra is presently Deputy Director and Professor at Amity Center for Astronomy & Astrophysics, Amity Institute of Applied Sciences, Amity University, Noida. Earlier, he was Director at the School of Physical Sciences, SRTM University, Nanded (Maharashtra); Director at the School of Physical Sciences; and Dean of Students at SMVD University, Katra (J&K). He is a Fellow of the Royal Astronomical Society (London), a Fellow of the Maharashtra Academy of Sciences (Pune), an Associate Member of the Third World Academy of Sciences (Italy), a Fellow of the Alexander von Humboldt Foundation (Germany), Life Member of the International Astronomical Union, and Life Member of several other national academic bodies.

He has published 152 research papers in journals of international repute, has written 20 books and 32 scientific articles, and has participated in 75 international/national conferences. He has successfully completed 13 research projects funded by various agencies in India.

Dr. Mohit Kumar Sharma is a National Postdoctoral Fellow of SERB, DST, New Delhi, at Amity Center for Astronomy & Astrophysics, Amity Institute of Applied Sciences, Amity University, Noida. Earlier, he worked as an Assistant Professor at ITM University, Gwalior. He has received the Young Scientist Award (2014) from the Indian Science Congress Association, Kolkata. As a brilliant student throughout, he earned his Ph.D. (Physics) from Jiwaji University, Gwalior. He is the co-author of eight textbooks and has published 22 research papers in journals of international repute. He is a Life Member of the Astronomical Society of India; Indian Science Congress Association, Kolkata; Indian Society of Atomic & Molecular Physics; Indian Association of Physics Teachers; and Astro-chemistry Society of India.

Chapter 1
Reflection

Discussion of optical phenomena is generally divided into two categories: (i) geometrical optics and (ii) physical optics. In the physical optics, we deal with the phenomena which arise due to the wave nature of light and they produce phenomenon such as the interference. In the geometrical optics, we deal with the phenomena where light is considered as a ray traveling along a straight line (perpendicular to the wavefront). Reflection and refraction can be easily explained with the help of the geometrical optics. However, the reflection and refraction can also be explained with the help of the wave nature. In this chapter, we deal with the reflection with the help of the geometrical optics.

1 Wavefronts

When some rays of light travel in a medium, a locus of their positions at a given time represents a wavefront of the group of rays. A wavefront may have one of the three shapes as shown in Fig. 1.

These are: (a) plane wavefront, (b) spherical wavefront, and (c) cylindrical wavefront. Here, we consider homogeneous[1] and isotropic[2] medium. For a plane wavefront, a source is situated at very large distance so that all the rays are moving parallel to one another. (A plane wavefront can also be produced with the help of a lens.) For a spherical wave front, a point source emanates rays in all directions in a medium. For a cylindrical wavefront, a line source emanates rays in all directions in a medium, perpendicular to the line.

[1] A medium whose properties do not change when one moves along a straight line is said to be a homogeneous medium.

[2] A medium whose properties do not change when the direction of motion is rotated through an angle is said to be an isotropic medium.

S. Chandra and M. K. Sharma, *A Textbook of Optics*,
https://doi.org/10.1007/978-3-031-55676-0_1

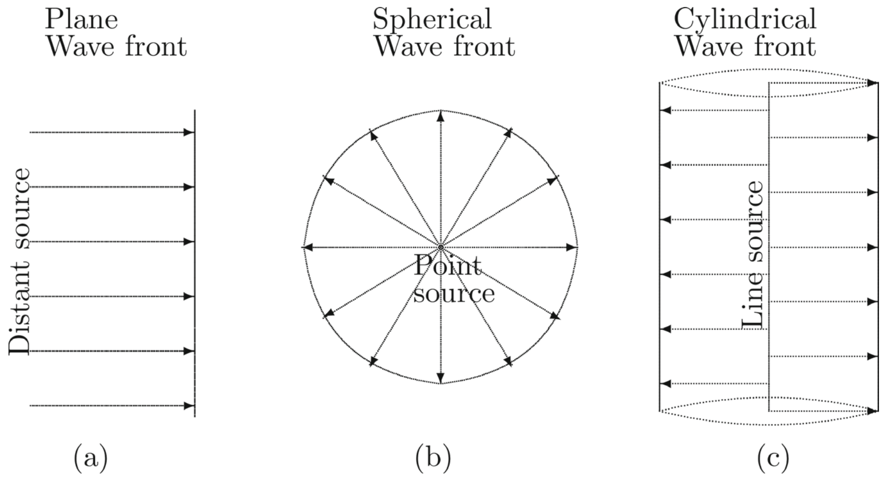


Fig. 1 Three possible shapes of a wave front

After moving a large distance, both the spherical wavefront and the cylindrical wavefront obviously convert into a plane wavefront. That is when a source of light (point source, line source, or any other shape) is situated at very large distance, we get only a plane wavefront.

Exercise 1 Light rays are coming from a point source situated at a distance of 100 light years. Assuming that the medium between the source and observer is homogeneous and isotropic, what is the shape of wavefront.

Solution The wavefront is spherical when the observer is close to the point source. After traveling a long distance, the wavefront is a plane.

2 Rectilinear Propagation of Light

We are aware that the shortest path between two given points is along a straight line joining the points. Therefore, in a homogeneous medium the time taken by a ray of light along a straight line is the minimum, as compared to that along any other path. Thus, the path of a ray is along a straight line. This phenomenon is known as the rectilinear propagation of light.

2.1 Refractive Index of Medium

According to the wave theory of light, if c is the speed of light in vacuum or free space and v the speed of light in a medium, the refractive index μ of the medium is expressed as

$$\mu = \frac{c}{v}$$

As c is the largest value for the speed of light, we have v always less than c and therefore the refractive index μ of a medium is always large than one. Often, for practical purposes, the air is taken equivalent to vacuum. The refractive index of vacuum (air) is 1 and that of any other medium is larger than 1.

Exercise 2 Velocity of light in air is 3×10^8 m/s. Calculate the velocity of light in the glass having refractive index 1.5.

Solution We have $c = 3 \times 10^8$ m/s. The velocity of light in the glass is

$$v = \frac{c}{\mu} = \frac{3 \times 10^8}{1.5} = 2 \times 10^8 \text{ m/s}$$

Exercise 3 Light from the sun reaches the earth in 8 min 20 s. Assuming that the medium between the sun and earth is homogeneous and isotropic air, calculate the distance of sun from the earth. The velocity of light in air is 3×10^8 m/s. How much time, the light from the sun to earth would take when the refractive index of the medium between the sun and earth is 1.2.

Solution The time taken by the light from sun to earth $t = 500$ s. The velocity of light in air (refractive index = 1) is $c = 3 \times 10^8$ m/s. Thus, the distance of sun from earth is

$$d = ct = c = 3 \times 10^8 \times 500 = 1.5 \times 10^{11} \text{ m}$$

When the medium between the sun and earth has refractive index $\mu = 1.2$. The velocity of light in that medium is

$$v = \frac{c}{\mu} = \frac{3 \times 10^8}{1.2} = 2.5 \times 10^8 \text{ m/s}$$

The time taken by the light for moving in that medium is

$$t' = \frac{d}{v} = \frac{1.5 \times 10^{11}}{2.5 \times 10^8} = 600 \text{ s} = 10 \text{ min}$$

2.2 Geometrical Path and Optical Path

Suppose, a ray of light travels in a medium of refractive index μ. The geometrical path between two points is the actual distance between the points. The optical path between two points is defined as the product of the geometrical path and the refractive index of the medium between the points.

2.3 Fermat's Principle

Fermat's principle plays important role in the geometrical optics. This principle states that when a ray of light travels from one point to another through a set of media, it always follows the path along which the time taken is the minimum.

3 Mirrors

A smooth surface, in a plane or spherical shape, polished by some specific material so that it reflects a beam of light falling on it, in one direction, is termed as a mirror. Figure 2 shows three surfaces. Surface (a) is plane in shape whereas the surfaces (b) and (c) are part of some sphere and are generally called a spherical surface. When the surface is plane, it is known as a plane mirror; when the surface is spherical, it is known as a spherical mirror.

Suppose, we polish each of these three surfaces on the left side of the surface so that the light falling from the left side is reflected back.[3] In our discussion, we consider an incident ray moving from left towards right which falls on a mirror.

The spherical mirrors are obviously of two types. The mirror (b) is known as a concave mirror whereas the mirror (c) is known as a convex mirror. Thus, a mirror is a polished surface which can reflect a beam of light, falling on it, in one direction without scattering in many directions or absorption.

In a homogeneous medium, a light ray is considered to travel along a straight line until it strikes on some solid surface or enters into another medium. On moving from one medium to another, the direction of motion of a ray changes depending on the refractive index of the second medium as compared to the first one. This phenomenon is known as the refraction and is discussed in the next chapter.

[3] In the present scenario, when the light falls on a mirror from right side, no phenomenon of interest is observed and it is not meaningful.

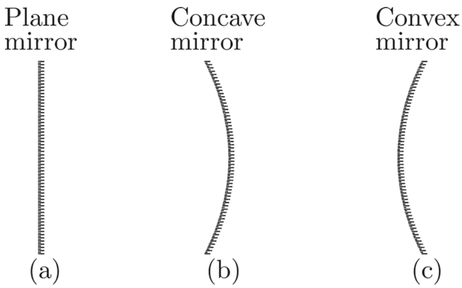


Fig. 2 Left side of each mirror is polished by some specific material

4 Reflection from a Plane Mirror

Let PQ be a plane mirror, polished on the upper surface. Suppose, a ray of light AB incident on the mirror (Fig. 3). At the point of incidence B, we draw a line BN normal to the surface. The $\angle$ ABN, denoted by i, is known as the angle of incidence. After reflection, the ray of light moves along the path BC. The angle $\angle$ NBC, denoted by r, is known as the angle of reflection. The incoming ray AB is generally known as the incident ray and the outgoing ray BC as the reflected ray.

4.1 Laws of Reflection

There are two laws of reflection.

1. The incident ray AB, normal BN to the plane mirror at the point B, and the reflected ray BC all lie in a common plane.
2. The angle of incidence i is equal to the angle of reflection r. That is, $i = r$.

Exercise 4 Using Fermat's principle show that for reflection by a plane mirror, the angle of incidence is equal to the angle of reflection.

Solution Suppose, an incident ray AB after reflection at the mirror PQ moves along the path BC. At the point B, BN is normal to the mirror. The perpendiculars from points A and C on the mirror intersect at the points M and L, respectively. If the separation between A and C (or between M and L) is d and that between M and B is x. The distance traveled by the incident ray AB is

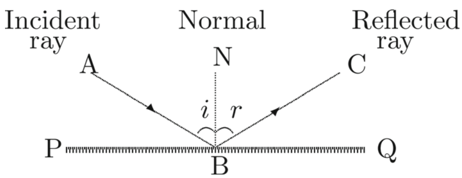
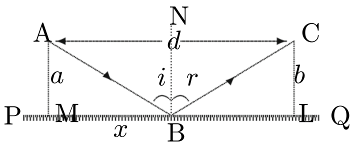


Fig. 3 Reflection of a ray from a plane mirror

$$\mathrm{AB} = \sqrt{a^2 + x^2}$$

and that traveled by the reflected ray BC is

$$\mathrm{BC} = \sqrt{b^2 + (d - x)^2}$$



Thus, the time of travel from A to C is

$$t = \frac{\mathrm{AB} + \mathrm{BC}}{c} = \frac{\sqrt{a^2 + x^2} + \sqrt{b^2 + (d - x)^2}}{c}$$

On differentiation of t with respect to x, we have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{c}\left[\frac{x}{\sqrt{a^2 + x^2}} - \frac{(d - x)}{\sqrt{b^2 + (d - x)^2}}\right]$$

According to Fermat's principle, the time of actual traverse should be minimum and therefore we should have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = 0 \qquad \text{or} \qquad \frac{1}{c}\left[\frac{x}{\sqrt{a^2 + x^2}} - \frac{(d - x)}{\sqrt{b^2 + (d - x)^2}}\right] = 0$$

It gives

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{(d - x)}{\sqrt{b^2 + (d - x)^2}} \qquad \text{or} \qquad \frac{\mathrm{AN}}{\mathrm{AB}} = \frac{\mathrm{CN}}{\mathrm{BC}}$$

Therefore,

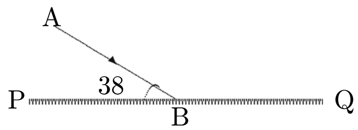
$$\sin i = \sin r \qquad \text{or} \qquad i = r$$

where i is the angle of incidence (the angle between the incidence ray AB and normal BN) and r is the angle of reflection (the angle between the reflected ray BC and normal BN). Second differentiation of t with respect to x gives

$$\frac{\mathrm{d}^2 t}{\mathrm{d}x^2} = \frac{1}{c}\left[\frac{a^2}{\sqrt{(a^2 + x^2)^{3/2}}} + \frac{d^2}{[b^2 + (d - x)^2]}^{3/2}\right]$$

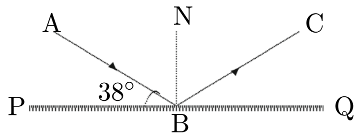
This is positive quantity for $i = r$. Hence, under the condition, $i = r$, the time along the path ABC is the minimum. Thus, Fermat's principle says that a ray always moves along a path for which the time taken is the minimum.

Exercise 5 A ray AB strikes a plane mirror PQ as shown in the following figure. Calculate the angle of reflection.



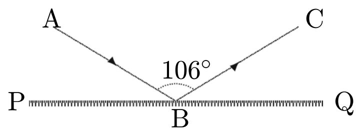
Solution Let us draw a line BN normal to the surface of the plane mirror and the reflected ray BC. The angle of incident is

$$\angle \text{ABN} = \angle \text{PBN} - \angle \text{ABP} = 90\text{–}38 = 52°$$

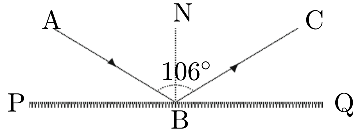


Following the law of reflection that the angle of incidence is equal to the angle of reflection, the angle of reflection is 52°.

Exercise 6 A ray AB strikes a plane mirror PQ and is reflected along BC as shown in the following figure. Calculate the angle of incidence and angle of reflection.

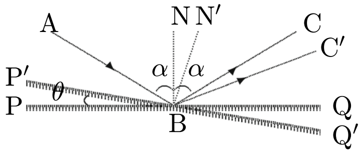


Solution Let us draw a line BN normal to the surface of the plane mirror PQ. Following the law of reflection, we have $\angle$ ABN = $\angle$ NBC. Therefore, we have $\angle$ ABN = 106/2 = 53°. Thus, the angle of incidence is 53° and the angle of reflection is 53°.



4.2 Effect of Rotation of Mirror

Let PQ be a plane mirror. The incident ray AB after reflection follows the path BC such that, following the law of reflection, $\angle$ ABN = $\angle$ NBC (say, α). Here, the normal at the point of incidence B is BN. Thus, $\angle$ ABC = 2 α.



Now, suppose the mirror is rotated through an angle θ and the new position of the mirror is P′Q′. The new position of the normal is BN′ such that $\angle$ NBN′ = θ. The incident ray AB after reflection with the mirror P′Q′ follows the path BC′. Now, the angle of incidence is $\angle$ ABN′ = $\alpha + \theta$. Thus, the $\angle$ ABC′ = 2 $(\alpha + \theta)$. Thus, we have

$$\angle \text{CBC}' = \angle \text{ABC}' - \angle \text{ABC} = 2(\alpha + \theta) - 2\alpha = 2\theta, \text{ for the same incident ray.}$$

It shows that if a mirror is rotated through an angle θ, the reflected ray is rotated through the angle 2θ.

Exercise 7 A ray reflected from a plane mirror is received by a person standing on the earth. If the mirror is rotated in the horizontal plane through an angle of 25°. How much the person should move in order to receive the reflected ray.

Solution Since on rotation of a plane mirror through an angle θ, the reflected ray rotates through the angle 2θ, the man should rotate through an angle $25 \times 2 = 50°$ to receive the reflected ray again.

4.3 Reflection from a Pair of Plane Mirrors

Let us consider two plane mirrors PQ and QR which are combined at a point Q and the angle between them is β. Suppose, the incident ray AB strikes at the mirror PQ with the angle of incidence α (Fig. 4). The line BN is perpendicular to the mirror PQ at the point B. Thus, the angle of reflection is

$$\angle \text{NBC} = \alpha$$

Therefore, $\angle$ QBC = $\angle$ NBQ − $\angle$ NBC = 90 $-\alpha$

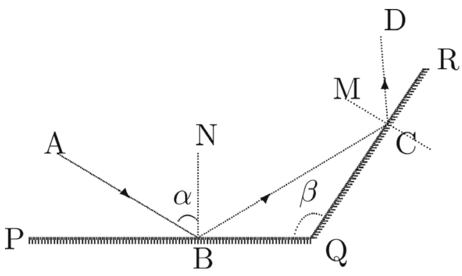
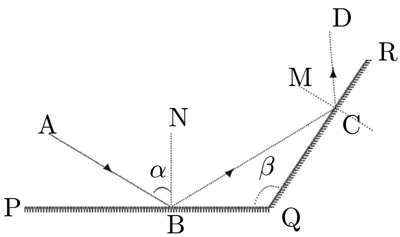


Fig. 4 Reflection from a pair of plane mirror

The ray BC strikes the mirror QR at the point C. The line CM is perpendicular to the mirror QR at the point C. In the triangle BCQ, the sum of three angles is 180°. Thus, $\angle$ BCQ= 180 − $\angle$ QBC − $\angle$ BQC = 180 − (90 −α) −β = 90 + α − β

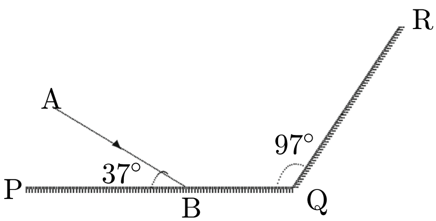


The $\angle$ BCM = $\angle$ QCM − $\angle$ BCQ = 90 − (90 + α − β) = β − α

This is the angle of incidence for the mirror QR. Following the law of reflection, the angle of reflection at the mirror QR is

$\angle$ MCD = β − α

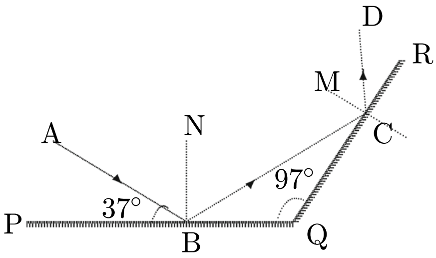
Exercise 8 Two plane mirrors PQ and QR are inclined at an angle of 97°. A ray AB incidents on a plane mirror PQ as shown in the following figure. Calculate the angle of reflection of the ray at the mirror QR.



Solution Let us draw a line BN perpendicular to the mirror PQ at the point B. Thus, the angle of incidence $\angle$ ABN = 90 − 37 = 53°. After reflection at the mirror PQ, the ray moves along the path BC and strikes at the point C on the mirror QR. Following the law of reflection, the ∠ NBC = 53°.

Thus, we have ∠ CBQ = 90 − 53 = 37°.

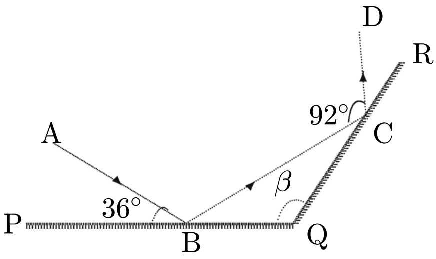
In the triangle BCQ, we have ∠ BCQ = 180 −∠ CBQ −∠ BQC = 180 − 37 − 97 = 46°.



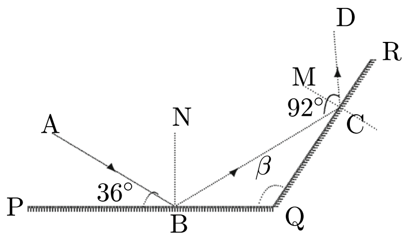
Let us draw a line CM normal to the surface of the plane mirror QR at the point C. Thus, we have ∠ MCB = 90 −∠ BCQ = 90 − 46 = 44°. Therefore, the angle of incidence at the mirror QR is 44°.

Following the law of reflection, we have ∠ MCD = 44°.

Exercise 9 Two plane mirrors PQ and QR are inclined at an angle of β. A ray AB incidents on a plane mirror PQ and after reflection moves along the path BC as shown in the following figure. This reflected ray strikes the plane mirror QR at the point C and is reflected along the path CD so that the ∠ BCD is 92°. Calculate the angle β between the mirrors PQ and QR.



Solution Let us draw a line BN perpendicular to the mirror PQ at the point B and line CM normal to the mirror QR at the point C. Following the law of reflection, the angle of incidence is equal to the angle of reflection. That is ∠ BCM = ∠ MCD. Therefore, ∠ BCM = 92/2 = 46°.



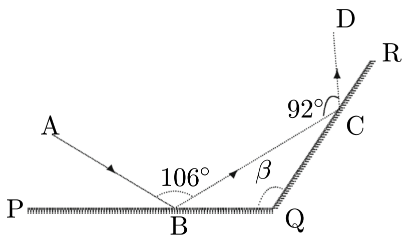
Now, we have ∠ BCQ = ∠ MCQ − ∠ MCB = 90 − 46 = 44°.

We have ∠ ABN = ∠ PBN − ∠ PBA = 90 − 36 = 54°. Following the law of reflection that the angle of incidence is equal to the angle of reflection, we have ∠ NBC = 54°.

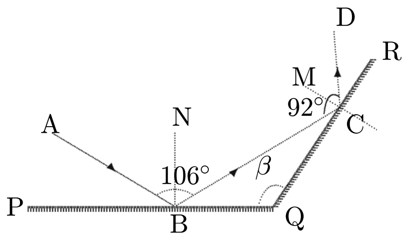
We have ∠ CBQ = ∠ NBQ − ∠ NBC = 90 − 54 = 36°.

In the triangle BCQ, we have ∠ BQC = 180 − ∠ CBQ − ∠ BCQ = 180 − 36 − 44 = 100°. Thus, the angle β is 100°.

Exercise 10 Two plane mirrors PQ and QR are inclined at an angle of β. A ray AB incidents on a plane mirror PQ and after reflection moves along the path BC so that the ∠ ABC is 106°. This reflected ray strikes the plane mirror QR at the point C and is reflected along the path CD so that the ∠ BCD is 92°. Calculate the angle β between the mirrors PQ and QR.



Solution Let us draw a line BN perpendicular to the mirror PQ at the point B and line CM normal to the mirror QR at the point C. Following the law of reflection, the angle of incidence is equal to the angle of reflection. That is ∠ ABN = ∠ NBC, and therefore ∠ NBC = 106/2 = 53°. Further, ∠ BCM = ∠ MCD, and therefore ∠ BCM = 92/2 = 46°.



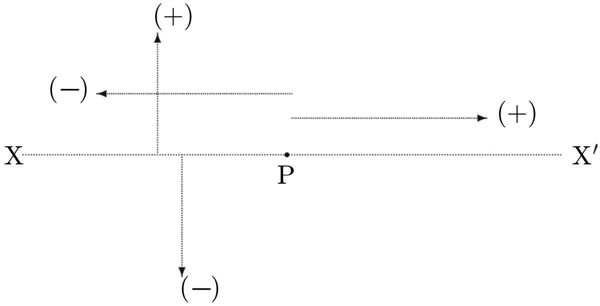
Now, we have ∠ BCQ = ∠ MCQ − ∠ MCB = 90 − 46 = 44°.

Further, we have ∠ CBQ = ∠ NBQ − ∠ NBC = 90 − 53 = 37°.

In the triangle BCQ, we have ∠ BQC = 180 − ∠ CBQ − ∠ BCQ = 180 − 37 − 44 = 99°. Thus, the angle β is 99°.

5 Sign Convention

For the formation of image of an object placed in front of a mirror, we consider a horizontal line, called the principal axis, which is perpendicular to the surface of the mirror. Obviously, in case of a spherical mirror, the principal axis passes through the center of curvature of the mirror. A mirror is placed so that its left surface is polished. An object is placed on the left side to a mirror. The point of intersection of principal axis and the surface of the mirror is known as the pole. Position of a pole depends on the principal axis drawn. For a mirror, we have various principal axes and corresponding to each principal axis, there is a pole. Now, for the distances measured, the sign convention is as follows:



(i) The distances measured from the pole in the left direction are taken as negative whereas those in the right direction as positive.

(ii) The distances measured from the principal axis in the downward direction are taken as negative whereas those in the upward direction as positive.

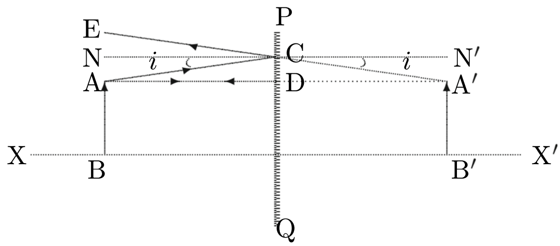
For example, in the following figure for a mirror, the principal axis is XX′. The point of intersection P of a mirror and principal axis is the pole. Now, a distance measured from P (also from mirror) in the left direction is taken as negative (−) whereas that in the right direction is taken as positive (+). Further, a distance measured from the principal axis in the upward direction is taken as positive (+) whereas that in the downward direction is taken as negative (−).

6 Formation of Image of an Object

For the formation of image of an object by a mirror (plane as well as spherical) or by a lens (to be discussed in the next chapter), we consider a point on the object. From this point, we consider at least two rays which are reflected from a mirror (refracted through a lens) and finally meet at a common point or appear to come from a common point. (When rays appear to come from a common point, that point is a virtual image of the point on the object.) This point is the image of the point on the object. Other points on the object also form the corresponding images. Now, the locus of all these image points is the image of the object. The formation of image of an object with the help of a mirror is discussed in this chapter and with the help of a lens in the next chapter.

7 Image Formation by a Plane Mirror

Let us consider a plane mirror PQ and the principal axis XX′. Suppose, an object AB is placed on the left side of the mirror.



Let us consider a point A on the object AB. A ray AD falls normally on the mirror and is reflected back along DA. The ray AC falling on the mirror is reflected back along CE so that ∠ ACN = ∠ NCE (law of reflection). These two reflected rays DA and CE appear to come from a common point A′. Thus, the point A′ is the image of the point A on the object. Similarly, one can find out images of other points on the object. A locus of all the image points is A′B′. Hence, the A′B′ is the image of the object AB. Such an image is virtual, as it cannot be matched with any real object. We have ∠ ACN = ∠ NCE = i.

Further, we have ∠ ECN = ∠ N′CA′.

We have ∠ ACD = ∠ NCD − ∠ NCA = 90 −i

and ∠ A′CD = ∠ N′CD − ∠ N′CA′ = 90 −i.

In the ΔACD and ΔA′CD, we have

∠ACD = ∠ A′CD

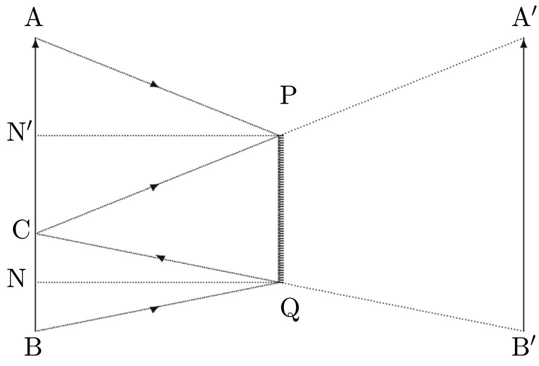
CD is common in both the triangles. Therefore, ΔACD and ΔA′CD are equal triangles.

Therefore, AD = A′D. It shows that the distance of image from the mirror is equal to that of the object from the mirror.

Since AA′ is parallel to the principal axis XX′. It shows that the length of the image is equal to that of the object.

7.1 Minimum Size of Mirror for a Complete Image

We know that with the help of a plane mirror, image of an object is formed of the same size as that of the object and the distance of the image from the mirror is equal to that of the object from the mirror.



Suppose, PQ is a plane mirror and AB an object placed in front of the mirror. The image of the object formed by the mirror is A′B′. For the formation of a complete image of an object, the rays AP and BQ, after reflection from the upper and lower edges of the mirror, should not cross each other. At the most, they can reach at a common point C. Thus, in the following figure, the size of the mirror is the minimum required for the formation of a complete image of the object. From the figure, we have AP = CP = A′P and BQ = QC = B′Q.

Thus, CP = CA′/2 and CQ = CB′/2. Consequently, PQ is parallel to A′B′ and the ΔCPQ and ΔCA′B′ are similar triangles. Consequently, we have PQ = A′B′/2.

Thus, the minimum size of the mirror for the formation of a complete image is half of the size of the object.

Exercise 11 A 1.64 m tall person is standing in front of a large plane mirror at a distance of 1.05 m so that a complete image of the person is formed. Write down the size of the image and its position.

Solution In case of a large plane mirror, a complete image is formed. In case of plane mirror, the size of image is equal to that of the object. The size of the image is 1.64 m. Further, the distance of the image from the mirror is equal to the distance of the object from the mirror. Hence, the image is formed at a distance 1.05 m from the mirror on the opposite side.

Exercise 12 A 1.7 cm tall person wants to see his complete image with the help of a plane mirror. What should be the minimum size of the mirror?

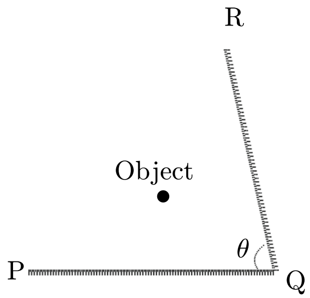
Solution We know that for the formation of a complete image of an object, the minimum size of a mirror should be half of the size of the object. Therefore, the minimum size of the mirror should be 1.7/2 = 0.85 m = 85 cm.

8 Object Between a Pair of Plane Mirrors

When an object is placed between two plane mirrors inclined at an angle θ, multiple virtual images are formed due to multiple reflections of light from the mirrors. Here, PQ and QR are two plane mirrors inclined at an angle θ. An object is placed between the mirrors. The number N of images formed depends on the angle θ and is

$$N = \frac{360}{\theta} - 1$$

Here, the angle θ is expressed in degrees. For a pair of parallel mirrors, we have $\theta = 0$ and infinite number of images are formed.



Exercise 13 An object is placed between two plane mirrors inclined at an angle of 60°. Calculate the number of images formed.

Solution Here, we have $\theta = 60°$. The number N of images is

$$N = \frac{360}{\theta} - 1 = \frac{360}{60} - 1 = 5$$

Exercise 14 What should be the angle between two plane mirrors so that we can see two images of an object?

Solution Here, $N = 2$. Thus, we have

$$2 = \frac{360}{\theta} - 1$$

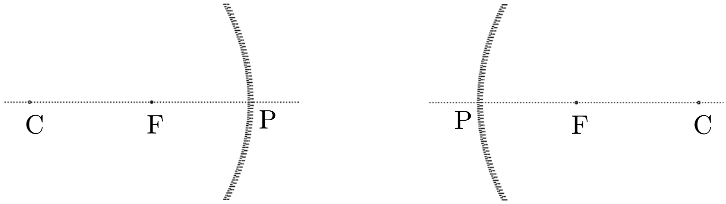
This equation gives $\theta = 120°$.

9 Fundamentals for Image Formation by a Spherical Mirror

Before discussing about formation of image of an object with the help of a spherical mirror, let us first understand some important aspects about the spherical mirrors.

9.1 Radius of Curvature and Focal Length

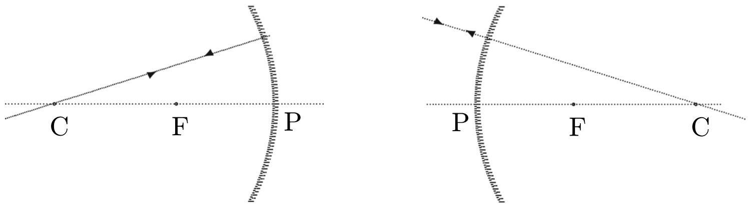
A spherical mirror is a part of some sphere of radius R. The center of that sphere is known as the center of curvature of the spherical mirror and R is the radius of curvature of the mirror. A line joining a point on the mirror and the center of curvature C is normal to the mirror. In each of the following figures, C is the center of curvature and CP is the radius of curvature R. For a horizontal principal axis passing through C, the point P is the pole. A mid-point, between P and C, is known as the focal point and is denoted by F. Obviously, we have PF = FC = $R/2$. This distance is often denoted by f so that $f = R/2$.



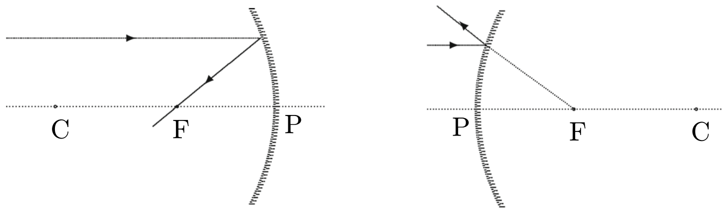
9.2 *Reflection of Rays*

For the formation of image of an object, we may consider some rays mentioned in the following discussion. An image is formed at the position where the reflected rays meet or appear to meet to each other.

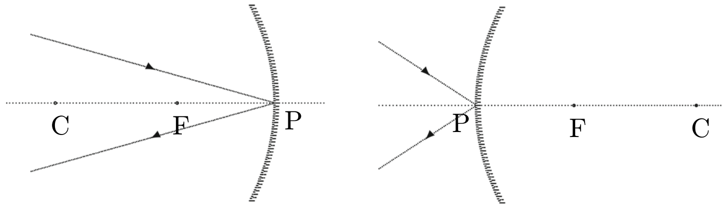
(i) A ray passing through the center of curvature C or appearing going towards C, after reflection from a spherical mirror, moves along the same path traversed in the reverse sense.



(ii) A ray moving parallel to the principal axis, after reflection from a spherical mirror, is reflected along a path passing through the focal point (concave mirror) or reflected along the path which appears as if coming from the focal point (convex mirror).

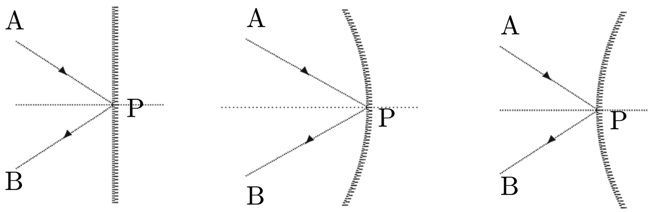


(iii) A ray which is incident at the pole is reflected back so that the angle of incidence is equal to the angle of reflection. Here, the line joining the pole and the center of curvature is the normal to surface of the mirror.



9.3 *Reversibility Theorem*

In the aforesaid discussion, we have seen that a ray of light after falling on a mirror (plane or spherical) is reflected in some direction. Let us consider the incident (AP) and reflected (PB) rays for three types of mirrors, shown in the following figure. The reversibility theorem states that if a ray is incident on a mirror along BP then the reflected ray will move along PA. That is, the path is traversed back. This theorem is valid for reflection of any ray by a plane or spherical mirror.

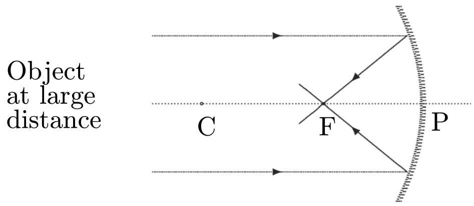


10 Formation of Image by a Concave Mirror

Position and size of an image of an object formed by a concave mirror depend on the position of the object in front of the mirror. There are six cases as discussed in the following.

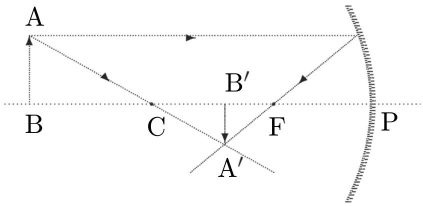
10.1 When an Object is at Infinite Distance

Let us consider an object placed at very large distance from a concave mirror (figure below). From the distant object, the coming rays are parallel to one another. The parallel rays after reflection at the mirror pass through the focal point of the mirror as shown in the following figure. Since the two rays from the object after reflection intersect to each other at the focal point, the image of the object is formed at the focal point of the mirror.



10.2 When an Object is Between Infinity and the Center of Curvature

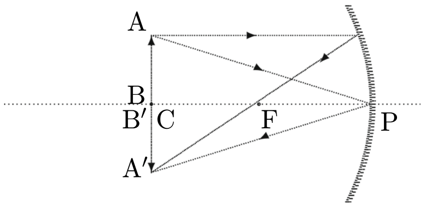
Let us consider an object AB placed at a point between infinity and the center of curvature C (figure below). A ray parallel to the principal axis from a point A after reflection at the mirror passes through the focal point F. Second ray passing through the center of curvature C strikes the mirror and is reflected back along the same line in the reverse sense. These two rays from the point A after reflection intersect at the point A′. This A′ is the image of A. Similarly, we can have images of other points on the object AB and we get an image A′B′ of the object AB. The image is formed between the center of curvature and the focal point.



10.3 When an Object is at the Center of Curvature

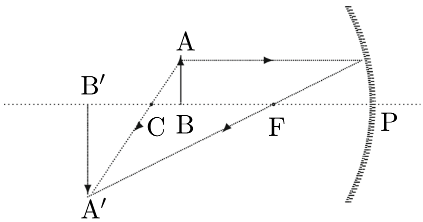
Let us consider an object AB placed at the center of curvature of the concave mirror (figure below).

The ray parallel to the principal axis from a point A on the object after reflection at the mirror passes through the focal point F. Second ray AP is reflected along PA′ so that the angle of incidence is equal to the angle of reflection. These two rays from the point A after reflection intersect at the point A′. This A′ is the image of A. Similarly, we can have images of other points on the object AB and we get an image A′B′ of the object AB. The image is formed at the center of curvature.



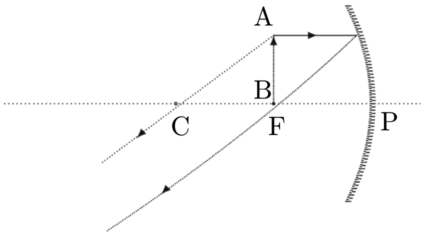
10.4 When an Object is Between the Center of Curvature and Focal Point

Let us consider an object AB placed between the center of curvature and focal point (figure below). A ray parallel to the principal axis from a point A on the object after reflection at the mirror passes through the focal point F of the mirror. Second ray AC passes through the center of curvature. These two rays from the point A after reflection intersect at the point A′. This A′ is the image of A. Similarly, we can have images of other points on the object AB and we get an image A′B′ of the object AB. The image is formed between the infinity and the center of curvature.



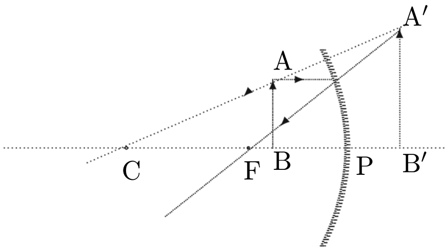
10.5 When an Object is at the Focal Point

Let us consider an object AB placed at the focal point (figure below). A ray parallel to the principal axis from a point A on the object after reflection at the mirror passes through the focal point F. Second ray AC passes through the center of curvature. As these two rays from the point A after reflection are parallel to each other, the image is formed at infinity.



10.6 When an Object is Between the Focal Point and Pole

Let us consider an object AB placed between the focal point and pole (figure below). A ray parallel to the principal axis from a point A on the object after reflection at the mirror passes through the focal point F of the mirror. Second ray AC passes through the center of curvature. As these two rays from the point A after reflection appear to meet at the point A′ on other side of the mirror. This A′ is the image of A. Similarly, we have images of other points on the object AB and we get an image A′B′ of the object AB. Virtual image is formed on the back side of the mirror.



Note These six positions of object, and the corresponding positions of image and its nature are summarized in the following table.

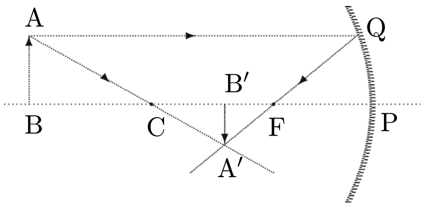
	Position of object	Position of image	Nature of image
1.	At infinity	Focal point	Real
2.	Between infinity and center of curvature	Between center of curvature and focal point	Real
3.	Center of curvature	Center of curvature	Real
4.	Center of curvature and focal point	Between infinity and center of curvature	Real
5.	Focal point	At infinity	Real
6.	Between focal point and pole	Behind the mirror	Virtual

10.7 Relation Between u, v, and f

Here, u denotes the distance of object from the pole and v denotes the distance of image from the pole. We know that f is the distance of focal point from the pole, which is half of the radius of curvature R. The relation between u, v, and f can be derived in any of the said six cases of concave mirror. For convenience, let us consider the following figure.

Let us consider the formation of image A′B′ of an object AB. Since AB and A′B′ are perpendicular to the principal axis, the ΔABC and ΔA′B′C are the similar triangles, and we have

$$\frac{AB}{A'B'} = \frac{BC}{B'C} \tag{1}$$



For a mirror with a large radius of curvature, PQ is almost perpendicular to the principal axis. As PQ and A′B′ are perpendicular to the principal axis, the ΔPQF and ΔA′B′F are the similar triangles, and we have

$$\frac{PQ}{A'B'} = \frac{PF}{B'F} \tag{2}$$

Since AB = PQ, from Eqs. (1) and (2), we have

$$\frac{\mathrm{BC}}{\mathrm{B'C}} = \frac{\mathrm{PF}}{\mathrm{B'F}} \qquad \text{or} \qquad \frac{\mathrm{BP} - \mathrm{PC}}{\mathrm{CP} - \mathrm{B'P}} = \frac{\mathrm{PF}}{\mathrm{B'P} - \mathrm{PF}}$$

Accounting for the sign convention, a distance measured left to the pole is negative, we have BP = $-u$, B′P = $-v$, FP = $-f$, and CP = $-R = -2f$. Using these values, we have

$$\frac{-u + 2f}{-2f + v} = \frac{-f}{-v + f}$$

On performing cross multiplication and solving, we get

$$uv = uf + vf$$

On dividing this equation by uvf, we get

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

This is the required relation between u, v, and f and is known as the mirror equation.

10.8 Magnification

The ratio of the size of image to the size of object is known as the magnification, denoted by m. Thus, with the sign convention, we have

$$m = -\frac{\mathrm{A'B'}}{\mathrm{AB}}$$

Using Eq. (1), we have

$$m = -\frac{\mathrm{B'C}}{\mathrm{BC}} = -\frac{\mathrm{CP} - \mathrm{B'P}}{\mathrm{BP} - \mathrm{PC}}$$

Using the values with sign convention, we get

$$m = -\frac{-2f + v}{-u + 2f}$$

From the mirror equation, we have $f = uv/(u + v)$. Using this relation, we have

$$m = -\frac{\frac{-2uv}{(u+v)} + v}{-u + \frac{2uv}{(u+v)}} = -\frac{-uv + v^2}{-u^2 + uv} = -\frac{v}{u}$$

Hence, the magnification is the ratio of the distance of image to that of the object from the pole.

Exercise 15 Image of a distant star is formed with the help of a concave mirror of radius of curvature 1.6 m. Find out the position of image formed.

Solution Star is at very large distance from us and therefore we have $u = \infty$. The focal length of the concave mirror is $f = R/2 = 1.6/2 = 0.8$ m. After accounting for the sign convention, we have $u = -\infty$ and $f = -0.8$ m. In the mirror equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

using the values, we have

$$\frac{1}{-\infty} + \frac{1}{v} = \frac{1}{-0.8} \qquad \text{or} \qquad \frac{1}{v} = -\frac{1}{0.8}$$

Thus, the image is formed at a distance of 0.8 m on the side of the star. Hence, the image is formed at the focal point of the mirror.

Exercise 16 An object of length 30 cm is placed in front of a concave mirror of radius of curvature 1.4 m at a distance 2.1 m. Find out the position of image formed and the magnification of the mirror.

Solution Focal length $f = R/2 = 1.4/2 = 0.7$ m. After accounting for the sign convention, we have $f = -0.7$ m and $u = -2.1$ m. In the mirror equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

using the values, we get

$$\frac{1}{-2.1} + \frac{1}{v} = \frac{1}{-0.7} \qquad \text{or} \qquad \frac{1}{v} = -\frac{10}{7} + \frac{10}{21}$$

On solving, we get $v = -1.05$ m. That the image is formed on the side of the object. The magnification is

$$m = -\frac{v}{u} = -\frac{-1.05}{-2.1} = -0.5$$

Thus, we have

$$\frac{\text{length of image}}{\text{length of object}} = -0.5 \qquad \text{or} \qquad \frac{\text{length of image}}{0.30} = -0.5$$

Hence, the length of image = −0.15 m = −15 cm. Thus, an inverted image of length of 15 cm is formed.

Exercise 17 An object of length 40 cm is placed in front of a concave mirror of radius of curvature 1.2 m at a distance 1.2 m. Find out the position of image formed and the magnification of the mirror.

Solution Focal length $f = R/2 = 1.2/2 = 0.6$ m. After accounting for the sign convention, we have $f = -0.6$ m and $u = -1.2$ m. In the mirror equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

using the values, we get

$$\frac{1}{-1.2} + \frac{1}{v} = \frac{1}{-0.6} \qquad \text{or} \qquad \frac{1}{v} = -\frac{10}{6} + \frac{10}{12}$$

On solving, we get $v = -1.2$ m. Thus, the image is formed on the side of the object. The magnification is

$$m = -\frac{v}{u} = -\frac{-1.2}{-1.2} = -1.0$$

Therefore, we have

$$\frac{\text{length of image}}{\text{length of object}} = -1.0 \qquad \text{or} \qquad \frac{\text{length of image}}{0.4} = -1.0$$

Thus, the length of the image = −0.4 m = −40 cm. Thus, inverted image of length of 40 cm is formed on the side of the object.

Exercise 18 An object of length 25 cm is placed in front of a concave mirror of radius of curvature 1.4 m at a distance of 80 cm. Find out the position of image formed and the magnification of the mirror.

Solution Focal length $f = R/2 = 1.4/2 = 0.7$ m. After accounting for the sign convention, we have $f = -0.7$ m and $u = -0.8$ m. In the mirror equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

using the values, we get

$$\frac{1}{-0.8} + \frac{1}{v} = \frac{1}{-0.7} \qquad \text{or} \qquad \frac{1}{v} = -\frac{10}{7} + \frac{10}{8}$$

On solving, we get $v = -5.6$ m. Thus, the image is formed on the side of the object. The magnification is

$$m = -\frac{v}{u} = -\frac{-5.6}{-0.8} = -7.0$$

Thus, we have

$$\frac{\text{length of image}}{\text{length of object}} = -7.0 \qquad \text{or} \qquad \frac{\text{length of image}}{0.25} = -7.0$$

Hence, the length of image = -1.75 m. Thus, inverted image of length of 1.75 m is formed on the side of the object.

Exercise 19 An object of length 25 cm is placed in front of a concave mirror of radius of curvature 1.4 m at a distance 70 cm. Find out the position of image formed and the magnification of the mirror.

Solution Focal length $f = R/2 = 1.4/2 = 0.7$ m. After accounting for the sign convention, we have $f = -0.7$ m and $u = -0.7$ m. In the mirror equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

using the values, we get

$$\frac{1}{-0.7} + \frac{1}{v} = \frac{1}{-0.7} \qquad \text{or} \qquad \frac{1}{v} = -\frac{10}{7} + \frac{10}{7}$$

On solving, we get $v = -\infty$. Thus, the image is formed on the side of the object placed at an infinite distance. The magnification is

$$m = -\frac{v}{u} = -\frac{-\infty}{-1.2} = -\infty$$

Thus, we have

$$\frac{\text{length of image}}{\text{length of object}} = -\infty \qquad \text{or} \qquad \frac{\text{length of image}}{0.25} = -\infty$$

Hence, the length of image = ∞. Thus, an inverted image of ∞ length is formed on the side of the object.

Exercise 20 An object of length 30 cm is placed in front of a concave mirror of radius of curvature 1.4 m at a distance of 50 cm. Find out the position of image formed and the magnification of the mirror.

Solution Focal length $f = R/2 = 1.4/2 = 0.7$ m. After accounting for the sign convention, we have $f = -0.7$ m and $u = -0.5$ m. In the mirror equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

using the values, we get

$$\frac{1}{-0.5} + \frac{1}{v} = \frac{1}{-0.7} \qquad \text{or} \qquad \frac{1}{v} = -\frac{10}{7} + \frac{10}{5}$$

On solving, we get $v = 1.75$ m. Thus, the image is formed on the opposite side of the object. The magnification is

$$m = -\frac{v}{u} = -\frac{1.75}{-0.5} = 3.5$$

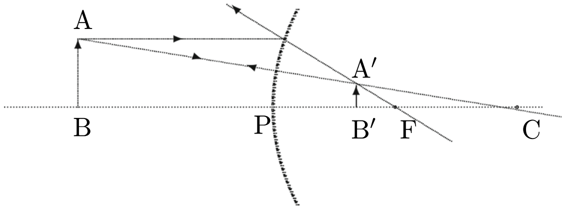
Thus, we have

$$\frac{\text{length of image}}{\text{length of object}} = 3.5 \qquad \text{or} \qquad \frac{\text{length of image}}{0.3} = 3.5$$

Hence, the length of image = 1.05 m. Thus, an image of length of 1.05 m is formed.

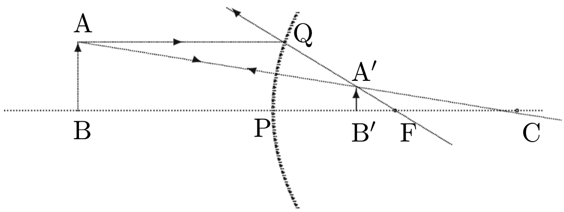
11 Formation of Image by a Convex Mirror

As in the case of a convex mirror, the focal point and center of curvature are on the other side as compared to that of the object, we have only one case as shown in the following figure. A ray parallel to the principal axis from a point A after reflection at the mirror appears to come from the focal point F. Second ray going in the direction of the center of curvature C strikes the mirror and is reflected back along the same line in the reverse sense. As these two rays from the point A after reflection appear to intersect at the point A′. This A′ is a virtual image of A. Similarly, we have images of other points on the object AB and we get image A′B′ of the object AB. The image is virtual.



11.1 Relation Between u, v, and f

Here, u denotes the distance of object from the pole whereas v denotes the distance of image from the pole. We know that f is the distance of the focal point from the pole which is half of the radius of curvature R. The relation between u, v, and f can be derived in the following manner.



Let us consider formation of image A′B′ of an object AB. Since AB and A′B′ are perpendicular to the principal axis, the ΔABC and ΔA′B′C are the similar triangles, and we have

$$\frac{\mathrm{AB}}{\mathrm{A'B'}} = \frac{\mathrm{BC}}{\mathrm{B'C}} \tag{3}$$

For a mirror with large radius of curvature, PQ is almost perpendicular to the principal axis. As PQ and A′B′ are perpendicular to the principal axis, the ΔPQF and ΔA′B′F are the similar triangles, and we have

$$\frac{\mathrm{PQ}}{\mathrm{A'B'}} = \frac{\mathrm{PF}}{\mathrm{B'F}} \tag{4}$$

Since AB = PQ, from Eqs. (3) and (4), we have

$$\frac{\mathrm{BC}}{\mathrm{B'C}} = \frac{\mathrm{PF}}{\mathrm{B'F}} \qquad \text{or} \qquad \frac{\mathrm{BP}+\mathrm{PC}}{\mathrm{CP}-\mathrm{B'P}} = \frac{\mathrm{PF}}{\mathrm{PF}-\mathrm{B'P}}$$

Accounting for the sign convention that a distance measured left to the pole is negative whereas to the right is positive, we have BP = $-u$, B′P = v, FP = f, and CP = $R = 2f$. Using these values, we have

$$\frac{-u+2f}{2f-v} = \frac{f}{f-v}$$

On performing cross multiplication and solving, we get

$$uv = uf + vf$$

On dividing this equation by uvf, we get

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

This is the required relation between u, v, and f and is known as the mirror equation.

11.2 Magnification

The ratio of the size of image to the size of object is known as the magnification, denoted by m. Thus, with the sign convention, we have

$$m = \frac{\mathrm{A'B'}}{\mathrm{AB}}$$

Using Eq. (3), we have

$$m = \frac{\mathrm{B'C}}{\mathrm{BC}} = \frac{\mathrm{CP} - \mathrm{B'P}}{\mathrm{BP} + \mathrm{PC}}$$

Using the values with sign convention, we get

$$m = \frac{2f-v}{-u+2f}$$

From the mirror equation, we have $f = uv/(u+v)$. Using this relation, we have

$$m = \frac{\frac{2uv}{(u+v)} - v}{-u + \frac{2uv}{(u+v)}} = \frac{uv - v^2}{-u^2 + uv} = -\frac{v}{u}$$

Exercise 21 An object of length 40 cm is placed in front of a convex mirror of radius of curvature 1.2 m at a distance of 1.2 m. Find out the position of the image formed and magnification of the mirror.

Solution Focal length $f = R/2 = 1.2/2 = 0.6$ m. After accounting for the sign convention, we have $f = 0.6$ m and $u = -1.2$ m. In the mirror equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

using the values, we get

$$\frac{1}{-1.2} + \frac{1}{v} = \frac{1}{0.6} \qquad \text{or} \qquad \frac{1}{v} = \frac{10}{6} + \frac{10}{12}$$

On solving, we get $v = 0.4$ m. Thus, the image is formed on the other side of the object. The magnification is

$$m = -\frac{v}{u} = -\frac{0.4}{-1.2} = 0.333$$

Therefore, we have

$$\frac{\text{length of image}}{\text{length of object}} = 0.333 \qquad \text{or} ce1.5cm \frac{\text{length of image}}{0.4} = 0.333$$

Hence, the length of image = 0.1332 m = 13.32 cm. Thus, an image of length of 13.23 cm is formed on the opposite side of the object.

12 Multiple Choice Questions

1. For the light coming from a very distant star, the wavefront is

   A. Circular B. Cylindrical C. Plane D. Depends on star

   Ans. C

2. A point source is radiating light in a laboratory, the wavefront is

   A. Circular B. Cylindrical C. Plane D. Depends on source

   Ans. A

3. A line source is radiating light in a laboratory, the wavefront is

   A. Circular B. Cylindrical C. Plane D. Depends on source

   Ans. B

4. The refractive index of a medium is always

A. Larger than 1 B. Equal to 1

C. Smaller than 1 D. Depends on medium

Ans. A

5. Suppose, c denotes the speed of light in vacuum. In a medium, the speed of light is always

A. Larger than c B. Equal to c

C. Smaller than c D. Depends on medium

Ans. C

6. For reflection of light in a medium, the angle of incidence i and the angle of reflection r have a relation

A. $i > r$ B. $i = r$ C. $i < r$ D. Depends on medium

Ans. B

7. The minimum size of a plane mirror required to see the complete image of a person of height h is

A. $h/4$ B. $h/2$ C. $3h/4$ D. h

Ans. B

8. Two plane mirrors are inclined at an angle of $60°$. The number of images formed of a object placed between them is

A. 3 B. 5 C. 6 D. 7

Ans. B

9. Two plane mirrors are inclined at an angle of θ. The number of images formed of a object placed between them is 5. The value of θ is

A. $36°$ B. $60°$ C. $72°$ D. $90°$

Ans. B

10. The light is coming form a distant star. The image of the star formed by a concave mirror is

    A. Between focus and pole
    B. At infinite distance
    C. Between focus and center of curvature
    D. At focal point

    Ans. D

11. An object is placed at the center of curvature of a concave mirror. The image of the object formed by the mirror is

    A. Between focal point and pole
    B. At center of curvature
    C. Between focal point and center of curvature
    D. At focal point

    Ans. B

12. An object is placed between infinity and center of curvature of a concave mirror. The image of the object formed by the mirror is

    A. Between focal point and pole
    B. At center of curvature
    C. Between focal point and center of curvature
    D. At infinity

    Ans. C

13. An object is placed between focal point and center of curvature of a concave mirror. The image of the object formed by the mirror is

    A. Between focal point and pole
    B. At center of curvature
    C. Between infinity and center of curvature
    D. At infinity

    Ans. C

14. An object is placed at focal point of a concave mirror. The image of the object formed by the mirror is

    A. Between focal point and pole
    B. At center of curvature
    C. Between infinity and center of curvature
    D. At infinity

    Ans. D

15. An object is placed between the focal point and pole of a concave mirror. The image of the object formed by the mirror is

    A. Between focal point and pole B. At center of curvature

    C. Behind the mirror D. At infinity

    Ans. C

16. Suppose, u and v denote the distances of object and image from the pole of a spherical mirror, respectively. For the focal length f of the mirror, the relation between u, v, and f is

    A. $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ B. $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

    C. $\frac{1}{v} + \frac{1}{f} = \frac{1}{u}$ D. $\frac{1}{u} + \frac{1}{f} = \frac{1}{v}$

    Ans. A

17. For a convex mirror, the sizes of an object and its image are 8 cm and 4 cm, respectively. The magnitude of magnification is

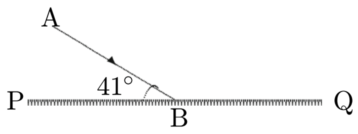
    A. 0.5 B. 2 C. 2.33 D. 8

    Ans. A

13 Problems and Questions

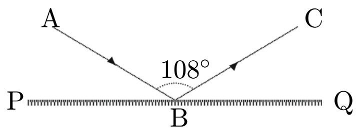
1. Define refractive index of a medium. What is the difference between the geometrical path and optical path?
2. Describe that the refractive index of a medium is always larger than 1.
3. Describe about the reflection of light. What are the laws of reflection?

4. What are the laws of reflection? A ray AB strikes a plane mirror PQ as shown in the figure. Calculate the angle of reflection.



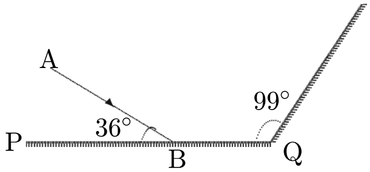
[Ans. 49°]

5. A ray AB strikes a plane mirror PQ and is reflected along BC as shown in the following figure. Calculate the angle of incidence and angle of reflection.



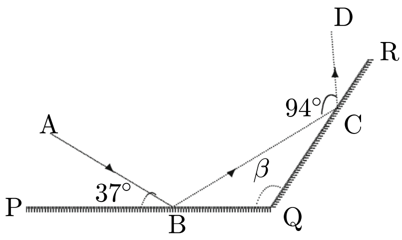
[Ans. 54°, 54°]

6. Show that on rotation of a mirror through an angle θ, the reflected ray is rotated through the angle 2θ, for the same incident ray.
7. Two plane mirrors PQ and QR are inclined at an angle of 99°. A ray AB incident on a plane mirror PQ is shown in the following figure. Calculate the angle of reflection of the ray at the mirror QR.



[Ans. 45°]

8. Two plane mirrors PQ and QR are inclined at an angle of β. A ray AB incident on a plane mirror PQ and after reflection moves along the path BC as shown in the following figure. This ray strikes the plane mirror QR at the point C and is reflected along the path CD so that the $\angle$ BCD is 94°. Calculate the angle β between the mirrors PQ and QR.

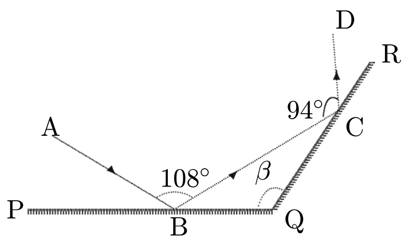


[Ans. 100°]

9. Show that for a plane mirror, image of an object is formed of the same size as that of the object and the distance of image from the mirror is equal to the distance of the object from the mirror.
10. Show that when a plane mirror is rotated through an angle θ, the reflected ray rotates through the angle 2θ.
11. A ray reflected from a plane mirror is received by a person standing on the earth. If the mirror is rotated in the horizontal plane through an angle of 32°. How much the person should move in order to receive the reflected ray.

[Ans. 64°]

12. An object is placed in front of a plane mirror. Show that the distance of image of the object from the mirror is equal to the distance of the object from the mirror.
13. An object is placed between two plane mirrors inclined at an angle of 60°. Calculate the number of images formed.
14. Two plane mirrors PQ and QR are inclined at an angle of β. A ray AB incident on a plane mirror PQ and after reflection moves along the path BC so that the $\angle$ ABC is 108°. This reflected ray strikes the plane mirror QR at the point C and is reflected along the path CD so that the $\angle$ BCD is 94°. Calculate the angle β between the mirrors PQ and QR.



[Ans. 101°]

15. An object of height 80 cm is placed in front of a large plane mirror at a distance of 67 cm so that a complete image of the object is formed. Write down the size of the image and its position.

[Ans. 80 cm, 67 cm]

16. What should be the angle between two plane mirrors so that we can see three images of an object?
17. A 174 cm tall person wants to see his complete image with the help of a plane mirror. What should be the minimum size of the mirror?

[Ans. 87 cm]

18. What happens when

   (i) a ray from an object falls on a spherical mirror after passing through the center of curvature of the mirror or appears to pass through the center of curvature of the mirror?
   (ii) a ray from an object falls on a spherical mirror after moving parallel to the principal axis?

19. Draw ray diagram for the formation of image by a concave mirror of an object placed at infinite distance.
20. Draw ray diagram for the formation of image by a concave mirror of an object placed between infinity and the center of curvature.
21. Draw ray diagram for the formation of image by a concave mirror of an object placed at the center of curvature.
22. Draw ray diagram for the formation of image by a concave mirror of an object placed between the center of curvature and the focal point.
23. Draw ray diagram for the formation of image by a concave mirror of an object placed at the focal point.
24. Draw ray diagram for the formation of image by a concave mirror of an object placed between the focal point and the pole.
25. Derive relation between u, v, and f for a concave mirror.
26. Derive relation between u, v, and f for a convex mirror.
27. Derive expression for the magnification for a concave mirror.
28. Derive expression for the magnification for a convex mirror.
29. Show that the magnification of a concave mirror is equal to the ratio of the distance of image to that of the object from the pole.
30. Show that the magnification of a convex mirror is equal to the ratio of the distance of image to that of object from the pole.
31. Write short notes on the following:

   (i) Mirrors
   (ii) Laws of reflection
   (iii) Angle of incidence
   (iv) Angle of reflection
   (v) Wavefronts
   (vi) Sign convention for a mirror
   (vii) Minimum size of mirror for a complete image
   (viii) Radius of curvature and focal length of a spherical mirror
   (ix) Reversibility theorem for the mirrors
   (x) Mirror equation for a concave mirror

(xi) Mirror equation for a convex mirror
(xii) Magnification for a concave mirror
(xiii) Magnification for a convex mirror
(xiv) Rectilinear propagation of light
(xv) Refractive index of a medium
(xvi) Geometrical path and optical path
(xvii) Fermat's principle.

Chapter 2
Refraction

In the last chapter, we discussed the reflection of rays of light by a smooth surface, plane as well as spherical in shape. In the reflection phenomenon, a ray after an incident on the surface is reflected back into the same medium. When the surface is not polished by a specific material, the rays enter into another medium and deviate from their original direction. This phenomenon of deviation of rays from their paths after moving from one medium to another is known as refraction. In this chapter, we have discussed the refraction phenomenon.

1 Refraction

When a ray of light moving through a transparent medium 1 incidents on a boundary separating another transparent medium 2 as shown in Fig. 1, a part of light is reflected back into the medium 1 and the rest enters into the medium 2. The ray which enters into the second medium is said to be refracted and the phenomenon is known as the refraction of light.

Obviously, when a ray goes from one medium into another, its path changes. Suppose, MN is the normal at the point of incidence B on the surface PQ separating the two media. The angle between the direction AB of the incident ray and the normal MN is known as the angle of incidence, denoted by i, and the angle between the refracted ray BD and normal MN is known as the angle of refraction, denoted by r.

Suppose, μ_1 and μ_2 are refractive indexes of the media 1 and 2, respectively. The angle of refraction depends on the refractive index of the medium 2 relative to that of the medium 1. When the medium 2 is denser than the medium 1 ($\mu_2 > \mu_1$), the angle of refraction is smaller than the angle of incidence ($r < i$). On the other side,

S. Chandra and M. K. Sharma, *A Textbook of Optics*,
https://doi.org/10.1007/978-3-031-55676-0_2

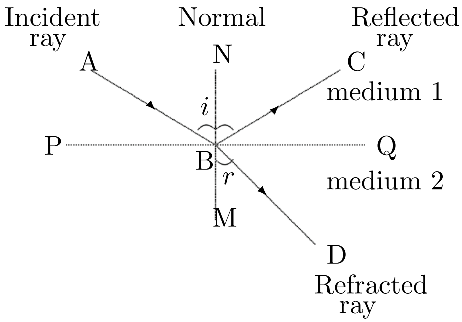


Fig. 1 Refraction of light at the boundary separating the two media 1 and 2

when the medium 2 is rarer than the medium 1 ($\mu_2 < \mu_1$), the angle of refraction is larger than the angle of incidence ($r > i$).

	Direction of motion of ray	Relation between angles
1.	From denser medium to rarer medium	Angle of refraction is larger than the angle of incidence
2.	From rarer medium to denser medium	Angle of refraction is smaller than the angle of incidence

1.1 Snell's Law

For the above situation, Snell's law states that the refractive index of medium 2 relative to that of medium 1, denoted by ${}_1\mu_2$, is

$$ {}_1\mu_2 = \frac{\mu_2}{\mu_1} = \frac{\sin i}{\sin r} $$

Note When a ray of light moves from medium 1 of refractive index μ_1 to the medium 2 of refractive index μ_2, the refractive index of medium 2 relative to that of medium 1 is

$$ {}_1\mu_2 = \frac{\mu_2}{\mu_1} $$

On the other side, when the ray of light moves from medium 2 to medium 1, the refractive index of medium 1 relative to that of medium 2 is

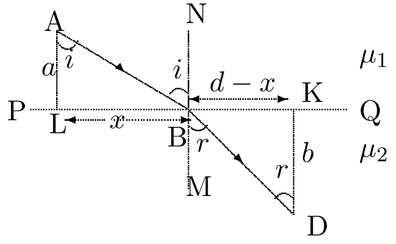
$$ {}_2\mu_1 = \frac{\mu_1}{\mu_2} $$

It obviously shows that

$$ {}_1\mu_2 = \frac{1}{{}_2\mu_1} $$

Exercise 1 Using Fermat's principle find out Snell's law for the refraction of a ray at the boundary separating the two media.

Solution For the refraction shown in the following figure, suppose the incident ray AB makes the incidence angle i and the refracted ray BD makes the refraction angle r. Let the refractive indexes of the media 1 and 2 are μ_1 and μ_2, respectively. Draw perpendiculars AL and DK on PQ from points A and D, respectively. Let AL = a and DK = b. Suppose, LB = x and LK = d so that BK = $(d - x)$.



Suppose, v_1 and v_2 be the velocities of a ray of light in the two media, respectively. The time taken by the radiation to move from A to D is

$$ t = \frac{\mathrm{AB}}{v_1} + \frac{\mathrm{BD}}{v_2} $$

Now, in Δ ABL and Δ BDK, we have

$$ \mathrm{AB} = \sqrt{a^2 + x^2} \qquad \text{and} \qquad \mathrm{BD} = \sqrt{b^2 + (d - x)^2} $$

Using these expressions for AB and BD, we have

$$ t = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d - x)^2}}{v_2} $$

On differentiating with respect to x, we get

$$ \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{v_1} \frac{x}{\sqrt{a^2 + x^2}} - \frac{1}{v_2} \frac{(d - x)}{\sqrt{b^2 + (d - x)^2}} $$

According to Fermat's principle, the time of the actual path between the two given points should be minimum and therefore we should have

$$ \frac{\mathrm{d}t}{\mathrm{d}x} = 0 \qquad \text{or} \qquad \frac{1}{v_1} \frac{x}{\sqrt{a^2 + x^2}} - \frac{1}{v_2} \frac{(d - x)}{\sqrt{b^2 + (d - x)^2}} = 0 $$

In Δ ABL and Δ BDK, we have

$$\sin i = \frac{x}{\sqrt{a^2 + x^2}} \qquad \text{and} \qquad \sin r = \frac{(d-x)}{\sqrt{b^2 + (d-x)^2}}$$

Using these relations, we have

$$\frac{\sin i}{v_1} - \frac{\sin r}{v_2} = 0 \qquad \text{or} \qquad \frac{v_1}{v_2} = \frac{\sin i}{\sin r}$$

The refractive indexes of the two media are

$$\mu_1 = \frac{c}{v_1} \qquad \text{and} \qquad \mu_2 = \frac{c}{v_2}$$

Using these relations, we have

$$\frac{\mu_2}{\mu_1} = \frac{v_1}{v_2} = \frac{\sin i}{\sin r} \qquad \text{or} \qquad {}_1\mu_2 = \frac{\sin i}{\sin r}$$

This proves Snell's law.

Exercise 2 Suppose, a radiation of frequency ν moves form medium 1 of refractive index μ_1 to another medium 2 of refractive index μ_2. Find out the refractive index of medium 2 relative to that of medium 1 (i) in terms of the velocities of radiation in the two media and (ii) in terms of the wavelengths of radiation in the two media.

Solution Suppose, v_1 and v_2 are the velocities of radiation in the media 1 and 2, respectively. We have

$$\mu_1 = \frac{c}{v_1} \qquad \text{and} \qquad \mu_2 = \frac{c}{v_2}$$

where c is the velocity of light in vacuum. The refractive index of medium 2 relative to that of medium 1 is ${}_1\mu_2$ is

$${}_1\mu_2 = \frac{\mu_2}{\mu_1} = \frac{c/v_2}{c/v_1} = \frac{v_1}{v_2}$$

This is the expression for the relative refractive index in terms of the velocities of radiation in the two media.

Frequency is the fundamental property of radiation. That is, when a radiation moves from one medium to another, its frequency does not change. Suppose, λ_1 and λ_2 are the wavelengths of radiation in the media 1 and 2, respectively, then we have

$$v_1 = \nu\lambda_1 \qquad \text{and} \qquad v_2 = \nu\lambda_2$$

Using these values, the refractive index of medium 2 relative to that of medium 1 is

$$_1\mu_2 = \frac{v_1}{v_2} = \frac{\nu\lambda_1}{\nu\lambda_2} = \frac{\lambda_1}{\lambda_2}$$

This is the expression for the relative refractive index in terms of the wavelengths of radiation in the two media.

Exercise 3 Suppose, λ_0 is the wavelength of a radiation in vacuum. Find out the wavelength of radiation in a medium having refractive index μ.

Solution Suppose, ν is the frequency of radiation. Being the fundamental characteristic, the frequency of radiation does not depend on the medium through which it moves. Suppose, λ_0 and λ_m be the wavelengths of radiation in the vacuum and in the medium of refractive index μ. We have

$$\mu = \frac{c}{v} = \frac{\nu\lambda_0}{\nu\lambda_m} = \frac{\lambda_0}{\lambda_m}$$

Hence, the wavelength of radiation in the medium is

$$\lambda_m = \frac{\lambda_0}{\mu}$$

Note Since the refractive index of a medium is always larger than that of the air (vacuum), the wavelength of radiation in the medium is always smaller than that in the air (vacuum).

Exercise 4 Wavelength of green light moving in a vacuum is 5500 Å. Calculate the wavelength of the light when it moves in a medium of refractive index 1.3.

Solution Refractive index of medium is $\mu = 1.3$ and wavelength of radiation in vacuum is $\lambda_0 = 5500$ Å. The wavelength of radiation moving in a medium of refractive μ is

$$\lambda_m = \frac{\lambda_0}{\mu} = \frac{5500}{1.3} = 4230.8\ \text{Å}$$

2 Total Internal Reflection

When a ray of light goes from a denser medium to a rarer medium, the angle of refraction is larger than the angle of incidence. If we go on increasing the angle of incidence, at some of its value, the refraction angle becomes equal to 90° and the ray does not enter into the rarer (second) medium (Fig. 2). This phenomenon is known as the total internal reflection and the corresponding angle of incidence is known as the critical angle, denoted by i_c. Thus, we have

$$_1\mu_2 = \mu = \frac{\sin i}{\sin r} = \frac{\sin i_c}{\sin 90} = \sin i_c$$

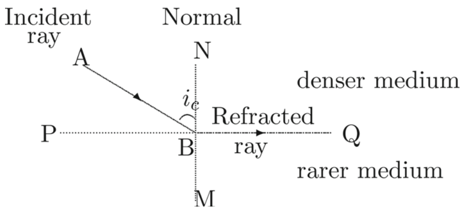


Fig. 2 Total internal reflection

Thus, the critical angle is

$$i_c = \sin^{-1} \mu$$

Exercise 5 A ray of light goes from the glass into the air. Calculate the critical angle for total reflection. The refractive index of glass relative to air is 1.5.

