Analytical Problems in Classical Mechanics

Complete Solutions

K. PRATHAPAN

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ANALYTICAL PROBLEMS IN
CLASSICAL MECHANICS

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Dedicated to

My wife Dhanya and sons Shivakrishna and Harikrishna

Preface

Classical mechanics is an integral part of curriculum for PG and UG courses in Physics in all universities in India and abroad. The branch of mechanics was developed by Isaac Newton and then different formulations are proposed by Lagrange, Hamilton, Jacobi and others. All these different formulations are usually discussed in Classical Mechanics.

Classical Mechanics is well known for its simplicity and various applications for solving many mechanical problems that we experience in our day-to-day life. The present book is written with a view that the students, both graduate and post graduate, must be able to understand the application of various formulations of classical mechanics and be equip themselves for solving mechanical problems after going through the book. The book is divided into seven chapters and theory notes are given at the beginning of each chapter.

This book contains problems of various difficulty levels. Most of the problems are discussed in detail for easy understanding the applications of various formulations. A limited number of problems are discussed in all formulations which will help the reader to make a comparison between different formulations. A large number of problems are selected from various university examinations as well as from competitive examinations like NET, JEST, GATE and Civil Services. Consequently, this book will be helpful to the students those who are preparing for such examinations. Also, a large number of practice problems are given at the end of each chapter. Suggestions for improving the content of this book will be appreciated.

For better understanding of this book, basic knowledge of mathematics and classical mechanics is required. Since many of the mechanical problems are to find the equation of motion, various types of integrations are to be performed, and a list of such integrals used in this is given in the Appendix.

I express my sincere gratitude to my teachers, especially to Dr. M K Satheesh Kumar and Prof. M M Madhusoodhanan who taught me Classical Mechanics during my post-graduation in Govt. Brennen College, Thalassery. I am also grateful to the teachers from other institutions who gave valuable suggestions and guidance received during his post-graduation and then in the career. Since joining the service, I am receiving cooperation, assistance and guidance from his colleagues, K Ratnakaran, K Muralidas, Lisha Damodaran, P Deneshan, T P Suresh and others; and their contributions are greatly appreciated.

I believe that this book is a result of blessings received from parents and the encouragement from wife and other family members. While preparing this book I consulted various books and other related publications, a list of which is included in the bibliography at the end of this book.

K. Prathapan

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CHAPTER

Foundations of Mechanics

CONCEPTS AND FORMULAE

1.1 NEWTON'S LAWS OF MOTION

(a) *First Law*: A body continues in its state of rest or uniform motion unless it is acted upon by an external force. This statement points out the inability of a body to change its state by itself and hence known as *law of inertia*.

(b) *Second Law*: The rate of change of linear momentum is directly proportional to the applied force and take place in the direction of the applied force. It is known as *law of causality*. The second law gives the measurement of the applied force.

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} = ma \tag{1.1}$$

(c) *Third Law*: Action and reaction are always equal and opposite and are directed along the same straight line. It is known as the *law of reciprocity*. Since action and reaction are acting on two different bodies they do not cancel each other. Newton's third law is valid only in inertial frames.

1.2 IMPULSE

(a) The first integral of a force with respect to time is its impulse.

$$\text{Impulse} = \int_1^2 F dt \tag{1.2}$$

(b) Impulsive force acts for a short time. The impulse is equal to the change in the momentum of the particle

1.3 CONSERVATION LAWS

(a) *Conservation of linear momentum*: In the absence of an external force the total linear momentum of a body is conserved. This statement is followed from the Newton's second law.

If $F = 0$, $\frac{dp}{dt} = 0$ and the momentum p is a constant.

For a system of particles, the linear momentum can be obtained as;

$$p = \sum_i m_i \frac{dr_i}{dt} \tag{1.3}$$

where m_i is the mass of i^{th} particle and r_i is its position vector.

(b) *Conservation of angular momentum*: In the absence of an external torque, the angular momentum of a particle is conserved. Torque is defined as the rate of change of angular momentum.

$\tau = \frac{dL}{dt}$, where L is the angular momentum.

If $\tau = 0$, $\frac{dL}{dt} = 0$ and the angular momentum L is a constant.

The angular momentum and the linear momentum of a particle is related through $L = r \times p$. For a system of particles, it is given by:

$$L = \sum_i r_i \times p_i \tag{1.4}$$

(c) *Conservation of Energy*: In a conservative force field, the sum of kinetic and potential energies of a system is conserved.

$T + V =$ Constant, where T is the kinetic energy and V, the potential energy.

(d) A force is said to be conservative if and only if it is time independent such that the work done by the force is independent of the path followed, that is, the work done by such a force depends only on the initial and final points. Then it follows that the work done by a conservative force around a closed path is zero.

$$\oint F \cdot dr = 0 \tag{1.5}$$

This is otherwise expressed as $\nabla \times F = 0$

(e) A conservative force can be expressed as the gradient of a scalar potential.

(f) If the potential depends explicitly on time and velocity, the total energy of the particle is not conserved.

1.4 WORK – ENERGY THEOREM

(a) Work done by an external force on a particle is equal to the change in kinetic energy of the particle.

$$\int_1^2 F \cdot dr = T_2 - T_1 \tag{1.6}$$

(b) Work done by a conservative force is equal to the change in the potential energy of the particle.

$$\int_1^2 F\cdot dr = V_1 - V_2 \tag{1.7}$$

(c) According to the law of conservation of energy, a decrease in potential energy appears as an increase in kinetic energy.

1.5 OSCILLATORY MOTION

(a) The equation of motion of a simple harmonic oscillation is

$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$; where frequency of oscillation, $\omega_0 = \sqrt{\frac{k}{m}}$, k is the force constant.

(b) When damping is present, the amplitude of oscillation decreases continuously and the force of damping is proportional to the velocity. The equation of motion is;

$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \omega_0^2 x = 0$, where γ is the damping coefficient.

(c) The frequency of oscillation of a damped harmonic oscillator is $\omega = \sqrt{\omega_0^2 - r^2}$

where $r = \frac{\gamma}{2m}$.

(d) The equation of motion of a forced damped harmonic oscillator is

$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \omega_0^2 x = f_o \sin \omega t$

1.6 VIRIAL THEOREM

(a) The time average of kinetic energy of a system of particles is equal to the *Virial of Clausius.*

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_i F_i \cdot r_i \right\rangle \tag{1.8}$$

RHS of the above equation is the Virial of Clausius.

(b) If the force can be derived from a potential, then the theorem becomes

$$\langle T \rangle = \frac{1}{2}\left\langle \sum_i \nabla_i V_i \cdot r_i \right\rangle \tag{1.9}$$

(c) For a single particle under a central force, the above equation becomes

$$\langle T \rangle = \frac{1}{2}\left\langle \frac{\partial V}{\partial r} r \right\rangle \tag{1.10}$$

1.7 CONSTRAINTS

(a) Constraints put geometrical restrictions upon the possible motion of a system. They result in the so-called forces of constraints.

(b) *Holonomic Constraints*: Constraints that can eliminate a degree of freedom are called holonomic constraints. This type of constraints can be expressed in the form of an equation that connects the position vectors of the particle, their derivatives and time.

$$G(r_1, r_2, \ldots\ldots, r_n, t) = 0 \tag{1.11}$$

Holonomic constraints are or can be made independent of velocity on integration.

(c) *Nonholonomic Constraints*: Constraints that cannot eliminate degrees of freedom are called nonholonomic constraints. Such constraints are expressed as inequalities

$$G(r_1, r_2, \ldots\ldots\ldots, r_n, t) \leq c \tag{1.12}$$

or are unintegrable. These relations are irreducible functions of velocities.

(d) *Rheonomic Constraints*: Constraint relations depend explicitly on time.

(e) *Scleronomic Constraints*: Constraint relations do not explicitly depend on time.

(f) *Conservative Constraints*: Force of constraint does not do any work and hence the total mechanical energy of the system remains conserved.

(g) *Dissipative Constraint*: Constraint forces work and hence the total mechanical energy is not conserved.

1.8 DEGREES OF FREEDOM

(a) Degree of freedom refers to the number of independent ways in which a system can move without violating the imposed constraints.

(b) In a system of N free particles, there are 3 N degrees of freedom. If k is the number of constraints imposed on the system, the degree of freedom is $3N - k$.

1.9 GENERALIZED COORDINATES

(a) A set of minimum number of independent coordinates required to specify the position and configuration of a system at a given time is known as its generalized coordinates. The generalized coordinates can be Cartesian coordinates, angles and a combination of them. They are usually denoted as $q_1, q_2, \ldots\ldots, q_j, \ldots\ldots q_n$.

(b) Introduction of generalized coordinates removes the difficulties in solving mechanical problems when constraints are present.

(c) *Generalized Force*: For a system with generalized coordinated $q_1, q_2, \ldots\ldots, q_j, \ldots\ldots q_n$, the generalized force is given by

$$Q_j = \sum_{i=1}^{N} F_i \cdot \frac{\partial r_i}{\partial q_j} \tag{1.13}$$

(d) For a conservative system, $Q_j = -\dfrac{\partial V}{\partial q_j}$ (1.14)

(e) *Generalized Momentum*: Generalized momentum is the product of mass and generalized velocity

$$p_j = m_j \dot{q}_j$$

For a conservative system, $p_j = \dfrac{\partial L}{\partial \dot{q}_j}$, where $L = T - V$ (1.15)

1.10 VIRTUAL DISPLACEMENT AND PRINCIPLE OF VIRTUAL WORK

(a) Infinitesimal change in the configuration of the particles of the system consistent with the forces and constraints acting on it is known as the virtual displacement (δr_i).

(b) The total virtual work done by all the applied forces acting on a system of particles in equilibrium is zero.

$$\sum_i F_i^e \cdot \delta r_i = 0 \tag{1.16}$$

(c) The virtual work done by a constraint force is always zero.

1.11 D'ALEMBERT'S PRINCIPLE

(a) During the course of motion of a system of particles, the work done by the effective forces is zero.

$$\sum_i (F_i^e - \dot{P}_i) \cdot \delta r_i = 0 \tag{1.17}$$

where, F_i^e is the external force and $\dot{P}_i$ is the reverse effective force.

(b) D'Alembert's principle does not contain the constraint forces and provides a complete solution to the holonomic and nonholonomic mechanical systems.

SOLVED PROBLEMS

EXAMPLE 1.1 Apply Newton's second law to show that the rate of falling under the action of gravity is same for all bodies.

Solution: In this problem, we have to show that the acceleration of the falling body is independent of its mass. The weight of a body of mass m is the force exerted by the earth on it.

i.e., $$W = F = -mg\hat{z} \tag{i}$$

where $\hat{z}$ is the unit vector pointing in the upward direction. Negative sign shows that the weight is acting in the downward direction.

According to Newton's second law

$$F = m\frac{d^2r}{dt^2} \tag{ii}$$

Therefore, the acceleration of the falling body is

$$\frac{d^2r}{dt^2} = -g\hat{z} \tag{iii}$$

Thus, the motion of a freely falling body is independent of the mass of the body. This was first demonstrated by Galileo through his experiment conducted from the top of the Leaning Tower of Pisa.

EXAMPLE 1.2 A particle is allowed to fall under gravity from height h. If the air damping on it is proportional to instantaneous velocity of the particle, find the velocity attained at time t.

Solution: Let v be the instantaneous velocity of the particle. Then the damping force is

$$F_d = -\gamma v \tag{i}$$

where, γ is the damping coefficient and negative sign shows that the damping force is opposite to the direction of motion.

Now, the equation of motion of the particle can be written as

$$m\frac{dv}{dt} = mg - \gamma v$$

or

$$\frac{dv}{g - \frac{\gamma}{m}v} = dt \tag{ii}$$

Take integral on both sides.

$$\int_0^v \frac{dv}{g - \frac{\gamma}{m}v} = \int_0^t dt$$

This would give $\log_e\left(\dfrac{g - \frac{\gamma}{m}v}{g}\right) = -\dfrac{\gamma}{m}t$ (iii)

Now take exponential on both sides, then we get

$$1 - \frac{\gamma}{mg}v = e^{-\gamma t/m}$$

or

$$v = \frac{mg}{\gamma}\left(1 - e^{-\gamma t/m}\right) \quad \text{(iv)}$$

which is the required expression.

EXAMPLE 1.3 A ball of mass m is thrown with velocity v_o on a horizontal surface where the retarding force is proportional to the square root of the instantaneous velocity. Calculate the velocity and the position of the ball as a function of time.

Solution: The retarding force is given as

$$F = m\frac{dv}{dt} = -kv^{1/2} \quad \text{(i)}$$

where, v is the velocity at a time t. This expression can be rearranged to get

$$\frac{dv}{v^{1/2}} = -\frac{k}{m}dt \quad \text{(ii)}$$

Taking integral on both sides, the above expression becomes

$$\int_{v_0}^{v} \frac{dv}{v^{1/2}} = -\int_{0}^{t} \frac{k}{m}dt \quad \text{(iii)}$$

After integration, applying the limit we get

$$2(\sqrt{v} - \sqrt{v_0}) = -\frac{k}{m}t$$

Then,

$$v = \left(\sqrt{v_0} - \frac{k}{2m}t\right)^2 \quad \text{(iv)}$$

On integration, the expression for velocity gives the displacement.

$$\int_{x_0}^{x} dx = \int_{0}^{t}\left(\sqrt{v_0} - \frac{k}{2m}t\right)^2 dt = \int_{0}^{t}\left(v_0 - \sqrt{v_0}\frac{k}{m}t + \frac{k^2t^2}{4m^2}\right)dt$$

On integration this yields

$$x - x_0 = v_0 t - \frac{1}{2}\sqrt{v_0}\frac{k}{m}t^2 + \frac{k^2}{12m^2}t^3 \quad \text{(v)}$$

Equations (iv) and (v) are the required expressions.

EXAMPLE 1.4 A body of mass m is thrown vertically upwards with an initial velocity v_0. Determine the maximum height attained by the body.

Solution: Let v be the velocity of the particle at a height x from the surface of the earth. Then the total energy of the body at a height x is

$$E = \frac{1}{2}mv^2 + mgx \tag{i}$$

The total energy can be determined from the initial conditions. Since the body is projected upwards with a velocity v_0 at $x = 0$, from the above equation, we get

$$E = \frac{1}{2}mv_0^2 \tag{ii}$$

Therefore,
$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgx$$

or
$$x = \frac{v_0^2 - v^2}{2g} \tag{iii}$$

At the maximum height, the velocity v becomes zero. Then the maximum height is

$$x_{\text{max}} = \frac{v_0^2}{2g} \tag{iv}$$

EXAMPLE 1.5 A particle of mass m is projected with a velocity v making an angle of 45° with the horizontal. Determine the magnitude of the angular momentum of the projectile about the point of projection when the particle is at its maximum height.

Solution: Let the maximum height attained by the particle be H. First let us find an expression for the maximum height in terms of the initial velocity.

Since the initial velocity v is at an angle 45° with the horizontal (say x-direction), at the maximum height, we have

$$v_x = v\cos 45 = \frac{v}{\sqrt{2}} \tag{i}$$

and
$$v_y = 0 = v\sin 45 - gT = \frac{v}{\sqrt{2}} - gT \tag{ii}$$

where T is the time taken by the particle to reach the maximum height

or
$$T = \frac{v}{\sqrt{2}g} \tag{iii}$$

Substituting these in the expression, $H = uT + \frac{1}{2}gT^2$, we get

$$H = \frac{v^2}{4g} \tag{iv}$$

The angular momentum of the particle about the point of projection is

$$L = mv_xH = m\frac{v}{\sqrt{2}}\frac{v^2}{4g} = m\frac{v^3}{4\sqrt{2}g} \tag{v}$$

Also from (iv), $v = 2\sqrt{gH}$, then

$$L = m\frac{8\left(\sqrt{gH}\right)^3}{4\sqrt{2}g} = m\left(2gH^3\right)^{\frac{1}{2}} \tag{vi}$$

EXAMPLE 1.6 Find the velocity and position as a function of time for a particle initially having velocity v_0 along the $+x$ axis and experiencing a linear retarding force $F_r(v) = -k\,v$.

Solution: The equation of motion of the particle can be written as

$$m\frac{d^2x}{dt^2} = -kv$$

or

$$\frac{d\dot{x}}{dt} = -\frac{k}{m}\dot{x}(t) \tag{i}$$

This equation can be solved by separating the variables and integrating.

$$\frac{d\dot{x}}{\dot{x}} = -\frac{k}{m}dt$$

Integrating on both sides, we get

$$\int_{\dot{x}_0}^{\dot{x}(t)} \frac{d\dot{x}}{\dot{x}} = -\int_0^t \frac{k}{m}dt$$

i.e.,

$$\log \dot{x}(t) - \log \dot{x}_0 = -\frac{k}{m}t$$

Taking exponential on both sides, we get the velocity as

$$\dot{x}(t) = \dot{x}_0 e^{-\frac{k}{m}t} \tag{ii}$$

It is clear that the velocity decreases exponentially.

Now, the position can be determined by integrating the expression for velocity.

We have

$$\frac{dx}{dt} = \dot{x}_0 e^{-\frac{k}{m}t}$$

$$x(t) = x_0 + \dot{x}_0 \int_0^t e^{-\frac{k}{m}t} dt$$

$$x(t) = x_0 + \frac{m\dot{x}_0}{k}\left(1 - e^{-\frac{k}{m}t}\right) \tag{iii}$$

This gives the position of the particle with time.

EXAMPLE 1.7 The potential energy of a vibrating diatomic molecule is given by $V(x) = V_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - V_0$.

where V_0, x_0, and δ are parameters chosen to describe the observed behaviour of a particular pair of atoms. The force that each atom exerts on the other is given by the derivative of this function with respect to x. Show that x_0 is the separation of the two atoms when the potential energy function is a minimum and that its value for that distance of separation is $V_{(x_0)} = -V_0$.

Solution: The potential energy of the diatomic molecule is minimum when its derivative with respect to x, the distance of separation, is zero. That is,

$$F(x) = -\frac{dV(x)}{dx} = 0 \tag{i}$$

Differentiating the given expression and equating to zero, we get

$$2\frac{V_0}{\delta}\left(1-e^{-(x-x_0)/\delta}\right)\left(e^{-(x-x_0)/\delta}\right) = 0$$

or

$$\left(1-e^{-(x-x_0)/\delta}\right) = 0$$

i.e.,

$$e^{-(x-x_0)/\delta} = 1 \tag{ii}$$

Taking logarithm on both sides, we get $\frac{-(x-x_0)}{\delta} = 0$ or $x = x_0$.

The minimum value of the potential energy can be obtained by setting $x = x_0$ in the given function $V(x)$. This gives $V(x_0) = -V_0$.

EXAMPLE 1.8 A particle of mass m is subjected to a force $F = a - 2bx$, where a and b are constants. Find the potential $V(x)$.

Solution: We know that force is the negative gradient of potential.

$$F = -\frac{dV}{dx} \tag{i}$$

Then the potential can be written as

$$V = -\int F dx$$

Substituting the expression for force in this equation and integrating, we get

$$V = -\int (a-2bx)dx = bx^2 - ax \tag{ii}$$

EXAMPLE 1.9 Two particles of mass m and $2m$ approach each other due to their mutual attractive force. Find velocity of their centre of mass.

Solution: This problem can be solved using the momentum conservation principle. Let v_1 and v_2 be the velocities of masses m and 2 m respectively. Since there is no external force, we can write

$$mv_1 = 2mv_2 \tag{i}$$

The position of the centre of mass can be expressed as

$$x_c = \frac{mx_1 - 2mx_2}{m+2m} \tag{ii}$$

Then, the velocity of the centre of mass is

$$v_c = \frac{mv_1 - 2mv_2}{m + 2m} = 0 \tag{iii}$$

EXAMPLE 1.10 Two particles of equal masses undergo an elastic collision between them. Show that if one of them is initially at rest, their final velocities are perpendicular to each other.

Solution: Let m be the mass of each particle. Let the second particle be at rest and u_1 be the initial velocity of the first particle. If v_1 and v_2 be their final velocities, according to the principle of momentum conservation

$$mu = mv_1 + mv_2$$

or

$$u = v_1 + v_2 \tag{i}$$

Now the energy conservation gives

$$\frac{1}{2}mu^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

or

$$u^2 = v_1^2 + v_2^2 \tag{ii}$$

Now, take the square of (i),

$$u^2 = (v_1 + v_2)^2 = v_1^2 + v_2^2 + 2v_1.v_2 \tag{iii}$$

From (ii) and (iii) one can see that

$$v_1 \cdot v_2 = 0$$

Therefore, the final velocities of the particles are perpendicular to each other.

EXAMPLE 1.11 In a one-dimensional elastic collision, the relative velocity of two particles after a collision is the negative of the relative velocity before the collision.

Solution: Let the masses of the colliding particles be m and M. Let v_i and V_i be the initial velocities, and v_f and V_f be the final velocities of the particles. The principles of conservation of momentum and energy give

$$mv_i + MV_i = mv_f + MV_f$$

$$\frac{1}{2}mv_i^2 + \frac{1}{2}MV_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}MV_f^2$$

Rearranging these expressions, we get

$$m(v_i - v_f) = M(V_f - V_i)$$

$$m(v_i^2 - v_f^2) = M(V_f^2 - V_i^2) \tag{i}$$

Dividing the second equation by the first equation gives

$$v_i + v_f = V_i + V_f$$

or

$$(v_i - V_i) = -(v_f - V_f) \tag{ii}$$

which is the required result.

EXAMPLE 1.12 A body of mass m splits into two masses m_1 and m_2 by an explosion. After the split, the masses move in the same direction with a total kinetic energy T. Determine the relative speed of the masses.

Solution: The body is assumed to be at rest initially so that the total initial momentum of the mass is zero. Therefore, by the principle of momentum conservation

$$m_1 v_1 + m_2 v_2 = 0 \qquad \text{or,} \quad \frac{v_2}{v_1} = -\frac{m_1}{m_2} \tag{i}$$

or

$$1 + \frac{v_2}{v_1} = 1 - \frac{m_1}{m_2} \qquad \text{or,} \quad v_1 + v_2 = \frac{(m_2 - m_1)}{m_2} v_1 \tag{ii}$$

Further, the total kinetic energy is

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Substituting for v_2 from (i) and simplifying, we get

$$v_1^2 = \frac{2T}{m_1} \frac{m_2}{(m_1 + m_2)}$$

Similarly, substitution for v_1 gives

$$v_2^2 = \frac{2T}{m_2} \frac{m_1}{(m_1 + m_2)}$$

Then,

$$v_1^2 - v_2^2 = 2T \frac{(m_2 - m_1)}{m_1 m_2} \tag{iii}$$

From (ii) and (iii), we can write the relative velocity as

$$v_1 - v_2 = \frac{2T}{m_1 v_1} = \frac{2T}{m_1} \sqrt{\frac{m_1}{2T} \frac{(m_1 + m_2)}{m_2}} = \sqrt{\frac{2Tm}{m_1 m_2}} \tag{iv}$$

EXAMPLE 1.13 A particle of mass m is subjected to two forces, a central force F_1 and a frictional force F_2 with $F_1 = f(r)$ and $F_2 = -\lambda v$, where λ is a positive quantity and v is the velocity of the particle. If the particle has an initial angular momentum L_0 about $r = 0$, find its angular momentum for all subsequent times.

Solution: Since the particle is acted upon by a central force, let us write the equations of motion in polar coordinates. Further, the equation of motion can be separated into a radial part and a tangential part. They are

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) - \lambda \dot{r} \tag{i}$$

and

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = -\lambda r\dot{\theta} \tag{ii}$$

Since we are interested in calculating the angular momentum, we rewrite the second equation as

$$\frac{1}{r} \frac{d(mr^2\dot{\theta})}{dt} = -\lambda r\dot{\theta} \tag{iii}$$

But, $mr^2\dot{\theta}$ is the angular momentum L. Therefore, the above equation becomes

$$\frac{dL}{dt} = -\lambda\frac{L}{m} \qquad \text{or} \quad \frac{dL}{L} = -\frac{\lambda}{m}dt$$

On integration this would yield

$$\log_e L = -\frac{\lambda}{m}t + C \tag{iv}$$

The integration constant can be evaluated using the initial condition and $C = \log_e L_0$. Therefore,

$$\log_e \frac{L}{L_0} = -\frac{\lambda}{m}t \qquad \text{or} \quad L = L_0 e^{-\frac{\lambda}{m}t} \tag{v}$$

This gives the variation of angular momentum with time.

EXAMPLE 1.14 The equation of motion of a point charge in a magnetic monopole of strength g is given by, $m\ddot{\vec{r}} = -ge\frac{\dot{\vec{r}}\times\vec{r}}{r^3}$. The monopole may be assumed as infinitely heavy and is placed at the origin. Show (a) that the kinetic energy $\frac{1}{2}m\dot{r}^2$ is a constant of motion; (b) and that $J = L + \frac{eg\vec{r}}{r}$ is also a constant of motion, where L is the angular momentum.

Solution:

(a) The kinetic energy of the charged particle can be written as $T = \frac{1}{2}m\dot{r}^2$

Now, take the time derivative of the kinetic energy

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = m\dot{r}.\ddot{r} \tag{i}$$

Use the given equation of motion in place of $m\ddot{r}$, then

$$\frac{dT}{dt} = \dot{r}.\left[-ge\frac{\dot{r}\times\vec{r}}{r^3}\right] = 0 \tag{ii}$$

i.e., kinetic energy T is a constant of motion.

(b) Given: $$J = L + \frac{eg\vec{r}}{r} \tag{iii}$$

or $$J = mr \times \dot{r} + \frac{eg\vec{r}}{r}$$

Now, take the time derivative, then

$$\dot{J} = \frac{d}{dt}\left(mr \times \dot{r} + \frac{eg\vec{r}}{r}\right)$$

$$= mr \times \ddot{r} + m\dot{r} \times \dot{r} + \frac{eg\dot{r}}{r} - \frac{eg\vec{r}}{r^2}\dot{r} \quad \left[\dot{r} = \frac{d}{dt}(\vec{r}.\vec{r})^{\frac{1}{2}} = \dot{r}.\frac{\vec{r}}{r}\right]$$

$$= mr \times \ddot{r} + m\dot{r} \times \dot{r} + \frac{eg\dot{r}}{r} - \frac{eg\vec{r}}{r^2}\left(\frac{\vec{r}.\dot{r}}{r}\right)$$

$$= mr \times \ddot{r} + \left[\frac{eg\dot{r}}{r} - \frac{eg\vec{r}(\vec{r}.\dot{r})}{r^3}\right]$$

$$= -ge\frac{\vec{r} \times (\dot{r} \times \vec{r})}{r^3} + \left\{\frac{eg\dot{r}}{r} - \frac{eg}{r^3}[\dot{r}(\vec{r}.\vec{r}) - \vec{r} \times (\dot{r} \times \vec{r})]\right\}$$

$$= -ge\frac{\vec{r} \times (\dot{r} \times \vec{r})}{r^3} + \left\{\frac{eg\dot{r}}{r} - \frac{eg\dot{r}}{r} + ge\frac{\vec{r} \times (\dot{r} \times \vec{r})}{r^3}\right\} = 0, \text{ hence proved.}$$

EXAMPLE 1.15 A smooth sphere rests on a horizontal plane. A point particle slides frictionlessly down the sphere starting from the top. Describe the path of the particle up to the time it strikes the plane.

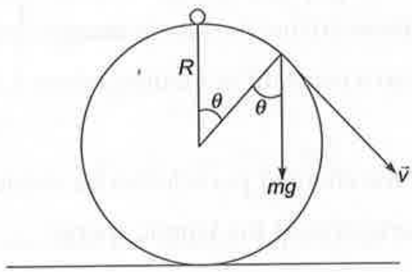

Fig. 1.1

Solution: Initially, the velocity of the particle is zero and its energy is potential energy. When it starts sliding it gains velocity and thereby its kinetic energy increases. From the figure using the principle of energy conservation we can write that gain in kinetic energy is equal to the decrease in potential energy.

$$\frac{1}{2}mv^2 = mgR(1-\cos\theta) \tag{i}$$

The radial force on the particle exerted by the sphere is

$$F = mg\cos\theta - \frac{mv^2}{R} \tag{ii}$$

The particle will leave the sphere when this force becomes zero. At this instant, we have

$$v^2 = Rg\cos\theta \tag{iii}$$

From the first equation, we get

$$v^2 = 2gR(1-\cos\theta)$$

This would yield $\cos\theta = \frac{2}{3}$ or, $\theta = 48.2°$ (iv)

Then, $$v = \sqrt{\frac{2gR}{3}} \tag{v}$$

Thus, the particle leaves the sphere with a velocity, $v = \sqrt{2gR/3}$ at an angle $\theta = 48.2°$. After leaving the sphere the particle follows a parabolic path until it reaches the horizontal plane.

EXAMPLE 1.16 A mechanical system consists of two particles P_1 and P_2 of masses m. Each is connected by a light rigid rod of length a. The particle P_1 is constrained to move along the horizontal direction and the second particle moves in the vertical plane. Find the particle velocities and kinetic energies for the given system.

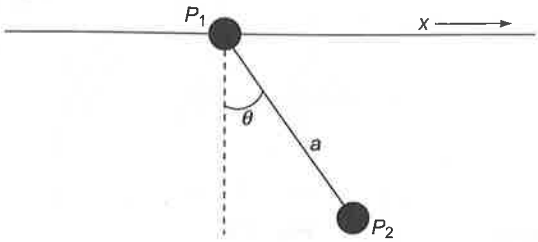

Fig. 1.2

Solution: Let the horizontal direction be the x-direction and the vertical direction be the z-direction. We identify x and θ as the generalized coordinates for the system under consideration.

For the particle P_1 the velocity can be written as

$$v_1 = \dot{x}\hat{i} \tag{i}$$

and the velocity of the second particle is

$$v_2 = \dot{x}\hat{i} + \left(a\cos\theta\hat{i} + a\sin\theta\hat{k}\right)\dot{\theta} \tag{ii}$$

The kinetic energy can be written as

$$T = \frac{1}{2}m(v_1.v_1) + \frac{1}{2}m(v_2.v_2)$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\dot{x}\hat{i} + \left(a\cos\theta\hat{i} + a\sin\theta\hat{k}\right)\dot{\theta}\right].\left[\dot{x}\hat{i} + \left(a\cos\theta\hat{i} + a\sin\theta\hat{k}\right)\dot{\theta}\right]$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + \left(a\dot{\theta}\right)^2 + 2a\dot{x}\dot{\theta}\cos\theta\right]$$

$$= m\dot{x}^2 + \frac{1}{2}ma^2\dot{\theta}^2 + (ma\cos\theta)\dot{x}\dot{\theta} \tag{iii}$$

This proves that the tension in the string decreases when the acceleration increases.

EXAMPLE 1.17 Two scale panes each of mass M are connected by a light string passing over a small pulley and in them are placed masses M_1 and M_2. Show that the reactions of the panes during the motion are $\dfrac{2M_1(M+M_2)}{M_1+M_2+2M}g$ and $\dfrac{2M_2(M+M_1)}{M_1+M_2+2M}g$.

Solution:

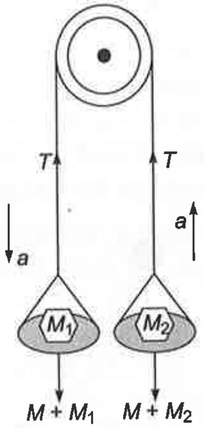

Fig. 1.3

Let us assume $M_1 > M_2$.

The equations of motion for the masses M_1 and M_2 can be written as

$$(M+M_1)g - T = (M+M_1)a \tag{i}$$

and

$$T - (M+M_2)g = (M+M_2)a \tag{ii}$$

Adding these two equations and rearranging, we get

$$a = \frac{(M_1 - M_2)}{(M_1 + M_2 + 2M)}g \tag{iii}$$

If R_1 is the reaction between M_1 and the scale pane, the force on the mass M_1 can be written as

$$M_1 a = M_1 g - R_1$$

or

$$R_1 = M_1(g - a)$$

Now, substitute for a from equation (iii), so that,

$$R_1 = M_1 g\left[1 - \frac{(M_1 - M_2)}{(M_1 + M_2 + 2M)}\right]$$

or $$R_1 = \frac{2M_1(M+M_2)}{M_1+M_2+2M} g \quad \text{(iv)}$$

Similarly, $$R_2 = \frac{2M_2(M+M_1)}{M_1+M_2+2M} g \quad \text{(v)}$$

EXAMPLE 1.18 A thin wheel of radius r is rotating about its axis. Determine the maximum number of rotations per second that the wheel can make without getting fractured.

Solution: Let n be the number of rotations per second. Then the angular velocity, $\omega = 2\pi n$ and the instantaneous speed is $v = r\omega = 2\pi rn$ (i)

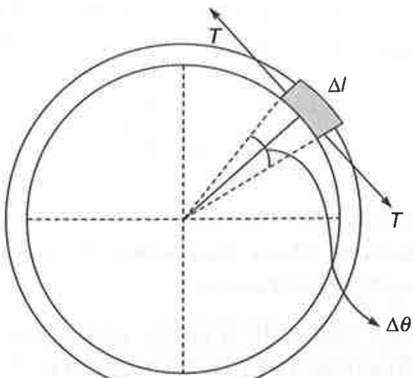

Fig. 1.4

Now, consider a small portion of length Δl of the wheel with mass Δm and let ρ be the density of the material of the wheel. Let T be the magnitude of the force acting at both ends of the segment. The segment subtends an angle $\Delta\theta$ at the centre of the wheel

$$\Delta\theta = \frac{\Delta l}{r}$$

Then from the geometry of the figure we get the component of each force T along the bisector of $\Delta\theta$ as, $T\sin\left(\frac{\Delta\theta}{2}\right)$. Then

$$2T\sin\left(\frac{\Delta\theta}{2}\right) = m\frac{v^2}{r} \quad \text{(ii)}$$

When $\Delta\theta \to 0$, $\sin\left(\frac{\Delta\theta}{2}\right) \approx \frac{\Delta\theta}{2}$, then

$$T\Delta\theta = m\frac{v^2}{r}$$

or
$$T\frac{\Delta l}{r} = m\frac{v^2}{r}$$

Then,
$$T\Delta l = mv^2 \tag{iii}$$

If A is the area of cross section of the segment, then

$$m = \rho A \Delta l$$

Therefore,
$$T = \rho A v^2 = 4\pi^2 r^2 n^2 \rho A \tag{iv}$$

This expression shows that as the number of rotations increases, the force also increases. Then, if S is the maximum tensile stress that the wheel can withstand, and correspond to a rotation n_{max}, we can write

$$n_{max} = \frac{1}{2\pi r}\left(\frac{S}{\rho}\right)^{\frac{1}{2}} \tag{v}$$

EXAMPLE 1.19 Find the acceleration of the masses and the tension of string of an Atwood machine. Show that when its acceleration increases the tension in the string decreases.

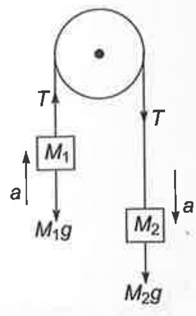

Fig. 1.5

Solution: A system of masses tied with a string and going over a pulley is called an Atwood machine. Let T be the tension in the string and a, the acceleration of the masses.

The equation of motion of the mass M_1 is

$$M_1 a = T - M_1 g \tag{i}$$

The equation of motion of the mass M_2 is

$$M_2 a = M_2 g - T \tag{ii}$$

Adding these two equations and rearranging, we get

$$a = \frac{(M_2 - M_1)}{(M_2 + M_1)} g \tag{iii}$$

Substituting the value of acceleration in either of the first two equations, we can write

$$T = \frac{2M_1 M_2}{M_1 + M_2} g \tag{iv}$$

Now we can write $a^2 = \dfrac{(M_2 - M_1)^2}{(M_2 + M_1)^2} g^2 = \dfrac{\left[(M_1 + M_2)^2 - 4M_1 M_2\right]}{(M_1 + M_2)^2} g^2$

Then, $$a^2 = \left(g^2 - \frac{2gT}{(M_1 + M_2)}\right) \tag{v}$$

EXAMPLE 1.20 A bead of mass m slides without friction on a wire sloped at an angle θ with respect to the horizontal. The wire rotates around a vertical axis at a constant angular velocity ω. Find the position of the bead with $r \neq 0$ such that it does not slide up or down the wire.

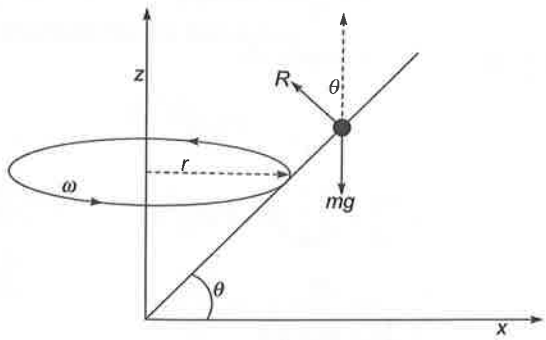

Fig. 1.6

Solution: Let the wire rotate around the z-axis with the angular velocity ω. The various forces acting on the bead are shown in the figure. It can easily be seen that the bead will not slide up or down if the centrifugal force of rotation is balanced by the x-component of the normal reaction.

Let us write the components of force acting on the bead that balances each other.

Along the horizontal the component of normal reaction is balanced by the centrifugal force

$$-R\sin\theta = -\frac{mv^2}{r} = -mr\omega^2 \tag{i}$$

Along the vertical the component of normal reaction is balanced by the weight of the bead

$$R\cos\theta = mg \tag{ii}$$

so that $$R = \frac{mg}{\cos\theta}$$

Then, equation (i) becomes

$$mg\tan\theta = mr\omega^2$$

or $$r = \frac{g\tan\theta}{\omega^2} \tag{iii}$$

Thus, as the bead reaches horizontal distance $r = \dfrac{g\tan\theta}{\omega^2}$ from the vertical axis, it will revolve around the vertical along with the wire without sliding up or down.

EXAMPLE 1.21 A particle has total energy E and the force on it is due to potential field $V(x)$. Show that the time taken by the particle to go from x_1 to x_2 is

$$t_2 - t_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} [E - V(x)]^{1/2} dx.$$

Solution: The total energy E of the moving particle can be written as

$$E = \frac{1}{2}mv^2 + V(x) \tag{i}$$

From this expression, we get the velocity as

$$v = \sqrt{\frac{2}{m}}[E - V(x)]^{1/2} \tag{ii}$$

or

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}}[E - V(x)]^{1/2}$$

or

$$\frac{dx}{[E - V(x)]^{1/2}} = \sqrt{\frac{2}{m}}dt \tag{iii}$$

Taking integral on both sides

$$\int_{x_1}^{x_2} \frac{dx}{[E - V(x)]^{1/2}} = \int_{t_1}^{t_2} \sqrt{\frac{2}{m}}dt$$

On integration RHS yields $\sqrt{\frac{2}{m}}(t_2 - t_1)$

Therefore, $t_2 - t_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} [E - V(x)]^{1/2} dx$, hence proved.

EXAMPLE 1.22 Solve the Kepler problem to obtain the equation of the orbit of a planet.

Solution: Kepler problem is the motion of a planet under the action of a central force, the gravitational force. Actually this is a two-body problem, but can be treated as one-body problem; the interaction of a reduced mass μ with an external central gravitational force.

Since the force is conservative, the potential energy can be written as

$$U(r) = -\frac{Gm_1m_2}{r} = -\frac{k}{r} \tag{i}$$

The total energy is

$$E = \frac{1}{2}\mu v^2 - \frac{k}{r} \tag{ii}$$

Now, the body is under the action of a central force and therefore the velocity has a radial and a tangential component. Then,

$$E=\frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2\right]-\frac{k}{r} \tag{iii}$$

The angular momentum of the body is $L=\mu r^2\frac{d\theta}{dt}$ or, $\frac{d\theta}{dt}=\frac{L}{\mu r^2}$. (iv)

Now the above equation can be modified as

$$E=\frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2+\left(\frac{L}{\mu r}\right)^2\right]-\frac{k}{r}$$

or

$$E=\frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2+\frac{L^2}{2\mu r^2}-\frac{k}{r} \tag{v}$$

Then,

$$\frac{dr}{dt}=\sqrt{\frac{2}{\mu}}\left(E-\frac{L^2}{2\mu r^2}+\frac{k}{r}\right)^{\frac{1}{2}} \tag{vi}$$

Solving this equation one can determine the position of the reduced mass as a function of time. However, in the case of planetary motion we usually describe the orbital by writing r as a function of θ. Dividing (iv) by (vi), we get

$$\frac{d\theta}{dr}=\frac{\frac{L}{\mu r^2}}{\sqrt{\frac{2}{\mu}}\left(E-\frac{L^2}{2\mu r^2}+\frac{k}{r}\right)^{\frac{1}{2}}}=\frac{\frac{L}{r^2}}{\sqrt{2\mu}\left(E-\frac{L^2}{2\mu r^2}+\frac{k}{r}\right)^{\frac{1}{2}}}$$

Then,

$$d\theta=\frac{\frac{L}{r^2}dr}{\sqrt{\left(2\mu E-\frac{L^2}{r^2}+\frac{2\mu k}{r}\right)}} \tag{vii}$$

Now, take $u=\frac{1}{r}$, so that $du=-\frac{dr}{r^2}$.

With this (vii) becomes

$$d\theta=-\frac{Lu}{\sqrt{\left(2\mu E-L^2u^2+2\mu ku\right)}} \tag{viii}$$

This can be integrated to obtain (neglecting the integration constant)

$$\theta = \cos^{-1}\left[\frac{\frac{L^2u}{\mu k}-1}{\sqrt{1+\frac{2EL^2}{\mu k^2}}}\right] \tag{ix}$$

Therefore,

$$\frac{L^2u}{\mu k} = 1+\sqrt{1+\frac{2EL^2}{\mu k^2}}\cos\theta$$

or

$$\frac{L^2}{\mu kr} = 1+\sqrt{1+\frac{2EL^2}{\mu k^2}}\cos\theta$$

that is,

$$\frac{1}{r} = \frac{\mu k}{L^2}\left(1+\sqrt{1+\frac{2EL^2}{\mu k^2}}\cos\theta\right) \tag{x}$$

This equation gives the position as function of θ. Equation given above can be rewritten as

$$\frac{1}{r} = \frac{1}{r_0}(1+\varepsilon\cos\theta) \tag{xi}$$

with $r_0 = \frac{L^2}{\mu k}$, the semilatus rectum and $\varepsilon = \sqrt{1+\frac{2EL^2}{\mu k^2}}$, the eccentricity of the orbit. When $\varepsilon = 0$ the orbit is circular with $r = r_0$, when $0<\varepsilon<1$, the orbit is ellipse and when $\varepsilon = 1$ the orbit is a hyperbola where bounded motion is not possible.

The semi-major axis of the ellipse is given by

$$a = \frac{r_0}{1-\varepsilon^2} = \frac{L^2}{\mu k}\frac{\mu k^2}{2L^2|E|} = \frac{k}{2|E|} \tag{xii}$$

EXAMPLE 1.23 A particle with unit mass is moving under the influence of a central force. Use the principle of energy conservation to obtain an expression for the path of motion.

Solution: This problem is similar to the Kepler problem discussed above. However, we adopt a slightly different procedure to arrive at the result. Since the motion is under a central force, it will be convenient to use the polar coordinates. The total energy of the particle is

$$E = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \tag{i}$$

Now the angular momentum $r^2\dot{\theta}$ is conserved, then; $r^2\dot{\theta} = L$, is a constant. This can be used to eliminate $\dot{\theta}$ from the expression for total energy to get

$$\frac{1}{2}\left(\dot{r}^2+\frac{L^2}{r^2}\right)-U(r)=E \tag{ii}$$

The total energy is also conserved. Now in the case of a central force the potential can be written as $U(r)=\frac{1}{r}$ and therefore

$$\frac{1}{2}\left(\dot{r}^2+\frac{L^2}{r^2}\right)-\frac{1}{r}=E$$

or
$$\dot{r}^2+\frac{L^2}{r^2}-\frac{2}{r}=2E \tag{iii}$$

To find the path of motion we will integrate the above expression (iii) with respect to θ with the aid of transformation $\frac{d}{dt}=\dot{\theta}\frac{d}{d\theta}$.

so that
$$\dot{r}=\dot{\theta}\frac{dr}{d\theta}=\frac{L}{r^2}\frac{dr}{d\theta}=-L\frac{d}{d\theta}\left(\frac{1}{r}\right)$$

Also we make a substitution; $\frac{1}{r}=u$, so that $\dot{r}=-L\frac{du}{d\theta}$ (iv)

With these the substitution equation (iii) becomes

$$L^2\left(\frac{du}{d\theta}\right)^2+L^2u^2-2u=2E$$

This can be rearranged to get

$$\left(\frac{du}{d\theta}\right)^2=\frac{2E}{L^2}+\frac{2u}{L^2}-u^2$$

$$=\frac{2EL^2+1}{L^4}-\left(\frac{1}{L^4}-\frac{2u}{L^2}+u^2\right)$$

or
$$\left(\frac{du}{d\theta}\right)^2=\frac{2EL^2+1}{L^4}-\left(u-\frac{1}{L^2}\right)^2 \tag{v}$$

Now we make another substitution

$$u-\frac{1}{L^2}=v$$

so that
$$\frac{du}{d\theta}=\frac{dv}{d\theta}$$

Then,
$$\left(\frac{dv}{d\theta}\right)^2=\left(\frac{\varepsilon}{L^2}\right)^2-v^2$$

or
$$\frac{\left(\frac{dv}{d\theta}\right)^2}{\left(\frac{\varepsilon}{L^2}\right)^2}+\frac{v^2}{\left(\frac{\varepsilon}{L^2}\right)^2}=1 \tag{vi}$$

with $\varepsilon = 2EL^2+1>0$.

The solution to equation (vi) can be written as
$$v=\frac{\varepsilon}{L^2}\cos(\theta-\theta_0) \tag{vii}$$

Now,
$$r=\frac{1}{u}=\frac{L^2}{L^2v+1}=\frac{L^2}{1+\varepsilon\cos(\theta-\theta_0)} \tag{viii}$$

This equation represents a conic section and the shape of the conic section is obvious from equation (vi).

EXAMPLE 1.24 For a planet revolving around the sun, show that the aerial velocity is a constant.

Solution: This is Kepler's second law of motion which states that the radius vector drawn from the sun to the planet sweeps out equal areas in equal interval of time.

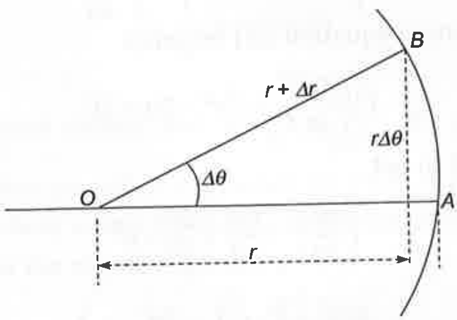

Fig. 1.7

From the geometry of the figure, the area of the region ABC is
$$\Delta A=\frac{1}{2}(r\Delta\theta)r+\frac{1}{2}(r\Delta\theta)\Delta r \tag{i}$$

Then the rate of change of area can be written as
$$\frac{\Delta A}{\Delta t}=\frac{1}{2}\frac{(r\Delta\theta)r}{\Delta t}+\frac{1}{2}\frac{(r\Delta\theta)\Delta r}{\Delta t} \tag{ii}$$

Now, in the limit $\Delta t\to 0;\Delta\theta\to 0$, $\Delta r\to 0$ and therefore the second term becomes negligible. Then,
$$\frac{dA}{dt}=\frac{1}{2}r^2\frac{d\theta}{dt} \tag{iii}$$

The angular momentum of the planet is given by

$$L = \mu r^2 \omega = \mu r^2 \frac{d\theta}{dt}$$

and, therefore,
$$\frac{dA}{dt} = \frac{L}{2\mu} \tag{iv}$$

Since the angular momentum of a planet is conserved, the RHS of (iv) is a constant which shows that the radial velocity $\frac{dA}{dt}$ is a constant.

EXAMPLE 1.25 A particle of mass m moves in a central force field by $\vec{F} = -\frac{k}{r^4}\vec{r}$. If E is the total energy supplied to the particle, show that its speed is given by $v = \sqrt{\left(\frac{k}{mr^2} + \frac{2E}{m}\right)}$.

Solution: In this problem we can make use of the expression for total energy; which is the sum of kinetic and potential energies, for arriving at the result.

From the given expression of force we can obtain the potential energy as

$$V(r) = -\int F dr = -\int \frac{k}{r^3} dr = -\frac{k}{2r^2} \tag{i}$$

where we used,
$$F = |\vec{F}| = \frac{k}{r^3}$$

Now, the total energy $E = T + V(r) = \frac{1}{2}mv^2 - \frac{k}{2r^2}$ (ii)

Therefore,
$$v = \sqrt{\left(\frac{k}{mr^2} + \frac{2E}{m}\right)} \tag{iii}$$

Hence, proved.

EXAMPLE 1.26 Show that the law of conservation of energy for a particle of mass m moves in a central force field can be expressed as $\frac{1}{2}m\frac{d}{dt}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{d}{dt}\int F(r)dr = E$.

Solution: Let the particle move in the xy-plane. The position of the particle at any time t can be represented by the polar coordinated (r, θ). In polar coordinates Newton's second law can be written as

$$m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}\right] = F(r)\hat{r} \tag{i}$$

Equating the radial and angular part on both sides, we get

$$m(\ddot{r} - r\dot{\theta}^2) = F(r) \tag{ii}$$

and
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{iii}$$

Now multiplying (ii) by $\dot{r}$ and (iii) by $r\dot{\theta}$ and adding the resulting equations, we get

$$m(\dot{r}\ddot{r} + r^2\dot{\theta}\ddot{\theta} + r\dot{r}\dot{\theta}^2) = F(r)\dot{r}$$

or
$$\frac{1}{2}m\frac{d}{dt}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \frac{d}{dt}\int F(r)dr$$

$$\frac{1}{2}m\frac{d}{dt}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{d}{dt}\int F(r)dr = \frac{1}{2}mv^2 + V = E\text{, the total energy.}$$

EXAMPLE 1.27 Estimate the Bohr radius of the first orbit of hydrogen atom using the idea of motion of a particle under a central force field.

Solution: Hydrogen is a two-particle system where an electron is revolving around a nucleus containing a single proton. This problem is mathematically equivalent to a system of reduced mass μ orbiting around a central point under the influence of a radially attractive force given by

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2} \tag{i}$$

where q_1 and q_2 are the charges of the particles, proton and electron and r is the separation between the particles. The reduced mass is given by

$$\mu = \frac{m_p m_e}{m_p + m_e} \tag{ii}$$

Then the total energy of the system can be written as

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \tag{iii}$$

This is the energy of the reduced body moving in two dimensions and can be reinterpreted as the energy of a reduced body moving in one dimension, the radial direction, in an effective potential energy given by two terms

$$U_{eff} = \frac{L^2}{2\mu r^2} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \tag{iv}$$

The effective kinetic energy associated with the one-dimensional motion is

$$T_{eff} = \frac{1}{2}\mu\dot{r}^2 \tag{v}$$

According to Bohr's theory, only discrete energy states or orbits are allowed. Assume the electrons are revolving in circular orbits, then the situation is described by a minimum of total energy which corresponds to the minimum of the effective potential energy. That is when

$$\frac{dU_{eff}}{dr} = 0 \text{ or, } -\frac{L^2}{\mu r^3} + \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = 0$$

This would give
$$r = \frac{4\pi\varepsilon_0 L^2}{\mu e^2} \tag{vi}$$

According to Bohr theory, the angular momentum L is quantized as
$$L = n\frac{h}{2\pi}$$

Then,
$$r_n = \frac{\varepsilon_0 n^2 h^2}{\pi e^2 \mu} \tag{vii}$$

Then the first Bohr radius of hydrogen atom can be estimated as
$$r_1 = \frac{8.85\times10^{-12}C^2N^{-1}m^{-2}\left(6.626\times10^{-34}kg.m^2.s^{-1}\right)^2}{3.14\times\left(1.6\times10^{-19}C\right)^2\times\left(9.1\times10^{-31}kg\right)} = 5.31\times10^{-11}m.$$

EXAMPLE 1.28 A particle moves under the influence of the potential $V(x) = -Cx^n e^{-ax}$. Find the frequency of small oscillations around the equilibrium point.

Solution: The frequency of an oscillator is given by the general expression, $\omega = \sqrt{\frac{k}{m}}$, where k is the force constant. Let x_0 be the equilibrium point. This point corresponds to the minimum value of the potential.

Differentiation of the expression for the potential w.r.t. x yields
$$V'(x) = -Ce^{-ax}x^{(n-1)}(n-ax) \tag{i}$$

At the equilibrium point x_0, $V'(x_0) = 0$

Therefore,
$$x_0 = \frac{n}{a} \tag{ii}$$

Now, the force constant k is equal to the second derivative of the potential evaluated at the equilibrium point. The second derivative of the potential is
$$V''(x) = -Ce^{-ax}x^{(n-2)}\left[(n-1-ax)(n-ax)-ax\right] \tag{iii}$$

Substituting the value of x_0 in this expression, we get
$$V''(x_0) = \frac{Ce^{-n}n^{(n-1)}}{a^{(n-2)}} \tag{iv}$$

Then,
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{V''(x_0)}{m}} = \sqrt{\frac{Ce^{-n}n^{(n-1)}}{ma^{(n-2)}}} \tag{v}$$

EXAMPLE 1.29 Obtain the equation of motion of a simple pendulum from the principle of conservation of energy.

Solution: Let l be the length of the pendulum and m be the mass of the bob. The tension on the string is T. Now the tension is perpendicular to the direction of velocity v of the bob. Therefore, the work done by the tension is zero.

$$W_T = \int T.vdt = 0 \tag{i}$$

Therefore, the only force doing the work is the weight mg of the bob. Since this is a conservative force, the total energy must also be conserved.

$$E = T + V = \frac{1}{2}mv^2 - mgl\cos\theta = \text{const} \tag{ii}$$

$$E = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = \text{const} \tag{iii}$$

On differentiation, we get

$$2\frac{1}{2}ml^2\ddot{\theta}\dot{\theta} + mgl\sin\theta\dot{\theta} = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{iv}$$

which is the required equation.

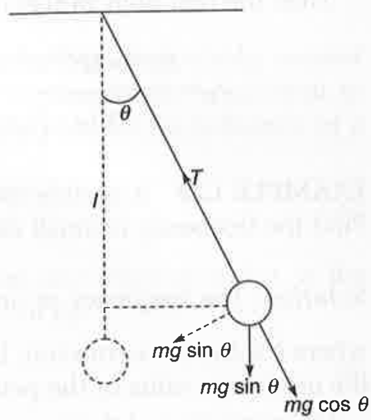

Fig. 1.8

EXAMPLE 1.30 The electrons in the ionosphere are acted upon by an oscillating electric field $E = E_0 \sin\omega t$. Obtain the equation of motion of the electrons.

Solution: The acceleration of an electron in an electric field is $a = -\frac{eE}{m}$, where e is the charge on the electron and m, its mass. Since the applied field varies sinusoidally, the acceleration is not uniform.

$$a(t) = -\frac{eE_0}{m}\sin\omega t = a_0 \sin\omega t \tag{i}$$

On integration, we get the velocity as

$$v(t) = \int_0^t a(t')dt' + v_0$$

$$= \int_0^t a_0 \sin\omega t' dt' + v_0 = v_0 - \frac{a_0}{\omega}(\cos\omega t - 1) \tag{ii}$$

Here, v_0 is the initial velocity and it appears as the integration constant.

On further integration (ii) gives

$$x(t) = \int_0^t v(t')dt' + x_0$$

$$= \int_0^t \left[v_0 - \frac{a_0}{\omega}(\cos\omega t' - 1) \right] dt' + x_0$$

$$= x_0 + \left(v_0 + \frac{a_0}{\omega} \right) t - \frac{a_0}{\omega^2} \sin\omega t \qquad \text{(iii)}$$

If we put $x_0 = v_0 = 0$, then

$$x(t) = \frac{a_0}{\omega} t - \frac{a_0}{\omega^2} \sin\omega t \qquad \text{(iv)}$$

The first term represents the motion of the electron with uniform velocity and the second term its oscillation at the frequency of the applied field.

EXAMPLE 1.31 For a forced damped harmonic oscillator $r = \frac{\omega_0}{4}$ and the driving force is $f = f_0 \cos\omega t$. Find a general solution.

Solution: The equation of motion of a driven harmonic oscillator with damping is given by the second order differential equation

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \omega_0^2 x = \frac{f_o}{m}\cos\omega t$$

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = \frac{f_o}{m}\cos\omega t, \qquad \text{(i)}$$

with $r = \frac{\gamma}{2m} = \frac{\omega_0}{4}$

Now, the frequency of oscillation of the oscillator is

$$\omega = \sqrt{(\omega_0^2 - r^2)} = \sqrt{\left(\omega_0^2 - \frac{\omega_0^2}{16} \right)} = 0.97\omega_0$$

The transient solution (complementary function) of the equation of motion,

$$x_c(t) = e^{-rt}\left(A_1 \cos\omega t + A_2 \sin\omega t \right)$$

in the present case becomes

$$x_c(t) = e^{-\frac{\omega_0}{4}t}\left(A_1 \cos 0.97t + A_2 \sin 0.97t \right) \qquad \text{(ii)}$$

Let the particular solution for the applied force be

$$x_i(t) = B_1 \cos\omega t + B_2 \sin\omega t$$

so that $\dot{x}_i(t) = -\omega B_1 \sin\omega t + \omega B_2 \cos\omega t$

and $\ddot{x}_i(t) = -\omega^2 B_1 \cos\omega t - \omega^2 B_2 \sin\omega t$

These three expressions can be substituted in the equation of motion to get;

$$\left(-\omega^2 B_1 - 2r\omega B_2 + \omega_0^2 B_1 \right)\cos\omega t + \left(-\omega^2 B_2 - 2r\omega B_1 + \omega_0^2 B_2 \right)\sin\omega t = \frac{f_0}{m}\cos\omega t$$

Equating the coefficients, we get

$$\left(\omega_0^2 - \omega^2\right)B_1 + 2r\omega B_2 = \frac{f_0}{m}$$

and $$-2r\omega B_1 + \left(\omega_0^2 - \omega^2\right)B_2 = 0$$

These equations can be solved to obtain the values of B_1 and B_2 in terms of f_o. The results are

$$B_1 = \frac{f_0\left(\omega_0^2 - \omega^2\right)}{m\left[\left(\omega_0^2 - \omega^2\right)^2 + 4r^2\omega^2\right]} \tag{iii}$$

and $$B_2 = \frac{2r\omega f_0}{m\left[\left(\omega_0^2 - \omega^2\right)^2 + 4r^2\omega^2\right]} \tag{iv}$$

Then the general solution is given by

$$x(t) = x_c(t) + x_i(t)$$

$$x(t) = e^{-\frac{\omega_0}{4}t}\left(A_1 \cos 0.97t + A_2 \sin 0.97t\right) + B_1 \cos \omega t + B_2 \sin \omega t$$

The coefficients B_1 and B_2 are given by the above expressions (iii) and (iv) and the coefficients A_1 and A_2 can be determined from the initial conditions.

EXAMPLE 1.32 The potential energy of a nonlinear oscillator is given by $V(x) = \frac{1}{2}kx^2 - \frac{1}{3}m\lambda x^3$, with λ small. Find the solution of the equation of motion to first order in λ (at $x = 0$, $t = 0$).

Solution: The equation of motion of the nonlinear oscillator is

$$m\frac{d^2x}{dt^2} = -\frac{dU(x)}{dx} = -kx + m\lambda x^2 \tag{i}$$

Neglecting $m\lambda x^2$, we get the zero order equation $m\frac{d^2x}{dt^2} + kx = 0$, whose solution is $x_{(0)} = A\sin(\omega t + \phi)$, with $\omega = \sqrt{\frac{k}{m}}$.

Since $x = 0$ at $t = 0$, the solution becomes; $x_{(0)} = A\sin\omega t$ (ii)

Now, let the solution to the first order equation have the form $x_{(1)} = x_{(0)} + \lambda x$

This can be substituted in the equation of motion (i), to get

$$m\left(\frac{d^2x_{(1)}}{dt^2}\right) = -kx_{(1)} + m\lambda x_{(1)}^2$$

or $$m\left(\frac{d^2x_{(0)}}{dt^2} + \lambda\frac{d^2x}{dt^2}\right) = -k(x_{(0)} + \lambda x) + m\lambda\left(x_{(0)} + \lambda x\right)^2$$

$$\left[m\frac{d^2x_{(0)}}{dt^2} + kx_{(0)}\right] + \lambda\left[m\frac{d^2x}{dt^2} + kx\right] = m\lambda\left(x_{(0)} + \lambda x\right)^2$$

$$\lambda\left[m\frac{d^2x}{dt^2} + kx\right] = m\lambda\left(x_{(0)}^2 + 2\lambda x_{(0)}x + \lambda^2x^2\right)$$

Neglecting the higher powers of λ, we get

$$\lambda\left[m\frac{d^2x}{dt^2} + kx\right] = m\lambda x_{(0)}^2$$

or

$$\frac{d^2x}{dt^2} + \omega^2x = x_{(0)}^2$$

$$\frac{d^2x}{dt^2} + \omega^2x = A^2\sin^2\omega t = \frac{A^2}{2}\{1 - \cos(2\omega t)\} \qquad \text{(iii)}$$

Now, for solving this equation, let us try a particular integral

$$x = B + C\cos(2\omega t) \qquad \text{(iv)}$$

Substituting this in (iii) and simplifying, we get

$$-3\omega^2C\cos(2\omega t) + \omega^2B = \frac{A^2}{2} - \frac{A^2}{2}\cos(2\omega t)$$

The coefficients can be compared to get

$$B = \frac{A^2}{2\omega^2} \text{ and } C = \frac{A^2}{6\omega^2}$$

Therefore, $$x = \frac{A^2}{2\omega^2} + \frac{A^2}{6\omega^2}\cos(2\omega t) \qquad \text{(v)}$$

The homogeneous equation is $\frac{d^2x}{dt^2} + \omega^2x = 0$, which has a solution;

$$x = D_1\sin\omega t + D_2\cos\omega t \qquad \text{(vi)}$$

Now, the complete solution $x_{(1)} = x_{(0)} + \lambda x$ becomes

$$x_{(1)} = (A\sin\omega t) + \lambda\left[\frac{A^2}{2\omega^2} + \frac{A^2}{6\omega^2}\cos(2\omega t) + D_1\sin\omega t + D_2\cos\omega t\right]$$

$$= (A + \lambda D_1)\sin\omega t + \left[\frac{A^2}{2\omega^2} + D_2\cos\omega t + \frac{A^2}{6\omega^2}\cos(2\omega t)\right]$$

Applying the initial condition, that $x = 0$ at $t = 0$, in this equation, we get

$D_2 = -\dfrac{2A^2}{3\omega^2}$, then;

$x_{(1)} = A'\sin\omega t + \dfrac{\lambda A^2}{\omega^2}\left[\dfrac{1}{2} - \dfrac{2}{3}\cos\omega t + \dfrac{1}{6}\cos(2\omega t)\right]$, which is the required solution. Here A' is an arbitrary constant. The constants A and A' can be determined from the displacement and velocity at the initial time.

EXAMPLE 1.33 A particle of mass m moves in two dimensions in a potential field of $V(r) = \frac{1}{2}k(x^2 + 4y^2)$. Obtain the equation of motion of the particle subjected to the initial conditions; at $t = 0; x = a, y = 0, \dot{x} = 0, \dot{y} = v_0$

Solution: From the given potential we can write the force as

$$F = -\nabla V$$

or

$$m\ddot{r} = -kx\hat{i} - 4ky\hat{j}$$

Thus, the components of force are

$$m\ddot{x} + kx = 0 \quad \text{and} \quad m\ddot{y} + 4ky = 0 \tag{i}$$

Therefore, the angular velocity is $\omega = \sqrt{\frac{k}{m}}$ for the motion in the x-direction and it is 2ω for the motion in the y-direction.

Now, the general solutions of (i) can be written as

$$x = A_1\cos\omega t + B_1\sin\omega t$$

and

$$y = A_2\cos 2\omega t + B_2\sin 2\omega t \tag{ii}$$

The coefficients A_1, B_1, A_2, and B_2 can be obtained from the initial conditions. For applying the initial conditions let us find the derivatives of equation (ii).

$$\dot{x} = -A_1\omega\sin\omega t + B_1\omega\cos\omega t$$

and

$$\dot{y} = -2A_2\omega\sin 2\omega t + 2B_2\omega\cos 2\omega t \tag{iii}$$

Putting the initial conditions in these equations and simplifying, we get

$$A_1 = a, \quad A_2 = B_1 = 0 \text{ and } B_2 = \frac{v_0}{2\omega}$$

Now, the equations of motion are

$$x = a\cos\omega t \text{ and } \quad y = \frac{v_0}{2\omega}\sin 2\omega t \tag{iv}$$

These are the required equations.

EXAMPLE 1.34 Show that for a one-dimensional conservative system of oscillator, the general solution can be written as $t - t_0 = \int \frac{dx}{\left\{\frac{2}{m}[E - V(x)]\right\}^{\frac{1}{2}}}$.

Solution: Since a one-dimensional oscillator has only one degree of freedom, its kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 \tag{i}$$

and potential energy is

$$V = V(x) \tag{ii}$$

Then the total energy is

$$E = T + V = \frac{1}{2}m\dot{x}^2 + V(x) \tag{iii}$$

This can be rearranged to get

$$\dot{x} = \frac{dx}{dt} = \left\{\frac{2}{m}[E - V(x)]\right\}^{\frac{1}{2}}$$

or

$$dt = \frac{dx}{\left\{\frac{2}{m}[E - V(x)]\right\}^{\frac{1}{2}}}$$

Then,

$$\int_{t_0}^{t} dt = \int \frac{dx}{\left\{\frac{2}{m}[E - V(x)]\right\}^{\frac{1}{2}}}$$

that is,

$$t - t_0 = \int \frac{dx}{\left\{\frac{2}{m}[E - V(x)]\right\}^{\frac{1}{2}}} \tag{iv}$$

This forms a general solution to the one-dimensional problem. The integral can be evaluated if $V(x)$ is known. This solution reveals some interesting features. For the motion to be oscillatory, there must be two points called the turning points, say x_1 and x_2 such that

$$V(x_1) = V(x_2) = E.$$

At all points $x_1 < x < x_2$ the integral must converge. Then at the turning points the velocity $\dot{x}$ vanishes. However, the force must not vanish at these points, otherwise the particle would remain stationary at these points.

EXAMPLE: 1.35 Using Virial theorem prove that for an inverse square law force $2\langle T\rangle + \langle V\rangle = 0$, where T is the kinetic energy and V the potential energy.

Solution: The mathematical statement of Virial theorem is

$$\langle T\rangle = -\frac{1}{2}\left\langle \sum_i \vec{F}_i.\vec{r}_i \right\rangle \tag{i}$$

where the RHS is the Virial of Clausius.

When the force is derivable from a potential

$$\langle T\rangle = \frac{1}{2}\left\langle \sum_i (\nabla_i V_i).\vec{r}_i \right\rangle$$

$$= \frac{1}{2}\frac{\partial V}{\partial r}r \tag{ii}$$

Now let us assume a general form for the potential energy as

$$V = kr^{n+1},$$

Then, $$\frac{\partial V}{\partial r} = (n+1)kr^n,$$

so that $$\langle T\rangle = \frac{1}{2}(n+1)kr^{n+1} = \frac{1}{2}(n+1)\langle V\rangle \tag{iii}$$

Now, for inverse square law, $n = -2$ and therefore

$$2\langle T\rangle + \langle V\rangle = 0$$

Hence, proved.

EXAMPLE 1.36 Estimate the average temperature of a star using Virial theorem.

Solution: Assuming star as a sphere of radius R and mass M, we can write the gravitational potential as

$$\langle V\rangle = -\frac{3}{5}\frac{GM^2}{R} \tag{i}$$

Now, the average value of kinetic energy of the atoms in a star can be obtained from equipartition of energy.

$$\langle T\rangle = \frac{3}{2}k_B\langle T\rangle, \text{ for each atom.}$$

If N is the total number of atoms in the star, then

$$\langle T\rangle = \frac{3}{2}Nk_B\langle T\rangle \tag{ii}$$

Since the gravitational force obeys inverse square law, the Virial theorem is expressed as $2\langle T\rangle+\langle V\rangle=0$. Then using (i) and (ii), we can write

$$2\frac{3}{2}Nk_B\langle T\rangle=\frac{3}{5}\frac{GM^2}{R}$$

or
$$\langle T\rangle=\frac{GM^2}{5k_BNR}=\frac{Gm}{5k_BR} \qquad \text{(iii)}$$

where, $m=\frac{M}{N}$, the average mass of an atom in a star.

Now, for the sun, $m\approx 2.2\times10^{-27}$ kg, $M=1.9\times10^{30}$ kg and $R=7\times10^{10}$ m. Substituting these values in (iii), we get

$$\langle T\rangle=\frac{6.67\times10^{-11}\times1.9\times10^{30}\times2.2\times10^{-27}}{5\times1.38\times10^{-23}\times7\times10^{8}}\approx10^7\,\mathrm{K}$$

EXAMPLE 1.37 Prove the Virial theorem for a classical system of point masses m_i with position vectors r_i on which a net force F_i is applied by assuming the quantity $\sum_i m_i\dot{r}_i.r_i$ has a finite value during the motion of the system.

Solution: Let us write the given quantity as

$$Q(t)=\sum_i m_i\dot{r}_i.r_i$$

Then,
$$\dot{Q}(t)=\sum_i m_i\dot{r}_i.\dot{r}_i+\sum_i m_i\ddot{r}_i.r_i$$

$$=\sum_i m_i\dot{r}_i^2+\sum_i F_i.r_i \qquad \text{(i)}$$

Now take the time average of the function $\dot{Q}(t)$.

$$\frac{1}{\tau}\int_0^\tau \dot{Q}(t)\mathrm{d}t=\frac{1}{\tau}\int_0^\tau\sum_i m_i\dot{r}_i^2dt+\frac{1}{\tau}\int_0^\tau\sum_i F_i.r_idt$$

or
$$\frac{1}{\tau}[Q(t)-Q(0)]=2\overline{T}+\overline{\sum_i F_i.r_i} \qquad \text{(ii)}$$

In the above expression τ is the time period if the motion of the system is periodic; otherwise it is infinity. However, in both cases the LHS of equation (ii) is equal to zero.

Therefore,
$$\overline{T}=-\frac{1}{2}\overline{\sum_i F_i.r_i} \qquad \text{(iii)}$$

Hence, proved.

EXAMPLE 1.38 Consider a fixed smooth cylinder held horizontally. Two heavy particles of masses M_1 and M_2 are connected by a light inextensible string and hang over the cylinder across its length. Find the condition of equilibrium of the system.

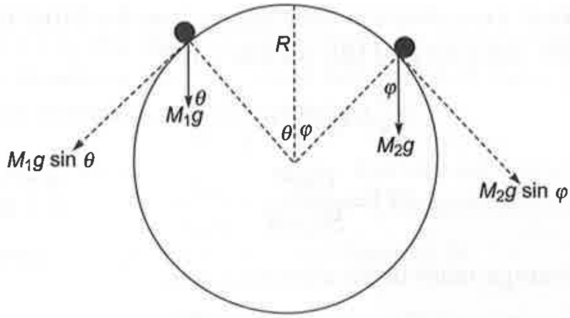

Fig. 1.9

Solution: This problem can be solved by applying the principle of virtual work. Let R be the radius of the cylinder.

For a system of particles the principle of virtual work gives

$$\sum_i \vec{F}_i . \delta\vec{r}_i = 0 \tag{i}$$

In the present problem, the forces that work are the components of M_1g and M_2g along the tangents, $M_1g\sin\theta$ and $M_2g\sin\phi$. Therefore,

$$M_1g\sin\theta \times \delta r_1 + M_2g\sin\phi \times \delta r_2 = 0 \tag{ii}$$

Now, $$\delta r_1 = R\delta\theta \quad \text{and} \quad \delta r_2 = R\delta\phi$$

Substituting in equation (ii), we get

$$M_1g\sin\theta \times R\delta\theta + M_2g\sin\phi \times R\delta\phi = 0$$

or $$M_1\sin\theta \times \delta\theta + M_2\sin\phi \times \delta\phi = 0 \tag{iii}$$

Since the string is inextensible, we have

$$\theta + \phi = \text{constant}$$

Therefore, $$\delta\theta + \delta\phi = 0 \quad \text{or,} \quad \delta\phi = -\delta\theta$$

Then, (iii) becomes

$$(M_1\sin\theta - M_2\sin\phi)\delta\theta = 0$$

$$M_1\sin\theta - M_2\sin\phi = 0$$

Therefore, the condition for equilibrium is

$$\frac{M_1}{M_2} = \frac{\sin\phi}{\sin\theta} \tag{iv}$$

EXAMPLE 1.39 Consider a bar of length L supported at its ends and loaded by a force F, a distance a from the left hand end. Reaction forces R_A and R_C act at the ends as shown in the figure. Using the concept of virtual work determines the reaction R_C so that the bar is in equilibrium.

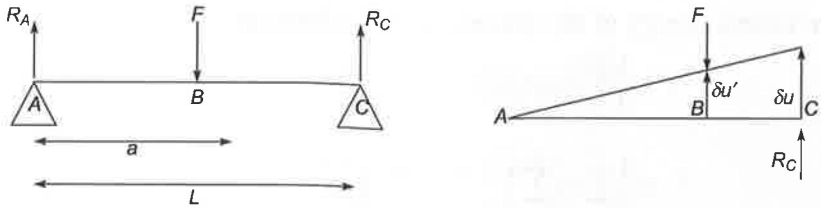

Fig. 1.10

Solution: Here we assume a **kinematically inadmissible** virtual displacement δu at the point C. (The term **kinematically admissible displacement** is used to mean one that does not violate the constraints and the work done by the reaction force is zero).

Let $\delta u'$ be the virtual displacement at the point where the force F is applied. Then using the two similar triangles in the second figure, we can write

$$\delta u' = \frac{a}{L}\delta u \tag{i}$$

The end A does not undergo any displacement. The total virtual work is

$$\delta W = R_C \delta u - F\frac{a}{L}\delta u \tag{ii}$$

The second term is negative because, at the point B the displacement is opposite to the force. Now, for the bar to be in equilibrium, the total virtual work must be equal to zero. Therefore,

$$R_C \delta u - F\frac{a}{L}\delta u = 0$$

or

$$R_C = F\frac{a}{L} \tag{iii}$$

EXAMPLE 1.40 Show that the kinetic energy of any holonomic system can be expressed in the form

$T = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}(q)\dot{q}_j\dot{q}_k$, where $q_1, q_2, \dots\dots\dots, q_n$ are the generalized coordinates.

Solution: Let us consider a system of particles subjected to holonomic constraints. In the system let r be the position vector and v the velocity of a typical particle.

Now from the chain rule, we have

$$v = \frac{\partial r}{\partial q_1}\dot{q}_1 + \frac{\partial r}{\partial q_2}\dot{q}_2 + \dots\dots\dots\dots + \frac{\partial r}{\partial q_n}\dot{q}_n = \sum_{j=1}^{n}\frac{\partial r}{\partial q_j}\dot{q}_j$$

so that

$$v.v = \left(\sum_{j=1}^{n}\frac{\partial r}{\partial q_j}\dot{q}_j\right)\cdot\left(\sum_{k=1}^{n}\frac{\partial r}{\partial q_k}\dot{q}_k\right)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial r}{\partial q_j}\cdot\frac{\partial r}{\partial q_k}\right)\dot{q}_j\dot{q}_k \tag{i}$$

Now the kinetic energy of the system can be written as

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i (v_i . v_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n} m_i \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial r}{\partial q_j}.\frac{\partial r}{\partial q_k}\right)\dot{q}_j\dot{q}_k$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{1}{2}\sum_{i=1}^{n} m_i \left(\frac{\partial r}{\partial q_j}.\frac{\partial r}{\partial q_k}\right)\right]\dot{q}_j\dot{q}_k = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}(q)\dot{q}_j\dot{q}_k$$

where, $$a_{jk} = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\frac{\partial r}{\partial q_j}.\frac{\partial r}{\partial q_k}\right) \qquad \text{(ii)}$$

EXAMPLE 1.41 D'Alembert's principle is given by $\sum_i (F_i^e - \dot{P}_i).\delta r_i = 0$ where, F_i^e is the external force and $\dot{P}_i$ is the reverse effective force. Show that the second part can be written as

$$\sum_i \dot{P}_i.\delta r_i = \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j .$$

Solution: Let the position vector of the particle, r be a function of q_j's and t. That is; $r_i = r_i(q_j, t)$ with $j = 1,2,\ldots\ldots, n$. Then,

$$\dot{r}_i = \frac{dr_i}{dt} = \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \qquad \text{(i)}$$

Further, $$\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} \qquad \text{(ii)}$$

The kinetic energy of the system is

$$T = \sum_i \frac{1}{2} m_i \dot{r}_i^2 \qquad \text{(iii)}$$

Using these expressions, we can write

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\right]\delta q_j = \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i \dot{r}_i^2\right)\right)\right]\delta q_j$$

$$= \sum_j \sum_i m_i \left[\frac{d}{dt}\left(\dot{r}_i . \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right)\right]\delta q_j$$

$$= \sum_j \sum_i m_i \left[\frac{d}{dt}\left(\dot{r}_i . \frac{\partial r_i}{\partial q_j}\right)\right]\delta q_j$$

$$= \sum_j \sum_i m_i \left(\ddot{r}_i . \frac{\partial r_i}{\partial q_j} + \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_j} \right) \delta q_j$$

$$= \sum_i m_i \left(\ddot{r}_i . \delta r_i + \dot{r}_i \delta \dot{r}_i \right)$$

where we used the relation $\delta r_i = \sum_j \frac{\partial r_i}{\partial q_j} \delta q_j$, then

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \right] \delta q_j = \sum_i \left(\dot{P}_i . \delta r_i + P_i . \delta \dot{r}_i \right) \quad \text{(iv)}$$

In a similar way, we can obtain

$$\sum_j \left(\frac{\partial T}{\partial q_j} \right) \delta q_j = \sum_i P_i . \delta \dot{r}_i \quad \text{(v)}$$

From equations (iv) and (v) it is easy to see that

$$\sum_i \dot{P}_i . \delta r_i = \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j$$

EXAMPLE 1.42 Show that the constraint forces do no work.

Solution: To prove the statement, let us consider the motion of a particle on a surface subjected to a constraint $f(x, y, z) = 0$ so that the constraint is holonomic and scleronomic.

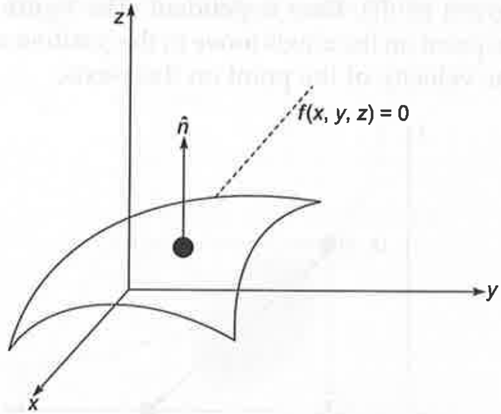

Fig. 1.11

Now the integrable (Pfaffian) and velocity forms of the constraints are

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0 \quad \text{(i)}$$

and
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} = 0 \qquad \text{(ii)}$$

Now, a vector normal to the surface can be written as

$$\vec{n} = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \qquad \text{(iii)}$$

Since the constraint force is normal to the surface, let us write it as

$$\vec{F} = \lambda \vec{n}$$

where, λ is such that the particle does not violate the constraint. An infinitesimal displacement of the particle on the surface that is consistent with the constraint can be written as

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

Then, the work done by the constraint force in this displacement is

$$dW = \vec{F}.\mathrm{d}\vec{r} = \lambda\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) = 0 \qquad \text{(iv)}$$

Thus, the constraint force does zero work.

EXAMPLE 1.43 A particle moves in the xy-plane under the constraint that its velocity vector is always directed towards a point on the x-axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is nonholonomic.

Solution: The abscissa is the distance of a point on the x-axis from the origin. In the present problem it is given as $f(t)$, time dependent. The figure shows the path when both the particle and the point on the x-axis move in the positive x-direction with particle velocity greater than the velocity of the point on the x-axis.

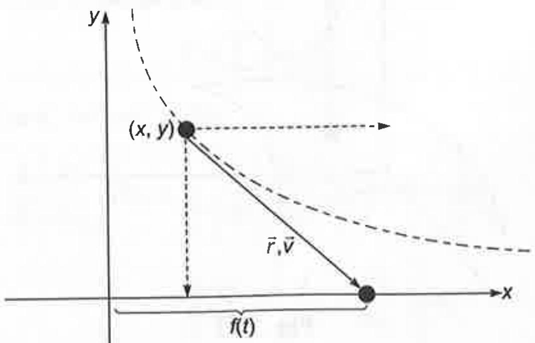

Fig. 1.12

The velocity components of the particle are given by

$$v_x = \frac{dx}{dt} \text{ and } v_y = \frac{dy}{dt} \tag{i}$$

Also, the components of the vector $\vec{r}$ that connect the particle to the point on the x-axis are

$$x' = f(t) - x(t) \text{ and } y' = y(t) \tag{ii}$$

Since the vectors $\vec{r}$ and $\vec{v}$ have the same direction, the ratio of the components of $\vec{r}$ must be equal to the ratio of the components of $\vec{v}$.

$$\frac{v_y}{v_x} = \frac{y'}{x'}$$

or
$$\frac{dy}{dx} = \frac{y(t)}{f(t) - x(t)}$$

Then,
$$\frac{dy}{y(t)} = \frac{dx}{f(t) - x(t)} \tag{iii}$$

This is not integrable since *f(t)* is arbitrary and hence the constraint is nonholonomic.

EXAMPLE 1.44 A disc is constraint to move on a horizontal plane such that its plane is always vertical.

Form the constraint equations and state whether the constraint is holonomic or nonholonomic.

Solution: Let us take the *xy*-plane as the horizontal plane and *z*-direction as the vertical. Then according to the constraint condition, the axis of the disc is always perpendicular to the *z*-axis.

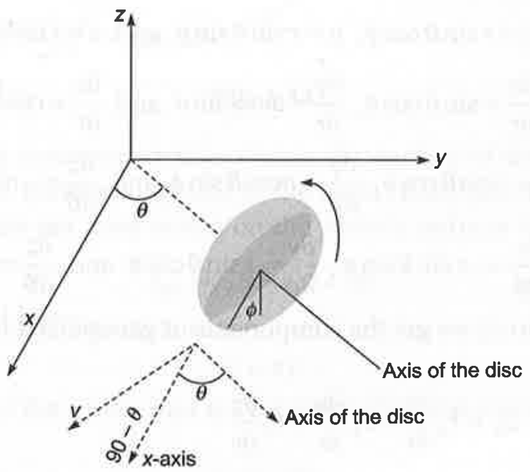

Fig. 1.13

Let θ be the angle that the axis of the disc makes with the x-axis and ϕ is the angle of rotation of the disc about its axis. If r is the radius of the disc, its velocity can be written as

$$v = r\frac{d\phi}{dt} \tag{i}$$

This velocity is directed perpendicular to the z-axis, i.e., in the xy-plane. Hence, we can resolve the velocity as

$$v_x = \frac{dx}{dt} = v\sin\theta \text{ and, } v_y = \frac{dy}{dt} = -v\cos\theta \tag{ii}$$

Using (i) we can rewrite (ii) as

$$\frac{dx}{dt} = r\frac{d\phi}{dt}\sin\theta \text{ and, } \frac{dy}{dt} = -r\frac{d\phi}{dt}\cos\theta \tag{iii}$$

or
$$dx - r\sin\theta d\phi = 0 \text{ and } dy + r\cos\theta\phi = 0 \tag{iv}$$

These equations are the constraint equations. We can see that these equations cannot be integrated until the problem is completely solved. Therefore, the given constraint is nonholonomic.

EXAMPLE 1.45 Calculate the generalized force components on a particle moving in space using spherical polar coordinates.

Solution: The generalized force corresponding to a coordinate q_j can be written as

$$Q_j = F_x\frac{\partial x}{\partial q_j} + F_y\frac{\partial y}{\partial q_j} + F_z\frac{\partial z}{\partial q_j} \tag{i}$$

In spherical polar coordinates

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi \text{ and } z = r\cos\theta$$

Then,
$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi,\ \frac{\partial y}{\partial r} = \sin\theta\sin\phi \text{ and } \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \text{ and } \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \text{ and } \frac{\partial z}{\partial \phi} = 0$$

Substituting these in (i) we get the components of generalized force in spherical polar coordinates as

$$Q_r = F_x\frac{\partial x}{\partial r} + F_y\frac{\partial y}{\partial r} + F_z\frac{\partial z}{\partial r}$$

$$= F_x\sin\theta\cos\phi + F_y\sin\theta\sin\phi + F_z\cos\theta \tag{ii}$$

Similarly, $$Q_\theta = F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} + F_z \frac{\partial z}{\partial \theta}$$

$$= F_x r \cos\theta \cos\phi + Fyr \cos\theta \sin\phi - F_z r \sin\theta \qquad \text{(iii)}$$

and $$Q_\phi = F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} + F_z \frac{\partial z}{\partial \phi}$$

$$= -F_x r \sin\theta \sin\phi + F_y r \sin\theta \cos\phi \qquad \text{(iv)}$$

Equations (ii), (iii) and (iv) are the required equations.

EXAMPLE 1.46 A particle of mass m is constrained to move on the surface of a sphere of radius R by an applied force $F(\theta, \phi)$. Obtain the equations of motion.

Solution: In spherical polar coordinates, the applied force can be written as

$$F = F_r \hat{r} + F_\theta \hat{\theta} + F_\phi \hat{\phi} \qquad \text{(i)}$$

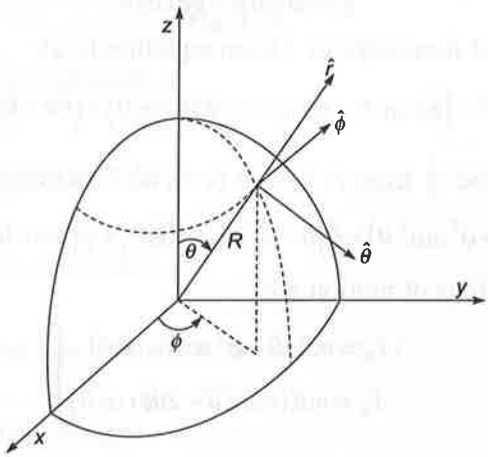

Fig. 1.14

Now, the particle is constrained to move on the surface of the sphere. As a result, there will be a force of reaction opposite to the radial component of the applied force, i.e., $-F_r \hat{r}$. Therefore, the net force acting on the particle reduces to

$$F_{net} = F_\theta \hat{\theta} + F_\phi \hat{\phi} = m\ddot{\vec{r}} \qquad \text{(ii)}$$

The position vector of the particle is

$$\vec{r} = R\hat{r}$$

where R is the radius of the sphere and is a constant. Then the acceleration of the particle is

$$a = \ddot{\vec{r}} = R\ddot{\hat{r}} \qquad \text{(iii)}$$

Now our task is to determine $\ddot{\hat{r}}$. For this let us write the unit vectors in spherical polar coordinates, $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ in terms of the unit vectors in rectangular coordinates, $\hat{i}, \hat{j}$ and $\hat{k}$.

$$\hat{r} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta \tag{iv a}$$

$$\hat{\theta} = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta \tag{iv b}$$

$$\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi \tag{iv c}$$

Then from (iv a), the first derivative is

$$\dot{\hat{r}} = \hat{i}\left(-\dot{\phi}\sin\theta in\phi + \dot{\theta}\cos\theta\cos\phi\right) + \hat{j}\left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right) - \hat{k}\dot{\theta}\sin\theta$$

$$= \left(-\hat{i}\sin\phi + \hat{j}\cos\phi\right)\dot{\phi}\sin\theta + \left(\hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta\right)\dot{\theta}$$

$$\dot{\hat{r}} = \hat{\phi}\dot{\phi}\sin\theta + \hat{\theta}\dot{\theta} \tag{v a}$$

Similarly,

$$\dot{\hat{\theta}} = -\hat{r}\dot{\theta} + \hat{\phi}\dot{\phi}\cos\theta \tag{v b}$$

and

$$\dot{\hat{\phi}} = \hat{r}\dot{\phi}\sin\theta - \hat{\theta}\dot{\phi}\cos\theta \tag{v c}$$

Now, take the second derivative of $\hat{r}$ from equation (v a).

$$\ddot{\hat{r}} = \left(\dot{\hat{\phi}}\dot{\theta}\sin\theta + \hat{\phi}\ddot{\phi}\sin\theta + \hat{\phi}\dot{\phi}\dot{\theta}\cos\theta\right) + \left(\dot{\hat{\theta}}\dot{\theta} + \hat{\theta}\ddot{\theta}\right)$$

Substituting for $\dot{\hat{\theta}}$ and $\dot{\hat{\phi}}$ from (v b) and (v c), and rearranging, we get

$$\ddot{\hat{r}} = -\hat{r}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \hat{\theta}\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right) + \hat{\phi}\left(\ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta\right) \tag{vi}$$

Therefore, the equations of motion are

$$\left.\begin{aligned} F_\theta &= mR\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right) \\ \text{and} \quad F_\phi &= mR\left(\ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta\right) \end{aligned}\right\} \tag{vii}$$

EXAMPLE 1.47 A horizontal frictionless table has a small hole in its centre. Block A of mass m on the table is connected to block B of mass M hanging beneath by a string of negligible mass which passes through the hole. Initially, B is held stationary and A rotates at constant radius r_0 with steady angular velocity ω_0. If B is released at $t = 0$, obtain the condition that the block B moves in the upward direction after release.

Solution: A schematic diagram representing the problem is given below. Let us consider the various forces acting on the system. On block A of mass m, the only force acting is the tension in the string and it is in the horizontal direction. On block B of mass M, its weight is acting in the downward direction and tension in the string in the upward direction. Let us take the vertical direction as the z-direction and write the various equations of motion.

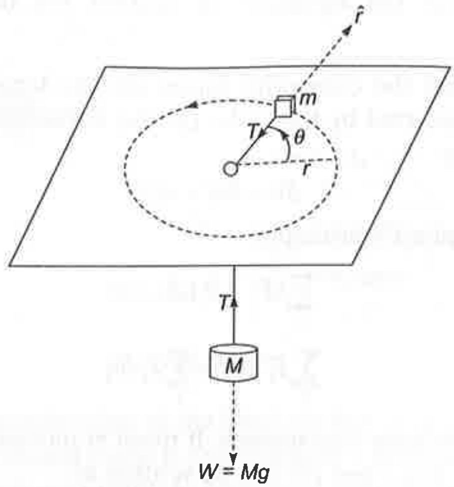

Fig. 1.15

For the radial movement of block A

$$m(\ddot{r} - r\dot{\theta}^2) = -T \tag{i}$$

For the tangential movement of block A

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{ii}$$

For the vertical movement of block B

$$M\,\ddot{z} = W - T \tag{iii}$$

The length of the string l is a constant, hence; $r + z = l$.

Therefore, $\ddot{r} = -\ddot{z}$

Using equation (i) in (iii), we get

$$M\,\ddot{z} = W + m(\ddot{r} - \dot{r}^2)$$

or $$M\,\ddot{z} = W + m(-\ddot{z} - \dot{r}^2)$$

or $$\ddot{z} = \frac{W - mr\dot{\theta}^2}{m + M} \tag{iv}$$

At the moment of dropping the block B, this becomes

$$\ddot{z} = \frac{W - mr_0\omega_0^2}{m + M} \tag{v}$$

Equation (v) gives the acceleration of block B. Depending upon the numerator, the acceleration can be positive or negative. Therefore, for block B to rise, the numerator should be negative and this will occur if the angular velocity of the block A, ω_0 is sufficiently large.

EXAMPLE 1.48 Obtain the equation of motion for the above problem from D' Alembert's principle.

Solution: In this problem the constraint forces are the tension in the string and the reaction on the mass m exerted by the table. Let the virtual displacement of the mass m be $\delta\vec{r}$, which is given by

$$\delta\vec{r} = \delta r\hat{r} + r\delta\theta\hat{\theta} \tag{i}$$

According to D' Alembert's principle

$$\sum_i (F_i^e - \dot{P}_i).\delta r_i = 0$$

or

$$\sum_i F_i^e.\delta r_i = \sum_i \dot{P}_i.\delta r_i \tag{ii}$$

In the present case we have two masses. If mass m moves radially through δr, mass M moves upward by $(-\delta r)$. Then (ii) can be written as

$$F_{1r}\delta r + rF_{1\theta}\delta\theta - F_2\delta r = \sum_{i=1}^{2} \dot{P}_i.\delta r_i \tag{iii}$$

where, F_1 is the external force acting on m and F_2 is the external force on M.

For mass m, we can write

$$\dot{P}_1 = m(a_r\hat{r} + a_\theta\hat{\theta}) = \mathrm{m}\left[\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}\right]$$

and for the mass M; $\dot{P}_2 = M\ddot{r}\hat{z}$.

There is no external force acting on the mass m and on M, the external force is the gravitational force. Therefore (iii) can be written as

$$-Mg\delta r = m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\delta r + r\delta\theta\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\right] + M\,\ddot{r}\delta r \tag{iv}$$

Now comparing the coefficients of δr and $\delta\theta$ on both sides, we get

$$-Mg = (m + M)\ddot{r} - mr\dot{\theta}^2$$

or

$$\ddot{r} = \frac{mr\dot{\theta}^2 - Mg}{(m+M)} \text{ or; } \ddot{z} = \frac{Mg - mr\dot{\theta}^2}{(m+M)} \tag{v}$$

EXAMPLE 1.49 A block of mass m is sliding over a frictionless inclined plane whose mass is M. The inclined plane is moving over a smooth frictionless horizontal surface with uniform acceleration A. If the angle of inclination is α, determine the acceleration of the inclined plane and the time taken by the block to reach the bottom of the plane starting from the top.

Solution: A schematic diagram of the problem is given below:

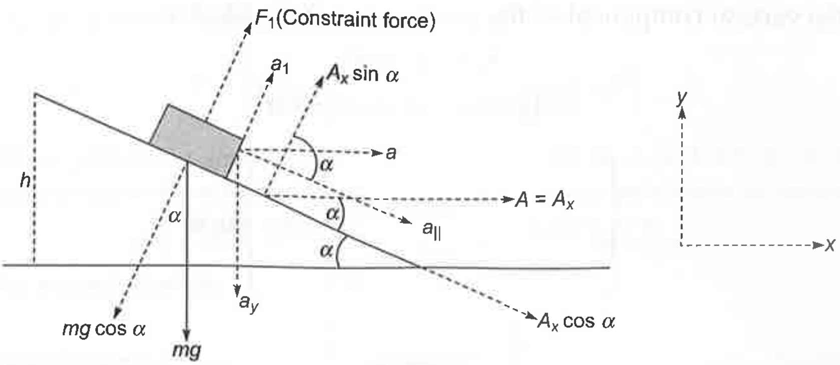

Fig. 1.16

Let a be the horizontal acceleration of the block so that $a_{||}$ is its component along the inclined plane. Since the inclined plane is moving with a horizontal acceleration $A = A_x$, the force acting on the block, according to Newton's second law is

$$F = m(a + A_x) \tag{i}$$

This can be resolved into two components, parallel to the inclined plane $(F_{||})$ and perpendicular to the inclined plane $(F_{\perp})$.

From the diagram

$$F_{||} = mg\sin\alpha = ma_{||} + mA_x\cos\alpha \tag{ii}$$

and
$$F_{\perp} = F_1 - mg\cos\alpha = ma_{\perp} + mA_x\sin\alpha \tag{iii}$$

where F_1 is the constraint force. Since the block is constraint to move only along the inclined plane; $a_{\perp} = 0$, then; from (iii) it follows that

$$F_1 = mg\cos\alpha + mA_x\sin\alpha \tag{iv}$$

Now, according to Newton's third law, there is a reaction $(-F_1)$ exerted by the block on the inclined plane, just opposite to the constraint force. The horizontal component of this force can be written as

$$-F_1\sin\alpha = MA_x \tag{v}$$

This expression can be used in (iv) to substitute for F_1 to get the expression for the acceleration of the inclined plane A_x as

$$A_x = -g\left(\frac{\sin\alpha\cos\alpha}{\sin^2\alpha + \dfrac{M}{m}}\right) \tag{vi}$$

The component of acceleration of the block along the inclined plane can be obtained from (ii) as

$$a_{||} = g\sin\alpha - A_x\cos\alpha \tag{vii}$$

Now, the vertical component of the acceleration of the block is

$$a_y = -a_{\parallel}\sin\alpha$$
$$= -\left(g\sin\alpha - A_x\cos\alpha\right)\sin\alpha$$
$$= -\left[g\sin\alpha + g\left(\frac{\sin\alpha\cos\alpha}{\sin^2\alpha + \dfrac{M}{m}}\right)\cos\alpha\right]\sin\alpha$$
$$= -g\sin^2\alpha\left[1 + \frac{\cos^2\alpha}{\sin^2\alpha + \dfrac{M}{m}}\right]$$
$$= -g\sin^2\alpha\left(\frac{M+m}{M+m\sin^2\alpha}\right) \quad \text{(viii)}$$

If h is the height of the inclined plane, the time taken by the block to reach the bottom of the plane is given by

$$-h = \frac{1}{2}a_y t^2$$

or

$$t = \sqrt{-2ha_y} \quad \text{(ix)}$$

EXAMPLE 1.50 Solve the above problem using the principle of virtual work.

Solution: As a first step to solve this problem we will find out the total kinetic energy of the block and the inclined plane in terms of the horizontal displacement of the inclined plane, X and the distance of the block from the top of the inclined plane, say d. Then

Kinetic energy of the inclined plane is

$$T_I = \frac{1}{2}M\dot{X}^2 \quad \text{(i)}$$

Since the displacement of the block is along the inclined plane its velocity can be resolved into two components

$$v_{xB} = \dot{X} + \dot{d}\cos\alpha \quad \text{and} \quad v_{yB} = -\dot{d}\sin\alpha$$

Therefore, the kinetic energy of the block is

$$T_B = \frac{1}{2}m\left(v_{xB}^2 + v_{yB}^2\right) = \frac{1}{2}m\left[\left(\dot{X} + \dot{d}\cos\alpha\right)^2 + \left(-\dot{d}\sin\alpha\right)^2\right]$$
$$= \frac{1}{2}m\left(\dot{X}^2 + 2\dot{d}\dot{X}\cos\alpha + \dot{d}^2\right) \quad \text{(ii)}$$

Then, the total kinetic energy is

$$T=\frac{1}{2}(m+M)\dot{X}^2+\frac{1}{2}m\left(2\dot{d}\dot{X}\cos\alpha+\dot{d}^2\right) \tag{iii}$$

Now, let us provide a small virtual displacement δd in d and δX in X. Then the virtual displacement of the inclined plane and the block can be written in vector form as

$$\delta\vec{R}_I=\delta X\hat{i} \text{ and } \delta\vec{r}_B=(\delta X+\delta d\cos\alpha)\hat{i}-\delta d\sin\alpha\hat{j} \tag{iv}$$

Then the virtual work is

$$\delta W=\vec{F}.\delta\vec{r} \tag{v}$$

The constraint force F_1 does not do any work as it is perpendicular to the displacements. Therefore, the forces that do virtual work are the gravitational forces acting on the inclined plane and the block and they are $(-mg\hat{j})$ and $(-Mg\hat{j})$ respectively. The negative sign is for the downward direction.

Then, using these expressions and (iv) in (v), the virtual work for the inclined pane and for the block can be written as

$$\delta W_I=0 \text{ and } \delta W_B=mg(\sin\alpha)\delta d \tag{vi}$$

Now, the D'Alembert's principle is

$$\delta W-\dot{\vec{P}}.\delta\vec{r}=0$$

$$\delta W-\left[\frac{d}{dt}\left(\vec{P}.\delta\vec{r}\right)-\vec{P}.\frac{d}{dt}(\delta\vec{r})\right]=0 \tag{vii}$$

To find the equation of motion let us find the expression for the total $\left(\vec{P}.\delta\vec{r}\right)$ adding together the contributions from the inclined plane and the block. Also we set $\frac{d}{dt}(\delta\vec{r})=0$ assuming $\delta\vec{r}$ as time independent constant. The displacement $\vec{r}$ is a function of d and X. (For inclined plane it is a function of X only.) Then,

$$\vec{P}.\delta\vec{r}=\vec{P}.\frac{\partial\vec{r}}{\partial d}\delta d+\vec{P}.\frac{\partial\vec{r}}{\partial X}\delta X \tag{viii}$$

In a similar way

$$\dot{\vec{r}}=\frac{\partial\vec{r}}{\partial d}\dot{d}+\frac{\partial\vec{r}}{\partial X}\dot{X} \tag{ix}$$

Now we replace $\frac{\partial\vec{r}}{\partial d}$ and $\frac{\partial\vec{r}}{\partial X}$ in (viii) by $\frac{\partial\dot{\vec{r}}}{\partial\dot{d}}$ and $\frac{\partial\dot{\vec{r}}}{\partial\dot{X}}$ respectively. Then,

$$\vec{P}.\delta\vec{r}=\vec{P}.\frac{\partial\dot{\vec{r}}}{\partial\dot{d}}\delta d+\vec{P}.\frac{\partial\dot{\vec{r}}}{\partial\dot{X}}\delta X \tag{x}$$

The components of momentum, $\vec{P}=m\dot{\vec{r}}$, in Cartesian coordinates can be written as

$$P_x=\frac{\partial T}{\partial\dot{x}},\ P_y=\frac{\partial T}{\partial\dot{y}} \text{ and } P_z=\frac{\partial T}{\partial\dot{z}} \tag{xi}$$

Putting these expressions in (x) and using the chain rule $\frac{\partial T}{\partial \dot{d}} = \frac{\partial T}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \dot{d}} + \frac{\partial T}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \dot{d}} + \frac{\partial T}{\partial \dot{z}}\frac{\partial \dot{z}}{\partial \dot{d}}$ and $\frac{\partial T}{\partial \dot{X}} = \frac{\partial T}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \dot{X}} + \frac{\partial T}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \dot{X}} + \frac{\partial T}{\partial \dot{z}}\frac{\partial \dot{z}}{\partial \dot{X}}$, we can obtain

$$\vec{P}.\delta\vec{r} = \frac{\partial T}{\partial \dot{d}}\delta d + \frac{\partial T}{\partial \dot{X}}\delta X \tag{xii}$$

Now put (xii) in (vii) and make use of equations (iii) and (vi), we get

$$\begin{aligned} mg(\sin\alpha)\delta d &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{d}}\right)\delta d + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}}\right)\delta X \\ &= \frac{d}{dt}\left(m\dot{X}\cos\alpha + m\dot{d}\right)\delta d + \frac{d}{dt}\left[(m+M)\dot{X} + m\dot{d}\cos\alpha\right]\delta X \\ &= m\left[\ddot{X}\cos\alpha + \ddot{d}\right]\delta d + \left[(m+M)\ddot{X} + m\ddot{d}\cos\alpha\right]\delta X \end{aligned} \tag{xiii}$$

Now, comparing the coefficients of δd, we get

$$mg\sin\alpha = \ddot{X}\cos\alpha + \ddot{d}$$

or

$$\ddot{d} = g\sin\alpha - \ddot{X}\cos\alpha \tag{xiv}$$

This is same as equation (vii) obtained in the previous problem, with $\ddot{d} = a_{||}$ and $\ddot{X} = A_x$.

On comparison of the coefficients of δX, we get

$$(m+M)\ddot{X} + m\ddot{d}\cos\alpha = 0$$

Substitute for $\ddot{d}$ from the above equation (xiv) and rearrange to get

$$\ddot{X} = -g\left(\frac{\sin\alpha\cos\alpha}{\sin^2\alpha + \frac{M}{m}}\right) \tag{xv}$$

Equations (xiv) and (xv) are the required equations of motion.

EXERCISES

1.1 A particle of mass m is subject to a force $F(t) = me^{-bt}$. The initial position and speed are zero. Find the position as a function of time.

1.2 A particle is allowed to fall under gravity from height h. If the air damping on it be proportional to instantaneous velocity of the particle, find the velocity attend at time t.

1.3 Two masses m_1 and m_2 are connected by a smooth string and pass over a pulley. The pulley is assumed to be frictionless so that it will not rotate. Assuming $m_2 > m_1$ obtain an expression for the acceleration of the masses and tension in the string.

1.4 Repeat the above question when the pulley is moving in the upward direction with a constant acceleration a_0.

1.5 For the double Atwood's machine shown in the figure below, obtain the acceleration of the masses.

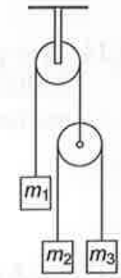

Fig. 1.17

1.6 A body of mass m is placed on an inclined plane that makes an angle θ with the horizontal. Obtain the equation of motion when the body moves over the surface in the upward direction.

1.7 A particle of mass m slides down a plane inclined at an angle θ under the influence of gravity. If the motion is resisted by a force $f = kmv^2$ show that the time required to move a distance d after starting from rest is

$$t = \frac{\cosh^{-1}\left(e^{kd}\right)}{kg\sin\theta}.$$

1.8 A ball is thrown at speed u from zero height on ground level. At what angle should it be thrown so that the area under the trajectory is maximum?

1.9 A ball is thrown at speed θ from zero height on ground level. Let θ_0 be the angle at which the ball should be thrown so that the distance travelled through the air is maximum. Show that θ_0 satisfies

$$\sin\theta_0\left[\ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right)\right] = 1$$

1.10 A particle of mass m is placed against the wall of a spinning drum. If the bottom of the drum is open, determine the minimum angular velocity required so that the particle stays pinned against the wall of the drum.

1.11 In example 1.10, suppose we want the drum to rotate at a speed of 2 revolutions per second and still be able to remove the bottom safely by: (a) changing the radius but keeping the coefficient of friction the same, and (b) changing the coefficient of friction, but keeping the radius the same. What are the values of the radius and coefficient of friction in the two cases?

1.12 Two blocks of masses m and M are connected by a string and pass over a frictionless pulley. Mass m hangs vertically and mass M moves on an inclined plane that makes an angle θ with the horizontal. If the coefficient of kinetic friction is μ, calculate the angle θ for which the blocks move with uniform velocity.

1.13 A particle of mass m slides on the surface of a fixed, smooth sphere of radius R. The particle starts at the top of the sphere with horizontal initial velocity u_x. Show that it leaves the sphere at an angular position θ (measured from the top of the sphere) given by

$$\theta = \cos^{-1}\left(\frac{2}{3} + \frac{u_x^2}{3Rg}\right).$$

1.14 Two particles of masses m_1 and m_2 at a distance R from each other are under the influence of an attractive force F. If the two masses undergo uniform circular motion about each other with an angular velocity ω, show that

$$F = \left(\frac{m_1 m_2}{m_1 + m_2}\right)\omega^2 R$$

1.15 (a) A particle is moving under the influence of a linear force, $F(x) = -kx$. Show that its equation of motion can be written as

$$t = \int \frac{dx}{\sqrt{\frac{2E}{m}\left(1 - \frac{k}{2E}x^2\right)}}$$

(b) Solve the equation to $x = A\sin(\omega t + \phi_0)$ [Hint: Put $\sqrt{\frac{k}{2E}}x = \sin\theta$ to solve the equation].

1.16 A particle of mass m moves in a central, isotropic force field $F(r)$. Show that the condition for a circular orbit of radius r_0 is, $F(r_0) = -\frac{L^2}{mr_0^3}$.

1.17 A particle of mass m describes an ellipse under a force of magnitude $\frac{m\mu}{r^2}$ directed towards the focus, where μ is a constant. Show that if the particle is moving with a speed v when it is at a distance x from the force centre, the period of motion is

$$T = \frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{x} - \frac{v^2}{\mu}\right]^{-3/2}.$$

1.18 A particle of mass m is oscillating in the potential $V = m\omega_0^2\left(\frac{x^2 - bx^4}{2}\right)$. Show that the period for oscillation of amplitude a is

$$T = \frac{2}{\omega_0}\int_{-a}^{a}\left(a^2 - x^2\right)^{-\frac{1}{2}}\left[1 - b\left(a^2 + x^2\right)\right]^{-\frac{1}{2}} dx.$$

1.19 For a head-on collision of two spheres, obtain the velocities after impact as a function of the initial velocities and masses of the spheres. Also determine the coefficient of restitution.

1.20 For a collision between two particles of masses m_1 and m_2, show that the energy released by the collision is $H=\frac{1}{2}\mu\left(1-e^2\right)\left(u_2-u_1\right)^2$, where e is the coefficient of restitution, u_1 and u_2 are the initial velocities of m_1 and m_2 respectively.

1.21 Show that in the limit of weak damping the energy of an underdamped oscillator is given by; $E(t)=E_0 e^{2t/\tau}$, where $E_0=\frac{1}{2}m\omega_0^2 A^2$.

1.22 The equation of motion of a driven damped harmonic oscillator is $\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=10\cos t$. Find the subsequent motion.

1.23 A particle of mass m is suspended from a fixed point O by a string of length a. A second particle of mass M is suspended from the first particle by a string of length b. If a horizontal velocity v is suddenly imparted to the mass m, show that the tension in the strings are immediately increased by amounts that are in the ratio, $\left[1+\frac{mb}{M(a+b)}\right]:1$

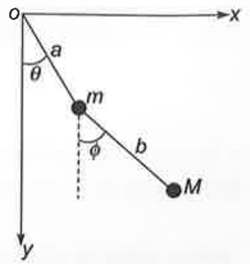

Fig. 1.18

1.24 A double pendulum consists of an inextensible string of negligible mass and length $2l$ with one end fixed and masses m attached at the midpoint and the other end. Show that for small planar oscillations the general solutions for the horizontal displacements of the masses from their equilibrium positions are

$$x_1(t)=A\cos(\omega_1 t+\phi_1)+B\cos(\omega_2 t+\phi_2) \text{ and}$$

$$x_2(t)=\left(\sqrt{2}+1\right)A\cos(\omega_1 t+\phi_1)-\left(\sqrt{2}-1\right)B\cos(\omega_2 t+\phi_2)$$

where $\quad \omega_0=\sqrt{\frac{l}{g}}$, $\omega_1=\sqrt{2-\sqrt{2}}\omega_0$ and $\omega_2=\sqrt{2+\sqrt{2}}\omega_0$.

1.25 For the previous problem obtain the expression for kinetic energy as a function of $\theta,\dot{\theta},\phi$ and $\dot{\phi}$.

1.26 A particle of mass m is connected to a fixed point P on a horizontal plane by an inextensible string of length a. The plane rotates with a constant angular velocity ω about a vertical axis passing through a point O on the plane such that $OP = b$. Find the equations of motion of the particle.

1.27 Show that the force field $F=e^y\hat{i}+\left(z+xe^y\right)\hat{j}+(1+y)\hat{k}$ is a conservative force field and determine the potential energy relative to the origin.

1.28 Consider two particles orbiting about one another and having masses m_1 and m_2. If the force between the two is given by, $\vec{F}=k^2\left(\vec{r}_1-\vec{r}_2\right)$. Show that the orbit of one particle about the other is an ellipse with one particle at the centre of the ellipse.

1.29 For a planet revolving around the sun show that the square of the time period of revolution is proportional to the cube of the semi-major axis (Kepler's third law).

1.30 A particle falls from a finite distance towards the centre of the potential $U = -\alpha r^{-n}$. Will it make a finite number of revolutions around the centre? Will it take finite time to fall towards the centre? Find the equation of the orbit.

1.31 Obtain ideal gas law from the Virial theorem.

1.32 Consider two frictionless blocks of masses m each connected with a rigid massless rod of length a as shown in the figure. Apply the principle of virtual work to determine the force F_2 if the system is in static equilibrium.

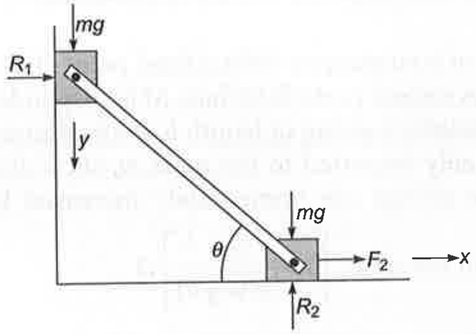

Fig. 1.19

1.33 A particle of mass m is suspended by a massless wire of length $r = a + b\cos\omega t$ to form a spherical pendulum. Find the equations of motion using *D' Alembert's* principle.

1.34 A thin uniform rod of mass m and length l is constrained to move in the xy-plane with the end A remaining on the x-axis. Taking (x, θ) as the generalized coordinates, find the expressions for the kinetic energy and the generalized momentum p_θ.

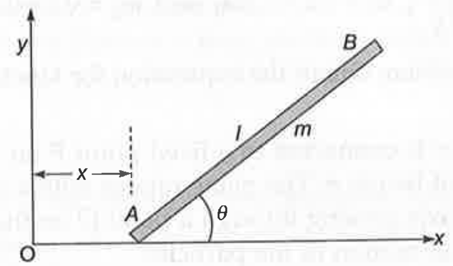

Fig. 1.20

1.35 Consider a uniaxial bar of length l with constant cross section A and Young's modulus Y, fixed at one end and subjected to a force F at the other. Use the principle of virtual work to show that the displacement at the loaded end is $\delta = \dfrac{Fl}{YA}$.