Solution The ray of light goes from glass to air and the refractive index of air relative to glass is

$${}_g\mu_a = \frac{1}{{}_a\mu_g} = \frac{1}{1.5} = 0.6667$$

Thus, the critical angle is

$$i_c = \sin^{-1}({}_g\mu_a) = \sin^{-1}(0.6667) = 41.8°$$

3 Refraction at a Spherical Convex Surface

Let us consider a spherical convex surface of radius of curvature R separating two media of refractive indexes μ_1 and μ_2 such that $\mu_2 > \mu_1$ (Fig. 3). Let C be the center of curvature and P the pole of the spherical surface. For convenience, suppose O be

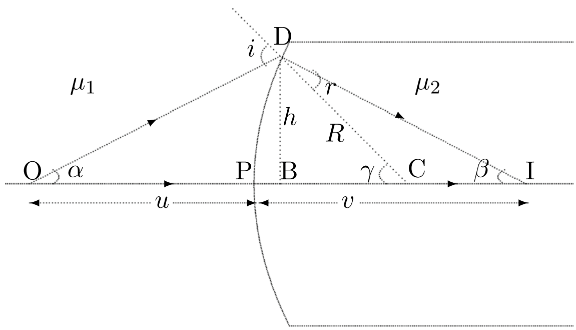


Fig. 3 Refraction of a ray at a spherical convex surface

a point object lying on the principal axis[1] in the rarer medium at a distance u from the pole. A ray of light OD incident at the point D on the spherical surface and after refraction bends towards the normal as $\mu_2 > \mu_1$ and moves along DI. Another ray from O along OP (normal incidence) moves along PI. Thus, at the point of intersection of two refracted rays DI and PI, the I is the image of the object O. Let PI be v. If i and r be the angle of incidence and angle of refraction (Fig. 3) and α, β, γ the angles at which the incident ray, the refracted ray and the normal at D make with the principal axis. According to Snell's law, we have

$$\frac{\mu_2}{\mu_1} = \frac{\sin i}{\sin r} \qquad \text{or} \qquad \mu_1 \sin i = \mu_2 \sin r$$

As the radius of curvature of spherical surface is very large, the point B is very close to the point P, the angles i and r are small and Snell's law can be expressed as

$$\mu_1 i = \mu_2 r$$

Further, we have

$$\text{OD} = \text{OP} = u \qquad \text{and} \qquad \text{ID} = \text{IP} = v$$

From Δ ODC, we have

$$i = \alpha + \gamma$$

In Δ IDC, we have

$$\gamma = r + \beta \qquad \text{or} \qquad r = \gamma - \beta$$

Substituting the vales of i and r in Snell's law, we get

$$\mu_1(\alpha + \gamma) = \mu_2(\gamma - \beta) \qquad \text{or} \qquad \mu_2\beta + \mu_1\alpha = \gamma(\mu_2 - \mu_1)$$

Denoting the perpendicular DB = h, we have

$$\alpha = \frac{h}{u} \qquad \beta = \frac{h}{v} \qquad \gamma = \frac{h}{R}$$

Following the sign convention, u is negative, and v and R are positive. Substituting the values of α, β and γ along with the sign convention, we get

[1] A line joining a point object and the center of curvature of the spherical surface is known as the principal axis. The point of intersection of the principal axis and the spherical surface is known as the pole.

$$\frac{\mu_2 h}{v} - \frac{\mu_1 h}{u} = \frac{h(\mu_2 - \mu_1)}{R} \qquad \text{or} \qquad \frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{(\mu_2 - \mu_1)}{R}$$

Generally, the medium outside the spherical surface is air, so that $\mu_1 = 1$ and $\mu_2 = \mu$, and therefore, we have

$$\frac{\mu}{v} - \frac{1}{u} = \frac{(\mu - 1)}{R}$$

Here, μ is the refractive index of the material (glass) of the lens with respect to the air, the outside medium.

4 Lenses of Various Shapes

When we have a medium (glass) enclosed by two surfaces, it is termed as a lens. Taking into account the shapes of two surfaces, the lenses may be of six shapes, shown in Fig. 4: (i) double convex lens (or simply convex lens), (ii) plano-convex lens, (iii) concavoconvex lens, (iv) double concave lens (or simply concave lens), (v) plano concave lens, and (vi) convexoconcave lens. In the present discussion, we mainly discuss the convex lens and concave lens. The thickness of each lens is very small. So, we deal with the thin lenses.

4.1 Principal Axis

The line connecting the centers of curvatures of two faces of a lens is known as the principal axis. For example for convex lens or concave lens shown below C_1 and C_2

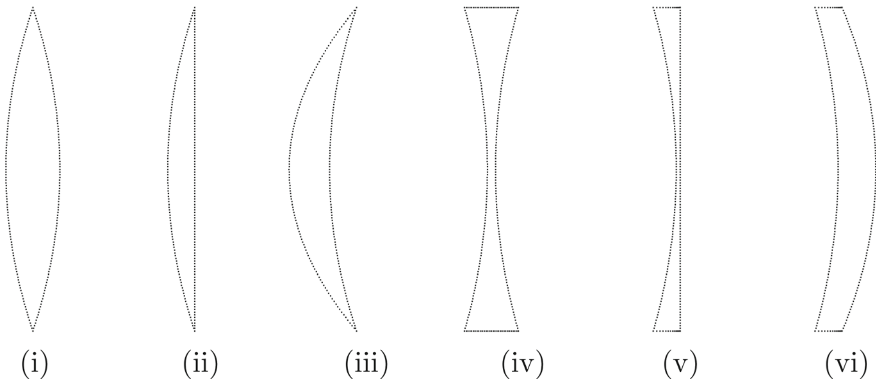
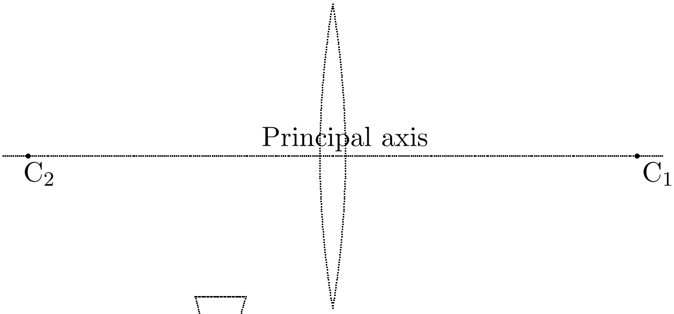
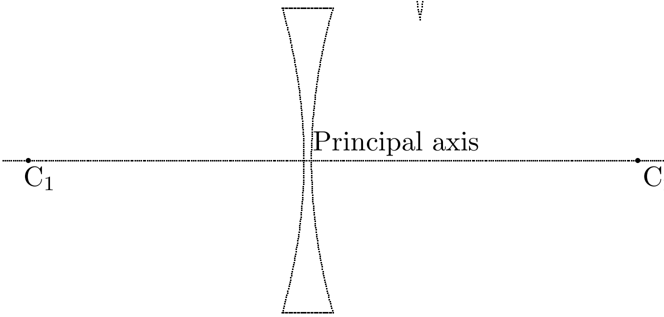


Fig. 4 Various shapes of lenses

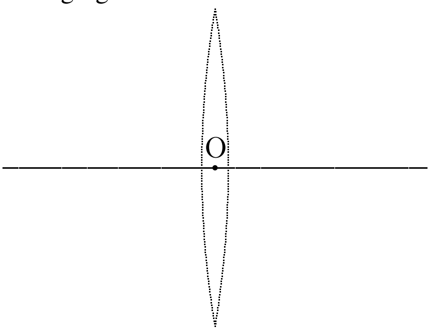
are centers of curvatures of two faces of the lens. The line joining C_1 and C_2 is the principal axis.

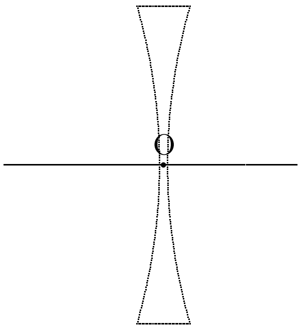




4.2 Optical Point

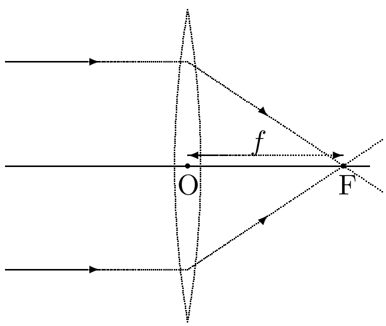
The point of the intersection of the principal axis with the lens is known as the optical point. For convex lens or a concave lens, the optical point denoted by O is shown in the following figures.

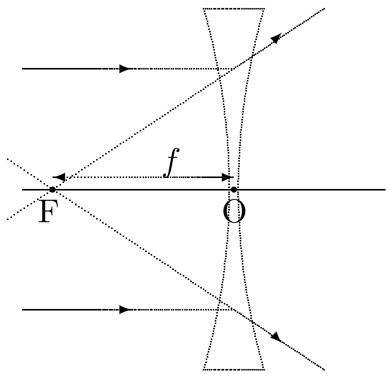


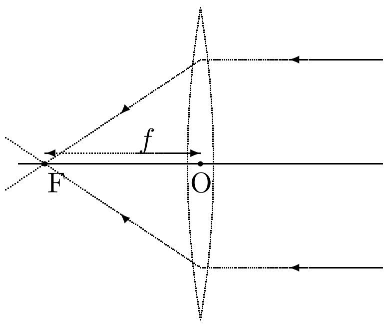


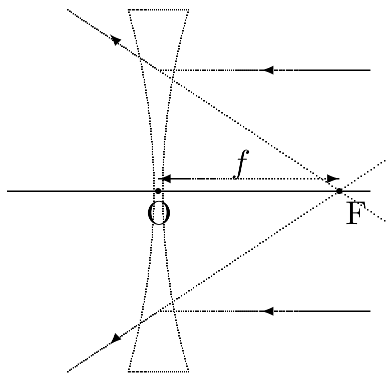
4.3 Focal Length

For a thin lens, the rays moving parallel to the principal axis of a lens after refraction through it pass through or appear to come from a point on the principal axis. This point on the principal axis is known as the focal point, denoted by F, and the distance of the focal point from the optical point O is known as the focal length, denoted by f of the lens. The following figures show the focal point and focal length of convex and concave lenses.





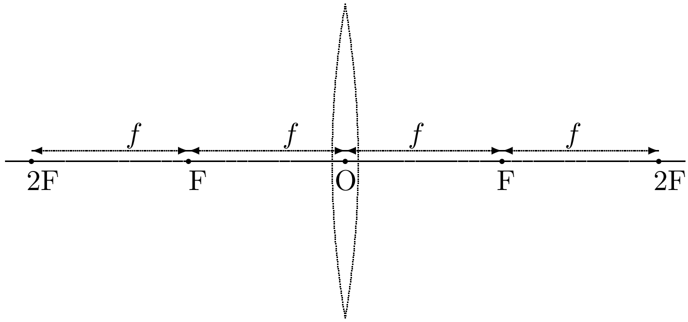


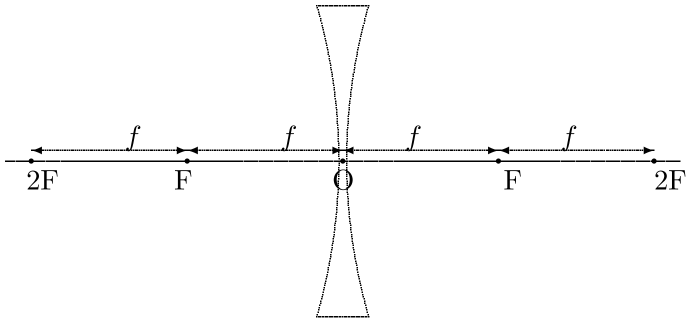


Note We have another point on the other side of the lens at a distance f from the focal point. That point also behaves as a focal point as the rays of light coming from the right (opposite) side will come to or appear to start from that point.

4.4 Point 2F

On each side of a lens, we have one point at a distance $2f$ from the optical point. This point is generally denoted by 2F and is known as the second focal point. The F and 2F for the convex and concave lenses are shown in the following figures.



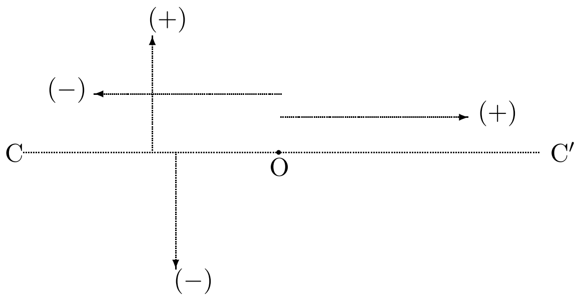


4.5 Sign Convention

For the formation of the image of an object placed in front of a lens, we consider the principal axis of the lens along a horizontal direction. An object is placed on the left side of the lens. Now, for the distances measured, the sign convention is as follows.

(i) The distances measured from the optical point in the left direction are taken as negative whereas those in the right direction as positive.
(ii) The distances measured from the principal axis in the downward direction are taken as negative whereas those in the upward direction as positive.

For example, in the following figure for a lens, the principal axis is CC′ and the optical point O. Now, a distance measured from O in the left direction is negative (−) whereas that in the right direction is positive (+). Further, a distance measured from the principal axis in the upward direction is positive (+) whereas that in the downward direction is negative (−).



Thus, the focal length of convex lens is positive whereas that of a concave lens is negative.

5 Expression for Focal Length of a Lens

For convenience, let us consider convex lens with surfaces A and B, radii of curvatures R_1 and R_2, respectively, of a thin double convex lens of refractive index μ_2. Let the refractive index of the medium (air) on the two sides of the lens is μ_1.

Consider a point object O placed on the principal axis in front of face A at a distance u (Fig. 5). If there had been a medium of refractive index μ_2 through out on the right side of the surface A (as discussed in the preceding section), let the image would have been I′ at a distance v' from A. Following the discussion in the preceding section we have

$$\frac{\mu_2}{v'} - \frac{\mu_1}{u} = \frac{(\mu_2 - \mu_1)}{R_1} \tag{1}$$

Since there is a medium of refractive index μ_1 on the right of the surface B, the rays are again refracted at the surface B. The image I′ is regarded as the object for the surface B and a final image is formed at I at a distance v from B (Fig. 5). As the rays

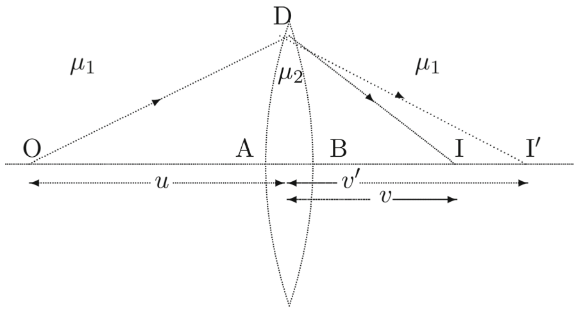


Fig. 5 Convex thin lens enclosed between two spherical surfaces A and B of radii of curvatures R_1 and R_2, respectively

are now passing from a medium of refractive index μ_2 in the medium of refractive index μ_1, the equation of refraction may be expressed as

$$\frac{\mu_1}{v} - \frac{\mu_2}{v'} = \frac{(\mu_1 - \mu_2)}{R_2} \tag{2}$$

On adding Eqs. (1) and (2), we get

$$\frac{\mu_1}{v} - \frac{\mu_1}{u} = (\mu_2 - \mu_1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

or

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{3}$$

where $\mu = \mu_2/\mu_1$ is the refractive index of the medium (glass) of the lens relative to the outside medium (air). We know when the object is at the infinite distance ($u = \infty$), the image is formed at the focal point ($v = f$), and Eq. (3) gives

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{4}$$

This is the expression for the focal length of a lens having two faces of radii of curvatures R_1 and R_2. As one has to account for the sign convention, this relation remains valid for all shapes of lenses.

Exercise 6 Radii of curvatures of two faces of convex lens placed in the air are 30 cm and 40 cm. If the refractive index of the material of the lens is 1.5, calculate the focal length of the lens.

Solution Considering the sign convention, for convex lens, we have $R_1 = 30$ cm and $R_2 = -40$ cm. For the refractive index $\mu = 1.5$, the focal length f of the lens is

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.5 - 1)\left(\frac{1}{30} - \frac{1}{-40}\right)$$

or

$$f = \frac{120}{0.5 \times 7} = 34.29 \text{ cm}$$

Thus, the focal length of the lens is 34.29 cm.

Exercise 7 Radii of curvatures of two faces of concave lens placed in the air are 40 cm and 50 cm, respectively. If the refractive index of the material of the lens is 1.5, calculate the focal length of the lens.

Solution Considering the sign convention, for concave lens, we have $R_1 = -40$ cm and $R_2 = 50$ cm. For the refractive index $\mu = 1.5$, the focal length f of the lens is

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.5 - 1)\left(\frac{1}{-40} - \frac{1}{50}\right)$$

or

$$f = -\frac{200}{0.5 \times 9} = -44.44 \text{ cm}$$

Thus, the focal length of the lens is 44.44 cm.

Exercise 8 If radii of curvatures of two faces of a concavoconvex (convexoconcave) lens are equal, showing that the lens behaves like a plane transparent plate.

Solution Let the radius of curvature of each surface of a concavoconvex (convexo-concave) lens is r. Then, on accounting for the sign convention, we have $R_1 = r$ and $R_2 = r$. The focal length f of the lens of refractive μ is

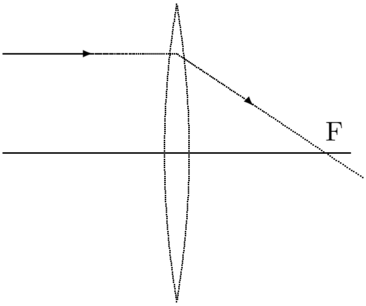
$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (\mu - 1)\left(\frac{1}{r} - \frac{1}{r}\right) = 0$$

Thus, the focal length $f = \infty$. It shows that the lens behaves like a plane transparent plate.

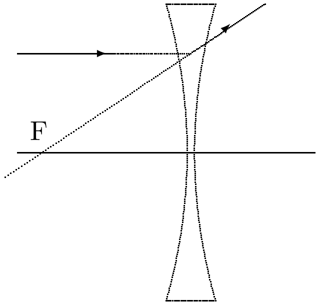
6 Refraction Through a Lens

Though a ray which incident on lens is refracted at both surfaces of the lens, but for convenience in a ray-diagram in the present presentation, we draw the incident ray up to the center of the lens and the emergent ray from the center of the lens. Before telling about the formation of the image of an object with the help of a lens, let us first understand some important aspects of a lens. For the formation of the image of an object, we consider the following rays.

(i) A ray moving parallel to the principal axis of a lens after refraction passes through the focal point F in case of convex lens (a) or appears to move away from the focal point F in case of concave lens (b).

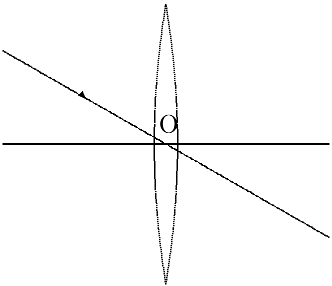


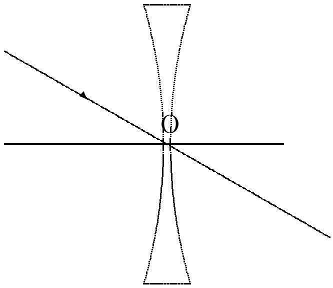
(a)



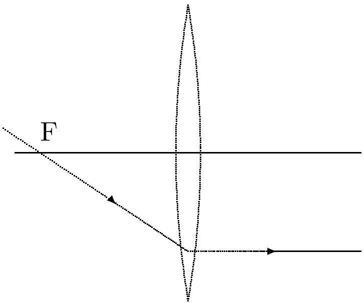
(b)

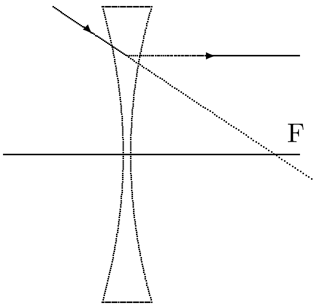
(ii) A ray moving towards the optical point is refracted along the same direction. That is, this ray moves without any deviation in its path. This is the case with both the convex lens as well as the concave lens.





(iii) A ray moving towards the focal point after refraction becomes parallel to the principal axis of the lens.





7 Formation of Image by a Convex Lens

In case of convex lens, the position of the image of an object depends on the position of the object. We can put an object at different distances from a lens, as discussed in the following.

7.1 *Object Is at Infinite Distance*

Suppose, an object is at infinite distance from the lens (Fig. 6). The coming rays are parallel to the principal axis of the lens. These rays after refraction through the lens reach the focal point F. Thus, the image of the object is formed at the focal point of the lens.

7.2 *Object Is Between Infinity and 2F*

Suppose, an object AB is placed between the infinity and 2F (Fig. 7). From point A on the object, a ray parallel to the principal axis of the lens after refraction passes through the focal point F. Other rays from point A passing through optical point O go in the same direction. These two rays form A intersect at the point A′. This point A′ is the image of point A. Similarly, we can consider other points on the object and get the corresponding images. All these images together form the image A′B′ of the object AB. In this case, the image is formed between points F and 2F on the other side.

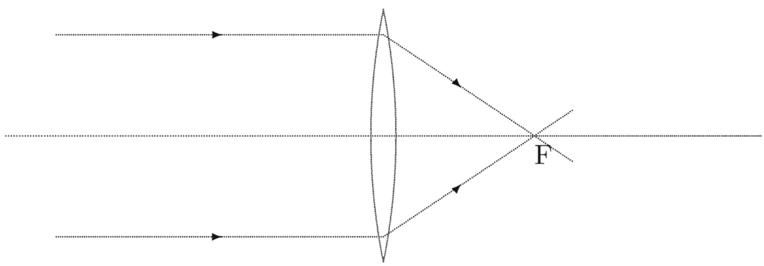


Fig. 6 Formation of image when an object is placed at infinite distance from a lens

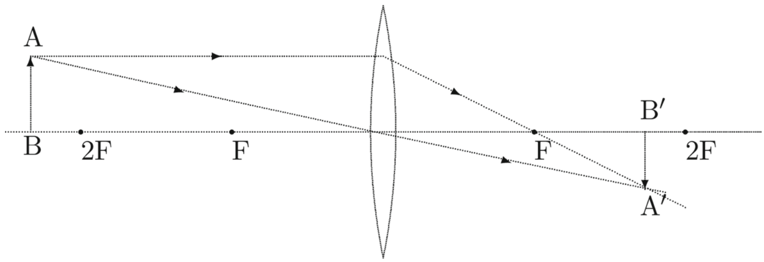


Fig. 7 Formation of image when an object is placed between infinity and 2 F

7.3 Object Is at 2F

Suppose, an object AB is placed at 2F (Fig. 8). From point A on the object, a ray parallel to the principal axis of the lens after refraction passes through the focal point F. Other rays from point A passing through optical point O go in the same direction. These two rays from A intersect at the point A′. This point A′ is the image of point A. Similarly, we can consider other points on the object and get the corresponding images. All these images together form the image A′B′ of the object AB. In this case, the image is formed at the point 2 F on the other side of the lens.

7.4 Object Is Between F and 2F

Suppose, an object AB is placed between F and 2F (Fig. 9). From point A on the object, a ray parallel to the principal axis of the lens after refraction passes through the focal point F. Other rays from point A passing through optical point O go in the same direction. These two rays from A intersect at the point A′. This point A′ is the image of point A.

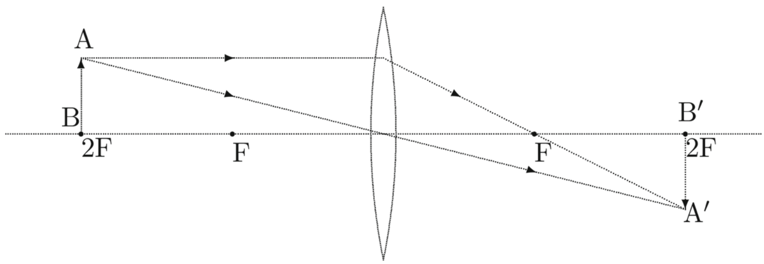


Fig. 8 Formation of image when an object is placed at 2F

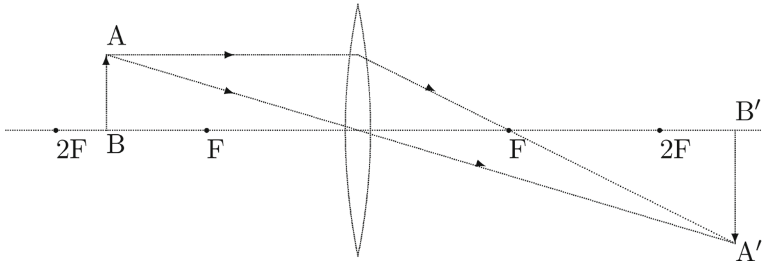


Fig. 9 Formation of image when an object is placed between the points F and 2 F

Similarly, we can consider other points on the object and get the corresponding images. All these images together form the image A′B′ of the object AB. In this case, the image is formed between the point 2 F, and infinity on the other side of the lens.

Notice that when an object is between the infinity and 2 F, the image is formed between the points F and 2 F and vice versa.

7.5 Object Is at the Focal Point F

Suppose, an object AB is placed at the focal point F (Fig. 10). From point A on the object, a ray parallel to the principal axis of the lens after refraction passes through the focal point F. Other rays from point A passing through optical point O go in the same direction. These two rays from A become parallel and do not intersect anywhere. Thus, the image of the object is formed at an infinite distance.

Notice that when an object is at infinite distance, the image is formed at the focal point and vice versa.

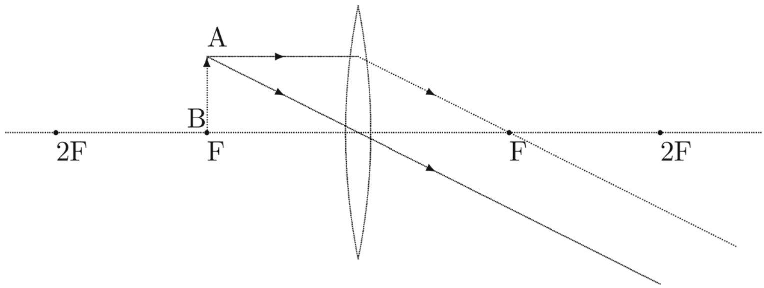


Fig. 10 Formation of image when an object is placed at the focal point

7.6 Object Is Between the Focal Point F and Optical Point O

Suppose, an object AB is placed between the focal point F and optical point O (Fig. 11). From point A on the object, a ray parallel to the principal axis of the lens after refraction passes through the focal point F. Other rays from point A passing through optical point O go in the same direction. These two rays from A appear to intersect at A′. This point A′ is virtual image of the point A.

Similarly, we can consider other points on the object and get the corresponding images. All these images together form the image A′B′ of the object AB. In this case, the image is formed on the same side of the lens. This image is not real but virtual (imaginary) as it is not formed due to a real intersection of refracted rays.

Note These six positions of an object, corresponding positions of an image of the object and nature of the image in the case of convex lens are summarized in Table 1.

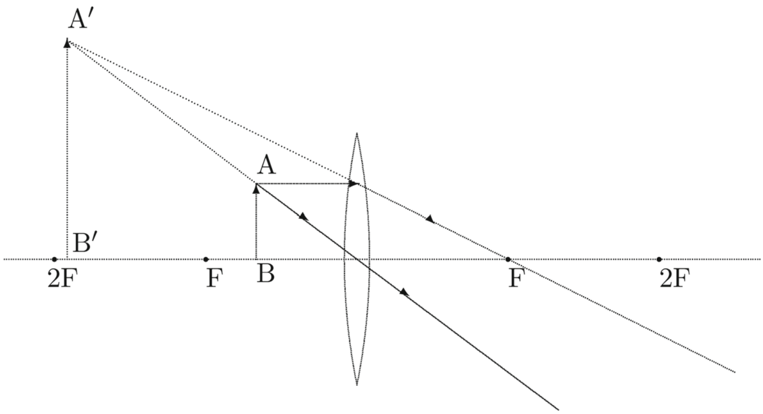


Fig. 11 Formation of image when an object is placed between the focal point F and optical point O

Table 1 Position of object, its image and nature of image

	Position of object	Position of image	Nature of image
1.	At infinity	Focal point F	Real
2.	Between infinity & 2F	Between 2F & F	Real
3.	At 2F	At 2F	Real
4.	Between 2F & F	Between 2F & Infinity	Real
5.	At focal point F	At infinity	Real
6.	Between F point O& Optical	In front of the lens	Virtual

7.7 Relation Between u, v and f

Here, u and v denote the distances of an object and its image from the optical point O of the lens, respectively. We know that f is the distance of the focal point from the optical point which is positive in the case of a convex lens. The relation between u, v and f can be derived in any of the said six cases discussed above of the convex lens. For convenience, let us consider the formation of the image as shown in Fig. 12 where a ray moving parallel to the principal axis of the lens after refraction passes through the focal point F.

Another ray going through the focal point F after refraction goes parallel to the principal axis of the lens. These two rays from point A on the object form the image at point A′. Similarly, we can consider other points on the object and get the corresponding images. All these images together form the image A′B′ of the object AB.

Since AB and OQ are perpendicular to the principal axis, the Δ ABF and Δ OQF are the similar triangles. In Δ ABF and Δ OQF, we have

$$\frac{\mathrm{AB}}{\mathrm{OQ}} = \frac{\mathrm{BF}}{\mathrm{FO}} \tag{5}$$

Since A′B′ and OP are perpendicular to the principal axis, the Δ FOP and Δ FA′B′ are the similar triangles. In Δ FOP and Δ FA′B′, we have

$$\frac{\mathrm{OP}}{\mathrm{A'B'}} = \frac{\mathrm{OF}}{\mathrm{FB'}} \tag{6}$$

Since AB = OP and OQ = A′B′, from Eqs. (5) and (6), we have

$$\frac{\mathrm{BF}}{\mathrm{FO}} = \frac{\mathrm{OF}}{\mathrm{FB'}} \quad \text{or} \quad \frac{\mathrm{BO}-\mathrm{OF}}{\mathrm{FO}} = \frac{\mathrm{OF}}{\mathrm{OB'}-\mathrm{FO}} \tag{7}$$

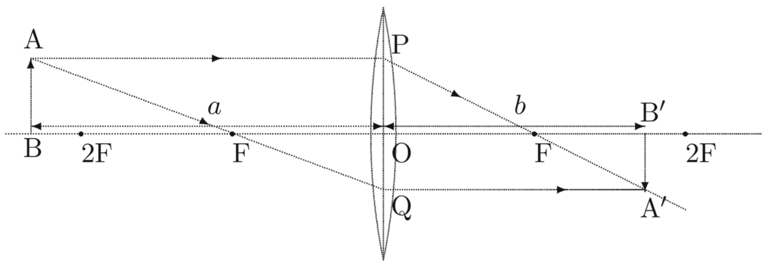


Fig. 12 Formation of image A′B′ of an object AB using a convex lens

Considering the sign convention, we have BO = $-u$, OB$'$ = v. Now, OF is the focal length which is positive in the case of convex lens and we have OF = f. Using these values in Eq. (7), we have

$$\frac{-u-f}{f}=\frac{f}{v-f}$$

On cross multiplication and rearrangement of this equation, we get

$$uf-vf=uv$$

On dividing this equation by uvf, we get

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

This is known as the equation of lens. This is the relation between u, v and f where the values are to be used along with the proper sign as per convention.

7.8 Magnification

The ratio of the size of the image to that of object is known as the magnification, denoted by m. Thus, we have

$$m=\frac{\mathrm{A'B'}}{\mathrm{AB}}$$

Notice that in Fig. 12, the size of image is in the downward direction whereas that of the object is in the upward direction. From Eq. (6), we have

$$\frac{\mathrm{A'B'}}{\mathrm{OP}}=\frac{\mathrm{FB'}}{\mathrm{OF}}$$

Since AB = OP, we have

$$\frac{\mathrm{A'B'}}{\mathrm{AB}}=\frac{\mathrm{FB'}}{\mathrm{OF}}=\frac{\mathrm{OB'}-\mathrm{OF}}{\mathrm{OF}}$$

Considering the sign convention, we have OB$'$ = $+b$ and OF = $+f$. Using these values, we have

$$\frac{\mathrm{A'B'}}{\mathrm{AB}}=\frac{b-f}{f}\tag{8}$$

From the equation of lens, we have

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \qquad \text{or} \qquad \frac{1}{b} + \frac{1}{a} = \frac{1}{f}$$

Here, sign convention is accounted for. Thus, we have $f = ab/(a+b)$. Using this expression for f in Eq. (8), we get

$$\frac{\mathrm{A'B'}}{\mathrm{AB}} = \frac{b - ab/(a+b)}{ab/(a+b)} = \frac{b}{a}$$

Hence, the magnification is

$$`m = \frac{b}{a}$$

Thus, the magnification of convex lens is the ratio of the distance of the image to that of the object for the lens.

Exercise 9 An object of length 6 cm is placed at a distance of 40 cm in front of convex lens of focal length 15 cm. Calculate the position and length of the image.

Solution We have $u = -40$ cm, $f = +15$ cm. From the equation of lens, we have

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

Using the values, we have

$$\frac{1}{v} + \frac{1}{40} = \frac{1}{15}$$

On solving, we get $v = 24$ cm. Thus, a real image is formed at a distance of 24 cm on the right side of the lens. Now, let h' be the height of the image and the size of the object $h = 6$ cm. We have

$$m = \frac{h'}{h} = \frac{24}{40}$$

Using the value of h here, we have

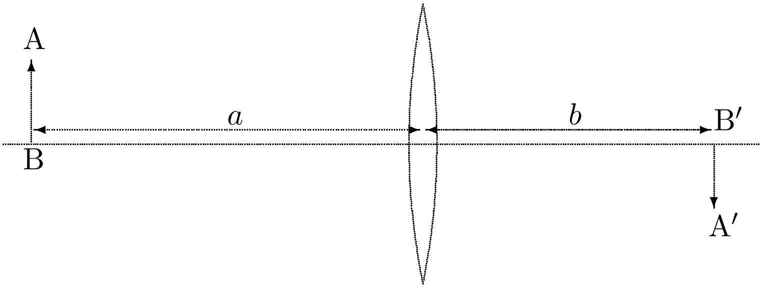
$$\frac{h'}{6} = \frac{24}{40} \qquad \text{or} \qquad h' = 3.6 \text{ cm}$$

Thus, an inverted image of 3.6 cm length is formed at a distance of 24 cm from the lens.

Exercise 10 Convex lens of focal length f produces a real image of an object. If the magnification of the image is m and the distance between the the object and image is d, show that

$$f = \frac{dm}{(1+m)^2}$$

Solution Suppose, for convex lens of focal length f, a is the distance of an object and b the distance of its real image as shown in the following figure.



We have magnification m and distance d as

$$m = \frac{b}{a} \qquad \text{and} \qquad d = a + b$$

From the equation of lens, we have

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \qquad \text{or} \qquad \frac{1}{b} + \frac{1}{a} = \frac{1}{f} \tag{9}$$

Multiplying Eq. (9) by b, we have

$$1 + \frac{b}{a} = \frac{b}{f} \qquad \text{or} \qquad 1 + m = \frac{b}{f} \tag{10}$$

Multiplying Eq. (9) by a, we have

$$\frac{a}{b} + 1 = \frac{a}{f} \qquad \text{or} \qquad \frac{1}{m} + 1 = \frac{a}{f} \tag{11}$$

Adding Eqs. (10) and (11), we get

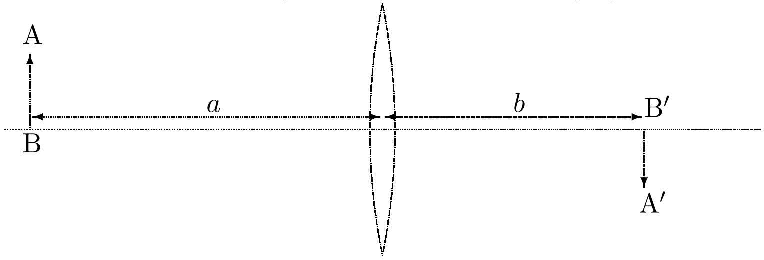
$$1 + m + \frac{1}{m} + 1 = \frac{b+a}{f} \qquad \text{or} \qquad \frac{m^2 + 2m + 1}{m} = \frac{d}{f}$$

Thus, we have

$$f = \frac{dm}{(1+m)^2}$$

Exercise 11 Show that the minimum distance between an object and its real image in convex lens is four times the focal length of the lens.

Solution Suppose, for convex lens of focal length f, a is the distance of an object and b the distance of its real image as shown in the following figure.



Suppose, d is the distance between the object and its image, then we have

$$d = a + b \qquad \text{or} \qquad a = d - b \tag{12}$$

The equation of lens is

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \qquad \text{or} \qquad \frac{1}{b} + \frac{1}{a} = \frac{1}{f} \tag{13}$$

Using the value of a from Eqs. (12) in (13), we have

$$\frac{1}{b} + \frac{1}{d-b} = \frac{1}{f} \qquad \text{or} \qquad \frac{d-b+b}{b(d-b)} = \frac{1}{f}$$

On simplification, we get

$$b^2 - bd + df = 0$$

This is a quadratic equation in b and its roots are

$$b = \frac{d \pm \sqrt{d^2 - 4df}}{2}$$

For b to be real, we should have

$$d^2 - 4df \geq 0 \qquad \text{or} \qquad d \geq 4f$$

It proves that for the formation of a real image of an object with a convex lens, the minimum distance between the object and its image should be four times the focal length of the lens.

8 Formation of Image by a Concave Lens

In case of a concave lens, the focal point is on the same side as the object, we therefore have only one case as shown in Fig. 13. Here, the image is always formed between the focal point F and optical point O. The ray from point A on the object moving parallel to the principal axis of the lens after refraction appears to come from the focal point F of the lens. The second ray from point A going in the direction of the optical point O moves in the same direction. As these two rays from point A after refraction appear to intersect at point A′. This A′ is the virtual image of point A. Similarly, we can consider other points on the object and get the corresponding images. All these images together form the image A′B′ of the object AB. The image is virtual and on the same side, as the object, of the lens.

8.1 Relation Between u, v and f

Here, u and v denote the distances of an object and its image from the optical point O of the lens, respectively. We know that f is the distance of the focal point from the optical point which is negative in the case of a concave lens. For deriving a relation between u, v and f, let us consider Fig. 14.

A ray moving parallel to the principal axis of the lens after refraction appears to come from the focal point F. Another ray passing through the optical point O goes in the same direction. These two rays from A appear to intersect at A′. This point A′ is the virtual image of point A. Similarly, we can consider other points on the object

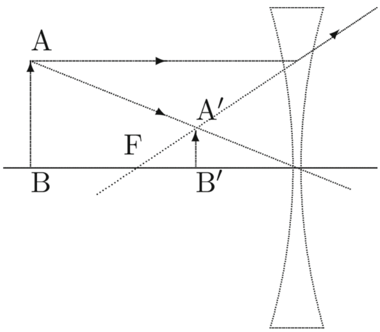


Fig. 13 Formation of image A′B′ of an object AB using a concave lens

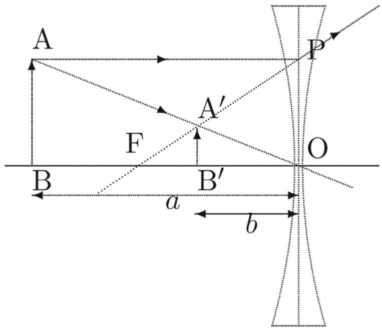


Fig. 14 Formation of image A′B′ of an object AB using a concave lens

and get the corresponding images. All these images together form the image A′B′ of the object AB.

Since AB and A′B′ are perpendicular to the principal axis, the Δ ABO and Δ A′B′O are the similar triangles. In Δ ABO and Δ A′B′O, we have

$$\frac{\mathrm{AB}}{\mathrm{A'B'}} = \frac{\mathrm{BO}}{\mathrm{B'O}} \tag{14}$$

and in Δ FOP and Δ FA′B′, we have

$$\frac{\mathrm{OP}}{\mathrm{A'B'}} = \frac{\mathrm{OF}}{\mathrm{FB'}} \tag{15}$$

Since AB = OP, from Eqs. (14) and (15), we have

$$\frac{\mathrm{BO}}{\mathrm{B'O}} = \frac{\mathrm{OF}}{\mathrm{FB'}} \quad \text{or} \quad \frac{\mathrm{BO}}{\mathrm{B'O}} = \frac{\mathrm{OF}}{\mathrm{OF} - \mathrm{B'O}} \tag{16}$$

Considering the sign convention, we have BO = $-u$, OB′ = $-v$. Now, OF is the focal length which is negative in the case of the concave lens and we have OF = $-f$. Using these values in Eq. (16), we have

$$\frac{-u}{-v} = \frac{-f}{-f + v}$$

On cross multiplication and rearrangement of this equation, we get

$$uf - vf = uv$$

On dividing this equation by uvf, we get

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

This is known as the equation of the lens. This is the relation between u, v and f where the values are to be used along with the proper sign as per convention.

8.2 Magnification

The ratio of the size of the image to that of object is known as the magnification, denoted by m. Thus, we have

$$m = \frac{\mathrm{A'B'}}{\mathrm{AB}}$$

Notice that in Fig. 14, the size of image as well as that of object is in the upward direction. From Eq. (15), we have

$$\frac{\mathrm{A'B'}}{\mathrm{OP}} = \frac{\mathrm{FB'}}{\mathrm{OF}}$$

Since AB = OP, we have

$$\frac{\mathrm{A'B'}}{\mathrm{AB}} = \frac{\mathrm{FB'}}{\mathrm{OF}} = \frac{\mathrm{OF} - \mathrm{OB'}}{\mathrm{OF}}$$

Considering the sign convention, we have $\mathrm{OB'} = -b$ and $\mathrm{OF} = -f$. Using these values, we have

$$\frac{\mathrm{A'B'}}{\mathrm{AB}} = \frac{-f + b}{-f} \tag{17}$$

From the equation of lens, we have

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \qquad \text{or} \qquad -\frac{1}{b} + \frac{1}{a} = -\frac{1}{f}$$

Here, the sign convention is accounted for. Thus, we have $f = ab/(a - b)$. Using this expression for f in Eq. (17), we get

$$\frac{\mathrm{A'B'}}{\mathrm{AB}} = \frac{-ab/(a-b) + b}{-ab/(a-b)} = \frac{b}{a}$$

Thus, the magnification is

$$m = \frac{b}{a}$$

Thus, the magnification of convex lens is the ratio of the distance of the image to that of the object for the lens.

9 Power of a Lens

Power of a lens is a measure of its ability to produce convergence (convex lens) or divergence (concave lens) of a parallel beam of light. When the focal length f is expressed in meter, the power P is

$$P = \frac{1}{f}$$

As the focal length of convex lens is positive and that of concave lens negative, the power of convex lens is positive and that of concave lens negative. The unit of power is diopter (D).

Exercise 12 Focal length of convex lens is 80 cm. Calculate the power of the lens.

Solution Focal length of convex lens is f = + 80 cm = 0.8 m. The power P of the lens is

$$P = \frac{1}{f} = \frac{1}{0.8} = 1.25\ \mathrm{D}$$

The power of convex lens is + 1.25 D.

Exercise 13 Focal length of concave lens is 66 cm. Calculate the power of the lens.

Solution Focal length of concave lens is f = −66 cm = −0.66 m. The power P of the lens is

$$P = \frac{1}{f} = \frac{1}{-0.66} = -1.515\ \mathrm{D}$$

The power of concave lens is −1.515 D.

10 Combination of Two Thin Lenses

For convenience, let us consider two thin convex lenses A and B of focal lengths f_1 and f_2, respectively, separated by a distance d and placed in such a manner that their principal axes coincide with each other. Consider a point object O placed on the principal axis at a distance a from the lens A. In the absence of lens B, the image

I′ of the object O is formed by the lens A at a distance b from A. From the equation of lens,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \tag{18}$$

we have

$$\frac{1}{b} + \frac{1}{a} = \frac{1}{f_1} \qquad \text{or} \qquad b = \frac{af_1}{a - f_1} \tag{19}$$

Here, sign convention us used and therefore $v = +b$, $u = -a$ and $f = +f_1$.

In the presence of both A and B lenses, the ray refracted through the lens A is again refracted by the lens B and finally a real image I of the object is formed at a distance c from the lens B. The image I′ at a distance $(b - d)$ from the lens B and behaves like an object for the lens B. The equation of lens (18) gives

$$\frac{1}{c} - \frac{1}{b - d} = \frac{1}{f_2} \qquad \text{or} \qquad (b - d) = \frac{cf_2}{f_2 - c} \tag{20}$$

Using the value of b from Eq. (19) in (20), we get

$$\frac{af_1}{a - f_1} - d = \frac{cf_2}{f_2 - c}$$

Multiplying this equation by $(a - f_1)(f_2 - c)$, we get

$$af_1(f_2 - c) - d(a - f_1)(f_2 - c) = cf_2(a - f_1)$$

After simplification, we get

$$ac(f_1 + f_2 - d) + a(df_2 - f_1 f_2) + c(df_1 - f_1 f_2) - df_1 f_2 = 0$$

Dividing this equation by $(f_1 + f_2 - d)$, we get

$$ac + \frac{a(df_2 - f_1 f_2)}{(f_1 + f_2 - d)} + \frac{c(df_1 - f_1 f_2)}{(f_1 + f_2 - d)} - \frac{df_1 f_2}{(f_1 + f_2 - d)} = 0 \tag{21}$$

Suppose, the equivalence of the arrangement of lenses A and B separated by a distance d is denoted by PQ in Fig. 15. Let the focal length of the equivalent lens PQ is f and the distance of object from PQ is $(a + \alpha)$ and the distance of image from PQ is $(c + \beta)$, as shown in Fig. 15. The equation of lens (18) gives

$$\frac{1}{c + \beta} + \frac{1}{a + \alpha} = \frac{1}{f}$$

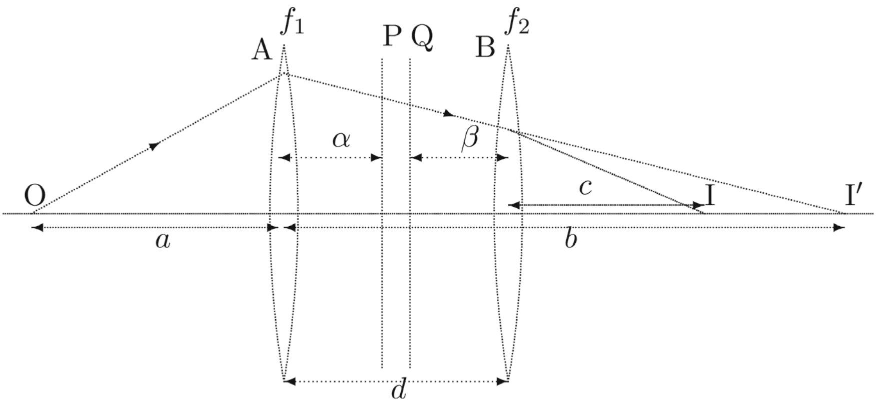


Fig. 15 Two convex lenses A and B are kept co-axially at a distance d from each other

Multiplying both sides of this equation by $(a+\alpha)(c+\beta)$, we get

$$(a+\alpha)+(c+\beta)=\frac{(a+\alpha)(c+\beta)}{f}$$

On simplification, we get

$$ac+a(\beta-f)+c(\alpha-f)+(\alpha\beta-\beta f-\alpha f)=0 \tag{22}$$

Equation (21) is for the arrangement of lenses A and B separated by a distance d whereas Eq. (22) is for the equivalent lens PQ. In principle, both the equations correspond to the same system. Comparing the coefficients of similar terms (i.e., the terms containing only a, only c, neither a nor c) in Eqs. (21) and (22), we get

$$(\beta-f)=\frac{(df_2-f_1f_2)}{(f_1+f_2-d)} \tag{23}$$

$$(\alpha-f)=\frac{(df_1-f_1f_2)}{(f_1+f_2-d)} \tag{24}$$

$$(\alpha\beta-\beta f-\alpha f)=-\frac{df_1f_2}{(f_1+f_2-d)} \tag{25}$$

On multiplying Eqs. (23) and (24), we get

$$(\beta-f)(\alpha-f)=\frac{(df_2-f_1f_2)(df_1-f_1f_2)}{(f_1+f_2-d)^2}$$

or

$$f^2 + \alpha\beta - \beta f - \alpha f = \frac{(df_2 - f_1 f_2)(df_1 - f_1 f_2)}{(f_1 + f_2 - d)^2} \tag{26}$$

Subtracting Eq. (25) from (26), we get

$$f^2 = \frac{df_1 f_2}{(f_1 + f_2 - d)} + \frac{(df_2 - f_1 f_2)(df_1 - f_1 f_2)}{(f_1 + f_2 - d)^2}$$

On simplification of this equation, we get

$$f^2 = \frac{f_1^2 f_2^2}{(f_1 + f_2 - d)^2} \qquad \text{or} \qquad f = \frac{f_1 f_2}{f_1 + f_2 - d}$$

This relation can be expressed as

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

This is the required relation for equivalent focal length f of a combination of two lenses of focal lengths f_1 and f_2 separated by a distance d.

Note This relation is valid for a combination of two lenses of any shapes, as we account for the sign convention.

10.1 Lenses in Contact

When the two thin lenses are in contact, we have $d = 0$ and

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} \qquad \text{or} \qquad f = \frac{f_1 f_2}{f_1 + f_2}$$

Exercise 14 For two thin convex lenses placed in contact, show that the power of the combination is equal to the sum of the individual powers of the lenses.

Solution Suppose, two thin lenses of focal lengths f_1 and f_2 are placed in contact. The powers p_1 and p_2 of these lenses are

$$p_1 = \frac{1}{f_1} \qquad \text{or} \qquad p_2 = \frac{1}{f_2}$$

The focal length f of the combination is

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

The power p of the combination is

$$p = \frac{1}{f}$$

Therefore, we have

$$p = \frac{1}{f_1} + \frac{1}{f_2} = p_1 + p_2$$

Exercise 15 Two thin convex lenses of focal lengths 8 cm and 12 cm are coaxial and separated by a distance of 4 cm from each other. Calculate the equivalent focal length of the combination.

Solution We are given $f_1 = +8$ cm, $f_2 = +12$ cm and $d = 4$ cm. The equivalent focal length f is

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} = \frac{1}{8} + \frac{1}{12} - \frac{4}{8 \times 12} = \frac{1}{6}$$

Thus, the equivalent focal length is +6 cm.

Exercise 16 One convex lens of focal lengths 8 cm and one concave lens of focal lengths 12 cm are coaxial and separated by a distance of 4 cm from each other. Calculate the equivalent focal length of the combination.

Solution We are given $f_1 = +8$ cm, $f_2 = -12$ cm and $d = 4$ cm. The equivalent focal length f is

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} = \frac{1}{8} - \frac{1}{12} + \frac{4}{8 \times 12} = \frac{1}{12}$$

Thus, the equivalent focal length is +12 cm.

11 Dispersion

It was found that the refractive index of a transparent medium depends on the wavelength of radiation passing through it. Here, we are concerned with the visible part of the electromagnetic spectrum. In the case of glass, the refractive index is found to decrease with the increase of wavelength of radiation. The refractive index for a

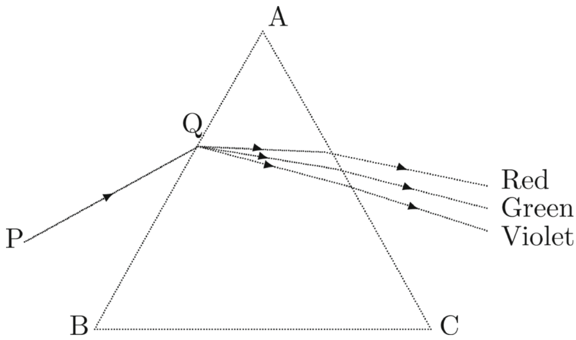


Fig. 16 A beam of white light after passing through a prism splits into seven colors, abbreviated as VIBGYOR. For convenience, here only three colors, red, green, and violet are shown

violet light is more than that for a red light. For other colors (indigo, blue, green, yellow, orange), it lies in between these two limits. Therefore, when a beam of white light PQ traveling in air is incident on a plane surface of the glass at an incident angle i, the violet light is found to show more deviation as compared to the red light (Fig. 16).

Thus, when a beam of white light is allowed to pass through a prism, the light splits into its constituents (wavelengths or colors). The phenomenon of breaking white light into several colors (wavelengths) is known as dispersion. The images thus formed (corresponding to various colors) on a screen form a pattern, generally known as the spectrum. For the clarity point of view, in Fig. 16, we have shown only three colors, red, green, and violet. However, there is a complete sequence of colors, namely, violet, indigo, blue, green, yellow, orange, and red, abbreviated as VIBGYOR. The deviation produced for the violet color of light is the maximum and that for the red is the minimum.

Consider a prism ABC made up of glass with refractive index μ relative to air surrounding the prism, as shown in Fig. 16. In this situation, the angle A is known as the angle of the prism.

Besides the visible part, the source of light may emit invisible parts of electromagnetic radiation. The region of wavelengths shorter than the violet color is known as the ultraviolet region and that of wavelengths longer than the red color is known as the infrared region. There are other regions of the electromagnetic spectrum beyond ultraviolet and infrared.

The refractive index of the material of a prism (or a lens) is different for different wavelengths (colors). The refractive index is large for the violet color and small for the red color. Consequently, the speed of the violet color is small and that of the red color is large.

11.1 Refraction Through a Prism

Consider a glass prism ABC, shown in Fig. 17. A monochromatic radiation PQ is incident on the surface AB of the prism at an angle i. After refraction, the ray moves along QR at the refraction angle r. The ray QR after refraction at the surface AC moves along RS. This ray RS is also known as the emergent ray. The angle between the incident ray PQ and the emergent ray RS is known as the deviation angle and is denoted by δ. The deviation angle δ depends on the incident angle i. The variation of δ as a function of i is shown in Fig. 18. Thus, if we start from a small value of i and increase it, the spectrum appears to move in the field of view. If we go on increasing the value of i, the situation comes when the spectrum does not move and later on starts to move in the opposite direction. The position when the spectrum does not move is known as the position of minimum deviation, and the deviation angle is denoted by δ_m. In the minimum deviation position, the emergent angle is equal to the incident angle. Hence, the ray QR is horizontal in the figure.

Here $A/2$ is half of the angle of the prism. Since AO and QT are parallel (in the vertical direction), we have $\angle$ BQT = $A/2$. Now, we have

$$\angle \text{BQT} + \angle \text{TQN} = 90°$$

and

$$\angle \text{TQN} + \angle \text{NQR} = 90°$$

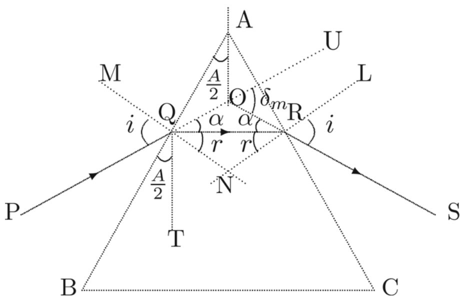


Fig. 17 Refraction of a monochromatic light through a prism ABC in the minimum deviation position

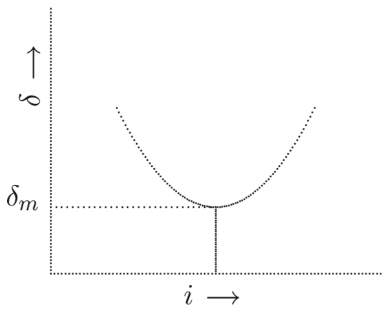


Fig. 18 Variation of the deviation angle δ versus incident angle i

Thus, we have

$$\angle \text{BQT} = \angle \text{NQR} \qquad \text{or} \qquad A/2 = r \tag{27}$$

In Δ OQR, we have

$$\delta_m = \alpha + \alpha \qquad \text{or} \qquad \alpha = \delta_m/2 \tag{28}$$

Now, we have

$$i = \alpha + r \tag{29}$$

Using Eqs. (27) and (28) in (29), we get

$$i = \frac{\delta_m}{2} + \frac{A}{2} = \frac{A + \delta_m}{2} \tag{30}$$

Suppose, μ_2 is the refractive index of glass of the prism and μ_1 of the medium around the prism. From Snell's law, we have

$$\mu_1 \sin i = \mu_2 \sin r \tag{31}$$

Using Eqs. (27) and (30) in (31), we get

$$\mu = \frac{\sin[(A + \delta_m)/2]}{\sin[A/2]} \tag{32}$$

where $\mu = \mu_2/\mu_1$ is the refractive index of the glass of the prism relative to that of the air around the prism.

Incidence of White Light on a Prism

As the refractive index of the material of the prism depends on the wavelength (color), Eq. (32) shows that the value of δ_m is different for different wavelengths (colors). Since the refractive index of the violet color is the largest and that of the red color is the smallest, therefore the deviation angle is the largest for the violet color and the smallest for the red color. Thus, when a white light is incident on a prism, there are several emergent rays depending on its wavelength (color). Consequently, we have a spectrum consisting of various colors, namely, violet, indigo, blue, green, yellow, orange, and red.

Thin Prism

When the angle of the prism is very small, the prism is said to be a thin prism. For a thin prism, the values of the angle of prism A and the deviation angle δ_m both are small. On expressing the angles in radians, Eq. (32) can be expressed as

$$\mu = \frac{A + \delta_m}{A} \qquad \text{or} \qquad \delta_m = (\mu - 1)A \tag{33}$$

The minimum deviations for the violet and red colors, respectively, can be expressed as

$$\delta_v = (\mu_v - 1)A \qquad \text{and} \qquad \delta_r = (\mu_r - 1)A \tag{34}$$

Here, μ_v and μ_r are, respectively, the refractive indices of the material of the prism for the violet and red colors. The total angle through which the spectrum is spread over is known as the angular dispersion. Thus, the angular dispersion is

$$\theta = \delta_v - \delta_r = (\mu_v - \mu_r)A \tag{35}$$

It shows that the angular dispersion depends on the nature of the material of the prism and the angle of the prism.

Dispersive power of a prism, denoted by ω, indicates the ability of material of the prism to disperse the light rays. It may be expressed as

$$\omega = \frac{\delta_v - \delta_r}{\delta} \tag{36}$$

where δ is the average deviation, or the deviation of the middle color. On using Eqs. (33) and (34) in (36), we get

$$\omega = \frac{\mu_v - \mu_r}{\mu - 1} \tag{37}$$

where μ is the refractive index for the middle color.

Exercise 17 Calculate the dispersive power for the crown and flint glasses from the data given in the following table.

	μ_r	μ	μ_v
Crown	1.5145	1.5170	1.5230
Flint	1.6444	1.6520	1.6637

Solution The dispersive power of crown glass is

$$\omega_c = \frac{\mu_v - \mu_r}{\mu - 1} = \frac{1.5230 - 1.5145}{1.5170 - 1} = 0.01644$$

The dispersive power of flint glass is

$$\omega_f = \frac{\mu_v - \mu_r}{\mu - 1} = \frac{1.6637 - 1.6444}{1.6520 - 1} = 0.02960$$

12 Aberrations

One of the basic problems of a lens is the imperfect quality of the image formed by it. The deviations in size, shape, and position of a color (wavelength) in the actual image produced by a lens in comparison to the object are known as the aberrations produced by the lens. The aberrations may be classified into two categories: (i) chromatic aberrations and (ii) monochromatic aberrations. The chromatic aberrations are the distortions of the image due to the dispersion of light in a lens used in an optical system when white light is used. Thus, chromatic aberration may be defined as a defect of a colored image formed by a lens while using a white light.

When a monochromatic light is used then the chromatic aberrations are automatically absent. Besides the chromatic aberrations, there are some defects that are present even when a monochromatic light is used. Such defects are known as monochromatic aberrations. These aberrations are due to (i) the large aperture of lenses, (ii) the large angle subtended by the rays with the principal axis, and (iii) the large size of the object. Examples of monochromatic aberrations are (i) spherical aberration, (ii) astigmatism, (iii) coma, (iv) curvature, and (v) distortion. In the present discussion, we discuss the chromatic aberration and spherical aberration.

12.1 Chromatic Aberration

The rays of light of different wavelengths (colors) have different velocities in a refracting medium. Thus, a refracting medium has a different refractive index for each color (wavelength). In the visible region of electromagnetic spectrum, the refractive index is the least for the red color and the maximum for the violet color.[2] It shows that the velocity of the red color is the largest and that of the violet color the smallest.[3]

[2] The refractive index varies with wavelength (color) as

$$\mu_v > \mu_i > \mu_b > \mu_g > \mu_y > \mu_o > \mu_r.$$

[3] The velocity of radiation varies with wavelength (color) as

$$v_v < v_i < v_b < v_g < v_y < v_o < v_r.$$

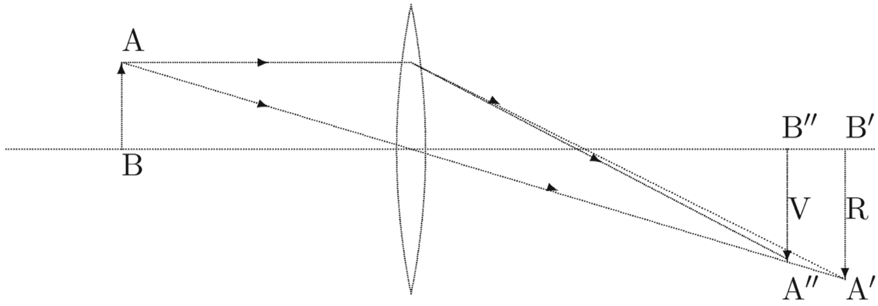


Fig. 19 Chromatic aberration

The focal length of a lens depends on the refractive index of the medium through the relation

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{38}$$

the focal length for the red color is the largest and that of the violet color the smallest.[4] Thus, when white light is used a single lens produces not only one image of the object but one image for each color (wavelength) at different positions and of different sizes. The variation of the image distance from the lens with refractive index measures the axial or longitudinal chromatic aberration.

In Fig. 19, AB is an object illuminated by a white light. Its images corresponding to Red and Violet colors are A′B′ and A″B″, respectively. The images corresponding to other colors are formed in between these two limiting images. The size of an image depends on the wavelength (color). Hence, the magnification depends on the wavelength (color). The distance x = B″B′ is the measure of the longitudinal chromatic aberration and the distance y = A′B′− A″B″ measures the lateral chromatic aberration.

Exercise 18 For a refracting medium, decide if the following statement is correct or not.

(i) The velocity of the red color is larger than that of the violet color.
(ii) The refractive index of the red color is larger than that of the violet color.
(iii) The Focal length of a lens for the red color is larger than that for the violet color.

Answer (i) Yes, (ii) No, (iii) Yes

---

[4] The focal length varies with wavelength (color) as

$$f_v < f_i < f_b < f_g < f_y < f_o < f_r.$$

12.2 *Chromatic Aberration of a Lens for an Object Placed at Infinity*

The chromatic aberration of a lens for an object placed at infinity can be calculated in the following manner.

Consider the images formed due to longitudinal aberration, as shown in Figs. 20 (convex lens) and 21 (concave lens). Let f_r, μ_r and f_v, μ_v be the focal length and refractive index for red and violet colors, respectively. Now, for the red color we have

$$\frac{1}{f_r} = (\mu_r - 1)\Big(\frac{1}{R_1} - \frac{1}{R_2}\Big) \tag{39}$$

and for the violet color we have

$$\frac{1}{f_v} = (\mu_v - 1)\Big(\frac{1}{R_1} - \frac{1}{R_2}\Big) \tag{40}$$

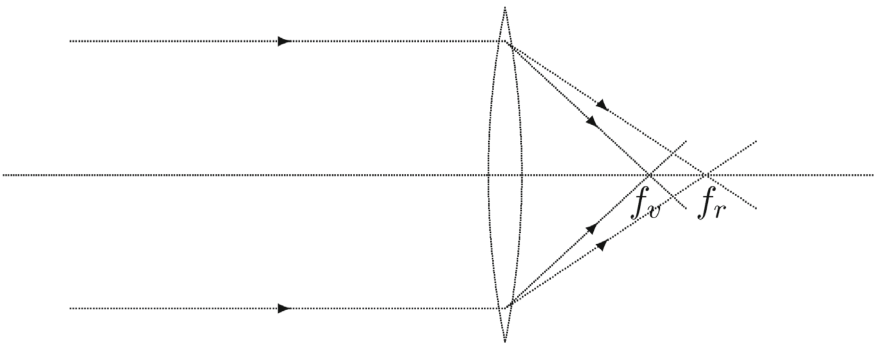
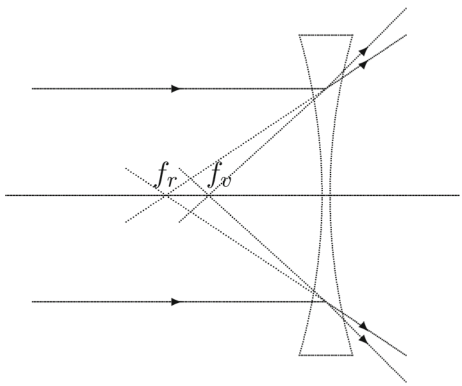


Fig. 20 Longitudinal aberration due to convex lens

Fig. 21 Longitudinal aberration due to concave lens



Here, R_1 and R_2 are radii of curvature of two faces of the lens. For the average focal length f and refractive index μ of the lens, we have

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{41}$$

From Eqs. (39) and (41), we have

$$\frac{1}{f_r} = \frac{(\mu_r - 1)}{f(\mu - 1)} \tag{42}$$

and from Eqs. (40) and (41), we have

$$\frac{1}{f_v} = \frac{(\mu_v - 1)}{f(\mu - 1)} \tag{43}$$

Subtracting Eq. (42) from (43), we get

$$\frac{1}{f_v} - \frac{1}{f_r} = \frac{(\mu_v - \mu_r)}{f(\mu - 1)} = \frac{\omega}{f} \tag{44}$$

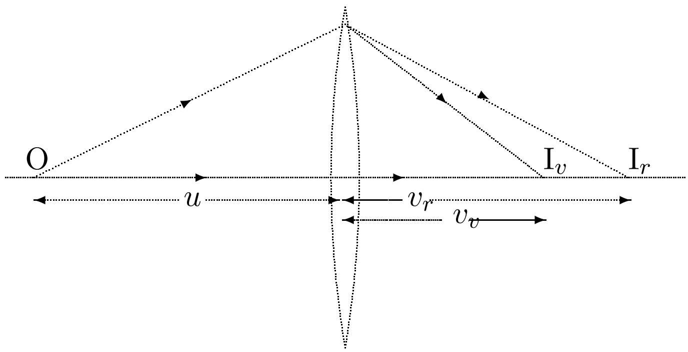
where $\omega = (\mu_v - \mu_r)/(\mu - 1)$ is known as the dispersive power of material of the lens. Equation (44) can be rearranged as

$$\frac{f_r - f_v}{f_r f_v} = \frac{\omega}{f} \qquad \text{or} \qquad f_r - f_v = \frac{\omega}{f} f^2 = \omega f \tag{45}$$

where $f = \sqrt{f_r\ f_v}$ is the average focal length of the lens. Thus, the longitudinal chromatic aberration $(f_r - f_v)$ is equal to the product of the dispersive power and the mean focal length of the lens, which may be taken as for the yellow light. It is obvious that a single lens with white light cannot produce an image that is free from chromatic aberration.

12.3 Chromatic Aberration of a Lens for an Object Placed at a Finite Distance

Let us consider a point object O illuminated by a white light and situated on the principal axis of the lens as shown in the following figure. Images corresponding to various colors are formed. The red I_r and violet I_v images are shown in the figure. Images for other colors are formed in between these two.



If u is the distance of the object from the lens, and v_v and v_r the distances of violet and red images, respectively. If f_v and f_r are the respective focal lengths for violet and red rays of light, then we have

$$\frac{1}{v_v} - \frac{1}{u} = \frac{1}{f_v} \qquad \text{and} \qquad \frac{1}{v_r} - \frac{1}{u} = \frac{1}{f_r} \tag{46}$$

so that

$$v_v = \frac{uf_v}{u + f_v} \qquad \text{and} \qquad v_r = \frac{uf_r}{u + f_r}$$

From Eq. (46), we have

$$\frac{1}{v_v} - \frac{1}{v_r} = \frac{1}{f_v} - \frac{1}{f_r} \qquad \text{or} \qquad \frac{v_r - v_v}{v_r v_v} = \frac{f_r - f_v}{f_r f_v}$$

Considering $f = \sqrt{f_r f_v}$ and $v = \sqrt{v_r v_v}$, we have

$$\frac{v_r - v_v}{v^2} = \frac{f_r - f_v}{f^2}$$

We know $(f_r - f_v) = \omega f$, where ω is the dispersive power and f the average focal length of the lens. Then, we have

$$v_r - v_v = \frac{\omega v^2}{f}$$

It shows that the longitudinal chromatic aberration depends on the distance of the image and hence on the distance of the object from the lens. Besides that it depends on the dispersive power ω and the focal length f of the lens.

12.4 Condition for Achromatism of Two Lenses Placed in Contact

Chromatic aberration may be eliminated by keeping two lenses, made up of different materials, in contact in such a way that all the rays of different colors are brought to focus by the combination at a common point. In practice, the chromatic aberration cannot be removed completely. Usually the achromatism is achieved for two prominent colors. An achromatic combination is made by placing in contact two lenses of different materials and of suitable focal lengths so that the focal length of the combination is the same for both the extreme colors, red and violet. The refraction formula for a combination of the lens in contact is

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{47}$$

where μ is the refractive index of the material of the lens, and R_1 and R_2 are radii of curvatures of the two surfaces of the lens. On differentiating Eq. (47), we get

$$-\frac{\mathrm{d}f}{f^2} = \mathrm{d}\mu\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{48}$$

Diving Eq. (48) by (47), we get

$$-\frac{\mathrm{d}f}{f^2} = \frac{\mathrm{d}\mu}{f(\mu - 1)} \tag{49}$$

If μ_v and μ_r are the refractive indices for violet and red colors, respectively, then $\mathrm{d}\mu$ is the change in the refractive index such that $\mathrm{d}\mu = \mu_v - \mu_r$. The dispersive power of the lens is

$$\omega = \frac{\mathrm{d}\mu}{\mu - 1} \tag{50}$$

Using Eq. (50) in (49), we have

$$-\frac{\mathrm{d}f}{f^2} = \frac{\omega}{f} \tag{51}$$

When two lenses of focal lengths f_1 and f_2 are placed in contact, the focal length f of the combination is

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

On differentiating this equation, we get

$$-\frac{\mathrm{d}f}{f^2} = -\frac{\mathrm{d}f_1}{f_1^2} - \frac{\mathrm{d}f_2}{f_2^2}$$

In order that the rays of different colors are brought to focus at a common point, the change in focal length of the combination should be zero, i.e., $\mathrm{d}f = 0$. Thus, we have

$$-\frac{\mathrm{d}f_1}{f_1^2} - \frac{\mathrm{d}f_2}{f_2^2} = 0$$

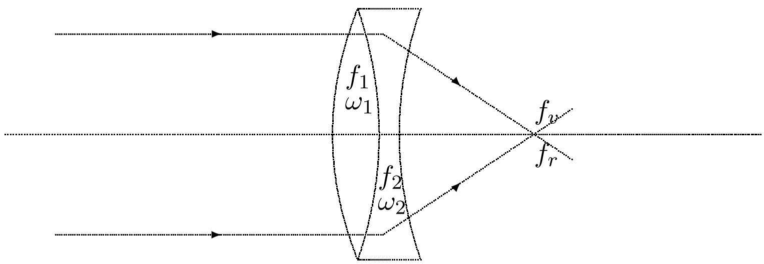
Using Eq. (51) here, we get

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0 \qquad (52)$$

where ω_1 and ω_2 are the dispersive powers for the materials of the lenses, respectively, and are always positive. Equation (52) shows that f_1 and f_2 must have opposite signs, i.e., the combination should consist of one concave lens and the other convex lens, generally made up of different materials. Such a combination is often known as the achromatic doublet.

Achromatic Doublet

An achromatic doublet consists of one convex lens and one concave lens which are placed in contact with each other, as shown in the following figure. One of them is made up of, say, crown glass, and the other of, say, flint glass. The dispersive powers of the materials of these lenses are ω_1 and ω_2, respectively.



From the condition of achromatism, we have

$$-\frac{f_2}{f_1} = \frac{\omega_2}{\omega_1}$$

A negative sign here appears as the focal lengths of the lenses are of opposite sign. We know that the focal length of the convex lens is positive whereas that of the concave lens is negative.

Exercise 19 Calculate the focal length of a lens having dispersive power 0.03 which should be kept in contact with a convex lens of focal length 80 cm and dispersive power 0.023 so that the combination is achromatic.

Solution Given, $f_1 = 80$ cm, $\omega_1 = 0.023$, $\omega_2 = 0.03$. For achromatism, we have

$$-\frac{f_2}{f_1} = \frac{\omega_2}{\omega_1}$$

Using the values here, we get

$$-\frac{f_2}{80} = \frac{0.03}{0.023} \qquad \text{so that} \qquad f_2 = -104.34 \text{ cm}$$

Thus, the required focal length of the concave lens is −104.34 cm.

12.5 Condition for Achromatism of Two Lenses Separated by a Finite Distance

Chromatic aberration may also be eliminated by keeping two lenses, made up of different materials, at a finite distance in such a way that all the rays of different colors are brought to focus on the combination at a common point. The refraction formula for a combination of two lens of focal lengths f_1 and f_2, separated by a distance d, is

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \tag{53}$$

On differentiation of this Eq. (53), we get

$$-\frac{\mathrm{d}f}{f^2} = -\frac{\mathrm{d}f_1}{f_1^2} - \frac{\mathrm{d}f_2}{f_2^2} + d\left[\frac{\mathrm{d}f_1}{f_1^2 f_2} + \frac{\mathrm{d}f_2}{f_1 f_2^2}\right] \tag{54}$$

For a single lens of focal length f and refractive index μ, we have

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{55}$$

On differentiation of this Eq. (55), we get

$$-\frac{\mathrm{d}f}{f^2} = \mathrm{d}\mu \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{56}$$

From Eqs. (55) and (56), we have

$$-\frac{\mathrm{d}f}{f^2} = \frac{\mathrm{d}\mu}{f(\mu - 1)} = \frac{\omega}{f} \tag{57}$$

where ω is the dispersive power of the lens. If ω_1 and ω_2 are the dispersive powers of the lenses of focal lengths f_1 and f_2, respectively, Eq. (54) can be expressed as

$$\frac{\omega}{f} = \frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} - d\left[\frac{\omega_1}{f_1 f_2} + \frac{\omega_2}{f_1 f_2}\right] \tag{58}$$

In order that the rays of different colors are brought to focus at a common point, the dispersive power ω of the combination of lenses should be zero. Thus, we have

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} - d\left[\frac{\omega_1}{f_1 f_2} + \frac{\omega_2}{f_1 f_2}\right] = 0 \tag{59}$$

or

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = \frac{d(\omega_1 + \omega_2)}{f_1 f_2}$$

This is the required condition for achromatism. When $d = 0$, we get

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0$$

It is the condition for two lenses in contact. If the two lenses are of the same material $\omega_1 = \omega_2 = \omega$, we have

$$\frac{\omega}{f_1} + \frac{\omega}{f_2} = \frac{d(\omega + \omega)}{f_1 f_2} \qquad \text{or} \qquad f_1 + f_2 = 2d$$

It shows that the two lenses of the same material when separated by a distance equal to the half of the sum of their focal lengths, they form an achromatic combination.

Exercise 20 Two convex lenses made up of the same material and focal lengths 30 cm and 40 cm are to be placed at a certain distance apart so that they form an image free from chromatic aberration. Calculate the distance between the lenses.

Solution Given, $f_1 = 30$ cm and $f_2 = 40$ cm. For the achromatism, the distance d between the two lenses is

$$d = \frac{f_1 + f_2}{2} = \frac{30 + 40}{2} = 35 \text{ cm}$$

Exercise 21 Two convex lenses made up of the same material and focal lengths f_1 and f_2 are to be placed at a certain distance apart so that they form an image free from chromatic aberration. Show that the effective focal length f of the combination is expressed as

$$\frac{1}{f} = \frac{1}{2f_1} + \frac{1}{2f_2}$$

Solution For achromatism, the distance d between the lenses made up of the same material is

$$d = \frac{f_1 + f_2}{2}$$

For the lenses of focal lengths f_1 and f_2, separated by a distance d, the effective focal length f is

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

Substituting the value of d, we have

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{f_1 + f_2}{2f_1 f_2} = \frac{f_1 + f_2}{2f_1 f_2}$$

Therefore, we have

$$\frac{1}{f} = \frac{1}{2f_1} + \frac{1}{2f_2}$$

12.6 Spherical Aberration

When a point object O is placed on the principal axis of a large lens, then the rays which are away from the axis are brought to focus at I_p and those near the axis are brought to focus at I_c, as shown in Fig. 22.

It is clear from the figure that the image formed by the central rays lies at a longer distance than that formed by the marginal rays. The image formed is, therefore, not sharp at any point on the axis. If a screen is placed perpendicular to the axis at I_p, the outer portions of the object are in focus, whereas at I_c, the inner portions of the object are in focus. The spherical aberration for the convex lens is positive. When the screen is placed in between I_p and I_c, the image appears to be a circular patch. This patch is known as the circle of least confusion and corresponds to the position of the best image.

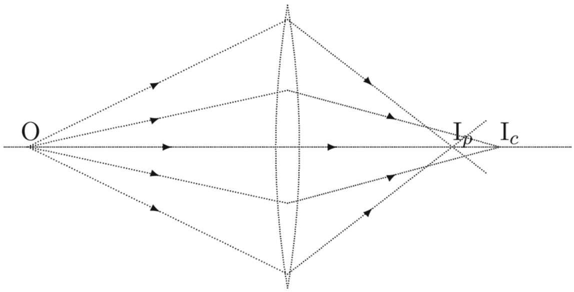


Fig. 22 Spherical aberration due to convex lens

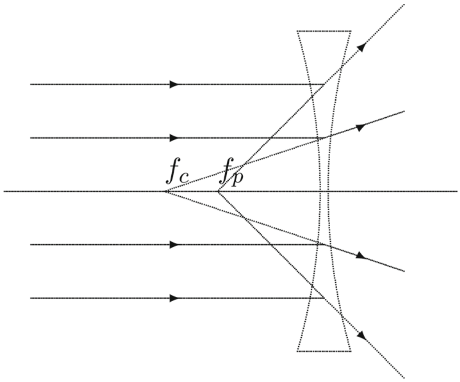


Fig. 23 Spherical aberration due to concave lens

The distance I_cI_p measures the axial or longitudinal spherical aberration. The radius of the circle of least confusion, lying between I_c and I_p, is a measure of the lateral spherical aberration.

It is obvious that for a point object O placed on the axis, the image extends over the length from I_p to I_c. This defect is known as spherical aberration and is due to the fact that different zones (parts) of the lens have different focal lengths.

The spherical aberration produced due to the concave lens is shown in Figure 23. When a parallel beam of light incidents on a concave lens, the rays which are away from the axis, after refraction appear to diverge from the point f_p and which are near the axis after refraction appear to diverge from the point f_c. The spherical aberration is negative for a concave lens.

Removal of Spherical Aberration

For a single lens, spherical aberration can be minimized under the following conditions.

(i) The spherical aberration depends on the aperture of the lens. It can therefore be reduced by reducing the aperture of the lens with the help of a coaxial aperture stop and therefore using only the central portion of the lens.

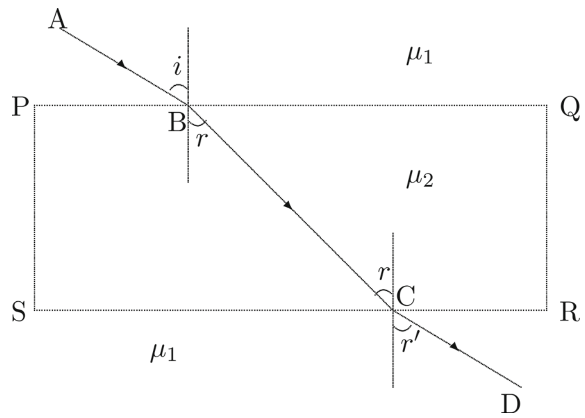


Fig. 24 A glass slab of refractive index μ_2 having parallel sides PQ and SR, and placed in a medium of refractive μ_1

(ii) By using a plano-convex lens with its convex surface towards the incident or emergent beam, whichever is more parallel to the lens.

(iii) By using a specially designed lens, called the crossed lens. The radii of curvature R_1 and R_2 of the two surfaces of the lens should satisfy the relation

$$\frac{R_2}{R_1} = -\frac{\mu(2\mu + 1)}{\mu(2\mu - 1) - 4}$$

where μ is the refractive index of the material of the lens relative to the outer medium (air).

12.7 Refraction Through a Glass Slab

Suppose, PQRS is a glass slab of refractive index μ_2 having parallel sides PQ and RS, and placed in a medium of refractive index μ_1 (Fig. 24). Thus, on every side of the slab, there is the medium of refractive index μ_1. The ray AB incident at the surface PQ at an incidence angle i and is refracted along the path BC into a medium of refractive index μ_2 so that angle of refraction is r. From Snell's law, we have

$$\mu_1 \sin i = \mu_2 \sin r \tag{60}$$

The ray BC incident at the surface RS at an angle of incidence r and is refracted into the medium with refractive index μ_1 at an angle of refraction r'. From Snell's law, we have

$$\mu_2 \sin r = \mu_1 \sin r' \tag{61}$$

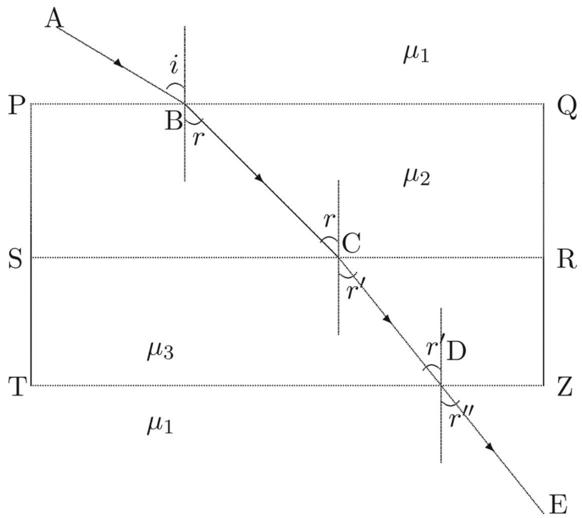


Fig. 25 Two-plane parallel slabs of different media of refractive indices μ_2 and μ_3 placed in a medium of refractive index μ_1

From Eqs. (60) and (61), we have

$$\mu_1 \sin i = \mu_1 \sin r' \qquad \text{or} \qquad i = r'$$

It shows that the angle of incidence i is the equation to the angle of transmission r'. It shows that for a parallel sided slab, the emergent ray is parallel to the incident ray.

12.8 Refraction Through a Pair of Slabs

Let us consider two-plane parallel slabs with refractive indices μ_2 and μ_3, as shown in Fig. 25. The ray AB incident at the surface PQ at an incidence angle i and is refracted along the path BC into a medium of refractive index μ_2 so that angle of refraction is r. From Snell's law, we have

$$\mu_1 \sin i = \mu_2 \sin r \tag{62}$$

The ray BC incident at the surface SR at an incidence angle r and is refracted along the path CD into a medium of refractive index μ_3 so that angle of refraction is r'. From Snell's law, we have

$$\mu_2 \sin r = \mu_3 \sin r' \tag{63}$$

The ray CD incident at the surface TZ at an incidence angle r' and is refracted along the path DE into a medium of refractive index μ_1 so that angle of refraction is r''. From Snell's law, we have

$$\mu_3 \sin r' = \mu_1 \sin r'' \tag{64}$$

From Eqs. (62) and (63), we get

$$\mu_1 \sin i = \mu_3 \sin r' \tag{65}$$

From Eqs. (65) and (64), we get

$$\mu_1 \sin i = \mu_1 \sin r'' \tag{66}$$

Showing that $i = r''$. That is the angle of incidence is equal to the transmission angle. From Eq. (62), we have

$${}_1\mu_2 = \frac{\sin i}{\sin r} \tag{67}$$

From Eq. (63), we have

$${}_2\mu_3 = \frac{\sin r}{\sin r'} \tag{68}$$

From Eq. (64), we have

$${}_3\mu_1 = \frac{\sin r'}{\sin r''} \tag{69}$$

From Eqs. (67), (68) and (69) we have

$${}_1\mu_2\ {}_2\mu_3 = \frac{1}{{}_3\mu_1} \qquad \text{or} \qquad {}_1\mu_2\ {}_2\mu_3 = {}_1\mu_3 \tag{70}$$

12.9 Apparent Depth

Suppose, an object O lies in a medium (say, water) of refractive index μ_2 and is seen almost normally in a medium (say, air) of refractive index μ_1, such that $\mu_2 > \mu_1$, as shown in Fig. 26. The Ray OA incident normally at the surface PQ, separating the two media, and therefore go without any deviation in its path. The ray OB incident at an angle i and after refraction goes into air with refraction angle r, such that $r > i$. These two refracted rays appear to meet at I. Thus, I is the image of O. According to Snell's law, we have

$$\mu_2 \sin i = \mu_1 \sin r \qquad \text{or} \qquad \mu_2 \frac{\text{AB}}{\text{OB}} = \mu_1 \frac{\text{AB}}{\text{IB}}$$

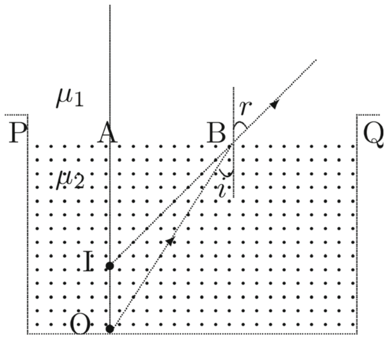


Fig. 26 Object is placed in a medium

It gives

$$IB = \frac{\mu_1}{\mu_2}\, OB$$

Since $\mu_2 > \mu_1$, we have IB < OB and the image I of an object O appears to be raised through a distance OI.

For an almost normal view, we have IB = AI and OB = OA. Let OA = t. Now, we have

$$\mathrm{OI} = \mathrm{OA} - \mathrm{AI} = t - \frac{\mu_1}{\mu_2}t = \left(1 - \frac{\mu_1}{\mu_2}\right)t$$

In case of viewing other than vertically, the position of the image changes with the angle, which the line of vision makes with the surface of separation of the two media.

13 Multiple Choice Questions

1. A monochromatic beam of light moves form a denser medium to a rarer medium, the angle of incidence i and the angle of refraction r a relation

   A. $i > r$ B. $i = r$ C. $i < r$ D. Depends on situation

   Ans. C

2. A monochromatic beam of light moves form a rarer medium to a denser medium, the angle of incidence i and the angle of refraction r a relation

   A. $i > r$ B. $i = r$ C. $i < r$ D. Depends on situation

   Ans. A

3. For refraction of light from medium 1 to medium 2, the angle of incidence is 30° whereas the angle of refraction is 45°. The refractive index of medium 2 relative to medium 1 is

   A. $\sqrt{2}$ B. $1/\sqrt{2}$ C. 0.67 D. 1.5

   Ans. B

4. For refraction of light from medium 1 to medium 2, the angle of incidence is 30° whereas the angle of refraction is 45°. The refractive index of medium 1 relative to medium 2 is
   A. $\sqrt{2}$ B. $1/\sqrt{2}$ C. 0.67 D. 1.5
   Ans. A
5. For the total internal reflection, the critical angle is always
   A. less than 90° B. equal to 90°
   C. greater than 90° D. equal to 45°
   Ans. A
6. The light is coming form a distant star. The image of the star formed by the convex lens is
   A. between F and optical point B. at F
   C. between F and 2F D. at infinite distance
   Ans. B
7. An object is placed at between infinity and 2F of convex lens. The image of the object formed by the lens is
   A. between infinity and 2F B. between 2F and F
   C. between F and optical point D. at F
   Ans. B
8. An object is placed at 2F of convex lens. The image of the object formed by the lens is
   A. between infinity and 2F B. between 2F and F
   C. between F and optical point D. at 2F
   Ans. D
9. An object is placed between F and 2F of convex lens. The image of the object formed by the lens is
   A. between infinity and 2F B. between 2F and F
   C. between F and optical point D. at 2F
   Ans. A
10. An object is placed at F of the convex lens. The image of the object formed by the lens is
    A. between infinity and 2F B. between 2F and F
    C. between F and optical point D. at infinity
    Ans. D
11. An object is placed between F and the optical point of the convex lens. The image of the object formed by the lens is
    A. between infinity and 2F B. between 2F and F
    C. in front of lens D. at infinity
    Ans. C

12. Suppose u and v denote the distances of the object and image from the optical point of a lens, respectively. For the focal length f of the lens, the relation between u, v and f is

A. $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ B. $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

C. $\frac{1}{v} + \frac{1}{f} = \frac{1}{u}$ D. $\frac{1}{u} + \frac{1}{f} = \frac{1}{v}$

Ans. B

13. For the formation of a real image of an object with the help of a convex lens of focal length 10 cm, the minimum distance between the object and image in cm should be
A. 20 B. 30 C. 40 D. 50
Ans. C

14. Two convex lenses, each of focal length 20 cm, are kept in contact. The focal length of the combination in cm is
A. 10 B. 20 C. 30 D. 40
Ans. A

15. A white light falls on a prism. The deviation angle is the largest for the color
A. red B. green C. yellow D. violet
Ans. D

16. A white light falls on a prism. The deviation angle is the minimum t for the color
A. red B. green C. yellow D. violet
Ans. A

14 Problems and Questions

1. Describe the refraction of light.
2. Wavelength of radiation moving in a vacuum is 5800 Å. Calculate the wavelength of the radiation if it moves in a medium with a refractive index of 1.43.
3. Derive an expression for the focal length of a lens.
4. Radii of curvatures of two faces of the convex lens placed in the air are 35 cm and 45 cm. If the refractive index of the material of the lens is 1.45, calculate the focal length of the lens.
5. Radii of curvatures of two faces of concave lens placed in air are 45 cm and 55 cm. If the refractive index of the material of the lens is 1.55, calculate the focal length of the lens.
6. What happens to a ray (i) moving parallel to the principal axis, (ii) moving towards the focal point, and (iii) moving towards the optical point of a lens.

7. Derive relation between u, v and f for convex lens.
8. Derive relation between u, v and f for concave lens.
9. Derive an expression for equivalent focal length for a combination of two thin convex lenses of focal lengths f_1 and f_2 placed coaxial to each other at d apart.
10. What are chromatic and monochromatic aberrations?
11. Can a single lens produce image, of a white light source, free from chromatic aberration.
12. Discuss about the chromatic aberration and the way for its rectification.
13. Show that the chromatic aberration can be eliminated by making a combination of two thin lenses made of different materials and kept in contact where one lens should be concave and the other convex.
14. When a monochromatic beam of light incident on a plane parallel slab of glass, show that the angle of incidence is equal to the angle of transmission.
15. Discuss about the chromatic aberration of a lens for an object placed at infinity.
16. Discuss about the chromatic aberration of a lens for an object placed at a finite distance.
17. Show that when two lenses of the same material are separated by a distance equal to half of the sum of their focal lengths, they form achromatic combination.
18. Show that the angular dispersion depends on the nature of the material of a prism and on the angle of the prism.
19. Write short notes on the following

    (i) Refraction
    (ii) Snell's law
    (iii) Total internal reflection
    (iv) Critical angle for total internal reflection
    (v) Various shapes of lenses
    (vi) Principal axis
    (vii) Optical point
    (viii) Sign convention for a lens
    (ix) Magnification with convex lens
    (x) Magnification with concave lens
    (xi) Aberrations
    (xii) Lateral chromatic aberration
    (xiii) Longitudinal chromatic aberration
    (xiv) Lateral spherical aberration
    (xv) Longitudinal spherical aberration
    (xvi) Dispersive power of a prism.

Chapter 3
Cardinal Points

We have learned about the formation of image by a single lens. Now, we want generalization of image formation by a complex optical system having a number of lenses and mirrors. In general, we have a number of refracting surfaces 1, 2, 3, …, m in sequence, each one of which forms image under well-defined laws of refraction. To find final image of an object after refractions through these surfaces, we have to apply m relations. Doing so is tedious and complex job. It is however possible to deal with m refractions together if we define some points, called the cardinal points. These cardinal points enable us to find final image of an object in a single step. In the present chapter, we have discussed about the cardinal points and their application for formation of image of an object.

1 Lateral or Transverse Magnification of a Spherical Refracting Surface

Let LM be principal section of a spherical refracting surface separating two media of refractive indices μ_1 and μ_2 (Fig. 1). C is the center of curvature of surface, P is the pole, F is the principal focus, and OPC is principal axis of refracting surface. Suppose, OO′ is a small object of size y_1 placed perpendicular to the principal axis, far away from P in the medium of refractive index μ_1. The distance of object from the pole is denoted by u.

To find image of point O, consider the rays starting from O. The ray OL after refraction at L follows the path LI and intersects the principal axis at the point I. Another ray OP towards the center of curvature C of surface goes undeviated. Both of these rays intersect at the point I and thus I is the image of O. To find image of point O′, consider the rays starting from O′. The ray O′L, parallel to the principal axis, after refraction at L follows the path LF. Another ray, O′S towards the center

S. Chandra and M. K. Sharma, *A Textbook of Optics*,
https://doi.org/10.1007/978-3-031-55676-0_3

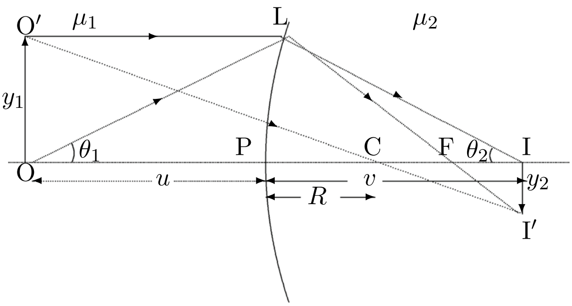


Fig. 1 Spherical refracting surface separating two media of refractive indices μ_1 and μ_2

of curvature C of surface goes undeviated. Both of these rays intersect at the point I′ and thus I′ is the image of O′. For any point between O and O′, the image is formed at a point between I and I′. Hence, II′ is image of OO′. Suppose, the size of image is y_2, perpendicular to principal axis. The distance of image from the pole is denoted by v.

Lateral or transverse magnification m_y is

$$m_y = \frac{\mathrm{II}'}{\mathrm{OO}'} = \frac{-y_2}{y_1} \tag{1}$$

In similar triangles OO′C and II′C, we have

$$\frac{\mathrm{II}'}{\mathrm{OO}'} = \frac{\mathrm{IC}}{\mathrm{OC}} \qquad \text{or} \qquad \frac{\mathrm{II}'}{\mathrm{OO}'} = \frac{\mathrm{IP} - \mathrm{PC}}{\mathrm{OP} + \mathrm{PC}}$$

Using the values along with sign convention, we have

$$\frac{-y_2}{y_1} = \frac{v - R}{-u + R} \qquad \text{or} \qquad \frac{y_2}{y_1} = \frac{v - R}{u - R} \tag{2}$$

For refraction at spherical surface, we have

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R} \qquad \text{or} \qquad \frac{\mu_2}{v} - \frac{\mu_2}{R} = \frac{\mu_1}{u} - \frac{\mu_1}{R}$$

$$\mu_2\left[\frac{R - v}{vR}\right] = \mu_1\left[\frac{R - u}{uR}\right] \qquad \text{or} \qquad \frac{v - R}{u - R} = \frac{\mu_1 v}{\mu_2 u} \tag{3}$$

Using Eq. (3) in (2), we get

$$\frac{y_2}{y_1} = \frac{\mu_1 v}{\mu_2 u} \tag{4}$$

Using Eq. (4) in (1), we get

$$m_y = -\frac{\mu_1}{\mu_2}\frac{v}{u}$$

This is expression for the lateral or transverse magnification.

2 Lagrange and Helmholtz Laws

Consider the refraction of ray OL in Fig. 1. The ray OL after refraction passes through the point I. If θ_1 and θ_2 are angles made by OL and LI with principal axis. The angular magnification m_θ is

$$m_\theta = \frac{\tan\theta_2}{\tan\theta_1}$$

When size of the refracting surface is small, we have

$$\tan\theta_1 = \frac{\text{PL}}{\text{PO}} \qquad \text{and} \qquad \tan\theta_2 = \frac{\text{PL}}{\text{PI}}$$

Therefore,

$$m_\theta = \frac{\tan\theta_2}{\tan\theta_1} = \frac{-\text{PL/PI}}{\text{PL/PO}} = -\frac{\text{PO}}{\text{PI}}$$

The sign negative appears as the angle θ_2 is traced in the clockwise direction. Putting the values along with the sign convention, we have

$$\frac{\tan\theta_2}{\tan\theta_1} = -\frac{-u}{v} = \frac{u}{v} \tag{5}$$

From Eqs. (4) and (5), we have

$$\frac{y_2}{y_1} = \frac{\mu_1\tan\theta_1}{\mu_2\tan\theta_2} \qquad \text{or} \qquad y_1\mu_1\tan\theta_1 = y_2\mu_2\tan\theta_2$$

This relation was first derived by Lagrange and is known as the Lagrange law.

If the angles are small and in radian, we have

$$y_1\mu_1\theta_1 = y_2\mu_2\theta_2$$

Suppose, there are $k-1$ coaxial refracting surfaces in succession separating the media of refracting indices μ_1, μ_2, ..., μ_k. Now the image formed by one refracting surface acts as object for the next refracting surface. Therefore, we have

$$y_1\mu_1\theta_1 = y_2\mu_2\theta_2$$

$$y_2\mu_2\theta_2 = y_3\mu_3\theta_3$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_{k-1}\mu_{k-1}\theta_{k-1} = y_k\mu_k\theta_k$$

On adding all these equations, we have

$$y_1\mu_1\theta_1 = y_k\mu_k\theta_k$$

This equation holds good for refraction at a system of coaxial refracting surfaces. This equation was first derived by Helmholtz and is known as the Helmholtz law for magnification.

3 Abbe's Sine Condition

The Lagrange law holds for very small aperture of refracting surface and it is not valid when the aperture is large. For full aperture, there is Abbe's sine condition which may be obtained as the following.

In Fig. 1, let the angles of incidence and refraction for rays OL and LI are i and r, respectively. In $\triangle$ OLC, we have

$$\frac{\sin(\pi - i)}{\sin\theta_1} = \frac{\text{OC}}{\text{CL}} \qquad \text{or} \qquad \frac{\sin i}{\sin\theta_1} = \frac{\text{OC}}{\text{CL}} \tag{6}$$

In $\triangle$ ILC, we have

$$\frac{\sin r}{\sin(-\theta_2)} = \frac{\text{IC}}{\text{CL}} \qquad \text{or} \qquad \frac{\sin r}{\sin\theta_2} = -\frac{\text{IC}}{\text{CL}} \tag{7}$$

From Eqs. (6) and (7), we have

$$\frac{\sin i}{\sin r} = -\frac{\text{OC}}{\text{CI}}\,\frac{\sin\theta_1}{\sin\theta_2}$$

Using Snell's law

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1}$$

we get

$$\frac{\mu_2}{\mu_1} = -\frac{\text{OC}}{\text{CI}}\frac{\sin\theta_1}{\sin\theta_2} \quad \text{or} \quad \frac{\text{CI}}{\text{OC}} = -\frac{\mu_1}{\mu_2}\frac{\sin\theta_1}{\sin\theta_2} \tag{8}$$

In similar triangles OO′C and II′C, we have

$$\frac{\text{CI}}{\text{OC}} = \frac{\text{II}'}{\text{OO}'} = \frac{-y_2}{y_1} \tag{9}$$

From Eqs. (8) and (9), we have

$$\frac{-y_2}{y_1} = -\frac{\mu_1}{\mu_2}\frac{\sin\theta_1}{\sin\theta_2} \quad \text{or} \quad y_1\mu_1\sin\theta_1 = y_2\mu_2\sin\theta_2$$

This relation is known as Abbe's sine condition.

4 Cardinal Points of a Coaxial Optical System

A coaxial optical system generally consists of two or more lenses separated from one another by a finite distance and having a common principal axis. The position and size of image formed by such a lens system can be determined by considering refraction at each surface of lens successively. This process though simple is very tedious.

In order to overcome this difficulty, Gauss in 1941 showed that if in a coaxial optical system, the positions of certain specific points are known, the system of lenses may be treated as a single unit. The position and size of image of an object may then be obtained directly by simple formulas for thin lens, whatever complicated the system may be. These specific points are known as the cardinal points or Gauss points of the optical system. The cardinal points of an optical system are six in number:

(i) Two focal points.
(ii) Two principal points.
(iii) Two nodal points.

4.1 Focal Points

A pair of points lying on the principal axis of optical system and conjugate to points at infinity are called the focal points. Let L_1L_2 be an optical system having principal axis AB (Fig. 2).

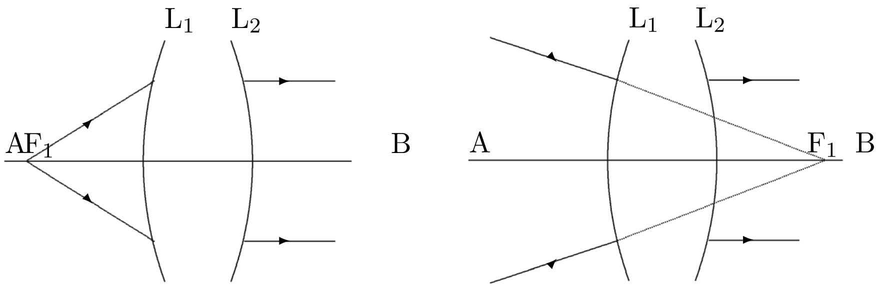
First focal point and first focal plane. F_1 is a point on the principal axis such that its point image by the optical system is formed at infinity (Fig. 3).

The first focal point of an optical system is defined as an object point on the principal axis for which the image point lies at infinity. It is denoted by F_1.

A set of rays starting from (in converging system) or directed towards (in diverging system) the axial point F_1 after refraction through an optical system becomes parallel to the principal axis. The point F_1 is called the first focal point and the plane passing through F_1 and perpendicular to the principal axis is called the first focal plane.

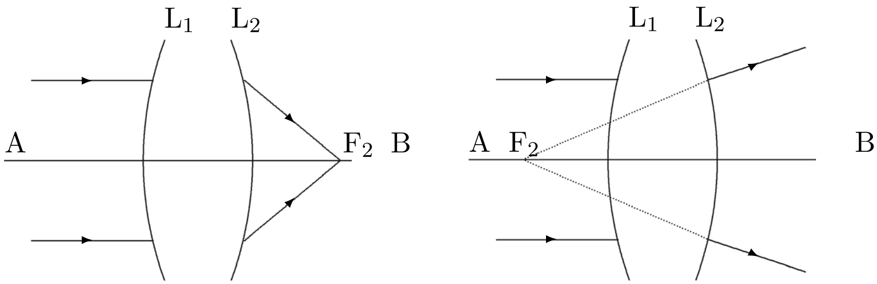
Second focal point and second focal plane. F_2 is a point on the principal axis such that image of a point object at infinity is formed by the optical system at F_2.

The second focal point of an optical system is defined as image point on the principal axis for which the object point lies at infinity. It is denoted by F_2.



(a) Convergent system (b) Divergent system

Fig. 2 First focal point



(a) Convergent system (b) Divergent system

Fig. 3 Second focal point

A set of rays parallel to principal axis incident and after refraction through the optical system passes through point F_2 in the converging system or appears to diverge from the point F_2 in the diverging system. The point F_2 is called the second focal point and the plane passing through F_2 and perpendicular to the principal axis is called the second focal plane.

Main property of focal planes. A first focal plane of optical system is also called the principal focal plane of the object space. Similarly, the second focal plane is called the principal focal plane of the image space. A set of rays starting from a point in the principal focal plane of object space (i.e., first focal plane) correspond to a set of conjugate parallel rays in the object space. Similarly, a set of parallel rays in the object space corresponds to a set of rays intersecting at a point in the principal focal plane of image space (i.e., second focal plane).

4.2 Principal Points

A pair of conjugate points on the principal axis of optical system having unit positive linear magnification is called principal points of the system.

Let F_1 and F_2 be the principal focal points of an optical system. In Fig. 4, an incident ray *a* parallel to the principal axis, after refraction through the optical system, passes through the second focal point F_2. The incident ray and emergent ray when produced meet at point A_2. The plane passing through A_2 and perpendicular to the principal axis of optical system is called the second principal plane and its point of intersection with the principal axis, H_2, is called the second principal point. Similarly, an incident ray *b* starting from first focal point F_1, after refraction through the optical system, emerges parallel to the principal axis at the same height as that of the ray *a* incident at the point P. These two rays (incident and emergent *b*) when produced intersect at the point A_1. The plane passing through A_1 and perpendicular to the principal axis of optical system is called the first principal plane and its point of intersection with the principal axis, H_1, is called the first principal point.

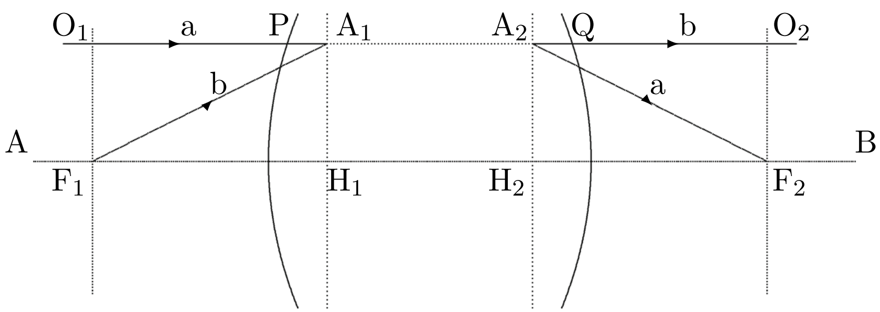


Fig. 4 Principal points

It is obvious from Fig. 4 that the incident rays a and b are converging towards point A_1 and after refraction the corresponding emergent rays appear to diverge from the point A_2. Hence A_2 is image of A_1, where $H_1A_1 = H_2A_2$. In this way, A_1 and A_2 are conjugate points and A_1H_1 and A_2H_2 is a pair of conjugate planes. This means that when an object is placed in the first principal plane, its image of the same size is formed by the optical system in its second principal plane. Obviously, the lateral magnification of the principal planes is $+1$.

Results.

(i) These two principal planes are called the two conjugate planes of unit positive lateral magnification or simply the unit planes. The distances H_1F_1 and H_2F_2 are called the first and second focal lengths, respectively, of the optical system.
(ii) If an incident ray passes through a point in the first principal plane at a given distance from the axis, the corresponding emergent ray certainly passes through a point in the second principal plane at the same distance from the principal axis on the same side.
(iii) If the medium on both the sides of optical system is the same, the first and second focal lengths of the optical system are equal numerically. That is, $f_1 = f_2$.

4.3 Nodal Points

A pair of conjugate points on the principal axis of optical system having unit positive angular magnification is called the nodal point of the system. This simply means that a ray of light directed towards one of these points. After refraction through the optical system, it appears to emerge from the second nodal point parallel to the original direction.

Suppose, H_1A_1 and H_2A_2 are first and second principal planes, and OF_1 and OF_2 are its first and second focal planes of the optical system, respectively.

Let O be a point on the first focal plane and OA is a ray of light parallel to the principal axis which intersects the first principal plane at A_1. The ray emerges out at the point A_2 on the second principal plane such that $A_1H_1 = A_2H_2$ and passes through second focal point F_2.

Consider another ray OP_1 parallel to A_2F_2 and intersecting the first principal plane at point P_1. This ray emerges out from P_2 on the second principal plane such that $H_1P_1 = H_2P_2$ and becomes parallel to A_2F_2, because both the rays start from a point O on the first focal plane. The points of intersection of incident ray OP_1 and the conjugate ray P_2I with the principal axis AB are the two nodal points, denoted by N_1 and N_2. It is obvious that N_1 and N_2 are a pair of conjugate points and the incident ray OP_1 is parallel to the corresponding emergent ray P_2I. Thus, the angular magnification is

$$m_\theta = \frac{\tan(-\theta_2)}{\tan(-\theta_1)} = +1$$

The planes passing through the nodal points N_1 and N_2 and normal to the principal axis are called the first and second nodal planes, respectively.

In the right-angled triangles $P_1N_1H_1$ and $P_2N_2H_2$, we have

$$P_1H_1 = P_2H_2 \quad \text{and} \quad \angle P_1N_1H_1 = \angle P_2N_2H_2$$

Thus, the triangles $P_1N_1H_1$ and $P_2N_2H_2$ are congruent and we have

$$H_1N_1 = H_2N_2 \tag{10}$$

Adding N_1H_2 on both sides, we have

$$H_1N_1 + N_1H_2 = H_2N_2 + N_1H_2 \quad \text{or} \quad H_1H_2 = N_1N_2$$

That is, the distance between two nodal points is equal to the distance between two principal points.

5 Coincidence of Principal Points and Nodal Points

In Fig. 5, consider the right triangles OF_1N_1 and $A_2F_2H_2$.

$$OF_1 = A_2H_2 \quad \text{and} \quad \angle ON_1F_1 = \angle A_2F_2H_2$$

Hence, triangles OF_1N_1 and $A_2F_2H_2$ are congruent. Therefore,

$$F_1N_1 = H_2F_2 \quad \text{or} \quad F_1H_1 + H_1N_1 = H_2F_2$$

Thus,

$$H_1N_1 = H_2F_2 - F_1H_1 \tag{11}$$

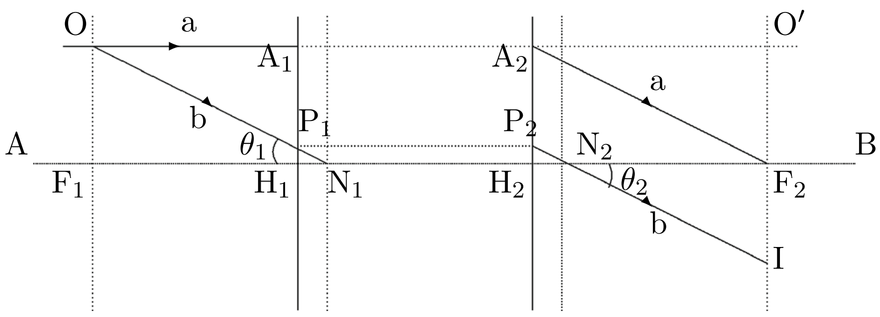


Fig. 5 Nodal points

Also $H_2F_2 = +f_2$ and $H_1F_1 = -f_1$. Thus, from Eq. (11), we have

$$H_1N_1 = f_2 + f_1 \tag{12}$$

If the mediums on two sides of the optical system are the same, we have $f_2 = -f_1$ and from Eq. (12), we have

$$H_1N_1 = 0$$

From Eq. (10), we have

$$H_2N_2 = H_1N_1 = 0$$

This means that when the mediums on both the sides of optical system are the same, the principal points coincide with the nodal points and these points are known as the equivalent points.

6 Graphical Construction of Image Using Cardinal Points

Having knowledge of cardinal points of an optical system, image of an object formed by the system can be traced by using the following rules:

1. An incident ray passing through a point on the first principal plane at a certain distance from the principal axis emerges from a point on the second principal plane at the same distance from the principal axis and on the same side of the axis.
2. An incident ray parallel to the principal axis, after refraction through the optical system, passes or appears to pass through the second focal point F_2 (Ray *a* in Fig. 6).
3. An incident ray passing through the first focal point F_1, after refraction through the optical system, becomes parallel to the principal axis (Ray *b* in Fig. 6).
4. An incident ray directed towards the first nodal point N_1 of the optical system emerges parallel to the original direction through the second nodal point N_2 (Ray *c* in Fig. 6).

Let F_1 and F_2 be the focal points, H_1 and H_2 the principal points, N_1 and N_2 the nodal points of an optical system (Fig. 6). OO' is an object on the principal axis AB. For getting the image of point O', we make the following construction:

(i) Draw a ray $O'A_1$ parallel to the principal axis and meeting the first principal plane at A_1. The conjugate ray emerges from the optical system through point A_2 on the second principal plane such that $H_1A_1 = H_2A_2$ and passes through the second focal point F_2.

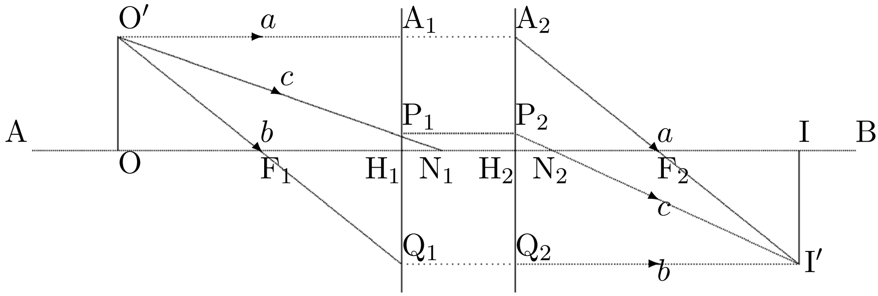


Fig. 6 Construction of image of coaxial converging lens system

(ii) Draw second ray $O'F_1Q_1$ passing through the first focal point F_1 and meeting the first principal plane at Q_1. This ray after refraction through the system emerges at point Q_2 on the second principal plane, such that $H_1Q_1 = H_2Q_2$, and becomes parallel to the principal axis.

(iii) Draw third ray $O'P_1N_1$ directed towards the first nodal point N_1 and meeting the first principal plane at P_1. This ray after refraction emerges at point P_2 on the second principal plane, such that $H_1P_1 = H_2P_2$, and parallel to the incident ray $O'P_1$.

The point of intersection I' of any two of the three emergent rays is the image of O'. The perpendicular II', drawn from I' on the principal axis, is image of the object OO'.

7 Newton's Formula

Suppose, AB be the principal axis of convergent coaxial system. Let F_1 and F_2 be the first and second focal points, H_1 and H_2 the first and second principal points, respectively, as shown in Fig. 7.

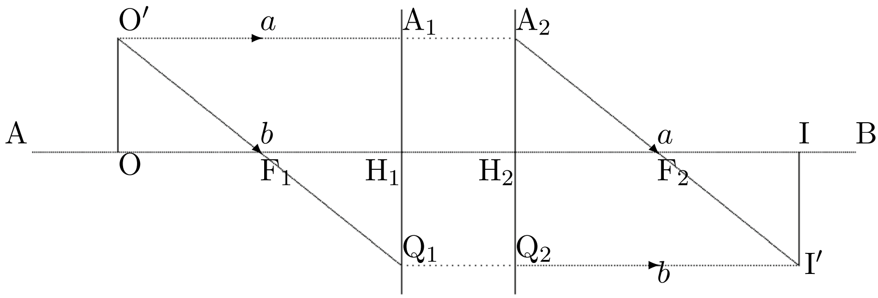


Fig. 7 Newton's formula

Suppose, an object OO′ is placed on the principal axis. An incident ray $O'A_1$ parallel to the principal axis meets the first principal plane at A_1. The conjugate ray emerges from the optical system through the point A_2 on the second principal plane such that $H_1A_1 = H_2A_2$ and passes through the second focal point F_2. Another incident ray $O'F_1Q_1$ passing through the first focal point F_1 meets the first principal plane at Q_1. This ray after refraction through the system emerges at point Q_2 on the second principal plane, such that $H_1Q_1 = H_2Q_2$, and becomes parallel to principal axis.

These two emergent rays intersect at I′ which is image of O′. The perpendicular II′, drawn from I′ on the principal axis, is image of the object OO′. Suppose, the lengths of object and image are y_1 and y_2, and f_1 and f_2 the first and second focal lengths of optical system, respectively. Then, following the sign convention, we have

$$\mathrm{OO'} = y_1, \quad \mathrm{II'} = -y_2, \quad \mathrm{H_1F_1} = -f_1 \quad \mathrm{H_2F_2} = f_2$$

Suppose, x_1 is the distance of object from the first focal point and x_2 the distance of image from the second focal point. Then, following the sign convention, we have

$$\mathrm{F_1O} = -x_1, \quad \mathrm{F_2I} = x_2$$

In similar triangles $H_1Q_1F_1$ and $OO'F_1$, we have

$$\frac{\mathrm{H_1Q_1}}{\mathrm{H_1F_1}} = \frac{\mathrm{OO'}}{\mathrm{OF_1}} \qquad \text{or} \qquad \frac{\mathrm{II'}}{\mathrm{H_1F_1}} = \frac{\mathrm{OO'}}{\mathrm{OF_1}}$$

Here, we have used $H_1Q_1 = II'$. Using the values, we get

$$\frac{-y_2}{-f_1} = \frac{y_1}{-x_1} \qquad \text{or} \qquad \frac{y_2}{y_1} = -\frac{f_1}{x_1} \tag{13}$$

In similar triangles $A_2F_2H_2$ and $II'F_2$, we have

$$\frac{\mathrm{A_2H_2}}{\mathrm{H_2F_2}} = \frac{\mathrm{II'}}{\mathrm{IF_2}} \qquad \text{or} \qquad \frac{\mathrm{OO'}}{\mathrm{H_2F_2}} = \frac{\mathrm{II'}}{\mathrm{IF_2}}$$

Here, we have used $A_2H_2 = OO'$. Using the values, we get

$$\frac{y_1}{f_2} = \frac{-y_2}{x_2} \qquad \text{or} \qquad \frac{y_2}{y_1} = -\frac{x_2}{f_2} \tag{14}$$

From Eqs. (13) and (14), we have

$$\frac{f_1}{x_1} = \frac{x_2}{f_2} \qquad \text{or} \qquad x_1x_2 = f_1f_2$$

This is Newton's formula for a coaxial lens system.

8 Relation Between f_1 and f_2

Suppose, AB be the principal axis of convergent coaxial system. Let F_1 and F_2 are the first and second focal points, H_1 and H_2 the first and second principal points, respectively, as shown in Fig. 8.

Let μ_1 and μ_2 be the refractive indices of initial and final media of the system, y_1 and y_2 are sizes of object and image, respectively. Following Helmholtz law, we have

$$\mu_1 y_1 \tan\theta_1 = \mu_2 y_2 \tan\theta_2 \qquad \text{or} \qquad \frac{y_1 \tan\theta_1}{y_2 \tan\theta_2} = \frac{\mu_2}{\mu_1} \tag{15}$$

If u and v are distances of object and image from the first and second principal points, respectively. Then, we have

$$\tan\theta_1 = \frac{A_1H_1}{OH_1} = \frac{y_1}{-u} = -\frac{y_1}{u}$$

and

$$\tan(-\theta_2) = \frac{A_2H_2}{IH_2} = \frac{y_1}{v} \qquad \text{or} \qquad \tan\theta_2 = -\frac{y_1}{v}$$

Thus, we have

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{v}{u} \tag{16}$$

From Fig. 8, we have

$$\frac{v}{f_2} = \frac{H_2I}{H_2F_2} = \frac{H_2F_2 + F_2I}{H_2F_2} = 1 + \frac{F_2I}{H_2F_2} \tag{17}$$

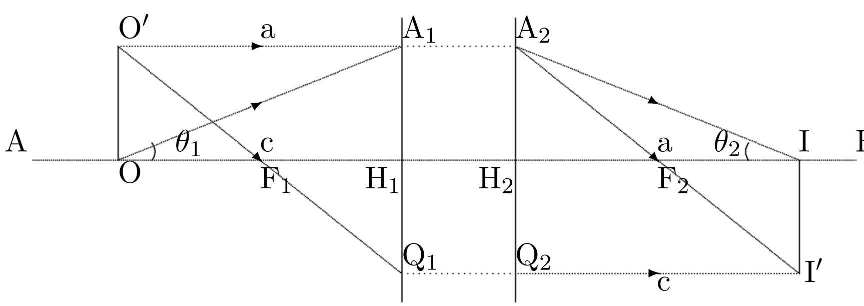


Fig. 8 Relation between focal points

In similar triangles $A_2H_2F_2$ and $II'F_2$, we have

$$\frac{F_2I}{H_2F_2} = \frac{II'}{A_2H_2} = \frac{-y_2}{y_1} = -\frac{y_2}{y_1} \tag{18}$$

From Eqs. (17) and (18), we have

$$\frac{v}{f_2} = 1 - \frac{y_2}{y_1} = \frac{y_1 - y_2}{y_1} \tag{19}$$

From Fig. 8, we have

$$\frac{-u}{-f_1} = \frac{OH_1}{H_1F_1} = \frac{OF_1 + F_1H_1}{H_1F_1} = 1 + \frac{OF_1}{H_1F_1} \tag{20}$$

In similar triangles $OO'F_1$ and $F_1H_1Q_1$, we have

$$\frac{O'F_1}{H_1F_1} = \frac{OO'}{H_1Q_1} = \frac{y_1}{-y_2} = -\frac{y_1}{y_2} \tag{21}$$

From Eqs. (20) and (21), we have

$$\frac{u}{f_1} = 1 - \frac{y_1}{y_2} = \frac{y_2 - y_1}{y_2} \tag{22}$$

On dividing Eq. (19) by (22), we have

$$\frac{v/f_2}{u/f_1} = -\frac{y_2}{y_1} \qquad \text{or} \qquad \frac{v}{u} = -\frac{y_2}{y_1}\frac{f_2}{f_1} \tag{23}$$

From Eqs. (16) and (23), we have

$$\frac{y_1 \tan\theta_1}{y_2 \tan\theta_2} = -\frac{f_2}{f_1} \tag{24}$$

From Eqs. (15) and (24), we have

$$-\frac{f_2}{f_1} = \frac{\mu_2}{\mu_1} \qquad \text{or} \qquad \frac{f_1}{f_2} = -\frac{\mu_1}{\mu_2}$$

If the mediums on both the sides are the same ($\mu_1 = \mu_2$), we have $f_1 = -f_2$.

9 Relation Between Formulas of Thin Lenses and Coaxial Lens System

Suppose in Fig. 6, u denotes the distance of object from the first principal plane and v the distance of image from the second principal plane. Following the sign system, we have

$$\mathrm{OH_1} = \mathrm{O'A_1} = -u \quad \mathrm{H_2I} = \mathrm{Q_2I'} = +v \quad \mathrm{OO'} = \mathrm{H_1A_1} = \mathrm{H_2A_2} = +y_1$$

$$\mathrm{II'} = \mathrm{H_2Q_2} = \mathrm{H_1Q_1} = -y_2 \quad \mathrm{F_1H_1} = -f_1 \quad \mathrm{F_2H_2} = f_2$$

In similar triangles $\mathrm{H_1Q_1F_1}$ and $\mathrm{Q_1O'A_1}$, we have

$$\frac{\mathrm{H_1Q_1}}{\mathrm{A_1Q_1}} = \frac{\mathrm{H_1F_1}}{\mathrm{O'A_1}} \qquad \text{or} \qquad \frac{\mathrm{H_1Q_1}}{\mathrm{A_1H_1} + \mathrm{H_1Q_1}} = \frac{\mathrm{H_1F_1}}{\mathrm{O'A_1}}$$

Using the values, we get

$$\frac{-y_2}{y_1 - y_2} = \frac{-f_1}{-u} \qquad \text{or} \qquad \frac{f_1}{u} = -\frac{y_2}{y_1 - y_2} \tag{25}$$

In similar triangles $\mathrm{A_2H_2F_2}$ and $\mathrm{A_2Q_2I'}$, we have

$$\frac{\mathrm{A_2H_2}}{\mathrm{A_2Q_2}} = \frac{\mathrm{H_2F_2}}{\mathrm{Q_2I'}} \qquad \text{or} \qquad \frac{\mathrm{A_2H_2}}{\mathrm{A_2H_2} + \mathrm{H_2Q_2}} = \frac{\mathrm{H_2F_2}}{\mathrm{Q_2I'}}$$

Using the values, we get

$$\frac{y_1}{y_1 - y_2} = \frac{f_2}{v} \qquad \text{or} \qquad \frac{f_2}{v} = \frac{y_1}{y_1 - y_2} \tag{26}$$

Adding Eqs. (25) and (26), we have

$$\frac{f_1}{u} + \frac{f_2}{v} = -\frac{y_2}{y_1 - y_2} + \frac{y_1}{y_1 - y_2} \qquad \text{or} \qquad \frac{f_1}{u} + \frac{f_2}{v} = 1 \tag{27}$$

When the lens system is placed in air, we have $f_1 = -f$ and $f_2 = f$. Therefore,

$$\frac{-f}{u} + \frac{f}{v} = 1 \qquad \text{or} \qquad \frac{1}{v} - \frac{1}{u} = \frac{1}{f} \tag{28}$$

This equation can also be deduced for divergent lens system. Equation (28) is the formula for thin lens and here it is derived for a coaxial lens system. It shows that the formulas for thin lens may be used for a coaxial lens system provided the distances are measured from the corresponding principal points.

10 Lateral, Longitudinal, and Angular Magnifications of a Coaxial Lens System

Lateral (or transverse) magnification of a lens system is defined as the ratio of the length of image to that of the object, both the lengths are measured perpendicular to the principal axis.

Suppose, y_1 and y_2 are the lengths of object and image measured perpendicular to the principal axis. The lateral magnification is

$$m_y = \frac{-y_2}{y_1} = -\frac{y_2}{y_1}$$

Following sign convention, the lateral magnification is positive for an erect image and negative for an inverted image.

Longitudinal magnification of a lens system is defined as the ratio of the extension of image to that of the object along the principal axis. Suppose, $\mathrm{d}x_1$ and $\mathrm{d}x_2$ are the extensions of object and image along the principal axis, respectively. The longitudinal magnification is

$$m_x = \frac{\mathrm{d}x_2}{\mathrm{d}x_1}$$

Angular magnification of a lens system is defined as the ratio of the slope of emergent ray and conjugate incident ray with the principal axis. Suppose, the incident ray and conjugate emergent ray make angles θ_1 and θ_2 with the principal axis. The angular magnification is

$$m_\theta = \frac{\tan(-\theta_2)}{\tan\theta_1} = -\frac{\tan\theta_2}{\tan\theta_1}$$

10.1 Relation Between Three Magnifications

Suppose, AB be the principal axis of convergent coaxial system. Let F_1 and F_2 be the first and second focal points, H_1 and H_2 the first and second principal points, respectively, as shown in Fig. 9.

Suppose, an object OO′ of small size y_1 is placed perpendicular to the principal axis. An incident ray $\mathrm{O'A_1}$ parallel to the principal axis meets the first principal plane at A_1. The conjugate ray emerges from the optical system through the point A_2 on

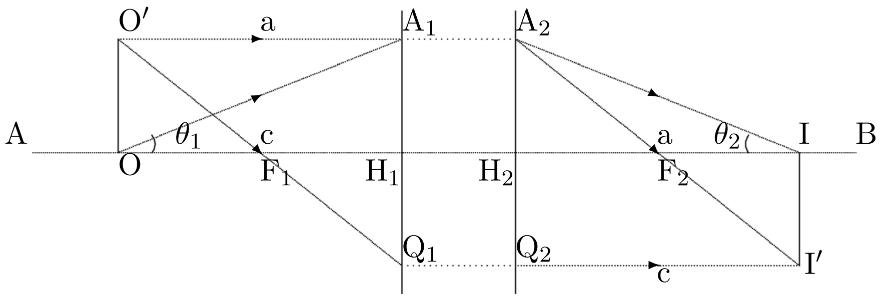


Fig. 9 Relation between three magnifications

the second principal plane such that $H_1A_1 = H_2A_2$ and passes through the second focal point F_2. Another incident ray $O'F_1Q_1$ passing through the first focal point F_1 meets the first principal plane at Q_1. This ray after refraction through the system emerges at the point Q_2 on the second principal plane, such that $H_1Q_1 = H_2Q_2$, and becomes parallel to principal axis. These two emergent rays intersect at I' which is image of O'. The perpendicular II' of size y_2, drawn from I' on the principal axis, is image of the object OO'.

Let f_1 and f_2 be the first and second focal lengths of lens system, and x_1 and x_2 the distances of object OO' and image II' from the focal points F_1 and F_2, respectively. According sign convention, we have

$$F_1H_1 = -f_1, \quad F_2H_2 = +f_2, \quad OO' = H_1A_1 = H_2A_2 = +y_1$$

$$II' = H_1Q_1 = H_2Q_2 = -y_2, \quad OF_1 = -x_1, \quad F_2I = +x_2$$

In similar triangles $H_1Q_1F_1$ and $OO'F_1$, we have

$$\frac{H_1Q_1}{H_1F_1} = \frac{OO'}{OF_1} \qquad \text{or} \qquad \frac{-y_2}{-f_1} = \frac{y_1}{-x_1}$$

Thus, the lateral magnification is

$$m_y = \frac{y_2}{y_1} = -\frac{f_1}{x_1} \tag{29}$$

In similar triangles $A_2F_2H_2$ and $II'F_2$, we have

$$\frac{A_2H_2}{H_2F_2} = \frac{II'}{IF_2} \qquad \text{or} \qquad \frac{y_1}{f_2} = \frac{-y_2}{x_2}$$

Thus, the lateral magnification is

$$m_y = \frac{y_2}{y_1} = -\frac{x_2}{f_2} \tag{30}$$

Comparing Eqs. (29) and (30), we have

$$-\frac{f_1}{x_1} = -\frac{x_2}{f_2} \qquad \text{or} \qquad x_1 x_2 = f_1 f_2 \tag{31}$$

The focal lengths f_1 and f_2 of a lens system are constant. The distances x_1 of object and x_2 of image may vary. On differentiation of Eq. (31), we get

$$x_1\,\mathrm{d}x_2 + x_2\,\mathrm{d}x_1 = 0 \qquad \text{or} \qquad \frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{x_2}{x_1}$$

Thus, the longitudinal magnification is

$$m_x = \frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{x_2}{x_1} \tag{32}$$

Let us now consider incident ray OA_1 meeting the first principal plane at A_1. The conjugate emergent ray is $\mathrm{A}_2\mathrm{I}$, as the point I is image of the point O. Suppose, θ_1 and θ_2 be the angles subtended by the rays OA_1 and $\mathrm{A}_2\mathrm{I}$ with the principal axis, respectively. Following sign convention θ_1 is positive and θ_2 is negative. Form Fig. 9, we have

$$\tan\theta_1 = \frac{\mathrm{A}_1\mathrm{H}_1}{\mathrm{OH}_1} = \frac{\mathrm{A}_1\mathrm{H}_1}{\mathrm{OF}_1 + \mathrm{F}_1\mathrm{H}_1} = \frac{y_1}{-x_1 - f_1} = -\frac{y_1}{x_1 + f_1} \tag{33}$$

We have

$$\tan(-\theta_2) = \frac{\mathrm{A}_2\mathrm{H}_2}{\mathrm{IH}_2} = \frac{\mathrm{A}_2\mathrm{H}_2}{\mathrm{IF}_2 + \mathrm{F}_2\mathrm{H}_2} = \frac{y_1}{x_2 + f_2}$$

Therefore,

$$\tan\theta_2 = -\frac{y_1}{x_2 + f_2} \tag{34}$$

Thus, the angular magnification is

$$m_\theta = \frac{\tan\theta_2}{\tan\theta_1} = \frac{-y_1/(x_2 + f_2)}{-y_1/(x_1 + f_1)} = \frac{x_1 + f_1}{x_2 + f_2} \tag{35}$$

Form Newton's formula, we have

$$x_1 x_2 = f_1 f_2 \qquad \text{or} \qquad x_2 = \frac{f_1 f_2}{x_1}$$

Using expression for x_2 in Eq. (35), we have

$$m_\theta = \frac{x_1}{f_2} \qquad \text{using Newton's formula} \qquad m_\theta = \frac{f_1}{x_2} \tag{36}$$

From Eqs. (30), (32), and (36), we get

$$m_x m_\theta = \left(-\frac{x_2}{x_1}\right)\left(\frac{x_1}{f_2}\right) = -\frac{x_2}{f_2} = m_y$$

$$\text{longitudinalmagnification} \times \text{angularmagnification} = \text{lateralmagnification}$$

This is the relation between three magnifications.

From Eqs. (29) and (36), we get

$$m_y m_\theta = \left(-\frac{f_1}{x_1}\right)\left(\frac{x_1}{f_2}\right) = -\frac{f_1}{f_2}$$

It shows that the product of lateral and angular magnifications of a lens system is constant, equal to the ratio first focal length to the second focal length of system.

11 Cardinal Points of a Coaxial Optical System of Two Thin Lenses

Let us consider a system of two lenses L_1 and L_2 placed co-axially. When the medium on either side of lens system and between the lenses is air, the nodal points coincide with the principal points. Thus, for such system, there are four cardinal points: (i) two principal points and (ii) two focal points.

Position of second principal point. In Fig. 10, the incident ray PA parallel to the principal axis and the final emergent ray BF_2 when produced intersect at A_2. By the property of principal planes, the plane passing through the point A_2 and perpendicular to the principal axis is the second principal plane and its point of intersection with the principal axis is the second principal point.

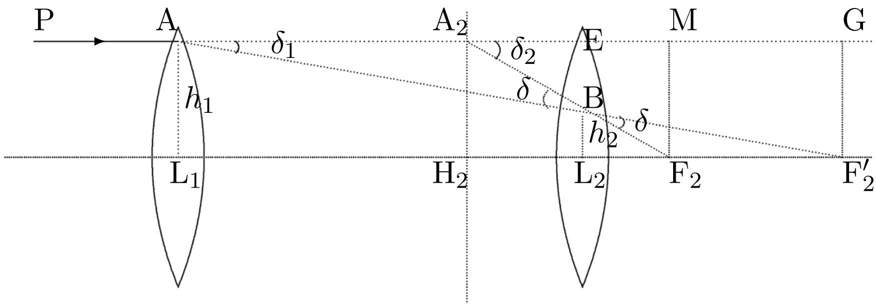


Fig. 10 Cardinal points of a coaxial optical system of two thin lenses

The F_2' is the focal point of lens L_1. The distance of second principal point from the second lens L_2 is

$$L_2H_2 = A_2E = \frac{BE}{\tan\delta_2} = \frac{AE\tan\delta_1}{\tan\delta_2} = \frac{d\delta_1}{\delta_2}$$

$$= \frac{dGF_2'/AG}{F_2M/A_2M} = \frac{d(h_1/f_1)}{h_1/H_2F_2} = H_2F_2\left(\frac{d}{f_1}\right)$$

But $H_2F_2 = F$ is the focal length of equivalent lens. Thus, we have

$$L_2H_2 = \frac{Fd}{f_1}$$

In Fig. 10, the point H_2 is to the left of L_2 and therefore according to the sign convention, the distance of second principal point from L_2 is

$$\alpha_2 = -L_2H_2 = -\frac{Fd}{f_1}$$

Position of second focal point. In Fig. 10, F_2 is the second focal point of the combination. If β_2 is the distance of second focus from the second lens L_2, we have

$$\beta_2 = L_2F_2 = H_2F_2 - L_2H_2 = F - \frac{Fd}{f_1} = F\left(1 - \frac{d}{f_1}\right)$$

Position of first principal point. Let a light ray PF_1 directed towards the first focal point F_1 of the combination is incident on lens L_1 at a height h_1 above the principal axis. This ray, after refraction through the lens L_1, follows the path AB and is incident on the lens L_2 at height h_2 above the principal axis. As the incident ray passes through the first focal point of combination, final emergent ray BR is parallel to the principal axis. The final emergent ray BR and the incident ray PF_1A when produced intersect at the point A_1. According to the property of principal planes, the plane passing through A_1 and normal to the principal axis is the first principal plane and its point of intersection H_1 with the principal axis is the first principal point. AB when produced intersects the principal axis at F_1''. As the emergent ray BR is parallel to the principal axis, the point F_1'' is the first focal point of the second lens L_2.

The distance of first principal point H_1 from the first lens L_1 is

$$\mathrm{L_1H_1} = \mathrm{A_1E} = \frac{\mathrm{AE}}{\tan\delta} = \frac{\mathrm{BE}\tan\delta_2}{\tan\delta} = \frac{d\delta_2}{\delta}$$

$$= \frac{d\mathrm{GF_1''/BG}}{\mathrm{F_1M/A_1M}} = \frac{d(\mathrm{MA_1})}{\mathrm{BG}} = \frac{d(\mathrm{H_1F_1})}{\mathrm{L_2F_1''}} = \frac{d(-F)}{-f_2} = \frac{dF}{f_2}$$

In Fig. 11, the point H_1 lies to the right of lens L_1, and therefore, according to sign convention, L_1H_1 is positive. Thus, the distance of first principal plane from lens L_1 is

$$\alpha_1 = \mathrm{L_1H_1} = \frac{Fd}{f_2}$$

Position of first focal point. In Fig. 11, F_1 is the first focal point of combination. The magnitude of distance of first focal point from the first lens is

$$|\mathrm{L_1F_1}| = |\mathrm{H_1F_1}| - |\mathrm{H_1L_1}|$$

$$= F - \frac{dF}{f_2} = F\left(1 - \frac{d}{f_2}\right)$$

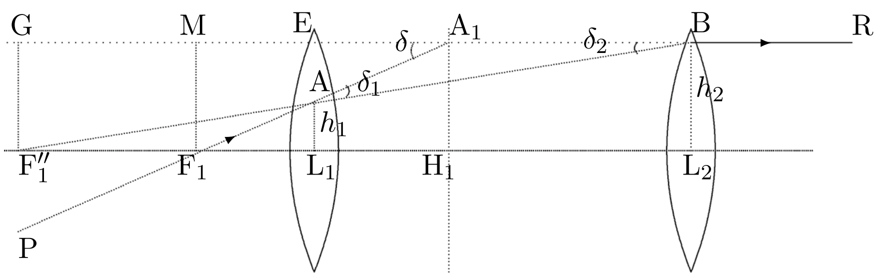


Fig. 11 Position of first principal point

In Fig. 11, the point F_1 lies to the left of lens L_1, and therefore, according to sign convention, L_1F_1 is negative. Thus, the distance of first focal point from lens L_1 is

$$\beta_1 = L_1F_1 = -F\left(1 - \frac{d}{f_2}\right)$$

Thus, we can summarize as

$$\alpha_1 = L_1H_1 = \frac{dF}{f_2} \qquad \alpha_2 = L_2H_2 = -\frac{dF}{f_1}$$

$$\beta_1 = L_1F_1 = -F\left(1 - \frac{d}{f_2}\right) \qquad \beta_2 = L_2F_2 = -F\left(1 - \frac{d}{f_1}\right)$$

Remarks If the medium on either side of the lenses is air and the medium between the lenses is one having refractive index μ. Then we can say that the rays emerging from the first lens are incident on the second lens as if they have traversed an apparent distance (d/μ) in air. Therefore, if we put (d/μ) in place of d in above results and they still hold true. Therefore, in such a case, we have

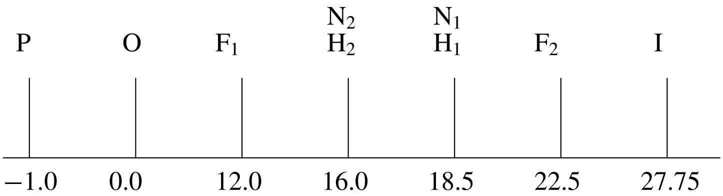
$$L_1H_1 = \alpha_1 = \frac{(d/\mu)F}{f_2} \qquad L_2H_2 = \alpha_2 = -\frac{(d/\mu)F}{f_1}$$

$$L_1F_1 = \beta_1 = -F\left(1 - \frac{(d/\mu)}{f_2}\right) \qquad L_2F_2 = \beta_2 = -F\left(1 - \frac{(d/\mu)}{f_1}\right)$$

Exercise 1 In a coaxial system of lenses, the focal points fall at 12 cm and 22.5 cm, and the principal points at 18.5 cm and 16 cm, respectively, when all measurements are made from a common origin. Deduce

(i) the focal lengths of object space and image space;
(ii) the position of image of an object placed at position -1 cm;
(iii) the position of nodal points of the system.

Solution Suppose, F_1 and F_2 are the focal points, and H_1 and H_2 are the principal points of a coaxial lens system. Suppose, O denotes the origin and P is the position of object.



(i) Focal length of object space is

$$f_1 = -|\mathrm{H_1F_1} = -(18.5 - 12.0) = -6.5 \text{ cm}$$

Focal length of image space is

$$f_2 = +|\mathrm{H_2F_2}| = +(22.5 - 16.0) = +6.5 \text{ cm}$$

(ii) The distance of object P from first principal point is

$$u = -|\mathrm{H_1P}| = -|18.5 + 1.0| = -19.5 \text{ cm}$$

With $f = f_2 = 6.5$, the relation

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u} \qquad \text{or} \qquad \frac{1}{v} = \frac{1}{f} + \frac{1}{u}$$

gives

$$\frac{1}{v} = \frac{1}{6.5} + \frac{1}{-18.5} = \frac{4}{39}$$

The distance of image from $\mathrm{H_2}$ is

$$v = \frac{39}{4} = 9.75 \text{ cm}$$

Position of image I = 16.0 + 9.75 = 25.75 cm

(iii) As the medium on either side of the lenses is the same, the nodal points $\mathrm{N_1}$ and $\mathrm{N_2}$ coincide with the principal points $\mathrm{H_1}$ and $\mathrm{H_2}$, respectively.

Exercise 2 Two thin convex lenses of focal lengths 2 cm and 5 cm are separated by a distance of 4 cm in air. Calculate the positions of cardinal points and show them on a sketch.

Solution We have $f_1 = 2$ cm, $f_2 = 6$ cm, and $d = 4$ cm. The focal length F of combination is

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \qquad \text{or} \qquad \frac{1}{F} = \frac{1}{2} + \frac{1}{6} - \frac{4}{2 \times 6}$$

giving $F = 3$ cm. The distance of first principal point H_1 from the first lens L_1 is

$$\alpha_1 = \frac{dF}{f_2} = \frac{4 \times 3}{6} = 2 \text{ cm}$$

The distance of second principal point H_2 from the second lens L_2 is

$$\alpha_2 = -\frac{dF}{f_1} = -\frac{4 \times 3}{2} = -6 \text{ cm}$$

The distance of first focal point F_1 from the first lens L_1 is

$$\beta_1 = -F\left(1 - \frac{d}{f_2}\right) = -3\left(1 - \frac{4}{6}\right) = -1 \text{ cm}$$

The distance of second focal point F_2 from the second lens L_2 is

$$\beta_2 = F\left(1 - \frac{d}{f_1}\right) = 3\left(1 - \frac{4}{2}\right) = -3 \text{ cm}$$

As the medium on either side of the lenses is the same, the nodal points N_1 and N_2 coincide with the principal points H_1 and H_2, respectively.

The positions of cardinal points are shown in Fig. 12.

Exercise 3 A telephoto lens has a thin convex lens of focal length 10 cm and a thin concave lens of focal length 20 cm, separated by 5 cm. Calculate the equivalent focal length of the telephoto lens. Locate the focal, principal, and nodal points of the telephoto lens and show them on a sketch.

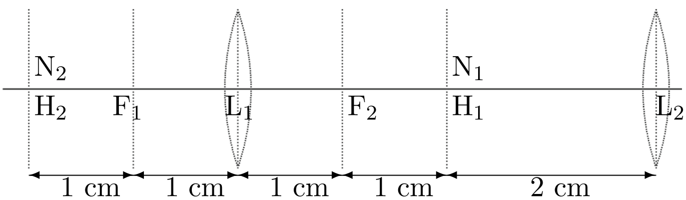


Fig. 12 Positions of cardinal points

Solution We have $f_1 = 10$ cm, $f_2 = -20$ cm, and $d = 5$ cm. The focal length F of combination is

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \qquad \text{or} \qquad \frac{1}{F} = \frac{1}{10} + \frac{1}{-20} - \frac{5}{10 \times (-20)}$$

giving $F = 40/3$ cm. The distance of first principal point H_1 from the first lens L_1 is

$$\alpha_1 = \frac{dF}{f_2} = \frac{5 \times (40/3)}{-20} = -10/3 \text{ cm}$$

The distance of second principal point H_2 from the second lens L_2 is

$$\alpha_2 = -\frac{dF}{f_1} = -\frac{5 \times (40/3)}{10} = -20/3 \text{ cm}$$

The distance of first focal point F_1 from the first lens L_1 is

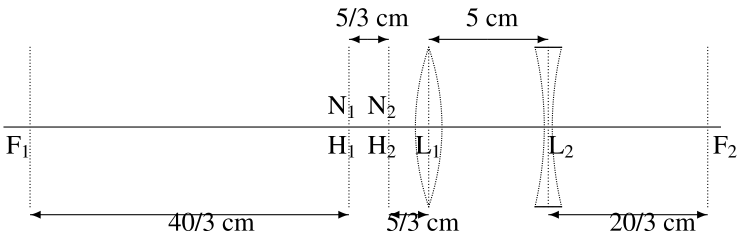
$$\beta_1 = -F\left(1 - \frac{d}{f_2}\right) = -\frac{40}{3}\left(1 - \frac{5}{-20}\right) = -50/3 \text{ cm}$$

The distance of second focal point F_2 from the second lens L_2 is

$$\beta_2 = F\left(1 - \frac{d}{f_1}\right) = \frac{40}{3}\left(1 - \frac{5}{10}\right) = 20/3 \text{ cm}$$

As the medium on either side of the lenses is the same, the nodal points N_1 and N_2 coincide with the principal points H_1 and H_2, respectively.

The positions of cardinal points are shown in figure below.



Exercise 4 Two thin convex lenses of focal length 10 cm each are placed co-axially 40 cm apart. Locate the positions of cardinal points and show them on a sketch.

An object of length 2 cm is placed at a distance of 20 cm from the nearer lens in the given optical system. Calculate the size and position of the image.