2
CHAPTER

Lagrangian Formulation

CONCEPTS AND FORMULAE

2.1 CONFIGURATION SPACE

The configuration of a system with n degrees of freedom can be specified by a set of n generalized coordinates. These sets of generalized coordinates form an n dimensional Cartesian space. Such a space is called a configuration space.

In a configuration space, a system is represented by a point called the system point. Then the motion of the system between any two instants of time is represented by the trajectory followed by the system point in the configuration space.

2.2 CALCULUS OF VARIATION

The technique of calculus of variation is mainly used in mechanics to determine the extremum path for a given mechanical problem. For the path to be an extremum, the function $f(x,y)$ representing the path must satisfy the following *Euler-Lagrangian* equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)-\frac{\partial f}{\partial y}=0 \tag{2.1}$$

2.3 LAGRANGIAN OF A SYSTEM

Lagrangian of a system is given by

$$L(q_j,\dot{q}_j)=T(q_j,\dot{q}_j)-V(q_j) \tag{2.2}$$

where, T is the kinetic energy and V, the potential energy of the system with q_j and $\dot{q}_j$ as the generalized coordinates and generalized velocity of the system.

2.4 HAMILTON'S VARIATIONAL PRINCIPLE

Hamilton's variational principle states that of all the kinematically possible motions that take a mechanical system from one given configuration to another within a given time interval, the actual motion is the one that minimises the time integral of the Lagrangian of the system. This can be expressed mathematically as

$$\delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j) dt = 0 \tag{2.3}$$

2.5 LAGRANGE'S UNDETERMINED MULTIPLIERS

Consider a function $f(x,y,z)$ and a constant equation $g(x,y,z)=0$, then; according to the Lagrange's method of undetermined multipliers, the function $f(x,y,z)$ has an extremum when

$$df + \lambda dg = \left(\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z}\right) dz = 0 \tag{2.4}$$

where, λ is Lagrange's undetermined multiplier. Here we select λ such that $\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0$.

Since x and y are independent

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$$

2.6 LAGRANGE'S EQUATIONS OF MOTION (FIRST AND SECOND KIND)

(a) Lagrange's equation of motion of first kind is given by

$$\sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \tag{2.5}$$

(b) For a conservative system, the potential is a function of coordinate only, then

$$Q_j = -\frac{\partial V}{\partial q_j} \tag{2.6}$$

(c) Lagrange's equations of the second kind is given by

$$\sum_j \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

or

$$\sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{2.7}$$

2.7 LAGRANGE'S EQUATIONS OF MOTION FOR A NONCONSERVATIVE SYSTEM

For a nonconservative system, the potential is a function of velocity and is called the generalized potential U. Then the generalized force is given by

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \tag{2.8}$$

Then, the Lagrange's equation of motion of a nonconservative system is

$$\sum_j \frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j} = 0 \tag{2.9}$$

2.8 LAGRANGE'S EQUATIONS OF MOTION FOR A DISSIPATIVE SYSTEM

(a) Rayleigh dissipation function ($\Im$) for a dissipative system is given by

$$\Im = \frac{1}{2}\sum_j k_j \dot{x}_j^2 \tag{2.10}$$

(b) The dissipative force is

$$Q_j^d = F_j^d = -\frac{\partial \Im}{\partial \dot{x}_j} \tag{2.11}$$

(c) Lagrange's equation of motion for a dissipative system is

$$\sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial qj} + \frac{\partial \Im}{\partial \dot{q}_j} = 0 \tag{2.12}$$

2.9 GENERALIZED MOMENTA

The generalized momenta p_j is given by

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{2.13}$$

Then,

$$\dot{p}_j = \frac{\partial L}{\partial q_j} \tag{2.14}$$

2.10 CYCLIC COORDINATES

(a) If the Lagrangian is independent of a coordinate, it is a cyclic coordinate.

(b) If the generalized coordinate q_j is cyclic in Lagrangian, then $\dot{p}_j = \frac{\partial L}{\partial q_j} = 0$, or p_j is a constant. Thus, if a coordinate is cyclic in Lagrangian, then the corresponding momenta is a constant of motion.

2.11 ELIMINATION OF CYCLIC COORDINATES

(a) The elimination of cyclic coordinates is achieved through the introduction of *Routhian function*. It does not contain generalized velocities corresponding to the cyclic coordinates.

(b) If the coordinates $q_1, q_2, ..., q_k$ are cyclic in Lagrangian, then the Routhian function is

$$R \equiv R(q_{k+1}, q_{k+2}, ..., q_n; \dot{q}_{k+1}, \dot{q}_{k+2}, ..., \dot{q}_n) \tag{2.15}$$

(c) Lagrange's equation, then becomes

$$\sum_{j=k+1}^{n} \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) - \frac{\partial R}{\partial q_j} = 0 \tag{2.16}$$

2.12 NOETHER'S THEOREM

Noether's theorem states that for each symmetry of the Lagrangian, there is a conserved quantity.

In the theorem the term *symmetry* means that if the coordinates are changed by some small quantities, then the Lagrangian has no first-order change in these quantities. That is the Lagrangian is invariant for an infinitesimal change in a coordinate given by

$$q_j' = q_j + \varepsilon \eta_j(q_i) \ , \ i = 1, 2, ..., n. \tag{2.17}$$

2.13 GAUGE INVARIANCE OF LAGRANGIAN

If two Lagrangian functions L and L' connected through gauge transformation given by

$$L' = L + \frac{dF}{dt} \tag{2.18}$$

with $F \equiv F(q_1, q_2, ..., q_n; t)$, then, L' also satisfies the Lagrange's equations of motion.

SOLVED PROBLEMS

EXAMPLE 2.1 Find the curve which represents the extremum of the integral $I = \int_1^2 \frac{\dot{y}^2}{4x} dx$.

Solution: The extremum of the given integral is the solution to the Euler-Lagrangian equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0 \tag{i}$$

For the given integral, $f = \frac{\dot{y}^2}{4x}$ and therefore

$$\frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{2x} \text{ and } \frac{\partial f}{\partial y} = 0 \tag{ii}$$

Then equation (i) becomes

$$\frac{d}{dx}\left(\frac{\dot{y}}{2x}\right) = 0$$

or $$\frac{\dot{y}}{2x} = c \text{, a constant} \tag{iii}$$

This can be rearranged to get

$$dy = 2cxdx$$

Then, $$y = \int 2cxdx = cx^2 + b \tag{iv}$$

This represents a family of parabola in the x-y plane.

EXAMPLE 2.2 From Fermat's principle, obtain Snell's law.

Solution: Fermat's principle states that of all the possible paths that a light ray might take between two fixed points, the actual path is the one that minimises the travel time of the ray.

Let us consider the general case of an inhomogeneous medium whose refractive index n varies continuously in the y-direction.

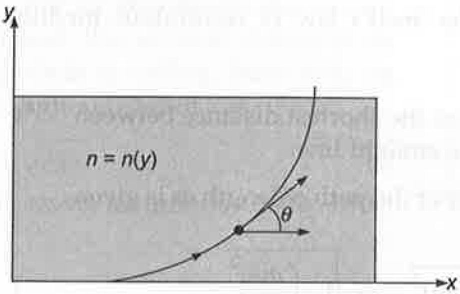

Fig. 2.1

A small segment of length ds on the path of the light is

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + \dot{y}^2} \tag{i}$$

Then the time taken to travel a distance ds is

$$dt = \frac{ds}{v} = \frac{n}{c}dx\sqrt{1 + \dot{y}^2} \tag{ii}$$

Time of travel through the medium can be obtained by integrating the above equation.

$$T = \frac{1}{c}\int_1^2 n\sqrt{1 + \dot{y}^2}\,dx = \frac{1}{c}\int_1^2 f dx \tag{iii}$$

According to Fermat's principle T is an extremum and hence the function, $f = n\sqrt{1 + \dot{y}^2}$ must satisfy the Euler-Lagrangian equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)-\frac{\partial f}{\partial y}=0 \qquad \text{(iv)}$$

Now,
$$\frac{\partial f}{\partial \dot{y}}=\frac{n\dot{y}}{\sqrt{1+\dot{y}^2}} \quad \text{and} \quad \frac{\partial f}{\partial y}=0$$

Then,
$$\frac{d}{dx}\left(\frac{n\dot{y}}{\sqrt{1+\dot{y}^2}}\right)=0$$

or
$$\frac{n\dot{y}}{\sqrt{1+\dot{y}^2}}=\text{Constant} \qquad \text{(v)}$$

Now, let us put $\dot{y}=\frac{dy}{dx}=\tan\theta$, as it follows from the figure, then equation (v) becomes

$$n\cos\theta = \text{ Constant.} \qquad \text{(vi)}$$

This is nothing but the Snell's law of continuous medium. Note that the angle of incidence here is $(90-\theta)$.

EXAMPLE 2.3 Show that the shortest distance between two points on a plane is a straight line.

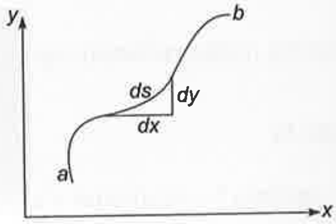

Fig. 2.2

Solution: A small segment of the path of length ds is given by

$$ds=\sqrt{(dx)^2+(dy)^2}=dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Now, let us denote, $\frac{dy}{dx}\equiv\dot{y}$, then

$$ds=\sqrt{1+\dot{y}^2}\,dx \qquad \text{(i)}$$

The total length of the path between the initial and final points can be represented by the integral

$$I=\int_a^b ds=\int_a^b \sqrt{1+\dot{y}^2}\,dx=\int_a^b f dx \qquad \text{(ii)}$$

For the path to be shortest the δ variation of the above integral must be equal to zero. In other words, the function $f=\sqrt{1+\dot{y}^2}$ must satisfy the equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)-\frac{\partial f}{\partial y}=0 \qquad \text{(iii)}$$

Now, we have $\frac{\partial f}{\partial \dot{y}}=\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$ and $\frac{\partial f}{\partial y}=0$. Substituting in (iii), we get

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right)=0$$

or

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = \text{a constant} \qquad \text{(iv)}$$

Then it follows that

$$\dot{y} = m\text{, a constant} \qquad \text{(v)}$$

On integration the above expression becomes

$$y = mx + c \qquad \text{(vi)}$$

which is the equation of a straight line.

EXAMPLE 2.4 A particle is falling from rest under the influence of gravity. Find the curve joining two points for which the time taken by the particle is minimum.

Solution: This problem is known popularly as the *branchistochrone problem.* Let us choose the horizontal direction as y-direction and the vertical direction as x-direction. Since the particle is falling from rest, its velocity when it falls through a distance x is

$$v = \sqrt{2gx} \qquad \text{(i)}$$

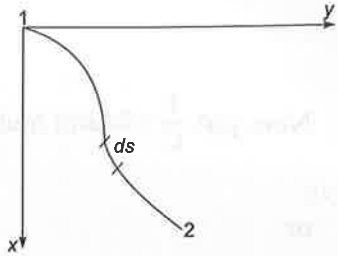

Fig. 2.3

Now, the time taken to travel an infinitesimally small distance ds is

$$dt = \frac{ds}{\sqrt{2gx}}$$

Then the time taken to travel from the initial to final point becomes

$$t = \int_1^2 dt = \int_1^2 \frac{ds}{\sqrt{2gx}} = \int_1^2 \frac{\sqrt{(dx)^2+(dy)^2}}{\sqrt{2gx}}$$

$$= \int_1^2 \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gx}}dx = \int_1^2 \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gx}}dx = \int_1^2 f dx \qquad \text{(ii)}$$

with
$$f = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gx}} \qquad \text{(iii)}$$

Now, for the time t to be minimum, the function f must satisfy the equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0 \qquad \text{(iv)}$$

From (iii), $\dfrac{\partial f}{\partial \dot{y}} = \dfrac{1}{\sqrt{2gx}} \dfrac{\dot{y}}{\sqrt{1+\dot{y}^2}}$ and $\dfrac{\partial f}{\partial y} = 0$. Then equation (iv) becomes

$$\frac{d}{dx}\left(\frac{1}{\sqrt{2gx}} \frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0$$

or

$$\frac{1}{\sqrt{2gx}} \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = const$$

On squaring

$$\frac{\dot{y}^2}{x\left(1+\dot{y}^2\right)} = C \tag{v}$$

This can be rearranged to get

$$\dot{y} = \frac{\sqrt{x}}{\sqrt{\dfrac{1}{C} - x}} \tag{vi}$$

Now, put $\dfrac{1}{C} = 2a$ and multiply numerator and denominator by $\sqrt{x}$, then we get

$$\dot{y} = \frac{dy}{dx} = \frac{x}{\sqrt{2ax - x^2}}$$

or

$$y = \int \frac{xdx}{\sqrt{2ax - x^2}} \tag{vii}$$

The integration can be performed by putting, $x = a(1-\cos\theta)$, and then we get

$$y = a(\theta - \sin\theta) \tag{viii}$$

Here, the integration constant is taken as zero by assuming the initial point as the origin. These are the parametric equations of a cycloid passing through the origin.

EXAMPLE 2.5 Use the technique of calculus of variation to find the curve for which the area of the surface of revolution is minimum.

Solution: Imagine a curve with end points (x_1, y_1) and (x_2, y_2) in the *x-y* plane. The surface of revolution is formed when this curve is revolved around (say) y-axis. A schematic representation is given below. Our aim is to find out the curve for which the surface area is a minimum.

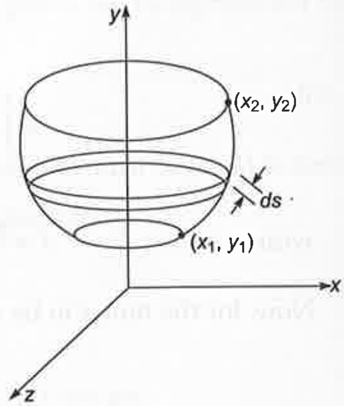

Fig. 2.4

Consider a small segment *ds* on the line of revolution. Now the revolution of the segment *ds* results in a strip whose length is $2\pi x$ and thickness dx.

Then, the area of revolution of the segment ds is

$$dA = 2\pi x ds = 2\pi x\sqrt{1+\dot{y}^2} \tag{i}$$

Therefore, the total area of the surface of revolution is

$$A = 2\pi\int_{x_1}^{x_2} x\sqrt{1+\dot{y}^2}\,dx \tag{ii}$$

This integral is an extremum if the function $f = x\sqrt{1+\dot{y}^2}$ satisfies the equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0 \tag{iii}$$

Now, $\dfrac{\partial f}{\partial \dot{y}} = \dfrac{x\dot{y}}{\sqrt{1+\dot{y}^2}}$ and $\dfrac{\partial f}{\partial y} = 0$, then

$$\frac{d}{dx}\left(\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0$$

or

$$\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = a\text{, a constant.} \tag{iv}$$

On rearranging the above expression, we get

$$\dot{y} = \frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}$$

that is, $$dy = \frac{a}{\sqrt{x^2-a^2}}dx$$

Then, $$y = \int\frac{a}{\sqrt{x^2-a^2}}dx = a\cosh^{-1}\frac{x}{a} + b \tag{v}$$

or $$x = a\cosh\left(\frac{y-b}{a}\right) \tag{vi}$$

where b is the constant of integration. Equation (vi) represents the curve for which the surface of revolution is minimum and the curve is a catenary.

EXAMPLE 2.6 Show that the shortest distance between two points on the surface of a sphere is the arc of the great circle connecting the two points.

Solution: The line that represents the shortest path between any two points when the path is restricted to a particular surface is called a *geodesic*.

We apply the technique of calculus of variation to solve the problem. Consider a sphere of radius r, then a small segment of the arc on the surface of the sphere can be written as

$$ds = \sqrt{r^2 (d\theta)^2 + r^2 \sin^2\theta (d\phi)^2} \quad \text{(i)}$$

Then, the distance between the two points on the surface of the sphere is

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left[r^2 + r^2 (\sin^2\theta)\dot{\phi}^2\right]} d\theta \quad \text{(ii)}$$

where $\dot{\phi} = \dfrac{d\phi}{d\theta}$.

For the length of the arc to be an extremum, the function $f(\theta, \dot{\phi}) = \sqrt{\left[r^2 + r^2 (\sin^2\theta)\dot{\phi}^2\right]}$ must satisfy the Euler-Lagrangian equation, i.e., $\dfrac{d}{d\theta}\left(\dfrac{\partial f}{\partial \dot{\phi}}\right) - \dfrac{\partial f}{\partial \phi} = 0$ (iii)

Now, $\dfrac{\partial f}{\partial \dot{\phi}} = \dfrac{r^2 \sin^2\theta \dot{\phi}}{\sqrt{\left[r^2 + r^2 (\sin^2\theta)\dot{\phi}^2\right]}}$ and $\dfrac{\partial f}{\partial \phi} = 0$.

Substituting these in (iii), we get

$$\frac{d}{d\theta}\left(\frac{r^2 \sin^2\theta \dot{\phi}}{\sqrt{\left[r^2 + r^2 (\sin^2\theta)\dot{\phi}^2\right]}}\right) = 0$$

or

$$\frac{r^2 \sin^2\theta \dot{\phi}}{\sqrt{\left[r^2 + r^2 (\sin^2\theta)\dot{\phi}^2\right]}} = \alpha \text{, a constant} \quad \text{(iv)}$$

Cross-multiplying and squaring the above equation and then rearranging, we get

$$\dot{\phi} = \frac{d\phi}{d\theta} = \frac{\alpha}{\sqrt{\left(r^2 \sin^4\theta - \alpha^2 \sin^2\theta\right)}}$$

Then,

$$d\phi = \frac{\alpha}{\sqrt{\left(r^2 \sin^4\theta - \alpha^2 \sin^2\theta\right)}} d\theta$$

or

$$\phi = \alpha \int \frac{1}{\sqrt{\left(r^2 \sin^4\theta - \alpha^2 \sin^2\theta\right)}} d\theta + \beta \quad \text{(v)}$$

where, β is the constant of integration.

Now, equation (v) can be rewritten as

$$\phi = \alpha \int \frac{\operatorname{cosec}^2\theta}{\sqrt{r^2 - \alpha^2 \operatorname{cosec}^2\theta}} d\theta + \beta \quad \text{(vi)}$$

Now, put $x = \alpha \cot\theta$ so that, $dx = -\alpha \operatorname{cosec}^2\theta d\theta$ (vii)

Also, $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \dfrac{x^2}{\alpha^2}$

With these substitutions, equation (vi) becomes

$$\phi = -\int \frac{dx}{\sqrt{r^2 - \alpha^2\left(1 + \dfrac{x^2}{\alpha^2}\right)}} + \beta$$

$$= -\int \frac{dx}{\sqrt{r^2 - \alpha^2 - x^2}} + \beta = -\int \frac{dx}{\sqrt{b^2 - x^2}} + \beta \qquad \text{(viii)}$$

where, $b^2 = r^2 - \alpha^2$.

Equation (vii) can be integrated to obtain

$$\phi = \cos^{-1}\left(\frac{x}{b}\right) + \beta$$

or $\qquad x = b\cos(\phi - \beta)$

But, $\qquad x = \alpha \cot\theta$ by (vii) and therefore

$$\alpha \cot\theta = b\cos(\phi - \beta)$$

$$\alpha \cos\theta = b\cos(\phi - \beta)\sin\theta$$

$$= b(\cos\phi\cos\beta + \sin\phi\sin\beta)\sin\theta \qquad \text{(ix)}$$

To interpret this result, let us use the relation between Cartesian and polar coordinate systems. These two systems are related through

$$x = r\cos\theta\sin\phi, \; y = r\sin\theta\sin\phi \text{ and } z = r\cos\theta$$

Substituting these in (ix) and rearranging, we get

$$\alpha z = (b\cos\beta)x + (b\sin\beta)y = Ax + By \qquad \text{(x)}$$

This equation represents a plane that passes through the centre of the sphere and cuts the surface of the sphere in a great circle. Thus, the shortest distance between two points on the surface of a sphere is the arc of the great circle connecting the two points.

EXAMPLE 2.7 From D'Alembert's principle, obtain Lagrange's equations of motion.

Solution: The mathematical statement of D'Alembert's principle is

$$\sum_i (F_i^e - \dot{P}_i).\delta r_i = 0 \qquad \text{(i)}$$

The first term can be written as

$$\sum_i F_i^e.\delta r_i = \sum_i F_i^e.\frac{\partial r_i}{\partial q_j}\delta q_j = \sum_i Q_j \delta q_j \qquad \text{(ii)}$$

The second term can be rewritten as

$$\sum_i \dot{P}_i.\delta r_i = \sum_{i,j} m_i \ddot{r}_i.\frac{\partial r_i}{\partial q_j}\delta q_j \tag{iii}$$

Now use, $\frac{d}{dt}\left(\dot{r}_i.\frac{\partial r_i}{\partial q_j}\right) = \ddot{r}_i.\frac{\partial r_i}{\partial q_j} + \dot{r}_i.\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)$ to rewrite the RHS of (iii). This would yield

$$\sum_i \dot{P}_i.\delta r_i = \sum_{i,j} m_i \left[\frac{d}{dt}\left(\dot{r}_i.\frac{\partial r_i}{\partial q_j}\right) - \dot{r}_i.\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)\right]\delta q_j$$

$$= \sum_{i,j} m_i \left[\frac{d}{dt}\left(\dot{r}_i.\frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) - \dot{r}_i.\left(\frac{\partial \dot{r}_i}{\partial q_j}\right)\right]\delta q_j$$

Now, from the expression for the kinetic energy, $T = \frac{1}{2}\sum_i m_i \dot{r}_i^2$, we can write

$$\frac{\partial T}{\partial q_j} = \sum_i m_i \dot{r}_i.\frac{\partial \dot{r}_i}{\partial q_j} \quad \text{and} \quad \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{r}_i.\frac{\partial \dot{r}_i}{\partial \dot{q}_j}$$

With these substitutions, (i) becomes

$$\sum_i \dot{P}_i.\delta r_i = \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j \tag{iv}$$

Equations (ii) and (iv) can be substituted in (i) to get

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right]\delta q_j = 0 \tag{v}$$

Now, for holonomic systems, the generalized coordinates are independent and hence the coefficients of δq_j in the above equation must vanish separately. Then,

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] = \sum_j Q_j \tag{vi}$$

For conservative system, the generalized force is derivable from a potential which is independent of velocity, i.e., $Q_j = -\frac{\partial V}{\partial q_j}$. Therefore, for a conservative system, the above equation becomes

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j}\right] = 0 \tag{vii}$$

or
$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right] = 0 \tag{viii}$$

where, $L = T - V$, the Lagrangian of the system. Equation (vii) is the Lagrange's equations of motion.

Note: For a nonconservative system, $L = T - U$, where U is the generalized potential with, $Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$.

EXAMPLE 2.8 From Hamilton's variational principle, obtain Lagrange's equations of motion.

Solution: Let $L(q_j, \dot{q}_j, t)$ be the Lagrangian of the system. According to Hamilton's principle the motion of a conservative system from an initial time t_1 to a final time t_2 is such that the integral, $I = \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt$ is an extremum. This is expressed mathematically as

$$\delta I = \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt = 0 \tag{i}$$

Now, let us introduce the parameter α to represent all possible paths from the initial point to the final point; the value of α being different for different paths. Then the integral becomes a function of α since $q_j \equiv q_j(\alpha, t)$. Now the integral I can be written as

$$I(\alpha) = \int_{t_1}^{t_2} L(q_j(\alpha,t), \dot{q}_j(\alpha,t), t)\,dt \tag{ii}$$

This expression can be integrated w.r.t. α to get

$$\frac{\partial I(\alpha)}{\partial \alpha} = \int_{t_1}^{t_2} \sum_j \left(\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \alpha} + \frac{\partial L}{\partial t}\frac{\partial t}{\partial \alpha} \right) dt \tag{iii}$$

Since there is no time variation along any path, $\frac{\partial t}{\partial \alpha} = 0$ for all paths. Then the above equation after multiplying with $d\alpha$ becomes

$$\frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} d\alpha dt + \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \alpha} d\alpha$$

$$= \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} d\alpha dt + \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial^2 q_j}{\partial t \partial \alpha} d\alpha$$

The second term can be integrated by parts to get

$$\frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial \alpha} d\alpha dt + \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \alpha} \Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \frac{\partial q_j}{\partial \alpha} d\alpha dt$$

The middle term on RHS is equal to zero since the end points are fixed. Therefore,

$$\frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial \alpha} d\alpha dt - \int_{t_1}^{t_2} \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \frac{\partial q_j}{\partial \alpha} d\alpha dt$$

$$= \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right] \frac{\partial q_j}{\partial \alpha} d\alpha dt$$

or $$\delta I(\alpha) = \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right] \delta q_j dt \qquad \text{(iv)}$$

Since the generalized coordinates are independent, δq_j will be independent and cannot be zero identically. Therefore; for $\delta I(\alpha) = 0$, the coefficients of δq_j must vanish. Then,

$$\sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right] = 0$$

or $$\sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = 0 \qquad \text{(v)}$$

These are the Lagrange's equations of motion for a conservative system.

EXAMPLE 2.9 Deduce Newton's second law of motion from Hamilton's variational principle.

Solution: Consider a particle of mass m moving under the influence of a force field F. The kinetic energy of the particle is

$$T = \frac{1}{2} m\dot{r}^2 \qquad \text{(i)}$$

Then, the variation in kinetic energy is

$$\delta T = m\dot{r}.\delta\dot{r} \qquad \text{(ii)}$$

The variation in the potential energy is

$$\delta V = -F.\delta r \qquad \text{(iii)}$$

Now, the Hamilton's variational principle is

$$\delta \int_{t}^{t_2} L dt = \delta \int_{t}^{t_2} (T - V) dt = \int_{t}^{t_2} (\delta T - \delta V) dt = 0$$

that is, $$\int_{t_1}^{t_2}(m\dot{r}.\delta\dot{r}+F.\delta r)dt=\int_{t_1}^{t_2}m\dot{r}.\delta\dot{r}dt+\int_{t_1}^{t_2}F.\delta rdt=0$$

The first term can be integrated by parts to get

$$m\dot{r}.\delta r\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}m\ddot{r}.\delta rdt+\int_{t_1}^{t_2}F.\delta rdt=0 \tag{iv}$$

The first term vanishes since the end points are fixed, $\delta r=0$ at the end points. Then,

$$\int_{t_1}^{t_2}(F-m\ddot{r}).\delta rdt=0 \tag{v}$$

In this equation δr cannot vanish identically since the equation is true for every virtual displacement δr. Then it follows that

$$F-m\ddot{r}=0 \quad \text{or,} \quad F=m\ddot{r} \tag{vi}$$

Equation (v_i) is Newton's second law of motion.

EXAMPLE 2.10 Assume that a particle travels along a sinusoidal path from a point $x=0$ to $x=x_1$ during a time interval Δt in a force free region of space. Use Hamilton's principle to show that the amplitude of the assumed sinusoidal path is zero, implying that the path the particle takes is really a straight line between the two points.

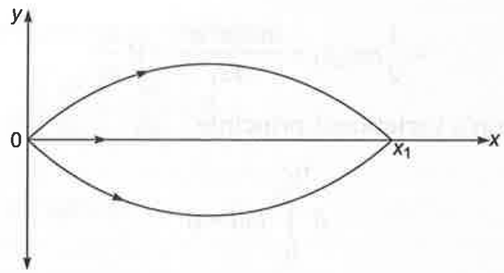

Fig. 2.5

Solution: Let the particle start from the origin and move with a velocity v_x in the x-direction. Since the particle is moving in a force free region of space, for all possible paths the time taken must be the same. That is,

$$\Delta t=\frac{x_1}{v_x} \tag{i}$$

The instantaneous amplitude of the path in the y-direction can be written as

$$y=\pm\alpha\sin\left(\pi\frac{v_x t}{x_1}\right) \tag{ii}$$

where α is the parameter that determines the amplitude of the sinusoidal paths. Different values of α define different paths.

Then, the kinetic energy of the particle can be written as sum of kinetic energies of linear motion and the kinetic energy of wave motion. That is,

$$T = \frac{1}{2}mv_x^2 + \frac{1}{2}m\left(\pi\frac{v_x}{x_1}\right)^2 \alpha^2 \cos^2\left(\pi\frac{v_x t}{x_1}\right) \tag{iii}$$

The Lagrangian of the particle is

$$L = T - V = \frac{1}{2}m\left[v_x^2 + \left(\pi\frac{\alpha v_x}{x_1}\right)^2 \cos^2\left(\pi\frac{v_x t}{x_1}\right)\right] - V \tag{iv}$$

Here the potential energy V is assumed to be constant as the particle is moving in a force free space.

Now,

$$I = \int_0^{x_1/v_x} L dt = \int_0^{x_1/v_x} \left\{\frac{1}{2}m\left[v_x^2 + \left(\pi\frac{\alpha v_x}{x_1}\right)^2 \cos^2\left(\pi\frac{v_x t}{x_1}\right)\right] - V\right\} dt$$

$$= \frac{1}{2}mv_x^2 \int_0^{x_1/v_x} dt + \frac{1}{2}m\left(\pi\frac{\alpha v_x}{x_1}\right)^2 \int_0^{x_1/v_x} \cos^2\left(\pi\frac{v_x t}{x_1}\right) - V\int_0^{x_1/v_x} dt$$

$$= \frac{1}{2}mv_x x_1 + \frac{mv_x \pi^2 \alpha^2}{4x_1} - V\frac{x_1}{v_x} \tag{v}$$

According to Hamilton's variational principle

$$\delta \int_0^{x_1/v_x} L dt = 0 \tag{vi}$$

Now, taking the variation of equation (v) w.r.t. α and using (vi), we get

$$\frac{\pi^2 m v_x \alpha}{2x_1}\delta\alpha = 0 \tag{vii}$$

Since $\delta\alpha$ cannot be equal to zero, for the above equation to be valid, we must have $\alpha = 0$. Then the actual path of the particle is represented by the equation $x = v_x t$, which represents a straight line between the initial and final points.

EXAMPLE 2.11 Obtain Hamilton's variational principle from Lagrange's equation of motion.

Solution: Consider a holonomic, conservative system. Then the Lagrangian of the system is $L = L(q_j, \dot{q}_j)$. Now, take the δ variation in the Lagrangian to get

$$\delta L = \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j \tag{i}$$

Now,
$$\delta\int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\right]dt$$

$$= \int_{t_1}^{t_2} \frac{\partial L}{\partial q_j}\delta q_j dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j dt \tag{ii}$$

The second term can be integrated by parts to get

$$\delta\int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial q_j}\delta q_j dt + \frac{\partial L}{\partial \dot{q}_j}\delta q_j\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j$$

The second term on the RHS is zero since the end points are fixed. Therefore,

$$\delta\int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial q_j}\delta q_j dt - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j$$

or

$$\delta\int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j dt \tag{iii}$$

Now, the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Therefore,
$$\delta\int_{t_1}^{t_2} L dt = 0 \tag{iv}$$

which is Hamilton's variational principle for a conservative system.

EXAMPLE 2.12 A particle of mass m is constrained to move inside a rectangular box of sides a, b and c.

The ground state energy of the particle is given by

$$E = \frac{\hbar^2}{8m}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

Find the shape of the box that minimizes the energy keeping the volume constant.

Solution: This problem can be solved using the method of Lagrange's undetermined multipliers as it involves a constraint.

The energy is given by the expression

$$E = \frac{\hbar^2}{8m}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \tag{i}$$

The constraint equation can be written as

$$xyz = V, \text{ a constant} \tag{ii}$$

According to the Lagrange's method of undetermined multipliers, the function E is an extremum if

$$\frac{\partial E}{\partial x} + \lambda \frac{\partial V}{\partial x} = 0$$

$$\frac{\partial E}{\partial y} + \lambda \frac{\partial V}{\partial y} = 0 \text{ and;}$$

$$\frac{\partial E}{\partial z} + \lambda \frac{\partial V}{\partial z} = 0$$

This gives

$$-\frac{\hbar^2}{4mx^3} + \lambda yz = 0, \tag{iii}$$

$$-\frac{\hbar^2}{4my^3} + \lambda xz = 0 \text{ and;} \tag{iv}$$

$$-\frac{\hbar^2}{4mz^3} + \lambda xy = 0 \tag{v}$$

Now, by multiplying (iii), (iv) and (v) with x, y and z respectively, we get

$$\lambda xyz = \frac{\hbar^2}{4mx^2} = \frac{\hbar^2}{4my^2} = \frac{\hbar^2}{4mz^2} \tag{vi}$$

This is true only if $x = y = z$, and therefore, the box should be a cube.

EXAMPLE 2.13 A point mass m is moving on an inclined plane which makes an angle α with the horizontal.

Obtain the equation of motion and the constraint force by the method of Lagrange's undetermined multipliers.

Solution: Let the inclined plane be in the xz-plane as shown in the figure. The x-direction is taken as the horizontal.

From the figure we can write a constraint equation as

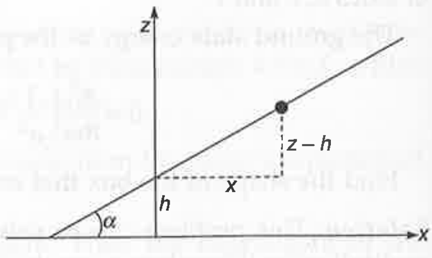

Fig. 2.6

$$z - h(t) = x\tan\alpha$$

or
$$f \equiv z - h(t) - x\tan\alpha = 0 \tag{i}$$

so that
$$\frac{\partial f}{\partial x} = -\tan\alpha\,,\ \frac{\partial f}{\partial y} = 0 \text{ and } \frac{\partial f}{\partial z} = 1$$

The constraint is holonomic and rheonomic. Now, the kinetic energy of the particle is $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$ and the potential energy is $V = mgz$. Therefore, the Lagrangian is;

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz \tag{ii}$$

The Lagrange's equations of motion are

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= \lambda\frac{\partial f}{\partial x} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= \lambda\frac{\partial f}{\partial y} \end{aligned}\right\}$$

and
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = \lambda\frac{\partial f}{\partial z} \tag{iii}$$

Using (i) and (ii) in (iii), we get

$$\left.\begin{aligned} m\ddot{x} &= -\lambda\tan\alpha \\ m\ddot{y} &= 0 \\ \text{and} \quad m\ddot{z} + mg &= \lambda \end{aligned}\right\} \tag{iv}$$

Now, to obtain the equation of motion, we try to eliminate λ. From (iv), we can write

$$\lambda = -m\ddot{x}\cot\alpha$$

and

$$\lambda = m(\ddot{z} + g)$$

Equating the RHS of these two equations, we get

$$\ddot{x}\cot\alpha + \ddot{z} + g = 0 \tag{v}$$

Now, from (i) by differentiating twice w.r.t. time, we get

$$\ddot{x}\tan\alpha - \ddot{z} + \ddot{h} = 0 \tag{vi}$$

Equations (v) and (vi) can be added to eliminate $\ddot{z}$ and to get

$$\ddot{x}(\cot\alpha + \tan\alpha) + g + \ddot{h} = 0$$

or
$$\ddot{x} = -\frac{\left(g + \ddot{h}\right)}{(\cot\alpha + \tan\alpha)} = -\left(g + \ddot{h}\right)\cos\alpha\sin\alpha \tag{vii}$$

This is the equation of motion and can be integrated to get the time dependence of h. Now, let us find λ. We have, $\lambda = -m\ddot{x}\cot\alpha$. Using (vii) in this expression, we get

$$\lambda = -m\ddot{x}\cot\alpha = m\left(g+\ddot{h}\right)\cos\alpha\sin\alpha\cot\alpha$$

or
$$\lambda = m\left(g+\ddot{h}\right)\cos^2\alpha \tag{viii}$$

EXAMPLE 2.14 A cylinder of radius r and mass m is rolling down without slipping on an inclined plane of length l and angle of inclination ϕ. Show that the constraint force is $\frac{1}{3}mg\sin\phi$.

Solution: Let us take x and θ as the generalized coordinates, where x is the distance along the inclined plane and θ is the angle of rotation of the cylinder.

Now a small displacement of the cylinder can be represented as; $dx = rd\theta$ which would give the constraint equation; which is nonholonomic, as

$$rd\theta - dx = 0 \tag{i}$$

The general constraint equation for a nonholonomic system with generalized coordinates x, θ is

$$a_x dx + a_\theta d\theta = 0 \tag{ii}$$

From (i) and (ii), we get

$$a_\theta = r \text{ and } a_x = -1 \tag{iii}$$

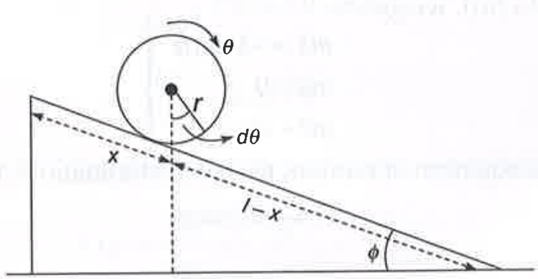

Fig. 2.7

Now, the kinetic energy of the cylinder can be written as the sum of translational and rotational kinetic energies. That is,

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 \tag{iv}$$

where, I is the moment of inertia of the cylinder about its axis.

The potential energy is

$$V = mgh = mg(l-x)\sin\phi \tag{v}$$

Then, the Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 - mg(l-x)\sin\phi \tag{vi}$$

This would give; $\dfrac{\partial L}{\partial \dot{x}} = m\dot{x}$; $\dfrac{\partial L}{\partial x} = mg\sin\phi$

and $$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \;;\; \frac{\partial L}{\partial \theta} = 0$$

Lagrange's equation of motion for x is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \lambda a_x$$

where λ is Lagrange's undetermined multiplier.

that is, $$\frac{d}{dt}(m\dot{x}) - mg\sin\phi = -\lambda$$

or

$$m\ddot{x} - mg\sin\phi = -\lambda \tag{vii}$$

Similarly, Lagrange's equation of motion for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda a_\theta$$

that is, $$\frac{d}{dt}\left(I\dot{\theta}\right) = \lambda r$$

or $$I\ddot{\theta} = \lambda r \text{ or; } \frac{1}{2}mr^2\ddot{\theta} = \lambda r \tag{viii}$$

Now, from (i), $r\ddot{\theta} = \ddot{x}$ and putting this in (viii), we get

$$\lambda = \frac{1}{2}m\ddot{x} \tag{ix}$$

Now, from (vii) $$m\ddot{x} + \lambda - mg\sin\phi = 0$$

or $$m\ddot{x} + \frac{1}{2}m\ddot{x} - mg\sin\phi = 0$$

or $$\frac{3}{2}\ddot{x} - g\sin\phi = 0 \Rightarrow \ddot{x} = \frac{2}{3}g\sin\phi \tag{x}$$

Using (x) in (ix), we get; $\lambda = \dfrac{1}{3}mg\sin\phi$, which is the required result.

EXAMPLE 2.15 A smooth sphere rests on a horizontal plane. A point particle slides frictionlessly down the sphere starting from the top. Calculate the reaction of the sphere on the particle. Also find the height when the particle falls off using Lagrange's undetermined multipliers.

Solution: Let R be the radius of the sphere and m be the mass of the particle. In this problem, so long as the particle remains in contact with the sphere, the constraint is $r = R$, so that $dr = 0$. Therefore, the constraint equation

$$a_r dr + a_\theta d\theta = 0 \tag{i}$$

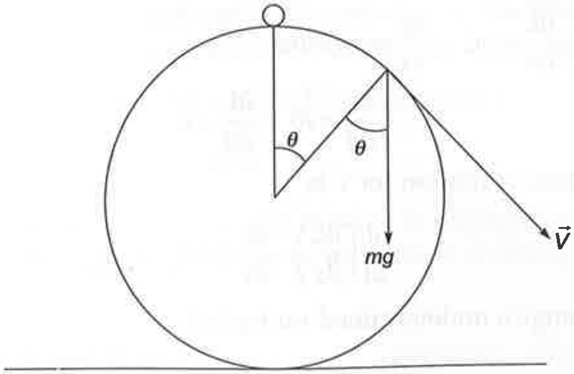

Fig. 2.8

$$a_r = 1 \text{ and } a_\theta = 0 \tag{ii}$$

The kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{iii}$$

and the potential energy is

$$V = mgr\cos\theta \tag{iv}$$

Here r is the height of the particle above the centre of the sphere and $r = R(1-\cos\theta)$. Now, the Lagrangian of the particle is

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta \tag{v}$$

so that $$\frac{\partial L}{\partial \dot{r}} = m\dot{r},\ \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta$$

and $$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta},\ \frac{\partial L}{\partial \theta} = mgr\sin\theta$$

Then, the Lagrange's equations of motion for r, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = a_r\lambda_r$ becomes

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + mg\cos\theta = \lambda_r$$

$$m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda_r \tag{vi}$$

Similarly, the Lagrange's equations of motion for θ, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = a_\theta\lambda_\theta$ becomes

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mgr\sin\theta = 0$$

$$r\ddot{\theta} - g\sin\theta = 0 \tag{vii}$$

Now, with $r=R$, $\dot{r}=0$ and (vi) becomes

$$mg\cos\theta - mR\dot{\theta}^2 = \lambda_r \tag{viii}$$

This equation represents the net inward force and is equal to the reaction.

From the law of conservation of energy, we can show that; $mR\dot{\theta}^2 = 2mg(1-\cos\theta)$, then the above equation (viii) becomes

$$mg\cos\theta - 2mg(1-\cos\theta) = \lambda_r$$

$$3mg\cos\theta - 2mg = \lambda_r \tag{ix}$$

The particle will leave the surface of the sphere when the reaction λ_r becomes zero. From (ix), we get, when the particle leaves the sphere

$$\cos\theta = \frac{2}{3}$$

Then, the height at which the particle leaves the surface is

$$h = R\cos\theta = \frac{2}{3}R \tag{x}$$

EXAMPLE 2.16 Consider a disc that has a string wrapped around it with one end attached to a fixed support and allowed to fall with the string unwinding as it falls. Find the equations of motion of the falling disc and the forces of constraint.

Solution: Let r be the radius and m be the mass of the disc. The generalized coordinates are the distance of fall y and the angle of rotation of the disc θ. The falling disc has translational as well as rotational motion. These two are related through

$$y = r\theta$$

and, therefore the constraint equation is

$$f = y - r\theta = 0 \tag{i}$$

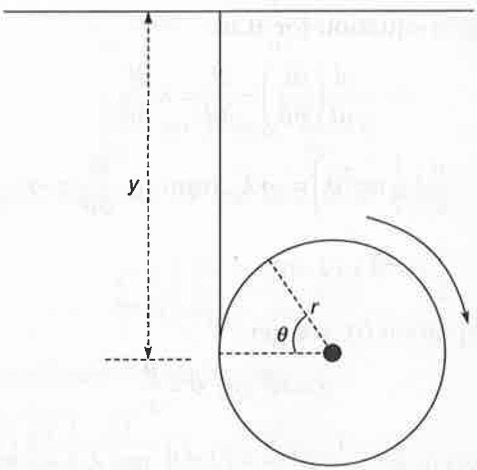

Fig. 2.9

Therefore, the kinetic energy of the falling disc can be written as

$$T=\frac{1}{2}m\dot{y}^2+\frac{1}{2}I\dot{\theta}^2$$

where I is the moment of inertia of the disc about the axis of rotation and $I=\frac{1}{2}mr^2$.

Then,
$$T=\frac{1}{2}m\dot{y}^2+\frac{1}{4}mr^2\dot{\theta}^2 \tag{ii}$$

Let the initial potential energy of the disc at the reference level is zero. Then the potential energy of the falling disc is

$$V=-mgy \tag{iii}$$

Therefore, the Lagrangian can be written as

$$L=T-V=\frac{1}{2}m\dot{y}^2+\frac{1}{4}mr^2\dot{\theta}^2+mgy \tag{iv}$$

From (iv),
$$\frac{\partial L}{\partial \dot{y}}=m\dot{y}\,,\quad \frac{\partial L}{\partial y}=mg$$

and
$$\frac{\partial L}{\partial \dot{\theta}}=\frac{1}{2}mr^2\dot{\theta}\,,\quad \frac{\partial L}{\partial \theta}=0$$

The Lagrange's equation of motion for y is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y}=\lambda\frac{\partial f}{\partial y}$$

that is,
$$\frac{d}{dt}(m\dot{y})-mg=\lambda\,;\ \text{from (i)}\ \frac{\partial f}{\partial y}=1$$

or
$$m\ddot{y}-mg-\lambda=0 \tag{v}$$

Similarly, the Lagrange's equation for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=\lambda\frac{\partial f}{\partial \theta}$$

that is,
$$\frac{d}{dt}\left(\frac{1}{2}mr^2\dot{\theta}\right)=-r\lambda\,,\ \text{from (i)}\ \frac{\partial f}{\partial \theta}=-r$$

or
$$\frac{1}{2}mr^2\ddot{\theta}+r\lambda=0 \tag{vi}$$

From the constraint equation (i), we get

$$\ddot{y}=r\ddot{\theta}\ \text{ or; }\ \ddot{\theta}=\frac{\ddot{y}}{r}$$

This can be used in (vi) to get; $\frac{1}{2}mr^2\frac{\ddot{y}}{r}+r\lambda=0$ or; $\lambda=-\frac{1}{2}m\ddot{y}$

From this, $m\ddot{y} = -2\lambda$ and use this in (v) to get

$$\lambda = -\frac{1}{3}mg \tag{vii}$$

Using (vii) in (v) and (vi), we get the equations of motion of the generalized coordinates as

$$\ddot{y} = \frac{2}{3}g \text{ and } \ddot{\theta} = \frac{2}{3}\frac{g}{r} \tag{viii}$$

The downward acceleration for a freely falling body is g, but here it is $\frac{2}{3}g$. Therefore, the constraint force in this case is the upward tension in the string and it is

$$Q_y = -\frac{1}{2}mg = \lambda \tag{ix}$$

Similarly, we have
$$Q_\theta = \frac{1}{3}mgr = r\lambda \tag{x}$$

This is the torque that forces the disc to rotate about its axis as it falls down.

EXAMPLE 2.17 Obtain the equation of motion of a free particle of mass m in a horizontal plane under the action of any force F.

Solution: In this problem we assume the particle is constrained to move in the xy-plane so that $z = 0$. Here, we use the polar coordinate system with generalized coordinates (r, ϕ). Then the kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \tag{i}$$

From this equation, we can write

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \text{ and, } \frac{\partial T}{\partial r} = mr\dot{\phi}^2 \tag{ii}$$

Similarly,
$$\frac{\partial T}{\partial \dot{\phi}} = mr^2\dot{\phi} \text{ and, } \frac{\partial T}{\partial \phi} = 0 \tag{iii}$$

Here we use Lagrange's equation of the first kind since the force is not defined. That is,

$$\sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \tag{iv}$$

For the generalized coordinate r thus becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r \text{ or; } \frac{d}{dt}(m\dot{r}) - mr\dot{\phi}^2 = Q_r$$

That is, $$m\ddot{r} - mr\dot{\phi}^2 = Q_r \tag{v}$$

Similarly, for the generalized coordinate ϕ, we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = Q_\phi$$

or

$$m\frac{d}{dt}\left(r^2\dot{\phi}\right) = Q_\phi \tag{vi}$$

Equations (v) and (vi) are the required equations of motion.

EXAMPLE 2.18 A particle of mass m moves on the xy–plane under the action of a force field given by $F = -(kr\cos\theta)r$, where, k is a constant and $\vec{r}$ is the radial vector. Show that the angular momentum of the particle is conserved. Also obtain the equation of motion.

Solution: Let us take (r, ϕ) as the polar coordinates of the particle, so that the kinetic energy of the particle can be written as

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \tag{i}$$

so that $$\frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \frac{\partial T}{\partial r} = mr\dot{\phi}^2 \text{ and; } \frac{\partial T}{\partial \dot{\phi}} = mr^2\dot{\phi}, \quad \frac{\partial T}{\partial \phi} = 0$$

Since the force is radial, the Lagrange's equation for ϕ can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = 0$$

or $$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0$$

that is, $$mr^2\dot{\phi} = c, \text{ a constant.} \tag{ii}$$

Now, $mr^2\dot{\phi} = L$, the angular momentum and therefore the angular momentum of the particle is a constant.

Now consider the Lagrange's equation for r; since the radial force is present it can be expressed as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r$$

or $$\frac{d}{dt}(m\dot{r}) - mr\dot{\phi}^2 = -(kr\cos\theta)$$

that is, $$m\ddot{r} - mr\dot{\phi}^2 + kr\cos\theta = 0 \tag{iii}$$

which is the required equation of motion.

EXAMPLE 2.19 A particle of mass m moves under the influence of a central force whose potential is given by $V=-\frac{k}{r}$. Obtain the equation of motion by Lagrange's method.

Solution: For the motion under a central force field, the generalized coordinates are (r,θ). Then the kinetic energy of the particle is;

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right) \tag{i}$$

The potential energy is given as

$$V=-\frac{k}{r} \tag{ii}$$

Then the Lagrangian of the particle is

$$L=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{k}{r} \tag{iii}$$

so that
$$\frac{\partial L}{\partial \dot{r}}=m\dot{r},\quad \frac{\partial L}{\partial r}=mr\dot{\theta}^2-\frac{k}{r^2}$$

and
$$\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta},\quad \frac{\partial L}{\partial \theta}=0$$

The Lagrange's equations of motion are

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0, \quad \text{or} \quad \frac{d}{dt}(m\dot{r})-mr\dot{\theta}^2+\frac{k}{r^2}=0 \\ \text{and} \quad &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0, \quad \text{or} \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right)=0 \end{aligned}\right\} \tag{iv}$$

This would yield

$$m\ddot{r}-mr\dot{\theta}^2+\frac{k}{r^2}=0 \tag{v}$$

and
$$2mr\dot{r}\dot{\theta}+mr^2\ddot{\theta}=0 \tag{vi}$$

These are the equations of motion of a particle moving under the influence of an attractive central force. Further, the equation $\frac{d}{dt}\left(mr^2\dot{\theta}\right)=0$, shows that

$$mr^2\dot{\theta}=L=\text{constant}$$

Therefore, the angular momentum of the particle remains constant when it moves under a central force.

EXAMPLE 2.20 A particle of mass m moves with an initial velocity v_0 in the x-y plane in a force free region along a circle whose radius increases with time. Obtain the equations of motion.

Solution: Let r_0 be the initial radius and then the particle is constrained to move on the xy-plane such that the instantaneous radius is given by

$$r^2(t) = x^2 + y^2 = r_0^2(1+kt)^2 \tag{i}$$

Since the motion is over a plane, we use the polar coordinates and therefore

$$x = r_0(1+kt)\cos\phi \text{ and } y = r_0(1+kt)\sin\phi \tag{ii}$$

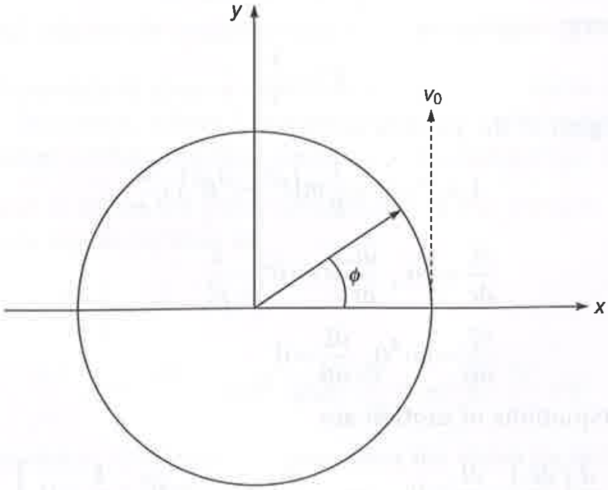

Fig. 2.10

Differentiating these equations with respect to time, we get

$$\dot{x} = r_0 k\cos\phi - r_0(1+kt)\dot{\phi}\sin\phi$$

and
$$\dot{y} = r_0\sin\phi + r_0(1+kt)\dot{\phi}\cos\phi$$

Then the kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$= \frac{1}{2}m\left\{\left[r_0 k\cos\phi - r_0(1+kt)\dot{\phi}\sin\phi\right]^2 + \left[r_0\sin\phi + r_0(1+kt)\dot{\phi}\cos\phi\right]^2\right\}$$

$$= mr_0^2\left[k^2 + (1+kt)^2\dot{\phi}^2\right] = mv^2 \tag{iii}$$

Then it follows that, $v^2 = r_0^2\left[k^2 + (1+kt)^2\dot{\phi}^2\right]$ (iv)

and when $t = 0$, $\quad v_0^2 = r_0^2\left(k^2 + \dot{\phi}^2\right)$ (v)

Now, differentiate (iii) w.r.t. $\dot{\phi}$ to get

$$\frac{\partial T}{\partial \dot{\phi}} = mr_0^2(1+kt)^2\dot{\phi} \qquad \text{(vi)}$$

Now, the Lagrange's equation for a force free region $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = 0$ becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{d}{dt}\left(mr_0^2(1+kt)^2\dot{\phi}\right) = 0$$

or $$mr_0^2(1+kt)^2\dot{\phi} = c\text{, a constant} \qquad \text{(vii)}$$

The constant c can be evaluated using the initial condition as follows. At $t=0, v=v_0$ and we have

$$v_0^2 = r_0^2\left(k^2+\dot{\phi}^2\right) = r_0^2k^2 + r_0^2\dot{\phi}^2$$

or $$r_0\dot{\phi} = \sqrt{v_0^2 - r_0^2k^2} \qquad \text{(viii)}$$

Now, from (vii) at $t=0$, we have $c = mr_0^2\dot{\phi} = mr_0\left(r_0\dot{\phi}\right) = mr_0\sqrt{v_0^2 - r_0^2k^2}$ (ix)

From (vii) and (ix), we can write

$$mr_0^2(1+kt)^2\dot{\phi} = mr_0\sqrt{v_0^2 - r_0^2k^2}$$

or

$$\frac{d\phi}{dt} = \frac{\sqrt{v_0^2 - r_0^2k^2}}{r_0(1+kt)^2} = A\frac{1}{(1+kt)^2} \qquad \text{(x)}$$

where, $$A = \frac{\sqrt{v_0^2 - r_0^2k^2}}{r_0} \qquad \text{(xi)}$$

From (x); $d\phi = A\frac{dt}{(1+kt)^2}$. This can be integrated to obtain

$$\phi = -\frac{A}{k}\frac{1}{(1+kt)} + c' \qquad \text{(xii)}$$

Since at $t=0, \phi=0$, we get, $c' = \frac{A}{k}$ and then (xii) becomes

$$\phi = \frac{At}{1+kt} \qquad \text{(xiii)}$$

Using (xiii) in (ii), we get the equations of motion as

$$x = r_0(1+kt)\cos\left(\frac{At}{1+kt}\right) \text{ and } y = r_0(1+kt)\sin\left(\frac{At}{1+kt}\right)$$

EXAMPLE 2.21 A particle of mass m is moving over the surface of the earth. If the gravity is directed along the negative z-direction obtain the equation of motion by Lagrange's approach.

Solution: The kinetic energy of a particle moving over the surface of the earth is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{i}$$

and the potential energy is

$$V = mgz \tag{ii}$$

So that the Lagrangian of the particle is

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz \tag{iii}$$

Now, the Lagrange's equation of motion for the variable x is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

that is,

$$\frac{d}{dt}(m\dot{x}) = 0 \text{ or; } m\dot{x} = \text{constant} \tag{iv}$$

Similarly, we get $\frac{d}{dt}(m\dot{y}) = 0$ or; $m\dot{y} = \text{constant}$ (v)

Equations (iv) and (v) show that the component of momentum along the x and y axes are conserved. Now we consider the equation of motion in the z-direction.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

that is,

$$\frac{d}{dt}(m\dot{z}) - mg = 0 \text{ or; } \quad \ddot{z} = g \tag{vi}$$

Equation (vi) can be integrated to obtain

$$\dot{z} = gt + c \tag{vii}$$

Integrating one more time, we get

$$z = \frac{1}{2}gt^2 + ct + d \tag{viii}$$

where c and d are integration constants.

EXAMPLE 2.22 A particle is moving in a plane under the influence of a central force given by

$$F = \frac{1}{r^2}\left(1 - \frac{\left(\dot{r}^2 - 2r\ddot{r}\right)}{c^2}\right)$$

where r is the distance of the particle from the centre of the force field. Find the generalized potential using Lagrange's equation of motion.

Solution: The Lagrangian of the particle is $L = T - U$, where U is the generalized potential. The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

that is,
$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{r}}\right) - \frac{\partial (T-U)}{\partial r} = 0 \qquad \text{(i)}$$

This can be separated to get

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right) - \frac{\partial U}{\partial r} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} \qquad \text{(ii)}$$

The RHS of (ii) is the generalized force and therefore

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right) - \frac{\partial U}{\partial r} = \frac{1}{r^2}\left(1 - \frac{\left(\dot{r}^2 - 2r\ddot{r}\right)}{c^2}\right)$$

$$= \frac{1}{r^2} - \frac{\dot{r}^2}{c^2 r^2} + \frac{2\ddot{r}}{c^2 r}$$

$$= \frac{1}{r^2} + \frac{\dot{r}^2}{c^2 r^2} - \frac{2\dot{r}^2}{c^2 r^2} + \frac{2\ddot{r}}{c^2 r}$$

$$= \frac{1}{r^2} + \frac{\dot{r}^2}{c^2 r^2} + \frac{d}{dt}\left(\frac{2\dot{r}}{c^2 r}\right)$$

$$= -\frac{\partial}{\partial r}\left(\frac{1}{r} + \frac{\dot{r}^2}{c^2 r}\right) + \frac{d}{dt}\frac{\partial}{\partial \dot{r}}\left(\frac{1}{r} + \frac{\dot{r}^2}{c^2 r}\right) \qquad \text{(iii)}$$

Then we get the generalized potential as

$$U = \frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right) \qquad \text{(iv)}$$

EXAMPLE 2.23 Obtain the equation of motion and the time period of a linear harmonic oscillator by Lagrange's method.

Solution: Consider a particle of mass m executing harmonic oscillations along the x-direction. The kinetic energy of the particle is

$$T = \frac{1}{2} m \dot{x}^2 \qquad \text{(i)}$$

and potential energy is

$$V = \frac{1}{2}kx^2 \tag{ii}$$

where k is the force constant.

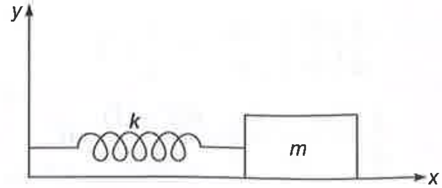

Fig. 2.11

Then the Lagrangian of the system can be written as

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{iii}$$

The Lagrange's equation of motion in one dimension is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{iv}$$

From (iii); $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $\frac{\partial L}{\partial x} = -kx$ so that the Lagrange's equation of motion becomes

$$\frac{d}{dt}(m\dot{x}) + kx = 0$$

that is,

$$m\ddot{x} + kx = 0 \quad \text{or,} \quad \ddot{x} + \frac{k}{m}x = 0 \tag{v}$$

This is the required equation of motion. This can be compared with the standard equation for simple harmonic motion, $\ddot{x} + \omega^2 x = 0$ to get the frequency of oscillation as

$$\omega = \sqrt{\frac{k}{m}} \tag{vi}$$

or the time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{vii}$$

EXAMPLE 2.24 Determine the time period of a simple pendulum by Lagrange's method.

Solution: For a simple pendulum we take the angular displacement θ as the generalized coordinate. Let l be the length of the pendulum and m be the mass of the pendulum bob.

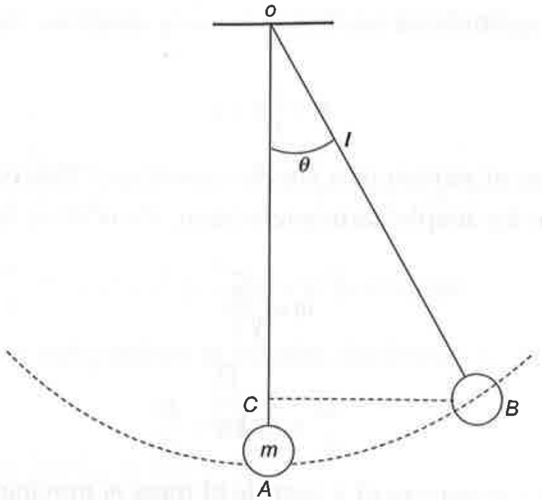

Fig. 2.12

The kinetic energy of the system in polar coordinate system is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2 \tag{i}$$

The potential energy is

$$V = mg(CA) = mgl(1-\cos\theta) \tag{ii}$$

Then the Lagrangian of the system can be written as

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta) \tag{iii}$$

The Lagrange's equation of motion for the generalized coordinate θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{iv}$$

From (iii), $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$ and $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$

Then the Lagrange's equation of motion becomes

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) - \left(-mgl\sin\theta\right) = 0$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{v}$$

We assume the amplitude of oscillation is very small so that; $\sin\theta \to \theta$ and (v) becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad \text{(vi)}$$

This is the equation of motion of a simple pendulum. This can be compared with the standard equation for simple harmonic motion, $\ddot{x} + \omega^2 x = 0$ to get the frequency of oscillation as

$$\omega = \sqrt{\frac{g}{l}} \qquad \text{(vii)}$$

and the time period as

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \text{(viii)}$$

EXAMPLE 2.25 The Lagrangian of a particle of mass m moving in one dimension is

$$L = e^{\alpha t}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right)$$

where α and k are positive constants. Obtain the equation of motion.

Solution: Given: $L = e^{\alpha t}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right)$ (i)

so that
$$\frac{\partial L}{\partial \dot{x}} = e^{\alpha t} m\dot{x} \text{ and } \frac{\partial L}{\partial x} = e^{\alpha t}(-kx) \qquad \text{(ii)}$$

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \text{(iii)}$$

Substituting the above results in this equation, we get

$$\frac{d}{dt}\left(e^{\alpha t} m\dot{x}\right) - (-e^{\alpha t}kx) = 0$$

or
$$e^{\alpha t} m\ddot{x} + \alpha e^{\alpha t} m\dot{x} + e^{\alpha t}kx = 0$$

that is,
$$\ddot{x} + \alpha\dot{x} + \frac{k}{m}x = 0 \qquad \text{(iv)}$$

which is the required equation of motion.

EXAMPLE 2.26 The Lagrangian of a particle of mass m is

$$L = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] - \frac{V}{2}(x^2 + y^2) + W\sin\omega t,$$

where, V, W and ω are constants. Show that the z-component of linear momentum is conserved.

Solution: Given:

$$L=\frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]-\frac{V}{2}\left(x^2+y^2\right)+W\sin\omega t$$

or

$$L=\frac{m}{2}\left[\dot{x}^2+\dot{y}^2+\dot{z}^2\right]-\frac{V}{2}\left(x^2+y^2\right)+W\sin\omega t \qquad \text{(i)}$$

Since the potential is independent of velocity, the energy is conserved.

Now, $$\frac{\partial L}{\partial \dot{z}}=m\dot{z} \text{ and } \frac{\partial L}{\partial z}=0 \qquad \text{(ii)}$$

Substituting in Lagrange's equation of motion; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)-\frac{\partial L}{\partial z}=0$, we get

$$m\ddot{z}=0 \qquad \text{(iii)}$$

that is, $F_z=0$ or, $\dot{P}_z=0$ or, P_z = constant

EXAMPLE 2.27 A particle of mass m is constrained to move in a vertical plane along a trajectory given by $x=A\cos\theta$ and $y=A\sin\theta$, where A is a constant. Construct the Lagrangian of the particle and obtain the equation of motion.

Solution: The kinetic energy of the particle is

$$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)=\frac{1}{2}mA^2\dot{\theta}^2 \qquad \text{(i)}$$

and the potential energy is

$$V=mgy=mgA\sin\theta \qquad \text{(ii)}$$

Then the Lagrangian is

$$L=\frac{1}{2}mA^2\dot{\theta}^2-mgA\sin\theta \qquad \text{(iii)}$$

Now, $$\frac{\partial L}{\partial \dot{\theta}}=mA^2\dot{\theta} \text{ and } \frac{\partial L}{\partial \theta}=-mgA\cos\theta \qquad \text{(iv)}$$

Substituting (iv) in the Lagrange's equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$, we get

$$mA^2\ddot{\theta}+mgA\cos\theta=0$$

or

$$\ddot{\theta}+\frac{g}{A}\cos\theta=0 \qquad \text{(v)}$$

EXAMPLE 2.28 Obtain the equations of motion of two coupled oscillators of mass m each whose

Lagrangian is $$L=\frac{1}{2}\left(\dot{x}_1^2+\dot{x}_2^2\right)-\frac{1}{2}m\omega_0^2\left(x_1^2+x_2^2\right)-m\omega_0^2\mu x_1 x_2 .$$

Solution: Given: $$L=\frac{1}{2}\left(\dot{x}_1^2+\dot{x}_2^2\right)-\frac{1}{2}m\omega_0^2\left(x_1^2+x_2^2\right)-m\omega_0^2\mu x_1 x_2 \quad \text{(i)}$$

Then, $$\frac{\partial L}{\partial \dot{x}_1}=m\dot{x}_1\ ;\ \frac{\partial L}{\partial x_1}=-m\omega_0^2 x_1+m\omega_0^2\mu x_2 \quad \text{(ii)}$$

and $$\frac{\partial L}{\partial \dot{x}_2}=m\dot{x}_2\ ;\ \frac{\partial L}{\partial x_2}=-m\omega_0^2 x_2+m\omega_0^2\mu x_1 \quad \text{(iii)}$$

Now, the Lagrange's equation for x_1 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=0$$

or $$m\ddot{x}_1+m\omega_0^2 x_1-m\omega_0^2\mu x_2=0$$

that is, $$\ddot{x}_1+\omega_0^2 x_1-\omega_0^2\mu x_2=0 \quad \text{(iv)}$$

Similarly, for x_2 the equation of motion is

$$\ddot{x}_2+\omega_0^2 x_2-\omega_0^2\mu x_1=0 \quad \text{(v)}$$

EXAMPLE 2.29 The Lagrangian of a system is given as $L=\frac{1}{2}\dot{q}^2+q\dot{q}-\frac{1}{2}q^2$. Show that the system executes simple harmonic motion.

Solution: Given: $$L=\frac{1}{2}\dot{q}^2+q\dot{q}-\frac{1}{2}q^2 \quad \text{(i)}$$

Now, $$\frac{\partial L}{\partial \dot{q}}=\dot{q}+q \text{ and } \frac{\partial L}{\partial q}=\dot{q}-q \quad \text{(ii)}$$

The Lagrange's equation of motion is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0$

Therefore, $$\frac{d}{dt}(\dot{q}+q)-(\dot{q}-q)=0$$

or $$\ddot{q}+q=0 \quad \text{(iii)}$$

which is the equation of simple harmonic motion.

EXAMPLE 2.30 A pendulum of mass m and length l attached to a massless block which is attached to a fixed wall by a massless spring of force constant k. The massless block moves without friction on a horizontal surface. Obtain the equation of motion of the system using Lagrange's method.

Solution: For the given system, the generalized coordinates are x and θ. The Cartesian coordinates of the pendulum can be written as

$$x = x_1 + l\sin\theta \text{ and } y = -l\cos\theta \tag{i}$$

so that the components of velocity are

$$v_x = \dot{x}_1 + (l\cos\theta)\dot{\theta} \text{ and } v_y = (l\sin\theta)\dot{\theta} \tag{ii}$$

Then, the kinetic energy of the pendulum is

$$T = \frac{1}{2}m\left(v_x^2 + v_y^2\right) = \frac{1}{2}m\left\{\left[\dot{x}_1 + (l\cos\theta)\dot{\theta}\right]^2 + \left[(l\sin\theta)\dot{\theta}\right]^2\right\}$$

$$= \frac{1}{2}m\left[\dot{x}_1^2 + l^2\dot{\theta}^2 + (2l\cos\theta)\dot{x}_1\dot{\theta}\right] \tag{iii}$$

Now, the potential energy of the system is equal to the sum of the gravitational potential energy and the potential energy of the stretched spring. Therefore,

$$V = \frac{1}{2}kx_1^2 - mgl\cos\theta \tag{iv}$$

Then, the Lagrangian of the system is

$$L = T - V$$

$$= \frac{1}{2}m\left[\dot{x}_1^2 + l^2\dot{\theta}^2 + (2l\cos\theta)\dot{x}_1\dot{\theta}\right] - \frac{1}{2}kx_1^2 + mgl\cos\theta \tag{v}$$

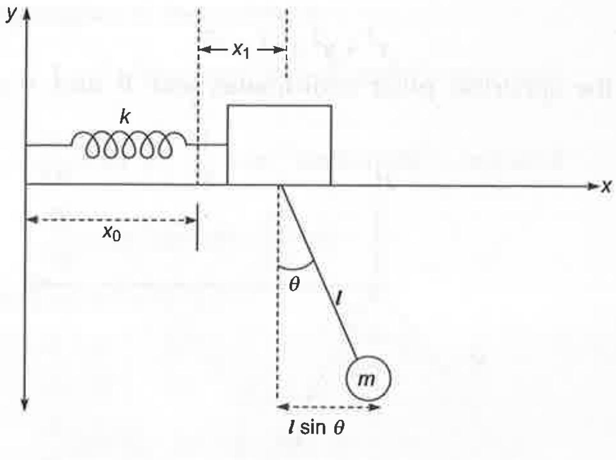

Fig. 2.13

Now, $\dfrac{\partial L}{\partial \dot{x}_1} = m\left[\dot{x}_1 + (l\cos\theta)\dot{\theta}\right]$ and $\dfrac{\partial L}{\partial x_1} = -kx_1$.

Therefore, the Lagrange's equation for x_1; $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right) - \dfrac{\partial L}{\partial x_1} = 0$ becomes

$$\frac{d}{dt}\left\{m\left[\dot{x}_1+(l\cos\theta)\dot{\theta}\right]\right\}-kx_1=0$$

or $$m\ddot{x}_1+kx_1=ml\left(\dot{\theta}^2\sin\theta-\ddot{\theta}\cos\theta\right) \quad \text{(vi)}$$

Similarly, $\frac{\partial L}{\partial\dot{\theta}}=ml^2\dot{\theta}+ml\dot{x}_1\cos\theta$ and $\frac{\partial L}{\partial\theta}=-ml\dot{x}_1\dot{\theta}\sin\theta-mgl\sin\theta$.

Therefore, the Lagrange's equation for θ; $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=0$ becomes

$$\frac{d}{dt}\left(ml^2\dot{\theta}+ml\dot{x}_1\cos\theta\right)+ml\dot{x}_1\dot{\theta}\sin\theta+mgl\sin\theta=0$$

or $$ml^2\ddot{\theta}+ml\left(\ddot{x}_1\cos\theta-\dot{x}_1\dot{\theta}\sin\theta\right)+ml\dot{x}_1\dot{\theta}\sin\theta+mgl\sin\theta=0$$

that is, $$ml^2\ddot{\theta}+ml\ddot{x}_1\cos\theta+mgl\sin\theta=0$$

or $$\ddot{\theta}+\frac{g}{l}\sin\theta=-\frac{\ddot{x}_1}{l}\cos\theta \quad \text{(vii)}$$

Equations (vi) and (vii) are the required equations of motion.

EXAMPLE 2.31 Consider a spherical pendulum with a small bob of mass m which is constrained to move on a spherical surface of radius r, which is equal to the length of the pendulum. Construct the Lagrangian and obtain the equation of motion of the system.

Solution: Since the bob is constrained to move over the surface of a sphere, the constraint equation is

$$x^2+y^2+z^2=r^2.$$

Here we use the spherical polar coordinates and θ and ϕ as the generalized coordinates.

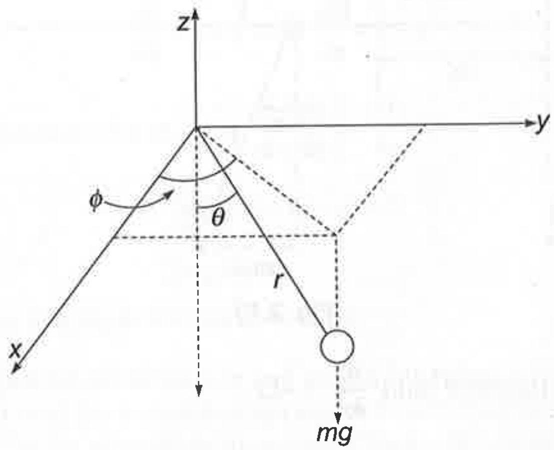

Fig. 2.14

The Cartesian coordinates of the bob can be written as

$$\left.\begin{aligned} x &= r\sin(\pi-\theta)\cos\phi = r\sin\theta\cos\phi \\ y &= r\sin(\pi-\theta)\sin\phi = r\sin\theta\sin\phi \end{aligned}\right\}$$

$$z = r\cos(\pi-\theta) = -r\cos\theta \tag{i}$$

Then,
$$\left.\begin{aligned} \dot{x} &= r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi \\ \dot{y} &= r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi \end{aligned}\right\}$$

$$\dot{z} = r\dot{\theta}\sin\theta \tag{ii}$$

Then, the kinetic energy of the pendulum can be written as

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}m\begin{bmatrix} \left(r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi\right)^2 + \\ \left(r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi\right)^2 + \left(r\dot{\theta}\sin\theta\right)^2 \end{bmatrix}$$

This can be simplified to obtain

$$T = \frac{1}{2}mr^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) \tag{iii}$$

The potential energy of the bob is

$$V = -mgr\cos\theta \tag{iv}$$

Therefore, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}mr^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgr\cos\theta \tag{v}$$

Now,
$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\,;\ \frac{\partial L}{\partial \theta} = mr^2\sin\theta\cos\theta\dot{\phi}^2 - mgr\sin\theta$$

and
$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}\,;\ \frac{\partial L}{\partial \phi} = 0$$

The Lagrange's equation for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or
$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mr^2\sin\theta\cos\theta\dot{\phi}^2 + mgr\sin\theta = 0$$

that is,
$$mr^2\ddot{\theta} - mr^2\sin\theta\cos\theta\dot{\phi}^2 + mgr\sin\theta = 0 \tag{vi}$$

The Lagrange's equation for ϕ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

or $$\frac{d}{dt}\left(mr^2 \sin^2 \theta\dot{\phi}\right) = 0$$

Therefore, $$mr^2 \sin^2 \theta\dot{\phi} = \text{constant} \tag{vii}$$

Equations (vi) and (vii) are the required equations of motion.

EXAMPLE 2.32 Consider a conical pendulum with a bob of mass m, moves in a horizontal circle. Construct the Lagrangian of the pendulum. Obtain the equation of motion and the expression for the time period of oscillation.

Solution: Since the bob moves on a horizontal plane, we use the polar coordinates (r, ϕ), then

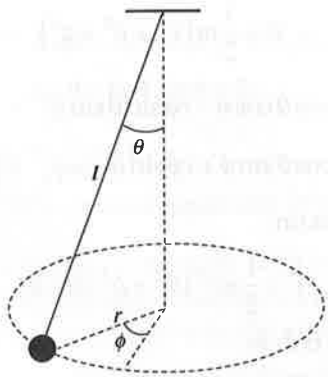

Fig. 2.15

$$x = r\cos\phi \text{ and } y = r\sin\phi$$

so that $$\dot{x} = -r\dot{\phi}\sin\phi \text{ and } \dot{y} = r\dot{\phi}\cos\phi$$

The kinetic energy of the bob is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}mr^2\dot{\phi}^2 \tag{i}$$

But, $r = l\sin\theta$ and therefore

$$T = \frac{1}{2}ml^2 \sin^2 \theta\dot{\phi}^2$$

Potential energy of the bob is

$$V = -mgl\cos\theta \tag{ii}$$

Therefore, the Lagrangian is

$$L = T - V = \frac{1}{2}ml^2 \sin^2 \theta\dot{\phi}^2 + mgl\cos\theta \tag{iii}$$

Then, $$\frac{\partial L}{\partial \dot{\phi}} = ml^2 \sin^2 \theta\dot{\phi}\,; \quad \frac{\partial L}{\partial \phi} = 0$$

and $$\frac{\partial L}{\partial \dot{\theta}}=0\,;\quad \frac{\partial L}{\partial \theta}=ml^2\sin\theta\cos\theta\dot{\phi}^2-mgl\sin\theta$$

Lagrange's equation for ϕ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=0$$

or

$$\frac{d}{dt}\left(ml^2\sin^2\theta\dot{\phi}\right)=0$$

that is, $$ml^2\sin^2\theta\dot{\phi}=\text{constant}$$

In other words, the angular velocity; $\dot{\phi}=\omega=\text{constant}$

Lagrange's equation for θ is; $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right)-\dfrac{\partial L}{\partial \theta}=0$

or

$$ml^2\sin\theta\cos\theta\dot{\phi}^2-mgl\sin\theta=0$$

that is, $$\dot{\phi}^2=\frac{g}{l\cos\theta}\quad\text{or,}\quad \dot{\phi}=\omega=\sqrt{\frac{g}{l\cos\theta}} \tag{iv}$$

Then the time period of oscillation is

$$T=2\pi\sqrt{\frac{l\cos\theta}{g}} \tag{v}$$

EXAMPLE 2.33 A mass m is suspended using a spring of force constant k. In addition to the longitudinal vibrations, the spring executes oscillatory motion in a vertical plane in the gravitational field. Obtain the Lagrangian of the system and derive the equations of motion.

Solution: In this problem we use the polar coordinates r and ϕ as the generalized coordinates. Therefore, we have

$$x=r\sin\phi \text{ and } y=r\cos\phi \tag{i}$$

so that $\dot{x}=\dot{r}\sin\phi-r\dot{\phi}\cos\phi$ and $\dot{y}=\dot{r}\cos\phi+r\dot{\phi}\sin\phi$ (ii)

The kinetic energy of the system is

$$T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right) \tag{iii}$$

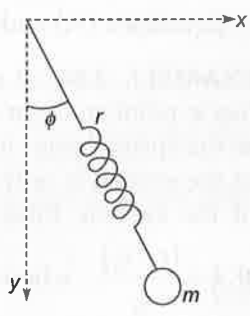

Fig. 2.16

Now, the potential energy of the system is the sum of gravitational potential energy and the potential energy due to the stretching of the spring. Let r_0 be the equilibrium length of the spring, then the potential energy can be written as

$$V=-mgy+\frac{1}{2}k\left(r-r_0\right)^2=-mgr\cos\phi+\frac{1}{2}k\left(r-r_0\right)^2 \tag{iv}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + mgr\cos\phi - \frac{1}{2}k(r - r_0)^2 \qquad \text{(v)}$$

Now, $$\frac{\partial L}{\partial \dot{r}} = m\dot{r}\,;\quad \frac{\partial L}{\partial r} = mr\dot{\phi}^2 + mg\cos\phi - k(r - r_0)$$

and $$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\,;\quad \frac{\partial L}{\partial \phi} = -mgr\sin\phi$$

The Lagrange's equation of motion for r is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

that is, $$\frac{d}{dt}(m\dot{r}) - \left[mr\dot{\phi}^2 + mg\cos\phi - k(r - r_0)\right] = 0$$

or $$m\ddot{r} = mr\dot{\phi}^2 + mg\cos\phi - k(r - r_0)$$

which gives $$\ddot{r} = r\dot{\phi}^2 + g\cos\phi - \frac{k}{m}(r - r_0) \qquad \text{(vi)}$$

Similarly, the Lagrange's equation of motion for ϕ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

that is, $$\frac{d}{dt}\left(mr^2\dot{\phi}\right) + mgr\sin\phi = 0$$

or $$mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} + mgr\sin\phi = 0$$

This would yield $$\ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi} - \frac{g}{r}\sin\phi \qquad \text{(vii)}$$

Equations (vi) and (vii) are the equations of motion.

EXAMPLE 2.34 A massless spring of rest length l_0 (with no tension) has a point mass m connected to one end and the other end fixed so the spring hangs in the gravity field as shown in figure. The motion of the system is only in one vertical plane. Construct the Lagrangian of the system. Find the Lagrange's equations using the variables $\theta, \lambda = \frac{(r - r_0)}{r_0}$, where r_0 is the equilibrium length of the spring with the hanging mass.

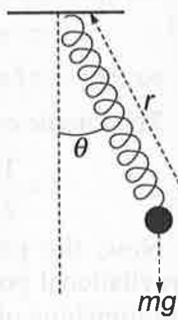

Fig. 2.17

Solution: Since the spring oscillates in a plane we use the polar coordinates (r, θ). The kinetic energy of the point mass is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad \text{(i)}$$

The potential energy can be expressed as

$$V = -mgr\cos\theta + \frac{1}{2}k(r - l_0)^2 \tag{ii}$$

Then the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta - \frac{1}{2}k(r - l_0)^2 \tag{iii}$$

From (iii), we have; $\dfrac{\partial L}{\partial \dot{r}} = m\dot{r}$; $\dfrac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta - k(r - l_0)$

and $\dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$; $\dfrac{\partial L}{\partial \theta} = -mgr\sin\theta$

The Lagrange's equation for r is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

or

$$\frac{d}{dt}(m\dot{r}) - \left[mr\dot{\theta}^2 + mg\cos\theta - k(r - l_0)\right] = 0$$

that is,

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + k(r - l_0) = 0 \tag{iv}$$

The Lagrange's equation for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) + mgr\sin\theta = 0$$

that is,

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0 \tag{v}$$

Equations (iv) and (v) are the Lagrange's equations of motion. However, we have to express (iv) in terms of λ instead of r using the relation $\lambda = \dfrac{(r - r_0)}{r_0}$. Given that the original length of the spring without mass attached is l_0 and the equilibrium length with suspended mass is r_0. Then according to Hooke's law,

$$k(r_0 - l_0) = mg \text{ or; } (r_0 - l_0) = \frac{mg}{k} \tag{vi}$$

Also,

$$\lambda r_0 = (r - r_0) \text{ or } r_0 = r - \lambda r_0 \tag{vii}$$

Using (vii) in (vi), we get

$$r - l_0 = \lambda r_0 + \frac{mg}{k} \tag{viii}$$

Now, $r = r_0(1+\lambda)$ so that; $\dot{r} = r_0\dot{\lambda}$ and $\ddot{r} = r_0\ddot{\lambda}$ (ix)

Substituting for $\ddot{r}, r$ and $(r - l_0)$ in equation (iv) using (viii) and (ix), we get

$$mr_0\ddot{\lambda} - mr_0(1+\lambda)\dot{\theta}^2 - mg\cos\theta + k(\lambda r_0 + \frac{mg}{k}) = 0$$

or
$$\ddot{\lambda} - (1+\lambda)\dot{\theta}^2 + \frac{k}{m}\lambda + \frac{g}{r_0}(1-\cos\theta) \qquad \text{(x)}$$

Substituting for $\ddot{r}, \dot{r}$ and r in (v) using (ix), we get

$$m\left[r_0(1+\lambda)\right]^2\ddot{\theta} + 2m\left[r_0(1+\lambda)\right]r_0\dot{\lambda}\dot{\theta} + mg\left[r_0(1+\lambda)\right]\sin\theta = 0$$

Dividing throughout by $mr_0^2(1+\lambda)$, we get

$$(1+\lambda)\ddot{\theta} + 2\dot{\lambda}\dot{\theta} + \frac{g}{r_0}\sin\theta = 0 \qquad \text{(xi)}$$

Equations (x) and (xi) are the required equations of motion.

EXAMPLE 2.35 Consider the two systems of axes shown in Figure 2.18. The unprimed system of axes is at rest in the laboratory, with the y-axis vertical and pointing downwards. The origin O' of the primed system oscillates with angular frequency ω and amplitude a along the y-axis. From the point O', a pendulum of length l is suspended. Obtain the equation of motion of the pendulum by Lagrange's method.

Solution: The instantaneous position of the bob is given by

$$x = l\sin\theta \text{ and } y = a\cos\omega t + l\cos\theta \qquad \text{(i)}$$

Then, the kinetic energy of the bob is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left[\left(l\dot{\theta}\cos\theta\right)^2 + \left(-a\omega\sin\omega t - l\dot{\theta}\sin\theta\right)^2\right]$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}^2\cos^2\theta + a^2\omega^2\sin^2\omega t + 2a\omega l\dot{\theta}\sin\omega t\sin\theta + l^2\dot{\theta}^2\sin^2\theta\right]$$

or
$$T = \frac{1}{2}m\left[l^2\dot{\theta}^2 + a^2\omega^2\sin^2\omega t + 2a\omega l\dot{\theta}\sin\omega t\sin\theta\right] \qquad \text{(ii)}$$

The potential energy is

$$V = -mgl\cos\theta \qquad \text{(iii)}$$

Then, the Lagrangian is

$$L = T - V = \frac{1}{2}m\left[l^2\dot{\theta}^2 + a^2\omega^2\sin^2\omega t + 2a\omega l\dot{\theta}\sin\omega t\sin\theta\right] + mgl\cos\theta \qquad \text{(iv)}$$

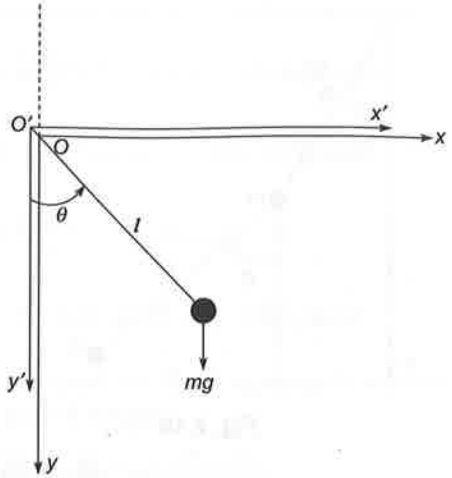

Fig. 2.18

Now, we make use of the gauge invariance of Lagrangian; $L = L' + \frac{dF}{dt}$. Assuming

$$F = -mal\omega \sin\omega t \cos\theta + \frac{a^2\omega}{8}\left[2\omega t - \sin(2\omega t)\right] \tag{v}$$

we can rewrite equation (iv) as

$$L' = \frac{1}{2}ml^2\dot{\theta}^2 + mal\omega^2 \cos\omega t \cos\theta + mgl\cos\theta \tag{vi}$$

Then, we have; $\frac{\partial L'}{\partial \dot{\theta}} = ml^2\dot{\theta}$ and $\frac{\partial L}{\partial \theta} = -mal\omega^2 \cos\omega t \sin\theta - mgl\sin\theta$

Now, the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

that is, $$\frac{d}{dt}\left(ml^2\dot{\theta}\right) + mal\omega^2 \cos\omega t \sin\theta + mgl\sin\theta = 0$$

or $$ml^2\ddot{\theta} + mal\omega^2 \cos\omega t \sin\theta + mgl\sin\theta = 0$$

can be simplified to obtain

$$\ddot{\theta} + \frac{1}{l}\left(g + a\omega^2 \cos\omega t\right)\sin\theta = 0 \tag{vii}$$

This is the required equation of motion.

EXAMPLE 2.36 For the double pendulum shown in Figure (2.19), obtain the equations of motion by Lagrange's method.

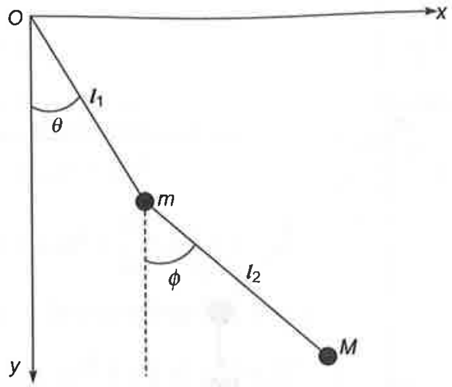

Fig. 2.19

Solution: The generalized coordinates for the given problem are θ and ϕ. The Cartesian coordinates of the mass m can be written as

$$x_1 = l_1 \sin\theta \text{ and } y_1 = l_1 \cos\theta$$

and that of the mass M are

$$x_2 = l_1 \sin\theta + l_2 \sin\phi \text{ and } y_2 = l_1 \cos\theta + l_2 \cos\phi$$

Differentiating these expressions with respect to time, we get

$$\left.\begin{aligned} &\dot{x}_1 = l_1\dot{\theta}\cos\theta && \dot{y}_1 = -l_1\dot{\theta}\sin\theta, \\ \text{and} \quad &\dot{x}_2 = l_1\dot{\theta}\cos\theta + l_2\dot{\phi}\cos\phi && \dot{y}_2 = -l_1\dot{\theta}\sin\theta - l_2\dot{\phi}\sin\phi \end{aligned}\right\} \tag{i}$$

The kinetic energy of the system is

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}M\left(\dot{x}_2^2 + \dot{y}_2^2\right) \tag{ii}$$

Using (i) in (ii) and simplifying, we get the kinetic energy of the system as

$$T = \frac{1}{2}ml_1^2\dot{\theta}^2 + \frac{1}{2}M\left[l_1^2\dot{\theta}^2 + l_2^2\dot{\phi}^2 + 2l_1l_2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right] \tag{iii}$$

To write the expression for the potential energy, we select a reference plane at a distance $(l_1 + l_2)$ below the point of suspension. Then, the potential energy of the system is

$$V = mg(l_1 + l_2 - l_1\cos\theta) + Mg\left[l_1 + l_2 - (l_1\cos\theta + l_2\cos\phi)\right] \tag{iv}$$

Therefore, the Lagrangian of the system is

$$L = T - V = \left\{\begin{aligned} &\frac{1}{2}ml_1^2\dot{\theta}^2 + \frac{1}{2}M\left[l_1^2\dot{\theta}^2 + l_2^2\dot{\phi}^2 + 2l_1l_2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right] \\ &-mg(l_1 + l_2 - l_1\cos\theta) - Mg\left[l_1 + l_2 - (l_1\cos\theta + l_2\cos\phi)\right] \end{aligned}\right\} \tag{v}$$

Then we can have

$$\frac{\partial L}{\partial \dot{\theta}} = ml_1^2\dot{\theta} + Ml_1^2\dot{\theta} + Ml_1l_2\dot{\phi}\cos(\theta-\phi)$$

$$\frac{\partial L}{\partial \theta} = -Ml_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) - mgl_1\sin\theta - Mgl_1\sin\theta$$

Similarly, $$\frac{\partial L}{\partial \dot{\phi}} = Ml_2^2\dot{\phi} + Ml_1l_2\dot{\theta}\cos(\theta-\phi)$$

and $$\frac{\partial L}{\partial \phi} = Ml_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) - Mgl_2\sin\phi$$

These expressions can be used in the Lagrange's equations of motion for the generalized coordinates θ and ϕ to get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

that is, $$\left\{\begin{array}{l}\dfrac{d}{dt}\left[ml_1^2\dot{\theta} + Ml_1^2\dot{\theta} + Ml_1l_2\dot{\phi}\cos(\theta-\phi)\right] + \\ Ml_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mgl_1\sin\theta + Mgl_1\sin\theta\end{array}\right\} = 0$$

$$\left\{\begin{array}{l}(m+M)l_1^2\ddot{\theta} + Ml_1l_2\ddot{\phi}\cos(\theta-\phi) - Ml_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) + Ml_1l_2\dot{\phi}^2\sin(\theta-\phi) \\ +Ml_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mgl_1\sin\theta + Mgl_1\sin\theta\end{array}\right\} = 0$$

or $$(m+M)l_1^2\ddot{\theta} + Ml_1l_2\ddot{\phi}\cos(\theta-\phi) + Ml_1l_2\dot{\phi}^2\sin(\theta-\phi) = -(m+M)gl_1\sin\theta \quad \text{(vi)}$$

Similarly, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

that is, $$\frac{d}{dt}\left[Ml_2^2\dot{\phi} + Ml_1l_2\dot{\theta}\cos(\theta-\phi)\right] - Ml_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) + Mgl_2\sin\phi = 0$$

or

$$\left\{\begin{array}{l}Ml_2^2\ddot{\phi} + Ml_1l_2\ddot{\theta}\cos(\theta-\phi) - Ml_1l_2\dot{\theta}^2\sin(\theta-\phi) + Ml_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) \\ -Ml_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) + Mgl_2\sin\phi\end{array}\right\} = 0$$

$$Ml_2^2\ddot{\phi} + Ml_1l_2\ddot{\theta}\cos(\theta-\phi) - Ml_1l_2\dot{\theta}^2\sin(\theta-\phi) = -Mgl_2\sin\phi \quad \text{(vii)}$$

Equations (vi) and (vii) are the required equations of motion.

Now, if $l_1 = l_2 \equiv l$ and $m = M \equiv m$; these equations reduce to

$$\left.\begin{array}{r}2l\ddot{\theta} + l\ddot{\phi}\cos(\theta-\phi) + l\dot{\phi}^2\sin(\theta-\phi) = -2g\sin\theta \\ \text{and} \quad l\ddot{\theta}\cos(\theta-\phi) + l\ddot{\phi} - l\dot{\theta}^2\sin(\theta-\phi) = -g\sin\phi\end{array}\right\} \quad \text{(viii)}$$

For small amplitude oscillations, these expressions can be further simplified using $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and neglecting terms containing $\dot{\theta}^2$ and $\dot{\phi}^2$, we obtain

$$\left.\begin{aligned} 2l\ddot{\theta} + l\ddot{\phi} &= -2g\theta \\ \text{and} \qquad l\ddot{\theta} + l\ddot{\phi} &= -g\phi \end{aligned}\right\} \tag{ix}$$

EXAMPLE 2.37 Consider a simple pendulum of mass m and length l. The point of suspension of the pendulum moves on a massless rim of radius r with an angular velocity ω. Obtain the equation of motion of the pendulum by Lagrange's method.

Solution: Let the origin of the coordinate system be at the centre of the rim. Now the position coordinates of the bob can be written as

$$x = r\cos\omega t + l\sin\theta$$

and $$y = r\sin\omega t - b\cos\theta$$

Then the velocity components are

$$\dot{x} = -r\omega\sin\omega t + l\dot{\theta}\cos\theta$$

and $$\dot{y} = r\omega\cos\omega t + l\dot{\theta}\sin\theta$$

The kinetic energy of the bob is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left[\left(-r\omega\sin\omega t + l\dot{\theta}\cos\theta\right)^2 + \left(r\omega\cos\omega t + l\dot{\theta}\sin\theta\right)^2\right]$$

This can be simplified to obtain

$$T = \frac{1}{2}m\left[r^2\omega^2 + l^2\dot{\theta}^2 + 2lr\dot{\theta}\omega\sin(\theta - \omega t)\right] \tag{i}$$

The potential energy of the bob is

$$V = mgy = mg\left(r\sin\omega t - b\cos\theta\right) \tag{ii}$$

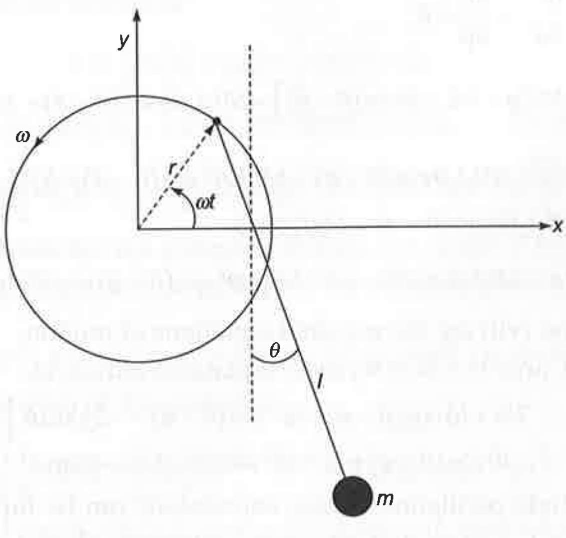

Fig. 2.20

Now, the Lagrangian of the system is

$$L = T - V = \left\{ \begin{array}{l} \frac{1}{2}m\left[r^2\omega^2 + l^2\dot{\theta}^2 + 2lr\dot{\theta}\omega\sin(\theta - \omega t)\right] \\ -mg(r\sin\omega t - b\cos\theta) \end{array} \right\} \quad \text{(iii)}$$

Then, $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + mlr\omega\sin(\theta - \omega t)$ and; $\frac{\partial L}{\partial \theta} = mlr\dot{\theta}\omega\cos(\theta - \omega t) - mgl\sin\theta$

Substituting these in the Lagrange's equation of motion, we get

$$\frac{d}{dt}\left[ml^2\dot{\theta} + mlr\omega\sin(\theta - \omega t)\right] - \left[mlr\dot{\theta}\omega\cos(\theta - \omega t) - mgl\sin\theta\right] = 0$$

that is, $ml^2\ddot{\theta} + mlr\omega\left(\dot{\theta} - \omega\right)\cos(\theta - \omega t) - mlr\dot{\theta}\omega\cos(\theta - \omega t) + mgl\sin\theta = 0$

or
$$ml^2\ddot{\theta} - mlr\omega^2\cos(\theta - \omega t) + mgl\sin\theta = 0$$

This would yield, $\ddot{\theta} = \frac{\omega^2 r}{l}\cos(\theta - \omega t) - \frac{g}{l}\sin\theta$ (iv)

This is the required equation of motion. Note that this equation will reduce to the equation of motion of a simple pendulum if $\omega = 0$.

EXAMPLE 2.38 Two particles of masses m_1 and m_2 are separated by a distance r. Obtain the Lagrangian of the system in the laboratory frame of reference and in the centre of mass frame. Also write the Lagrange's equation of motion in the centre of mass frame.

Solution: Let r_1, r_2 and R be the position vectors of m_1, m_2 and centre of mass respectively. The total kinetic energy of the system can be written as

$$T = \frac{1}{2}\left[m_1|\dot{r}_1|^2 + m_2|\dot{r}_2|^2\right] \quad \text{(i)}$$

Let us take the potential energy of the system as $V(r_1, r_2) = V(r_1 - r_2)$. Then the Lagrangian of the independent two-particle system is

$$L = T - V = \frac{1}{2}\left[m_1|\dot{r}_1|^2 + m_2|\dot{r}_2|^2\right] - V(r_1 - r_2) \quad \text{(ii)}$$

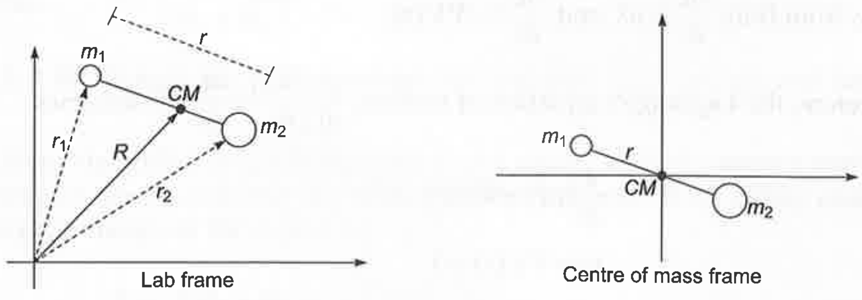

Fig. 2.21

Now, the position vector of the centre of mass can be written as

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \tag{iii}$$

and the inter-particle position vector is; $r = r_1 - r_2$. Then the position vectors of each particle can be expressed as

$$r_1 = R + \frac{m_2 r}{(m_1 + m_2)} \text{ and } r_2 = R - \frac{m_1 r}{(m_1 + m_2)} \tag{iv}$$

Differentiating equation (iv) w.r.t. time, we get

$$\dot{r}_1 = \dot{R} + \frac{m_2 \dot{r}}{(m_1 + m_2)} \text{ and } \dot{r}_2 = \dot{R} - \frac{m_1 \dot{r}}{(m_1 + m_2)} \tag{v}$$

Using (v) in (ii) and simplifying, we get

$$L = \frac{(m_1 + m_2)}{2}|\dot{R}|^2 + \frac{m_1 m_2}{2(m_1 + m_2)}|\dot{r}|^2 - V(r)$$

$$= \frac{(m_1 + m_2)}{2}|\dot{R}|^2 + \frac{\mu}{2}|\dot{r}|^2 - V(r) \tag{vi}$$

where, $\mu = \frac{m_1 m_2}{(m_1 + m_2)}$, the reduced mass of the system.

Equation (vi) is the Lagrangian of the system in the laboratory frame of reference. In order to obtain the Lagrangian of the system in the centre of mass frame, the origin of the coordinate system is to be moved to the centre of mass. That is, the centre of mass is defined by the condition $R = 0$. Then equation (vi) reduces to

$$L(r, \dot{r}) = \frac{\mu}{2}|\dot{r}|^2 - V(r) \tag{vii}$$

which is the Lagrangian in the centre of mass frame.

Now, from (vii); $\frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$ and $\frac{\partial L}{\partial r} = -\nabla V(r)$.

Therefore, the Lagrange's equation of motion, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$ becomes

$$\frac{d}{dt}(\mu \dot{r}) + \nabla V(r) = 0$$

or

$$\mu \ddot{r} + \nabla V(r) = 0 \tag{viii}$$

EXAMPLE 2.39 Find the acceleration of the masses m_1 and m_2 on the string of an Atwood machine by Lagrange's approach assuming a frictionless pulley.

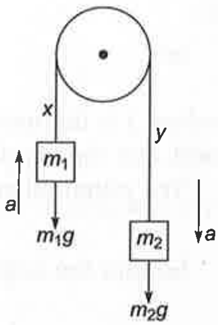

Fig. 2.22

Solution: Since the pulley is frictionless it will not rotate while the masses move over it. Let us take the generalized coordinates as x and y are subjected to the constraint

$$x + y = l \tag{i}$$

so that $\dot{x} = -\dot{y}$ and $\ddot{x} = -\ddot{y}$.

Now, the kinetic energy of the system can be written as

$$T = \frac{1}{2}M_1\dot{x}^2 + \frac{1}{2}M_2\dot{y}^2 = \frac{1}{2}(M_1 + M_2)\dot{x}^2 \tag{ii}$$

The potential energy of the system can be written as

$$V = -M_1gx - M_2gy = -M_1gx - M_2g(l - x) \tag{iii}$$

The Lagrangian of the system is

$$L(x,\dot{x}) = T - V = \frac{1}{2}(M_1 + M_2)\dot{x}^2 + M_1gx + M_2g(l - x) \tag{iv}$$

The Lagrange's equation of motion is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

From (iv); $\frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\dot{x}$ and $\frac{\partial L}{\partial x} = (M_1 - M_2)g$.

Then the Lagrange's equation of motion becomes

$$\frac{d}{dt}\left[(M_1 + M_2)\dot{x}\right] - (M_1 - M_2)g = 0$$

or $$(M_1 + M_2)\ddot{x} - (M_1 - M_2)g = 0$$

Therefore, $$\ddot{x} = \frac{(M_1 - M_2)g}{(M_1 + M_2)} \tag{v}$$

EXAMPLE 2.40 Repeat the above problem for a pulley with friction and having a radius r.

Solution: Since the pulley is not frictionless, it will rotate while the masses move over it. Therefore, we have to consider the rotational kinetic energy of the pulley also. Then the total kinetic energy of the system is

$$T = \frac{1}{2}(M_1 + M_2)\dot{x}^2 + \frac{1}{2}I\omega^2$$

or $$T = \frac{1}{2}(M_1 + M_2)\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{r}\right)^2 \qquad \text{(i)}$$

where I is the moment of inertia of the pulley about an axis passing through its centre and ω is the angular velocity of the pulley.

The potential energy is

$$V = -M_1 gx - M_2 g(l - x) \qquad \text{(ii)}$$

So that the Lagrangian of the system is

$$L = T - V = \frac{1}{2}(M_1 + M_2)\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{r}\right)^2 + M_1 gx + M_2 g(l - x) \qquad \text{(iii)}$$

Now, $$\frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\dot{x} + I\frac{\dot{x}}{r} \text{ and } \frac{\partial L}{\partial x} = (M_1 - M_2)g.$$

Then the Lagrange's equation of motion becomes

$$\frac{d}{dt}\left[(M_1 + M_2)\dot{x} + I\frac{\dot{x}}{r^2}\right] - (M_1 - M_2)g = 0$$

or $$(M_1 + M_2)\ddot{x} + I\frac{\ddot{x}}{r^2} - (M_1 - M_2)g = 0$$

Then, $$\ddot{x} = \frac{(M_1 - M_2)g}{\left(M_1 + M_2 + \frac{I}{r^2}\right)} \qquad \text{(iv)}$$

EXAMPLE 2.41 A bead of mass m is sliding on a uniformly rotating wire in a force free region. Obtain the equation of motion by Lagrange's approach.

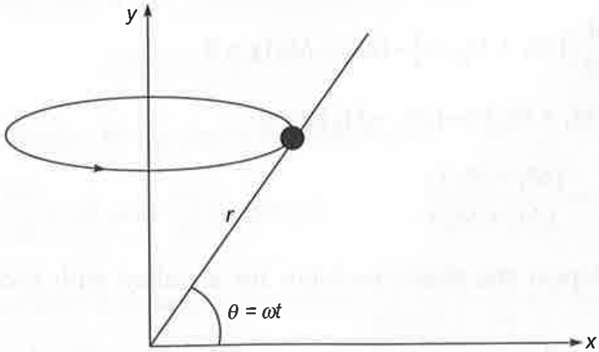

Fig. 2.23

Solution: In this problem, the bead is constrained to move over the wire which is rotating with a constant angular velocity. Therefore, the constraint is time dependent. The instantaneous position of the bead can be written as

$$x = r\cos\theta \text{ and } y = r\sin\theta \tag{i}$$

Therefore, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$ and $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

The kinetic energy of the bead is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{ii}$$

Since the bead is moving in a force free region, the generalized force is zero and we use the Lagrange's equation of the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r \text{ and } \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

From (ii), $\dfrac{\partial T}{\partial \dot{r}} = m\dot{r}$; $\dfrac{\partial T}{\partial r} = mr\dot{\theta}^2$ and $\dfrac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$; $\dfrac{\partial T}{\partial \theta} = 0$

Then the Lagrange's equation for r becomes

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 = 0 \text{ or; } m\ddot{r} - mr\dot{\theta}^2 = 0$$

or $$\ddot{r} = r\dot{\theta}^2 = r\omega^2 \tag{iii}$$

This shows that the bead moves outward due to the centrifugal acceleration.

The Lagrange's equation for θ is

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \text{ or } mr^2\dot{\theta} = \text{constant} \tag{iv}$$

Here, $mr^2\dot{\theta}$ is the angular momentum and it remains a constant.

EXAMPLE 2.42 A bead of mass m is sliding on a uniformly rotating wire in gravitational field. Obtain the equation of motion by Lagrange's approach.

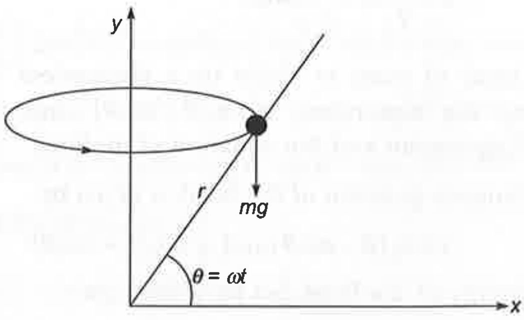

Fig. 2.24

Solution: The kinetic energy of the bead is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{i}$$

The potential energy is

$$V = mgy = mgr\sin\theta \tag{ii}$$

The Lagrangian can be written as

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\sin\theta \tag{iii}$$

Now, $$\frac{\partial L}{\partial \dot{r}} = m\dot{r}\,; \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\sin\theta$$

and $$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\,; \quad \frac{\partial L}{\partial \theta} = -mgr\cos\theta\dot{\theta}$$

The Lagrange's equation of motion for r is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + mg\sin\theta = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta = 0$$

that is, $$\ddot{r} = r\dot{\theta}^2 - g\sin\theta \tag{iv}$$

The Lagrange's equation for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) + mgr\cos\theta\dot{\theta} = 0$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos\theta\dot{\theta} = 0$$

that is, $$\ddot{\theta} = \frac{2}{r}\dot{r}\dot{\theta} + \frac{g}{r}\cos\theta\dot{\theta} \tag{v}$$

EXAMPLE 2.43 A bead of mass m slides on a frictionless wire whose shape is a cycloid described by the equations, $x = r_0(\theta - \sin\theta)$ and $y = r_0(1+\cos\theta)$, where $0 \le \theta \le 2\pi$. Find the Lagrangian and the equation of motion.

Solution: The instantaneous position of the bead is given by

$$x = r_0(\theta - \sin\theta) \text{ and; } y = r_0(1+\cos\theta) \tag{i}$$

Then, the kinetic energy of the bead can be written as

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$= \frac{1}{2}mr_0^2\left[\left(\dot{\theta} - \dot{\theta}\cos\theta\right)^2 + \left(-\dot{\theta}\sin\theta\right)^2\right]$$

$$= mr_0^2\dot{\theta}^2(1-\cos\theta) \tag{ii}$$

The potential energy is

$$V = mgy = mgr_0(1+\cos\theta) \tag{iii}$$

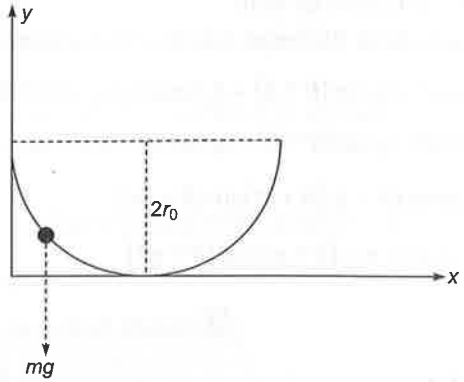

Fig. 2.25

Therefore, the Lagrangian of the particle is

$$L = T - V = mr_0^2\dot{\theta}^2(1-\cos\theta) - mgr_0(1+\cos\theta) \tag{iv}$$

The Lagrange's equation of motion for the generalized coordinate θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{v}$$

Now, $$\frac{\partial L}{\partial \dot{\theta}} = 2mr_0^2\dot{\theta}(1-\cos\theta) \text{ and } \frac{\partial L}{\partial \theta} = mr_0^2\dot{\theta}^2\sin\theta + mgr_0\sin\theta$$

Then, (vi) $$\Rightarrow \frac{d}{dt}\left[2mr_0^2\dot{\theta}(1-\cos\theta)\right] - \left[mr_0^2\dot{\theta}^2\sin\theta + mgr_0\sin\theta\right] = 0$$

or $$2mr_0^2\ddot{\theta}(1-\cos\theta) + 2mr_0^2\dot{\theta}^2\sin\theta - mr_0^2\dot{\theta}^2\sin\theta - mgr_0\sin\theta = 0$$

that is, $$2mr_0^2\ddot{\theta}(1-\cos\theta) + mr_0^2\dot{\theta}^2\sin\theta - mgr_0\sin\theta = 0$$

Dividing throughout by $2mr_0^2$, we get

$$\ddot{\theta}(1-\cos\theta) + \frac{1}{2}\dot{\theta}^2\sin\theta - \frac{g}{r_0}\sin\theta = 0 \tag{vi}$$

This is the required equation of motion.

EXAMPLE 2.44 A bead of mass m slides on a smooth frictionless circular wire of radius r_0. The circular loop itself is rotating in a horizontal plane with constant angular velocity ω about the origin. Show that the bead executes simple harmonic motion about the diameter OA of the circular loop.

Solution: Let us assume that the wire loop is rotating in the xy-plane. Since the loop is rotating with uniform angular velocity ω, we have, $\phi = \omega t$, which is a function of time. The bead is also rotating over the loop so that θ, the position of the bead w.r.t. the diameter OA is also a function of time.

The instantaneous position of the bead can be represented by the equations

$$x = r_0 \cos\phi + r_0 \cos(\theta + \phi) = r_0 \cos\omega t + r_0 \cos(\theta + \omega t)$$

and $$y = r_0 \sin\phi + r_0 \sin(\theta + \phi) = r_0 \sin\omega t + r_0 \sin(\theta + \omega t)$$

so that $$\dot{x} = -r_0\omega \sin\omega t - r_0 (\dot{\theta} + \omega)\sin(\theta + \omega t)$$

and $$\dot{y} = r_0\omega \cos\omega t + r_0 (\dot{\theta} + \omega)\cos(\theta + \omega t)$$

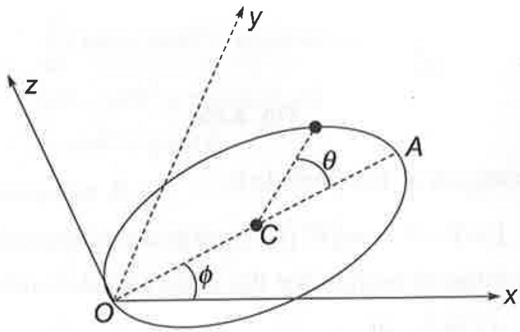

Fig. 2.26

Now, the kinetic energy of the particle is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left[-r_0\omega \sin\omega t - r_0 (\dot{\theta} + \omega)\sin(\theta + \omega t)\right]^2$$

$$+\left[r_0\omega \cos\omega t + r_0 (\dot{\theta} + \omega)\cos(\theta + \omega t)\right]^2$$

This can be simplified to obtain

$$T = \frac{1}{2}mr_0^2\left[(\dot{\theta} + \omega)^2 + 2\dot{\theta}\omega\cos(\theta + \omega t) + \omega^2\right] \tag{i}$$

We have to determine the motion of the bead and therefore we write the Lagrange's equation for θ, assuming a force free motion as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0 \tag{ii}$$

From (i), we get; $\frac{\partial T}{\partial \dot{\theta}} = mr_0^2\left(\dot{\theta} + \omega + \omega\cos\theta\right)$ and $\frac{\partial T}{\partial \theta} = -mr_0^2\omega\left(\dot{\theta} + \omega\right)\sin\theta$

Therefore, the Lagrange's equation becomes

$$\frac{d}{dt}\left[mr_0^2\left(\dot{\theta} + \omega + \omega\cos\theta\right)\right] + mr_0^2\omega\left(\dot{\theta} + \omega\right)\sin\theta = 0$$

$$mr_0^2\left[\ddot{\theta} + \dot{\omega} + \dot{\omega}\cos\theta - \omega\dot{\theta}\sin\theta\right] + mr_0^2\omega\left(\dot{\theta} + \omega\right)\sin\theta = 0$$

But, $\dot{\omega} = 0$ and therefore

$$mr_0^2\left[\ddot{\theta} - \omega\dot{\theta}\sin\theta\right] + mr_0^2\omega\left(\dot{\theta} + \omega\right)\sin\theta = 0$$

or $$\ddot{\theta} + \omega^2\sin\theta = 0 \tag{iii}$$

When θ is small; $\sin\theta \to \theta$, and therefore

$$\ddot{\theta} + \omega^2\theta = 0 \tag{iv}$$

This equation represents a simple harmonic motion with frequency ω.