Solution We have $f_1 = 10$ cm, $f_2 = 10$ cm, and $d = 40$ cm. The focal length F of combination is

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \qquad \text{or} \qquad \frac{1}{F} = \frac{1}{10} + \frac{1}{10} - \frac{40}{10 \times 10}$$

giving $F = -5$ cm. The distance of first principal point H_1 from the first lens L_1 is

$$\alpha_1 = \frac{dF}{f_2} = \frac{40 \times (-5)}{10} = -20 \text{ cm}$$

The distance of second principal point H_2 from the second lens L_2 is

$$\alpha_2 = -\frac{dF}{f_1} = -\frac{40 \times (-5)}{10} = 20 \text{ cm}$$

The distance of first focal point F_1 from the first lens L_1 is

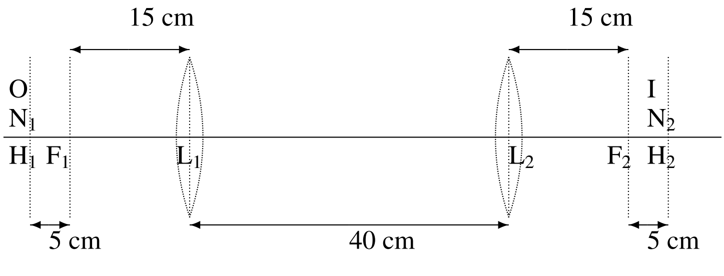
$$\beta_1 = -F\left(1 - \frac{d}{f_2}\right) = -(-5)\left(1 - \frac{40}{10}\right) = -15 \text{ cm}$$

The distance of second focal point F_2 from the second lens L_2 is

$$\beta_2 = F\left(1 - \frac{d}{f_1}\right) = -5\left(1 - \frac{40}{10}\right) = 15 \text{ cm}$$

As the medium on either side of the lenses is the same, the nodal points N_1 and N_2 coincide with the principal points H_1 and H_2, respectively.

The positions of cardinal points are shown in figure below.



The distance of object from the lens L_1 is -20 cm. Hence, the distance of object from the first principal point is

$$u = L_1O - L_1H_1 = (-20) - (-20) = 0 \text{ cm}$$

This means that the object is placed in the first principal plane. By definition of principal planes, the image is formed in the second principal plane. The image is erect and its size is the same as that of the object.

size of image = + 2 cm

distance of image = + 20 cm to the right of lens L_2

Exercise 5 A thin convex lens of focal length 30 cm is placed at a distance of 20 cm from a thin concave lens of focal length 50 cm. Calculate the equivalent focal length and locate the cardinal points of the system. If an object is at a distance of 40 cm from the convex lens, locate the position final image and its magnification.

Solution We have $f_1 = 30$ cm, $f_2 = -50$ cm, and $d = 20$ cm. The focal length F of combination is

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \qquad \text{or} \qquad \frac{1}{F} = \frac{1}{30} + \frac{1}{-50} - \frac{20}{30 \times (-50)}$$

giving $F = 37.5$ cm. The distance of first principal point H_1 from the first lens L_1 is

$$\alpha_1 = \frac{dF}{f_2} = \frac{20 \times 37.5}{-50} = -15 \text{ cm}$$

The distance of second principal point H_2 from the second lens L_2 is

$$\alpha_2 = -\frac{dF}{f_1} = -\frac{20 \times 37.5}{30} = -25 \text{ cm}$$

The distance of first focal point F_1 from the first lens L_1 is

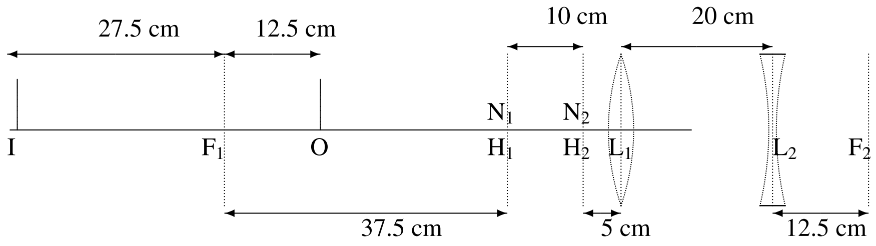
$$\beta_1 = -F\left(1 - \frac{d}{f_2}\right) = -37.5\left(1 - \frac{20}{-50}\right) = -52.5 \text{ cm}$$

The distance of second focal point F_2 from the second lens L_2 is

$$\beta_2 = F\left(1 - \frac{d}{f_1}\right) = 37.5\left(1 - \frac{20}{30}\right) = 12.5 \text{ cm}$$

As the medium on either side of the lenses is the same, the nodal points N_1 and N_2 coincide with the principal points H_1 and H_2, respectively.

The positions of cardinal points are shown in figure below.



The distance of object from the first principal point is

$u = -OH_1 = -(OL_1 - H_1L_1) = -(40 - 15) = -25$ cm

From the relation

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u} \qquad \text{or} \qquad \frac{1}{v} = \frac{1}{f} + \frac{1}{u}$$

we have

$$\frac{1}{v} = \frac{1}{37.5} + \frac{1}{-25}$$

It gives $v = -75$ cm. The distance of image from H_2 is 75 cm to the left. The magnification is

$$m = \frac{v}{u} = \frac{-75}{-25} = 3$$

Exercise 6 An equiconvex lens of refraction index 1.5 with radii 4 cm is located at a distance of 4 cm from an equiconcave lens of refraction index 1.6 with radii 8 cm. The lenses are thin and the medium between them is water with refractive index 4/3 and on both sides of the lenses the medium is air. Locate the cardinal points of the system.

Solution For equiconvex lens, we have $\mu_1 = 1.5$, $R_1 = 4$ cm, and $R_2 = -4$ cm. The focal length f_1 of the lens is

$$\frac{1}{f_1} = (\mu_1 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.5 - 1)\left(\frac{1}{4} - \frac{1}{-4}\right)$$

It gives $f_1 = 4$ cm. For equiconcave lens, we have $\mu_2 = 1.6$, $R_1 = -4$ cm, and $R_2 = 8$ cm. The focal length f_2 of the lens is

$$\frac{1}{f_2} = (\mu_2 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.6 - 1)\left(\frac{1}{-4} - \frac{1}{8}\right)$$

It gives $f_2 = -20/3$ cm. The medium on both sides of the lens system is air and the medium between the lenses is water of refractive index $\mu = 4/3$. The separation between the lenses is $d = 4$ cm. The equivalent focal length F is

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d/\mu}{f_1 f_2} \qquad \text{or} \qquad \frac{1}{F} = \frac{1}{4} + \frac{1}{-6.67} - \frac{4/(4/3)}{4 \times (-6.67)}$$

It gives $F = 80/17$ cm. The distance of first principal point H_1 from the first lens L_1 is

$$\alpha_1 = \frac{(d/\mu)F}{f_2} = \frac{3 \times (80/17)}{-20/3} = -2.118 \text{ cm}$$

The distance of second principal point H_2 from the second lens L_2 is

$$\alpha_2 = -\frac{(d/\mu)F}{f_1} = -\frac{3 \times (80/17)}{4} = -3.53 \text{ cm}$$

The distance of first focal point F_1 from the first lens L_1 is

$$\beta_1 = -F\left(1 - \frac{d/\mu}{f_2}\right) = -\frac{80}{17}\left(1 - \frac{3}{-20/3}\right) = -6.823 \text{ cm}$$

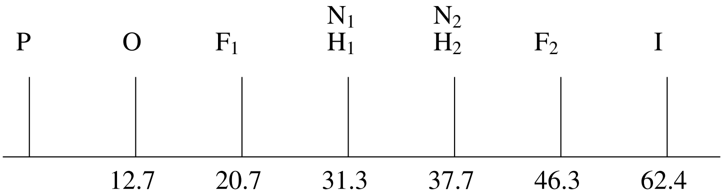
The distance of second focal point F_2 from the second lens L_2 is

$$\beta_2 = F\left(1 - \frac{d/\mu}{f_1}\right) = \frac{80}{17}\left(1 - \frac{3}{4}\right) = 1.176 \text{ cm}$$

Exercise 7 In a given optical system, with identical media on both sides, the focal points F_1 and F_2 are located at 20.7 cm and 48.3 cm, respectively, from an arbitrary origin. For an object located at 12.7 cm, the image is formed at 62.4 cm. Deduce

(i) locations of principal points H_1 and H_2;
(ii) locations of principal points N_1 and N_2;
(iii) lateral magnification m_y;
(iv) axial magnification m_x;
(v) angular magnification m_θ.

Solution The given points are given in figure below. The distances are taken from arbitrary origin P.



We have the distance of object from first focal point which is

$x_1 = F_1O = -(20.7 - 12.7) = -8.0$ cm

Distance of image from second focal point is

$x_2 = F_2I = (62.4 - 48.3) = 14.1$ cm

As the mediums on both sides of optical system are the same, the focal length of object space is equal in magnitude to the focal length of image space. That is

$f_1 = -f \qquad f_2 = f$

From Newton's formula $x_1 x_2 = f_1 f_2$, we have

$(-8.0)(14.1) = -f^2 \qquad f = 10.6$ cm

Therefore, we have

$f_1 = H_1F_1 = -10.6$ cm $\qquad f_2 = H_2F_2 = +10.6$ cm

Thus, F_1 lies at a distance of 10.6 cm on the left of H_1 and F_2 lies at a distance of 10.6 cm on the right of H_2.

The distance of H_1 from P is 20.7 + 10.6 = 31.3 cm

The distance of H_2 from P is 48.3 − 10.6 = 37.7 cm

(ii) As the mediums on both sides of the system are the same, the nodal points N_1 and N_2 coincide with the principal points H_1 and H_2.

(iii) Lateral magnification is

$$m_y = \frac{I}{O} = -\frac{f_2}{x_1} = -\frac{10.6}{8} = -1.3$$

(iv) Axial magnification is

$$m_x = \frac{x_2}{x_1} = -\frac{14.1}{-8} = 1.8$$

(v) Angular magnification is

$$m_\theta = \frac{m_y}{m_x} = \frac{-1.3}{1.8} = -0.7$$

12 Multiple Choice Questions

1. The number of cardinal points is
   A. 2 B. 4 C. 6 D. 8
   Ans. C
2. When mediums on both the sides of an optical system are the same
   A. focal points coincide with corresponding principal points
   B. nodal points coincide with corresponding principal points
   C. focal points coincide with corresponding nodal points
   D. focal points, nodal points, and principal points coincide with one another.
   Ans. B
3. Suppose, m_x denotes axial magnification, m_y denotes lateral magnification, and m_θ denotes angular magnification. Which of the following relations is valid?
   A. $m_x m_y m_\theta = 1$ B. $m_x m_\theta = m_y$
   C. $m_y m_\theta = m_x$ D. $m_x m_y = m_\theta$
   Ans. B
4. A system of two convex thin lenses of focal lengths f_1 and f_2 separated by a distance d behaves as a divergent system when
   A. $(f_1 + f_2) < d$ B. $(f_1 + f_2) > d$ C. $f_2 > (f_1 + d)$ D. $f_1 > (f_2 + d)$
   Ans. A
5. A system of two convex thin lenses of focal lengths f_1 and f_2 separated by a distance d behaves as a convergent system when
   A. $(f_1 + f_2) < d$ B. $(f_1 + f_2) > d$ C. $f_2 > (f_1 + d)$ D. $f_1 > (f_2 + d)$
   Ans. B
6. Suppose, x_1 and x_2 are distances of object and image from first and second focal points of a lens system, and f_1 and f_2 are focal lengths of object and image spaces, respectively. Which of the following relations is correct?
   A. $\frac{f_1}{f_2} = \frac{x_1}{x_2}$ B. $x_1 x_2 = f_1 f_2$ C. $\frac{f_1^2}{f_2} = \frac{x_1^2}{x_2}$ D. $\frac{f_1}{f_2} = -\frac{x_2}{x_1}$
   Ans. B

7. Suppose, f_1 and f_2 are focal lengths and μ_1 and μ_2 the refractive indices in the object and image spaces, respectively. Which of the following relations is correct?

A. $\dfrac{\mu_1}{f_1}+\dfrac{\mu_2}{f_2}=1$ B. $\dfrac{f_1}{f_2}=-\dfrac{\mu_1}{\mu_2}$ C. $\dfrac{f_1}{f_2}=\sqrt{\dfrac{\mu_1}{\mu_2}}$ D. $\dfrac{f_1}{f_2}=-\dfrac{\mu_2}{\mu_1}$

Ans. B

8. The planes characterized by unit linear (lateral) magnification are known as
   A. focal planes B. principal planes
   C. nodal planes D. none of these

   Ans. B
9. The planes characterized by unit angular magnification are known as
   A. focal planes B. principal planes
   C. nodal planes D. none of these

   Ans. C
10. To apply the formula $\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$ for a lens system, the distances are measured form
    A. focal planes B. principal planes
    C. nodal planes D. none of these

    Ans. B

13 Problems and Questions

1. Describe about the lateral or transverse magnification of a spherical refracting surface.
2. Describe about the Lagrange and Helmholtz laws for refraction between two mediums.
3. Derive Abbe's sine condition for refraction between two mediums.
4. Show that formulas for thin lens can be used for a coaxial lens system provided the distances are measured from the corresponding principal points.
5. Discuss about the Cardinal points of a coaxial optical system.
6. Show that principal points and nodal points of an optical system coincide when the mediums on two sides are the same.
7. Describe about the graphical construction of image using cardinal points of an optical system.
8. Derive relation between the focal lengths f_1 and f_2 of an optical system when the mediums on the two sides have refractive indices μ_1 and μ_2, respectively.
9. Derive Newton's formula for an optical system.
10. Describe about the lateral, longitudinal, and angular magnifications of a coaxial lens system and obtain relations between them.

11. Derive expressions for positions of cardinal points of a coaxial optical system of two thin lenses.
12. Write short notes on the following:

    (i) Cardinal points of a coaxial lens system
    (ii) Lateral, longitudinal, and angular magnifications of a coaxial lens system

Chapter 4
Interference

The phenomena of reflection and refraction of light were explained successfully using the wave nature of light. One more phenomenon that could be understood with the help of the wave nature of light is the **interference**. For explaining the interference in optics, two or more waves are considered to superimpose on each other simultaneously. Such superposition of various waves provides a new wave pattern that is different from all the waves participating in the superposition. For superposition of waves, there is no condition for their frequencies and wavelengths. However, for understanding the interference, we have considered the waves having the same frequencies. Such waves can be produced from a common source. We consider two waves propagating in the same direction and superimposing on each other. This results in a new wave pattern, and the phenomenon is known as the interference of light. More precisely, the interference may be defined as the interaction between the two or more waves having the same or very close frequencies emitted from two sources. Thomas Young was the first to explain the phenomenon of interference in light in 1802 using the double slit experiment, where two waves having the same frequency and propagating in the same direction were produced. Before proceeding to the phenomenon of interference, it is interesting first to understand the wave nature of light.

1 Wave Nature of Light

For explaining some phenomena produced by light, in 1678, Huygens considered the light as a wave. A wave is considered a disturbance traveling with time through the space, where, in general, the energy is transferred from one place to another, with no permanent displacement of any particle in the medium. Under this concept, the particles of the medium only oscillate about their equilibrium positions. There can be two modes of propagation of waves, known as the longitudinal and the transverse.

S. Chandra and M. K. Sharma, *A Textbook of Optics*,
https://doi.org/10.1007/978-3-031-55676-0_4

In the longitudinal propagation of wave, the particles of the medium oscillate in the direction of the propagation of the wave, and the wave is said to be longitudinal. In the transverse mode, the particles of the medium oscillate in the direction that is perpendicular to the direction of propagation of the wave, and the wave is said to be transverse.

Using the wave nature, Huygens was successful for explaining some observed phenomena of light. Based on this theory, he could explain successfully the phenomena of reflection, refraction, interference, and polarization of light. Huygens started with the longitudinal waves, which could not explain the phenomenon of polarization of light, for which he considered the transverse nature of waves. Huygens proposed a hypothesis for geometrical construction of position of a wavefront at any instant when the propagation of waves takes place in a medium. The wavefront is an imaginary surface, constructed by joining the points in the same phase in a wave propagating through the medium. The manner in which the wavefront propagates through a medium can be understood with the help of the Huygens' principle. In the concept of the propagation of waves, following assumptions were made:

(i) Each point on a wavefront of wave acts as a source of secondary wavelet.
(ii) The secondary wavelets from various points on the wavefront travel through the space in all directions with the equal velocity, called the velocity of light.
(iii) An imaginary surface touching the secondary wavelets tangentially in the forward direction at any given time provides a new wavefront at that time. It is known as the secondary wavefront.

In order to demonstrate Huygens' principle, we consider the propagation of a spherical wavefront (Fig. 1a) or of a plane wavefront (Fig. 1b) in a homogeneous and isotropic medium emerging from a point source of light S or line source, respectively. At some time, suppose AB is a section of primary wavefront obtained from a respective source. To find position of a wavefront after an interval t, we consider points denoted by 1, 2, 3, … on the primary wavefront AB. Now, considering Huygens' principle, these points act as the sources of secondary wavelets. Taking each point as center, we construct spheres, each of radius ct, where c denotes the speed of light. These spherical surfaces represent the positions of secondary wavelets at time t. Now, we draw a surface A_1B_1 that touches tangentially all these secondary wavelets in the forward direction. The surface A_1B_1 is the secondary wavefront. Other surface A_2B_2 in the backward direction is not considered, as there is no backward flow of energy during the propagation of waves.

2 Young's Double Slit Experiment

The interference phenomenon was explained with the help of Fig. 2 where two point sources S_1 and S_2 were obtained from a common source. Suppose, the sources S_1 and S_2 are close to each other. As each source emits waves in all directions, the

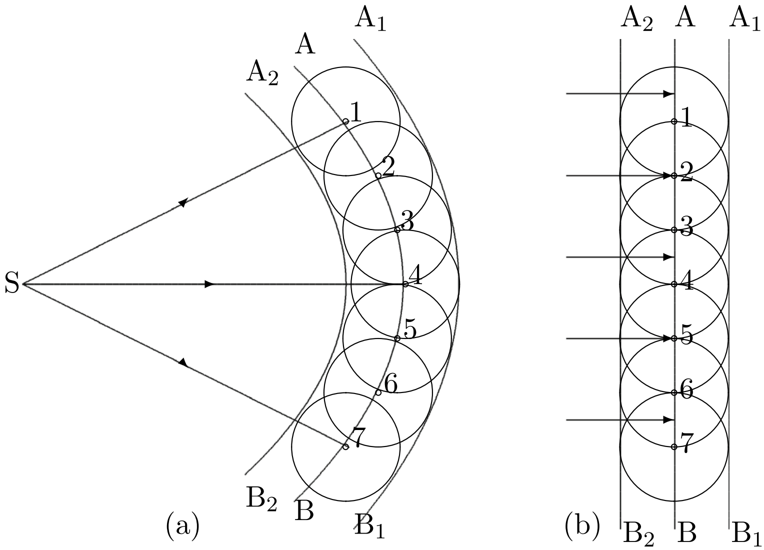
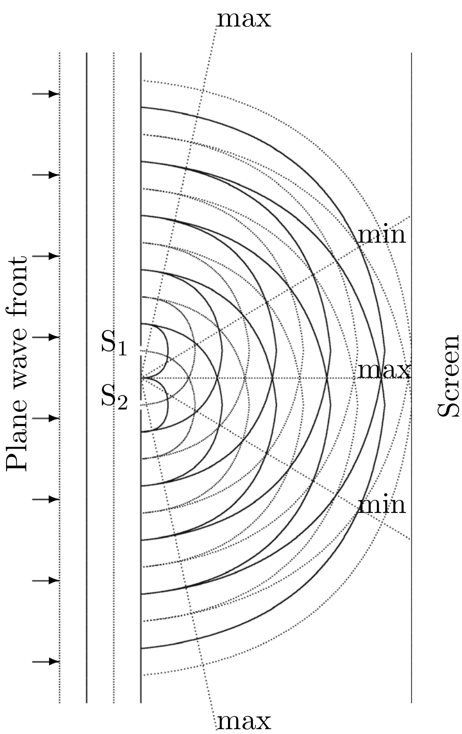


Fig. 1 Huygens' principle

Fig. 2 Double slit experiment

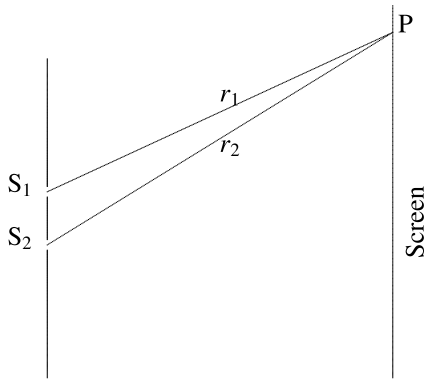


spherical waves from S_1 and S_2 expand through the space. The crests of the waves are shown by the continuous arcs and the troughs by the dotted arcs. It is known that the constructive interference takes place at the points where a crest due to one wave meets a crest due to other wave or where a trough due to one wave meets a trough due to other wave. For the constructive interference, the resultant amplitude is the sum of the amplitudes of the two individual waves.

Consequently the intensity of light is the maximum at these points of constructive interference. On the other side, where a crest due to one wave meets a trough due to other wave or vice versa, the resultant amplitude is the difference of the amplitudes of individual waves. At these points, the intensity of light is the minimum, and there is the destructive interference. Thus, due to the interaction of two waves from S_1 and S_2, alternate bright (constructive interference) and dark (destructive interference) fringes are seen on the screen placed on the right side. These fringes are obtained due to the phenomenon of interference of light.

3 Path Difference and Phase Difference

Let us consider the phenomenon of interference due to two waves from two sources S_1 and S_2 having the same frequency and the same wavelength. The interference pattern is formed on the screen. Suppose, the two waves from the sources S_1 and S_2 are reaching a point P. If the distances of the sources S_1 and S_2 from the point P are r_1 and r_2, respectively. Then, the path difference between the two waves is given by $\delta = r_2 - r_1$.



The phase difference ϕ between the two waves is related to the path difference through the following relation:

$$\phi = \frac{2\pi}{\lambda}\,\delta$$

where λ is the wavelength of the waves. A constructive interference is obtained when the phase difference is an integer multiple of 2π, i.e., $\phi = 2m\pi$, where m is an integer. Obviously, the path difference between the two waves is integer multiple of the wavelength of the wave. Further, a destructive interference is obtained when the phase difference is a half-integer multiple of 2π, i.e., $\phi = (2m + 1)\pi$, where m is an integer. Obviously, the path difference between the two waves is half-integer multiple of the wavelength of the wave.

4 Coherent Sources

Coherence is a property of waves which helps for getting the stationary interference. Two sources of light are said to be coherent when they emit waves of the same frequency (wavelength) and maintain a constant phase difference between them. The LASER is a good example of coherent source. In actual sense, it is not possible to have two independent sources that are coherent. It can be understood as the following.

A source of light has a large number of atoms. The theory of atoms says that the energy levels in an atom are quantized, and in the state of equilibrium, the atoms tend to remain in the lowest energy states. After receiving sufficient energy from some source, an electron in an atom moves to a higher energy level. This state of the atom is called the excited state, and the process is known as the excitation.

Generally, the electron lives in an excited state for $\sim 10^{-8}$ s and then returns back to a lower energy level. During this process, called the deexcitation, the atom radiates energy in the form of light. Out of a large number of atoms, some of them emit light at any instant of time and at the next instant other atoms do so, and so on. Thus, the light waves are emitted with different phases. Hence, it is difficult to obtain a coherent light from different parts of the same source. Therefore, such two independent sources of light can never act as coherent sources. Further, the light from a general source is not monochromatic, and has several wavelengths.

It is possible to have monochromatic sources. LASER is however an example of monochromatic source. From a monochromatic source, two coherent sources can be obtained either by producing them from the wavefront of the monochromatic source (division of wavefront) or by dividing the amplitude (intensity) of monochromatic source (division of amplitude).

In absence of LASER source, in our laboratories, a sodium lamp is used as a source of approximate monochromatic light. A sodium lamp, in principle, emits two wavelengths of 5890 and 5896 Å, which are very close to each other. Though the sodium lamp is a bichromatic source, it may be approximated as a monochromatic source with wavelength 5893 Å.

In the experiment of Fresnel's biprism, the interference is obtained between the two sources obtained by the division of wavefront. In the experiments of thin film, Newton's rings, Michelson's interferometer, the phenomenon of interference is obtained between the two sources obtained by the division of amplitude.

5 Mathematical Treatment of Interference

Let us consider superposition of two waves having the same frequency ω and a constant phase difference ϕ. Here, both the waves are traveling in the same direction. Suppose, the amplitudes of the waves are A_1 and A_2, respectively. The displacements of the waves are expressed as

$$y_1 = A_1 \sin \omega t \tag{1}$$

and

$$y_2 = A_2 \sin(\omega t + \phi) \tag{2}$$

Following the superposition of these two waves, the resultant displacement y is the algebraic sum of the two displacements. Therefore, we have

$$y = y_1 + y_2 = A_1 \sin \omega t + A_2 \sin(\omega t + \phi)$$

$$= A_1 \sin \omega t + A_2[\sin \omega t \cos \phi + \cos \omega t \sin \phi]$$

$$= \sin \omega t (A_1 + A_2 \cos \phi) + \cos \omega t (A_2 \sin \phi)$$

Substituting,

$$(A_1 + A_2 \cos \phi) = A' \cos \theta \tag{3}$$

and

$$A_2 \sin \phi = A' \sin \theta \tag{4}$$

we have

$$y = \sin \omega t \ A' \cos \theta + \cos \omega t \ A' \sin \theta = A' \sin(\omega t + \theta)$$

Squaring Eqs. (3) and (4), and adding them, we have

$$A'^2 = (A_1 + A_2 \cos \phi)^2 + (A_2 \sin \phi)^2$$

$$= A_1^2 + A_2^2 + 2A_1 A_2 \cos \phi \tag{5}$$

Dividing Eq. (4) by (3), we have

$$\tan \theta = \frac{A_2 \sin \phi}{A_1 + A_2 \cos \phi} \tag{6}$$

For $A_1 = A_2 = A$, we have

$$\tan\theta = \frac{\sin\phi}{1+\cos\phi} = \tan\phi/2 \qquad \text{or} \qquad \theta = \phi/2 \tag{7}$$

(i) For $\phi = 2m\pi$, where m is an integer, we have

$$\tan\theta = \frac{\sin 2m\pi}{1+\cos 2m\pi} = 0 \qquad \text{or} \qquad \theta = 0$$

(ii) For $\phi = (2m+1)\pi$, where m is an integer, we have

$$\tan\theta = \frac{\sin(2m+1)\pi}{1+\cos(2m+1)\pi} = \frac{0}{0}$$

It shows that the value cannot be determined. For dealing with such situation (differentiating numerator and denominator of Eq. 7 separately), we have

$$\tan\theta = \frac{\cos\phi}{-\sin\phi} = \infty \qquad \text{or} \qquad \theta = \pi/2$$

(iii) For $\phi = (4m+1)\pi/2$, where m is an integer, we have

$$\tan\theta = \frac{\sin[(4m+1)\pi/2]}{1+\cos[(4m+1)\pi/2]} = 1 \qquad \text{or} \qquad \theta = \pi/4$$

(iv) For $\phi = (4m-1)\pi/2$, where m is an integer, we have

$$\tan\theta = \frac{\sin[(4m-1)\pi/2]}{1+\cos[(4m-1)\pi/2]} = -1 \qquad \text{or} \qquad \theta = 3\pi/4$$

5.1 Intensity of Resultant Wave

We know that the intensity of a wave is proportional to square of its amplitude. Thus, the intensity of the interference pattern is given by

$$I = A'^2 = A_1^2 + A_2^2 + 2A_1A_2\,\cos\phi$$

5.2 Condition for the Constructive Interference

Equation (5) shows that the intensity I is the maximum at the points where the value of $\cos\phi$ is +1, i.e., the phase angle ϕ is $2m\pi$, where m is an integer. The maximum intensity is given by

$$I_{max} = (A_1 + A_2)^2$$

For $A_1 = A_2 = A$, we have $I_{max} = 4A^2$.

5.3 Condition for the Destructive Interference

Equation (5) shows that the intensity I is the minimum at the points where the value of $\cos\phi$ is -1, i.e., the phase angle ϕ is $(2m+1)\pi$, where m is an integer. The minimum intensity is

$$I_{min} = (A_1 - A_2)^2$$

For $A_1 = A_2 = A$, we have $I_{min} = 0$.

Exercise 1 Two monochromatic light sources having equal frequency and the intensity ratio 16 : 1 are producing the interference fringes. Calculate the ratio of intensities of the maximum and minimum in the fringe pattern.

Solution Suppose, the amplitudes of two waves participating in the interference are A_1 and A_2. Their intensities, respectively, are $I_1 = A_1^2$ and $I_2 = A_2^2$. We have $I_1 : I_2 = 16 : 1$. Thus, we have $A_1^2 : A_2^2 = 16 : 1$, and therefore

$$\frac{A_1^2}{A_2^2} = \frac{16}{1} \qquad \text{or} \qquad A_1 = 4A_2$$

In the interference system, the ratio of the maximum intensity I_{max} and the minimum intensity I_{min} is given by

$$\frac{I_{max}}{I_{min}} = \frac{(A_1 + A_2)^2}{(A_1 - A_2)^2}$$

Using $A_1 = 4A_2$, we get

$$\frac{I_{max}}{I_{min}} = \frac{(4A_2 + A_2)^2}{(4A_2 - A_2)^2} = \frac{25}{9}$$

Thus, we have $I_{max} : I_{min} = 25 : 9$.

5.4 Average Energy for Interference

For the interference of two waves having equal frequency ω, constructive interference and destructive interference are obtained regularly. For the waves (1) and (2), the maximum and minimum intensities, respectively, are given by Eqs. (6) and (7). Thus, the average intensity is given by

$$I_{av} = \frac{I_{max} + I_{min}}{2}$$

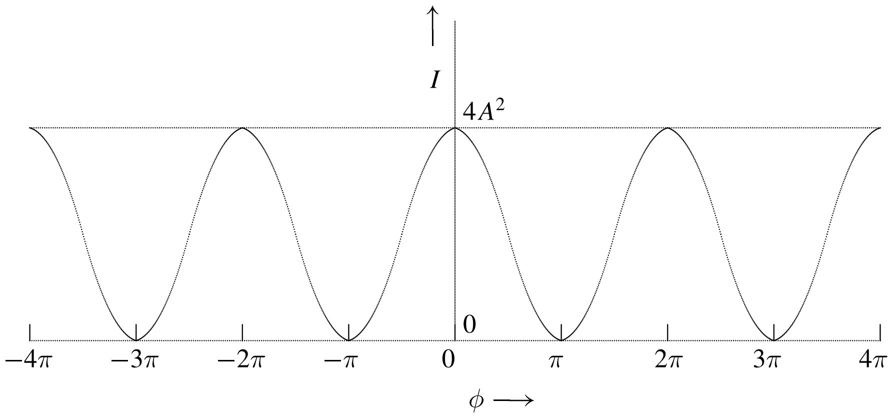
Using the expressions for I_{max} and I_{min}, we have

$$I_{av} = \frac{(A_1 + A_2)^2 + (A_1 - A_2)^2}{2} = A_1^2 + A_2^2$$

When $A_1 = A_2 = A$, we have $I_{av} = 2A^2$.

5.5 Variation of Intensity

For the case $A_1 = A_2 = A$, the variation of intensity I as a function of ϕ in the interference is shown in figure below. There is variation of intensity between 0 and $4A^2$.



5.6 Conditions for a Sustained Interference

We naturally want to have a sustained interference which does not vary with time. In order to have sustained interference, the following conditions should be satisfied:

(i) Both the sources participating in the interference should be monochromatic.
(ii) Both the sources should emit the waves having equal frequency (wavelength).

(iii) The waves from the sources should propagate along the same direction with equal speed, which is the speed of light.

(iv) The phase difference between the two waves should be constant and not varying with time.

(v) The two sources should be kept very close to each other. For large separation between the sources, clear maxima and minima do not appear in the spectrum pattern.

(vi) A reasonable distance between the sources and the screen should be kept, as the maxima and minima appear quite close when the distance is small. A large distance, however, reduces the intensity of the pattern.

(vii) To have distinct maxima and minima with good contrast, the amplitudes of the two interfering waves should be equal. Then the intensity of minima becomes zero that of the maxima becomes the maximum.

6 Fresnel's Biprism

In Young's double slit experiment, we have seen that the two coherent sources S_1 and S_2 were obtained from a common wavefront. Such situation is called as the division of wavefront. Another example of the division of wavefront is Fresnel's biprism. In the arrangement of Fresnel's biprism, we obtain two virtual coherent sources S_1 and S_2 of light. Fresnel's biprism is a combination of two acute-angle prisms which are joined with their bases in a manner that one angle becomes obtuse angle θ' of about 179° and the remaining two angles are acute angles, each of them of about 0.5°, as shown in Fig. 3. Here, the triangle abc represents the biprism and the angle abc is obtuse angle.

Let a monochromatic light of wavelength λ from a slit S be allowed to fall on the biprism placed at a small distance form S. For the light falling on the upper part of the biprism, it bends in the down direction and appears as it is coming from the source S_1. Similarly, for the light falling on the lower part of the biprism bends in the upward direction and appears as it is coming from the source S_2. These coherent sources S_1 and S_2 act as two virtual coherent sources of light (Fig. 3). In this situation on placing the screen XY on right side of the biprism, we obtain interference fringes (pattern) which are alternate bright and dark fringes.

6.1 Theory of Fringes

Let S_1 and S_2 be two virtual images, obtained from the source S, separated by a distance $2d$ from each other. The screen XY on which the interference fringes are obtained is separated by a distance D from the coherent sources S_1 and S_2, as shown in the following figure. From the point S, the perpendicular on the screen XY is the point C. The point C is obviously equidistant from both the S_1 and S_2. Thus, the path

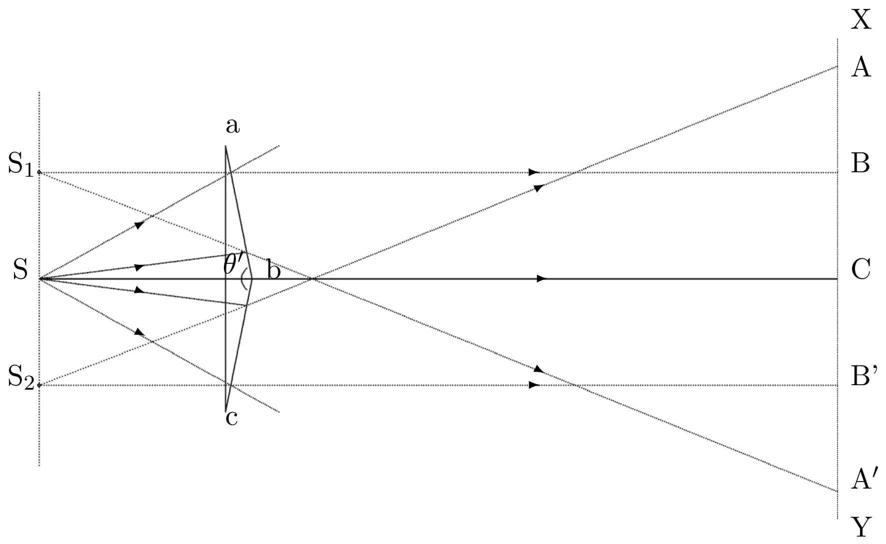
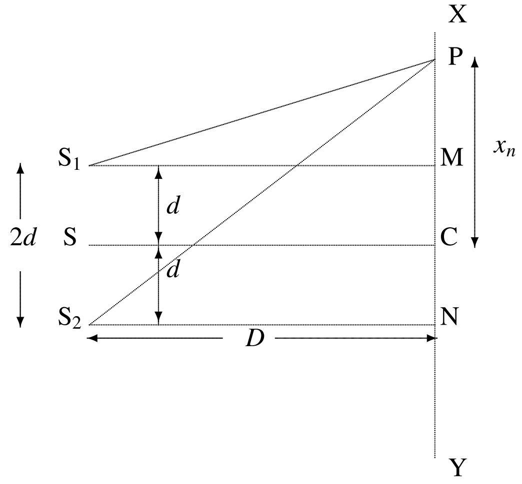


Fig. 3 Interference pattern with Fresnel's biprism

difference between the two waves from S_1 and S_2 reaching at C is zero. Hence, the point C is the center of a bright fringe. On both the sides of C on the screen XY, alternatively dark and bright fringes are produced.



We can draw perpendiculars S_1M and S_2N on the screen XY. Suppose, the distance of a point P on the screen from the point C be x_n. From the figure, we get

$$PM = x_n - d \qquad \text{and} \qquad PN = x_n + d$$

From the right-angled triangle S_1MP, we get

$$S_1P^2 = S_1M^2 + MP^2$$
$$= D^2 + (x_n - d)^2 = D^2\left[1 + \frac{(x_n - d)^2}{D^2}\right]$$

or

$$S_1P = D\left[1 + \frac{(x_n - d)^2}{D^2}\right]^{1/2}$$

As $(x_n - d) << D$, we get

$$S_1P = \left[D + \frac{(x_n - d)^2}{2D}\right]$$

The higher order terms, which are negligibly small in comparison to the first two terms, are neglected here. From the right-angled triangle S_2NP, we have

$$S_2P^2 = S_2N^2 + NP^2$$
$$= D^2 + (x_n + d)^2 = D^2\left[1 + \frac{(x_n + d)^2}{D^2}\right]$$

or

$$S_2P = D\left[1 + \frac{(x_n + d)^2}{D^2}\right]^{1/2}$$

As $(x_n + d) << D$, we get

$$S_2P = \left[D + \frac{(x_n + d)^2}{2D}\right]$$

Here, the higher order terms, which are negligibly small in comparison to the first two terms, are neglected. The path difference between the two waves from S_1 and S_2 reaching at P is

$$\Delta = S_2P - S_1P$$
$$= \left[D + \frac{(x_n + d)^2}{2D}\right] - \left[D + \frac{(x_n - d)^2}{2D}\right] = \frac{2x_nd}{D}$$

(i) Condition for the Bright Fringes

For constructive interference between the two coherent waves from S_1 and S_2 reaching at P, the path difference of the waves should be integral multiple of the wavelength λ of the wave. Thus, we have

$$\frac{2x_n d}{D} = n\lambda \qquad \text{or} \qquad x_n = \frac{n\lambda D}{2d}$$

where n is an integer. Here x_n is the distance of the nth bright fringe from the point C. The distance of the next $(n+1)$th bright fringe from the point C is given by

$$x_{n+1} = \frac{(n+1)\lambda D}{2d}$$

Hence, the separation between the two successive bright fringes, called the fringe width, is given by

$$\beta = x_{n+1} - x_n = \frac{(n+1)\lambda D}{2d} - \frac{n\lambda D}{2d} = \frac{\lambda D}{2d}$$

(ii) Condition for the Dark Fringes

For destructive interference between the two coherent waves reaching at the point P the path difference of the waves should be half-integral multiple of the wavelength λ of the wave. Thus, we have

$$\frac{2x_n d}{D} = \frac{(2n+1)}{2}\lambda \qquad \text{or} \qquad x_n = \frac{(2n+1)\lambda D}{4d}$$

where n is an integer. Here x_n is the distance of the nth dark fringe from the point C. The distance of the next $(n+1)$th dark fringe from the point C is given by

$$x_{n+1} = \frac{(2n+3)\lambda D}{4d}$$

Thus, the separation between two successive dark fringes, called the fringe width, is given by

$$\beta = x_{n+1} - x_n = \frac{(2n+3)\lambda D}{4d} - \frac{(2n+1)\lambda D}{4d} = \frac{\lambda D}{2d}$$

It shows that the fringe width for both the dark and bright fringes is equal.

Exercise 2 In a biprism experiment, the separation between two virtual sources producing the interference is 0.12 mm and the fringe width is 6.3 mm. When the distance between the screen and the monochromatic source is 1.1 m, calculate the wavelength of light used in the experiment.

Solution We have fringe width $\beta = 6.3 \times 10^{-3}$ m; separation between two virtual sources $2d = 0.12 \times 10^{-3}$ m; distance between the source and the screen $D = 1.1$ m. The wavelength of light used is given by

$$\lambda = \frac{2d\beta}{D} = \frac{0.12 \times 10^{-3} \times 6.3 \times 10^{-3}}{1.1} = 6.873 \times 10^{-7}\ \text{m}$$

The wavelength of light used in the experiment is $\lambda = 6873$ Å.

Exercise 3 In a biprism experiment, the eye was placed at a distance of 1.3 m from the monochromatic source. Calculate the wavelength of light, when the eye is required to move through a distance of 2.1 cm for 20 fringes. The distance between the two virtual sources is 0.5 mm.

Solution We have $x_n = 2.1 \times 10^{-2}$ m for $n = 20$; separation between the virtual sources $2d = 0.5 \times 10^{-3}$ m; distance between the screen (eye) and the source $D = 1.3$ m. The fringe width is given by

$$\beta = \frac{x_n}{n} = \frac{2.1 \times 10^{-2}}{20} = 1.05 \times 10^{-3}\ \text{m}$$

Thus, the wavelength of light used in the experiment is given by

$$\lambda = \frac{2d\beta}{D} = \frac{0.5 \times 10^{-3} \times 1.05 \times 10^{-3}}{1.3} = 4.038 \times 10^{-7}\ \text{m}$$

Thus, the wavelength of light used in the experiment is $\lambda = 4038$ Å.

Exercise 4 In a biprism experiment, when a light of wavelength 5890 Å is used, one finds 40 fringes in the field of view. Calculate the number of fringes in the filed of view when light of 4450 Å is used.

Solution In the biprism experiment, the fringe width is given by

$$\beta = \frac{\lambda D}{2d}$$

When in the field of view of width x, we observe N fringes then we have

$$x = N\beta = \frac{N\lambda D}{2d}$$

We have $N_1 = 40$ when $\lambda_1 = 5890$ Å. For the given field of view, we want to find the number of fringes N_2 when $\lambda_2 = 4450$ Å. Thus, we have

$$\frac{N_1\lambda_1 D}{2d} = \frac{N_2\lambda_2 D}{2d} \qquad \text{or} \qquad N_1\lambda_1 = N_2\lambda_2$$

On substituting the values, we have

$$N_2 = \frac{N_1\lambda_1}{\lambda_2} = \frac{40 \times 5890}{4450} = 52.9$$

Thus, 53 fringes are seen in the field of view when light of 4450 Å is used in the experiment.

7 Interference in a Uniform Thin Film

Consider a uniform transparent thin film of thickness t and refractive index $\mu > 1$ as shown in Fig. 4. The upper surface of the film is PQ and the lower surface is RS. A wave of light AB is incident on the upper surface PQ of the film at an angle i. At the point B, the wave is partly reflected along BC and partly refracted along BF. The angle of refraction at B is r. Inside the film, the wave moves along BF and incident at the lower surface RS at the point F. At F, the angle of incidence is r. The wave at F is partly reflected along FD and partly refracted along FK. The angle of refraction at F is i.

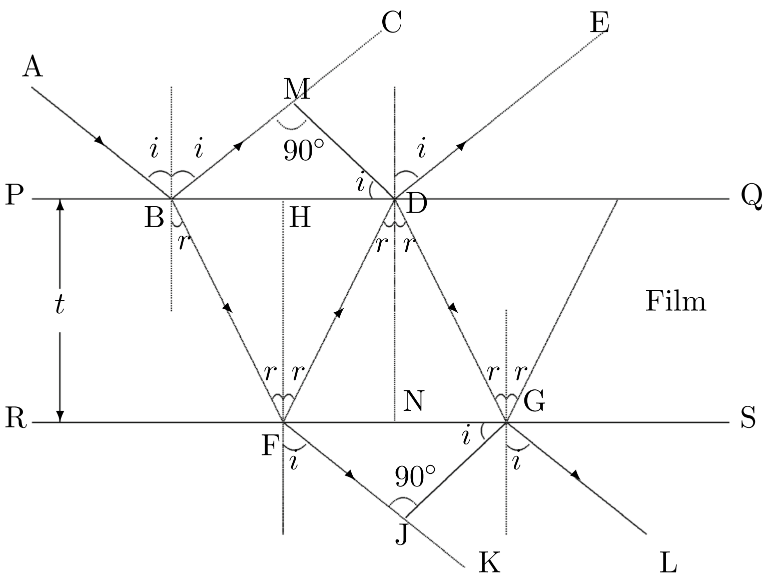


Fig. 4 Interference in a uniform thin film

The phenomena of reflection and refraction at the surfaces PQ and RS go on. Here, the interference occurs between the waves BC and DE (above the film), and between the waves FK and GL (below the film). The waves going in the upward direction from the surface PQ are generally known as the reflected waves, and the waves going in the downward direction from the surface RS are generally known as the transmitted waves. Draw the perpendicular DM from the point D on BC and perpendicular GJ from the point G on FK.

7.1 *Interference Between the Reflected Waves*

For the reflected waves BC and DE, beyond the line MD, both the waves travel equal distance. The path difference between the two waves is introduced during the journey from B to D through F and from B to M. Thus, the path difference is given by

$$\Delta = (\mathrm{BF} + \mathrm{FD})_{\text{in film}} - (\mathrm{BM})_{\text{in air}}$$

$$= \mu(\mathrm{BF} + \mathrm{FD}) - \mathrm{BM}$$

As BF = FD, we get

$$\Delta = 2\mu\,\mathrm{BF} - \mathrm{BM} \tag{8}$$

In the triangle BFH, we get

$$\cos r = \frac{t}{\mathrm{BF}} \qquad \text{or} \qquad \mathrm{BF} = \frac{t}{\cos r} \tag{9}$$

$$\tan r = \frac{\mathrm{BH}}{t} \qquad \text{or} \qquad \mathrm{BH} = t\tan r$$

and

$$\mathrm{BD} = 2 \times \mathrm{BH} = 2t\tan r \tag{10}$$

In the triangle BMD, we have

$$\sin i = \frac{\mathrm{BM}}{\mathrm{BD}} \qquad \text{or} \qquad \mathrm{BM} = \mathrm{BD}\ \sin i$$

Using here the value of BD, we get

$$\mathrm{BM} = 2t\tan r\ \sin i \tag{11}$$

From Eqs. (8), (9), (10), and (11), we have

$$\Delta = 2\mu \frac{t}{\cos r} - 2t \tan r \ \sin i \tag{12}$$

Now, we know

$$\mu = \frac{\sin i}{\sin r} \qquad \text{or} \qquad \sin i = \mu \ \sin r \tag{13}$$

Using Eq. (13) in (12), we get

$$\Delta = \frac{2\mu t}{\cos r} - 2t \frac{\sin r}{\cos r} \mu \ \sin r = \frac{2\mu t}{\cos r} (1 - \sin^2 r) = 2\mu t \ \cos r \tag{14}$$

Equation (14) represents only the apparent path difference and it does not represent the effective path difference. When the light is reflected from a surface of an optically denser medium, a phase change which is equivalent to the path difference of $\lambda/2$ is introduced. Thus, the total path difference is given by

$$\Delta = 2\mu t \ \cos r + \lambda/2$$

Thus, the path difference depends on the thickness t of the film. It should be noted that the interference pattern is not perfect as the intensities of the rays BC and DE are not the same and their amplitudes are different.

Condition for Maxima

For a maximum at a given point, the two waves should arrive at the point in phase. That is, the path difference should be integer multiple of the wavelength λ of the wave. Thus, we have

$$\Delta = n\lambda$$

where n is an integer. Thus, we have

$$2\mu t \ \cos r + \lambda/2 = n\lambda \qquad \text{or} \qquad 2\mu t \ \cos r = (2n - 1)\lambda/2$$

Condition for Minima

For a minimum at a given point, the two waves should arrive completely out of phase at the point. That is, the path difference should be half-integer multiple of the wavelength λ of the wave. Thus, we have

$$\Delta = (n + 1/2)\lambda$$

where n in an integer. Thus, we have

$$2\mu t\ \cos r + \lambda/2 = (2n+1)\lambda/2 \qquad \text{or} \qquad 2\mu t\ \cos r = n\lambda$$

7.2 Interference Between the Transmitted Waves

For the transmitted waves FK and GL, beyond the line GJ, both the waves travel equal distance. The path difference between the two waves is introduced during the journey from F to G through D and from F to J. Thus, the path difference is given by

$$\Delta = (\text{FD} + \text{DG})_{\text{in film}} - (\text{FJ})_{\text{in air}}$$

$$= \mu(\text{FD} + \text{DG}) - \text{FJ}$$

As FD = DG, we get

$$\Delta = 2\mu\text{FD} - \text{FJ} \tag{15}$$

In the triangle FDN, we get

$$\cos r = \frac{t}{\text{FD}} \qquad \text{or} \qquad \text{FD} = \frac{t}{\cos r} \tag{16}$$

$$\tan r = \frac{\text{FN}}{t} \qquad \text{or} \qquad \text{FN} = t\tan r$$

and

$$\text{FG} = 2\times\text{FN} = 2t\tan r \tag{17}$$

In the triangle FJG, we have

$$\sin i = \frac{\text{FJ}}{\text{FG}} \qquad \text{or} \qquad \text{FJ} = \text{FG}\ \sin i$$

Using here the value of FG, we get

$$\text{FJ} = 2t\tan r\ \sin i \tag{18}$$

From Eqs. (15), (16), (17), and (18), we have

$$\Delta = 2\mu\frac{t}{\cos r} - 2t\tan r\ \sin i \tag{19}$$

Now, we know

$$\mu = \frac{\sin i}{\sin r} \qquad \text{or} \qquad \sin i = \mu \ \sin r \tag{20}$$

Using Eq. (20) in (19), we have

$$\Delta = \frac{2\mu t}{\cos r} - 2t \ \frac{\sin r}{\cos r} \ \mu \ \sin r = \frac{2\mu t}{\cos r} \ (1 - \sin^2 r) = 2\mu t \ \cos r \tag{21}$$

Equation (21) represents the effective path difference. For reflection at the rarer medium, additional path difference is not introduced.

Condition for Maxima

For a maximum at a given point, the two waves should arrive at the point in phase. That is, the path difference between the waves should be integer multiple of the wavelength λ of the wave. Thus, we have

$$\Delta = n\lambda$$

where n is an integer. Thus, we have

$$2\mu t \ \cos r = n\lambda$$

Condition for Minima

For a minimum at a given point, the two waves should arrive at the point completely out of phase. That is, the path difference between the waves should be half-integer multiple of the wavelength λ of the wave. Thus, we have

$$\Delta = (n + 1/2)\lambda$$

where n is an integer. Thus, we have

$$2\mu t \ \cos r = (2n + 1)\lambda/2$$

Again, the interference pattern is not perfect as the intensities of the waves FK and GL are not the same and their amplitudes are different. It should be noted that the conditions for the interference between the reflected waves are opposite to those for the interference between transmitted waves. It shows that the interference patterns between the reflected waves and between the transmitted waves compliment with each other.

Exercise 5 Light of 5893 Å is reflected at normal incidence from a soap film whose refractive index is $\mu = 1.4$. Calculate the least thickness of the film that appears (i) bright and (ii) dark.

Solution For the normal incidence on a thin film, we have $i = 0$ and $r = 0$. For observation of a bright fringe, we have

$$2\mu t = (2n - 1)\lambda/2 \qquad \text{or} \qquad t = \frac{(2n-1)\lambda}{4\mu}$$

The minimum thickness for a bright fringe is given by

$$t = \frac{\lambda}{4\mu} = \frac{5893 \times 10^{-10}}{4 \times 1.4} = 1.05 \times 10^{-7}\ \text{m}$$

The minimum thickness of film to have a bright fringe is 1.05×10^{-7} m. For the normal incidence on a thin film, for observation of a dark fringe, we have

$$2\mu t = n\lambda \qquad \text{or} \qquad t = \frac{n\lambda}{2\mu}$$

The minimum thickness for a dark fringe is given by

$$t = \frac{\lambda}{2\mu} = \frac{5893 \times 10^{-10}}{2 \times 1.4} = 2.1 \times 10^{-7}\ \text{m}$$

The minimum thickness of film to have a dark fringe is 2.1×10^{-7} m.

Exercise 6 Calculate the minimum thickness of a soap film (refractive index $\mu = 1.46$) that results in a constructive interference in the reflected light when the film is illuminated normally with a light having wavelength of 5893 Å.

Solution We have $\lambda = 5893 \times 10^{-10}$ m and $\mu = 1.46$. For the normal incidence, the thickness of soap film for the constructive interference is given by

$$t = \frac{(2n-1)\lambda}{4\mu}$$

The smallest thickness of the soap film for the constructive interference is

$$t = \frac{\lambda}{4\mu} = \frac{5893 \times 10^{-10}}{4 \times 1.46} = 1.0 \times 10^{-7}\ \text{m}$$

Exercise 7 A wave of light having wavelength 5893 Å incidents on a thin glass plate (refractive index $\mu = 1.5$) such that the angle of refraction in the plate is 60°.

Calculate the smallest thickness of glass plate which appears dark in the reflected light.

Solution We have $\lambda = 5893 \times 10^{-10}$ m, $\mu = 1.5$ and the angle of refraction $r = 60°$. The thickness of the soap film for the destructive interference is given by

$$2\mu t \cos r = n\lambda \qquad \text{or} \qquad t = \frac{n\lambda}{2\mu \cos r}$$

The smallest thickness of the glass plate is given by

$$t = \frac{\lambda}{2\mu \cos r} = \frac{5893 \times 10^{-10}}{2 \times 1.5 \times \cos 60°} = 3.93 \times 10^{-7} \text{ m}$$

8 Interference in a Wedge-Shaped Thin Film

Let us consider a thin film bounded by two plane surfaces PQ and PR, inclined at an angle θ (Fig. 5). The thickness of the film obviously increases from left to right. A wave of monochromatic light AB is incident on the upper surface PR. At the point B, the light is partly reflected along BK and partly refracted along BC. The angle of refraction at B is r. Inside the film, the wave moves along BC. In the triangle PBT, the $\angle$ PBT is 90°, we therefore have $\angle$ PTB $= 90 - \theta$. Now, the $\angle$ CTM $= 90 - \theta$.

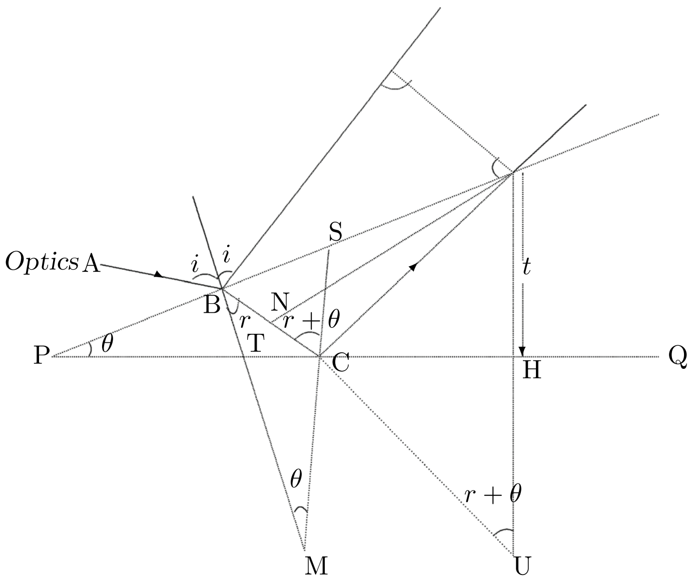


Fig. 5 Interference in a wedge-shaped thin film

In the triangle TCM, the angle TMC = θ. The angle PCS is therefore $r + \theta$. At the point C, the angle of incidence is $r + \theta$ and the light is partly reflected along CD and is partly refracted through the surface PQ. At the point D, the light is partly reflected back into the film and partly refracted along DL.

From the point D, draw the line DH perpendicular on the line PQ. The lines BC and DH intersect at the point U. As the lines SM and DU are parallel to each other, the angle CUH $= r + \theta$. From the point D, draw the line DE perpendicular on the line BK. From the geometry of the figure, angles at various points are shown. The waves going in the upward direction from the surface PR are the reflected waves, and those going in the downward direction from the surface PQ are the transmitted waves.

8.1 Interference Between the Reflected Waves

For the two reflected waves BK and DL, beyond the line DE, both the waves travel equal distance. The path difference between the two waves is introduced during the journey from B to D through C and from B to E. Thus, the path difference is given by

$$\Delta = (\mathrm{BC} + \mathrm{CD})_{\text{in film}} - (\mathrm{BE})_{\text{in air}}$$

$$= \mu(\mathrm{BC} + \mathrm{CD}) - \mathrm{BE}$$

As CD = CU, we have

$$\Delta = \mu(\mathrm{BC} + \mathrm{CU}) - \mathrm{BE} = \mu\mathrm{BU} - \mathrm{BE} \tag{22}$$

In the right-angled triangle BED, we have

$$\sin i = \frac{\mathrm{BE}}{\mathrm{BD}} \tag{23}$$

In the right-angled triangle BND, we have

$$\sin r = \frac{\mathrm{BN}}{\mathrm{BD}} \tag{24}$$

From Eqs. (23) and (24), we get

$$\mu = \frac{\sin i}{\sin r} = \frac{\mathrm{BE}}{\mathrm{BN}} \qquad \text{or} \qquad \mathrm{BE} = \mu\mathrm{BN} \tag{25}$$

Using Eq. (25) in (22), we have

$$\Delta = \mu\mathrm{BU} - \mu\mathrm{BN} = \mu\mathrm{NU} \tag{26}$$

We have

$$\mathrm{DU} = \mathrm{DH} + \mathrm{HU} = t + t = 2t$$

In the right-angled triangle DNU, we have

$$\cos(r+\theta) = \frac{\mathrm{NU}}{\mathrm{DU}} = \frac{\mathrm{NU}}{2t} \qquad \text{or} \qquad \mathrm{NU} = 2t\cos(r+\theta) \tag{27}$$

Using Eq. (27) in (26), we have

$$\Delta = 2\mu t\cos(r+\theta) \tag{28}$$

Equation (28) represents only the apparent path difference and does not represent the effective path difference. For reflection of light from a surface of an optically denser medium, a phase change equivalent to the path difference of $\lambda/2$ is introduced. Thus, the total path difference is given by

$$\Delta = 2\mu t\cos(r+\theta) + \lambda/2$$

Thus, the path difference depends on the thickness t of the film and the angle θ. It should be noted that the interference pattern is not perfect as the intensities of the waves BC and DE are not the same and their amplitudes are different. At $t = 0$, the path difference is

$$\Delta = \lambda/2$$

It shows that at $t = 0$, there is a dark fringe.

Condition for Maxima

For a maximum at a given point, the two waves should arrive at the point in phase. Thus, the path difference should be integer multiple of the wavelength λ of the wave. Therefore, we have

$$\Delta = n\lambda$$

where n is an integer. Thus, we have

$$2\mu t\cos(r+\theta) + \lambda/2 = n\lambda \qquad \text{or} \qquad 2\mu t\cos(r+\theta) = (2n-1)\lambda/2$$

Condition for Minima

For a minimum at a given point, the two waves should arrive at the point completely out of phase. Thus, the path difference should be half-integer multiple of the wavelength λ of the wave. Therefore, we have

$$\Delta = (n + 1/2)\lambda$$

where n is an integer. Thus, we get

$$2\mu t \cos(r + \theta) + \lambda/2 = (2n + 1)\lambda/2 \qquad \text{or} \qquad 2\mu t \cos(r + \theta) = n\lambda$$

8.2 *Normal Incidence*

For the normal incidence, we have $i = 0$ and therefore $r = 0$. The condition for a maxima is

$$2\mu t \cos\theta = (2n - 1)\lambda/2$$

and the condition for a minima is

$$2\mu t \cos\theta = n\lambda$$

Further, when the angle θ of the film is small, we have $\cos\theta \approx 1$. Now, the condition for maxima is

$$2\mu t = (2n - 1)\lambda/2$$

and the condition for a minima is

$$2\mu t = n\lambda$$

For nth and $(n + 1)$th bright fringes formed at the thicknesses t_1 and t_2, respectively, we have

$$2\mu t_1 \cos\theta = (2n - 1)\lambda/2 \qquad \text{or} \qquad 2\mu t_2 \cos\theta = (2n + 1)\lambda/2$$

Thus, we get

$$2\mu(t_2 - t_1) \cos\theta = \lambda$$

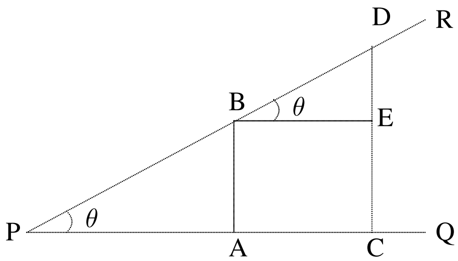
Similarly, for nth and $(n+1)$th dark fringes formed at the thicknesses t_1' and t_2', respectively, we have

$$2\mu t_1' \cos\theta = n\lambda \qquad \text{or} \qquad 2\mu t_2' \cos\theta = (n+1)\lambda$$

Thus, we have

$$2\mu(t_2' - t_1') \cos\theta = \lambda$$

Let us try to understand the situation with the help of the following figure where AB and CD are positions of two successive bright (or dark) fringes. Thus, AC is the separation between the two fringes and DE is $(t_2 - t_1)$ [or $(t_2' - t_1')$].



In the triangle BED, we have

$$\tan\theta = \frac{\text{DE}}{\text{BE}} \qquad \text{or} \qquad \text{DE} = \text{BE}\ \tan\theta$$

Using this relation in the condition of maxima (or minima), we get

$$2\mu\,\text{BE}\ \tan\theta\ \cos\theta = \lambda \qquad \text{or} \qquad 2\mu\,\text{BE}\ \sin\theta = \lambda$$

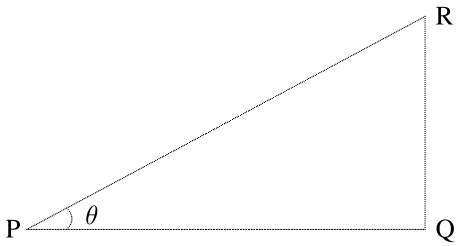
For small angle θ, we have $\sin\theta \approx \theta$ and therefore

$$\text{BE} = \frac{\lambda}{2\mu\theta}$$

It shows that the dependence of the separation between two successive bright (or dark) fringes on λ, μ, and θ.

Exercise 8 Two plane glass surfaces in contact at one edge are separated at the opposite edge with the help of a thin wire. For the normal incidence of sodium light of wavelength $\lambda = 5893$ Å one finds 20 fringes between these edges. Calculate the thickness of the wire.

Solution Two plane glass surfaces are PQ and PR. Here, QR represents the wire. Thus, the width of 20 fringes is equal to the length PQ. We have the wavelength of light used $\lambda = 5893$ Å and $\mu = 1$.



For the normal incidence, the width w of a fringe is given by

$$w = \frac{\lambda}{2\mu\theta}$$

Now, PQ = 20 w. For small angle θ, we have

$$\theta = \tan\theta = \frac{\text{QR}}{\text{PQ}}$$

Therefore,

$$\text{QR} = \text{PQ}\,\theta = 20\;w\theta$$

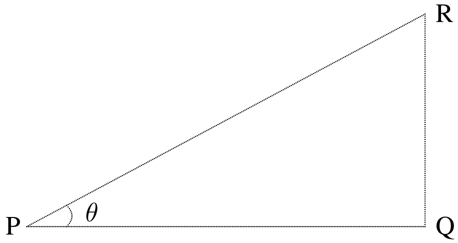
Using the expression for w, we have

$$\text{QR} = 20\theta\,\frac{\lambda}{2\theta} = 10\lambda = 10 \times 5893 \times 10^{-10} = 5.893 \times 10^{-6}\ \text{m}$$

The thickness of wire is 5.893×10^{-6} m.

Exercise 9 Between two glass plates, a wedge-shaped film is formed, where the plates are in contact along one edge and are separated at the other edge using a wire of diameter 6×10^{-6} m at a distance of 15 cm from the edge. For the normal incidence of light of wavelength $\lambda = 6000$ Å, calculate the fringe width of interference pattern formed in the reflected light.

Solution In the wedge-shaped film shown below we are given, QR $= 6 \times 10^{-6}$ m and PQ $= 15$ cm $= 15 \times 10^{-2}$ m, $\lambda = 6000$ Å and $\mu = 1$.



For the normal incidence, the width w of a fringe is given by

$$w = \frac{\lambda}{2\theta}$$

For small value of angle θ, we have

$$\theta = \tan\theta = \frac{\text{QR}}{\text{PQ}} = \frac{6 \times 10^{-6}}{15 \times 10^{-2}} = 4 \times 10^{-5}$$

Thus, we get

$$w = \frac{6000 \times 10^{-10}}{2 \times 4 \times 10^{-5}} = 7.5 \times 10^{-3}\ \text{m}$$

The fringe width obtained is 7.5×10^{-3} m.

Exercise 10 A wedge-shaped film having angle 60 arc sec is incident normally by a monochromatic light of wavelength $\lambda = 6000$ Å. Calculate the fringe width due to the interference between the reflected waves.

Solution For the air, $\mu = 1$ and the fringe width w is given by

$$w = \frac{\lambda}{2\theta}$$

Now, we have

$$\theta = 60'' = \frac{60}{3600}\,\frac{\pi}{180} = 2.9 \times 10^{-4}\ \text{rad}$$

Thus, we get

$$w = \frac{6000 \times 10^{-10}}{2 \times 2.9 \times 10^{-4}} = 1.03 \times 10^{-3}\ \text{m}$$

The fringe width obtained is 1.03×10^{-3} m.

Exercise 11 Interference fringes are obtained for normal incidence of a monochromatic light of wavelength $\lambda = 6500$ Å on a wedge-shaped film of refractive index 1.5. If the separation between two successive fringes is 0.15 mm, calculate the angle of the film.

Solution We have wavelength $\lambda = 6500 \times 10^{-10}$ m, refractive index $\mu = 1.5$, and the fringe width $w = 0.15 \times 10^{-3}$ m. For the normal incident, the interference between the reflected waves, the fringe width is given by

$$w = \frac{\lambda}{2\mu\theta} \qquad \text{or} \qquad \theta = \frac{\lambda}{2\mu w}$$

Using the values, we get

$$\theta = \frac{\lambda}{2\mu w} = \frac{6500 \times 10^{-10}}{2 \times 1.5 \times 0.15 \times 10^{-3}} = 1.44 \times 10^{-3} \text{ rad}$$

$$= 1.44 \times 10^{-3} \times \frac{180}{\pi} = 0.0825^\circ = 4.95'$$

9 Newton's Rings

Consider a plano-convex lens placed in such a manner that its curved surface lies on a plane glass plate, so that an air film of gradually increasing thickness is formed between the two surfaces. When a monochromatic light is allowed to fall normally on the film and viewed with the help of microscope, as shown in Fig. 6, alternating dark and bright circular fringes are found. These fringes are formed due to the interference between the reflected waves from the top and bottom of the air film. These fringes are circular, because of the circular symmetry of the air film and the thickness of the film corresponding to each fringe is the same throughout the circle. These fringes are known as **Newton's rings**, as they were first investigated by Newton.

Light from a monochromatic source S, after passing through a convex lens L, moves as a horizontal wave of light. This light falls on a plane glass plate PQ inclined at an angle of 45° with the horizontal (as well as with the vertical). This plate reflects the light in the vertically downward direction, and falls normally on the air film formed between the surface of the plano-convex lens and the plane glass plate. The light reflected from the lower and upper surfaces of the air film produces interference fringes which are seen with the help of a microscope.

For the normal incidence, from the preceding section, the path difference between the two reflected waves is given by

$$\Delta = 2\mu t \cos\theta + \lambda/2$$

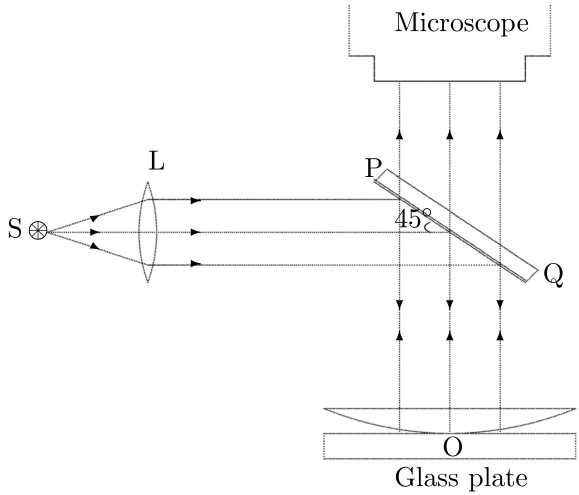


Fig. 6 Newton's ring arrangement

where μ denotes the refractive index of the medium between the plano-convex lens and the glass plate. In case of air between the plano-convex lens and the glass plate, the refractive index $\mu = 1$. For small angle of film, we have $\theta \approx 0$ and thus we have

$$\Delta = 2\mu t + \lambda/2$$

At the place where the thick $t = 0$, we have

$$\Delta = \lambda/2$$

It corresponds to the minimum intensity. Therefore, the central spot of the interference rings is dark.

Condition for the Maxima

For a maximum at a given point, the two waves should arrive at the point in phase. That is, the path difference should be integer multiple of the wavelength λ of the wave. Thus, we have

$$2\mu t + \lambda/2 = n\lambda \qquad \text{or} \qquad 2\mu t = (2n-1)\lambda/2$$

where n is an integer.

Condition for the Minima

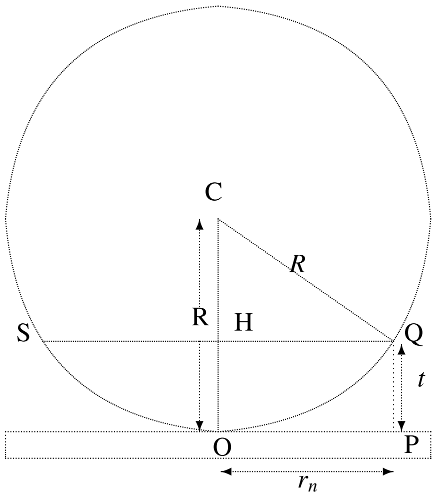
For a minimum at a given point, the two waves should arrive at the point completely out of phase. That is, the path difference should be half-integer multiple of the wavelength λ of the wave. Thus, we have

$$2\mu t + \lambda/2 = (2n + 1)\lambda/2 \qquad \text{or} \qquad 2\mu t = n\lambda$$

where n an assumed integer.

9.1 Diameter of Dark and Bright Rings

The following figure has a sphere of radius R out of which the plano-convex lens is formed. Here, SOQ is the plano-convex lens, placed on the plane glass plate, so that a wedge-shaped film is formed between the lens and the plate. Suppose at the point Q, the thickness of air film is t and the radius of the fringe is r_n.



We have

$$\text{CO} = \text{CQ} = R; \qquad \text{HO} = t; \qquad \text{CH} = \text{CO} - \text{HO} = R - t; \qquad \text{HQ} = r_n$$

Here, r_n is the radius of nth ring. In the right-angled triangle CHQ, we have

$$\text{CQ}^2 = \text{CH}^2 + \text{HQ}^2 \qquad \text{or} \qquad R^2 = (R - t)^2 + r_n^2$$

Thus, we get

$$r_n^2 = 2Rt - t^2 = t(2R - t)$$

Since the value of R is very large and t can be neglected in comparison to R. Then, we have

$$r_n^2 = 2Rt \qquad \text{or} \qquad D_n^2 = (2r_n)^2 = 8Rt$$

where D_n is the diameter of the ring.

(i) Bright Rings

For a bright ring, we have

$$2\mu t = (2n - 1)\lambda/2$$

and therefore

$$D_n^2 = 2R(2n - 1)\lambda/\mu \qquad \text{or} \qquad D_n = \sqrt{2R(2n - 1)\lambda/\mu}$$

(ii) Dark Rings

For a dark ring, we have

$$2\mu t = n\lambda$$

and therefore

$$D_n^2 = 4Rn\lambda/\mu \qquad \text{or} \qquad D_n = 2\sqrt{Rn\lambda/\mu}$$

9.2 Determination of Wavelength of Light

We know that the diameter D_n of the nth dark ring in Newton's ring method is

$$D_n^2 = 4Rn\lambda/\mu$$

Accordingly, the diameter D_{n+p} of the $(n + p)$th dark ring is given by

$$D_{n+p}^2 = 4R(n + p)\lambda/\mu$$

Thus, we get

$$D_{n+p}^2 - D_n^2 = 4Rp\lambda/\mu \qquad \text{or} \qquad \lambda = \frac{\mu(D_{n+p}^2 - D_n^2)}{4Rp}$$

Hence, the determination of the diameters of nth and $(n + p)$th dark rings along with the radius of curvature of plano-convex lens gives us the wavelength of monochromatic source.

Similarly, the expression for the wavelength λ can also be obtained for the bright rings as the following. We know that the diameter D_n of the nth bright ring in Newton's ring method is

$$D_n^2 = 2R(2n - 1)\lambda/\mu$$

Similarly, the diameter D_{n+p} of the $(n + p)$th bright ring is given by

$$D_{n+p}^2 = 2R(2n + 2p - 1)\lambda/\mu$$

From the above, we have

$$D_{n+p}^2 - D_n^2 = 4Rp\lambda/\mu \qquad \text{or} \qquad \lambda = \frac{\mu(D_{n+p}^2 - D_n^2)}{4Rp}$$

In this manner, the determination of the diameters of nth and $(n + p)$th bright rings along with the radius of curvature of plano-convex lens provides the wavelength of the monochromatic source.

9.3 Determination of Radius of Curvature of Plano-Convex Lens

The above derivation gives

$$R = \frac{\mu(D_{n+p}^2 - D_n^2)}{4p\lambda}$$

Hence, the determination of diameters of nth and $(n + p)$th dark rings along with the wavelength of monochromatic source used provides the radius of curvature of the plano-convex lens.