EXAMPLE 2.45 A particle of mass m is constrained to move on the smooth paraboloid given by the equation, $z = c\left(x^2 + y^2\right); c > 0$ under the action of gravity. Construct the Lagrangian and obtain the equation of motion.

Solution: In this problem we use the cylindrical coordinate system so that the Cartesian coordinates of the particle are

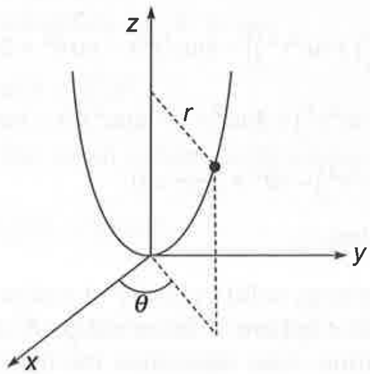

Fig. 2.27

$$x = r\cos\theta, y = r\sin\theta \text{ and } z = c\left(x^2 + y^2\right) = cr^2 \tag{i}$$

Then, $$\left.\begin{array}{l}\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta; \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta\end{array}\right\}$$

and $$\dot{z} = 2cr\dot{r} \tag{ii}$$

Then the kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$\frac{1}{2}m\begin{bmatrix}\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)^2 + \\ \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)^2 + (2cr\dot{r})^2\end{bmatrix}$$

This can be simplified to obtain

$$T = \frac{1}{2}m\left[\dot{r}^2\left(1 + 4c^2r^2\right) + r^2\dot{\theta}^2\right] \tag{iii}$$

Now, the potential energy of the particle is

$$V = mgz = mgcr^2 \tag{iv}$$

Then, the Lagrangian of the particle is

$$L = T - V = \frac{1}{2}m\left[\dot{r}^2\left(1 + 4c^2r^2\right) + r^2\dot{\theta}^2\right] - mgcr^2 \tag{v}$$

This gives

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}\left(1 + 4c^2r^2\right) \text{ and; } \frac{\partial L}{\partial r} = 4mc^2\dot{r}^2r + mr\dot{\theta}^2 - 2mgcr$$

Using these in the Lagrange's equation for r, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$, we get

$$\frac{d}{dt}\left(m\dot{r}\left(1 + 4c^2r^2\right)\right) - 4mc^2\dot{r}^2r - mr\dot{\theta}^2 + 2mgcr = 0$$

or
$$m\ddot{r}\left(1 + 4c^2r^2\right) + 4mc^2\dot{r}^2r - 4mc^2\dot{r}^2r - mr\dot{\theta}^2 + 2mgcr = 0$$

that is,
$$\ddot{r}\left(1 + 4c^2r^2\right) - r\dot{\theta}^2 + 2gcr = 0 \tag{vi}$$

This is the required equation.

EXAMPLE 2.46 A homogeneous solid cylinder of radius r rolls without slipping on the inner side of a larger hollow sphere of inner radius R. Assuming the large sphere at rest find the equation of motion. Also determine the time period of oscillations about the stable equilibrium position.

Solution: The generalized coordinate for the problem is the angle ϕ which the line joining the centres of the smaller and larger sphere makes with the vertical. The angular displacement of the smaller sphere is θ.

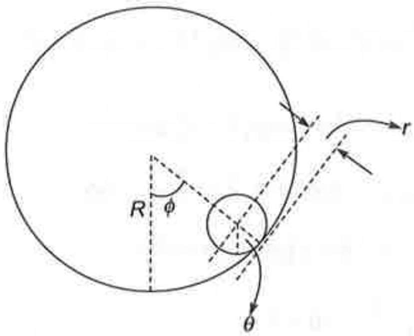

Fig. 2.28

The kinetic energy of the smaller sphere is the sum of the translational kinetic energy of the centre of mass of the smaller sphere and the rotational kinetic energy of the smaller sphere.

That is, $$T=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2=\frac{1}{2}\left[m(R-r)^2\dot{\phi}^2+I\dot{\theta}^2\right]$$

$$=\frac{1}{2}\left[m(R-r)^2\dot{\phi}^2+\frac{1}{2}mr^2\dot{\theta}^2\right] \tag{i}$$

Since the sphere is rolling without slipping, we have

$$r\theta=(R-r)\phi \quad \text{or;} \quad r\dot{\theta}=(R-r)\dot{\phi}$$

This can be used in (i) to substitute for $\dot{\theta}$ to get

$$T=\frac{3}{4}m(R-r)^2\dot{\phi}^2 \tag{ii}$$

The potential energy of the small sphere with respect to the centre of the large sphere is

$$V=-mg(R-r)\cos\phi \tag{iii}$$

Then the Lagrangian is

$$L=T-V=\frac{3}{4}m(R-r)^2\dot{\phi}^2+mg(R-r)\cos\phi \tag{iv}$$

Then, $$\frac{\partial L}{\partial\dot{\phi}}=\frac{3}{2}m(R-r)^2\dot{\phi} \quad \text{and} \quad \frac{\partial L}{\partial\phi}=-mg(R-r)\sin\phi$$

Now, the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)-\frac{\partial L}{\partial\phi}=0$$

or $$\frac{d}{dt}\left(\frac{3}{2}m(R-r)^2\dot{\phi}\right)+mg(R-r)\sin\phi=0$$

that is, $$\frac{3}{2}m(R-r)^2\ddot{\phi}+mg(R-r)\sin\phi=0 \qquad \text{(v)}$$

Since for small oscillations; $\sin\phi\to\phi$ and therefore

$$\frac{3}{2}(R-r)^2\ddot{\phi}+g(R-r)\phi=0$$

$$\ddot{\phi}+\frac{g}{\frac{3}{2}(R-r)}\phi=0 \qquad \text{(vi)}$$

This equation represents a simple harmonic oscillation with frequency

$$\omega=\sqrt{\frac{2g}{3(R-r)}} \qquad \text{(vii)}$$

or, time period, $$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{3(R-r)}{2g}} \qquad \text{(viii)}$$

EXAMPLE 2.47 A particle of mass m slides without friction inside a tube which is bent in the form of a circle of radius r. The circular tube also rotates about a vertical axis passing through its diameter with a constant angular velocity ω. Construct the Lagrangian and obtain the Lagrange's equation of motion of the system.

Solution: There are two types of motion associated with the system. One is the motion of the particle inside the circular tube. The second one is the rotation of the circular tube about a vertical axis passing through its diameter. As the tube rotates, the particle also executes circular motion about the vertical axis with angular velocity ω along a circle of radius $r\sin\theta$. Therefore, the kinetic energy is,

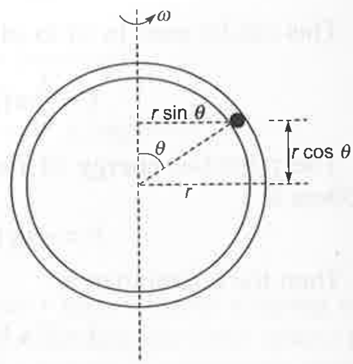

Fig. 2.29

$$T=\frac{1}{2}I\dot{\theta}^2+\frac{1}{2}m(r\sin\theta)^2\omega^2 \qquad \text{(i)}$$

where I is the moment of inertia of the particle about the centre of the circular tube.

Therefore, $I=mr^2$ so that

$$T=\frac{1}{2}mr^2\dot{\theta}^2+\frac{1}{2}m(r\sin\theta)^2\omega^2$$

$$=\frac{1}{2}mr^2\left[\dot{\theta}^2+\omega^2\sin^2\theta\right] \qquad \text{(ii)}$$

The potential energy of the particle with respect to the centre of the circular tube is

$$V = mgr\cos\theta \tag{iii}$$

Then the Lagrangian of the particle is

$$L = T - V = \frac{1}{2}mr^2\left[\dot{\theta}^2 + \omega^2\sin^2\theta\right] - mgr\cos\theta \tag{iv}$$

so that $\dfrac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta}$ and, $\dfrac{\partial L}{\partial\theta} = mr^2\omega^2\sin\theta\cos\theta + mgr\sin\theta$. Then the Lagrange's equation of motion for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

that is, $$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - \left(mr^2\omega^2\sin\theta\cos\theta + mgr\sin\theta\right) = 0$$

or $$mr^2\ddot{\theta} - mr^2\omega^2\sin\theta\cos\theta - mgr\sin\theta = 0 \tag{v}$$

This is the required equation of motion.

EXAMPLE 2.48 A bead of mass M is initially at rest on a smooth horizontal wire and is attached to a point on the wire by a massless spring of force constant k and unstretched length a. A rod of mass m and length $2l$ is freely suspended from the bead. Assuming harmonic vibration for the spring, obtain the equation of motion of the system.

Solution: A schematic representation of the problem is given in the Figure (2.30) below.

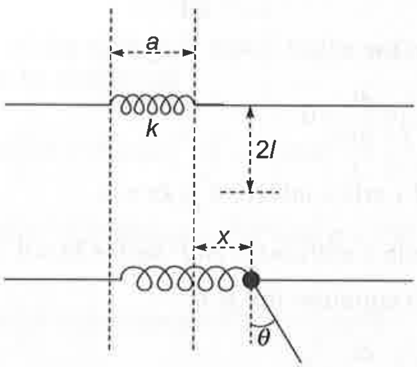

Fig. 2.30

We take x and θ as the generalized coordinates of the system. The kinetic energy is the sum of the kinetic energies of the bead and the rod. Therefore, it is given by

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left\{\left[\frac{d}{dt}(x + l\sin\theta)\right]^2 + \left[\frac{d}{dt}(l\cos\theta)\right]^2\right\} + \frac{1}{2}ml^2\dot{\theta}^2$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}m\left(-l\dot{\theta}\sin\theta\right)^2 + \frac{1}{2}ml^2\dot{\theta}^2$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) + \frac{1}{2}ml^2\dot{\theta}^2$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + ml^2\dot{\theta}^2 + ml\dot{x}\dot{\theta}\cos\theta \tag{i}$$

Here, the first term is the kinetic energy of the bead; second and third terms represent the kinetic energy of the rod. The kinetic energy of the rod is calculated by considering the movement of the centre of mass of the rod.

The potential energy of the system is

$$V = \frac{1}{2}kx^2 - mgl\cos\theta \tag{ii}$$

Therefore, the Lagrangian of the system is

$$L = T - V = \begin{bmatrix} \frac{1}{2}(M+m)\dot{x}^2 + ml^2\dot{\theta}^2 + ml\dot{x}\dot{\theta}\cos\theta \\ -\frac{1}{2}kx^2 + mgl\cos\theta \end{bmatrix} \tag{iii}$$

Then, $\quad \dfrac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\dot{\theta}\cos\theta\,; \quad \dfrac{\partial L}{\partial x} = -kx$

and $\quad \dfrac{\partial L}{\partial \dot{\theta}} = 2ml^2\dot{\theta} + ml\dot{x}\cos\theta\,; \quad \dfrac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta$

The Lagrange's equation for x is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

that is, $\quad \dfrac{d}{dt}\left[(M+m)\dot{x} + ml\dot{\theta}\cos\theta\right] + kx = 0$

or $\quad (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + kx = 0 \qquad$ (iv)

Similarly, the Lagrange's equation for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

that is,

$$\frac{d}{dt}\left(2ml^2\dot{\theta} + ml\dot{x}\cos\theta\right) + ml\dot{x}\dot{\theta}\sin + mgl\sin\theta = 0$$

or $\quad 2ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - ml\dot{x}\dot{\theta}\sin\theta + ml\dot{x}\dot{\theta}\sin\theta + mgl\dot{\theta}\sin\theta = 0$

$$2ml^2\ddot{\theta} + ml\ddot{x}\cos\theta + mgl\dot{\theta}\sin\theta = 0 \tag{v}$$

Equations (iv) and (v) are the required equations of motion.

EXAMPLE 2.49 A sphere of mass m and radius r rolls without slipping down a triangular block of mass M that is free to move on a frictionless horizontal surface. Construct the Lagrangian and obtain Lagrange's equations for this system subject to the force of gravity at the surface of the earth. Given that all objects are initially at rest and the sphere's centre is at a distance H above the surface.

Solution: Let θ be the angle of rotation of the sphere. At $t = 0; x = 0, \theta = 0, \xi = \xi_0, y = H$. Let the triangular block move in the x-direction and its position is represented by the variable x. Since the sphere rolls down without slipping, we can represent the coordinate of the centre of the sphere as

$$(x', y') = \{[x + (\xi_0 + r\theta)\cos\phi], [H - r\theta\sin\phi]\} \tag{i}$$

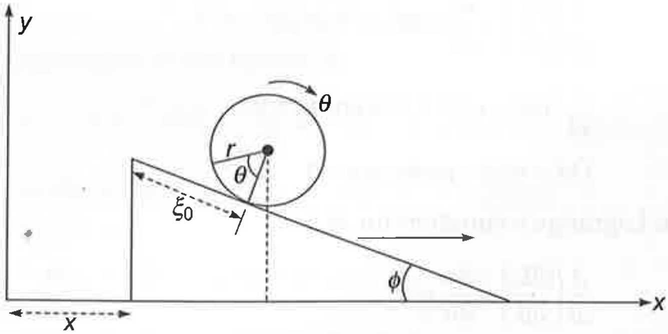

Fig. 2.31

Then the kinetic energy of the system is equal to the sum of kinetic energies of the block and the sphere. It can be written as

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + r\dot{\theta}\cos\phi)^2 + (-r\dot{\theta}\sin\phi)^2\right] + \frac{1}{2}\left(\frac{2}{5}mr^2\dot{\theta}^2\right)$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2r\dot{x}\dot{\theta}\cos\phi + r^2\dot{\theta}^2\cos^2\phi + r^2\dot{\theta}^2\sin^2\phi\right) + \frac{1}{5}mr^2\dot{\theta}^2$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2r\dot{x}\dot{\theta}\cos\phi + r^2\dot{\theta}^2\right) + \frac{1}{5}mr^2\dot{\theta}^2 \tag{ii}$$

The first term represents the kinetic energy of the moving triangular block, second term the translational kinetic energy of the sphere and third term the rotational kinetic energy of the sphere.

The potential energy is

$$V = mg(H - r\theta\sin\theta) \tag{iii}$$

Therefore, the Lagrangian of the system is

$$L = T - V = \begin{bmatrix} \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2r\dot{x}\dot{\theta}\cos\phi + r^2\dot{\theta}^2\right) + \frac{1}{5}mr^2\dot{\theta}^2 \\ -mg\left(H - r\theta\sin\phi\right) \end{bmatrix} \quad \text{(iv)}$$

Then, $$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + mr\dot{\theta}\cos\phi\,; \quad \frac{\partial L}{\partial x} = 0$$

and $$\frac{\partial L}{\partial \dot{\theta}} = mr\dot{x}\cos\phi + \frac{7}{5}mr^2\dot{\theta}\,; \quad \frac{\partial L}{\partial \theta} = mgr\sin\phi$$

The Lagrange's equation for x is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

or $$\frac{d}{dt}\left[(M+m)\dot{x} + mr\dot{\theta}\cos\phi\right] = 0$$

that is, $$(M+m)\ddot{x} + mr\ddot{\theta}\cos\phi = 0 \quad \text{(v)}$$

Similarly, the Lagrange's equation for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or

$$\frac{d}{dt}\left(mr\dot{x}\cos\phi + \frac{7}{5}mr^2\dot{\theta}\right) - mgr\sin\phi = 0$$

that is, $$mr\ddot{x}\cos\phi + \frac{7}{5}mr^2\ddot{\theta} - mgr\sin\phi = 0$$

or $$\ddot{x}\cos\phi + \frac{7}{5}r\ddot{\theta} - g\sin\phi = 0 \quad \text{(vi)}$$

Equations (v) and (vi) are the Lagrange's equations of motion.

EXAMPLE 2.50 A small block of mass m is held stationary on a frictionless inclined plane of a block of mass M. The angle of inclination of the plane is θ. The block rests on a frictionless horizontal surface. If the small block is released on the inclined plane, determine the horizontal acceleration of the large block.

Solution Let x_1 be the horizontal coordinate of the inclined block and x_2 that of the small block. Initially, assume that the block is at the top edge of the inclined plane so that $x_1 = x_2 = 0$. As the small block slides down, the relative distance between the plane and the small block increases and is equal to $x_1 + x_2$. Therefore, the height fallen by the block is $(x_1 + x_2)\tan\theta$.

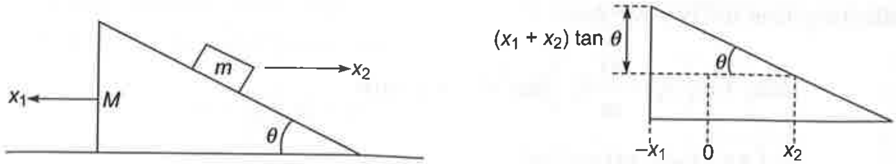

Fig. 2.32

Then the kinetic energy of the system can be written as

$$T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\left[\dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2 \tan^2\theta\right] \qquad \text{(i)}$$

The potential energy is

$$V = -mg(x_1 + x_2)\tan\theta \qquad \text{(ii)}$$

Therefore, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\left[\dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2 \tan^2\theta\right] + mg(x_1 + x_2)\tan\theta \qquad \text{(iii)}$$

Now, $\quad \dfrac{\partial L}{\partial \dot{x}_1} = M\dot{x}_1 + m(\dot{x}_1 + \dot{x}_2)\tan^2\theta$ and $\dfrac{\partial L}{\partial x_1} = mg\tan\theta$.

Similarly, $\quad \dfrac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 + m(\dot{x}_1 + \dot{x}_2)\tan^2\theta$ and $\dfrac{\partial L}{\partial x_2} = mg\tan\theta$

Now, the Lagrange's equation of motion for x_1 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

that is, $\quad \dfrac{d}{dt}\left[M\dot{x}_1 + m(\dot{x}_1 + \dot{x}_2)\tan^2\theta\right] - mg\tan\theta = 0$

or, $\quad M\ddot{x}_1 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta = mg\tan\theta \qquad \text{(iv)}$

Similarly, for the Lagrange's equation for x_2 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

or

$$\frac{d}{dt}\left[m\dot{x}_2 + m(\dot{x}_1 + \dot{x}_2)\tan^2\theta\right] - mg\tan\theta = 0$$

that is, $\quad m\ddot{x}_2 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta = mg\tan\theta \qquad \text{(v)}$

Equations (iv) and (v) are Lagrange's equations of motion.

Now we have to determine the horizontal acceleration of the inclined block, i.e., $\ddot{x}_1$. Now take the difference between (iv) and (v) to get

$$M\ddot{x}_1 - m\ddot{x}_2 = 0 \text{ or, } \ddot{x}_2 = \frac{M}{m}\ddot{x}_1$$

Substituting this in (iv), we get

$$M\ddot{x}_1 + m\left(\ddot{x}_1 + \frac{M}{m}\ddot{x}_1\right)\tan^2\theta = mg\tan\theta$$

$$\ddot{x}_1\left[M + (m+M)\tan^2\theta\right] = mg\tan\theta$$

Then, $$\ddot{x}_1 = \frac{mg\tan\theta}{\left[M + (m+M)\tan^2\theta\right]}$$

Dividing numerator and denominator by $\tan\theta$, we get

$$\ddot{x}_1 = \frac{mg}{\left[\dfrac{M}{\tan\theta} + (m+M)\tan\theta\right]} = \frac{mg}{\left[\dfrac{M\cos\theta}{\sin\theta} + (m+M)\dfrac{\sin\theta}{\cos\theta}\right]}$$

This can be simplified to get

$$\ddot{x}_1 = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} \tag{vi}$$

This is the required result.

EXAMPLE 2.51 Consider a coupled system of two equal masses m and three springs of force constant k, enclosed between two rigid supports as shown in Figure 2.23. Obtain the Lagrange's equations of motion.

Solution: In the diagram x_{10} and x_{20} are the equilibrium positions of the masses. Their displacements from the equilibrium position are given as; x_1 and x_2. This is a one-dimensional problem and hence the kinetic energy can be written as

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) \tag{i}$$

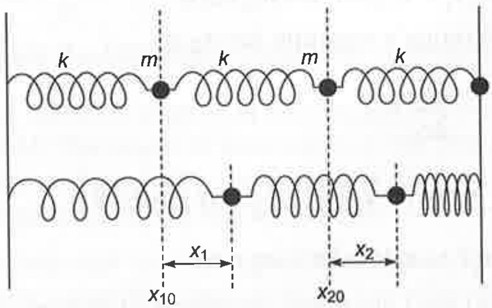

Fig. 2.33

The two external springs act like external forces and hence their potential energies are proportional to the square of the displacements x_1 and x_2. However, the middle spring acts like internal force and its potential energy is proportional to the square of $|x_1 - x_2|$. Therefore, the total potential energy is

$$V = \frac{1}{2}k\left[x_1^2 + x_2^2 + (x_1 - x_2)^2\right]$$

$$= k\left(x_1^2 + x_2^2 - x_1x_2\right) \tag{ii}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - k\left(x_1^2 + x_2^2 - x_1x_2\right) \tag{iii}$$

so that $$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1, \quad \frac{\partial L}{\partial x_1} = -k(2x_1 - x_2)$$

and $$\frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2, \quad \frac{\partial L}{\partial x_2} = -k(2x_2 - x_1).$$

Now, the Lagrange's equation of motion for x_1 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

or $$\frac{d}{dt}(m\dot{x}_1) + k(2x_1 - x_2) = 0$$

that is, $$m\ddot{x}_1 + k(2x_1 - x_2) = 0 \tag{iv}$$

Similarly, the Lagrange's equation of motion for x_2 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

or $$\frac{d}{dt}(m\dot{x}_2) + k(2x_2 - x_1) = 0$$

that is, $$m\ddot{x}_2 + k(2x_2 - x_1) = 0 \tag{v}$$

Equations (iv) and (v) are the required equations.

EXAMPLE 2.52 Three massless perfectly elastic springs *AB*, *BC* and *CD* are attached in a horizontal line as shown (Fig. 2.34). The ends at *A* and *D* are fixed. The objects of mass *m* are located where the springs join at *B* and *C*. The two outer springs *AB* and *CD* have natural lengths a_1 and force constant k_1, while *BC* has natural length a_2 and force constant k_2. The distance l between the fixed end points *AD* is greater than the total natural lengths of the springs: $l > 2a_1 + a_2$. Construct the Lagrangian and obtain the Lagrange's equations of motion.

Solution: Let x_1 and x_2 be the generalized coordinates of the first and second mass respectively. Then the kinetic energy of the system is

$$T = \frac{1}{2} m\left(\dot{x}_1^2 + \dot{x}_2^2\right) \tag{i}$$

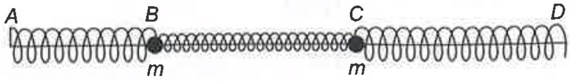

Fig. 2.34

The potential energy is the sum of the potential energies of the three springs. It is given by

$$V = \frac{1}{2} k_1 (x_1 - a_1)^2 + \frac{1}{2} k_2 (x_2 - x_1 - a_2)^2 + \frac{1}{2} k_1 (l - x_2 - a_1)^2 \tag{ii}$$

Then, the Lagrangian of the system is

$$L = T - V = \left[\begin{array}{l} \frac{1}{2} m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2} k_1 (x_1 - a_1)^2 \\ -\frac{1}{2} k_2 (x_2 - x_1 - a_2)^2 - \frac{1}{2} k_1 (l - x_2 - a_1)^2 \end{array}\right] \tag{iii}$$

so that $\quad \dfrac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \; ; \; \dfrac{\partial L}{\partial x_1} = -k_1 (x_1 - a_1) + k_2 (x_2 - x_1 - a_2)$

and $\quad \dfrac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 ; \; \dfrac{\partial L}{\partial x_2} = -k_2 (x_2 - x_1 - a_2) + k_1 (l - x_2 - a_1)$

The Lagrange's equation for x_1 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

that is, $\quad \dfrac{d}{dt}(m\dot{x}_1) + k_1 (x_1 - a_1) - k_2 (x_2 - x_1 - a_2) = 0$

or

$$m\ddot{x}_1 + k_1 (x_1 - a_1) - k_2 (x_2 - x_1 - a_2) = 0 \tag{iv}$$

Similarly, Lagrange's equation for x_2 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

that is, $\quad \dfrac{d}{dt}(m\dot{x}_2) + k_2 (x_2 - x_1 - a_2) - k_1 (l - x_2 - a_1) = 0$

or

$$m\ddot{x}_2 + k_2 (x_2 - x_1 - a_2) - k_1 (l - x_2 - a_1) = 0 \tag{v}$$

Equations (iv) and (v) are the required equations.

EXAMPLE 2.53 Obtain the equation for motion of a compound pendulum by Lagrange's method. Determine the time period.

Solution: Consider a compound pendulum of mass m. Let O be the point of suspension and G, the centre of gravity. Let the distance between O and G be l. The plane of oscillation of the pendulum is the xy-plane.

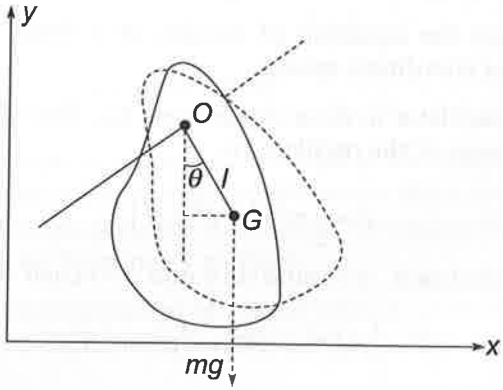

Fig. 2.35

The generalized coordinate for the motion of the compound pendulum is θ. If I is the moment of inertia of the body about the axis of rotation, the kinetic energy of motion is;

$$T = \frac{1}{2} I \dot{\theta}^2 \tag{i}$$

Potential energy relative to a horizontal plane through O is

$$V = -mgl\cos\theta \tag{ii}$$

Then, the Lagrangian is $\quad L = T - V = \frac{1}{2} I \dot{\theta}^2 + mgl\cos\theta \quad$ (iii)

so that $\dfrac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$ and, $\dfrac{\partial L}{\partial \theta} = -mgl\sin\theta$

Then, the Lagrange's equation of motion for θ, $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = 0$ becomes

$$\frac{d}{dt}(I\dot{\theta}) + mgl\sin\theta = 0$$

or

$$I\ddot{\theta} + mgl\sin\theta = 0$$

that is,

$$\ddot{\theta} + \frac{mgl}{I}\sin\theta = 0 \tag{iv}$$

When the amplitude of oscillation of motion is small, $\sin\theta \approx \theta$, therefore

$$\ddot{\theta} + \frac{mgl}{I}\theta = 0 \tag{v}$$

This is the equation of motion of a compound pendulum. Comparing this with the standard equation for simple harmonic motion we get the time period of a compound pendulum as

$$T = 2\pi\sqrt{\frac{I}{mgl}} \tag{vi}$$

EXAMPLE 2.54 Obtain the equation of motion of a three dimensional isotropic oscillator spherical polar coordinate system.

Solution: An isotropic oscillator in three dimensions has three degrees of freedom and therefore the kinetic energy of the oscillator is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{i}$$

Now, we have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. Using this in (i), we get

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) \tag{ii}$$

The potential energy of the isotropic oscillator is

$$V = \frac{1}{2}kr^2 \tag{iii}$$

Then, the Lagrangian is

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) - \frac{1}{2}kr^2 \tag{iv}$$

From (iv) we get

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\dot{\phi}^2\sin^2\theta - kr$$

Similarly,
$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = mr^2\dot{\phi}^2\sin\theta\cos\theta$$

and
$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta, \quad \frac{\partial L}{\partial \phi} = 0$$

Now, the Lagrange's equations of motion for the generalized coordinate r is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot{r}) - \left(mr\dot{\theta}^2 + mr\dot{\phi}^2\sin^2\theta - kr\right) = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 - mr\dot{\phi}^2\sin^2\theta + kr = 0 \tag{v}$$

Similarly, the Lagrange's equations of motion for the generalized coordinate θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mr^2\dot{\phi}^2 \sin\theta\cos\theta = 0$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\dot{\phi}^2 \sin\theta\cos\theta = 0 \qquad \text{(vi)}$$

Finally, the Lagrange's equations of motion for the generalized coordinate ϕ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(mr^2\dot{\phi}\sin^2\theta\right) = 0 \qquad \text{(vii)}$$

EXAMPLE 2.55 A particle of mass m is attached to the midpoint of a weightless rod of length l. The ends of the rod are constrained to move along the x and y axes. A uniform gravitational field acts in the negative y-direction. Assuming there is no friction, obtain the equation of motion by Lagrange's method.

Solution: A schematic representation of the problem is given (Figure 2.36.) Let us take θ as the generalized coordinate. The coordinates of the mass at any instant is given by

$$x = \frac{l}{2}\sin\theta \text{ and } y = \frac{l}{2}\cos\theta \qquad \text{(i)}$$

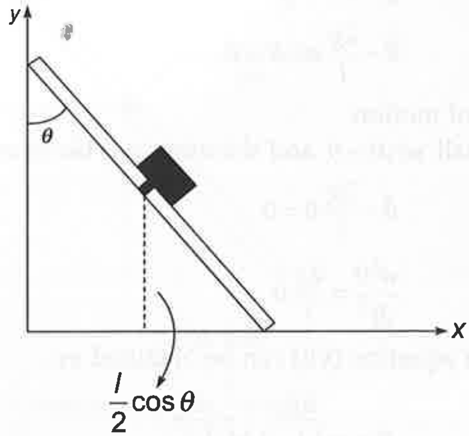

Fig. 2.36

Then the kinetic energy of the mass is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \qquad \text{(ii)}$$

From (i), $\dot{x} = \frac{l}{2}\dot{\theta}\cos\theta$ and $\dot{y} = -\frac{l}{2}\dot{\theta}\sin\theta$. Then, (ii) becomes

$$T = \frac{1}{2}m\left[\left(\frac{l}{2}\dot{\theta}\cos\theta\right)^2 + \left(-\frac{l}{2}\dot{\theta}\sin\theta\right)^2\right] = \frac{1}{8}ml^2\dot{\theta}^2 \quad \text{(iii)}$$

The potential energy is

$$V = mg\left(\frac{l}{2}\cos\theta\right) \quad \text{(iv)}$$

Therefore, the Lagrangian is

$$L = T - V = \frac{1}{8}ml^2\dot{\theta}^2 - mg\left(\frac{l}{2}\cos\theta\right) \quad \text{(v)}$$

Then, $$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{4}ml^2\dot{\theta} \text{ and } \frac{\partial L}{\partial \theta} = \frac{1}{2}mgl\sin\theta$$

The Lagrange's equation for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

that is, $$\frac{d}{dt}\left(\frac{1}{4}ml^2\dot{\theta}\right) - \frac{1}{2}mgl\sin\theta = 0$$

or $$\frac{1}{4}ml^2\ddot{\theta} - \frac{1}{2}mgl\sin\theta = 0$$

or $$\ddot{\theta} - \frac{2g}{l}\sin\theta = 0 \quad \text{(vi)}$$

This is the equation of motion.

Now, if θ is very small $\sin\theta \approx \theta$ and therefore (vi) becomes

$$\ddot{\theta} - \frac{2g}{l}\theta = 0$$

or $$\frac{d^2\theta}{dt^2} = \frac{2g}{l}\theta \quad \text{(vii)}$$

A general solution to equation (vii) can be obtained as

$$\theta = ae^{\sqrt{\frac{2g}{l}}t} + be^{-\sqrt{\frac{2g}{l}}t} \quad \text{(viii)}$$

where the constants a and b are to be determined from the initial conditions.

If we assume the initial conditions as $\theta|_{t=0} = \theta_0$ and $\left.\frac{d\theta}{dt}\right|_{t=0} = 0$, we get; $a = b = \frac{\theta_0}{2}$ and then (viii) becomes

$$\theta = \theta_0 \cosh\left(\sqrt{\frac{2g}{l}}t\right) \quad \text{(ix)}$$

This gives the solution of equation (vii)

EXAMPLE 2.56 A uniform ladder of mass M and length l slides without friction from wall or floor.

(a) Obtain the equation of motion, assuming the ladder remains in contact with the wall.

(b) If the ladder is initially at rest at an angle α_0 with the floor, at what angle, if any, will it break contact with the wall?

Solution:

(a) The coordinates of the centre of mass can be written as

$$x = \frac{l}{2}\cos\alpha \text{ and } y = \frac{l}{2}\sin\alpha \tag{i}$$

This equation serves as the constraint so long as the ladder is in contact with the wall. Now, the kinetic energy involves two parts; kinetic energy of motion of the centre of mass and kinetic energy of motion about the centre of mass. Therefore,

$$T = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\alpha}^2 \tag{ii}$$

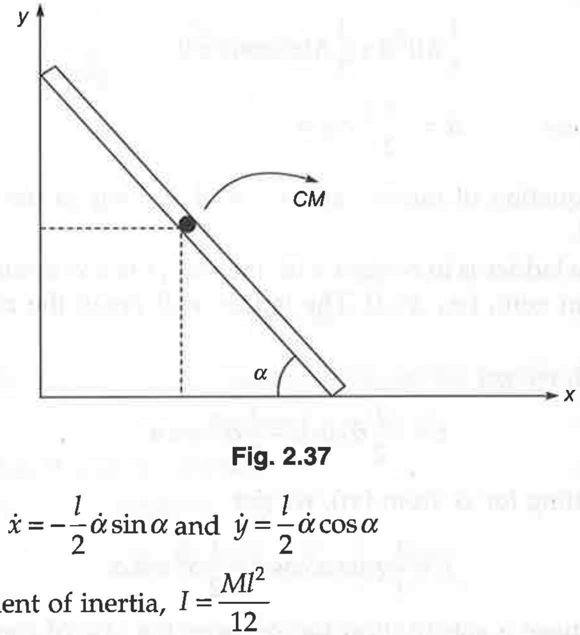

Fig. 2.37

Now, from (i), $\dot{x} = -\frac{l}{2}\dot{\alpha}\sin\alpha$ and $\dot{y} = \frac{l}{2}\dot{\alpha}\cos\alpha$

Also, the moment of inertia, $I = \frac{Ml^2}{12}$

With these substitutions, (ii) becomes

$$T = \frac{1}{2}M\left[\left(-\frac{l}{2}\dot{\alpha}\sin\alpha\right)^2 + \left(\frac{l}{2}\dot{\alpha}\cos\alpha\right)^2\right] + \frac{1}{2}\frac{Ml^2}{12}\dot{\alpha}^2$$

or
$$T = \frac{1}{6}Ml^2\dot{\alpha}^2 \tag{iii}$$

Potential energy is

$$V = Mg\frac{l}{2}\sin\alpha \tag{iv}$$

Therefore, the Lagrangian is

$$L = T - V = \frac{1}{6}Ml^2\dot{\alpha}^2 - \frac{1}{2}Mgl\sin\alpha \tag{v}$$

Then, $$\frac{\partial L}{\partial \dot{\alpha}} = \frac{1}{3}Ml^2\dot{\alpha} \text{ and } \frac{\partial L}{\partial \alpha} = -\frac{1}{2}Mgl\cos\alpha$$

Now, the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) - \frac{\partial L}{\partial \alpha} = 0$$

or

$$\frac{d}{dt}\left(\frac{1}{3}Ml^2\dot{\alpha}\right) + \frac{1}{2}Mgl\cos\alpha = 0$$

that is, $$\frac{1}{3}Ml^2\ddot{\alpha} + \frac{1}{2}Mgl\cos\alpha = 0$$

This would give $$\ddot{\alpha} = -\frac{3}{2}\frac{g}{l}\cos\alpha \tag{vi}$$

This is the equation of motion and is valid as long as the ladder is in contact with the wall.

(b) As long as the ladder is in contact with the wall, the acceleration in the x-direction is greater than zero, i.e., $\ddot{x} > 0$. The ladder will break the contact with the wall when $\ddot{x} = 0$.

Now, from (i), we get

$$\ddot{x} = -\frac{l}{2}\ddot{\alpha}\sin\alpha - \frac{l}{2}\dot{\alpha}^2\cos\alpha$$

Now substituting for $\ddot{\alpha}$ from (vi), we get

$$\ddot{x} = \frac{3}{4}g\sin\alpha\cos\alpha - \frac{l}{2}\dot{\alpha}^2\cos\alpha \tag{vii}$$

Now, let us make a substitution for $\dot{\alpha}^2$ from the law of conservation of energy. The energy conservation gives

$$\frac{1}{6}Ml^2\dot{\alpha}^2 + \frac{1}{2}Mgl\sin\alpha = \frac{1}{2}Mgl\sin\alpha_0$$

or $$\dot{\alpha}^2 = 3\frac{g}{l}(\sin\alpha_0 - \sin\alpha)$$

With this substitution (vii) becomes

$$\ddot{x} = \frac{3}{4} g \sin\alpha \cos\alpha - \frac{l}{2} 3 \frac{g}{l} (\sin\alpha_0 - \sin\alpha)\cos\alpha$$

On simplification, we get

$$\ddot{x} = \frac{3}{4} g \cos\alpha (3\sin\alpha - 2\sin\alpha_0) \quad \text{(viii)}$$

From (viii) it is clear that the ladder will break the contact with the wall when, $3\sin\alpha = 2\sin\alpha_0$.

EXAMPLE 2.57 Two blocks of equal masses m are connected by a rigid rod of length l. The system is constrained to move in the x-y plane as shown in (Figure 2.38.) Taking θ as the generalized coordinate, obtain the equation of motion by Lagrange's method.

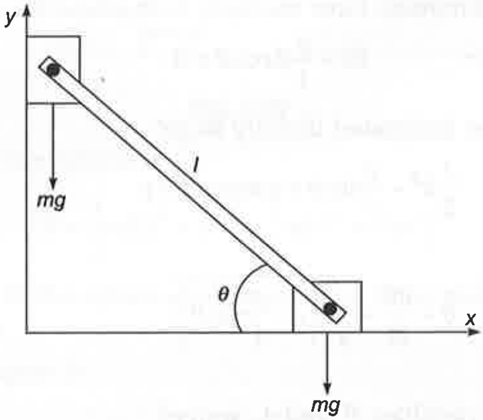

Fig. 2.38

Solution: From the figure, the relative distance between the two blocks can be written as

$$x = l\cos\theta \text{ and } y = l\sin\theta \quad \text{(i)}$$

Therefore, $\dot{x} = -l\dot{\theta}\sin\theta$ and $\dot{y} = l\dot{\theta}\cos\theta$

The kinetic energy of the system is

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} ml^2\dot{\theta}^2 \quad \text{(ii)}$$

Potential energy is

$$V = mgy = mgl\sin\theta \quad \text{(iii)}$$

Therefore, the Lagrangian of the system is

$$L = T - V = \frac{1}{2} ml^2\dot{\theta}^2 - mgl\sin\theta \quad \text{(iv)}$$

so that $$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \text{ and } \frac{\partial L}{\partial \theta} = -mgl\cos\theta$$

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$$

that is, $$\frac{d}{dt}\left(ml^2\dot{\theta}\right)+mgl\cos\theta=0$$

or $$ml^2\ddot{\theta}+mgl\cos\theta=0$$

On simplification, we get

$$\ddot{\theta}+\frac{g}{l}\cos\theta=0 \tag{v}$$

This is the equation of motion. Now multiply both sides of equation (v) with $\dot{\theta}$ to get

$$\ddot{\theta}\dot{\theta}+\frac{g}{l}\dot{\theta}\cos\theta=0$$

This expression can be integrated directly to get

$$\frac{1}{2}\dot{\theta}^2+\frac{g}{l}\sin\theta=\text{constant}=c$$

This would give $$\dot{\theta}=\frac{d\theta}{dt}=\sqrt{2\left(c-\frac{g}{l}\sin\theta\right)} \tag{vi}$$

On separation of the variables θ and t, we get

$$dt=\frac{d\theta}{\sqrt{2\left(c-\frac{g}{l}\sin\theta\right)}}$$

This can be integrated to get

$$t-t_0=\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{2\left(c-\frac{g}{l}\sin\theta\right)}} \tag{vii}$$

The constants c and t_0 can be determined from the initial conditions.

EXAMPLE 2.58 A sphere of mass m moves in a tube that rotates in the x-y plane about the z-axis with a constant angular velocity ω. Obtain the equation of motion by Lagrange's method.

Solution: This problem involves only one degree of freedom and then needs only one generalized coordinate to describe the motion. It is the distance of the sphere from the centre of rotation and let it be r. Now, the instantaneous position of the sphere is given by;

$$x = r\cos\omega t \text{ and } y = r\sin\omega t \tag{i}$$

Then, $\dot{x} = \dot{r}\cos\omega t - r\omega\sin\omega t$ and $\dot{y} = \dot{r}\sin\omega t + r\omega\cos\omega t$

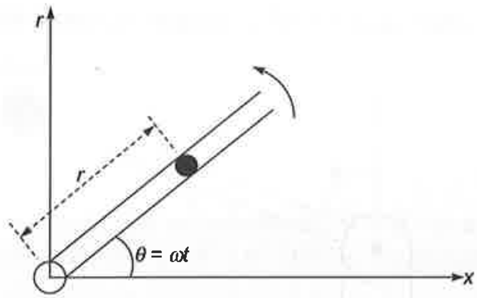

Fig. 2.39

The kinetic energy of the sphere is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right) \tag{ii}$$

The potential energy of the sphere can be assumed to be zero as it is rotating in the *xy*-plane inside the tube.

Therefore, the Lagrangian is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right) \tag{iii}$$

Now, $$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \text{ and } \frac{\partial L}{\partial r} = m\omega^2 r$$

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

that is, $$\frac{d}{dt}(m\dot{r}) - m\omega^2 r = 0$$

or $$\ddot{r} - \omega^2 r = 0 \tag{iv}$$

This equation has a general solution of the form

$$r(t) = Ae^{\omega t} + Be^{-\omega t}$$

which is non-oscillatory.

EXAMPLE 2.59 A point mass m is fixed on one end of a massless rod of length l whose other end is fixed on a hinge. The hinge oscillates in a vertical direction according to $h(t) = h_0 \cos\omega t$. This arrangement forms an upright pendulum. Assuming the angle between the bar and the vertical is θ, determine the equation of motion of the system by Lagrange's method.

Solution: The instantaneous position of the point mass m is given by

$$x = l\sin\theta \text{ and } y = h(t) + l\cos\theta = h_0\cos\omega t + l\cos\theta \tag{i}$$

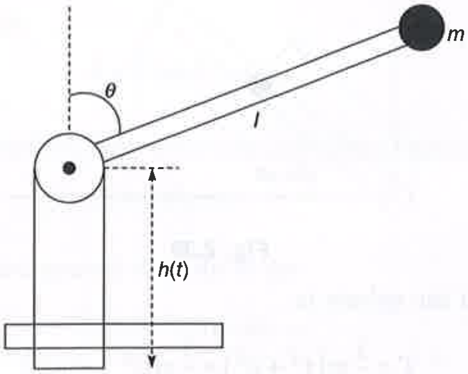

Fig. 2.40

From (i) we get

$$\dot{x} = l\dot{\theta}\cos\theta \text{ and } \dot{y} = -\left(h_0\omega\sin\omega t + l\dot{\theta}\sin\theta\right) \tag{ii}$$

Then, the kinetic energy of the system is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left[\left(l\dot{\theta}\cos\theta\right)^2 + \left(h_0\omega\sin\omega t + l\dot{\theta}\sin\theta\right)^2\right]$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}^2 + h_0^2\omega^2\sin^2\omega t + 2h_0\omega l\dot{\theta}\sin\omega t\sin\theta\right] \tag{iii}$$

The potential energy is

$$V = mhy = mg\left(h_0\cos\omega t + l\cos\theta\right) \tag{iv}$$

Then, the Lagrangian of the system is

$$L = T - V = \left\{\begin{array}{l}\frac{1}{2}m\left[l^2\dot{\theta}^2 + h_0^2\omega^2\sin^2\omega t + 2h_0\omega l\dot{\theta}\sin\omega t\sin\theta\right]\\ -mg\left(h_0\cos\omega t + l\cos\theta\right)\end{array}\right\} \tag{v}$$

From (v) we can have

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} + mh_0\omega l\sin\omega t\sin\theta$$

and $$\frac{\partial L}{\partial \theta} = mh_0\omega l\dot{\theta}\sin\omega t\cos\theta + mgl\sin\theta$$

Therefore, the Lagrange's equation of motion, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$ becomes

$$\frac{d}{dt}\left(ml^2\dot{\theta} + mh_0\omega l\sin\omega t\sin\theta\right) - \left(mh_0\omega l\dot{\theta}\sin\omega t\cos\theta + mgl\sin\theta\right) = 0$$

or $$ml^2\ddot{\theta} + mh_0\omega^2 l\cos\omega t\sin\theta + mh_0\omega l\dot{\theta}\sin\omega t\cos\theta - \left(mh_0\omega l\dot{\theta}\sin\omega t\cos\theta + mgl\sin\theta\right) = 0$$

$$l^2\ddot{\theta} + h_0\omega^2 l\cos\omega t\sin\theta + h_0\omega l\dot{\theta}\sin\omega t\cos\theta - h_0\omega l\dot{\theta}\sin\omega t\cos\theta - gl\sin\theta = 0$$

that is, $$l^2\ddot{\theta} + h_0\omega^2 l\cos\omega t\sin\theta - gl\sin\theta = 0$$

or $$l\ddot{\theta} + \left(h_0\omega^2\cos\omega t - g\right)\sin\theta = 0 \tag{vi}$$

Since the pendulum is upright, let us substitute; $\theta' = \theta - \pi$ so that $\sin\theta = -\sin\theta'$. If the amplitude of oscillation is small; $\sin\theta' \approx \theta'$, hence, the equation of motion becomes

$$l\ddot{\theta}' + \left(g - h_0\omega^2\cos\omega t\right)\theta' = 0 \tag{vii}$$

It is interesting to note that if the hinge does not oscillate, the equation of motion becomes

$$\ddot{\theta}' + \frac{g}{l}\theta' = 0 \tag{viii}$$

which is the equation of motion of an ordinary simple pendulum.

EXAMPLE 2.60 Four equal point masses m are constrained to move along a circle of radius R. Each mass is coupled to its neighbours by a spring of force constant k. Construct the Lagrangian of the system and obtain the equations of motion.

Solution: Let s_n denote the displacement of each mass from their equilibrium positions. Now, we can write; $s_n = R\theta_n$; so that the generalized coordinates can be taken as θ_n.

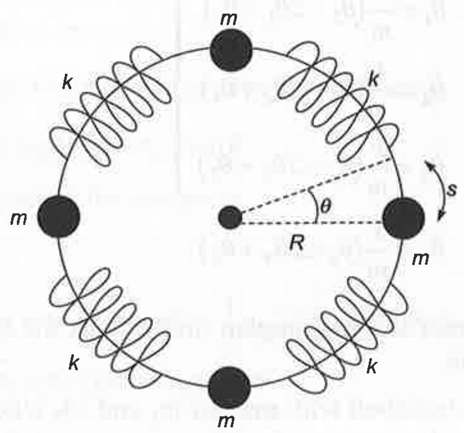

Fig. 2.41

Then, the kinetic energy of the system is

$$T = \frac{1}{2} m \sum_{n=1}^{4} \dot{s}_n^2 = \frac{1}{2} mR^2 \sum_{n=1}^{4} \dot{\theta}_n^2 \tag{i}$$

The potential energy corresponding to the displacement from the mean position is

$$V = \frac{1}{2} k \sum_{n=1}^{4} (s_{n+1} - s_n)^2 = \frac{1}{2} kR^2 \sum_{n=1}^{4} \left[(\theta_n - \theta_{n+1})^2 + (\theta_n - \theta_{n-1})^2 \right] \tag{ii}$$

Note that; $\theta_{4+1} = \theta_1$ and n^{th} mass is attached to both $(n+1)^{th}$ and $(n-1)^{th}$ masses.
Now, the Lagrangian of the system is

$$L = T - V = \frac{1}{2} mR^2 \sum_{n=1}^{4} \dot{\theta}_n^2 - \frac{1}{2} kR^2 \sum_{n=1}^{4} (\theta_{n+1} - \theta_n)^2 \tag{iii}$$

From this we can write

$$\frac{\partial L}{\partial \dot{\theta}_n} = mR^2 \dot{\theta}_n \text{ and } \frac{\partial L}{\partial \theta_n} = -kR^2 (-\theta_{n+1} + 2\theta_n - \theta_{n-1})$$

Then, the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_n} \right) - \frac{\partial L}{\partial \theta_n} = 0$$

that is,

$$\frac{d}{dt}\left(mR^2 \dot{\theta}_n \right) + kR^2 (-\theta_{n+1} + 2\theta_n - \theta_{n-1}) = 0$$

or

$$\ddot{\theta}_n = \frac{k}{m} (\theta_{n+1} - 2\theta_n + \theta_{n-1}) \tag{iv}$$

Therefore, the equations of motion of the masses are

$$\left.\begin{aligned} \ddot{\theta}_1 &= \frac{k}{m} (\theta_2 - 2\theta_1 + \theta_4) \\ \ddot{\theta}_2 &= \frac{k}{m} (\theta_3 - 2\theta_2 + \theta_1) \\ \ddot{\theta}_3 &= \frac{k}{m} (\theta_4 - 2\theta_3 + \theta_2) \\ \ddot{\theta}_4 &= \frac{k}{m} (\theta_1 - 2\theta_4 + \theta_3) \end{aligned}\right\} \tag{v}$$

EXAMPLE 2.61 Construct the Lagrangian and obtain the equation of motion of a dumbbell in the x-y plane.

Solution: We consider a dumbbell with masses m_1 and m_2 whose centres are separated by a distance l. Let (x_1, y_1) and (x_2, y_2) be the coordinates of the centres of masses m_1

and m_2 respectively. Let θ be the angle which the line joining the centres of the masses makes with x-axis. Then we can write

$$x_2 = x_1 + l\cos\theta \text{ and } y_2 = y_1 + l\sin\theta \qquad \text{(i)}$$

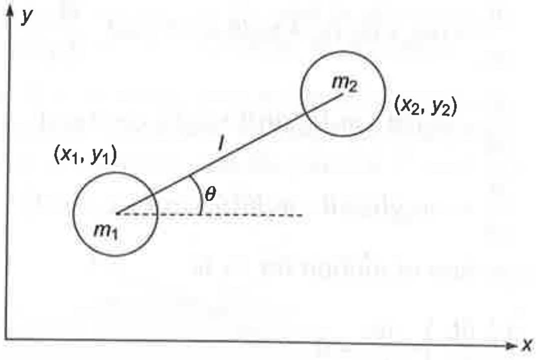

Fig. 2.42

so that $\quad \dot{x}_2 = \dot{x}_1 - l\dot{\theta}\sin\theta$ and $\dot{y}_2 = \dot{y}_1 + l\dot{\theta}\cos\theta \qquad$ (ii)

The kinetic energy of the system is

$$T = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$= \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left[\left(\dot{x}_1 - l\dot{\theta}\sin\theta\right)^2 + \left(\dot{y}_1 + l\dot{\theta}\cos\theta\right)^2\right]$$

This can be expanded and simplified to get

$$T = \frac{1}{2}(m_1 + m_2)\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(l^2\dot{\theta}^2 - 2l\dot{x}_1\dot{\theta}\sin\theta + 2l\dot{y}_1\dot{\theta}\cos\theta\right) \qquad \text{(iii)}$$

The potential energy is

$$V = m_1 g y_1 + m_2 g y_2 = m_1 g y_1 + m_2 g\left(y_1 + l\sin\theta\right)$$

$$= (m_1 + m_2) g y_1 + m_2 g l \sin\theta \qquad \text{(iv)}$$

Therefore, the Lagrangian of the system is

$$L = T - V$$

$$= \begin{bmatrix} \frac{1}{2}(m_1 + m_2)\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(l^2\dot{\theta}^2 - 2l\dot{x}_1\dot{\theta}\sin\theta + 2l\dot{y}_1\dot{\theta}\cos\theta\right) \\ -(m_1 + m_2) g y_1 - m_2 g l \sin\theta \end{bmatrix} \qquad \text{(v)}$$

From the expression for Lagrangian, we can have

$$\frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1 - m_2 l\dot{\theta}\sin\theta \text{ and, } \frac{\partial L}{\partial x_1} = 0$$

Similarly, $$\frac{\partial L}{\partial \dot{y}_1} = (m_1 + m_2)\dot{y}_1 + m_2 l\dot{\theta}\cos\theta \text{ and, } \frac{\partial L}{\partial y_1} = -(m_1 + m_2)g$$

and $$\frac{\partial L}{\partial \dot{\theta}} = m_2 l^2\dot{\theta} - m_2 l\dot{x}_1\sin\theta + m_2 l\dot{y}_1\cos\theta \text{ and}$$

$$\frac{\partial L}{\partial \theta} = -m_2 g l\cos\theta - m_2 l\dot{\theta}(\dot{y}_1\sin\theta + \dot{x}_1\cos\theta)$$

Now, Lagrange's equation of motion for x_1 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

that is, $$\frac{d}{dt}\left[(m_1 + m_2)\dot{x}_1 - m_2 l\dot{\theta}\sin\theta\right] = 0$$

or $$(m_1 + m_2)\ddot{x}_1 - m_2 l\ddot{\theta}\sin\theta - m_2 l\dot{\theta}^2\cos\theta = 0 \tag{vi}$$

Similarly, the Lagrange's equation of motion for y_1 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) - \frac{\partial L}{\partial y_1} = 0$$

that is, $$\frac{d}{dt}\left[(m_1 + m_2)\dot{y}_1 + m_2 l\dot{\theta}\cos\theta\right] + (m_1 + m_2)g = 0$$

or $$(m_1 + m_2)\ddot{y}_1 + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta + (m_1 + m_2)g = 0 \tag{vii}$$

Finally, the Lagrange's equation of motion for the generalized coordinate θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

that is, $$\frac{d}{dt}\left(m_2 l^2\dot{\theta} - m_2 l\dot{x}_1\sin\theta + m_2 l\dot{y}_1\cos\theta\right) + m_2 g l\cos\theta + m_2 l\dot{\theta}(\dot{y}_1\sin\theta + \dot{x}_1\cos\theta) = 0$$

$$\begin{bmatrix} m_2 l^2\ddot{\theta} - m_2 l\ddot{x}_1\sin\theta - m_2 l\dot{x}_1\dot{\theta}\cos\theta + m_2 l\ddot{y}_1\cos\theta - m_2 l\dot{y}_1\dot{\theta}\sin\theta \\ +m_2 g l\cos\theta + m_2 l\dot{\theta}(\dot{y}_1\sin\theta + \dot{x}_1\cos\theta) \end{bmatrix} = 0$$

On simplification, we get

$$m_2 l^2\ddot{\theta} - m_2 l\ddot{x}_1\sin\theta + m_2 l\ddot{y}_1\cos\theta + m_2 g l\cos\theta = 0 \tag{viii}$$

Equations (vi), (vii) and (viii) are the required equations.

EXAMPLE 2.62 Write down Lagrange's equations for a system of two point particles P_1 of mass m_1 and P_2 of mass m_2, with P_1 constrained to move on a circle of radius R and centre O, P_2 constrained to move along the line OP_1, in the presence of the forces F_1 and F_2. The force F_1 with constant magnitude is applied to P_1, and tangential to the circle. The force F_2, is applied to P_2, and is of constant magnitude, parallel to F_1 but with opposite orientation. Hence, obtain the equation of motion.

Solution: The particle P_1 is moving over a circle of constant radius R and can be described by the generalized coordinate θ. However, the particle P_2 is moving along the line joining the centre of the circle and the particle P_1 and therefore the generalized coordinates are (r,θ).

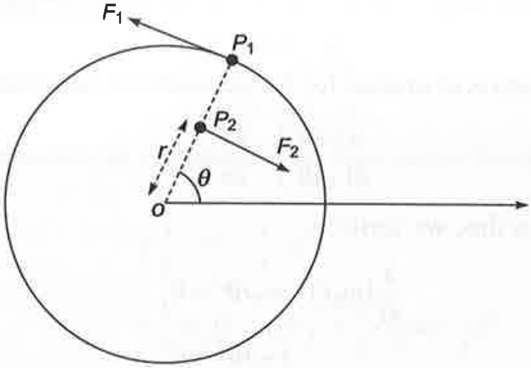

Fig. 2.43

The Cartesian coordinates of m_1 are

$$x_1 = R\cos\theta \text{ and } y_1 = R\sin\theta$$

so that $$\dot{x}_1 = -R\dot{\theta}\sin\theta \text{ and } \dot{y}_1 = R\dot{\theta}\cos\theta \tag{i}$$

Similarly, the Cartesian coordinates of m_2 are

$$x_2 = r\cos\theta \text{ and } y_2 = r\sin\theta$$

Then, $$\dot{x}_2 = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \text{ and } \dot{y}_2 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{ii}$$

The kinetic energy of the system can be written as

$$T = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

Substituting (ii) in this expression and simplifying, we get

$$T = \frac{1}{2}m_1R^2\dot{\theta}^2 + \frac{1}{2}m_2\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{iii}$$

This would yield

$$\left.\begin{aligned} &\frac{\partial T}{\partial \dot{r}} = m_2\dot{r} \text{ and } \frac{\partial T}{\partial r} = m_2 r\dot{\theta}^2 \\ &\text{Similarly,} \quad \frac{\partial T}{\partial \dot{\theta}} = m_1R^2\dot{\theta} + m_2r^2\dot{\theta} \quad \text{and} \quad \frac{\partial T}{\partial \theta} = 0 \end{aligned}\right\} \qquad \text{(iv)}$$

Now, let us consider the components of forces along the radial and tangential directions. Evidently, the component of force along the radial direction is zero; that is

$$F_r = 0 \qquad \text{(v)}$$

However, there is a net force in the tangential direction and the net force in the tangential direction with respect to the centre of the circle can be written as

$$F_\theta = RF_1 - rF_2 \qquad \text{(vi)}$$

The Lagrange's equation of motion for the generalized coordinate r can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = F_r$$

Using (iv) and (v) in this, we write

$$\frac{d}{dt}(m_2\dot{r}) - m_2r\dot{\theta}^2 = 0$$

or

$$\ddot{r} - r\dot{\theta}^2 = 0 \qquad \text{(vii)}$$

Similarly, the Lagrange's equation for θ is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = F_\theta$$

Using (iv) and (vi) we can write

$$\frac{d}{dt}\left(m_1R^2\dot{\theta} + m_2r^2\dot{\theta}\right) = RF_1 - rF_2$$

that is,

$$\left(m_1R^2 + m_2r^2\right)\ddot{\theta} + 2m_2r\dot{r}\dot{\theta} = RF_1 - rF_2 \qquad \text{(viii)}$$

Equations (vii) and (viii) are the required result.

EXAMPLE 2.63 A solid cylinder with centre G and radius a rolling on the rough inside surface of a fixed cylinder with centre O and radius $b > a$. Find the Lagrange equation of motion and deduce the period of small oscillations about the equilibrium position.

Solution: A schematic representation of the problem is given below. From Figure 2.44 it is clear that the angular velocity of the sphere inside the cylinder is $\dot{\theta}$ so that the linear velocity is $(b-a)\dot{\theta}$ and the angular velocity of rotation of the sphere is $\dot{\phi}$.

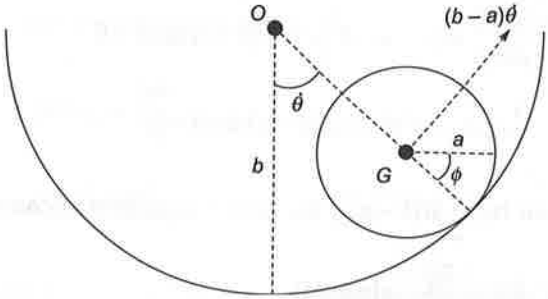

Fig. 2.44

Assuming $\phi = 0$ when $\theta = 0$ we get the constraint equation as;

$$(b-a)\theta - a\phi = 0 \tag{i}$$

This would yield $$(b-a)\dot{\theta} - a\dot{\phi} = 0 \tag{ii}$$

Taking θ as the generalized coordinate, we can write the kinetic energy of the system as

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}m\left[(b-a)\dot{\theta}\right]^2 + \frac{1}{2}\left(\frac{1}{2}ma^2\right)\dot{\phi}^2$$

Now, from (ii) $\dot{\phi} = \left(\dfrac{b-a}{a}\right)\dot{\theta}$ and using this, the expression for kinetic energy becomes

$$T = \frac{1}{2}m\left[(b-a)\dot{\theta}\right]^2 + \frac{1}{2}\left(\frac{1}{2}ma^2\right)\left(\frac{b-a}{a}\right)^2\dot{\theta}^2$$

This can be simplified to get

$$T = \frac{3}{4}m(b-a)^2\dot{\theta}^2 \tag{iii}$$

The potential energy of the sphere with respect to the centre of the cylinder is

$$V = -mg(b-a)\cos\theta \tag{iv}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{3}{4}m(b-a)^2\dot{\theta}^2 + mg(b-a)\cos\theta \tag{v}$$

Then, $$\frac{\partial L}{\partial \dot{\theta}} = \frac{3}{2}m(b-a)^2\dot{\theta} \quad \text{and} \quad \frac{\partial L}{\partial \theta} = -mg(b-a)\sin\theta$$

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

that is, $$\frac{d}{dt}\left(\frac{3}{2}m(b-a)^2\dot{\theta}\right)+mg(b-a)\sin\theta=0$$

or

$$\frac{3}{2}m(b-a)^2\ddot{\theta}+mg(b-a)\sin\theta=0$$

Dividing throughout by $\frac{3}{2}m(b-a)^2$, the above equation becomes

$$\ddot{\theta}+\frac{2g}{3(b-a)}\sin\theta=0 \tag{vi}$$

When the amplitude of oscillation is small we can put, $\sin\theta\approx\theta$ and therefore

$$\ddot{\theta}+\frac{2g}{3(b-a)}\theta=0 \tag{vii}$$

Equation (vii) is same as the equation of motion of a simple pendulum with length, $l=\frac{3}{2}(b-a)$.

Therefore, the period of oscillation of the sphere inside the cylinder can be written as

$$\tau=2\pi\sqrt{\frac{3(b-a)}{2g}} \tag{viii}$$

EXAMPLE 2.64 A particle of mass m and charge q can move freely in the static electric field, $E = E(r)$ and the static magnetic field, $B = B(r)$. The electric and magnetic fields exert a force on the charged particle which is given by the Lorentz force formula. (i) Show that the force can be represented by a velocity dependent potential and (ii) construct the Lagrangian of the system.

Solution:

(i) We start with the Lorentz formula for the force on a charged particle when it is in an electromagnetic field. It is given by

$$F=q\left[E+(v\times B)\right] \tag{i}$$

The electric and magnetic fields can be expressed in terms of a scalar potential Φ and a vector potential A through

$$E=-\nabla\Phi-\frac{\partial A}{\partial t} \quad \text{and} \quad B=\nabla\times A \tag{ii}$$

Then, $$F=q\left[-\nabla\Phi-\frac{\partial A}{\partial t}+(v\times\nabla\times A)\right] \tag{iii}$$

Now, let us take the x-component of each term on the RHS of the above equation. Then,

$$(\nabla\Phi)_x = \frac{\partial \Phi}{\partial x}$$

$$(v\times\nabla\times A)_x = v_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-v_x\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)$$

Expanding the RHS and adding and subtracting the terms $v_x\dfrac{\partial A_x}{\partial x}$, we get

$$(v\times\nabla\times A)_x = \left(v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x}\right)-\left(v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}(v.A)-\left(v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z}\right) \qquad \text{(iv)}$$

Now, the second term on the RHS can be rewritten by considering the total time derivative of the x-component of the vector potential A. It is given by

$$\frac{dA_x}{dt}=\frac{\partial A_x}{\partial t}+\frac{\partial A_x}{\partial x}\frac{dx}{dt}+\frac{\partial A_x}{\partial y}\frac{dy}{dt}+\frac{\partial A_x}{\partial z}\frac{dz}{dt}$$

$$=\frac{\partial A_x}{\partial t}+\frac{\partial A_x}{\partial x}v_x+\frac{\partial A_x}{\partial y}v_y+\frac{\partial A_x}{\partial z}v_z \qquad \text{(v)}$$

Using (v) in (iv), we get

$$(v\times\nabla\times A)_x=\frac{\partial}{\partial x}(v.A)-\frac{dA_x}{dt}+\frac{\partial A_x}{\partial t} \qquad \text{(vi)}$$

Then, the x-component of the Lorentz force can be written as

$$F_x = q\left[-\frac{\partial \Phi}{\partial x}-\frac{\partial A_x}{\partial t}+\frac{\partial}{\partial x}(v.A)-\frac{dA_x}{dt}+\frac{\partial A_x}{\partial t}\right]$$

$$=q\left[-\frac{\partial}{\partial x}(\Phi-v.A)-\frac{dA_x}{dt}\right] \qquad \text{(vii)}$$

Now, the second term on the RHS can be written as

$$\frac{dA_x}{dt}=\frac{d}{dt}\left(\frac{\partial}{\partial v_x}(v.A)\right)=-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}(\Phi-v.A)\right) \qquad \text{(viii)}$$

The insertion of Φ in the above expression is possible since it is independent of velocity.

In view of (viii) we can rewrite (vii) as

$$F_x = q\left[-\frac{\partial}{\partial x}(\Phi - v.A) + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}(\Phi - v.A)\right)\right] = q\left[-\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right)\right] \tag{ix}$$

where, $U = q(\Phi - v.A)$.

Thus, equation (ix) shows that the electromagnetic force on a charged particle can be derived from a velocity dependent potential U.

(ii) Now, our system is a non-conservative one. For such a system, the Lagrange's equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

where, Q_j is the generalized force given by (ix). Therefore,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

On rearranging, this would yield

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T - U)\right) - \frac{\partial}{\partial q_j}(T - U) = 0 \tag{x}$$

Comparing this with the Lagrange's equation of motion we see that the Lagrangian of the system is

$$L = T - U = T - q(\Phi - v.A) \tag{xi}$$

EXAMPLE 2.65 A particle of mass m and charge q is injected with a velocity $v_{0y}\hat{j}$ into a uniform magnetic field $B = B_0\hat{k}$. The particle is released at the origin and motion of the charged particle is in the xy – plane. Obtain the equation of motion of the charged particle and show that the path followed by the particle in this case is a circle. Assume the scalar potential in the region is zero.

Solution: Given that $\Phi = 0$; and $B = \nabla \times A = B_0\hat{k}$

Let us take the possible vector potential that would give $\nabla \times A = B_0\hat{k}$ as $A = -B_0 y\hat{i}$ (one can find other possible forms for the vector potential that would give the same result).

Now, the Lagrangian of the particle can be written as

$$L = T - V = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) - \left(-qv_x A\right)$$

$$= \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) - qB_0 y v_x \tag{i}$$

Since the magnetic field is in the z-direction, there will not be any force due to the magnetic field in the z-direction. Also the magnetic force is in the x-direction and hence, only the x-component of the velocity will vary. Therefore,

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial v_x} = mv_x - qB_0 y \text{ and } \frac{\partial L}{\partial x} = 0 \qquad \text{(ii)}$$

Then the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial v_x}\right) - \frac{\partial R}{\partial x} = \frac{d}{dt}(mv_x - qB_0 y) = 0$$

This would give; $\quad mv_x - qB_0 y = \text{constant} \qquad$ (iii)

The constant can be evaluated from the initial condition at $t = 0; y = 0$ and $v_x = 0$. Therefore, we get the value of the constant as zero. Then,

$$mv_x - qB_0 y = 0$$

$$mv_x = qB_0 y \text{ or; } \dot{x} = \frac{qB_0 y}{m} = \omega_c y \qquad \text{(iv)}$$

where, $\omega_c = \frac{qB_0}{m}$, the cyclotron frequency. For motion in the y-direction

$$\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial v_y} = mv_y \text{ and } \frac{\partial L}{\partial y} = -qB_0 v_x$$

Then, the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial v_y}\right) - \frac{\partial R}{\partial y} = \frac{d}{dt}(mv_y) + qB_0 v_x = 0$$

$$\dot{v}_y + \frac{qB_0}{m} v_x = \ddot{y} + \omega_c v_x = 0$$

that is, $\quad \ddot{y} + \omega_c \dot{x} = 0 \qquad$ (v)

Using (iv) in (v) we get $\ddot{y} + \omega_c^2 y = 0 \qquad$ (vi)

Equation (vi) represents a simple harmonic motion and we have the general solution to equation (vi) as

$$y(t) = A\cos\omega_c t + B\sin\omega_c t \qquad \text{(vii)}$$

The coefficients A and B can be obtained from the initial conditions. When $t = 0$; $y = 0$ hence, $A = 0$. Similarly, when $t = 0$, $\dot{y} = v_{0y}$ so that $B = \frac{v_{0y}}{\omega_c}$ so that

$$y(t) = \frac{v_{0y}}{\omega_c}\sin\omega_c t \qquad \text{(viii)}$$

Using (viii) in (iv), we get

$$\dot{x} = v_{0y} \sin \omega_c t$$

This can be integrated to get

$$x = C - \frac{v_{0y}}{\omega_c} \cos \omega_c t \tag{ix}$$

where, C is the constant of integration. The initial condition that at $t = 0,\ x = 0$ would give the value of the integration constant as; $C = \frac{v_{0y}}{\omega_c}$. Then, (ix) becomes

$$\left(x - \frac{v_{0y}}{\omega_c} \right) = -\frac{v_{0y}}{\omega_c} \cos \omega_c t \tag{x}$$

Equations (viii) and (x) represent the motion in y and x directions respectively. Squaring and adding these two equations, we get

$$\left(x - \frac{v_{0y}}{\omega_c} \right)^2 + y^2(t) = \frac{v_{0y}^2}{\omega_c^2} \tag{xi}$$

Clearly (xi) represents a circle whose radius is $\frac{v_{0y}}{\omega_c}$ and centred at $\left(\frac{v_{0y}}{\omega_c}, 0 \right)$.

EXAMPLE 2.66 A velocity dependent potential is given by; $U = q\left[\Phi(r,t) - \dot{r}.A(r,t)\right]$.

(a) Show that this would satisfy the Lorentz force equation $F = q(E + v \times B)$ that acts on a charge q moving with velocity v in the general electrodynamic field by Lagrange's method.

(b) Show that the potentials $\Phi = 0$ and $A = tz\hat{i}$ generate a field that satisfies all four Maxwell equations in free space.

(c) A particle of mass m and charge q moves in this field. Find the Lagrangian of the particle in terms of Cartesian coordinates. Show that x and y are cyclic coordinates and find the conserved momenta P_x, P_y.

Solution:

(a) Given that, $U = q\left[\Phi(r,t) - \dot{r}.A(r,t)\right]$

$$= q\left[\Phi(r,t) - \left(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z\right)\right]$$

Then $\frac{\partial U}{\partial \dot{x}} = -A_x$ and $\frac{\partial U}{\partial x} = \frac{\partial \Phi}{\partial x} - \dot{x}\frac{\partial A_x}{\partial x} - \dot{y}\frac{\partial A_y}{\partial x} - \dot{z}\frac{\partial A_z}{\partial x}$ (i)

The above result is written by dropping q for a moment. Using chain rule, we can write

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}} \right) = -\dot{x}\frac{\partial A_x}{\partial x} - \dot{y}\frac{\partial A_x}{\partial y} - \dot{z}\frac{\partial A_x}{\partial z} - \frac{\partial A_x}{\partial t} \tag{ii}$$

Then,

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right)-\frac{\partial U}{\partial x}=\begin{bmatrix} -\dot{x}\dfrac{\partial A_x}{\partial x}-\dot{y}\dfrac{\partial A_x}{\partial y}-\dot{z}\dfrac{\partial A_x}{\partial z}-\dfrac{\partial A_x}{\partial t} \\ -\left(\dfrac{\partial \Phi}{\partial x}-\dot{x}\dfrac{\partial A_x}{\partial x}-\dot{y}\dfrac{\partial A_y}{\partial x}-\dot{z}\dfrac{\partial A_z}{\partial x}\right) \end{bmatrix}$$

$$=-\dot{x}\frac{\partial A_x}{\partial x}-\dot{y}\frac{\partial A_x}{\partial y}-\dot{z}\frac{\partial A_x}{\partial z}-\frac{\partial A_x}{\partial t}-\frac{\partial \Phi}{\partial x}+\dot{x}\frac{\partial A_x}{\partial x}+\dot{y}\frac{\partial A_y}{\partial x}+\dot{z}\frac{\partial A_z}{\partial x}$$

$$=-\frac{\partial A_x}{\partial t}-\frac{\partial \Phi}{\partial x}+\dot{y}\left(\frac{\partial A_x}{\partial y}+\frac{\partial A_y}{\partial x}\right)-\dot{z}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)$$

$$=-(\nabla\Phi)_x-\left(\frac{\partial A}{\partial t}\right)_x+\dot{y}(\nabla\times A)_z-\dot{z}(\nabla\times A)_y$$

$$=-(\nabla\Phi)_x-\left(\frac{\partial A}{\partial t}\right)_x+\dot{r}\times(\nabla\times A)_x$$

$$=E_x+(\dot{r}\times B)_x \qquad \text{(iii)}$$

The LHS of (iii) is the generalized force and hence RHS on multiplication with q represents the x-component of the Lorentz force. Similar argument is valid for other components also.

(b) When $\Phi=0$ and $A=tz\hat{i}$

$$E=-\nabla\Phi-\frac{\partial A}{\partial t}=-z\hat{i} \qquad \text{(iv)}$$

and $$B=\nabla\times A=\nabla\times\left(tz\hat{i}\right)=(\nabla tz)\times\hat{i}=t\left(\hat{k}\times\hat{i}\right)=t\,\hat{j} \qquad \text{(v)}$$

Then, using these two expressions let us try to obtain the Maxwell equations. Thus, we obtain

$$\nabla.E=0$$

$$\nabla\times E=\nabla\times\left(-z\hat{i}\right)=-(\nabla z)\times\hat{i}=-\hat{k}\times\hat{i}=-\hat{j}=-\frac{\partial B}{\partial t}$$

$$\nabla.B=0$$

$$\nabla\times B=\nabla\times t\,\hat{j}=0=\frac{\partial E}{\partial t}$$

Thus, all the four Maxwell's equations for free space are satisfied.

(c) For the particle with mass m and charge q moving in this field, one can write the Lagrangian as

$$L=T-U=\frac{1}{2}m\dot{r}^2-q\Phi+q(\dot{r}.A)$$

$$=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+q\left(\dot{x}\hat{i}+\dot{y}\hat{j}+\dot{z}\hat{k}\right).tz\hat{i}$$

$$=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+qtz\dot{x} \tag{vi}$$

Thus from the above expression it is clear that x and y are cyclic coordinates. Then the corresponding momenta are;

$$P_x=\frac{\partial L}{\partial \dot{x}}=m\dot{x}+etz \tag{vii}$$

and
$$P_y=\frac{\partial L}{\partial \dot{y}}=m\dot{y} \tag{viii}$$

These are the required results.

EXAMPLE 2.67 A particle of mass m and charge q is moving in xy-plane under an electric and magnetic field described by the potentials $\Phi=-E_0y$ and

$A=-B_0\left[\alpha y\hat{i}-(1-\alpha)x\hat{j}\right]$, where α is a parameter; E_0 and B_0 are constants.

(a) Deduce Lagrange's equations of motion.
(b) If, at $t=0$, the particle is at rest at the origin, obtain the equations of motion when $\alpha=1$.

Solution:

(a) Since the particle is moving in the xy-plane, the Lagrangian of the particle is;

$$L=T-U=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+q(\Phi-v.A) \tag{i}$$

Now, $v.A=-B_0\left(\dot{x}\hat{i}+\dot{y}\hat{j}\right).\left[\alpha y\hat{i}-(1-\alpha)x\hat{j}\right]=-B_0\left[\alpha y\dot{x}-(1-\alpha)x\dot{y}\right]$, then;

$$L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+qE_oy-qB_0\left[\alpha y\dot{x}-(1-\alpha)x\dot{y}\right] \tag{ii}$$

From (ii) we get

$$\frac{\partial L}{\partial \dot{x}}=m\dot{x}-qB_0\alpha y \text{ and } \frac{\partial L}{\partial x}=qB_0(1-\alpha)\dot{y} \tag{iii}$$

Similarly,
$$\frac{\partial L}{\partial \dot{y}}=m\dot{y}+qB_0(1-\alpha)x \text{ and } \frac{\partial L}{\partial y}=-qB_0\alpha\dot{x} \tag{iv}$$

Now, the Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y}=0$$

Substituting from (iii) and (iv), Lagrange's equation becomes

$$\frac{d}{dt}\left(m\dot{x} - qB_0\alpha y\right) - qB_0(1-\alpha)\dot{y} = 0$$

or $$m\ddot{x} - qB_0\alpha\dot{y} - qB_0(1-\alpha)\dot{y} = 0$$

that is, $$m\ddot{x} - qB_0\dot{y} = 0 \tag{v}$$

Similarly, the equation for y is

$$\frac{d}{dt}\left(m\dot{y} + qB_0(1-\alpha)x\right) + qB_0\alpha\dot{x} = 0$$

or $$m\ddot{y} + qB_0(1-\alpha)\dot{x} + qB_0\alpha\dot{x} = 0$$

that is, $$m\ddot{y} + qB_0\dot{x} = 0 \tag{vi}$$

Equations (v) and (vi) are the Lagrange's equations. It is to be noted that both are independent of α.

(b) With $\alpha = 1$, the Lagrangian of the particle becomes

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + qE_o y - qB_0 y\dot{x} \tag{vii}$$

Then, $$\frac{\partial L}{\partial \dot{x}} = m\dot{x} - qB_0 y \text{ and } \frac{\partial L}{\partial x} = 0.$$

Similarly, $$\frac{\partial L}{\partial \dot{y}} = m\dot{y} \text{ and } \frac{\partial L}{\partial y} = qE_0 - qB_0\dot{x}$$

These can be substituted in the Lagrange's equations of motion to get

$$\frac{d}{dt}\left(m\dot{x} - qB_0 y\right) = 0\text{; or } \left(m\dot{x} - qB_0 y\right) = constant \tag{viii}$$

Since $x = y = \dot{x} = \dot{y} = 0$ at $t = 0$; we get the value of the constant as zero. Then;

$$m\dot{x} = qB_0 y \tag{ix}$$

The equation for y is

$$\frac{d}{dt}(m\dot{y}) - qE_0 + qB_0\dot{x} = 0$$

or

$$m\ddot{y} - qE_0 + qB_0\dot{x} = 0$$

Substituting for $\dot{x}$ from (ix) it becomes

$$m\ddot{y} - qE_0 + \frac{q^2B_0^2}{m}y = 0$$

or

$$\ddot{y} - \frac{qE_0}{m} + \frac{q^2B_0^2}{m^2}y = 0$$

This can be written as

$$\ddot{y}+\omega_0^2 y=\frac{qE_0}{m} \tag{x}$$

where, $\omega_0=\dfrac{qB_0}{m}$. The solution of (x) can be obtained as

$$y(t)=A\cos(\omega_0 t+\delta)+\frac{qE_0}{m\omega_0^2} \tag{xi}$$

Applying the initial conditions, we get $A=-\dfrac{qE_0}{m\omega_0^2}$ and $\delta=0$; so that (xi) becomes;

$$y(t)=\frac{qE_0}{m\omega_0^2}[1-\cos\omega_0 t] \tag{xii}$$

From (ix) $$\dot{x}=\omega_0 y=\frac{qE_0}{m\omega_0}[1-\cos\omega_0 t]$$

On integration with respect to $\omega_0 t$, we get

$$x(t)=\frac{qE_0}{m\omega_0^2}[\omega_0 t-\sin\omega_0 t] \tag{xiii}$$

Equations (xii) and (xiii) are the required equations of motion. These equations represent cycloids. Thus, the motion of a charged particle in an electromagnetic field is a cycloid motion. The particle can confine in a circular orbit if the motion is perpendicular to the magnetic field.

EXAMPLE 2.68 A particle of mass m is projected with an initial velocity u at an angle α with the horizontal. Obtain the equation of motion describing the path of the particle by Lagrange's method.

Solution: In this problem, let us neglect the air resistance. Assume the plane of motion as the xy-plane and $P(x,y)$ be the position of the particle at a given instant t. Then the kinetic energy of the particle can be written as

$$T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \tag{i}$$

Taking y as the vertical direction, the potential energy is

$$V=mgy \tag{ii}$$

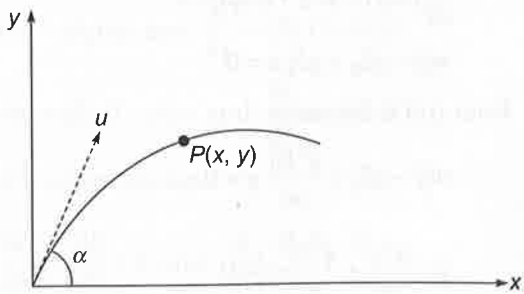

Fig. 2.45

Now, the Lagrangian of the system becomes

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy \tag{iii}$$

From (iii) we get $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $\frac{\partial L}{\partial \dot{y}} = m\dot{y}$ and, similarly, $\frac{\partial L}{\partial x} = 0$; $\frac{\partial L}{\partial y} = -mg$

These expressions can be substituted in the Lagrange's equations; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$ to get

$$\ddot{x} = 0 \text{ and } \ddot{y} + g = 0 \tag{iv}$$

This shows that there is no acceleration in the x–direction and the acceleration along the vertical is equal to the acceleration due to gravity. This is obvious as there is no other force acting on the particle except the gravitational force.

Now, the expressions in (iv) can be integrated to get

$$\dot{x} = A \tag{v}$$

and
$$\dot{y} = gt + C \tag{vi}$$

Differentiating once again, we get

$$x = At + B \tag{vii}$$

and
$$y = -\frac{1}{2}gt^2 + Ct + D \tag{viii}$$

The coefficients A, B, C and D are the integration constants and can be determined from the initial conditions.

At $t = 0$ we have $x = 0$, $\dot{x} = u\cos\alpha$, $y = 0$ and $\dot{y} = u\sin\alpha$

Using the first two conditions in (v) and (vii), we get $A = u\cos\alpha$ and $B = 0$ so that

$$x = (u\cos\alpha)t \tag{ix}$$

Similarly, using the second two initial conditions in (vi) and (viii) we get; $C = u\sin\alpha$ and $D = 0$ so that

$$y = (u\sin\alpha)t - \frac{1}{2}gt^2 \tag{x}$$

From (ix); $t = \frac{x}{u\cos\alpha}$ and using this in (x) we obtain the equation of motion as

$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha} \tag{xi}$$

This equation represents a parabola and hence the path of a projectile is a parabola.

EXAMPLE 2.69 Consider the motion of a particle of mass m in one dimension. The Lagrangian of the particle is given by; $L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x)$. Obtain the equation of motion by Lagrange's method.

Solution: Given:
$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x) \tag{i}$$

Then,
$$\frac{\partial L}{\partial \dot{x}} = \frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x) \text{ and } \frac{\partial L}{\partial x} = m\dot{x}^2\frac{\partial V}{\partial x} - 2V(x)\frac{\partial V}{\partial x}$$

These can be substituted in the Lagrange's equation of motion
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{ii}$$

to get,
$$\frac{d}{dt}\left(\frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x)\right) - \left(m\dot{x}^2\frac{\partial V}{\partial x} - 2V(x)\frac{\partial V}{\partial x}\right) = 0$$

or
$$m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V(x) + 2m\dot{x}\frac{\partial V}{\partial t} - \left(m\dot{x}^2\frac{\partial V}{\partial x} - 2V(x)\frac{\partial V}{\partial x}\right) = 0$$

that is,
$$m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V(x) + 2m\dot{x}\frac{\partial V}{\partial x}\frac{dx}{dt} - m\dot{x}^2\frac{\partial V}{\partial x} + 2V(x)\frac{\partial V}{\partial x} = 0$$

or
$$m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V(x) + \left(2m\dot{x}^2\frac{\partial V}{\partial x} - m\dot{x}^2\frac{\partial V}{\partial x}\right) + 2V(x)\frac{\partial V}{\partial x} = 0$$

This can be written as
$$m\dot{x}^2\left(m\ddot{x} + \frac{\partial V}{\partial x}\right) + 2V(x)\left(m\ddot{x} + \frac{\partial V}{\partial x}\right) = 0$$

or
$$\left(m\ddot{x} + \frac{\partial V}{\partial x}\right)\left(m\dot{x}^2 + 2V(x)\right) = 0$$

or
$$2\left(m\ddot{x} + \frac{\partial V}{\partial x}\right)\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) = 0 \tag{iii}$$

Here, $\frac{1}{2}m\dot{x}^2 + V(x)$ is the total energy of the system which is a nonzero quantity. Therefore, the equation of motion is
$$m\ddot{x} + \frac{\partial V}{\partial x} = 0 \tag{iv}$$

EXAMPLE 2.70 The kinetic and potential energies of a dynamical system are given respectively as; $T=\frac{1}{2}\left[(1+2k)\dot{\theta}^2+2\dot{\theta}\dot{\phi}+\dot{\phi}^2\right]$ and $V=\frac{\eta^2}{2}\left[(1+k)\theta^2+\phi^2\right]$ where, θ and ϕ are coordinates and, k and η are positive quantities. Obtain the equation of motion by Lagrange's method and show that $(\ddot{\theta}-\ddot{\phi})+\eta^2\left(\frac{1+k}{k}\right)(\theta-\phi)=0$.

Solution: Given: $T=\frac{1}{2}\left[(1+2k)\dot{\theta}^2+2\dot{\theta}\dot{\phi}+\dot{\phi}^2\right]$ and $V=\frac{\eta^2}{2}\left[(1+k)\theta^2+\phi^2\right]$ so that the Lagrangian of the system is

$$L=T-V=\frac{1}{2}\left[(1+2k)\dot{\theta}^2+2\dot{\theta}\dot{\phi}+\dot{\phi}^2\right]-\frac{\eta^2}{2}\left[(1+k)\theta^2+\phi^2\right] \tag{i}$$

Then, $$\frac{\partial L}{\partial \dot{\theta}}=\frac{1}{2}\left[(1+2k)2\dot{\theta}+2\dot{\phi}\right]=(1+2k)\dot{\theta}+\dot{\phi}$$

and $$\frac{\partial L}{\partial \theta}=-\eta^2(1+k)\theta$$

Therefore, the Lagrange's equation of motion for the coordinate θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=\frac{d}{dt}\left[(1+2k)\dot{\theta}+\dot{\phi}\right]+\eta^2(1+k)\theta=0$$

that is, $$(1+2k)\ddot{\theta}+\ddot{\phi}+\eta^2(1+k)\theta=0 \tag{ii}$$

Also, $$\frac{\partial L}{\partial \dot{\phi}}=\dot{\theta}+\dot{\phi} \quad \text{and} \quad \frac{\partial L}{\partial \phi}=-\eta^2\phi$$

so that the Lagrange's equation for the coordinate ϕ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=\frac{d}{dt}(\dot{\theta}+\dot{\phi})+\eta^2\phi=0$$

or $$\ddot{\theta}+\ddot{\phi}+\eta^2\phi=0 \tag{iii}$$

Equations (ii) and (iii) are the required equations of motion. Now multiply (iii) with $(1+k)$ and subtract from (ii), we get

$$\left[(1+2k)\ddot{\theta}+\ddot{\phi}+\eta^2(1+k)\theta\right]-\left[(\ddot{\theta}+\ddot{\phi}+\eta^2\phi)(1+k)\right]=0$$

This can be simplified to obtain

$$k(\ddot{\theta}-\ddot{\phi})+\eta^2(1+k)(\theta-\phi)=0 \tag{iv}$$

Hence proved.

EXAMPLE 2.71 Prove that if F does not depend on time explicitly, then, $F-\dot{y}\dfrac{\partial F}{\partial \dot{y}}$ is a constant of motion. Here, $\dot{y}=\dfrac{dy}{dt}$.

Solution: We consider the Euler-Lagrangian equation for the function $F(y,\dot{y})$. It is given by

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)-\frac{\partial F}{\partial y}=0 \tag{i}$$

Multiplying by $\dot{y}$, and adding and subtracting $\ddot{y}\dfrac{\partial F}{\partial \dot{y}}$ the above equation becomes

$$\dot{y}\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)-\dot{y}\frac{\partial F}{\partial y}+\ddot{y}\frac{\partial F}{\partial \dot{y}}-\ddot{y}\frac{\partial F}{\partial \dot{y}}=0$$

or

$$\left[\dot{y}\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)+\ddot{y}\frac{\partial F}{\partial \dot{y}}\right]-\dot{y}\frac{\partial F}{\partial y}-\ddot{y}\frac{\partial F}{\partial \dot{y}}=0$$

This can be written as

$$\frac{d}{dt}\left(\dot{y}\frac{\partial F}{\partial \dot{y}}\right)-\dot{y}\frac{\partial F}{\partial y}-\ddot{y}\frac{\partial F}{\partial \dot{y}}=0 \tag{ii}$$

Now, add and subtract $\dfrac{\partial F}{\partial t}$ to the above expression

$$\frac{d}{dt}\left(\dot{y}\frac{\partial F}{\partial \dot{y}}\right)-\dot{y}\frac{\partial F}{\partial y}-\ddot{y}\frac{\partial F}{\partial \dot{y}}+\frac{\partial F}{\partial t}-\frac{\partial F}{\partial t}=0$$

or

$$\frac{d}{dx}\left(\dot{y}\frac{\partial F}{\partial \dot{y}}\right)-\left(\ddot{y}\frac{\partial F}{\partial \dot{y}}+\dot{y}\frac{\partial F}{\partial y}+\frac{\partial F}{\partial x}\right)+\frac{\partial F}{\partial x}=0$$

that is,

$$\frac{d}{dt}\left(\dot{y}\frac{\partial F}{\partial \dot{y}}\right)-\frac{dF}{dt}+\frac{\partial F}{\partial t}=0$$

Since $\dfrac{\partial F}{\partial t}=0$, we have; $\dfrac{d}{dt}\left(\dot{y}\dfrac{\partial F}{\partial \dot{y}}\right)-\dfrac{dF}{dt}=0$

or

$$\frac{d}{dt}\left(\dot{y}\frac{\partial F}{\partial \dot{y}}-F\right)=0$$

This means, $\dot{y}\dfrac{\partial F}{\partial \dot{y}}-F=\text{constant}$, hence proved.

EXAMPLE 2.72 If L is the Lagrangian of a system of n degrees of freedom, satisfying Lagrange's equation of motion, show that $L' = L + \frac{dF}{dt}$ also satisfies the Lagrange's equation of motion, where $F \equiv F(q_1, q_2, \ldots\ldots, q_n, t)$ is an arbitrary differentiable function of its arguments.

Solution: Given: $L' = L + \frac{dF}{dt}$. Now, let us make a substitution in the Lagrange's equation of motion and see if it would yield zero so that the statement is proved.

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{i}$$

Now,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t}\right)\right) - \frac{\partial}{\partial q_j}\left(L + \frac{\partial F}{\partial t}\right)$$

$$= \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right\} + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial F}{\partial t}\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t}\right)$$

Now add and subtract the term $\frac{\partial}{\partial q_j}\left(\sum_i \frac{\partial F}{\partial q_i}\frac{dq_i}{dt}\right)$ to the above expression. Then,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right\} + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial F}{\partial t}\right)\right) + \frac{\partial}{\partial q_j}\left(\sum_i \frac{\partial F}{\partial q_i}\frac{dq_i}{dt}\right)$$

$$- \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t}\right) - \frac{\partial}{\partial q_j}\left(\sum_i \frac{\partial F}{\partial q_i}\frac{dq_i}{dt}\right)$$

that is,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right\} + \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q_i}\frac{dq_i}{dt}\right)\right\}$$

$$- \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q_i}\frac{dq_i}{dt}\right) \tag{ii}$$

In this expression the first curly bracket on RHS is equal to zero. Then,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\left(\frac{\partial F}{\partial t}\right) + \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{\partial F}{\partial q_i}\frac{dq_i}{dt}\right)\right\} - \frac{\partial^2 F}{\partial q_j \partial t} - \frac{\partial}{\partial q_j}\left(\sum_i \frac{\partial F}{\partial q_i}\frac{dq_i}{dt}\right)$$

$$= \frac{\partial^2 F}{\partial q_j \partial t} + \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i - \frac{\partial^2 F}{\partial q_j \partial t} - \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i$$

$$= 0$$

Hence proved.

Alternate Method: The total time derivative of the function F is given by

$$\frac{dF}{dt}=\frac{\partial F}{\partial t}+\sum_j \frac{\partial F}{\partial q_j}\dot{q}_j \tag{i}$$

Now $L' = L + \frac{dF}{dt}$ so that the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right)-\frac{\partial L'}{\partial q_j}=\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(L+\frac{dF}{dt}\right)\right)-\frac{\partial}{\partial q_j}\left(L+\frac{dF}{dt}\right)=0$$

Let us find each term separately.

$$\frac{\partial L'}{\partial \dot{q}_k}=\frac{\partial}{\partial \dot{q}_k}\left(L+\frac{\partial F}{\partial t}+\sum_j \frac{\partial F}{\partial q_j}\dot{q}_j\right)$$

and

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_k}\right)=\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(L+\frac{\partial F}{\partial t}+\sum_j \frac{\partial F}{\partial q_j}\dot{q}_j\right)\right]$$

$$=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)+\frac{\partial^2 F}{\partial q_k \partial t}+\sum_j \dot{q}_j \frac{\partial^2 F}{\partial q_j \partial q_k} \tag{ii}$$

Similarly,

$$\frac{\partial L'}{\partial q_k}=\frac{\partial}{\partial q_k}\left(L+\frac{\partial F}{\partial t}+\sum_j \frac{\partial F}{\partial q_j}\dot{q}_j\right)$$

$$=\frac{\partial L}{\partial q_k}+\frac{\partial^2 F}{\partial q_k \partial t}+\sum_j \dot{q}_j \frac{\partial^2 F}{\partial q_k \partial q_j} \tag{iii}$$

From (ii) and (iii) it can be easily seen that

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_k}\right)-\frac{\partial L'}{\partial q_k}=0$$

Hence, the new Lagrangian also satisfies the Lagrange's equation of motion. This proves that Lagrangian is not unique and is invariant under a gauge transformation.

EXAMPLE 2.73 A particle of mass m is confined to move on the surface of a cone of apex angle α such that $\sqrt{x^2+y^2}=z\tan\alpha$ under the gravitational field. Construct the Lagrangian of the particle and obtain the equation of motion.

Solution: A schematic diagram of the problem is given in Figure 2.46. The instantaneous position of the particle is given by

$$r=\sqrt{x^2+y^2+z^2}=\sqrt{z^2\tan^2\alpha+z^2}=z\sqrt{1+\tan^2\alpha}=z\sec\alpha$$

The Lagrangian for this constrained mechanical system is expressed in terms of the generalized coordinates (r,ϕ) where r denotes the distance from the cone's apex and ϕ is the standard polar angle in the (x, y)-plane. Then the kinetic and potential energies of the particle can be written as

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\sin^2\alpha\right)$$

and $$V=mgz=mgr\cos\alpha$$

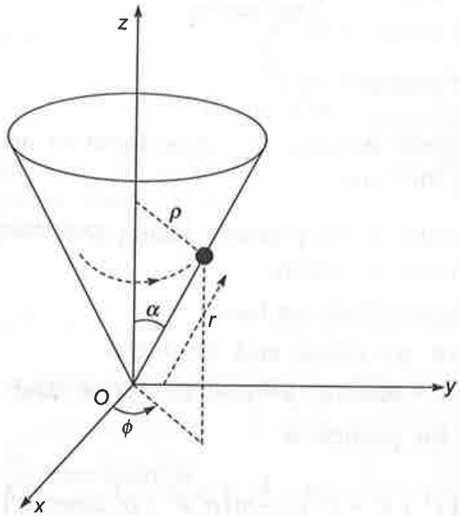

Fig. 2.46

Therefore, the Lagrangian of the particle is

$$L=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\sin^2\alpha\right)-mgr\cos\alpha \tag{i}$$

It is to be noted that the Lagrangian is independent of the polar angle ϕ so that the corresponding canonical angular momentum p_ϕ is conserved.

But, $$p_\phi=\frac{\partial L}{\partial\dot{\phi}}=mr^2\dot{\phi}\sin^2\alpha \tag{ii}$$

From (i) we get,

$$\frac{\partial L}{\partial\dot{r}}=m\dot{r} \text{ and } \frac{\partial L}{\partial r}=mr\dot{\phi}^2\sin^2\alpha-mg\cos\alpha$$

This can be substituted in the Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right)-\frac{\partial L}{\partial r}=0 \tag{iii}$$

we get $$\frac{d}{dt}(m\dot{r}) - \left(mr\dot{\phi}^2 \sin^2\alpha - mg\cos\alpha\right) = 0$$

or $$m\ddot{r} - mr\dot{\phi}^2 \sin^2\alpha + mg\cos\alpha = 0$$

that is, $$\ddot{r} + g\cos\alpha = r\dot{\phi}^2 \sin^2\alpha \qquad \text{(iv)}$$

We can make a substitution for $\dot{\phi}$ from (ii) so that (iv) becomes

$$\ddot{r} + g\cos\alpha = r\left(\frac{p_\phi}{mr^2\sin^2\alpha}\right)^2 \sin^2\alpha$$

or $$\ddot{r} + g\cos\alpha = \frac{p_\phi^2}{m^2 r^3 \sin^2\alpha} \qquad \text{(v)}$$

In this expression $g\cos\alpha$ denotes the component of acceleration due to gravity parallel to the surface of the cone.

EXAMPLE 2.74 Repeat the above problem taking $\rho = z\tan\alpha$, the projection of r on the z–axis as the generalized coordinate.

Solution: Referring to Figure (2.46) we have

$$x = \rho\cos\phi,\ y = \rho\sin\phi \ \text{ and } \ z = \rho\cot\alpha$$

so that $$\dot{x} = \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi,\ \dot{y} = \dot{\rho}\sin\phi + \rho\cos\phi \ \text{ and } \ \dot{z} = \dot{\rho}\cot\alpha$$

The kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\rho^2\dot{\phi}^2 + \dot{\rho}^2 \operatorname{cosec}^2\alpha\right)$$

and potential energy is

$$V = mgz = mg\rho\cot\alpha$$

Now, the Lagrangian of the particle is

$$L = T - V = \frac{1}{2}m\left(\rho^2\dot{\phi}^2 + \dot{\rho}^2 \operatorname{cosec}^2\alpha\right) - mg\rho\cot\alpha \qquad \text{(i)}$$

Then, $$\frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}\operatorname{cosec}^2\alpha \ \text{ and } \ \frac{\partial L}{\partial \rho} = m\rho\dot{\phi}^2 - mg\cot\alpha$$

The Lagrange's equation of motion for ρ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 0 \qquad \text{(ii)}$$

that is, $$\frac{d}{dt}\left(m\dot{\rho}\operatorname{cosec}^2\alpha\right) - \left(m\rho\dot{\phi}^2 - mg\cot\alpha\right) = 0$$

or $$\ddot{\rho}\operatorname{cosec}^2\alpha - \rho\dot{\phi}^2 + g\cot\alpha = 0 \qquad \text{(iii)}$$

Similarly, $\dfrac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi}$ and $\dfrac{\partial L}{\partial \phi} = 0$

Therefore, the Lagrange's equation of motion for the coordinate ϕ becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=\frac{d}{dt}\left(m\rho^2\dot{\phi}\right)-0=0$$

or $$m\rho^2\dot{\phi}=\text{constant} \tag{iv}$$

But, $\dfrac{\partial L}{\partial \dot{\phi}}=p_\phi$ is the generalized momentum and from (iv) it is clear that the angular momentum of the particle about the z–axis is conserved.

EXAMPLE 2.75 A particle of mass m is confined to move on the surface of a cone of apex angle α such that $\sqrt{x^2+y^2}=z\tan\alpha$ under the gravitational field. Obtain the equation of motion by Routh's method of elimination of cyclic coordinate.

Solution: In Routh's procedure, the cyclic coordinates are eliminated by introducing their corresponding momenta. Thus, a reduced Lagrangian called Routhian is defined.

Referring problem (2.71) and figure (2.46) we can write the Lagrangian of the system as

$$L=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\sin^2\alpha\right)-mgr\cos\alpha \tag{i}$$

Since the Lagrangian is cyclic in ϕ, the corresponding momentum p_ϕ is conserved and it is given by

$$p_\phi=\frac{\partial L}{\partial \dot{\phi}}=mr^2\dot{\phi}\sin^2\alpha \tag{ii}$$

Now, the Routhian of the particle is defined as

$$R\left(r,\dot{r},p_\phi\right)=L\left(r,\dot{r},\dot{\phi}\right)-p_\phi\dot{\phi} \tag{iii}$$

that is, $$R\left(r,\dot{r},p_\phi\right)=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\sin^2\alpha\right)-mgr\cos\alpha-p_\phi\dot{\phi}$$

Now, we can substitute for $\dot{\phi}$ from (ii) to get

$$R\left(r,\dot{r},p_\phi\right)=\frac{1}{2}m\left[\dot{r}^2+r^2\left(\frac{p_\phi}{mr^2\sin^2\alpha}\right)^2\sin^2\alpha\right]-mgr\cos\alpha-\frac{p_\phi^2}{mr^2\sin^2\alpha}$$

$$=\frac{1}{2}m\dot{r}^2-\left(mgr\cos\alpha+\frac{p_\phi^2}{2mr^2\sin^2\alpha}\right) \tag{iv}$$

The above expression can be written in terms of an effective potential V given by

$$V(r)=mgr\cos\alpha+\frac{p_\phi^2}{2mr^2\sin^2\alpha} \tag{v}$$

Therefore, $$R(r,\dot{r},p_\phi) = \frac{1}{2}m\dot{r}^2 - V(r) \tag{vi}$$

Now, from (vi) we get $\dfrac{\partial R}{\partial \dot{r}} = m\dot{r}$ and $\dfrac{\partial S}{\partial r} = -V'(r)$

The Lagrange's equation in terms of Routhian is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = 0 \tag{vii}$$

or

$$\frac{d}{dt}(m\dot{r}) + V'(r) = 0$$

that is, $$m\ddot{r} + V'(r) = 0 \tag{viii}$$

From (v) one can determine the value of r corresponding to the minimum of the effective potential and it is given by

$$r_0 = \left(\frac{p_\phi^2}{m^2 g \sin^2\alpha\cos\alpha}\right)^{1/3} \tag{ix}$$

And the minimum value of the potential is

$$V_0 \equiv V(r_0) = \frac{3}{2}mgr_0\cos\alpha \tag{x}$$

This is obtained from (v) by putting $r = r_0$ and eliminating p_ϕ using (ix).

EXAMPLE 2.76 According to Noether's theorem, for each symmetry of the Lagrangian, there is a conserved quantity. Prove this statement assuming the Lagrangian is invariant under a coordinate transformation given by; $q_j \to q_j + \varepsilon K_j(q)$, where ε is an infinitesimal first order parameter and $K_j(q)$ may be a function of all q_j's.

Solution: Given that the Lagrangian is invariant under a coordinate transformation

$$q_j \to q_j + \varepsilon K_j(q) \tag{i}$$

The Lagrangian is invariant at the first order in ε means, $\dfrac{dL}{d\varepsilon} = 0$. Assuming there is no explicit time dependence, this condition can be written as

$$\sum_j \left(\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \varepsilon} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \varepsilon}\right) = 0 \tag{ii}$$

The condition (i) can be used to rewrite (ii) as

$$\sum_j \left(\frac{\partial L}{\partial q_j}K_j + \frac{\partial L}{\partial \dot{q}_j}\dot{K}_j\right) = 0 \tag{iii}$$

The Euler-Lagrangian equation is

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] = 0 \tag{iv}$$

Now make a substitution for $\frac{\partial L}{\partial q_j}$ from (iii) so that the Euler-Lagrangian equation becomes

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) K_j + \frac{\partial L}{\partial \dot{q}_j} \dot{K}_j \right] = 0$$

or

$$\frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j} K_j \right) = 0 \tag{v}$$

That is, the quantity $\sum_j \frac{\partial L}{\partial \dot{q}_j} K_j$ does not change with time. It is known as the conserved momentum and need not have the dimensions of linear momentum. Thus, for each symmetry of the Lagrangian, there is a conserved quantity. Hence, the theorem is proved.

EXAMPLE 2.77 A massless rod of length l is hinged at the extremity of a vertical spring that is fixed to the ground. A point mass m rests on the rod. Assuming harmonic motion of the spring, construct the Lagrangian, of the system and obtain the equation of motion of the point mass.

Solution: Let $y(t)$ be the height of the spring at any instant and y_0 its height when $t = 0$. Since the spring is executing harmonic vibrations, we have

$$y(t) = y_0 \cos \omega t$$

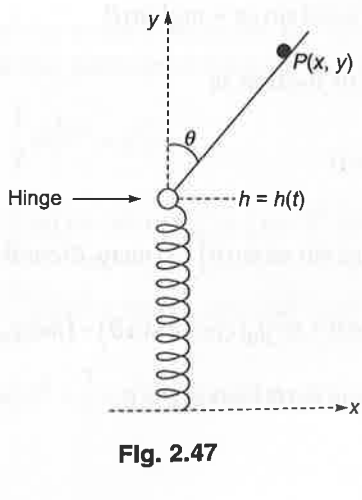

Fig. 2.47

The coordinates of the point mass are given by

$$x = l\sin\theta \text{ and } y = y_0\cos\omega t + l\cos\theta$$

so that $\dot{x} = l\dot{\theta}\cos\theta$ and $\dot{y} = -\left(\omega y_0\sin\omega t + l\dot{\theta}\sin\theta\right)$

Now, the kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) = \frac{1}{2}m\left[\left(l\dot{\theta}\cos\theta\right)^2 + \left(\omega y_0\sin\omega t + l\dot{\theta}\sin\theta\right)^2\right]$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}^2\cos^2\theta + \omega^2 y_0^2\sin^2\omega t + l^2\dot{\theta}^2\sin^2\theta + 2\omega y_0 l\dot{\theta}\sin\omega t\sin\theta\right]$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}^2 + \omega^2 y_0^2\sin^2\omega t + 2\omega y_0 l\dot{\theta}\sin\omega t\sin\theta\right] \tag{i}$$

The potential energy of the particle is

$$V = mgy = mg\left(y_0\cos\omega t + l\cos\theta\right) \tag{ii}$$

Then, the Lagrangian of the particle is

$$L = T - V$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}^2 + \omega^2 y_0^2\sin^2\omega t + 2\omega y_0 l\dot{\theta}\sin\omega t\sin\theta\right]$$

$$-mg\left(y_0\cos\omega t + l\cos\theta\right) \tag{iii}$$

The canonical momentum is

Then, $$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} + m\omega y_0 l\sin\omega t\sin\theta = m\left(l^2\dot{\theta} + \omega y_0 l\sin\omega t\sin\theta\right)$$

$$\frac{\partial L}{\partial\theta} = m\omega y_0 l\dot{\theta}^2\cos\theta\sin\omega t + mgl\sin\theta$$

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}\left[m\left(l^2\dot{\theta} + \omega y_0 l\sin\omega t\sin\theta\right)\right] - \left(m\omega y_0 l\dot{\theta}\cos\theta\sin\omega t + mgl\sin\theta\right) = 0$$

$$m\left(l^2\ddot{\theta} + \omega y_0 l\dot{\theta}\sin\omega t\cos\theta + \omega^2 y_0 l\cos\omega t\sin\theta\right) - \left(m\omega y_0 l\dot{\theta}\cos\theta\sin\omega t + mgl\dot{\theta}\sin\theta\right) = 0$$

that is, $$l^2\ddot{\theta} + \omega^2 y_0 l\cos\omega t\sin\theta - gl\sin\theta = 0 \tag{iv}$$

This is the equation of motion.

Note: If the hinge is fixed, $y_0 = 0$, then, the above equation reduces to

$$\ddot{\theta} - \frac{g}{l}\sin\theta = 0$$

This represents a simple harmonic motion.

EXAMPLE 2.78 A point mass m slides without friction, under the influence of gravity, along a massive ring of radius a and mass M. The ring is affixed by horizontal springs to two fixed vertical surfaces, as shown in Fig. 2.48. All motions are within the plane of the figure. Obtain the Lagrangian, canonical momenta and the equation of motion of the system.

Solution: Here we take x and θ as the generalized coordinates of the system and k, the force constant of the spring. Also assume at $x = 0$ both springs are at equilibrium. The coordinates of the point mass can be written as

$$X = x + a\sin\theta \text{ and } Y = -a\cos\theta$$

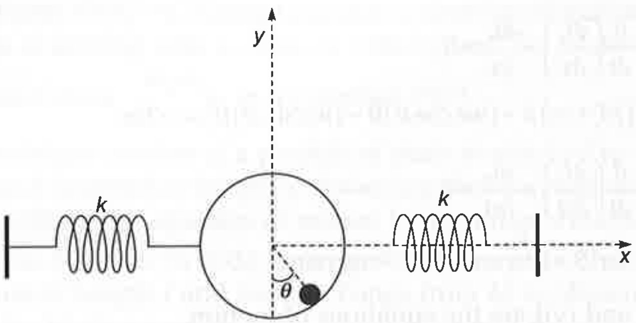

Fig. 2.48

Now, the kinetic energy of the system includes the kinetic energy of the sphere of mass M and kinetic energy of the particle of mass m. It is given by

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x} + a\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}ma^2\dot{\theta}^2\sin^2\theta$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2a\dot{x}\dot{\theta}\cos\theta + a^2\dot{\theta}^2\cos^2\theta\right) + \frac{1}{2}ma^2\dot{\theta}^2\sin^2\theta$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ma^2\dot{\theta}^2 + ma\dot{x}\dot{\theta}\cos\theta \qquad \text{(i)}$$

Potential energy of the system is the sum of potential energies of the two springs and the particle. It is given by

$$V = kx^2 - mga\cos\theta \tag{ii}$$

Therefore, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ma^2\dot{\theta}^2 + ma\dot{x}\dot{\theta}\cos\theta - \left(kx^2 - mga\cos\theta\right) \tag{iii}$$

The canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ma\dot{\theta}\cos\theta \quad \text{and}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} + ma\dot{x}\cos\theta$$

Then, $$\dot{x} = \frac{p_x - ma\dot{\theta}\cos\theta}{(M+m)} \quad \text{and} \quad \dot{\theta} = \frac{p_\theta - ma\dot{x}\cos\theta}{ma^2} \tag{iv}$$

The Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

that is, $$(M+m)\ddot{x} + (ma\cos\theta)\ddot{\theta} - (ma\sin\theta)\dot{\theta}^2 = -2kx \tag{v}$$

Similarly, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

that is, $$ma^2\ddot{\theta} + (ma\cos\theta)\ddot{x} = -mga\sin\theta \tag{vi}$$

Equations (v) and (vi) are the equations of motion.

EXERCISES

2.1 Show that the time required for a particle to move without friction to the minimum point of the cycloid solution of the Brachistochrone problem is $\pi\sqrt{\frac{a}{g}}$.

2.2 A light ray travels in a medium with refractive index $n(y) = n_0 e^{-\beta y}$, where n_0 is the refractive index at $y = 0$ and β is a positive constant. Show that the path of the light ray can be expressed as $y(x,\beta) = \frac{1}{\beta}\ln\left[\frac{\cos(\beta x - \phi)}{\cos\phi}\right]$, where ϕ is the initial direction of the ray measured from the x-axis.

2.3 Using Hamilton's variational principle, find the equations of motion of a particle of mass m moving on a plane in a conservative field.

2.4 Obtain the Lagrangian and the equation of motion for an electrical circuit having inductance L and capacitance C in series.

2.5 A particle of mass m is constrained to slide down a curve $y = V(x)$ under the action of gravity without friction. Show that the Euler-Lagrange equation for this system yields the equation $\ddot{x} = -V'(g + \ddot{V})$, where, $\dot{V} = \dot{x}V'$ and $\ddot{V} = \ddot{x}V' + \dot{x}^2 V''$. Also V' and V'' are the first and second derivatives of V w.r.t. x.