Exercise 12 In Newton's ring experiment, the diameters of 8th and 18th rings are, respectively, 0.36 and 0.60 mm. If the radius of the plano-convex lens is 100 cm, calculate the wavelength of the light used.

Solution We have $D_8 = 0.36 \times 10^{-3}$ m, $D_{18} = 0.60 \times 10^{-3}$ m, $R = 1$ m, and $p = 18 - 8 = 10$. The wavelength of the light used is given by

$$\lambda = \frac{D_{n+p}^2 - D_n^2}{4Rp} = \frac{(0.60 \times 10^{-3})^2 - (0.36 \times 10^{-3})^2}{4 \times 100 \times 10}$$

$$= 5.76 \times 10^{-7} \text{ m} = 5760 \text{ Å}$$

Exercise 13 Two light sources of wavelengths $\lambda_1 = 6000$ Å and $\lambda_2 = 4800$ Å are used in Newton's ring experiment. It is found that the nth dark ring of λ_1 coincides with $(n+1)$th dark ring of λ_2. If the radius of curvature of plano-convex lens is 0.96 m. Calculate the diameter of $(n+1)$th dark ring of λ_2.

Solution We have $\lambda_1 = 6000 \times 10^{-10}$ m, $\lambda_2 = 4800 \times 10^{-10}$ m and $R = 0.96$ m. The diameter of nth dark ring of λ_1 is given by

$$D_n^2 = 4n\lambda_1 R$$

and of $(n+1)$th dark ring of λ_2 is given by

$$D_{n+1}^2 = 4(n+1)\lambda_2 R$$

On equating these two diameters, we have

$$4n\lambda_1 R = 4(n+1)\lambda_2 R \qquad \text{or} \qquad \frac{(n+1)}{n} = \frac{\lambda_1}{\lambda_2}$$

Hence, we get

$$1 + \frac{1}{n} = \frac{\lambda_1}{\lambda_2} \qquad \text{or} \qquad n = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

For λ_1 and λ_2, we get

$$n = \frac{\lambda_2}{\lambda_1 - \lambda_2} = \frac{4800 \times 10^{-10}}{6000 \times 10^{-10} - 4800 \times 10^{-10}} = 4$$

Therefore, the fourth dark fringe of λ_1 coincides with the fifth dark fringe of λ_2. Hence, the diameter of fifth dark ring of λ_2 is given by

$$D_5^2 = 4 \times 5\ \lambda_2 R = 20 \times 4800 \times 10^{-10} \times 0.96 = 9.216 \times 10^{-6}$$

Therefore, we have $D_5 = \sqrt{9.216 \times 10^{-6}} = 3.04 \times 10^{-3}$ m.

Exercise 14 In Newton's ring experiment, a source is emitting two wavelengths $\lambda_1 = 6050$ Å and $\lambda_2 = 5940$ Å. There the nth dark ring of the first wavelength coincides with the $(n+1)$th dark ring of the second wavelength. Calculate the diameter of nth dark ring due to the first wavelength when the radius of curvature of the plano-convex lens is 90 cm.

Solution We have $\lambda_1 = 6050 \times 10^{-10}$ m, $\lambda_2 = 5940 \times 10^{-10}$ m, and $R = 0.9$m. The diameter of nth dark ring of λ_1 is given by

$$D_n^2 = 4n\lambda_1 R$$

and of $(n + 1)$th dark ring of λ_2 is given by

$$D_{n+1}^2 = 4(n + 1)\lambda_2 R$$

On equating these two diameters, we have

$$4n\lambda_1 R = 4(n + 1)\lambda_2 R\lambda \qquad \text{or} \qquad \frac{(n + 1)}{n} = \frac{\lambda_1}{\lambda_2}$$

Thus, we have

$$1 + \frac{1}{n} = \frac{\lambda_1}{\lambda_2} \qquad \text{or} \qquad n = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

Using λ_1 and λ_2, we get

$$n = \frac{\lambda_2}{\lambda_1 - \lambda_2} = \frac{5940 \times 10^{-10}}{6050 \times 10^{-10} - 5940 \times 10^{-10}} = 54$$

Hence, the diameter of the nth dark ring of λ_1 is given by

$$D_n^2 = 4n\lambda_1 R = 4 \times 54 \times 6050 \times 10^{-10} \times 0.90 = 1.176 \times 10^{-4} \text{ m}$$

Thus, we get $D_n = 1.084 \times 10^{-2}$ m.

Exercise 15 Newton's rings are obtained by using monochromatic light of wavelength 5893 Å. A liquid is placed between the plano-convex lens and the glass plate. If the diameter of 6th bright ring is 0.4 cm and the radius of curvature of the lens is 94 cm. Calculate the refractive index of the liquid.

Solution We have $\lambda = 5893 \times 10^{-10}$ m, $D_6 = 0.4 \times 10^{-2}$ m, $R = 0.94$ m, and $n = 6$. The diameter of nth bright ring of wavelength λ is given by

$$D_n^2 = \frac{2R(2n - 1)\lambda}{\mu} \qquad \text{or} \qquad \mu = \frac{2R(2n - 1)\lambda}{D_n^2}$$

Using the values, we have

$$\mu = \frac{2 \times 0.94(2 \times 6 - 1)5893 \times 10^{-10}}{(0.4 \times 10^{-2})^2} = 0.76$$

Exercise 16 Newton's rings are obtained for two different media between the plano-convex lens and the glass plate. The ratio of diameters of nth ring is 11 : 8. Calculate the ratio of the refractive indexes of the two media.

Solution Suppose, D'_n and D''_n are diameters of nth ring, say dark ring, when two different media of refractive indexes μ' and μ'', respectively, between the lens and the plate are introduced. Given, $D'_n : D''_n = 11 : 8$. Now, we have

$$D'^2_n = \frac{4Rn\lambda}{\mu'} \qquad \text{and} \qquad D''^2_n = \frac{4Rn\lambda}{\mu''}$$

Thus, we have

$$\frac{D'^2_n}{D''^2_n} = \frac{\mu''}{\mu'} \qquad \text{or} \qquad \frac{\mu'}{\mu''} = \frac{D''^2_n}{D'^2_n} = \frac{64}{121}$$

Hence, the ratio of the refractive indexes $\mu' : \mu''$ is 64 : 121.

10 Michelson's Interferometer

Michelson's interferometer has two highly polished mirrors M_1 and M_2 and two identical (same thickness and same material) plane glass plates PQ and RS placed parallel to each other (Fig. 7). The glass plate PQ is half-silvered on its back surface (right side) and is inclined at an angle of 45° to the wave of incident monochromatic light. The mirrors M_1 and M_2 are silvered on their front surfaces and mounted on two arms at right angle to each other. Position of M_1 can be changed with the help of a fine screw.

Monochromatic light from a source S, rendered parallel by a lens L, falls on the glass plate PQ. The semi-silvered plate PQ divides the incident wave into two parts of nearly equal intensities, called the reflected and transmitted waves. The reflected wave moves towards the mirror M_1 and incidents normally on it and therefore is reflected back to the plate PQ through the same path traveled in the reverse direction, and finally enters the telescope T. The transmitted wave moves towards the mirror M_2 and incidents normally on it and therefore is reflected back to the plate PQ through equal path traveled in the opposite direction. At the plate PQ, the wave is reflected towards the telescope T. As the waves reaching the telescope are obtained from the same incident wave, these two waves participate in the interference phenomenon.

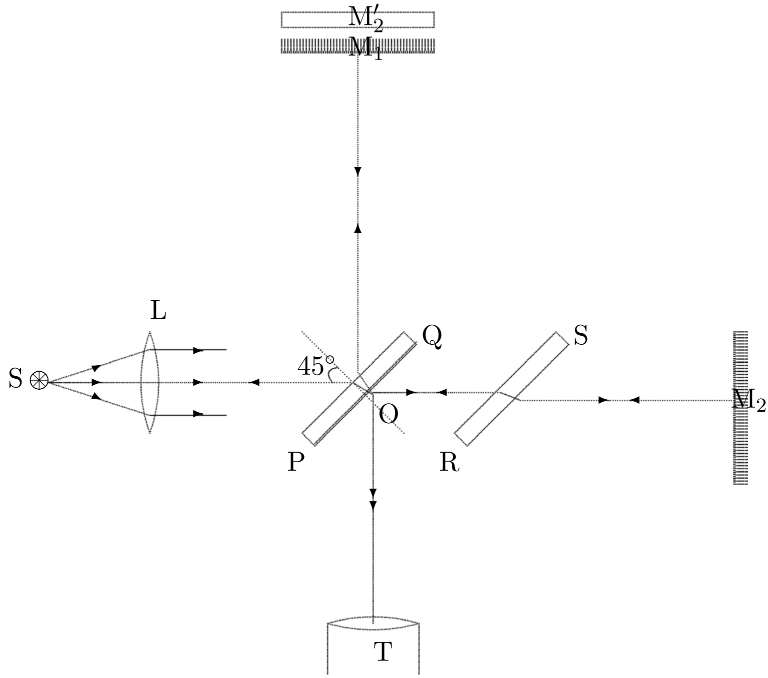


Fig. 7 Michelson's interferometer

For the following reason, an identical plate RS parallel to PQ is introduced. The wave going towards the mirror M_1 and reflected back passes through the plate PQ three times. After introduction of the plate RS, the transmitted wave also passes the plate PQ one time and plate RS two times. It neutralizes the phase changes in the paths of the reflected and transmitted waves due to the plates. Because of this reason, the plate RS is also known as the compensatory plate.

10.1 Kinds of Fringes

In Michelson's interferometer, the shape of fringes depends on the relative inclination of M_1 and M_2. Suppose, M_2' is the image of M_2 produced by the reflection at the half-silvered surface of plate PQ. The interference fringes may be considered as formed by the light reflected from the surfaces M_1 and M_2'. Thus, this arrangement is equivalent to an air film enclosed between the reflection surfaces M_1 and M_2'. It shows that the path difference between the two waves produced by the reflecting surfaces M_1 and M_2' is equal to the twice of the thickness of the film M_1M_2'. This path difference can be varied by moving M_1 forward or backward parallel to itself.

For the monochromatic light, the pattern of bright and dark fringes is formed. The shape of the fringes depends on the inclination of M_1 and M_2.

For the case of M_1 to be exactly perpendicular to M_2, the reflecting surfaces M_1 and M_2' are parallel and therefore the air film between M_1 and M_2' is of constant thickness t, so that we have circular fringes [Fig. 8d]. These fringes are known as Haidinger's fringes which can be seen in the field of view of a telescope. When the distance between the mirrors M_1 and M_2 (or between M_1 and M_2') is decreased, the circular fringes shrink and vanish at the center. A ring (fringe) disappears each time when the path difference $2t$ decreases by λ.

When M_1 is not perfectly perpendicular to M_2, a wedge-shaped film is formed between M_1 and M_2'. Then we get almost straight line fringes of equal thickness in the field of view of the telescope, as the radius of curvature of the fringes is very large.

As the vertical wave (wave towards M_1) first is reflected at the inner surface of PQ (internal reflection) and then from the front surface of M_1 (external reflection) a phase change of π radiation (equivalent to path difference of $\lambda/2$) is introduced. The horizontal wave (wave towards M_2) first is reflected from the front surface of M_2 (external reflection) and then at the inner surface of the glass plate PQ (external reflection), so there is no net phase change, as it is $\pi + \pi = 2\pi$ radiation (equivalent to the path difference of λ). Hence, the total path difference is given by

$$\Delta = 2t\cos\theta + \lambda/2$$

where θ is the angle between M_1 and M_2'. Thus, for a bright fringe, we have

$$\Delta = n\lambda \qquad \text{or} \qquad 2t\cos\theta = (n - 1/2)\lambda$$

and for a dark fringe, we have

$$\Delta = (n + 1/2)\lambda \qquad \text{or} \qquad 2t\cos\theta = n\lambda$$

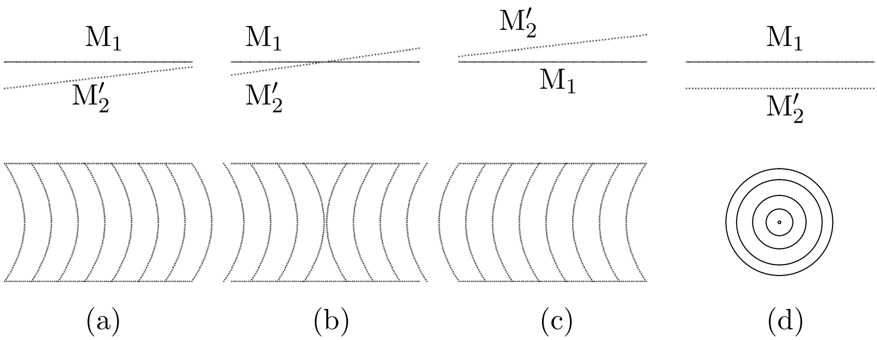


Fig. 8 Shape of fringes in Michelson's interferometer

Here, n is an integer. On decreasing t, a limit is attained when M_1 and M_2' coincide with each other, i.e., $t = 0$. At that place, the field of view is perfectly dark. On moving M_1 further, the fringes appear again. Various shapes of the fringes are shown in Fig. 8.

10.2 Determination of Wavelength of Light

For determination of wavelength of monochromatic light, first of all, Michelson's interferometer is set for circular fringes with the central bright spot, which is seen when M_1 is perfectly perpendicular to M_2. Suppose, t is the thickness of film formed between the two mirrors M_1 and M_2' and n the order of the spot obtained, we have

$$2t + \lambda/2 = n\lambda \qquad \text{or} \qquad 2t = (n - 1/2)\lambda$$

On moving M_1 through a distance $\lambda/2$ away from M_2', then an additional path difference of λ is introduced, and thus $(n + 1)$th bright spot appears at the center of the field of view. Thus, when each time M_1 is moved through a distance $\lambda/2$, a new bright spot appears at the center. Suppose, M_1 is moved from a position x_1 to x_2 and N new spots are moved at the center, then we have

$$x_2 - x_1 = N\lambda/2 \qquad \text{or} \qquad \lambda = \frac{2(x_2 - x_1)}{N}$$

The separation $(x_2 - x_1)$ and the number N of the spots appeared at the center are obtained. By using these values, we can determine the wavelength of monochromatic light.

10.3 Determination of Difference Between Two Wavelengths of Light

Michelson's interferometer can also be used for the determination of difference between two close wavelengths of light. For example, two wavelengths 5890 Å and 5896 Åare in the sodium lamp. Suppose, the source of light has two close wavelengths λ_1 and λ_2 such that $\lambda_1 > \lambda_2$. Each of these two wavelengths forms their separate fringe patterns. As λ_1 and λ_2 are very close to each other and thickness of the air film is small, the two patterns coincide with each other. On moving the mirror M_1 slowly, the two patterns separate slowly and when the thickness of the air film is such that the dark fringe of λ_1 coincides with the bright fringe of λ_2, the result is the maximum indistinctness. Now, the mirror M_1 is moved further, say through a distance x, so that the next indistinct position is obtained. In this position, if n fringes of λ_1 appear at the center, then $(n + 1)$ fringes of λ_2 should appear at the center. Thus, we have

$$x = n\,\frac{\lambda_1}{2} \qquad \text{and} \qquad x = (n+1)\,\frac{\lambda_2}{2}$$

We can write

$$n = \frac{2x}{\lambda_1} \qquad \text{and} \qquad (n+1) = \frac{2x}{\lambda_2}$$

On eliminating n from these two relations, we get

$$(n+1) - n = \frac{2x}{\lambda_2} - \frac{2x}{\lambda_1} \qquad \text{or} \qquad 1 = \frac{2x(\lambda_1 - \lambda_2)}{\lambda_1 \lambda_2}$$

Hence, we get

$$(\lambda_1 - \lambda_2) = \frac{\lambda^2}{2x}$$

where $\lambda = \sqrt{\lambda_1 \lambda_2}$ is the square root mean of λ_1 and λ_2.

Exercise 17 When a movable mirror of Michelson interferometer is moved through a distance of 0.07 mm, 260 fringes are found to cross the field of view. Calculate the wavelength of monochromatic light used.

Solution A fringe crosses the field of view when the movable mirror of the Michelson interferometer is moved through a distance $\lambda/2$. We have $x = 0.07 \times 10^{-3}$ m and $N = 260$. Hence, we get

$$x = N\lambda/2 \qquad \text{or} \qquad \lambda = \frac{2x}{N}$$

Using the values, we have

$$\lambda = \frac{2 \times 0.07 \times 10^{-3}}{260} = 5.385 \times 10^{-7}\ \text{m} = 5385\ \text{Å}$$

Exercise 18 Calculate the separation between the successive positions of movable mirror of Michelson interferometer giving the best fringes in case of sodium sources having the wavelengths 5890 and 5896 Å.

Solution We have $\lambda_1 = 5896\ \times 10^{-10}$ m, $\lambda_2 = 5890\ \times 10^{-10}$ m. When x is the separation between the successive positions, we have

$$\Delta\lambda = \lambda_1 - \lambda_2 = \frac{\lambda_1 \lambda_2}{2x} \qquad \text{or} \qquad x = \frac{\lambda_1 \lambda_2}{2(\lambda_1 - \lambda_2)}$$

Using the values, we have

$$x = \frac{5896 \times 10^{-10} \times 5890 \times 10^{-10}}{2(5896 \times 10^{-10} - 5890 \times 10^{-10})} = 2.894 \times 10^{-4}\ \mathrm{m}$$

11 Multiple Choice Questions

1. For two electromagnetic waves of wavelength λ, the phase difference of π radian between them is equivalent to the path difference
   A. 2λ B. $3\lambda/2$ C. λ D. $\lambda/2$
   Ans. D
2. For two electromagnetic waves of wavelength λ, the path difference of $\lambda/2$ between them is equivalent to the phase difference in radian
   A. $\pi/2$ B. π C. $3\pi/2$ D. 2π
   Ans. B
3. For two electromagnetic waves of wavelength λ, the path difference of $\lambda/2$ between them is equivalent to the phase difference in radian
   A. $\pi/2$ B. π C. $3\pi/2$ D. 2π
   Ans. B
4. For a Fresnel's biprism, $2d$ is the separation between two virtual sources, λ is the wavelength of monochromatic light used, and D is the distance of screen from the biprism. The fringe width is proportional to
   A. λ only B. D only C. λ and D both D. d
   Ans. C
5. For a Fresnel's biprism, $2d$ is the separation between two virtual sources, λ is the wavelength of monochromatic light used, and D is the distance of screen from the biprism. The fringe width is inversely proportional to
   A. λ only B. D only C. λ and D both D. d
   Ans. D
6. For a Fresnel's biprism, a monochromatic light of wavelength λ is used. For a constructive interference, the path difference between two interfering waves is (n is integer)
   A. $n\lambda$ B. $\frac{(2n+1)\lambda}{2}$ C. $\frac{(2n-1)\lambda}{2}$ D. $\frac{(2n+1)\lambda}{4}$
   Ans. A

7. For a Fresnel's biprism, a monochromatic light of wavelength λ is used. For a destructive interference, the path difference between two interfering waves is (n is integer)

   A. $n\lambda$ B. $\dfrac{(2n+1)\lambda}{2}$ C. $(2n-1)\lambda$ D. $\dfrac{(2n+1)\lambda}{4}$

   Ans. B
8. In Michelson's interferometer, the fringes are circular when the angle between the mirror M_1 and the image of M_2 is
   A. 0° B. 45° C. 90° D. 180°
   Ans. A
9. In Michelson's interferometer, the fringes are circular when the angle between the two mirrors is
   A. 0° B. 45° C. 90° D. 180°
   Ans. C

12 Problems and Questions

1. Explain in brief about the wave concept of light.
2. Describe about Young's double slit experiment.
3. Two simple harmonic waves having the same frequency but phase difference ϕ arrive at a common point. Show that the waves produce constructive interference when $\phi = 2m\pi$ and destructive interference when $\phi = (2m+1)\pi$. Here, m is a positive integer.
4. Two waves $y_1 = A \sin \omega t$ and $y_2 = A \sin(\omega t + \phi)$ produce an interference pattern. Discuss the variation of intensity as a function of ϕ in the pattern.
5. Describe the phenomenon of interference in Fresnel's biprism.
6. In the experiment of Fresnel's biprism, a monochromatic source of wavelength λ produces interference pattern on a screen placed at a distance D. Obtain an expression for the fringe width.
7. In the biprism experiment, the separation between two virtual sources producing interference is 0.1 mm and the fringe width is 6 mm. If the distance between the screen and the monochromatic source is 1 m, calculate the wavelength of light used in the experiment. [Ans. 6000 Å]
8. In the biprism experiment, the eye was placed at a distance of 1.2 m from the monochromatic source. Calculate the wavelength of light, if the eye is required to move through a distance of 2 cm for 16 fringes. The distance between two virtual sources is 0.6 mm.
9. In the biprism experiment, when wavelength of 6300 Å is used one finds 42 fringes are observed in the field of view. Calculate the number of fringes in the field of view when light of 4534 Å is used. [Ans. 58 fringes]

10. When a light wave of wavelength λ is reflected from a surface of an optically denser medium, how much phase difference is introduced? [Ans: $\lambda/2$]
11. What happens when a wave is reflected back form dense medium?
12. Discuss about interference between two waves reflected from a thin film of thickness t and refractive index $\mu > 1$.
13. Discuss about interference between two waves transmitted through a thin film of thickness t and refractive index $\mu > 1$.
14. Light of wavelength 5900 Å is reflected at normal incidence from a soap film of refractive index $\mu = 1.3$. Calculate the least thickness of film that appears (i) bright and (ii) dark.
15. Calculate the smallest thickness of a soap film (refractive index $\mu = 1.4$) that results in a constructive interference in the reflected light when the film is illuminated normally with light whose wavelength is 5800 Å.
16. A wave of light of wavelength 6000 Å incidents on a thin glass plate (refractive index $\mu = 1.4$) such that the angle of refraction in the plate is 60°. Calculate the smallest thickness of the glass plate which appears dark in the reflected light.
17. Discuss about interference between two waves reflected from a wedge- shaped thin film of refractive index $\mu > 1$.
18. A monochromatic wave of light incident normally on a wedge-shaped thin film of refractive index $\mu > 1$. Show that the maxima occur when
$$2\mu t \cos\theta = (2n - 1)\lambda/2$$
where n is an integer, θ is the angle between two surfaces of the film, and t is the thickness of film at that point.
19. A monochromatic wave of light incident normally on a wedge-shaped thin film of refractive index $\mu > 1$. Show that the minima occur when
$$2\mu t \cos\theta = n\lambda$$
where n is an integer, θ is the angle between two surfaces of the film, and t is the thickness of the film at that point.
20. Two plane glass surfaces in contact at one edge are separated at the opposite edge with the help of a thin wire. For the normal incidence of sodium light of wavelength $\lambda = 5800$ Å one finds 18 fringes between these edges. Calculate the thickness of the wire.
21. A wedge-shaped film is enclosed between two glass plates in contact along one edge and separated at the opposite edge with help of a wire of diameter 6×10^{-6} m at a distance of 14 cm from the edge. For the normal incidence of light of wavelength $\lambda = 5800$ Å, calculate the width of interference pattern formed in the reflected light.
22. A wedge-shaped film having angle $60''$ is incident normally by a monochromatic light of wavelength $\lambda = 5893$ Å. Calculate the fringe width due to the interference between the reflected waves.

23. Interference fringes are produced when monochromatic light of wavelength $\lambda =$ 5800 Å is incident normally on a wedge-shaped film of refractive index 1.46. If the separation between the two successive fringes is 0.014 mm, calculate the angle of the film.
24. Describe the formation of Newton's rings. Discuss the conditions for bright and dark fringes.
25. For Newton's rings, obtain expression for the diameter of bright and dark rings. How the wavelength of monochromatic light can be determined with the help of Newton's rings.
26. For Newton's rings, obtain expression for the diameter of bright and dark rings. How the radius of curvature of plano-convex lens is determined with the help of Newton's rings.
27. In Newton's ring experiment, the diameters of 6th and 16th rings are found, respectively, 0.35 mm and 0.62 mm. If the radius of plano-convex lens is 98 cm, calculate the wavelength of light used.
28. In Newton's ring experiment, two light sources of wavelengths $\lambda_1 = 5800$ Å and $\lambda_2 = 4600$ Å are used. It is found that the nth dark ring of λ_1 coincides with $(n + 1)$th dark ring of λ_2. If the radius of curvature of plano-convex lens is 0.98 m. Calculate the diameter of $(n + 1)$th dark ring of λ_2.
29. In Newton's ring experiment, a source is emitting two wavelengths $\lambda_1 = 6150$ Å and $\lambda_2 = 5740$ Å. It is found that nth dark ring due to the first wavelength coincides with the $(n + 1)$th dark ring due to the second wavelength. Calculate the diameter of nth dark ring due to the first wavelength if the radius of curvature of the plano-convex lens is 94 cm.
30. Newton's rings are obtained by using a monochromatic light of wavelength 5890 Å. A liquid is placed between the plano-convex lens and the glass plate. If the diameter of 6th bright ring is 0.36 cm and the radius of curvature of the lens is 95 cm. Calculate the refractive index of the liquid.
31. Newton's rings are obtained for two different media between the lens and the plate. The ratio of the diameters of nth ring is 10 : 7. Calculate the ratio of the refractive indexes of the two media.
32. Describe the working of Michelson's interferometer.
33. Describe about various types of fringes obtained in Michelson's interferometer.
34. How Michelson's interferometer can be used for the determination of wavelength of monochromatic light and difference between the two close wavelengths.
35. If a movable mirror of Michelson interferometer is moved through a distance of 0.05 mm, 210 fringes are found to cross the field of view. Find the wavelength of monochromatic light used.
36. Calculate the separation between two successive positions of movable mirror of Michelson interferometer giving the best fringes in case of sources having wavelengths 6072 Å and 6064 Å.

37. Write short notes on the following:

(i) Huygens' principle
(ii) Interference
(iii) Phase difference between two waves for interference
(iv) Constructive interference and destructive interference
(v) Conditions for sustained interference
(vi) Newton's rings.

Chapter 5
Fresnel Diffraction

In the Huygens theory, the light appears to travel in a straight line. However, when a wave encounters an obstacle (or an opening), it bends around the edges of the obstacle (or opening). This phenomenon of bending of wave (light) around an edge of the obstacle (or opening) is known as the **diffraction**. The diffraction is one of the phenomena shown by the light. We have discussed earlier about the reflection, refraction, and interference phenomena shown by the light. When the size of the obstacle (or opening) is larger than the wavelength of the wave, the bending is not noticeable. But, for the small size of the obstacle (or opening), the bending around an edge is remarkable. In the diffraction also we see the spectrum with maximum and minimum intensities. We have already discussed the bright and dark bands in the interference phenomenon. The main differences between interference and diffraction may be as the following.

	Interference	Diffraction
1.	Produced due to the interaction between different wavefronts originated from a common source	Produced due to interaction between different parts of a common wavefront
2.	Fringe widths may or may not be equal	Fringe widths are not equal
3.	Minimum intensity may or may not be perfectly dark	Minimum intensity is not perfectly dark
4.	All bright lines are of equal intensity	Different bright lines are of varying intensities with the maximum intensity for the central maximum

Diffraction phenomena are two types: (i) Fresnel diffraction, and (ii) Fraunhoffer diffraction. In the Fresnel diffraction, the source of light is at a finite distance from the obstacle or aperture. An incident wavefront is either spherical (point-source) or cylindrical (line-source) in shape, but not plane. (After traveling a long distance, both the spherical and cylindrical wavefronts convert into a plane wavefront.) On the

S. Chandra and M. K. Sharma, *A Textbook of Optics*,
https://doi.org/10.1007/978-3-031-55676-0_5

other side, in the Fraunhoffer diffraction, the source of light is at an infinite distance from the obstacle or aperture. Thus, the incident wavefront is plane. We know that a spherical wavefront is generated by a point-source and a cylindrical wavefront is generated by a line-source. Here, we consider the Fresnel diffraction and in the next chapter, we have considered the Fraunhoffer diffraction.

1 Fresnel Assumptions

The source of light, in the Fresnel diffraction, is at a finite distance from an obstacle or aperture. Hence, the incident wavefront is either spherical (in the case of a point-source) or cylindrical (in the case of a line-source). According to the Huygens principle, various points on a wavefront can be treated as the sources of secondary waves.

All secondary waves generated from different points on a wavefront are not in the same phase when they reach a common point on a screen placed at a finite distance from the obstacle or aperture. The resultant amplitude due to the secondary waves varies from point to point on the screen, giving a pattern of maximum and minimum intensities.

In Fig. 1, S is a source of monochromatic light generating a wavefront QQ′. S may be a point-source (generating a spherical wavefront) or a line-source (generating a cylindrical wavefront). MN is a small aperture (linear or circular in shape) [Fig. 1a] or a small obstacle (linear or circular in shape) [Fig. 1b] which is at a finite distance from the source S and XY is a screen which is at a finite distance from MN. In order to have a resultant effect at a point P′ on the screen, Fresnel made the following assumptions:

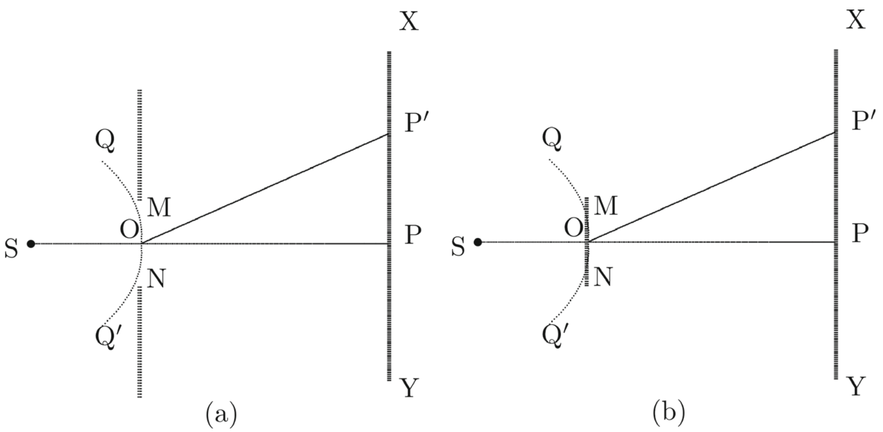


Fig. 1 A source S and screen XY are at finite distances from MN which is a small opening in (intensity) or obstacle in (intensity)

(i) A wavefront can be divided into a large number of small elements (linear or circular in shape), called Fresnel's zones, each of a small area. The resultant effect at any point P′ on the screen depends on the net effect of all the secondary waves produced from these zones.
(ii) The effect at the point P′ due to a zone depends on its distance from the zone.
(iii) The effect at the point P′ also depends on the obliquity of the point with respect to the zone. The effect at a point due to the obliquity is proportional to $(1 + \cos\theta)$ where θ is the angle subtended by the lines OP′ and OP. Hence, the effect is maximum at the point P, where $\theta = 0$. When $\theta = 90°$, the obliquity effect is one half of that at the point P.

In the following sections, we have discussed how the half-period zones are constructed for spherical and cylindrical wavefronts.

2 Diffraction with Spherical Wavefront

A spherical wavefront is produced form a point-source of light. Let us consider a point-source of monochromatic light having wavelength λ. Suppose, ABCD is a part of the spherical wavefront of radius a, as shown in Fig. 2. Let P be an external point and the line joining the point-source (situated at the center of the spherical wavefront) and P intersects the wavefront at the point O. Suppose, OP $= b$. Now, for applying Fresnel's method to calculate the resultant intensity at the point P due to the wavefront ABCD, we construct half-period zones, also called Fresnel's half-period zones.

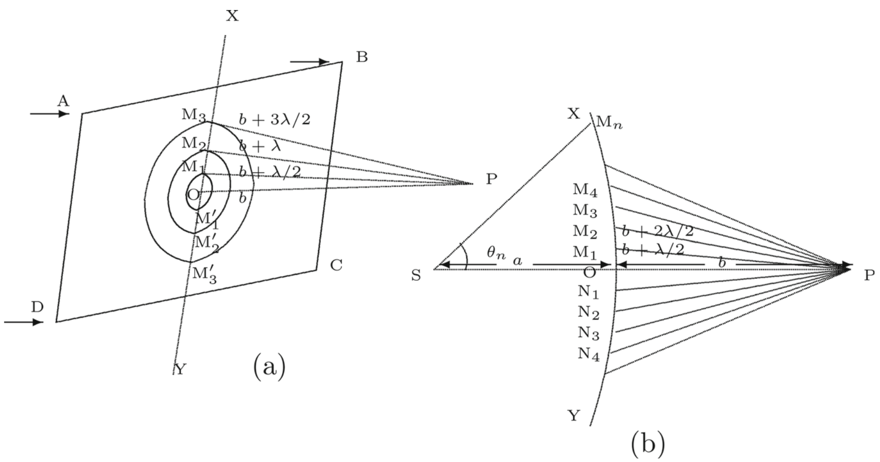


Fig. 2 Construction of half-period zones in case of a spherical wavefront

For making the half-period zones, let us consider spheres with a center at P and of radii equal to $b + \lambda/2$, $b + \lambda$, $b + 3\lambda/2$, and so on. These spheres cut the surfaces ABCD in the circles at the points M_1, M_2, M_3, and so on the wavefront, as shown in Fig. 2b. From Fig. 2b, we have

$$\mathrm{PM}_n = b + \frac{n\lambda}{2}$$

In the triangle $\mathrm{PM}_n\mathrm{S}$, we have

$$\mathrm{PM}_n^2 = \mathrm{SM}_n^2 + \mathrm{PS}^2 - 2\,\mathrm{P'S}\,.\,\mathrm{SM}_n \cos\theta_n$$

Using the values here, we get

$$\left(b + \frac{n\lambda}{2}\right)^2 = a^2 + (a+b)^2 - 2(a+b)a\cos\theta_n \tag{1}$$

As the angle θ_n is in radian and small, we have

$$\cos\theta_n = 1 - \frac{\theta_n^2}{2!} + \frac{\theta_n^4}{4!} - \frac{\theta_n^6}{6!} + \cdots \qquad \text{or} \qquad \cos\theta_n = 1 - \frac{\theta_n^2}{2!} \tag{2}$$

where we have neglected higher order terms. Using Eqs. (2) in (1), we have

$$b^2 + bn\lambda + \frac{n^2\lambda^2}{4} = 2a^2 + b^2 + 2ab - 2a(a+b)\left(1 - \frac{\theta_n^2}{2!}\right) \tag{3}$$

Since λ is much smaller than b, neglecting the term $n^2\lambda^2/4$ and rearranging Eq. (3), we get

$$bn\lambda = a(a+b)\theta_n^2 \qquad \text{or} \qquad \theta_n = k\sqrt{n}$$

where

$$k = \sqrt{\frac{b\lambda}{a(a+b)}}$$

As the arc

$$\mathrm{OM}_n = a\theta_n = ak\sqrt{n}$$

we have

$$\mathrm{OM}_1 = ak \qquad \mathrm{OM}_2 = ak\sqrt{2} \qquad \mathrm{OM}_3 = ak\sqrt{3}$$

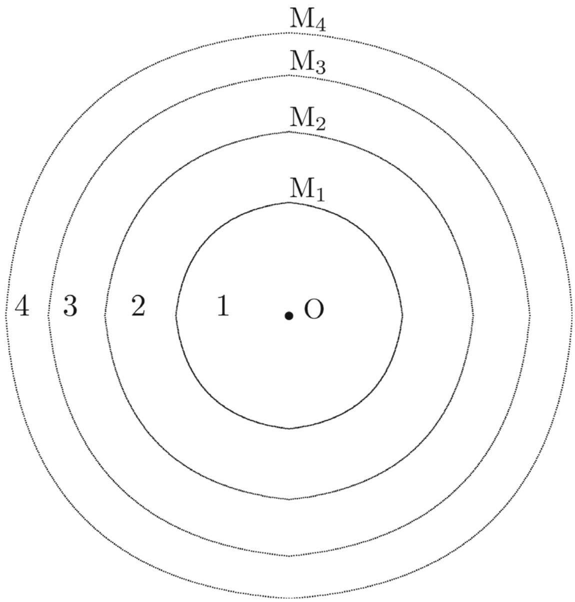


Fig. 3 Half-period zones for spherical wavefront

$$\mathrm{OM}_4 = 2ak \qquad \mathrm{OM}_5 = ak\sqrt{5} \qquad \text{and so on}$$

These annular areas between two consecutive circles are known as the half-period zones or half-period elements (Fig. 3). Each of the zones differs from its neighbor by a path difference of $\lambda/2$ or phase difference of π radians. Thus, the secondary waves beginning from two consecutive half-period zones and reaching the point P differ by a path difference of $\lambda/2$ or phase difference of π radians. For example, the secondary waves from the first and second half-period zones reach the point P with a path difference of $\lambda/2$ or phase difference of π radians. Thus, the secondary waves from the first and third half-period zones reach the point P with a path difference of λ or phase difference of 2π radians.

The areas A_1, A_2, A_3, etc. of first, second, third, etc. half-period zones, respectively, are given as the following.

$$A_1 = \pi\mathrm{OM}_1^2 = \pi a^2k^2 \qquad A_2 = \pi\mathrm{OM}_2^2 - \pi\mathrm{OM}_1^2 = \pi a^2k^2$$

$$A_3 = \pi\mathrm{OM}_3^2 - \pi\mathrm{OM}_2^2 = \pi a^2k^2 \qquad \text{and so on}$$

Thus, all the zones are of equal area πa^2k^2. The area of each zone is

$$A = \pi a^2k^2 = \pi a^2\frac{b\lambda}{a(a+b)} = \frac{\pi ab\lambda}{(a+b)}$$

Though the area of each zone is equal, the distance of the zone from the point P and the obliquity (angle θ) both increase with the increase of the order of zone. Therefore, the amplitudes m_1, m_2, m_3, etc. of the secondary waves from the first, second, third, etc. zones decrease with the increase of the order of zone.

3 Diffraction with Cylindrical Wavefront

Suppose, a long and narrow slit S_1S_2 is placed perpendicular to the plane of the paper and illuminated by a monochromatic light of wavelength λ, as shown in Fig. 4a. It gives rise to a cylindrical wavefront with the slit at the axis. For finding the effect of the wavefront at an external point P, the wavefront may be divided into half-period strips or elements. For the construction of half-period strips, draw a perpendicular PS from the point P on the S_1S_2, intersecting the wavefront at O. The point O is the pole of the wavefront with respect to the point P. Consider an equatorial section AOB through O in the plane of the paper. Suppose, OP $= b$.

With the point P as the center and radii $(b + \lambda/2)$, $(b + \lambda)$, $(b + 3\lambda/2)$, etc. draw circles which cut the section AOB of the wavefront at the points (M_1, N_1), (M_2, N_2), (M_3, N_3), etc. as shown in Fig. 4b. Through the points M_1, N_1, M_2, N_2, M_3, N_3, etc. draw lines parallel to the axis S_1S_2 to obtain the half-period zones as shown in Fig. 5. The first half-period zone is the area between the vertical lines passing through M_1 and through N_1; the second half-period zone is the area between the vertical lines passing through M_1 and through M_2, and between the vertical lines passing through N_1 and through N_2; the third half-period zone is the areas between the vertical lines passing through M_2 and through M_3, and between the vertical lines passing through N_2 and through N_3; and so on.

As the vertical lengths of all strips are the same, the areas of successive strips are proportional to the arcs OM_1, M_1M_2, M_2M_3, etc. To calculate the widths of the

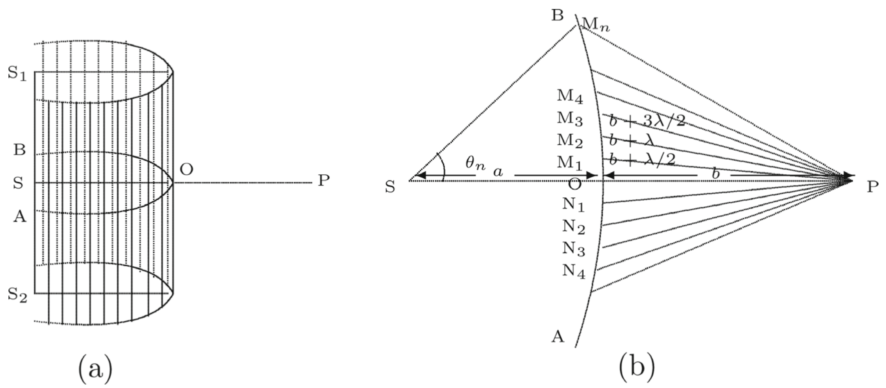
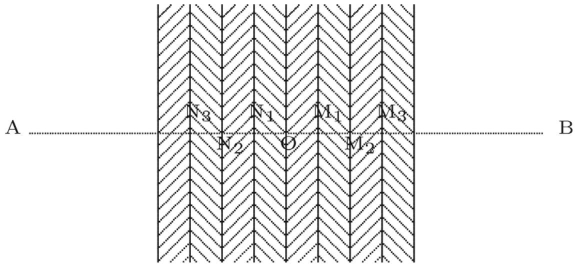


Fig. 4 Diffraction with cylindrical wavefront

Fig. 5 Half-period strips



strips, from Fig. 4b, we have

$$\mathrm{PM}_n = b + \frac{n\lambda}{2}$$

In the triangle $\mathrm{PM}_n\mathrm{S}$, we have

$$\mathrm{PM}_n^2 = \mathrm{SM}_n^2 + \mathrm{PS}^2 - 2\,\mathrm{PS}\;.\;\mathrm{SM}_n \cos\theta_n$$

Using the values, we get

$$\left(b + \frac{n\lambda}{2}\right)^2 = a^2 + (a+b)^2 - 2(a+b)a\cos\theta_n \tag{4}$$

As the angle θ_n is in radian and small, we have

$$\cos\theta_n = 1 - \frac{\theta_n^2}{2!} + \frac{\theta_n^4}{4!} - \frac{\theta_n^6}{6!} + \cdots \qquad \text{or} \qquad \cos\theta_n = 1 - \frac{\theta_n^2}{2!} \tag{5}$$

where we have neglected higher order terms. On using Eqs. (5) in (4), we get

$$b^2 + bn\lambda + \frac{n^2\lambda^2}{4} = 2a^2 + b^2 + 2ab - 2a(a+b)\left(1 - \frac{\theta_n^2}{2!}\right) \tag{6}$$

As λ is much smaller than b, neglecting the term $n^2\lambda^2/4$ and rearranging Eq. (6), we get

$$bn\lambda = a(a+b)\theta_n^2 \qquad \text{or} \qquad \theta_n = k\sqrt{n}$$

where

$$k = \sqrt{\frac{b\lambda}{a(a+b)}}$$

Now, the arc

$$\mathrm{OM}_n = a\theta_n = ak\sqrt{n}$$

Thus, we have

$$\mathrm{OM}_1 = ak \qquad \mathrm{OM}_2 = ak\sqrt{2} \qquad \mathrm{OM}_3 = ak\sqrt{3}$$

$$\mathrm{OM}_4 = 2ak \qquad \mathrm{OM}_5 = ak\sqrt{5} \qquad \text{and so on}$$

Now, the thicknesses of the strips are

$$\mathrm{OM}_1 = ak$$

$$\mathrm{M}_1\mathrm{M}_2 = ak(\sqrt{2} - 1) = 0.414ak$$

$$\mathrm{M}_2\mathrm{M}_3 = ak(\sqrt{3} - \sqrt{2}) = 0.318ak$$

$$\mathrm{M}_3\mathrm{M}_4 = ak(2 - \sqrt{3}) = 0.268ak$$

$$\mathrm{M}_4\mathrm{M}_5 = ak(\sqrt{5} - 2) = 0.236ak \qquad \text{and so on}$$

It shows that the areas of the strips decrease rapidly first and then slowly with the increase of the order. Thus, the successive higher order strips cancel the effect of each other in pairs, being in opposite phases and having practically equal amplitudes. Hence, the effect at point P is only due to the first few half-period elements. Now, the distance of the zone from the point P and the obliquity (angle θ) both increase with the increase of the order of zone. Therefore, the amplitudes m_1, m_2, m_3, etc. of secondary waves from the first, second, third, etc. zones decrease with the increase of the order of zone.

4 Rectilinear Propagation of Light

To explain the rectilinear propagation of light, we find out the resultant effect of whole wavefront QQ′ (spherical or cylindrical) at the point P (Fig. 6). For doing so, the wavefront is divided into the half-period zones or strips. The problem is thus reduced to find the resultant effect of disturbances originating from various half-period zones.

As we have discussed, the effect of a half-period zone at an external point P depends on: (i) distance of the point P from the wavefront, (ii) area of the zone, and

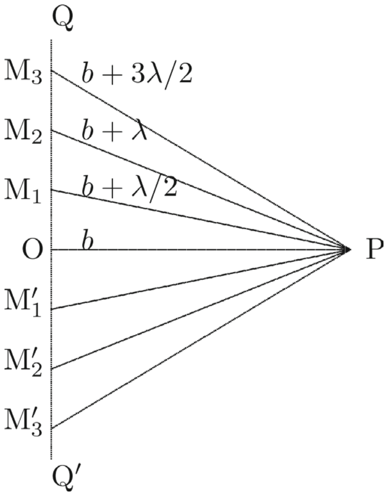


Fig. 6 Half-period zones constructed on a wavefront QQ′

(iii) obliquity factor depending on the angle between the normal to the wavefront and the line joining the half-period zone and the point P. As we have discussed in the preceding sections, the amplitudes of secondary waves generated from the first, second, third, etc. half-period zones decrease with the increase of the order of zone. The secondary waves from various zones, reaching the point P is regularly out of phase and in phase with respect to the first half-period zone.

Suppose, $m_1, m_2, m_3, \ldots, m_n$ represent the amplitudes of secondary waves generated from the first, second, third, …, nth zones, respectively, at the point P, as shown in Fig. 7. Hence, m_1 is greater than m_2, m_2 is greater than m_3, and so on. Due to the phase difference of π between any two consecutive zones, when the displacements of either particles due to odd numbered zones are in the positive direction, then due to even numbered zones the displacements are in the negative direction at the same

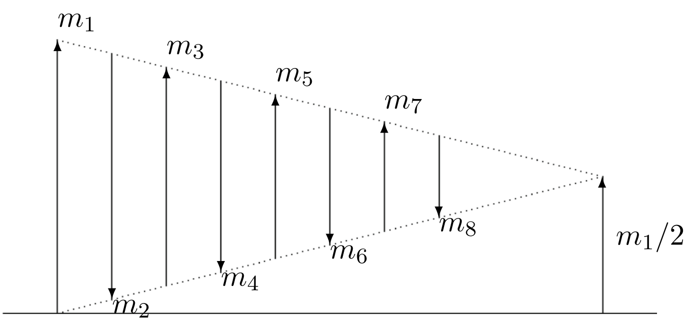


Fig. 7 Amplitudes of waves from the first, second, third, etc. half-period zones

time. As the amplitudes are of gradually decreasing magnitude, the amplitude of vibration at P due to any zone may be approximately taken as a mean of amplitudes due to the zones preceding and succeeding it. Thus, we have

$$m_i = \frac{m_{i+1} + m_{i-1}}{2}$$

The resultant amplitude at P an any instant is

$$A = m_1 - m_2 + m_3 - m_4 + m_5 - m_6 + \cdots + m_n \tag{7}$$

when n is odd, we have

$$A = m_1 - m_2 + m_3 - m_4 + m_5 - m_6 + \cdots - m_n \tag{8}$$

when n is even. Equation (7) can be rearranged as

$$A = \frac{m_1}{2} + \left[\frac{m_1}{2} - m_2 + \frac{m_3}{2}\right] + \left[\frac{m_3}{2} - m_4 + \frac{m_5}{2}\right] + \cdots + \frac{m_n}{2}$$

$$= \frac{m_1}{2} + \frac{m_n}{2} \tag{9}$$

Equation (8) can be rearranged as

$$A = \frac{m_1}{2} + \left[\frac{m_1}{2} - m_2 + \frac{m_3}{2}\right] + \left[\frac{m_3}{2} - m_4 + \frac{m_5}{2}\right] + \ldots + \frac{m_{n-1}}{2} - \frac{m_n}{2}$$

$$= \frac{m_1}{2} + \frac{m_{n-1}}{2} - \frac{m_n}{2} \tag{10}$$

When the whole wavefront ABCD is unobstructed, the number of half-period zones that can be constructed with reference to the point P is infinite ($n = \infty$). As the amplitudes are of gradually decreasing magnitude, m_n and m_{n-1} tend to zero, and the resultant amplitude at P due to the whole wavefront is

$$A = \frac{m_1}{2}$$

The intensity at a point is proportional to the square of amplitude and thus, we get

$$I = \frac{m_1^2}{4}$$

Thus, the resultant intensity at P is equal to one-fourth of that due to the first half-period zone.

5 Diffraction Due to a Straight Edge

Consider a narrow slit S illuminated by a source of monochromatic light of wavelength λ. The length of the slit is perpendicular to the plane of the paper. In Fig. 8, AD is a straight edge and the length of the edge is parallel to the length of the slit. QQ′ is the incident cylindrical wavefront. P is a point on the screen and SOP is perpendicular to the screen XY. The screen XY is perpendicular to the plane of the paper. Below the point P is the geometrical shadow and above P is the illuminated portion.

Suppose, the distance OP is b. With reference to the point P, the wavefront can be divided into a number of half-period zones (or strips) as shown in Fig. 9. AB is the wavefront, O is the pole of the wavefront, and OM_1, M_1M_2, M_2M_3, etc. measure thickness of the first, second, third, etc. half-period strips. With the increase in order, the area of the strip decreases.

From Fig. 8, we have

$$OP = b, \qquad PM_1 = b + \lambda/2, \qquad PM_2 = b + 2\lambda/2,$$

$$PM_3 = b + 3\lambda/2, \qquad \ldots, \qquad PM_n = b + n\lambda/2$$

Suppose P′ is a point on the screen in the illuminated part (Fig. 10). To calculate the resultant intensity at the point P′ due to the wavefront QQ′, let us join S to P′. This line meets the wavefront at O′. This point O′ is the pole of the wavefront with reference to the point P′ and the intensity at P′ depends mainly on the number of half-period strips enclosed between O and O′. The effect at P′ due to the wavefront

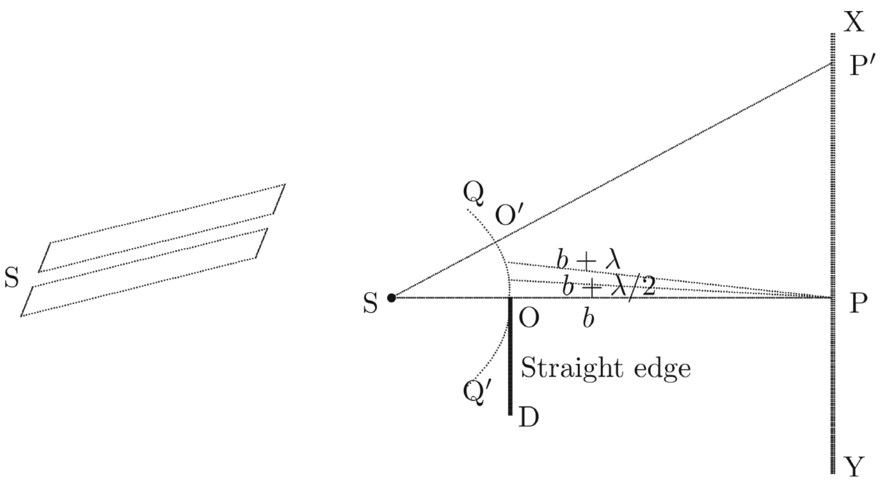


Fig. 8 Cylindrical wavefront QQ′ is incident on a straight edge AD

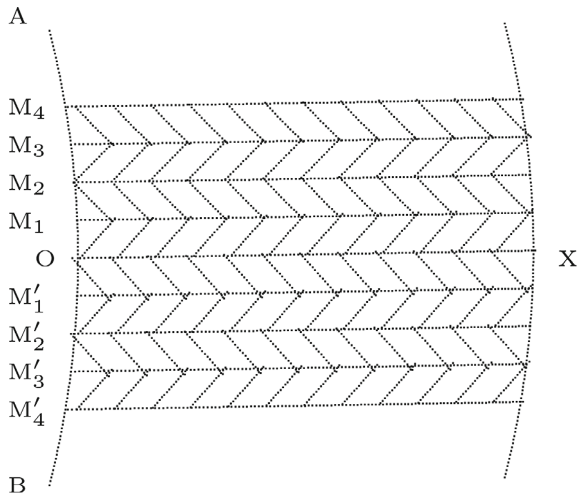


Fig. 9 Half-period strips

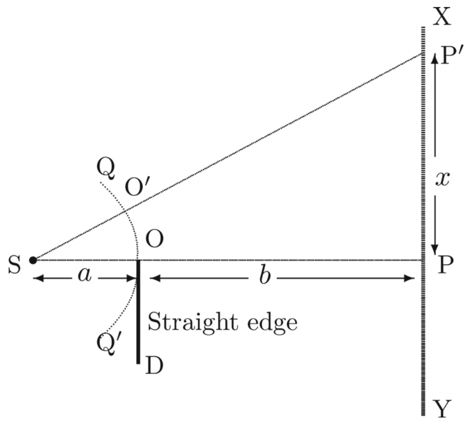


Fig. 10 Positions of the maximum and minimum intensities

above O′ is the same at all the points on the screen whereas it is different at different points due to the wavefront between the points O and O′. The point P′ is of maximum intensity when the number of half-period strips enclosed between the points O and O′ is odd and of minimum intensity when the number of half-period strips enclosed between the points O and O′ is even.

Positions of maximum and minimum intensity

Suppose, the distance between the source and the straight edge is a and the distance between the straight edge and the screen is b. Let PP′ be x. The path difference is given by

$$\delta = \mathrm{OP}' - \mathrm{O}'\mathrm{P}' = (b^2 + x^2)^{1/2} - [\mathrm{SP}' - \mathrm{SO}']$$

$$= (b^2 + x^2)^{1/2} - \left[\sqrt{(a+b)^2 + x^2} - a\right]$$

$$= b\left(1 + \frac{x^2}{b^2}\right)^{1/2} - (a+b)\left[1 + \frac{x^2}{(a+b)^2}\right]^{1/2} + a$$

$$= b\left(1 + \frac{x^2}{2b^2}\right) - (a+b)\left[1 + \frac{x^2}{2(a+b)^2}\right] + a$$

As x is smaller than b, we have neglected higher order terms, and we have

$$\delta = \frac{x^2}{2}\left(\frac{1}{b} - \frac{1}{a+b}\right) = \frac{x^2 a}{2b(a+b)}$$

The point P$'$ has maximum intensity when $\delta = (2n+1)\lambda/2$, where n is an integer[1] and we have

$$\frac{(2n+1)\lambda}{2} = \frac{x^2 a}{2b(a+b)} \qquad \text{or} \qquad x^2 = \frac{(2n+1)\lambda b(a+b)}{a}$$

Hence, the distance of nth bright fringe is

$$x_n = \sqrt{\frac{(2n+1)\lambda b(a+b)}{a}}$$

The point P$'$ has the minimum intensity when $\delta = n\lambda$, where n is an integer and we have

$$n\lambda = \frac{x^2 a}{2b(a+b)} \qquad \text{or} \qquad x^2 = \frac{2n\lambda b(a+b)}{a}$$

---

[1] Note that here conditions for maxima and minima are reversed as compared to those in case of the interference. It is because we measure the path difference between two waves one from O and other from a point on the wavefront. The path difference between the waves from O and M_1 is $\lambda/2$ and we have one half-period zone giving maximum intensity at P; the path difference between the waves from the point O and M_2 is λ and we have two half-period zone giving maximum intensity at the point P; and so on. Thus, the condition for the maxima is

$$\mathrm{M}_n P - OP = \frac{(2n+1)\lambda}{2}$$

and for the minima is

$$\mathrm{M}_n P - OP = n\lambda$$

Hence, the distance of the nth dark fringe is given by

$$x_n = \sqrt{\frac{2n\lambda b(a+b)}{a}}$$

5.1 Intensity Inside the Geometrical Region

Suppose, P′ is a point in the geometrical region (below P) as shown in Fig. 11 and O′ is pole of the wavefront with reference to the point P′, then the half-period strips below O′ are cut off by the obstacle and only the uncovered half-period strips above O′ are effective in generating the illumination at P′. As the point P′ moves away from P, more number of half-period strips above O′ is also cut off and the intensity gradually decreases. Thus, in the geometrical shadow, the intensity gradually decreases depending on the position of the point P′ with respect to the point P as shown in Fig. 12.

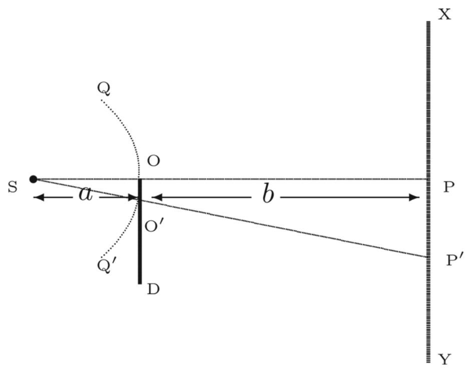


Fig. 11 Intensity inside the geometrical region

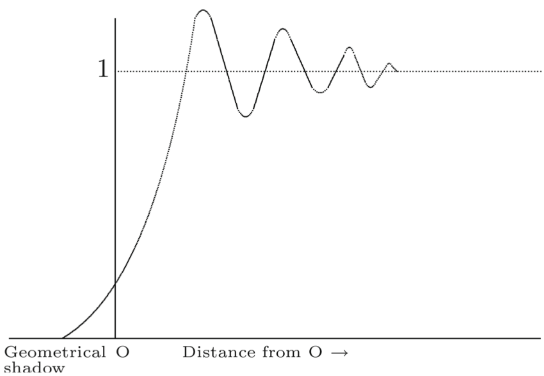


Fig. 12 Variation of intensity

5.2 Determination of Wavelength

Suppose a straight edge, say a sharp razor blade is set with its edge parallel to a slit, which is illuminated with a monochromatic light. Diffraction bands of unequal widths and decreasing intensity are observed in the field of view. The position of the first maximum band is given by

$$x_1 = \sqrt{\frac{\lambda b(a+b)}{a}}$$

For the nth maximum, the position is given by

$$x_n = \sqrt{\frac{(2n+1)\lambda b(a+b)}{a}}$$

Hence, the separation between the first and nth maxima is given by

$$\begin{aligned} x_n - x_1 &= \sqrt{\frac{(2n+1)\lambda b(a+b)}{a}} - \sqrt{\frac{\lambda b(a+b)}{a}} \\ &= \sqrt{\frac{\lambda b(a+b)}{a}}\left[\sqrt{(2n+1)} - 1\right] \end{aligned}$$

The values of a and b are known. By using the values of x_1 and x_n in this expression, the wavelength of monochromatic light can be calculated.

Exercise 1 A narrow slit illuminated by light of wavelength 5893 Å is at a distance of 0.1 m from a straight edge. Calculate the separation between the first and second bright bands, measured at a distance of 0.5 m from a straight edge.

Solution Given, $a = 0.1$ m, $b = 0.5$ m, $\lambda = 5893 \times 10^{-10}$ m. For the nth bright band, we have

$$x_n = \sqrt{\frac{(2n-1)\lambda b(a+b)}{a}}$$

Thus, we have

$$x_2 - x_1 = \sqrt{\frac{\lambda b(a+b)}{a}}\left[\sqrt{3} - 1\right]$$

$$= \sqrt{\frac{5893 \times 10^{-10} \times 0.5(0.1+0.5)}{0.1}}\left[\sqrt{3} - 1\right]$$

$$= 9.734 \times 10^{-4} \text{ m}$$

6 Diffraction at a Wire

Suppose S is a narrow rectangular slit placed parallel to a wire of thickness MN and perpendicular to the plane of the paper as shown in Fig. 13. This Produces a geometrical shadow AB on the screen XY. The effect at a point outside the geometrical shadow is the same as that due to a straight edge at M and thus, the diffraction bands of unequal width and intensity are seen above point A and similarly below point B. These bands are independent to the thickness of the wire, as on either side, the effect of the other half of the wave is negligible, as the most important half-period elements are cut off due to the finite width of the wire.

The interference fringes are observed, in the geometrical shadow. The effect due to the portion MQ of the cylindrical wavefront, at any point P′ in the geometrical shadow, is entirely due to few half-period elements at X, so that it may be regarded as a small luminous source at the point M. Similarly, the effect due to the portion NQ′ is equal to a small luminous source at the point N.

The effect at a point P′ in the geometrical shadow AB depends on the path difference NP′ - MP′. The point P′ is bright when NP′ - MP′ = $n\lambda$ and dark when NP′ - MP′ = $(2n + 1)\lambda/2$, where n is an integer. These fringes are of equal width with the fringe-width $\beta = D\lambda/d$, where D is the distance of screen from the wire, d the thickness of the wire. The center P is bright as at the point P, the waves from M and N always meet in phase.

When the diameter of the wire increases, the fringe-width decreases, and when the thickness of the wire is sufficiently large, the interference fringes disappear and only the diffraction bands outside the limits of geometrical shadow are visible.

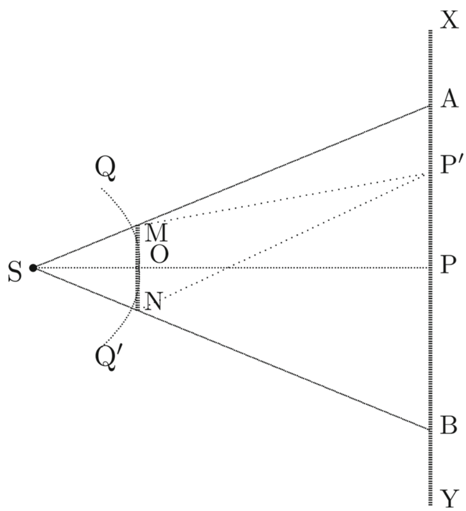


Fig. 13 A monochromatic cylindrical wavefront QQ′ is incident on a wire MN

6.1 Measurement of Thickness of Wire

For small thickness of wire, the interference fringes are generated in the region of the geometrical shadow. These fringes are of equal width to the fringe-width

$$\beta = \frac{D\lambda}{d} \qquad \text{or} \qquad d = \frac{D\lambda}{\beta}$$

For known values of D, λ, β, we can find the thickness of the wire.

7 Diffraction Due to a Narrow Slit

Suppose, MN is width of a rectangular slit which is placed perpendicular to the plane of the paper, and S the source of monochromatic light placed parallel to the slit as shown in Fig. 14. QQ′ is the section of cylindrical wavefront generated from the light source S. The part AB, on the screen, is illuminated.

When the slit is wide enough so that it allows about 100 half-period elements from each half of the wavefront to pass through the slit, the edges M and N act like independent straight edges. It results in the production of diffraction bands of unequal width at A and B inside the bright region, similar to the bands produced by a straight edge. When we go from A towards B, the general illumination increases, and the visibility of the bands decreases, so that there is a uniform illuminating about P. Beyond A and B, the illumination rapidly decreases to zero.

When the slit is narrow so that it allows only a few half-period elements, say five (odd), from each half of the wavefront to pass through the slit, the illumination at the point P is again maximum. When the number of half-period elements contained in each half of the wavefront to pass through the slit is even, the point P has minimum intensity.

On a fix screen, when we move to a point P′ such that four half-period elements from the upper half of the wavefront and six half-period elements from the lower half of the wavefront pass through the slit, the resultant amplitude of the secondary waves from the upper half of the wavefront is $m_1 - m_2 + m_3 - m_4 = 0$ and that of the secondary waves from the lower half of the wavefront is $m_1 - m_2 + m_3 - m_4 + m_5 - m_6 = 0$. Thus, the illumination at the point P′ is the minimum.

On moving the point P′ a little further so that three half-period elements from the upper half of the wavefront and seven half-period elements from the lower half of the wavefront pass through the slit, the resultant amplitude of the secondary waves from the upper half of the wavefront is $m_1 - m_2 + m_3 = m_1$ and that of the secondary waves from the lower half of the wavefront is $m_1 - m_2 + m_3 - m_4 + m_5 - m_6 + m_7 = m_1$. Thus, the resultant amplitude is $m_1 + m_1 = 2m_1$ which produces a maximum intensity.

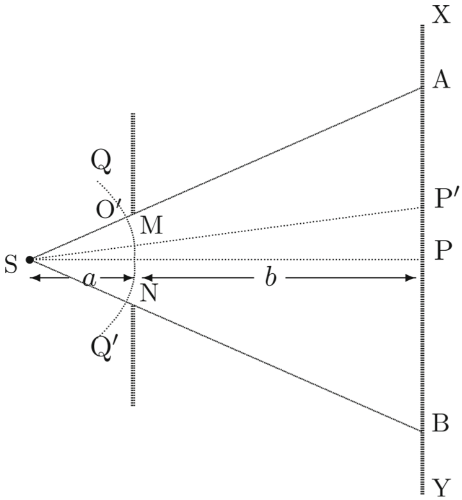


Fig. 14 A monochromatic cylindrical wavefront QQ′ illuminates a narrow slit MN

8 Diffraction at a Circular Aperture

Suppose, MN is a small circular aperture (say a pin-hole) of radius r with center at O (Fig. 15). For a point-source S of monochromatic light placed at a distance a from the aperture. XY is a screen perpendicular to the plane of the paper placed at a distance b from the aperture. P is a point on the screen. QQ′ ′ is the incident spherical wavefront and with reference to the point P, the pole on the wavefront is the point O (Fig. 15).

To find intensity at the point P, half-period zones can be constructed with P as center and radii $b + \lambda/2$, $b + 2\lambda/2$, $b + 3\lambda/2$, ..., on the exposed wavefront MON. Depending on the distance of P from the aperture (i.e., the distance b) the number of half-period zones that can be constructed may be odd or even. When the distance b is such that only one half-period zone can be constructed, then the intensity at the point P is proportional to m_1^2 (where m_1 is the amplitude due to the first zone at P). On other side, when a large number of half-period zones are constructed, the resultant amplitude at the point P is $m_1/2$ and the intensity is proportional to $m_1^2/4$.

The position of the screen can be changed to construct 2, 3, or more half-period zones for the same area of the aperture. In the case of only two half-period zones, the resultant amplitude at the point P is $m_1 - m_2$ (minimum) and in case of three half-period zones, the resultant amplitude at the point P is $m_1 - m_2 + m_3$ (maximum) and so on. Thus, by continuously changing the value of b (i.e., the position of the screen), the point P becomes alternately bright and dark depending on whether the odd or even number of half-period zones are constructed on the aperture MN.

Consider a point P′ on the screen XY (Fig. 15). Let the line joining the points S and P′ meet the wavefront at the point O′. We construct half-period zones with the point O′ as the pole of the wavefront. The upper half of the wavefront is cut off by

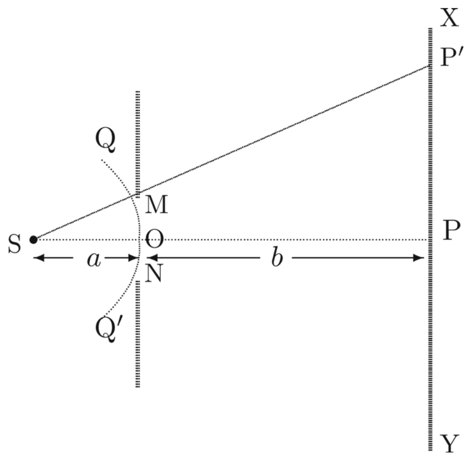


Fig. 15 A monochromatic spherical wavefront QQ′, a circular aperture MN is illuminated

the obstacle. If the first two zones are cut off by the obstacle between the points O′ and M and only third, fourth, and fifth zones are exposed by the aperture MN, then the intensity at the point P′ is the maximum. On the other side, if only the third and fourth zones are exposed by the aperture MN, then the intensity at the point P′ is the minimum. Thus, when an odd number of half-period zones are exposed, the point P′ has the maximum intensity and when an even number of half-period zones are exposed, the point P′ has the minimum intensity.

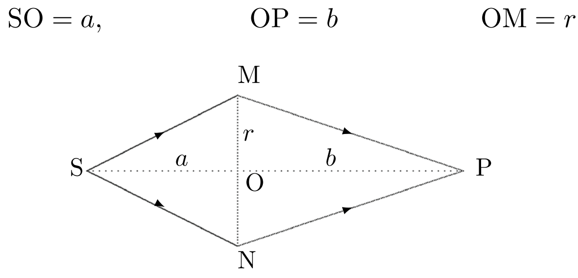
When the distance of P′ from P increases the intensity of maxima and minima gradually decreases. It is because with the increase of the distance of P′ from the point P, the most effective central half-period zones are cut off by the obstacle between the points O′ and M. With the outer zones, the obliquity increases with reference to the point P′ and hence the intensity of maxima and minima also is less. When the point P′ has the maximum intensity, then the points lying on a circle of radius PP′ on the screen also is of the maximum intensity. Thus, with a circular aperture, the diffraction pattern is the concentric bright and dark rings with the center at P bright or dark depending on the distance b. The width of the rings decreases continuously.

8.1 Mathematical Treatment

Figure 16 shows a monochromatic source S, circular aperture MN having radius r, point P on the screen XY and point O at the center of the aperture. The line SOP is perpendicular to the aperture MN. The screen XY is perpendicular to the plane of the paper. We have

$$\mathrm{SO} = a, \qquad \mathrm{OP} = b \qquad \mathrm{OM} = r$$

Fig. 16 Diffraction at a circular aperture



Suppose, δ is the path difference for the waves reaching the point P along the paths of SMP and SOP. Then, we have

$$\delta = \mathrm{SM} + \mathrm{MP} - \mathrm{SOP}$$

$$= (a^2 + r^2)^{1/2} + (b^2 + r^2)^{1/2} - (a + b)$$

$$= a\left(1 + \frac{r^2}{a^2}\right)^{1/2} + b\left(1 + \frac{r^2}{b^2}\right)^{1/2} - (a + b)$$

$$= a\left(1 + \frac{r^2}{2a^2}\right) + b\left(1 + \frac{r^2}{2b^2}\right) - (a + b)$$

The higher order terms are neglected, as r is much smaller than both a and b. Thus, we have

$$\delta = \frac{r^2}{2a} + \frac{r^2}{2b} = \frac{r^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

Thus, we have

$$\frac{1}{a} + \frac{1}{b} = \frac{2\delta}{r^2} \tag{11}$$

When the position of screen is such that n full number of half-period zones can be constructed on the aperture, then the path difference is $\delta = n\lambda/2$. Using this value of δ in Eq. (11), we get

$$\frac{1}{a} + \frac{1}{b} = \frac{n\,\lambda}{r^2} \tag{12}$$

The point P has the maximum or minimum intensity, depending on whether n is odd or even. When the source is at infinite distance (for an incident plane wavefront), then $a = \infty$ and we get

$$\frac{1}{b} = \frac{n\,\lambda}{r^2} \qquad \text{or} \qquad b = \frac{r^2}{n\,\lambda}$$

8.2 Intensity at a Point Away from the Center

Consider a point P′ at a distance x from the point P as shown in Fig. 17. Suppose, r is the radius of the circular aperture MN. The path difference between the secondary waves from the points M and N, and reaching at P′ is given by

$$\delta = NP' - MP' = \sqrt{b^2 + (x+r)^2} - \sqrt{b^2 + (x-r)^2}$$

$$= b\left[1 + \frac{(x+r)^2}{b^2}\right]^{1/2} - b\left[1 + \frac{(x-r)^2}{b^2}\right]^{1/2}$$

$$= b\left[1 + \frac{(x+r)^2}{2b^2}\right] - b\left[1 + \frac{(x-r)^2}{2b^2}\right]$$

Higher order terms are neglected, as $x + r$ is much smaller than b. Then, we get

$$\delta = b\left[1 + \frac{(x+r)^2}{2b^2}\right] - b\left[1 + \frac{(x-r)^2}{2b^2}\right] = \frac{2xr}{b} \tag{13}$$

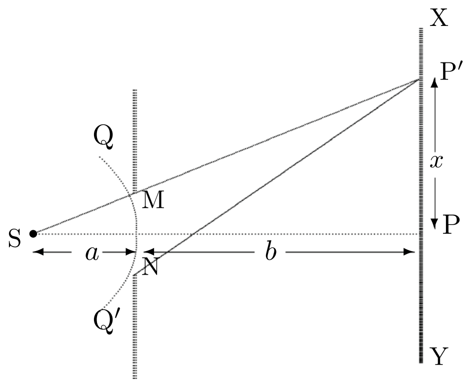


Fig. 17 Diffraction at a circular aperture

For a bright point P′, we should have $\delta = (2n + 1)\lambda/2$, where n is an integer.[2] Use of the value of δ in Eq. (13) gives

$$\frac{(2n+1)\lambda}{2} = \frac{2xr}{b} \qquad \text{or} \qquad x = \frac{(2n+1)\lambda b}{4r}$$

Thus, the radius of nth bright ring is given by

$$x_n = \frac{(2n+1)\lambda b}{4r}$$

The point P′ is dark when $\delta = n\lambda$, where n is an integer. Using the value of δ in Eq. (13), we have

$$n\lambda = \frac{2xr}{b} \qquad \text{or} \qquad x = \frac{n\lambda b}{2r}$$

Thus, the radius of nth dark ring is given by

$$x_n = \frac{n\lambda b}{2r}$$

9 Diffraction at a Circular Disc

Suppose, a monochromatic spherical wavefront QQ′, produced from a point-source S, incident on a circular disc MN with center at O and radius r as shown in Fig. 18. On the screen, XY placed perpendicular to the plane of the paper, AB represents the geometrical shadow, which is circular in shape. Suppose, P is a point on the screen such that the line SOP is perpendicular to the disc. With reference to the point P, the wavefront QQ′ can be divided into the half-period zones, considering O as a pole. Suppose, m_1, m_2, m_3, etc. are the amplitudes of waves produced from first, second, third, etc., zones, respectively. The intensity at the point P depends on the number of zones covered by the disc. In case only one half-period zone is covered by the disc, the rest of the zones are exposed to point P, and the resultant amplitude at the point P is given by

[2] Note that here the conditions for maxima and minima are opposite as compared to those in case of the interference. The reason for that is because of measuring the path difference between the two waves, one from the point N and other from a point on the wavefront. When the path difference between the waves from the point O and other point is $\lambda/2$, we have one half-period zone giving the maximum intensity at P′; when the path difference between the waves from the point O and other point is λ, we have two half-period zones giving the maximum intensity at the point P′; and so on.

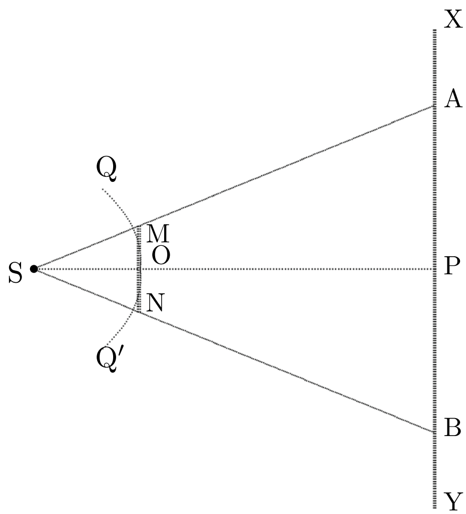


Fig. 18 A monochromatic spherical wavefront QQ′ is incident on a circular disc MN

$$A = m_2 - m_3 + m_4 - m_5 + m_6 - \cdots$$
$$= \frac{m_2}{2} + \left[\frac{m_2}{2} - m_3 + \frac{m_4}{2}\right] + \left[\frac{m_4}{2} - m_5 + \frac{m_6}{2}\right] + \cdots +$$

Since the amplitudes are decreasing gradually, the resultant amplitude at the point P is given by

$$A = \frac{m_2}{2}$$

and the intensity is given by

$$I = \frac{m_2^2}{4}$$

Similarly, when two half-period zones are covered by the disc, the rest of the zones are exposed to the point P and the resultant amplitude at P is

$$A = m_3 - m_4 + m_5 - m_6 + m_7 - \cdots$$

$$= \frac{m_3}{2} + \left[\frac{m_3}{2} - m_4 + \frac{m_5}{2}\right] + \left[\frac{m_5}{2} - m_6 + \frac{m_7}{2}\right] + \cdots +$$

Since the amplitudes are decreasing gradually, the resultant amplitude at the point P is

$$A = \frac{m_3}{2}$$

and thus, the intensity is given by

$$I = \frac{m_3^2}{4}$$

Hence, the point P is always bright, but the intensity at P decreases with the increase in the diameter of the disc. That is, for a large diameter of the disc, the most effective central zones are cut off by the disc and the exposed outer zones are more oblique the reference to point P. Thus, the center (at the point P) of the geometrical shadow is as bright as when the disc was absent. The diffraction pattern consists of a central bright spot surrounded by alternate bright and dark rings.

10 Multiple Choice Questions

1. What cannot be the shape of a wavefront for a Fresnel diffraction?

   A. circular B. cylindrical C. plane D. either of above

   Ans. C

2. What cannot be the shape of a wavefront for a Fraunhoffer diffraction?

   A. circular B. cylindrical C. plane D. either of above

   Ans. C

3. For a spherical wavefront of radius a, the radius of nth half-period zone is proportional to

   A. $\sqrt{n}$ B. n C. $n^{3/2}$ D. n^2

   Ans. A

4. For a spherical wavefront of radius a, the area of nth half-period zone is proportional to

   A. $\sqrt{n}$ B. n C. a D. a^2

   Ans. D

5. For a cylindrical wavefront of radius a, the width of nth half-period zone is proportional to

   A. $\sqrt{n}$ B. n C. $\sqrt{n} - \sqrt{n-1}$ D. $\sqrt{n} + \sqrt{n-1}$

   Ans. C

6. In the Fresnel diffraction, a circular aperture is illuminated by a monochromatic light of wavelength λ. A screen is placed at a distance b from the aperture. A point on the screen is bright when its distance from the center is proportional to

A. λ/b B. b/λ C. $b\lambda$ D. $1/b\lambda$

Ans. C

11 Problems and Questions

1. Define the phenomenon of diffraction. What is the difference between diffraction and interference.
2. Describe the Huygens-Fresnel theory for diffraction of light.
3. For a plane wavefront of monochromatic light, discuss the procedure for the construction of half-period zones.
4. Show that according to the Fresnel hypothesis, the resultant intensity at an external point P due to a complete wavefront is equal to one-fourth of that due to the first half-period zone.
5. In the Fresnel diffraction, describe the diffraction with spherical wavefront.
6. In the Fresnel diffraction, describe the diffraction with cylindrical wavefront.
7. In the Fresnel diffraction, describe the rectilinear propagation of light.
8. In the Fresnel diffraction, describe the diffraction due to a straight edge.
9. In the Fresnel diffraction, describe the diffraction at a wire.
10. In the Fresnel diffraction, describe the diffraction due to a narrow slit.
11. In the Fresnel diffraction, describe the diffraction at a circular aperture.
12. In the Fresnel diffraction, describe the diffraction at a circular disc.
13. Write short notes on the following

    (i) Diffraction of light
    (ii) Fresnel's assumptions for diffraction
    (iii) Half-period zones.

Chapter 6
Fraunhofer Diffraction

For obtaining the Fraunhofer diffraction, the incident wavefront must be plane, and the diffracted waves are collected on a screen with the help of an achromatic lens. Hence, the source of light should be either at a large distance from the slit or a collimation lens must be used.

1 Fraunhofer Diffraction at a Single Slit

Let us consider a plane wavefront of monochromatic light of wavelength λ incident on a rectangular slit AB of width b, as shown in Fig. 1. The slit is placed perpendicular to the plane of the paper. We can divide this wavefront AB into a large number of points, each producing a large number of secondary waves of equal amplitude a. These secondary waves get diffracted and then interfere to produce a diffraction pattern on the screen. The secondary waves traveling along the direction of the incident beam are focused at the point C whereas those inclined at an angle θ with the direction of the incident beam is focused at a point P.

In order to find out the resultant intensity at the point P, we draw a perpendicular AK on BK. Figure 1 shows that the optical paths of all the waves traveled after the plane AK to the point P are equal. However, the optical paths of the waves produced from the points on the slit AB (from A towards B) and reaching at the point P increases gradually. Thus, the phase difference between them gets larger, as shown in Fig. 2a for 4 parts, each of amplitude a. This figure has a phase difference of ϕ between two successive waves. The resultant magnitude of these waves at the point P is denoted by R. Here, $\alpha = 4\phi$.

The total path difference between the wave originating for the extreme points A and B of the silt is BK = AB $\sin\theta = b\sin\theta$. The path difference between the waves originating from various points on AB varies from zero to $b\sin\theta$. The value is zero

S. Chandra and M. K. Sharma, *A Textbook of Optics*,
https://doi.org/10.1007/978-3-031-55676-0_6

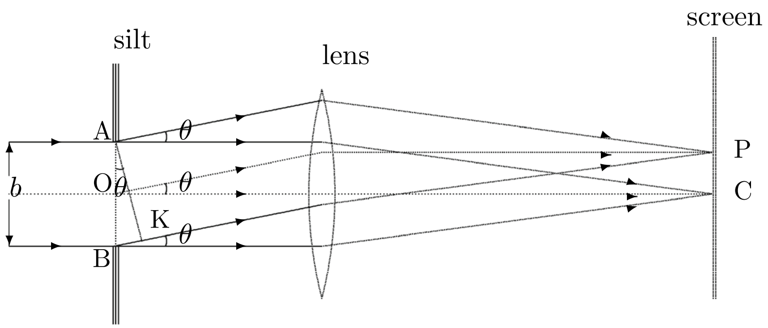


Fig. 1 Fraunhofer diffraction by a single slit

for the wave originating from point A and $b \sin \theta$ for the wave originating from point B. The phase difference corresponding to the path difference $b \sin \theta$ is

$$\frac{2\pi}{\lambda}\, b \sin \theta$$

When the slit AB is divided into n equal parts, the phase difference between the waves generated from the two successive parts is

$$\frac{1}{n}\,\frac{2\pi}{\lambda}\, b \sin \theta = \phi \ \ \text{(say)}$$

The resultant amplitude and intensity at the point P due to all these secondary waves can be obtained with the help of the vector polygon method. Suppose, α be the phase difference between the initial direction and the direction of the resultant, as shown in Fig. 2a. Then, 2α is the total phase difference between the secondary waves originating from the two extreme points A and B, as shown in Fig. 2b.

Here, it is assumed that due to a large number n of the parts, the phase difference ϕ between the two successive waves is very small and the amplitudes of the waves constitute an arc of a circle of radius r. We have $\angle$ POQ = α and because of symmetry $\angle$ OPQ = α. Therefore, we have $\angle$ OQP = $\pi - 2\alpha$. OMP is the arc of the circle of radius r and having center at C. We know

$$\angle OQP + \angle OCP = \pi$$

Therefore, $\angle$ OCP = 2α. The chord OP gives the resultant amplitude at the point P due to all the secondary waves. In Δ OCN, we have

$$\sin \alpha = \frac{\text{ON}}{\text{OC}} = \frac{\text{ON}}{r} \qquad \text{or} \qquad \text{ON} = r \sin \alpha$$

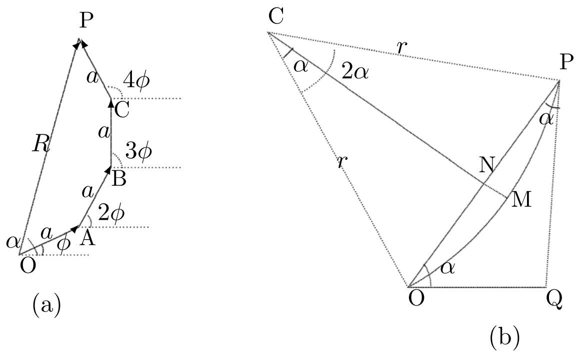


Fig. 2 **a** Is the vector polygon for $n = 4$. When the value of n is very large, the polygon is like an arc of a circle

The chord ONP = 2 ON = $2r \sin\alpha$. Since the chord OP is the resultant amplitude R, we have

$$R = 2r \sin\alpha \tag{1}$$

The length of the arc OMP = na where n is the number of equal parts of AB and a the amplitude of each secondary wave, originated from a point on AB. We know that

$$\angle \text{PCO} = \frac{\text{arc OMP}}{\text{radius}} \qquad 2\alpha = \frac{na}{r} \qquad 2r = \frac{na}{\alpha} \tag{2}$$

Using the value of $2r$ from Eq. (2) in (1), we have

$$R = \frac{na}{\alpha} \sin\alpha \qquad \text{or} \qquad R = A\,\frac{\sin\alpha}{\alpha}$$

where $A = na$. The resultant intensity at the point P is

$$I = R^2 = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2$$

Here, $I_0 = A^2$ represents the intensity at $\theta = 0$. It shows that the magnitude of the resultant intensity at a point P depends on α and therefore on the slit width b. As the phase difference of 2α is introduced due to the path difference of $b \sin\theta$, we have

$$2\alpha = \frac{2\pi}{\lambda}\, b \sin\theta \qquad \text{or} \qquad \alpha = \frac{\pi}{\lambda}\, b \sin\theta$$

It shows that α depends on the angle of diffraction θ and therefore, $\sin^2\alpha/\alpha^2$ gives the intensity at different values of θ.

1.1 Position of Central (Principal) Maximum

For the central point C on the screen, we have $\theta = 0$ and therefore, $\alpha = 0$. Thus, we have

$$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1$$

and the intensity at C is $I = I_0$.

1.2 Position of Minima

The intensity is minimal when

$$\frac{\sin \alpha}{\alpha} = 0$$

such that $\sin \alpha = 0$, but $\alpha \neq 0$. Therefore, we have

$$\alpha = \pm m\pi \qquad \text{where } m = 1, 2, 3, \ldots$$

Hence, we have

$$\frac{\pi}{\lambda}\, b \sin \theta = \pm m\pi \qquad \text{or} \qquad b \sin \theta = \pm m\lambda$$

This is the condition for m-th order minimum.

1.3 Secondary Maxima

The direction of m-th order maxima is expressed as

$$b \sin \theta = \pm\left(m + \frac{1}{2}\right)\lambda$$

This is equivalent to

$$\alpha = \frac{\pi}{\lambda}\, b \sin \theta = \pm\frac{\pi}{\lambda}\left(m + \frac{1}{2}\right)\lambda = \pm\left(m + \frac{1}{2}\right)\pi$$

Therefore, we have

$$\alpha = \pm\frac{3\pi}{2}, \quad \pm\frac{5\pi}{2}, \quad \pm\frac{7\pi}{2}, \ldots$$

where the secondary maxima are situated. The intensity of the first secondary maximum is

$$I_1 = I_0\left(\frac{\sin 3\pi/2}{3\pi/2}\right)^2 = \frac{4}{9\pi^2}\ I_0$$

The intensity of the second secondary maximum is

$$I_2 = I_0\left(\frac{\sin 5\pi/2}{5\pi/2}\right)^2 = \frac{4}{25\pi^2}\ I_0$$

1.4 Ratio of Intensities of Secondary Maxima

The ratio of intensities of the successive maxima is

$$I_0 : \frac{4}{9\pi^2}\ I_0 : \frac{4}{25\pi^2}\ I_0 : \frac{4}{49\pi^2}\ I_0 : \ldots \qquad \text{or} \qquad 1 : \frac{4}{9\pi^2} : \frac{4}{25\pi^2} : \frac{4}{49\pi^2} : \ldots$$

1.5 Width of Central Maximum

Suppose, y is the distance of the first minimum from the center of the principal maximum. The width of the central maxima is $w = 2y$. For small angle θ, we have

$$\tan\theta = \sin\theta = \frac{y}{D}$$

The condition for the first minimum is

$$b\sin\theta = \pm\lambda \qquad \text{or} \qquad \sin\theta = \pm\frac{\lambda}{b}$$

Thus, we have

$$\frac{y}{D} = \pm\frac{\lambda}{b} \qquad \text{or} \qquad y = \pm\frac{\lambda D}{b}$$

Thus, the width of the central maxima is

$$w = 2y = \frac{2\lambda D}{b}$$

It shows that the size of the central maximum increases with the increase of distance of the screen from the slit, and with the decrease of the width of the slit.

1.6 Effect of the Slit Width

We have

$$\sin\theta = \pm\frac{\lambda}{b}$$

(i) When b is large, $\sin\theta$ is small, and consequently, the angle θ is small. It shows that the maxima and minima are close to the central maximum.

(ii) When b is small, $\sin\theta$ is large, and consequently, the angle θ is large. It shows that the maxima and minima are quite distinct and clear.

Exercise 1 A light of wavelength 5500 Å falls normally on a slit of width 2.3 μ m. Calculate the angular displacement of the second and third minima.

Solution Given, $b = 2.3 \times 10^{-6}$ m and $\lambda = 5500$ Å = 5.5×10^{-7} m. For the second minima, the angular displacement θ is expressed as

$$\sin\theta = \frac{2\lambda}{b} = \frac{2 \times 5.5 \times 10^{-7}}{2.3 \times 10^{-6}} = 0.4783$$

or

$$\theta = \sin^{-1}(0.4783) = 28.57^\circ$$

For the third minima, the angular displacement θ is expressed as

$$\sin\theta = \frac{3\lambda}{b} = \frac{3 \times 5.5 \times 10^{-7}}{2.3 \times 10^{-6}} = 0.7174$$

or

$$\theta = \sin^{-1}(0.7174) = 45.84^\circ$$

The angular displacements for second and third order maxima are 28.57° and 45.84°.

Exercise 2 In the Fraunhofer diffraction, a narrow slit of width 2.4 μ m is illuminated by monochromatic light of wavelength 5600 Å. Calculate the half angular width of the central maximum.

Solution Given, $b = 2.4 \times 10^{-6}$ m and $\lambda = 5600$ Å = 5.6×10^{-7} m. The first minimum in the diffraction pattern is in the direction of the half angular width of the central maximum. Thus, we have

$$\sin\theta = \frac{m\lambda}{b} = \frac{5.6 \times 10^{-7}}{2.4 \times 10^{-6}} = 0.2333$$

or

$$\theta = \sin^{-1}(0.2333) = 13.49^\circ$$

Thus, the half angular width of the central maximum is 13.49°.

Exercise 3 In the Fraunhofer diffraction, a beam of monochromatic light of wavelength 5600 Å incident normally on a slit. The central maximum fans out at 15° on both sides of the central position. Calculate the width of the slit.

Solution Given, $\lambda = 5600$ Å = 5.6×10^{-7} m, $\theta = 15^\circ$ for the first minimum. The width of the slit is

$$b = \frac{\lambda}{\sin\theta} = \frac{5.6 \times 10^{-7}}{\sin 15^\circ} = \frac{5.6 \times 10^{-7}}{0.2588}$$

$$= 2.16 \times 10^{-6} \text{ m} = 2.16\ \mu\text{m}$$

Exercise 4 In the Fraunhofer diffraction, a beam of monochromatic light of wavelength 5700 Å incident normally on a slit of width 0.1 mm. Calculate the angular displacement and linear size of the central maximum formed on a screen placed 120 cm away from the slit.

Solution Given, $b = 0.1$ mm = 0.1×10^{-3} m and $\lambda = 5700$ Å = 5.7×10^{-7} m, $x_0 = 120$ cm = 1.2 m. For the first minimum, we have

$$\sin\theta = \frac{\lambda}{b} = \frac{5.7 \times 10^{-7}}{0.1 \times 10^{-3}} = 5.7 \times 10^{-3}$$

or

$$\theta = \sin^{-1}(5.7 \times 10^{-3}) = 0.3265^\circ$$

Thus, the total spread of the central maximum is

$$2\theta = 2 \times 0.3265 = 0.653^\circ$$

For the linear width z of the half of the central maximum is

$$z = D\ \sin\theta = 1.2 \times 5.7 \times 10^{-3} = 6.84 \times 10^{-3}\ \mathrm{m}$$

Therefore, the total linear width of the central maximum is

$$2z = 2 \times 6.84 \times 10^{-3} = 1.368 \times 10^{-3}\ \mathrm{m}$$

Exercise 5 In the Fraunhofer diffraction, a slit is illuminated by a light having two wavelengths λ_1 and λ_2. It is found that the first minimum obtained for λ_1 coincides with the second minimum obtained for λ_2. Find out the relation between λ_1 and λ_2.

Solution For the first minimum due to λ_1, we have

$$b \sin\theta = \lambda_1$$

For the second minimum due to λ_2, we have

$$b \sin\theta = 2\lambda_2$$

Therefore, we have

$$\lambda_1 = 2\lambda_2$$

Exercise 6 Calculate the angular width of central bright maximum in the Fraunhofer diffraction pattern of a slit of width 2.6 μ m illuminated by a monochromatic light of wavelength 5800 Å.

Solution Given, $b = 2.6 \times 10^{-6}$ m and $\lambda = 5800$ Å = 5.8×10^{-7} m. For the first minimum, we have

$$\sin\theta = \frac{\lambda}{b} = \frac{5.8 \times 10^{-7}}{2.6 \times 10^{-6}} = 0.223 \qquad \text{or} \qquad \theta = \sin^{-1}(0.223) = 12.89^\circ$$

Therefore, the total angular width of the central maximum is

$$2\theta = 2 \times 12.89 = 25.78^\circ$$

2 Fraunhofer Diffraction at a Double Slit

Suppose, a parallel, collimated beam of monochromatic light of wavelength λ be incident normally on two parallel slits AB and DE as shown in Fig. 3. For convenience, we consider the slits of equal width b (= AB = DE) and separated by an opaque distance d (= BD). The distance between two corresponding points of two slits is $(b + d)$. Suppose, the diffracted light is focused with the help of a convex lens on the screen placed in the focal plane of the lens. Obviously, all the secondary waves produced at the slits S_1 and S_2, and propagating parallel to MC (the direction of the incident wave) get focused at a point C. Hence, the point C corresponds to the position of the central bright maximum.

Here, we consider that the two slits are equivalent to two coherent sources placed at the middle points S_1 and S_2 of the slits AB and DE, respectively. From the discussion of a single slit, we know that the amplitude R' due to each slit of width b is

$$R' = \frac{A \ \sin\alpha}{\alpha} \tag{3}$$

at a point P on the screen, making an angle θ with MC. Here,

$$\alpha = \frac{\pi b \sin\theta}{\lambda}$$

and A is a constant. We may consider that each slit is sending a wave of amplitude $A \sin\alpha/\alpha$ and these two waves produce an interference pattern. The resultant amplitude due to interference having a phase difference of ϕ can be calculated in the following manner. Draw a perpendicular S_1K from the point S_1 on S_2K. Thus, the path difference between the two rays reaching at the point P is

$$S_2K = (b + d)\sin\theta$$

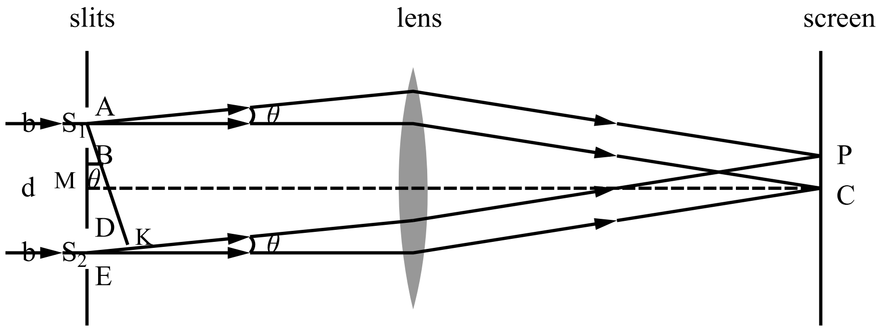


Fig. 3 Fraunhofer diffraction at double slit

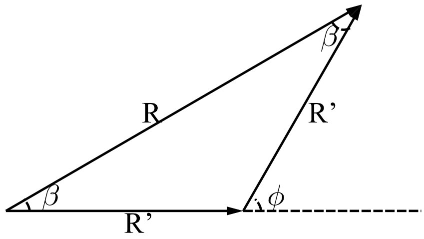


Fig. 4 Vector addition method for finding the resultant amplitude. R' and R' are amplitudes of the waves from the two slits and ϕ the phase difference between the two waves

The phase difference between them is

$$\phi = \frac{2\pi}{\lambda}\,(b + d)\sin\theta \tag{4}$$

The resultant amplitude R at the point P can be determined with the help of the vector addition method as shown in Fig. 4. Since both the slits are of the same size and therefore send the waves of the same amplitude R', we can write

$$\vec{R} = \vec{R}' + \vec{R}'$$

On squaring this equation, we get

$$R^2 = R'^2 + R'^2 + 2\,\vec{R}'\,.\,\vec{R}' = 2R'^2 + 2R'^2\cos\phi$$

$$= 2R'^2(1 + \cos\phi) = 4R'^2\cos^2(\phi/2) \tag{5}$$

From Fig. 4, we have

$$\phi = \beta + \beta \qquad \text{or} \qquad \beta = \phi/2 \tag{6}$$

From Eqs. (4) and (6), we have

$$\beta = \frac{\pi}{\lambda}\,(b + d)\sin\theta$$

Using Eqs. (3) and (6) in (5), we get

$$R^2 = 4A^2\,\frac{\sin^2\alpha}{\alpha^2}\,\cos^2\beta$$

Thus, the intensity I is expressed as

$$I = R^2 = 4I_0\,\frac{\sin^2\alpha}{\alpha^2}\,\cos^2\beta \tag{7}$$

where $I_0 = A^2$. Equation (7) shows that the resultant intensity depends on two factors:

(i) The factor $I_0 \sin^2 \alpha/\alpha^2$, which gives the diffraction pattern due to a single slit.
(ii) The factor $\cos^2 \beta$, which gives the interference pattern due to the waves from the two slits.

Thus, the resultant intensity I is a product of intensities obtained from the double slit interference and the single slit diffraction. Equation (7) shows that the maximum intensity is $4I_0$ when $\alpha = 0$ and $\beta = 0$. This intensity is four times that obtained in the case of a single slit.

When we analyze the diffraction factor $\sin^2 \alpha/\alpha^2$, we find that there is principal maximum in the direction of $\theta = 0$ on the screen at the point C. The positions of minima can be obtained in the direction $\sin \alpha = 0$ when $\alpha \neq 0$. Therefore,

$$\alpha = m\pi \qquad \frac{\pi b \sin\theta}{\lambda} = m\pi \qquad b\sin\theta = m\lambda \tag{8}$$

where $m = \pm 1, \pm 2, \pm 3, \ldots$. The factor $\sin^2 \alpha/\alpha^2$ gives secondary maxima at the points where

$$\alpha = \frac{(2m+1)\pi}{2} \qquad \text{or} \qquad b\sin\theta = \frac{(2m+1)}{2}\lambda$$

where $m = \pm 1, \pm 2, \pm 3, \ldots$. Notice that $|\alpha| \neq \pi/2$ as the first minimum is at $|\alpha| = \pi$.

When we analyze the variation of intensity due to the factor $\cos^2 \beta$. It gives the minimum intensity when

$$\cos^2\beta = 0 \qquad \cos\beta = 0 \qquad \beta = \frac{(2n+1)\pi}{2}$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Using the expression for β, we get

$$\frac{\pi}{\lambda}(b+d)\sin\theta = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad (b+d)\sin\theta = \frac{(2n+1)\lambda}{2}$$

For the maximum intensity in the interference factor, we have

$$\cos^2\beta = 1 \qquad \beta = n\pi$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Using the expression for β, we get

$$\frac{\pi}{\lambda}(b+d)\sin\theta = n\pi \qquad \text{or} \qquad (b+d)\sin\theta = n\lambda$$

2.1 Missing Orders of Diffraction Pattern

In the diffraction pattern due to a double slit, the width of each slit is b and the separation between the slits as d. When the width b is kept constant, the diffraction pattern remains the same. Keeping the value of b as constant and varying the value of d, the position of the interference maxima changes. Depending on the relative values of d and b, certain orders of interference maxima are missing in the resultant pattern. The directions of the interference maxima are expressed as

$$(b + d) \sin \theta = n\lambda \tag{9}$$

where n is an integer. The directions of diffraction minima are expressed as

$$b \sin \theta = m\lambda \tag{10}$$

where m is an integer. When the values of d and b are such that both the Eqs. (9) and (10) are satisfied simultaneously for the same value of θ, then the positions of certain interference maxima corresponds to the diffraction minima at the same position on the screen. Those interference maxima are missing in the diffraction pattern. On dividing Eq. (9) by (10), we get

$$\frac{b + d}{b} = \frac{n}{m}$$

(i) Let $b = d$, then we have

$$\frac{d + d}{d} = \frac{n}{m} \qquad \text{or} \qquad n = 2m$$

Therefore, for

$$m = 1, 2, 3, \ldots \qquad \text{we have} \qquad n = 2, 4, 6, \ldots$$

Hence, the orders 2, 4, 6, …of the interference maxima are missing in the diffraction pattern.

(ii) Let $d = 2b$, then we have

$$\frac{2b + b}{b} = \frac{n}{m} \qquad \text{or} \qquad n = 3m$$

Therefore, for

$$m = 1, 2, 3, \ldots \qquad \text{we have} \qquad n = 3, 6, 9, \ldots$$

Hence, the orders 3, 6, 9, ...of the interference maxima are missing in the diffraction pattern.

(iii) Let $d = 0$, then the two silts join and form a single slit of width $2b$. Moreover, we have $n = m$. Now, all the orders of the interference maximum are missing.

Exercise 7 In the Fraunhofer diffraction due to a narrow slit, a pattern is obtained on a screen placed 2 m away from the slit. The slit width is 0.2 mm and the first minima lies 5 mm away from the central maximum on either side. Find the wavelength of monochromatic light used in the experiment.

Solution We have b = 0.2 mm = 0.02 cm, z = 5 mm = 0.5 cm, $x_0 = 2$ m = 200 cm. In case of Fraunhofer diffraction at a narrow slit, the minima are expressed as

$$b\ \sin\theta = m\lambda$$

For the first minimum ($m = 1$), we have

$$\sin\theta = \frac{\lambda}{b}$$

We have

$$\sin\theta = \frac{z}{x_0}$$

Therefore, we have

$$\frac{z}{x_0} = \frac{\lambda}{b} \qquad \text{or} \qquad \lambda = \frac{zb}{x_0}$$

Using the values, we get

$$\lambda = \frac{zb}{x_0} = \frac{0.02 \times 0.5}{200} = 5 \times 01^{-5}\ \text{cm} = 5000\ \text{Å}$$

Exercise 8 In the Fraunhofer diffraction due to a narrow slit, a pattern is obtained on a screen placed at a large distance from the slit. The first minima on either side of the central maximum is observed in the direction making an angle of $\pm 12°$. For the monochromatic light of 5500 Å used in the experiment, calculate the width of the slit.

Solution For the first minimum in the Fraunhofer diffraction due to a narrow slit, we have

$$b\ \sin\theta = \lambda \qquad \text{or} \qquad b = \frac{\lambda}{\sin\theta}$$

Using the values, we get

$$b = \frac{\lambda}{\sin\theta} = \frac{5500 \times 10^{-8}}{\sin 12^\circ} = \frac{5500 \times 10^{-8}}{0.2079} = 2.65 \times 10^{-4}\ \text{cm} = 2.65\ \mu\text{m}$$

Exercise 9 In the Fraunhofer diffraction due to a narrow slit, calculate the relative intensities of the first, second, and third secondary maxima.

Solution In the Fraunhofer diffraction due to a narrow slit, the intensity of secondary maxima is expressed as

$$I = I_0\left(\frac{\sin^2\alpha}{\alpha^2}\right)^2 \qquad \text{where} \qquad \alpha = \frac{(2n+1)\pi}{2}$$

For the first, second and third secondary maxima, we have, respectively,

$$\alpha = \frac{3\pi}{2}, \qquad \alpha = \frac{5\pi}{2}, \qquad \alpha = \frac{7\pi}{2}$$

Relative to the intensity I_0 of the central maximum, the intensity of the first secondary maximum is

$$\frac{I_1}{I_0} = \left(\frac{\sin 3\pi/2}{3\pi/2}\right)^2 = 4.50 \times 10^{-2}$$

Relative to I_0, the intensity of the second secondary maximum is

$$\frac{I_2}{I_0} = \left(\frac{\sin 5\pi/2}{5\pi/2}\right)^2 = 1.62 \times 10^{-2}$$

Relative to I_0, the intensity of the third secondary maximum is

$$\frac{I_3}{I_0} = \left(\frac{\sin 7\pi/2}{7\pi/2}\right)^2 = 8.27 \times 10^{-3}$$

Exercise 10 Calculate the missing orders in a double slit Fraunhofer diffraction pattern when the width of each slit is 0.06×10^{-5} m and they are 0.03×10^{-5} m apart.

Solution The directions of interference maxima are expressed as

$$(b + d)\sin\theta = n\lambda$$

where n is an integer. The directions of diffraction minima are expressed as

$$b \sin\theta = m\lambda$$

where m is an integer. For missing maxima, we have

$$\frac{b+d}{b} = \frac{n}{m}$$

Using the values of b and d, we have

$$\frac{b+d}{b} = \frac{n}{m} \qquad \text{or} \qquad \frac{0.03 \times 10^{-5} + 0.06 \times 10^{-5}}{0.06 \times 10^{-5}} = 1.5 = \frac{n}{m}$$

Thus, we have $n = 1.5$ m.

3 Diffraction Grating

The Fraunhofer diffraction at two slits consists of diffraction maxima and minima given by $\sin^2\alpha/\alpha^2$ and the interference maxima and minima in each diffraction maximum governed by the factor $\cos^2\beta$. We have also seen that the maximum intensity produced by a double slit is four times larger than that of a single slit. Therefore, it is expected that a device having a large number of slits produces large intensity.

Let us consider a device having N slits. For convenience, we consider that each of the N slits is of equal width b and they are separated by equal opaque part of size d. Such a device is known as the transmission grating. It can be constructed by ruling a large number of fine, equidistant lines on a transparent plane glass plate with the help of a fine diamond point. The ruled lines are opaque to light whereas the space between any two lines is transparent to the light and acts as a slit. There are about 5000 lines per cm or more in such a grating.

Suppose, a parallel, collimated beam of monochromatic light of wavelength λ incident normally on diffraction grating having N slits, each of width b and separated by an opaque distance d. The factor $(b+d)$ is known as the grating element. Obviously, in the two successive slits, the middle points are separated by $(b+d)$. Let the diffracted light be focused by a convex lens on the screen placed in the focal plane of the lens. All the secondary waves from the slits traveling in the direction parallel to the direction of the incident wave are brought to focus at point C (Fig. 5), which corresponds to the position of the central bright maximum. The waves making an angle θ with the direction of incidence is focused at a point P.

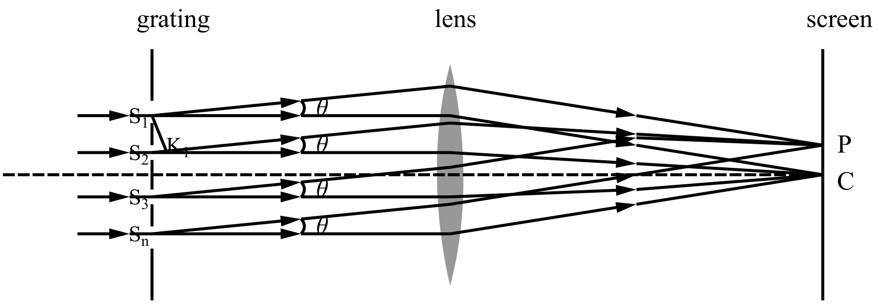


Fig. 5 Fraunhofer diffraction at transmission grating

We may assume that each slit in the grating is equivalent to an individual coherent source placed at the mid-point of each slit and sending a wave of amplitude

$$A' = A\,\frac{\sin\alpha}{\alpha} \tag{11}$$

at an angle θ with the direction of incident wave. Here, we have

$$\alpha = \frac{\pi}{\lambda}\,b\sin\theta$$

Let S_1K_1 be the perpendicular on S_2K_1. Then the path difference between the waves produced from S_1 and S_2 is

$$S_2K_1 = (b+d)\sin\theta$$

and the corresponding phase difference is

$$\frac{2\pi}{\lambda}\,(b+d)\sin\theta = 2\beta \quad \text{(say)} \qquad \text{or} \qquad \beta = \frac{\pi}{\lambda}\,(b+d)\sin\theta$$

To find the intensity I, we have to superimpose N waves each of amplitude A' differing in phase with near by wave by 2β.

Figure 6 shows the phasor-addition of N phasors. The angle between the two successive phasors is 2β. Therefore, the angle between the first and last phasors is $2N\beta$. The resultant of the phasors is OP. As the number N is large, the polynomial of phasors may be regarded as an arc of a circle with a center at C. We have

$$\mathrm{OP} = 2\,\mathrm{CO}\ \sin N\beta = 2r\sin N\beta$$

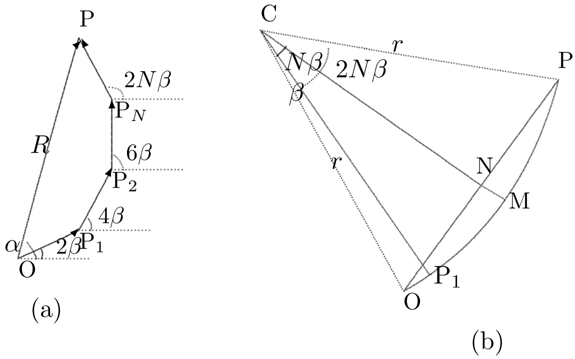


Fig. 6 The phasor diagram of N waves

For a single phasor OP_1, we have

$$OP_1 = 2\,CO\ \sin\beta = 2r\sin\beta$$

Therefore, we have

$$\frac{OP}{OP_1} = \frac{\sin N\beta}{\sin\beta}$$

Here, OP is the resultant disturbance and OP_1 disturbance of single slit, and therefore, we have

$$A_0 = A'\,\frac{\sin N\beta}{\sin\beta}$$

Using the expression for A' from Eq. (11), we have

$$A_0 = A\,\frac{\sin\alpha}{\alpha}\,\frac{\sin N\beta}{\sin\beta}$$

The resultant intensity is

$$I = A_0^2 = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2\left(\frac{\sin N\beta}{\sin\beta}\right)^2 \tag{12}$$

where $I_0 = A^2$. Equation (12) shows that the resultant intensity depends on two factors:

(i) The factor $I_0\sin^2\alpha/\alpha^2$, which gives the diffraction pattern due to a single slit.
(ii) The factor $\sin N\beta/\sin\beta$, which gives the interference pattern in the waves from N slits.

3.1 Principal Maximum

The intensity is the maximum when $\beta = n\pi$. For $\beta = n\pi$, we have

$$\frac{\sin N\beta}{\sin \beta} = \frac{\sin Nn\pi}{\sin n\pi} = \frac{0}{0} \quad \text{(indeterminate)}$$

Therefore, the value can be obtained after differentiation and we have

$$\lim_{\beta \to n\pi} \frac{\sin N\beta}{\sin \beta} = \lim_{\beta \to n\pi} \frac{N \cos N\beta}{\cos \beta} = \frac{N \cos Nn\pi}{\cos n\pi} = \pm N$$

So, the intensity at the central maximum is

$$I = I_0 N^2 \left(\frac{\sin \alpha}{\alpha}\right)^2 \tag{13}$$

For $\beta = n\pi$, we have

$$\frac{\pi}{\lambda}(b+d)\sin\theta = n\pi \qquad \text{or} \qquad (b+d)\sin\theta = n\lambda$$

For $n = 0$, we have zeroth order maximum and for $n = \pm 1, \pm 2, \ldots$, we have first, second, …order principal maximum, respectively.

3.2 Secondary Minima

The intensity is zero when $\sin N\beta = 0$ and $\sin\beta \neq 0$. For $N\beta = m\pi$, we have

$$N\frac{\pi}{\lambda}(b+d)\sin\theta = m\pi \qquad \text{or} \qquad N\,(b+d)\sin\theta = m\lambda$$

where $m = \pm 1, \pm 2, \ldots, \pm(N-1)$. If $m = 0$ gives the principal maximum and $m = N$ also gives the principal maximum. The $m = \pm 1, \pm 2, \ldots, \pm(N-1)$ gives minima. There are $(N-1)$ minima between two principal maxima.

3.3 Secondary Maxima

As there are $(N-1)$ minima between two maxima, there must be $(N-2)$ maxima between the two principal maxima. To find out the positions of these secondary maxima, we differentiate equation (12) with respect to β and equate to zero.

$$\frac{dI}{d\beta} = I_0 \left(\frac{\sin\alpha}{\alpha}\right)^2 \frac{\sin N\beta}{\sin\beta}\left[\frac{N\sin\beta\ \cos N\beta - \sin N\beta\ \cos\beta}{\sin^2\beta}\right] = 0$$

It gives

$$N\sin\beta\ \cos N\beta - \sin N\beta\ \cos\beta = 0 \qquad \text{or} \qquad \tan N\beta = N\tan\beta$$

We have

$$\sec N\beta = \sqrt{1+\tan^2 N\beta} = \sqrt{1+N^2\tan^2\beta}$$

It gives

$$\cos N\beta = \frac{1}{\sqrt{1+N^2\tan^2\beta}}$$

Thus, we have

$$\sin N\beta = \tan N\beta \cos N\beta = \frac{N\tan\beta}{\sqrt{1+N^2\tan^2\beta}}$$

Therefore, we have

$$\frac{\sin^2 N\beta}{\sin^2\beta} = \frac{(N^2\tan^2\beta)/(1+N^2\tan^2\beta)}{\sin^2\beta}$$

$$= \frac{N^2}{\cos^2\beta + N^2\sin^2\beta} = \frac{N^2}{1+(N^2-1)\sin^2\beta}$$

The intensity at the secondary maxima is

$$I = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2 \frac{N^2}{1+(N^2-1)\sin^2\beta} \tag{14}$$

From Eqs. (13) and (14), we get

$$\frac{\text{Intensity of secondary maxima}}{\text{Intensity of principal maximum}} = \frac{1}{1+(N^2-1)\sin^2\beta}$$

When N is large, the intensity of the secondary maxima is very small. Since in the transmission grating, N is very large, the secondary maxima are not visible in the transmission grating.

3.4 Angular Half-Width of Principal Maxima

The condition for the nth principal maximum in the direction θ_n in diffracting grating is

$$(b + d)\sin\theta_n = n\lambda \tag{15}$$

Let $(\theta_n + \mathrm{d}\theta_n)$ be the direction of the first secondary maximum on the two sides of the nth primary maximum as per positive and negative signs. Then we have

$$(b + d)\sin(\theta_n + \mathrm{d}\theta_n) = n\lambda \pm \frac{\lambda}{N} \tag{16}$$

On dividing Eq. (16) by (15), we get

$$\frac{\sin(\theta_n + \mathrm{d}\theta_n)}{\sin\theta_n} = \frac{n\lambda \pm \lambda/N}{n\lambda}$$

or

$$\frac{\sin\theta_n \ \cos\mathrm{d}\theta_n + \cos\theta_n \ \sin\mathrm{d}\theta_n}{\sin\theta_n} = 1 \pm \frac{1}{nN}$$

For the small value of $\mathrm{d}\theta_n$, we have $\sin\mathrm{d}\theta_n = \mathrm{d}\theta_n$ and $\cos\mathrm{d}\theta_n = 1$. Then, we have

$$\frac{\sin\theta_n + \cos\theta_n \ \mathrm{d}\theta_n}{\sin\theta_n} = 1 \pm \frac{1}{nN} \qquad \text{or} \qquad 1 + \frac{\cos\theta_n \ \mathrm{d}\theta_n}{\sin\theta_n} = 1 \pm \frac{1}{nN}$$

It gives

$$\mathrm{d}\theta_n = \frac{1}{nN\cot\theta_n}$$

Here, $\mathrm{d}\theta_n$ relates to half angular width of the principal maximum.

3.5 Missing Orders

The resultant intensity is

$$I = I_0 \left(\frac{\sin\alpha}{\alpha}\right)^2 \left(\frac{\sin N\beta}{\sin\beta}\right)^2 \tag{17}$$

where

$$\beta = \frac{\pi}{\lambda}\,(b + d)\sin\theta \qquad \text{and} \qquad \alpha = \frac{\pi b \sin\theta}{\lambda}$$

The direction of minima for a single slit pattern is expressed as

$$b \sin\theta = m\lambda \tag{18}$$

The direction of principal maxima in grating spectra is expressed as

$$(b + d)\sin\theta = n\lambda \tag{19}$$

If two conditions expressed by (18) and (19) are satisfied simultaneously then a particular maximum of nth order is absent in the grating spectrum. These are known as the absent spectra or missing order spectra. On dividing Eq. (19) by (18), we get

$$\frac{(b + d)}{b} = \frac{n}{m}$$

This is the condition for the missing order spectra in the grating pattern.

(i) When $d = b$, we have

$$\frac{n}{m} = 2 \qquad \text{or} \qquad n = 2m$$

For $m = 1, 2, 3, \ldots$, we have $n = 2, 4, 6, \ldots$. That is, the second, fourth, sixth, …order maxima are missing.

(ii) When $d = 2b$, we have

$$\frac{n}{m} = 3 \qquad \text{or} \qquad n = 3m$$

For $m = 1, 2, 3, \ldots$, we have $n = 3, 6, 9, \ldots$. That is, the third, sixth, ninth …order maxima are missing.

Exercise 11 Light of wavelength 500 nm falls normally on a plane transmission grating having 5000 line per cm. Find the angle of diffraction for the maximum intensity in the first order. Given $\sin^{-1} 0.25 = 14°47'$.

Solution Given, $\lambda = 500 \times 10^{-9}$ m $= 5 \times 10^{-5}$ cm, $N = 5000$. We have

$$(b + d) = \frac{1}{N} = \frac{1}{5000}$$

The direction of the first maximum is expressed as

$$(b+d)\sin\theta = \lambda \qquad \text{or} \qquad \sin\theta = \frac{\lambda}{(b+d)} = \frac{5\times 10^{-5}}{1/5000} = 0.25$$

The angle of diffraction is $\theta = \sin^{-1} 0.25 = 14°47'$.

4 Resolving Power of a Plane Diffraction Grating

Consider a parallel beam of light of wavelengths λ and $(\lambda + \mathrm{d}\lambda)$ incident normally on a plane transmission grating having grating element $(b+d)$ and total number of rulings N. The direction of n-th principal maximum for the wavelength λ is

$$(b+d)\sin\theta = n\lambda \tag{20}$$

The direction of n-th principal maximum for the wavelength $(\lambda + \mathrm{d}\lambda)$ is

$$(b+d)\sin(\theta + \mathrm{d}\theta) = n(\lambda + \mathrm{d}\lambda) \tag{21}$$

The equation of minima for wavelength λ is

$$N(b+d)\sin\theta = m\lambda \tag{22}$$

where m has all the integer values except 0, N, $2N$, $\ldots nN$, as for these values of m, the conditions for the maxima are satisfied. Thus, the first minimum adjacent to the n-th principal maximum in the direction $(\theta + \mathrm{d}\theta)$ can be obtained by substituting the value of m as $(nN + 1)$ in Eq. (22). Thus, the first minimum in the direction $(\theta + \mathrm{d}\theta)$ is

$$N(b+d)\sin(\theta + \mathrm{d}\theta) = (nN + 1)\lambda$$

or

$$(b+d)\sin(\theta + \mathrm{d}\theta) = n\lambda + \frac{\lambda}{N} \tag{23}$$

From Eqs. (21) and (23), we get

$$n(\lambda + \mathrm{d}\lambda) = n\lambda + \frac{\lambda}{N} \qquad \text{or} \qquad \frac{\lambda}{\mathrm{d}\lambda} = nN$$

Thus, the resolving power of grating is nN. This says that the number of rulings per centimeter of a grating should be larger in order to increase its resolving power.

Exercise 12 Light incident normally on a grating of total ruled width of 5.1×10^{-3} m with 2520 lines in all. Find out if two sodium lines with wavelengths 5890 Å and 5896 Å can be seen distinctly by the grating.

Solution Given, wavelengths λ_1 = 5890 Å = 5.89 $\times 10^{-7}$ cm, λ_2 = 5896 Å = 5.896 $\times 10^{-7}$ cm, N = 2520, $n = 1$ and width of ruling 5.1×10^{-3} m. Therefore, we have

$$(b+d) = \frac{5.1 \times 10^{-3}}{2520} = 2.024 \times 10^{-6} \text{ m}$$

For the first order ($n = 1$) of first line, we have

$$\sin\theta_1 = \frac{n\lambda_1}{(b+d)} = \frac{1(5.89 \times 10^{-7})}{2.024 \times 10^{-6}} = 0.2910$$

so that

$$\theta_1 = \sin^{-1}(0.2910) = 16.9178^\circ$$

For the first order ($n = 1$) of second line, we have

$$\sin\theta_2 = \frac{n\lambda_2}{(b+d)} = \frac{1(5.896 \times 10^{-7})}{2.024 \times 10^{-6}} = 0.2913$$

so that

$$\theta_2 = \sin^{-1}(0.2913) = 16.9358^\circ$$

The angular separation between the two lines is

$$\Delta\theta = \theta_2 - \theta_1 = 16.9358 - 16.9178 = 0.018^\circ$$

The resolving power of the grating is

$$\frac{\lambda}{\mathrm{d}\lambda} = \frac{5893 \times 10^{-10}}{6 \times 10^{-10}} = 982$$

As $nN = 2520$ is more than 982, the two lines of sodium are seen distinctly by the grating.

5 Dispersion Power of a Plane Diffraction Grating

We have seen that the light of different wavelengths gets dispersed/diffracted by a grating at different angles. In view of this, the angular dispersive power of a diffraction grating is defined as the rate of change of the angle of diffraction with the wavelength of light. It is expressed as

$$D = \frac{\mathrm{d}\theta}{\mathrm{d}\lambda}$$

For a plane transmission grating, the condition for the principal maxima is

$$(b + d) \sin\theta = n\lambda$$

On differentiation, we get

$$(b + d) \cos\theta \, \mathrm{d}\theta = n \, \mathrm{d}\lambda \qquad \text{or} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \frac{n}{(b + d)\cos\theta}$$

Thus, the dispersion power D of a plane diffraction grating is

$$D = \frac{n}{(b + d)\cos\theta}$$

Exercise 13 What is the highest order of spectrum which can be seen with a monochromatic light of wavelength 5500 Å by means of a diffraction grating having 4000 lines per cm.

Solution Given, $\lambda = 5500$ Å $= 5.5 \times 10^{-7}$ m, $N = 4000$ lines per cm. Therefore,

$$(b + d) = \frac{1}{4000} \text{ cm} = 2.5 \times 10^{-6} \text{ m}$$

For the maximum number of orders to be seen, we can have $\sin\theta = 1$. Thus, we have

$$n = \frac{(b + d)}{\lambda} = \frac{2.5 \times 10^{-6}}{5.5 \times 10^{-7}} = 4.55$$

The highest order to be seen is 4.

Exercise 14 How many orders are observed by a grating having 4260 lines per cm when it is illuminated by a white light having the range from 4000 Å to 7000 Å in wavelength.

Solution Given,

$$(b + d) = \frac{1}{4260}\ \text{cm} = \frac{10^{-2}}{4260}\ \text{m}$$

For the maximum number of orders to be seen, we can have $\sin\theta = 1$. Thus, we have

$$(b + d)\sin\theta = n\lambda \qquad \text{or} \qquad n = \frac{(b + d)}{\lambda}$$

For the wavelength $\lambda = 4000$ Å = 4×10^{-7} m, we have

$$n = \frac{(b + d)}{\lambda} = \frac{10^{-2}}{4260 \times 4 \times 10^{-7}} = 5.87$$

For the wavelength $\lambda = 7000$ Å = 7×10^{-7} m, we have

$$n = \frac{(b + d)}{\lambda} = \frac{10^{-2}}{4260 \times 7 \times 10^{-7}} = 3.35$$

The maximum number of order to be seen varies from 3 to 5 depending on the wavelength.

6 Multiple Choice Questions

1. In Fraunhofer diffraction, the incident wavefront should be
   A. elliptical B. spherical C. cylindrical D. plane
   Ans. D
2. In Fraunhofer diffraction at a single slit illuminated by a monochromatic light of wavelength λ, the width of the central maximum, observed at a screen placed at a distance D, is
   A. $2\lambda b/D$ B. $2\lambda D/b$ C. $2bD/\lambda$ D. $2D/b\lambda$
   Ans. B
3. In Fraunhofer diffraction a single slit of width b is illuminated by a monochromatic light of wavelength λ. The condition for minima is
   A. $b\cos\theta = \pm m\lambda$ B. $b\sin\theta = \pm m\lambda$
   C. $b\cos\theta = \pm(2m + 1)\lambda/2$ D. $b\sin\theta = \pm(2m + 1)\lambda/2$
   Ans. B
4. In Fraunhofer diffraction at a single slit when I_0 is the intensity of the central maximum, the intensity of the first secondary maximum is
   A. $I/4$ B. $4I_0/9\pi^2$ C. $I_0/2$ D. I_0
   Ans. B

5. In Fraunhofer diffraction at a single slit when I_0 is the intensity of the central maximum, the intensity of the first secondary maximum is
   A. $I/4$ B. $4I_0/9\pi^2$ C. $I_0/2$ D. I_0
   Ans. B
6. In Fraunhofer diffraction at a single slit when I_0 is the intensity of the central maximum, the intensity of the second secondary maximum is
   A. $I/4$ B. $4I_0/25\pi^2$ C. $I_0/2$ D. $I_0/8$
   Ans. B
7. In Fraunhofer diffraction two slits, each of width b, are separated by a distance d. When $b = d$, the order of missing minima is
   A. 1 B. 2 C. 3 D. none of these
   Ans. B
8. In Fraunhofer diffraction two slits, each of width b, are separated by a distance d. When $d = 2b$, the order of missing minima is
   A. 1 B. 2 C. 3 D. none of these
   Ans. C

7 Problems and Questions

1. Describe the Fraunhofer diffraction at a single slit.
2. Describe the Fraunhofer diffraction at the double slit.
3. Describe the resolving power of a plane diffraction grating.
4. Describe the dispersion power of a plane diffraction grating.
5. Describe the diffraction grating.
6. For diffraction grating, discuss the intensities of central maximum, secondary minima, and secondary maxima.
7. Show that the resolving power of a plane diffraction grating increases with the increase of the number of rulings in the grating.
8. Using the expression for intensity I

$$I = I_0 \left(\frac{\sin \alpha}{\alpha}\right)^2 \left(\frac{\sin N\beta}{\sin \beta}\right)^2$$

   for a diffraction grating having N slits, discuss the reason for absence of secondary maxima.
9. On what factors the dispersion power of a grating depends?
10. Write short notes on the following

    (i) Fraunhofer diffraction
    (ii) Resolving power of a plane diffraction grating
    (iii) Dispersion power of a plane diffraction grating.

Chapter 7
Polarization

Interference and diffraction phenomena in light, discussed in the preceding chapters, proved that the light is a wave motion and also enabled us to determine the wavelength of light. However, the phenomena were not found capable to decide whether the light waves are longitudinal or transverse in nature, and whether the vibrations are linear or circular.

Maxwell developed an electromagnetic theory of light and suggested that the light waves are electromagnetic waves which are transverse in nature. An electromagnetic wave consists of electric and magnetic fields vibrating perpendicular to each other and perpendicular to the direction of propagation of the wave. The vibrating electric vector $\vec{E}$ and the direction of propagation of the wave form a plane, known as the plane of polarization. In an unpolarized light, the electric vector vibrates in all the possible directions and corresponding to each electric vector there is a magnetic vector in the perpendicular direction.

The concept of transverse nature leads to the concept of polarization. Polarization cannot be found in case of a longitudinal wave. Light coming from a general source of light is unpolarized. The state of polarization however cannot be detected by unaided eyes. Polarized light has many important applications in industry and engineering. One of the most important applications is in the liquid crystal displays (LCDs) which are widely used in calculators, televisions, computers, watches, etc.

1 Polarized Light

A polarized light is an electromagnetic wave in which the direction of vibration of electric vector $\vec{E}$ is strictly confined to a single direction in a plane, perpendicular to the direction of propagation of the wave. Thus, a plane polarized light is a wave in which the electric vector $\vec{E}$ is everywhere confined to a single plane. A linearly polarized light is a wave in which the electric vector $\vec{E}$ oscillates in a given constant

S. Chandra and M. K. Sharma, *A Textbook of Optics*,
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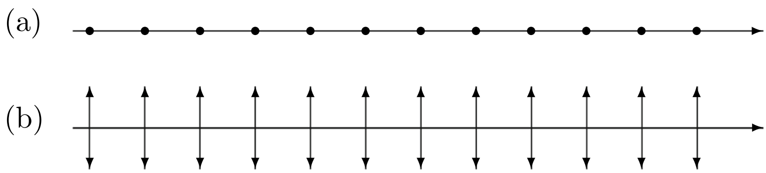


Fig. 1 Linearly polarized light

orientation. Linearly polarized light is shown in Fig. 1. In (a), the direction of oscillation of electric vector $\vec{E}$ is perpendicular to the plane of paper whereas in (b) it is in the plane of paper.

2 Production of Linearly Polarized Light

Linearly polarized light may be produced from an unpolarized light using the following optical phenomena:

(i) Reflection.
(ii) Refraction.
(iii) Scattering.
(iv) Selective absorption (dichroism).
(v) Double refraction.

2.1 *Polarization by Reflection*

Malus in 1808 discovered polarization of natural light by reflection from a surface of glass. He noticed that when a natural light falls on a plane sheet of glass at a certain angle, the reflected beam is plane polarized.

Let us consider a glass plate with refractive index μ relative to the air around the plate, as shown in Fig. 2. Suppose, a beam of light AB incidents on the plate at an incident angle i. The normal NBM defines the plane of incidence (which is the plane of paper). The electric vector E of the incident ray AB can be resolved into two components, one perpendicular to the plane of paper and the other lying in the plane of paper.

The perpendicular component is generally represented by dots and is known as the s-component. The parallel component is represented by arrows and is known as the p-component. In case of a completely unpolarized light, both the components are of equal magnitude. When a natural light is incident on the surface of plate, the two components in the reflected as well as in the refracted rays are of different extents.

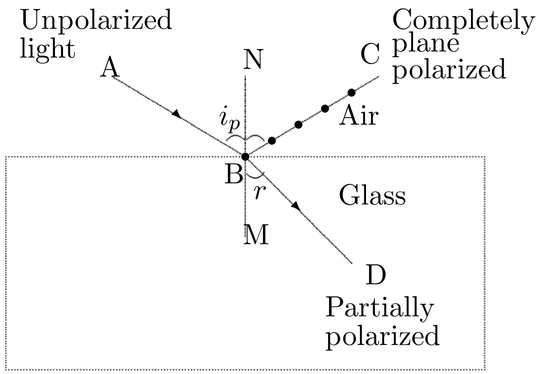


Fig. 2 Polarization by reflection

The reflected ray BC has a predominance of s-component and as such it is partially polarized. Obviously, the refracted ray has predominance of p-component.

When we vary the incidence angle, the degree of polarization of both the reflected as well as refracted rays changes. At a particular value i_p for the incident angle, the reflected ray becomes completely plane polarized. This angle is known as the polarizing angle. However, the refracted ray always remains partially polarized.

Brewster's Law

Based on a series of experiments on the polarization of light by reflection at a number of surfaces, Brewster found that the polarizing angle depends on the refractive index of the material of the reflecting surface. If i_p is the polarizing angle and μ the refractive index of the material of the reflecting surface, Brewster's law may be expressed as

$$\mu = \tan i_p$$

This angle i_p is also known as the Brewster angle. Brewster's law is equivalent to say that when a reflected ray is polarized, the angle between the reflected ray and the refracted ray is 90°. If r be the angle of refraction, for the maximum polarization of the reflected ray, we have

$$i_p + r = 90^\circ \qquad \text{or} \qquad r = 90 - i_p$$

According to Snell's law, we have

$$\mu = \frac{\mu_2}{\mu_1} = \frac{\sin i_p}{\sin r}$$

Using the value of r here, we have

$$\mu = \frac{\sin i_p}{\sin(90 - i_p)} = \frac{\sin i_p}{\cos i_p} = \tan i_p$$

It is Brewster's law for polarization by reflection.

Exercise 1 Refractive index of glass is 1.5. Calculate the polarizing angle for polarization by reflection.

Solution The polarizing angle i_p and the refractive index μ are related as

$$\mu = \tan i_p$$

For the given value $\mu = 1.5$, the polarizing angle i_p is

$$i_p = \tan^{-1} \mu = \tan^{-1} 1.5 = 56.3^\circ$$

Thus, when an unpolarized ray is incident at an angle of 56.3°, the reflected ray is plane polarized.

2.2 Polarization by Refraction

When an unpolarized light incidents at Brewster's angle i_p, the reflected light is totally polarized whereas the refracted light is partially polarized as shown in Fig. 3. If natural light is transmitted through a single plate, the transmitted beam is only partially polarized.

If a stack of glass plates made up of the same material is used instead of a single plate, the reflections from the successive surfaces occur leading to the filtering of the s-component in the transmitted ray. That is, in each reflection, the s-component is reflected and the transmitted ray has the p-component. Finally, the transmitted ray consists of p-component alone. Suppose, I_p and I_s are the intensities of parallel and perpendicular components in the refracted light, the degree of polarization of the transmitted light is

$$P = \frac{I_p - I_s}{I_p + I_s} = \frac{m}{m + [2\mu/(1 - \mu^2)]^2}$$

where m is the number of plates required and μ is the refractive index of the material. It is found that a stack of about 15 glass plates is sufficient for this purpose (Fig. 4).

The glass plates are placed in a tube and are kept inclined at an angle of about $(90^\circ - i_p)$ to the axis of the tube. Such an arrangement is known as a pile of plates. Unpolarized light enters the tube and incidents on the plates at the angle i_p, and the transmitted light is polarized.

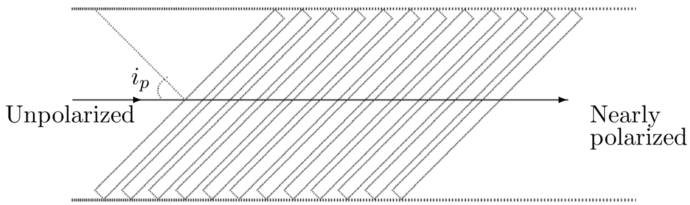
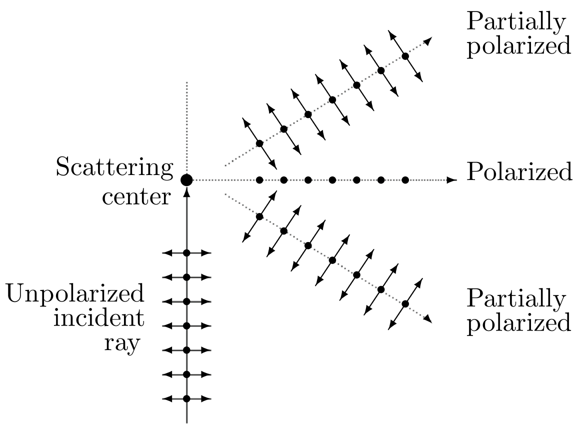


Fig. 3 Polarization by refraction

Fig. 4 Polarization by scattering



2.3 Polarization by Scattering

When a narrow beam of natural light incidents on a transparent medium containing a suspension of ultra-microscopic particles, the scattered light is partially polarized. The degree of polarization depends on the angle of scattering. The ray scattered at 90° with respect to the direction of incident ray is nearly polarized. When the direction of propagation of unpolarized light is in the plane of the paper, the direction of vibration of electric vector $\vec{E}$ in the scattered light is perpendicular to the plane of the paper.

Sunlight scattered by air molecules is polarized. The maximum effect is observed on a clear day when the sun is near the horizon. The light reaching an observer on the ground from directly overhead is polarized to the extent of 70% to 80%.

2.4 Polarization by Selective Absorption

In 1815, Biot discovered that certain mineral crystals absorb light selectively. When an unpolarized light incidents on a crystal such as tourmaline, it is split into two components, which are linearly polarized in the mutually perpendicular planes. The crystal absorbs light which is polarized in a direction parallel to a particular plane

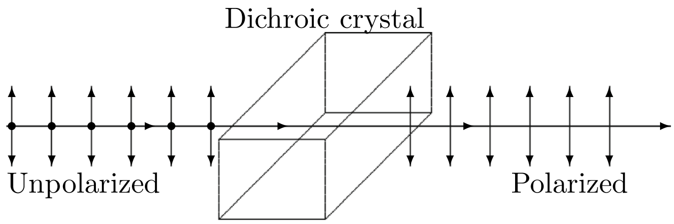


Fig. 5 Polarization by selective absorption

in the crystal but allows free transmission of other component which is polarized in the perpendicular direction.

Such difference in the absorption for the rays is known as the selective absorption or dichroism. When the crystal is of proper thickness, one of the components is totally absorbed and the other component emerging from the crystal is linearly polarized. Polarization by selective absorption is shown in Fig. 5. A good example of dichroic crystal is a tourmaline crystal.

2.5 Polarization by Double Refraction

The phenomenon of double refraction was discovered in 1669 by Erasmus Bartholin during his studies on calcite ($CaCO_3$) crystals. When a light incidents on a calcite crystal, it is split into two refracted rays, called the o-ray (ordinary ray) and e-ray (extraordinary ray) as shown in Fig. 6.

The o-ray and e-ray differ in their properties. The o-ray has the same velocity in all directions in the crystal whereas the velocity of e-ray depends on the direction of propagation. Both the o-ray and e-ray have equal velocity along only one direction, called the optical axis. The phenomenon of producing two refracted rays by a crystal is known as the birefringence or double refraction. Such a crystal is said to be birefringent. The two rays produced in double refraction are linearly polarized with their planes of polarization in the mutually perpendicular directions. One of these

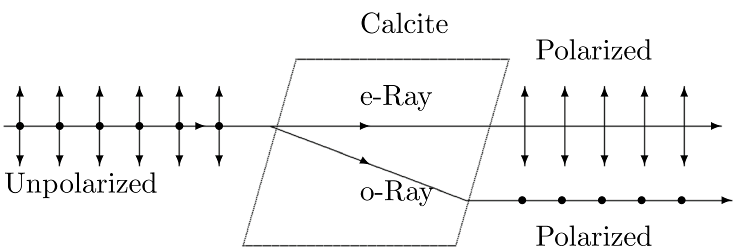


Fig. 6 Polarization by double refraction

refracted rays, called the o-ray or ordinary ray, obeys Snell's law of refraction. But the other ray, called the e-ray or extraordinary ray, does not obey Snell's law of refraction.

Finally, when one of these two refracted rays is eliminated, the light transmitted through the crystal is plane polarized.

3 Polarizer and Analyzer

An optical device which is used for polarization of an unpolarized light is known as the polarizer. Figure 7 shows that when an unpolarized light incidents on a polarizer, the emergent light is polarized. When the emergent light is linearly polarized, it is known as a linear polarizer. A linear polarizer is associated with a specific direction, called the transmission axis of the polarizer. When unpolarized (natural) light is incident on a linear polarizer, only that vibration which is parallel to the transmission axis is allowed to pass through the polarizer. On the other side, the vibration which is perpendicular to the transmission axis is totally stopped by the polarizer.

In Fig. 7, suppose we rotate the polarizer about the direction of propagation of light, the transmission axis and the direction of vibration of plane polarized light are rotated correspondingly.

An analyzer is a device which is used to identify the direction of vibration of linearly polarized light as shown in Fig. 8. When a polarized light incidents on an analyzer, the transmitted light may be plane polarized. When we rotate the analyzer about the direction of propagation of light, the intensity of the transmitted light varies. The intensity is the maximum when the direction of vibrations in the incident light is parallel to the transmission axis and the minimum when the direction of vibrations in the incident light is perpendicular to the transmission axis.

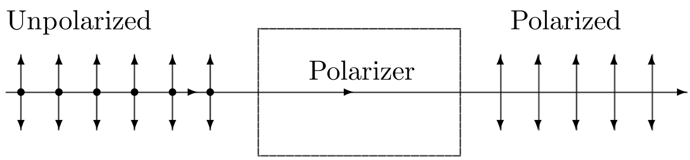


Fig. 7 Polarization of unpolarized light

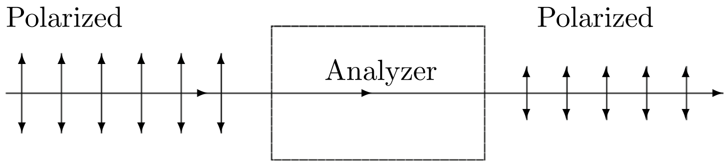


Fig. 8 Analyzing of a polarized light

A polarizer and an analyzer are fabricated in the same way and have the same effect on the incident light. That is, they are the same and different names are given depending on its use.

4 Nicol Prism

For practical purpose, a linear polarizer or analyzer is manufactured by using a birefringent crystal or dichroic crystal. Nicol prism is widely used for the construction of a polarizer or analyzer.

A Nicol prism is made up of calcite crystal. It was designed by William Nicol in 1820. A rhomb of calcite crystal whose length is about three times of its thickness is obtained by cleavage from an original crystal. The ends of the rhombohedron are grounded until they subtend an angle of 68° instead of 71° with the longitudinal edges.

This piece is then cut into two parts along a plane perpendicular to both the principal axis and the new end surfaces. The two parts of the crystal are then cemented together with Canada balsam, whose refractive index lies between the refractive indices of calcite for the o-ray and e-ray. The refractive indices of calcite for the o-ray and e-ray are $\mu_o = 1.66$ and $\mu_e = 1.486$. The refractive index of Canada balsam is $\mu_{cb} = 1.55$. The position of the optic axis AB is shown in Fig. 9.

The refractive index for o-ray is the same in all directions in the crystal whereas that for the e-ray depends on the direction of propagation in the crystal. The refractive index for both the o-ray and e-ray is the same along the optic axis. The difference between the refractive index of o-ray and that of e-ray goes on increasing with the angle between the direction of propagation of the ray and the optic axis.

An unpolarized light is made to fall on a crystal as shown in Fig. 9 at an angle of about 15°. The incident ray after entering the crystal suffers double refraction and splits into o-ray and e-ray. Both of these rays are linearly polarized. However, their planes of polarization are perpendicular to each other. The values of refractive

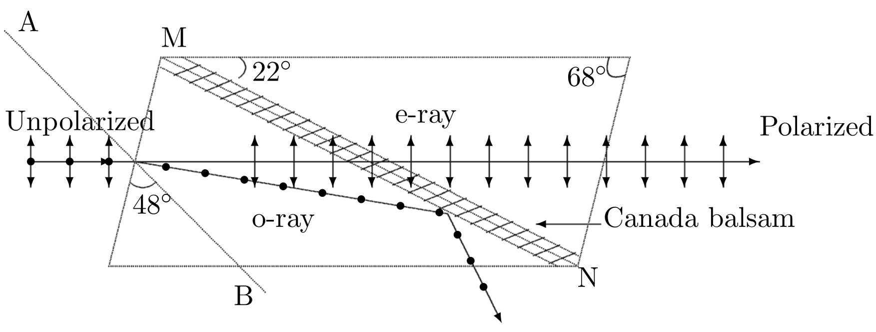


Fig. 9 Nicol prism

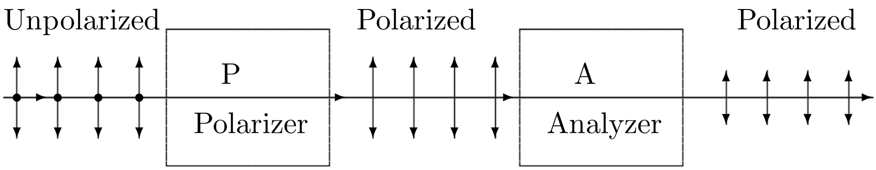


Fig. 10 Working of polarizer and analyzer

indices and the angles of incidence at the Canada balsam layer are such that the e-ray is transmitted through the crystal whereas the o-ray is reflected and propagates in other direction. The face where the o-ray incidents is blackened so that the o-ray is completely absorbed. Thus, we get only the linearly polarized e-ray emerging out of the Nicol prism with the direction of vibration as shown in Fig. 9. This way, the Nicol prism works as a polarizer.

The Nicol prism is widely used as a polarizing device. Nicol prisms are good polarizers, but they are expensive and have a limited field of view of about 28°. For the study of optical properties of transparent substances, two Nicol prisms are used—one as polarizer and the other as analyzer.

4.1 Working

Nicol prism can be used for production and for analysis of linearly polarized light. These components are appropriately mounted in a metal ring structure such that the transmission axis can be rotated and the amount of rotation may be measured with the help of the scale attached to the ring.

When two Nicol prisms P (polarizer) and A (analyzer) are placed adjacent to each other as shown in Fig. 10, one of them acts as a polarizer and the other acts as an analyzer. When an unpolarized ray of light incidents on the Nicol prism P, a linearly polarized e-ray emerges out from P with its vibration in the direction lying in the principal section of P. The state of polarization of the light emerging from P can be examined with the help of the Nicol prism A. This prism A is known as the analyzer.

Let the ray of light emerging from P incident on the other Nicol prism A, whose principal section is parallel to that of P. The vibration direction of the ray is in the principal section of A and therefore it is transmitted unhindered through the analyzer A.

When the Nicol prism A is gradually rotated, the intensity of the light emerging out from A decreases in accordance with Malus law. When its principal section becomes perpendicular to that of P, the intensity of the light emerging out from A becomes zero. On further rotation of A, the intensity of the light emerging out from A goes on increasing and finally becomes the maximum. This is the situation when the principal sections of both P and A are parallel to each other.

Thus, the prism P produces linearly polarized light and the prism A analyzes it. Therefore, the prism P is known as the polarizer and A the analyzer.

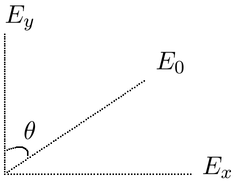
Exercise 2 An unpolarized light of intensity I_0 passes through a polarizer. Show that the intensity of light transmitted through the polarizer is $I_0/2$.