2.6 A system of two degrees of freedom is described by a Lagrangian $L = \frac{1}{2}m(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{1}{2}k(ax^2 + 2bxy + cy^2)$, where, a, b and c are constants with $b^2 \neq ac$. Write down the Lagrange's equations of motion and identify the system.

2.7 The Lagrangian of two particles of masses m_1 and m_2 and coordinates r_1 and r_2 interacting under a potential $V(r_1 - r_2)$ is $L = \frac{1}{2}m_1|\dot{r}_1|^2 + \frac{1}{2}m_2|\dot{r}_2|^2 - V(r_1 - r_2)$. Rewrite the Lagrangian in terms of centre of mass coordinate $R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$ and relative coordinate $r = r_1 - r_2$. Using Lagrange's equation of motion, show that the centre of mass is moving with a constant velocity and the relative motion is like that of a reduced mass $\frac{m_1 m_2}{m_1 + m_2}$ in a potential $V(r)$.

2.8 An elastic pendulum consists of a particle of mass m attached to an elastic string of stiffness k and unstretched length l_0. Assuming that the motion takes place in a vertical plane, obtain the equation of motion by Lagrange's method.

2.9 A block of mass M is free to slide down a frictionless plane inclined at an angle α. A pendulum of length l and mass m hangs from M as shown in Figure 2.49. Assume that the block extends a short distance beyond the side of the plane, so the pendulum can hang down. Find the equations of motion.

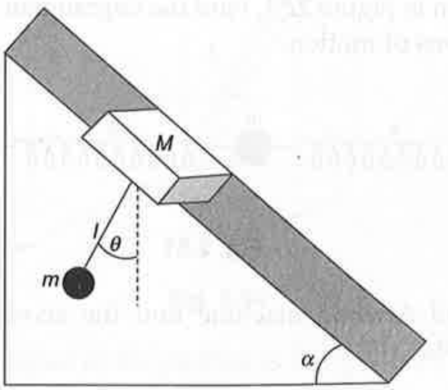

Fig. 2.49

2.10 Four massless rods of length L are hinged together at their ends to form a rhombus. A particle of mass M is attached to each vertex. The opposite corners are joined by springs of spring constant k. In the square configuration, the strings are unstretched. The motion is confined to a plane, and the particles move only along the diagonals of the rhombus. Introduce suitable generalized coordinates and find the Lagrangian of the system. Deduce the equations of motion.

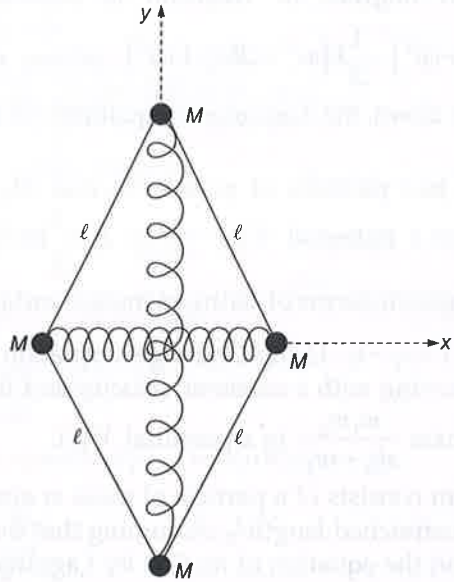

Fig. 2.50

2.11 Two identical beads of mass m each can move without friction along a horizontal wire and are connected to a fixed wall with two identical springs of spring constant k as shown in Figure 2.51. Find the Lagrangian for this system and derive from it the equations of motion.

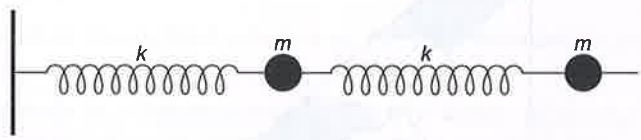

Fig. 2.51

2.12 For the compound Atwood machine find the acceleration of the masses by Lagrange's method.

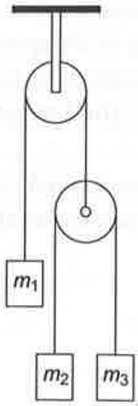

Fig. 2.52

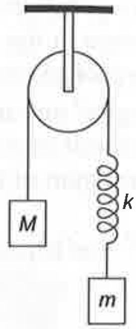

Fig. 2.53

2.13 An Atwood machine is composed of two masses m and M attached by means of a massless rope into which a massless spring (with constant k) is inserted as shown in Figure 2.53. When the spring is in a relaxed state, the spring-rope length is l. Construct the Lagrangian and obtain the equations of motion.

2.14 A particle of mass m is confined to move inside an open cylinder of radius R and bounder to the origin by a spring of force constant k as shown in Figure 2.54.

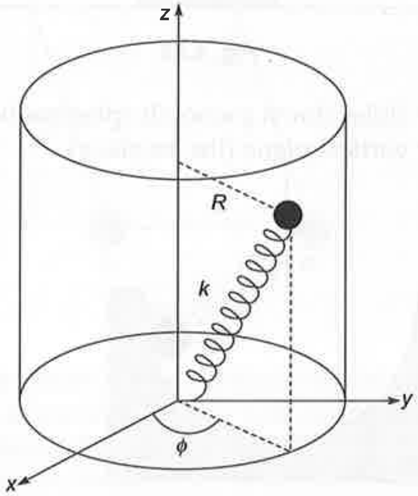

Fig. 2.54

Show that the Lagrangian of the particle is $L = \frac{1}{2}m\left[(R\dot{\theta})^2 + \dot{z}^2\right] - \frac{1}{2}k\left(R^2 + z^2\right)$.

Find the conjugate momenta and obtain the equations of motion of the particle.

2.15 A box of mass M slides horizontally on a frictionless surface. The distance of the box's centre of mass from the origin is denoted by X. Suspended from inside the centre of the box is a pendulum of length l at the bottom of which is a mass m. All the motion takes place in the XY plane. Construct the Lagrangian of the system. Obtain the equations of motion.

2.16 A pendulum of length l and mass m is mounted on a block of mass M. The block can move freely without friction on a horizontal surface as shown in Figure 2.55. Show that the Lagrangian of the system is;

$$L=\left(\frac{M+m}{2}\right)\dot{x}^2+ml\dot{x}\dot{\theta}\cos\theta+\frac{m}{2}l^2\dot{\theta}^2+mgl\cos\theta$$

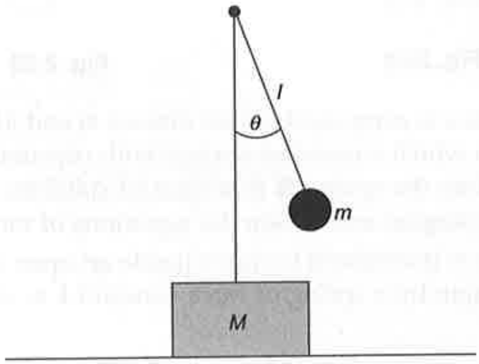

Fig. 2.55

2.17 A particle of mass m slides down a smooth spherical bowl, as in Figure 2.56. The particle remains in a vertical plane (the xz-plane).

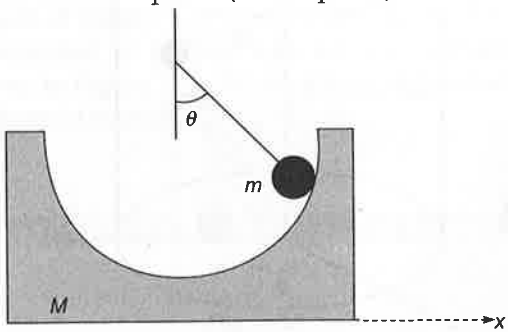

Fig. 2.56

(a) Assuming that the bowl is at rest write the Lagrangian, taking the angle θ with respect to the vertical direction as the generalized coordinate. Hence, derive the equation of motion for the particle.

(b) Obtain the Lagrangian and the equations of motion when the bowl of mass M slides frictionlessely on a horizontal surface.

2.18 A block of mass m is attached to a wedge of mass M by a spring with spring constant k. The inclined frictionless surface of the wedge makes an angle α to the horizontal. The wedge is free to slide on a horizontal frictionless surface as shown in Figure 2.57. Construct the Lagrangian and obtain the equation of motion.

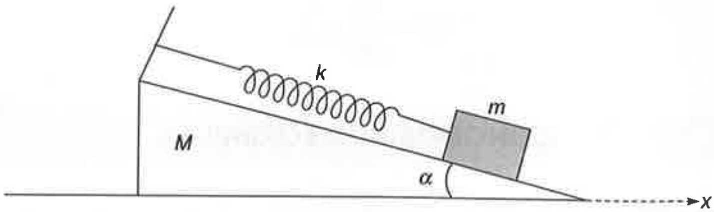

Fig. 2.57

2.19 Obtain the Lagrangian and the equation of motion of the coupled pendulum, where the bobs are connected with a rigid rod as shown in Figure 2.58. Neglect the mass of the rod.

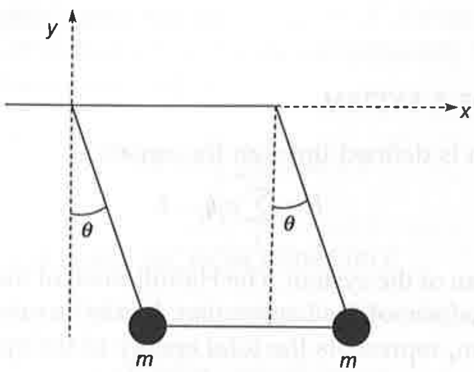

Fig. 2.58

2.20 A particle of mass m is placed inside a hollow ring of radius R and mass M. If the ring is able to roll on a horizontal plane without friction, obtain the Lagrangian and equations of motion of the system.

3

CHAPTER

Hamiltonian Formulation

CONCEPTS AND FORMULAE

3.1 PHASE SPACE

Equations of motion are second order differential equations. When we integrate these equations we get the velocity and position. The integration constants are, then, the initial velocity and the initial position. As a result, we can assume that the motion of dynamical systems is not taking place in ordinary space, instead in a space where the position and velocity (and hence momentum) are the coordinates. Such a space is known as the phase space.

3.2 HAMILTONIAN OF A SYSTEM

Hamiltonian of a system is defined through the equation

$$H = \sum_j p_j \dot{q}_j - L \tag{3.1}$$

where, L is the Lagrangian of the system. The Hamiltonian of the system can be obtained through a Legendre transform of the Lagrangian. It is to be noted that the Hamiltonian, for a conservative system, represents the total energy of the system. That is,

$$H = T + V \tag{3.2}$$

3.3 HAMILTON'S CANONICAL EQUATIONS OF MOTION

Hamilton's equations of motion are

$$\dot{q}_j = \frac{\partial H}{\partial p_j}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \text{ and}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{3.3}$$

These equations are called canonical equations as they are basic, but describe the motion of the system completely.

3.4 EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

According to Euler's theorem, for any homogeneous function F, we have

$$\sum q_j \frac{\partial F}{\partial \dot{q}_j} = 2F \tag{3.4}$$

3.5 ACTION

For a dynamical system, the action or the action integral is defined as

$$A = \int_{t_1}^{t_2} 2T dt \tag{3.5}$$

where, T is the kinetic energy of the system.

3.6 PRINCIPLE OF LEAST ACTION

The principle of least action states that the variation of A with time i.e., ΔA will become zero on the actual path of motion, provided the Hamiltonian, H, is constant throughout that actual path. It can be represented as

$$\Delta A = \Delta \int_{t_1}^{t_2} \sum_j p_j \dot{q}_j dt = 0 \tag{3.6}$$

3.7 JACOBI'S FORM OF LEAST ACTION PRINCIPLE

The Jacobi's form of the principle of least action is given by

$$\Delta \int \sqrt{2\left[H - V(q)\right]}\ d\rho = 0 \tag{3.7}$$

where, $d\rho$ is the differential cross section and is the variable of integration.

3.8 LIOUVILLE'S THEOREM

The Liouville's theorem states that the volume of an arbitrary region in phase space is conserved if the points of its boundary move according to the canonical equations.

SOLVED PROBLEMS

EXAMPLE 3.1 Deduce Hamilton's canonical equations from variational principle.

Solution: Let us start with the Hamilton's principle which states that of all the kinematically possible motions that take a mechanical system from one given configuration to another within a given time interval, the actual motion is the one that minimises the time integral of the Lagrangian of the system. This can be expressed mathematically as

$$\delta I = \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j) dt = 0$$

The Lagrangian L can be expressed in terms of the Hamiltonian H through the relation

$$L = \sum_j p_j \dot{q}_j - H(q_j, p_j, t)$$

And the resulting equation is referred to as the modified Hamilton's principle, which is

$$\delta I = \delta \int_{t_1}^{t_2} \left[\sum_j p_j \dot{q}_j - H(q_j, p_j, t) \right] dt = 0 \tag{i}$$

Now, let us find the variation of the above integral w.r.t. α, which is a parameter that represents various possible paths between the initial and final points in the configuration space. Then,

$$\delta I = d\alpha \frac{\partial I}{\partial \alpha} = d\alpha \frac{\partial}{\partial \alpha} \int_{t_1}^{t_2} \left[\sum_j p_j \dot{q}_j - H(q_j, p_j, t) \right] dt = 0$$

$$= d\alpha \int_{t_1}^{t_2} \frac{\partial}{\partial \alpha} \left[\sum_j p_j \dot{q}_j - H(q_j, p_j, t) \right] dt = 0 \tag{ii}$$

The inclusion of the derivative inside the integral is justified through the fact that the end points are same for all paths and hence the limits are independent of the parameter α. Then,

$$\delta I = d\alpha \int_{t_1}^{t_2} \sum_j \left(\frac{\partial p_j}{\partial \alpha} \dot{q}_j + \frac{\partial \dot{q}_j}{\partial \alpha} p_j - \frac{\partial H}{\partial q_j} \frac{\partial q_j}{\partial \alpha} - \frac{\partial H}{\partial p_j} \frac{\partial p_j}{\partial \alpha} - \frac{\partial H}{\partial t} \frac{\partial t}{\partial \alpha} \right) dt = 0 \tag{iii}$$

In the above expression $\frac{\partial t}{\partial \alpha} = 0$, since the time required to travel through all paths is the same. Therefore,

$$\delta I = d\alpha \int_{t_1}^{t_2} \sum_j \left(\frac{\partial p_j}{\partial \alpha} \dot{q}_j + \frac{\partial \dot{q}_j}{\partial \alpha} p_j - \frac{\partial H}{\partial q_j} \frac{\partial q_j}{\partial \alpha} - \frac{\partial H}{\partial p_j} \frac{\partial p_j}{\partial \alpha} \right) dt = 0 \tag{iv}$$

Now,
$$\int_{t_1}^{t_2} \frac{\partial \dot{q}_j}{\partial \alpha} p_j dt = \int_{t_1}^{t_2} p_j \frac{\partial}{\partial t} \left(\frac{\partial q_j}{\partial \alpha} \right) dt$$

$$= p_j \frac{\partial q_j}{\partial \alpha} \Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_j \frac{\partial q_j}{\partial \alpha} dt = - \int_{t_1}^{t_2} \dot{p}_j \frac{\partial q_j}{\partial \alpha} dt \tag{v}$$

Here, the first term in the above expression vanishes since the end points are same for all paths. Then (iv) becomes

$$\delta I = d\alpha \int_{t_1}^{t_2} \sum_j \left(\frac{\partial p_j}{\partial \alpha} \dot{q}_j - \dot{p}_j \frac{\partial q_j}{\partial \alpha} - \frac{\partial H}{\partial q_j} \frac{\partial q_j}{\partial \alpha} - \frac{\partial H}{\partial p_j} \frac{\partial p_j}{\partial \alpha} \right) dt = 0 \tag{vi}$$

Using the notation, $\delta \to d\alpha \dfrac{\partial}{\partial \alpha}$, this expression can be rewritten as

$$\delta I = \int_{t_1}^{t_2} \sum_j \left(\delta p_j \dot{q}_j - \dot{p}_j \delta q_j - \frac{\partial H}{\partial q_j} \delta q_j - \frac{\partial H}{\partial p_j} \delta p_j \right) dt = 0$$

$$= \int_{t_1}^{t_2} \sum_j \left\{ \delta p_j \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right) - \left(\dot{p}_j + \frac{\partial H}{\partial q_j} \right) \delta q_j \right\} dt = 0 \tag{vii}$$

From this expression we can directly obtain the Hamilton's canonical equations by directly equating the coefficients to zero. This is possible since q_j and p_j are independent variables and so δq_j and δp_j also. Then,

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \text{ and } \dot{p}_j = -\frac{\partial H}{\partial q_j} \tag{viii}$$

These are the Hamilton's canonical equations.

EXAMPLE 3.2 Express Hamilton's canonical equations in a cylindrical coordinate system.

Solution: The coordinates of a cylindrical coordinate system (r, ϕ, z) and that of the Cartesian coordinate system are related through

$$x = r\cos\phi\,,\ y = r\sin\phi \text{ and } z = z$$

The expression for the kinetic energy of a particle is

$$T = \frac{1}{2} m\left(x^2 + y^2 + z^2\right) = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) \tag{i}$$

We consider a conservative system so that $p_j = \dfrac{\partial T}{\partial \dot{q}_j}$ and therefore

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}\,,\ p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mr^2\dot{\phi} \text{ and } p_z = \frac{\partial T}{\partial z} = m\dot{z}$$

Now, the Hamiltonian of the system is

$$H = T + V = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) + V(r, \phi, z) \tag{ii}$$

Using the above expressions, (ii) can be rewritten as

$$H = \frac{1}{2m}\left(p_r^r + \frac{p_\phi^r}{r^2} + p_z^2 \right) + V(r,\phi,z) \tag{iii}$$

Hamilton's canonical equations are

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{1}{mr^3}p_\phi^2 - \frac{\partial V}{\partial r} \quad \text{and,}\ \dot{r} = \frac{p_r}{m}$$

Similarly, $$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi} \quad \text{and,}\ \dot{\phi} = \frac{p_\phi}{mr^2}$$

and $$\dot{p}_z = \frac{-\partial H}{\partial z} = -\frac{\partial V}{\partial z} \quad \text{and,}\ \dot{z} = \frac{p_z}{m}$$

These are Hamilton's canonical equations in a cylindrical coordinate system.

EXAMPLE 3.3 Express Hamilton's canonical equations in a spherical polar coordinate system.

Solution: The coordinates of a spherical polar coordinate system (r,θ,ϕ) and that of the Cartesian coordinate system are related through

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi \text{ and } z = r\cos\theta$$

The kinetic energy of a particle can be written as

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \tag{i}$$

Again, for a conservative system, $p_j = \frac{\partial T}{\partial \dot{q}_j}$ and therefore

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}\,,\ p_\theta = mr^2\dot{\theta} \text{ and } p_\phi = mr^2\sin^2\theta\dot{\phi} \tag{ii}$$

Now, the Hamiltonian of the system is

$$H = T + V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + V(r,\theta,\phi) \tag{iii}$$

Using (ii) in (iii) we get the Hamiltonian of the system as

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta} \right) + V(r,\theta,\phi) \tag{iv}$$

Then, the Hamilton's canonical equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \text{ and } \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{1}{mr^3}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^{2\theta}}\right) - \frac{\partial V}{\partial r} \tag{v}$$

Similarly,
$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2}{mr^2 \sin^3\theta} - \frac{\partial V}{\partial \theta} \tag{vi}$$

and
$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^2\theta} \text{ and } \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi} \tag{vii}$$

Equations (v), (vi) and (vii) are the required equations.

EXAMPLE 3.4 Express Hamilton's canonical equations in a polar coordinate system.

Solution: Polar coordinate system is used when the motion of the particle is confined to a plane. For a polar coordinate system, we have

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Then,
$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \text{ and } \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

The kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{i}$$

Let the system be conservative so that $p_j = \dfrac{\partial T}{\partial \dot{q}_j}$ and therefore,

$$p_r = m\dot{r} \text{ and } p_\theta = mr^2\dot{\theta} \tag{ii}$$

The Hamiltonian of the system is

$$H = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r,\theta) \tag{iii}$$

Using (ii) in (iii), we get

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r,\theta) \tag{iv}$$

Now, Hamilton's canonical equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \text{ and } \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \tag{v}$$

Similarly,
$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial V}{\partial \theta} \tag{vi}$$

Equations (v) and (vi) are the required equations.

EXAMPLE 3.5 Obtain the equation of motion of a linear harmonic oscillator by Hamilton's method.

Solution: As an example for a linear harmonic oscillator, consider the spring of force constant k loaded with a mass m.

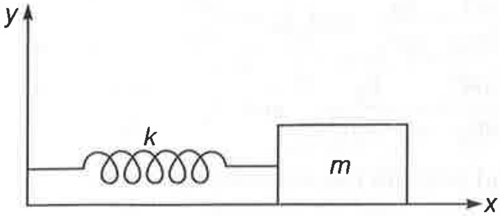

Fig. 3.1

For a linear harmonic oscillator, the kinetic and potential energies are given by

$$T = \frac{1}{2}m\dot{x}^2 \text{ and } V = \frac{1}{2}kx^2 \tag{i}$$

So that the Lagrangian of the system,

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

and the canonical momentum,

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

Hamiltonian of the system is given by

$$H = \sum_j p_j\dot{q}_j - L = p_x\dot{x} - \left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right] = m\dot{x}^2 - \left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right]$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{ii}$$

Then, Hamilton's canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \text{ and } \dot{p}_x = -\frac{\partial H}{\partial x} = -kx \tag{iii}$$

Using the above two expressions one can get

$$p_x = m\dot{x}$$

or $\quad \dot{p}_x = m\ddot{x} = -kx$

that is, $\ddot{x} + \dfrac{k}{m}x = 0$, which is the required equation of motion.

EXAMPLE 3.6 Obtain the equation of motion of a two-dimensional isotropic harmonic oscillator by Hamilton's method in Cartesian and polar coordinate systems.

Solution:

(1) In Cartesian Coordinate System For two-dimensional isotropic oscillator, the kinetic energy and potential energies are given by $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$ and $V = \frac{1}{2}k\left(x^2 + y^2\right)$ respectively. Then the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}k\left(x^2 + y^2\right) \tag{i}$$

From this, we can have

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{or,} \quad \dot{x} = \frac{p_x}{m}$$

and
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \quad \text{or,} \quad \dot{y} = \frac{p_y}{m}$$

Now, the Hamiltonian of the system is

$$\begin{aligned} H &= \sum_j p_j \dot{q}_j - L \\ &= p_x\dot{x} + p_y\dot{y} - \left[\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}k\left(x^2 + y^2\right)\right] \\ &= \frac{p_x^2}{m} + \frac{p_y^2}{m} - \frac{1}{2}\left(\frac{p_x^2}{m} + \frac{p_y^2}{m}\right) + \frac{1}{2}k\left(x^2 + y^2\right) \\ &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}k\left(x^2 + y^2\right) \end{aligned} \tag{ii}$$

Hamilton's canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad \text{and} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -kx \tag{iii}$$

Similarly,
$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \quad \text{and} \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -ky \tag{iv}$$

From (iii)

$$p_x = m\dot{x} \quad \text{so that,} \quad \dot{p}_x = m\ddot{x}$$

that is,
$$m\ddot{x} = -kx$$

or
$$\ddot{x} + \frac{k}{m}x = 0 \tag{v}$$

Similarly,
$$\ddot{y} + \frac{k}{m}y = 0 \tag{vi}$$

Equations (v) and (vi) are the equations of motion in Cartesian coordinate system. It represents simple harmonic motion.

(2) In Polar Coordinate System The Lagrangian of the system in polar coordinate system can be written as

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}kr^2 \tag{i}$$

Then, $$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{or, } \dot{r} = \frac{p_r}{m}$$

Similarly, $$p_\theta = mr^2\dot{\theta} \qquad \text{or } \dot{\theta} = \frac{p_\theta}{mr^2}$$

The Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L$$

$$= p_r\dot{r} + p_\theta\dot{\theta} - \left[\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}kr^2\right]$$

$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{m} - \frac{1}{2}\left(\frac{p_r^2}{m} + \frac{p_\theta^2}{m}\right) + \frac{1}{2}kr^2$$

$$= \frac{1}{2}\left(\frac{p_r^2}{m} + \frac{p_\theta^2}{m}\right) + \frac{1}{2}kr^2 \tag{ii}$$

Hamilton's canonical equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \quad \text{and } \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - kr$$

Similarly, $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad \text{and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

Note that θ is cyclic in Hamiltonian and therefore the corresponding momentum p_θ is a constant of motion and $\dot{p}_\theta = 0$.

EXAMPLE 3.7 Obtain the equation of motion of a simple pendulum by Hamilton's method.

Solution: For a simple pendulum, θ is the generalized coordinate. The kinetic energy of the simple pendulum is

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

Also, the potential energy is

$$V = mgl(1 - \cos\theta)$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) \tag{i}$$

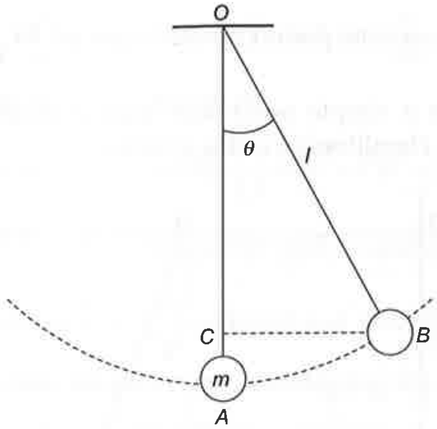

Fig. 3.2

and the canonical momentum is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \tag{ii}$$

The Hamiltonian of the system can be obtained as

$$H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta} - \left[\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\right]$$

$$= ml^2\dot{\theta}^2 - \left[\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\right]$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$= \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta) \tag{iii}$$

Now, Hamilton's canonical equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \tag{iv}$$

Again, from (ii); $\dot{p}_\theta = ml^2\ddot{\theta}$ and using (iv), we have

$$ml^2\ddot{\theta} = -mgl\sin\theta$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta \tag{v}$$

which is the equation of motion of a simple pendulum. Now, when the value of θ is very small, $\sin\theta \approx \theta$ and therefore,

$$\ddot{\theta} + \frac{g}{l}\theta \tag{vi}$$

This expression leads to a time period of oscillation of $2\pi\sqrt{l/g}$.

EXAMPLE 3.8 Consider a simple pendulum with a moving support as shown in Figure. 3.3. Construct the Hamiltonian of the system.

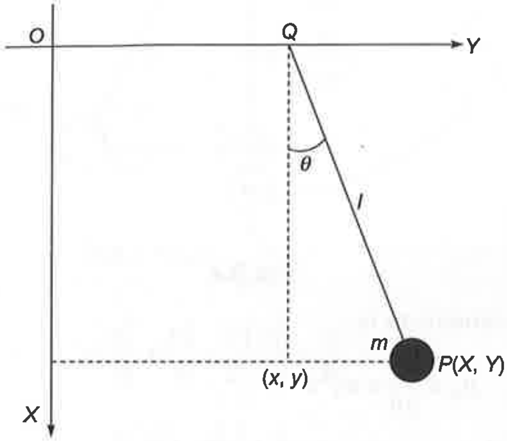

Fig. 3.3

Solution: For the simple pendulum given above, the support Q is moving in a horizontal direction and the pendulum is oscillating in a vertical plane. Let l be the length of the pendulum. Then,

$$X = l\cos\theta \text{ and } Y = y + l\sin\theta$$

The kinetic energy of the pendulum bob is

$$T = \frac{1}{2}m\left(\dot{X}^2 + \dot{Y}^2\right) = \frac{1}{2}m\left(\dot{y}^2 + l^2\dot{\theta}^2 + 2l\dot{y}\dot{\theta}\cos\theta\right) \tag{i}$$

The potential energy is $\quad V = mgl(1-\cos\theta)$ (ii)

The Lagrangian of the system can be written as

$$L = T - V = \frac{1}{2}m\left(\dot{y}^2 + l^2\dot{\theta}^2 + 2l\dot{y}\dot{\theta}\cos\theta\right) - mgl(1-\cos\theta) \tag{iii}$$

The Hamiltonian can be obtained as follows:

$$H = \sum_j p_j\dot{q}_j - L$$

$$= p_y\dot{y} + p_\theta\dot{\theta} - \left[\frac{1}{2}m\left(\dot{y}^2 + l^2\dot{\theta}^2 + 2l\dot{y}\dot{\theta}\cos\theta\right) - mgl(1-\cos\theta)\right] \tag{iv}$$

From (iii) we get $p_y = \frac{\partial L}{\partial \dot{y}} = m\left(\dot{y} + l\dot{\theta}\cos\theta\right)$ and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml\left(l\dot{\theta} + \dot{y}\cos\theta\right)$ (v)

Using (v) in (iv), we get the Hamiltonian as

$$H = m\left(\dot{y}^2 + l\dot{y}\dot{\theta}\cos\theta\right) + ml\left(l\dot{\theta}^2 + \dot{y}\dot{\theta}\cos\theta\right) - \left[\frac{1}{2}m\left(\dot{y}^2 + l^2\dot{\theta}^2 + 2l\dot{y}\dot{\theta}\cos\theta\right) - mgl(1-\cos\theta)\right]$$

or
$$H = \frac{1}{2}m\left(\dot{y}^2 + l^2\dot{\theta}^2 + 2l\dot{y}\dot{\theta}\cos\theta\right) + mgl(1-\cos\theta) \quad \text{(vi)}$$

Further, from (v) we can have; $\dot{y} = \frac{p_y}{m} - l\dot{\theta}\cos\theta$ and $\dot{\theta} = \frac{p_\theta}{ml^2} - \frac{\dot{y}}{l}\cos\theta$

Inserting one equation into the other and rearranging, we get

$$\dot{y} = \frac{1}{m\sin^2\theta}\left(p_y - \frac{p_\theta}{l}\cos\theta\right) \text{ and } l\dot{\theta} = -\frac{1}{m\sin^2\theta}\left(p_y\cos\theta - \frac{p_\theta}{l}\right) \quad \text{(vii)}$$

With this substitution, the Hamiltonian becomes;

$$H = \frac{1}{2m\sin^2\theta}\left(p_y^2 + \frac{p_\theta^2}{l^2} - \frac{2p_y p_\theta}{l}\cos\theta\right) + mgl(1-\cos\theta) \quad \text{(viii)}$$

It is to be noted that y is cyclic in Hamiltonian and hence the corresponding momentum p_y is a constant of motion.

EXAMPLE 3.9 A simple pendulum is hanging from the ceiling of a lift moving in the vertical direction. Construct the Hamiltonian of the simple pendulum.

Solution: Let z be the vertical direction, then the coordinates of the bob of the pendulum can be written as

$$x = l\sin\theta \text{ and } z = z(t) + l\cos\theta \quad \text{(i)}$$

The kinetic energy of the bob can be written as

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) = \frac{1}{2}m\left[\left(l\dot{\theta}\cos\theta\right)^2 + \left(\dot{z} - l\dot{\theta}\sin\theta\right)^2\right]$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}^2\cos^2\theta + \dot{z}^2 - 2l\dot{z}\dot{\theta}\sin\theta + l^2\dot{\theta}^2\sin^2\theta\right]$$

$$= \frac{1}{2}m\left(l^2\dot{\theta}^2 + \dot{z}^2 - 2l\dot{z}\dot{\theta}\sin\theta\right) \quad \text{(ii)}$$

The potential energy is

$$V = -mg(z + l\cos\theta) \quad \text{(iii)}$$

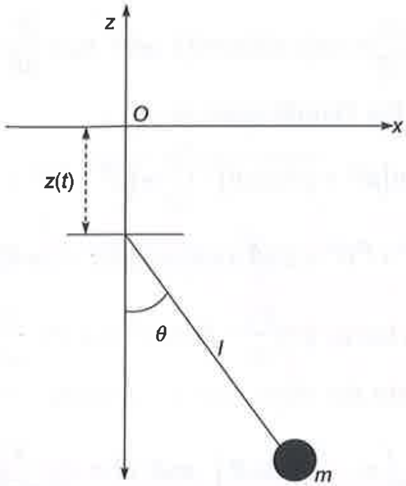

Fig. 3.4

Therefore, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(l^2\dot{\theta}^2 + \dot{z}^2 - 2l\dot{z}\dot{\theta}\sin\theta\right) + mg\left(z + l\cos\theta\right) \tag{iv}$$

Then, the canonical momentum corresponding to θ is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\left(l^2\dot{\theta} - l\dot{z}\sin\theta\right)$$

This would give

$$\dot{\theta} = \frac{p_\theta}{l^2 m} + \frac{\dot{z}\sin\theta}{l} \tag{v}$$

Now, the Hamiltonian is given by

$$H = \sum_j p_j \dot{q}_j - L = p_\theta\dot{\theta} - \left[\frac{1}{2}m\left(l^2\dot{\theta}^2 + \dot{z}^2 - 2l\dot{z}\dot{\theta}\sin\theta\right) + mg\left(z + l\cos\theta\right)\right]$$

$$= p_\theta\dot{\theta} - \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{z}\dot{\theta}\sin\theta - \frac{1}{2}m\dot{z}^2 - mgz - mgl\cos\theta$$

$$= p_\theta\dot{\theta} - \left(\frac{1}{2}ml^2\dot{\theta} + ml\dot{z}\sin\theta\right)\dot{\theta} - \frac{1}{2}m\dot{z}^2 - mgz - mgl\cos\theta$$

$$= p_\theta\dot{\theta} - \left(\left(ml^2\dot{\theta} + ml\dot{z}\sin\theta\right) - \frac{1}{2}ml^2\dot{\theta}\right)\dot{\theta} - \frac{1}{2}m\dot{z}^2 - mgz - mgl\cos\theta$$

$$= p_\theta\dot{\theta} - \left(p_\theta - \frac{1}{2}ml^2\dot{\theta}\right)\dot{\theta} - \frac{1}{2}m\dot{z}^2 - mgz - mgl\cos\theta$$

$$= p_\theta \dot{\theta} - p_\theta \dot{\theta} + \frac{1}{2} ml^2 \dot{\theta}^2 - \frac{1}{2} m\dot{z}^2 - mgz - mgl\cos\theta$$

$$= \frac{1}{2} ml^2 \dot{\theta}^2 - \frac{1}{2} m\dot{z}^2 - mgz - mgl\cos\theta$$

Substituting for $\dot{\theta}$ from (v), the above expression becomes

$$H = \frac{1}{2} ml^2 \left[\frac{p_\theta}{l^2 m} + \frac{\dot{z}\sin\theta}{l} \right]^2 - \frac{1}{2} m\dot{z}^2 - mgz - mgl\cos\theta$$

$$= \frac{(p_\theta + ml\dot{z}\sin\theta)^2}{2ml^2} - \frac{1}{2} m\dot{z}^2 - mgz - mgl\cos\theta \qquad \text{(vi)}$$

This is the required result.

EXAMPLE 3.10 Obtain the equation of motion of a compound pendulum by Hamilton's method.

Solution: Consider a body of mass M suspended about an axis passing through the point O as shown in Figure 3.5. For a compound pendulum, the kinetic energy is given by

$$T = \frac{1}{2} I\dot{\theta}^2$$

where I is the moment of inertia about the axis of rotation. The potential energy about the centre of suspension O is

$$V = -Mgl\cos\theta$$

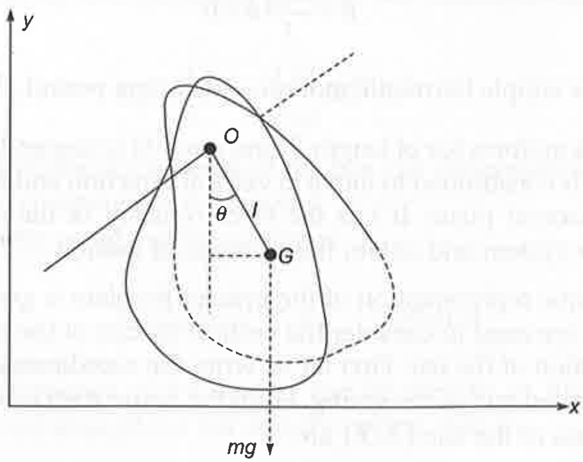

Fig. 3.5

Now, the Lagrangian of the compound pendulum is

$$L = T - V = \frac{1}{2} I\dot{\theta}^2 + Mgl\cos\theta \tag{i}$$

Then, the generalized momentum, $p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$

The Hamiltonian is given by

$$H = \sum_j p_j \dot{q}_j - L = p_\theta \dot{\theta} - \left[\frac{1}{2} I\dot{\theta}^2 + Mgl\cos\theta\right]$$

Using $\dot{\theta} = \dfrac{p_\theta}{I}$ in this expression, we get

$$H = \frac{p_\theta^2}{2I} - Mgl\cos\theta \tag{ii}$$

Hamilton's canonical equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -Mgl\sin\theta \tag{iii}$$

Then, $p_\theta = I\dot{\theta}$ or, $\dot{p}_\theta = I\ddot{\theta}$, using (iii) we have

$$I\ddot{\theta} = -Mgl\sin\theta$$

or
$$\ddot{\theta} + \frac{Mgl}{I}\sin\theta = 0 \tag{iv}$$

This is the equation of motion of a compound pendulum. Now, when the amplitude of oscillation is very small, $\sin\theta \approx \theta$ and the above equation becomes;

$$\ddot{\theta} + \frac{Mgl}{I}\theta = 0 \tag{v}$$

This represents a simple harmonic motion with a time period, $T = 2\pi\sqrt{I/Mgl}$.

EXAMPLE 3.11 A uniform bar of length $2l$ and mass M is suspended from one end by a spring. The spring is constrained to move in vertical direction and the bar is constrained to rotate in a horizontal plane. If k is the force constant of the spring, construct the Hamiltonian of the system and obtain the equation of motion.

Solution: A schematic representation of the present problem is given below.

In this problem, we need to consider the vertical motion of the loaded spring as well as the rotatory motion of the bar. First let us write the coordinates of centre of mass of the bar w.r.t. the loaded end of the spring. From the figure it is clear that the coordinates of the centre of mass of the bar (X, Y) are

$$X = l\sin\theta \quad \text{and} \quad Y = y + l\cos\theta$$

so that
$$\dot{X} = l\dot{\theta}\cos\theta \quad \text{and} \quad \dot{Y} = \dot{y} - l\dot{\theta}\sin\theta$$

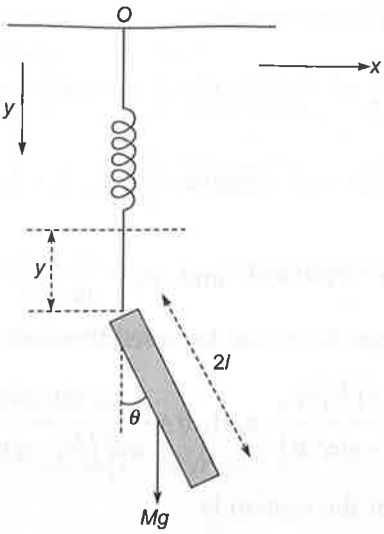

Fig. 3.6

Then, the kinetic energy of motion of the loaded spring is

$$T_{tra} = \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2\right) = \frac{1}{2}M\left[\left(l\dot{\theta}\cos\theta\right)^2 + \left(\dot{y} - l\dot{\theta}\sin\theta\right)^2\right]$$

$$= \frac{1}{2}M\left(l^2\dot{\theta}^2 + \dot{y}^2 - 2l\dot{y}\dot{\theta}\sin\theta\right) \qquad \text{(i)}$$

Since the bar is rotating in a horizontal plane, it has rotational kinetic energy also. It is given by

$$T_{rot} = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{3}Ml^2\right)\dot{\theta}^2 \qquad \text{(ii)}$$

(While writing the moment of inertia of the bar, the dimension of the cross section is assumed negligible compared to the length.)

Then, the total kinetic energy of the system is

$$T = \frac{1}{2}M\left(\frac{4}{3}l^2\dot{\theta}^2 + \dot{y}^2 - 2l\dot{y}\dot{\theta}\sin\theta\right) \qquad \text{(iii)}$$

Now, the potential energy of the system can be written as the sum of potential energies of the centre of mass of the bar and the spring, that is,

$$V(y,\theta) = -Mg\left(y + l\cos\theta\right) + \frac{1}{2}ky^2 \qquad \text{(iv)}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}M\left(\frac{4}{3}l^2\dot{\theta}^2 + \dot{y}^2 - 2l\dot{y}\dot{\theta}\sin\theta\right) - \left(-Mg\left(y + l\cos\theta\right) + \frac{1}{2}ky^2\right)$$

$$= \frac{1}{2}M\left(\frac{4}{3}l^2\dot{\theta}^2 + \dot{y}^2 - 2l\dot{y}\dot{\theta}\sin\theta\right) + Mg\left(y + l\cos\theta\right) - \frac{1}{2}ky^2 \qquad \text{(v)}$$

Then, $\quad p_y = \dfrac{\partial L}{\partial \dot{y}} = M\dot{y} - Ml\dot{\theta}\sin\theta \quad$ and, $p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = \dfrac{4}{3}Ml^2\dot{\theta} - Ml\dot{y}\sin\theta$

These two expressions can be solved between themselves to get the velocities as

$$\dot{y} = \frac{p_\theta\sin\theta + \left(4/3\right)lp_y}{Ml\left[4/3 - \sin^2\theta\right]} \quad \text{and, } \dot{\theta} = \frac{p_\theta + lp_y\sin\theta}{Ml^2\left(4/3 - \sin^2\theta\right)} \qquad \text{(vi)}$$

Now, the Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L$$

$$= p_y\dot{y} + p_\theta\dot{\theta} - \left[\frac{1}{2}M\left(\frac{4}{3}l^2\dot{\theta}^2 + \dot{y}^2 - 2l\dot{y}\dot{\theta}\sin\theta\right) + Mg\left(y + l\cos\theta\right) - \frac{1}{2}ky^2\right]$$

Substituting for $\dot{y}$ and $\dot{\theta}$ from (vi) and simplifying, we get

$$H = \frac{2p_y^2}{3M\left(4/3 - \sin^2\theta\right)} + \frac{p_\theta^2}{2Ml^2\left(4/3 - \sin^2\theta\right)}$$

$$+ \frac{p_yp_\theta\sin\theta}{Ml\left(4/3 - \sin^2\theta\right)} - Mg\left(y + l\cos\theta\right) + \frac{1}{2}ky^2 \qquad \text{(vii)}$$

Hamilton's canonical equations are

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{4p_y}{3M\left(4/3 - \sin^2\theta\right)} + \frac{p_\theta\sin\theta}{Ml\left(4/3 - \sin^2\theta\right)} \text{ and,}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = Mg - ky$$

Similarly, $\quad \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{Ml^2\left(4/3 - \sin^2\theta\right)} + \dfrac{p_y\sin\theta}{Ml\left(4/3 - \sin^2\theta\right)}$ and,

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \left(\frac{p_\theta^2}{2Ml^2} + \frac{2p_y^2}{3M}\right)\frac{\sin 2\theta}{\left(4/3 - \sin^2\theta\right)^2} + \frac{p_yp_\theta}{Ml}\left[\frac{\left(4/3 - \sin^2\theta\right)\cos\theta + \sin 2\theta}{\left(4/3 - \sin^2\theta\right)^2}\right] + Mgl\sin\theta$$

These are the required equations.

EXAMPLE 3.12 Obtain the equation of motion of a particle moving under the influence of a central force field by Hamilton's method.

Solution: To solve the problem, we use the polar coordinate system. For a central force, it obeys the inverse square law, that is, $F=-\frac{k}{r^2}=-\frac{\partial V}{\partial r}$ and hence, the potential is $V(r)=-\frac{k}{r}$.

The kinetic energy of a particle of mass m moving under a central force is

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)$$

Then, the Lagrangian of the system is

$$L=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{k}{r} \tag{i}$$

Now, $$p_r=\frac{\partial L}{\partial \dot{\theta}}=m\dot{r} \text{ and } p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$$

Therefore, $$\dot{r}=\frac{p_r}{m} \text{ and } \dot{\theta}=\frac{p_\theta}{mr^2} \tag{ii}$$

Hamiltonian of the system is

$$H=\sum_j p_j\dot{q}_j-L=p_r\dot{r}+p_\theta\dot{\theta}-\left[\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{k}{r}\right]$$

Substituting the values of $\dot{r}$ and $\dot{\theta}$ from (ii) and simplifying, we get

$$H=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)-\frac{k}{r} \tag{iii}$$

Hamilton's canonical equations are

$$\dot{r}=\frac{\partial H}{\partial p_r}=\frac{p_r}{m} \text{ and } \dot{p}_r=-\frac{\partial H}{\partial r}=\frac{p_\theta}{mr^3}-\frac{k}{r^2} \tag{iv}$$

Similarly, $$\dot{\theta}=\frac{\partial H}{\partial p_\theta}=\frac{p_\theta}{mr^2} \text{ and } \dot{p}_\theta=-\frac{\partial H}{\partial \theta}=0 \tag{v}$$

It is clear that θ cyclic in Hamiltonian and p_θ (the angular momentum) is conserved. Using (iv), we get; $p_r=m\dot{r}$ and so $\dot{p}_r=m\ddot{r}$.

Then, $$m\ddot{r}=\frac{p_\theta}{mr^3}-\frac{k}{r^2} \text{ or, } m\ddot{r}-\frac{p_\theta}{mr^3}+\frac{k}{r^2}=0 \tag{vi}$$

Equation (vi) is the equation of motion of a particle moving under a central force field.

EXAMPLE 3.13 In the previous problem we considered a conservative force field where, the potential is independent of velocity and depends only on the position coordinate. Now let us consider a situation where the potential is velocity dependent. Weber's force between two charges is given by $F=\frac{1}{r^2}\left(1-\frac{\dot{r}^2-2r\ddot{r}}{c^2}\right)$ and the corresponding potential is a velocity dependent potential of the form, $U=\frac{1}{r}+\frac{\dot{r}^2}{rc^2}$. Write the Hamiltonian and obtain Hamilton's equations.

Solution: Since the particle is moving under a central force, the kinetic energy is,

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)$$

Given that the potential energy is $U=\frac{1}{r}+\frac{\dot{r}^2}{rc^2}$

Therefore, the Lagrangian of the system is

$$L=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-\frac{1}{r}-\frac{\dot{r}^2}{rc^2} \tag{i}$$

Then, $$p_r=\frac{\partial L}{\partial \dot{r}}=m\dot{r}-\frac{2\dot{r}}{rc^2} \quad \text{and} \quad p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$$

Therefore, $$\dot{r}=\frac{p_r}{\left(m-\frac{2}{rc^2}\right)} \text{ and } \dot{\theta}=\frac{p_\theta}{mr^2} \tag{ii}$$

Now, the Hamiltonian of the system is

$$H=\sum_j p_j\dot{q}_j-L=p_r\dot{r}+p_\theta\dot{\theta}-\left[\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-\frac{1}{r}-\frac{\dot{r}^2}{rc^2}\right]$$

$$=mr^2\dot{\theta}^2+m\dot{r}^2-\frac{2\dot{r}^2}{rc^2}-\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{1}{r}+\frac{\dot{r}^2}{rc^2}$$

$$=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{1}{r}-\frac{\dot{r}^2}{rc^2} \tag{iii}$$

Substituting for $\dot{r}$ and $\dot{\theta}$ using (ii) and rearranging, the above expression becomes,

$$H=\frac{p_r^2}{2\left[m-\left(\frac{2}{rc^2}\right)\right]}+\frac{p_\theta^2}{2mr^2}+\frac{1}{r} \tag{iv}$$

Hamilton's canonical equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m - \left(\frac{2}{rc^2}\right)} \quad \text{and}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_r^2}{r^2 c^2 \left[m - \left(\frac{2}{rc^2}\right)\right]^2} + \frac{p_\theta^2}{mr^3} + \frac{1}{r^2}$$

Similarly, $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

Since θ is a cyclic coordinate, p_θ is a constant of motion.

EXAMPLE 3.14 Consider a massless spring of force constant k which is used to suspend a mass m. If the point of suspension of the spring is moving in the upward direction with a constant acceleration, find the equation of motion by Hamilton's method.

Solution: Let Y-axis represent the upward direction and a_0, the upward acceleration. Therefore, the net acceleration in the upward direction is $(a_0 - g)$. The potential energy of the system is the sum of the potential energy of the loaded spring and gravitational potential energy of the loaded mass. Therefore,

$$V = \frac{1}{2}ky^2 + m(a_0 - g)y$$

and kinetic energy is

$$T = \frac{1}{2}k\dot{y}^2$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\dot{y}^2 - \left(\frac{1}{2}ky^2 + m(a_0 - g)y\right) \tag{i}$$

The canonical momentum is

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \quad \text{and} \quad \dot{y} = \frac{p_y}{m} \tag{ii}$$

The Hamiltonian of the system

$$H = \sum_j p_j \dot{q}_j - L = p_y \dot{y} - \left[\frac{1}{2}m\dot{y}^2 - \left(\frac{1}{2}ky^2 + m(a_0 - g)y\right)\right]$$

$$= \frac{p_y^2}{m} - \left[\frac{1}{2}\frac{p_y^2}{m} - \left(\frac{1}{2}ky^2 + m(a_0 - g)y\right)\right]$$

$$= \frac{p_y^2}{2m} + \frac{1}{2}ky^2 + m(a_0 - g)y \tag{iii}$$

Hamilton's canonical equations are

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \text{ and } \dot{p}_y = -\frac{\partial H}{\partial y} = -ky - m(a_0 - g) \tag{iv}$$

The equation of motion can be obtained from (iv).

We have $\quad p_y = m\dot{y}$ and therefore; $\dot{p}_y = m\ddot{y}$.

Then, $\quad m\ddot{y} = -ky - m(a_0 - g)$

or

$$\left[\ddot{y} - (a_0 - g)\right] + \frac{k}{m} y = 0 \tag{v}$$

This is the required equation of motion.

EXAMPLE 3.15 Obtain the Hamiltonian and equations of motion of a particle moving near the surface of the earth.

Solution: Consider a particle of mass m moving near the surface of the earth. The kinetic energy of the particle can be written as

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Let z-axis be along the vertical direction so that the potential energy of the particle is;

$$V = mgz$$

Then, the Lagrangian of the particle is

$$L = T - V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \tag{i}$$

so that $\quad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$ and $p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$

The Hamiltonian of the system is

$$H = \sum_j p_j \dot{q}_j - L = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \left[\frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz\right]$$

$$= \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz$$

$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz \tag{ii}$$

Hamilton's canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \text{ and } \dot{p}_x = -\frac{\partial H}{\partial x} = 0$$

Similarly, $\quad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$ and $\dot{p}_y = -\frac{\partial H}{\partial y} = 0$

and $$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \text{ and } \dot{p}_z = -\frac{\partial H}{\partial z} = -mg$$

From these expressions we can easily get the equations of motion as

$$\ddot{x} = \frac{\dot{p}_x}{m} = 0, \; \ddot{y} = \frac{\dot{p}_y}{m} = 0 \text{ and } \ddot{z} = \frac{\dot{p}_z}{m} = -g$$

This shows that the only acceleration of the particle is the acceleration due to gravity.

EXAMPLE 3.16 Obtain the Hamiltonian and the equations of a charged particle moving in an electromagnetic field.

Solution: Let q be the charge and m be the mass of the particle moving in the electromagnetic field. The kinetic energy of the particle is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}\sum_j mv_j^2, \quad \text{where } j = x, y \text{ and } z$$

The potential of the charged particle is

$$U = q(\phi - v.A) = q\phi - q\sum_j v_j A_j$$

where, ϕ is the scalar potential and A is the vector potential.
Then, the Lagrangian of the particle is

$$L = T - U = \frac{1}{2}\sum_j mv_j^2 - \left(q\phi - q\sum_j v_j A_j\right) = \frac{1}{2}\sum_j mv_j^2 + q\sum_j v_j A_j - q\phi \tag{i}$$

The generalized momentum is

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial v_j} = mv_j + qA_j \tag{ii}$$

The Hamiltonian of the system is

$$H = \sum_j p_j \dot{q}_j - L = \sum_j (mv_j + qA_j)v_j - L$$

$$= \sum_j mv_j^2 + q\sum_j v_j A_j - \left(\frac{1}{2}\sum_j mv_j^2 + q\sum_j v_j A_j - q\phi\right)$$

$$= \frac{1}{2}\sum_j mv_j^2 + q\phi \tag{iii}$$

From (ii), $v_j = \dfrac{p_j - qA_j}{m}$ and with this substitution (iii) becomes

$$H = \sum_j \frac{1}{2m}(p_j - qA_j)^2 + q\phi \tag{iv}$$

Hamilton's canonical equations can be obtained as

$$v_j = \frac{\partial H}{\partial p_j} = \frac{1}{m}\left(p_j - qA_j\right) \text{ and } \dot{p}_j = -\frac{\partial H}{\partial q_j} = q\nabla\left(\vec{v}.\vec{A}\right) - q\nabla\phi \tag{v}$$

Note: The gradient in the above expression denotes the three dimensional spatial derivatives. While obtaining $\dot{p}_j$, the Hamiltonian is used as in (iii) with one v_j replaced by $v_j = \dfrac{p_j - qA_j}{m}$ and the spatial derivative of p_j is zero.

EXAMPLE 3.17 A particle of mass m is constrained to move on the surface of a cylinder. The particle is subjected to a force directed towards the origin and proportional to the distance of the particle from the origin. Obtain the Hamiltonian of the particle and Hamilton's equations of motion.

Solution: Since the particle is constrained to move on the surface of the cylinder, we have

$$x^2 + y^2 = R^2$$

where, R is the radius of the cylinder.

Since the force is central, the potential energy of the particle is

$$V = \frac{1}{2}k\left(x^2 + y^2 + z^2\right) = \frac{1}{2}k\left(R^2 + z^2\right)$$

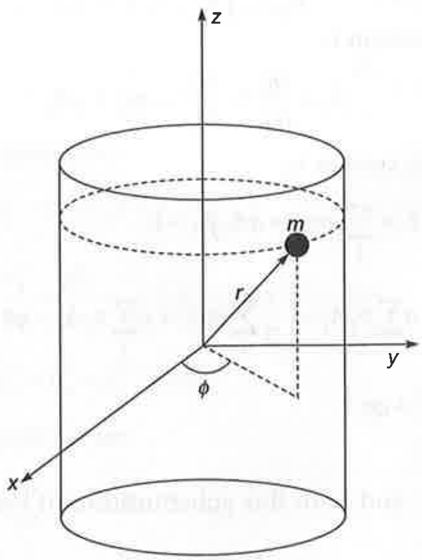

Fig. 3.7

Kinetic energy of the particle can be expressed in cylindrical coordinates as;

$$T = \frac{1}{2}m\left(r^2\dot{\phi}^2 + \dot{z}^2\right)$$

The kinetic energy contains only two terms since the particle is restricted to move only over the surface of the cylinder.

The Lagrangian of the particle is

$$L = T - V = \frac{1}{2}m\left(r^2\dot{\phi}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right) \qquad \text{(i)}$$

Then, the canonical momenta are

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \text{ and } p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad \text{(ii)}$$

This would give

$$\dot{\phi} = \frac{p_\phi}{mr^2} \text{ and } \dot{z} = \frac{p_z}{m} \qquad \text{(iii)}$$

The Hamiltonian of the particle is

$$H = \sum_j p_j\dot{q}_j - L = p_\phi\dot{\phi} + p_z\dot{z} - \left[\frac{1}{2}m\left(r^2\dot{\phi}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right)\right]$$

Substituting for $\dot{\phi}$ and $\dot{z}$ from (iii) the above expression for Hamiltonian becomes

$$H = \frac{p_\phi^2}{mr^2} + \frac{p_z^2}{m} - \left[\frac{1}{2m}\left(\frac{p_\phi^2}{r^2} + p_z^2\right) - \frac{1}{2}k\left(R^2 + z^2\right)\right]$$

$$= \frac{p_\phi^2}{2mr^2} + \frac{p_z^2}{2m} + \frac{1}{2}k\left(R^2 + z^2\right) \qquad \text{(iv)}$$

Hamilton's canonical equations of motion are

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} \text{ and } \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \qquad \text{(v)}$$

Similarly, $$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \text{ and } \dot{p}_z = -\frac{\partial H}{\partial z} = -kz \qquad \text{(vi)}$$

From (v) we see that the angular momentum is conserved and from (vi) we obtain the equation of motion in the z-direction as $\ddot{z} + \frac{k}{m}z = 0$, which represents a simple harmonic motion.

EXAMPLE 3.18 A point mass m is placed on a frictionless plane that is tangential to the earth's surface. Construct the Hamiltonian and obtain the canonical equations of motion.

Solution: The system can be assumed to be conservative since the mass is moving on a frictionless plane. An illustration is given below.

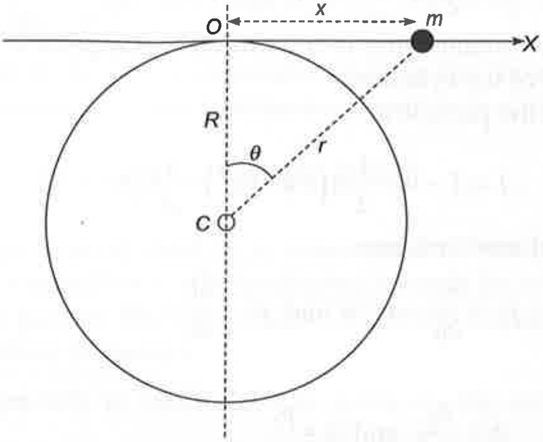

Fig. 3.8

The kinetic energy of the particle is

$$T = \frac{1}{2} m\dot{x}^2$$

The potential energy of the particle is the gravitational potential energy at a height equal to $(r - R)$ above the surface of the earth. It is given by

$$V = mg(r - R)$$

The Lagrangian of the system is

$$L = T - V = \frac{1}{2} m\dot{x}^2 - mg(r - R)$$

$$= \frac{1}{2} m\dot{x}^2 - mg\left(\sqrt{x^2 + R^2} - R\right) \quad \text{(i)}$$

The canonical momentum is

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{(ii)}$$

The Hamiltonian of the system is

$$H = \sum_j p_j \dot{q}_j - L = p_x \dot{x} - \left[\frac{1}{2} m\dot{x}^2 - mg\left(\sqrt{x^2 + R^2} - R\right)\right]$$

$$= \frac{p_x^2}{m} - \left[\frac{1}{2}\frac{p_x^2}{m} - mg\left(\sqrt{x^2 + R^2} - R\right)\right]$$

$$= \frac{p_x^2}{2m} + mg\left(\sqrt{x^2 + R^2} - R\right) \tag{iii}$$

The canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \text{ and } \dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{mgx}{\sqrt{x^2 + R^2}} \tag{iv}$$

Then, $\quad p_x = m\dot{x}$ and $\dot{p}_x = m\ddot{x}$

The equation of motion is

$$\ddot{x} + \frac{gx}{\sqrt{x^2 + R^2}} = 0 \tag{v}$$

EXAMPLE 3.19 Repeat the above problem taking θ as the generalized coordinate.

Solution: In terms of θ, we have;

$$l = \frac{R}{\cos\theta} = R\sec\theta \quad \text{and} \quad x = R\tan\theta \tag{i}$$

The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\left(R\dot{\theta}\sec\theta\right)^2$$

and, the potential energy is

$$V = mg(r - R) = mgR(\sec\theta - 1)$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m\left(R\dot{\theta}\sec\theta\right)^2 - mgR(\sec\theta - 1) \tag{ii}$$

The canonical momentum is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}\sec^4\theta \tag{iii}$$

and $$\dot{\theta} = \frac{p_\theta}{mR^2\sec^4\theta} \tag{iv}$$

The Hamiltonian of the particle is

$$H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta} - \left[\frac{1}{2}m\left(R\dot{\theta}\sec\theta\right)^2 - mgR(\sec\theta - 1)\right]$$

Substituting for $\dot{\theta}$ from (iv), and simplifying, equation (v) becomes

$$H = \frac{p_\theta^2}{2mR^2\sec^4\theta} + mgR(\sec\theta - 1) \tag{v}$$

Hamilton's canonical equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2 \sec^4 \theta}$$

and
$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{2p_\theta^2 \sin\theta}{mR^2 \sec^3 \theta} - mgR \sec\theta \tan\theta$$

EXAMPLE 3.20 A horizontal frictionless table has a small hole in its centre. Block A of mass m on the table is connected to block B of mass M hanging beneath by a string of negligible mass which passes through the hole. Initially, B is held stationary and A rotates at constant radius r_0 with steady angular velocity ω_0. If B is released at $t = 0$, it moves in the downward direction, still the rotation of A is continuing its rotation, but with decreasing r. Set up the Hamiltonian of the system and find the canonical equations of motion.

Solution: The illustration of the problem is given below.

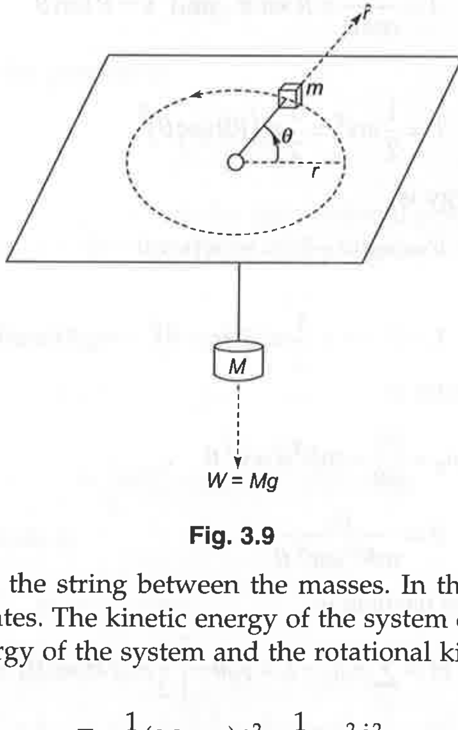

Fig. 3.9

Let l be the length of the string between the masses. In this problem, r and θ are the generalized coordinates. The kinetic energy of the system consists of two parts, the translational kinetic energy of the system and the rotational kinetic energy of the mass m and is given by

$$T = \frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

The potential energy of the block with respect to the horizontal plane is

$$V = -Mg(l-r)$$

The Lagrangian of the system is

$$L = T - V = \frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + Mg(l-r) \tag{i}$$

The canonical momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = (M+m)\dot{r} \text{ and } p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{ii}$$

which gives,

$$\dot{r} = \frac{p_r}{(M+m)} \text{ and } \dot{\theta} = \frac{p_\theta}{mr^2} \tag{iii}$$

The Hamiltonian of the system is

$$L = T - V = \sum_j p_j \dot{q}_j - L$$

$$= p_r\dot{r} + p_\theta\dot{\theta} - \left[\frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + Mg(l-r)\right]$$

Substituting for $\dot{r}$ and $\dot{\theta}$ from (iii) the above expression becomes

$$H = \frac{p_r^2}{(M+m)} + \frac{p_\theta^2}{mr^2} - \left[\frac{1}{2}\frac{p_r^2}{(M+m)} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + Mg(l-r)\right]$$

$$= \frac{p_r^2}{2(M+m)} + \frac{p_\theta^2}{2mr^2} - Mg(l-r) \tag{iv}$$

Hamilton's canonical equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{M+m} \text{ and } \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr} - Mg \tag{v}$$

Also, $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \tag{vi}$$

From (v) $$\dot{p}_r = (M+m)\ddot{r} = \frac{p_\theta^2}{mr} - Mg$$

or $$\ddot{r} = \frac{p_\theta^2}{(M+m)mr} - \frac{Mg}{(M+m)} \tag{vii}$$

This represents the equation of motion of the mass m. For the mass M, the equation of motion is

$$\ddot{y} = \frac{Mg}{(M+m)} - \frac{p_\theta^2}{(M+m)mr} \tag{viii}$$

This follows, since the constraint equation is $y + r = l$, and therefore, $\ddot{r} = -\ddot{y}$.

EXAMPLE 3.21 The Hamiltonian of a system is given by,

$H=\frac{p_x^2}{2}+\frac{\omega_0^2x^2}{2}+\lambda\left(\frac{p_x^2}{2}+\frac{\omega_0^2x^2}{2}\right)^2$. Obtain Hamilton's canonical equations and the equation of motion.

Solution: Given, the Hamiltonian is $H=\frac{p_x^2}{2}+\frac{\omega_0^2x^2}{2}+\lambda\left(\frac{p_x^2}{2}+\frac{\omega_0^2x^2}{2}\right)^2$ (i)

Hamilton's canonical equations are

$$\dot{x}=\frac{\partial H}{\partial p_x}=p_x+2\lambda p_x\left(\frac{p_x^2}{2}+\frac{\omega_0^2x^2}{2}\right) \tag{ii}$$

and

$$\dot{p}_x=-\frac{\partial H}{\partial x}=-\left[\omega_0^2x+2\lambda\omega_0^2x\left(\frac{p_x^2}{2}+\frac{\omega_0^2x^2}{2}\right)\right] \tag{iii}$$

Now let us find the equation of motion.

From (ii) we have $\frac{\dot{x}}{p_x}=\left[1+2\lambda\left(\frac{p_x^2}{2}+\frac{\omega_0^2x^2}{2}\right)\right]$

and from (iii) we get; $\frac{\dot{p}_x}{\omega_0^2x}=-\left[1+2\left(\frac{p_x^2}{2}+\frac{\omega_0^2x^2}{2}\right)\right]$

so that, $$\frac{\dot{x}}{p_x}+\frac{\dot{p}_x}{\omega_0^2x}=0$$

or $$p_x\dot{p}_x+\omega_0^2x\dot{x}=0$$

that is, $$\frac{d}{dt}\left(p_x^2+\omega_0^2x^2\right)=0 \text{ or; } p_x^2+\omega_0^2x^2=constant=k \tag{iv}$$

Now; (ii) and (iii) becomes

$$\dot{x}=p_x+k\lambda p_x \text{ and } \dot{p}_x=-\left[\omega_0^2x+k\lambda\omega_0^2x\right]$$

Then, $$\ddot{x}=\dot{p}_x(1+k\lambda)=-(1+k\lambda)\left(\omega_0^2x+k\lambda\omega_0^2x\right)$$

$$=-(1+k\lambda)^2\,\omega_0^2x=-\omega^2x$$

that is, $$\ddot{x}+\omega^2x=0 \tag{v}$$

where, $\omega=(1+k\lambda)\omega_0$. Thus, (v) represents a simple harmonic motion with a solution; $x=A\cos(\omega t+\phi)$, where, A is the amplitude and ϕ is the initial phase angle.

Now, we have to find the constant k.

We have $$\dot{x}=p_x+k\lambda p_x$$

or

$$p_x = \frac{\dot{x}}{1+k\lambda} = \frac{-A\omega\sin(\omega t+\phi)}{1+k\lambda} = -A\omega_0\sin(\omega t+\phi)$$

Now, the expression, $p_x^2 + \omega_0^2 x^2 = k$ can be rewritten as

$$\left[-A\omega_0\sin(\omega t+\phi)\right]^2 + \omega_0^2\left[A\cos(\omega t+\phi)\right]^2 = k$$

On simplification, we get $$k = A^2\omega_0^2 \tag{vi}$$

Now, the frequency of oscillation, $\omega = (1+k\lambda)\omega_0$ becomes

$$\omega = \left(1+\lambda A^2\omega_0^2\right)\omega_0$$

This can be expressed in terms of the total energy of the system, $E_0 = \frac{1}{2}A^2\omega_0^2$ as

$$\omega = (1+2\lambda E_0)\omega_0 \tag{vii}$$

EXAMPLE 3.22 Obtain the equation of motion of a projectile in space by using Hamilton's method.

Solution: The projectile is moving in space. Let z be the vertical direction. Also assume the coordinate axes are attached to earth. The only force acting on the projectile is the gravitational force of the earth; vertically downward direction.

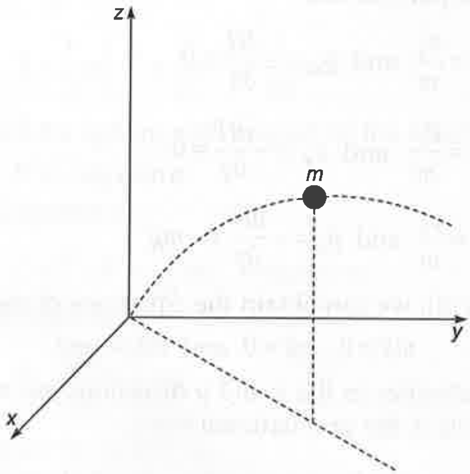

Fig. 3.10

The kinetic energy of the particle is given by

$$T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) \tag{i}$$

The potential energy is

$$V = mgz \tag{ii}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz \tag{iii}$$

The generalized momenta, then, are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}\,,\ p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \text{ and } p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

Then, $$\dot{x} = \frac{p_x}{m}\,,\ \dot{y} = \frac{p_y}{m} \text{ and } \dot{z} = \frac{p_z}{m} \tag{iv}$$

The Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = \left(p_x\dot{x} + p_y\dot{y} + p_z\dot{z}\right) - \left[\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz\right]$$

Substituting for $\dot{x}, \dot{y}$ and $\dot{z}$ from (iv), the above expression becomes

$$H = \left(\frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m}\right) - \left(\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right) + mgz$$

$$= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgz \tag{v}$$

Hamilton's canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \text{ and } \dot{p}_x = -\frac{\partial H}{\partial x} = 0 \tag{vi}$$

Similarly, $$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \text{ and } \dot{p}_y = -\frac{\partial H}{\partial y} = 0 \tag{vii}$$

Also, $$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \text{ and } \dot{p}_z = -\frac{\partial H}{\partial z} = -mg \tag{viii}$$

From, (vi), (vii) and (viii), we can obtain the equations of motion as

$$m\ddot{x} = 0,\ m\ddot{y} = 0 \text{ and } m\ddot{z} = -mg \tag{ix}$$

The components of velocities in the x and y directions are unaffected since the only force acting on the particle is the gravitational force.

EXAMPLE 3.23 A sphere rolls down a rough inclined plane. If x is the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration of the sphere by using Hamilton's method.

Solution: A schematic representation of the problem is given below. Let α be the angle of inclination of the sphere.

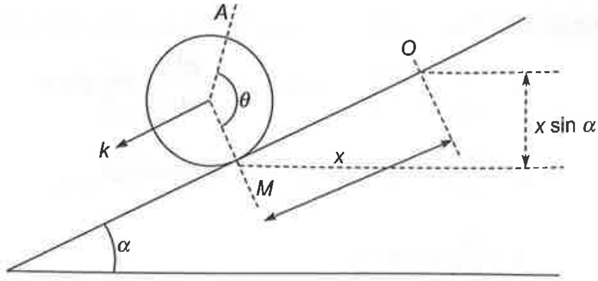

Fig. 3.11

The kinetic energy of the sphere consists the kinetic energy of translation over the plane as well as the rotational kinetic energy. Let m be the mass of the sphere, then

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mK^2\dot{\theta}^2$$

where, K is the radius of gyration of the sphere. It is given by $K^2 = \frac{2}{5}a^2$ where a is the radius of the sphere. Thus;

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\frac{2}{5}a^2\dot{\theta}^2 = \frac{1}{2}m\left(\dot{x}^2 + \frac{2}{5}a^2\dot{\theta}^2\right) \quad \text{(i)}$$

But from the figure, $x = \widehat{OM} = a\theta$, and therefore, $\dot{x} = a\dot{\theta}$. Then the kinetic energy becomes

$$T = \frac{1}{2}m\left(\dot{x}^2 + \frac{2}{5}\dot{x}^2\right) = \frac{7}{10}m\dot{x}^2 \quad \text{(ii)}$$

The potential energy of the sphere with respect to the fixed point O is

$$V = -mgx\sin\alpha \quad \text{(iii)}$$

The Lagrangian of the system is

$$L = T - V = \frac{7}{10}m\dot{x}^2 + mgx\sin\alpha \quad \text{(iv)}$$

The generalized momentum is

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5}m\dot{x}\,,\ \text{or}\ \dot{x} = \frac{5p_x}{7m} \quad \text{(v)}$$

Now, the Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = p_x\dot{x} - \left(\frac{7}{10}m\dot{x}^2 + mgx\sin\alpha\right)$$

$$= p_x\frac{5p_x}{7m} - \left[\frac{7}{10}m\left(\frac{5p_x}{7m}\right)^2 + mgx\sin\alpha\right]$$

$$= \frac{5}{14}\frac{p_x^2}{m} - mgx\sin\alpha \quad \text{(vi)}$$

Hamilton's canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{5}{7}\frac{p_x}{m} \quad \text{and} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = mg\sin\alpha$$

or
$$p_x = \frac{7}{5}m\dot{x} \quad \text{and} \quad \dot{p}_x = \frac{7}{5}m\ddot{x} = mg\sin\alpha$$

Then,
$$\ddot{x} = \frac{5}{7}g\sin\alpha \tag{vii}$$

This is the required equation of motion.

EXAMPLE 3.24 The Lagrangian of a charged particle moving in an electromagnetic field is given by $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{q}{c}\frac{B}{2}r^2\dot{\theta}$. If the particle follows a circular path, find the radius of the path and the frequency of revolution.

Solution: Given, the Lagrangian of the charged particle is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{q}{c}\frac{B}{2}r^2\dot{\theta} \tag{i}$$

Then, the canonical momenta can be written as

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{and} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} - \frac{qB}{2c}r^2 \tag{ii}$$

Now, let $\omega = \frac{qB}{mc}$ and from the above equations, we have

$$\dot{r} = \frac{p_r}{m} \quad \text{and} \quad \dot{\theta} = \frac{p_\theta}{mr^2} + \frac{\omega}{2} \tag{iii}$$

The Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = p_r\dot{r} + p_\theta\dot{\theta} - \left[\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{q}{c}\frac{B}{2}r^2\dot{\theta}\right]$$

$$= p_r\dot{r} + p_\theta\dot{\theta} - \left[\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mr^2\omega\dot{\theta}\right]$$

Substituting for $\dot{r}$ and $\dot{\theta}$ from (iii), the above expression becomes

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{1}{2}p_\theta\omega + \frac{1}{8}mr^2\omega^2 \tag{iv}$$

In the expression for Hamiltonian, θ is a cyclic coordinate and hence p_θ is a constant of motion. Since the charged particle follows a circular path, we must have $p_r = \dot{p}_r = 0$. From Hamilton's canonical equations, we have

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{1}{4}mr\omega^2 \tag{v}$$

This equation gives the radius of the circular orbit with parameters p_θ and ω as

$$r = \sqrt{\frac{2p_\theta}{m\omega}} \tag{vi}$$

Again, $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} + \frac{\omega}{2} = \frac{p_\theta}{m\left(\frac{2p_\theta}{m\omega}\right)} + \frac{\omega}{2} = \omega$$

Therefore, ω represents the angular velocity and is given by, $\omega = \frac{qB}{mc}$.

EXAMPLE 3.25 In a horizontal plane a homogeneous rod AB, of length l and mass M is constrained to rotate around its centre O. A point particle P of mass m can move on the rod and is attracted by the point O with an elastic force of constant k. Assuming the constraints are frictionless, obtain Hamilton's equations.

Solution: The schematic representation of the problem is given below.

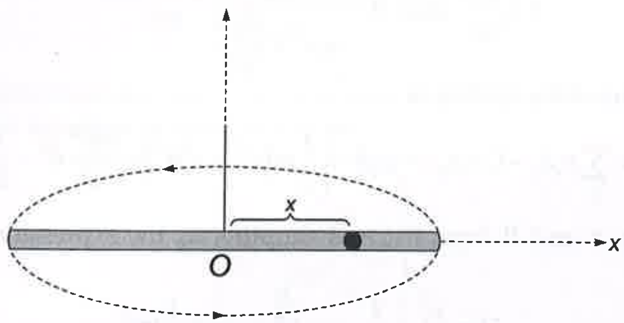

Fig. 3.12

The kinetic energy of the system consists of two parts, the kinetic energy of the point mass and the rotational kinetic energy of the rod. The kinetic energy of the point mass, in a polar coordinate system, can be written as;

$$T_m = \frac{1}{2}m\left(\dot{x}^2 + x^2\dot{\theta}^2\right)$$

Similarly, the rotational kinetic energy of the rod is

$$T_M = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(\frac{Ml^2}{12}\right)\dot{\theta}^2 = \frac{Ml^2}{24}\dot{\theta}^2$$

where, $I = \frac{Ml^2}{12}$ is the moment of inertia of the rod about an axis passing through its centre and normal to the length. Then, the total kinetic energy of the system is

$$T = \frac{1}{2}m\left(\dot{x}^2 + x^2\dot{\theta}^2\right) + \frac{Ml^2}{24}\dot{\theta}^2 \tag{i}$$

The potential energy can be written as

$$V = \frac{1}{2}kx^2 \tag{ii}$$

Now, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + x^2\dot{\theta}^2\right) + \frac{Ml^2}{24}\dot{\theta}^2 - \frac{1}{2}kx^2 \tag{iii}$$

The canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mx^2\dot{\theta} + \frac{Ml^2}{12}\dot{\theta}$$

This would give

$$\dot{x} = \frac{p_x}{m} \quad \text{and} \quad \dot{\theta} = \frac{p_\theta}{mx^2 + \frac{Ml^2}{12}} \tag{iv}$$

The Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = p_x\dot{x} + p_\theta\dot{\theta} - \left[\frac{1}{2}m\left(\dot{x}^2 + x^2\dot{\theta}^2\right) + \frac{Ml^2}{24}\dot{\theta}^2 - \frac{1}{2}kx^2\right]$$

Substituting for $\dot{x}$ and $\dot{\theta}$ from (iv) and simplifying, the expression for Hamiltonian becomes

$$H = \frac{p_x^2}{2m} + \frac{1}{2}\frac{p_\theta^2}{\left(mx^2 + \frac{Ml^2}{12}\right)} + \frac{1}{2}kx^2 \tag{v}$$

Hamilton's canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{2m} \quad \text{and} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = \frac{mxp_\theta^2}{\left(mx^2 + \frac{Ml^2}{12}\right)^2} - kx \tag{vi}$$

and

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\left(mx^2 + \frac{Ml^2}{12}\right)} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \tag{vii}$$

That is, p_θ, the angular momentum about the centre of the rod is a constant of motion.

From (vi) $p_x = 2m\dot{x}$ and $\dot{p}_x = 2m\ddot{x} = \dfrac{mxp_\theta^2}{\left(mx^2 + \frac{Ml^2}{12}\right)^2} - kx$

Then, the equation of motion is

$$\ddot{x} + \frac{k}{m}x - \frac{p_\theta^2}{\left(mx^2 + \frac{Ml^2}{12}\right)^2}x = 0 \tag{viii}$$

Note: For small oscillations of the mass about the centre of the rod, $\frac{1}{\left(mx^2 + \frac{Ml^2}{12}\right)}$ in the expression for Hamiltonian can be approximated as

$$\left(mx^2 + \frac{Ml^2}{12}\right)^{-1} = \frac{1}{\frac{1}{12}Ml^2}\left(1 + \frac{12m}{Ml^2}x^2\right)^{-1} \simeq \frac{1}{\frac{1}{12}Ml^2}\left(1 - \frac{12m}{Ml^2}x^2\right)$$

and the Hamiltonian, first order in x becomes

$$H = \frac{p_x^2}{2m} + \frac{1}{2}\frac{p_\theta^2}{\frac{1}{12}Ml^2}\left(1 - \frac{12m}{Ml^2}x^2\right) + \frac{1}{2}kx^2 \tag{ix}$$

Now we can determine the canonical equations and the equation of motion as in the previous case. The equation of motion will be,

$$\ddot{x} + \left(\frac{k}{m} - \frac{p_\theta^2}{\left(Ml^2/12\right)^2}\right)x = 0 \tag{x}$$

and the frequency of oscillation will be, $\omega = \left(\frac{k}{m} - \frac{p_\theta^2}{\left(Ml^2/12\right)^2}\right)^{1/2}$.

EXAMPLE 3.26 Find the acceleration of the masses string of an Atwood machine by Hamilton's approach assuming a frictionless pulley.

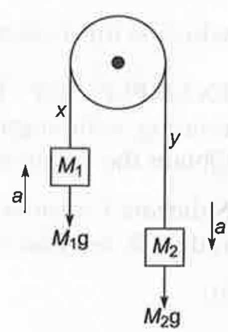

Fig. 3.13

Solution: Since the pulley is frictionless it will not rotate while the masses move over it. Let us take the generalized coordinates as x and y subjected to the constraint,

$$x + y = l \tag{i}$$

so that $\dot{x} = -\dot{y}$ and $\ddot{x} = -\ddot{y}$.

Now, the kinetic energy of the system can be written as

$$T=\frac{1}{2}M_1\dot{x}^2+\frac{1}{2}M_2\dot{y}^2=\frac{1}{2}(M_1+M_2)\dot{x}^2 \tag{ii}$$

The potential energy of the system can be written as

$$V=-M_1gx-M_2gy=-M_1gx-M_2g(l-x) \tag{iii}$$

The Lagrangian of the system is;

$$L=T-V=\frac{1}{2}(M_1+M_2)\dot{x}^2+M_1gx+M_2g(l-x) \tag{iv}$$

From (iv) the canonical momentum can be obtained as

$$p_x=\frac{\partial L}{\partial \dot{x}}=(M_1+M_2)\dot{x}\text{, so that }\dot{x}=\frac{p_x}{M_1+M_2}$$

The Hamiltonian of the system is

$$H=\sum_j p_j\dot{q}_j-L=p_x\dot{x}-\left[\frac{1}{2}(M_1+M_2)\dot{x}^2+M_1gx+M_2g(l-x)\right]$$

Substituting for $\dot{x}$ and rearranging, the above expression becomes

$$H=\frac{p_x^2}{2(M_1+M_2)}-M_1gx-M_2g(l-x) \tag{v}$$

Hamilton's canonical equations are

$$\dot{x}=\frac{\partial H}{\partial p_x}=\frac{p_x}{(M_1+M_2)}\quad\text{and}\quad \dot{p}_x=-\frac{\partial H}{\partial x}=(M_1-M_2)g$$

Then, $$p_x=(M_1+M_2)\dot{x}\text{ or, }\dot{p}_x=(M_1+M_2)\ddot{x}=(M_1-M_2)g$$

Therefore, $$\ddot{x}=\frac{(M_1-M_2)g}{(M_1+M_2)} \tag{vi}$$

which is the acceleration of masses.

EXAMPLE: 3.27 Consider a bead of mass m sliding freely on a hoop of radius R rotating with angular velocity ω in a constant gravitational field with acceleration g. Obtain the equation of motion by using Hamilton's method.

Solution: Consider the figure given below. Since the bead is moving over a surface of radius R, we take θ and ϕ as the generalized coordinates.

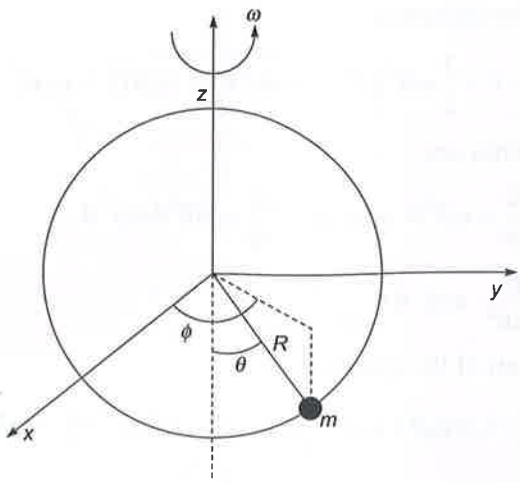

Fig. 3.14

The position coordinates of the bead are

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = -R\cos\theta$$

where, $\phi = \omega t + \phi_0$

Then,
$$\dot{x} = R\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)$$
$$\dot{y} = R\left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right) \text{ and,}$$
$$\dot{z} = R\dot{\theta}\sin\theta$$

The kinetic energy of the bead is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Using the above expressions, we get

$$T = \frac{1}{2}mR^2\left[\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)^2 + \left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right)^2 + \dot{\theta}^2\sin^2\theta\right]$$

$$= \frac{1}{2}mR^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) \quad \text{(i)}$$

The potential energy of the bead is

$$V = mgR(1 - \cos\theta) \quad \text{(ii)}$$

The Lagrangian of the system is

$$L = T - V = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - mgR(1-\cos\theta) \tag{iii}$$

The canonical momenta are

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta} \text{ and } p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mR^2\dot{\phi}\sin^2\theta$$

or $$\dot{\theta} = \frac{p_\theta}{mR^2} \text{ and } \dot{\phi} = \frac{p_\phi}{mR^2\sin^2\theta} \tag{iv}$$

Now, the Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{1}{2}mR^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - mgR(1-\cos\theta)$$

Using (iv), we get

$$H = \frac{p_\theta^2}{mR^2} + \frac{p_\phi^2}{mR^2\sin^2\theta} - \frac{1}{2}\left(\frac{p_\theta^2}{mR^2} + \frac{p_\phi^2}{mR^2\sin^2\theta}\right) - mgR(1-\cos\theta)$$

$$= \frac{1}{2}\left(\frac{p_\theta^2}{mR^2} + \frac{p_\phi^2}{mR^2\sin^2\theta}\right) - mgR(1-\cos\theta) \tag{v}$$

Note that ϕ is cyclic in Hamiltonian and p_ϕ is a constant of motion. Eliminating p_ϕ from Hamiltonian, we get

$$H = \frac{1}{2}\left(\frac{p_\theta^2}{mR^2} + mR^2\dot{\phi}^2\sin^2\theta\right) - mgR(1-\cos\theta) \tag{vi}$$

Hamilton's canonical equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial\theta} = mR^2\dot{\phi}^2\sin\theta\cos\theta - mgR\sin\theta$$

Therefore, $$mR^2\ddot{\theta} = mR^2\dot{\phi}^2\sin\theta\cos\theta - mgR\sin\theta$$

that is, $$\ddot{\theta} + \sin\theta\left(\frac{g}{R} - \dot{\phi}^2\cos\theta\right) = 0 \tag{vii}$$

which is the equation of motion.

EXAMPLE 3.28 A particle of mass m is constrained to move on the surface of a cone of apex angle α in the presence of a gravitational field as shown (Figure 3.15).

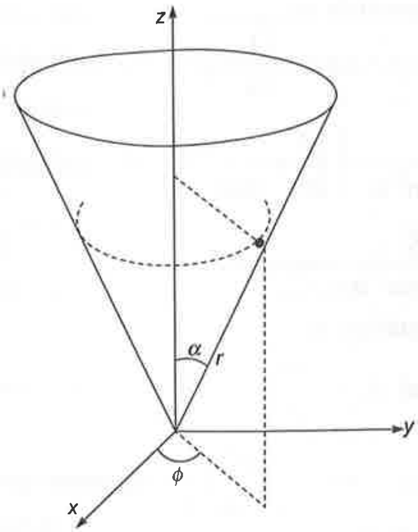

Fig. 3.15

Construct the Hamiltonian of the particle and obtain the equation of motion.

Solution: The coordinates of the mass in terms of r and the polar coordinate ϕ are

$$x = r\sin\alpha\cos\phi\,,\ y = r\sin\alpha\sin\phi \text{ and } z = r\cos\alpha$$

Then,
$$\dot{x} = \dot{r}\sin\alpha\cos\phi - r\dot{\phi}\sin\alpha\sin\phi$$

$$\dot{y} = \dot{r}\sin\alpha\sin\phi + r\dot{\phi}\sin\alpha\cos\phi \text{ and}$$
$$\dot{z} = \dot{r}\cos\alpha$$

The kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha\right) \tag{i}$$

Potential energy is

$$V = mgz = mgr\cos\alpha \tag{ii}$$

Therefore, the Lagrangian of the particle is

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha\right) - mgr\cos\alpha \tag{iii}$$

The canonical momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \text{ and } p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\alpha$$

or
$$\dot{r} = \frac{p_r}{m} \text{ and } \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\alpha} \tag{iv}$$

The Hamiltonian of the particle is

$$H=\sum_j p_j\dot{q}_j - L = p_r\dot{r}+p_\phi\dot{\phi}-\left(\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\sin^2\alpha\right)-mgr\cos\alpha\right)$$

$$=\frac{p_r^2}{m}+\frac{p_\phi^2}{mr^2\sin^2\alpha}-\left(\frac{p_r^2}{2m}+\frac{p_\phi^2}{2mr^2\sin^2\alpha}-mgr\cos\alpha\right)$$

$$=\frac{p_r^2}{2m}+\frac{p_\phi^2}{2mr^2\sin^2\alpha}+mgr\cos\alpha \tag{v}$$

Hamilton's canonical equation is

$$\dot{r}=\frac{\partial H}{\partial p_r}=\frac{p_r}{m} \text{ and } \dot{p}_r=-\frac{\partial H}{\partial r}=\frac{p_\phi^2}{mr^3\sin^2\alpha}-mg\cos\alpha \tag{vi}$$

Therefore, $p_r = m\dot{r}$ and $\dot{p}_r = m\ddot{r} = \dfrac{p_\phi^2}{mr^3\sin^2\alpha} - mg\cos\alpha$

Thus, the equation of motion is

$$\ddot{r}+g\cos\alpha=\frac{p_\phi^2}{m^2r^3\sin^2\alpha}=\frac{L^2}{m^2r^3\sin^2\alpha} \tag{vii}$$

where, $L = p_\phi$ is the angular momentum and is a constant of motion. Also $g\cos\alpha$ is the component of acceleration due to gravity along the surface of the cone.

EXAMPLE 3.29 For the spherical pendulum shown below, find the Hamiltonian and the equation of motion.

Solution: The coordinates of the spherical pendulum in terms of the generalized coordinate θ and the polar angle ϕ, w.r.t. the fixed point can be written as

$x = l\sin\theta\cos\phi$, $y = l\sin\theta\sin\phi$ and $z = -l\cos\theta$

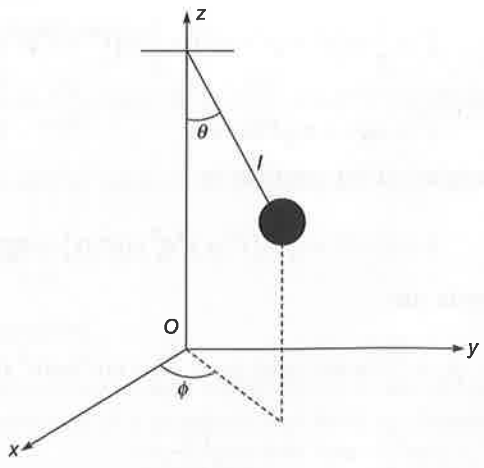

Fig. 3.16

so that $$\dot{x} = l\dot{\theta}\cos\theta\cos\phi - l\dot{\phi}\sin\theta\sin\phi$$
$$\dot{y} = l\dot{\theta}\cos\theta\sin\phi + l\dot{\phi}\sin\theta\cos\phi \text{ and,}$$
$$\dot{z} = -l\dot{\theta}\sin\theta$$

The kinetic energy of the pendulum bob is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Substituting for $\dot{x}, \dot{y}$ and $\dot{z}$, and simplifying, we get

$$T = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) \tag{i}$$

The potential energy of the bob w.r.t. the fixed point is

$$V = mgz = -mgl\cos\theta \tag{ii}$$

Therefore, the Lagrangian is

$$L = T - V = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta \tag{iii}$$

The canonical momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \text{ and } p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta$$

and $$\dot{\theta} = \frac{p_\theta}{ml^2} \text{ and } \dot{\phi} = \frac{p_\phi}{ml^2\sin^2\theta} \tag{iv}$$

The Hamiltonian is

$$H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta} + p_\phi\dot{\phi} - \left(\frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta\right)$$

Substituting for $\dot{\theta}$ and $\dot{\phi}$ from (iv) and simplifying, the Hamiltonian becomes

$$H = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta \tag{v}$$

Hamilton's canonical equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \tag{vi-a}$$

Similarly, $$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{ml^2\sin^2\theta} \text{ and } \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \tag{vi-b}$$

Therefore, p_ϕ is a constant of motion, or the angular momentum is conserved.

From (vi-a), $p_\theta = ml^2\dot{\theta}$ or, $\dot{p}_\theta = ml^2\ddot{\theta} = -mgl\sin\theta$

that is, $$\ddot{\theta}+\frac{g}{l}\sin\theta=0 \tag{vii}$$

which is the equation of motion. For small values of θ, the motion will be simple harmonic.

EXAMPLE 3.30 A simple pendulum is constrained to swing in a plane while the plane is rotating with an angular velocity Ω. Find the Hamiltonian and obtain the equation of motion near the equilibrium point.

Solution: A schematic representation of the problem is given below.

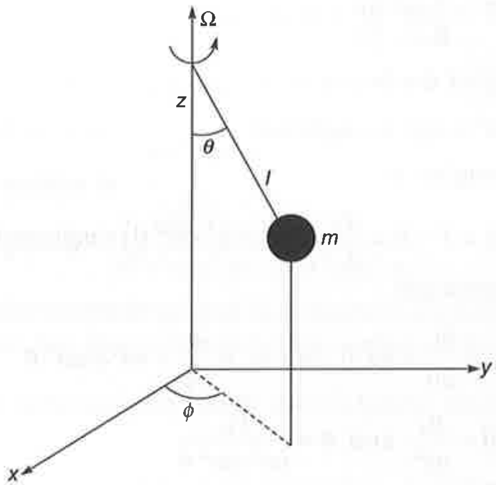

Fig. 3.17

The pendulum is constrained to rotate in a plane which is described by the generalized coordinate θ. However, the plane is rotating with an angular velocity Ω that makes the polar angle ϕ to change, i.e., $\Omega=\dot{\phi}$.

The coordinates of the spherical pendulum in terms of the generalized coordinate θ and the polar angle ϕ, w.r.t. the fixed point can be written as;

$$x=l\sin\theta\cos\phi,\ y=l\sin\theta\sin\phi \text{ and } z=-l\cos\theta$$

so that
$$\dot{x}=l\dot{\theta}\cos\theta\cos\phi-l\dot{\phi}\sin\theta\sin\phi$$
$$\dot{y}=l\dot{\theta}\cos\theta\sin\phi+l\dot{\phi}\sin\theta\cos\phi \text{ and,}$$
$$\dot{z}=-l\dot{\theta}\sin\theta$$

The kinetic energy of the pendulum bob is

$$T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$

Substituting for $\dot{x}, \dot{y}$ and $\dot{z}$, and simplifying, we get

$$T = \frac{1}{2} ml^2 \left(\dot{\theta}^2 + \Omega^2 \sin^2\theta\right) \tag{i}$$

The potential energy of the bob w.r.t. the fixed point is

$$V = mgz = -mgl\cos\theta \tag{ii}$$

Therefore, the Lagrangian is

$$L = T - V = \frac{1}{2} ml^2 \left(\dot{\theta}^2 + \Omega^2 \sin^2\theta\right) + mgl\cos\theta \tag{iii}$$

When the angular velocity Ω is a constant, we can choose θ as the single generalized coordinate.

The canonical momentum is $p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$, so that $\dot{\theta} = \dfrac{p_\theta}{ml^2}$ (iv)

Then, the Hamiltonian of the system is

$$H = \sum_j p_j \dot{q}_j - L = p_\theta \dot{\theta} - \left(\frac{1}{2} ml^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + mgl\cos\theta\right)$$

Substituting for $\dot{\theta}$ from (iv), Hamiltonian becomes

$$H = \frac{p_\theta^2}{2ml^2} - \frac{1}{2} ml^2 \Omega^2 \sin^2\theta - mgl\cos\theta \tag{v}$$

Note that the Hamiltonian does not represent the total energy, but is a constant in time if Ω is considered a constant. Near the equilibrium point, the Hamiltonian can be approximated as

$$H \cong \frac{p_\theta^2}{2ml^2} + \frac{1}{2} mgl\left(1 - \frac{\Omega^2 l}{g}\right)\theta^2 - mgl \tag{vi}$$

Here we used the approximations $\sin\theta \cong \theta$ and $\cos\theta \cong 1 - \dfrac{\theta^2}{2}$, when θ is very small. Now, Hamilton's canonical equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\left(1 - \frac{\Omega^2 l}{g}\right)\theta \tag{vii}$$

Then, $$\ddot{\theta} = \frac{\dot{p}_\theta}{ml^2} = \frac{-mgl\left(1 - \dfrac{\Omega^2 l}{g}\right)\theta}{ml^2} = -\frac{g}{l}\left(1 - \frac{\Omega^2 l}{g}\right)\theta$$

that is, $$\ddot{\theta} + \frac{g}{l}\left(1 - \frac{\Omega^2 l}{g}\right)\theta = 0$$

The equation of motion, then, is

$$\ddot{\theta}+\left(\omega^2-\Omega^2\right)\theta=0 \tag{viii}$$

where, $\omega=\sqrt{\frac{l}{g}}$ the frequency of oscillation of the simple pendulum.

EXAMPLE 3.31 The Hamiltonian of a dynamical system is given by, $H=qp^2-qp+kp$, where k is a constant. Obtain the equation of motion of the system.

Solution: Given, the Hamiltonian of the system as

$$H=qp^2-qp+kp \tag{i}$$

Hamilton's canonical equations are

$$\dot{q}=\frac{\partial H}{\partial p}=(2p-1)q+k \tag{ii}$$

and
$$\dot{p}=-\frac{\partial H}{\partial q}=-\left(p^2-p\right) \tag{iii}$$

From (iii), we can have

$$\dot{p}=\frac{dp}{dt}=-\left(p^2-p\right) \text{ or; } \frac{dp}{\left(p^2-p\right)}=-dt$$

or
$$\left(\frac{1}{p}-\frac{1}{p-1}\right)dp=dt$$

On integration, we get

$$\log p-\log(p-1)=t+c_1, \text{ where } c_1 \text{ is the constant of integration.}$$

or
$$\log\frac{p}{p-1}=t+c_1$$

Taking exponential on both sides, we get

$$\frac{p}{p-1}=e^{t+c_1} \text{ or, } p=\frac{e^{t+c_1}}{e^{t+c_1}-1}=\frac{1}{2}\left[1+\coth\left(\frac{t+c_1}{2}\right)\right] \tag{iv}$$

Now, equation (iv) can be substituted in (ii) to get

$$\dot{q}=\frac{dq}{dt}=q\coth\left(\frac{t+c_1}{2}\right)+k$$

or
$$\frac{dq}{dt}-q\coth\left(\frac{t+c_1}{2}\right)=k$$

Now, multiplying both sides with $e^{-\int \coth\left(\frac{t+c_1}{2}\right)dt}$, we get

$$\frac{dq}{dt}e^{-\int \coth\left(\frac{t+c_1}{2}\right)dt} - q\coth\left(\frac{t+c_1}{2}\right)e^{-\int \coth\left(\frac{t+c_1}{2}\right)dt} = ke^{-\int \coth\left(\frac{t+c_1}{2}\right)dt}$$

or $$\frac{d}{dt}\left[qe^{-\int \coth\left(\frac{t+c_1}{2}\right)dt}\right] = ke^{-\int \coth\left(\frac{t+c_1}{2}\right)dt}$$

Taking integral on both sides, we get

$$qe^{-\int \coth\left(\frac{t+c_1}{2}\right)dt} = \int ke^{-\int \coth\left(\frac{t+c_1}{2}\right)dt}dt + c_2,$$ where c_2 is the constant of integration.

Now, $e^{-\int \coth\left(\frac{t+c_1}{2}\right)dt} = cosech^2\left(\frac{t+c_1}{2}\right)$ and therefore, the above equation becomes

$$q\,cosech^2\left(\frac{t+c_1}{2}\right) = k\int cosech^2\left(\frac{t+c_1}{2}\right) + c_2 = -2k\coth\left(\frac{t+c_1}{2}\right)$$

Then, $$q = c_2\sinh^2\left(\frac{t+c_1}{2}\right) - 2k\coth\left(\frac{t+c_1}{2}\right)\sinh^2\left(\frac{t+c_1}{2}\right) \quad \text{(v)}$$

which is the equation of motion.

EXAMPLE 3.32 The Hamiltonian of a dynamical system is given by, $H = q_1p_1 - q_2p_2 - aq_1^2 + bq_2^2$, where a and b are constants. Solve the problem. Also show that the functions $F_1 = \frac{p_1 - aq_1}{q_2}$ and $F_2 = q_1q_2$ are constants of motion.

Solution: The Hamiltonian of the system is given by

$$H = q_1p_1 - q_2p_2 - aq_1^2 + bq_2^2 \quad \text{(i)}$$

Hamilton's canonical equations are

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = q_1 \text{ and } \dot{p}_1 = -\frac{\partial H}{\partial p_1} = -(p_1 - 2aq_1) \quad \text{(ii)}$$

Similarly, $$\dot{q}_2 = \frac{\partial H}{\partial p_2} = -q_2 \text{ and } \dot{p}_2 = -\frac{\partial H}{\partial q_2} = p_2 - 2bq_2 \quad \text{(iii)}$$

From (ii), $\dot{q}_1 = \frac{dq_1}{dt} = q_1$, and we get; $\frac{dq_1}{q_1} = dt$ and on integration this becomes;

$$\log q_1 = t + c_1 \text{ or; } q_1 = e^{t+c_1} \quad \text{(iv)}$$

Similarly, from (iii) we can have $q_2 = e^{t+c_2}$ (v)

where c_1 and c_2 are constants of integration.

Again, from(ii) $\dot{p}_1 = \dfrac{dp_1}{dt} = -(p_1 - 2aq_1)$ or, $\dfrac{dp_1}{dt} + p_1 = 2aq_1 = 2ae^{t+c_1}$

Now, multiply both sides with e^t to get;

$$\frac{dp_1}{dt}e^t + p_1e^t = 2ae^{2t+c_1}$$

This can be written as

$$\frac{d}{dt}\left(p_1e^t\right) = 2ae^{2t+c_1}$$

On integration, we get

$$p_1e^t = 2a\int e^{2t+c_1}dt + e^{c_3} \text{ or; } p_1 = 2ae^{t+c_1} + e^{c_3-t} \qquad \text{(vi)}$$

Similarly, from (iii), we get; $p_2 = -2be^{t+c_2} + e^{t+c_4}$ (vii)

where e^{c_3} and e^{c_4} are taken as the constants of integration. Equations (iv), (v), (vi) and (vii) are the required solutions.

Now, let us consider the second part of the problem. Given that, $F_1 = \dfrac{p_1 - aq_1}{q_2}$. Let us take the time derivative of the function F_1. That is,

$$\frac{dF_1}{dt} = \frac{d}{dt}\left(\frac{p_1 - aq_1}{q_2}\right) = \frac{q_2(\dot{p}_1 - a\dot{q}_1) - \dot{q}_2(p_1 - aq_1)}{q_2^2}$$

Now put, $\dot{q}_1 = q_1$, $\dot{q}_2 = -q_2$ and $\dot{p}_1 = -(p_1 - 2aq_1)$ so that;

$$\frac{dF_1}{dt} = \frac{d}{dt}\left(\frac{p_1 - aq_1}{q_2}\right) = \frac{\left[-(p_1 - 2aq_1) - aq_1\right] + (p_1 - aq_1)}{q_2}$$

$$= \frac{(-p_1 + aq_1) + (p_1 - aq_1)}{q_2} = 0$$

that is, $F_1 = \dfrac{p_1 - aq_1}{q_2}$ is a constant of motion.

Similarly, taking the time derivative of the function, $F_2 = q_1q_2$, we get

$$\frac{dF_2}{dt} = \frac{d}{dt}(q_1q_2) = \dot{q}_1q_2 + \dot{q}_2q_1 = q_1q_2 - q_2q_1 = 0$$

Therefore, $F_2 = q_1q_2$ is also a constant of motion.

EXAMPLE 3.33 A particle of mass m is sliding on a wire bent in the form of parabola. Ignore the friction so that the particle is acted upon only by the gravitational force. Obtain the equation of motion of the particle.

Solution: A schematic representation of the problem is shown below. Let the wire be placed in the x-y plane.

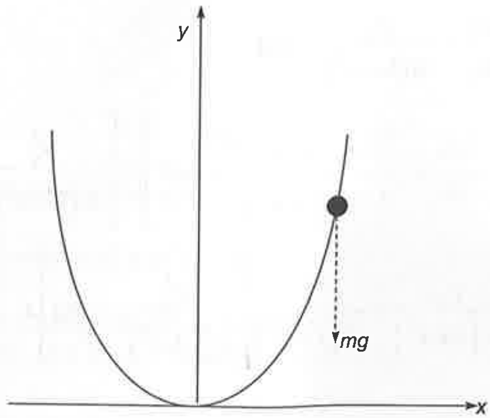

Fig. 3.18

The kinetic energy of the particle is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

The potential energy is $V = mgy$

Now, the parabola can be represented by $y = \frac{1}{2}x^2$ so that, $\dot{y} = x\dot{x}$ and then the kinetic and potential energies are, $T = \frac{1}{2}m\left(1+x^2\right)\dot{x}^2$, and $V = \frac{1}{2}mgx^2$ respectively. Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(1+x^2\right)\dot{x}^2 - \frac{1}{2}mgx^2 \tag{i}$$

The canonical momentum is

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\left(1+x^2\right)\dot{x} \text{ or; } \dot{x} = \frac{p_x}{m\left(1+x^2\right)} \tag{ii}$$

The Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = p_x\dot{x} - \left[\frac{1}{2}m\left(1+x^2\right)\dot{x}^2 - \frac{1}{2}mgx^2\right]$$

Substituting for $\dot{x}$ from (ii), the Hamiltonian becomes

$$H = \frac{p_x^2}{2m(1+x^2)} + \frac{1}{2}mgx^2 \tag{iii}$$

The Hamilton's canonical equation is

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m(1+x^2)} \quad \text{and}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\left[-\frac{p_x^2 x}{m(1+x^2)^2} + mgx\right] = \left[\frac{p_x^2}{m(1+x^2)^2} - mg\right]x$$

Therefore,
$$\ddot{x} = \frac{\dot{p}_x}{m(1+x^2)} = \frac{1}{m(1+x^2)}\left[\frac{p_x^2}{m(1+x^2)^2} - mg\right]x$$

$$= \left[\frac{p_x^2}{m^2(1+x^2)^3} - \frac{g}{(1+x^2)}\right]x$$

or
$$\ddot{x} - \left[\frac{p_x^2}{m^2(1+x^2)^3} - \frac{g}{(1+x^2)}\right]x = 0 \tag{iv}$$

This is the required equation of motion.

EXAMPLE 3.34 Two masses and two springs are configured linearly and externally driven to rotate with angular velocity ω about a fixed point on a horizontal surface, as shown in Figure 3.19. The unstretched length of each spring is a. Obtain the expressions for the conserved quantities. If the system is not externally driven, and that the angular coordinate ϕ is a dynamical variable obtain the Hamiltonian and the equation of motion of the system.

Solution: For the first part of the problem, let us take r_1 and r_2 as the generalized coordinates. Then the kinetic energy of the system can be written as

$$T = \frac{1}{2}m\left(\dot{r}_1^2 + \dot{r}_2^2 + \omega^2 r_1^2 + \omega^2 r_2^2\right) \tag{i}$$

The potential energy of the system is

$$V = \frac{1}{2}k(r_1 - a)^2 + \frac{1}{2}k(r_2 - r_1 - a)^2 \tag{ii}$$

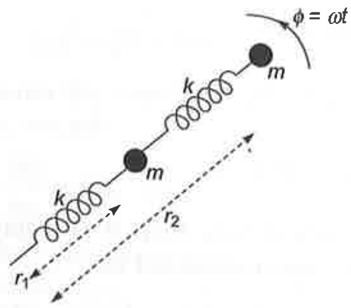

Fig. 3.19

Therefore, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(\dot{r}_1^2 + \dot{r}_2^2 + \omega^2 r_1^2 + \omega^2 r_2^2\right) - \frac{1}{2}k(r_1 - a)^2 - \frac{1}{2}k(r_2 - r_1 - a)^2 \qquad \text{(iii)}$$

Note that the kinetic energy is not a homogeneous function of order 2 in generalized velocity and hence, $H \neq T + V$. However, if ω is kept constant, Hamiltonian is a conserved quantity.

Now, the canonical momenta are

$$p_{r_1} = \frac{\partial L}{\partial \dot{r}_1} = m\dot{r}_1 \text{ and } p_{r_2} = \frac{\partial L}{\partial \dot{r}_2} = m\dot{r}_2$$

so that $$\dot{r}_1 = \frac{p_{r_1}}{m} \text{ and } \dot{r}_2 = \frac{p_{r_2}}{m} \qquad \text{(iv)}$$

The Hamiltonian of the system is given by

$$H = \sum_j p_j \dot{q}_j - L$$

$$= p_{r_1}\dot{r}_1 + p_{r_2}\dot{r}_2 - \left[\frac{1}{2}m\left(\dot{r}_1^2 + \dot{r}_2^2 + \omega^2 r_1^2 + \omega^2 r_2^2\right) - \frac{1}{2}k(r_1 - a)^2 - \frac{1}{2}k(r_2 - r_1 - a)^2\right]$$

Using (iv), the expression for the Hamiltonian becomes

$$H = \frac{1}{2m}\left(p_{r_1}^2 + p_{r_2}^2\right) - \frac{1}{2}m\omega^2\left(r_1^2 + r_2^2\right) + \frac{1}{2}k(r_1 - a)^2 + \frac{1}{2}k(r_2 - r_1 - a)^2 \qquad \text{(v)}$$

The Hamilton's canonical equations of motion are

$$\dot{r}_1 = \frac{\partial H}{\partial p_{r_1}} = \frac{p_{r_1}}{m} \text{ and}$$

$$\dot{p}_{r_1} = -\frac{\partial H}{\partial r_1} = -m\omega^2 r_1 - k(r_1 - a) + k(r_2 - r_1 - a) = -m\omega^2 r_1 + k(r_2 - 2r_1)$$

From the first canonical equation

$$\ddot{r}_1 = \frac{\dot{p}_{r_1}}{m} = -\omega^2 r_1 + \frac{k}{m}(r_2 - 2r_1)$$

or
$$\ddot{r}_1 + \omega^2 r_1 - \frac{k}{m}(r_2 - 2r_1) = 0 \tag{vi}$$

If the system is not externally driven, then ϕ is a generalized coordinate and then, the Lagrangian of the system can be obtained as

$$L = \frac{1}{2}m\left(\dot{r}_1^2 + \dot{r}_2^2 + r_1^2\dot{\phi}^2 + r_2^2\dot{\phi}^2\right) - \frac{1}{2}k(r_1 - a)^2 - \frac{1}{2}k(r_2 - r_1 - a)^2 \tag{vii}$$

Now, the kinetic energy is a homogeneous function of order 2 in generalized velocity and hence, $H = T + V$.

The generalized momentum corresponding to ϕ is

$$p_\phi = \frac{\partial L}{\partial \phi} = m\left(r_1^2 + r_2^2\right)\dot{\phi} \text{ and } \dot{\phi} = \frac{p_\phi}{m\left(r_1^2 + r_2^2\right)} \tag{viii}$$

Now, the Hamiltonian is

$$H = \frac{1}{2}m\left(\dot{r}_1^2 + \dot{r}_2^2 + r_1^2\dot{\phi}^2 + r_2^2\dot{\phi}^2\right) + \frac{1}{2}k(r_1 - a)^2 + \frac{1}{2}k(r_2 - r_1 - a)^2$$

Substituting for $\dot{r}_1, \dot{r}_2$ and $\dot{\phi}$, the Hamiltonian becomes

$$H = \frac{1}{2m}\left(p_{r_1}^2 + p_{r_2}^2\right) + \frac{p_\phi^2}{2m\left(r_1^2 + r_2^2\right)} + \frac{1}{2}k(r_1 - a)^2 + \frac{1}{2}k(r_2 - r_1 - a)^2 \tag{ix}$$

Note that ϕ is a cyclic coordinate and hence p_ϕ is conserved during motion. Again the motion of the system is governed by equation (vi) with $\omega = \dot{\phi}$.

EXAMPLE 3.35 The Hamiltonian of a system having two degrees of freedom is $H = \frac{1}{2}\left(p_1^2 q_1^4 + p_2^2 q_1^2 - 2\alpha q_1\right)$ where α is a constant. Show that, q_1 varies sinusoidally with q_2.

Solution: The given Hamiltonian is

$$H = \frac{1}{2}\left(p_1^2 q_1^4 + p_2^2 q_1^2 - 2\alpha q_1\right) \tag{i}$$

Since the Hamiltonian does not include time explicitly, H is a constant of motion. Therefore, let

$$\frac{1}{2}\left(p_1^2 q_1^4 + p_2^2 q_1^2 - 2\alpha q_1\right) = \frac{\lambda}{2}$$

or
$$p_1^2 q_1^4 + p_2^2 q_1^2 - 2\alpha q_1 = \lambda$$

or
$$p_1 q_1^2 = \sqrt{\lambda + 2\alpha q_1 - p_2^2 q_1^2} \tag{ii}$$

Since q_2 is a cyclic coordinate, the corresponding momenta p_2 is constant. From Hamilton's canonical equation, we get

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = p_1 q_1^4 \text{ and } \dot{q}_2 = \frac{\partial H}{\partial p_1} = p_2 q_1^2$$

Dividing one equation by the other, we get

$$\frac{dq_1}{dq_2} = \frac{p_1 q_1^2}{p_2} = \frac{\sqrt{\lambda + 2\alpha q_1 - p_2^2 q_1^2}}{p_2}$$

or
$$dq_2 = \frac{dq_1}{\left(\sqrt{\frac{\lambda}{p_2^2} + \frac{2\alpha q_1}{p_2^2} - q_1^2}\right)} = \frac{dq_1}{\left(\sqrt{\frac{\lambda}{p_2^2} + \frac{\alpha^2}{p_2^4} + \frac{2\alpha q_1}{p_2^2} - q_1^2 - \frac{\alpha^2}{p_2^4}}\right)}$$

$$= \frac{dq_1}{\left(\sqrt{A^2 - (q_1 - \varepsilon)^2}\right)} \tag{iii}$$

where, $A = \left(\frac{\lambda}{p_2^2} + \frac{\alpha^2}{p_2^4}\right)$ and $\varepsilon = \frac{\alpha}{p_2^2}$.