Solution Let E_0 be the amplitude of vibration of one of the waves of unpolarized ray which incidents on the polarizer. The intensity of unpolarized light is

$$I_0 = E_0^2$$

Let E_0 make an angle θ with the transmission axis of the polarizer. The E_0 may be resolved into its components E_x and E_y along the x- and y-axes, so that

$$E_x = E_0 \sin\theta \qquad \text{and} \qquad E_y = E_0 \cos\theta$$



The direction of propagation of the ray of light is perpendicular to the xy-plane. Let the polarizer block the component E_x and transmit the component E_y.[1]
The intensity of the transmitted light is

[1] Suppose, the polarizer blocks the component E_y and transmits the component E_x. The intensity of the transmitted light is

$$I = E_x^2 = E_0^2 \sin^2\theta = I_0 \sin^2\theta$$

In an unpolarized light, all the values of θ, ranging from 0 to 2π, are equally probable. Thus, the intensity of the transmitted light is

$$I = I_0 \langle \sin^2\theta \rangle$$

The average value of $\sin^2\theta$ is

$$\langle \sin^2\theta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2} \right) d\theta$$

$$= \frac{1}{4\pi} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} = \frac{1}{2}$$

Thus, the intensity of the transmitted light is

$$I = \frac{I_0}{2}$$

$$I = E_y^2 = E_0^2 \cos^2\theta = I_0 \cos^2\theta$$

In an unpolarized light, all the values of θ, ranging from 0 to 2π, are equally probable. Thus, the intensity of transmitted light is

$$I = I_0 \left\langle \cos^2\theta \right\rangle$$

The average value of $\cos^2\theta$ is

$$\left\langle \cos^2\theta \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{4\pi}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{1}{2}$$

Thus, the intensity of the transmitted light is

$$I = \frac{I_0}{2}$$

4.2 *Malus Law*

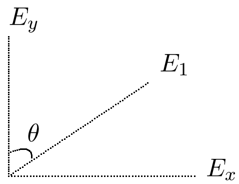
When an unpolarized light incidents on a polarizer, the transmitted light is linearly polarized. Suppose, this polarized light further incidents on a analyzer, the intensity of light transmitted through the analyzer varies with the angle between the transmission axis of the polarizer and that of the analyzer. Malus studied the variation of intensity of light emerging out of the analyzer and formulated a law which states that the intensity of the polarized light transmitted through an analyzer is proportional to cosine square of the angle between the plane of transmission of the analyzer and the plane of transmission of the polarizer. This is known as the Malus law.

Let E_1 be the amplitude of vibration of linearly polarized light incidents on the analyzer. The intensity of the polarized light is

$$I_0 = E_1^2$$

Let E_1 make an angle θ with the transmission axis of the polarizer. The E_1 may be resolved into its components E_x and E_y along the x- and y-axes, so that

$$E_x = E_1 \sin\theta \qquad \text{and} \qquad E_y = E_1 \cos\theta$$



Suppose, the analyzer blocks the component E_x and transmits the component E_y. The intensity of the light transmitted through the analyzer is

$$I = E_y^2 = E_1^2 \cos^2\theta = I_0 \cos^2\theta$$

The polarized light falling on the analyzer has only one direction of polarization. This relation is known as the Malus law.

Exercise 3 Four perfect polarizing plates are staked together so that the axis of each one is turned through 30° clockwise with respect to the preceding one. How much of the intensity of an incident unpolarized light is transmitted?

Solution For the incident intensity I_0, the intensity I of emergent light is

$$I = \frac{I_0}{2}\ \cos^2(30^\circ)\ \cos^2(30^\circ)\ \cos^2(30^\circ) = \frac{I_0}{2}\frac{3}{4}\frac{3}{4}\frac{3}{4} = \frac{27 I_0}{128}$$

5 Multiple Choice Questions

1. An unpolarized light has electric vector $\vec{E}$ and magnetic vector $\vec{H}$. Which of the following statements is wrong?

   A. $\vec{E}$ and $\vec{H}$ both are perpendicular to the direction of propagation of light
   B. $\vec{E}$ and $\vec{H}$ both are perpendicular to each other
   C. $\vec{H}$ is along whereas $\vec{E}$ is perpendicular to the direction of propagation of light
   D. None of these

   Ans. C

2. An unpolarized light has electric vector $\vec{E}$ and magnetic vector $\vec{H}$. Which of the following statements is wrong?

   A. $\vec{E}$ and $\vec{H}$ both are perpendicular to the direction of propagation of light
   B. $\vec{E}$ and $\vec{H}$ both are perpendicular to each other
   C. $\vec{E}$ is along whereas $\vec{H}$ is perpendicular to the direction of propagation of light
   D. None of these

   Ans. C

3. For polarization of light through reflection, the polarizing angle i_p and the refractive index of medium μ are related as

A. $\mu = \sin i_p$ B. $\mu = \cos i_p$ C. $\mu = \tan i_p$ D. $\mu = \cot i_p$

Ans. C

4. For polarization of light through reflection, the relation between the polarizing angle i_p and the angle of refraction r is

A. $i_p = 90 + r$ B. $i_p = 90 - r$ C. $i_p = r$ D. $i_p = 180 - r$

Ans. B

5. In a transmitted light, I_p and I_s are the intensities of parallel and perpendicular components. The degree of polarization P is expressed as

A. $P = \dfrac{I_p + I_s}{I_p - I_s}$ B. $P = \dfrac{I_p - I_s}{I_p + I_s}$

C. $P = \dfrac{I_p}{I_p + I_s}$ D. $P = \dfrac{I_s}{I_p + I_s}$

Ans. B

6. In the polarization by scattering in a medium with refractive index μ, the angle i_p between the polarized light and incident light is
   A. expressed as $i_p = \tan^{-1} \mu$ B. 90°
   C. expressed as $i_p = \tan^{-1}(1/\mu)$ D. 45°

Ans. B

7. In a birefringent, the o-ray (ordinary ray) and e-ray (extraordinary ray) are produced. Which of the following statements is wrong?
   A. Along the optic axis, both o-ray and e-ray travel with the same speed
   B. The o-ray follows Snell's law
   C. The e-ray follows Snell's law
   D. The speed of o-ray does not depend on the direction of propagation

Ans. C

8. In a birefringent, the o-ray (ordinary ray) and e-ray (extraordinary ray) are produced. Which of the following statements is wrong?
   A. Along the optic axis, both o-ray and e-ray travel with the same speed
   B. The o-ray follows Snell's law
   C. The e-ray does not follow Snell's law

D. The speed of o-ray depends on the direction of propagation

Ans. D

9. In a birefringent, the o-ray (ordinary ray) and e-ray (extraordinary ray) are produced. Which of the following statements is wrong?

   A. Along the optic axis, both o-ray and e-ray travel with different speeds
   B. The o-ray follows Snell's law
   C. The e-ray does not follow Snell's law
   D. The speed of o-ray does not depend on the direction of propagation

   Ans. A

10. In a Nicol prism, if the refractive indices for the o-ray (ordinary ray) and e-ray (extraordinary ray) are 1.66 and 1.486, respectively. The refractive index μ of Canada balsam used is
    A. $\mu > 1.66$ B. $1.486 < \mu < 1.66$ C. $\mu < 1.486$
    D. depends on the situation

    Ans. B
11. An unpolarized light of intensity I_0 falls on a polarizer. The intensity of polarized light is
    A. I_0 B. $I_0/4$ C. $I_0/2$ D. $2I_0$

    Ans. C
12. Suppose, I_0 be the intensity of polarized light falling on an analyzer and θ is the angle between the planes of transmission of polarizer and analyzer. The intensity of light transmitted through the analyzer varies as
    A. $\cos\theta$ B. $\sin\theta$ C. $\cos^2\theta$ D. $\sin^2\theta$

    Ans. C

6 Problems and Questions

1. Is it possible to have polarization of a longitudinal wave?
2. Show that for polarization by reflection, Brewster's law $\mu = \tan i_p$ is equivalent to say that the reflected and refracted rays should be at right angle to each other.
3. Describe the construction and working of a Nicol prism.
4. State and derive the Malus law.
5. An unpolarized light of intensity I_0 passes through a polarizer. Show that the intensity of the light transmitted through the polarizer is $I_0/2$.
6. Write short notes on the following:

   (i) Plane of polarization
   (ii) Polarization by reflection

(iii) Polarization by refraction
(iv) Polarization by scattering
(v) Polarization by selective absorption
(vi) Polarization by double refraction
(vii) Polarizer and analyzer
(viii) Nicol prism
(ix) Malus law.

Chapter 8
Lasers and Holography

The word LASER stands for the Light Amplification by Stimulated Emission of Radiation. When the stimulated emission occurs in the microwave region of the electromagnetic spectrum, it is termed as the MASER, which stands for the Microwave Amplification by Stimulated Emission of Radiations. While considering the equilibrium between matter and radiation, Einstein found that the usual absorption and emission (now called, spontaneous emission) processes alone were not sufficient to explain the equilibrium. Then, he predicted a third process, called the stimulated emission. This work was paid little attention until 1954 when Townes and Gordon constructed a MASER by using ammonia (NH_3).

In 1958, Schawlow and Townes showed that the principle of maser action could be extended into the visible region, and in 1960, Maiman developed the first laser by using ruby as the active medium. Soon after, Ali Javan (Bell Telephone Laboratories, USA) in 1960 constructed the first gas laser (He-Ne laser). Nowadays, we have a large number of lasers. These lasers have wide applications in our lives. In this chapter, we discuss the lasers.

It may also be worthwhile to state here that besides the lasers and masers developed in our laboratories on the earth, scientists have found some radiations from the interstellar medium, whose intensities are so large that they can only be understood in terms of the masing action. After the discovery of the first molecule OH through its 18 cm radiation in the interstellar medium in 1963, this radiation of 18 cm in 1965 was found to show the maser action. After the OH molecule, H_2O molecule through its transition at 1.35 cm was found in 1969 to show the maser action. There are other molecules, such as SiO, CH_3OH etc. which show the maser action.

S. Chandra and M. K. Sharma, *A Textbook of Optics*,
https://doi.org/10.1007/978-3-031-55676-0_8

1 Energy Levels

In an atom or ion, the electrons are bound. For example, from the Bohr theory of the hydrogen atom, we know that an electron revolves around the nucleus. In a bound system, the energy levels are quantized. That is, the energy of a system can have some discrete values. For example, the energies of three levels in a system are shown in Fig. 1. The levels are labeled as 1, 2, 3. The spacings between two adjacent levels are different from one another.

1.1 Excitation

When a system (electron) moves from a lower level to an upper level, the process is known as excitation. In excitation, obviously, the energy is supplied to the system from outside. This energy may be in the form of a photon (packet of electromagnetic energy), heat, or kinetic energy (due to collision).

1.2 Deexcitation

When a system (electron) moves from an upper level to a lower level, the process is known as deexcitation. In the deexcitation, obviously, the energy is released out from the system. This energy may be in the form of a photon, heat, or kinetic energy given to the colliding partner.

2 Radiative Transitions

A transition (excitation or deexcitation) between two levels is known as the radiative transition when a photon is involved. That is, in the excitation process, a photon

Fig. 1 Three levels in a system having discrete energies

of proper frequency (energy) is absorbed. In the deexcitation process, a photon of proper frequency (energy) is released. The radiative transitions are governed by some selection rules. Further, the energy of the photon is exactly equal to the energy difference between the two levels. Suppose, E_l and E_u are the energies of the lower and upper levels, respectively, for a transition. The proper frequency ν corresponding to the transition between these levels is

$$\nu = \frac{E_u - E_l}{h}$$

where h is the Planck constant. In a laser, there are radiative transitions. The kinds of radiative transitions are: (i) Stimulated absorption, (ii) Spontaneous emission, and (iii) Stimulated emission.

2.1 Stimulated Absorption

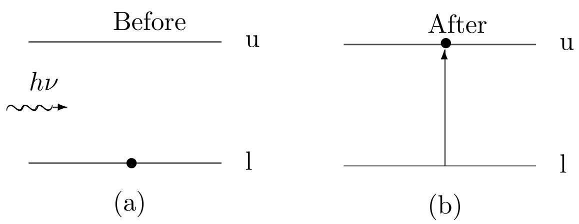
Suppose, a system is in a lower state l, as shown in Fig. 2a. After absorption of a photon of proper frequency ν (such that $h\nu = E_u - E_l$, where E_u and E_l are the energies of upper and lower levels, respectively), a system can go from a lower level l to an upper level u, provided the transition is radiatively allowed. Such absorption of photons is known as stimulated absorption or simply, absorption. Here, a photon of proper frequency stimulates the system to go from the lower level l to upper level u, as shown in Fig. 2b. Owing to the absorption of photons, the population density of the lower level decreases whereas that of the upper level increases.

The population of the upper level increases and that of the lower level decreases. The rate of increase of the population of the upper level is proportional to the population density n_l of the lower level and is proportional to the density $\rho(\nu)$ of radiations having frequency ν. Thus, the rate of increase of the population of the upper level is

$$\frac{\mathrm{d}n_u}{\mathrm{d}t} \propto n_l \rho(\nu) \qquad \text{or} \qquad \frac{\mathrm{d}n_u}{\mathrm{d}t} = B_{lu} n_l \rho(\nu)$$

Here, B_{lu} is the Einstein B-coefficient for absorption. It is also known as the radiative transition probability for absorption of a photon. It represents the probability of absorption of a photon of frequency ν.

Fig. 2 Stimulated absorption process



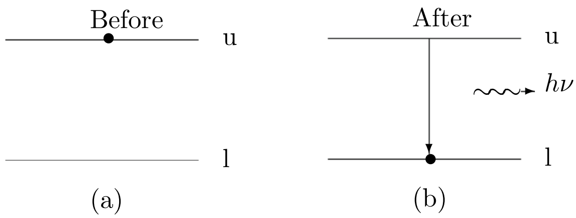


Fig. 3 Spontaneous emission process

2.2 *Spontaneous Emission*

Suppose, a system is in an upper (excited) state u, as shown in Fig. 3a. The system after staying there for a period, called the radiative lifetime of the level, can deexcite to a lower level, as shown in Fig. 3b, provided the transition is radiatively allowed. Since there is no interaction from outside, the probability of such deexcitation is proportional to the population density n_u of the upper state. In this process, a photon of proper frequency ν is emitted. The distribution of emission is spherically symmetric. That is, the probability of emission is equal in all the directions.

In the process, the population of the upper level decreases, and that of the lower level increases. The rate of decrease in population density at the upper level is proportional to the density of the upper level. That is,

$$-\frac{\mathrm{d}n_u}{\mathrm{d}t} \propto n_u \qquad \text{or} \qquad -\frac{\mathrm{d}n_u}{\mathrm{d}t} = A_{ul} n_u$$

where A_{ul} is the Einstein A-coefficient for spontaneous emission. It is also known as the radiative transition probability for spontaneous emission of a photon. It represents the probability of the emission of photons in the absence of any external agent. The unit of the Einstein A-coefficient is s^{-1}.

2.2.1 Radiative Life-Time of Level

For an optically allowed transition, the radiative lifetime T_u of the upper level is the inverse of Einstein A-coefficient for the transition. Thus, we have[1]

$$T_u = \frac{1}{A_{ul}}$$

Exercise 1 Einstein A-coefficient for an optically allowed transition is $4 \times 10^{-5}\ \mathrm{s}^{-1}$. Calculate the radiative lifetime of the upper level of the transition.

[1] When there is more than one downward radiative transition from the upper level, we have

$$T_u = \frac{1}{\sum_l A_{ul}}.$$

Solution Given, Einstein A-coefficient $A = 4 \times 10^{-5}$ s^{-1}. The radiative lifetime T_u of upper level of transition is

$$T_u = \frac{1}{A_{ul}} = \frac{1}{4 \times 10^{-5}} = 2.5 \times 10^4 \text{ s}$$

Exercise 2 The radiative life-time of a level is 10^{-8} s. Calculate the value of Einstein A-coefficient for the radiative transition.

Solution We have $T_u = 10^{-8}$ s. So the Einstein A-coefficient A_{ul} is

$$A_{ul} = \frac{1}{T_u} = \frac{1}{10^{-8}} = 10^8 \text{ s}^{-1}$$

Exercise 3 The radiative life-time of a metastable level is 10^{-3} s. Calculate the value of Einstein A-coefficient for the radiative transition.

Solution We have $T_u = 10^{-3}$ s. So the Einstein A-coefficient A_{ul} is

$$A_{ul} = \frac{1}{T_u} = \frac{1}{10^{-3}} = 10^3 \text{ s}^{-1}$$

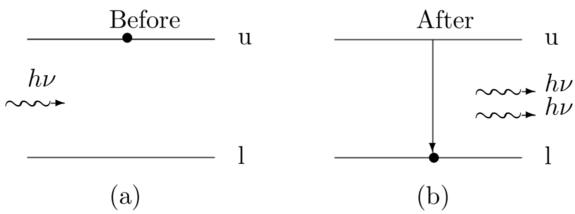
2.3 Stimulated Emission

Suppose, a system is in an upper (excited) state u, as shown in Fig. 4a. The system can be stimulated by a photon of proper frequency ν to deexcite to a lower level, as shown in Fig. 4b, provided the radiative lifetime of the upper level is sufficiently large and the transition is radiatively allowed. In this process, we get two photons of the same frequency. The population density of the upper level decreases and that of the lower level increases.

The rate of decrease of population density of the upper level is proportional to the population density of the upper level and proportional to the density $\rho(\nu)$ of radiations having frequency ν. Thus, we have

$$-\frac{dn_u}{dt} \propto n_u \rho(\nu) \qquad \text{or} \qquad -\frac{dn_u}{dt} = B_{ul} n_u \rho(\nu)$$

Fig. 4 Stimulated emission process



Here, B_{ul} is the Einstein B-coefficient for stimulated emission. It is also known as the radiative transition probability for the stimulated emission of a photon. It represents the probability of emission in the presence of photons of proper frequency.

In the case of stimulated emission, the emitted photon moves parallel to the direction of the incident photon, and both photons are in phase with each other.

2.4 Relation Between Einstein A and B Coefficients

In the last sections, we have introduced the Einstein A and B coefficients. Considering all the three processes, namely, absorption, spontaneous emission and stimulated emission, simultaneously, the rate of increase in population density of upper level u can be expressed as

$$\frac{\mathrm{d}n_u}{\mathrm{d}t} = B_{lu} n_l \rho(\nu) - B_{ul} n_u \rho(\nu) - A_{ul} n_u \tag{1}$$

Considering a system of atoms inside a cavity at a kinetic temperature T, the energy density of radiation with frequency ν is expressed as (Planck's formula for black-body radiation)

$$\rho(\nu) = B_\nu(T) = \frac{8\pi h \nu^3}{c^3} \frac{1}{\exp(h\nu/kT) - 1} \tag{2}$$

where h is the Planck constant and k the Boltzmann constant. When we consider a system with only two levels, upper level u and lower level l, we have the rate of increase of population density of the upper level equal to the rate of decrease of population density of lower level, and vice versa. That is, we have

$$\frac{\mathrm{d}n_u}{\mathrm{d}t} = -\frac{\mathrm{d}n_l}{\mathrm{d}t}$$

At an equilibrium, we have

$$\frac{\mathrm{d}n_u}{\mathrm{d}t} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}n_l}{\mathrm{d}t} = 0$$

and Eq. (1) gives

$$\rho(\nu)\left[\frac{n_l}{n_u} B_{lu} - B_{ul}\right] = A_{ul} \qquad \text{or} \qquad \rho(\nu) = \frac{A_{ul}}{n_l B_{lu}/n_u - B_{ul}} \tag{3}$$

Following the Boltzmann law, the populations n_u and n_l at the thermal equilibrium temperature T have relation[2]

$$\frac{n_u}{n_l} = \frac{\exp[-E_u/kT]}{\exp[-E_l/kT]} = \exp[-(E_u - E_l)/kT]$$

where E_u and E_l are the energies of upper and lower levels, respectively. Therefore, we have

$$\frac{n_u}{n_l} = \exp(-h\nu/kT) \tag{4}$$

where $h\nu = (E_u - E_l)$ is the energy difference between the two levels. Einstein assumed $B_{lu} = B_{ul}$. Using Eqs. (4) in (3), we get

$$\rho(\nu) = \frac{A_{ul}}{\exp(h\nu/kT)\; B_{ul} - B_{ul}} = \frac{A_{ul}}{B_{ul}} \frac{1}{[\exp(h\nu/kT) - 1]} \tag{5}$$

On comparing Eqs. (2) and (5), we get

$$\frac{8\pi h\nu^3}{c^3} \frac{1}{\exp(h\nu/kT) - 1} = \frac{A_{ul}}{B_{ul}} \frac{1}{[\exp(h\nu/kT) - 1]}$$

This relation gives

$$A_{ul} = \frac{8\pi h\nu^3}{c^3} B_{ul}$$

The relations between the Einstein A and B coefficients say that by knowing the A-coefficients, the B-coefficients can be calculated and vice versa.

Exercise 4 For a He-Ne laser operating at 632.8 nm, calculated the ratio of stimulated emission to spontaneous emission coefficients.

Solution The ratio of stimulated emission to spontaneous emission coefficients is

$$\frac{B_{ul}}{A_{ul}} = \frac{c^3}{8\pi h\nu^3} = \frac{\lambda^3}{8\pi h}$$

Using the values, we have

$$\frac{B_{ul}}{A_{ul}} = \frac{(632.8 \times 10^{-9})^3}{8 \times 3.14 \times 6.626 \times 10^{-34}} = 1.52 \times 10^{13}\ \text{m}^3/\text{Js}$$

[2] Here, we are considering non-degenerate energy levels. In fact the energy levels are degenerate.

2.5 Ratio of Rates

The ratio of the spontaneous emission rate to the stimulated emission rate is

$$\frac{\text{Spontaneous emission rate}}{\text{Stimulated emission rate}} = \frac{A_{ul}n_u}{B_{ul}n_u\rho(\nu)}$$

Using Eq. (5), we get

$$\frac{\text{Spontaneous emission rate}}{\text{Stimulated emission rate}} = \mathrm{e}^{h\nu/kT} - 1$$

When $h\nu >> kT$, the value of $(\exp(h\nu/kT) - 1)$ is very large and the spontaneous emission exceeds the stimulated emission. At the ordinary temperatures, this happens in the visible region. The stimulated emission becomes important when $h\nu \approx kT$, and may dominate when $h\nu << kT$. At ordinary temperatures, this happens in the microwave region. Thus, Townes tried the first fabrication of a maser (microwave amplification of stimulated emission of radiation).

Exercise 5 Calculate the ratio of spontaneous emission to stimulated emission at a temperature 27 °C for the radiations having wavelength (i) 6000 Å and (ii) 1.5 cm.

Solution Given, temperature $T = 273 + 27 = 300$ K. (i) For wavelength $\lambda = 6000$ Å, we have

$$\frac{h\nu}{kT} = \frac{hc}{\lambda kT} = \frac{(6.62 \times 10^{-34})(3 \times 10^8)}{(6000 \times 10^{-10})(1.38 \times 10^{-23})300} = 79.95$$

Thus, we have

$$\frac{\text{Spontaneous emission rate}}{\text{Stimulated emission rate}} = \mathrm{e}^{h\nu/kT} - 1 = \mathrm{e}^{79.95} - 1 = 5.27 \times 10^{34}$$

(ii) For wavelength λ = 1.5 cm, we have

$$\frac{h\nu}{kT} = \frac{hc}{\lambda kT} = \frac{(6.62 \times 10^{-34})(3 \times 10^8)}{(1.5 \times 10^{-2})(1.38 \times 10^{-23})300} = 0.0032$$

Thus, we have

$$\frac{\text{Spontaneous emission rate}}{\text{Stimulated emission rate}} = \mathrm{e}^{h\nu/kT} - 1 = \mathrm{e}^{0.0032} - 1 = 0.0032$$

It shows that the spontaneous emission is much smaller than the stimulated emission.

Exercise 6 Find out the condition under which the spontaneous emission rate is equal to the stimulated emission rate.

Solution We have

$$\frac{\text{Spontaneous emission rate}}{\text{Stimulated emission rate}} = e^{h\nu/kT} - 1$$

Thus, we have

$$1 = e^{h\nu/kT} - 1 \qquad \text{or} \qquad e^{h\nu/kT} = 2$$

It gives

$$\frac{h\nu}{kT} = \ln(2) = 0.693$$

It shows that

$$\frac{\nu}{T} = \frac{0.693k}{h} = \frac{0.693(1.38 \times 10^{-23})}{6.62 \times 10^{-34}} = 1.44 \times 10^{10}\ \mathrm{K^{-1}s^{-1}}$$

This is the required condition.

Exercise 7 Determine which type of emission dominates for radiation of wavelength 10^{-6} m at a temperature of 100 K. Given, Boltzmann constant $k = 1.38 \times 10^{-23}$ J/K and Planck constant $h = 6.62 \times 10^{-34}$ Js.

Solution We have wavelength $\lambda = 10^{-6}$ m, temperature $T = 100$ K. Thus,

$$h\nu = \frac{hc}{\lambda} = \frac{(6.62 \times 10^{-34})(3 \times 10^{8})}{10^{-6}} = 1.99 \times 10^{-19}\ \mathrm{J}$$

and

$$kT = (1.38 \times 10^{-23})100 = 1.38 \times 10^{-21}\ \mathrm{J}$$

Therefore,

$$\frac{h\nu}{kT} = \frac{1.99 \times 10^{-19}}{1.38 \times 10^{-21}} = 144.2$$

We have

$$\frac{\text{Spontaneous emission rate}}{\text{Stimulated emission rate}} = e^{h\nu/kT} - 1 = e^{144.2} - 1 = 4.22 \times 10^{62}$$

It shows that for the radiation, the spontaneous emission dominates the stimulated emission.

3 Non-radiative Transitions

A transition (excitation or deexcitation) between two levels is known as the non-radiative transition when no photon is involved. In such a transition, the heat may be involved or it may be due to collision (transfer of kinetic energy) with some particle. There are no selection rules for the collisional transitions.

In a collision process, a system (atom) A in an initial state i collides with a particle p having kinetic energy E_1. After collision, the system goes to a final state f and the particle p moves with the kinetic energy E_2. This process can be expressed as

$$A(i) + p(E_1) \rightarrow A(f) + p(E_2)$$

3.1 Collisional Excitation

When the final level f lies higher than the initial level i, the process is known as the collisional excitation. Here, the energy E_1 of the incident particle must be larger than the energy difference between the two states. After giving the energy ΔE to the system, the particle p moves with the remaining energy E_2, such that

$$E_2 = E_1 - \Delta E$$

Since E_2 is to be positive, one requires $E_1 \geq \Delta E$.

3.2 Collisional Deexcitation

When the final level f lies lower than the initial level i, the process is known as the collisional deexcitation. Here, the incident particle can have any amount of energy E_1, as no energy is given to the system. After taking energy ΔE from the system, the colliding particle p moves with the energy E_2, such that

$$E_2 = E_1 + \Delta E$$

Obviously, in the deexcitation process, the final energy of the colliding particle is larger than the initial energy.

Exercise 8 In a system, energies of upper and lower levels are 3.2 eV and 1.4 e V, respectively. A particle having an energy of 1.9 eV, deexcites the system from the upper level to the lower level. Calculate the energy of the particle after collision.

Solution The energy emitted from the system during deexcitation $= 3.2 - 1.4 = 1.8$ eV.

The energy of the particle after collision = $1.9 + 1.8 = 3.7$ eV.

Exercise 9 In a system, energies of upper and lower levels are 2.9 eV and 1.3 e V, respectively. A particle having an energy of 1.9 eV collides with the system. Can the particle excite the system from the lower level to the upper level? If so, calculate the energy of the particle after collision.

Solution The energy required for the excitation of the system = $2.9 - 1.3 = 1.6$ eV.

Since the required energy is less than the energy of the particle, the particle can excite the system.

The energy of the particle after collision = $1.9 - 1.6 = 0.3$ eV.

4 Population Inversion

In the local thermal equilibrium (LTE),[3] the population (number of systems) in an upper level is smaller than that in a lower level. But, for the laser action, it is essential that the population of the upper level is larger than that of the lower level. In other words, the number of systems in the upper level must be larger than the number of systems in the lower level. This situation is known as the *population inversion*. Such a state is known as non-local thermal equilibrium (NLTE).

In a system, let us have upper level u and lower level l having energies E_u and E_l, respectively. Let us consider the relation

$$\frac{n_u}{n_l} = \exp[-(E_u - E_l)/kT]$$

On taking logarithm, we have

$$\ln\left(\frac{n_u}{n_l}\right) = -\frac{(E_u - E_l)}{kT} \tag{6}$$

In the state of population inversion, we have

$n_u > n_l$ and therefore, $\ln(n_u/n_l)$ is positive.

Now, $(E_u - E_l)$ is positive and the Boltzmann constant k is positive. Thus, for the satisfaction of relation (6), the only possibility is that the temperature T (in Kelvin) is negative.

The kinetic temperature can never be negative. Then, what is this T in the relation (6). The point is that, in the LTE, the T in Eq. (6) is known as the kinetic temperature, and for the given value of the kinetic temperature, the populations of the levels can

[3] In the LTE, the transitions between the levels are governed by the collisions, and the probability of radiative transitions is negligibly small.

be calculated. However, the situation of population inversion is the NLTE (not LTE), where the collisional rates are comparable to the radiative transition probabilities.

Under the NLTE, the populations of the levels are obtained, and for the given values of the level populations, the T is calculated, and now the T is known as the *excitation temperature*. That is, in the state of the population inversion, the excitation temperature is negative.

Thus, we have a kinetic temperature that can be measured with the help of a thermometer. The excitation temperature corresponds to the populations of a pair of levels producing radiation. Each pair of levels (producing radiation) has a different value for the excitation temperature. In the limiting situation (LTE), all the excitation temperatures converge to the kinetic temperature.

In the situation of population inversion, the stimulated emission can produce the laser action. In the stimulated emission, each photon produces one additional photon. Thus, corresponding to each photon, we have two photons of the same frequency and the same phase. These two photons produce two additional photons. Thus, we have four photons of the same frequency and the same phase. Continuation of this process produces a large number of photons of the same frequency and the same phase.

5 Pumping Mechanisms

The process by which one can produce and maintain the state of population inversion is known as pumping. The pump is an external source that supplies energy and helps in maintaining the population inversion. In this process, it is necessary that the systems must be continuously promoted to the upper level of the laser transition. The promotion may be indirect via other levels. As a result of the promotion, the population inversion between the lower and upper levels of the laser transition is maintained. In simple words, we can say that the process by which the population inversion between the two levels of the laser transition is maintained is known as the pumping.

There are different methods by which the energy can be supplied to the system and population inversion between the two levels of laser transition is maintained. Some of them are discussed in the following sections.

5.1 Optical Pumping

In the optical pumping, a photon of large frequency (larger than the laser frequency) is absorbed. This absorption of photons takes the system to a level, higher than the upper level of laser transition. From this higher level, the system cascades down to the upper level of laser transition. Consequently, the population inversion is achieved.

Figure 5 shows the upper level u and lower level l of laser transition, along with a level p lying higher than the level u. In the optical pumping, by absorption of a

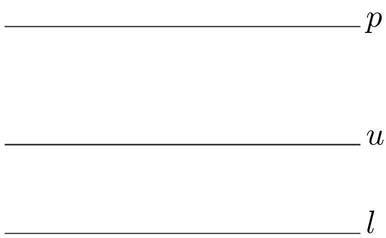


Fig. 5 System may be excited to a level p lying above the upper level u

photon of large frequency, the system is excited to the level p. From the level p, the system is transferred to the level u.

5.2 Electrical Discharge

In this type of pumping, the direct excitation of the system occurs through an electric discharge. This method is preferred in gaseous ion lasers, such as argon-ion lasers. In the case of a gas laser, a high-voltage pulse initially ionizes the gas so that it conducts electricity. An electric current flowing through the gas excites the atoms to the upper level of the laser transition. In case the discharge takes the system to a higher level (higher than the upper level of the laser transition), the system may cascade down to the upper level of the laser transition.

5.3 Collisional Pumping

In this type of pumping, the system is excited through a collision with a particle to a high level (higher than the upper level of laser transition). Here, the kinetic energy of the colliding particle is transferred to the system (collisional excitation). From this high level, the system may cascade down to the upper level of laser transition.

5.4 Chemical Pumping

In a chemical laser, radiations come out of a chemical reaction. For example, when hydrogen combines with fluorine, the heat energy is generated as

$$H_2 + F_2 \rightarrow 2HF + \text{heat}$$

This heat energy is enough to pump a CO_2 laser.

5.5 Direct Conversion

In a light-emitting diode (LED) and semiconductor, the electrons combine with the holes producing laser light. In this manner, a direct conversion of electrical energy into radiation takes place.

6 Quenching

For population inversion, it is required that the population of the upper level must be larger than that of the lower level. So, it is also required that the population of lower levels must not be allowed to increase. The process of removing the population of lower levels of laser transition is known as quenching.

7 Characteristics of a Laser Beam

The main characteristics shown by a laser beam are as follows.

(i) Monochromatic
(ii) Coherent
(iii) Directional
(iv) Bright

Now, we discuss them in brief.

7.1 Monochromatic

A laser beam is highly monochromatic. That is, all the photons in the beam have the same wavelength (frequency). It is due to the fact that all the emitted photons are produced due to the transitions between two given energy levels. Hence, all the photons have almost exactly the same frequency. However, the energy levels are not sharp; there is always a small spread for the frequency distribution, which may easily cover several discrete frequencies. As a consequence of the finite width of energy levels, some closely spaced frequencies may appear in a laser beam. This shows that the laser beam, in principle, is not monochromatic.

In order to obtain the optimum monochromatic beam, generally, an etalon is placed within the laser cavity and an arrangement is made in such a manner that only a well-defined wavelength can travel back and forth in between the plane mirrors enclosing the active medium.

The monochromatic characteristic of a laser beam is expressed in terms of the quality factor Q, which is defined as the ratio of the radiation frequency ν to the line-width $\Delta\nu$ or as the ratio of the radiation wavelength λ to the line-width $\Delta\lambda$. Thus, we have

$$Q = \frac{\nu}{\Delta\nu} \qquad \text{or} \qquad Q = \frac{\lambda}{\Delta\lambda}$$

Here, ν stands for the central frequency and the corresponding wavelength is λ. For the monochromatic nature, the value of the quality factor Q should be very large.

The monochromaticity is expressed as the inverse of the quality factor Q. For a source having wavelength λ = 5000 Å, the frequency is $\nu = 6 \times 10^{14}$ Hz. For a conventional source, we have $\Delta\nu = 10^{10}$ Hz and therefore,

$$\frac{\Delta\nu}{\nu} = \frac{10^{10}}{6 \times 10^{14}} = 1.67 \times 10^{-5}$$

For a laser source, we have $\Delta\nu$ = 500 Hz and therefore,

$$\frac{\Delta\nu}{\nu} = \frac{500}{6 \times 10^{14}} = 8.33 \times 10^{-13}$$

Thus, the quality factor Q for a laser source is much larger than that for a conventional source.

Exercise 10 For a laser beam, the wavelength is 5634 Å and its width is 0.01 Å. Calculate the quality factor of the beam.

Solution We are given, λ = 5634 Åand $\Delta\lambda$ = 0.01 Å. The quality factor is

$$Q = \frac{\lambda}{\Delta\lambda} = \frac{5634}{0.01} = 5.634 \times 10^4$$

Exercise 11 For a laser beam, the frequency 6×10^{14} Hz and its width is 2×10^{10} Hz. Calculate the quality factor of the beam.

Solution We are given, $\nu = 6 \times 10^{14}$ Hz and $\Delta\nu = 2 \times 10^{10}$ Hz. The quality factor is

$$Q = \frac{\nu}{\Delta\nu} = \frac{6 \times 10^{14}}{2 \times 10^{10}} = 5.634 \times 10^4$$

7.2 Coherent

A wave which appears to be a pure sine-wave for an infinitely long period of time, or in an infinitely extended space, is known as a perfectly coherent wave. For such a wave, there is a definite relationship between the phase of the wave at a given time and at a certain time later, or at a given point and at a point which is at a certain distance away. No actual light source, however, emits a perfectly coherent wave. Light waves which are pure sine-waves for a limited period of time or in a limited space are partially coherent waves. There are two different criterion of coherence:

(i) temporal coherence (criterion of time)
(ii) spatial coherence (criterion of space)

7.2.1 Temporal Coherence

Here, we discuss the variation with time. While the phase of the oscillating electric field E of radiation would vary linearly with time, a perfectly coherent light wave would have a constant amplitude of vibration at any given time. A time variation of the electric field would appear as an ideal sinusoidal wave, as shown in Fig. 6.

However, no light emitted by an actual source produces an ideal sinusoidal field at all values of time. This is because when an excited atom returns to the lower state, it emits a pulse of short duration, such as of the order of 10^{-10} s for sodium atom, for example. Thus, the field remains sinusoidal for the time-intervals of the order of 10^{-10} s, after that the phase changes abruptly. Thus, the field due to an actual light source is as shown in Fig. 7. The average time-interval for which the field remains sinusoidal (i.e., till the definite phase relationship exists) is known as the time of coherence and is denoted by τ_c. The distance l_c for which the field is sinusoidal is expressed as

$$l_c = c\,\tau_c$$

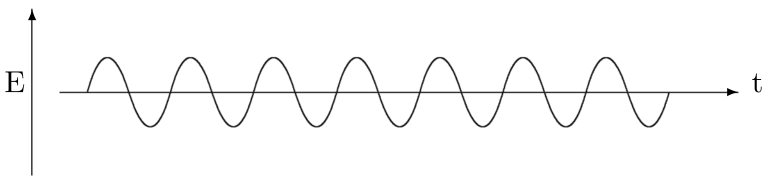


Fig. 6 Variation of electric field E of radiation with time in an ideal sinusoidal wave

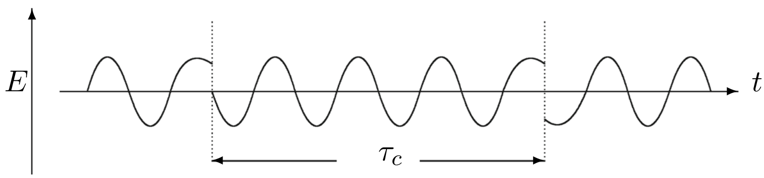


Fig. 7 Temporal coherence over the time τ_c

where c is the speed of light and l_c is known as the coherence length of light beam.

Exercise 12 A sodium lamp has a coherence time 10^{-10} s. Calculate the coherence length.

Solution For the coherence time τ_c, the coherence length l_c is

$$l_c = c\,\tau_c = 3 \times 10^{10} \times 10^{-10} = 3 \text{ cm}$$

Exercise 13 For a sodium light of wavelength 5890 Å, the coherence length is 2.5 $\times 10^{-2}$ m. Calculate

(i) the number of oscillations corresponding to the coherence length.
(ii) the coherence time.

Solution Given, $\lambda = 5890 \times 10^{-10}$ m and coherence length $l_c = 2.5 \times 10^{-2}$ m. (i) The number of oscillations in the length l_c is

$$n = \frac{l_c}{\lambda} = \frac{2.5 \times 10^{-2}}{5890 \times 10^{-10}} = 4.310 \times 10^4$$

(ii) Coherence time

$$\tau_c = \frac{l_c}{c} = \frac{2.5 \times 10^{-2}}{3 \times 10^8} = 8.333 \times 10^{-11} \text{ s}$$

Note: The coherence time is often taken as the inverse of the line width. The coherence time of a laser beam is of the order of millisecond. For a coherence time of 2 ms, the line-width $\Delta\nu$ is

$$\Delta\nu = \frac{1}{\tau_c} = \frac{1}{2 \times 10^{-3}} = 500 \text{ Hz}$$

Exercise 14 The coherence time for laser radiation is 2.5 m s and its wavelength is 5450 Å. Calculate the monochromaticity of the radiation.

Solution Given, coherence time $\tau_c = 2.5 \times 10^{-3}$ s, wavelength $\lambda = 5450 \times 10^{-10}$ m. The line-width $\Delta\nu$ is

$$\Delta\nu = \frac{1}{\tau_c} = \frac{1}{2.5 \times 10^{-3}} = 400 \text{ Hz}$$

The frequency ν of radiation is

$$\nu = \frac{c}{\lambda} = \frac{3 \times 10^8}{5450 \times 10^{-10}} = 5.505 \times 10^{14} \text{ Hz}$$

Hence, the monochromaticity of the radiation is

$$\frac{\Delta \nu}{\nu} = \frac{400}{5.505 \times 10^{14}} = 7.27 \times 10^{-13}$$

7.2.2 Spatial Coherence

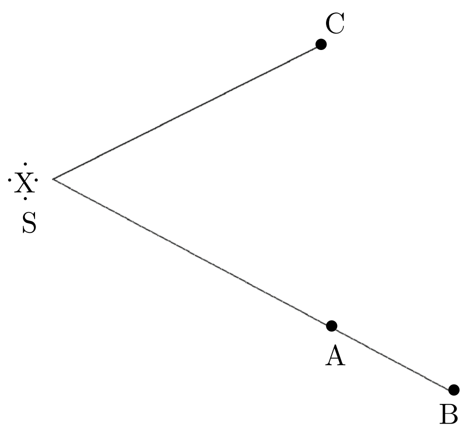
Here, we discuss the variation with space. The spatial coherence is the phase relationship between the radiations at different points in space. In order to understand the spatial coherence, let us consider light waves emitted from a point-source S, as shown in Fig. 8. Suppose, A and B are two points lying on a line joining them with S. The phase relationship between points A and B depends on the distance AB and on the temporal coherence of the beam. When AB $< l_c$, where l_c is the coherence length, there is a definite phase relationship between the points A and B. On the other side, when AB $> l_c$, there is no coherence between points A and B.

Let us now consider two points A and C, which are equidistant from the point source S (SA = SC). Then the waves from S reach points A and C in exactly the same phase, that is, the two points have perfect (spatial) coherence. However, in the case of an extended source, points A and C are not in coherence, as the light reaches from different points on the extended source. Thus, the spatial coherence is related to the size of the source.

7.3 Directional

The photons produced in the stimulated emission are almost in the direction of propagation of incident photons. The mirrors selectively amplify the axial beam.

Fig. 8 Diagram for explaining spatial coherence



A laser therefore emits a narrow parallel beam from its output mirror. The beam divergence is essentially determined by the diffraction limit of the output aperture. The diffraction theory gives the divergence angle θ as

$$\theta = \frac{\beta\lambda}{D}$$

where λ and D are, respectively, the wavelength of the laser and diameter of the laser source, and β is a coefficient, whose value is about unity.

Exercise 15 A laser beam has a wavelength of 6.5×10^{-7} m and circular aperture of 5×10^{-3} m. The laser beam is sent to the moon which is situated at a distance of 4×10^{8} m from the earth. Calculate

(i) the angular spread of the beam.
(ii) the area of spread when it strikes at the moon.

Solution (i) The angular spread is

$$\theta = \frac{\lambda}{D} = \frac{6.5 \times 10^{-7}}{5 \times 10^{-3}} = 1.3 \times 10^{-4} \text{ rad}$$

(ii) The linear spread d is

$$d = (4 \times 10^{8})(1.3 \times 10^{-4}) = 5.2 \times 10^{4} \text{ m}$$

Thus, the area A of spread is

$$A = \frac{\pi d^2}{4} = \frac{3.14 \times (5.2 \times 10^{4})^2}{4} = 2.12 \times 10^{9} \text{ m}^2$$

Exercise 16 Calculate the angular spread of the laser beam of wavelength 6930 Å due to diffraction if the beam emerges through a 3 mm diameter mirror. Calculate the diameter of the beam when it strikes a satellite situated at a distance of 250 km from the earth.

Solution The angular spread $\Delta\theta$ is

$$\Delta\theta = \frac{\lambda}{D} = \frac{6930 \times 10^{-10}}{3 \times 10^{-3}} = 2.31 \times 10^{-4} \text{ rad}$$

The diameter a of beam on the satellite is

$$a = \Delta\theta \times d = (2.31 \times 10^{-4})(250 \times 10^{3}) = 57.75 \text{ m}$$

7.4 Intensity

The laser beam is extremely intense. The energy in a laser falls on a very small area and therefore the intensity becomes very large. It is estimated that light from a 1 mW laser is thousands of times more intense than the intensity of light of the sun falling on the earth's surface.

Exercise 17 Calculate the intensity of a 10 mW laser having a circular aperture of diameter 1.5 mm. Assume that the intensity is distributed uniformly across the beam.

Solution Given, power $P = 10 \times 10^{-3}$ W and diameter $d = 1.5 \times 10^{-3}$ m. The intensity I is

$$I = \frac{P}{\pi d^2/4} = \frac{10 \times 10^{-3}}{3.14(1.5 \times 10^{-3})^2/4} = 5.66 \times 10^3 \text{ W/m}^2$$

Exercise 18 A He-Ne laser having a wavelength of 6328 Å is focused on an area equal to the square of its wavelength. Calculate the intensity of the laser when it radiates energy at the rate of 5 mW.

Solution Given, wavelength $\lambda = 6328 \times 10^{-10}$ m, power $P = 5 \times 10^{-3}$ W.

Therefore, area $A = \lambda^2 = (6328 \times 10^{-10})^2 = 4.004 \times 10^{-13}$ m^2. The intensity I is

$$I = \frac{P}{A} = \frac{5 \times 10^{-3}}{4.004 \times 10^{-13}} = 1.25 \times 10^{10} \text{ W/m}^2$$

Exercise 19 A ruby laser having wavelength of 6940 Å emits 1.25 J pulses. Calculate the minimum number of Cr^{3+} ions in the ruby laser.

Solution Given, wavelength $\lambda = 6940 \times 10^{-10}$ m. Suppose, n be the minimum number of Cr^{3+} ions in the ruby laser. The power p is

$$P = \frac{nhc}{\lambda} \qquad \text{therefore} \qquad n = \frac{P\lambda}{hc}$$

On using the values, we have

$$n = \frac{1.25(6940 \times 10^{-10})}{(6.62 \times 10^{-34})(3 \times 10^8)} = 4.37 \times 10^{10}$$

8 Main Components of a Laser

The main components of a laser are as the following.

(i) Pumping mechanism
(ii) Active system
(iii) Resonant cavity

8.1 Pumping Mechanisms

The process through which the ratio of the populations of upper to lower levels of a laser transition increases as compared to that according to the Boltzmann law is known as the *pumping mechanism*. Through a pumping process, we transfer the system indirectly to the upper level of the lasing transition. A general mechanism for pumping is shown in Fig. 9. The pumping generates population inversion, which is a precondition for the stimulated emission. Two important pumping mechanisms are (discussed earlier):

(i) Radiative pumping
(ii) Collisional pumping

Here, u and l are, respectively, the upper and lower levels of a laser transition. In the pumping process, the system is excited to a level p, which exists above the level u. The pumping can be from the lower level l or from any other level. The system in the level p cascades down to the upper level u. In this manner, a population inversion between the levels u and l is obtained.

8.2 Active System

A system in which the population inversion is achieved is known as the active system or the gain medium for a laser. A laser system is named according to the gain medium, which may be in the form of a gas, liquid, or solid. For example, in the He=Ne laser, the active system is in the gaseous form.

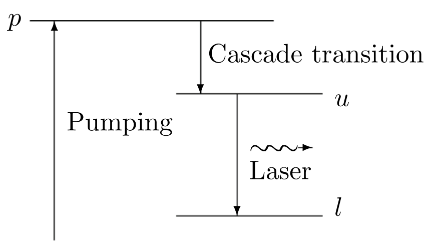


Fig. 9 Pumping mechanism

8.3 *Resonant Cavity*

The resonant cavity of a laser consists of a gain medium enclosed in between two plane parallel mirrors M_1 and M_2, where M_1 is perfectly reflecting and M_2 is partially reflecting. The surfaces of M_1 and M_2 are perpendicular to the axis of the gain medium. In each passage through the material, the intensity increases. On the other side, for each reflection, a part of energy is lost. When the electromagnetic radiation crosses a length l through the active material, it is amplified. After the successive reflections at M_1 and M_2, the radiation moves in its initial direction as shown in Fig. 10.

Thus, in between the two mirrors M_1 and M_2, there are waves propagating along both directions. The waves are reflected back and forth between the mirrors and form a standing wave pattern. For that, the total phase change suffered by the wave in one complete round trip must be an integral multiple of 2π. Thus, we should have

$$\frac{2\pi}{\lambda}(2l) = 2m\pi \qquad \text{or} \qquad m = \frac{2l}{\lambda}$$

where m is a positive integer and λ the wavelength of radiation. For example, for a laser cavity of length $l = 50$ cm, operating at an optical wavelength of 5000 Å we have

$$m = \frac{2l}{\lambda} = \frac{2 \times 50}{5000 \times 10^{-8}} = 2 \times 10^6$$

9 Schemes of Energy Levels in a Laser

Since for getting the required condition of population inversion a pumping mechanism is required, it is not possible to have laser action in a system having only two levels. Therefore, for a laser action between the two levels, a system must have at least three levels. Here, we have discussed the mechanism for a laser having (i) three levels and (ii) four levels. In the three-level system, the lower level of the laser transition is involved in the pumping mechanism whereas in the four-level system, the lower level of the laser transition is not involved in the pumping mechanism.

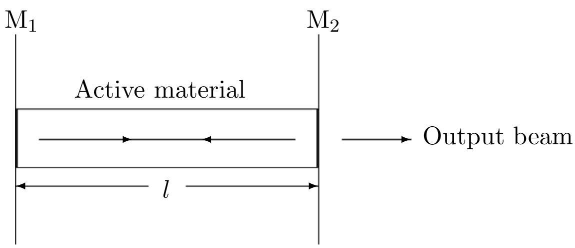


Fig. 10 Schematic diagram for laser cavity

9.1 Three-Level System

Let us consider a three-level system with three levels, denoted by 1, 2, and 3, as shown in Fig. 11. In this scheme, the system is raised from the lowest level 1 to the highest level 3 through a pumping mechanism. From the level 3, the system decays to the level 2 very rapidly. The radiative lifetime of level 2 is quite large, of the order of 10^{-3} s. (Generally, the radiative lifetime of an energy level is of the order of 10^{-8} s.) Such a level with a large radiative lifetime is known as the *metastable level*. Because of the large radiative lifetime, the population density of 2 starts increasing and a population inversion between levels 1 and 2 is achieved. Ruby laser is an example of the three-level system.

Exercise 20 Laser action occurs by stimulated emission from an upper level to a level at 25 eV. If the wavelength of radiation emitted is 6700 Å, calculate the energy of the upper level.

Solution Energy E of laser radiation emitted is

$$E = h\nu = \frac{hc}{\lambda} = \frac{(6.62 \times 10^{-34})(3 \times 10^{8})}{6700 \times 10^{-10}} = 2.964 \times 10^{-19}\ \text{J}$$

$$= \frac{2.964 \times 10^{-19}}{1.6 \times 10^{-19}} = 1.85\ \text{eV}$$

The energy of upper level $= 25 + 1.85 = 26.85$ eV.

9.2 Four-Level System

Though in the three-level scheme, a population inversion can be achieved, but in some systems, we find a four-level scheme. Let us consider a system having four levels, denoted by 1, 2, 3, and 4, as shown in Fig. 12. In this scheme, the system

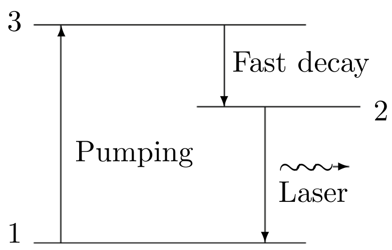


Fig. 11 Three-level system for a laser action

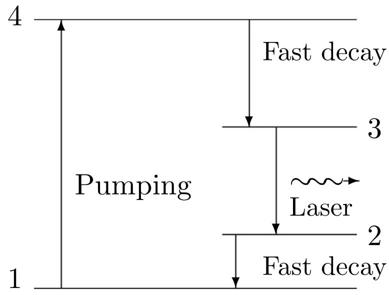


Fig. 12 Four-level system for laser action

is raised from the lowest level 1 to level 4 through a pumping mechanism. The radiative lifetime of level 4 is of the order of 10^{-8} s. From the level 4, the system decays rapidly to the level 3. Level 3 is a metastable level with a radiative lifetime of the order of 10^{-3} s. Because of the large radiative lifetime, the population density of level 3 starts to increase. The level 2 is almost empty, as it is quite high relative to 1 and its lifetime is quite small. Thus, the population inversion between the levels 2 and 3 can be achieved. For maintaining the population inversion between 2 and 3, it is necessary that the population of level 2 should quickly decay to level 1, so that level 2 remains more or less empty. This process is known as quenching.

10 He-Ne Laser

He-Ne laser is the first gas laser proposed and developed by Ali Javan in 1960. The working material is a mixture of helium and neon gases filled at 1 mm Hg and 0.1 mm Hg pressures, respectively, in a discharge tube of length 100 cm and inner diameter of 1.5 cm, as shown in Fig. 13.

The density of He atoms is larger than that of the Ne atoms. The discharge tube is kept in between the two mirrors M_1 and M_2 which are flat and almost perpendicular to the axis of the tube. The mirror M_1 is highly silvered and behaves as a perfect reflector. The other mirror M_2 is partially silvered so that the intense beam of radiation could emerge through it. A high frequency and high voltage electromagnetic field applied between the two electrodes D_1 and D_2 imparts energy to the electrons inside the tube. The He and Ne atoms in the tube are excited due to collisions with the energetic electrons (collisional excitation).

Figure 14 shows the scheme of energy levels in the He and Ne atoms. The transitions involved in the He-Ne laser are also shown. The He atom with electronic configuration $1s^2$ has one term ^1S, which is the ground state. On excitation, the electronic configuration becomes $1s2s$ and that has two terms, ^1S and ^3S, which are metastable, as there is no radiatively allowed transition in the downward direction from these levels. The Ne atom with electronic configuration $1s^22s^22p^6$ has one term ^1S, which is the ground state. The excited states of Ne atom may have configurations

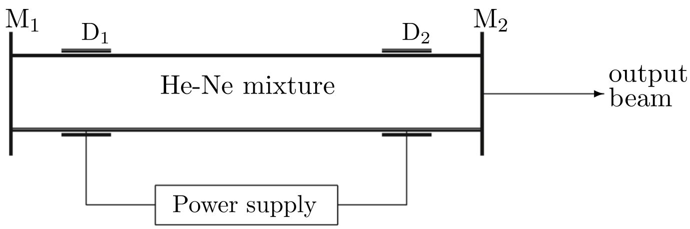


Fig. 13 Schematic diagram of the He-Ne laser

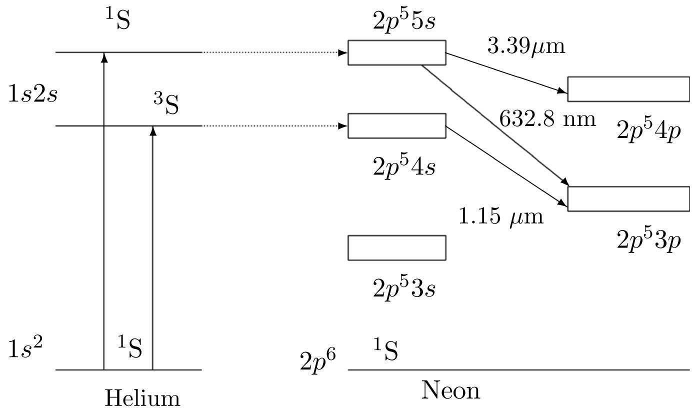


Fig. 14 Scheme of energy levels in the He and Ne atoms, and the He-Ne laser transitions

$2p^5ns$ and $2p^5np$ with the integer $n > 2$. The lifetimes of the states corresponding to $2p^5ns$ configurations are somewhat longer (10^{-7} s) than those of the configurations $2p^5np$ ($\sim 10^{-8}$ s).

The states arising from the $2p^55s$ configuration of Ne atom have nearly equal energy to that of $1s2s\ ^1S$ state of He atom. Similarly, the states arising from the $2p^54s$ configuration of Ne atom lie just below the $1s2s\ ^3S$ state of He atom. An electric discharge is created in the He-Ne gas mixture by applying either a DC voltage or with the help of microwave radiation. In such a discharge, some gas atoms are ionized, creating a mixture of neutral atoms, positively charged ions, and free electrons. The applied electric field accelerates the light-mass electrons more effectively than the heavy-mass ions.

These accelerated electrons collide with the He atoms and excite them from $1s^2$ ^1S state to $1s2s\ ^1S$ and ^3S states. As these transitions are radiatively forbidden, they cannot return back to the ground state $1s^2\ ^1S$. The Ne atoms are not very strongly excited by electron impact. The lifetime of He $1s2s\ ^1S$ and ^3S states is $\sim 10^{-4}$ s. These excited states of He are very close in energy to the excited states $2p^54s$ and $2p^55s$ of Ne. Thus, once the He atoms are lifted to $1s2s\ ^1S$ and ^3S states, there is a fair chance that before it decays by other means, it collides with a Ne atom in the ground state. If this occurs, there is finite probability for the transfer of energy from He to Ne atom.

Thus, the Ne atoms are excited to $2p^5 4s$ and $2p^5 5s$ levels. Now, the inversions can be obtained between the pairs of levels: (i) $2p^5 5s$ and $2p^5 4p$, (ii) $2p^5 5s$ and $2p^5 3p$ and (iii) $2p^5 4s$ and $2p^5 3p$. Though a large number of laser transitions take place between these levels, the predominant ones have the wavelengths 1.15 μm, 632.8 nm and 3.39 μm. The lifetime of p states of Ne atom is very small and therefore, the atoms in these states decay to $2p^5 3s$ state through the spontaneous transition. From here, the system deexcites to the ground state $2p^6\ {}^1S$. From there, the Ne atom is again excited through the collision with the He atom.

Exercise 21 A He-Ne laser emits a beam of wavelength 632.8 nm and has a power of 4 mW. Calculate the number of photons emitted per second by the laser.

Solution Energy E of a photon of wavelength 632.8 nm is

$$E = \frac{hc}{\lambda} = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{632.8 \times 10^{-9}} = 3.141 \times 10^{-19}\ \text{J}$$

Suppose, n is the number of photons emitted per second. The energy of these photons is 4 mW. Thus, we have

$$n = \frac{4 \times 10^{-3}}{3.141 \times 10^{-19}} = 1.273 \times 10^{16}\ \text{s}^{-1}$$

The number of photons emitted per second is 1.273×10^{16}.

Exercise 22 A He-Ne laser is capable of emitting laser radiations at several different wavelengths, the prominent one being 3.39 μm. Calculate the energy difference between the upper and lower levels of the laser transition.

Solution The wavelength λ of laser transition is 3.39×10^{-6} m. The energy difference E between the upper and lower levels of the laser transition is

$$E = \frac{hc}{\lambda} = \frac{(6.62 \times 10^{-34})(3 \times 10^8)}{3.39 \times 10^{-6}} = 5.858 \times 10^{-20}\ \text{J}$$

11 Ruby Laser

Ruby is a three-level solid-state laser. It was first demonstrated in 1960 by Maiman. It consists of a crystal of Al_2O_3 doped with about 0.05% by weight of Cr_2O_3. The concentration of Cr^{3+} atoms is so low that the interaction between them may be neglected, and they behave like independent atoms. That is, the Cr^{3+} atoms are separated far away from one another. The Al_2O_3 behaves as a host crystal. In the crystal, some of the aluminum atoms are replaced by chromium atoms. These chromium atoms produce laser action and are the source of red light (λ = 6943 Å) emitted by the ruby laser.

11.1 Construction

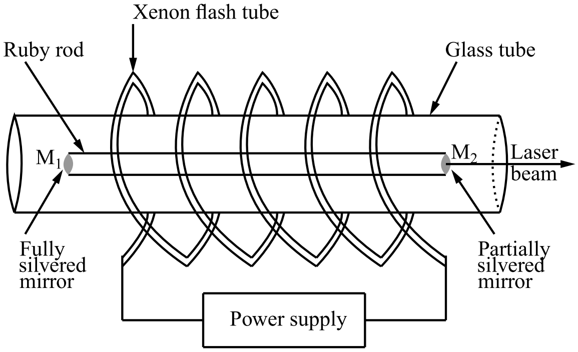
A schematic diagram of a ruby laser is shown in Fig. 15. A cylindrical ruby rod of length 4 cm and diameter 0.5 cm is used. The end-faces M_1 and M_2 of the ruby are flat and almost perpendicular to the axis of the rod. The end M_1 is highly silvered and behaves as a perfect reflector. The other end M_2 is partially silvered so that an intense beam of radiation could emerge out through the mirror. The rod is surrounded by a helical flash tube filled with xenon. This xenon flash tube produces white light when activated by a power supply. For the cooling of the cavity, there is a provision of circulating coolant around the ruby rod.

11.2 Working

The energy levels diagram of a ruby laser is a three-level system as shown in Fig. 16. Suppose, E_1, E_2, E_3 are energies of the levels 1, 2, 3, respectively. The chromium atoms are pumped (excited) by an external optical source. When the lamp is flashed, a radiation of wavelength $\lambda = 6600$ Å from the flash lamp are absorbed by the chromium atoms and the atoms excite from the ground state 1 to the excited state 3.

The chromium atoms in the excited state 3 have a larger probability to deexcite spontaneously to level 2 as compared to that to level 1. Thus, the atom is transferred from the level 1 to the level 2 via the level 3. The energy $(E_3 - E_2)$ released during the deexcitation is absorbed by the lattice of the ruby crystal and therefore no electromagnetic radiation corresponding to the energy $(E_3 - E_2)$ is emitted out.

Fig. 15 Schematic diagram of ruby laser



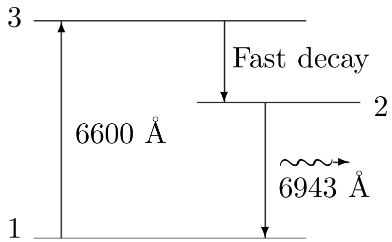


Fig. 16 Energy level diagram of ruby laser

Level 2 is a metastable state because its radiative lifetime (few milli second) is much larger than that of a normal level ($\sim 10^{-8}$ s). Hence, the atoms can stay at level 2. In this way, the population of level 1 can be transferred to level 2 via level 3 with the help of a flash lamp. Thus, the population inversion is established between the levels 1 and 2.

The energy $(E_2 - E_1)$ corresponds to a red light of wavelength $\lambda = 6943$ Å. A radiation of wavelength $\lambda = 6943$ Å stimulates the atom in the level 2 to deexcite to the level 1. Consequently, two radiations of wavelength $\lambda = 6943$ Å for each stimulating photon are emitted.

These photons move parallel to the axis of the ruby rod and are reflected back and forth between the silvered end-faces M_1 and M_2 of the rod. These photons stimulate the atoms in the metastable level 2 and produce more and more photons of wavelength $\lambda = 6943$ Å. This process continues repeatedly. When the beam of radiation becomes intense enough to emerge out of the half-silvered end-face M_2 of the rod and a highly intense, monochromatic, coherent and unidirectional beam of wavelength $\lambda = 6943$ Å is obtained. Since the pumping source in the ruby laser is a flash lamp (which is not continuous), the output beam of wavelength $\lambda = 6943$ Å is not continuous. The ruby laser is a pulse laser.

The ruby laser operates at about 1% efficiency. It may produce a laser beam of 1–25 mm in diameter. The beam obtained is in the form of pulses. The construction of this laser is simple and the operation is very easy. For this reason, this laser is known as the practical laser.

Note: In numerical, quite often energy is given in electron-volts (eV) and the radiation is expressed in nanometers (nm). 1 eV = 1.6×10^{-19} J; 1 nm = 10^{-9} m. We have $h = 6.63 \times 10^{-34}$ Js and $c = 3 \times 10^8$ m.

$$hc = \frac{(6.63 \times 10^{-34})(3 \times 10^8)(10^9)}{1.6 \times 10^{-19}} = 1243 \text{ nm eV}$$

Exercise 23 A pulsed laser is constructed with a ruby crystal as the active element. The ruby rod contains a total of 2.5×10^{19} Cr^{3+} ions. When the laser emits at 6940 Å wavelength, calculate

(i) the energy of one photon emitted.
(ii) the total energy available per laser pulse.

Solution The wavelength λ of laser transition is 6940×10^{-10} m. (i) The energy E of a photon emitted is

$$E = \frac{hc}{\lambda} = \frac{(6.62 \times 10^{-34})(3 \times 10^{8})}{6940 \times 10^{-10}} = 2.862 \times 10^{-19}\ \text{J}$$

(ii) We have

$$\text{Energy per pulse} = (\text{Energy of one photon})(\text{Total number of photons})$$

$$= (\text{Energy of one photon})(\text{Total number of atoms in the excited state})$$

$$= (2.862 \times 10^{-19})(2.5 \times 10^{19}) = 7.155\ \text{J}$$

Exercise 24 The pulse duration of a laser of wavelength 9640 Å is 20 ms. If the average power output per pulse is 0.9 W, calculate (i) the energy released per pulse and (ii) the number of photons in each pulse.

Solution (i) The energy E released per pulse is

$$E = P \times \Delta t = (0.9)(20 \times 10^{-3}) = 0.018\ \text{J}$$

(ii) Number n of photons in a pulse is

$$n = \frac{E}{h\nu} = \frac{E\lambda}{hc} = \frac{(0.018)(9640 \times 10^{-10}}{(6.62 \times 10^{-34})(3 \times 10^{8})} = 8.74 \times 10^{16}$$

12 Nd:YAG Laser

Neodymium is a rare earth material, which works as a four-level system. The pump thresholds of these systems for both the pulsed and continuous operations are much lower than for the ruby laser. Hence, the Nd:YAG laser is a four-level, solid-state laser. It was first demonstrated by J.E. Geusic and his coworkers in 1964. Here, Nd stands for *neodymium* and YAG stands for *Yittium Aluminum Garnet* ($Y_3Al_5O_{12}$). A schematic diagram of an Nd-YAG laser is shown in Fig. 17. YAG is an optically isotropic crystal. In this laser, some of the Y^{3+} ions in the crystal are replaced by Nd^{3+} ions. It is the neodymium ion which provides the lasing activity in the crystal.

This laser is capable of producing very high power emissions, as a result of its lasing medium that operates as a four-level system. The lasing medium in Nd:YAG laser

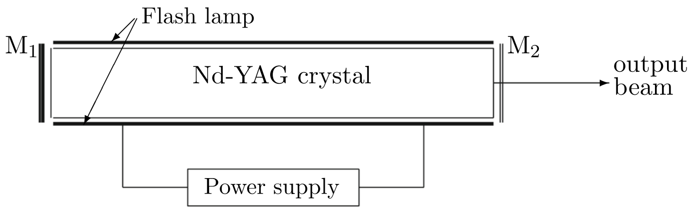


Fig. 17 Schematic diagram of a Nd:YAG laser

is a colorless, isotropic crystal, called Yittium Aluminum Garnet (YAG, $Y_3Al_5O_{12}$). The main dopant in the lasing medium is neodymium (Nd^{3+}). When it is used in a laser, neodymium replaces 1% of Yittium and consequently, the crystal takes a light blue color. The YAG has relatively high thermal conductivity, which improves the thermal dissipation in the thermal cavity.

The energy level diagram for Nd:YAG laser is shown in Fig. 18. Owing to the interaction between electrons, the levels $^4F_{5/2}$ and $^2H_{9/2}$ are broadened. Fluorescence occurs from the level $^4F_{3/2}$ to four multiplets (not shown in the figure) of the ground state. The transitions between the levels F (having orbital quantum number 3) and the levels I (having orbital quantum number 6) are not optically allowed, as the orbital quantum number changes by three (more than 1). Thus, the F levels are metastable levels.

The probability of transition to the level $^4I_{11/2}$ is an order of magnitude higher than that of the other members of the multiplets. This level is about 2000 cm^{-1} (frequency $\nu = 6 \times 10^{13}$ Hz) above the ground state and therefore at room temperature T the $^4I_{11/2}$ level is virtually empty (as $h\nu >> kT$). Consequently, any population in the level $^4F_{3/2}$ gives rise to a population inversion. Transition from the level $^4I_{11/2}$ to the lowest level takes place by non-radiative processes and is quite rapid. The laser transition $^4F_{3/2} \rightarrow ^4 I_{11/2}$, corresponding to a wavelength of 1.06 μm. The radiative life-time of upper level is 0.23×10^{-3} s.

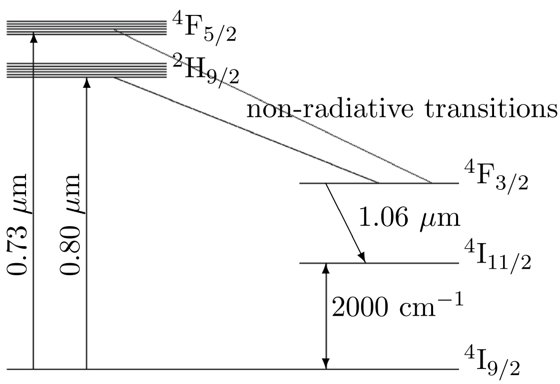


Fig. 18 Energy level diagram of Nd:YAG laser

Nd:YAG lasers are usually excited with the help of a krypton lamp.

In some cases, YAG crystals are doped with chromium ions in addition to Nd^{3+} ions and xenon lamps are used for pumping. Chromium ions in YAG have two broad absorption bands at 0.43 and 0.59 μm which fall in the range of the spectrum of xenon lamp. The chromium ions in the excited state transfer energy to the neodymium ions. However, the transfer takes a comparatively longer time, and therefore the method is restricted only to the continuous operation of the laser.

13 Carbon Dioxide Laser

Carbon dioxide (CO_2) molecule has three vibrational modes:

(i) Symmetric stretching at 1337 cm^{-1}
(ii) Bending at 667 cm^{-1}
(iii) Asymmetric stretching at 2349 cm^{-1}

The carbon dioxide (CO_2) laser is one of the most powerful and efficient lasers. It has a four-level system where vibrational-rotational transitions take place. It produces a light of infrared (IR) of 10.6 and 9.4 μm wavelength.

13.1 Construction

A schematic diagram of the carbon dioxide laser is shown in Fig. 19. There is a discharge tube having a bore of cross-section 1.5 cm^2 and a length of about 26 cm. The tube is filled with a mixture of carbon dioxide, nitrogen , and helium gases in 3 : 12 : 70 proportions, respectively. A high value of DC voltage is applied for the electric discharge in the tube due to which the CO_2 breaks into CO and O.

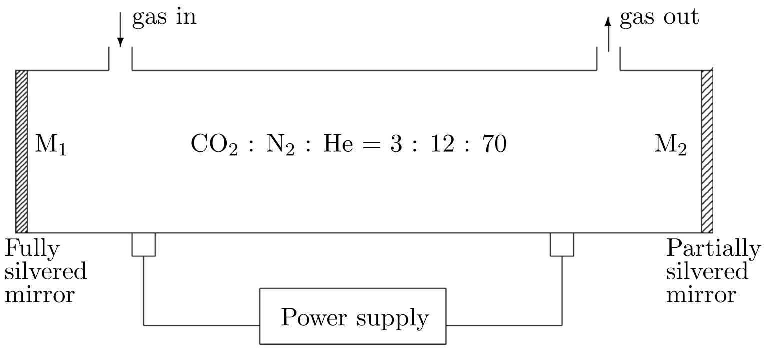


Fig. 19 Schematic diagram of CO_2 laser

13.2 Working

The energy levels involved in the working of the CO_2 laser are shown in Fig. 20. The nitrogen molecule is excited from the ground state to the excited state through the electric discharge. The lowest vibrational level of N_2 has nearly as much energy as the asymmetric stretching mode of CO_2 molecule. Thus, the nitrogen molecule can easily transfer energy to CO_2 molecule. Hence, the CO_2 molecules are excited to the energy level E_5.

Corresponding to the different states of vibrations, the transitions are allowed from E_5 to E_4 and from E_5 to E_3 to produce the photons of wavelengths 10.6 µm and 9.6 µm, respectively. These are the laser transitions. Now, the CO_2 molecules at the levels E_4 and E_3 deexcite to the level E_2 through the collisions with the helium atoms. The helium atoms deplete the level E_2 through collisions. Since the level E_2 is very close to the level E_1, so there is a maximum chance that this level may be populated through thermal deexcitations. Helium is also used to keep the CO_2 cool.

14 Semiconductor Laser

A semiconductor laser is a specially fabricated p-n junction device which emits coherent light when it is forward-biased. A semiconductor laser has almost the same features as found in a solid-state laser, except for one major difference in the solid state lasers only 1% of the active material participates in the process of laser action whereas semiconductor lasers, the whole material is active. For the semiconductor lasers, the following conditions are essential:

(i) The semiconductor must have a very high transition probability between the conduction and the valence bands.

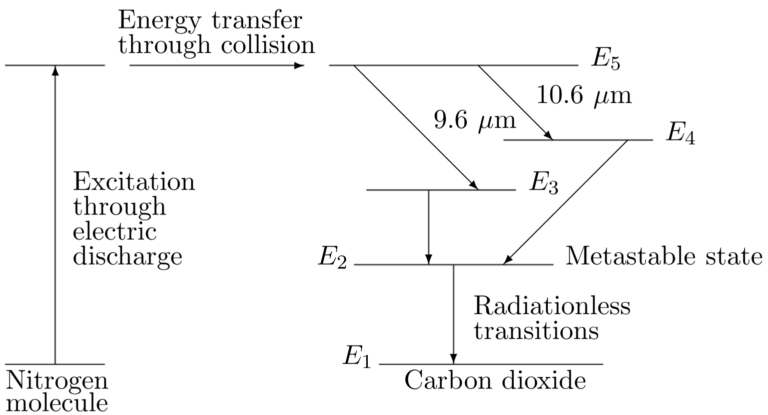


Fig. 20 Energy levels scheme of the carbon dioxide laser

(ii) The excess population can be maintained across the laser transition.