Integrating equation (iii) we get

$$\int dq_2 = \int \frac{dq_1}{\left(\sqrt{A^2 - (q_1 - \varepsilon)^2}\right)}$$

That is, $q_2 = \sin^{-1}\left(\frac{q_1 - \varepsilon}{A}\right) - \phi$, where $-\phi$ is the constant of integration. This can be rearranged to get

$$q_1 = \varepsilon + A\sin(q_2 + \phi) \tag{iv}$$

That is, q_1 varies sinusoidally with q_2.

EXAMPLE 3.36 Consider the system shown in Figure 3.20 that consists of two particles P_1 and P_2 connected by a light inextensible string of length a. The particle P_1 is also constrained to move along a fixed horizontal rail and the whole system moves in the vertical plane through the rail. Take the variables x and θ as generalized coordinates. Obtain the Hamiltonian and verify if it satisfies the equations $\dot{x} = \frac{\partial H}{\partial p_x}$ and $\dot{\theta} = \frac{\partial H}{\partial p_\theta}$.

Solution: The kinetic energy of the system consists of the kinetic energy of the two particles. The velocity of the particle $P2$ should be taken as the resultant of the two velocities $\dot{x}$ and $a\dot{\theta}$. In terms of x and θ it can be written as

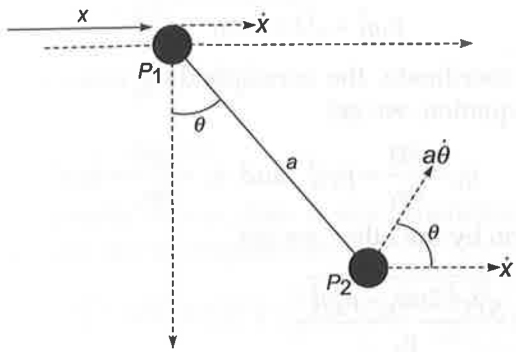

Fig. 3.20

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta\right) \tag{i}$$

The potential energy of the system is

$$V = -mga\cos\theta \tag{ii}$$

The Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta\right) + mga\cos\theta$$

$$= \frac{1}{2}m\left(2\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta\right) + mga\cos\theta \tag{iii}$$

The canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\left(2\dot{x} + a\dot{\theta}\cos\theta\right) \text{ and}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma\left(a\dot{\theta} + \dot{x}\cos\theta\right)$$

These two equations can be solved between themselves to get

$$\dot{x} = \frac{p_x - \frac{p_\theta}{a}\cos\theta}{m\left(2 - \cos^2\theta\right)} \text{ and}$$

$$a\dot{\theta} = \frac{2\left(\frac{p_\theta}{a}\right) - p_x\cos\theta}{m\left(2 - \cos^2\theta\right)}$$

The Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = p_x\dot{x} + p_\theta\dot{\theta} - \left[\frac{1}{2}m\left(2\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta\right) + mga\cos\theta\right]$$

Substituting for $\dot{x}$ and $\dot{\theta}$ from the above equations and simplifying, the expression for Hamiltonian becomes

$$H = \frac{p_x^2 + 2\left(\frac{p_\theta}{a}\right)^2 - \frac{2}{a} p_x p_\theta \cos\theta}{2m\left(2 - \cos^2\theta\right)} - mga\cos\theta \tag{iv}$$

Hamilton's canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x - \frac{p_\theta}{a}\cos\theta}{m\left(2 - \cos^2\theta\right)}$$

Similarly, $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{\left(\frac{4p_\theta}{a^2} - \frac{2}{a} p_x \cos\theta\right)}{2m\left(2 - \cos^2\theta\right)} = \frac{2\left(\frac{p_\theta}{a}\right) - p_x\cos\theta}{ma\left(2 - \cos^2\theta\right)}$$

Hence, verified.

EXAMPLE 3.37 A particle of mass m is suspended from a support by a light inextensible string which passes through a small fixed ring vertically below the support. The particle moves in a vertical plane with the taut string. At the same time, the support is made to move vertically having an upward displacement $z(t)$ at time t. The effect is that the particle oscillates like a simple pendulum whose string length at time t is $a - z(t)$, where a is a positive constant. Construct the Hamiltonian of the system and obtain the Hamilton's canonical equations of motion.

Solution: A schematic representation of the problem is given in Figure 3.21. The constant a can be regarded as the initial length of the pendulum below the point O.

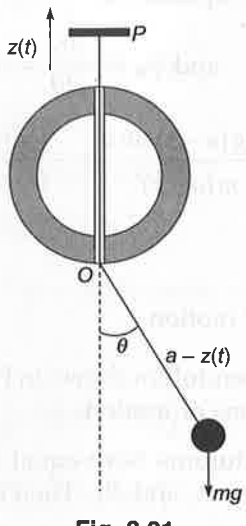

Fig. 3.21

The generalized coordinate is θ and the kinetic energy is given by

$$T=\frac{1}{2}m\left[\dot{z}^2+(a-z)^2\dot{\theta}^2\right] \tag{i}$$

The potential energy w.r.t. the point O is

$$V=-mg(a-z)\cos\theta \tag{ii}$$

Then, the Lagrangian of the system is

$$L=T-V=\frac{1}{2}m\left[\dot{z}^2+(a-z)^2\dot{\theta}^2\right]+mg(a-z)\cos\theta \tag{iii}$$

The canonical momentum is

$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=m(a-z)^2\dot{\theta} \quad \text{so that} \quad \dot{\theta}=\frac{p_\theta}{m(a-z)^2}$$

Now, the Hamiltonian of the system is

$$H=\sum_j p_j\dot{q}_j-L=p_\theta\dot{\theta}-\left[\frac{1}{2}m\left[\dot{z}^2+(a-z)^2\dot{\theta}^2\right]+mg(a-z)\cos\theta\right]$$

$$=\frac{p_\theta^2}{m(a-z)^2}-\left\{\frac{1}{2}m\left[\dot{z}^2+(a-z)^2\left(\frac{p_\theta}{m(a-z)^2}\right)^2\right]+mg(a-z)\cos\theta\right\}$$

$$=\frac{p_\theta^2}{2m(a-z)^2}-\frac{1}{2}m\dot{z}^2-mg(a-z)\cos\theta \tag{iv}$$

Now, the Hamilton's canonical equation is

$$\dot{\theta}=\frac{\partial H}{\partial p_\theta}=\frac{p_\theta}{m(a-z)^2} \quad \text{and} \quad \dot{p}_\theta=-\frac{\partial H}{\partial\theta}=-mg(a-z)\sin\theta$$

Then, $$\ddot{\theta}=\frac{\dot{p}_\theta}{m(a-z)^2}=-\frac{mg(a-z)\sin\theta}{m(a-z)^2}=-\frac{g\sin\theta}{(a-z)}$$

that is, $$\ddot{\theta}+\frac{g}{(a-z)}\sin\theta=0 \tag{v}$$

which is the required equation of motion.

EXAMPLE 3.38 For the double pendulum shown in Figure 3.22, obtain the Hamiltonian and Hamilton's canonical equations of motion.

Solution: Assume that both pendulums have equal length l and bob has equal mass m. The generalized coordinates are θ_1 and θ_2. Then the coordinates of the bobs can be written as;

For the first bob, $x_1 = l\sin\theta_1$ and $y_1 = -l\cos\theta_1$

For the second bob, $x_2 = l\sin\theta_1 + l\sin\theta_2$ and $y_2 = -l\cos\theta_1 - l\cos\theta_2$

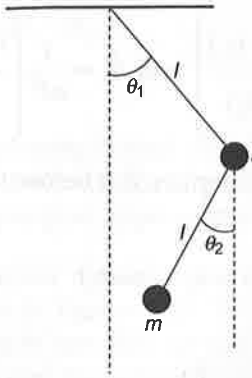

Fig. 3.22

so that $\dot{x}_1 = l\dot{\theta}_1\cos\theta_1$ and $\dot{y}_1 = l\dot{\theta}_1\sin\theta_1$

and $\quad \dot{x}_2 = l\dot{\theta}_1\cos\theta_1 + l\dot{\theta}_2\cos\theta_2$ and $\dot{y}_2 = l\dot{\theta}_1\sin\theta_1 + l\dot{\theta}_2\sin\theta_2$

Then, the kinetic energy of the system is given by

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2\right)$$

$$T = \frac{1}{2}m\left[\left(l\dot{\theta}_1\cos\theta_1\right)^2 + \left(l\dot{\theta}_1\cos\theta_1 + l\dot{\theta}_2\cos\theta_2\right)^2 + \left(l\dot{\theta}_1\sin\theta_1\right)^2 + \left(l\dot{\theta}_1\sin\theta_1 + l\dot{\theta}_2\sin\theta_2\right)^2\right]$$

On simplification, this would yield

$$T = \frac{1}{2}ml^2\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right] \tag{i}$$

The potential energy of the system is

$$V = mg(y_1 + y_2) = -mgl(2\cos\theta_1 + \cos\theta_2) \tag{ii}$$

Therefore, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}ml^2\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right] + mgl(2\cos\theta_1 + \cos\theta_2) \tag{iii}$$

The canonical momenta are

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = ml^2\left[2\dot{\theta}_1 + \dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$$

and $$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = ml^2\left[\dot{\theta}_2 + \dot{\theta}_1\cos(\theta_1 - \theta_2)\right]$$

These two expressions can be solved themselves to obtain the expressions for $\dot{\theta}_1$ and $\dot{\theta}_2$ as

$$\dot{\theta}_1 = \frac{1}{2ml^2}\left[\frac{p_{\theta_1} - p_{\theta_2}\cos(\theta_1-\theta_2)}{1-\frac{1}{2}\cos^2(\theta_1-\theta_2)}\right] \text{ and; } \dot{\theta}_2 = \frac{1}{ml^2}\left[\frac{p_{\theta_2} - \frac{p_{\theta_1}}{2}\cos(\theta_1-\theta_2)}{1-\frac{1}{2}\cos^2(\theta_1-\theta_2)}\right] \tag{iv}$$

The next step is to find the Hamiltonian. But before that let us rewrite the Lagrangian in terms of canonical momenta to get

$$L = \frac{1}{2}\left(p_{\theta_1}\dot{\theta}_1 + p_{\theta_2}\dot{\theta}_2\right) + mgl(2\cos\theta_1 + \cos\theta_2) \tag{v}$$

Now, the Hamiltonian is

$$H = \sum_j p_j \dot{q}_j - L = p_{\theta_1}\dot{\theta}_1 + p_{\theta_2}\dot{\theta}_2 - \left[\frac{1}{2}\left(p_{\theta_1}\dot{\theta}_1 + p_{\theta_2}\dot{\theta}_2\right) + mgl(2\cos\theta_1 + \cos\theta_2)\right]$$

$$= \frac{1}{2}\left(p_{\theta_1}\dot{\theta}_1 + p_{\theta_2}\dot{\theta}_2\right) + mgl(2\cos\theta_1 + \cos\theta_2) \tag{vi}$$

Substituting for $\dot{\theta}_1$ and $\dot{\theta}_2$ from (iv) and simplifying, the Hamiltonian becomes

$$H = \frac{1}{4ml^2}\left[\frac{p_{\theta_1}^2 + 2p_{\theta_2}^2 - 2p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)}{1-\frac{1}{2}\cos^2(\theta_1-\theta_2)}\right] - mgl(2\cos\theta_1 + \cos\theta_2) \tag{vii}$$

From Hamilton's canonical equations, we get

$$\dot{p}_{\theta_1} = -\frac{\partial H}{\partial \theta_1} = \frac{1}{2ml^2}\left[\frac{p_{\theta_1}p_{\theta_2}\sin(\theta_1-\theta_2)}{1-\frac{1}{2}\cos^2(\theta_1-\theta_2)}\right]$$

$$-\frac{1}{4ml^2}\left[\frac{p_{\theta_1}^2 + 2p_{\theta_2}^2 - 2p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)}{\left(1-\frac{1}{2}\cos^2(\theta_1-\theta_2)\right)^2}\right]\cos(\theta_1-\theta_2)\sin(\theta_1-\theta_2) + 2mg\,l\sin\theta_1 \tag{viii}$$

Similarly, we can have

$$\dot{p}_{\theta_2} = -\frac{\partial H}{\partial \theta_2} = -\frac{1}{2ml^2}\left[\frac{p_{\theta_1}p_{\theta_2}\sin(\theta_1-\theta_2)}{1-\frac{1}{2}\cos^2(\theta_1-\theta_2)}\right]$$

$$+\frac{1}{4ml^2}\left[\frac{p_{\theta_1}^2+2p_{\theta_2}^2-2p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)}{\left(1-\frac{1}{2}\cos^2(\theta_1-\theta_2)\right)^2}\right]\cos(\theta_1-\theta_2)\sin(\theta_1-\theta_2)$$

$$+mgl\sin\theta_2 \tag{ix}$$

Note that the expression is too complicated and we could not gain any information from Hamiltonian approach. But for the same problem, we could get a much more simpler and useful result in Lagrangian approach.

EXAMPLE 3.39 A flyball governor consists of two masses m connected to arms of length l and a mass M as shown in Figure 3.23. The assembly is constrained to rotate around a shaft on which the mass M can slide up and down without friction. Neglecting the mass of the arms, air friction, and assuming that the diameter of the mass M is small, obtain the Hamiltonian of the system.

Solution: The schematic representation of the problem is given in Figure 3.23. O is a fixed point and is taken as the origin of the coordinate system.

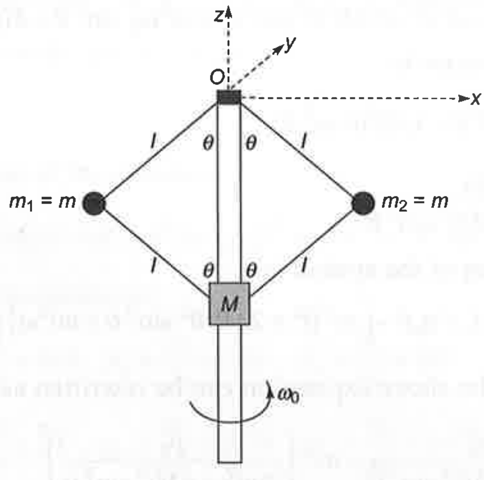

Fig. 3.23

First let us write the coordinates of the three masses. For the mass m_1 the coordinates are $(-l\sin\theta, 0, -l\cos\theta)$; for the mass m_2, the coordinates are, $(l\sin\theta, 0, -l\cos\theta)$ and that of M are, $(0, 0, -2l\cos\theta)$.

Now, the kinetic energy of the system can be obtained as the sum of the translational kinetic energies of the three masses and rotational kinetic energies of the masses m_1 and m_2.

The translational kinetic energy can be written as

$$T_{tra} = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2 + \dot{z}_1^2 + \dot{z}_2^2\right) + \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right)$$

Using the coordinate given above we can find the corresponding velocities and can be substituted in the above expression to get

$$T_{tra} = ml^2\dot{\theta}^2 + 2Ml^2\dot{\theta}^2\sin^2\theta$$

The rotational kinetic energy of each mass m is equal to $\frac{1}{2}ml^2\omega_0^2\sin^2\theta$ so that the total rotational kinetic energy is

$$T_{rot} = ml^2\omega_0^2\sin^2\theta$$

Now, the total kinetic energy of the system is

$$T = ml^2\dot{\theta}^2 + 2Ml^2\dot{\theta}^2\sin^2\theta + ml^2\omega_0^2\sin^2\theta \tag{i}$$

The potential energy of the system is

$$V = -mgl\cos\theta - mgl\cos\theta - 2Mgl\cos\theta = -2(m+M)l\cos\theta \tag{ii}$$

Now, the Lagrangian of the system is

$$L = T - V = ml^2\dot{\theta}^2 + 2Ml^2\dot{\theta}^2\sin^2\theta + ml^2\omega_0^2\sin^2\theta + 2(m+M)l\cos\theta \tag{iii}$$

The canonical momentum is

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = 2ml^2\dot{\theta} + 4Ml^2\dot{\theta}\sin^2\theta \tag{iv}$$

so that $$\dot{\theta} = \frac{p_\theta}{2ml^2 + 4Ml^2\sin^2\theta} \tag{v}$$

Then, the Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta} - \left[ml^2\dot{\theta}^2 + 2Ml^2\dot{\theta}^2\sin^2\theta + ml^2\omega_0^2\sin^2\theta + 2(m+M)l\cos\theta\right]$$

Using equation (v), the above expression can be rewritten as

$$H = \frac{p_\theta^2}{2ml^2 + 4Ml^2\sin^2\theta} - ml^2\left(\frac{p_\theta}{2ml^2 + 4Ml^2\sin^2\theta}\right)^2$$

$$-2Ml^2\left(\frac{p_\theta}{2ml^2 + 4Ml^2\sin^2\theta}\right)^2\sin^2\theta - ml^2\omega_0^2\sin^2\theta - 2(m+M)l\cos\theta$$

This expression can be simplified to obtain the Hamiltonian as

$$H = \frac{p_\theta^2}{4l^2\left(m + 2M\sin^2\theta\right)} - ml^2\omega_0^2\sin^2\theta - 2(m+M)l\cos\theta \tag{vi}$$

One of the Hamilton's canonical equations, we already have obtained in equation (v). The second one is

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\theta^2 \sin 2\theta}{2l^2\left(m+2M\sin^2\theta\right)^2} - ml^2\omega_0^2 \sin 2\theta + 2(m+M)l\sin\theta \qquad \text{(vii)}$$

EXAMPLE 3.40 The interaction between two inert gas atoms each of mass m is given by the potential $V(r) = -\frac{2A}{r^6} + \frac{B}{r^{12}}$, where A and B are positive constants and $r = r_1 - r_2$ is the separation between the atoms. Obtain the Hamiltonian of the system and show that the energy of the lowest energy state is $H_{\min} = -\frac{A^2}{B}$.

Solution: Let r_1 and r_2 be the position vectors of the atoms and R that of the centre of mass. Let the coordinate of the centre of mass be (x, y, z). The kinetic energy of the system can be written as (refer problem 2.38)

$$T = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) \qquad \text{(i)}$$

where, $M = 2m$ the total mass and $\mu = \frac{mm}{m+m} = \frac{m}{2}$ the reduced mass of the system.

The potential energy of the system is given as

$$V = -\frac{2A}{r^6} + \frac{B}{r^{12}} \qquad \text{(ii)}$$

So that the Lagrangian of the system is

$$L = T - V = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) + \frac{2A}{r^6} - \frac{B}{r^{12}} \qquad \text{(iii)}$$

Now, the canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = M\dot{x}\ ,\ p_y = \frac{\partial L}{\partial \dot{y}} = M\dot{y}\ ,\ p_z = \frac{\partial L}{\partial \dot{z}} = M\dot{z}$$

and

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}\ ,\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}\ ,\ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2\dot{\phi}\sin^2\theta$$

The Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L$$

Using the expressions for canonical momenta, the Hamiltonian can be obtained as

$$H = \frac{1}{2M}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2\mu}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{2A}{r^6} + \frac{B}{r^{12}} \qquad \text{(iv)}$$

Now, let us consider the second part of the problem. The lowest energy state corresponds to the zero value of canonical momenta and an equilibrium separation r_0 that minimizes the potential. The equilibrium separation can be obtained by setting

$$\frac{d}{dt}\left(-\frac{2A}{r^6}+\frac{B}{r^{12}}\right)=0$$

that is, $-2A\times -6\times r^{-7}\dfrac{dr}{dt}-12Br^{-13}\dfrac{dr}{dt}=0$

Upon simplification, we get

$$r_0=\left(\frac{B}{A}\right)^{1/6} \tag{v}$$

Using this in (iv) and setting all canonical momenta to zero, we get the minimum value of energy as

$$H_{\min}=-\frac{A^2}{B} \tag{vi}$$

Hence, proved.

EXAMPLE 3.41 A particle of mass m is constrained to move on the surface of a sphere of radius R. Obtain the Hamiltonian of the particle when no external force is acting on the particle. Also show that the motion of the particle is along a great circle of the sphere.

Solution: Since the particle is constrained to move on the surface of the sphere, there are two degrees of freedom for the particle and we need two generalized coordinates. Since the radius of the sphere is a constant, we take (θ,ϕ) of the spherical polar coordinates as the generalized coordinates.

Therefore, the kinetic energy of the particle can be written as

$$T=\frac{1}{2}mR^2\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta\right) \tag{i}$$

The external force acting on the particle is zero and hence the potential energy of the particle is zero. Then, the Lagrangian of the system is

$$L=T-V=\frac{1}{2}mR^2\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta\right) \tag{ii}$$

The canonical momenta are

$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=mR^2\dot{\theta}\ \text{ and }\ p_\phi=\frac{\partial L}{\partial\dot{\phi}}=mR^2\dot{\phi}\sin^2\theta$$

This would give

$$\dot{\theta}=\frac{p_\theta}{mR^2}\ \text{ and }\ \dot{\phi}=\frac{p_\phi}{mR^2\sin^2\theta} \tag{iii}$$

The Hamiltonian of the particle is

$$H = \sum_j p_j \dot{q}_j - L = p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} mR^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

$$= \frac{p_\theta^2}{mR^2} + \frac{p_\phi^2}{mR^2 \sin^2\theta} - \frac{1}{2} mR^2 \left[\left(\frac{p_\theta}{mR^2}\right)^2 + \left(\frac{p_\phi}{mR^2 \sin^2\theta}\right)^2 \sin^2\theta\right]$$

$$= \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{2mR^2 \sin^2\theta} = \frac{1}{2mR^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) \qquad \text{(iv)}$$

Note that the Hamiltonian is not an explicit function of time and hence it is a constant of motion.

To solve the second part of the problem, we consider the Hamilton's canonical equation

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

This means $p_\phi = mR^2 \dot{\phi} \sin^2\theta$ is a constant.

or $$\dot{\phi} \sin^2\theta = constant$$

We can choose the set of coordinates (θ, ϕ) so that the initial condition is $\dot{\phi} = 0$ at $t = 0$. Then the above constant is zero at all time $\dot{\phi} \sin^2\theta = 0$. Since θ cannot be zero at all time, $\dot{\phi} = 0$, or $\phi =$ constant, that is, the motion of the particle is along a great circle of the sphere.

EXAMPLE 3.42 A nonrelativistic electron of mass m, charge $-e$ in a cylindrical magneton moves between a wire of radius a at a negative electric potential $-\phi_0$ and a concentric cylindrical conductor of radius R at zero potential. There is a uniform constant magnetic field B parallel to the axis. Given that the electric and magnetic vector potentials can be written as

$$\phi = -\phi_0 \frac{\ln(r/R)}{\ln(a/R)} \text{ and } A = \frac{1}{2} B r \hat{\theta}$$

Obtain the Hamiltonian of the system. Also show that there are three constants of the motion.

Solution: Since the problem has a cylindrical symmetry, we use the cylindrical coordinate system. The kinetic energy of the charged particle in cylindrical coordinate system can be written as

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \qquad \text{(i)}$$

The potential energy of the particle is

$$V = -e\phi + e(v.A) = -e\phi + \frac{1}{2} eBr^2\dot{\theta} \qquad \text{(ii)}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) + e\phi - \frac{1}{2}eBr^2\dot{\theta} \tag{iii}$$

The canonical momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}\ ,\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} - \frac{1}{2}eBr^2 \ \text{ and } \ p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \tag{iv}$$

The inverse relations are

$$\dot{r} = \frac{p_r}{m},\quad \dot{\theta} = \frac{p_\theta + \frac{1}{2}eBr^2}{mr^2} \quad \text{and,}\ \dot{z} = \frac{p_z}{m} \tag{v}$$

The Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta} + p_\phi\dot{\phi} + p_z\dot{z} - \left(\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) + e\phi - \frac{1}{2}eBr^2\dot{\theta}\right)$$

Using equation (iv), the Hamiltonian can be rewritten as

$$H = \frac{1}{2m}\left[p_r^2 + \left(\frac{p_\theta}{r} + \frac{1}{2}eBr\right)^2 + p_z^2\right] - e\phi \tag{vi}$$

Now, let us solve the second part of the problem. Note that the Hamiltonian is not an explicit function of time and hence, it is a constant of motion. Again, the coordinates, θ and ϕ are cyclic in Hamiltonian and therefore the corresponding momenta are constants of motion. That is, p_θ and p_ϕ are constants of motion. They are given through the expression (iv).

EXAMPLE 3.43 A system of two particles of equal mass m are connected by a rope of length l. One mass is constrained to stay on the surface of an upright cone of half-angle α and the other mass is hanging freely inside the cone, the rope is passing through a hole at the top of the cone. Neglecting the friction, obtain the Hamiltonian of the system. Also find the angular frequency of the mass which is outside and is moving in a circular orbit.

Solution: A schematic representation of the problem is given below. First let us find the coordinates of the two masses. Let us use the spherical polar coordinate system with origin at the vertex of the cone.

The coordinates of the mass inside the cone are (r,θ,ϕ), where the symbols have their usual meaning. The coordinates of the mass outside the cone are $(l-r,\pi-\alpha,\beta)$ where $(\pi-\alpha)$ is the angle which the line of length $(l-r)$ makes with the z-axis and β is the azimuthal angle for the mass outside of the cone.

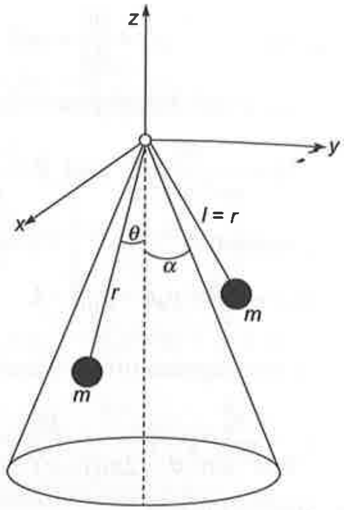

Fig. 3.24

The kinetic energy of the mass inside the cone is

$$T_1 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta\right)$$

Similarly, the kinetic energy of the mass outside the cone is

$$T_2 = \frac{1}{2}m\left[\dot{r}^2 + 0 + (l-r)^2\dot{\beta}^2 \sin^2(\pi-\alpha)\right]$$

Therefore, the total kinetic energy of the system is

$$T = \frac{1}{2}m\left[2\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta + (l-r)^2\dot{\beta}^2\sin^2(\pi-\alpha)\right] \tag{i}$$

The potential energy of the system is the sum of the potential energies of the two masses, and is given by

$$V = -mgr\cos\theta - mg(l-r)\cos\alpha \tag{ii}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left[2\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta + (l-r)^2\dot{\beta}^2\sin^2(\pi-\alpha)\right]$$
$$+mgr\cos\theta + mg(l-r)\cos\alpha \tag{iii}$$

The canonical momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = 2m\dot{r}, \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta, \qquad p_\beta = \frac{\partial L}{\partial \dot{\beta}} = m(l-r)^2\dot{\beta}\sin^2\alpha$$

From these expressions one can write the inverse relations as

$$\dot{r} = \frac{p_r}{2m},\ \dot{\theta} = \frac{p_\theta}{mr^2},\ \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta} \text{ and } \dot{\beta} = \frac{p_\beta}{m(l-r)^2\sin^2\alpha} \tag{iv}$$

Now, the Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} + p_\beta\dot{\beta} - L$$

Using equations (iii) and (iv), the expression for Hamiltonian becomes

$$H = \frac{p_r^2}{4m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + \frac{p_\beta^2}{2m(l-r)^2\sin^2\alpha}$$
$$-mgr\cos\theta - mg(l-r)\cos\alpha \tag{v}$$

Note that the canonical momenta p_ϕ and p_β are constants of motion since ϕ and β are cyclic coordinates.

Now, if the mass outside the cone is rotating in a circular orbit of constant radius, its angular velocity can be obtained from Hamilton's canonical equation as

$$\dot{\beta} = \frac{\partial H}{\partial p_\beta} = \frac{p_\beta}{m(l-r)^2\sin^2\alpha}$$

EXAMPLE 3.44 If the Hamiltonian and Lagrangian of a dynamical system is a function of some parameter α, prove that $\left(\frac{\partial H}{\partial \alpha}\right)_{p,q} = -\left(\frac{\partial L}{\partial \alpha}\right)_{\dot{q},q}$.

Solution: The Lagrangian of a system, in the present case, is a function of the generalized coordinate and generalized velocity and the parameter α. That is,

$$L \equiv L(q_j, \dot{q}_j, \alpha)$$

The total derivative of Lagrangian can be written as

$$dL = \sum_j \frac{\partial L}{\partial q_j}dq_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}d\dot{q}_j + \frac{\partial L}{\partial \alpha}d\alpha$$
$$= \sum_j \dot{p}_j dq_j + \sum_j p_j d\dot{q}_j + \frac{\partial L}{\partial \alpha}d\alpha \tag{i}$$

Now, $d\left(\sum_j p_j\dot{q}_j\right) = \sum_j p_j d\dot{q}_j + \sum_j \dot{q}_j dp_j$

Using this in (i), we get

$$dL = \sum_j \dot{p}_j dq_j + d\left(\sum_j p_j \dot{q}_j\right) - \sum_j \dot{q}_j dp_j + \frac{\partial L}{\partial \alpha} d\alpha$$

This can be rearranged to get

$$dL - d\left(\sum_j p_j \dot{q}_j\right) = \sum_j \dot{p}_j dq_j - \sum_j \dot{q}_j dp_j + \frac{\partial L}{\partial \alpha} d\alpha$$

or $$d\left(L - \sum_j p_j \dot{q}_j\right) = \sum_j \dot{p}_j dq_j - \sum_j \dot{q}_j dp_j + \frac{\partial L}{\partial \alpha} d\alpha$$

that is, $$-dH = \sum_j \dot{p}_j dq_j - \sum_j \dot{q}_j dp_j + \frac{\partial L}{\partial \alpha} d\alpha \qquad \text{(ii)}$$

where, $\sum_j p_j \dot{q}_j - L = H$,the Hamiltonian of the system.

Now, using Hamilton's canonical equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \text{ and } \dot{p}_j = -\frac{\partial H}{\partial q_j},$$

we can rewrite equation (ii) as

$$dH = \sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial p_j} dp_j - \frac{\partial L}{\partial \alpha} d\alpha \qquad \text{(iii)}$$

From a comparison of (i) and (iii) it can be seen that

$$\left(\frac{\partial H}{\partial \alpha}\right)_{p,q} = -\left(\frac{\partial L}{\partial \alpha}\right)_{\dot{q},q},$$

Hence, proved.

EXAMPLE 3.45 Consider a system of n particles, each of mass m and a single particle of mass M. Obtain the Hamiltonian of the system, excluding the motion of the centre of mass.

Solution: In the problem (2.38), we have obtained the Lagrangian of a system of two particles separated by a distance as

$$L = \frac{(m_1 + m_2)}{2}|\dot{R}|^2 + \frac{\mu}{2}|\dot{r}|^2 - V(r)$$

where, R is the position vector of the centre of mass and r, the separation between the particles and μ is the reduced mass of the system. In the present problem, the Lagrangian of the system of particles can be written as

$$L = \frac{1}{2}M|\dot{R}|^2 + \frac{1}{2}m\sum_{i=1}^{n}\dot{r}_i^2 - V \qquad \text{(i)}$$

where, R is the position vector of the mass M and r_i, those of the particles of mass m.

Now, let $r_i = R_i - R$ or, $R_i = R + r_i$ and taking the origin of the coordinate system at the centre of mass, we can write, the total moment

$$MR + \sum_i mR_i = 0$$

or
$$MR + \sum_i m(R + r_i) = 0$$

that is,
$$MR + \sum_i mR + \sum_i mr_i = 0$$

This would give
$$R = -\frac{\sum_i mr_i}{M + \sum_i m} = -\frac{m\sum_i r_i}{M + nm} = -\frac{m}{\mu}\sum_i r_i \tag{ii}$$

where, $\mu = M + nm$

Substituting these, the expression for kinetic energy becomes

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}m\sum_i \dot{R}_i^2$$
$$= \frac{1}{2}M\dot{R}^2 + \frac{1}{2}m\sum_i (\dot{R} + \dot{r}_i)^2$$
$$= \frac{1}{2}\mu\dot{R}^2 + \frac{1}{2}m\sum_i \dot{r}_i^2 + m\dot{R}\sum_i \dot{r}_i \tag{iii}$$

Using (ii) the above equation can be modified as

$$T = \frac{1}{2}\frac{m^2}{\mu}\left(\sum_i \dot{r}_i\right)^2 + \frac{1}{2}m\sum_i \dot{r}_i^2 - \frac{m^2}{\mu}\left(\sum_i \dot{r}_i\right)^2$$
$$= \frac{1}{2}m\sum_i \dot{r}_i^2 - \frac{m^2}{2\mu}\left(\sum_i \dot{r}_i\right)^2 \tag{iv}$$

Then, the Lagrangian of the system is

$$L = \frac{1}{2}m\sum_i \dot{r}_i^2 - \frac{m^2}{2\mu}\left(\sum_i \dot{r}_i\right)^2 - V \tag{v}$$

The generalized momentum is

$$p_i = \frac{\partial L}{\partial \dot{r}_i} = m\dot{r}_i - \frac{m^2}{\mu}\sum_j \dot{r}_j \tag{vi}$$

Then,
$$\sum_i p_i = \left(m\sum_i \dot{r}_i - n\frac{m^2}{\mu}\sum_j \dot{r}_j\right) = m\left(1 - n\frac{m}{\mu}\right)\sum_i \dot{r}_i$$

Using, $\mu = M + nm$, this expression becomes

$$\sum_i p_i = \frac{Mm}{\mu}\sum_i \dot{r}_i \tag{vii}$$

or

$$\sum_i \dot{r}_i = \frac{\mu}{Mm}\sum_i p_i \tag{viii}$$

Here, from (vi); $\dot{r}_i = \frac{p_i}{m} + \frac{m}{\mu}\sum_j \dot{r}_j = \frac{p_i}{m} + \frac{1}{M}\sum_j p_j$ (ix)

where we used equation (vii) to rewrite the second term.

Now, let us rewrite the expression for kinetic energy using the equations (viii) and (ix).

Then,

$$T = \frac{1}{2}m\sum_i\left(\frac{p_i}{m} + \frac{1}{M}\sum_j p_j\right)^2 - \frac{m^2}{2\mu}\left(\frac{\mu}{Mm}\sum_i p_i\right)^2$$

$$= \frac{1}{2}m\sum_i\left[\frac{p_i^2}{m^2} + \frac{1}{M^2}\left(\sum_j p_j\right)^2 + \frac{2p_i}{Mm}\sum_j p_j\right] - \frac{\mu}{2M^2}\left(\sum_i p_i\right)^2$$

$$= \frac{1}{2m}\sum_i p_i^2 + \frac{nm}{2M^2}\left(\sum_i p_i\right)^2 + \frac{1}{M}\left(\sum_i p_i\right)^2 - \frac{\mu}{2M^2}\left(\sum_i p_i\right)^2$$

$$= \frac{1}{2m}\sum_i p_i^2 + \left(\frac{nm}{2M^2} + \frac{1}{M} - \frac{\mu}{2M^2}\right)\left(\sum_i p_i\right)^2$$

$$= \frac{1}{2m}\sum_i p_i^2 + \frac{1}{2M}\left(\sum_i p_i\right)^2 \tag{x}$$

Since the kinetic energy is a homogeneous second order function in velocity, the Hamiltonian of the system is

$$H = T + V = \frac{1}{2m}\sum_i p_i^2 + \frac{1}{2M}\left(\sum_i p_i\right)^2 + V \tag{xi}$$

This is the required result.

EXAMPLE 3.46 Consider the motion of a particle of mass m is constrained to move on the surface of a perfectly smooth sphere of radius R rotating with angular velocity Ω about the z-axis. Suppose the force on the particle is given by an effective potential $V(\theta,\phi)$, where θ and ϕ are the latitude and longitude respectively. Obtain the Hamiltonian in a frame rotating with the planet, taking the generalized coordinates to be the latitude and longitude.

Solution: Since the generalized coordinates are θ and ϕ, the latitude and longitude, the position of the particle can be written in terms of these quantities as

$$x = R\cos\theta\cos(\phi+\Omega t),\ y = R\cos\theta\sin(\phi+\Omega t) \text{ and } z = R\sin\theta$$

so that
$$\dot{x} = -R\dot{\theta}\sin\theta\cos(\phi+\Omega t) - R(\Omega+\dot{\phi})\cos\theta\sin(\phi+\Omega t),$$
$$\dot{y} = -R\dot{\theta}\sin\theta\sin(\phi+\Omega t) + R(\Omega+\dot{\phi})\cos\theta\cos(\phi+\Omega t) \text{ and;}$$
$$\dot{z} = R\dot{\theta}\cos\theta$$

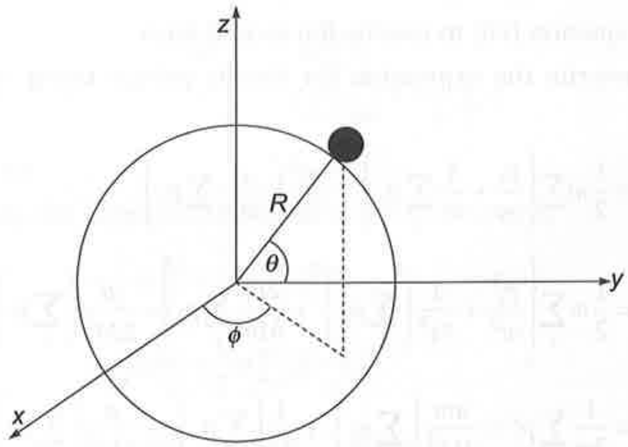

Fig. 3.25

Then, the kinetic energy of the particle is given by

$$T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$
$$= \frac{1}{2}m\left\{\begin{array}{l}\left(-R\dot{\theta}\sin\theta\cos(\phi+\Omega t) - R(\Omega+\dot{\phi})\cos\theta\sin(\phi+\Omega t)\right)^2 \\ +\left(-R\dot{\theta}\sin\theta\sin(\phi+\Omega t) + R(\Omega+\dot{\phi})\cos\theta\cos(\phi+\Omega t)\right)^2 \\ +\left(R\dot{\theta}\cos\theta\right)^2\end{array}\right\}$$

On simplification, this would yield

$$T = \frac{1}{2}mR^2\left[\dot{\theta}^2 + (\Omega+\dot{\phi})^2\cos^2\theta\right] \tag{i}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}mR^2\left[\dot{\theta}^2 + (\Omega+\dot{\phi})^2\cos^2\theta\right] - V(\theta,\phi) \tag{ii}$$

The canonical momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \text{ and, } p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\left(\Omega + \dot{\phi}\right)\cos^2\theta \tag{iii}$$

The inverse relations are

$$\dot{\theta} = \frac{p_\theta}{mR^2} \text{ and } \dot{\phi} = \frac{p_\phi}{mR^2\cos^2\theta} - \Omega \tag{iv}$$

The Hamiltonian of the system is given by

$$H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta} + p_\phi\dot{\phi} - \left\{\frac{1}{2}mR^2\left[\dot{\theta}^2 + \left(\Omega + \dot{\phi}\right)^2\cos^2\theta\right] - V(\theta,\phi)\right\}$$

Using, equation (iv) in the above expression and simplifying, we get

$$H = \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{2mR^2\cos^2\theta} - \Omega p_\phi + V(\theta,\phi) \tag{v}$$

Note that the Hamiltonian $H \neq T + V$, but differ by a quantity, Ωp_ϕ. However, it is a constant of motion as it does not depend explicitly on time.

EXAMPLE: 3.47 Consider a charged particle q constrained to move on a non-rotating smooth insulating sphere, placed in a uniform magnetic field $B = B_z$, on which the electrostatic potential is a function of latitude and longitude. Obtain the Hamiltonian of the charged particle.

Solution: This problem is similar to the above problem with $\Omega = 0$. The coordinates of the charged particle can be written as;

$$x = R\cos\theta\cos\phi,\ y = R\cos\theta\sin\phi \text{ and } z = R\sin\theta$$

so that

$$\dot{x} = -R\dot{\theta}\sin\theta\cos\phi - R\dot{\phi}\cos\theta\sin\phi,$$

$$\dot{y} = -R\dot{\theta}\sin\theta\sin\phi + R\dot{\phi}\cos\theta\cos\phi \text{ and}$$

$$\dot{z} = R\dot{\theta}\cos\theta$$

So that the kinetic energy of the particle is

$$T = \frac{1}{2}mR^2\left[\dot{\theta}^2 + \left(\Omega + \dot{\phi}\right)^2\cos^2\theta\right] \tag{i}$$

We know that the Lagrangian of a charged particle in an electromagnetic field is given by

$$L = T + q(\dot{r}.A) - q\Phi \tag{ii}$$

Now, we select a vector potential A such that $B_z = \frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = B$. This can be achieved by setting

$$A_x = -\frac{1}{2}Bx,\ A_y = \frac{1}{2}By \text{ and } A_z = 0. \text{ Then}$$

$$q(\dot{r}.A)=q\left[\left(\dot{x}\hat{i}+\dot{y}\hat{j}+\dot{z}\hat{k}\right).\left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)\right]$$

$$=\frac{1}{2}qB(x\dot{y}-y\dot{x})$$

$$=\frac{1}{2}qBR^2\cos^2\theta\begin{bmatrix}\cos\phi\left(-\dot{\theta}\sin\theta\sin\phi+\dot{\phi}\cos\theta\cos\phi\right)+\\ \sin\phi\left(\dot{\theta}\sin\theta\cos\phi+\dot{\phi}\cos\theta\sin\phi\right)\end{bmatrix}$$

$$=\frac{1}{2}qBR^2\dot{\phi}\cos^2\theta$$

Now, the Lagrangian is

$$L=\frac{1}{2}mR^2\left[\dot{\theta}^2+\dot{\phi}^2\cos^2\theta\right]+\frac{1}{2}qBR^2\dot{\phi}\cos^2\theta-q\Phi(\theta,\phi) \tag{iii}$$

Now, adding and subtracting $\frac{q^2B^2R^2}{8m}\cos^2\theta$ to the above expression and simplifying, we get

$$L=\frac{1}{2}mR^2\left[\dot{\theta}^2+\dot{\phi}^2\cos^2\theta\right]+\frac{1}{2}qBR^2\dot{\phi}\cos^2\theta+\frac{q^2B^2R^2}{8m}\cos^2\theta-\frac{q^2B^2R^2}{8m}\cos^2\theta-q\Phi(\theta,\phi)$$

$$=\frac{1}{2}mR^2\left[\dot{\theta}^2+\left(\dot{\phi}^2\cos^2\theta+\frac{qB\dot{\phi}\cos^2\theta}{m}+\frac{q^2B^2}{4m^2}\cos^2\theta\right)\right]-\frac{q^2B^2R^2}{8m}\cos^2\theta-q\Phi(\theta,\phi)$$

$$=\frac{1}{2}mR^2\left[\dot{\theta}^2+\left(\dot{\phi}^2+\frac{qB\dot{\phi}}{m}+\frac{q^2B^2}{4m^2}\right)\cos^2\theta\right]-\frac{q^2B^2R^2}{8m}\cos^2\theta-q\Phi(\theta,\phi)$$

$$=\frac{1}{2}mR^2\left[\dot{\theta}^2+\left(\dot{\phi}+\frac{qB}{2m}\right)^2\cos^2\theta\right]-\frac{q^2B^2R^2}{8m}\cos^2\theta-q\Phi(\theta,\phi) \tag{iv}$$

Note that on a comparison with the previous problem, we can see that $\Omega=\frac{qB}{2m}$ and the potential is

$$V(\theta,\phi)=\frac{q^2B^2R^2}{8m}\cos^2\theta+q\Phi(\theta,\phi) \tag{v}$$

Now, the canonical momenta are

$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=mR^2\dot{\theta} \text{ and } p_\phi=\frac{\partial L}{\partial\dot{\phi}}=mR^2\left(\dot{\phi}+\frac{qB}{2m}\right)\cos^2\theta \tag{vi}$$

The inverse relations are

$$\dot{\theta}=\frac{p_\theta}{mR^2} \text{ and } \dot{\phi}=\frac{p_\phi}{mR^2\cos^2\theta}-\frac{qB}{2m} \tag{vii}$$

Now, the Hamiltonian of the particle is

$$H = \sum_j p_j \dot{q}_j - L$$
$$= p_\theta \dot{\theta} + p_\phi \dot{\phi} - \left\{ \frac{1}{2} mR^2 \left[\dot{\theta}^2 + \left(\dot{\phi} + \frac{qB}{2m} \right)^2 \cos^2\theta \right] - \frac{q^2B^2R^2}{8m} \cos^2\theta - q\Phi(\theta,\phi) \right\}$$

Using equation (vii) in the above expression and simplifying, we get

$$H = \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{2mR^2\cos^2\theta} - \frac{qB}{2m} p_\phi + \frac{q^2B^2R^2}{8m}\cos^2\theta + q\Phi(\theta,\phi) \qquad \text{(viii)}$$

In this problem, we see that $H = T + V$ and hence the Hamiltonian represents the total energy.

EXAMPLE 3.48 A spherical pendulum of length l and mass m carries a positive charge q and moves under the action of a constant gravitational field and a constant magnetic field $B = -B\hat{z}$. Construct the Hamiltonian and obtain Hamilton's canonical equations of motion.

Solution: A schematic representation of the problem is shown below. Let us take the point of suspension of the pendulum as the origin of the coordinate system.

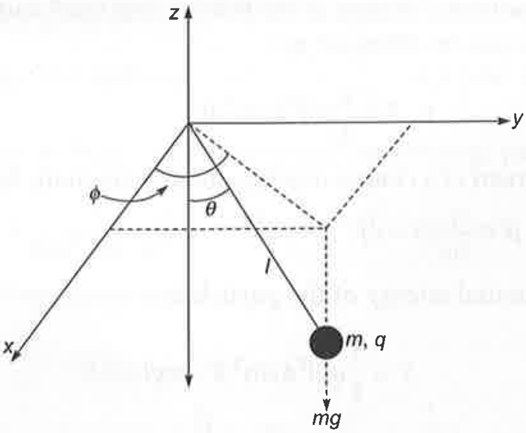

Fig. 3.26

Then, the coordinates of the pendulum bob can be written as

$$x = l\sin\theta\cos\phi,\ y = l\sin\theta\sin\phi \text{ and } z = -l\cos\theta$$

so that

$$\dot{x} = l\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)$$
$$\dot{y} = l\left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right) \text{ and;}$$
$$\dot{z} = l\dot{\theta}\sin\theta$$

The kinetic energy of the bob is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}ml^2\begin{bmatrix}\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)^2 + \\ \left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right)^2 + \dot{\theta}^2\sin^2\theta\end{bmatrix}$$

On further simplification, this would yield

$$T = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) \tag{i}$$

Now, we must include a term, $q(\dot{r}.A)$ in the expression for Lagrangian.
As in the previous problem we can show that

$$q(\dot{r}.A) = -\frac{1}{2}qBl^2\dot{\phi}\sin^2\theta. \tag{ii}$$

Also, the particle has two forms of potential energy, gravitational potential energy and the magnetic potential energy.

The gravitational potential energy of the bob is $-mgl\cos\theta$ and the magnetic potential energy of the particle can be obtained as

$$\mu_z.B = \frac{1}{2}qBl^2\dot{\phi}\sin^2\theta \tag{iii}$$

where, magnetic moment of a charge moving about a magnetic field line can be obtained from the expression $\mu = \frac{q}{2m}(x \times v)$.

Now, the total potential energy of the particle is

$$V = \frac{1}{2}qBl^2\dot{\phi}\sin^2\theta - mgl\cos\theta \tag{iv}$$

Then, the Lagrangian of the particle is

$$L = T + q(\dot{r}.A) - V = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - qBl^2\dot{\phi}\sin^2\theta + mgl\cos\theta$$

$$= ml^2\left(\frac{\dot{\theta}^2}{2} + \frac{\dot{\phi}^2}{2}\sin^2\theta\right) - qBl^2\dot{\phi}\sin^2\theta + mgl\cos\theta$$

$$= ml^2\left(\frac{\dot{\theta}^2}{2} + \frac{\dot{\phi}^2}{2}\sin^2\theta\right) - \frac{mqB}{m}l^2\dot{\phi}\sin^2\theta + mgl\cos\theta$$

$$= ml^2\left(\frac{\dot{\theta}^2}{2}+\frac{\dot{\phi}^2}{2}\sin^2\theta-\frac{qB}{m}\dot{\phi}\sin^2\theta\right)+mgl\cos\theta$$

$$= ml^2\left[\frac{\dot{\theta}^2}{2}+\left(\frac{\dot{\phi}^2}{2}-\frac{qB}{m}\dot{\phi}\right)\sin^2\theta\right]+mgl\cos\theta$$

$$= ml^2\left[\frac{\dot{\theta}^2}{2}+\left(\frac{\dot{\phi}^2}{2}-\omega_c\dot{\phi}\right)\sin^2\theta\right]+mgl\cos\theta \tag{v}$$

where, $\omega_c = \dfrac{qB}{m}$, the cyclotron frequency.

Now, the canonical momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \text{ and } p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\sin^2\theta\left(\dot{\phi}-\omega_c\right) \tag{vi}$$

The inverse relations are

$$\dot{\theta} = \frac{p_\theta}{ml^2} \text{ and } \dot{\phi} = \frac{p_\phi}{ml^2\sin^2\theta}+\omega_c \tag{vii}$$

The Hamiltonian of the particle is

$$H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta}+p_\phi\dot{\phi}-ml^2\left[\frac{\dot{\theta}^2}{2}+\left(\frac{\dot{\phi}^2}{2}-\omega_c\dot{\phi}\right)\sin^2\theta\right]-mgl\cos\theta$$

Substituting for $\dot{\theta}$ and $\dot{\phi}$ from equation (v) and simplifying, we get

$$H = \frac{p_\theta^2}{2ml^2}+\frac{1}{2ml^2\sin^2\theta}\left(p_\phi - ml^2\omega_c\sin^2\theta\right)^2 - mgl\cos\theta \tag{viii}$$

Hamilton's canonical equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \text{ and;}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = ml^2\sin\theta\cos\theta\left[\left(\frac{p_\phi}{ml^2\sin^2\theta}\right)^2-\omega_c^2\right]-mgl\sin\theta$$

Similarly, $\dot{\phi} = \dfrac{\partial H}{\partial p_\phi} = \dfrac{p_\phi}{ml^2\sin^2\theta}+\omega_c$ and $\dot{p}_\phi = -\dfrac{\partial H}{\partial \phi} = 0$

So, the angular momentum of the particle remains constant during the motion. Now let us find the equation of motion. From the first pair of equations, we get

$$\ddot{\theta} = \frac{\dot{p}_\theta}{ml^2} = \frac{1}{ml^2}\left\{ml^2\sin\theta\cos\theta\left[\left(\frac{p_\phi}{ml^2\sin^2\theta}\right)^2-\omega_c^2\right]-mgl\sin\theta\right\}$$

that is, $$\ddot{\theta}+\frac{g}{l}\sin\theta=\sin\theta\cos\theta\left[\left(\frac{p_\phi}{ml^2\sin^2\theta}\right)^2-\omega_c^2\right] \tag{ix}$$

This is the equation of motion.

EXAMPLE 3.49 A pendulum is mounted on the edge of a disc with radius a, rotating with constant angular velocity ω. Obtain the Hamiltonian of the system.

Solution: Consider Figure 3.27 shown below.

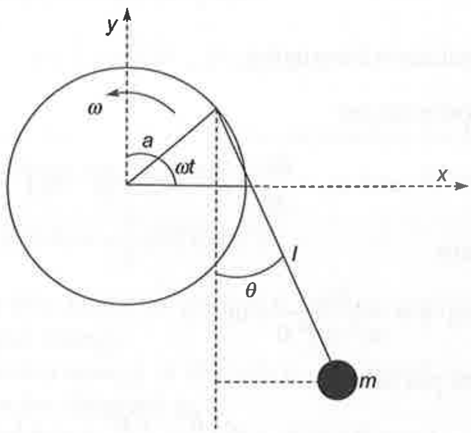

Fig. 3.27

The disc is rotating in the counterclockwise with a constant angular velocity ω. If the point of support is in the horizontal position at $t=0$. The position of the point of support at any time t is $\phi=\omega t$.

The coordinates of the pendulum bob are given by

$$x=a\cos\omega t+l\sin\theta \text{ and } y=a\sin\omega t-l\cos\theta$$

Then, $$\dot{x}=-a\omega\sin\omega t+l\dot{\theta}\cos\theta \text{ and } \dot{y}=a\omega\cos\omega t+l\dot{\theta}\sin\theta$$

The kinetic energy of the system is given by

$$T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)=\frac{1}{2}m\left[\left(-a\omega\sin\omega t+l\dot{\theta}\cos\theta\right)^2+\left(a\omega\cos\omega t+l\dot{\theta}\sin\theta\right)^2\right]$$

$$=\frac{1}{2}m\left[\begin{array}{l}\left(a^2\omega^2\sin^2\omega t+l^2\dot{\theta}^2\cos^2\theta-2a\omega\sin\omega t l\dot{\theta}\cos\theta\right)\\+\left(a^2\omega^2\cos^2\omega t+l^2\dot{\theta}^2\sin^2\theta+2a\omega\cos\omega t l\dot{\theta}\sin\theta\right)\end{array}\right]$$

$$=\frac{1}{2}m\left[a^2\omega^2+l^2\dot{\theta}^2+2a\omega l\dot{\theta}\left(\sin\theta\cos\omega t-\cos\theta\sin\omega t\right)\right]$$

$$=\frac{1}{2}m\left[a^2\omega^2+l^2\dot{\theta}^2+2a\omega l\dot{\theta}\sin(\theta-\omega t)\right] \tag{i}$$

The potential energy of the system is

$$V = mgy = mg(a\sin\omega t - l\cos\theta) \tag{ii}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left[a^2\omega^2 + l^2\dot{\theta}^2 + 2a\omega l\dot{\theta}\sin(\theta - \omega t)\right] - mg(a\sin\omega t - l\cos\theta) \tag{iii}$$

The canonical momentum is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + 2a\omega l\sin(\theta - \omega t) \tag{iv}$$

The inverse relation is

$$\dot{\theta} = \frac{p_\theta - 2a\omega l\sin(\theta - \omega t)}{ml^2} \tag{v}$$

The Hamiltonian of the system is

$$H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta} - \frac{1}{2}m\left[a^2\omega^2 + l^2\dot{\theta}^2 + 2a\omega l\dot{\theta}\sin(\theta - \omega t)\right] + mg(a\sin\omega t - l\cos\theta)$$

$$= p_\theta\left(\frac{p_\theta - 2a\omega l\sin(\theta - \omega t)}{ml^2}\right) - \frac{1}{2}m\left[a^2\omega^2 + l^2\left(\frac{p_\theta - 2a\omega l\sin(\theta - \omega t)}{ml^2}\right)^2\right]$$

$$-\frac{1}{2}m\left[2a\omega l\left(\frac{p_\theta - 2a\omega l\sin(\theta - \omega t)}{ml^2}\right)\sin(\theta - \omega t)\right] + mg(a\sin\omega t - l\cos\theta)$$

On simplification, we get

$$H = \frac{p_\theta^2}{2ml^2} - \frac{1}{2}ma^2\omega^2 + 2a^2\omega^2l^2\left(1 - \frac{1}{m}\right)\sin^2(\theta - \omega t)$$

$$-\frac{a\omega p_\theta}{l}\sin(\theta - \omega t) + mg(a\sin\omega t - l\cos\theta) \tag{vi}$$

EXAMPLE 3.50 A light, uniform U-shaped tube is partially filled with mercury with total mass M and mass per unit length p as shown in Figure 3.28. The tube is mounted so that it can rotate about one of the vertical legs. Neglecting friction, the mass and moment of inertia of the glass tube, and the moment of inertia of the mercury column on the axis of rotation, obtain the Hamiltonian and equation of motion.

Solution: Since the U-shaped tube is rotating about one of its arm, let us take the angle of rotation θ and the elevation/depression of the liquid z from the equilibrium position as the generalized coordinates.

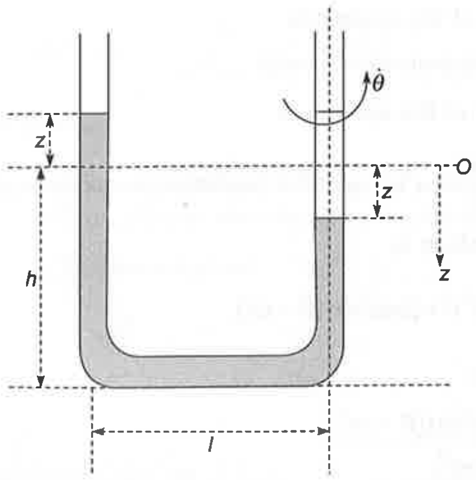

Fig. 3.28

Now, the kinetic energy of the liquid can be written as the sum of three parts.

For the liquid in the left arm of the tube, the kinetic energy is the sum of translational and rotational kinetic energies due to the oscillation of the liquid column and rotation of the tube. It is given by

$$T_1 = \frac{1}{2}m\left(\dot{z}^2 + l^2\dot{\theta}^2\right) = \frac{1}{2}\rho(h+z)\left(\dot{z}^2 + l^2\dot{\theta}^2\right)$$

For the horizontal portion of the tube the kinetic energy can be written as

$$T_2 = \frac{1}{2}\rho\int_0^l \left(\dot{z}^2 + x^2\dot{\theta}^2\right)dx = \frac{1}{2}\rho\left(\dot{z}^2 l + \frac{l^3}{3}\dot{\theta}^2\right)$$

For the liquid in the right arm of the tube, the kinetic energy is only due to the oscillation of the liquid column. It is given by

$$T_3 = \frac{1}{2}\rho(h-z)\dot{z}^2$$

Therefore, the total kinetic energy of the system is

$$T = \frac{1}{2}\rho(h+z)\left(\dot{z}^2 + l^2\dot{\theta}^2\right) + \frac{1}{2}\rho\left(\dot{z}^2 l + \frac{l^3}{3}\dot{\theta}^2\right) + \frac{1}{2}\rho(h-z)\dot{z}^2$$

$$= \frac{1}{2}\rho(l+2h)\dot{z}^2 + \frac{1}{2}\rho\left(\frac{l}{3} + h + z\right)l^2\dot{\theta}^2$$

$$= \frac{1}{2}\rho s\dot{z}^2 + \frac{1}{2}\rho\left(\frac{l}{3} + h + z\right)l^2\dot{\theta}^2 \qquad \text{(i)}$$

where, $s = l + 2h$.

Now the potential energy is the work done by the external force acting on the liquid surfaces at the open ends. It is given by

$$W = \int F dz = \int (2z\rho g) dz = z^2 \rho g \tag{ii}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}\rho s \dot{z}^2 + \frac{1}{2}\rho\left(\frac{l}{3} + h + z\right) l^2 \dot{\theta}^2 - z^2 \rho g \tag{iii}$$

Now, the canonical momenta are

$$p_z = \frac{\partial L}{\partial \dot{z}} = \rho s \dot{z} \quad \text{and} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \rho\left(\frac{l}{3} + h + z\right) l^2 \dot{\theta} \tag{iv}$$

The inverse relations are

$$\dot{z} = \frac{p_z}{\rho s} \quad \text{and} \quad \dot{\theta} = \frac{p_\theta}{\rho\left(\frac{l}{3} + h + z\right) l^2} \tag{v}$$

Then, the Hamiltonian of the system is

$$H = \sum_j p_j \dot{q}_j - L = p_z \dot{z} + p_\theta \dot{\theta} - \left[\frac{1}{2}\rho s \dot{z}^2 + \frac{1}{2}\rho\left(\frac{l}{3} + h + z\right) l^2 \dot{\theta}^2 - z^2 \rho g\right]$$

$$= p_z\left(\frac{p_z}{\rho s}\right) + p_\theta\left(\frac{p_\theta}{\rho\left(\frac{l}{3} + h + z\right) l^2}\right)$$

$$-\left[\frac{1}{2}\rho s\left(\frac{p_z}{\rho s}\right)^2 + \frac{1}{2}\rho\left(\frac{l}{3} + h + z\right) l^2 \left(\frac{p_\theta}{\rho\left(\frac{l}{3} + h + z\right) l^2}\right)^2 - z^2 \rho g\right]$$

On simplification, this would give

$$H = \frac{p_z^2}{2\rho s} + \frac{p_\theta^2}{2\rho\left(\frac{l}{3} + h + z\right) l^2} + \rho g z^2 \tag{vi}$$

Now, Hamilton's canonical equations are

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{\rho s} \quad \text{and} \quad \dot{p}_z = -\frac{\partial H}{\partial z} = \frac{p_\theta^2}{2\rho\left(\frac{l}{3} + h + z\right)^2 l^2} - 2\rho g z \tag{vii}$$

Similarly, $\dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{\rho\left(\dfrac{l}{3}+h+z\right)l^2}$ and $\dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = 0$ (viii)

Since θ is a cyclic coordinate, the corresponding momentum is conserved. The equation of motion can be obtained from (vii) as;

$$\ddot{z} = \frac{\dot{p}_z}{\rho s} = \frac{p_\theta^2}{2s\rho^2\left(\dfrac{l}{3}+h+z\right)^2 l^2} - \frac{2gz}{s}$$

$$s\ddot{z} = \frac{p_\theta^2}{2\rho^2\left(\dfrac{l}{3}+h+z\right)^2 l^2} - 2gz$$

$$s\ddot{z} = \frac{A}{\left(\dfrac{l}{3}+h+z\right)^2} - 2gz \tag{ix}$$

where, $A = \dfrac{p_\theta}{2\rho^2 l^2}$, which is a constant since p_θ is a constant. Equation (ix) is the required equation of motion.

EXAMPLE 3.51 A particle with mass m is constrained to move on a sphere with radius R, and is connected to a spring with spring constant k which can slide without friction along the z-axis and rotate without friction about the same axis. Obtain the Hamiltonian of the system.

Solution: A schematic representation of the problem is given in Figure 3.29. We use spherical polar coordinate system. The coordinates of the mass are given by;

$$x = R\sin\theta\cos\phi,\ y = R\sin\theta\sin\phi \text{ and } z = R\cos\theta$$

Then,

$$\dot{x} = R\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right),$$

$$\dot{y} = R\left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right) \text{ and}$$

$$\dot{z} = -R\dot{\theta}\sin\theta$$

The kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$

Substituting for $\dot{x}$, $\dot{y}$ and $\dot{z}$, we get the kinetic energy as

$$T = \frac{1}{2}mR^2\left[\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)^2 + \left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right)^2 + \left(-R\dot{\theta}\sin\theta\right)^2\right]$$

$$= \frac{1}{2}mR^2[(\dot{\theta}^2\cos^2\theta\cos^2\phi + \dot{\phi}^2\sin^2\theta\sin^2\phi - 2\dot{\theta}\dot{\phi}\cos\theta\cos\phi\sin\theta\sin\phi)$$

$$+(\dot{\theta}^2\cos^2\theta\sin^2\phi + \dot{\phi}^2\sin^2\theta\cos^2\phi + 2\dot{\theta}\dot{\phi}\cos\theta\sin\phi\sin\theta\cos\phi)^2 + \dot{\theta}^2\sin^2\theta]$$

On further simplification, we get

$$T = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) \tag{i}$$

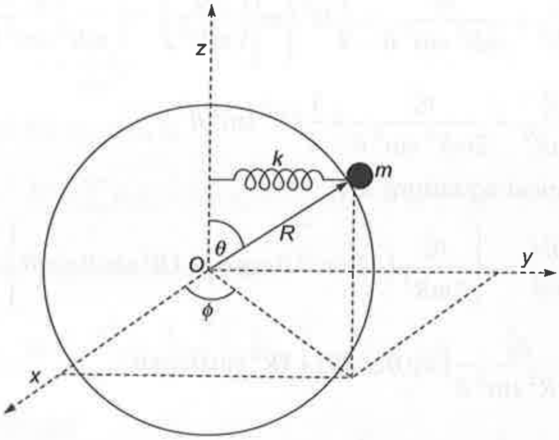

Fig. 3.29

The potential energy is that of the spring. As the particle moves over the surface of the sphere, the spring undergoes elongation and compression in a horizontal plane since it can slide along the z direction. Then, its potential energy can be written as

$$V = \frac{1}{2}k\left(x^2 + y^2\right) = \frac{1}{2}k\left[(R\sin\theta\cos\phi)^2 + (R\sin\theta\sin\phi)^2\right]$$

$$= \frac{1}{2}kR^2\left(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi\right) = \frac{1}{2}kR^2\sin^2\theta \tag{ii}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - \frac{1}{2}kR^2\sin^2\theta$$

$$= \frac{1}{2}R^2\left[m\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - k\sin^2\theta\right] \tag{iii}$$

The canonical momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \quad \text{and} \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}\sin^2\theta \tag{iv}$$

The inverse relations are

$$\dot{\theta} = \frac{p_\theta}{mR^2} \text{ and } \dot{\phi} = \frac{p_\phi}{mR^2 \sin^2\theta} \qquad \text{(v)}$$

Now, the Hamiltonian is

$$H = \sum_j p_j \dot{q}_j - L = p_\theta \dot{\theta} + p_\phi \dot{\phi} - \left[\frac{1}{2} R^2 \left[m\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - k \sin^2\theta\right]\right]$$

$$= \frac{p_\theta^2}{mR^2} + \frac{p_\phi^2}{mR^2 \sin^2\theta} - \frac{1}{2} R^2 \left\{ m\left[\left(\frac{p_\theta}{mR^2}\right)^2 + \left(\frac{p_\phi}{mR^2 \sin^2\theta}\right)^2 \sin^2\theta\right] - k \sin^2\theta \right\}$$

$$= \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{2mR^2 \sin^2\theta} + \frac{1}{2} kR^2 \sin^2\theta \qquad \text{(vi)}$$

Hamilton's canonical equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\left[\frac{p_\phi^2}{2mR^2}\left(-2\sin^{-3}\theta \cos\theta\right) - kR^2 \sin\theta \cos\theta\right]$$

$$= \frac{p_\phi^2}{mR^2 \sin^4\theta}(\sin\theta \cos\theta) + kR^2 \sin\theta \cos\theta$$

$$= \left(\frac{p_\phi^2}{mR^2 \sin^4\theta} + kR^2\right) \sin\theta \cos\theta \qquad \text{(vii)}$$

From (v) $\ddot{\theta} = \dfrac{\dot{p}_\theta}{mR^2} = \left(\dfrac{p_\phi^2}{m^2R^4 \sin^4\theta} + \dfrac{k}{m}\right) \sin\theta \cos\theta$ (viii)

This is one of the equations of motion. If we use (v) in this equation it can be written in terms of $\dot{\phi}$ as

$$\ddot{\theta} = \left(\dot{\phi}^2 + \frac{k}{m}\right) \sin\theta \cos\theta \qquad \text{(ix)}$$

The other canonical equation is; $\dot{p}_\phi = -\dfrac{\partial H}{\partial \phi} = 0$

This follows, since the Hamiltonian is independent of ϕ and therefore, the angular momentum is conserved.

EXAMPLE 3.52 The Lagrangian of a dynamical systemis given by, $L = \frac{1}{2} m\left(\dot{x}^2 - \omega^2 x^2\right) e^{\gamma t}$. Obtain the canonical momentum and the Hamiltonian of the system. Determine these quantities again after introducing a new coordinate defined by $q = xe^{\gamma t/2}$. Also obtain the equation of motion.

Solution: For the first part of the problem, we have, the Lagrangian as

$$L = \frac{1}{2} m\left(\dot{x}^2 - \omega^2 x^2\right) e^{\gamma t} \tag{i}$$

The canonical momentum is given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} e^{\gamma t} \tag{ii}$$

The inverse relation is

$$\dot{x} = \frac{p_x}{m e^{\gamma t}} \tag{iii}$$

The Hamiltonian of the system is

$$H = \sum_j p_j \dot{q}_j - L = p_x \dot{x} - \frac{1}{2} m\left(\dot{x}^2 - \omega^2 x^2\right) e^{\gamma t}$$

Using (iii), the above expression becomes

$$H = \frac{p_x^2}{m e^{\gamma t}} - \frac{1}{2} m\left(\left(\frac{p_x}{m e^{\gamma t}}\right)^2 - \omega^2 x^2\right) e^{\gamma t}$$

On simplification, we get

$$H = \frac{p_x^2}{2m e^{\gamma t}} + \frac{1}{2} m\omega^2 x^2 e^{\gamma t} \tag{iv}$$

Note that the Hamiltonian is an explicit function of time and hence, it is not a constant of motion.

Now, let us consider the second part of the problem. Given that

$$q = x e^{\gamma t/2} \text{ so that; } x = q e^{-\gamma t/2} \text{ and}$$

$$\dot{q} = \dot{x} e^{\gamma t/2} + x\frac{\gamma}{2} e^{\gamma t/2} \text{ so that; } \dot{x} = \dot{q} e^{-\gamma t/2} - x\frac{\gamma}{2} = \dot{q} e^{-\gamma t/2} - q e^{-\gamma t/2}\frac{\gamma}{2} = \left(\dot{q} - q\frac{\gamma}{2}\right) e^{-\gamma t/2}$$

$$\text{Then, } \dot{x}^2 = \left[\left(\dot{q} - q\frac{\gamma}{2}\right) e^{-\gamma t/2}\right]^2 = \left(\dot{q} - q\frac{\gamma}{2}\right)^2 e^{-\gamma t}$$

$$= \left(\dot{q}^2 + \frac{q^2\gamma^2}{4} - q\dot{q}\gamma\right) e^{-\gamma t} \tag{v}$$

Using the expressions for x and $\dot{x}$ in equation (i), the Lagrangian of the system becomes

$$L=\frac{1}{2}m\left(\dot{x}^2-\omega^2x^2\right)e^{\gamma t}=\frac{1}{2}m\left[\left(\dot{q}^2+\frac{q^2\gamma^2}{4}-q\dot{q}\gamma\right)e^{-\gamma t}-\omega^2qe^{-\gamma t}\right]e^{\gamma t}$$

$$=\frac{1}{2}m\left[\dot{q}^2+\left(\frac{\gamma^2}{4}-\omega^2\right)q^2-q\dot{q}\gamma\right] \tag{vi}$$

Now, the canonical momentum is

$$p_q=\frac{\partial L}{\partial \dot{q}}=m\left(\dot{q}-\frac{\gamma q}{2}\right) \tag{vii}$$

The inverse relation is

$$\dot{q}=\frac{p_q}{m}+\frac{\gamma q}{2} \tag{viii}$$

The Hamiltonian is

$$H=\sum_j p_j\dot{q}_j-L=p_q\dot{q}-\left\{\frac{1}{2}m\left[\dot{q}^2+\left(\frac{\gamma^2}{4}-\omega^2\right)q^2-q\dot{q}\gamma\right]\right\}$$

$$=p_q\left(\frac{p_q}{m}+\frac{\gamma q}{2}\right)-\left\{\frac{1}{2}m\left[\left(\frac{p_q}{m}+\frac{\gamma q}{2}\right)^2+\left(\frac{\gamma^2}{4}-\omega^2\right)q^2-q\left(\frac{p_q}{m}+\frac{\gamma q}{2}\right)\gamma\right]\right\}$$

This can be simplified to obtain

$$H=\frac{p_q^2}{2m}+\gamma\frac{p_q q}{2}+\frac{1}{2}m\omega^2q^2 \tag{ix}$$

Note that there is no explicit time dependence in the Lagrangian and hence, the Hamiltonian is a constant of motion.

Hamilton's canonical equations are

$$\dot{q}=\frac{\partial H}{\partial p_q}=\frac{p_q}{m}+\gamma\frac{q}{2} \text{ and } \dot{p}_q=-\frac{\partial H}{\partial q}=\gamma\frac{p_q}{2}+m\omega^2q \tag{x}$$

Then, $$\ddot{q}=\frac{\dot{p}_q}{m}+\gamma\frac{\dot{q}}{2}$$

Using the expressions (viii) and (x), we get

$$\ddot{q}=\frac{1}{m}\left(\gamma\frac{p_q}{2}+m\omega^2q\right)+\gamma\frac{1}{2}\left(\frac{p_q}{m}+\frac{\gamma q}{2}\right)$$

$$=\gamma\frac{p_q}{m}+\left(\omega^2+\frac{\gamma^2}{4}\right)q \tag{xi}$$

This is the equation of motion. Now, using the expression (vii), we can rewrite the equation of motion as

$$\ddot{q}-\gamma\dot{q}+\left(\frac{\gamma^2}{4}-\omega^2\right)q=0 \tag{xii}$$

EXAMPLE 3.53 A particle is moving in a central force field given by the potential, $V=-k\frac{e^{-ar}}{r}$, where, k and a are positive constants. Construct the Hamiltonian of the particle and obtain the equation of motion.

Solution: Let m be the mass of the particle. In polar coordinate system, the kinetic energy of the particle can be written as;

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right) \tag{i}$$

The potential is given as

$$V=-k\frac{e^{-ar}}{r} \tag{ii}$$

Therefore, the Lagrangian of the system is

$$L=T-V=T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+k\frac{e^{-ar}}{r} \tag{iii}$$

The canonical momenta are

$$p_r=\frac{\partial L}{\partial \dot{r}}=m\dot{r} \quad \text{and} \quad p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$$

The inverse relations are

$$\dot{r}=\frac{p_r}{m} \quad \text{and} \quad \dot{\theta}=\frac{p_\theta}{mr^2} \tag{iv}$$

The Hamiltonian of the system is

$$H=\sum_j p_j\dot{q}_j-L=p_r\dot{r}+p_\theta\dot{\theta}-\left(\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+k\frac{e^{-ar}}{r}\right)$$

$$=\frac{p_r^2}{m}+\frac{p_\theta^2}{mr^2}-\left\{\frac{1}{2}m\left[\left(\frac{p_r}{m}\right)^2+r^2\left(\frac{p_\theta}{mr^2}\right)^2\right]+k\frac{e^{-ar}}{r}\right\}$$

$$=\frac{p_r^2}{2m}+\frac{p_\theta^2}{2mr^2}-k\frac{e^{-ar}}{r} \tag{v}$$

Hamilton's canonical equations are

$$\dot{r}=\frac{\partial H}{\partial p_r}=\frac{p_r}{m} \quad \text{and} \quad \dot{p}_r=-\frac{\partial H}{\partial r}=\frac{p_\theta^2}{mr^3}-k\frac{e^{-ar}}{r}\left(a+\frac{1}{r}\right)$$

Then, the equation of motion can be written as

$$\ddot{r} = \frac{\dot{p}_r}{m} = \frac{1}{m}\left[\frac{p_\theta^2}{mr^3} - k\frac{e^{-ar}}{r}\left(a + \frac{1}{r}\right)\right]$$

$$= \frac{p_\theta^2}{m^2r^3} - \frac{k}{m}\frac{e^{-ar}}{r}\left(a + \frac{1}{r}\right)$$

$$= \frac{p_\theta^2}{m^2r^3} - \frac{k}{m}(1 + ar)\frac{e^{-ar}}{r^2} \tag{vi}$$

Similarly, the other set of equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

Therefore, the angular momentum p_θ is conserved.

EXAMPLE 3.54 Obtain the equation of the orbit for the Kepler problem using Jacobi's form of the principle of lease action.

Solution: For the Kepler problem, the kinetic energy in polar coordinate system is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{i}$$

The potential energy is

$$V = -\frac{k}{r} \tag{ii}$$

Now, find the Lagrangian and the Hamiltonian of the system. It can be shown that the Hamiltonian is equal to the total energy of the system. That is,

$$H = E = T + V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r}$$

From this expression, we get

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = E + \frac{k}{r} \tag{iii}$$

Now, the Jacobi's form of the Least Action principle is

$$\Delta\int\sqrt{2[H - V]}\; d\rho = 0 \tag{iv}$$

Here, the differential cross section is given by the expression $d\rho = d\theta\sqrt{m\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}$, the proof is available in many standard books. Then, equation (vi) becomes

$$\Delta\int\sqrt{2m(H-V)\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta=0$$

that is,
$$\Delta\int\sqrt{2m\left(E+k/_r\right)\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta=0$$

This can be written as

$$\Delta\int f\left(r,\frac{dr}{d\theta}\right)d\theta=\Delta\int f(r,r')d\theta=0 \tag{v}$$

where,
$$f(r,r')=\sqrt{2m\left(E+k/_r\right)\left(r^2+r'^2\right)}\text{ and } r'=\frac{dr}{d\theta} \tag{vi}$$

Now,
$$\frac{\partial f}{\partial r'}=\sqrt{\frac{2m\left(E+k/_r\right)}{\left(r^2+r'^2\right)}}.r' \tag{vii}$$

When the Lagrangian does not involve time explicitly, we have $\sum_j\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j-L=H$. Similarly, in the present case the function $f(r,r')$ does not involve the coordinate θ explicitly and hence, we can write

$$\frac{\partial f}{\partial r'}r'-f(r,r')=l \tag{viii}$$

where, l is a constant and later we will identify it as the angular momentum. Using the expressions (vi) and (vii) in (viii), we get

$$\sqrt{\frac{2m\left(E+k/_r\right)}{\left(r^2+r'^2\right)}}.r'^2-\sqrt{2m\left(E+k/_r\right)\left(r^2+r'^2\right)}=l$$

that is,
$$\sqrt{2m\left(E+k/_r\right)}\left[\frac{r'^2}{\sqrt{\left(r^2+r'^2\right)}}-\sqrt{\left(r^2+r'^2\right)}\right]=l$$

or
$$\sqrt{2m\left(E+k/_r\right)}\left(\frac{r^2}{\sqrt{\left(r^2+r'^2\right)}}\right)=-l$$

Squaring both sides, we get

$$\frac{2m\left(E+k/_r\right)}{\left(r^2+r'^2\right)}r^4=l^2$$

or $$2m\left(E+{}^{k}\!/_{r}\right)r^4=l^2\left(r^2+r'^2\right)$$

This would give

$$r'^2=\frac{2mr^4}{l^2}\left(E+{}^{k}\!/_{r}\right)-r^2$$

or

$$r'^2=\frac{2mr^2}{l^2}\left(Er^2+kr\right)-r^2=\frac{2mr^2}{l^2}\left(Er^2+kr-\frac{l^2}{2m}\right)$$

But, $r'=\dfrac{dr}{d\theta}$ and hence

$$\frac{dr}{d\theta}=\sqrt{2m}\,\frac{r}{l}\left(Er^2+kr-\frac{l^2}{2m}\right)^{1/2} \tag{ix}$$

From this expression, we can have

$$d\theta=\frac{l}{\sqrt{2m}}\frac{dr}{r\left[Er^2+kr-\left({}^{l^2}\!/_{2m}\right)\right]^{1/2}}$$

Now, we choose the reference direction for measuring θ at the position of minimum value of r which is r_0. At this point, θ is taken as $\theta_0=0$. Therefore,

$$\int_0^{\theta}d\theta=\frac{l}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr}{r\left[Er^2+kr-\left({}^{l^2}\!/_{2m}\right)\right]^{1/2}}$$

This integral can be evaluated to get the value of the generalized coordinate θ as

$$\theta=\sin^{-1}\left[-\frac{kr-{}^{l^2}\!/_{m}}{r\sqrt{\left(k^2+{}^{2El^2}\!/_{m}\right)}}\right]-\frac{\pi}{2} \tag{x}$$

Then, $$-\frac{kr-{}^{l^2}\!/_{m}}{r\sqrt{\left(k^2+{}^{2El^2}\!/_{m}\right)}}=\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta$$

or $$-kr+{}^{l^2}\!/_{m}=r\sqrt{\left(k^2+{}^{2El^2}\!/_{m}\right)}\cos\theta$$

or $${}^{l^2}\!/_{m}=kr+r\sqrt{\left(k^2+{}^{2El^2}\!/_{m}\right)}\cos\theta=kr\left\{1+\sqrt{\left[1+\left({}^{2El^2}\!/_{mk^2}\right)\right]}\cos\theta\right\}$$

From this expression one can get

$$r = \frac{l^2/mk}{\left\{1 + \sqrt{\left[1 + \left(2El^2/mk^2\right)\right]} \cos\theta\right\}} = \frac{p}{1 + \varepsilon \cos\theta} \tag{xi}$$

where, $\quad p = l^2/mk$ and $\varepsilon = \sqrt{\left[1 + \left(2El^2/mk^2\right)\right]}$ (xii)

Equation (xi) is the equation of motion for the Kepler problem and represents the possible orbits. Here, ε is the eccentricity of the orbit. The path is an ellipse, parabola or hyperbola depending on ε is less than, equal to or greater than unity. From (xii), we get, when, $E < 0$, the orbit is an ellipse, when $E = 0$ the orbit is a parabola and when $E > 0$ the orbit is a hyperbola.

Now, let us show that l represents the angular momentum. We make use of an intermediate equation from the above solution, which is

$$\sqrt{2m\left(E + k/r\right)}\left(\frac{r^2}{\sqrt{\left(r^2 + r'^2\right)}}\right) = -l$$

Now let us make use of $\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = r'\dot{\theta}$ in the above expression to rewrite r' and to get

$$\sqrt{2m\left(E + k/r\right)}\left(\frac{r^2}{\sqrt{\left(r^2 + \dot{r}^2/\dot{\theta}^2\right)}}\right) = -l$$

This can be rearranged as

$$\sqrt{\frac{2m\left(E + k/r\right)}{\left(r^2\dot{\theta}^2 + \dot{r}^2\right)}}\, r^2\dot{\theta}^2 = -l$$

From equation (iii) we have $\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \frac{2}{m}\left(E + k/r\right)$, so that

$$\sqrt{\frac{2m\left(E + k/r\right)}{\frac{2}{m}\left(E + k/r\right)}}\, r^2\dot{\theta}^2 = -l$$

or, $mr^2\dot{\theta}^2 = -l$, which is the angular momentum of the particle.

EXAMPLE 3.55 Derive the equation of motion of a projectile using Jacob's form of least action principle.

Solution: We start from the expression for kinetic and potential energies of a projectile in the Cartesian coordinate system.

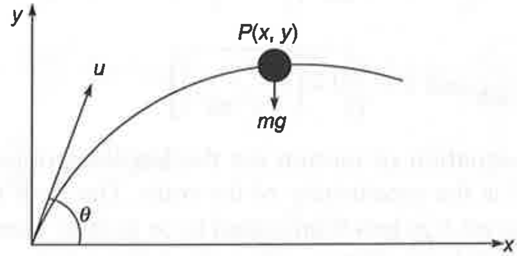

Fig. 3.30

The kinetic energy is

$$T = \frac{1}{2m}\left(p_x^2 + p_y^2\right)$$

and, the potential energy is

$$V = mgy$$

The Hamiltonian of the particle can be obtained as

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + mgy \tag{i}$$

Least action principle in Jacobi's form (in the case of a single particle) is given by the expression

$$\Delta\int\sqrt{2m\left[H - V(r)\right]}\ ds = 0 \tag{ii}$$

Now, the length $ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + y'^2}$, where $y' = \frac{dy}{dx}$. Then equation (ii) becomes

$$\Delta\int\sqrt{2m(H - mgy)\left(1 + y'^2\right)}.dx = 0$$

or $$\Delta\int f(y, y').dx = 0$$

where, $f = \sqrt{2m(H - mgy)\left(1 + y'^2\right)}$. Since f is not an explicit function of x, we can have

$$\frac{\partial f}{\partial y'}y' - f = k \tag{iii}$$

where, k is a constant.

Now, $$\frac{\partial f}{\partial y'}=\sqrt{2m(H-mgy)}\frac{y'}{\sqrt{1+y'^2}}$$

Using the expressions for f and $\frac{\partial f}{\partial y'}$ in (iii), we get

$$\sqrt{2m(H-mgy)}\frac{y'^2}{\sqrt{1+y'^2}}-\sqrt{2m(H-mgy)\left(1+y'^2\right)}=k$$

that is, $$-\sqrt{\frac{2m(H-mgy)}{1+y'^2}}=k \tag{iv}$$

To find k, consider the above equation

$$\sqrt{\frac{2m(H-mgy)}{1+y'^2}}=\sqrt{\frac{2m(H-mgy)}{1+\left({}^{dy}\!/_{dx}\right)^2}}=\sqrt{\frac{2m(H-mgy)}{1+\left({}^{\dot{y}}\!/_{\dot{x}}\right)^2}}=-k$$

that is, $$\sqrt{\frac{2m(H-mgy)}{\left(\dot{x}^2+\dot{y}^2\right)}}\dot{x}=-k \tag{v}$$

Now, from the Hamiltonian, $H=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+mgy$, we get $\left(\dot{x}^2+\dot{y}^2\right)=\frac{2}{m}(H-mgy)$, then, the above equation becomes

$$k=-m\dot{x} \tag{vi}$$

Since u is the initial velocity of the projectile

$$\dot{x}=u\cos\theta \text{ and } \dot{y}=u\sin\theta$$

Therefore, $$k=-mu\cos\theta \tag{vii}$$

Now, from equation (iv), after substituting for k and squaring, we get

$$\frac{2m(H-mgy)}{1+y'^2}=m^2u^2\cos^2\theta$$

or $$1+y'^2=\frac{2(H-mgy)}{mu^2\cos^2\theta}$$

or $$\left(\frac{dy}{dx}\right)^2=\frac{2(H-mgy)}{mu^2\cos^2\theta}-1 \tag{viii}$$

Now, we have $H = E = \frac{1}{2}mu^2$, using this in the above expression, we get

$$\left(\frac{dy}{dx}\right)^2 = \frac{2(E-mgy)-2E\cos^2\theta}{2E\cos^2\theta} = \frac{(E-mgy)-E\cos^2\theta}{E\cos^2\theta}$$

$$= \frac{E\left(1-\cos^2\theta\right)-mgy}{E\cos^2\theta} = \frac{E\sin^2\theta - mgy}{E\cos^2\theta}$$

Then,
$$\frac{dy}{dx} = \left(\frac{E\sin^2\theta - mgy}{E\cos^2\theta}\right)^{1/2}$$

or
$$\frac{dy}{\sqrt{E\sin^2\theta - mgy}} = \frac{dx}{\sqrt{E\cos^2\theta}}$$

Using the expression, $E = \frac{1}{2}mu^2$, the above expression can be modified as

$$\sqrt{\frac{2}{m}}\frac{dy}{\sqrt{u^2\sin^2\theta - 2gy}} = \sqrt{\frac{2}{m}}\frac{dx}{u\cos\theta}$$

On integration, we get

$$\int -\sqrt{\frac{2}{m}}\left(\frac{1}{2g}\right)\frac{2g\,dy}{\sqrt{u^2\sin^2\theta - 2gy}} = \sqrt{\frac{2}{m}}\frac{x}{u\cos\theta} + C$$

or
$$-\sqrt{\frac{2}{m}}\left(\frac{2}{2g}\right)\sqrt{u^2\sin^2\theta - 2gy} = \sqrt{\frac{2}{m}}\frac{x}{u\cos\theta} + C \tag{ix}$$

The constant of integration C can be obtained from the initial conditions, that is, at the origin, $x = y = 0$, then

$$C = -\sqrt{\frac{2}{m}}\left(\frac{u\sin\theta}{g}\right)$$

The equation (ix) becomes

$$-\left(\frac{1}{g}\right)\sqrt{u^2\sin^2\theta - 2gy} = \frac{x}{u\cos\theta} - \left(\frac{u\sin\theta}{g}\right)$$

or
$$-\sqrt{u^2\sin^2\theta - 2gy} = \frac{gx}{u\cos\theta} - u\sin\theta$$

Squaring both sides we get;

$$u^2 \sin^2\theta - 2gy = \left(\frac{gx}{u\cos\theta} - u\sin\theta\right)^2$$

$$= \frac{g^2x^2}{u^2\cos^2\theta} + u^2\sin^2\theta - 2gx\tan\theta$$

that is,
$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2 \tag{x}$$

Equation (x) is the equation of motion of a projectile.

EXAMPLE 3.56 A particle of unit mass is projected so that its total energy is E in a field it is moving and its potential energy is $V(r)$, where r is the distance of the particle from the origin, which is the point of projection. Using the principle of least action, show that the path of the particle can be represented by the differential equation, $k\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] = r^4\left[E - V(r)\right]$.

Solution: We start from the definition of action given by the expression; $A = \int_0^t 2T dt$, where T is the kinetic energy of the particle.

But, $T = \frac{1}{2}mv^2 = \frac{1}{2}v^2$ so that $2T = v^2$. Now the action integral can be written as

$$A = \int_0^t v^2 dt = \int_0^t v.\frac{ds}{dt}dt = \int_0^s v.ds \tag{i}$$

Given that the total energy is E and the potential energy is $V(r)$, then the kinetic energy can be written as

$$T = \frac{1}{2}v^2 = E - V(r)$$

so that
$$v = \sqrt{2\left[E - V(r)\right]} \tag{ii}$$

Now, the principle of least action gives

$$\Delta A = \int_0^s \sqrt{2\left[E - V(r)\right]}.ds = 0 \tag{iii}$$

Now, we need to substitute for ds and for this we make use of the expression for velocity in polar coordinate system. That is,

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

so that
$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

Dividing throughout by $\left(\frac{d\theta}{dt}\right)^2$ we get

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2 = r'^2 + r^2$$

where, $r' = \frac{dr}{d\theta}$. From the above expression, we have

$$ds = d\theta\sqrt{\left(r^2 + r'^2\right)} \qquad \text{(iv)}$$

Using equation (iv), equation (iii) can be written as

$$\Delta A = \int_0^s \sqrt{2[E - V(r)]\left(r^2 + r'^2\right)}.d\theta = 0$$

$$\Delta A = \int_0^s f(r, r').d\theta = 0$$

where, $$f = \sqrt{2[E - V(r)]\left(r^2 + r'^2\right)} \qquad \text{(v)}$$

and is independent of θ. Therefore, we can write

$$\frac{\partial f}{\partial r'}r' - f = k \qquad \text{(vi)}$$

From (v), we get

$$\frac{\partial f}{\partial r'} = \sqrt{2[E - V(r)]}.2r'.\frac{1}{2\sqrt{r^2 + r'^2}}$$

$$= \sqrt{\frac{2[E - V(r)]}{r^2 + r'^2}}.r' \qquad \text{(vii)}$$

Using equations (v) and (vii) in (vi), we get

$$\sqrt{\frac{2[E - V(r)]}{r^2 + r'^2}}.r'^2 - \sqrt{2[E - V(r)]\left(r^2 + r'^2\right)} = k$$

or $$\sqrt{2[E - V(r)]}\left[\frac{r'^2}{\sqrt{r^2 + r'^2}} - \sqrt{\left(r^2 + r'^2\right)}\right] = k$$

that is, $$\sqrt{2[E - V(r)]}\left(-\frac{r^2}{\sqrt{r^2 + r'^2}}\right) = k$$

On squaring this would yield

$$\frac{2[E-V(r)]}{r^2+r'^2}r^4 = k^2$$

or $$2[E-V(r)]r^4 = k^2(r^2+r'^2)$$

that is, $$2[E-V(r)]r^4 = k^2\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right],$$ Hence, proved.

EXERCISES

3.1 Obtain Hamilton's canonical equations in cylindrical and spherical polar coordinate systems.

3.2 Consider a particle of mass m moving in two dimensions, subject to a force $F=-kx\hat{i}+K\hat{j}$ where k and K are positive constants. Obtain the Hamiltonian and Hamilton's equations of motion of the particle. Solve the equation of motion and describe the type of motion.

3.3 A point mass m moves in a cylindrically symmetric potential $V(r,z)$. Obtain the Hamiltonian of the particle and hence find the canonical equations of motion in Cartesian coordinate system and in the cylindrical coordinate system.

3.4 A particle of mass m is moving in the upward direction in a uniform gravitational field. Assuming a damping force with magnitude cv^2, where c is a constant, obtain the equation of motion by Hamilton's method.

3.5 A particle of mass m can slide without friction inside a circular tube of radius R. The tube is capable of rotation about a vertical axis. Obtain the equation of motion by Hamilton's method.

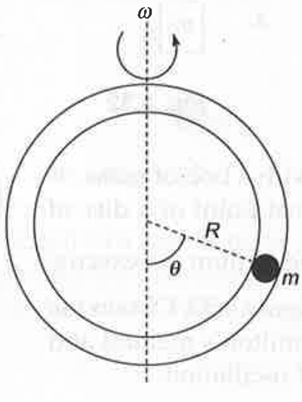

Fig. 3.31

3.6 A particle in a three-dimensional harmonic oscillator potential has a natural frequency ω_0. Obtain the equation of motion by Hamilton's method if the particle is charged and simultaneously acted on by uniform electric and magnetic field.

3.7 A particle of mass m moves in one dimension under the influence of a central force $F = \frac{k}{x^2} e^{-t/\tau}$, where k and τ are positive constants. Obtain the Lagrangian and Hamiltonian of the particle. Find the equation of motion.

3.8 A particle of mass m is subjected to a central attractive force given by $F = -\frac{ke^{-\alpha t}}{r^2} \hat{r}$, where k and α are positive constants. Find the Hamiltonian and the canonical equations of the particle.

3.9 Use Hamilton's equations to obtain the equations of motion of a uniform heavy rod of mass M and length $2l$ turning about one end which is fixed.

3.10 Obtain the equation of motion of the double pulley system shown in Figure (3.32) by Hamilton's method.

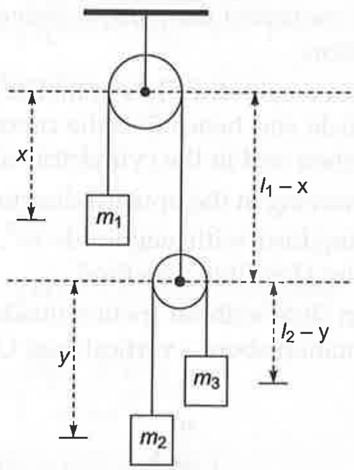

Fig. 3.32

3.11 A pendulum of length l with a bob of mass m is connected to the top most point of a disc of radius R with $R < \frac{\pi}{l}$. The pendulum can execute oscillations as shown in Figure 3.33. Obtain the equation of motion by Hamilton's method and then find the frequency of oscillation.

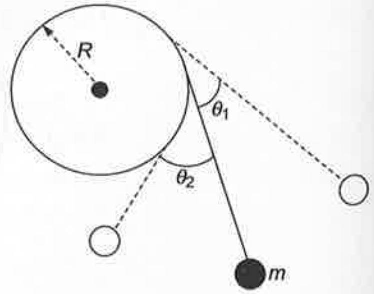

Fig. 3.33

3.12 A particle of mass m slides down over a smooth circular wedge of mass M. The wedge is placed on a smooth frictionless horizontal surface. Obtain the equation of motion of both masses by Hamilton'smethod.

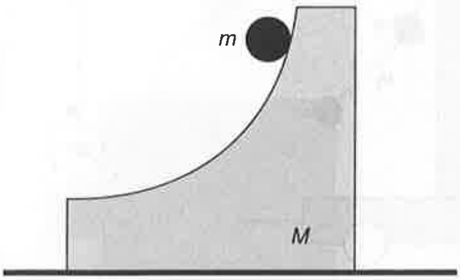

Fig. 3.34

3.13 The potential of a harmonic oscillator is given by $V=\frac{1}{2}kx^2+\frac{1}{4}\beta x^4$, where, k and β are positive constants. Obtain Hamilton's canonical equations of motion.

3.14 A pendulum of length l with a bob of mass m is connected to the centre of a disc rolling on a horizontal surface as shown in Figure 3.35. Obtain the equation of motion.

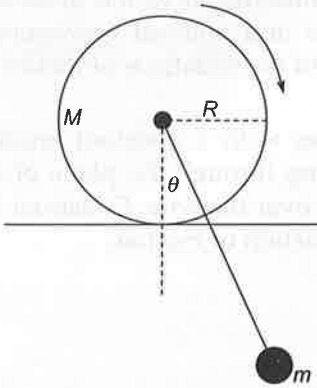

Fig. 3.35

3.15 A double pendulum is attached to a cart of mass M as shown in Figure 3.36. The pendula are of equal length and the bobs are of equal masses. The cart is moving with a velocity v m/s. Construct the Hamiltonian and obtain the equations of motion of the system.

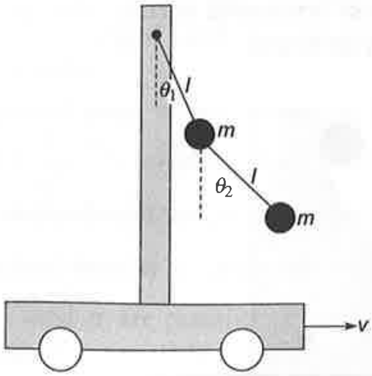

Fig. 3.36

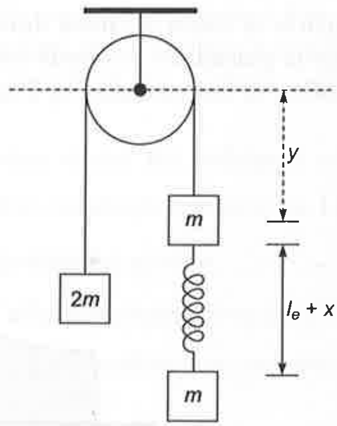

Fig. 3.37

3.16 Figure 3.37 shows a modified Atwood's machine. The equilibrium length of the spring of force constant k is taken as l_e and the extension as x. Obtain the Hamilton's canonical equations of motion if at equilibrium $y = y_0$ and at $t = 0$, $x = x_0$.

3.17 The bearing of a rigid pendulum of mass m is forced to rotate uniformly with angular velocity ω. The angle between the rotation axis and the pendulum is θ. Neglect the inertia of the bearing and of the rod connecting it to the mass and friction. If the system is placed in a uniform gravitational field, obtain the Hamiltonian and the equations of motion of the system.

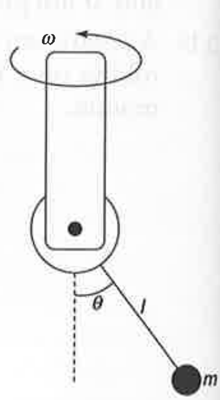

Fig. 3.38

3.18 A smooth circular wire rotates with a constant angular velocity ω about an axis passing through the plane of the circle. A bead of mass m slides over the wire. Construct the Hamiltonian and obtain the equation of motion.

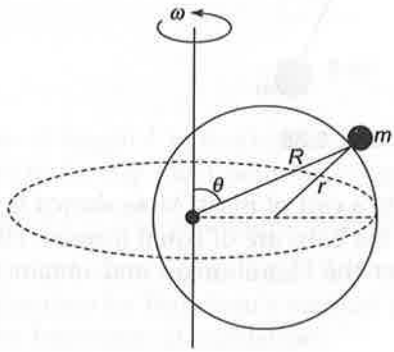

Fig. 3.39

3.19 Obtain the equation of motion for the system shown in Figure 3.40. Assuming there is no frictional force present.

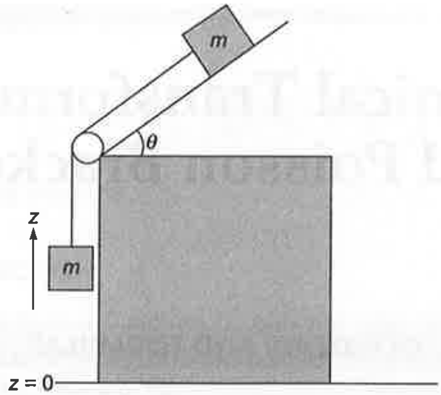

Fig. 3.40

3.20 Prove Liouville's theorem in the $2n$-dimensional phase space of a system with n degrees of freedom.

3.21 A particle falls from a height y_0 in a time $t_0 = \sqrt{2y_0/g}$ and the distance travelled in a time t is given as $y = \alpha t + \beta t^2$, where α and β are constants whose values are adjusted such that the time to fall through y_0 is always t_0. Show that the integral $\int_0^t L dt$ is an extremum for the real values of the coefficients only if $\alpha = 0$ and, $\beta = \dfrac{g}{2}$.

4

CHAPTER

Canonical Transformation and Poisson Brackets

CONCEPTS AND FORMULAE

4.1 TRANSFORMATION

For a given dynamical system, we can have more than one set of generalized coordinates. Among these different sets, we choose the most convenient one to solve the problem. Transformation simply denotes the switching from one set of generalized coordinates to a more convenient set of generalized coordinates.

4.2 POINT TRANSFORMATION

Transformation from one set of coordinates q_j to a new set of coordinates Q_j is called point transformation. Such a transformation can be represented as;

$$Q_j = Q_j\left(q_j, t\right) \tag{4.1}$$

Point transformation is considered the transformation of configuration space as it provides information about the position coordinate only.

4.3 CANONICAL OR CONTACT TRANSFORMATION

In a canonical transformation, we transform both the coordinate and momentum to new coordinate and momentum. As a result, such a transformation is considered a transformation of phase space. An important property of canonical transformation is that, it leaves the form of Hamilton's canonical equations unmodified.

That is,
$$\dot{Q}_j = \frac{\partial H'}{\partial P_j} \text{ and } \dot{P}_j = -\frac{\partial H'}{\partial Q_j} \tag{4.2}$$

where, H' is the transformed Hamiltonian.

A canonical transformation can be represented in the form

$$Q_j = Q_j(q,p,t) \text{ and } P_j = P_j(q,p,t) \tag{4.3}$$

The canonical transformation is given by

$$\left(\sum_j p_j \dot{q}_j - H\right) - \left(\sum_j P_j \dot{Q}_j - H'\right) = \frac{dF}{dt} \tag{4.4}$$

where, F is the generating function.

4.4 GENERATING FUNCTION

Generating function (say F) is a function of old and new sets of coordinates that can cause a canonical transformation from the old set to the new set. The form of the generating function can be selected depending upon the particular problem. The four forms of generating function that are possible are:

$$F_1(q,Q,t),\ F_2(q,P,t),\ F_3(p,Q,t) \text{ and } F_4(p,P,t) \tag{4.5}$$

In a canonical transformation, the generating function must always satisfy the condition

$$\delta\left[F(q,p,t)\right]_{t_1}^{t_2} = 0 \tag{4.6}$$

4.5 CONDITION FOR A TRANSFORMATION TO BE CANONICAL

A transformation from an old set of coordinates (q_j, p_j, t) to a new set of variables (Q_j, P_j, t) is canonical if $\sum_j (p_j dq_j - P_j dQ_j)$ is an exact differential.

That is,

$$\sum_j (p_j dq_j - P_j dQ_j) = dF \tag{4.7}$$

where, dF is the exact differential of F.

4.6 INFINITESIMAL CANONICAL TRANSFORMATION

In an infinitesimal canonical transformation, the old and new sets of coordinates differ only by an infinitesimal amount. An infinitesimal canonical transformation is carried out by a generating function

$$F_2 \equiv F = \sum_j q_j P_j + \varepsilon G(q_j, P_j) \tag{4.8}$$

where, ε is the infinitesimal parameter and G is an arbitrary function.

The transformation equations are

$$\delta q_j = Q_j - q_j = \varepsilon \frac{\partial G}{\partial P_j} \quad \text{and} \quad \delta p_j = P_j - p_j = -\varepsilon \frac{\partial G}{\partial q_j} \tag{4.9}$$

4.7 POISSON BRACKETS

If the dynamical variables F and G of a system is a function of q_j and p_j, then the Poisson bracket of F and G is defined as

$$[F,G]_{q_j,p_j} = \sum_j \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial p_j} \right) \tag{4.10}$$

4.8 PROPERTIES OF POISSON BRACKETS

Poisson bracket is not commutative

$$[F,G] = -[G,F] \tag{4.11}$$

Poisson bracket of a function with itself vanishes

$$[F,F] = 0 \tag{4.12}$$

Poisson bracket has distributive property

$$[F,G+A] = [F,G] + [F,A] \tag{4.13}$$

$$[F,G.A] = G[F,A] + A[F,G] \tag{4.14}$$

Poisson bracket is invariant under canonical transformation

$$[F,G]_{q,p} = [F,G]_{Q,P} \tag{4.15}$$

4.9 EQUATIONS OF MOTION IN TERMS OF POISSON BRACKETS

$$\dot{q}_j = [q_j, H] \quad \text{and} \quad \dot{p}_j = [p_j, H] \tag{4.16}$$

where H is the Hamiltonian of the system

4.10 POISSON'S THEOREM

If the Poisson bracket of a function F with Hamiltonian vanishes, i.e. $[F,H] = 0$, then the function is a constant of motion.

4.11 JACOBI-POISSON THEOREM OR POISSON'S SECOND THEOREM

If X,Y and Z are three functions of the q_j and p_j, then

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{4.17}$$

This is also known as Jacobi's identity.

4.12 ANGULAR MOMENTUM POISSON BRACKETS

If L_x, L_y and L_z be the components of angular momentum, then

$$\left[L_x, L_y\right] = L_z \ , \ \left[L_y, L_z\right] = L_x \text{ and } \left[L_z, L_x\right] = L_y \tag{4.18}$$

4.13 LAGRANGE'S BRACKET

Lagrange's bracket of two functions *f*, *g* with respect to the basis q_j and p_j is defined as;

$$\{f, g\} = \sum_j \left(\frac{\partial q_j}{\partial f}\frac{\partial p_j}{\partial g} - \frac{\partial q_j}{\partial g}\frac{\partial p_j}{\partial f} \right) \tag{4.19}$$

Lagrange's bracket is not commutative. It is invariant in a canonical transformation.

4.14 LIOUVILLE'S THEOREM

Liouville's theorem consists of two parts:

1. Conservation of density in phase space, $\dfrac{d\rho}{dt} = 0$
2. Conservation of extension in free space, $\dfrac{d}{dt}(\delta V) = 0$

SOLVED PROBLEMS

EXAMPLE 4.1 Obtain Hamilton's equations of motion from Lagrange's equation of motion using Legendre transform.

Solution: Legendre transform is a mathematical technique to transform a function $f(x, y)$ with basis x, y to another function $g(u, y)$ with basis u, y.

Let us see the technique of Legendre transform first. The total derivative of the given function $f(x, y)$ is given by;

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= udx + vdy \tag{i}$$

where, $$u = \frac{\partial f}{\partial x} \text{ and } v = \frac{\partial f}{\partial y}.$$

The function $g(u, y)$ is defined as

$$g(u, y) = ux - f(x, y) \tag{ii}$$

Now, $$dg = udx + xdu - df$$

Using equation (i), the above expression becomes

$$dg = udx + xdu - udx - vdy$$
$$= xdu - vdy \tag{iii}$$

Now, we put

$$x = \frac{\partial g}{\partial u} \text{ and } v = -\frac{\partial g}{\partial y} \tag{iv}$$

With this equation, (iii) becomes

$$dg = \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial y}dy \tag{v}$$

This equation can be obtained from the function $g(u,y)$ by applying Euler's theorem and hence, it is the desired form of transformation. Thus, if $u = \frac{\partial f}{\partial x}$, then the relation, $g(u,y) = ux - f(x,y)$, changes the basis from x, y to the basis u, y.

Now we consider our problem. Here, $f(x,y) = L(\dot{q}_j, q_j)$ and $g(u,y) = H(p_j, q_j)$. Then using equation (ii), we can write

$$H(p_j, q_j) = \sum_j p_j \dot{q}_j - L(\dot{q}_j, q_j) \tag{vi}$$

Taking the derivative of this expression with respect to q_j, p_j and $\dot{q}_j$, we get

$$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j}\ ;\ \frac{\partial H}{\partial p_j} = \dot{q}_j \text{ and } \frac{\partial H}{\partial \dot{q}_j} = 0 = p_j - \frac{\partial L}{\partial \dot{q}_j} \tag{vii}$$

Using the last expression in the Lagrange's equation of motion $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$, we get

$$\frac{d}{dt}(p_j) - \frac{\partial L}{\partial q_j} = 0$$

or

$$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j} \tag{viii}$$

From the equations (vii) and (viii), we get

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \text{ and } \dot{p}_j = -\frac{\partial H}{\partial q_j} \tag{ix}$$

These are the Hamilton's canonical equations of motion.

EXAMPLE 4.2 Show that a canonical transformation, in general, can be represented by the expression $\left(\sum_j p_j \dot{q}_j - H\right) - \left(\sum_j P_j \dot{Q}_j - H'\right) = \frac{dF}{dt}$, where, H is the old Hamiltonian, H' is the new Hamiltonian and F is the generating function.

Solution: We have $H = H\left(q_j, p_j, t\right)$ and $H' = H'\left(Q_j, P_j, t\right)$. Since the oldest of variables are canonical, we have, from Hamilton's principle

$$\delta \int_{t_1}^{t_2} \left[\sum_j p_j \dot{q}_j - H\left(q_j, p_j, t\right) \right] dt = 0 \tag{i}$$

The new variables must also be canonical and therefore,

$$\delta \int_{t_1}^{t_2} \left[\sum_j P_j \dot{Q}_j - H'\left(Q_j, P_j, t\right) \right] dt = 0 \tag{ii}$$

This is known as Hamilton's modified principle. Since the right hand sides of equations (i) and (ii) are equal to zero. We can write

$$\delta \int_{t_1}^{t_2} \left[\left(\sum_j p_j \dot{q}_j - H \right) - \left(\sum_j P_j \dot{Q}_j - H' \right) \right] dt = 0 \tag{iii}$$

Note that the integrands need not be equal to zero.

Now let us subtract the total derivative of a function $F\left(q_j, p_j, t\right)$ from the left hand side of the above expression. This is justified because, since the end points are fixed;

$$\delta \int_{t_1}^{t_2} \frac{dF}{dt} dt = \frac{\partial F}{\partial q_j} \delta q_j \Bigg|_{t_1}^{t_2} + \frac{\partial F}{\partial p_j} \delta p_j \Bigg|_{t_1}^{t_2} = 0$$

Equation (iii) then becomes

$$\delta \int_{t_1}^{t_2} \left[\left(\sum_j p_j \dot{q}_j - H \right) - \left(\sum_j P_j \dot{Q}_j - H' \right) - \frac{dF}{dt} \right] dt = 0$$

It follows that,

$$\left(\sum_j p_j \dot{q}_j - H \right) - \left(\sum_j P_j \dot{Q}_j - H' \right) = \frac{dF}{dt} \tag{iv}$$

This is the general transformation equation for canonical transformation.

EXAMPLE 4.3 Obtain the transformation equations in terms of the generating function $F_1(q, Q, t)$.

Solution: The general transformation equation is

$$\left(\sum_j p_j \dot{q}_j - H \right) - \left(\sum_j P_j \dot{Q}_j - H' \right) = \frac{dF_1}{dt} \tag{i}$$

Now, $F_1 = F_1(q,Q,t)$ and its total derivative is

$$\frac{dF_1}{dt} = \sum_j \frac{\partial F_1}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial F_1}{\partial Q_j}\dot{Q}_j + \frac{\partial F_1}{\partial t} \tag{ii}$$

This can be substituted in (i) to get

$$\sum_j p_j\dot{q}_j - H = \sum_j P_j\dot{Q}_j - H' + \sum_j \frac{\partial F_1}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial F_1}{\partial Q_j}\dot{Q}_j + \frac{\partial F_1}{\partial t}$$

This can be rearranged to get

$$\sum_j\left(\frac{\partial F_1}{\partial q_j} - p_j\right)\dot{q}_j + \sum_j\left(P_j + \frac{\partial F_1}{\partial Q_j}\right)\dot{Q}_j + H - H' + \frac{\partial F_1}{\partial t} = 0 \tag{iii}$$

The coordinates, q_j and Q_j are to be treated as independent variables and hence equation (iii) will be valid only if

$$p_j = \frac{\partial F_1}{\partial q_j} \tag{iv}$$

$$P_j = -\frac{\partial F_1}{\partial Q_j} \tag{v}$$

and $$H' = H + \frac{\partial F_1}{\partial t} \tag{vi}$$

Equations (iv), (v) and (vi) are the transformation equations.

EXAMPLE 4.4 Obtain the transformation equations in terms of the generating function $F_2(q,P,t)$.

Solution: Let us start from the expression

$$\left(\sum_j p_j\dot{q}_j - H\right) - \left(\sum_j P_j\dot{Q}_j - H'\right) = \frac{dF_1}{dt} \tag{i}$$

In the present problem F_2 is not a function of Q. Therefore, a change of basis from (q,Q) to (q,P) is required. In other words, we need to find a transformation equation between F_1 and F_2. This can be carried out through Legendre transform.

Put, $f = F_1$, $g = F_2$, $u = -P_j$, $x = Q_j$ and $y = q_j$ in the equation for the Legendre transform $g(u,y) = f(x,y) - ux$, we get

$$F_1(q_j,Q_j,t) = F_2(q_j,P_j,t) - \sum_j P_jQ_j \tag{ii}$$

Now, equation (i) can be written as

$$\left(\sum_j p_j\dot{q}_j - H\right) - \left(\sum_j P_j\dot{Q}_j - H'\right) = \frac{d}{dt}\left[F_2\left(q_j, P_j, t\right) - \sum_j P_jQ_j\right]$$

or $$\sum_j p_j\dot{q}_j - H = \sum_j P_j\dot{Q}_j - H' + \sum_j \frac{\partial F_2}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial F_2}{\partial P_j}\dot{P}_j + \frac{\partial F_2}{\partial t} - \sum_j P_j\dot{Q}_j - \sum_j \dot{P}_jQ_j$$

or $$= \sum_j \frac{\partial F_2}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial F_2}{\partial P_j}\dot{P}_j + \frac{\partial F_2}{\partial t} - \sum_j \dot{P}_jQ_j - H'$$

that is, $$\sum_j\left(\frac{\partial F_2}{\partial q_j} - p_j\right)\dot{q}_j + \sum_j\left(\frac{\partial F_2}{\partial P_j} - Q_j\right)\dot{P}_j + H + \frac{\partial F_2}{\partial t} - H' = 0 \tag{iii}$$

Since q_j and P_j are independent variables, from the above expression, we get

$$p_j = \frac{\partial F_2}{\partial q_j} \tag{iv}$$

$$Q_j = \frac{\partial F_2}{\partial P_j} \tag{v}$$

and $$H' = H + \frac{\partial F_2}{\partial t} \tag{vi}$$

Equations (iv), (v) and (vi) are the transformation equations.

EXAMPLE 4.5 Obtain the transformation equations in terms of the generating function $F_3(p, Q, t)$.

Solution: We start from the transformation equation

$$\left(\sum_j p_j\dot{q}_j - H\right) - \left(\sum_j P_j\dot{Q}_j - H'\right) = \frac{dF_1}{dt} \tag{i}$$

and we seek a Legendre transform that relates $F_1(q, Q, t)$ and $F_3(p, Q, t)$. This can be achieved by putting; $f = F_1$, $g = F_3$, $u = p_j$, $x = q_j$ and $y = Q_j$ in the equation for the Legendre transform $g(u, y) = f(x, y) - ux$. Then, we get

$$F_1\left(q_j, Q_j, t\right) = F_3\left(p_j, Q_j, t\right) + \sum_j p_jq_j \tag{ii}$$

Using this in (i), we get

$$\left(\sum_j p_j\dot{q}_j - H\right) - \left(\sum_j P_j\dot{Q}_j - H'\right) = \frac{d}{dt}\left[F_3\left(p_j, Q_j, t\right) + \sum_j p_jq_j\right] \tag{iii}$$

or $$\sum_j p_j\dot{q}_j - H = \sum_j P_j\dot{Q}_j - H' + \sum_j \frac{\partial F_3}{\partial p_j}\dot{p}_j + \sum_j \frac{\partial F_3}{\partial Q_j}\dot{Q}_j + \frac{\partial F_3}{\partial t} + \sum_j p_j\dot{q}_j + \sum_j \dot{p}_j q_j$$

or $$\sum_j\left(q_j + \frac{\partial F_3}{\partial p_j}\right)\dot{p}_j + \sum_j\left(P_j + \frac{\partial F_3}{\partial Q_j}\right)\dot{Q}_j + H + \frac{\partial F_3}{\partial t} - H' = 0 \qquad \text{(iv)}$$

Since p_j and Q_j are independent variables; from the above expression, we get

$$q_j = -\frac{\partial F_3}{\partial p_j} \qquad \text{(v)}$$

$$P_j = -\frac{\partial F_3}{\partial Q_j} \qquad \text{(vi)}$$

and $$H' = H + \frac{\partial F_3}{\partial t} \qquad \text{(vii)}$$

Equations (v), (vi) and (vii) are the transformation equations.

EXAMPLE 4.6 Obtain the transformation equations in terms of the generating function $F_4(p,P,t)$.

Solution: The canonical transformation in terms of the generating function of the form $F_1(q,Q,t)$ is given by

$$\left(\sum_j p_j\dot{q}_j - H\right) - \left(\sum_j P_j\dot{Q}_j - H'\right) = \frac{dF_1}{dt} \qquad \text{(i)}$$

In this case, the transformation from the basis (q_j, Q_j) to (p_j, P_j) can be achieved by a double Legendre transformation, i.e., from F_1 to F_3 and then from F_3 to F_4. Here we get the transformation equations as

$$F_3(p_j, Q_j, t) = F_1(q_j.Q_j, t) - \sum_j p_j q_j$$

and $$F_4(p_j, P_j, t) = F_3(p_j, Q_j, t) + \sum_j P_j Q_j$$

Using these expressions, we get

$$F_1(q_j.Q_j, t) = F_4(p_j, P_j, t) + \sum_j p_j q_j - \sum_j P_j Q_j \qquad \text{(ii)}$$

Now, equation (i) becomes

$$\left(\sum_j p_j\dot{q}_j - H\right) - \left(\sum_j P_j\dot{Q}_j - H'\right) = \frac{d}{dt}\left[F_4(p_j, P_j, t) + \sum_j p_j q_j - \sum_j P_j Q_j\right]$$

or
$$\sum_j p_j\dot{q}_j - H = \sum_j P_j\dot{Q}_j - H' + \sum_j \frac{\partial F_4}{\partial p_j}\dot{p}_j + \sum_j \frac{\partial F_4}{\partial P_j}\dot{P}_j + \frac{\partial F_3}{\partial t} + \sum_j p_j\dot{q}_j + \sum_j \dot{p}_j q_j - \sum_j P_j\dot{Q}_j - \sum_j \dot{P}_j Q_j$$

that is,
$$\sum_j\left(q_j + \frac{\partial F_4}{\partial p_j}\right)\dot{p}_j + \sum_j\left(\frac{\partial F_4}{\partial P_j} - Q_j\right)\dot{P}_j + H + \frac{\partial F_3}{\partial t} - H' = 0 \tag{iii}$$

Since p_j and P_j are independent variables, from the above expression, we get

$$q_j = -\frac{\partial F_4}{\partial p_j} \tag{iv}$$

$$Q_j = \frac{\partial F_4}{\partial P_j} \tag{v}$$

and
$$H' = H + \frac{\partial F_4}{\partial t} \tag{vi}$$

Equations (iv), (v) and (vi) are the transformation equations.

EXAMPLE 4.7 Solve the simple harmonic oscillator problem by the method of canonical transformation. Given that the generating function is: $F_1 = \mu q^2 \cot Q$, where, μ is a constant.

Solution: In this problem we perform a canonical transformation from the old set of variables (q, p) to the new set of variables, (Q, P) such that the new coordinate Q is cyclic and the corresponding momentum is a constant.

The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \tag{i}$$

For the generating function, $F_1(q_j, Q_j, t)$, we have the transformation equations as

$$p_j = \frac{\partial F_1}{\partial q_j} \text{ so that, } p = 2\mu q \cot Q \tag{ii}$$

$$P_j = -\frac{\partial F_1}{\partial Q_j} \text{ so that, } P = \mu q^2 cosec^2 Q \tag{iii}$$

and
$$H' = H + \frac{\partial F_1}{\partial t}, \text{ so that, } H' = H \tag{iv}$$

From equation (iii), we get

$$q = \sqrt{\frac{P}{\mu}}\sin Q \tag{v}$$

Using (v) in (ii),we get

$$p=\sqrt{4\mu P}\cos Q \tag{vi}$$

Now, the new Hamiltonian is

$$H'=H=\frac{p^2}{2m}+\frac{1}{2}kq^2=\frac{1}{2m}\left(4\mu P\cos^2 Q\right)+\frac{1}{2}k\left(\frac{P\sin^2 Q}{\mu}\right)$$

$$=\frac{2\mu P\cos^2 Q}{m}+\frac{1}{2}k\left(\frac{P\sin^2 Q}{\mu}\right)$$

This expression can be rearranged to get

$$H'=\frac{kP}{2\mu}\left(\frac{4\mu^2}{mk}\cos^2 Q+\sin^2 Q\right) \tag{vii}$$

Equation (vii) gives the Hamiltonian in terms of new variables. Now, let

$$\mu=\frac{\sqrt{mk}}{2}$$

Then the expression for Hamiltonian becomes

$$H'=\frac{kP}{2\mu}\left(\cos^2 Q+\sin^2 Q\right)=\frac{kP}{2\mu}=P\sqrt{\frac{k}{m}} \tag{viii}$$

Note that Q is now a cyclic coordinate and therefore the corresponding momentum is a constant of motion. Therefore,

$$\dot{P}=-\frac{\partial H'}{\partial Q}=0$$

and we take, $P=\alpha$, a constant.

Also, $$\dot{Q}=\frac{\partial H'}{\partial P}=\sqrt{\frac{k}{m}}\text{, a constant.}$$

On integration, we get

$$Q=\left(\sqrt{\frac{k}{m}}\right)t+\beta \tag{ix}$$

Here, β is the constant of integration. Using (ix) in (v), we get

$$q=\sqrt{\frac{P}{\mu}}\sin\left[\left(\sqrt{\frac{k}{m}}\right)t+\beta\right]$$

But, $P=\alpha$ and therefore

$$q=\sqrt{\frac{\alpha}{\mu}}\sin\left[\left(\sqrt{\frac{k}{m}}\right)t+\beta\right] \tag{x}$$

Equation (x) gives the solution of a simple harmonic oscillator.

EXAMPLE 4.8 For a certain canonical transformation, it is known that $Q=\sqrt{q^2+p^2}$ and the generating function is given by the expression $F=\frac{1}{2}\left(q^2+p^2\right)\tan^{-1}\left(\frac{q}{p}\right)+\frac{1}{2}qp$. Deduce the expression for $P(q,p)$ and $F(q,Q)$.

Solution: Given that

$$Q=\sqrt{q^2+p^2} \tag{i}$$

and

$$F=\frac{1}{2}\left(q^2+p^2\right)\tan^{-1}\left(\frac{q}{p}\right)+\frac{1}{2}qp \tag{ii}$$

From (i), we get; $p=\sqrt{Q^2-q^2}$ and using this in (ii), we get

$$F=\frac{1}{2}Q^2\tan^{-1}\left(\frac{q}{\sqrt{Q^2-q^2}}\right)+\frac{1}{2}q\sqrt{Q^2-q^2} \tag{iii}$$

Now, let $\theta=\tan^{-1}\left(\frac{q}{\sqrt{Q^2-q^2}}\right)$ so that, $\tan\theta=\frac{q}{\sqrt{Q^2-q^2}}$

Then, $\sec^2\theta=1+\tan^2\theta=\frac{Q^2}{Q^2-q^2}$

Then, $\cos^2\theta=\frac{Q^2-q^2}{Q^2}$ and $\sin^2\theta=\frac{q^2}{Q^2}$. Therefore, $\theta=\sin^{-1}\left(\frac{q}{Q}\right)$

Using this in (iii), we get

$$F=\frac{1}{2}Q^2\sin^{-1}\left(\frac{q}{Q}\right)+\frac{1}{2}q\sqrt{Q^2-q^2} \tag{iv}$$

Now, using the transformation equation $P=-\frac{\partial F_1}{\partial Q}$, we can write

$$P=-\frac{\partial F}{\partial Q}=-\frac{\partial}{\partial Q}\left[\frac{1}{2}Q^2\sin^{-1}\left(\frac{q}{Q}\right)+\frac{1}{2}q\sqrt{Q^2-q^2}\right]$$

$$=-\left[Q\sin^{-1}\left(\frac{q}{Q}\right)-\frac{1}{2}\frac{qQ}{\sqrt{Q^2-q^2}}+\frac{1}{2}\frac{qQ}{\sqrt{Q^2-q^2}}\right]$$

that is, $$P(q,Q) = -Q\sin^{-1}\left(\frac{q}{Q}\right)$$

so that $$P(q,p) = -\left(\sqrt{q^2+p^2}\right)\sin^{-1}\left(\frac{q}{\sqrt{q^2+p^2}}\right) \tag{v}$$

This can be further simplified to get

$$P(q,p) = -\left(\sqrt{q^2+p^2}\right)\tan^{-1}\left(\frac{q}{p}\right) \tag{vi}$$

EXAMPLE 4.9 Show that the transformation $P = \frac{1}{2}(q^2+p^2)$ and $Q = \tan^{-1}\left(\frac{q}{p}\right)$ represent a canonical transformation.

Solution: Here the old set of variables is canonical and therefore it must satisfy the Hamilton's canonical equations. That is,

$$\dot{q} = \frac{\partial H}{\partial p} \text{ and } \dot{p} = -\frac{\partial H}{\partial q} \tag{i}$$

Now, $q = q(Q,P)$ and $p = p(Q,P)$, therefore

$$\dot{q} = \frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial P}\dot{P} \text{ and } \dot{p} = \frac{\partial p}{\partial Q}\dot{Q} + \frac{\partial p}{\partial P}\dot{P} \tag{ii}$$

Since the old and new Hamiltonians are independent of time, $H(q,p) = H'(Q,P)$ and therefore,

$$\frac{\partial H}{\partial q} = \frac{\partial H'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H'}{\partial P}\frac{\partial P}{\partial q} \text{ and } \frac{\partial H}{\partial p} = \frac{\partial H'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H'}{\partial P}\frac{\partial P}{\partial p} \tag{iii}$$

Now the transformation equations are given as

$$P = \frac{1}{2}(q^2+p^2) \text{ and } Q = \tan^{-1}\left(\frac{q}{p}\right) \text{ or } \tan Q = \frac{q}{p} \tag{iv}$$

Differentiating these expressions with respect to the old variables q and p, we get

$$\frac{\partial P}{\partial q} = q \text{ and } \frac{\partial P}{\partial p} = p \tag{v}$$

Now, $$\frac{\partial}{\partial Q}(\tan Q).\frac{\partial Q}{\partial q} = \frac{1}{p} \text{ or } \sec^2 Q.\frac{\partial Q}{\partial q} = \frac{1}{p}$$

that is, $$\frac{\partial Q}{\partial q} = \frac{1}{p\sec^2 Q} = \frac{1}{p\left(1+\tan^2 Q\right)} = \frac{1}{p\left(1+\frac{q^2}{p^2}\right)} = \frac{p}{q^2+p^2} \tag{vi}$$

Similarly, $$\frac{\partial Q}{\partial p} = -\frac{q}{q^2+p^2} \tag{vii}$$

Again differentiating the transformation equations with respect to the new variables Q and P, we get

$$\frac{\partial P}{\partial Q}=0=q\frac{\partial q}{\partial Q}+p\frac{\partial p}{\partial Q} \text{ and} \tag{viii}$$

$$\frac{\partial P}{\partial P}=1=q\frac{\partial q}{\partial P}+p\frac{\partial p}{\partial P} \tag{ix}$$

Now, $$\frac{\partial}{\partial Q}(\tan Q)=\sec^2 Q.\frac{\partial Q}{\partial Q}=\frac{p\frac{\partial q}{\partial Q}-q\frac{\partial p}{\partial Q}}{p^2}$$

or $$\frac{\partial Q}{\partial Q}=1=\frac{\left(p\frac{\partial q}{\partial Q}-q\frac{\partial p}{\partial Q}\right)}{p^2\sec^2 Q}=\frac{\left(p\frac{\partial q}{\partial Q}-q\frac{\partial p}{\partial Q}\right)}{p^2\left(1+\tan^2 Q\right)}=\frac{\left(p\frac{\partial q}{\partial Q}-q\frac{\partial p}{\partial Q}\right)}{p^2\left(1+\frac{q^2}{p^2}\right)}$$

that is, $$1=\frac{\left(p\frac{\partial q}{\partial Q}-q\frac{\partial p}{\partial Q}\right)}{q^2+p^2} \tag{x}$$

Similarly, we get

$$\frac{\partial Q}{\partial P}=0=\frac{\left(p\frac{\partial q}{\partial P}-q\frac{\partial p}{\partial P}\right)}{p^2} \tag{xi}$$

Now, multiply (ix) with q and equation (xi) with p to get

$$q^2\frac{\partial q}{\partial P}+qp\frac{\partial p}{\partial P}=q \text{ and } \frac{\left(p^2\frac{\partial q}{\partial P}-qp\frac{\partial p}{\partial P}\right)}{p^2}=0$$

Adding these two expressions, we get

$$\left(q^2+p^2\right)\frac{\partial q}{\partial P}=q \text{ or } \frac{\partial q}{\partial P}=\frac{q}{\left(q^2+p^2\right)} \tag{xii}$$

Similarly, $$\frac{\partial p}{\partial P}=\frac{p}{\left(q^2+p^2\right)} \tag{xiii}$$

Now, multiply equation (viii) with q and equation (x) with p to get

$$q^2\frac{\partial q}{\partial Q}+qp\frac{\partial p}{\partial Q}=0 \text{ and } p^2\frac{\partial q}{\partial Q}-qp\frac{\partial p}{\partial Q}=p\left(q^2+p^2\right)$$

Adding these two expressions, we get

$$\left(q^2+p^2\right)\frac{\partial q}{\partial Q}=p\left(q^2+p^2\right) \text{ or, } \frac{\partial q}{\partial Q}=p \tag{xiv}$$

Similarly, we can have $$\frac{\partial p}{\partial Q} = -q \tag{xv}$$

Using (xii), (xiii), (xiv) and (xv) in (ii) and (iii), we get

$$\dot{q} = p\dot{Q} + \frac{q}{\left(q^2+p^2\right)}\dot{P} \text{ and } \dot{p} = -q\dot{Q} + \frac{p}{\left(q^2+p^2\right)}\dot{P} \tag{xvi}$$

$$\frac{\partial H}{\partial q} = -\dot{p} = \left(\frac{p}{q^2+p^2}\right)\frac{\partial H'}{\partial Q} + q\frac{\partial H'}{\partial P} \text{ and,}$$

$$\frac{\partial H}{\partial p} = \dot{q} = \left(-\frac{q}{q^2+p^2}\right)\frac{\partial H'}{\partial Q} + p\frac{\partial H'}{\partial P}$$

Using (xvi) in these expressions, we have

$$q\dot{Q} - \frac{p}{\left(q^2+p^2\right)}\dot{P} = \left(\frac{p}{q^2+p^2}\right)\frac{\partial H'}{\partial Q} + q\frac{\partial H'}{\partial P} \tag{xvii}$$

$$p\dot{Q} + \frac{q}{\left(q^2+p^2\right)}\dot{P} = p\frac{\partial H'}{\partial P} - \left(\frac{q}{q^2+p^2}\right)\frac{\partial H'}{\partial Q} \tag{xviii}$$

Multiplying (xvii) by p and (xviii) by q and then adding, we get

$$\dot{P} = -\frac{\partial H'}{\partial Q}$$

Similarly, multiplying (xvii) by q and (xviii) by p and adding, we get

$$\dot{Q} = \frac{\partial H}{\partial P}$$

Therefore, P and Q are canonical and the transformation is a canonical transformation.

EXAMPLE 4.10 Show that the transformation given by $Q = \left(\sqrt{2q}\right)e^{\alpha}\cos p$ and $P = \left(\sqrt{2q}\right)e^{-\alpha}\sin p$ represent a canonical transformation.

Solution: Given that:

$$Q = \left(\sqrt{2q}\right)e^{\alpha}\cos p \text{ and } P = \left(\sqrt{2q}\right)e^{-\alpha}\sin p \tag{i}$$

For a transformation to be canonical, the quantity $PdQ - pdq$ must be an exact differential.

Now, from the transformation equation, we have

$$dQ = \frac{e^{\alpha}}{\sqrt{2q}}(\cos p)dq - \left(\sqrt{2q}\right)e^{\alpha}(\sin p)dp \tag{ii}$$

$$\text{Now, } PdQ - pdq = \left(\sqrt{2q}\right)e^{-\alpha}\sin p\left[\frac{e^{\alpha}}{\sqrt{2q}}(\cos p)dq - \left(\sqrt{2q}\right)e^{\alpha}(\sin p)dp\right] - pdq$$

$$= (\sin p\cos p)dq - 2q\left(\sin^2 p\right)dp - pdq$$

$$= (\sin p\cos p - p)dq - 2q\left(\sin^2 p\right)dp$$

$$= \left(\frac{1}{2}\sin 2p - p\right)dq - 2q\left(\sin^2 p\right)dp$$

$$= \frac{\partial}{\partial q}\left(\frac{1}{2}q\sin 2p - qp\right)dq - \frac{\partial}{\partial p}\left(\frac{1}{2}q\sin 2p - qp\right)dp$$

$$= \frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial p}dp = dF$$

where, $F = \frac{1}{2}q\sin 2p - qp$. Therefore, $PdQ - pdq$ is an exact differential and hence, the transformation is canonical.

EXAMPLE 4.11 Prove the bilinear invariant condition.

Solution: Bilinear invariant condition states that if a transformation from an old set of variables (q_j, p_j) to a new set of variables (Q_j, P_j), the set being canonical, then the bilinear form $\sum_j \left(\delta p_j dq_j - \delta q_j dp_j\right)$ remains invariant. That is, we have to prove

$$\sum_j \left(\delta p_j dq_j - \delta q_j dp_j\right) = \sum_j \left(\delta P_j dQ_j - \delta Q_j dP_j\right) \qquad \text{(i)}$$

To prove this result, we start from Hamilton's canonical equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \text{ or, } dq_j = \frac{\partial H}{\partial p_j}dt$$

and

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \text{ or, } dp_j = -\frac{\partial H}{\partial q_j}dt$$

Similarly,

$$dQ_j = \frac{\partial H}{\partial P_j}dt \text{ and } dP_j = -\frac{\partial H}{\partial Q_j}dt$$

Using these results, we can write

$$\sum_j \left[\delta p_j\left(dq_j - \frac{\partial H}{\partial p_j}dt\right) - \delta q_j\left(dp_j + \frac{\partial H}{\partial q_j}dt\right)\right] = 0 \qquad \text{(ii)}$$

The expression (ii) is justified since the delta variations are arbitrary; for the equation to be satisfied their coefficients must vanish. Now, Equation (ii) can be written as:

$$\sum_j \left(\delta p_j dq_j - \delta q_j dp_j\right) - \sum_j \left(\frac{\partial H}{\partial p_j}\delta p_j + \frac{\partial H}{\partial q_j}\delta q_j\right) dt = 0$$

$$\sum_j \left(\delta p_j dq_j - \delta q_j dp_j\right) - \delta H dt = 0 \tag{iii}$$

Now, for a transformation in which the generating function F is independent of time, we can have $H' = H$, through the same procedure as above we can have

$$\sum_j \left(\delta P_j dQ_j - \delta Q_j dP_j\right) - \delta H dt = 0 \tag{iv}$$

From the equations (iii) and (iv), we get

$$\sum_j \left(\delta p_j dq_j - \delta q_j dp_j\right) = \sum_j \left(\delta P_j dQ_j - \delta Q_j dP_j\right)$$

Hence, proved.

EXAMPLE 4.12 Show that the transformation $P = q\cot p$ and $Q = \log\left(\frac{\sin p}{q}\right)$ is canonical and the generating function is $F = e^{-Q}\left(1 - q^2 e^{2Q}\right)^{1/2} + q\sin^{-1}\left(qe^Q\right)$.

Solution: To prove that the transformation is canonical, we will show that $(PdQ - pdq)$ is an exact differential. Given that,

$$Q = \log\left(\frac{\sin p}{q}\right) \tag{i}$$

$$P = q\cot p \tag{ii}$$

Now,
$$dQ = \frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp$$

$$= \frac{\partial}{\partial q}\left[\log\left(\frac{\sin p}{q}\right)\right]dq + \frac{\partial}{\partial p}\left[\log\left(\frac{\sin p}{q}\right)\right]dp$$

$$= -\frac{1}{\left(\frac{\sin p}{q}\right)}\frac{\sin p}{q^2}dq + \frac{1}{\left(\frac{\sin p}{q}\right)}\frac{\cos p}{q}dp$$

$$= -\frac{dq}{q} + (\cot p)dp$$

$$PdQ - pdq = q\cot p\left[-\frac{dq}{q} + (\cot p)dp\right] - pdq$$

$$= -(\cot p)dq + q\left(\cot^2 p\right)dp - pdq$$

$$= -\left[(p + \cot p)dq\right] + q\left(\cot^2 p\right)dp$$

$$= -\left[(p + \cot p)dq - q\left(\cot^2 p\right)dp\right]$$

$$= -\left[\frac{\partial}{\partial q}(qp + q\cot p)dq + \frac{\partial}{\partial p}(qp + q\cot p)dp\right]$$

$$= -\left[\frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial p}dp\right] \tag{iii}$$

This is an exact differential with $F = (qp + q\cot p)$ and hence the transformation is canonical.

Now, let us try the second part of the problem. Here F is the generating function of the transformation.

$$F = (qp + q\cot p) \tag{iv}$$

We have $\quad Q = \log\left(\frac{\sin p}{q}\right)$ or, $\sin p = qe^Q$.

Then, $\quad \cos p = \sqrt{1 - q^2 e^{2Q}}$ and $\cot p = \frac{\sqrt{1 - q^2 e^Q}}{qe^Q}$ (v)

Using this in (iv), we get

$$F = \left(q\sin^{-1}\left(qe^Q\right) + q\frac{\sqrt{1 - q^2 e^Q}}{qe^Q}\right)$$

$$= \left(q\sin^{-1}\left(qe^Q\right) + e^Q\sqrt{1 - q^2 e^Q}\right) \tag{vi}$$

This is the generating function and is of the first kind, where, $F = F_1(q, Q)$.

EXAMPLE 4.13 Show that the transformation $Q = \frac{1}{p}$ and $P = qp^2$ is canonical by verifying the bilinear invariance.

Solution: The bilinear invariance is expressed as

$$\sum_j \left(\delta p_j dq_j - \delta q_j dp_j\right) = \sum_j \left(\delta P_j dQ_j - \delta Q_j dP_j\right) \tag{i}$$

Gives, the transformation equations as

$$Q = \frac{1}{p} \text{ and } P = qp^2 \tag{ii}$$

From this transformation equation, first we calculate the quantity on the right hand side of equation (i). For that, we have

$$dQ = \frac{\partial}{\partial q}\left(\frac{1}{p}\right)dq + \frac{\partial}{\partial p}\left(\frac{1}{p}\right)dp = -\frac{dp}{p^2}$$

$$\delta Q = \frac{\partial}{\partial q}\left(\frac{1}{p}\right)\delta q + \frac{\partial}{\partial p}\left(\frac{1}{p}\right)\delta p = -\frac{\delta p}{p^2}$$

In a similar way,

$$dP = \frac{\partial}{\partial q}\left(qp^2\right)dq + \frac{\partial}{\partial p}\left(qp^2\right)dp = p^2 dq + 2qpdp$$

and $$\delta P = \frac{\partial}{\partial q}(qp)\delta q + \frac{\partial}{\partial p}(qp)\delta p = p^2\delta q + 2qp\delta p$$

Now, the RHS of the equation (i) becomes

$$\delta PdQ - \delta QdP = \left(p^2\delta q + 2qp\delta p\right)\left(-\frac{dp}{p^2}\right) - \left(-\frac{\delta p}{p^2}\right)\left(p^2 dq + 2qpdp\right)$$
$$= \delta pdq - \delta qdp \tag{iii}$$

This is the bilinear form of invariant and therefore, the transformation is canonical.

EXAMPLE 4.14 Find the generating function for the transformation, $q = PQ^2$ and $p = \frac{1}{Q}$.

Solution: The generating function can be easily determined through the exact differential condition. Given that

$$q = PQ^2 \text{ and } p = \frac{1}{Q} \tag{i}$$

Then, $$dq = \frac{\partial q}{\partial Q}dQ + \frac{\partial q}{\partial P}dP$$
$$= \frac{\partial}{\partial Q}\left(PQ^2\right)dQ + \frac{\partial}{\partial P}\left(PQ^2\right)dP$$
$$= 2PQdQ + Q^2 dP \tag{ii}$$

Now, $$pdq = \frac{1}{Q}\left(2PQdQ + Q^2 dP\right) = 2PdQ + QdP \tag{iii}$$

The condition that must be satisfied for a canonical transformation is that $pdq - PdQ$ must be an exact differential. Now,

$$pdq - PdQ = 2PdQ + QdP - PdQ = PdQ + QdP$$

$$= \frac{\partial}{\partial Q}(QP)dQ + \frac{\partial}{\partial P}(QP)dp = \frac{\partial F}{\partial Q}dQ + \frac{\partial F}{\partial P}dp$$

$$= dF, \text{ an exact differential.}$$

where $F = PQ$ is the generating function. Note that this expression involves only the new set of variables. But a generating function must involve both the old and new variables. Therefore, in the expression of F, we substitute for P from equation (i). Then, we get

$$F = \frac{q}{Q} \tag{iv}$$

Hence shown.

EXAMPLE 4.15 A transformation between an old and new sets of coordinates is defined by, $P = 2\left(1 + q^{1/2}\cos p\right)q^{1/2}\sin p$ and $Q = \log\left(1 + q^{1/2}\cos p\right)$. Show that the transformation is canonical. Also find the generating function.

Solution: The condition for a transformation to be canonical, $pdq - PdQ$ must be an exact differential. Given that

$$P = 2\left(1 + q^{1/2}\cos p\right)q^{1/2}\sin p \tag{i}$$

$$Q = \log\left(1 + q^{1/2}\cos p\right) \tag{ii}$$

Now, $$dQ = \frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp$$

$$= \frac{q^{-1/2}\cos p}{2\left(1 + q^{1/2}\cos p\right)}dq - \frac{q^{1/2}\sin p}{\left(1 + q^{1/2}\cos p\right)}$$

$$= \frac{(\cos p)dq - 2q(\sin p)dp}{2q^{1/2}\left(1 + q^{1/2}\cos p\right)} \tag{iii}$$

Now, $$pdq - PdQ = pdq - 2\left(1 + q^{1/2}\cos p\right)q^{1/2}\sin p.\frac{(\cos p)dq - 2q(\sin p)dp}{2q^{1/2}\left(1 + q^{1/2}\cos p\right)}$$

$$= \left(p - \frac{1}{2}\sin 2p\right)dq + q(1 - \cos 2p)dp$$

$$= pdq - (\sin p \cos p)dq + \left(2q\sin^2 p\right)dp$$

$$= d\left[q\left(p - \frac{1}{2}\sin 2p\right)\right] \tag{iv}$$

This is an exact differential and hence, the transformation is canonical.
To determine the generating function, consider

$$Q = \log\left(1 + q^{1/2}\cos p\right)$$

or $$e^Q = \left(1 + q^{1/2}\cos p\right) \text{ so that } q = \frac{\left(e^Q - 1\right)^2}{\cos^2 p} \tag{v}$$

Using (v) in the equation (i), we get

$$P = 2\left[1 + \left(\frac{\left(e^Q - 1\right)}{\cos p}\right)\cos p\right]\left(\frac{\left(e^Q - 1\right)}{\cos p}\right)\sin p$$

$$= 2e^Q(e^Q - 1)\tan p \tag{vi}$$

We have the transformation equations generated by the function F_3 as

$$q = -\frac{\partial F_3}{\partial p} \text{ and } P = -\frac{\partial F_3}{\partial Q} \tag{vii}$$

From the equations, (v), (vi) and (vii), we get

$$\frac{\partial F_3}{\partial p} = -\frac{\left(e^Q - 1\right)^2}{\cos^2 p} = -\left(e^Q - 1\right)^2 \sec^2 p \tag{viii}$$

and $$\frac{\partial F_3}{\partial Q} = -2e^Q\left(e^Q - 1\right)\tan p \tag{ix}$$

On integration (viii) becomes

$$F_3 = -\int \left(e^Q - 1\right)^2 \left(\sec^2 p\right)dp + \text{constant}$$

$$= -\left(e^Q - 1\right)^2 \tan p + \text{constant}$$

For particular cases, the constant of integration can be equal to zero and therefore,

$$F_3 = -\left(e^Q - 1\right)^2 \tan p \tag{x}$$

Again, integrating (ix), we get

$$F_3 = -\int 2e^Q\left(e^Q - 1\right)(\tan p)dQ + \text{constant}$$

$$F_3 = -\left(e^Q - 1\right)^2 (\tan p) + \text{constant}$$

Again, for particular cases, we can neglect the constant of integration and, therefore,

$$F_3 = -\left(e^Q - 1\right)^2 \tan p \tag{xi}$$

Equations (x) and (xi) gives the same expression for F_3 and hence, the generating function is $F_3 = -\left(e^Q - 1\right)^2 \tan p$.

EXAMPLE 4.16 Find the values of α and β such that the transformation given by the equations, $Q = q^\alpha \cos\beta p$ and $P = q^\alpha \sin\beta p$ represent a canonical transformation. Find the generating function F_3.

Solution: Given that

$$Q = q^\alpha \cos\beta p \text{ and } P = q^a \sin\beta p \tag{i}$$

Now, $$dQ = aq^{a-1}\cos\beta p dq - \beta q^\alpha \sin\beta p dp \tag{ii}$$

For the transformation to be canonical, $(pdq - PdQ)$ must be an exact differential. That is,

$$dF_3 = (pdq - PdQ)$$

Now, $$pdq - PdQ = pdq - \left(q^\alpha \sin\beta p\right)\left(aq^{\alpha-1}\cos\beta p dq - \beta q^\alpha \sin\beta p dp\right)$$

$$= \left(p - \alpha q^{2\alpha-1}\sin\beta p\cos\beta p\right)dq - \beta q^{2\alpha}\sin^2\beta p dp \tag{iii}$$

The Euler reciprocity relation for a function F is, if $dF = Mdx + Ndy$, then $\left(\frac{\partial M}{\partial y}\right)_x = \left(\frac{\partial N}{\partial x}\right)_y$.

Then, the right hand side of equation (iii) is an exact differential, if

$$\frac{\partial}{\partial p}\left(p - \alpha q^{2\alpha-1}\sin\beta p\cos\beta p\right) = \frac{\partial}{\partial q}\left(\beta q^{2\alpha}\sin^2\beta p\right)$$

or $$\left[1 - \alpha\beta q^{2\alpha-1}\left(\cos^2\beta p - \sin^2\beta p\right)\right] = 2\alpha\beta q^{2\alpha-1}\sin^2\beta p$$

$$\left[1 - \alpha\beta q^{2\alpha-1}\left(1 - 2\sin^2\beta p\right)\right] = 2\alpha\beta q^{2\alpha-1}\sin^2\beta p$$

that is, $$\alpha\beta q^{2\alpha-1} = 1$$

This is possible, if $2\alpha - 1 = 0$ and $\alpha\beta = 1$

so that $$\alpha = \frac{1}{2} \text{ and } \beta = 2 \tag{iv}$$

Then we get, $$Q = q^{1/2}\cos 2p \text{ and } P = q^{1/2}\sin 2p \tag{v}$$

Substituting (v) in equation (iii), we get

$$dF_3 = \left(p - \frac{1}{2}q^0 \sin 2p \cos 2p\right)dq - 2q^1 \sin^2 2p dp$$

$$= \left(p - \frac{1}{2}\sin 2p \cos 2p\right)dq - 2q \sin^2 2p dp$$

$$= d\left(pq - \frac{1}{2}q \sin 2p \cos 2p\right) \quad \text{(vi)}$$

Therefore, the generating function is

$$F_3 = pq - \frac{1}{2}q \sin 2p \cos 2p \quad \text{(vii)}$$

Equation (vii) is the required result.

EXAMPLE 4.17 The generating function of a canonical transformation is given by, $F_1 = \frac{1}{2}q^2 \cot Q$. Show that, $p^2 + q^2 = 2P$.

Solution: Given that the generating function is

$$F_1 = \frac{1}{2}q^2 \cot Q \quad \text{(i)}$$

Using the transformation equations of generating function F_1, we can write

$$p = \frac{\partial F_1}{\partial q} = q \cot Q \quad \text{(ii)}$$

and $$P = -\frac{\partial F_1}{\partial Q} = \frac{1}{2}q^2 cosec^2 Q = \frac{q^2}{2\sin^2 Q} \quad \text{(iii)}$$

From (iii) we get, $q^2 = 2P\sin^2 Q$

or $$q = \left(\sqrt{2P}\right)\sin Q \quad \text{(iv)}$$

Using (iv) in (ii), we can have

$$p = \left(\sqrt{2P}\right)\sin Q \cot Q = \left(\sqrt{2P}\right)\cos Q \quad \text{(v)}$$

Squaring and adding (iv) and (v), we get

$$p^2 + q^2 = 2P$$

Hence, proved.

EXAMPLE 4.18 Show that the transformation from an old set of variables to a new set of variables is canonical if there exists a function F such that, $\frac{dF}{dt} = L - L'$, where L and L' are the Lagrangians in the old and new sets of coordinates.

Solution: We know that, under a canonical transformation, Hamilton's principle remains unchanged. Therefore, we must have

$$\delta\int_{t_1}^{t_2} L dt = 0 \tag{i}$$

and $$\delta\int_{t_1}^{t_2} L' dt = 0 \tag{ii}$$

Subtracting (ii) from (i), we get

$$\delta\int_{t_1}^{t_2} (L - L') dt = 0 \tag{iii}$$

Given that $L - L' = \dfrac{dF}{dt}$ and, therefore,

$$\delta\int_{t_1}^{t_2} \frac{dF}{dt} dt = \delta\left[F(t_2) - F(t_1)\right] dt = 0 \tag{iv}$$

From the definition for the generating function, we have $\delta\left[F(q,p,t)\right]_{t_1}^{t_2} = 0$ and it follows that the transformation is canonical.

EXAMPLE 4.19 Show that the Poisson bracket is invariant under a canonical transformation.

Solution: The invariance of Poisson bracket under a canonical transformation can be represented mathematically as

$$[F,G]_{q,p} = [F,G]_{Q,P} \tag{i}$$

Now, $$[F,G]_{Q,P} = \sum_j \left(\frac{\partial F}{\partial Q_j}\frac{\partial G}{\partial P_j} - \frac{\partial F}{\partial P_j}\frac{\partial G}{\partial Q_j} \right)$$

$$= \sum_{j,i} \left[\frac{\partial F}{\partial Q_j}\left(\frac{\partial G}{\partial q_i}\frac{\partial q_i}{\partial P_j} - \frac{\partial G}{\partial p_i}\frac{\partial p_i}{\partial P_j} \right) - \frac{\partial F}{\partial P_j}\left(\frac{\partial G}{\partial q_i}\frac{\partial q_i}{\partial Q_j} - \frac{\partial G}{\partial p_i}\frac{\partial p_i}{\partial Q_j} \right) \right]$$

$$= \sum_i \frac{\partial G}{\partial q_i} \sum_j \left(\frac{\partial F}{\partial Q_j}\frac{\partial q_i}{\partial P_j} - \frac{\partial F}{\partial P_j}\frac{\partial q_i}{\partial Q_j} \right) + \sum_i \frac{\partial G}{\partial p_i} \sum_j \left(\frac{\partial F}{\partial Q_j}\frac{\partial p_i}{\partial P_j} - \frac{\partial F}{\partial P_j}\frac{\partial p_i}{\partial Q_j} \right)$$

$$= \sum_i \left\{ \frac{\partial G}{\partial q_i}[F,q_i]_{Q,P} + \frac{\partial G}{\partial p_i}[F,p_i]_{Q,P} \right\} \tag{ii}$$

Since the Poisson brackets are noncommutative

$$[F,q_i]_{Q,P} = -[q_i,F]_{Q,P}$$

$$= -\sum_k\left(\frac{\partial q_i}{\partial Q_k}\frac{\partial F}{\partial P_k} - \frac{\partial q_i}{\partial P_k}\frac{\partial F}{\partial Q_k}\right)$$

$$= -\sum_{k,l}\left[\frac{\partial q_i}{\partial Q_k}\left(\frac{\partial F}{\partial q_l}\frac{\partial q_l}{\partial P_k} - \frac{\partial F}{\partial p_l}\frac{\partial p_l}{\partial P_k}\right) - \frac{\partial q_i}{\partial P_k}\left(\frac{\partial F}{\partial q_l}\frac{\partial q_l}{\partial Q_k} - \frac{\partial F}{\partial p_l}\frac{\partial p_l}{\partial Q_k}\right)\right]$$

$$= -\sum_l \frac{\partial F}{\partial q_l}\sum_k\left(\frac{\partial q_i}{\partial Q_k}\frac{\partial q_l}{\partial P_k} - \frac{\partial q_i}{\partial P_k}\frac{\partial q_l}{\partial Q_k}\right) + \sum_l \frac{\partial F}{\partial p_l}\sum_k\left(\frac{\partial q_i}{\partial Q_k}\frac{\partial p_l}{\partial P_k} - \frac{\partial q_i}{\partial P_k}\frac{\partial p_l}{\partial Q_k}\right)$$

$$= -\sum_l\left\{\frac{\partial F}{\partial q_l}[q_i,q_l]_{Q,P} + \frac{\partial F}{\partial p_l}[q_i,p_l]_{Q,P}\right\}$$

$$= -\sum_l \frac{\partial F}{\partial p_l}[q_i,p_l]_{Q,P} = -\sum_l \frac{\partial F}{\partial p_l}\delta_{il} = -\frac{\partial F}{\partial p_i} \quad \text{(iii)}$$

Similarly, we can show that

$$[F,p_i]_{Q,P} = \frac{\partial F}{\partial q_i} \quad \text{(iv)}$$

Using (iii) and (iv) in equation (ii), we get

$$[F,G]_{Q,P} = \sum_i\left[-\frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i} + \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}\right] = [F,G]_{q,p} \quad \text{(v)}$$

Therefore, the Poisson bracket is invariant under a canonical transformation.

EXAMPLE 4.20 Show that the transformation given by $q = (\sqrt{2P})\sin Q$ and $p = (\sqrt{2P})\cos Q$ is canonical using Poisson bracket method.

Solution: Given that

$$q = (\sqrt{2P})\sin Q \quad \text{and} \quad p = (\sqrt{2P})\cos Q \quad \text{(i)}$$

These equations can be written as

$$\tan Q = \frac{q}{p} \quad \text{(ii)}$$

and $$P = \frac{1}{2}\left(q^2 + p^2\right) \quad \text{(iii)}$$

From the properties of the Poisson brackets, we have

$$[Q,Q] = [P,P] = 0 \quad \text{and} \quad [Q,P] = 1 \quad \text{(iv)}$$

From (ii) and (iii), by differentiating with respect to q we get

$$\sec^2 Q \frac{\partial Q}{\partial q} = \frac{1}{p} \text{ and } \frac{\partial P}{\partial q} = q \tag{v}$$

Similarly, by differentiating with respect to p we get

$$\sec^2 Q \frac{\partial Q}{\partial p} = -\frac{q}{p^2} \text{ and } \frac{\partial P}{\partial p} = p \tag{vi}$$

The Poisson bracket between Q and P is defined as

$$[Q,P]_{q,p} = \left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right)$$

$$= \left(\frac{1}{p\sec^2 Q}p + \frac{q}{p^2 \sec^2 Q}q\right)$$

$$= \frac{1}{\sec^2 Q}\left(1+\frac{q^2}{p^2}\right) = \cos^2 Q\left(1+\frac{q^2}{p^2}\right) \tag{vii}$$

Now, substitute for q and p from (i) to get

$$[Q,P]_{q,p} = \cos^2 Q\left(1+\frac{q^2}{p^2}\right) = \cos^2 Q\left(1+\frac{\sin^2 Q}{\cos^2 Q}\right)$$

$$= \cos^2 Q + \sin^2 Q = 1$$

Therefore, the transformation is canonical.

EXAMPLE 4.21 A canonical transformation is given by, $Q = \ln\left(\sin\frac{p}{q}\right)$ and $P = q\cot p$. Find the Poisson bracket $[Q,P]_{q,p}$. Also show that $\left[H,[Q,P]\right] = 0$.

Solution: Given that

$$Q = \ln\left(\sin\frac{p}{q}\right) \text{ and } P = q\cot p \tag{i}$$

This can be written as,

$$e^Q = \sin\frac{p}{q} \tag{ii}$$

and

$$P = q\cot p \tag{iii}$$

Since Q and P are canonical, we have

$$[Q,Q]_{q,p} = [P,P]_{q,p} = 0 \text{ and } [Q,P]_{q,p} = 1$$

In the present problem we need to prove that; $[Q,P]_{q,p}=1$

Differentiating (ii) with respect to q and p, we get

$$e^Q\frac{\partial Q}{\partial q}=-\frac{\sin p}{q^2}$$

and $$e^Q\frac{\partial Q}{\partial p}=\frac{\cos p}{q}$$

Since $e^Q=\sin\frac{p}{q}$, the above two expressions become

$$\frac{\partial Q}{\partial q}=-\frac{1}{q} \text{ and } \frac{\partial Q}{\partial p}=\cot p \qquad \text{(iv)}$$

Again, differentiating (iii) with respect to q and p, we get

$$\frac{\partial P}{\partial q}=\cot p \text{ and } \frac{\partial P}{\partial p}=-q cosec^2 p \qquad \text{(v)}$$

The Poisson bracket $[Q,P]_{q,p}$ is defined as;

$$[Q,P]_{q,p}=\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

Using (iv) and (v), this can be written as

$$[Q,P]_{q,p}=\left(-\frac{1}{q}\right)\left(-q cosec^2 p\right)-\cot p\cot p$$

$$=cosec^2 p-\cot^2 p=1 \qquad \text{(vi)}$$

Now, $$[H,[Q,P]]=[H,1]=0$$

This is evident since the Poisson bracket of a function with a constant is zero.

EXAMPLE 4.22 Show that the transformation given by $Q=e^{-q}\sqrt{1-p^2e^{2q}}$ and $P=\tan^{-1}\left(\frac{e^{-q}\sqrt{1-p^2e^{q}}}{p}\right)$ is canonical by using Poisson brackets.

Solution: The transformation equations are given by

$$Q=e^{-q}\sqrt{1-p^2e^{2q}} \text{ and } P=\tan^{-1}\left(\frac{e^{-q}\sqrt{1-p^2e^{q}}}{p}\right) \qquad \text{(i)}$$

This can be rearranged to get

$$Q=\sqrt{e^{-2q}-p^2} \text{ and } \tan P=\frac{Q}{p} \qquad \text{(ii)}$$

Using (ii) we can write

$$1+\tan^2 P = 1+\frac{Q^2}{p^2}$$

or $$\sec^2 P = \frac{Q^2+p^2}{p}$$

or $$\cos^2 P = \frac{p^2}{Q^2+p^2} \text{ and } \cos P = \frac{p}{\sqrt{Q^2-p^2}}$$

Substituting for Q from (ii), we get

$$\cos P = \frac{p}{\sqrt{e^{-2q}-p^2+p^2}} = pe^q$$

Now, the transformation equations are modified as

$$Q = \sqrt{e^{-2q}-p^2} \text{ and } P = \cos^{-1}\left(pe^q\right) \tag{iii}$$

To show that the transformation is canonical, we need to prove the relation, $[Q,P]_{q,p} = 1$. Now, let us take the derivatives of Q and P with respect to q and p. That is,

$$\frac{\partial Q}{\partial q} = \frac{-e^{-2q}}{\sqrt{e^{-2q}-p^2}} \text{ and } \frac{\partial Q}{\partial p} = \frac{-p}{\sqrt{e^{-2q}-p^2}} \tag{iv}$$

Similarly, $$\frac{\partial P}{\partial q} = \frac{-p}{\sqrt{e^{-2q}-p^2}} \text{ and } \frac{\partial P}{\partial p} = \frac{-1}{\sqrt{e^{-2q}-p^2}} \tag{v}$$

Now, the Poisson bracket is defined as

$$[Q,P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

Using (iv) and (v), we get

$$[Q,P]_{q,p} = \left(\frac{-e^{-2q}}{\sqrt{e^{-2q}-p^2}}\right)\left(\frac{-1}{\sqrt{e^{-2q}-p^2}}\right) - \left(\frac{-p}{\sqrt{e^{-2q}-p^2}}\right)\left(\frac{-p}{\sqrt{e^{-2q}-p^2}}\right)$$

$$= \frac{e^{-2q}}{e^{-2q}-p^2} - \frac{p^2}{e^{-2q}-p^2} = \frac{e^{-2q}-p^2}{e^{-2q}-p^2} = 1$$

Further, $[Q,Q]_{q,p} = [P,P]_{q,p} = 0$ and hence, the transformation is canonical.

EXAMPLE 4.23 Using Poisson bracket, show that the transformation given by $Q=\ln\left(1+q^{\frac{1}{2}}\cos p\right)$ and $P=2\left(1+q^{\frac{1}{2}}\cos p\right)\sin p$ is canonical.

Solution: Given that

$$Q=\ln\left(1+q^{\frac{1}{2}}\cos p\right) \tag{i}$$

and $$q^{\frac{1}{2}}P=2\left(1+q^{\frac{1}{2}}\cos p\right)\sin p \tag{ii}$$

Equation (ii) can be modified as

$$P=2q^{\frac{1}{2}}\sin p+q\sin 2p \tag{iii}$$

To show that the transformation is canonical, we need to prove the relation, $[Q,P]_{q,p}=1$.

Now, take the derivative of (i) with respect to q and p. Then,

$$\frac{\partial Q}{\partial q}=\frac{1}{2}\frac{q^{-\frac{1}{2}}\cos p}{\left(1+q^{\frac{1}{2}}\cos p\right)} \text{ and, } \frac{\partial Q}{\partial p}=-\frac{q^{\frac{1}{2}}\sin p}{\left(1+q^{\frac{1}{2}}\cos p\right)} \tag{iv}$$

Taking the derivative of (iii) with respect to q and p, we get

$$\frac{\partial P}{\partial q}=q^{-\frac{1}{2}}\sin p+\sin 2p \text{ and, } \frac{\partial P}{\partial p}=2q^{\frac{1}{2}}\cos p+2q\cos 2p \tag{v}$$

Now, $$[Q,P]_{q,p}=\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \tag{vi}$$

Using the above derivatives (iv) and (v) in equation (vi), we get

$$[Q,P]_{q,p}=\left[\frac{1}{2}\frac{q^{-\frac{1}{2}}\cos p}{\left(1+q^{\frac{1}{2}}\cos p\right)}\right]\left[2q^{\frac{1}{2}}\cos p+2q\cos 2p\right]$$

$$-\left[-\frac{q^{\frac{1}{2}}\sin p}{\left(1+q^{\frac{1}{2}}\cos p\right)}\right]\left[q^{-\frac{1}{2}}\sin p+\sin 2p\right]$$

$$=\frac{1}{\left(1+q^{\frac{1}{2}}\cos p\right)}\left(\cos^2 p+q^{\frac{1}{2}}\cos 2p\cos p+\sin^2 p+q^{\frac{1}{2}}\sin 2p\sin p\right)$$

$$= \frac{\left[1+q^{1/2}\left(\cos 2p\cos p+\sin 2p\sin p\right)\right]}{\left(1+q^{1/2}\cos p\right)}$$

$$= \frac{\left(1+q^{1/2}\cos p\right)}{\left(1+q^{1/2}\cos p\right)} = 1$$

Hence, the transformation is canonical.

EXAMPLE 4.24 Using Poisson bracket, obtain the values of α and β such that the transformation given by $Q = q^{\alpha}\cos\beta p$ and $P = q^{\alpha}\sin\beta p$. Also find the generating function $F_3(p,Q)$.

Solution: For the transformation to be canonical, we must have, $[Q,P]_{q,p} = 1$. Now, the Poisson bracket is defined as

$$[Q,P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \tag{i}$$

Now, let us find the partial derivatives of Q and P with respect to q and p. Given that

$$Q = q^{\alpha}\cos\beta p \tag{ii}$$

and $$P = q^{\alpha}\sin\beta p \tag{iii}$$

From (ii) we have

$$\frac{\partial Q}{\partial q} = \alpha q^{\alpha-1}\cos\beta p \quad \text{and} \quad \frac{\partial Q}{\partial p} = -\beta q^{\alpha}\sin\beta p \tag{iv}$$

and from (iii), we get

$$\frac{\partial P}{\partial q} = \alpha q^{\alpha-1}\sin\beta p \quad \text{and} \quad \frac{\partial P}{\partial p} = \beta q^{\alpha}\cos\beta p \tag{v}$$

Substituting (iv) and (v) in (i), we get

$$\begin{aligned}[Q,P]_{q,p} &= \left(\alpha q^{\alpha-1}\cos\beta p\right)\left(\beta q^{\alpha}\cos\beta p\right) - \left(-\beta q^{\alpha}\sin\beta p\right)\left(\alpha q^{\alpha-1}\sin\beta p\right)\\ &= \alpha\beta q^{2\alpha-1}\cos^2\beta p + \alpha\beta q^{2\alpha-1}\sin^2\beta p\\ &= \alpha\beta q^{2\alpha-1}\end{aligned}$$

For the transformation to be canonical

$$\alpha\beta q^{2\alpha-1} = 1 \tag{vi}$$

This is possible only if

$$2\alpha-1=0 \text{ and } \alpha\beta=1$$

Therefore, on solving, we get

$$\alpha=\frac{1}{2} \text{ and } \beta=2 \tag{vii}$$

Therefore, the transformation equations are

$$Q=q^{1/2}\cos 2p \text{ and } P=q^{1/2}\sin 2p \tag{viii}$$

Now, let us find the generating function $F_3(p,Q)$. For that we make use of the transformation equations of F_3 given by

$$\frac{\partial F_3}{\partial p}=-q \text{ and } \frac{\partial F_3}{\partial Q}=-P \tag{ix}$$

Integrating the first expression, we get

$$F_3=-\int q\,dp \tag{x}$$

From, $Q=q^{1/2}\cos 2p$, we get $q=Q^2\sec^2 2p$ and use this in (x) to get

$$F_3=-\int Q^2\sec^2 2p\,dp=-\frac{1}{2}Q^2\tan^2 2p \tag{xi}$$

Equation (xi) gives the generating function.

EXAMPLE 4.25 Using Poisson bracket, show that the transformation given by, $Q=q\cos\alpha-p\sin\alpha$ and $P=q\sin\alpha+p\cos\alpha$ is a canonical transformation.

Solution: The given transformation equations are

$$Q=q\cos\alpha-p\sin\alpha \tag{i}$$

and

$$P=q\sin\alpha+p\cos\alpha \tag{ii}$$

For the transformation to be canonical, we must have, $[Q,P]_{q,p}=1$. Now, the Poisson bracket is defined as;

$$[Q,P]_{q,p}=\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \tag{iii}$$

From (i), by differentiating with respect to q and p, we get

$$\frac{\partial Q}{\partial q}=\cos\alpha \text{ and } \frac{\partial Q}{\partial p}=-\sin\alpha \tag{iv}$$

Similarly, by differentiating (ii) with respect to q and p, we get

$$\frac{\partial P}{\partial q}=\sin\alpha \text{ and } \frac{\partial P}{\partial p}=\cos\alpha \tag{v}$$

Using (iv) and (v) in (iii), we get

$$[Q,P]_{q,p} = \cos\alpha\cos\alpha + \sin\alpha\sin\alpha$$
$$= \cos^2\alpha + \sin^2\alpha = 1 \qquad \text{(vi)}$$

Therefore, the given transformation is canonical.

EXAMPLE 4.26 Show that, $(ad - bc) = 1$ for the transformation $Q = aq + bp$ and $p = cq + dp$ to be canonical.

Solution: The given transformation equations are

$$Q = aq + bp \qquad \text{(i)}$$

and
$$P = cq + dp \qquad \text{(ii)}$$

For the transformation to be canonical, we must have, $[Q,P]_{q,p} = 1$. Now, the Poisson bracket is defined as

$$[Q,P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \qquad \text{(iii)}$$

From (i), by differentiating with respect to q and p, we get

$$\frac{\partial Q}{\partial q} = a \quad \text{and} \quad \frac{\partial Q}{\partial p} = b \qquad \text{(iv)}$$

Similarly, by differentiating (ii) with respect to q and p, we get

$$\frac{\partial P}{\partial q} = c \quad \text{and} \quad \frac{\partial P}{\partial p} = d \qquad \text{(v)}$$

Using (iv) and (v) in (iii), we get

$$[Q,P]_{q,p} = ad - bc$$

For the transformation to be canonical this must be equal to unity. Therefore,

$$(ad - bc) = 1 \qquad \text{(vi)}$$

Hence, proved.

EXAMPLE 4.27 Express the Hamilton's canonical equations of motion in Poisson bracket form.

Solution: Consider a function defined by, $F = F(q_j, p_j, t)$

The total derivative of the function is

$$\frac{dF}{dt} = \sum_j \left(\frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial p_j}\dot{p}_j\right) + \frac{\partial F}{\partial t} \qquad \text{(i)}$$

Now, Hamilton's canonical equations are

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \text{ and } \dot{p}_j = -\frac{\partial H}{\partial q_j} \tag{ii}$$

Using (ii) in (i), we get

$$\frac{dF}{dt} = \sum_j \left(\frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial H}{\partial q_j} \right) + \frac{\partial F}{\partial t}$$

$$= [F, H]_{q_j, p_j} + \frac{\partial F}{\partial t} \tag{iii}$$

This is the equation of motion of F in Poisson bracket form. Note that if q_j and p_j do not depend explicitly on time, by replacing F with q_j and p_j, we get

$$\dot{q}_j = \left[q_j, H\right] \text{ and } \dot{p}_j = \left[p_j, H\right] \tag{iv}$$

Equation (iv) gives the canonical equation of motion in Poisson bracket form.

EXAMPLE 4.28 The Hamiltonian of a three dimensional harmonic isotropic oscillator is given by the expression $H = \frac{1}{2m}\sum_{n=1}^{3} p_n^2 + \frac{m\omega^2}{2}\sum_{n=1}^{3} q_n^2$ and consider the function $T_{ij} = p_i p_j + (m\omega)^2 q_i q_j$ for $i, j = 1,2,3$. Show that these functions are the constants of motion for all k and l.

Solution: Let us start with the general equation of motion of a function in terms of Poisson bracket. It is given by

$$\frac{dF}{dt} = [F, H]_{q_j, p_j} + \frac{\partial F}{\partial t} \tag{i}$$

Now for the function T_{ij} this becomes

$$\frac{dT_{ij}}{dt} = \left[T_{ij}, H\right] + \frac{\partial T_{ij}}{\partial t} \tag{ii}$$

Given that, $T_{ij} = p_i p_j + (m\omega)^2 q_i q_j$ and it does not have explicit time dependence. Hence, equation (ii) reduces to

$$\frac{dT_{ij}}{dt} = \left[T_{ij}, H\right] \tag{iii}$$

Now, let us calculate the Poisson bracket $\left[T_{ij}, H\right]_{q_j, p_j}$.

$$\left[T_{ij}, H\right] = \left[\left(p_i p_j + m^2\omega^2 q_i q_j\right), \frac{1}{2m}\sum_{n=1}^{3} p_n^2 + \frac{m\omega^2}{2}\sum_{n=1}^{3} q_n^2 \right]$$

$$= \frac{1}{2m}\sum_{n=1}^{3}\left[p_ip_j, p_np_n\right] + \frac{(m\omega)^2}{2}\frac{m\omega^2}{}\sum_{n=1}^{3}\left[q_iq_j, q_nq_n\right]$$

$$+\frac{(m\omega)^2}{2m}\sum_{n=1}^{3}\left[q_iq_j, p_np_n\right] + \frac{m\omega^2}{2}\sum_{n=1}^{3}\left[p_ip_j, q_nq_n\right] \quad \text{(iv)}$$

We have the general relations

$$\left[q_i, q_j\right] = \left[p_i, p_j\right] = 0 \text{ and } \left[q_i, p_j\right] = -\left[p_j, q_i\right] = \delta_{ij}$$

Using these relations, we can show that the first two terms in equation (iv) are equal to zero. (For the evaluation see the next step.)

Now, consider the Poisson bracket in the third term.

$$\left[q_iq_j, p_np_n\right] = \sum_k\left(\frac{\partial q_iq_j}{\partial q_k}\frac{\partial p_np_n}{\partial p_k} - \frac{\partial q_iq_j}{\partial p_k}\frac{\partial p_np_n}{\partial q_k}\right) \quad \text{(v)}$$

But $\frac{\partial p_np_n}{\partial q_k} = \frac{\partial p_n}{\partial q_k}p_n + p_n\frac{\partial p_n}{\partial q_k}$ and $\frac{\partial p_np_n}{\partial p_k} = \frac{\partial p_n}{\partial p_k}p_n + p_n\frac{\partial p_n}{\partial p_k}$ (vi)

This can be substituted in (v) to get

$$\left[q_iq_j, p_np_n\right] = \left[q_iq_j, p_n\right]p_n + p_n\left[q_iq_j, p_n\right] \quad \text{(vii)}$$

In the same way, we can prove that

$$\left[q_iq_j, p_n\right] = \left[q_i, p_n\right]q_j + q_i\left[q_j, p_n\right]$$
$$= \delta_{in}q_j + \delta_{jn}q_i \quad \text{(viii)}$$

Substituting this in (vii), we get

$$\left[q_iq_j, p_np_n\right] = \left(\delta_{in}q_j + \delta_{jn}q_i\right)p_n + p_n\left(\delta_{in}q_j + \delta_{jn}q_i\right)$$
$$= q_jp_i + q_ip_j + p_iq_j + p_jq_i \quad \text{(ix)}$$

For the last term also we can proceed through these steps and finally arrive at

$$\left[p_ip_j, q_nq_n\right] = -q_ip_j - q_jp_i - p_jq_i - p_iq_j \quad \text{(x)}$$

Using the equations, (ix) and (x) in (iv), we get

$$\left[T_{ij}, H\right] = 0 \quad \text{(xi)}$$

Therefore, T_{ij} is a constant of motion.

EXAMPLE 4.29 Show that the transformation on 2n-dimensional phase space associated with a coordinate transformation on configuration space, is given by; $q_i \to Q_i(q)$ and $p_i \to P_i(q,p) = \sum_j p_j \frac{\partial q_j}{\partial Q_i}$ is a canonical transformation.

Solution: We know that for a transformation to be canonical, the new variable must also satisfy the Poisson bracket relations as the old variables. That is,

$$[Q_i, Q_j] = [P_i, P_j] = 0 \quad \text{and} \quad [Q_i, P_j] = \delta_{ij} \tag{i}$$

Let us evaluate these in the present case. Given that

$$Q_i = q_i \quad \text{and} \quad P_i(q,p) = \sum_j p_j \frac{\partial q_j}{\partial Q_i} \tag{ii}$$

Now,
$$[Q_i, Q_j] = \sum_j \left(\frac{\partial Q_i}{\partial q_k}\frac{\partial Q_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k}\frac{\partial Q_j}{\partial q_k} \right) = 0 \tag{iii}$$

This follows because, the new coordinates Q's are independent of p.

Also,
$$[P_i, P_j] = \sum_j \left(\frac{\partial P_i}{\partial q_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial P_i}{\partial p_k}\frac{\partial P_j}{\partial q_k} \right) \tag{iv}$$

Now,
$$\frac{\partial P_i}{\partial q_k} = \sum_l p_l \frac{\partial^2 q_l}{\partial q_k \partial Q_i} \quad \text{and} \quad \frac{\partial P_i}{\partial p_k} = \frac{\partial q_k}{\partial Q_i}$$

Similarly,
$$\frac{\partial P_j}{\partial q_k} = \sum_l p_l \frac{\partial^2 q_l}{\partial q_k \partial Q_j} \quad \text{and} \quad \frac{\partial P_j}{\partial p_k} = \frac{\partial q_k}{\partial Q_j}$$

Using these expressions in (iv) we get

$$[P_i, P_j] = \sum_j \left[\left(\sum_l p_l \frac{\partial^2 q_l}{\partial q_k \partial Q_i} \right) \frac{\partial q_k}{\partial Q_j} - \frac{\partial q_k}{\partial Q_i} \left(\sum_l p_l \frac{\partial^2 q_l}{\partial q_k \partial Q_j} \right) \right]$$

$$= \sum_j \left[\left(\sum_l p_l \frac{\partial^2 q_l}{\partial Q_i \partial Q_j} \right) \frac{\partial q_k}{\partial q_k} - \frac{\partial q_k}{\partial q_k} \left(\sum_l p_l \frac{\partial^2 q_l}{\partial Q_j \partial Q_i} \right) \right]$$

$$= \sum_j \sum_l p_l \left(\frac{\partial^2 q_l}{\partial Q_i \partial Q_j} - \frac{\partial^2 q_l}{\partial Q_j \partial Q_i} \right) = 0 \tag{v}$$

Now, let us find $[Q_i, P_j]$.

$$[Q_i, P_j] = \sum_k \left(\frac{\partial Q_i}{\partial q_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k}\frac{\partial P_j}{\partial q_k} \right)$$

Since $\dfrac{\partial Q_i}{\partial p_k} = 0$ as Q is independent of p, we get

$$\left[Q_i, P_j\right] = \sum_k \left(\frac{\partial Q_i}{\partial q_k} \frac{\partial q_k}{\partial Q_j} - 0 \right) = \frac{\partial Q_i}{\partial Q_j} = \delta_{ij} \qquad \text{(vi)}$$

Therefore, the new variables, Q and P satisfies the fundamental Poisson bracket relations and hence, the transformation is canonical.

EXAMPLE 4.30 Using Poisson bracket, show that the transformation given by $Q = \left(\sqrt{2q}\right)e^{t} \cos p$ and $P = \left(\sqrt{2q}\right)e^{-t} \sin p$.

Solution: Given that;

$$Q = \left(\sqrt{2q}\right)e^{t} \cos p \qquad \text{(i)}$$

and

$$P = \left(\sqrt{2q}\right)e^{-t} \sin p \qquad \text{(ii)}$$

For a transformation to be canonical, we must have

$$[Q,P]_{q,p} = 1$$

Now, from (i) we get

$$\frac{\partial Q}{\partial q} = (2q)^{-1/2} e^{t} \cos p \text{ and } \frac{\partial Q}{\partial p} = -(2q)^{1/2} e^{t} \sin p \qquad \text{(iii)}$$

Similarly, from (ii) we can have

$$\frac{\partial P}{\partial q} = (2q)^{-1/2} e^{-t} \sin p \text{ and } \frac{\partial P}{\partial p} = (2q)^{1/2} e^{-t} \cos p \qquad \text{(iv)}$$

Now, $$[Q,P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

Using (iii) and (iv) the above expression becomes

$$[Q,P]_{q,p} = \left[(2q)^{-1/2} e^{t} \cos p\right]\left[(2q)^{1/2} e^{-t} \cos p\right] - \left[-(2q)^{1/2} e^{t} \sin p\right]\left[(2q)^{-1/2} e^{-t} \sin p\right]$$

$$= \cos^2 p + \sin^2 p = 1 \qquad \text{(v)}$$

Hence, the given transformation is a canonical transformation.

EXAMPLE 4.31 Using Poisson bracket, show that the transformation given by $Q = q\cos\theta - \dfrac{p}{m\omega}\sin\theta$ and $P = m\omega q \sin\theta + p\cos\theta$ is a canonical transformation (a) by using Poisson bracket, and (b) by direct method. Also find the generating function of the form $F_2(q, P,)$.

Solution:

(a) Given that

$$Q = q\cos\theta - \frac{p}{m\omega}\sin\theta \tag{i}$$

and $$P = m\omega q\sin\theta + p\cos\theta \tag{ii}$$

For the transformation to be canonical, we must have

$$[Q,P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1 \tag{iii}$$

From (i), by differentiating with respect to q and p, we get

$$\frac{\partial Q}{\partial q} = \cos\theta \text{ and } \frac{\partial Q}{\partial p} = -\frac{\sin\theta}{m\omega} \tag{iv}$$

Similarly, differentiating (ii) with respect to q and p, we get

$$\frac{\partial P}{\partial q} = m\omega\sin\theta \text{ and } \frac{\partial P}{\partial p} = \cos\theta \tag{v}$$

Using (iv) and (v), equation (iii) becomes

$$\begin{aligned}[Q,P]_{q,p} &= \cos\theta\cos\theta + \sin\theta\sin\theta \\ &= \cos^2\theta + \sin^2\theta = 1\end{aligned} \tag{vi}$$

Similarly, $$[Q,Q]_{q,p} = \left(\frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q}\right) = 0$$

Also, $$[P,P]_{q,p} = \left(\frac{\partial P}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P}{\partial q}\right) = 0$$

Therefore, the new variables, Q and P satisfy the fundamental Poisson bracket relations and hence, the transformation is canonical.

(b) In the direct method we show that the expression $(pdq - PdQ)$ is an exact differential.

From (i), $$Q = q\cos\theta - \frac{p}{m\omega}\sin\theta$$

or $$p = \frac{m\omega}{\sin\theta}(q\cos\theta - Q) = m\omega\left(q\cot\theta - \frac{Q}{\sin\theta}\right) \tag{vii}$$

Now use this expression in (ii) to substitute for p. Then we get

$$\begin{aligned}P &= m\omega q\sin\theta + p\cos\theta \\ &= m\omega q\sin\theta + \left[m\omega\left(q\cot\theta - \frac{Q}{\sin\theta}\right)\right]\cos\theta\end{aligned}$$

$$= m\omega q \sin\theta + m\omega q \cot\theta \cos\theta - m\omega Q \cot\theta$$

$$= m\omega q \sin\theta + m\omega q \frac{\cos^2\theta}{\sin\theta} - m\omega Q \cot\theta$$

$$= m\omega \frac{q}{\sin\theta}\left(\sin^2\theta + \cos^2\theta\right) - m\omega Q \cot\theta$$

$$= m\omega\left(\frac{q}{\sin\theta} - Q\cot\theta\right) \qquad \text{(viii)}$$

Using (vii) and (viii) the condition for exact differential can be written as

$$pdq - PdQ = m\omega\left(q\cot\theta - \frac{Q}{\sin\theta}\right)dq - m\omega\left(\frac{q}{\sin\theta} - Q\cot\theta\right)dQ$$

$$= d\left[\frac{1}{2}m\omega\left(q^2 + Q^2\right)\cot\theta - m\omega\frac{qQ}{\sin\theta}\right] \qquad \text{(ix)}$$

The RHS of the equation (ix) is an exact differential and hence, the given transformation is canonical.

(c) Now, let us consider the third part of the problem. The generating function of the first kind is given by

$$F_1(q,Q) = \frac{1}{2}m\omega\left(q^2 + Q^2\right)\cot\theta - m\omega\frac{qQ}{\sin\theta} \qquad \text{(x)}$$

The generating function of the second type, $F_2(q,P)$ can be obtained from $F_1(q,Q)$ by setting

$$F_2(q,P) = F_1(q,Q) + PQ$$

$$= \frac{1}{2}m\omega\left(q^2 + Q^2\right)\cot\theta - m\omega\frac{qQ}{\sin\theta} + PQ$$

$$= \frac{1}{2}m\omega\left(q^2 + Q^2\right)\cot\theta - m\omega\frac{qQ}{\sin\theta} + \left[m\omega\left(\frac{q}{\sin\theta} - Q\cot\theta\right)\right]Q$$

This can be simplified to get,

$$F_2(q,P) = \frac{1}{2}m\omega\left(q^2 - Q^2\right)\cot\theta \qquad \text{(xi)}$$

Since F_2 should be a function of q and P, we need to eliminate Q from the above expression using equation (viii). That is,

$$P = m\omega\left(\frac{q}{\sin\theta} - Q\cot\theta\right) \text{ or; } Q = \frac{q}{\cos\theta} - \frac{P}{m\omega}\tan\theta \qquad \text{(xii)}$$

Using this in (xi), we get

$$F_2(q,P)=\frac{1}{2}m\omega\left[q^2-\left(\frac{q}{\cos\theta}-\frac{P}{m\omega}\tan\theta\right)^2\right]\cot\theta$$

$$=\frac{1}{2}m\omega\left[q^2-\frac{q^2}{\cos^2\theta}+2\frac{q}{\cos\theta}\frac{P}{m\omega}\tan\theta-\frac{P^2}{m^2\omega^2}\tan^2\theta\right]\cot\theta$$

$$=\frac{qP}{\cos\theta}+\frac{1}{2}m\omega\left[\left(q^2\cot\theta-\frac{q^2\cot\theta}{\cos^2\theta}\right)-\frac{P^2}{m^2\omega^2}\tan\theta\right]$$

$$=\frac{qP}{\cos\theta}+\frac{1}{2}m\omega\left[q^2\left(\frac{\cos^2\theta-1}{\cos^2\theta}\right)\cot\theta-\frac{P^2}{m^2\omega^2}\tan\theta\right]$$

$$=\frac{qP}{\cos\theta}+\frac{1}{2}m\omega\left[q^2\left(-\frac{\sin^2\theta}{\cos^2\theta}\right)\cot\theta-\frac{P^2}{m^2\omega^2}\tan\theta\right]$$

$$=\frac{qP}{\cos\theta}-\frac{1}{2}m\omega\left(q^2\tan\theta+\frac{P^2}{m^2\omega^2}\tan\theta\right) \tag{xiii}$$

Equation (xiii) is the required result.

EXAMPLE 4.32 The canonical variables q and p in the above problem are supposed to be the canonical variables for a simple harmonic oscillator. If so, find the Hamiltonian H' in terms of the new variables, Q and P assuming that the parameter θ is some function of time. Also show that we can choose $\theta(t)$ such that $H'=0$.

Solution: In the above problem, we obtained the canonical variables q and p as

$$q=Q\cos\theta+\frac{P}{m\omega}\sin\theta \tag{i}$$

and $$p=P\cos\theta-m\omega Q\sin\theta \tag{ii}$$

Note that the same can be obtained through the transformation equations involving the generating function.

Now, the old and new Hamiltonians are related through the transformation equation;

$$H'(Q,P,t)=H(q,p,t)+\left(\frac{\partial F_2}{\partial t}\right)_{q,P} \tag{iii}$$

where, the generating function $F_2(q,P,t)$ is given by

$$F_2(q,P,t)=\frac{qP}{\cos\theta}-\frac{1}{2}m\omega\left(q^2+\frac{P^2}{m^2\omega^2}\right)\tan\theta \tag{iv}$$

We have, the Hamiltonian of a simple harmonic motion given as

$$H(q,p)=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$$

This can be rewritten in terms of the new canonical variables Q and P as

$$\begin{aligned}
H(q,p)&=\frac{1}{2m}(P\cos\theta-m\omega Q\sin\theta)^2+\frac{1}{2}m\omega^2\left(Q\cos\theta+\frac{P}{m\omega}\sin\theta\right)^2\\
&=\frac{1}{2m}\left(P^2\cos^2\theta+m^2\omega^2Q^2\sin^2\theta-2PQm\omega\sin\theta\cos\theta\right)\\
&\quad+\frac{1}{2}m\omega^2\left(Q^2\cos^2\theta+\frac{P^2}{m^2\omega^2}\sin^2\theta+2\frac{QP}{m\omega}\sin\theta\cos\theta\right)
\end{aligned}$$

This expression on further simplification reduces to

$$H(q,p)=\frac{P^2}{2m}+\frac{1}{2}m\omega^2Q^2=H(Q,P) \tag{v}$$

Now, let us find $\left(\frac{\partial F_2}{\partial t}\right)_{q,P}$ by differentiation equation (iv) with respect to time. Then

$$\begin{aligned}
\left(\frac{\partial F_2}{\partial t}\right)_{q,P}&=\frac{\partial}{\partial t}\left[\frac{qP}{\cos\theta}-\frac{1}{2}m\omega\left(q^2+\frac{P^2}{m^2\omega^2}\right)\tan\theta\right]\\
&=\frac{\partial}{\partial t}\left[qP\cos^{-1}\theta-\frac{1}{2}m\omega\left(q^2+\frac{P^2}{m^2\omega^2}\right)\tan\theta\right]\\
&=\left[-qP\cos^{-2}\theta(-\sin\theta)\dot{\theta}\right]-\frac{1}{2}m\omega\left(q^2+\frac{P^2}{m^2\omega^2}\right)(\sec^2\theta)\dot{\theta}\\
&=\left[\frac{(qP\sin\theta)\dot{\theta}}{\cos^2\theta}\right]-\frac{1}{2}m\omega\left(q^2+\frac{P^2}{m^2\omega^2}\right)\left(\frac{1}{\cos^2\theta}\right)\dot{\theta}\\
&=\left[qP\sin\theta-\frac{1}{2}m\omega\left(q^2+\frac{P^2}{m^2\omega^2}\right)\right]\left(\frac{\dot{\theta}}{\cos^2\theta}\right)
\end{aligned} \tag{vi}$$

Now substituting the expressions (i) and (ii) in (vi) for q and p respectively, we get

$$\begin{aligned}
\left(\frac{\partial F_2}{\partial t}\right)_{q,P}&=\left(Q\cos\theta+\frac{P}{m\omega}\sin\theta\right)P\sin\theta\left(\frac{\dot{\theta}}{\cos^2\theta}\right)\\
&\quad-\frac{1}{2}m\omega\left(\left(Q\cos\theta+\frac{P}{m\omega}\sin\theta\right)^2+\frac{P^2}{m^2\omega^2}\right)\left(\frac{\dot{\theta}}{\cos^2\theta}\right)
\end{aligned}$$

This can be expanded and simplified to get

$$\left(\frac{\partial F_2}{\partial t}\right)_{q,P} = -\left(\frac{P^2}{2m\omega} + \frac{1}{2}m\omega Q^2\right)\dot{\theta}$$

$$= -\left(\frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2\right)\left(\frac{\dot{\theta}}{\omega}\right) = -H(Q,P)\left(\frac{\dot{\theta}}{\omega}\right) \quad \text{(vii)}$$

The new Hamiltonian is

$$H'(Q,P,t) = H(q,p,t) + \left(\frac{\partial F_2}{\partial t}\right)_{q,P} = H(Q,P,t) + -H(Q,P,t)\left(\frac{\dot{\theta}}{\omega}\right) \quad \text{(viii)}$$

$$= H(Q,P,t) + -H(Q,P,t)\left(\frac{\dot{\theta}}{\omega}\right) = H(Q,P,t)\left[1 - \left(\frac{\dot{\theta}}{\omega}\right)\right] \quad \text{(ix)}$$

Now, the new Hamiltonian can be reduced to zero, if we take $\theta = \omega t$

EXAMPLE 4.33 Show that the following transformations are canonical by using Poisson brackets. $Q_1 = \frac{1}{\sqrt{2}}\left(q_1 + \frac{p_2}{m\omega}\right)$, $P_1 = \frac{1}{\sqrt{2}}(p_1 - m\omega q_2)$, $Q_2 = \frac{1}{\sqrt{2}}\left(q_1 - \frac{p_2}{m\omega}\right)$, $P_2 = \frac{1}{\sqrt{2}}(p_1 + m\omega q_2)$, where $m\omega$ is a constant. Also find a generating function $F(q_1, q_2, Q_1, P_2)$ for this transformation.

Solution: Given that

$$Q_1 = \frac{1}{\sqrt{2}}\left(q_1 + \frac{p_2}{m\omega}\right), \quad P_1 = \frac{1}{\sqrt{2}}(p_1 - m\omega q_2), \quad \text{(i)}$$

and
$$Q_2 = \frac{1}{\sqrt{2}}\left(q_1 - \frac{p_2}{m\omega}\right), \quad P_2 = \frac{1}{\sqrt{2}}(p_1 + m\omega q_2) \quad \text{(ii)}$$

In the first part of the solution let us evaluate the fundamental Poisson brackets to decide whether the transformation is canonical or not. That is,

(i) $$[Q_1, Q_2]_{q,p} = \left(\frac{\partial Q_1}{\partial q_1}\frac{\partial Q_2}{\partial p_1} - \frac{\partial Q_1}{\partial p_1}\frac{\partial Q_2}{\partial q_1}\right) + \left(\frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial p_2} - \frac{\partial Q_1}{\partial p_2}\frac{\partial Q_2}{\partial q_2}\right) = 0$$

(ii) $$[Q_1, P_1]_{q,p} = \left(\frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1}\frac{\partial P_1}{\partial q_1}\right) + \left(\frac{\partial Q_1}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2}\frac{\partial P_1}{\partial q_2}\right)$$

$$= \left(\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} - 0\right)\left(0 + \frac{1}{m\omega\sqrt{2}}\frac{m\omega}{\sqrt{2}}\right) = 1$$

(iii) $$[Q_1, P_2]_{q,p} = \left(\frac{\partial Q_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial Q_1}{\partial p_1}\frac{\partial P_2}{\partial q_1}\right) + \left(\frac{\partial Q_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial Q_1}{\partial p_2}\frac{\partial P_2}{\partial q_2}\right)$$

$$= \left(\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}} - 0\right)\left(0 - \frac{1}{m\omega\sqrt{2}}\frac{m\omega}{\sqrt{2}}\right) = 0$$

(iv) $$[Q_2, P_1]_{q,p} = \left(\frac{\partial Q_2}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial Q_2}{\partial p_1}\frac{\partial P_1}{\partial q_1}\right) + \left(\frac{\partial Q_2}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial Q_2}{\partial p_2}\frac{\partial P_1}{\partial q_2}\right)$$

$$= \left(\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}} - 0\right)\left(0 - \frac{1}{m\omega\sqrt{2}}\frac{m\omega}{\sqrt{2}}\right) = 0$$

(v) $$[Q_2, P_2]_{q,p} = \left(\frac{\partial Q_2}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1}\frac{\partial P_2}{\partial q_1}\right) + \left(\frac{\partial Q_2}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial Q_2}{\partial p_2}\frac{\partial P_2}{\partial q_2}\right)$$

$$= \left(\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}} - 0\right)\left(0 + \frac{1}{m\omega\sqrt{2}}\frac{m\omega}{\sqrt{2}}\right) = 1$$

(vi) $$[P_1, P_2]_{q,p} = \left(\frac{\partial P_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_1}{\partial p_1}\frac{\partial P_2}{\partial q_1}\right) + \left(\frac{\partial P_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_1}{\partial p_2}\frac{\partial P_2}{\partial q_2}\right) = 0$$

The fundamental Poisson brackets are verified and hence the transformation is canonical.

Now, to find the generating function, we use the exact differential condition of canonical transformation, that is, we evaluate the expression, $(p_1dq_1 + p_2dq_2 - P_1dQ_1 + Q_2dP_2)$. The last term is chosen so as to get the desired result.

From the given transformation equations, we can have

$$p_1 = \sqrt{2}P_2 - m\omega q_2,\ p_2 = m\omega\left(\sqrt{2}Q_1 - q_1\right) \tag{iii}$$

Similarly, $$P_1 = P_2 - \sqrt{2}m\omega q_2 \text{ and } Q_2 = \sqrt{2}q_1 - Q_1 \tag{iv}$$

Now, $$p_1dq_1 + p_2dq_2 - P_1dQ_1 + Q_2dP_2 = \left(\sqrt{2}P_2 - m\omega q_2\right)dq_1 + m\omega\left(\sqrt{2}Q_1 - q_1\right)dq_2$$
$$-\left(P_2 - \sqrt{2}m\omega q_2\right) + \left(\sqrt{2}q_1 - Q_1\right)$$

This expression can be rearranged to get

$$p_1dq_1 + p_2dq_2 - P_1dQ_1 + Q_2dP_2 = d\left(\sqrt{2}q_1P_2 - m\omega q_1q_2 + \sqrt{2}m\omega q_2Q_1 - Q_1P_2\right)$$

The RHS is an exact differential and hence, the transformation is canonical. From this expression we get the generating function as

$$F(q_1, q_2, Q_1, P_2) = \sqrt{2}q_1P_2 - m\omega q_1q_2 + \sqrt{2}m\omega q_2Q_1 - Q_1P_2 \tag{v}$$

EXAMPLE 4.34 Prove Poisson's first theorem.

Solution: Poisson's first theorem states that if a dynamic variable F is a constant of motion, then its Poisson bracket with Hamiltonian vanishes.

The equation of motion of a dynamic variable F is given by

$$\frac{dF}{dt}=[F,H]+\frac{\partial F}{\partial t} \tag{i}$$

If F is a constant of motion, then $\frac{dF}{dt}=0$. Therefore, equation (i) becomes

$$[F,H]+\frac{\partial F}{\partial t}=0 \tag{ii}$$

Also, if F is a constant of motion, it does not contain time explicitly. That is, $\frac{\partial F}{\partial t}=0$. Hence, the necessary condition for a dynamic variable to be a constant of motion is

$$[F,H]=0 \tag{iii}$$

Hence, proved.

EXAMPLE 4.35 Prove Poisson's second theorem (Jacobi-Poisson theorem).

Solution:

Poisson's second theorem is otherwise known as Jacobi's identity. It is given by

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0 \tag{i}$$

To prove this, let us evaluate the expression $[X,[Y,Z]]-[Y,[Z,X]]$.

$$[X,[Y,Z]]-[Y,[X,Z]]=\left[X,\sum_j\left(\frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j}-\frac{\partial Y}{\partial p_j}\frac{\partial Z}{\partial q_j}\right)\right]-\left[Y,\sum_j\left(\frac{\partial X}{\partial q_j}\frac{\partial Z}{\partial p_j}-\frac{\partial X}{\partial p_j}\frac{\partial Z}{\partial q_j}\right)\right]$$

Now using the property $[A,(B-C)]=[A,B]-[A,C]$, we can expand the above expression as

$$[X,[Y,Z]]-[Y,[X,Z]]=\left[X,\sum_j\frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j}\right]-\left[X,\sum_j\frac{\partial Y}{\partial p_j}\frac{\partial Z}{\partial q_j}\right]$$

$$-\left[Y,\sum_j\frac{\partial X}{\partial q_j}\frac{\partial Z}{\partial p_j}\right]+\left[Y,\sum_j\frac{\partial X}{\partial p_j}\frac{\partial Z}{\partial q_j}\right] \tag{ii}$$

Using the property $[A,BC]=[A,B]C+[A,C]B$, we can expand the above expression to get

$$[X,[Y,Z]]-[Y,[X,Z]]=\left[X,\sum_j\frac{\partial Y}{\partial q_j}\right]\sum_j\frac{\partial Z}{\partial p_j}+\left[X,\sum_j\frac{\partial Z}{\partial p_j}\right]\sum_j\frac{\partial Y}{\partial q_j}$$

$$-\left[X,\sum_j \frac{\partial Y}{\partial p_j}\right]\sum_j \frac{\partial Z}{\partial q_j} - \left[X,\sum_j \frac{\partial Z}{\partial q_j}\right]\sum_j \frac{\partial Y}{\partial p_j}$$

$$-\left[Y,\sum_j \frac{\partial X}{\partial q_j}\right]\sum_j \frac{\partial Z}{\partial p_j} - \left[Y,\sum_j \frac{\partial Z}{\partial p_j}\right]\sum_j \frac{\partial X}{\partial q_j}$$

$$+\left[Y,\sum_j \frac{\partial X}{\partial p_j}\right]\sum_j \frac{\partial Z}{\partial q_j} + \left[Y,\sum_j \frac{\partial Z}{\partial q_j}\right]\sum_j \frac{\partial X}{\partial p_j} \qquad \text{(iii)}$$

This expression can be rewritten as

$$[X,[Y,Z]]-[Y,[X,Z]] = \sum_j \left\{ \frac{\partial Z}{\partial q_j}\left(\left[Y,\frac{\partial X}{\partial p_j}\right] - \left[X,\frac{\partial Y}{\partial p_j}\right] \right) \right\}$$

$$+\sum_j \left\{ \frac{\partial Z}{\partial p_j}\left(\left[\frac{\partial X}{\partial q_j},Y\right] + \left[X,\frac{\partial Y}{\partial q_j}\right] \right) \right\}$$

$$+\sum_j \left\{ \frac{\partial Y}{\partial q_j}\left[X,\frac{\partial Z}{\partial p_j}\right] - \frac{\partial Y}{\partial p_j}\left[X,\frac{\partial Z}{\partial q_j}\right] \right\}$$

$$-\sum_j \left\{ \frac{\partial X}{\partial q_j}\left[Y,\frac{\partial Z}{\partial p_j}\right] - \frac{\partial X}{\partial p_j}\left[Y,\frac{\partial Z}{\partial p_j}\right] \right\} \qquad \text{(iv)}$$

The last two terms will become zero on expansion and then our expression reduces to

$$[X,[Y,Z]]-[Y,[X,Z]] = -\sum_j \left\{ \frac{\partial Z}{\partial q_j}\left(\left[\frac{\partial X}{\partial p_j},Y\right] + \left[X,\frac{\partial Y}{\partial p_j}\right] \right) \right\}$$

$$+\sum_j \left\{ \frac{\partial Z}{\partial p_j}\left(\left[\frac{\partial X}{\partial q_j},Y\right] + \left[X,\frac{\partial Y}{\partial q_j}\right] \right) \right\} \qquad \text{(v)}$$

Now, we use the identity, $\frac{\partial}{\partial x}[X,Y] = \left[\frac{\partial X}{\partial x},Y\right] + \left[X,\frac{\partial Y}{\partial x}\right]$ to rewrite the above expression.

Then,

$$[X,[Y,Z]]-[Y,[X,Z]] = \sum_j \left\{ -\frac{\partial Z}{\partial q_j}\frac{\partial}{\partial p_j}[X,Y] + \frac{\partial Z}{\partial p_j}\frac{\partial}{\partial q_j}[X,Y] \right\}$$

$$= -[Z,[X,Y]]$$

or $$[X,[Y,Z]]-[Y,[X,Z]]+[Z,[X,Y]]=0$$

or $$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0 \tag{vi}$$

Hence, Jacobi's identity is proved.

EXAMPLE 4.36 Obtain the canonical equations of motion under an infinitesimal canonical transformation. Show that the Hamiltonian is the generator of the motion of the system.

Solution: Here we start with a generating function that generates identity transformation, that is, $q \to Q$ and $p \to P$. The simplest form of such a function is $F_2 = \sum_j q_j P_j$. Now we introduce an ε infinitesimal parameter which is independent of q and p that can cause an infinitesimal change in the canonical variables q and p through the expression

$$F_2 = \sum_j q_j P_j + \varepsilon G(q_j, P_j) \tag{i}$$

Now, we have the transformation equations generated by the function $F_2(q,P)$ as $p_j = \dfrac{\partial F_2}{\partial q_j}$ and $Q_j = \dfrac{\partial F_2}{\partial P_j}$, and in the present case this would give

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j + \varepsilon \frac{\partial G}{\partial q_j} \tag{ii}$$

and $$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \varepsilon \frac{\partial G}{\partial P_j} \tag{iii}$$

From (ii) and (iii) we can write the infinitesimal change in the canonical variables as

$$\delta q_j = Q_j - q_j = \varepsilon \frac{\partial G}{\partial P_j} \tag{iv}$$

and $$\delta p_j = P_j - p_j = -\varepsilon \frac{\partial G}{\partial q_j} \tag{v}$$

Note that the change in the canonical variables is infinitesimal and hence we can write $G(q_j, P_j) = G(q_j, p_j)$. With this equations (iv) and (v) becomes

$$\delta q_j = \varepsilon \frac{\partial}{\partial P_j} G(q_j, p_j) \tag{vi}$$

$$\delta p_j = -\varepsilon \frac{\partial}{\partial q_j} G(q_j, p_j) \tag{vii}$$

Equations (vi) and (vii) are the equations for infinitesimal canonical transformation. Such a transformation is generated by a function $G(q_j, p_j)$.

Now, to solve the second part of the problem, we make the substitutions $\varepsilon = dt$ and $G(q_j, p_j) = H(q_j, p_j)$. With this substitution (vi) and (vii) becomes

$$\delta q_j = dt \frac{\partial}{\partial p_j} H(q_j, p_j) = \dot{q}_j dt = dq_j \tag{viii}$$

$$\delta p_j = -dt \frac{\partial}{\partial q_j} H(q_j, p_j) = \dot{p}_j dt = dp_j \tag{ix}$$

where we used the Hamilton's canonical equations. Equations (viii) and (ix) represent the changes occurring in the canonical variables during the motion of the system in a small time interval dt. Then, such a change can be regarded as an infinitesimal canonical transformation generated by the Hamiltonian.

EXAMPLE 4.37 Express infinitesimal canonical transformation in terms of Poisson brackets.

Solution: We know that an infinitesimal canonical transformation can be carried out by a generating function given by

$$F_2 = \sum_j q_j P_j + \varepsilon G(q_j, P_j) \tag{i}$$

Then, the infinitesimal changes produced in the canonical variables are given by

$$\delta q_j = \varepsilon \frac{\partial G}{\partial p_j} \quad \text{and} \quad \delta p_j = -\varepsilon \frac{\partial G}{\partial q_j} \tag{ii}$$

Now, consider a dynamic variable $X = X(q_j, p_j)$. An infinitesimal change in the variables q_j and p_j will produce a corresponding change in the variable X. It is given by

$$\delta X = \sum_j \frac{\partial X}{\partial q_j} \delta q_j + \sum_j \frac{\partial X}{\partial p_j} \delta p_j \tag{iii}$$

Using (ii) in (iii), we get

$$\delta X = \varepsilon \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial G}{\partial q_j} \right) = \varepsilon [X, G] \tag{iv}$$

Now, let $G = H$ and $\varepsilon = dt$ in the above expression, it becomes

$$\delta X = dt[X, H] \tag{v}$$

Then, the canonical equations of motion can be obtained by putting $X = q_j$ and $X = p_j$. If, $X = q_j$, we have

$$\delta q_j = dt[q_j, H]$$

$$= dt \dot{q}_j = dt \frac{dq_j}{dt} = dq_j \tag{vi}$$

and if $X = p_j$, we get

$$\delta p_j = dt\left[p_j, H\right]$$

$$= dt\dot{p}_j = dt\frac{dp_j}{dt} = dp_j \qquad \text{(vii)}$$

Equations are the canonical equations of motion.

Note: While writing equation (iii) the derivative with respect to time was not included. If we consider the total derivative of $X = X\left(q_j, p_j\right)$, we get

$$\frac{dX}{dt} = [X, H] + \frac{\partial X}{\partial t}$$

or
$$[X, H] = \frac{dX}{dt} - \frac{\partial X}{\partial t}$$

Using this in equation (v), we get

$$\delta X = dt\left[\frac{dX}{dt} - \frac{\partial X}{\partial t}\right] \neq dX$$

This shows that $\delta X = dX$ only if $\frac{\partial X}{\partial t} = 0$. That is, an infinitesimal canonical transformation will result in the actual change in X, only if it does not depend explicitly on time.

EXAMPLE 4.38 Show that the volume in phase space is invariant under a canonical transformation.

Solution: This can be proved using Poincare's theorem of integral invariant. The integral is due to Poincare and is defined as

$$J_1 = \iint_s \sum_j dq_j dp_j \qquad \text{(i)}$$

The integral is taken over an arbitrary two-dimensional surface s of $2n$ the dimensional phase space. Poincare showed that this integral is invariant under a canonical transformation and is known as Poincare's integral invariant.

The invariance of this integral under a canonical transformation means

$$\iint_s \sum_j dq_j dp_j = \iint_s \sum_j dQ_j dP_j \qquad \text{(ii)}$$

Now, let $q_j = q_j(u, v)$ and $p_j = p_j(u, v)$, where u and v are two parameters that completely specify the points on the two-dimensional surface, here the surface in phase space. To carry out the transformation of the integral to new set of variables, we start with the relation

$$dq_j dp_j = \frac{\partial\left(q_j, p_j\right)}{\partial(u, v)} du dv \qquad \text{(iii)}$$

In this expression,

$$\frac{\partial(q_j,p_j)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial q_j}{\partial u} & \dfrac{\partial p_j}{\partial u}\\ \dfrac{\partial q_j}{\partial v} & \dfrac{\partial p_j}{\partial v}\end{vmatrix} \tag{iv}$$

is called Jacobian. Equation (iv) can be used in (ii) to get

$$\iint_s \sum_j \frac{\partial(q_j,p_j)}{\partial(u,v)}dudv=\iint_s \sum_j \frac{\partial(Q_j,P_j)}{\partial(u,v)}dudv$$

Now, since the surface is arbitrary, if the new coordinates are obtained through a canonical transformation from the old variables, the integrands must be equal. That is

$$\sum_j \frac{\partial(q_j,p_j)}{\partial(u,v)}=\sum_j \frac{\partial(Q_j,P_j)}{\partial(u,v)} \tag{v}$$

To show that the integral is invariant under a canonical transformation, now it is sufficient to show that the sum of the Jacobians in (v) is invariant.

Let us prove this through a transformation from the basis (q,p) to a new basis (Q,P) through the generating function $F_2(q,P,t)$.

We have

$$Q_j=\frac{\partial F_2}{\partial P_j}\ \text{ and }\ p_j=\frac{\partial F_2}{\partial q_j}$$

so that
$$\frac{\partial p_j}{\partial u}=\frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial q_j}\right)=\sum_i \frac{\partial^2 F_2}{\partial q_i\partial q_j}\frac{\partial q_i}{\partial u}+\sum_i \frac{\partial^2 F_2}{\partial P_i\partial q_j}\frac{\partial P_i}{\partial u}$$

$$\frac{\partial p_j}{\partial v}=\frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial q_j}\right)=\sum_i \frac{\partial^2 F_2}{\partial q_i\partial q_j}\frac{\partial q_i}{\partial v}+\sum_i \frac{\partial^2 F_2}{\partial P_i\partial q_j}\frac{\partial P_i}{\partial v}$$

Now, the LHS of equation (v) becomes

$$\sum_j \frac{\partial(q_j,p_j)}{\partial(u,v)}=\sum_j \begin{vmatrix}\dfrac{\partial q_j}{\partial u} & \displaystyle\sum_i \frac{\partial^2 F_2}{\partial q_i\partial q_j}\frac{\partial q_i}{\partial u}+\sum_i \frac{\partial^2 F_2}{\partial P_i\partial q_j}\frac{\partial P_i}{\partial u}\\ \dfrac{\partial q_j}{\partial v} & \displaystyle\sum_i \frac{\partial^2 F_2}{\partial q_i\partial q_j}\frac{\partial q_i}{\partial v}+\sum_i \frac{\partial^2 F_2}{\partial P_i\partial q_j}\frac{\partial P_i}{\partial v}\end{vmatrix}$$

$$=\sum_i\sum_j \frac{\partial^2 F_2}{\partial q_i\partial q_j}\begin{vmatrix}\dfrac{\partial q_j}{\partial u} & \dfrac{\partial q_i}{\partial u}\\ \dfrac{\partial q_j}{\partial v} & \dfrac{\partial q_i}{\partial v}\end{vmatrix}+\sum_i\sum_j \frac{\partial^2 F_2}{\partial P_i\partial q_j}\begin{vmatrix}\dfrac{\partial q_j}{\partial u} & \dfrac{\partial P_i}{\partial u}\\ \dfrac{\partial q_j}{\partial v} & \dfrac{\partial P_i}{\partial v}\end{vmatrix}$$

Since the first term on RHS is anti-symmetric, its value will be zero and hence,

$$\sum_j \frac{\partial(q_j, p_j)}{\partial(u,v)} = \sum_i \sum_j \frac{\partial^2 F_2}{\partial P_i \partial q_j} \begin{vmatrix} \frac{\partial q_j}{\partial u} & \frac{\partial P_i}{\partial u} \\ \frac{\partial q_j}{\partial v} & \frac{\partial P_i}{\partial v} \end{vmatrix} \tag{vi}$$

Now, we have

$$\sum_i \sum_j \frac{\partial^2 F_2}{\partial P_i \partial P_j} \begin{vmatrix} \frac{\partial P_j}{\partial u} & \frac{\partial P_i}{\partial u} \\ \frac{\partial P_j}{\partial v} & \frac{\partial P_i}{\partial v} \end{vmatrix} = 0$$

Therefore, this can be added to the RHS of the equation (vi) to get

$$\sum_j \frac{\partial(q_j, p_j)}{\partial(u,v)} = \sum_i \sum_j \frac{\partial^2 F_2}{\partial P_i \partial q_j} \begin{vmatrix} \frac{\partial q_j}{\partial u} & \frac{\partial P_i}{\partial u} \\ \frac{\partial q_j}{\partial v} & \frac{\partial P_i}{\partial v} \end{vmatrix} + \sum_i \sum_j \frac{\partial^2 F_2}{\partial P_i \partial P_j} \begin{vmatrix} \frac{\partial P_j}{\partial u} & \frac{\partial P_i}{\partial u} \\ \frac{\partial P_j}{\partial v} & \frac{\partial P_i}{\partial v} \end{vmatrix}$$

$$= \sum_i \begin{vmatrix} \sum_j \frac{\partial^2 F_2}{\partial P_i \partial q_j} \frac{\partial q_j}{\partial u} + \sum_j \frac{\partial^2 F_2}{\partial P_i \partial P_j} \frac{\partial P_j}{\partial u} & \frac{\partial P_i}{\partial u} \\ \sum_j \frac{\partial^2 F_2}{\partial P_i \partial q_j} \frac{\partial q_j}{\partial v} + \sum_j \frac{\partial^2 F_2}{\partial P_i \partial P_j} \frac{\partial P_j}{\partial v} & \frac{\partial P_i}{\partial v} \end{vmatrix} \tag{vii}$$

Now, we have

$$\frac{\partial Q_i}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial P_i}\right) = \sum_j \frac{\partial^2 F_2}{\partial P_i \partial q_j} \frac{\partial q_j}{\partial u} + \sum_j \frac{\partial^2 F_2}{\partial P_i \partial P_j} \frac{\partial P_j}{\partial u}$$

Similarly, $$\frac{\partial Q_i}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial P_i}\right) = \sum_j \frac{\partial^2 F_2}{\partial P_i \partial q_j} \frac{\partial q_j}{\partial v} + \sum_j \frac{\partial^2 F_2}{\partial P_i \partial P_j} \frac{\partial P_j}{\partial v}$$

With these substitutions, the expression (vii) becomes

$$\sum_j \frac{\partial(q_j, p_j)}{\partial(u,v)} = \sum_i \begin{vmatrix} \frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial P_i}\right) & \frac{\partial P_i}{\partial u} \\ \frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial P_i}\right) & \frac{\partial P_i}{\partial v} \end{vmatrix}$$

$$= \sum_i \begin{vmatrix} \frac{\partial Q_i}{\partial u} & \frac{\partial P_i}{\partial u} \\ \frac{\partial Q_i}{\partial v} & \frac{\partial P_i}{\partial v} \end{vmatrix} = \sum_j \frac{\partial(Q_i, \partial P_i)}{\partial P_i (u,v)} \tag{viii}$$

Thus, we have shown that

$$\sum_j \frac{\partial(q_j, p_j)}{\partial(u,v)} = \sum_j \frac{\partial(Q_j, P_j)}{\partial(u,v)} \tag{ix}$$

Therefore, the integral J_1 is invariant under a canonical transformation. In a similar way we can show that

$$J_2 = \iiiint_s \sum dq_i dp_i dq_j dp_j \tag{x}$$

is invariant under a canonical transformation, where s is a four-dimensional surface in the $2n$ dimensional phase space. This procedure can be expanded to the integral

$$J_n = \int ....\int dq_1 dq_2 ......dq_n dp_1 dp_2 ......dp_n \tag{xi}$$

The invariance of the integral J_n under a canonical transformation shows that the phase space volume is invariant under a canonical transformation.

EXAMPLE 4.39 Show that the Lagrange's bracket is invariant under a canonical transformation.

Solution: The Lagrange's bracket $\{u,v\}$ with respect to the basis (q_j, p_j) is defined as

$$\{u,v\} = \sum_j \left(\frac{\partial q_j}{\partial u}\frac{\partial p_j}{\partial v} - \frac{\partial p_j}{\partial u}\frac{\partial q_j}{\partial v} \right) \tag{i}$$

We need to prove this is an invariant under a canonical transformation. We start with Poincare's theorem of integral invariant which states that the integral

$$J_1 = \iint_s \sum_j dq_j dp_j \tag{ii}$$

is invariant under a canonical transformation. The integral is taken over an arbitrary two-dimensional surface s of $2n$ dimensional phase space.

The invariance of this integral under a canonical transformation means

$$\iint_s \sum_j dq_j dp_j = \iint_s \sum_j dQ_j dP_j \tag{iii}$$

Now, let $q_j = q_j(u,v)$ and $p_j = p_j(u,v)$, where u and v are two parameters that completely specify the points on the two dimensional surface, here the surface in phase space. To carry out the transformation of the integral to new set of variables, we start with the relation

$$dq_j dp_j = \frac{\partial(q_j, p_j)}{\partial(u,v)} du dv \tag{iv}$$

In this expression,

$$\frac{\partial(q_j,p_j)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial q_j}{\partial u} & \dfrac{\partial p_j}{\partial u}\\ \dfrac{\partial q_j}{\partial v} & \dfrac{\partial p_j}{\partial v}\end{vmatrix} \tag{v}$$

is called Jacobian. Equation (iv) can be used in (iii) to get

$$\iint_s \sum_j \frac{\partial(q_j,p_j)}{\partial(u,v)}dudv=\iint_s \sum_j \frac{\partial(Q_j,P_j)}{\partial(u,v)}dudv \tag{vi}$$

Now, since the surface is arbitrary, if the new coordinates are obtained through a canonical transformation from the old variables, the integrands must be equal. That is,

$$\sum_j \frac{\partial(q_j,p_j)}{\partial(u,v)}=\sum_j \frac{\partial(Q_j,P_j)}{\partial(u,v)} \tag{vii}$$

that is,

$$\sum_j \begin{vmatrix}\dfrac{\partial q_j}{\partial u} & \dfrac{\partial p_j}{\partial u}\\ \dfrac{\partial q_j}{\partial v} & \dfrac{\partial p_j}{\partial v}\end{vmatrix}=\sum_j \begin{vmatrix}\dfrac{\partial Q_j}{\partial u} & \dfrac{\partial P_j}{\partial u}\\ \dfrac{\partial Q_j}{\partial v} & \dfrac{\partial P_j}{\partial v}\end{vmatrix}$$

On expanding the determinants on both sides, we get

$$\sum_j\left(\frac{\partial q_j}{\partial u}\frac{\partial p_j}{\partial v}-\frac{\partial p_j}{\partial u}\frac{\partial q_j}{\partial v}\right)=\sum_j\left(\frac{\partial Q_j}{\partial u}\frac{\partial P_j}{\partial v}-\frac{\partial P_j}{\partial u}\frac{\partial Q_j}{\partial v}\right)$$

or

$$\{u,v\}_{q_j,p_j}=\{u,v\}_{Q_j,P_j}$$

Thus, Lagrange's brackets are invariant under a canonical transformation.

EXAMPLE 4.40 Show that; $\sum_{l=1}^{2n}\{u_l,u_i\}[u_l,u_j]=\delta_{ij}$, where, $\{u_l,u_i\}$ is the Lagrange's bracket and $[u_l,u_j]$ is the Poisson bracket.

Solution: We have the Lagrange's bracket defined as

$$\{u,v\}=\sum_{k=1}^{n}\left(\frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i}-\frac{\partial p_k}{\partial u_l}\frac{\partial q_k}{\partial u_i}\right) \tag{i}$$

and the Poisson bracket as

$$[u_l,u_j]=\sum_{m=1}^{n}\left(\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m}-\frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\right) \tag{ii}$$

Then $$\sum_{l=1}^{2n}\{u_l,u_i\}\left[u_l,u_j\right]=\sum_{l=1}^{2n}\left(\sum_{k=1}^{n}\left(\frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i}-\frac{\partial p_k}{\partial u_l}\frac{\partial q_k}{\partial u_i}\right)\sum_{m=1}^{n}\left(\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m}-\frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\right)\right)$$

On multiplying and expanding this becomes

$$\sum_{l=1}^{2n}\{u_l,u_i\}\left[u_l,u_j\right]=\sum_{l=1}^{2n}\left(\begin{array}{l}\displaystyle\sum_{k,m=1}^{n}\left(\frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i}\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m}-\frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i}\frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\right)\\ \displaystyle+\sum_{k,m=1}^{n}\left(\frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\frac{\partial p_k}{\partial u_l}\frac{\partial q_k}{\partial u_i}-\frac{\partial p_k}{\partial u_l}\frac{\partial q_k}{\partial u_i}\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m}\right)\end{array}\right) \tag{iii}$$

The first term on the RHS of the equation (iii) is

$$\begin{aligned}\sum_{l=1}^{2n}\sum_{k,m=1}^{n}\frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i}\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m}&=\sum_{k,m=1}^{n}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\sum_{l=1}^{2n}\frac{\partial q_k}{\partial u_l}\frac{\partial u_l}{\partial q_m}\\&=\sum_{k,m=1}^{n}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\frac{\partial q_k}{\partial q_m}=\sum_{k,m=1}^{n}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\delta_{km}\end{aligned} \tag{iv}$$

But δ_{km} can be written as $\delta_{km}=\dfrac{\partial p_m}{\partial p_k}$ and using this in equation (iv), we get

$$\begin{aligned}\sum_{l=1}^{2n}\sum_{k,m=1}^{n}\frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i}\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m}&=\sum_{k,m=1}^{n}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\frac{\partial p_k}{\partial p_m}\\&=\sum_{k,m=1}^{n}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\frac{\partial p_m}{\partial p_k}=\sum_{k=1}^{n}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_k}\end{aligned} \tag{v}$$

In a similar way we can show that the third term on RHS of equation (iii) can be written as

$$\sum_{l=1}^{2n}\sum_{k,m=1}^{n}\frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\frac{\partial p_k}{\partial u_l}\frac{\partial q_k}{\partial u_i}=\sum_{k=1}^{n}\frac{\partial q_k}{\partial u_i}\frac{\partial u_j}{\partial q_k} \tag{vi}$$

The second term can be evaluated as

$$\begin{aligned}-\sum_{l=1}^{2n}\sum_{k,m=1}^{n}\frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i}\frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}&=-\sum_{k,m=1}^{n}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}\sum_{l=1}^{2n}\frac{\partial q_k}{\partial u_l}\frac{\partial u_l}{\partial p_m}\\&=-\sum_{k,m=1}^{n}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}\frac{\partial q_k}{\partial p_m}=0\end{aligned} \tag{vii}$$

This is because $\dfrac{\partial q_k}{\partial p_m}=0$. Similarly, the last term on the RHS of equation (iii) also vanishes. Therefore,

$$\sum_{l=1}^{2n}\{u_l,u_i\}[u_l,u_j]=\sum_{k=1}^{n}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_k}+\sum_{k=1}^{n}\frac{\partial q_k}{\partial u_i}\frac{\partial u_j}{\partial q_k}$$

$$=\sum_{k=1}^{n}\frac{\partial u_j}{\partial p_k}\frac{\partial p_k}{\partial u_i}+\frac{\partial u_j}{\partial q_k}\frac{\partial q_k}{\partial u_i}=\frac{\partial u_j(q_k,p_k)}{\partial u_i}=\delta_{ij}$$

Thus, $\sum_{l=1}^{2n}\{u_l,u_i\}[u_l,u_j]=\delta_{ij}$, hence proved. This actually gives the relationship between the Lagrange's and Poisson brackets.

EXAMPLE 4.41 Show that the density of phase space points is conserved. (Show that $\frac{d\rho}{dt}=0$, where ρ is the number density of system points in phase space.)