The first semiconductor laser was manufactured in 1962 by Hall and his coworkers in the USA. In this laser, Gallium Arsenide (Ga-As) is used, and it is usually operated at low temperatures and emits laser light in the near-infrared region. Room-temperature semiconductor lasers were fabricated in 1970. Nowadays, the p-n junction (diode) lasers are made to emit light almost in any part of the electromagnetic spectrum, from UV to IR.

Diode lasers are remarkable due to their small size (0.1 mm long) and high efficiency of the order of 40%. The main advantage of a semiconductor laser is that it is a portable and easily controlled source of coherent radiation. Because of the rapid advances in semiconductor technology, semiconductor lasers have a variety of applications in optical fiber communications, CD audio players, CD-ROM drives, optical reading, high-speed laser printing, etc.

15 Applications of Lasers

The monochromatic property, directionality, and high intensity of a laser beam have made possible a number of investigations in various scientific fields, such as Physics, Chemistry, Biology, and Medicine. Many investigations in these fields would have not been possible without a laser beam. Laser has an important contribution in the areas of communication, industry, warfare, etc. Here, we discuss some of the laser applications.

15.1 In Science and Research

After the discovery of the laser beam, the Michelson–Morley experiment has been improved a lot. The original experiment performed in 1881 showed that light is unaffected by the earth's motion through space with an accuracy of about 150 mm/s. That is, the resolution of the Michelson–Morley experiment has been 150 mm/s. The new version of an experiment where two He-Ne lasers are used has improved the accuracy to about 3 mm/s. That is, the resolution has been improved to 3 mm/s.

A laser beam being much collimated is used in the measurements of long distances. The distance between the earth and the moon is determined to an accuracy of 15 cm.

Laser-torch is being used to see the objects at very large distances. Laser has an important application in three-dimensional lens-less photography (generally called the holography), in the study of Raman, ESRA, and MR spectroscopy.

15.2 In Isotope Separation

Lasers (particularly tunable lasers) are successful in separating efficiently the isotopic species of an element in an isotopic mixture. For example, natural uranium ore contains mainly the isotope U^{238} and only 0.7% of U^{235}. The U^{235} is used as fuel for nuclear reactors and therefore needs to be separated out of the ore. There are other applications of pure isotopes in medicine, science, and technology. Such work of separation of isotopes is being done successfully with the help of a laser beam.

The mechanism of separation of isotopes is based on the fact that atoms of different isotopes of the same element have slightly different energy levels due to the difference in their nuclear masses. This difference is known as the isotopic shift. Hence, the radiations of a certain frequency may be absorbed by one isotope, but not by the others. Thus, when the extremely monochromatic light emerging from a properly tuned laser is allowed to fall on a mixture of various isotopes, the atoms of only one of the isotopes are excited. These excited atoms may be separated from the mixture with the help of the methods available for the purpose.

15.3 In Laser-Induced Fusion

In the fusion process, two or more light nuclei fuse together and form a heavy nucleus, whose mass is less than the sum of the masses of the original nuclei. The lost mass appears as energy released in the process. According to the energy-mass equivalence relation, 1 gram mass is equivalent to 9×10^{13} J of energy.

There are two main problems in initiating and sustaining a thermonuclear fusion reaction:

(i) The fusing material needs to be heated to a high temperature, of the order of 10^7 K, so that the nuclei have kinetic energies, high enough to fuse together against their mutual electrostatic repulsion.
(ii) At such a high temperature, the matter is in a fully ionized state. It is necessary to confine the hot plasma for a long enough time to sustain a fusion reaction.

A laser is capable of producing very high temperature and pressure required to initiate a fusion reaction and to confine the fusion material through inertial forces generated when an intense laser pulse interacts with the fusion material.

15.4 In Chemistry

The laser beam can be used for getting information about the nature of chemical bonds in a molecule. Ultraviolet lasers are used for the purpose of depositing metallic films and to add dopant in a semiconductor substrate with a spatial resolution of about 1 μm. Infrared CO_2 laser can deposit metal structures of about 50 μm wide with the help of laser-induced chemical vapor deposition.

15.5 In Communication

The use of laser beams in conjunction with optical fiber has shown a great revolution in the area of communication. The frequencies of the optical waves ($\sim 10^{15}$ Hz) are very high as compared to those in the radio waves ($\sim 10^6$ Hz) and microwaves ($\sim 10^9$ Hz). Therefore, the optical waves acting as carrier waves are capable of transmitting a large amount of information as compared to radio waves. A typical laser beam can transmit $\sim 10^{12}$ speech signals over the same channel.

15.6 In the Study of Atmosphere

A laser is generally used for remote probing of the atmosphere for detection of pollutant gases as well as for measuring the water vapor concentration and temperature in the atmosphere.

15.7 In Biology

The ability of laser beam to concentrate high power density of light at a point enables the study of biological samples which are available in very small quantities.

15.8 In Medicine

A laser beam is used in delicate eye surgery, like cornea grafting. With the help of a laser beam, the surgical operation is completed in a much shorter time. A laser is also used in the treatment of kidney stones, cancer, and tumors and in cutting and sealing the small blood vessels in the brain operation.

15.9 In Industry

A laser beam can be used for cutting fabric, for clothing purposes, and for cutting steel sheets. It can drill extremely fine holes even in paper clips, single human hair, and hard materials including teeth and diamonds. Extremely thin wires used in the cables are drawn through the diamond hole. Metallic roads can be melted and joined together with the use of a laser beam. It can also be used to vaporize unwanted material during the manufacture of electronic circuits on semiconductor chips.

15.10 In War

During wartime, a laser beam is used for detection and then to destroy enemy missiles. Nowadays, laser rifles, laser pistols, and laser bombs are also made which can be aimed at the enemy during the night times as well.

15.11 In Space

A laser can be used to control the rockets and satellites. It is also used in the directional radio communication such as fiber-optics telephony.

15.12 Lidar

The function of a Lidar is very similar to that of a radar. The word Lidar is an acronym for Light Detection And Ranging. In this technique, a laser pulse is generated and transmitted into the atmosphere. This laser pulse after striking the object, situated at a distance R, is scattered back and arrives at the receiver of the Lidar after a time-interval t $(= 2R/c)$. From this time interval, the range of the object is obtained. The range resolution ΔR is given by the duration of laser pulse t_p, expressed as $\Delta R = t_p c/2$. As the light travels about 30 cm in 1 ns, thus the segment of the atmosphere about 1 m in depth can be probed with the help of the Lidar.

16 Holography

In 1948, Dennis Gabor devised a technique, known as holography, in which a lens-less three-dimensional image of an object is recorded and reproduced. The word holography originated from the Greek words, ‘holos’ (complete) and ‘graphos’ (writing). It is a technique to record a complete picture of an object and to reconstruct a three-dimensional image of the object.

In the conventional photography, a two-dimensional image of the picture is recorded as it records only the intensity distribution. On the other side, in the holography, both the intensity and phase of the light wave are recorded. In the holography, the light waves reflected from an object are recorded. These records have information about the intensity and phase of light waves, and a record is known as the hologram. The hologram has no resemblance to the original object, but it contains all the information about the object in an optical code. A hologram is prepared through a process, called the recording process. The formation of the three-dimensional image from the hologram is done through a process called the reconstruction process. Thus, the photography has two processes:

(i) Preparation of a hologram
(ii) Reconstruction of image from the hologram

16.1 Preparation of a Hologram

The recording of a hologram is based on the phenomenon of interference. Figure 21 shows a schematic diagram of the process of recording a hologram. It requires a laser source, a plane mirror (beam splitter), an object , and a photographic plate.

A laser beam from a laser source incident on a plane mirror (beam splitter). As the name suggests, the beam splitter splits the laser beam into two parts. One part of the beam, after reflection from the beam splitter reaches the photographic plate. This beam is known as the reference beam. The second part of the beam (transmitted through the beam splitter) after reflection from various points of the object reaches the photographic plate. This beam is known as the object beam.

The object beam reflected from the object interferes with the reference beam when both the beams reach the photographic plate. The supervision of these two beams

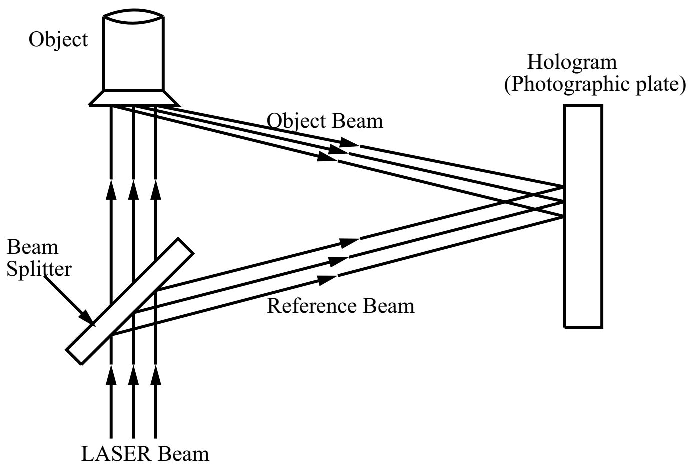


Fig. 21 Recording of hologram

produces an interference pattern (in the form of dark and bright fringes) and this pattern is recorded on the photographic plate. The photographic plate along with the recorded pattern is known as the hologram. The photographic plate is also known as the Gabor zone plate in honor of Dennis Gabor who developed the phenomenon of holography.

Each and every part of a hologram receives light from various points of the object. Thus, even if a hologram is broken into parts, each part of the hologram is capable of reconstructing the image of the whole object.

16.2 Reconstruction of Image

A schematic diagram of the reconstruction of the image of an object is shown in Fig. 22. In the reconstruction process, the hologram is illuminated by a laser beam, and the beam is known as the reconstruction beam. This beam is identical to the reference beam, which is used in the construction of holograms.

The hologram acts as a diffraction grating. The reconstruction beam undergoes diffraction during the passage through the hologram. The reconstruction beam after passing through the hologram produces a real image and a virtual image of the object.

One group of the diffracted rays emerging from the hologram appears to diverge from an apparent object when projected back as shown in Fig. 22. Thus, a virtual image is formed behind the hologram at the original site of the object and a real image of the object is formed in front of the hologram. Thus, an observer sees light waves diverging from the virtual image and the image is identical to the object. When the observer moves around the virtual image then the other sides of the object which

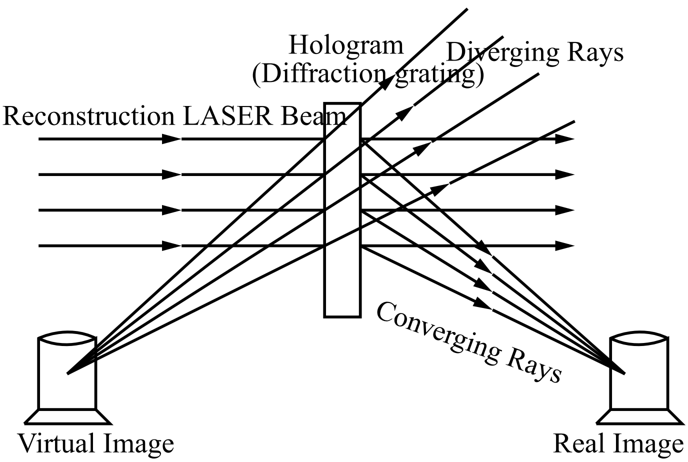


Fig. 22 Reconstruction of image

were not noticed earlier are observed. Hence, the virtual image shows all the true three-dimensional characteristics. The real image can be recorded on a photographic plate.

Thus, the holography is a two-step process. The first step is the image recording on a hologram in the form of an interference pattern. In the second step, the hologram acts as a diffraction grating for the reconstruction of the image of the object.

16.3 Difference Between Holography and Photography

(i) In the photography, only the intensity of light waves is recorded and therefore the photography produces a two-dimensional image (picture)] of an object. On the other side, in the holography, both the intensity and phase of light waves are recorded and therefore, the holography produces three-dimensional image (picture)] of an object.
(ii) In photography, the negative is prepared first whereas in the holography, no negative is required and hologram behaves as a negative.
(iii) If a hologram is broken into parts, each part is capable of reconstruction of the entire object. In photography, the destruction of even a very small part of the negative results in irreparable loss of information.
(iv) In the photography, the presence of the object is not required, but in the holography, the presence of the object is essential.
(v) The holography has high information capacity as compared to the photography.

17 Multiple Choice Questions

1. A LASER source is
   A. monochromatic B. bi-chromatic
   C. white D. none of these
   Ans. A
2. The principle of LASER was given by
   A. Maiman B. Planck C. Einstein D. Townes
   Ans. C
3. The working of LASER is based on
   A. stimulated absorption B. spontaneous emission
   C. stimulated emission D. None of these
   Ans. C
4. Temporal coherence is responsible for
   A. directionality of laser
   B. brightness of laser
   C. monochromaticity of laser
   D. intensity of laser
   Ans. C

5. Spatial coherence is responsible for
   A. directionality of laser
   B. brightness of laser
   C. monochromaticity of laser
   D. intensity of laser

   Ans. A

6. For non-degenerate energy levels, the transition probability for stimulated emission B_{ul} is related to the transition probability for stimulated absorption B_{lu} as
   A. $B_{ul} < B_{lu}$ B. $B_{ul} = B_{lu}$
   C. $B_{ul} > B_{lu}$ D. $B_{ul} \geq B_{lu}$

   Ans. B

7. Nd:YAG laser is
   A. two-level laser B. three-level laser
   C. four-level laser D. None of these

   Ans. C

8. Highly monochromatic laser is
   A. He-Ne laser B. CO_2 laser
   C. Nd:YAG laser D. semiconductor laser

   Ans. A

9. A LASER transition is
   A. collisional transition B. radiative transition
   C. thermal transition D. none of these

   Ans. B

10. Which of the following is a non-radiative transition?
    A. Stimulated emission B. Collisional transition
    C. Spontaneous emission D. Stimulated emission

    Ans. B

11. Which of the following relations between Einstein A and B coefficients for lower level l and upper level u is correct.

    A. $A_{ul} = \dfrac{8\pi h\nu^3}{c^3} B_{ul}$ B. $A_{ul} = \dfrac{8\pi h\nu^3}{c^3} B_{lu}$

    C. $A_{ul} = \dfrac{c^3}{8\pi h\nu^3} B_{ul}$ D. $A_{ul} = \dfrac{c^3}{8\pi h\nu^3} B_{lu}$

    Ans. A

12. In a system, three levels A, B, and C have energies of 0.5 eV, 1.2 eV and 2.6 eV, respectively. The excitation from the lower level B to the upper level C can be done by a photon having energy
    A. 1.2 eV B. 2.6 eV C. 1.4 eV D. 2.1 eV

    Ans. C

13. In a He-Ne laser, the most favorable ratio of helium to neon for satisfactory laser action is
A. 1 : 4 B. 4 : 1 C. 1 : 7 D. 7 : 1
Ans. D
14. The number of energy levels used in the Ruby laser is
A. two B. three C. four D. five
Ans. B
15. The number of energy levels used in the He-Ne laser is
A. two B. three C. four D. five
Ans. C
16. The number of energy levels used in the CO_2 laser is
A. two B. three C. four D. five
Ans. C
17. Ground state of energy levels is the lower level of laser transition in
A. two-level laser B. three-level laser
C. four-level laser D. None of these
Ans. B
18. Ground state of energy levels is never the lower level of laser transition in
A. two-level laser B. three-level laser
C. four-level laser D. None of these
Ans. C
19. Suppose, n_u and n_l are, respectively, the number of atoms in the non-degenerate upper and lower levels of a laser transition. For the population inversion, we have
A. $n_u < n_l$ B. $n_u > n_l$ C. $n_l = n_2$ D. $n_l = 2n_u$
Ans. B
20. The image produced by holography is
A. one-dimensional B. two-dimensional
C. three-dimensional D. four-dimensional
Ans. C
21. A hologram contains the information of the object about
A. amplitude only B. amplitude and phase both
C. phase only D. neither amplitude nor phase
Ans. B
22. Hologram is the consequence of
A. interference of object and reference beam.
B. polarization of object and reference beam.
C. diffraction of object and reference beam.
D. both interference and polarization of object and reference beam.
Ans. A
23. The information carrying capacity of a hologram is
A. less than an ordinary photograph.
B. equal to an ordinary photograph.
C. more than an ordinary photograph.
D. none of these.
Ans. C

24. A part of a hologram contains information about
    A. particular part of the object.
    B. entire object.
    C. important parts of the object.
    D. front side of the object. Ans. B
25. In the collisional excitation process, the initial energy E_1 and the final energy E_2 of a colliding particle has the relation.
    A. $E_1 < E_2$ B. $E_1 = E_2$ C. $E_1 > E_2$ D. none of them
    Ans. C
26. In the collisional deexcitation process, the initial energy E_1 and the final energy E_2 of a colliding particle has the relation.
    A. $E_1 < E_2$ B. $E_1 = E_2$ C. $E_1 > E_2$ D. none of them
    Ans. A
27. For a laser beam, the unit of quality factor is
    A. Angstrom B. Hertz C. none D. Watt
    Ans. C
28. Laser action is not possible when the system has the following number of levels.
    A. Two B. Three C. Four D. Five
    Ans. A
29. For a system to have laser action, the minimum number of levels required is
    A. Two B. Three C. Four D. Five
    Ans. B
30. The system having the following number of levels is used in the ruby laser.
    A. Two B. Three C. Four D. Five
    Ans. B
31. The radiative lifetime of a metastable level is of the order of
    A. 10^{-10} s B. 10^{-8} s C. 10^{-3} s D. 10^{3} s
    Ans. C
32. Nd-YAG laser is known as the system of following a number of levels.
    A. Two B. Three C. Four D. Five
    Ans. C
33. Carbon dioxide laser is known as the system of following a number of levels.
    A. Two B. Three C. Four D. Five
    Ans. C
34. Which of the following lasers is in the form of pulses?
    A. He-Ne B. Ruby C. CO_2 D. Nd:YAK
    Ans. B
35. Recording of a hologram is governed by the phenomenon of
    A. polarization B. diffraction C. interference D. refraction
    Ans. C
36. For the reconstruction of an image from a hologram, which of the following phenomenon is used.
    A. polarization B. diffraction C. interference D. refraction
    Ans. B

37. In the holography, for the formation of images,
    A. the presence of object is essential.
    B. the object may be present or absent.
    C. the object is absent.
    D. use of laser is not essential.

    Ans. A

38. In holography, in a hologram, we record
    A. only intensity of light wave.
    B. only phase of light wave.
    C. both intensity and phase of light wave.
    D. neither intensity nor phase of light wave.

    Ans. C

39. In conventional photography, we record
    A. only intensity of light wave.
    B. only phase of light wave.
    C. both intensity and phase of light wave.
    D. neither intensity nor phase of light wave.

    Ans. A

40. The temperature corresponding to population inversion between two levels is known as
    A. kinetic temperature B. excitation temperature
    C. brightness temperature D. election temperature

    Ans. B

41. In the He-Ne laser, the form of an active system is
    A. plasma B. gas C. liquid D. solid

    Ans. B

42. In the ruby laser, the form of an active system is
    A. plasma B. gas C. liquid D. solid

    Ans. D

43. In the Nd: YAG laser, the form of the active system is
    A. plasma B. gas C. liquid D. solid

    Ans. D

44. In the CO_2 laser, the form of active system is
    A. plasma B. gas C. liquid D. solid

    Ans. B

45. In the semiconductor laser, the form of the active system is
    A. plasma B. gas C. liquid D. solid

    Ans. D

18 Problems and Questions

1. What for the abbreviation LASER stands?
2. What is a laser? Explain its principle.

3. What are the essential requirements for the laser action? Describe the important features of stimulated emission of radiation.
4. Describe the radiative and non-radiative transitions.
5. Describe the phenomena of (i) radiative absorption, (ii) Spontaneous emission and (iii) Stimulated emission. Obtain the relations between Einstein A and B coefficients.
6. Describe the phenomena of population inversion. Comment on the following relation where there is population inversion.

$$\frac{n_u}{n_l} = \exp\left[-\frac{E_u - E_l}{kT}\right]$$

7. What are the Einstein coefficients for radiative transitions? Find the relations between them.
8. Explain the phenomenon of population inversion.
9. Discuss the characteristics of the laser beam.
10. What is a laser? Explain the construction and working of the ruby laser with a necessary diagram.
11. Explain the construction and working of the He-Ne laser with a necessary diagram.
12. Explain the construction and working of CO_2 laser with a necessary diagram.
13. Describe the essential conditions for a semiconductor laser.
14. Discuss the population inversion in a three-level system, when all the levels are non-degenerate.
15. Describe the working of the He-Ne laser
16. Describe the working of the Ruby laser
17. Discuss the population inversion in a four-level system, when all the levels are non-degenerate.
18. What are the important applications of lasers?
19. Define a hologram. Discuss the principle of holography with suitable diagram.
20. What are the salient features of holography?
21. Discuss the construction and reconstruction of the image on a hologram.
22. Describe the reason that the construction of a maser is easier than that of a laser.
23. Discuss the difference between holography and photography.
24. Write short notes on the following

    (i) LASER
    (ii) Excitation process
    (iii) Deexcitation process
    (iv) Radiative transitions
    (v) Non-radiative transitions
    (vi) Stimulated absorption
    (vii) Spontaneous emission
    (viii) Stimulated emission
    (ix) Population inversion

(x) Three-level system of laser action
(xi) Four-level system of laser action
(xii) Spatial coherence
(xiii) Temporal coherence
(xiv) Laser cavity
(xv) Laser applications
(xvi) He-Ne laser
(xvii) Ruby laser
(xviii) Nd-YAG laser
(xix) Carbon dioxide laser
(xx) Semiconductor laser
(xxi) Holography
(xxii) Excitation temperature
(xxiii) Quenching
(xxiv) Metastable level
(xxv) Construction of hologram
(xxvi) Reconstruction of image from a hologram

Chapter 9
Fiber Optics

1 Introduction

For sending information from one place to another, a traditional method has been to use the radio waves (or microwaves) as a carrier. The discovery of laser in 1960 brought a big revolution in the field of telecommunication, as the laser is monochromatic and coherent source of light waves. The frequency of a laser wave is about 10^5 times larger than that of a radio wave. (The wavelength of a laser wave is about 10^5 times smaller than that of a radio wave.) Hence, with the help of the laser waves, about 10^5 times more information can be carried out as compared to that with the help of the radio waves.

However, the scattering of a laser wave is about 10^{20} times larger than that of a radio wave.[1] Consequently, the energy of a laser wave gets dissipated very easily in an open atmosphere. Hence, the laser waves cannot travel long distances in an open atmosphere. Therefore, we require a medium in which there is no loss of energy during propagation of laser waves from one place to another. In such arrangement, the waves need to be guided also. This requirement for the laser waves has been achieved with the use of an optical fiber.

An optical fiber is a very thin glass or plastic conduit designed to guide the light waves along its length. The working of an optical fiber is based on the phenomenon of total internal reflection. Fiber optics is a technology which uses glass, plastic, or such material for transmission of data. A cable for fiber optics consists of a bundle of fibers which are protected by an outer jacket, made up of treated paper, PVC, or metal.

Optical fiber has a number of advantages over the copper wire used to make connections electrically. For example, an optical fiber, being made up of glass or plastic, is protected from electromagnetic interference such as that caused by thunderstorms. A single optical fiber has three parts, as shown in Fig. 1:

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[1] The scattering of an electromagnetic wave is inversely proportional to the fourth power of its wavelength.

S. Chandra and M. K. Sharma, *A Textbook of Optics*,
https://doi.org/10.1007/978-3-031-55676-0_9

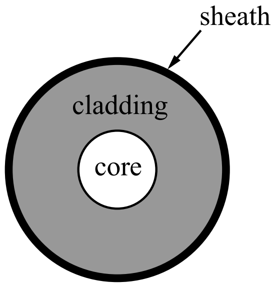

Fig. 1 An optical fiber has a core, cladding, and sheath

(i) core,
(ii) cladding, and
(iii) sheath (protecting layer).

The shape of a core is cylindrical with small radius. It is situated at the center of the fiber along its axis and is made up of glass. Cladding is a thin cylindrical shell surrounding the core and is made up of an optical material. The refractive index of the material of cladding is smaller than that of the material of core, so that total internal reflection at the interface of the core and cladding could take place when the angle of incidence is larger than the critical angle. Sheath is a plastic coating that protects the fibers from damage and moisture.

In order to understand about the advantages of fiber optics, it is necessary to know about the bandwidth, which is described as the difference between the upper and lower cut-off frequencies of a filter, a communication channel, or a signal spectrum. It is expressed in Hertz. In the case of a low-pass filter or base band signal, the bandwidth is equal to its upper cut-off frequency. In the radio communications, bandwidth is the range of frequencies occupied by a modulated carrier wave. For example, an FM radio receiver's tuner spans over a limited range of frequencies. In optics, it is the width of an individual spectral line or the entire spectral range.

Fiber optics has many advantages as compared to the traditional metal communication lines. Some of them are as follows:

(i) Fiber-optic cables can carry more data as their bandwidth is greater as compared to that of a metal cable.
(ii) Fiber-optic cables are less susceptible to interference as compared to the metal cables.
(iii) Fiber-optic cables are much thinner and lighter as compared to the metal cables.
(iv) Through the fiber-optic cables, the data can be transmitted digitally rather than analogically.
(v) Attenuation through fiber-optic cables is very low in transmitting the data over a long distance, so there is no need of repeaters.

2 Basic Concepts Used in Optical Fiber

In an optical fiber, we employ light waves for carrying the digital signals from one place to another. This technology is based on the concept of total internal reflection. The digital signal being carried by the light wave is reflected inside the optical cable and hence transfers the information from one place to another. The main concepts of physics that are involved in the optical fibers are the refraction, refractive indices, critical angle, and total internal reflection.

In the refraction, the light wave bends away from the normal when it propagates from a medium having higher refractive index to a medium having lower refractive index. The phenomenon of total internal reflection takes place when the angle of refraction becomes 90°. The incident angle for which the angle of refraction is 90° is known as the critical angle. When a light ray propagating from a higher refractive index medium to a lower refractive index medium has a sufficiently large incident angle, i.e., greater than the critical angle, the light gets reflected back into the same medium. For a particular case of an optical fiber whose core is made of glass which is bounced by a plastic cladding the critical angle is 82°. Thus, when the light hits the plastic cladding at an angle more than 82°, then it is reflected back in the same medium, i.e., back to the core.

3 Total Internal Reflection

When a ray of light propagates from a denser medium to a rarer one, the angle of refraction is larger than the angle of incidence. If we go on increasing the angle of incidence, at some of its value, the refraction angle becomes equal to 90° and the ray does not enter into the rarer (second) medium, as shown in Fig. 2. This phenomenon is known as the total internal reflection and the corresponding angle of incidence is known as the critical angle, denoted by i_c. Thus, we have

$$ {}_1\mu_2 = \mu = \frac{\sin i}{\sin r} = \frac{\sin i_c}{\sin 90} = \sin i_c $$

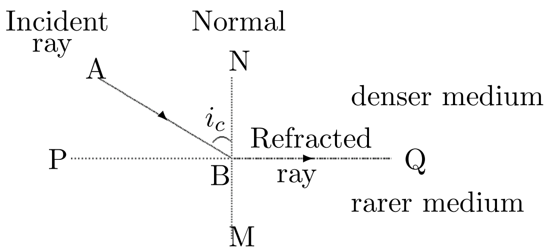

Fig. 2 Total internal reflection

Thus, the critical angle is

$$i_c = \sin^{-1} \mu$$

With the further increase of the angle of incidence, the incident ray is reflected back into the first medium.

Exercise 1 A ray of light propagates from the glass to the air. Calculate the critical angle for total reflection. Refractive index of glass relative to air is 1.5.

Solution The ray of light goes from glass to air and the refractive index of air relative to glass is

$$_g\mu_a = \frac{1}{_a\mu_g} = \frac{1}{1.5} = 0.6667$$

Thus, the critical angle i_c is

$$i_c = \sin^{-1}(_g\mu_a) = \sin^{-1}(0.6667) = 41.8^\circ$$

4 Types of Optical Fibers

Based on the transmission properties and structure, optical fibers are, in general, classified into two categories:

(i) single-mode fiber,
(ii) multi-mode fiber.

As the name implies, a single-mode fiber sustains only one mode of propagation, whereas a multi-mode fiber sustains multiple modes. The basic difference from the structural point of view between the two categories is the size of the core. The core diameter of a single-mode fiber is about 10 μm and that of a multi-mode fiber ranges from 50 μm to 100 μm, depending on the fiber. When the refractive indices of the core as well as of the cladding are constant, the optical fiber is said to be step index fiber. On the other side, when the refractive index of the core or/and of the cladding varies, the optical fiber is said to be graded index fiber.

4.1 Salient Features of Single-Mode Fibers

(i) For a single-mode step index fiber, the core diameter is about 8–12 μm with a cladding thickness of about 125 μm. For a graded index single-mode fiber, the core diameter is slightly larger as compared in a step index single-mode fiber. This makes handling, connecting, and coupling of a graded index fiber a bit easier as compared to those of a step index fiber.

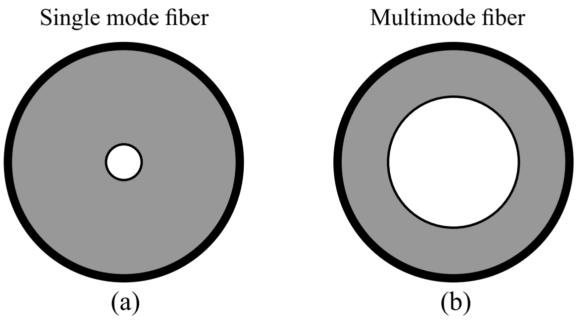

Fig. 3 **a** Single-mode fiber; **b** Multi-mode fiber

(ii) For a single-mode fiber, the losses are less.
(iii) In a single-mode fiber, the cladding is relatively thick. By making the cladding thick, the field at the cladding-air boundary is minimized.
(iv) Both the relative refractive index $\Delta\mu_r$ and numerical aperture N_a, for a single-mode fiber are very small. Low N_a means a very small acceptance angle. Thus, the incident ray in a single-mode fiber must be nearly perpendicular to the fiber edge.
(v) As only one mode propagates in a single-mode fiber, no intermodal dispersion exists there. This makes the single-mode fibers suitable for their use with high data rates.
(vi) Manufacturing a single-mode fiber is more expensive and difficult. To launch light energy into the fiber, costly laser diodes are required. However, when the high data rate is considered, it becomes cost-effective.
(vii) Single-mode fibers are becoming more popular for other specialized applications.

4.2 Salient Features of Multi-mode Fibers

(i) For a multi-mode step index fiber, the core diameter is about 50–250 μm with a cladding thickness of about 125–400 μm. For a multi-mode index fiber, the core diameter is about 50–100 μm with a cladding thickness of about 125–140 μm.
(ii) Both the relative refractive index $\Delta\mu_r$ and numerical aperture N_a for a multi-mode fiber are large. Thus, the acceptance angle is much larger as compared to that for a single-mode fiber. Therefore, the launching of light signal into a multi-mode fiber is much easier.
(iii) An LED (Light-Emitting Diode), which is comparatively less expensive, can be used for launching of light into a multi-mode fiber.
(iv) Handling, connecting, and coupling of multi-mode fibers are easier as compared to those of a single-mode fiber.

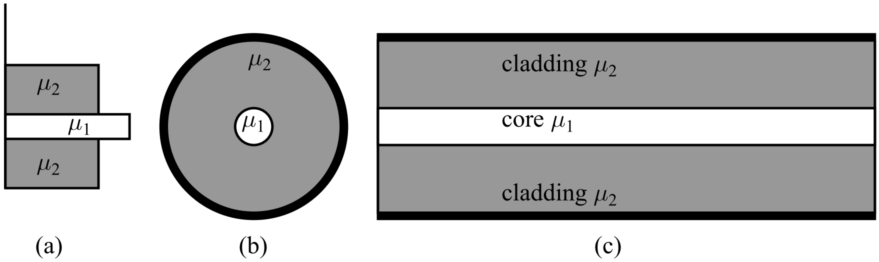

Fig. 4 **a** Profile of refractive index of single-mode step index fiber; **b** Cross-sectional view of single-mode fiber; **c** Longitudinal view of single-mode fiber

(v) The major disadvantage in case of a multi-mode fiber is the intermodal dispersion, which limits the data transmission rates.

4.3 Single-Mode Step Index Fiber

In this class of optical fibers, the refractive indices of the materials of the core (μ_1) and of cladding (μ_2) are both constant, such that $\mu_1 > \mu_2$. When we proceed from the core to the cladding, the refractive index changes in a single step from μ_1 to μ_2. At the interface between the core and cladding, the refractive index changes suddenly in a step, as shown in Fig. 4a. The diameter of the core is about 10 μm whereas the outer diameter of the cladding is about 125 μm, as shown in Fig. 4b. Owing to the small diameter of the core, this fiber allows only one mode to propagate through the core at a time.

As compared to a multi-mode fiber, a single-mode fiber has a lower signal loss and higher information capacity or bandwidth. These fibers are capable of transferring larger amount of data due to low fiber dispersion. In these fibers, the wavelength can increase or decrease the losses caused by the fiber bending. In general, single-mode fibers are considered to be low-loss fibers, which increase system bandwidth and length. Therefore, these fibers are most useful for large bandwidth applications. Since these fibers are more resistant to attenuation, they can also be used in significantly longer cable runs.

4.4 Multi-mode Fiber

Contrary to a single-mode fiber, a multi-mode fiber (MMF) allows more than one mode to propagate through the core at a time. For example, over 100 modes can propagate through the core of the fiber at a time. The size of its core is typically about 50 μm whereas the outer diameter of the cladding is about 125 μm, as shown in Fig. 3b. The multi-mode fibers are of two categories:

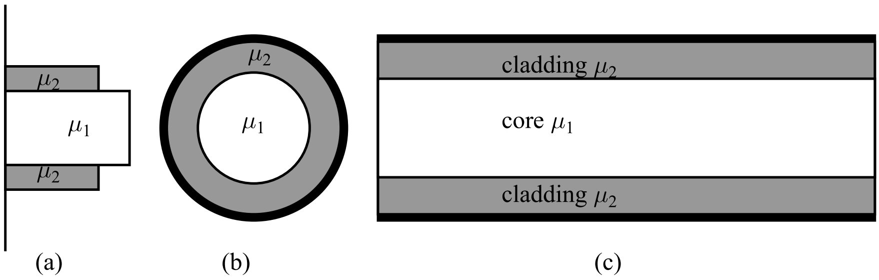

Fig. 5 **a** Profile of refractive index of multi-mode index fiber; **b** Cross-sectional view of multi-mode fiber; **c** Longitudinal view of multi-mode fiber

(i) Multi-mode step index fiber.
(ii) Multi-mode graded index fiber.

(i) Multi-mode Step Index Fiber

In this class, the refractive indices of the materials of the core μ_1 and cladding μ_2 are both constant. At the interface between the core and cladding, the refractive index changes suddenly in a step, as shown in Fig. 5a. In this type of optical fiber, the number of propagating modes depends on the ratio of core diameter to the wavelength. This ratio is inversely proportional to the numerical aperture (abbreviated as Na and discussed later on). Typically the core diameter is 50 μm to 100 μm and Na varies from 0.20 to 0.29, respectively. Multi-mode fiber is used in short lengths, such as those used in Local Area Networks (LANs) and Storage Area Networks (SLANs).

Because the multi-mode optical fiber has larger Na and the larger core size, fiber connections and launching of light are very easy. Multi-mode fibers permit the use of light emitting diodes (LEDs). In such fibers, core-to-core alignment is less critical during the fiber spacing. However, due to several modes, the effect of dispersion gets increased, i.e., the modes arrive at the fiber end at slightly different times and therefore spreading of pulses takes place. This dispersion of the modes affects the system bandwidth. Hence, the core diameter, Na, and index profile properties of multi-mode fibers are optimized to maximize the system bandwidth.

(ii) Multi-mode Graded Index Fiber

In a multi-mode graded index fiber, the refractive index of the core decreases with increasing the radial distance from the fiber axis, which is the imaginary central axis running along the length of the fiber, as shown in Fig. 6. The value of the refractive index is the highest at the center of the core and decreases to a value at the edge of the core that equals the refractive index of the cladding. Hence, the light rays in the outer zones of the core travel faster than those in the center of the core. Hence, the dispersion of the modes is compensated by this type of fiber design.

Under this situation, the light rays follow sinusoidal paths along the fiber. In such fibers, the most common profile of the refractive index is very nearly parabolic that

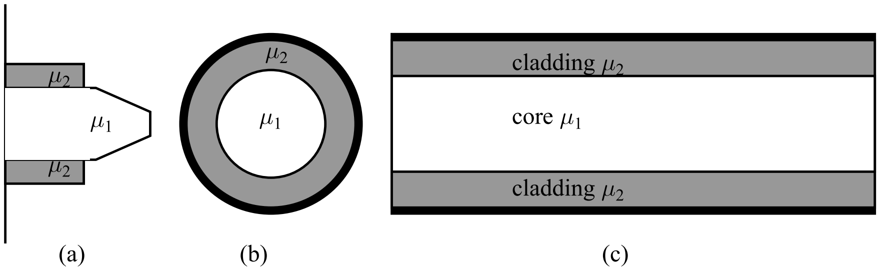

Fig. 6 **a** Profile of refractive index of multi-mode graded index fiber; **b** Cross-sectional view of multi-mode fiber; **c** Longitudinal view of multi-mode

results in the continual refocusing of the rays in the core, and minimizing modal dispersion. Standard graded index fibers typically have a core diameter of 50 μm or 62.5 μm and a cladding diameter of 125 μm. It is typically used for transmitting the information to the distance to a couple of kilometers. The advantage of the multi-mode graded index fiber in comparison with the multi-mode step index fiber is the considerable decrease in the modal dispersion.

5 Parameters of Fiber Optics

As mentioned earlier, the main phenomenon used in the fiber optics technique is the total internal reflection at the interface between the core and cladding in a fiber cable. For total internal reflection, the light must travel from a medium of large refractive index to another medium of low refractive index and the angle of incidence must be larger than the critical angle. Suppose, the refractive indices of the materials of the core and cladding are μ_1 and μ_2, respectively, such that $\mu_1 > \mu_2$. Some parameters pertaining to the fiber optics are as follows:

(i) Relative refractive index.
(ii) Acceptance angle.
(iii) Acceptance cone.
(iv) Numerical aperture.
(v) Skip distance.

5.1 Relative Refractive Index

When the refractive indices of the materials of the core and cladding in an optical fiber are μ_1 and μ_2, respectively, the relative refractive index $\Delta\mu_r$ is expressed as

$$\Delta\mu_r = \frac{\mu_1 - \mu_2}{\mu_1} = 1 - {}_1\mu_2$$

where ${}_1\mu_2$ is the refractive index of cladding relative to that of in the core. It is also known as the fractional refractive index. Obviously, it is a measure of the difference between the refractive indices of the materials of core and cladding. Larger the difference, larger the relative refractive index.

5.2 Acceptance Angle

Let us consider a cylindrical optical fiber (Fig. 7) where μ_1 and μ_2 are the refractive indices of the materials of core and cladding, respectively, such that $\mu_1 > \mu_2$. Suppose, μ_0 is refractive index of medium (which is generally the air) outside the fiber, such that $\mu_0 < \mu_1$.

At one end of the optical fiber, suppose the light incidents at an angle θ_i. The angle of refraction is θ_r such that $\theta_r < \theta_i$. Then, the light travels in the core and incident at the core-cladding interface at an angle θ so that $\theta_r + \theta = 90°$. When the angle θ is larger than the critical angle θ_c, the light undergoes the total internal reflection at the interface between the core and cladding. As long as the angle θ is larger than the critical angle θ_c, the light is reflected back into the core. Let us now compute the incident angle θ_i for which $\theta \geq \theta_c$. Following Snell's law, we have

$$\mu_0 \sin\theta_i = \mu_1 \sin\theta_r \qquad \text{or} \qquad \frac{\sin\theta_i}{\sin\theta_r} = \frac{\mu_1}{\mu_0} \tag{1}$$

With the increase of the angle θ_i, the angle θ_r increases, and consequently the angle θ decreases. For the fiber optics technique, obviously, we cannot afford to increase the value of θ_i beyond a certain limit denoted by θ_m. This value θ_m of the incidence angle is known as the acceptance angle. Corresponding to θ_m, we have $\theta = \theta_c$. From the ΔABC, it is seen that

$$\sin\theta_r = \sin(90° - \theta) = \cos\theta \tag{2}$$

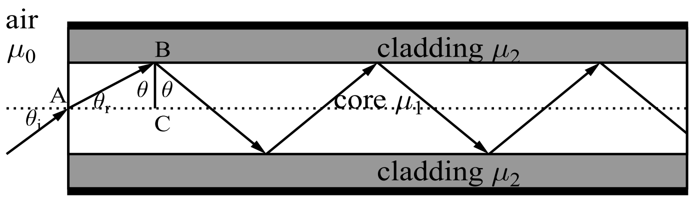

Fig. 7 At one end of an optical fiber, the light incidents at an angle θ_i

From Eqs. (1) and (2), we have

$$\frac{\sin\theta_i}{\cos\theta} = \frac{\mu_1}{\mu_0} \quad \text{or} \quad \sin\theta_i = \frac{\mu_1}{\mu_0}\cos\theta$$

When $\theta = \theta_c$, we have $\theta_i = \theta_m$ and therefore

$$\sin\theta_m = \frac{\mu_1}{\mu_0}\cos\theta_c \tag{3}$$

At the critical angle θ_c, from Snell's law, we have

$$\mu_1 \sin\theta_c = \mu_2 \sin 90^\circ \quad \text{or} \quad \sin\theta_c = \frac{\mu_2}{\mu_1}$$

We have

$$\cos\theta_c = \sqrt{1-\sin^2\theta_c} = \sqrt{\frac{\mu_1^2-\mu_2^2}{\mu_1^2}} \tag{4}$$

Using Eq. (4) in (3), we have

$$\sin\theta_m = \frac{\sqrt{\mu_1^2-\mu_2^2}}{\mu_0} \tag{5}$$

When the incident ray of light is launched from air medium (for which $\mu_0 = 1$), we have

$$\sin\theta_m = \sqrt{\mu_1^2-\mu_2^2} \quad \text{or} \quad \theta_m = \sin^{-1}(\sqrt{\mu_1^2-\mu_2^2})$$

The angle θ_m is known as the acceptance angle for the optical fiber. The value of θ_m is always less than 90°. That is when the incidence angle θ_i is less than θ_m, the light at the interface between the core and cladding shows the phenomenon of total internal reflection.

5.3 Acceptance Cone

An optical fiber is cylindrical in shape. Obviously, the light can incident from any direction around the axis of the fiber. Thus, all these directions form a cone, as shown in Fig. 8. When the incidence angle θ_i is equal to the acceptance angle θ_m, the corresponding cone is known as the acceptance cone.

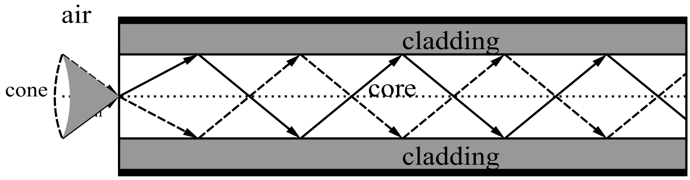

Fig. 8 Acceptance cone for a fiber cable

Thus, a light ray lying in the acceptance cone shows the phenomenon of total internal reflection. Obviously, a light ray lying outside the acceptance cone is refracted into the cladding and the energy is lost.

5.4 Numerical Aperture

Numerical aperture N_a is another parameter of an optical fiber. It is a measure of light-gathering ability of the optical fiber and is expressed as

$$N_a = \mu_0 \sin\theta_m$$

where μ_0 is the refractive index of the medium outside the optical fiber and θ_m the acceptance angle. It is a dimensionless quantity. When the medium outside the fiber is air ($\mu_0 = 1$), the value of N_a is less than one, as the value of $\sin\theta_m$ is always less than one. Using the value of $\sin\theta_m$, we get

$$N_a = \sqrt{\mu_1^2 - \mu_2^2}$$

This relation shows that the light-gathering ability of an optical fiber increases with its numerical aperture. As the value of $\sin\theta_m$ is always less than 1, the value of the numerical aperture is less than 1. The value of N_a varies in the range from 0.14 to 0.5.

5.5 Skip Distance

As mentioned earlier, the propagation of light in an optical fiber is based on the phenomenon of total internal reflection. A light ray gets reflected at the interface between the core and cladding. The distance along the length of the fiber, between two successive reflections of a ray of light propagating in the fiber, is known as the skip distance L_s. In Fig. 9, the distance BD is the skip distance and is expressed as

$$L_s = d\cot\theta_r \tag{6}$$

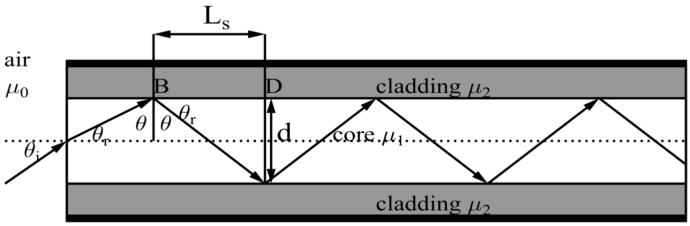

Fig. 9 Skip distance for a fiber cable

where d is the diameter of the core and θ_r is the angle of refraction in the core. When μ_0 and μ_1 are refractive indices of outside medium and of core, respectively, Snell's law is

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{\mu_1}{\mu_0} \qquad \text{or} \qquad \sin\theta_r = \frac{\mu_0}{\mu_1}\,\sin\theta_i$$

Thus, we have

$$\cos\theta_r = \sqrt{1 - \left(\frac{\mu_0\sin\theta_i}{\mu_1}\right)^2}$$

and

$$\cot\theta_r = \frac{\cos\theta_r}{\sin\theta_r} = \sqrt{\left(\frac{\mu_1}{\mu_0\,\sin\theta_i}\right)^2 - 1} \tag{7}$$

Using Eq. (7) in (6), we have

$$L_s = d\sqrt{\left(\frac{\mu_1}{\mu_0\,\sin\theta_i}\right)^2 - 1}$$

It is obvious that the inverse of the skip distance, i.e., $1/L_s$ gives the total number of reflections made by the light ray per unit length of the fiber. For an fiber of length L, the number of reflections N_r is

$$N_r = \frac{L}{L_s} = \frac{L}{d\sqrt{(\mu_1/\mu_0\,\sin\theta_i)^2 - 1}}$$

Exercise 2 The refractive indices of the materials of core and cladding of a step index optical fiber are 1.52 and 1.40, respectively. Calculate the (i) critical angle, (ii) numerical aperture, (iii) acceptance angle, and (iv) fractional refractive index.

Solution Given, refractive index of the material of core μ_1 = 1.52 and that of the cladding μ_2 = 1.40. (i) The critical angle is

$$\theta_c = \sin^{-1}(\mu_2/\mu_1) = \sin^{-1}(1.40/1.52)$$

$$= \sin^{-1}(0.9211) = 67.09°$$

(ii) The numerical aperture is

$$N_a = \sqrt{\mu_1^2 - \mu_2^2} = \sqrt{(1.52)^2 - (1.40)^2} = 0.5919$$

(iii) The acceptance angle is

$$\theta_0 = \sin^{-1}\sqrt{(\mu_1^2 - \mu_2^2)} = \sin^{-1}\sqrt{(1.52)^2 - (1.40)^2}$$

$$= \sin^{-1}(0.5919) = 36.29°$$

(iv) The fractional refractive index is

$$\Delta\mu_r = \frac{\mu_1 - \mu_2}{\mu_1} = \frac{1.52 - 1.40}{1.52} = 0.0789$$

Exercise 3 A light ray enters from air to a optical fiber. The refractive index of air is 1.0. The refractive indices of the materials of core and cladding of a step index optical fiber are 1.51 and 1.46, respectively. Calculate the (i) critical angle, (ii) numerical aperture, (iii) acceptance angle, and (iv) fractional refractive index.

Solution Given, refractive index of air $\mu_0 = 1$, that of the material of core $\mu_1 = 1.51$ and that of the material of cladding $\mu_2 = 1.46$. (i) The critical angle is

$$\theta_c = \sin^{-1}(\mu_2/\mu_1) = \sin^{-1}(1.46/1.51)$$

$$= \sin^{-1}(0.9669) = 75.21°$$

(ii) The numerical aperture is

$$N_a = \sqrt{\mu_1^2 - \mu_2^2} = \sqrt{(1.51)^2 - (1.46)^2} = 0.3854$$

(iii) The acceptance angle is

$$\theta_0 = \sin^{-1}\sqrt{(\mu_1^2 - \mu_2^2)} = \sin^{-1}\sqrt{(1.51)^2 - (1.46)^2}$$

$$= \sin^{-1}(0.3854) = 22.67°$$

(iv) The fractional refractive index is

$$\Delta\mu_r = \frac{\mu_1 - \mu_2}{\mu_1} = \frac{1.51 - 1.46}{1.51} = 0.0331$$

Exercise 4 A glass clad fiber has refractive index 1.5 and the cladding is doped to give a fractional refractive index of 0.0005. Determine the (i) cladding refractive index, (ii) critical refraction angle, (iii) acceptance angle, and (iv) numerical aperture.

Solution We have refractive index of core $\mu_1 = 1.5$ and the fractional refractive index is

$$\frac{\mu_1 - \mu_2}{\mu_1} = 0.0005$$

Using $\mu_1 = 1.5$, we have

$$\frac{1.5 - \mu_2}{1.5} = 0.0005 \qquad \text{or} \qquad \mu_2 = 1.5 - 1.5 \times 0.0005 = 1.49925$$

When θ_r is the angle of refraction and light enters the core from outside, for total internal reflection we have

$$\mu_1 \sin(90 - \theta_r) = \mu_2 \sin 90 \qquad \text{or} \qquad \cos\theta_r = \frac{1.49925}{1.5} = 0.9995$$

It gives $\theta_r = 1.81°$. Thus, the angle for critical refraction is $90 - 1.81 = 88.19°$. For the incidence angle θ_i, we have

$$\mu_0 \sin\theta_i = \mu_1 \sin\theta_r \qquad \text{or} \qquad \sin\theta_i = \frac{1.5}{1.0}\sqrt{1 - \cos^2\theta_r}$$

Thus, we have

$$\sin\theta_i = 1.5\sqrt{1 - (0.9995)^2} = 0.04743 \qquad \text{or} \qquad \theta_i = 2.72°$$

Hence, the critical acceptance angle is 2.72°. The numerical aperture is

$$N_a = \sqrt{\mu_1^2 - \mu_2^2} = \sqrt{(1.5)^2 - (1.49925)^2} = 2.24 \times 10^{-3}$$

Exercise 5 The refractive indices of the materials of the core and cladding of a step index optical fiber of diameter 0.062 mm are 1.54 and 1.38, respectively. Calculate the (i) critical angle, (ii) numerical aperture, (iii) acceptance angle, (iv) fractional refractive index, and (v) number of reflections in 80 cm of fiber for a ray at the maximum incidence angle.

Solution Given, $d = 0.062$ mm, $L = 80\,\text{cm}$, refractive index of air $\mu_0 = 1$, that of the material of core $\mu_1 = 1.54$ and that of the material of cladding $\mu_2 = 1.38$. (i) The critical angle is

$$\theta_c = \sin^{-1}(\mu_2/\mu_1) = \sin^{-1}(1.38/1.54) = \sin^{-1}(0.8961) = 63.65^\circ$$

(ii) The numerical aperture is

$$N_a = \sqrt{\mu_1^2 - \mu_2^2} = \sqrt{(1.54)^2 - (1.38)^2} = 0.6835$$

(iii) The acceptance angle is

$$\theta_0 = \sin^{-1}\sqrt{(\mu_1^2 - \mu_2^2)} = \sin^{-1}\sqrt{(1.54)^2 - (1.38)^2}$$

$$= \sin^{-1}(0.6835) = 43.12^\circ$$

(iv) The fractional refractive index is

$$\Delta\mu_r = \frac{\mu_1 - \mu_2}{\mu_1} = \frac{1.54 - 1.38}{1.54} = 0.1039$$

(v) The number of reflections

$$N_r = \frac{L}{d\sqrt{(\mu_1/\mu_0\ \sin\theta_i)^2 - 1}}$$

$$= \frac{80 \times 10^{-2}}{0.062 \times 10^{-3}\sqrt{(1.54/1.0 \times 0.6835)^2 - 1}} = 6390.79$$

The number of reflections is 6390.

6 V-Number of a Fiber

The V-number of a fiber is defined as

$$V = \frac{2\pi a}{\lambda}\,(\mu_1^2 - \mu_2^2)^{1/2}$$

where a is the radius of fiber core and λ the operating wavelength. The V-number is dimensionless and often referred to as the normalized frequency. The parameter V is related to the number of modes a fiber can support. Detailed calculations give the number of modes M supported in a multi-mode step index fiber as

$$M_{SI} = \frac{1}{2}\frac{(2\pi a)^2}{\lambda^2}\left(\mu_1^2 - \mu_2^2\right) = \frac{V^2}{2}$$

When two possible polarizations are accounted for, the number of modes gets

$$M_{SI} = 2 \times \frac{V^2}{2} = V^2$$

The number of modes in a graded index fiber is about half of that in a step index fiber. Thus, the number of modes in a graded index fiber is

$$M_{GI} = \frac{M_{SI}}{2} = \frac{V^2/2}{2} = \frac{V^2}{4}$$

On account of the two possible polarizations, we have

$$M_{GI} = 2 \times \frac{V^2}{4} = \frac{V^2}{2}$$

7 Fiber Optics Communication

A block diagram of a communication system often used before the development of optical fiber communication system is shown in Fig. 10. The essential components of the system are a modulator or transmitter, transmission medium, and the demodulator or receiver.

The optical fibers have replaced the copper coaxial cables due to their various advantages as they are very light weight, thin conduit cables which provide large communication capacity with lower loss. A block diagram of a fiber optic communication system from the signal source to the signal output is shown in Fig. 11. Here, the information that is to be transmitted is first converted into an optical signal from an electrical signal. Then the optical signal is further converted into an electrical after transmission by an optical fiber.

Independent of the original nature of the signal, a fiber provides the choice format of transmission as analog and digital because these two formats are convertible from one into the other. Therefore, the signal in analog or digital form is impressed into the carrier wave with the help of a modular. The carrier wave is generated from the carrier source which may be either light emitting diode (LED) or laser diode (LD). This carrier wave is modulated using various techniques, *viz.*, frequently modulation,

Fig. 10 Block diagram of a communication system used in the early days

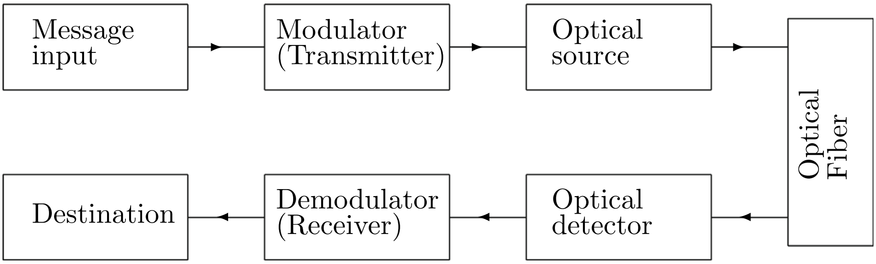

Fig. 11 Block diagram of fiber-optic communication system

amplitude modulation, and digital modulation. The carrier source output into the optical fiber is represented by a single pulse. When a pulse is passed through a fiber, then it is attenuated and distorted due to several mechanisms, for example, by intermodal distortion.

Therefore, repeaters and regenerators are used to amplify the light signal at several positions of the fiber. And after that the light signal at several positions of the fiber. And after that the light is coupled into a detector that may be a semiconductor device or most commonly used as a PIN diode at the end of a fiber. This changes the optical signal back into an electrical signal. The response of a detector should be well matched with the optical frequency of the signal received. The output of the detector then passes through the signal processor, which is used to capture the original electrical signal from the carrier by using the process of filtering, amplification, and an analog-to-digital conversion. The signal output is finally communicated by the cathode ray tube (when it is a video signal), by the loudspeaker (when it is a audio signal), or by the computer input (when it is a digital signal).

7.1 Advantages of Optical Communication

The prime goal of a communication system is to transfer information (data) from one place to another in an effective manner. Thus, the performance of any communication system may be assessed in terms of the amount of data transferred and the effectiveness of the way of the communication. In the light of these, the merits of the optical fiber communication over the conventional (radio and microwave) methods may be considered as follows:

(i) **High capacity for carrying information**: As we know that the light waves have extremely high capacity for carrying information from one place to another. Thus, with the invent of optical fibers, the bandwidth available is extremely large.

(ii) **Small size and small weight**: Optical fibers are so thin that their diameter is less than that of human hair. Thus, even along with the protective coating, the optical fibers are much lighter than the copper cables used in the conventional communication. Hence, the optical fibers occupy very little space in the already crowded ducts in the cities.

(iii) **Immunity to interference**: As an optical fiber is a dielectric waveguide, it has no effect on the electromagnetic pulses and can be in an environment where electrical noise is present. Thus, the fiber cables require no shielding from electromagnetic interference. It is free from the lightning strikes when used overhead.

(iv) **Negligible cross talk**: As the fields are confined to a fiber, the optical interference between the fibers is negligible. The cross talk is negligible even when a number of fibers are cabled together.

(v) **Low transmission loss**: The optical fibers have been developed with losses as low as 0.2 dB/km or less. It allows the implementation of communication links with extremely large repeater spacing.

(vi) **Ruggedness and flexibility**: Because of the protective coatings and proper cabling, it is possible to construct the optical fibers having high tensile strength. In terms of storage, transportation handling, and installation, optical cables are generally superior to copper cables.

(vii) **Signal security**: As the light from the optical fibers do not radiate significantly, it is difficult to tap information during its transmission.

(viii) **System reliability**: The reliability of optical components is considered from 20 to 30 years. This value of reliability is much better than that with the conventional electrical conductor systems.

(ix) **Low cost**: The cost of semiconductor lasers, photodiodes, connectors, and couplers is on the higher side. The number of repeater stations required is less. Taking all the factors such as the information carrying capacity, system reliability, predicted lifetime, and security, the overall system cost is very low.

Owing to the aforesaid advantages, light wave communication is becoming rapidly an important mode of transmission of information from one place to another.

7.2 High Bit Rate Fiber Communication

A high bit rate signal carried on copper wire transmission line is generally needed to be amplified after every 300 m. However, high bit rate signals when carried on an optical fiber, the need of such amplification is required after every 100 km or so. As discussed earlier, a detector changes the optical signal back into an electrical signal, the light signal is coupled to the detector at the remote end of the fiber. This is done effectively when the response of the detector is well matched with the optical frequency of the signal received. Then a signal processor handles the detector output. The function of the signal processor is to recapture the original signal from the carrier. The process involves the filtration and amplification, and a digital-to-analog conversion.

7.3 Allowed Modes of Normalized Frequency

It appears from the theory of acceptance cone that every ray propagates successfully once it enters the fiber within its acceptance cone. However, this is not the case always and only certain ray directions or modes are allowed to propagate successfully. Actually a ray represents plane waves which move up and down in the fiber. Evidently, such waves overlap and interfere with one another and only those waves sustain which satisfy the condition of resonance. Considering this point, we can derive a relation for a parameter m_m in terms of the core diameter d, numerical aperture N_a, and the wavelength λ as

$$m_m = \frac{1}{2}\left(\frac{\pi d N_a}{\lambda}\right)^2$$

The largest integer that is less than the parameter m_m gives the maximum number of modes which can propagate successfully through the fiber. Therefore, it is clear that the number of possible modes is larger for the higher value of the ratio d/λ. So, a fiber of big diameter allows large number of modes to propagate through it. For this reason, they are known as the multi-mode fiber. However, when d/λ is small such that m_m is less than 2, the fiber allows only one mode to propagate through it. Such type of fiber is therefore known as the single-mode fiber or mono-mode fiber. The condition $m_m < 2$ for a single-mode fiber can be achieved when

$$\frac{d}{\lambda} < \frac{2}{\pi N_a}$$

The above condition, related to the diameter, guarantees the performance of single-mode fiber. However, a more careful analysis shows that the single-mode performance can be achieved even when

$$\frac{d}{\lambda} < \frac{2.4}{\pi N_a}$$

It is evident that the parameter m_m decides the number of possible modes. Since this parameter depends on core diameter d and the numerical aperture N_a, the number of allowed modes would be different for fibers of different core diameters. The word "number" intuitively adds a concept of normalized frequency, expressed as

$$\nu_n = \frac{\pi d N_a}{\lambda} = \frac{\pi d}{\lambda}\sqrt{\mu_1^2 - \mu_2^2}$$

It shows that the normalized frequency is nothing but the factor carried by the parentheses of the parameter m_m. Thus, in terms of the normalized frequency ν_n, we have

$$m_m = \frac{\nu_n^2}{2}$$

Exercise 6 A graded optical fiber has a core diameter of 0.06 mm and numerical aperture of 0.23 at a wavelength 8600 Å. Calculate the (i) normalized frequency and (ii) number of modes guided in the core.

Solution Given, $d = 0.06$ mm, numerical aperture $N_a = 0.23$, and wavelength $\lambda = 8600$ Å. (i) The normalized frequency

$$\nu_n = \frac{\pi d N_a}{\lambda} = \frac{3.14 \times 0.06 \times 10^{-3} \times 0.23}{8600 \times 10^{-10}} = 50.39$$

(ii) The number of modes guided in the core.

$$m_m = \frac{1}{2}\,\nu_n^2 = \frac{1}{2}\,(50.39)^2 = 1269.58$$

The number of modes guided in the core is 1269.

8 Fiber Losses

The following three losses may occur in an optical fiber:

(i) Absorption.
(ii) Geometric effects.
(iii) Rayleigh scattering.

8.1 Absorption

Even a pure glass absorbs light of specific wavelength. Strong electronic absorption occurs in the UV region and vibrational absorption occurs in the IR region of wavelengths from 7 µm to 12 µm. These losses are attributed due to inherent property of glass and is known as the intrinsic absorption. Further, this loss is insignificant.

Impurities are major extrinsic sources of losses in an optical fiber. Hydroxyl radical OH and transitions metals like Ni, Cr, Cu, Mn, etc. have electronic absorption near visible range of spectrum. These impurities should be kept away, as far as possible, from the fiber. Intrinsic as well as extrinsic losses are found minimum around 1.3 μm.

8.2 Geometric Effects

These losses may occur due to manufacturing defects like irregularities in fiber dimensions during the drawing process or during the coating, cabling, or insulation processes.

8.3 Rayleigh Scattering

As glass has disordered structure having a local microscopic variation in density which may also cause the variation in the refractive indices. Thus, the light propagating through these structures may suffer scattering losses due to Rayleigh scattering, which is inversely proportional to the fourth power of wavelength of radiation. It shows that the Rayleigh scattering sets a lower limit on wavelength that can be transmitted by a glass fiber at 0.8 μm below for which the scattering loss is appreciably high.

9 Dispersion

We know that a pulse launched with a fiber gets attenuated due to the losses in a fiber. Further, the incoming pulse also spreads during the transit through the fiber. Therefore, a pulse at the output is wider as compared to that at the input. Hence, the pulse gets distorted as it moves through the fiber. This distortion of pulse is due to the dispersion effects which is measured in terms of the nanoseconds per km.

There are three phenomena that may contribute towards the distortion effects as follows:

(i) Material dispersion.
(ii) Waveguide dispersion.
(iii) Intermodal dispersion.

9.1 Material Dispersion

Material dispersion occurs as the refractive index of material varies with the wavelength of radiation. Since the group velocity V_g of a mode is a function of refractive index, various spectral components of a given mode travel at different speeds, depending on the wavelength. Material dispersion is therefore an intramodal dispersion effect. For the refractive index $\mu(\lambda)$, the material dispersion is expressed as

$$D_{mat} = \lambda \frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2} \tag{8}$$

9.2 Waveguide Dispersion

The effect of waveguide dispersion on the pulse spreading can be approximated by assuming that the refractive index of the material is independent of wavelength.

Waveguide dispersion arises due to the guiding properties of fiber. Amount of waveguide dispersion may also be expressed in a manner similar to Eq. (8) where the material refractive index is replaced by the effective refractive index. The effective refractive index for any mode of propagation varies with the wavelength which may cause pulse spreading in the same manner as the variation of refractive index in case of material dispersion.

9.3 Intermodal Dispersion

A ray of light passing through a fiber follows a zig-zag path and when a number of modes are moving through a fiber, they move with different net velocities with respect to the fiber axis. It shows that some modes arrive at the output earlier than the others. That is, there is spread of the input pulse. This process is known as the intermodal dispersion. It may be noted that this kind of dispersion does not depend on the spectral width of the source. That is, a light pulse from a pure monochromatic source is still giving intermodal dispersion.

In case of MMF, all the three spreading mechanisms are observed simultaneously whereas in case of SMF, only material and waveguide dispersion are observable.

In the case of fibers with low numerical aperture, smaller dispersion is observed while for a fiber with high numerical aperture, large dispersion is observed. Dispersion may be restricted by the use of a low numerical aperture and a narrow spread width source.

10 Transmission Windows for Optical Fibers

The wavelength range in which the fiber communication takes place is from 0.7 to 1.7 μm. As we stated earlier, the major sources responsible for the loss of light in a fiber during transmission are: (i) impurities in the material, (ii) scattering of light, and (iii) geometry of the transmission line. Taking into account all kinds of possible losses, there are some regions where the net loss is small. These regions are used for the transmission purpose and are known as the transmission windows. The ranges of these windows are as follows:

(i) from 0.80 to 0.90 μm region, called the first window;
(ii) from 1.26 to 1.34 μm region, called the second window;
(iii) from 1.50 to 1.63 μm region, called the third window.

Suppose, P_i is the input power to a fiber and P_0 is the output power at a distance z. The fiber attenuation (loss), denoted by α, is expressed as

$$\alpha = \frac{10}{z} \log\left(\frac{P_i}{P_0}\right)$$

Generally, a low-loss fiber may have average loss of 1 dB/km at 900 nm. In the present days, commonly used fibers have losses as low as 0.1 dB/km.

11 Multiple Choice Questions

1. Which of the followings is not part of the optical fiber?

   A. Core B. Copper wire C. Cladding D. Sheath

   Ans. B

2. By increasing which of the following, the information capacity of a wave can be increased.

   A. Wavelength B. Amplitude
   C. Frequency D. Intensity

   C

3. Which of the following waves can carry more information?

   A. Microwaves B. Radio waves
   C. Light waves D. Ultrasonic waves

   C

4. In an optical fiber, the light is guided through

   A. Malus' law B. Brewster's law
   C. Snell's law D. Total internal reflection

   D.

5. Which of the followings is wrong? Optical fiber can be

   A. Single mode B. Multi-mode
   C. Graded index D. None of them

   D

6. In an optical fiber, the acceptance angle is

   A. the maximum angle for light to be guided through the fiber
   B. the minimum angle for light to be guided through the fiber'
   C. the angle equal to the critical angle
   D. the angle greater than the critical angle

   A

7. Which of the followings is wrong? The fiber-optic communication consists of

   A. Optical transmitter B. Optical fiber
   C. Optical receiver D. None of these

   D

8. Rayleigh scattering causes the following losses in an optical fiber:

   A. Dispersion B. Radiative
   C. Absorptive D. All of these

   B

9. For the increase of bandwidth, which of the followings is correct?

   A. Information capacity decreases
   B. Information capacity increases
   C. Information capacity may decrease or increase
   D. None of these

   B

10. For an optical fiber having core refractive index 1.48 and cladding refractive index 1.45, the acceptance angle and numerical aperture, respectively, are

    A. 10° and 0.25 B. 12° and 0.20
    C. 17° and 0.30 D. 20° and 0.35

    C

11. The V-parameter of an optical fiber is a measure for

    A. Acceptance angle B. Total loss
    C. Total number of modes
    D. Possibility of being used for the communication

    C

12. An optical fiber is multi-moded when the V-parameter is

    A. < 2.4045 B. $<< 1$ C. < 2.3512 D. $>> 1$

    D

13. Which of the followings is not relevant to the fiber optics?

    A. Reflection B. Critical angle
    C. Refraction D. Interference

    Ans. D

14. For the total internal reflection,

    A. Angle of refraction is 90°.
    B. Incidence angle is 90°.
    C. Both the angle of refraction and incidence angle are 90°.
    D. Both the angle of refraction and incidence angle are 45°.

    Ans. A

15. Total internal reflection takes place when the

    A. light travels from a medium with low refractive index to that with high refractive index.
    B. light travels from a medium with high refractive index to that with low refractive index.
    C. incidence angle is less than the critical angle.
    D. incidence angle is 90°.

    Ans. B

16. Carrier waves used in the optical fiber communication are

    A. Ordinarily light B. Laser waves
    C. Radio-wave D. Microwaves

    Ans. B

17. Optical fiber communication is based on the phenomenon of

    A. Refraction B. Polarization
    C. Diffraction D. Total internal reflection

    Ans. D

18. In a single-mode fiber, the diameter of core is nearly equal to

    A. 10 μm B. 50 μm C. 100 μm D. 125 μm

    Ans. A

19. The inner most part of an optical fiber is known as the

    A. core B. cladding C. sheath D. optical fiber axis

    Ans. A

20. The refractive indices of the materials of core (μ_1) and cladding (μ_2) in an optical fiber satisfy the relation

    A. $\mu_1 > \mu_2$ B. $\mu_1 = \mu_2$ C. $\mu_1 < \mu_2$ D. None of them

    Ans. A

21. In a graded index optical fiber, the refractive index of the core

    A. increases towards the axis of the core.
    B. is non-uniform
    C. is the same at core-cladding interface
    D. decreases towards the axis of the core.

    Ans. A.

22. The acceptance angle in terms of the refractive indices of the materials of core (μ_1) and cladding (μ_2), when the end face of an optical fiber is exposed by the air, is

A. $\cos^{-1}(\mu_1^2 - \mu_2^2)$ B. $\sin^{-1}(\mu_1^2 - \mu_2^2)$
C. $\cos^{-1}\sqrt{(\mu_1^2 - \mu_2^2)}$ D. $\sin^{-1}\sqrt{(\mu_1^2 - \mu_2^2)}$

Ans. D

23. Suppose, d_1 is the diameter of a single-mode fiber and d_2 of a multi-mode fiber. We have

A. $d_1 = d_2$ B. $d_1 > d_2$ C. $d_1 < d_2$ D. Nothing is definite

Ans. C

24. Suppose, α_1 is the acceptance angle for a single-mode fiber and α_2 for a multi-mode fiber. We have

A. $\alpha_1 = \alpha_2$ B. $\alpha_1 > \alpha_2$ C. $\alpha_1 < \alpha_2$ D. Nothing is definite

Ans. C

25. In the step index fiber the

A. refractive index of core varies and refractive index of the cladding is constant.
B. refractive index of core is constant and refractive index of the cladding varies.
C. refractive indices of both the core and cladding are constant.
D. refractive indices of both the core and cladding vary.

Ans. C

26. The refractive indices of materials used for the core and cladding of a fiber cable are 1.5 and 1.35, respectively. The relative refractive index is

A. 0.1 B. 0.11 C. 0.15 D. 2.85

Ans. A

27. The refractive indices of materials used for the core and cladding of a fiber cable are 1.4 and 1.2, respectively. For acceptance angle θ, the value of $\sin\theta$ is

A. 0.2 B. 0.447 C. 0.52 D. 0.721

Ans. D

28. The refractive indices of materials used for core and cladding of a fiber cable are 1.4 and 1.2, respectively. The numerical aperture for the fiber cable is

A. 0.2 B. 0.447 C. 0.52 D. 0.721

Ans. D

12 Problems and Questions

1. Describe about various parts of an optical fiber.
2. Describe about an optical fiber and explain the terms: (a) Acceptance angle, (b) Acceptance cone, (c) Numerical aperture, (d) Relative refractive index, (e) Propagation modes, and (f) Normalized frequency
3. Write in brief about the advantages of fiber optics as compared to the traditional metal communication lines.
4. What are single-mode, multi-mode, and graded index fibers? Also, describe in detail the difference in the structures of single-mode step index and multi-mode graded index fibers.
5. Discuss the physical significance of numerical aperture. How does it depend on the refractive indices of the core and cladding?
6. Describe the allowed modes of an optical fiber. How they are related to the normalized frequency.
7. Discuss the propagation mechanisms of higher waves in an optical fiber.
8. Describe schematically the basic elements of an optical fiber communication system.
9. Explain why does a fraction of power of a signal is lost due to bending of fiber.
10. Describe about the transmission windows for optical fibers.
11. Write short notes on the following:

    (i) Fiber optics
    (ii) Numerical aperture and its physical significance
    (iii) Core and cladding
    (iv) Acceptance angle
    (v) Single-mode step index fiber optics
    (vi) Multi-mode fiber
    (vii) Skip distance
    (viii) Transmission windows for optical fibers.

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https://doi.org/10.1007/978-3-031-55676-0

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https://doi.org/10.1007/978-3-031-55676-0