Solution: This is the first part of the Liouville's theorem and can be mathematically represented as; $\frac{\partial \rho}{\partial t}=0$, where ρ is the number density of the phase space points. To prove this consider an elemental volume in phase space enclosed between

$$q_1,(q_1+\delta q_1),q_2,(q_2+\delta q_2)+\,.......+q_n,(q_n+\delta q_n)$$

and

$$p_1,(p_1+\delta p_1),p_2,(p_2+\delta p_2)+\,.......+p_n,(p_n+\delta p_n).$$

Then the elemental volume is

$$\delta V=\delta q_1\delta q_2......\delta q_n\delta p_1\delta p_2.......\delta p_n \tag{i}$$

The number of systems enclosed in this volume element changes continuously since the systems are entering and leaving the volume continuously. The change in the number of systems in the elemental volume in a time dt seconds is given by

$$\delta N=\left(\frac{\partial \rho}{\partial t}\right)dt\delta V$$

$$=\left(\frac{\partial \rho}{\partial t}\right)dt\delta q_1\delta q_2......\delta q_n\delta p_1\delta p_2.......\delta p_n \tag{ii}$$

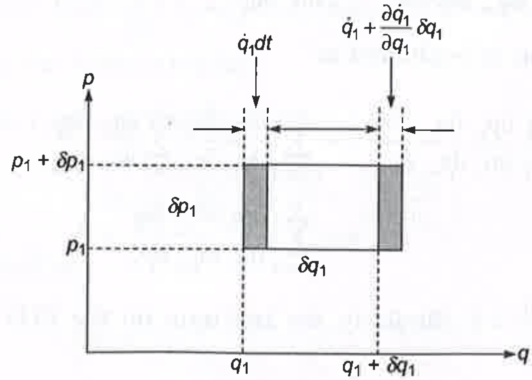

Fig. 4.1

Now, consider two opposite surfaces of the hypothetical elemental volume normal to the coordinate axes. The number of phase space points entering the first surface in time dt can be written as

$$dN_1 = \rho(\dot{q}_1 dt)\delta q_2 \ldots\ldots \delta q_n \delta p_1 \delta p_2 \ldots\ldots \delta p_n \tag{iii}$$

where, $\dot{q}_1$ is the component of velocity of phase space points in the direction of q. Similarly, the number of points entering the second surface which is at $(q_1 + \delta q_1)$ can be written as

$$dN_1' = \left(\rho + \frac{\partial \rho}{\partial q_1}\delta q_1\right)\left(\dot{q}_1 + \frac{\partial \dot{q}_1}{\partial q_1}\delta q_1\right) dt\, \delta q_2 \ldots\ldots \delta q_n \delta p_1 \delta p_2 \ldots\ldots \delta p_n$$

$$\cong \left[\rho \dot{q}_1 + \left(\dot{q}_1 \frac{\partial \rho}{\partial q_1} + \rho \frac{\partial \dot{q}_1}{\partial q_1}\right)\delta q_1\right] dt\, \delta q_2 \ldots\ldots \delta q_n \delta p_1 \delta p_2 \ldots\ldots \delta p_n \tag{iv}$$

Subtracting (iii) from (ii), we get the total increase in the number of system points in the direction normal to q_1.

$$(\delta N)_{q_1} = -\left(\dot{q}_1 \frac{\partial \rho}{\partial q_1} + \rho \frac{\partial \dot{q}_1}{\partial q_1}\right) dt\, \delta q_1 \delta q_2 \ldots\ldots \delta q_n \delta p_1 \delta p_2 \ldots\ldots \delta p_n \tag{v}$$

Similarly, for the p_1 coordinate, we can have

$$(\delta N)_{p_1} = -\left(\dot{p}_1 \frac{\partial \rho}{\partial p_1} + \rho \frac{\partial \dot{p}_1}{\partial p_1}\right) dt\, \delta q_1 \delta q_2 \ldots\ldots \delta q_n \delta p_1 \delta p_2 \ldots\ldots \delta p_n \tag{vi}$$

In a similar way, the increase in the number of system points normal to all other coordinates $q_2, q_3, \ldots\ldots, q_n$ and $p_2, p_3, \ldots\ldots, p_n$ can also be determined. Then the net increase in the number of system points in the elemental volume in a time dt is given by

$$\delta N = -\sum_{j=1}^{n}\left[\rho\left(\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j}\right) + \dot{q}_j \frac{\partial \rho}{\partial q_j} + \dot{p}_j \frac{\partial \rho}{\partial p_j}\right] dt\, \delta q_1 \delta q_2, \ldots\ldots, \delta q_n \delta p_1 \delta p_2, \ldots\ldots, \delta p_n \tag{vii}$$

Comparing this expression with equation (ii), we get

$$\frac{\partial \rho}{\partial t} = -\sum_{j=1}^{n}\left[\rho\left(\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j}\right) + \dot{q}_j \frac{\partial \rho}{\partial q_j} + \dot{p}_j \frac{\partial \rho}{\partial p_j}\right] \tag{viii}$$

Now, we have Hamilton's canonical equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

This can be used in (viii) to get

$$\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} = \frac{\partial}{\partial q_j}\left(\frac{\partial H}{\partial p_j}\right) - \frac{\partial}{\partial p_j}\left(\frac{\partial H}{\partial q_j}\right) = 0$$

Therefore, equation (viii) reduces to

$$\frac{\partial \rho}{\partial t} = -\sum_{j=1}^{n}\left(\frac{\partial \rho}{\partial q_j}\dot{q}_j + \dot{p}_j\frac{\partial \rho}{\partial p_j}\right) \tag{ix}$$

Equation (ix) is known as Liouville's theorem.
Now, let us rewrite equation (ix) as

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{n}\left(\frac{\partial \rho}{\partial q_j}\dot{q}_j + \dot{p}_j\frac{\partial \rho}{\partial p_j}\right) = 0$$

or

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{n}\left(\frac{\partial \rho}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial \rho}{\partial p_j}\frac{dp_j}{dt}\right) = 0 \tag{x}$$

If $\rho = \rho(q_j, p_j, t)$; also q_j and p_j are time dependent, then the total time derivative of ρ is given by

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{j=1}^{n}\left(\frac{\partial \rho}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial \rho}{\partial p_j}\frac{dp_j}{dt}\right) \tag{xi}$$

Comparing the equations (x) and (xi), we get

$$\frac{d\rho}{dt} = 0 \tag{xii}$$

This result can be regarded as the principle of conservation of density in phase space.

EXAMPLE 4.42 Show that $\frac{d}{dt}(\delta V) = 0$, where δV is the elemental volume in phase space.

Solution: The number of system points in the volume element δV can be written as;

$$\delta N = \rho \delta V \tag{i}$$

Differentiating this expression with respect to time, we get

$$\frac{d}{dt}(\delta N) = \frac{d\rho}{dt}\delta V + \rho\frac{d}{dt}(\delta V) \tag{ii}$$

The LHS of the equation (ii) must be zero, since the number of system points in a given region of phase space must remain unchanged. Then,

$$\frac{d}{dt}(\delta N) = 0$$

or
$$\frac{d\rho}{dt}\delta V+\rho\frac{d}{dt}(\delta V)=0$$

that is,
$$\rho\frac{d}{dt}(\delta V)=0 \tag{iii}$$

since $\dfrac{d\rho}{dt}=0$.

Now, ρ cannot be equal to zero identically, we get

$$\frac{d}{dt}(\delta V)=0 \tag{iv}$$

Hence, proved.

EXERCISES

4.1 Show that the generating function $G=\sum_j q_jP_j$ generates the identity transformation and the function $G=\sum_j q_jQ_j$ generates the exchange transformation.

4.2 Prove the properties of the Poisson brackets.

4.3 Find the generating functions F_1, F_2, F_3 and F_4 for the canonical transformation given by $Q=\ln\left(\dfrac{\sin p}{q}\right)$ and $P=q\cot p$.

4.4 A charged particle moves in a constant magnetic field $\vec{B}$ whose potential is given by $\vec{A}=\dfrac{1}{2}\vec{B}\times\vec{r}$. Find the Poisson bracket between the x and y components of its velocity.

4.5 Show that the transformation given by $Q=\tan^{-1}\left(\dfrac{\alpha q}{p}\right)$ and $P=\dfrac{\alpha q^2}{2}\left(1+\dfrac{p^2}{\alpha^2q^2}\right)$ is canonical for a one-dimensional system for any arbitrary value of α. Also find the generating functions F_1 and F_2.

4.6 Show that the transformation given by the expressions $q=\left(\dfrac{P}{2\pi^2 m\nu}\right)^{1/2}\cos(2\pi Q)$ and $p=-(2m\nu P)^{1/2}\sin(2\pi Q)$ is canonical. Show that the transformed Hamiltonian is a function of the generalized momentum P only. Also show that the equation of motion is $q=A\left(\dfrac{1}{2\pi\nu}\right)^{1/2}\cos(2\pi\nu t+\alpha)$, where A and α are constants.

4.7 The motion of a particle of mass m undergoing constant acceleration a in one dimension is described by $x=X+\dfrac{P_x}{m}t+\dfrac{1}{2}at^2$ and $p_x=P_x+mat$, where, (x,p_x)

and (X,P_x) are respectively the old and new sets of variables. Show that the transformation is canonical. Find the generating function F_1.

4.8 Show that the Hamiltonian of a simple pendulum is invariant under a canonical transformation.

4.9 Show that the transformation given by $Q=\tan q$ and $P=(p-mv_0)\cos^2 q$ is a canonical transformation. Find the generating function F_1.

4.10 A particle of mass m moves in the xy-plane under the action of a potential given by $V=ky$. For a homogeneous point transformation $Q_1=xy$ and $Q_2=\frac{1}{2}(x^2-y^2)$. Find the expressions for P_1, P_2, the generating function $F_2(q,P)$, and the new Hamiltonian.

4.11 Show that, $Q_1=q_1q_2$, $Q_2=\frac{1}{2}(q_1^2-q_2^2)$, $P_1=\frac{q_1p_2+q_2p_1}{q_1^2+q_2^2}$ and $P_1=\frac{q_1p_1-q_2p_2}{q_1^2+q_2^2}$ represent a canonical transformation.

4.12 Determine whether the transformation $Q_1=q_1q_2$, $Q_2=q_1+q_2$, $P_1=1+\frac{p_1-p_2}{q_2-q_1}$ and $P_2=1+\frac{q_2p_2-q_1p_1}{q_2-q_1}$ is canonical or not.

4.13 Show that the transformations $x=\frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1+P_2\right)$, $p_x=\frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1-Q_2\right)$, $y=\frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1+Q_2\right)$ and $p_y=\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1-P_2\right)$ are canonical, where α is a fixed parameter.

4.14 A canonical transformation is given by $Q_1=q_1-v_0t$, $Q_2=\sqrt{2p_2e^{-t}}\sin q_2$, $P_1=p_1-v_0$ and $P_2=\sqrt{2p_2e^{t}}\cos q_2$. The Hamiltonian is $H=\frac{1}{2}\left(p_1^2+p_2^2+q_1^2\right)$. Find the generating function $F_2(q,P)$, the new Hamiltonian $H'(Q,P)$ and the canonical equations of motion in terms of the new variables.

4.15 Find the canonical transformation relating to the Hamiltonian formulations obtained from two Lagrangians which differ by a total derivative term: L and $L+\frac{dF}{dt}$.

4.16 The Hamiltonian of a system with canonical variables q and p is $H=\frac{p^2}{2m}$. Perform a canonical transformation from the old set of variables (q,p) to the new set of variables (Q,P) using the generating function $F_2(q,P)=(q-at)(P+b)$, where a and b are positive constants.

4.17 Find the generating function for the canonical transformation $(r,\theta,\phi,p_r,p_\theta,p_\phi)$ to (x,y,x,p_x,p_y,p_x).

4.18 Perform canonical transformation of $H = \frac{1}{2m}\left[p_1^2 + (p_2 - \lambda x_1)^2 + p_3^2\right]$ using the generating function $F_2(r, P) = x_3 P_3 + P_1 P_2 + \sqrt{\lambda}\,(x_1 P_2 + x_2 P_1) + \lambda x_1 x_2$. Obtain the Hamilton's equation of motion for the new Hamiltonian and solve it.

4.19 A one-dimensional damped oscillator with coordinate q satisfies the equation $\ddot{q} + 4\dot{q} + 3q = 0$, which is equivalent to a first order system with $\dot{q} = v$ and $\dot{v} = -3q - 4v$. Show that the area $A(t)$ of any region of points moving in (q, v) space has the time variation $A(t) = A(0)e^{-4t}$.

4.20 Use Liouville's theorem to show that any autonomous system with n degrees of freedom and $n\,(n-1)$ cyclic coordinates must be integrable.

5

CHAPTER

Hamilton-Jacobi Formulation and Action-Angle Variables

CONCEPTS AND FORMULAE

5.1 HAMILTON-JACOBI METHOD

In Hamilton-Jacobi method, one seeks a canonical transformation from the old set of coordinate and momentum (q,p) to a new set of quantities (Q,P) which are constants. These constant quantities may be the initial values (q_0,p_0) at $t=0$. Hamilton-Jacobi method has the advantage that while obtaining the transformation equations, we arrive at the solution.

5.2 HAMILTON-JACOBI EQUATION

Hamilton-Jacobi equation is given by

$$H(q_1,q_2,\ldots\ldots,q_n;p_1,p_2,\ldots\ldots,p_n;t)+\frac{\partial F(q,P,t)}{\partial t}=0 \tag{5.1}$$

Also, in this case, $p_j=\frac{\partial F}{\partial q_j}$ and the above equation becomes

$$H\left(q_1,q_2,\ldots\ldots,q_n;\frac{\partial F}{\partial q_1},\frac{\partial F}{\partial q_2},\ldots\ldots,\frac{\partial F}{\partial q_n};t\right)+\frac{\partial F(q,P,t)}{\partial t}=0 \tag{5.2}$$

5.3 HAMILTON'S PRINCIPAL FUNCTION

Hamilton's principal function (S) is the solution of the Hamilton-Jacobi equation. It is the generating function that generates a canonical transformation from old set of coordinate and momentum (q,p) to a new constant coordinate and momentum.
It is defined as

$$S\equiv S\left(q_j,\alpha_j,t\right) \tag{5.3}$$

where, $\alpha_j=P_j$ are the new momenta which are constants.

In terms of Hamilton's principal function, the Hamilton-Jacobi equation is given by

$$H\left(q_j, \frac{\partial S}{\partial q_j}\right) + \frac{\partial S}{\partial t} = 0 \tag{5.4}$$

5.4 PHYSICAL SIGNIFICANCE OF HAMILTON'S PRINCIPAL FUNCTION

Hamilton's principal function itself has no physical significance, however, we have

$$S = \int L dt + \text{constant} \tag{5.5}$$

Note that, this differs from Hamilton's variational principle by a constant since the integral is indefinite. Therefore, the same integral in definite form represents the Hamilton's variational principle. Also this integral is of no practical use since the integration can be performed only if q_j and p_j are known, i.e., only after solving the problem.

5.5 HAMILTON'S CHARACTERISTIC FUNCTION

For a conservative system, the Hamiltonian represents the total energy and is a constant. In this case we can separate the time dependent part of the Hamilton's principal function to write

$$S(q_j, \alpha_j, t) = W(q_j, \alpha_j) + \alpha(t) \tag{5.6}$$

Here, $W(q_j, \alpha_j)$ is known as Hamilton's characteristic function and is independent of time.

5.6 PHYSICAL SIGNIFICANCE OF HAMILTON'S CHARACTERISTIC FUNCTION

We have, $W = W(q_j, \alpha_j)$, and it can be shown that

$$W = \int \left(\sum_j p_j \dot{q}_j \right) dt \tag{5.7}$$

The above integral is known as action. Therefore, Hamilton's characteristic function can be identified as the action (A).

5.7 ACTION OF PHASE INTEGRAL AND ANGLE VARIABLE

The action of phase integral that is conjugate to the coordinate q_j is defined as

$$J_j = \oint p_j dq_j \tag{5.8}$$

We can see that the action has the dimension of angular momentum and hence, the coordinate that is conjugate to action is an angle and is known as angle variable (θ). Thus, action and angle form a pair of canonically conjugate variables to solve mechanical problems.

In terms of the action and angle variables, Hamilton's canonical equations become

$$\dot{\theta} = \frac{\partial H}{\partial J} \quad \text{and} \quad \dot{J} = \frac{\partial H}{\partial \theta} \tag{5.9}$$

SOLVED PROBLEMS

EXAMPLE 5.1 Deduce Hamilton-Jacobi equation.

Solution: In Hamilton-Jacobi method, we perform a transformation from the old set of coordinate and momentum (q,p) to a new set of coordinate and momentum which are constants. This requires that the new Hamiltonian H' must be zero identically. We have the Hamilton's canonical equations in terms of the new variables as

$$\dot{Q}_j = \frac{\partial H}{\partial P_j}; \; \dot{P}_j = -\frac{\partial H}{\partial Q_j} \quad \text{and} \quad H' = H + \frac{\partial F}{\partial t} \tag{i}$$

where, F is the generating function of the transformation. For the new coordinate and momenta are to be constants, we must have

$$\dot{Q}_j = \frac{\partial H}{\partial P_j} = 0; \quad \dot{P}_j = -\frac{\partial H}{\partial Q_j} = 0 \tag{ii}$$

This is possible if the new Hamiltonian H' is zero identically. Then from equation (i) we get

$$H(q_j, p_j, t) + \frac{\partial F}{\partial t} = 0 \tag{iii}$$

Let us choose the generating function of the transformation as

$$F \equiv F(q_j, P_j, t)$$

Then, the transformation equation can be written as

$$p_j = \frac{\partial F(q_j, P_j, t)}{\partial q_j} \tag{iv}$$

With this equation (iii) becomes

$$H\left(q_j, \frac{\partial F}{\partial q_j}, t\right) + \frac{\partial F(q_j, P_j, t)}{\partial t} = 0 \tag{v}$$

Equation (v) is the Hamilton-Jacobi equation.

Note that, equation (v) has only $(n+1)$ independent variables whereas, equation (iii) has $(2n+1)$ independent variables.

EXAMPLE 5.2 Show that $S = \int L dt + \text{constant}$, where S is Hamilton's principal function.

Solution: Hamilton's principal function is the solution of Hamilton-Jacobi equation and we have; $S = S(q_j, \alpha_j, t)$, with $\alpha_j = P_j = \text{constant}$. Taking the total time derivative of S, we get

$$\frac{dS}{dt} = \sum_j \frac{\partial S}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial S}{\partial \alpha_j}\dot{\alpha}_j + \frac{\partial S}{\partial t} \tag{i}$$

But, we have

$$p_j = \frac{\partial F(q_j, P_j, t)}{\partial q_j} = \frac{\partial S}{\partial t}$$

Then, equation (i) becomes

$$\frac{dS}{dt} = \sum_j p_j \dot{q}_j + \frac{\partial S}{\partial t} \tag{ii}$$

Now, we have Hamilton-Jacobi equation as

$$H + \frac{\partial S}{\partial t} = 0 \quad \text{or,} \quad \frac{\partial S}{\partial t} = -H$$

Using this in (ii), we get

$$\frac{dS}{dt} = \sum_j p_j \dot{q}_j - H = L \tag{iii}$$

Integrating this expression, we get

$$S = \int L dt + constant \tag{iv}$$

This expression differs from Hamilton's variational principle by a constant since the integral is indefinite. Therefore, the same integral in definite form represents the Hamilton's variational principle. Also this integral is of no practical use since the integration can be performed only if q_j and p_j are known, i.e., only after solving the problem.

EXAMPLE 5.3 Discuss the motion of a particle falling vertically under the influence of uniform gravitational field using Hamilton-Jacobi method.

Solution: We start with the Hamiltonian of a particle falling freely under the influence of a gravitational field. It is given by

$$H = \frac{p^2}{2m} + mgz \tag{i}$$

where m is the mass of the particle and z direction is taken as the vertical. In Hamilton-Jacobi method, the generating function is Hamilton's principal function and we have

$$p_j = \frac{\partial F}{\partial q_j} = \frac{\partial S}{\partial q_j}$$

which in the present case is

$$p = \frac{\partial S}{\partial z} \tag{ii}$$

Using this in (i), we get

$$H = \frac{1}{2m}\left(\frac{\partial S}{\partial z}\right)^2 + mgz \tag{iii}$$

Now, the Hamilton Jacobi equation is

$$H + \frac{\partial S}{\partial t} = 0 \tag{iv}$$

Substituting for the Hamiltonian from the equation (iii) the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left(\frac{\partial S}{\partial z}\right)^2 + mgz + \frac{\partial S}{\partial t} = 0 \tag{v}$$

Now, let us write the Hamilton's principal function in terms of Hamilton's characteristic function as

$$S(z,\alpha,t) = W(z,\alpha) - \alpha t \tag{vi}$$

From this expression, we get

$$\frac{\partial S}{\partial z} = \frac{\partial W}{\partial z} \text{ and } \frac{\partial S}{\partial t} = -\alpha$$

Substituting this in equation (v), we get

$$\frac{1}{2m}\left(\frac{\partial W}{\partial z}\right)^2 + mgz - \alpha = 0$$

or

$$\left(\frac{\partial W}{\partial z}\right)^2 = 2m(\alpha - mgz)$$

and

$$\frac{\partial W}{\partial z} = \sqrt{2m(\alpha - mgz)}$$

On integration, we get

$$W = \sqrt{2m}\int \sqrt{(\alpha - mgz)}dz + c \tag{vii}$$

Now, equation (vi) becomes

$$S = \sqrt{2m}\int \sqrt{(\alpha - mgz)}dz + c - \alpha t \tag{viii}$$

Differentiating this with respect to α we get

$$\beta = \frac{\partial S}{\partial \alpha} = \sqrt{2m}\frac{\partial}{\partial \alpha}\left(\int \sqrt{(\alpha - mgz)}dz\right) - t$$

$$= \sqrt{2m}\left(\int \frac{1}{2\sqrt{(\alpha - mgz)}}dz\right) - t$$

$$= \sqrt{\frac{m}{2}}\left(\int \frac{1}{\sqrt{(\alpha - mgz)}}dz\right) - t$$

Now, we can perform the integration also to get

$$\beta = \sqrt{\frac{m}{2}}\left(\int \frac{mgdz}{mg\sqrt{(\alpha - mgz)}}\right) - t$$

$$= \sqrt{\frac{m}{2}}\frac{2}{mg}\sqrt{(\alpha - mgz)} - t$$

$$= \sqrt{\frac{2}{m}}\frac{1}{g}\sqrt{(\alpha - mgz)} - t$$

or

$$(\beta + t)^2 = \frac{2}{mg^2}(\alpha - mgz)$$

On rearranging we get

$$z = -\frac{1}{2}g(\beta + t)^2 + \frac{\alpha}{mg} \tag{ix}$$

Now, let at $t = 0$; $z = z_0$ and $p = 0$ so that

$$p = \left.\frac{\partial S}{\partial z}\right|_{z=0} = \left.\frac{\partial W}{\partial z}\right|_{z=0} = \sqrt{2m(\alpha - mgz_0)} = 0$$

and this would give

$$\alpha = mgz_0 \tag{x}$$

Substituting this in (ix), we get

$$z = z_0 - \frac{1}{2}g(\beta + t)^2 \tag{xii}$$

Also at $t = 0$; since $z = z_0$, from (xi), we get $\beta = 0$

Therefore,

$$z = z_0 - \frac{1}{2}gt^2 \tag{xii}$$

This expression describes the motion of a particle falling freely under the influence of gravitational force.

EXAMPLE 5.4 Solve the harmonic oscillator problem by using Hamilton-Jacobi method.

Solution: We start with the Hamiltonian of a linear harmonic oscillator as a function of the variables (q,p,t) and is given by

$$H(q,p,t)=\frac{p^2}{2m}+\frac{1}{2}kq^2 \tag{i}$$

Now, we have one of the transformation equations

$$p=\frac{\partial S}{\partial q}$$

where, $S(q,P,t)=S(q,\alpha,t)$ is the Hamilton's principal function and α is a constant. Hamilton's principal function is the generating function for the transformation such that the new Hamiltonian H' is zero.

Now, the Hamiltonian becomes

$$H(q,p,t)=\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{1}{2}kq^2 \tag{ii}$$

The Hamilton-Jacobi equation is

$$H+\frac{\partial S}{\partial t}=0 \tag{iii}$$

Using (ii) in (iii), we get

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{1}{2}kq^2+\frac{\partial S}{\partial t}=0 \tag{iv}$$

Now, we separate the variables and rewrite the solution as

$$S(q,\alpha,t)=W(q,\alpha)-\alpha t \tag{v}$$

Then, $$\frac{\partial S}{\partial q}=\frac{\partial W}{\partial q} \text{ and } \frac{\partial S}{\partial t}=-\alpha$$

With these substitutions, equation (iv) becomes

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2+\frac{1}{2}kq^2-\alpha=0$$

Multiplying throughout by $2m$, we get

$$\left(\frac{\partial W}{\partial q}\right)^2 + mkq^2 - 2m\alpha = 0$$

This expression can be rearranged to get

$$\frac{\partial W}{\partial q} = \sqrt{\left(2m\alpha - mkq^2\right)} = \sqrt{mk}\sqrt{\frac{2\alpha}{k} - q^2}$$

This can be integrated to get

$$W = \sqrt{mk}\int\left(\sqrt{\frac{2\alpha}{k} - q^2}\right)dq + c \tag{vi}$$

Now, equation (v) becomes

$$S(q,\alpha,t) = \sqrt{mk}\int\left(\sqrt{\frac{2\alpha}{k} - q^2}\right)dq - \alpha t + c \tag{vii}$$

In this equation c is the constant of integration and it will not affect the transformation. Now, the second transformation equation is $Q = \frac{\partial S}{\partial P}$ which can be written as

$$\beta = \frac{\partial S}{\partial \alpha} \tag{viii}$$

Equation (vii) can be differentiated with respect to α so that

$$\beta = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{k}}\int\frac{dq}{\sqrt{\frac{2\alpha}{k} - q^2}} - t$$

$$= \sqrt{\frac{m}{k}}\cos^{-1}\left[q\Big/\left(\sqrt{\frac{2\alpha}{k}}\right)\right] - t$$

$$= \sqrt{\frac{m}{k}}\cos^{-1}\left(q\sqrt{\frac{k}{2\alpha}}\right) - t$$

or
$$\beta + t = \sqrt{\frac{m}{k}}\cos^{-1}\left(q\sqrt{\frac{k}{2\alpha}}\right)$$

that is,
$$\sqrt{\frac{k}{m}}(\beta + t) = \cos^{-1}\left(q\sqrt{\frac{k}{2\alpha}}\right)$$

or
$$q = \sqrt{\frac{2\alpha}{k}} \cos\left[\sqrt{\frac{k}{m}}(\beta + t)\right]$$

$$= \sqrt{\frac{2\alpha}{k}} \cos\left[\omega(\beta + t)\right] \tag{ix}$$

where, $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency. Equation (ix) is the solution to the Harmonic oscillator problem.

In the solution we have two unknown quantities α and β. We now evaluate these quantities in terms of initial conditions. Let at time $t = 0$, the particle be at rest and the initial displacement is q_0. Then,

$$p_0 = \left.\frac{\partial S}{\partial q}\right|_{t=0} = \left.\frac{\partial W}{\partial q}\right|_{t=0} = 0$$

Then, we have

$$\left.\frac{\partial W}{\partial q}\right|_{t=0} = \sqrt{mk}\sqrt{\frac{2\alpha}{k} - q_0^2} = 0$$

that is,
$$\frac{2\alpha}{k} - q_0^2 = 0, \text{ so that;}$$

$$\alpha = \frac{1}{2}kq_0^2 \tag{x}$$

Thus, α represents the total energy of the harmonic oscillator. Putting the value of α in (ix), we get

$$q = q_0 \cos\left[\omega(\beta + t)\right] \tag{xi}$$

Now, at $t = 0$, we have $q = q_0$ from (xi), we get

$$\beta = 0$$

Then, the final solution of the harmonic oscillator problem is $q = q_0 \cos\omega t$. Note that the new momentum which is a constant is identified as the total energy of the system.

EXAMPLE 5.5 Discuss the Kepler problem in Hamilton-Jacobi method to obtain the path of motion.

Solution: Kepler problem, in general, deals with the path of a particle moving in a central force field that obeys inverse square law of force. In the present case we consider the motion of an electron of mass m and charge e around a nucleus. Taking r and θ as the generalized coordinates, we can write the expression for the Hamiltonian of the electron as

$$H = T + V = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{Ze^2}{r} \tag{i}$$

where Z is the atomic number and Ze is the charge of the nucleus, p_r and p_θ are the canonical momenta along r and θ directions respectively. Also, we assumed a conservative system. Then, for a conservative system, Hamiltonian represents the total energy of the system. That is,

$$H = E \equiv \alpha_1$$

Therefore,

$$\frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{Ze^2}{r} = E = \alpha_1$$

or

$$p_r^2 + \frac{p_\theta^2}{r^2} = 2mE + \frac{2mZe^2}{r} \tag{ii}$$

Now, let $S(r,\theta,\alpha_1,\alpha_2,t)$ be the generating function that generates the canonical transformation and is known as the Hamilton's principal function. Here α_1 and α_2 are the new canonical momenta P_r and P_θ respectively and are constants. The transformation equations, in general, are

$$p_j = \frac{\partial S}{\partial q_j} \text{ and } Q_j = \beta_j = \frac{\partial S}{\partial P_j} = \frac{\partial S}{\partial \alpha_j} \tag{iii}$$

Since the Hamiltonian has no explicit time dependence, we can separate the variables and the Hamilton's principal function can be written as

$$S(r,\theta,\alpha_1,\alpha_2,t) = W(r,\theta,\alpha_1,\alpha_2) - \alpha t \tag{iv}$$

Then, we get

$$p_r = \frac{\partial S}{\partial r} = \frac{\partial W}{\partial r} \text{ and } p_\theta = \frac{\partial S}{\partial \theta} = \frac{\partial W}{\partial \theta} \tag{v}$$

This can be substituted in equation (ii), to get

$$\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 = 2mE + \frac{2mZe^2}{r} \tag{vi}$$

Further, we write; $W = W_r(r) + W_\theta(\theta)$ so that the above expression becomes

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 = 2mE + \frac{2mZe^2}{r} \tag{vii}$$

Since we are considering the motion under a central force field, the angular momentum is conserved. That is,

$$p_\theta = \frac{\partial S}{\partial \theta} = \frac{\partial W}{\partial \theta} = \alpha_2, \text{ a constant}$$

On integration, we get

$$W = \alpha_2\theta + constant \tag{viii}$$

With this substitution (vii) becomes

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_2^2}{r^2} = 2mE + \frac{2mZe^2}{r}$$

or
$$\frac{\partial W_r}{\partial r} = \left(2mE + \frac{2mZe^2}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2}$$

Then,
$$W_r = \int \left(2mE + \frac{2mZe^2}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2} dr + \text{constant} \tag{ix}$$

Using (viii) and (ix), the Hamilton's characteristic function can be written as

$$W = W_r(r) + W_\theta(\theta)$$
$$= \alpha_2\theta + \int \left(2mE + \frac{2mZe^2}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2} dr + \text{constant} \tag{x}$$

Now, we have $H = \alpha_1$, so that the canonical transformation is generated by the function $W(r, \theta, \alpha_1, \alpha_2)$. Then the new Hamiltonian is

$$H' = H + \frac{\partial W}{\partial t} = \alpha_1 + \frac{\partial W(r, \theta, \alpha_1, \alpha_2)}{\partial t} = \alpha_1 \tag{xi}$$

Hamilton's canonical equations of motion gives

$$\dot{Q}_1 = \frac{\partial H'}{\partial \alpha_1} = \frac{\partial \alpha_1}{\partial \alpha_1} = 1$$

that is,
$$\frac{dQ_1}{dt} = 1; \quad \text{or, } Q_1 = t + \beta_1 \tag{xii}$$

Similarly,
$$\dot{Q}_2 = \frac{\partial H'}{\partial \alpha_2} = \frac{\partial \alpha_1}{\partial \alpha_2} = 0$$

that is,
$$\frac{dQ_2}{dt} = 0; \quad \text{or, } Q_2 = \beta_2 \tag{xiii}$$

Also from the transformation equations (iii), we get

$$Q_1 = \frac{\partial W}{\partial \alpha_1} \quad \text{and} \quad Q_2 = \frac{\partial W}{\partial \alpha_2} \tag{xiv}$$

From the equations (xii), (xiii) and (xiv), we get

$$\frac{\partial W}{\partial \alpha_1} = t + \beta_1 \quad \text{and} \quad \frac{\partial W}{\partial \alpha_2} = \beta_2 \tag{xv}$$

Now, equation (x) with $E = \alpha_1$ gives

$$\frac{\partial W}{\partial \alpha_1} = t + \beta_1 = \int \frac{m\,dr}{\left(2mE + \dfrac{2mZe^2}{r} - \dfrac{\alpha_2^2}{r^2}\right)^{1/2}} \tag{xvi}$$

and

$$\frac{\partial W}{\partial \alpha_2} = \beta_2 = \theta - \int \frac{\alpha_2\,dr}{r^2\left(2mE + \dfrac{2mZe^2}{r} - \dfrac{\alpha_2^2}{r^2}\right)^{1/2}} \tag{xvii}$$

Equation (xvi) can be integrated to get the position as a function of time. Now, let us perform the integration in (xvii) by putting

$$r = \frac{1}{u} \text{ and } dr = -\frac{du}{u^2}$$

With this substitution, (xvii) becomes

$$\theta - \beta_2 = -\int \frac{\alpha_2\,dr}{r^2\left(2mE + \dfrac{2mZe^2}{r} - \dfrac{\alpha_2^2}{r^2}\right)^{1/2}} = -\int \frac{\alpha_2\,dr}{r^2\left(\dfrac{2mE}{\alpha_2^2} + \dfrac{2mZe^2}{\alpha_2^2 r} - \dfrac{1}{r^2}\right)^{1/2}}$$

$$= -\int \frac{du}{\left(\dfrac{2mE}{\alpha_2^2} + \dfrac{2mZe^2}{\alpha_2^2}u - u^2\right)^{1/2}}$$

$$= \cos^{-1}\frac{\alpha_2^2 u - mZe^2}{\left(2mE\alpha_2^2 + \dfrac{Z^2m^2e^4}{\alpha_2^4}\right)^{1/2}} = \cos^{-1}\frac{\dfrac{\alpha_2^2}{r} - mZe^2}{\left(2mE\alpha_2^2 + \dfrac{Z^2m^2e^4}{\alpha_2^4}\right)^{1/2}}$$

Then,

$$\frac{\dfrac{\alpha_2^2}{r} - mZe^2}{\left(2mE\alpha_2^2 + \dfrac{Z^2m^2e^4}{\alpha_2^4}\right)^{1/2}} = \cos(\theta - \beta_2)$$

or

$$\frac{\alpha_2^2}{r} - mZe^2 = \left(2mE\alpha_2^2 + Z^2m^2e^4\right)^{1/2}\cos(\theta - \beta_2)$$

that is,

$$\frac{\alpha_2^2}{r} = \frac{mZe^2}{\alpha_2^2} + \left(2mE\alpha_2^2 + \frac{Z^2m^2e^4}{\alpha_2^4}\right)^{1/2}\cos(\theta - \beta_2)$$

or
$$\frac{1}{r}=\frac{mZe^2}{\alpha_2^2}\left[1+\left(1+\frac{2E\alpha_2^2}{Z^2me^4}\right)^{1/2}\cos(\theta-\beta_2)\right] \tag{xviii}$$

This can be written as

$$\frac{1}{r}=\frac{1}{l\varepsilon}\left[1-\varepsilon\cos(\theta-\beta_2)\right] \tag{xix}$$

where, $\varepsilon=\left(1+\frac{2E\alpha_2^2}{Z^2me^4}\right)^{1/2}$ the eccentricity of the conic section and $l\varepsilon=\frac{\alpha_2^2}{mZe^2}$

Thus, it is clear that, if $E<0$, the eccentricity $\varepsilon<1$ and the path of the particle is an ellipse. If $E=0$, $\varepsilon=1$ so that the path is a parabola and if $E>0$, $\varepsilon>1$; the path will be a hyperbola.

EXAMPLE 5.6 A particle of mass m is thrown vertically upwards with an initial velocity u. Obtain the equation of motion by Hamilton-Jacobi method.

Solution: Let us start with the Hamiltonian of the particle which is thrown in the upward direction. It is given by

$$H=\frac{p_z^2}{2m}+mgz \tag{i}$$

where, p_z is the canonical momentum and z is the vertical direction.

The Hamilton-Jacobi equation is given by

$$H+\frac{\partial S}{\partial t}=0 \tag{ii}$$

where $S(z,P_z,t)$ is Hamilton's principal function and the generator of the transformation.

Now, the transformation equations are

$$p_z=\frac{\partial S}{\partial z} \text{ and } Z=\frac{\partial S}{\partial P_z} \tag{iii}$$

Now, let us write the Hamilton's principal function in terms of Hamilton's characteristic function as

$$S(z,\alpha,t)=W(z,\alpha)-\alpha t \tag{iv}$$

From this expression, we get

$$\frac{\partial S}{\partial z}=\frac{\partial W}{\partial z} \text{ and } \frac{\partial S}{\partial t}=-\alpha \tag{v}$$

Now, the Hamilton-Jacobi equation becomes

$$\frac{p_z^2}{2m}+mgz+\frac{\partial S}{\partial t}=0$$

or
$$\frac{1}{2m}\left(\frac{\partial W}{\partial z}\right)^2 + mgz - \alpha = 0 \tag{vi}$$

This can be rearranged to get

$$\frac{\partial W}{\partial z} = \sqrt{2m(\alpha - mgz)}$$

On integration, this would give

$$W = \int\left[\sqrt{2m(\alpha - mgz)}\right]dz \tag{vii}$$

Now, the transformation equation $Z = \frac{\partial S}{\partial P_z}$ gives

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial \alpha} - t$$

or
$$\beta + t = \frac{\partial W}{\partial \alpha} \tag{viii}$$

From (vii) we get

$$\frac{\partial W}{\partial \alpha} = \int \frac{mdz}{\sqrt{2m(\alpha - mgz)}}$$

Then (viii) becomes

$$\beta + t = \int \frac{mdz}{\sqrt{2m(\alpha - mgz)}}$$

This can be integrated directly to get

$$\beta + t = -\frac{1}{g}\sqrt{\frac{2}{m}(\alpha - mgz)} \tag{ix}$$

Also, we gave $H = \alpha = \frac{1}{2}mu^2$, the total energy of the particle. Using this in (ix), we get

$$\beta + t = -\frac{1}{g}\sqrt{\frac{2}{m}\left(\frac{1}{2}mu^2 - mgz\right)} = -\frac{1}{g}\sqrt{u^2 - 2gz} \tag{x}$$

At $t = 0$, we have $x = 0$ and therefore; $\beta = -\frac{u}{g}$, so that

$$-\frac{u}{g} + t = -\frac{1}{g}\sqrt{u^2 - 2gz}$$

or
$$-u + gt = -\sqrt{u^2 - 2gz}$$

Squaring both sides, we get

$$u^2 - 2ugt + g^2t^2 = u^2 - 2gz$$

or
$$-2ut + gt^2 = -2z$$

that is,
$$z = ut - \frac{1}{2}gt^2 \tag{xi}$$

Equation (xi) is the required result.

EXAMPLE 5.7 Use Hamilton-Jacobi method to discuss the motion of a particle in one dimension.

Solution: We consider the particle of mass m moving in the x direction. Let p_x be the momentum of the particle and $V(x)$ the potential energy. In general, the potential energy may depend on other coordinates also. Then, the Hamiltonian of the particle is

$$H = T + V = \frac{p_x^2}{2m} + V(x) \tag{i}$$

Now we introduce the Hamilton's principal function S as the generator of the canonical transformation in which the new coordinate and momentum are constants and the new Hamiltonian vanishes identically. Then the Hamilton-Jacobi equation gives

$$H + \frac{\partial S}{\partial t} = 0 \text{ and } H' = 0 \tag{ii}$$

The general transformation equations are; $p_j = \dfrac{\partial S}{\partial q_j}$ and $Q_j = \dfrac{\partial S}{\partial P_j}$ which in the present case gives

$$p_x = \frac{\partial S}{\partial x} \tag{iii}$$

Now, the Hamiltonian becomes

$$H = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right) + V(x)$$

and the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right) + V(x) + \frac{\partial S}{\partial t} = 0 \tag{iv}$$

Now, let us write the Hamilton's principal function as

$$S(x, P_x, t) = W(x, P_x) - \alpha t \tag{v}$$

where, W is Hamilton's characteristic function, P_x is the new momentum which is a constant and α is a constant.

From (v), we have

$$\frac{\partial S}{\partial x} = \frac{\partial W}{\partial x} \text{ and } \frac{\partial S}{\partial t} = -\alpha$$

With this substitution, Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)+V(x)-\alpha=0$$

or
$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)+V(x)=\alpha\equiv E \tag{vi}$$

where E is the total energy of the system. From (vi) on rearrangement, we get

$$\frac{\partial W}{\partial x}=\left\{2m\left[E-V(x)\right]\right\}^{1/2}$$

and
$$W=\int_0^x\left\{2m\left[E-V(x)\right]\right\}^{1/2}dx \tag{vii}$$

Therefore,
$$S=\int_0^x\left\{2m\left[E-V(x)\right]\right\}^{1/2}dx-Et \tag{viii}$$

Further, we have; the new coordinate $X=\beta$ given by

$$\beta=\frac{\partial S}{\partial\alpha}=\frac{\partial S}{\partial E}$$

From (viii) by differentiating with respect to E, we get

$$\beta=\int_0^x\frac{mdx}{\left\{2m\left[E-V(x)\right]\right\}^{1/2}}-t \tag{ix}$$

The integration can be performed if we know the dependence of the potential on the coordinate

Now,
$$p_x=\frac{\partial S}{\partial x}=\left\{2m\left[E-V(x)\right]\right\}^{1/2} \tag{x}$$

Equations (ix) and (x) describe the motion of the particle.

EXAMPLE 5.8 A particle of mass m is projected with an initial velocity u making an angle θ with the horizontal. Obtain the equations of motion by using Hamilton-Jacobi method.

Solution: Let us assume the motion of the projectile in the $x-y$ plane, x the horizontal and y the vertical, so that the Hamiltonian of the particle can be written as

$$H=\frac{1}{2m}\left(p_x^2+p_y^2\right)+mgy \tag{i}$$

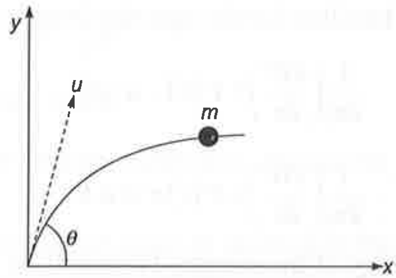

Fig. 5.1

We assume a conservative system so that the Hamiltonian represents the total energy. Also, for a conservative system Hamilton's principal function S can be separated into a time independent and a time dependent part. The time independent part is Hamilton's characteristic function W. Now, the Hamilton's characteristic function itself is a function of the coordinates x and y. That is,

$$W = W(x) + W(y)$$

Now, we have the transformation equations; $p_j = \frac{\partial S}{\partial q_j}$ and $Q_j = \frac{\partial S}{\partial P_j}$, from which we get

$$p_x = \frac{\partial S}{\partial x} = \frac{\partial W_x}{\partial x} \text{ and } p_y = \frac{\partial S}{\partial y} = \frac{\partial W_y}{\partial y}$$

Then, the Hamiltonian becomes

$$H = \frac{1}{2m}\left[\left(\frac{\partial W_x}{\partial x}\right)^2 + \left(\frac{\partial W_y}{\partial y}\right)^2\right] + mgy = E$$

or

$$\left(\frac{\partial W_y}{\partial y}\right)^2 = 2m(E - mgy) - \left(\frac{\partial W_x}{\partial x}\right)^2 = 2m(E - mgy) - p_x^2$$

so that $$\frac{\partial W_y}{\partial y} = \left[2m(E - mgy) - p_x^2\right]^{1/2}$$

and $$W_y = \int \left[2m(E - mgy) - p_x^2\right]^{1/2} dy \qquad \text{(ii)}$$

Similarly, $$W_x = p_x x \qquad \text{(ii)}$$

Then, $$W = p_x x + \int \left[2m(E - mgy) - p_x^2\right]^{1/2} dy \qquad \text{(iii)}$$

Now, Hamilton's principal function can be written as

$$S = W - Et$$

$$= p_x x + \int \left[2m(E - mgy) - p_x^2\right]^{1/2} dy - Et \qquad \text{(iv)}$$

and Hamilton-Jacobi equation is

$$H + \frac{\partial S}{\partial t} = 0 \quad \text{or,} \quad H = -\frac{\partial S}{\partial t} = E \tag{v}$$

Now, we have

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{\partial S}{\partial E} = \int \frac{m\,dy}{\left[2m(E - mgy) - p_x^2\right]^{1/2}} - t$$

or

$$\beta + t = \int \frac{m\,dy}{\left[2m(E - mgy) - p_x^2\right]^{1/2}} \tag{vi}$$

Again, we have

$$\beta_x = \frac{\partial W}{\partial p_x} = x - \int \frac{p_x\,dy}{\left[2m(E - mgy) - p_x^2\right]^{1/2}}$$

On integration, this would yield

$$\beta_x = x + \frac{p_x}{m^2 g}\left[2m(E - mgy) - p_x^2\right]^{1/2} \tag{vii}$$

Since β_x is a constant, we can put $\beta_x = x_0$, the initial coordinate. Then,

$$x_0 = x + \frac{p_x}{m^2 g}\left(2mE - 2m^2 gy - p_x^2\right)^{1/2}$$

or

$$(x - x_0)^2 = \frac{p_x^2}{m^4 g^2}\left(2mE - 2m^2 gy - p_x^2\right)$$

This can be rearranged to get

$$\frac{1}{2} m^2 g \left(\frac{x - x_0}{p_x}\right)^2 = \frac{1}{2m^2 g}\left(2mE - p_x^2\right) - y \quad = y_0 - y$$

or

$$y = y_0 - \frac{1}{2} m^2 g \left(\frac{x - x_0}{p_x}\right)^2 \tag{viii}$$

where, $y_0 = \frac{1}{2m^2 g}\left(2mE - p_x^2\right)$ and is a constant.

We have $p_x = m\dot{x} = mu\cos\theta$ and using this in (viii), we get

$$y = y_0 - \frac{1}{2} g \left(\frac{x - x_0}{u\cos\theta}\right)^2 \tag{ix}$$

This represents a parabola and therefore, the path of a projectile is a parabola. The motion along the horizontal can be described as

$$m\dot{x} = p_x \qquad \text{or,} \quad x = \int_{t_0}^{t} \frac{p_x}{m}\,dt$$

or $$x = x_0 + \frac{p_x}{m}(t - t_0) \tag{x}$$

Equations (ix) and (x) describes the motion of the projectile.

EXAMPLE 5.9 A particle of mass m is thrown horizontally from the top of a building of height h with an initial velocity u. Obtain the equation of motion by using Hamilton-Jacobi method.

Solution: We start with the expression for the Hamiltonian of the particle, and is the sum of kinetic and potential energies.

The kinetic energy of the particle at any instant is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2m}\left(p_x^2 + p_y^2\right)$$

Potential energy of the particle with respect to the top of the building is

$$V = -mg(h - y)$$

Then, the Hamiltonian of the particle is

$$H = T + V = \frac{1}{2m}\left(p_x^2 + p_y^2\right) - mg(h - y) \tag{i}$$

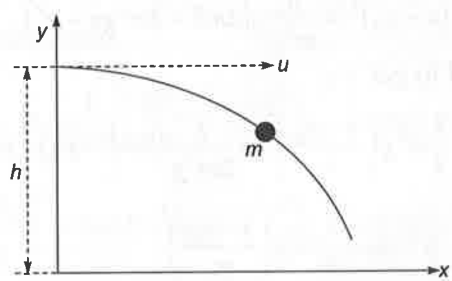

Fig. 5.2

Note that x is cyclic in Hamiltonian and hence, $p_x \equiv p = mu$ is a constant.

From Hamilton-Jacobi equation, we have;

$$H = -\frac{\partial S}{\partial t}$$

Since $$S = W - \alpha t, \quad \frac{\partial S}{\partial t} = -\alpha$$

or $$H = \alpha, \text{ is a constant.} \tag{ii}$$

Now, we have the transformation equations $p_j = \dfrac{\partial S}{\partial q_j}$ and $Q_j = \dfrac{\partial S}{\partial P_j}$, from which, we get

$$p_x = \frac{\partial S}{\partial x} = \frac{\partial W}{\partial x} \text{ and } p_y = \frac{\partial S}{\partial y} = \frac{\partial W}{\partial y}$$

However, we will not substitute for p_x since it is a constant. Substituting for p_y and using (ii), equation (i) becomes

$$\frac{p^2}{2m} + \frac{1}{2m}\left(\frac{\partial W}{\partial y}\right) - mg(h-y) = \alpha$$

or

$$\left(\frac{\partial W}{\partial y}\right)^2 = 2m\left[\left(\alpha - \frac{p^2}{2m}\right) + mg(h-y)\right]$$

On further rearrangement, we get

$$W = \int \sqrt{2m}\left[\left(\alpha - \frac{p^2}{2m}\right) + mg(h-y)\right]^{1/2} dy \qquad \text{(iii)}$$

Now, the new constant coordinate β is given by

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial \alpha} - t$$

Then,

$$\beta + t = \frac{\partial W}{\partial \alpha} = \sqrt{\frac{m}{2}}\int \left[\left(\alpha - \frac{p^2}{2m}\right) + mg(h-y)\right]^{-1/2} dy \qquad \text{(iv)}$$

Now, $H = \alpha \equiv E = \dfrac{1}{2}mu^2 = \dfrac{p^2}{2m}$, then the above expression reduces to

$$\beta + t = \sqrt{\frac{m}{2}}\int \frac{1}{\left[mg(h-y)\right]^{1/2}} dy$$

On integration, we get

$$\beta + t = -\sqrt{\frac{2(h-y)}{g}} \qquad \text{(v)}$$

The value of β can be obtained from the initial condition that at $t = 0$; $y = h$. Then, we get $\beta = 0$. Therefore,

$$t = -\sqrt{\frac{2(h-y)}{g}}$$

Squaring both sides and rearranging, we get

$$t^2 = \frac{2}{g}(h-y) \tag{vi}$$

Now, we have; $x = ut$

or
$$x^2 = \frac{2u^2}{g}(h-y) \tag{vii}$$

Equation (vii) represents a parabola and hence, the path of the projected particle is a parabola.

EXAMPLE 5.10 A particle of mass m is executing damped harmonic oscillations. Solve the equation of motion by Hamilton-Jacobi method.

Solution: The solution to this problem is not straightforward, because, the Hamiltonian of the particle is not a constant of motion and it does not represent the total energy of the system. This is due to the damping and the energy decreases continuously. Therefore, first we seek a canonical transformation from the old set of variables to a new set of variables such that the new Hamiltonian is a constant of motion. Thereafter we solve the Hamilton-Jacobi equation.

In general, the equation of motion of a damped harmonic oscillator can be written as

$$m\ddot{q} + \lambda\dot{q} + kq = 0 \tag{i}$$

where, λ is the damping coefficient. The Lagrangian of a damped harmonic oscillator is

$$L = e^{\lambda t/m}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right)$$

So that the canonical momentum is; $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}e^{\lambda t/m}$

The Hamiltonian of the particle is

$$H = e^{\lambda t/m}\frac{p^2}{2m} + \frac{1}{2}kq^2 e^{-\lambda t/m} \tag{ii}$$

Note that the Hamiltonian has explicit time dependence and is not a constant of motion. Also it cannot be considered the total energy of the system. We now try a transformation generated by the function given by

$$F_2(q,P,t) = qPe^{\lambda t/2m} \tag{iii}$$

and the transformation equations given by

$$Q = qe^{\lambda t/2m} \quad \text{and} \quad P = pe^{\lambda t/2m} \tag{iv}$$

so that the new Hamiltonian becomes a constant of motion.

The new Hamiltonian is

$$H'(Q,P,t) = H(q,p,t) + \frac{\partial F_2}{\partial t}$$

$$= e^{\lambda t/m}\frac{p^2}{2m} + \frac{1}{2}kq^2 e^{-\lambda t/m} + \frac{\partial F_2}{\partial t} \tag{v}$$

Substituting for q, p and for $\frac{\partial F_2}{\partial t}$ using equations (iii) and (iv), we get

$$H'(Q,P,t) = \frac{P^2}{2m} + \frac{1}{2}kQ^2 + \frac{\lambda}{2m}QP \tag{vi}$$

Now, the new Hamiltonian is independent of time and hence, is a constant and is equal to the total energy, $H' = E = \alpha$. Therefore, we can apply Hamilton-Jacobi equation to the new Hamiltonian to get

$$H' + \frac{\partial S}{\partial t} = 0$$

or

$$\frac{P^2}{2m} + \frac{1}{2}kQ^2 + \frac{\lambda}{2m}QP + \frac{\partial S}{\partial t} = 0 \tag{vii}$$

where, $S(Q,\alpha,t)$ is Hamilton's principal function. The transformation equation is

$$P = \frac{\partial S}{\partial Q}$$

so that the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left(\frac{\partial S}{\partial Q}\right)^2 + \frac{1}{2}kQ^2 + \frac{\lambda}{2m}Q\frac{\partial S}{\partial Q} + \frac{\partial S}{\partial t} = 0 \tag{viii}$$

Now, let us write Hamilton's principal function as the sum of Hamilton's characteristic function and a time dependent part as

$$S(Q,\alpha,t) = W(Q,\alpha) - \alpha t$$

so that $\qquad \dfrac{\partial S}{\partial Q} = \dfrac{\partial W}{\partial Q}$ and $\dfrac{\partial S}{\partial t} = -\alpha$

With this equation (viii) becomes

$$\frac{1}{2m}\left(\frac{\partial W}{\partial Q}\right)^2 + \frac{1}{2}kQ^2 + \frac{\lambda}{2m}Q\frac{\partial W}{\partial Q} - \alpha = 0 \tag{ix}$$

Now put $Q = \dfrac{x}{(mk)^{1/4}}$ and $dQ = \dfrac{dx}{(mk)^{1/4}}$ in the above equation. Then,

$$\frac{1}{2m}\left(\frac{dW}{dx}\right)^2 (mk)^{1/2} + \frac{1}{2}k\left(\frac{x}{(mk)^{1/4}}\right)^2 + \frac{\lambda}{2m}\left(\frac{x}{(mk)^{1/4}}\right)\frac{dW}{dx}(mk)^{1/4} - \alpha = 0$$

or
$$\sqrt{\frac{k}{m}}\left(\frac{dW}{dx}\right)^2 + \sqrt{\frac{k}{m}}x^2 + \frac{\lambda}{m}x\frac{dW}{dx} - 2\alpha = 0$$

or
$$\left(\frac{dW}{dx}\right)^2 + x^2 + \frac{\lambda}{\sqrt{km}}\frac{dW}{dx}x - 2\sqrt{\frac{m}{k}}\alpha = 0$$

that is,
$$\left(\frac{dW}{dx}\right)^2 + \frac{\lambda}{\sqrt{km}}\frac{dW}{dx}x + \left(x^2 - 2\sqrt{\frac{m}{k}}\alpha\right) = 0 \tag{x}$$

Now put $\dfrac{\lambda}{\sqrt{km}} = a$ and $2\sqrt{\dfrac{m}{k}}\alpha = b$ so that equation (x) becomes

$$\left(\frac{dW}{dx}\right)^2 + a\frac{dW}{dx}x + \left(x^2 - b\right) = 0 \tag{xi}$$

On integration this would give

$$W(x) = -\frac{\alpha x^2}{4} + \int dx\sqrt{b - \left(1 - \frac{a^2}{4}\right)x^2} = -\frac{\alpha x^2}{4} + \int dx\sqrt{b - c^2x^2} \tag{xii}$$

where, $c^2 = 1 - \dfrac{a^2}{4}$.

Then, the nature of damped oscillation is determined by the value of a. If $a > 2$, it is an overdamped oscillation. If $a = 2$, the oscillation is critically damped and if $a < 2$ the oscillation is underdamped.

Now, Hamilton's principal function becomes

$$S = -\frac{\alpha x^2}{4} + \int dx\sqrt{b - cx^2} - \alpha t \tag{xiii}$$

We have
$$\beta = \frac{\partial S}{\partial \alpha} = -t + \sqrt{\frac{m}{k}}\int \frac{dx}{\sqrt{b - c^2x^2}}$$

or
$$\beta + t = \sqrt{\frac{m}{k}}\int \frac{dx}{c\sqrt{\frac{b}{c^2} - x^2}} = \frac{1}{c}\sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{cx}{\sqrt{b}}\right)$$

or $$x=\frac{\sqrt{b}}{c}\sin\left[c\sqrt{\frac{k}{m}}(\beta+t)\right]$$

or $$(mk)^{1/4}Q=\frac{\sqrt{b}}{c}\sin\left[c\sqrt{\frac{k}{m}}(\beta+t)\right]$$

that is, $$Q=(mk)^{-1/4}\frac{\sqrt{b}}{c}\sin\left[c\omega(\beta+t)\right] \tag{xiv}$$

From (iv), we have

$$q=Qe^{-\lambda t/2m}$$
$$=(mk)^{-1/4}\frac{\sqrt{b}}{c}\sin\left[c\omega(\beta+t)\right]e^{-\lambda t/2m}$$

Since a, b and c are constants, we can write

$$q=Ae^{-\lambda t/2m}\sin[\omega' t+\delta] \tag{xv}$$

with $$A=(mk)^{-1/4}\frac{\sqrt{b}}{c},\ \omega'=c\omega \text{ and } \delta=\omega'\beta.$$

Equation (xv) is the equation representing damped harmonic oscillations. The amplitude is $Ae^{-\lambda t/2m}$ and decreases with time.

EXAMPLE 5.11 Using Hamilton-Jacobi method show that the motion of a free particle of mass m in a gravitational field in the z-direction is described completely by the equation; $z=\frac{\alpha_3}{mg}-\frac{g}{2}(\beta_3+t)^2$, where α_3 and β_3 are the new momentum and coordinate respectively corresponding to the z-coordinate.

Solution: Let us start with the Hamiltonian of the particle moving in an arbitrary direction in the gravitational field. It is given by

$$H=\frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right)+mgz \tag{i}$$

Now, the Hamilton's principal function is given by

$$S=S(x,y,z;\alpha_1,\alpha_2,\alpha_3;t)$$

where, α_j is the new canonical momentum.

Then, the transformation equations $p_j=\frac{\partial S}{\partial q_j}$ and $Q_j=\beta_j=\frac{\partial S}{\partial \alpha_j}$ gives

$$p_x=\frac{\partial S}{\partial x},\ p_y=\frac{\partial S}{\partial y} \text{ and } p_z=\frac{\partial S}{\partial z} \tag{ii}$$

This can be substituted in (i) to get

$$H=\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2\right]+mgz \tag{iii}$$

The Hamilton-Jacobi equation is

$$H+\frac{\partial S}{\partial t}=0$$

or

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2\right]+mgz+\frac{\partial S}{\partial t}=0 \tag{iv}$$

Since the Hamiltonian does not contain time explicitly, we can separate the variables and write the Hamilton's principal function as

$$S=W_1(x)+W_2(y)+W_3(z)-\alpha t \tag{v}$$

Using this in (iv), we get

$$\frac{1}{2m}\left[\left(\frac{\partial W_1(x)}{\partial x}\right)^2+\left(\frac{\partial W_2(y)}{\partial y}\right)^2+\left(\frac{\partial W_3(z)}{\partial z}\right)^2\right]+mgz=\alpha \tag{vi}$$

Since the RHS of equation (vi) is a constant, and LHS contains three terms, we can write the constant α as; $\alpha=\alpha_1+\alpha_2+\alpha_3$, with

$$\alpha_1=\frac{1}{2m}\left(\frac{\partial W_1(x)}{\partial x}\right)^2$$

$$\alpha_2=\frac{1}{2m}\left(\frac{\partial W_2(y)}{\partial y}\right)^2 \quad \text{and;}$$

$$\alpha_3=\frac{1}{2m}\left(\frac{\partial W_2(z)}{\partial z}\right)^2+mgz$$

From these expressions, we can have

$$\left(\frac{\partial W(x)}{\partial x}\right)^2=2m\alpha_1 \quad \text{or; } W_1(x)=x\sqrt{2m\alpha_1} \tag{vii}$$

$$\left(\frac{\partial W_2(y)}{\partial y}\right)^2=2m\alpha_2 \quad \text{or; } W_2(y)=x\sqrt{2m\alpha_2} \tag{viii}$$

$$\left(\frac{\partial W_2(z)}{\partial z}\right)^2=2m(\alpha_3-mgz) \quad \text{or; } \frac{\partial W_2(z)}{\partial z}=\sqrt{2m(\alpha_3-mgz)}$$

$$W_2(z)=\sqrt{2m}\,\frac{2}{3mg}(\alpha_3-mgz)^{3/2}$$

$$=\sqrt{\frac{8}{9mg^2}}(\alpha_3-mgz)^{3/2} \tag{ix}$$

Now, using the second transformation $Q_j=\beta_j=\dfrac{\partial S}{\partial \alpha_j}$, we get

$$\beta_1=\frac{\partial S}{\partial \alpha_1}=\frac{\partial W_1(x)}{\partial \alpha_1}-t=x\sqrt{\frac{m}{2\alpha_1}}-t$$

Also, $$\beta_2=\frac{\partial S}{\partial \alpha_2}=\frac{\partial W_2(x)}{\partial \alpha_2}-t=y\sqrt{\frac{m}{2\alpha_2}}-t$$

and $$\beta_3=\frac{\partial S}{\partial \alpha_3}=\frac{\partial W_3(x)}{\partial \alpha_3}-t=y\sqrt{\frac{2(\alpha_2-mgz)}{mg^2}}-t$$

From these expressions, by rearranging, we get

$$x=(\beta_1+t)\sqrt{\frac{2\alpha_1}{m}}, \tag{x}$$

$$y=(\beta_2+t)\sqrt{\frac{2\alpha_2}{m}} \tag{xi}$$

and $$z=\sqrt{\frac{mg^2}{2(\alpha_2-mgz)}}(\beta_3+t) \tag{xii}$$

Equations (x), (xi) and (xii) describes the motion of a free particle in a gravitational field completely. The value of the constants α_j and β_j depends on the initial values of the old coordinates (x,y,z) at $t=0$.

EXAMPLE 5.12 Obtain Hamilton's principal function $S(z,t,z_0,t_0)$ for a particle of mass m moving vertically in the uniform gravitational field near the surface of the earth by integrating the Lagrangian along the actual path which joins the end points.

Solution: Let z be the vertical direction. Then the Hamiltonian of the particle is given by;

$$L(\dot{z},z)=\frac{1}{2}m\dot{z}^2-mgz \tag{i}$$

The Hamilton's principal function is given by

$$S(z,t,z_0,t_0)=\int_{t_0}^{t}L(\dot{z},z)\,dt=\int_{t_0}^{t}\left(\frac{1}{2}m\dot{z}^2-mgz\right)dt \tag{ii}$$

Now, we have the instantaneous position and velocity of the particle as

$$z(t') = z_0 + v_0(t'-t_0) - \frac{1}{2}g(t'-t_0)^2 \tag{iii}$$

and $$\dot{z}(t') = v_0 - g(t'-t_0) \tag{iv}$$

Here, z_0 and v_0 are the initial position and velocity respectively. If the initial velocity is chosen such that the particle reaches the end point z at time t. This requires

$$v_0 = \frac{z-z_0}{t-t_0} + \frac{1}{2}g(t-t_0) \tag{v}$$

Using equations (iii) and (iv) in (ii), we get

$$S(z,t,z_0,t_0) = \int_{t_0}^{t}\left\{\frac{1}{2}m\left[v_0 - g(t-t_0)\right]^2 - mg\left[z_0 + v_0(t-t_0) - \frac{1}{2}g(t-t_0)^2\right]\right\}dt$$

This can be simplified to get

$$S(z,t,z_0,t_0) = \left(\frac{1}{2}mv_0^2 - mgz_0\right)(t-t_0) - mgv_0(t-t_0)^2 + \frac{1}{3}mg^2(t-t_0)^3 \tag{vi}$$

Now, we substitute for v_0 in equation (vi) to get

$$S(z,t,z_0,t_0) = \frac{m}{2}\frac{z-z_0}{(t-t_0)} - \frac{1}{2}mg(z+z_0)(t-t_0) - \frac{1}{24}mg^2(t-t_0)^3 \tag{vii}$$

Equation (vii) is the required result.

EXAMPLE 5.13 Using Hamilton-Jacobi method, obtain the equation of motion of a three-dimensional isotropic harmonic oscillator.

Solution: We start with the Hamiltonian of a three-dimensional isotropic harmonic oscillator. It is given by the expression

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}kr^2$$
$$= \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}k\left(x^2 + y^2 + z^2\right) \tag{i}$$

Now, the Hamilton-Jacobi equation is

$$H + \frac{\partial S}{\partial t} = 0$$

and the Hamilton's principal function is given by

$$S = S\left(x, y, z; \alpha_x, \alpha_y, \alpha_z; t\right)$$

where, α_j are the new canonical momenta.

Then, the transformation equations $p_j = \dfrac{\partial S}{\partial q_j}$ and $Q_j = \beta_j = \dfrac{\partial S}{\partial \alpha_j}$ gives

$$p_x = \frac{\partial S}{\partial x},\ p_y = \frac{\partial S}{\partial y} \text{ and } p_z = \frac{\partial S}{\partial z} \tag{ii}$$

This can be substituted in (i) to get

$$H = \frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] + \frac{1}{2}k\left(x^2 + y^2 + z^2\right) \tag{iii}$$

Since the Hamiltonian does not depend on time explicitly, we can separate the variables as

$$S = W - \alpha t$$

and write the time independent Hamilton-Jacobi equation as

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right] + \frac{1}{2}k\left(x^2 + y^2 + z^2\right) = \alpha \tag{iv}$$

Now, we assume; $W = X(x) + Y(y) + Z(z)$ so that the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left[\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2\right] + \frac{1}{2}k\left(x^2 + y^2 + z^2\right) = \alpha$$

or

$$\left[\frac{1}{2m}\left(\frac{\partial X}{\partial x}\right)^2 + \frac{1}{2}kx^2\right] + \left[\frac{1}{2m}\left(\frac{\partial Y}{\partial y}\right)^2 + \frac{1}{2}ky^2\right] + \left[\frac{1}{2m}\left(\frac{\partial Z}{\partial z}\right)^2 + \frac{1}{2}kz^2\right] = \alpha$$

Since the RHS of the above equation is a constant and the LHS is the sum of three terms, each term on LHS is equal to a constant. Thus, we write

$$\left[\frac{1}{2m}\left(\frac{\partial X}{\partial x}\right)^2 + \frac{1}{2}kx^2\right] = \alpha_x \quad \text{or;} \quad X = \int\sqrt{\left(2m\alpha_x - mkx^2\right)}dx$$

$$\left[\frac{1}{2m}\left(\frac{\partial Y}{\partial y}\right)^2 + \frac{1}{2}ky^2\right] = \alpha_y \quad \text{or;} \quad Y = \int\sqrt{\left(2m\alpha_y - mky^2\right)}dy$$

and

$$\left[\frac{1}{2m}\left(\frac{\partial Z}{\partial z}\right)^2 + \frac{1}{2}kz^2\right] = \alpha_z \quad \text{or;} \quad Z = \int\sqrt{\left(2m\alpha_z - mkz^2\right)}dz$$

with $\alpha_x + \alpha_y + \alpha_z = \alpha$.

Now, the Hamilton's characteristic function becomes

$$W = \int \sqrt{\left(2m\alpha_x - mkx^2\right)}dx + \int \sqrt{\left(2m\alpha_y - mky^2\right)}dy + \int \sqrt{\left(2m\alpha_z - mkz^2\right)}dz \tag{v}$$

The new coordinate β_x is obtained by a canonical transformation generated by W and is given by;

$$\beta_x = \frac{\partial W}{\partial \alpha_x} = m\int \frac{dx}{\sqrt{\left(2m\alpha_x - mkx^2\right)}} - t$$

or
$$\beta_x + t = m\int \frac{dx}{\sqrt{\left(2m\alpha_x - mkx^2\right)}} \tag{vi}$$

To perform the integration, we may put

$$x = \sqrt{\frac{2\alpha_x}{k}}\sin\theta_x \quad \text{and} \quad dx = \sqrt{\frac{2\alpha_x}{k}}\cos\theta_x d\theta_x$$

Using this in (vi), we get

$$\beta_x + t = m\int \left(\frac{2\alpha_x}{k}\right)^{1/2} \frac{\cos\theta_x d\theta_x}{\left[2m\alpha_x - mk\left(\sqrt{\frac{2\alpha_x}{k}}\sin\theta_x\right)^2\right]^{1/2}}$$

or
$$\beta_x + t = m\int \left(\frac{2\alpha_x}{k}\right)^{1/2} \frac{\cos\theta_x d\theta_x}{\left(2m\alpha_x\right)^{1/2}\left[1 - \sin^2\theta_x\right]^{1/2}}$$

$$\beta_x + t = \int \left(\frac{m}{k}\right)^{1/2} d\theta_x = \left(\frac{m}{k}\right)^{1/2}\theta_x$$

or
$$\theta_x = \sqrt{\frac{k}{m}}\left(\beta_x + t\right)$$

so that
$$x = \sqrt{\frac{2\alpha_x}{k}}\sin\sqrt{\frac{k}{m}}\left(\beta_x + t\right) \tag{vii}$$

In a similar way, we get

$$y = \sqrt{\frac{2\alpha_y}{k}}\sin\sqrt{\frac{k}{m}}\left(\beta_y + t\right) \tag{viii}$$

and
$$z = \sqrt{\frac{2\alpha_z}{k}}\sin\sqrt{\frac{k}{m}}\left(\beta_z + t\right) \tag{ix}$$

Equations (vii), (viii) and (ix) are the required equations of motion.

EXAMPLE 5.14 Using Hamilton-Jacobi method obtain the equations of motion of a particle in a dipole field with a non-central potential $V = \dfrac{k\cos\theta}{r^2}$ in spherical polar coordinate system.

Solution: The Hamiltonian of a particle moving in a dipole field in spherical polar coordinate system is given by

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2\right) + \frac{k\cos\theta}{r^2} \tag{i}$$

Now, the Hamilton-Jacobi equation is

$$H + \frac{\partial S}{\partial t} = 0$$

and the Hamilton's principal function is given by

$$S = S(r,\theta,\phi;\alpha_r,\alpha_\theta,\alpha_\phi;t)$$

where, α_j is the new canonical momentum.

Then, the transformation equations $p_j = \dfrac{\partial S}{\partial q_j}$ and $Q_j = \beta_j = \dfrac{\partial S}{\partial \alpha_j}$ gives;

$$p_r = \frac{\partial S}{\partial r},\ p_\theta = \frac{\partial S}{\partial \theta} \text{ and } p_\phi = \frac{\partial S}{\partial \phi} \tag{ii}$$

This can be substituted in (i) to get

$$H = \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\right] + \frac{k\cos\theta}{r^2} \tag{iii}$$

Since the Hamiltonian does not depend on time explicitly, we can separate the variables as

$$S = W - Et$$

and write the time independent Hamilton-Jacobi equation as

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2\right] + \frac{k\cos\theta}{r^2} = E \tag{iv}$$

Now, we try a solution to this equation of the form

$$W = R(r) + \Theta(\theta) + \Phi(\phi)$$

Substituting this in equation (iv), we get

$$\frac{1}{2m}\left[\left(\frac{\partial R}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \Theta}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial \Phi}{\partial \phi}\right)^2\right] + \frac{k\cos\theta}{r^2} = E$$

or

$$\left(\frac{\partial\Phi}{\partial\phi}\right)^2 = r^2\sin^2\theta\left[2m\left(E-\frac{k\cos\theta}{r^2}\right)-\left(\frac{\partial R}{\partial r}\right)^2-\frac{1}{r^2}\left(\frac{\partial\Theta}{\partial\theta}\right)^2\right]$$

$$= 2mr^2\sin^2\theta\left[\left(E-\frac{k\cos\theta}{r^2}\right)-\frac{1}{2m}\left(\frac{\partial R}{\partial r}\right)^2-\frac{1}{2mr^2}\left(\frac{\partial\Theta}{\partial\theta}\right)^2\right] \tag{v}$$

Now, LHS of equation (v) is a function of ϕ only and RHS is a function of r and θ. For this both sides must be equal to a constant separately. Therefore, we put

$$\left(\frac{\partial\Phi}{\partial\phi}\right)^2 = L_z^2 \tag{vi}$$

and $$2mr^2\sin^2\theta\left[\frac{1}{2m}\left(\frac{\partial R}{\partial r}\right)^2+\frac{1}{2mr^2}\left(\frac{\partial\Theta}{\partial\theta}\right)^2+\frac{k\cos\theta}{r^2}-E\right] = -L_z^2 \tag{vii}$$

From (vi), on integration, we get

$$\Phi = L_z\phi \tag{viii}$$

From (vii), we have

$$2mr^2\left[\frac{1}{2m}\left(\frac{\partial R}{\partial r}\right)^2-E\right] = -\left[\left(\frac{\partial\Theta}{\partial\theta}\right)^2+\frac{L_z^2}{\sin^2\theta}+2mk\cos\theta\right] \tag{ix}$$

The LHS of the above expression is a function of r only and RHS is a function of θ only. Since these two are equal, both of them must be equal to a constant separately. Therefore, we write

$$\left(\frac{\partial\Theta}{\partial\theta}\right)^2+\frac{L_z^2}{\sin^2\theta}+2mk\cos\theta = \alpha \tag{x}$$

and then; $$2mr^2\left[\frac{1}{2m}\left(\frac{\partial R}{\partial r}\right)^2-E\right] = -\alpha \tag{xi}$$

Now, from (x) we get

$$\frac{\partial\Theta}{\partial\theta} = \left(\alpha-\frac{L_z^2}{\sin^2\theta}-2mk\cos\theta\right)^{1/2}$$

or $$\Theta = \int\left(\alpha-\frac{L_z^2}{\sin^2\theta}-2mk\cos\theta\right)^{1/2} d\theta \tag{xii}$$

Similarly, from (xi), we get

$$\frac{\partial R}{\partial r} = \left(2mE-\frac{\alpha}{r^2}\right)^{1/2} \quad \text{or;} \quad R = \int\left(2mE-\frac{\alpha}{r^2}\right)^{1/2} dr \tag{xiii}$$

Now, the complete solution can be written as

$$W = \int\left(2mE - \frac{\alpha}{r^2}\right)^{1/2} dr + \int\left(\alpha - \frac{L_z^2}{\sin^2\theta} - 2mk\cos\theta\right)^{1/2} d\theta + L_z\phi \tag{xiv}$$

Using the transformation equation, $p_j = \dfrac{\partial S}{\partial q_j} = \dfrac{\partial W}{\partial q_j}$, we get

$$\left.\begin{aligned} p_r &= \frac{\partial W}{\partial r} = \left(2mE - \frac{\alpha}{r^2}\right)^{1/2} \\ p_\theta &= \frac{\partial W}{\partial \theta} = \left(\alpha - \frac{L_z^2}{\sin^2\theta} - 2mk\cos\theta\right)^{1/2} \\ \text{and} \quad p_\phi &= \frac{\partial W}{\partial \phi} = L_z \end{aligned}\right\} \tag{xv}$$

Therefore, p_ϕ is the z–component of the angular momentum.

The second transformation equation, $Q_j = \beta_j = \dfrac{\partial W}{\partial \alpha_j}$, we get

$$\beta_E = \frac{\partial W}{\partial E} = m\int \frac{dr}{\left(2mE - \frac{\alpha}{r^2}\right)^{1/2}} - t$$

$$\text{or} \quad \beta_E + t = \frac{\partial W}{\partial E} = m\int \frac{dr}{\left(2mE - \frac{\alpha}{r^2}\right)^{1/2}} \tag{xvi}$$

$$\text{Also,} \quad \beta_\alpha = \frac{\partial W}{\partial \alpha} = -\frac{1}{2}\int \frac{dr}{r^2\left(2mE - \frac{\alpha}{r^2}\right)^{1/2}} + \frac{1}{2}\int \frac{d\theta}{\left(\alpha - \frac{L_z^2}{\sin^2\theta} - 2mk\cos\theta\right)^{1/2}} \tag{xvii}$$

$$\text{and} \quad \beta_{L_z} = \frac{\partial W}{\partial L_z} = -\int \frac{L_z\, d\theta}{\left(\alpha - \frac{L_z^2}{\sin^2\theta} - 2mk\cos\theta\right)^{1/2}} + \phi \tag{xviii}$$

Now, equation (xvi) can be integrated directly to get

$$\beta_E + t = m\int \frac{dr}{\left(2mE - \frac{\alpha}{r^2}\right)^{1/2}} = m\int \frac{r\,dr}{\left(2mEr^2 - \alpha\right)^{1/2}}$$

$$= \frac{1}{2E}\left(2mEr^2 - \alpha\right)^{1/2}$$

or $$2mEr^2 = \left[2E(\beta_E + t)\right]^2 + \alpha$$

so that $$r^2 = \frac{2E}{m}(\beta_E + t)^2 + \frac{\alpha}{2mE} \tag{xix}$$

This equation represents the radial motion of the particle.

EXAMPLE 5.15 The Hamiltonian of a charged particle of mass m moving in a magnetic field pointing in the z-direction is given by $H = \frac{1}{2m}\left[\left(p_x + \frac{eB}{c}y\right)^2 + p_y^2 + p_z^2\right]$ in a gauge in which the vector potential is $A = (-B_y, 0, 0)$. Write down the time independent Hamilton-Jacobi equation, separate the variables and obtain the complete integral representing the Hamilton's principal function W. Using W obtain the expression for the Cartesian coordinates as a function of time.

Solution: The Hamiltonian of the charged particle in a magnetic field, which is in the z-direction is given as

$$H = \frac{1}{2m}\left[\left(p_x + \frac{eB}{c}y\right)^2 + p_y^2 + p_z^2\right] \tag{i}$$

The Hamilton-Jacobi equation is

$$H + \frac{\partial S}{\partial t} = 0 \tag{ii}$$

where S is the Hamilton's principal function and is the generator of the canonical transformation with the transformation equations

$$p_j = \frac{\partial S}{\partial q_j} \text{ and } Q_j = \beta_j = \frac{\partial S}{\partial \alpha_j}$$

Using (i) in (ii), we get

$$\frac{1}{2m}\left[\left(p_x + \frac{eB}{c}y\right)^2 + p_y^2 + p_z^2\right] + \frac{\partial S}{\partial t} = 0 \tag{iii}$$

Since the Hamiltonian does not depend explicitly on time we can separate the variables and write the Hamilton's principal function as

$$S = W - \alpha t \tag{iv}$$

where, W is the Hamilton's characteristic function.
Then, the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x} + \frac{eB}{c}y\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right] - \alpha = 0 \tag{v}$$

Now, we try a solution of the form

$$W = X(x) + Y(y) + Z(z) \tag{vi}$$

Then, equation (v) becomes

$$\frac{1}{2m}\left[\left(\frac{\partial X}{\partial x} + \frac{eB}{c}y\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2\right] - \alpha = 0 \tag{vii}$$

This can be rearranged to get

$$\left(\frac{\partial X}{\partial x} + \frac{eB}{c}y\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 - 2m\alpha = -\left(\frac{\partial Z}{\partial z}\right)^2$$

Now, RHS of the above expression is a function of z-only and LHS is a function of x and y. Then both sides must separately be equal to a constant. Therefore, we put

$$\left(\frac{\partial Z}{\partial z}\right)^2 = \alpha_z^2 \tag{viii}$$

and
$$\left(\frac{\partial X}{\partial x} + \frac{eB}{c}y\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 - 2m\alpha = -\alpha_z^2 \tag{ix}$$

From (viii), by integration, we get

$$Z = \alpha_z z \tag{x}$$

Again, we rewrite equation (ix) as

$$\frac{\partial X}{\partial x} = -\frac{eB}{c}y + \left[2m\alpha - \left(\frac{\partial Y}{\partial y}\right)^2 - \alpha_z^2\right]^{1/2} \tag{xi}$$

Now, LHS of (xi) is a function of x and RHS is a function of y only. Since they are equal, they must be separately equal to a constant. Then, we write

$$\frac{\partial X}{\partial x} = \alpha_x \tag{xii}$$

and
$$-\frac{eB}{c}y + \left[2m\alpha - \left(\frac{\partial Y}{\partial y}\right)^2 - \alpha_z^2\right]^{1/2} = \alpha_x$$

or
$$\left(\frac{\partial Y}{\partial y}\right)^2 = 2m\alpha - \alpha_z^2 - \left(\alpha_x + \frac{eB}{c}y\right)^2 \tag{xiii}$$

From, (xii), through integration, we get

$$X = \alpha_x x \tag{xiv}$$

Similarly, from, (xiii) we get

$$Y = \int \left[2m\alpha - \alpha_z^2 - \left(\alpha_x + \frac{eB}{c} y \right)^2 \right]^{1/2} dy \tag{xv}$$

Now, the Hamilton's characteristic function becomes

$$W = \alpha_x x + \int \left[2m\alpha - \alpha_z^2 - \left(\alpha_x + \frac{eB}{c} y \right)^2 \right]^{1/2} dy + \alpha_z z \tag{xvi}$$

Now, we write the transformation equations generated by the Hamilton's characteristic function. From the transformation equation, $p_j = \frac{\partial S}{\partial q_j}$, we get

$$\left.\begin{aligned} p_x &= \frac{\partial W}{\partial x} = \alpha_x \\ p_y &= \frac{\partial W}{\partial y} = \left[2m\alpha - \alpha_z^2 - \left(\alpha_x + \frac{eB}{c} y \right)^2 \right]^{1/2} \quad \text{and;} \\ p_z &= \frac{\partial W}{\partial z} = \alpha_z \end{aligned}\right\} \tag{xvii}$$

Similarly, using the transformation equation; $Q_j = \beta_j = \frac{\partial S}{\partial \alpha_j}$, we get

$$\left.\begin{aligned} \beta_x &= \frac{\partial W}{\partial \alpha_x} = x - \int \frac{\alpha_x + \frac{eB}{c} y dy}{\left[2m\alpha - \alpha_z^2 - \left(\alpha_x + \frac{eB}{c} y \right)^2 \right]^{1/2}} \\ \beta_z &= \frac{\partial W}{\partial \alpha_z} = z - \int \frac{\alpha_z dy}{\left[2m\alpha - \alpha_z^2 - \left(\alpha_x + \frac{eB}{c} y \right)^2 \right]^{1/2}} \\ \beta_\alpha + t &= \frac{\partial W}{\partial \alpha_x} = \int \frac{m dy}{\left[2m\alpha - \alpha_z^2 - \left(\alpha_x + \frac{eB}{c} y \right)^2 \right]^{1/2}} \end{aligned}\right\} \tag{xviii}$$

The first expression in equation (xviii) can be integrated directly to get

$$\beta_x = x - \frac{c}{eB} \left[2m\alpha - \alpha_z^2 - \left(\alpha_x + \frac{eB}{c} y \right)^2 \right]^{1/2} \tag{xix}$$

This can be rearranged to get

$$(x-\beta_x)^2+\left(y+\frac{c\alpha_x}{eB}\right)^2=\frac{c^2}{e^2B^2}\left(2m\alpha-\alpha_z^2\right) \tag{xx}$$

This expression represents a circle with centre $\left(\beta_x,-\frac{c\alpha_x}{eB}\right)$

Then, the second and third equations can be solved among themselves to get

$$z(t)=\beta_z+\frac{\alpha_z}{m}(\beta_\alpha+t) \tag{xxi}$$

This expression shows that the motion along the z-direction is with a constant velocity $\frac{\alpha_z}{m}$. Now, we perform the integration of the third expression in equation (xviii) by making a substitution

$$y=-\frac{c\alpha_x}{eB}+\frac{c}{eB}\left(2m\alpha-\alpha_z^2\right)\sin\Omega \text{ and } dy=\frac{c}{eB}\left(2m\alpha-\alpha_z^2\right)\cos\Omega d\Omega$$

Then, $$\beta_\alpha+t=\frac{\partial W}{\partial\alpha_x}=\int\frac{m\frac{c}{eB}\left(2m\alpha-\alpha_z^2\right)\cos\Omega d\Omega}{\left\{2m\alpha-\alpha_z^2-\left[\alpha_x+\frac{eB}{c}\left(-\frac{c\alpha_x}{eB}+\frac{c}{eB}\left(2m\alpha=\alpha_z^2\right)\sin\Omega\right)\right]^2\right\}^{1/2}}$$

This can be rearranged to get

$$\beta_\alpha+t=\int\frac{m\frac{c}{eB}\left(2m\alpha-\alpha_z^2\right)\cos\Omega d\Omega}{\left\{\left(2m\alpha-\alpha_z^2\right)-\left(2m\alpha-\alpha_z^2\right)\sin^2\Omega\right\}^{1/2}}=m\frac{c}{eB}\int d\Omega=m\frac{c}{eB}\Omega$$

$$\Omega=\frac{eB}{mc}(\beta_\alpha+t)=\omega_c(\beta_\alpha+t) \tag{xxii}$$

where, $\omega_c=\frac{eB}{mc}$ is the cyclotron frequency.

Now, $$y=-\frac{c\alpha_x}{eB}+\frac{c}{eB}\left(2m\alpha-\alpha_z^2\right)\sin\left[\omega_c(\beta_\alpha+t)\right] \tag{xxiii}$$

Substituting this in (xix) and simplified to get

$$x(t)=\beta_x-\frac{c}{eB}\left(2m\alpha-\alpha_z^2\right)\cos\left[\omega_c(\beta_\alpha+t)\right] \tag{xxiv}$$

Equations (xxi), (xxiii) and (xxiv) are the required expressions of the Cartesian coordinates.

EXAMPLE 5.16 Consider a particle of unit mass moving with a Hamiltonian $H = \frac{p^2}{2}$. (i) Solve the Hamilton-Jacobi equation and obtain the canonical transformation equations. (ii) If there is a perturbing Hamiltonian $H_1 = \frac{q^2}{2}$, then show that the perturbed solution is simple harmonic.

Solution:

(i) Given that the unperturbed Hamiltonian is

$$H = \frac{p^2}{2} \tag{i}$$

The Hamilton-Jacobi equation is

$$H + \frac{\partial S}{\partial t} = 0 \tag{ii}$$

where, S is the Hamilton's principal function and is the generator of the canonical transformation with the transformation equations

$$p_j = \frac{\partial S}{\partial q_j} \text{ and } Q_j = \beta_j = \frac{\partial S}{\partial \alpha_j} \tag{iii}$$

Using equations (i) and (ii), we can rewrite equation (ii) as

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t} = 0 \tag{iv}$$

Now, we can directly apply the method of separation of variable to the above equation since the Hamiltonian does not contain time explicitly. Let

$$S = S_1(q) + S_2(t)$$

Then we write the above equation as

$$\frac{1}{2}\left(\frac{\partial S_1}{\partial q}\right)^2 = -\frac{\partial S_2}{\partial t}$$

Now, the LHS is a function of q and RHS is a function of t alone. Since they are equal, they must be equal to the same constant and we take the constant as γ. Then,

$$\frac{1}{2}\left(\frac{\partial S_1}{\partial q}\right)^2 = \gamma \quad \text{and;} \quad \frac{\partial S_2}{\partial t} = -\gamma$$

From these expressions, through integration, we get

$$S_1 = q\sqrt{2\gamma} \quad \text{and} \quad S_2 = -\gamma t$$

Then, $$S = q\sqrt{2\gamma} - \gamma t \tag{v}$$

Now we put, $\sqrt{2\gamma} = \alpha$, so that equation (v) takes the form

$$S = \alpha q - \frac{1}{2}\alpha^2 t \tag{vi}$$

which is the generating function for the canonical transformation. Now, let us write the transformation equations as it follows from equation (iii) and we get

$$p = \frac{\partial S}{\partial q} = \alpha \tag{vii}$$

and $$Q = \beta = \frac{\partial S}{\partial P} = \frac{\partial S}{\partial \alpha} = q - \alpha t \tag{viii}$$

Equations (vii) and (viii) are the required transformation equations.

(ii) For the second part of the problem we need to consider the perturbation also. Now, the perturbed Hamiltonian of the particle is

$$H = \frac{p^2}{2} + \frac{q^2}{2} \tag{ix}$$

Note that, since the Hamiltonian is perturbed, the new momentum α no longer remains constant and the new Hamiltonian will not vanish. Therefore,

$$H' = H + \frac{\partial S}{\partial t} = \frac{p^2}{2} + \frac{q^2}{2} + \frac{\partial S}{\partial t}$$

Using (vi), (vii) and (viii) in this expression, we get

$$H' = \frac{\alpha^2}{2} + \frac{1}{2}(\beta + \alpha t)^2 - \frac{\alpha^2}{2} = \frac{1}{2}(\beta + \alpha t)^2 \tag{x}$$

We have the Hamilton's canonical equations as

$$\dot{Q} = \frac{\partial H'}{\partial P} \text{ and } \dot{P} = -\frac{\partial H}{\partial Q}$$

Then, from (x), we get;

$$\dot{\beta} = \frac{\partial H'}{\partial \alpha} = (\beta + \alpha t)t \quad \text{and; } \dot{\alpha} = -(\beta + \alpha t) \tag{xi}$$

Differentiating the second expression with respect to time, we get

$$\ddot{\alpha} + \alpha = 0$$

This shows that α is harmonic and can be written as

$$\alpha = \alpha_0 \sin(t + \phi) \tag{xii}$$

where α_0 and ϕ are constants.

Again from (xi), we have

$$\dot{\beta} = -\dot{\alpha} - \alpha t$$

and using (xii) in this expression, we get

$$\dot{\beta} = -\alpha_0\left[\cos(t+\phi) + t\sin(t+\phi)\right] \tag{xiii}$$

Now, from the transformation equations (vii) and (viii), we get

$$p = \alpha = \alpha_0 \sin(t+\phi) \quad \text{and} \quad q = \beta + \alpha t = -\dot{\alpha} = -\alpha_0 \cos(t+\phi) \tag{xix}$$

The expressions in (xix) shows that, the solution of the perturbed system is harmonic.

EXAMPLE 5.17 The motion of a free particle of mass m in a plane is described by the Hamiltonian $H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right)$, where the symbols have their usual meaning. Solve the Hamilton-Jacobi equation to find the complete integral W and use the result to find r and ϕ as a function of time.

Solution: Given that the Hamiltonian of the particle is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) \tag{i}$$

The canonical momenta can be expressed in terms of the Hamilton's principal function as

$$p_r = \frac{\partial S}{\partial r} \quad \text{and} \quad p_\phi = \frac{\partial S}{\partial \phi}$$

With this, the Hamiltonian becomes

$$H = \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2\right] \tag{ii}$$

The Hamilton-Jacobi equation is

$$H + \frac{\partial S}{\partial t} = 0$$

or

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2\right] + \frac{\partial S}{\partial t} = 0 \tag{iii}$$

Since the Hamiltonian does not depend on time explicitly, we can separate the variables and write

$$S = W - \alpha t$$

So that the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \phi}\right)^2\right] = \alpha \qquad \text{(iv)}$$

Now, we try a solution of the form

$$W = R(r) + \Phi(\phi) \qquad \text{(v)}$$

Using (v) in (iv), we get

$$\frac{1}{2m}\left[\left(\frac{\partial R}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \Phi}{\partial \phi}\right)^2\right] = \alpha$$

or

$$r^2\left(\frac{\partial R}{\partial r}\right)^2 - 2mr^2\alpha = -\left(\frac{\partial \Phi}{\partial \phi}\right)^2$$

that is,

$$2mr^2\left[\left(\frac{\partial R}{\partial r}\right)^2 - \alpha\right] = -\left(\frac{\partial \Phi}{\partial \phi}\right)^2 \qquad \text{(vi)}$$

Now, the LHS of (vi) is a function of r only and RHS is a function of ϕ only. Since they are equal, they must be equal to the same constant. Therefore, we can put

$$\left(\frac{\partial \Phi}{\partial \phi}\right)^2 = L^2 \text{ and;} \qquad \text{(vii)}$$

$$2mr^2\left[\left(\frac{\partial R}{\partial r}\right)^2 - \alpha\right] = -L^2$$

or

$$\frac{1}{2m}\left(\frac{\partial R}{\partial r}\right)^2 + \frac{L^2}{2mr^2} = \alpha \qquad \text{(viii)}$$

From (vi) and (viii), through integration, we get

$$\Phi = L\phi \qquad \text{(ix)}$$

and

$$R = \int\left(2m\alpha - \frac{L^2}{r^2}\right)^{1/2} dr \qquad \text{(x)}$$

Substituting (ix) and (x) in (v), we get

$$W = \int\left(2m\alpha - \frac{L^2}{r^2}\right)^{1/2} dr + L\phi \qquad \text{(xi)}$$

Now, the Hamilton's principal function $W = W(r,\phi,\alpha,L)$ (where α and L are the new canonical momenta) is the generator of the canonical transformation. The canonical transformation equations are

$$p_r = \frac{\partial S}{\partial r} = \frac{\partial W}{\partial r} = \left(2m\alpha - \frac{L^2}{r^2}\right)^{1/2} \text{ and } p_\phi = \frac{\partial S}{\partial \phi} = \frac{\partial W}{\partial \phi} = L \tag{xii}$$

Also,
$$\beta_\alpha = \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial \alpha} - t = \int \frac{m\,dr}{\left(2m\alpha - \frac{L^2}{r^2}\right)^{1/2}} - t$$

$$\left.\begin{aligned} &\text{or} \quad \beta_\alpha + t = \int \frac{m\,dr}{\left(2m\alpha - \frac{L^2}{r^2}\right)^{1/2}} \\ &\text{and} \quad \beta_L = \frac{\partial S}{\partial L} = \frac{\partial W}{\partial L} = -\int \frac{L\,dr}{r^2\left(2m\alpha - \frac{L^2}{r^2}\right)^{1/2}} + \phi \end{aligned}\right\} \tag{xiii}$$

The integration in the last expression can be carried out by putting

$$\frac{1}{r} = \frac{\sqrt{2m\alpha}}{L}\cos\theta \text{ and } \frac{dr}{r^2} = \frac{\sqrt{2m\alpha}}{L}\sin\theta\, d\theta$$

Then,
$$\beta_L = -\int \frac{L}{\left(2m\alpha - L^2 \frac{2m\alpha}{L^2}\cos^2\theta\right)^{1/2}} \frac{\sqrt{2m\alpha}}{L}\sin\theta\, d\theta + \phi$$

$$\beta_L = -\int \frac{\sin\theta\, d\theta}{\left(1-\cos^2\theta\right)^{1/2}} + \phi$$

or
$$\beta_L = \phi - \theta \tag{xiv}$$

Using this in the expression $\frac{1}{r} = \frac{\sqrt{2m\alpha}}{L}\cos\theta$, we get the equation for the trajectory as

$$r\cos(\phi - \beta_L) = \frac{L}{\sqrt{2m\alpha}} \tag{xv}$$

This equation represents a straight line in the polar coordinate system.
Now, the first equation in (xiii) can be rearranged to get

$$\beta_\alpha + t = \left(\frac{m}{2\alpha}\right)^{1/2} \int \frac{r\,dr}{\left(r^2 - \frac{L^2}{2m\alpha}\right)^{1/2}}$$

This can be integrated directly to obtain

$$\beta_\alpha + t = \left(\frac{m}{2\alpha}\right)^{1/2}\left(r^2 - \frac{L^2}{2m\alpha}\right)^{1/2}$$

This on rearrangement becomes

$$r^2 = \frac{L^2}{2m\alpha} + \left(\frac{2\alpha}{m}\right)(\beta_\alpha + t)^2 \tag{xvi}$$

This expression shows that the particle moves with a velocity $V = \left(\frac{2\alpha}{m}\right)^{1/2}$ along the trajectory. The quantity $\frac{L}{\sqrt{2m\alpha}} = b$ can be regarded as the distance of closest approach to the origin. Then, equation (xvi) can be written as

$$r^2 = b^2 + v^2(\beta_\alpha + t)^2 \tag{xvii}$$

And from (xv), we get

$$b = r\cos(\phi - \beta_L) \tag{xviii}$$

Equations (xvii) and (xviii) are the required results.

EXAMPLE 5.18 The Hamiltonian of a particle is given by; $H = q + p$. Show that the Hamilton's principal function is given by; $S = \alpha q - \frac{1}{2}q^2 - \alpha t$ when the system is conservative.

Solution: Given that, the Hamiltonian of the system is

$$H = q + p \tag{i}$$

Now, the Hamilton's principal function is $S = S(q, \alpha, t)$, where α is the new constant momentum. From transformation equations generated by the principal function, we have

$$p = \frac{\partial S}{\partial q} \tag{ii}$$

Using this in (i), we get

$$H = q + \frac{\partial S}{\partial q} \tag{iii}$$

The Hamilton Jacobi equation can be written as

$$q + \frac{\partial S}{\partial q} + \frac{\partial S}{\partial t} = 0 \tag{iv}$$

Now, we can separate the variables since the Hamiltonian does not have explicit dependence on time. Therefore, we write

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t \tag{v}$$

So that $\frac{\partial S}{\partial q} = \frac{\partial W}{\partial q}$ and $\frac{\partial S}{\partial t} = -\alpha$. Then, equation (iv) becomes

$$q + \frac{\partial W}{\partial q} - \alpha = 0$$

or

$$\frac{\partial W}{\partial q} = \alpha - q$$

This, on integration will give

$$W = \alpha q - \frac{1}{2}q^2 \tag{vi}$$

This can be substituted in (v) to get

$$S = \alpha q - \frac{1}{2}q^2 - \alpha t \tag{vii}$$

Hence, proved.

EXAMPLE 5.19 Solve the Harmonic oscillator problem by the action-angle variable method. Obtain the expression for the frequency of oscillation. Express the canonical coordinate and momentum in terms of the action and angle variables.

Solution: We solve the problem by considering a general coordinate q and a corresponding canonical momentum p. We write the Hamiltonian as a function of the variables (q,p,t) and is given by

$$H(q,p,t) = \frac{p^2}{2m} + \frac{1}{2}kq^2 \tag{i}$$

Now, we have one of the transformation equations

$$p = \frac{\partial S}{\partial q}$$

where, $S(q,P,t) = S(q,\alpha,t)$ is the Hamilton's principal function and α is a constant. Hamilton's principal function is the generating function for the transformation such that the new Hamiltonian H' is zero.

Now, the Hamiltonian becomes

$$H(q,p,t) = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}kq^2 \tag{ii}$$

The Hamilton-Jacobi equation is

$$H + \frac{\partial S}{\partial t} = 0 \tag{iii}$$

Using (ii) in (iii) we get

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}kq^2 + \frac{\partial S}{\partial t} = 0 \tag{iv}$$

Now, we separate the variables and rewrite the solution as

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t \tag{v}$$

Then, $$\frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} \text{ and } \frac{\partial S}{\partial t} = -\alpha$$

With these substitutions, equation (iv) becomes

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}kq^2 - \alpha = 0$$

Multiplying throughout by $2m$, we get

$$\left(\frac{\partial W}{\partial q}\right)^2 + mkq^2 - 2m\alpha = 0$$

This expression can be rearranged to get

$$\frac{\partial W}{\partial q} = \sqrt{\left(2m\alpha - mkq^2\right)} = \sqrt{mk}\sqrt{\frac{2\alpha}{k} - q^2} \qquad \text{(vi)}$$

Now, we define the action variable for the Harmonic oscillator problem as

$$J = \oint p\,dq = \oint \frac{\partial W(q,\alpha)}{\partial q}dq$$

Using (vi), this expression can be written as

$$J = \oint \sqrt{mk}\left(\frac{2\alpha}{k} - q^2\right)^{1/2} dq \qquad \text{(vii)}$$

In equation (vii), we make a substitution as

$$q = \left(\frac{2\alpha}{k}\right)^{1/2}\sin\theta \text{ and } dq = \left(\frac{2\alpha}{k}\right)^{1/2}\cos\theta\, d\theta$$

Therefore, we get

$$J = \oint \sqrt{mk}\left(\frac{2\alpha}{k} - \frac{2\alpha}{k}\sin^2\theta\right)^{1/2}\left(\frac{2\alpha}{k}\right)^{1/2}\cos\theta\, d\theta$$

$$= \oint \sqrt{mk}\left(\frac{2\alpha}{k}\right)\left(1 - \sin^2\theta\right)^{1/2}\cos\theta\, d\theta$$

$$= \oint 2\alpha\sqrt{\frac{m}{k}}\cos^2\theta\, d\theta$$

This expression for a complete cycle of oscillation can be written as

$$J = 2\alpha\sqrt{\frac{m}{k}}\int_0^{2\pi}\cos^2\theta\, d\theta$$

On integration, we get

$$J = 2\pi\alpha\sqrt{\frac{m}{k}}$$

or

$$\alpha = \frac{J}{2\pi}\sqrt{\frac{k}{m}} \tag{viii}$$

From, the Hamilton-Jacobi equation for a conservative system, we have, $H = \alpha$ and therefore

$$H = \frac{J}{2\pi}\sqrt{\frac{k}{m}} \tag{ix}$$

The transformation equation gives

$$\dot{\theta} = \frac{\partial H}{\partial J} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

This can be identified as the frequency of oscillation of the harmonic oscillator. Therefore,

$$\dot{\theta} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{\omega}{2\pi} = \nu \tag{x}$$

This expression can be integrated to get

$$\theta = \nu t + \beta = \frac{\omega}{2\pi}t + \beta = \omega' t + \beta \tag{xi}$$

Then, the expression for the generalized coordinate is

$$q = \left(\frac{2\alpha}{k}\right)^{1/2} \sin(\omega' t + \beta) = \left(\frac{2\alpha}{k}\right)^{1/2} \sin\theta$$

Substituting for α from (viii), we get

$$q = \left(\frac{2}{k}\frac{J}{2\pi}\sqrt{\frac{k}{m}}\right)^{1/2} \sin\theta = \left(\frac{J}{\pi m\omega}\right)^{1/2} \sin\theta \tag{xii}$$

Now, we have

$$p = \frac{\partial W}{\partial q} = \sqrt{\left(2m\alpha - mkq^2\right)}$$

Substituting for α and q from (viii) and (xii), we get

$$p = \frac{\partial W}{\partial q} = \sqrt{\left(2m\alpha - mk\left(\frac{2\alpha}{k}\right)\sin^2\theta\right)}$$

$$= \sqrt{2m\alpha}\sqrt{\left[1-\sin^2\theta\right]} = \sqrt{2m\alpha}\cos\theta$$

$$= \sqrt{\frac{Jm\omega}{\pi}}\cos\theta \qquad \text{(xiii)}$$

Equations (xii) and (xiii) give the canonical coordinate and momentum (q,p) in terms of the action and angle variables, (J,θ).

EXAMPLE 5.20 Obtain an expression for the frequency and time period for the Kepler problem by applying action-angle variable method.

Solution: For the Kepler problem, the Hamiltonian is given by

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{k}{r} \qquad \text{(i)}$$

For a conservative system, Hamiltonian represents the total energy of the system, that is,

$$H = E \equiv \alpha_1$$

Therefore,

$$\frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{k}{r} = E = \alpha_1$$

or
$$p_r^2 + \frac{p_\theta^2}{r^2} = 2m\alpha_1 + \frac{2mk}{r} \qquad \text{(ii)}$$

Now, let $S(r,\theta,\alpha_1,\alpha_2,t)$ be the generating function that generates the canonical transformation and is known as the Hamilton's principal function. Here α_1 and α_2 are the new canonical momenta P_r and P_θ respectively and are constants. The transformation equations, in general, are

$$p_j = \frac{\partial S}{\partial q_j} \text{ and } Q_j = \beta_j = \frac{\partial S}{\partial P_j} = \frac{\partial S}{\partial \alpha_j} \qquad \text{(iii)}$$

Since the Hamiltonian has no explicit time dependence, we can separate the variables and the Hamilton's principal function can be written as

$$S(r,\theta,\alpha_1,\alpha_2,t) = W(r,\theta,\alpha_1,\alpha_2) - \alpha t \qquad \text{(iv)}$$

Then, we get

$$p_r = \frac{\partial S}{\partial r} = \frac{\partial W}{\partial r} \text{ and } p_\theta = \frac{\partial S}{\partial \theta} = \frac{\partial W}{\partial \theta} \qquad \text{(v)}$$

This can be substituted in equation (ii) to get

$$\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 = 2m\alpha_1 + \frac{2mk}{r} \qquad \text{(vi)}$$

Further, we write $W = W_r(r) + W_\theta(\theta)$ so that the above expression becomes

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 = 2m\alpha_1 + \frac{2mk}{r} \tag{vii}$$

Since we are considering the motion under a central force field, the angular momentum is conserved, that is,

$$p_\theta = \frac{\partial S}{\partial \theta} = \frac{\partial W}{\partial \theta} = \alpha_2, \text{ a constant}$$

On integration, we get

$$W = \alpha_2\theta + \text{constant} \tag{viii}$$

With this substitution (vii) becomes

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_2^2}{r^2} = 2m\alpha_1 + \frac{2mk}{r}$$

or
$$\frac{\partial W_r}{\partial r} = \left(2m\alpha_1 + \frac{2mk}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2}$$

Then,
$$W_r = \int \left(2m\alpha_1 + \frac{2mk}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2} dr + \text{constant} \tag{ix}$$

Using (viii) and (ix), the Hamilton's characteristic function can be written as

$$W = W_r(r) + W_\theta(\theta)$$
$$= \alpha_2\theta + \int \left(2m\alpha_1 + \frac{2mk}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2} dr + \text{constant} \tag{x}$$

Also, from the transformation equations, we get

$$\frac{\partial W}{\partial \alpha_1} = t + \beta_1 = \int \frac{m\,dr}{\left(2m\alpha_1 + \frac{2mk}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2}} \tag{xi}$$

and
$$\frac{\partial W}{\partial \alpha_2} = \beta_2 = \theta - \int \frac{\alpha_2\,dr}{r^2\left(2m\alpha_1 + \frac{2mk}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2}} \tag{xii}$$

or
$$\theta - \beta_2 = \int \frac{\alpha_2\,dr}{r^2\left(2m\alpha_1 + \frac{2mk}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2}}$$

Now, the maximum and minimum values of r is given by the roots of the quadratic equation, $2m\alpha_1 + \frac{2mk}{r} - \frac{\alpha_2^2}{r^2} = 0$, or $2m\alpha_1 r^2 + 2mkr - \alpha_2^2 = 0$. It is given by

$$r = -\frac{\lambda}{2\alpha_1} \pm \frac{1}{2\alpha_1}\left(\lambda^2 - \frac{2\alpha_1\alpha_2^2}{m}\right)^{1/2} = \frac{\lambda}{2\alpha_1}\left[-1 \pm \left(1 + \frac{2\alpha_1\alpha_2^2}{m}\right)^{1/2}\right] \tag{xiii}$$

Note that the system must execute a complete cycle of motion for applying the method of action-angle variables.

In the present problem, the generalized coordinates are r and θ, and the corresponding action variables can be defined as

$$J_\theta = \oint p_\theta d\theta, \quad \text{and } J_r = \oint p_r dr \tag{xiv}$$

Also, $\quad J_\theta = \oint p_\theta d\theta = \oint \frac{\partial W}{\partial \theta} d\theta = \oint \alpha_2 d\theta$

that is, $\quad J_\theta = \int_0^{2\pi} \alpha_2 d\theta = 2\pi\alpha_2 \tag{xv}$

Similarly, $J_r = \oint p_r dr = \oint \frac{\partial W}{\partial r} dr$

$$= \oint \left(2m\alpha_1 + \frac{2mk}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2} dr$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{1}{r}\left(2m\alpha_1 r^2 + 2mkr - \alpha_2^2\right)^{1/2} dr$$

In the above expression, the factor 2 is to complete the integration over a complete cycle. The above expression can be rearranged to get

$$J_r = 2\int_{r_{min}}^{r_{max}} \frac{1}{r}\left(2m\alpha_1 r^2 + 2mkr - \alpha_2^2\right)\frac{dr}{\left(2m\alpha_1 r^2 + 2mkr - \alpha_2^2\right)^{1/2}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{(2m\alpha_1 r + 2mk)dr}{\left(2m\alpha_1 r^2 + 2mkr - \alpha_2^2\right)^{1/2}} - 2\alpha_2^2\int_{r_{min}}^{r_{max}} \frac{dr}{r\left(2m\alpha_1 r^2 + 2mkr - \alpha_2^2\right)^{1/2}}$$

Now, let us split the first term into two and write

$$J_r = 2\int_{r_{min}}^{r_{max}} \frac{(4m\alpha_1 r + 2mk)dr}{\left(2m\alpha_1 r^2 + 2mkr - \alpha_2^2\right)^{1/2}} + \frac{2mk}{(2m\alpha_1)}\int_{r_{min}}^{r_{max}} \frac{dr}{\left\{\left(r + \frac{\lambda}{2\alpha_1}\right)^2 - \frac{mk^2 + 2\alpha_1\alpha_2^2}{4m\alpha_1^2}\right\}}$$

$$-2\alpha_2^2\int_{r_{min}}^{r_{max}} \frac{dr}{r\left(2m\alpha_1 r^2 + 2mkr - \alpha_2^2\right)^{1/2}}$$

Then, we put $r = \frac{1}{u}$ and $dr = -\frac{du}{u^2}$, in the last term to get

$$J_r = 2\int_{r_{min}}^{r_{max}} \frac{(4m\alpha_1 r + 2mk)dr}{\left(2m\alpha_1 r^2 + 2mkr - \alpha_2^2\right)^{1/2}} + \frac{2mk}{(2m\alpha_1)}\int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{\left\{\left(r + \frac{\lambda}{2\alpha_1}\right)^2 - \frac{mk^2 + 2\alpha_1\alpha_2^2}{4m\alpha_1^2}\right\}}}$$

$$-2\alpha_2^2 \int_{r_{min}}^{r_{max}} \frac{-\frac{du}{u^2}}{\frac{1}{u}\left(\frac{2m\alpha_1}{u^2} + \frac{2mk}{u} - \alpha_2^2\right)^{1/2}}$$

Now, we can integrate the first two terms and rearranging the third term, to get

$$J_r = 2\left[\left(2m\alpha_1 r^2 + 2mkr - \alpha_2^2\right)^{1/2}\right]_{r_{max}}^{r_{min}}$$

$$+k\left(\frac{2m}{\alpha_1}\right)^{1/2}\left[\log\left(r + \frac{\lambda}{2\alpha_1}\right) + \left\{\left(r + \frac{\lambda}{2\alpha_1}\right)^2 - \frac{m\lambda^2 + 2\alpha_1\alpha_2^2}{4m\alpha_1^2}\right\}^{1/2}\right]_{r_{min}}^{r_{max}}$$

$$+2\alpha_2^2 \int_{r_{min}}^{r_{max}} \frac{du}{\left(2m\alpha_1 + 2mku - \alpha_2^2 u^2\right)^{1/2}} \qquad \text{(xvi)}$$

Substituting the expressions for the maximum and minimum values of r from equation (xiii), the first term on RHS becomes zero. Similarly, the second is equal to $\frac{2\pi mk}{\sqrt{(-2m\alpha_1)}}$

Then, $$J_r = \frac{2\pi mk}{\sqrt{(-2m\alpha_1)}} + 2\alpha_2^2 \int_{r_{min}}^{r_{max}} \frac{du}{\left(2m\alpha_1 + 2mku - \alpha_2^2 u^2\right)^{1/2}}$$

$$= \frac{2\pi mk}{\sqrt{(-2m\alpha_1)}} + 2\alpha_2 \int_{r_{min}}^{r_{max}} \frac{du}{\left(\frac{2m\alpha_1}{\alpha_2^2} + \frac{2mku}{\alpha_2^2} - u^2\right)^{1/2}}$$

$$= \frac{2\pi mk}{\sqrt{(-2m\alpha_1)}} + 2\alpha_2 \int_{r_{min}}^{r_{max}} \frac{du}{\left[\frac{2m\alpha_1}{\alpha_2^2} + \frac{m^2k^2}{\alpha_2^2} - \left(u - \frac{mk}{\alpha_2^2}\right)^2\right]^{1/2}}$$

$$= \frac{2\pi mk}{\sqrt{(-2m\alpha_1)}} + 2\alpha_2 \int_{r_{\min}}^{r_{\max}} \frac{du}{\left[\dfrac{2m\alpha_1\alpha_2^2 + m^2k^2}{\alpha_2^2} - \left(u - \dfrac{mk}{\alpha_2^2}\right)^2\right]^{1/2}}$$

$$= \frac{2\pi mk}{\sqrt{(-2m\alpha_1)}} + 2\alpha_2 \sin^{-1}\left[\frac{u - \dfrac{mk}{\alpha_2^2}}{\left(\dfrac{2m\alpha_1\alpha_2^2 + m^2k^2}{\alpha_2^4}\right)^{1/2}}\right]_{r_{\min}}^{r_{\max}}$$

$$= \frac{2\pi mk}{\sqrt{(-2m\alpha_1)}} + 2\alpha_2 \sin^{-1}\left[\frac{u - \dfrac{mk}{\alpha_2^2}}{\left(\dfrac{2m\alpha_1\alpha_2^2 + m^2k^2}{\alpha_2^4}\right)^{1/2}}\right]_{r_{\min}}^{r_{\max}}$$

$$= \frac{2\pi mk}{\sqrt{(-2m\alpha_1)}} + 2\alpha_2 \sin^{-1}\left[\frac{u\alpha_2^2 - mk}{\left(2m\alpha_1\alpha_2^2 + m^2k^2\right)^{1/2}}\right]_{r_{\min}}^{r_{\max}}$$

Now, we substitute $u = \dfrac{1}{r}$ in the above expression to get

$$J_r = \frac{2\pi mk}{\sqrt{(-2m\alpha_1)}} + 2\alpha_2 \sin^{-1}\left[\frac{\alpha_2^2 - mkr}{r\left(2m\alpha_1\alpha_2^2 + m^2k^2\right)^{1/2}}\right]_{r_{\min}}^{r_{\max}}$$

Substituting for the maximum and minimum values of r, the above expression becomes

$$J_r = \frac{2\pi mk}{\sqrt{(-2m\alpha_1)}} - 2\pi\alpha_2 \tag{xvii}$$

Then, from the equations (xv) and (xvii), we get

$$J_\theta + J_r = \frac{2\pi mk}{\sqrt{(-2m\alpha_1)}} \tag{xviii}$$

We have $\alpha_1 = E$, the total energy and therefore

$$J_\theta + J_r = \frac{2\pi mk}{\sqrt{(-2mE)}} \quad \text{or,} \quad E = \alpha_1 = -\frac{2\pi^2 mk^2}{(J_r + J_\theta)^2} \tag{xix}$$

Also, we know that $\alpha_1 = H(J_r, J_\theta)$ and hence, we have the frequencies as

$$\nu_r = \frac{\partial H}{\partial J_r} = \frac{\partial}{\partial r}\left[-\frac{2\pi^2 mk^2}{(J_r + J_\theta)^2}\right] = \frac{4\pi^2 mk^2}{(J_r + J_\theta)^3} \tag{xx}$$

and $$\nu_\theta = \frac{\partial H}{\partial J_\theta} = \frac{\partial}{\partial \theta}\left[-\frac{2\pi^2 mk^2}{(J_r + J_\theta)^2}\right] = \frac{4\pi^2 mk^2}{(J_r + J_\theta)^3} \tag{xxi}$$

The time period around the orbit is given by

$$T = \frac{1}{\nu_r} = \frac{(J_r + J_\theta)^3}{4\pi^2 mk^2}$$

Substituting for $(J_r + J_\theta)$ from (xix), we get

$$T = \left(-\frac{2\pi^2 mk^2}{E}\right)^{3/2}\left(4\pi^2 mk^2\right)^{-1} = \pi k\left(-\frac{m}{2E^3}\right)^{1/2} \tag{xxii}$$

Equation (xxii) gives the time period of the particle around the orbit.

EXAMPLE 5.21 A particle of mass m moves in two dimensions in a rectangular infinite square well potential. Find the action variable I_x and I_y. Also the frequencies ω_x and ω_y; and write down the conditions for periodic trajectory.

Solution: The infinite square well potential can be written as

$$V = 0 \text{ for } 0 < x < a \text{ and } 0 < y < b; \text{ and } V = \infty \text{ elsewhere.}$$

The Hamiltonian of the particle moving in such a squire well potential is;

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) \text{ with } 0 < x < a \text{ and } 0 < y < b \tag{i}$$

The Hamilton-Jacobi equation

$$H + \frac{\partial S}{\partial t} = 0 \text{ or, } \frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + \frac{\partial S}{\partial t} = 0$$

in terms of the Hamilton's characteristic function becomes

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] - \alpha = 0$$

or $$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] = E \tag{ii}$$

Here we have used $S = W - \alpha t$ and $\alpha = E$, the total energy

Now, we try a solution of the form

$$W = X(x) + Y(y) \tag{iii}$$

Then, equation (ii) becomes

$$\frac{1}{2m}\left[\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2\right] = E$$

or

$$\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 = 2mE \tag{iv}$$

The first term on the RHS is a function of x only and the second term is a function of y only. Since their sum is equal to a constant, both of them are constants. Thus, we write

$$\left(\frac{\partial X}{\partial x}\right)^2 = \alpha_x^2; \quad \text{and} \quad \left(\frac{\partial Y}{\partial y}\right)^2 = \alpha_y^2 \tag{v}$$

α_x and α_y are the separation constants. Then, from equation (iv), we get

$$E = \frac{1}{2m}\left(\alpha_x^2 + \alpha_y^2\right) \tag{vi}$$

On integration, equation (v) gives

$$X = \pm\alpha_x x \text{ and } Y = \pm\alpha_y y$$

and

$$W = \pm\alpha_x x \pm \alpha_y y \tag{vii}$$

Then, the transformation equation gives

$$p_x = \frac{\partial W}{\partial x} = \pm\alpha_x \quad \text{and} \quad p_y = \frac{\partial W}{\partial y} = \pm\alpha_y \tag{viii}$$

From equation (viii) it is clear that when each time the trajectory reaches the boundary, the momentum changes its sign.

Now, we can find the action variables as

$$J_x = \oint p_x dx = \int_0^a \alpha_x dx + \int_a^0 (-\alpha_x) dx = 2\alpha_x a \tag{ix}$$

and

$$J_y = \oint p_y dy = \int_0^b \alpha_y dy + \int_b^0 (-\alpha_y) dy = 2\alpha_y b \tag{x}$$

Then, the separation constants can be written in terms of the action variables as

$$\alpha_x = \frac{J_x}{2a} \text{ and } \alpha_y = \frac{J_y}{2a}$$

Then, from (vi), the expression for energy is

$$E = \frac{1}{8m}\left(\frac{J_x^2}{a^2} + \frac{J_y^2}{b^2}\right) \tag{xi}$$

The frequencies corresponding to the action variables can be written as

$$\omega_x = \frac{\partial E}{\partial J_x} = \frac{J_x}{4ma^2} \quad \text{and} \quad \omega_y = \frac{\partial E}{\partial J_y} = \frac{J_y}{4ma^2} \tag{xii}$$

Periodic motion occurs for the trajectories that satisfy the condition

$$\frac{\omega_x}{\omega_y} = \frac{J_x b^2}{J_y a^2} = \frac{n_x}{n_y} \tag{xiii}$$

where, n_x and n_y are integers.

EXAMPLE 5.22 A particle of mass m moves in a three-dimensional isotropic oscillator well given by the expression, $V = \frac{1}{2}m\omega^2\left(x^2 + y^2 + z^2\right)$. Separate the Hamilton-Jacobi equation and find the action variables. Express the Hamiltonian in terms of the action variables. Find the frequencies of oscillation.

Solution: For the three-dimensional isotropic oscillator discussed in 5.13, we obtained the Hamilton's characteristic function as

$$W = \int\sqrt{\left(2m\alpha_x - mkx^2\right)}dx + \int\sqrt{\left(2m\alpha_y - mky^2\right)}dy + \int\sqrt{\left(2m\alpha_z - mkz^2\right)}dz$$

Then, the canonical momenta can be obtained as

$$p_x = \frac{\partial W}{\partial x} = \sqrt{\left(2m\alpha_x - mkx^2\right)}$$

$$p_y = \frac{\partial W}{\partial y} = \sqrt{\left(2m\alpha_y - mky^2\right)}$$

and $$p_z = \frac{\partial W}{\partial z} = \sqrt{\left(2m\alpha_z - mkz^2\right)}$$

Also we have seen that α_x, α_y and α_z are the energies associated with each degree of freedom and $\alpha_x + \alpha_y + \alpha_z = E$.

As in the case of one-dimensional oscillator we can find the action variable corresponding to each degrees of freedom as

$$J = \oint p\,dq = \oint \frac{\partial W(q,\alpha)}{\partial q}dq$$

which finally gives (refer Example 5.19),

$$J_x = 2\pi\alpha_x\sqrt{\frac{m}{k}}, \quad J_y = 2\pi\alpha_y\sqrt{\frac{m}{k}} \quad \text{and} \quad J_x = 2\pi\alpha_z\sqrt{\frac{m}{k}} \tag{i}$$

Then, the total energy is

$$E = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\left(J_x + J_y + J_z\right) \tag{ii}$$

Then, the frequencies can be obtained as

$$\omega_x = \frac{\partial E}{\partial J_x} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\omega_y = \frac{\partial E}{\partial J_y} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ and}$$

$$\omega_z = \frac{\partial E}{\partial J_z} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

These are the required results.

EXAMPLE 5.23 A particle of mass m moves in a two-dimensional non-isotropic simple harmonic oscillator well given by the potential $V(x,y) = \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2\right)$. Find the action variables and express the energy in terms of them. Also determine the corresponding angle variables and express the Cartesian coordinates in their terms.

Solution: Given that the potential is

$$V(x,y) = \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2\right)$$

The Hamiltonian of a particle moving in such a potential can be written as

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2\right) \tag{i}$$

As usual we can substitute for the canonical momentum in terms of the Hamilton's principal function S as

$$p_x = \frac{\partial S}{\partial x} \text{ and } p_y = \frac{\partial S}{\partial y}$$

Then write the Hamilton-Jacobi equation and separate the variables by introducing Hamilton's characteristic function W through $S = W - \alpha t = W - Et$, where, $\alpha \equiv E$, the total energy of the system. Thus, we get the time independent Hamilton-Jacobi equation as

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2\right) = E \tag{ii}$$

Now, we try a solution, $W = X(x) + Y(y)$ and with this, the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left[\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2\right] + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2\right) = E$$

or $$\left[\frac{1}{2m}\left(\frac{\partial X}{\partial x}\right)^2+\frac{1}{2}m\omega_x^2x^2\right]+\left[\frac{1}{2m}\left(\frac{\partial Y}{\partial y}\right)^2+\frac{1}{2}m\omega_y^2y^2\right]=E \qquad \text{(iii)}$$

Now, the first term on LHS is a function of x and the second term is a function of y only. Since their sum is equal to a constant, both terms must be equal to a constant. Therefore, we write

$$\frac{1}{2m}\left(\frac{\partial X}{\partial x}\right)^2+\frac{1}{2}m\omega_x^2x^2=\alpha_x \text{ and } \frac{1}{2m}\left(\frac{\partial Y}{\partial y}\right)^2+\frac{1}{2}m\omega_y^2y^2=\alpha_y \qquad \text{(iv)}$$

where, α_x and α_y are separation constants, and $\alpha_x+\alpha_y=E$.
From, the expressions in (iv), we can have

$$X=\int\left(2m\alpha_x-m^2\omega_x^2x^2\right)^{1/2}dx \text{ and } Y=\int\left(2m\alpha_y-m^2\omega_y^2y^2\right)^{1/2}dy \qquad \text{(v)}$$

Then the Hamilton's characteristic function becomes

$$W=\int\left(2m\alpha_x-m^2\omega_x^2x^2\right)^{1/2}dx+\int\left(2m\alpha_y-m^2\omega_y^2y^2\right)^{1/2}dy \qquad \text{(vi)}$$

Now, the canonical momenta can be obtained as

$$p_x=\frac{\partial W}{\partial x}=\left(2m\alpha_x-m^2\omega_x^2x^2\right)^{1/2} \text{ and } p_y=\frac{\partial W}{\partial y}=\left(2m\alpha_y-m^2\omega_y^2y^2\right)^{1/2} \qquad \text{(vii)}$$

The action variables can be obtained from the expression, $J=\oint p\,dq=\oint\frac{\partial W(q,\alpha)}{\partial q}dq$ and are

$$J_x=\oint\left(2m\alpha_x-m^2\omega_x^2x^2\right)^{1/2}dx \text{ and } J_y=\oint\left(2m\alpha_y-m^2\omega_y^2y^2\right)^{1/2}dy \qquad \text{(viii)}$$

The integrals in the above expression are the same as that for a one-dimensional harmonic oscillator discussed in Example 5.19. Therefore, we get

$$J_x=\frac{\alpha_x}{\omega_x} \text{ and } J_y=\frac{\alpha_y}{\omega_y} \qquad \text{(ix)}$$

The total energy can be written as

$$E=J_x\omega_x+J_y\omega_y \qquad \text{(x)}$$

Then, the Hamilton's characteristic function is

$$W=\int\left(2mJ_x\omega_x-m^2\omega_x^2x^2\right)^{1/2}dx+\int\left(2mJ_y\omega_y-m^2\omega_y^2y^2\right)^{1/2}dy$$

The angle variables can be obtained as

$$\theta_x=\frac{\partial W}{\partial J_x}=\int\frac{dx}{\left(\frac{2J_x}{m\omega_x}-x^2\right)^{1/2}}=\sin^{-1}\left(x\Big/\sqrt{\frac{2J_x}{m\omega_x}}\right) \qquad \text{(xi)}$$

and $$\theta_y = \frac{\partial W}{\partial J_y} = \int \frac{dy}{\left(\frac{2J_y}{m\omega_y} - y^2\right)^{1/2}} = \sin^{-1}\left(y \Big/ \sqrt{\frac{2J_y}{m\omega_y}}\right) \tag{xii}$$

Now, the Cartesian coordinate can be represented in terms of the action and angle variables as

$$x = \sqrt{\frac{2J_x}{m\omega_x}}\sin\theta_x \quad \text{and} \quad y = \sqrt{\frac{2J_y}{m\omega_y}}\sin\theta_y \tag{xiii}$$

Now, the Hamilton's canonical equations for the action-angle variable gives

$$\dot{\theta}_x = \frac{\partial E}{\partial I_x} = \omega_x \quad \text{and} \quad \dot{J}_x = -\frac{\partial E}{\partial \theta_x} = 0 \tag{xiv}$$

Similarly, $$\dot{\theta}_y = \frac{\partial E}{\partial I_y} = \omega_y \quad \text{and} \quad \dot{J}_y = -\frac{\partial E}{\partial \theta_y} = 0 \tag{xv}$$

From the above two expressions, on integration, we get

$$\theta_x = \omega_x t + \theta_{x0} \quad \text{and} \quad \theta_y = \omega_y t + \theta_{y0}$$

Then, the Cartesian coordinate given in equation (xiii) becomes

$$x = \sqrt{\frac{2J_x}{m\omega_x}}\sin(\omega_x t + \theta_{x0}) \quad \text{and} \quad y = \sqrt{\frac{2J_y}{m\omega_y}}\sin(\omega_y t + \theta_{y0}) \tag{xvi}$$

These are the required results.

EXAMPLE 5.24 A particle of mass m is constrained to move on a curve in a vertical plane defined by the parametric equations $x = l(2\phi + \sin 2\phi)$ and $y = l(1 - \cos 2\phi)$, where, y is taken as the vertical direction. Using the method of action-angle variables, find the frequency of oscillation such that $\phi_{\max} \le \frac{\pi}{4}$.

Solution: First let us find the Hamiltonian of the system. The potential energy can be written as

$$V = mgy = mgl(1 - \cos 2\phi) = 2mgl\sin^2\phi$$

The kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}ml^2\left[\left(2\dot{\phi} + 2\dot{\phi}\cos 2\phi\right)^2 + \left(2\dot{\phi}\sin 2\phi\right)^2\right]$$

$$= 8ml^2\dot{\phi}^2\cos^2\phi$$

Then the Lagrangian is

$$L = 8ml^2\dot{\phi}^2\cos^2\phi - 2mgl\sin^2\phi \tag{i}$$

And the canonical momentum is given by

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = 16ml^2\dot{\phi}\cos^2\phi$$

Now, the Hamiltonian is

$$H = p_\phi\dot{\phi} - L = 16ml^2\dot{\phi}^2\cos^2\phi - \left[8ml^2\dot{\phi}^2\cos^2\phi - 2mgl\sin^2\phi\right]$$

$$= 8ml^2\dot{\phi}^2\cos^2\phi + 2mgl\sin^2\phi$$

$$= \frac{p_\phi^2}{32ml^2\cos^2\phi} + 2mgl\sin^2\phi \qquad \text{(ii)}$$

Hence, the Hamiltonian represents the total energy of the system, $H = E$. Then from (ii), we can have

$$E = \frac{p_\phi^2}{32ml^2\cos^2\phi} + 2mgl\sin^2\phi = \frac{p_\phi^2}{32ml^2\cos^2\phi} + 2mgl\sin^2\phi$$

or, $$p_\phi^2 = 32ml^2\cos^2\phi\left(E - 2mgl\sin^2\phi\right)$$

This can be simplified to get

$$p_\phi = 4l\sqrt{2mE}\left[1 - \frac{2mgl}{E}\sin^2\phi\right]^{1/2}\cos\phi \qquad \text{(iii)}$$

Now, the action variable for the periodic motion is

$$J = \oint p_\phi d\phi = 16l\sqrt{2mE}\int_0^{\phi_{\max}}\left[1 - \frac{2mgl}{E}\sin^2\phi\right]^{1/2}\cos\phi d\phi \qquad \text{(iv)}$$

In the above integral we have inserted a multiplication factor 4 to get the integral over a complete cycle. The integration can be performed, by putting

$$u = \sqrt{\frac{2mgl}{E}}\sin\phi \quad \text{so that } du = \sqrt{\frac{2mgl}{E}}\cos\phi d\phi \text{ and } d\phi = \frac{1}{\cos\phi}\sqrt{\frac{E}{2mgl}}du$$

Then, $$J = 16E\sqrt{\frac{l}{g}}\int_0^{u_{\max}}\left(1 - u^2\right)^{1/2} du \qquad \text{(v)}$$

where the upper limit of integration is

$$u_{\max} = \sqrt{\frac{2mgl}{E}}\sin\phi_{\max}$$

From equation (i), we see that $2mgl\sin^2\phi_{\max}$ is the maximum value of the potential energy and is equal to the total energy. Therefore, we get; $u_{\max} = 1$. Then,

$$J = 16E\sqrt{\frac{l}{g}}\int_0^1\left(1 - u^2\right)^{1/2} du \qquad \text{(vi)}$$

Integration can be performed using the product rule by taking $(1-u^2)^{1/2}$ as the first function and du as the second function. Then, we get

$$J = 8E\sqrt{\frac{l}{g}}\int_0^1\left|u(1-u^2)^{1/2} + \sin^{-1} u\right|_0^1 = 4\pi E\sqrt{\frac{l}{g}} \tag{vii}$$

Now, the Hamiltonian can be written as

$$H = E = \frac{J}{4\pi}\sqrt{\frac{g}{l}} \tag{viii}$$

Then, the frequency is

$$\omega = \frac{\partial H}{\partial J} = \frac{1}{4\pi}\sqrt{\frac{g}{l}} \tag{ix}$$

This is the required result.

EXERCISES

5.1 Obtain the Hamilton's principal function for the simple harmonic oscillator whose Lagrangian is given by $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2$ along the actual path between the end points.

5.2 The Hamiltonian describing one-dimensional motion of a particle is given by, $H = \frac{p_x^2}{2m} - Amtx$, where, A is a constant and the potential varies linearly with time. Solve the problem by the Hamilton-Jacobi method with initial conditions given by at $t = 0$, $x = 0$ and $p_x = mv_0$.

5.3 Using Hamilton-Jacobi method show that the orbit of a particle with Hamiltonian $H = \frac{1}{2}\left(\frac{p_x^2 + p_y^2}{x^2 + y^2}\right) + \frac{1}{x^2 + y^2}$ is a conic in the x-y plane.

5.4 Use the Hamilton-Jacobi equation to obtain an equation for the orbit of a particle moving in a two-dimensional potential $V = \frac{1}{2}\alpha r^2$, describing the motion in terms of the coordinates u and v defined by $x = \cosh u \cos v$ and $y = \sinh u \sin v$.

5.5 A particle of mass m moves in a field which is a superposition of a Coulomb field with potential $-\frac{k}{r}$ and a constant field F in the z-direction with potential $-Fz$. The total potential is $V = -\frac{k}{r} - Fz$. Set up the time independent Hamilton-Jacobi equation and obtain the Hamilton's characteristic function.

5.6 A particle of mass m and charge q is moving in the field of a fixed electric dipole with dipole moment p. In spherical polar coordinates, the potential energy is given by $V = \frac{1}{4\pi\varepsilon_0}\frac{qp}{r^2}\cos\theta$. Obtain the complete integral of the Hamilton's characteristic function.

5.7 A particle of mass m moves in one dimension, subjected to a potential given by the expression $V = \frac{a}{\sin^2(x/x_0)}$. Obtain the complete integral for the Hamilton's characteristic function.

5.8 A particle of mass m is constrained to remain on a smooth moving wire represented by the equation $y = x + \frac{1}{2}at^2$. Initially, when $t = 0$, the particle is at rest at the point $x = y = 0$. Find the path of the particle using Hamilton-Jacobi method.

5.9 A particle of mass m moves in a potential, which in spherical polar coordinate system is given by $V = f(r) + \frac{g(\theta)}{r^2}$, where, $f(r)$ and $g(\theta)$ are some functions of r and θ respectively. Obtain a general solution to the orbit by Hamilton-Jacobi method.

5.10 A particle executes periodic motion in one dimension under the influence of a potential $V(x) = kx$, where, k is a constant. Find the action and angle variables and then find the frequency of motion.

5.11 Use the action and angle variable method to describe the motion of a particle moving in a plane under the action of a central force whose potential is given by $V(r) = -\frac{k}{r} - \frac{\beta}{r^2}$, where k and β are positive constants and r is the distance from the force centre. Also $\beta << kr$. Work out the problem in polar coordinate system.

5.12 A particle moves in two dimensions (r, ϕ) in a circular infinite well potential given by $V = 0$ for $r < a$ and $V = \infty$ for $r \geq a$. Find the action and angle variables. Also determine the frequencies of motion.

5.13 A particle moves in one dimension in a potential $V = V_0 \tan^2\left(\frac{\pi x}{2a}\right)$, where V_0 and a are constants. Obtain the action and angle variables and express the total energy in terms of them. Also find the frequency.

5.14 A particle of mass m moves in a three-dimensional isotropic oscillator well whose potential is given by; $V = \frac{1}{2}m\omega^2\left(\rho^2 + z^2\right)$. Find the action and angle variables and express the Hamiltonian in terms of them. Also find the frequencies.

5.15 Obtain the action and angle variables for a simple harmonic oscillator that has a time dependent frequency of oscillation. Also obtain the Hamilton's canonical equations of motion for the action and variables.

5.16 A particle of mass m moves in the field of a fixed central force whose potential is $V(r)$. Find the action and angle variables of the particle in the spherical polar coordinate system. Then obtain the frequency of motion.

5.17 A particle of mass m is constrained to move on a cylinder of radius R held vertically in a uniform gravitational field of strength g with an additional constraint $z \geq 0$. When it reaches the bottom it bounces elastically and reverses the direction of the vertical component of its velocity while the horizontal component remains unchanged. Use the action-angle variable method to determine the frequency of motion.

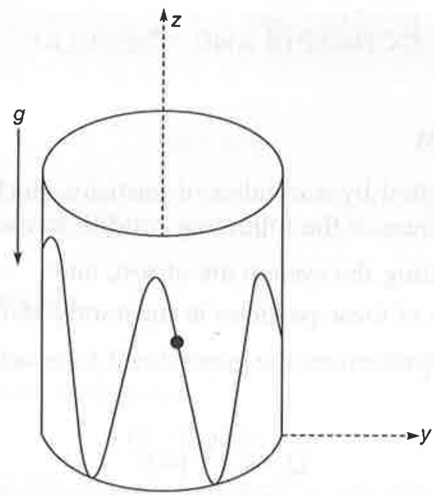

Fig. 5.3

5.18 A three-dimensional harmonic oscillator has the force constant k_1 and k_2 in the x- and y- directions and k_3 in the z-direction. Using cylindrical coordinates (with the axis of the cylinder in the z direction), describe the motion in terms of the corresponding action-angle variables, and determine the frequencies.

5.19 Solve the three-dimensional Kepler problem using the action-angle variable method.

5.20 Find the frequencies of a three-dimensional harmonic oscillator with unequal force constants using the method of action-angle variables. Obtain the solution for each Cartesian coordinate and conjugate momentum as functions of the action-angle variables.

6

CHAPTER

Small Oscillations

CONCEPTS AND FORMULAE

6.1 STATIC EQUILIBRIUM

Consider a system constituted by a number of particles. Such a system of particles is said to be in static equilibrium, if the following conditions are satisfied:

- the particles constituting the system are at rest, and
- the net force on each of these particles is constantly zero.

When the system is in equilibrium, the generalized force acting on the system is zero, that is,

$$Q_j = \left(\frac{\partial V}{\partial q_j}\right) = 0 \tag{6.1}$$

6.2 STABLE AND UNSTABLE EQUILIBRIUM

For a system in equilibrium, if a small oscillation about the equilibrium position causes bounded motion about the equilibrium position, then it is called a stable equilibrium. On the other hand, such an oscillation causes an unbound motion of the system, then it is called unstable equilibrium.

6.3 CONDITION FOR BOUNDED MOTION

For a system to execute bounded oscillation, there must be a local minimum for the potential energy. If there is a local maximum for the potential energy, then the motion will be unbounded. This can be put mathematically as;

$V(x_0)=0$ and $V''(x)<0$, for bounded motion and,

$V(x_0)=0$ and $V''(x)>0$, for unbound motion.

6.4 LAGRANGE'S EQUATION OF MOTION FOR SMALL OSCILLATION

The equation of motion for small oscillations can be obtained from the Lagrange's equation of motion and is given by

$$T\ddot{\eta} + V\eta = 0 \tag{6.2}$$

where, η is the displacement vector and represents the deviation from the equilibrium position, $\left[T_{j,k}\right]$ and, $\left[V_{j,k}\right]$ are symmetric and constant matrices representing the kinetic and potential energies respectively.

6.5 EIGENVALUE EQUATION FOR SMALL OSCILLATION

The equation of motion given in (6.2) can be converted into an eigenvalue equation and is given as

$$A\eta = \omega^2 \eta \tag{6.3}$$

where, $A = T^{-1}V$ and ω is the eigenvalue and is called the *eigenfrequency*. Each oscillation with a definite frequency is called a normal mode of oscillation.

6.6 NORMAL COORDINATES

We can introduce a coordinate, say Q_j along the direction of the eigenvector η_j, so that the equation of motion can be written as

$$\ddot{Q}_j + \omega_j^2 Q_j = 0 \tag{6.4}$$

The quantity Q_j is called the normal coordinate of the system.

SOLVED PROBLEMS

EXAMPLE 6.1 Show that the eigenvalue equation of small oscillations can be expressed in the form $A\eta = \omega^2\eta$, where, $A = T^{-1}V$ and ω is the eigenfrequency.

Solution: We consider a system which is in stable equilibrium so that a small displacement from the equilibrium position causes bounded oscillatory motion. Let q_j with $j = 1, 2, 3......$ be the generalized coordinate, then the displacement from the equilibrium position can be written as

$$\eta_j = q_j - q_{0j} \tag{i}$$

where, q_{0k} is the coordinate of the equilibrium position.

Now we expand the potential energy in a Tylor series about the equilibrium position. that is,

$$V(q_1, q_2, q_3, ...) = V(q_{01}, q_{02}, q_{03}, ...) + \sum_j \left(\frac{\partial V}{\partial q_j}\right)_0 \eta_j + \frac{1}{2}\sum_{j,k}\left(\frac{\partial^2 V}{\partial q_j \partial q_k}\right)_0 \eta_j \eta_k + ...$$

$$V(q_1, q_2, q_3, \ldots\ldots) = V_0 + \sum_j \left(\frac{\partial V}{\partial q_j}\right)_0 \eta_j + \frac{1}{2}\sum_{j,k}\left(\frac{\partial^2 V}{\partial q_j \partial q_k}\right)_0 \eta_j \eta_k + \ldots\ldots\ldots \tag{ii}$$

The first term on RHS of (ii) is a constant and can be taken as zero. The second term represents the generalized force and since the system is in equilibrium it is equal to zero. Further, we neglect the terms containing higher powers and write

$$V = \frac{1}{2}\sum_{j,k}\left(\frac{\partial^2 V}{\partial q_j \partial q_k}\right)_0 \eta_j \eta_k = \frac{1}{2}\sum_{j,k} V_{jk}\eta_j\eta_k \tag{iii}$$

where,
$$V_{jk} = \left(\frac{\partial^2 V}{\partial q_j \partial q_k}\right)_0 = V_{kj} \tag{iv}$$

The kinetic energy of the system can be written as

$$T = \frac{1}{2}\sum_{j,k} T_{jk}\dot{\eta}_j\dot{\eta}_k \tag{v}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}\sum_{j,k} T_{jk}\dot{\eta}_j\dot{\eta}_k - \frac{1}{2}\sum_{j,k} V_{jk}\eta_j\eta_k$$

$$= \frac{1}{2}\sum_{j,k}\left(T_{jk}\dot{\eta}_j\dot{\eta}_k - V_{jk}\eta_j\eta_k\right) \tag{vi}$$

Using this expression of Lagrangian in the Lagrange's equation of motion, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$, we get n equations of motion as

$$\sum_k T_{jk}\ddot{\eta}_k + \sum_k V_{jk}\eta_k = 0 \quad \text{with } k = 1, 2, 3, \ldots\ldots, n \tag{vii}$$

In general, this expression can be written as

$$T\ddot{\eta} + V\eta = 0 \tag{viii}$$

In this expression, $T = \left[T_{jk}\right]$ and $V = \left[V_{jk}\right]$ are two symmetry and constant matrices. Equation (viii) represents a set of simultaneous differential equations representing a set of coupled oscillations.

Now, let $A = T^{-1}V$ so that equation (viii) becomes

$$\ddot{\eta} + A\eta = 0 \tag{ix}$$

We assume a trial solution to the above equation as $A = \omega^2 I$, where I is the identity matrix. Then; from (ix), we get

$$\ddot{\eta} + \omega^2\eta = 0 \tag{x}$$

where ω^2 is the eigenvalue or eigenfrequency.

Now, using the equations (ix) and (x), we can write the equation of a simple harmonic motion as

$$A\eta = \omega^2 \eta \text{ or, } \left(A - \omega^2 I\right)\eta = 0 \tag{xi}$$

This is the required result. The eigenvalues can be evaluated as

$$\left|A - \omega^2 I\right| = 0 \tag{xii}$$

Using $A = T^{-1}V$, we can rewrite (xii) as

$$\left|V - \omega^2 T\right| = 0 \tag{xiii}$$

Note that, the matrices V and T are real and therefore, the eigenvalues are real. Further, we can try solutions to equation (viii) as

$$\eta_k = Ca_k e^{-i\omega t} \tag{xiv}$$

and get a set of linear homogeneous equations of the variables a_k' s given by

$$V_{jk}a_k - \omega^2 T_{jk}a_k = 0 \tag{xv}$$

These equations have a nontrivial solution if the determinant of the coefficients vanishes. This can be solved for ω^2 and then for a_k's.

EXAMPLE 6.2 Show that the free vibration of a diatomic molecule is simple harmonic in nature and obtain the expression frequency vibration.

Solution: We consider a diatomic molecule composed of two atoms with masses m_1 and m_2. Assume that the atoms are connected through a spring of elastic constant k. Let the equilibrium separation between the atoms be a. The displacement of the atoms from their equilibrium position is represented by x_1 and x_2; respectively that of m_1 and m_2.

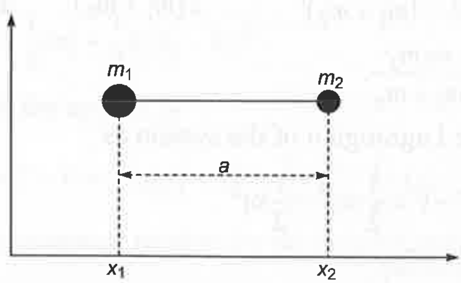

Fig. 6.1

Then the extension of the spring from its equilibrium length is given by

$$q = x_2 - x_1 - a \tag{i}$$

The kinetic and potential energies of the molecule can be written as

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \tag{ii}$$

and
$$V = \frac{1}{2}kq^2 \tag{iii}$$

Now, to write the Lagrangian of the system, we need to write the kinetic energy in terms of the generalized velocity $\dot{q}$.

From (i), we get

$$\dot{q} = \dot{x}_2 - \dot{x}_1 \tag{iv}$$

To obtain, $\dot{x}_1$ and $\dot{x}_2$ we make use of the law of conservation of linear momentum. That is,

$$m_1\dot{x}_1 + m_2\dot{x}_2 = 0 \tag{v}$$

Note that the RHS of equation (v) is made zero by the choice of suitable origin.

Now, we can solve the equations (iv) and (v) to get

$$\dot{x}_1 = \frac{m_2}{m_1+m_2}\dot{q} \text{ and } \dot{x}_2 = -\frac{m_1}{m_1+m_2}\dot{q} \tag{vi}$$

Then, the expression for kinetic energy becomes

$$T = \frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2\dot{q}^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\right)^2\dot{q}^2$$

$$= \frac{1}{2}\frac{m_1m_2^2}{(m_1+m_2)^2}\dot{q}^2 + \frac{1}{2}\frac{m_2m_1^2}{(m_1+m_2)^2}\dot{q}^2$$

$$= \frac{1}{2}\frac{m_1m_2(m_1+m_2)}{(m_1+m_2)^2}\dot{q}^2 = \frac{1}{2}\frac{m_1m_2}{(m_1+m_2)}\dot{q}^2 = \frac{1}{2}m\dot{q}^2 \tag{vii}$$

where, $m = \dfrac{m_1m_2}{m_1+m_2}$.

Now, we can write the Lagrangian of the system as

$$L = T - V = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \tag{viii}$$

Then, $\dfrac{\partial L}{\partial \dot{q}} = m\dot{q}$ and $\dfrac{\partial L}{\partial q} = -kq$.

These can be used in the Lagrange's equation of motion $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) - \dfrac{\partial L}{\partial q} = 0$, we get

$$\frac{d}{dt}(m\dot{q}) + kq = 0, \text{ or } m\ddot{q} + kq = 0$$

or
$$m\ddot{q} + kq = 0$$

that is,
$$\ddot{q}+\frac{k}{m}q=0 \tag{ix}$$

Equation (ix) represents a simple harmonic motion whose frequency is given by, $\omega=\sqrt{\frac{k}{m}}$.

EXAMPLE 6.3 Two spheres of equal masses m are connected to each other with a spring of force constant k. One of the spheres is connected to a wall using a spring of the same force constant. The spheres are restricted to move in a straight line in a horizontal frictionless surface. Obtain the equation of motion for small amplitude oscillation of the spheres and the normal frequencies.

Solution: Consider the schematic diagram given below. The instantaneous positions of the masses from an arbitrary origin are taken as, x_1 and x_2.

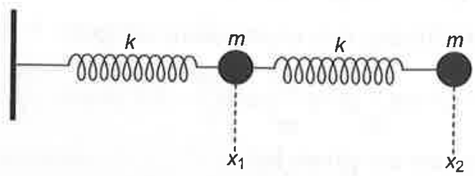

Fig. 6.2

The kinetic energy of the system can be written as

$$T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2=\frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2\right) \tag{i}$$

The potential energy is given by

$$V=\frac{1}{2}kx_1^2+\frac{1}{2}k(x_2-x_1)^2 \tag{ii}$$

Now, the Lagrangian of the system is

$$L=T-V=\frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2\right)-\frac{1}{2}kx_1^2-\frac{1}{2}k(x_2-x_1)^2 \tag{iii}$$

Then, the Lagrange's equation of motion for the coordinates x_1 is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=0$$

or
$$\frac{d}{dt}(m\dot{x}_1)+kx_1-k(x_2-x_1)=0$$

that is,
$$m\ddot{x}_1+kx_1-k(x_2-x_1)=0$$

or $$m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \tag{iv}$$

Similarly, for the coordinate, the Lagrange's equation of motion gives

$$m\ddot{x}_2 - kx_1 + kx_2 = 0 \tag{v}$$

Now, let, $x_1 = \alpha e^{i\omega t}$ and $x_2 = \beta e^{i\omega t}$, so that the equations (iv) and (v) become

$$-m\omega^2\alpha + 2k\alpha - k\beta = 0 \tag{vi}$$

and $$-m\omega^2\beta - k\alpha + k\beta = 0 \tag{vii}$$

For a nontrivial solution, the determinant of the coefficients of the equations (vi) and (vii) must vanish. That is,

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = 0$$

or $$\left(2k - m\omega^2\right)\left(k - m\omega^2\right) - k^2 = 0$$

On expanding and simplifying this expression, we get

$$\omega^4 - 3\frac{k}{m}\omega^2 + \frac{k^2}{m^2} = 0 \tag{viii}$$

Then, the eigenfrequencies are given by

$$\omega_1^2 = \frac{1}{2}\left(3 + \sqrt{5}\right)\frac{k}{m} \text{ and } \omega_2^2 = \frac{1}{2}\left(3 - \sqrt{5}\right)\frac{k}{m} \tag{ix}$$

Equation (ix) is the required result.

EXAMPLE 6.4 Two identical simple pendula of mass m and length l are connected by a massless spring of force constant k. Obtain the equation of motion, frequencies of normal modes and normal coordinates for small oscillations executed by the system.

Solution: A schematic representation of the problem is given below in Figure 6.3.

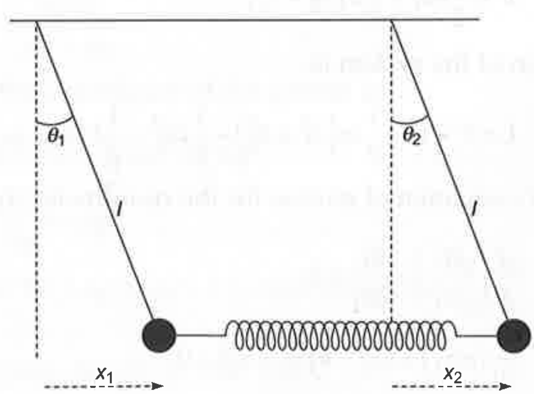

Fig. 6.3

Let x_1 and x_2 be the displacements of the bobs from their equilibrium position. Then, the kinetic energy of the system is

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) = \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) \tag{i}$$

and in matrix form

$$T_{jk} = \begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 \end{bmatrix} \tag{ii}$$

Then, the potential energy of the system is

$$\begin{aligned} V &= mgl(1-\cos\theta_1) + mgl(1-\cos\theta_2) + \frac{1}{2}k(x_2 - x_1)^2 \\ &= mgl(1-\cos\theta_1) + mgl(1-\cos\theta_2) + \frac{1}{2}kl^2(\sin\theta_2 - \sin\theta_1)^2 \end{aligned} \tag{iii}$$

For small oscillations, θ is very small and hence, we can write

$$\cos\theta_1 \cong 1 - \frac{\theta_1^2}{2}, \ \cos\theta_2 \cong 1 - \frac{\theta_2^2}{2} \text{ and } \sin\theta_2 - \sin\theta_1 \cong \theta_2 - \theta_1$$

Then equation (iii) becomes

$$\begin{aligned} V &= mgl\frac{\theta_1^2}{2} + mgl\frac{\theta_2^2}{2} + \frac{1}{2}kl^2(\theta_2 - \theta_1)^2 \\ &= mgl\frac{\theta_1^2}{2} + mgl\frac{\theta_2^2}{2} + \frac{1}{2}kl^2\left(\theta_2^2 - 2\theta_2\theta_1 + \theta_1^2\right) \\ &= \frac{1}{2}\left(mgl + kl^2\right)\theta_1^2 + \frac{1}{2}\left(mgl + kl^2\right)\theta_2^2 - kl^2\theta_2\theta_1 \end{aligned} \tag{iv}$$

In matrix form this can be written as

$$V = \begin{bmatrix} mgl + kl^2 & -kl^2 \\ -kl^2 & mgl + kl^2 \end{bmatrix} \tag{v}$$

The eigenfrequencies can be obtained by evaluating the determinant $\left|V - \omega^2 T\right| = 0$. Then,

$$\begin{vmatrix} mgl + kl^2 - \omega^2 ml^2 & -kl^2 \\ -kl^2 & mgl + kl^2 - \omega^2 ml^2 \end{vmatrix} = 0 \tag{vi}$$

or $$\left(mgl + kl^2 - \omega^2 ml^2\right)^2 - k^2 l^4 = 0$$

or $$\left(\frac{mg}{l} + k - \omega^2 m\right)^2 - k^2 = 0$$

This can be written as

$$\left(\frac{mg}{l}+2k-\omega^2 m\right)^2-\left(\frac{mg}{l}-\omega^2 m\right)=0 \tag{vii}$$

From this expression, we get

$$\omega^2=\frac{g}{l} \quad \text{or,}\ \omega^2=\frac{g}{l}+\frac{2k}{m}$$

that is,

$$\omega_1=\sqrt{\frac{g}{l}} \quad \text{and}\ \omega_2=\sqrt{\frac{g}{l}+\frac{2k}{m}} \tag{viii}$$

Equation (vii) gives the normal frequencies or the eigenfrequencies.

Now, we determine the eigenvectors by using the determinant equation (vi). We write

$$\left(mgl+kl^2-\omega^2 ml^2\right)a_{11}-kl^2 a_{12}=0$$

or

$$\left(\frac{mg}{l}+k-\omega^2 m\right)a_{11}-ka_{12}=0$$

For the frequency $\omega_1=\sqrt{\frac{g}{l}}$ this expression becomes

$$\left[\frac{mg}{l}+k-\left(\frac{g}{l}\right)m\right]a_{11}-ka_{12}=0$$

or

$$ka_{11}-ka_{12}=0$$

Then,

$$a_{11}=a_{12}\equiv\alpha \tag{ix}$$

Again, we have

$$\left(mgl+kl^2-\omega^2 ml^2\right)a_{21}-kl^2 a_{22}=0$$

or

$$\left(\frac{mg}{l}+k-\omega^2 m\right)a_{21}-ka_{22}=0$$

For the frequency $\omega_2=\sqrt{\frac{g}{l}+\frac{2k}{m}}$, the above equation becomes

$$\left[\frac{mg}{l}+k-\left(\frac{g}{l}+\frac{2k}{m}\right)m\right]a_{21}-ka_{22}=0$$

$$-ka_{21}-ka_{22}=0$$

Then,

$$a_{21}=-a_{22} \tag{x}$$

Now, the eigenvectors can be represented in a matrix form as

$$A=\begin{bmatrix}\alpha & \alpha\\ \beta & -\beta\end{bmatrix} \tag{xi}$$

Now, α and β can be evaluated using the relation $A^T TA = I$, then

$$\begin{bmatrix}\alpha & \beta\\ \alpha & -\beta\end{bmatrix}\begin{bmatrix}ml^2 & 0\\ 0 & ml^2\end{bmatrix}\begin{bmatrix}\alpha & \alpha\\ \beta & -\beta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

This can be simplified to get

$$\alpha=\beta=\frac{1}{\sqrt{2ml^2}} \tag{xii}$$

Then, equation (xi) becomes

$$A=\begin{bmatrix}\dfrac{1}{\sqrt{2ml^2}} & \dfrac{1}{\sqrt{2ml^2}}\\ \dfrac{1}{\sqrt{2ml^2}} & -\dfrac{1}{\sqrt{2ml^2}}\end{bmatrix} \tag{xiii}$$

Now, let Q_1 and Q_2 be the normal coordinates so that the transformation equation can be written as

$$\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\sqrt{2ml^2}} & \dfrac{1}{\sqrt{2ml^2}}\\ \dfrac{1}{\sqrt{2ml^2}} & -\dfrac{1}{\sqrt{2ml^2}}\end{bmatrix}\begin{bmatrix}Q_1\\ Q_2\end{bmatrix}$$

On expansion, we get

$$\theta_1=\frac{Q_1+Q_2}{\sqrt{2ml^2}} \text{ and, } \theta_2=\frac{Q_1-Q_2}{\sqrt{2ml^2}} \tag{xiv}$$

This can be solved to obtain

$$Q_1=\sqrt{\frac{ml^2}{2}}(\theta_1+\theta_2) \text{ and } Q_2=\sqrt{\frac{ml^2}{2}}(\theta_1-\theta_2) \tag{xv}$$

Note that, for the Q_1 mode, we take $Q_2=0$ and hence $\theta_1=\theta_2$, that is the two pendula are oscillating in phase. On the other hand for Q_2 mode, we take $Q_1=0$ and hence $\theta_1=-\theta_2$, then the two pendula are oscillating out of phase.

EXAMPLE 6.5 Two masses m_1 and m_2 are attached to the ends of a massless spring having a force constant k. Determine the frequency of oscillation of the masses.

Solution: Let x_1 and x_2 be the displacements of the masses m_1 and m_2 respectively. Then the kinetic energy of the system can be written as

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \tag{i}$$

and the potential energy is

$$V = \frac{1}{2}k(x_2 - x_1)^2 \tag{ii}$$

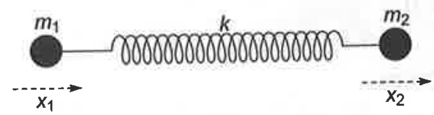

Fig. 6.4

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1)^2 \tag{iii}$$

Now, we write the Lagrange's equations of motion for the coordinates x_1 and x_2. For the coordinate x_1 it is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

that is,

$$\frac{d}{dt}(m_1\dot{x}_1) - k(x_2 - x_1) = 0$$

or

$$m_1\ddot{x}_1 - k(x_2 - x_1) = 0 \quad \text{or,} \quad \ddot{x}_1 - \frac{k}{m_1}(x_2 - x_1) = 0 \tag{iv}$$

Similarly, for the coordinate x_2, we get

$$\ddot{x}_2 + \frac{k}{m_2}(x_2 - x_1) = 0 \tag{v}$$

Subtracting (iv) from (v), we get

$$(\ddot{x}_2 - \ddot{x}_1) + k\left(\frac{1}{m_2} + \frac{1}{m_1}\right)(x_2 - x_1) = 0 \tag{vi}$$

Now, let $\xi = x_2 - x_1$ so that, $\ddot{x}_2 - \ddot{x}_1 = \ddot{\xi}$. Then equation (vi) becomes

$$\ddot{\xi} + k\left(\frac{m_1 + m_2}{m_1 m_2}\right)\xi = 0 \tag{vii}$$

Equation (vii) represents a simple harmonic motion with an eigenfrequency $\omega=\sqrt{k\left(\frac{m_1+m_2}{m_1 m_2}\right)}$.

EXAMPLE 6.6 The Lagrangian of a vibrating system is given by $L=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)-\frac{1}{2}\omega_0^2\left(x^2+y^2\right)+\alpha\dot{x}\dot{y}$. Show that the normal modes of vibration exist only if $\alpha\neq 1$.

Solution: Given that the Lagrangian of the system is

$$L=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)-\frac{1}{2}\omega_0^2\left(x^2+y^2\right)+\alpha\dot{x}\dot{y} \qquad \text{(i)}$$

First, let us find the Lagrange's equations of motion corresponding to the coordinates x and y. For the coordinate x, it is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

that is,

$$\frac{d}{dt}\left(\dot{x}+\alpha\dot{y}\right)+\omega_0^2 x=0$$

or

$$\ddot{x}+\alpha\ddot{y}+\omega_0^2 x=0 \qquad \text{(ii)}$$

Similarly, for the coordinate y, we get the equation of motion as

$$\ddot{y}+\alpha\ddot{x}+\omega_0^2 y=0 \qquad \text{(iii)}$$

Now, we try a solution of the form

$$x=x_0 e^{\omega t} \quad \text{and} \quad y=y_0 e^{\omega t} \qquad \text{(iv)}$$

Then,

$$\ddot{x}=x_0\omega^2 e^{\omega t} \text{ and } \ddot{y}=y_0\omega^2 e^{\omega t}$$

These can be substituted in equation (ii) to get

$$x_0\omega^2 e^{\omega t}+\alpha y_0\omega^2 e^{\omega t}+\omega_0^2 x_0 e^{\omega t}=0$$

or

$$x_0\left(\omega^2+\omega_0^2\right)+\alpha\omega^2 y_0=0 \qquad \text{(v)}$$

Similarly, from (iii), we get

$$y_0\left(\omega^2+\omega_0^2\right)+\alpha\omega^2 x_0=0 \qquad \text{(vi)}$$

Equations (v) and (vi) have a nontrivial solution if the determinant of their coefficient vanishes.

That is, if

$$\begin{vmatrix} \omega^2+\omega_0^2 & \alpha\omega^2 \\ \alpha\omega^2 & \omega^2+\omega_0^2 \end{vmatrix}=0$$

This can be expanded to get

$$\left(\omega^2 + \omega_0^2\right)^2 - \left(\alpha\omega^2\right)^2 = 0$$

or $$\omega^4 + \omega_0^4 + 2\omega_0^2\omega^2 - \alpha^2\omega^4 = 0 \tag{vii}$$

From (vii) we see that if $\alpha = 1$,

$$\omega_0^2 + 2\omega^2 = 0$$

This is never possible since as per the above expression, the frequency of oscillation is a negative quantity. Therefore, α cannot be equal to unity. Hence, proved.

EXAMPLE 6.7 The Lagrangian of an oscillating system is given by $L = \left(\dot{q}_1^2 + \dot{q}_2^2\right) + \dot{q}_1\dot{q}_2 - \left(q_1^2 + q_2^2\right)$. Obtain the eigenfrequencies.

Solution: Given that the Lagrangian is

$$L = \left(\dot{q}_1^2 + \dot{q}_2^2\right) + \dot{q}_1\dot{q}_2 - \left(q_1^2 + q_2^2\right) \tag{i}$$

Now, we find the Lagrange's equations of motion for the coordinates q_1 and q_2. For the coordinate q_1 the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0$$

that is, $$\frac{d}{dt}(2\dot{q}_1 + \dot{q}_2) + 2q_1 = 0$$

or $$2\ddot{q}_1 + \ddot{q}_2 + 2q_1 = 0 \tag{ii}$$

Similarly for the coordinate q_2 the Lagrange's equation becomes

$$2\ddot{q}_2 + \ddot{q}_1 + 2q_2 = 0 \tag{iii}$$

Now we try a solution to the above equations in the form,

$$q_1 = \alpha e^{i\omega t} \quad \text{and} \quad q_2 = \beta e^{i\omega t} \tag{iv}$$

Then, $$\ddot{q}_1 = -\alpha\omega^2 e^{i\omega t} \quad \text{and} \quad \ddot{q}_2 = -\beta\omega^2 e^{i\omega t}$$

Substituting these in equation (ii), we get

$$-2\alpha\omega^2 e^{i\omega t} - \beta\omega^2 e^{i\omega t} + 2\alpha e^{i\omega t} = 0$$

or $$-2\alpha\omega^2 - \beta\omega^2 + 2\alpha = 0$$

or $$\left(2 - 2\omega^2\right)\alpha - \omega^2\beta = 0 \tag{v}$$

Also from (iii), we get

$$-\omega^2\alpha + \left(2 - 2\omega^2\right)\beta = 0 \tag{vi}$$

Equations (v) and (vi) have a nontrivial solution if the determinant of their coefficient vanishes. That is,

$$\begin{vmatrix} 2-2\omega^2 & -\omega^2 \\ -\omega^2 & 2-2\omega^2 \end{vmatrix} = 0$$

This can be expanded to get

$$\left(2-2\omega^2\right)^2 - \omega^4 = 0$$

or $$3\omega^4 - 8\omega^2 + 4 = 0$$

or $$3\omega^4 - 6\omega^2 - 2\omega^2 + 4 = 0$$

that is, $$3\omega^2\left(\omega^2-2\right) - 2\left(\omega^2-2\right) = 0$$

or $$\left(3\omega^2-2\right)\left(\omega^2-2\right) = 0 \qquad \text{(vii)}$$

From (vii), we get the eigenfrequencies as

$$\omega_1 = \sqrt{2} \quad \text{and} \quad \omega_2 = \sqrt{2/3} \qquad \text{(viii)}$$

These are the required results.

EXAMPLE 6.8 A particle of mass m is attached to the ends of two springs as shown in Figure 6.5. The other end of one spring is fixed at $(0,l)$ and the other end of the second spring is fixed at $\left(\frac{l}{\sqrt{2}}, \frac{l}{\sqrt{2}}\right)$. The equilibrium position of the particle is at $(0,0)$. Obtain the eigenfrequencies of small oscillations executed by the system.

Solution: We first find the extension (displacement) of both springs. For the spring whose fixed end is at $(0,l)$, the extension can be written as

$$\begin{aligned} d_1 &= \sqrt{x^2 + (y-l)^2} - l \\ &= \sqrt{x^2 + y^2 - 2yl + l^2} - l \\ &= l + \frac{x^2+y^2-2ly}{2l} - \frac{1}{2}\frac{y^2}{l} - l \\ &= -y + \frac{x^2}{2l} \end{aligned} \qquad \text{(i)}$$

The extension of the second spring can be obtained as

$$d_2 = \sqrt{\left(x - \frac{l}{\sqrt{2}}\right)^2 + \left(y - \frac{l}{\sqrt{2}}\right)^2} - l$$

$$= \sqrt{l^2 + x^2 + y^2 - \sqrt{2}l(x+y)} - l$$

$$= \frac{x^2+y^2}{2l} - \frac{xl+yl}{2l} - \frac{1}{4}\frac{x^2+y^2+2x}{l} \qquad \text{(ii)}$$

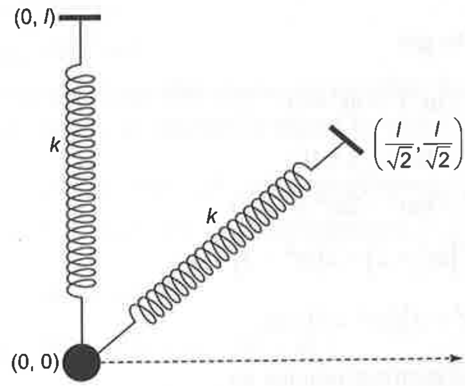

Fig. 6.5

In the above two expressions we keep only the terms in the first order and write the potential energy of the system as

$$V = \frac{1}{2}ky^2 + \frac{1}{2}k\frac{(x+y)^2}{4}$$

$$= \frac{1}{2}ky^2 + \frac{1}{2}k\left(\frac{x^2+2xy+y^2}{4}\right) \qquad \text{(iii)}$$

The kinetic energy of the system is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \qquad \text{(iv)}$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}ky^2 - \frac{1}{2}k\left(\frac{x^2+2xy+y^2}{4}\right) \qquad \text{(v)}$$

The Lagrange's equation of motion for the coordinates can be obtained as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

that is, $$\frac{d}{dt}(m\dot{x}) + \frac{1}{4}k(x+y) = 0$$

or $$m\ddot{x}+\frac{1}{4}k(x+y)=0 \qquad \text{(vi)}$$

Similarly, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y}=0$$

that is, $$\frac{d}{dt}(m\dot{y})+ky+\frac{1}{4}k(x+y)=0$$

or $$m\ddot{y}+ky+\frac{1}{4}k(x+y)=0$$

or $$m\ddot{y}+\frac{5}{4}ky+\frac{1}{4}kx=0 \qquad \text{(vii)}$$

Now, we try a solution of the form

$$x=\alpha e^{i\omega t} \quad \text{and} \quad y=\beta e^{i\omega t} \qquad \text{(viii)}$$

Then, $$\ddot{x}=-\alpha\omega^2 e^{i\omega t} \quad \text{and} \quad \ddot{y}=-\beta\omega^2 e^{i\omega t}$$

Substituting these in (vi), we get

$$-m\alpha\omega^2 e^{i\omega t}+\frac{1}{4}k\left(\alpha e^{i\omega t}+\beta e^{i\omega t}\right)=0$$

or $$-m\alpha\omega^2+\frac{1}{4}k(\alpha+\beta)=0$$

or $$\left(\frac{1}{4}k-m\omega^2\right)\alpha+\frac{1}{4}k\beta=0 \qquad \text{(ix)}$$

Then, from (viii) we get

$$-m\beta\omega^2 e^{i\omega t}+\frac{5}{4}k\beta e^{i\omega t}+\frac{1}{4}k\alpha e^{i\omega t}=0$$

or $$-m\beta\omega^2+\frac{5}{4}k\beta+\frac{1}{4}k\alpha=0$$

or $$\frac{1}{4}k\alpha+\left(\frac{5}{4}k-m\omega^2\right)\beta=0 \qquad \text{(x)}$$

Now, equations (ix) and (x) have nontrivial solutions, if the determinant of their coefficients vanishes. That is,

$$\begin{vmatrix} \frac{1}{4}k-m\omega^2 & \frac{1}{4}k \\ \frac{1}{4}k & \frac{5}{4}k-m\omega^2 \end{vmatrix}=0$$

or $$\left(\frac{1}{4}k - m\omega^2\right)\left(\frac{5}{4}k - m\omega^2\right) - \frac{k^2}{16} = 0$$

or $$m^2\omega^4 - \frac{6k}{4}m\omega^2 + \frac{k^2}{4} = 0$$

Dividing throughout by m^2 the above expression becomes

$$\omega^4 - \frac{6}{4}\frac{k}{m}\omega^2 + \frac{1}{4}\frac{k^2}{m^2} = 0 \tag{xi}$$

Now, we put, $\omega_0 = \sqrt{\frac{k}{m}}$ and equation (xi) becomes

$$\omega^4 - \frac{6}{4}\omega_0^2\omega^2 + \frac{1}{4}\omega_0^4 = 0$$

or $$4\omega^4 - 6\omega_0^2\omega^2 + \omega_0^4 = 0$$

The solution to this equation are the eigenfrequencies and is given by

$$\omega = \left(\frac{6 \pm 2\sqrt{5}}{8}\right)^{1/2} \omega_0 \tag{xii}$$

EXAMPLE 6.9 Consider the Lagrangian, $L = \frac{1}{2}\left(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2\right) - a^2\left(\eta_1^2 + \eta_2^2 + \eta_3^2 - \eta_1\eta_3\right)$ of a three-particle system, where a is a real quantity. Obtain the normal frequencies of small oscillations.

Solution: Given that

$$L = \frac{1}{2}\left(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2\right) - a^2\left(\eta_1^2 + \eta_2^2 + \eta_3^2 - \eta_1\eta_3\right) \tag{i}$$

We have the matrix representing the kinetic energy as

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

and in general, $$T = \frac{1}{2}\left(T_{11}\dot{\eta}_1^2 + T_{12}\dot{\eta}_1\dot{\eta}_2 + T_{13}\dot{\eta}_1\dot{\eta}_3 + T_{21}\dot{\eta}_2\dot{\eta}_1 + T_{22}\dot{\eta}_2^2 + \ldots\ldots\ldots\right) \tag{ii}$$

Comparing (ii) with (i), we get

$$T_{11} = T_{22} = T_{33} = 1 \text{ and } T_{ij} = 0 \text{ for } i \neq j$$

Then, $$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(iii)}$$

Similarly, the matrix representing the potential energy is

$$V = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}$$

Now, $$V_{11} = \left(\frac{\partial^2 V}{\partial \eta_1^2}\right) = 2a^2\ ,\ V_{22} = \left(\frac{\partial^2 V}{\partial \eta_2^2}\right) = 2a^2\ ,\ V_{33} = \left(\frac{\partial^2 V}{\partial \eta_3^2}\right) = 2a^2$$

Also, $$V_{13} = \left(\frac{\partial^2 V}{\partial \eta_1 \partial \eta_3}\right) = -a^2 \text{ and } V_{31} = \left(\frac{\partial^2 V}{\partial \eta_3 \partial \eta_1}\right) = -a^2$$

Then, $$V = \begin{bmatrix} 2a^2 & 0 & -a^2 \\ 0 & 2a^2 & 0 \\ -a^2 & 0 & 2a^2 \end{bmatrix} \quad \text{(iv)}$$

The normal frequencies can be determined by evaluating the determinant, $\left|V - \omega^2 T\right| = 0$

that is, $$\begin{vmatrix} 2a^2 - \omega^2 & 0 & -a^2 \\ 0 & 2a^2 - \omega^2 & 0 \\ -a^2 & 0 & 2a^2 - \omega^2 \end{vmatrix} = 0$$

$$(2a^2 - \omega^2)(2a^2 - \omega^2)(2a^2 - \omega^2) - a^2\left[a^2(2a^2 - \omega^2)\right] = 0$$

or $$(2a^2 - \omega^2)\left[(2a^2 - \omega^2)^2 - a^4\right] = 0 \quad \text{(v)}$$

From this equation, it is clear that one of the normal frequencies is given by

$$\omega^2 = 2a^2 \quad \text{(vi)}$$

The other frequencies can be obtained from;

$$(2a^2 - \omega^2)^2 - a^4 = 0$$

or $$2a^2 - \omega^2 = a^2 \text{ or, } \omega = \pm a \quad \text{(vii)}$$

Equations (vi) and (vii) are the required results.

EXAMPLE 6.10 Determine the normal frequencies of two coupled oscillators whose Lagrangian is given by $L=\frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2\right)-\frac{1}{2}m\omega_0^2\left(x_1^2+x_2^2\right)+m\omega_0^2\mu x_1x_2$, where m is the mass of the oscillators.

Solution: The Lagrangian of the system is given by

$$L=T-V=\frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2\right)-\frac{1}{2}m\omega_0^2\left(x_1^2+x_2^2\right)+m\omega_0^2\mu x_1x_2 \tag{i}$$

From this we may write the matrix representing the kinetic energy as

$$T=\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \tag{ii}$$

Similarly, the matrix representing the potential energy is

$$V=\begin{bmatrix} m\omega_0^2 & -m\omega_0^2\mu \\ -m\omega_0^2\mu & m\omega_0^2 \end{bmatrix} \tag{iii}$$

The eigenfrequencies can be obtained from the condition

$$\left|V-\omega^2T\right|=0$$

that is,

$$\begin{vmatrix} m\omega_0^2-m\omega^2 & -m\omega_0^2\mu \\ -m\omega_0^2\mu & m\omega_0^2-m\omega^2 \end{vmatrix}=0$$

or

$$\left(m\omega_0^2-m\omega^2\right)\left(m\omega_0^2-m\omega^2\right)-m^2\omega_0^4\mu^2=0$$

or

$$\left(\omega_0^2-\omega^2\right)^2-\omega_0^4\mu^2=0$$

or

$$\omega^4+\omega_0^4-2\omega_0^2\omega^2-\omega_0^4\mu^2=0$$

or

$$\omega^4-2\omega_0^2\omega^2-\left(\mu^2-1\right)\omega_0^4=0 \tag{iv}$$

The solution to this equation can be written as

$$\omega^2=\frac{2\omega_0^2\pm\sqrt{4\omega_0^4-4\left(\mu^2-1\right)\omega_0^4}}{2}$$

$$=\omega_0^2\pm\sqrt{\mu^2\omega_0^4}=\omega_0^2\pm\mu\omega_0^2$$

$$=\omega_0^2\left(1\pm\mu\right)$$

or

$$\omega=\pm\omega_0\sqrt{1\pm\mu}$$

that is,

$$\omega=\pm\omega_0\sqrt{1+\mu}\ \text{ and }\ \omega=\pm\omega_0\sqrt{1-\mu} \tag{v}$$

EXAMPLE 6.11 Find the normal frequencies and normal coordinates of the system whose Lagrangian is given by, $L=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)-\frac{1}{2}\left(\omega_1^2x^2+\omega_2^2y^2\right)+\alpha xy$.

Solution: Given that

$$L=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)-\frac{1}{2}\left(\omega_1^2x^2+\omega_2^2y^2\right)+\alpha xy \tag{i}$$

We first write the Lagrange's equations of motion for the coordinates x and y. That is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

or
$$\frac{d}{dt}(\dot{x})+\omega_1^2x-\alpha y=0$$

or
$$\ddot{x}+\omega_1^2x-\alpha y=0 \tag{ii}$$

Similarly, for the coordinate y, we get

$$\ddot{y}+\omega_2^2y-\alpha x=0 \tag{iii}$$

Now, we try a solution of the form

$$x=x_0e^{i\omega t} \text{ and } y=y_0e^{i\omega t} \tag{iv}$$

Then,

$$\ddot{x}=-\omega^2x \text{ and } \ddot{y}=-\omega^2y$$

Using these in equations (ii) and (iii), we get

$$\left(\omega_1^2-\omega^2\right)x-\alpha y=0 \text{ or, } x=\frac{\alpha y}{\left(\omega_1^2-\omega^2\right)}$$

or
$$\frac{x}{y}=\frac{\alpha}{\left(\omega_1^2-\omega^2\right)} \tag{v}$$

Similarly,
$$\left(\omega_2^2-\omega^2\right)y-\alpha x=0 \text{ or,}$$

$$\frac{x}{y}=\frac{\left(\omega_2^2-\omega^2\right)}{\alpha} \tag{vi}$$

From equations (iv) and (v), we get

$$\frac{\alpha}{\left(\omega_1^2-\omega^2\right)}=\frac{\left(\omega_2^2-\omega^2\right)}{\alpha}$$

or
$$\left(\omega_1^2-\omega^2\right)\left(\omega_2^2-\omega^2\right)-\alpha^2=0$$

or
$$\omega^4-\left(\omega_1^2+\omega_2^2\right)\omega^2+\left(\omega_1^2\omega_2^2-\alpha^2\right)=0 \tag{vii}$$

Solution to this equation can be written as

$$\omega^2 = \frac{\left(\omega_1^2+\omega_2^2\right) \pm \sqrt{\left(\omega_1^2-\omega_2^2\right)+4\alpha^2}}{2} \tag{viii}$$

Equation (viii) gives the normal frequencies of small oscillations. Note that if $\alpha = 0$, the normal frequencies will be ω_1 and ω_2.

Now, we can get the normal coordinates by solving equation (iv) for x and y. Then we get

$$x = A_1 \cos\left(\omega t + \varepsilon_1\right) \text{ and } y = A_2 \cos\left(\omega t + \varepsilon_2\right) \tag{ix}$$

where, A_1, A_2, ε_1 and ε_2 are arbitrary constants.

EXAMPLE 6.12 Find the normal frequencies and normal modes for a double pendulum: (i) by considering bobs of different masses and different lengths of the pendulum, (ii) by considering bobs of equal masses and equal lengths of the pendulum. Find the eigenvectors in the second case.

Solution:

(i) Let m_1, m_2 be the masses and l_1, l_2 be the lengths of the pendula. From Figure 6.6, we get the coordinates of the bobs of the pendula as

$$x_1 = l_1 \sin\theta \text{ and } y_1 = l_1 \cos\theta \tag{i}$$

Similarly, $x_2 = l_1 \sin\theta + l_2 \sin\phi$ and $y_2 = l_1 \cos\theta + l_2 \cos\phi$

Let us take the reference point as the point of suspension. Then the total potential energy of the system is

$$V = -m_1 g y_1 - m_2 g y_2 = -m_1 g l_1 \cos\theta - m_2 g \left(l_1 \cos\theta + l_2 \cos\phi\right) \tag{ii}$$

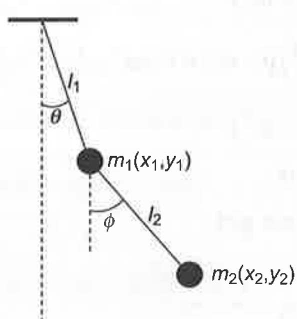

Fig. 6.6

For small values of θ and ϕ, we have, $\cos\theta = 1 - \frac{\theta^2}{2}$ and $\cos\phi = 1 - \frac{\phi^2}{2}$. Then the potential energy becomes

$$V = -m_1 g l_1\left(1-\frac{\theta^2}{2}\right) - m_2 g l_1\left(1-\frac{\theta^2}{2}\right) - m_2 g l_2\left(1-\frac{\phi^2}{2}\right)$$

$$= -m_1 g l_1 + m_1 g l_1 \frac{\theta^2}{2} - m_2 g l_1 + m_2 g l_1 \frac{\theta^2}{2} - m_2 g l_2 + m_2 g l_2 \frac{\phi^2}{2}$$

$$= -m_1 g l_1 - m_2 g\left(l_1 + l_2\right) + \frac{1}{2} m_1 g l_1 \theta^2 + \frac{1}{2} m_2 g\left(l_1 \theta^2 + l_2 \phi^2\right) \tag{iii}$$

Now, we write the matrix element of the potential energy as

$$V_{11} = \frac{\partial^2 V}{\partial^2 \theta} = \left(m_1 + m_2\right) g l_1$$

$$V_{12} = V_{21} = \frac{\partial^2 V}{\partial \theta \partial \phi} = 0 \quad \text{and}$$

$$V_{22} = \frac{\partial^2 V}{\partial^2 \phi} = m_2 g l_2$$

Then the matrix representing the potential energy is

$$V = \begin{bmatrix} \left(m_1 + m_2\right) g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix} \tag{iv}$$

The kinetic energy of the system is

$$T = \frac{1}{2} m_1\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{1}{2} m_2\left(\dot{y}_1^2 + \dot{y}_2^2\right)$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}^2 + \frac{1}{2} m_2\left[l_1^2 \dot{\theta}^2 + l_2^2 \dot{\phi}^2 + 2 l_1 l_2 \dot{\theta}\dot{\phi}\cos(\theta - \phi)\right]$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}^2 + \frac{1}{2} m_2\left(l_1^2 \dot{\theta}^2 + l_2^2 \dot{\phi}^2 + 2 l_1 l_2 \dot{\theta}\dot{\phi}\right)$$

$$= \frac{1}{2}\left(m_1 + m_2\right) l_1^2 \dot{\theta}^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}^2 + m_2 l_1 l_2 \dot{\theta}\dot{\phi} \tag{v}$$

The matrix elements of the kinetic energy are

$$T_{11} = \left(m_1 + m_2\right) l_1^2, \ T_{12} = T_{21} = m_2 l_1 l_2 \quad \text{and} \quad T_{22} = m_2 l_2^2$$

Then the matrix representing the kinetic energy is

$$T = \begin{bmatrix} \left(m_1 + m_2\right) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix} \tag{vi}$$

The eigenfrequencies can be obtained by evaluating the determinant $\left|V - \omega^2 T\right| = 0$.

or
$$\begin{bmatrix} \left(m_1 + m_2\right) g l_1 - \omega^2\left(m_1 + m_2\right) l_1^2 & -\omega^2 m_2 l_1 l_2 \\ -\omega^2 m_2 l_1 l_2 & m_2 g l_2 - \omega^2 m_2 l_2^2 \end{bmatrix} = 0 \tag{vii}$$

This can be simplified to get

$$\omega^4 m_2 l_1 l_2 - \omega^2 (m_1 + m_2)(l_1 + l_2) g + (m_1 + m_2) g^2 = 0 \qquad \text{(viii)}$$

Now we divide throughout by m_2 and then put $\lambda = \frac{m_1}{m_2}$. Then we get

$$\omega^4 l_1 l_2 - \omega^2 (1+\lambda)(l_1 + l_2) g + (1+\lambda) g^2 = 0 \qquad \text{(ix)}$$

Solution to this equation can be written as

$$\omega^2 = \frac{1}{2l_1 l_2}\left[(1+\lambda)(l_1+l_2)g \pm \sqrt{g^2(1+\lambda)^2(l_1+l_2)^2 - 4g^2(1+\lambda)l_1 l_2}\right] \qquad \text{(x)}$$

From this equation it is clear that the eigenfrequencies are real.

(ii) Now, we attempt the second part of the problem which is a special case of the first part. Here we assume that $l_1 = l_2 \equiv l$ and $m_1 = m_2 \equiv m$ so that $\lambda = 1$. Then equation (ix) reduces to

$$\omega^2 = \frac{1}{2l^2}\left(4gl \pm \sqrt{16g^2l^2 - 8g^2l^2}\right) = \frac{g}{l}\left(2 \pm \sqrt{2}\right) \qquad \text{(xi)}$$

So we get the eigenfrequencies as

$$\omega_1^2 = \frac{g}{l}\left(2+\sqrt{2}\right) \text{ and } \omega_2^2 = \frac{g}{l}\left(2-\sqrt{2}\right) \qquad \text{(xii)}$$

The eigenvectors can be determined by the equation: $V_{jk}a_k - \omega^2 T_{jk}a_k = 0$, which in the present case is

$$\begin{bmatrix} 2g - 2\omega^2 l & -\omega^2 l \\ -\omega^2 l & g - \omega^2 l \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = 0 \qquad \text{(xiii)}$$

Here, we have used equation (vi) with the condition $l_1 = l_2 \equiv l$ and $m_1 = m_2 \equiv m$.

For the eigenfrequency, $\omega_1^2 = \frac{g}{l}\left(2+\sqrt{2}\right)$ this expression becomes

$$\begin{bmatrix} \left(2-2\sqrt{2}\right)g & -\left(2+\sqrt{2}\right)g \\ -\left(2+\sqrt{2}\right)g & -\left(1+\sqrt{2}\right)g \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = 0 \qquad \text{(xiv)}$$

From (xiv) we can form two equations as follows:

$$\left(2-2\sqrt{2}\right)ga_{11} - \left(2+\sqrt{2}\right)ga_{21} = 0$$

and

$$-\left(2+\sqrt{2}\right)ga_{11} - \left(1+\sqrt{2}\right)ga_{21} = 0$$

From the second of these equations, we get

$$\frac{a_{21}}{a_{11}} = -\frac{2+\sqrt{2}}{1+\sqrt{2}}$$

So we take, $a_{11} = \left(1+\sqrt{2}\right)\alpha$ and $a_{21} = -\left(2+\sqrt{2}\right)\alpha$ (xv)

For the eigenfrequency $\omega_2^2 = \frac{g}{l}\left(2-\sqrt{2}\right)$ the matrix equation (xiii) becomes

$$\begin{bmatrix} \left(-2+2\sqrt{2}\right)g & -\left(2-\sqrt{2}\right)g \\ -\left(2-\sqrt{2}\right)g & \left(-1+\sqrt{2}\right)g \end{bmatrix}\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = 0 \tag{xvi}$$

This would yield

$$\frac{a_{22}}{a_{12}} = -\frac{2-\sqrt{2}}{1-\sqrt{2}}$$

or, we can write $a_{12} = \left(1-\sqrt{2}\right)\beta$ and $a_{22} = -\left(2-\sqrt{2}\right)\beta$ (xvii)

Therefore, we get

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \left(1+\sqrt{2}\right)\alpha & \left(1-\sqrt{2}\right)\beta \\ -\left(2+\sqrt{2}\right)\alpha & -\left(2-\sqrt{2}\right)\beta \end{bmatrix} \tag{xviii}$$

Now, the matrix representing the kinetic energy, with the condition $l_1 = l_2 \equiv l$ and $m_1 = m_2 \equiv m$ becomes

$$T = \begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & 2ml^2 \end{bmatrix} = ml^2\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{xix}$$

To find the values of α and β we use the condition; $A^T T A = I$. That is,

$$ml^2\begin{bmatrix} \left(1+\sqrt{2}\right)\alpha & -\left(2+\sqrt{2}\right)\alpha \\ \left(1-\sqrt{2}\right)\beta & -\left(2-\sqrt{2}\right)\beta \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \left(1+\sqrt{2}\right)\alpha & \left(1-\sqrt{2}\right)\beta \\ -\left(2+\sqrt{2}\right)\alpha & -\left(2-\sqrt{2}\right)\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On simplification, we get

$$ml^2\begin{bmatrix} \left(4+2\sqrt{2}\right)\alpha^2 & 0 \\ 0 & \left(4-2\sqrt{2}\right)\beta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{xx}$$

Then, we get

$$\alpha = \frac{1}{l\sqrt{m\left(4+2\sqrt{2}\right)}} \quad \text{and} \quad \beta = \frac{1}{l\sqrt{m\left(4-2\sqrt{2}\right)}} \tag{xxi}$$

Therefore, the eigenvectors are

$$a_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \frac{1}{l\sqrt{m\left(4+2\sqrt{2}\right)}} \begin{bmatrix} 1+\sqrt{2} \\ -\left(2+\sqrt{2}\right) \end{bmatrix} \tag{xxii}$$

and $$a_2 = \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix} = \frac{1}{l\sqrt{m\left(4-2\sqrt{2}\right)}} \begin{bmatrix} 1-\sqrt{2} \\ -\left(2-\sqrt{2}\right) \end{bmatrix} \tag{xxiii}$$

EXAMPLE 6.13 A triple pendulum consists of three bobs of equal masses m and three identical rigid rods of length l as shown in Figure 6.7. Find the normal frequencies and the eigenvectors.

Solution: First we write the Cartesian coordinates of the masses. They are

$$x_1 = l\sin\theta_1 \quad \text{and} \quad y_1 = -l\cos\theta_1$$
$$x_2 = l\sin\theta_1 + l\sin\theta_2 \quad \text{and} \quad y_2 = -l\cos\theta_1 - l\cos\theta_2$$
$$x_3 = l\sin\theta_1 + l\sin\theta_2 + l\sin\theta_3 \quad \text{and} \quad y_3 = -l\cos\theta_1 - l\cos\theta_2 - l\cos\theta_3$$

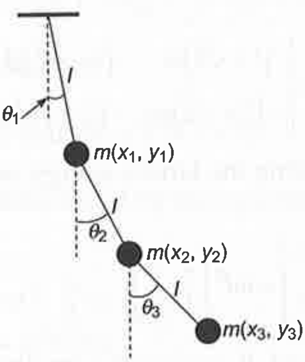

Fig. 6.7

The kinetic energy of the system is given by

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\right)$$

The time derivatives of the coordinates can be determined using the equations given above and can be substituted in the expression for kinetic energy to get

$$T = \frac{1}{2}ml^2 \begin{bmatrix} 3\dot{\theta}_1^2 + 2\dot{\theta}_2^2 + \dot{\theta}_3^2 + 4\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) \\ +2\dot{\theta}_1\dot{\theta}_3\cos\left(\theta_1 - \theta_3\right) + 2\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2 - \theta_3\right) \end{bmatrix} \tag{i}$$

In the case of small oscillations, we take $\cos\left(\theta_1 - \theta_2\right) = \cos\left(\theta_1 - \theta_3\right) = \cos\left(\theta_2 - \theta_1\right) \cong 1$ so that the expression for the kinetic energy reduces to

$$T = \frac{1}{2} ml^2 \left(3\dot{\theta}_1^2 + 2\dot{\theta}_2^2 + \dot{\theta}_3^2 + 4\dot{\theta}_1\dot{\theta}_2 + 2\dot{\theta}_1\dot{\theta}_3 + 2\dot{\theta}_2\dot{\theta}_3\right) \tag{ii}$$

Then, the matrix representing the kinetic energy can be written as

$$T = ml^2 \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{iii}$$

Then, the potential energy of the system is

$$V = mg(y_1 + y_2 + y_3)$$

Taking the point of suspension as the reference point, we can write

$$V = -mg(3l\cos\theta_1 + 2l\cos\theta_2 + l\cos\theta_3) \tag{iv}$$

Then, the matrix representing the potential energy can be written as

$$V = mgl \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{v}$$

Now, the eigenfrequencies are given by the roots of the equation $|V - \omega^2 T| = 0$.

$$\begin{vmatrix} 3mgl - 3ml^2\omega^2 & -2ml^2\omega^2 & -ml^2\omega^2 \\ -2ml^2\omega^2 & 2mgl - 2ml^2\omega^2 & -ml^2\omega^2 \\ -ml^2\omega^2 & -ml^2\omega^2 & mgl - ml^2\omega^2 \end{vmatrix} = 0$$

or

$$ml^2 \begin{vmatrix} 3\left(\frac{g}{l} - \omega^2\right) & -2\omega^2 & -\omega^2 \\ -2\omega^2 & 2\left(\frac{g}{l} - \omega^2\right) & -\omega^2 \\ -\omega^2 & -\omega^2 & \left(\frac{g}{l} - \omega^2\right) \end{vmatrix} = 0$$

This can be written as

$$\begin{vmatrix} 3(\omega^2 - \omega_0^2) & 2\omega^2 & \omega^2 \\ 2\omega^2 & 2(\omega^2 - \omega_0^2) & \omega^2 \\ \omega^2 & \omega^2 & (\omega^2 - \omega_0^2) \end{vmatrix} = 0 \tag{vi}$$

where, $\omega_0 = \sqrt{\frac{g}{l}}$.

The determinant can be expanded to get

$$3\left(\omega^2 - \omega_0^2\right)\left[2\left(\omega^2 - \omega_0^2\right)^2 - \omega^4\right] - 2\omega^2\left[2\left(\omega^2 - \omega_0^2\right) - \omega^4\right] + \omega^2\left[2\omega^4 - 2\omega^2\left(\omega^2 - \omega_0^2\right)\right] = 0$$

This can be simplified to get

$$\omega^6 - 9\omega_0^2\omega^4 + 18\omega_0^4\omega^2 - 6\omega_0^6 = 0 \tag{vii}$$

This is a cubic equation in ω^2 and can be solved by setting $\omega^2 = \lambda\omega_0^2$. With this equation (vii) becomes

$$\lambda^3 - 9\lambda^2 + 18\lambda - 6 = 0 \tag{viii}$$

Solving this equation is a simple mathematical exercise and one can easily obtain the three roots of this equation as

$$\lambda_1 = 0.42, \lambda_2 = 2.29 \text{ and } \lambda_3 = 6.29 \tag{ix}$$

Then, the eigenfrequencies are

$$\omega_1^2 = 0.42\omega_0^2\ ,\ \omega_2^2 = 2.29\omega_0^2 \text{ and } \omega_3^2 = 6.29\omega_0^2 \tag{x}$$

The eigenvectors corresponding to the three eigenfrequencies can be obtained from (vi) through the matrix equation

$$\begin{bmatrix} 3\left(\omega_j^2 - \omega_0^2\right) & 2\omega_j^2 & \omega_j^2 \\ 2\omega_j^2 & 2\left(\omega_j^2 - \omega_0^2\right) & \omega_j^2 \\ \omega_j^2 & \omega_j^2 & \left(\omega_j^2 - \omega_0^2\right) \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix} = 0$$

Then for $\omega_j^2 = \omega_1^2 = 0.42\omega_0^2$, we get the eigenvectors as

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.29 \\ 1.63 \end{bmatrix}$$

Similarly, for the other two frequencies, we get the eigenvectors as

$$\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ 0.25 \\ -2.4 \end{bmatrix} \text{ and } \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ -1.65 \\ 0.77 \end{bmatrix}$$

Thus, the complete matrix representing the eigenvectors can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1.29 & 0.25 & -1.65 \\ 1.63 & -2.4 & 0.77 \end{bmatrix}$$

EXAMPLE 6.14 Obtain the eigenfrequency and eigenvectors corresponding to the small oscillations of a linear triatomic molecule.

Solution: We consider two atoms of mass m each placed at same distance on either side of a third atom of mass M. Further, we assume that the atoms are connected through springs of force constant k and there is no interaction between atoms of mass m.

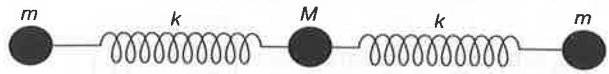

Fig. 6.8

Let x_1, x_2 and x_3 be the displacements of the atoms from their equilibrium position. Then, the kinetic energy of the system is

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_3^2\right) + \frac{1}{2}M\dot{x}_2^2 \tag{i}$$

Here we have taken x_2 as the displacement of the central atom with mass M. Thus, we have; $T_{11} = T_{33} = m$, $T_{22} = M$ and all other elements are zero.

Then, the matrix representing the kinetic energy is

$$T = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \tag{ii}$$

The potential energy of the system is

$$V = \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_2 - x_3)^2$$

$$= \frac{1}{2}k\left(x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3\right) \tag{iii}$$

From this expression, we have the matrix representing the potential energy as

$$V = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \tag{iv}$$

Using the expressions (ii) and (iii) we write the characteristic equation $\left|V - \omega^2 T\right| = 0$ as

$$\begin{vmatrix} k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 M & -k \\ 0 & -k & k - \omega^2 m \end{vmatrix} = 0 \tag{v}$$

The determinant can be evaluated to get

$$\left(k-\omega^2 m\right)\left(-2km\omega^2-kM\omega^2+mM\omega^4\right)=0$$

or

$$\omega^2\left(k-\omega^2 m\right)\left[-k(2m+M)+mM\omega^2\right]=0 \tag{vi}$$

This would yield three roots of ω and are

$$\omega_1=0,\ \omega_2=\sqrt{\frac{k}{m}} \text{ and } \omega_3=\sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)} \tag{vii}$$

Now we discuss the three cases in detail and determine the eigenvectors in each case.

Case 1: $\omega=\omega_1=0$

Let the eigenvectors be (a_{11}, a_{21}, a_{31}). Then from (v)

$$\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}=0$$

This would give three equations and are

$$ka_{11}-ka_{21}=0$$

$$-ka_{11}+2ka_{21}-ka_{31}=0$$

and

$$-ka_{21}+ka_{31}=0\cdot$$

These equations can be solved to get

$$a_{11}=a_{21}=a_{31}=\alpha \text{ (say)} \tag{viii}$$

Therefore, the eigenvector corresponding to $\omega=\omega_1=0$ is

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}=\begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} \tag{ix}$$

This shows that each atom is equally displaced in the same direction.

Case 2: $\omega=\omega_2=\sqrt{\dfrac{k}{m}}$

We assume the eigenvectors as (a_{12}, a_{22}, a_{32}), then from (v), we get

$$\begin{bmatrix} 0 & -k & 0 \\ -k & 2k-k\dfrac{M}{m} & -k \\ 0 & -k & 0 \end{bmatrix}\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}=0$$

Then we get the equations

$$-ka_{22} = 0 \quad \text{and} \quad -ka_{12} + k\left(2 - \frac{M}{m}\right)a_{22} - ka_{32} = 0$$

From these equations, we get

$$a_{22} = 0 \text{ and } a_{12} = -a_{32} = \beta \text{ (say)} \tag{x}$$

Then the eigenvector corresponding to the frequency, $\omega = \omega_2 = \sqrt{\frac{k}{m}}$ is

$$\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} \beta \\ 0 \\ \beta \end{bmatrix} \tag{xi}$$

This means that the central atom remains at rest and the atoms on either side are equally displaced in opposite directions.

Case 3: $\omega = \omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$

Let the eigenvectors be (a_{13}, a_{23}, a_{33}). Now equation (v), with $\omega = \omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$ becomes

$$\begin{bmatrix} -2k\frac{m}{M} & -k & 0 \\ -k & -k\frac{M}{m} & -k \\ 0 & -k & -2k\frac{m}{M} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = 0$$

Then the three sets of equations are

$$-2k\frac{m}{M}a_{13} - ka_{23} = 0$$

$$-ka_{13} - k\frac{M}{m}a_{23} - ka_{33} = 0$$

and

$$-ka_{23} - 2k\frac{m}{M}a_{33} = 0$$

Solving these three equations, we get

$$a_{13} = a_{33} = \gamma \text{ (say) and } a_{23} = -2\frac{m}{M}\gamma \tag{xii}$$

Then the eigenvector corresponding to the frequency, $\omega = \omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$ is

$$\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} \gamma \\ -2\frac{m}{M}\gamma \\ \gamma \end{bmatrix} \tag{xiii}$$

Then the eigenvector in complete form can be represented as

$$A=\begin{bmatrix}\alpha & \beta & \gamma\\ \alpha & 0 & -2\dfrac{m}{M}\gamma\\ \alpha & \beta & \gamma\end{bmatrix} \tag{xiv}$$

Now, as a final step we evaluate α,β and γ for which we make use of the identity $A^TTA=I$. Then

$$\begin{bmatrix}\alpha & \alpha & \alpha\\ \beta & 0 & -\beta\\ \gamma & -2\dfrac{m}{M}\gamma & \gamma\end{bmatrix}\begin{bmatrix}m & 0 & 0\\ 0 & M & 0\\ 0 & 0 & m\end{bmatrix}\begin{bmatrix}\alpha & \beta & \gamma\\ \alpha & 0 & -2\dfrac{m}{M}\gamma\\ \alpha & \beta & \gamma\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

This can be simplified to get

$$\begin{bmatrix}(2m+M)\alpha^2 & 0 & 0\\ 0 & 2m\beta^2 & 0\\ 0 & 0 & \left(2m+\dfrac{4m^2}{M}\right)\gamma^2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Therefore, $(2m+M)\alpha^2=1$ or $\alpha=\dfrac{1}{\sqrt{(2m+M)}}$

Similarly, $2m\beta^2=1$ or, $\beta=\dfrac{1}{\sqrt{2m}}$

and $\left(2m+\dfrac{4m^2}{M}\right)\gamma^2=1$ or $\gamma=\dfrac{1}{\sqrt{2m\left(1+\dfrac{2m}{M}\right)}}$

Thus, equation (xiv) with the values of α,β and γ given above, gives the eigenvectors. The normal coordinates Q_j can be determined as follows:

$$Q_1=\alpha(x_1+x_2+x_3)=\frac{1}{\sqrt{2m+M}}(x_1+x_2+x_3)$$

$$Q_2=\beta(x_1-x_2)=\frac{1}{\sqrt{2m}}(x_1-x_2)$$

and $$Q_3 = \gamma\left(x_1 - \frac{2m}{M}x_2 + x_3\right) = \frac{1}{\sqrt{2m\left(1+\dfrac{2m}{M}\right)}}\left(x_1 - \frac{2m}{M}x_2 + x_3\right)$$

EXAMPLE 6.15 A light string of length $4a$ is held taut with a tension F between two fixed points. Three equal masses m are attached at equidistant points on the string. Determine the eigenfrequencies and eigenvectors of small oscillations.

Solution: Let y_1, y_2 and y_3 be the transverse displacements of the masses attached to the string. Then, the kinetic energy can be written as

$$T = \frac{1}{2}m\left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2\right) \tag{i}$$

The potential energy of each mass is equal to the work done in displacing the masses from their equilibrium positions. Now, the downward component of the force on the first mass is $F\sin\theta$, where $\sin\theta = \dfrac{y_1}{\sqrt{a^2 + y_1^2}} \cong \dfrac{y_1}{a}$.

Then, the potential energy of the first mass is $\dfrac{Fy_1}{a}\cdot\dfrac{1}{2}y_1 = \dfrac{Fy_1^2}{2a}$

Similarly, the potential energy of the third mass is $\dfrac{Fy_3^2}{2a}$

On the second mass two forces are acting in the downward direction and hence the potential energy of the second mass is $\dfrac{F(y_2 - y_1)^2}{2a} + \dfrac{F(y_2 - y_3)^2}{2a}$.

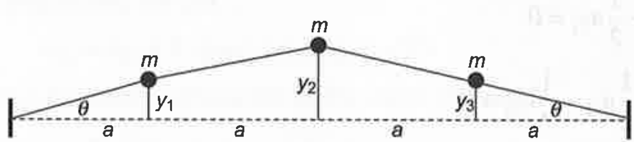

Fig. 6.9

Therefore, the total potential energy of the system is

$$V = \frac{F}{2a}\left[y_1^2 + (y_2 - y_1)^2 + (y_2 - y_3)^2 + y_3^2\right]$$

On simplification, we get

$$V = \frac{F}{a}\left(y_1^2 - y_1y_2 + y_2^2 - y_2y_3 + y_3^2\right) \tag{ii}$$

Then, the matrices representing the kinetic and potential energies are

$$T = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \text{ and } V = \frac{F}{2}\begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix} \tag{iii}$$

Now, eigenfrequencies can be obtained by evaluating the determinant $\left|V - \omega^2 T\right| = 0$.

That is,

$$\begin{bmatrix} 1-\omega^2 m & -1/2 & 0 \\ -1/2 & 1-\omega^2 m & -1/2 \\ 0 & -1/2 & 1-\omega^2 m \end{bmatrix} = 0 \tag{iv}$$

or

$$\left(1-\omega^2 m\right)\left[\left(1-\omega^2 m\right)^2 - \frac{1}{2}\right] = 0 \tag{v}$$

This would give the eigenfrequencies as

$$\omega_1^2 = \frac{1}{m},\ \omega_2^2 = \frac{\sqrt{2}+1}{\sqrt{2}m} \text{ and } \omega_3^2 = \frac{\sqrt{2}-1}{\sqrt{2}m} \tag{vi}$$

Case 1: $\omega^2 = \omega_1^2 = \dfrac{1}{m}$

Let the eigenvectors be (a_{11}, a_{21}, a_{31}). Then from (iv), we get

$$\begin{bmatrix} 0 & -1/2 & 0 \\ -1/2 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = 0$$

This will give $-\dfrac{1}{2}a_{21} = 0$

$$-\frac{1}{2}a_{11} - \frac{1}{2}a_{31} = 0$$

Then,

$$a_{11} = -a_{31} = \alpha \text{ (say) and } a_{21} = 0 \tag{vii}$$

Case 2: $\omega^2 = \omega_2^2 = \dfrac{\sqrt{2}+1}{\sqrt{2}m}$

We assume the eigenvectors as (a_{12}, a_{22}, a_{32}), and from (iv), we get

$$\begin{bmatrix} 1/\sqrt{2} & -1/2 & 0 \\ -1/2 & 1/\sqrt{2} & -1/2 \\ 0 & -1/2 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = 0$$

that is, $$\frac{1}{\sqrt{2}}a_{12} - \frac{1}{2}a_{22} = 0$$

$$-\frac{1}{2}a_{12} + \frac{1}{\sqrt{2}}a_{22} - \frac{1}{2}a_{32} = 0$$

and $$-\frac{1}{2}a_{22} + \frac{1}{\sqrt{2}}a_{32} = 0$$

Solving these three equations, we get

$$a_{12} = a_{32} = \beta \text{ (say) and } a_{22} = -\sqrt{2}\beta \tag{viii}$$

Case 3: $\omega^2 = \omega_2^2 = \dfrac{\sqrt{2}-1}{\sqrt{2}m}$

Let the eigenvectors be (a_{13}, a_{23}, a_{33}). From (iv), we get

$$\begin{bmatrix} -1/\sqrt{2} & -1/2 & 0 \\ -1/2 & -1/\sqrt{2} & -1/2 \\ 0 & -1/2 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = 0$$

that is, $$-\frac{1}{\sqrt{2}}a_{12} - \frac{1}{2}a_{22} = 0$$

$$-\frac{1}{2}a_{12} - \frac{1}{\sqrt{2}}a_{22} - \frac{1}{2}a_{32} = 0$$

and $$-\frac{1}{2}a_{22} - \frac{1}{\sqrt{2}}a_{32} = 0$$

Solving these equations, we get

$$a_{13} = a_{33} = \gamma \text{ (say) and } a_{23} = \sqrt{2}\gamma \tag{ix}$$

Thus, the complete matrix representing the eigenvectors is

$$A = \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & -\sqrt{2}\beta & \sqrt{2}\gamma \\ -\alpha & \beta & \gamma \end{bmatrix} \tag{x}$$

Further, the values of α, β and γ can be evaluated using the identity $A^T T A = I$

EXAMPLE 6.16 The Lagrangian of a system is given by $L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}m\omega_0^2\left(x^2 + y^2\right) - \frac{1}{2}m\alpha^2(x-y)^2$. Determine the normal frequencies and eigenvectors of oscillation.

Solution: Given that, $L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-\frac{1}{2}m\omega_0^2\left(x^2+y^2\right)-\frac{1}{2}m\alpha^2\left(x-y\right)^2$ (i)

Then, the matrix representing the kinetic energy is

$$T=\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \tag{ii}$$

Similarly, the matrix representing the potential energy is

$$V=m\begin{bmatrix} \alpha^2+\omega_0^2 & -\alpha^2 \\ -\alpha^2 & \alpha^2+\omega_0^2 \end{bmatrix} \tag{iii}$$

Now, the eigenfrequencies can be determined by evaluating the determinant equation $\left|V-\omega^2 T\right|=0$. Then,

$$\begin{vmatrix} \alpha^2+\omega_0^2-\omega^2 & -\alpha^2 \\ -\alpha^2 & \alpha^2+\omega_0^2-\omega^2 \end{vmatrix}=0 \tag{iv}$$

that is, $\left(\alpha^2+\omega_0^2-\omega^2\right)^2-\alpha^4=0$

or $\left(\alpha^2+\omega_0^2\right)^2-2\left(\alpha^2+\omega_0^2\right)\omega^2+\omega^4-\alpha^4=0$

or $\alpha^4+2\alpha^2\omega_0^2+\omega_0^4-2\alpha^2\omega^2-2\omega_0^2\omega^2+\omega^4-\alpha^4=0$

or $2\alpha^2\left(\omega_0^2-\omega^2\right)+\left(\omega_0^2-\omega^2\right)^2=0$

or $\left(\omega_0^2-\omega^2\right)\left[\left(2\alpha^2+\omega_0^2\right)-\omega^2\right]=0$ (v)

Thus, the eigenfrequencies are

$$\omega_1=\omega_0 \text{ and } \omega_2=\sqrt{2\alpha^2+\omega_0^2} \tag{vi}$$

Case 1: When $\omega=\omega_1=\omega_0$

Let the eigenvectors be (a_{11},a_{21}) and from (iv), we get

$$\begin{bmatrix} \alpha^2 & -\alpha^2 \\ -\alpha^2 & \alpha^2 \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}=0$$

or $\alpha^2 a_{11}-\alpha^2 a_{21}=0$ or $a_{11}=a_{21}=\lambda$ (say) (vii)

Case 2: When $\omega=\omega_2=\sqrt{2\alpha^2+\omega_0^2}$

Let the eigenvectors be (a_{12}, a_{22}). From (iv), we have

$$\begin{bmatrix} -\alpha^2 & -\alpha^2 \\ -\alpha^2 & -\alpha^2 \end{bmatrix}\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = 0$$

or $\quad -\alpha^2 a_{12} - \alpha^2 a_{22} = 0$ or, $a_{12} = -a_{22} = \mu$ (say) $\quad$ (viii)

Then the matrix representing the eigenvectors is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ \lambda & -\mu \end{bmatrix} \quad \text{(ix)}$$

The values of λ and μ can be obtained by evaluating $A^T T A = I$. That is

$$\begin{bmatrix} \lambda & \lambda \\ \mu & -\mu \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \lambda & \mu \\ \lambda & -\mu \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On simplifying, we get $\quad 2m\begin{bmatrix} \lambda^2 & 0 \\ 0 & \mu^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then we get $\quad \lambda = \mu = \dfrac{1}{\sqrt{2m}} \quad$ (x)

Therefore, $\quad A = \begin{bmatrix} \dfrac{1}{\sqrt{2m}} & 0 \\ 0 & -\dfrac{1}{\sqrt{2m}} \end{bmatrix} \quad$ (xi)

EXAMPLE 6.17 Two spheres and three springs are connected together as shown in Figure 6.10. Assuming the unstretched position of the springs as the equilibrium position, obtain the eigenfrequencies and eigenvectors of small oscillations of the system.

Solution: We first write the expression for the kinetic energy of the system. We assume that x_1 and x_2 are the displacements of the spheres of masses $2m$ and m respectively from their equilibrium position. Then the kinetic energy of the system is given by;

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \quad \text{(i)}$$

Therefore, the matrix representing the kinetic energy is

$$T = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \quad \text{(ii)}$$

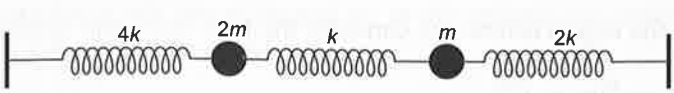

Fig. 6.10

The potential energy of the system can be written as

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_2x_2^2$$

$$= \frac{1}{2}4kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}2kx_2^2$$

$$= \frac{1}{2}4kx_1^2 + \frac{1}{2}k\left(x_2^2 + x_1^2 - 2x_1x_2\right) + \frac{1}{2}2kx_2^2$$

$$= \frac{1}{2}\left(5kx_1^2 - 2kx_1x_2 + 3kx_2^2\right) \tag{iii}$$

Then the matrix representing the potential energy is

$$V = \begin{bmatrix} 5k & -k \\ -k & 3k \end{bmatrix} \tag{iv}$$

Now, the characteristic equation can be written as

$$\left|V - \omega^2 T\right| = 0$$

or
$$\begin{vmatrix} 5k - 2m\omega^2 & -k \\ -k & 3k - m\omega^2 \end{vmatrix} = 0 \tag{v}$$

or
$$\left(5k - 2m\omega^2\right)\left(3k - m\omega^2\right) - k^2 = 0$$

or
$$2m^2\omega^4 - 11km\omega^2 + 14k^2 = 0$$

or
$$2\omega^4 - 11\frac{k}{m}\omega^2 + 14\frac{k^2}{m^2} = 0$$

or
$$2\omega^4 - 11\omega_0^2\omega^2 + 14\omega_0^4 = 0, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}$$

The above expression can be written as

$$2\omega^2\left(\omega^2 - 2\omega_0^2\right) - 7\omega_0^2\left(\omega^2 - 2\omega_0^2\right) = 0$$

or
$$\left(\omega^2 - 2\omega_0^2\right)\left(2\omega^2 - 7\omega_0^2\right) = 0 \tag{vi}$$

Then the roots of this equation gives the eigenfrequencies, and are

$$\omega_1 = \sqrt{2}\omega_0 = \sqrt{\frac{2k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{7}{2}}\omega_0 = \sqrt{\frac{7k}{2m}} \tag{vii}$$

To determine the eigenvectors, we consider the two cases separately.

Case 1: $\omega = \omega_1 = \sqrt{2}\omega_0 = \sqrt{\frac{2k}{m}}$

We assume the eigenvectors as (a_{11}, a_{21}) and from equation (v), we get

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = 0$$

or $$ka_{11} - ka_{21} = 0 \text{ or } a_{11} = a_{21} = \alpha \text{ (say)} \tag{viii}$$

Case 2: $\omega = \omega_2 = \sqrt{\frac{7}{2}}\omega_0 = \sqrt{\frac{7k}{2m}}$

Let the eigenvectors be (a_{12}, a_{22}), then from (v) with the given frequency, we get

$$\begin{bmatrix} -2k & -k \\ -k & -k/2 \end{bmatrix}\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = 0$$

or $$-2ka_{12} - ka_{22} = 0 \text{ or, } a_{12} = -\frac{1}{2}a_{22} = \beta \text{ (say)} \tag{ix}$$

From (viii) and (ix), we can write the matrix representing the eigenvectors as

$$A = \begin{bmatrix} \alpha & \beta \\ \alpha & -2\beta \end{bmatrix} \tag{x}$$

Now, α and β can be evaluated using the identity, $A^T TA = I$

that is, $$\begin{bmatrix} \alpha & \alpha \\ \beta & -2\beta \end{bmatrix}\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \alpha & -2\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This can be simplified to get;

$$\begin{bmatrix} 3m\alpha^2 & 0 \\ 0 & 6m\beta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, $$\alpha = \frac{1}{\sqrt{3m}} \text{ and } \beta = \frac{1}{\sqrt{6m}} \tag{xi}$$

Equation (x) with the values of α and β given by (xi) is the eigenvector.

EXAMPLE 6.18 Three masses m_1, m_2 and m_3 on a frictionless hoop of radius R are shown in Figure 6.11. All the three springs have the same force constant k. Determine the eigenfrequencies and eigenvectors of small oscillations.

Solution: In this problem, we use ϕ_1, ϕ_2 and ϕ_3 as the generalized coordinates with the condition that; $\phi_1 \le \phi_2 \le \phi_3 \le (2\pi + \phi_1)$.

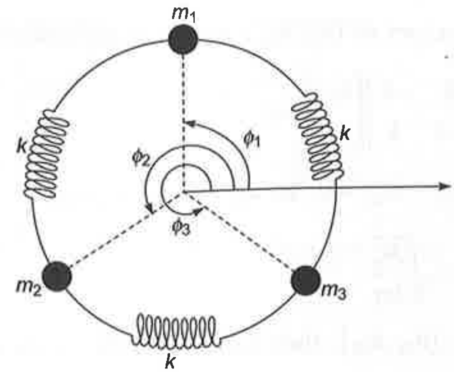

Fig. 6.11

The kinetic energy of the system can be written as

$$T = \frac{1}{2}R^2\left(m_1\dot{\phi}_1^2 + m_2\dot{\phi}_2^2 + m_3\dot{\phi}_3^2\right) \tag{i}$$

Then the matrix representing the kinetic energy is

$$T = \begin{bmatrix} m_1R^2 & 0 & 0 \\ 0 & m_2R^2 & 0 \\ 0 & 0 & m_3R^2 \end{bmatrix} \tag{ii}$$

Now, let $R\theta$ be the equilibrium length of each segment. Then the potential energy of the system can be written as

$$V = \frac{1}{2}kR^2\left[(\phi_2 - \phi_1 - \theta)^2 + (\phi_3 - \phi_2 - \theta)^2 + (2\pi + \phi_1 - \phi_3 - \theta)^2\right]$$

$$= \frac{1}{2}kR^2\begin{bmatrix}(\phi_2 - \phi_1)^2 - 2(\phi_2 - \phi_1) + \theta^2 + (\phi_3 - \phi_2)^2 - 2(\phi_3 - \phi_2) + \theta^2 \\ +(2\pi + \phi_1 - \phi_3)^2 - 2(2\pi + \phi_1 - \phi_3)\theta + \theta^2\end{bmatrix}$$

$$= \frac{1}{2}kR^2\left[(\phi_2 - \phi_1)^2 + (\phi_3 - \phi_2)^2 + (2\pi + \phi_1 - \phi_3)^2 + 3\theta^2 - 4\pi\theta\right]$$

This can be further simplified to get

$$V = \frac{1}{2}kR^2\left[\left(2\dot{\phi}_1^2 + 2\dot{\phi}_2^2 + 2\dot{\phi}_3^2 - 2\phi_1\phi_2 - 2\phi_2\phi_3 - 2\phi_1\phi_3\right) + 4\pi^2 + 4\pi(\phi_3 - \phi_1)\right] \tag{iii}$$

The last two terms in this expression are simply an additive constant in view of the condition given and can be neglected.

Then the matrix representing the potential energy is

$$V = \begin{bmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{bmatrix} \tag{iv}$$

Then the characteristic equation $\left|V - \omega^2 T\right| = 0$ in the present case can be written as

$$\begin{vmatrix} 2kR^2 - m_1 R^2 \omega & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 - m_2 R^2 \omega & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 - m_3 R^2 \omega \end{vmatrix} = 0$$

or

$$kR^2 \begin{vmatrix} \dfrac{\omega^2}{\Omega_1^2} - 2 & 1 & 1 \\ 1 & \dfrac{\omega^2}{\Omega_2^2} - 2 & 1 \\ 1 & 1 & \dfrac{\omega^2}{\Omega_3^2} - 2 \end{vmatrix} = 0$$

where $\Omega_i = \dfrac{k}{m_i}$. The determinant can be expanded to get

$$\frac{\omega^6}{\Omega_1^2 \Omega_2^2 \Omega_3^2} - 2\left(\frac{1}{\Omega_1^2 \Omega_2^2} + \frac{1}{\Omega_2^2 \Omega_3^2} + \frac{1}{\Omega_3^2 \Omega_3^2}\right)\omega^4 + 3\left(\frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2} + \frac{1}{\Omega_3^2}\right)\omega^2 = 0 \tag{v}$$

From this expression it is clear that one of the roots is $\omega = \omega_1 = 0$. This equation is quadratic in ω^2 when we divide throughout by ω^2. That is,

$$\frac{\omega^4}{\Omega_1^2 \Omega_2^2 \Omega_3^2} - 2\left(\frac{1}{\Omega_1^2 \Omega_2^2} + \frac{1}{\Omega_2^2 \Omega_3^2} + \frac{1}{\Omega_3^2 \Omega_3^2}\right)\omega^2 + 3\left(\frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2} + \frac{1}{\Omega_3^2}\right) = 0 \tag{vi}$$

Then, the two other roots can be obtained as

$$\omega_2^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \sqrt{\frac{1}{2}\left[\left(\Omega_1^2 - \Omega_2^2\right)^2 + \left(\Omega_2^2 - \Omega_3^2\right)^2 + \left(\Omega_3^2 - \Omega_1^2\right)^2\right]} \tag{vii}$$

and

$$\omega_3^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 - \sqrt{\frac{1}{2}\left[\left(\Omega_1^2 - \Omega_2^2\right)^2 + \left(\Omega_2^2 - \Omega_3^2\right)^2 + \left(\Omega_3^2 - \Omega_1^2\right)^2\right]} \tag{viii}$$

Now, we try to find a general expression for the eigen vectors by setting

$$\begin{bmatrix} \dfrac{\omega_j^2}{\Omega_1^2}-2 & 1 & 1 \\ 1 & \dfrac{\omega_j^2}{\Omega_2^2}-2 & 1 \\ 1 & 1 & \dfrac{\omega_j^2}{\Omega_3^2}-2 \end{bmatrix}\begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix}=0$$

Then we get three characteristic equations as

$$\left(\frac{\omega_j^2}{\Omega_1^2}-2\right)a_{1j}+a_{2j}+a_{3j}=0$$

$$a_{1j}+\left(\frac{\omega_j^2}{\Omega_2^2}-2\right)a_{2j}+a_{3j}=0$$

and

$$a_{1j}+a_{2j}+\left(\frac{\omega_j^2}{\Omega_3^2}-2\right)a_{3j}=0$$

These equations can be solved to get

$$\left(\frac{\omega_j^2}{\Omega_1^2}-3\right)a_{1j}=\left(\frac{\omega_j^2}{\Omega_2^2}-3\right)a_{2j}=\left(\frac{\omega_j^2}{\Omega_3^2}-3\right)a_{3j}=\alpha \text{ (say)} \tag{ix}$$

Then the eigenvector can be represented as

$$a_j=\alpha\begin{bmatrix} \left(\dfrac{\omega_j^2}{\Omega_1^2}-3\right)^{-1} \\ \left(\dfrac{\omega_j^2}{\Omega_2^2}-3\right)^{-1} \\ \left(\dfrac{\omega_j^2}{\Omega_3^2}-3\right)^{-1} \end{bmatrix} \tag{x}$$

Finally, the value of α can be obtained from the relation $A^T TA = I$.

EXAMPLE 6.19 Four equal masses m are connected using four springs of same force constants k and are constrained to move on a circle of radius R as shown in Figure 6.12. Determine the eigenfrequencies of small oscillations.

Solution: We first select the generalized coordinates for our problem. As the masses are moving along the circle it is the length of the arc between the masses that are changing. So the same can be taken as the generalized coordinates in the present problem.

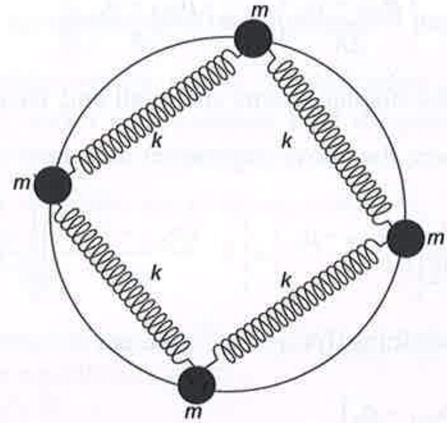

Fig. 6.12

Let, ρ_1, ρ_2, ρ_3 and ρ_4 be the changes in length of the arcs of the four masses due to the displacement of masses from their equilibrium position. Therefore, the kinetic energy of the system can be written as

$$T = \frac{1}{2} m\left(\dot{\rho}_1^2 + \dot{\rho}_2^2 + \dot{\rho}_3^2 + \dot{\rho}_4^2\right) \tag{i}$$

Then, the matrix representing the kinetic energy is

$$T = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \tag{ii}$$

Since the springs and masses are identical, we assume that at the equilibrium, the lengths of the arcs between the masses are the same. In other words, the arc between two neighbouring masses subtend an angle $\frac{\pi}{2}$ at the centre of the circle. Therefore, the equilibrium length of the arc can be written as

$$\rho = 2\left(R \sin \frac{\pi}{4}\right)$$

Then, the change in length of the arc between adjacent masses, say, n^{th} and $(n+1)^{th}$ is given by

$$\rho_n' = 2R \sin\left[\frac{1}{2}\left(\frac{\rho_{n+1} - \rho_n}{R} + \frac{\pi}{2}\right)\right] - 2\left(R \sin \frac{\pi}{4}\right)$$

$$= 2R\left[\sin\left(\frac{\rho_{n+1}-\rho_n}{2R}\right)\cos\frac{\pi}{4}+\cos\left(\frac{\rho_{n+1}-\rho_n}{2R}\right)\sin\frac{\pi}{4}\right]-2\left(R\sin\frac{\pi}{4}\right)$$

$$= 2R\frac{1}{\sqrt{2}}\left[\sin\left(\frac{\rho_{n+1}-\rho_n}{2R}\right)+\cos\left(\frac{\rho_{n+1}-\rho_n}{2R}\right)\right]-2R\frac{1}{\sqrt{2}}$$

For small oscillations, the displacements are small and we can approximate $\sin\theta$ to θ and, $\cos\theta$ to $1-\frac{\theta^2}{2}$. Then, the above expression becomes

$$\rho_n' = 2R\frac{1}{\sqrt{2}}\left[\left(\frac{\rho_{n+1}-\rho_n}{2R}\right)+\left(1-\frac{(\rho_{n+1}-\rho_n)^2}{8R^2}\right)\right]-2R\frac{1}{\sqrt{2}}$$

Neglecting the term containing $(\rho_{n+1}-\rho_n)^2$, we get

$$\rho_n' = \frac{1}{\sqrt{2}}(\rho_{n+1}-\rho_n) \tag{iii}$$

Now, the potential energy of the system is

$$V=\frac{1}{2}k\left\{\frac{1}{2}\left[(\rho_2-\rho_1)^2+(\rho_3-\rho_2)^2+(\rho_4-\rho_3)^2+(\rho_1-\rho_4)^2\right]\right\}$$

$$=\frac{1}{2}k\left(\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2-\rho_1\rho_2-\rho_2\rho_3-\rho_3\rho_4-\rho_4\rho_1\right) \tag{iv}$$

The matrix representing the potential energy is

$$V=\begin{bmatrix} k & -k/2 & 0 & -k/2 \\ -k/2 & k & -k/2 & 0 \\ 0 & -k/2 & k & -k/2 \\ -k/2 & 0 & -k/2 & k \end{bmatrix} \tag{v}$$

The eigenfrequencies can be obtained by solving the determinant equation, $|V-\omega^2 T|=0$.

$$\begin{vmatrix} k-m\omega^2 & -k/2 & 0 & -k/2 \\ -k/2 & k-m\omega^2 & -k/2 & 0 \\ 0 & -k/2 & k-m\omega^2 & -k/2 \\ -k/2 & 0 & -k/2 & k-m\omega^2 \end{vmatrix}=0 \tag{vi}$$

This determinant can be solved to get

$$\omega^4\left(\omega^2-\frac{k}{m}\right)\left(\omega^2-\frac{2k}{m}\right)=0 \tag{vii}$$

Obviously, the four roots of the equation are

$$\omega_1 = 0\,,\ \omega_2 = 0\,,\ \omega_3 = \sqrt{\frac{k}{m}}\ \text{and}\ \omega_4 = \sqrt{\frac{2k}{m}} \tag{viii}$$

which are the eigenfrequencies of small oscillations.

EXAMPLE 6.20 A particle moves under the influence of a potential given by $V(x) = -Cx^n e^{-ax}$, where C and a are constants. Find the point of equilibrium and the frequency of small oscillations.

Solution: In this problem, we make use of the equation

$$\omega = \sqrt{\frac{1}{m}V''(x_0)} \tag{i}$$

To determine the frequency of oscillation. Here $V''(x_0)$ is the second derivative of the potential evaluated at the equilibrium point.

Given that; $V(x) = -Cx^n e^{-ax}$ (ii)

The point of equilibrium is given by the condition $V'(x_0) = 0$. Differentiating equation (ii) with respect to x, we get

$$V'(x) = \frac{d}{dx}\left(-Cx^n e^{-ax}\right) = -Cnx^{n-1}e^{-ax} + Cax^n e^{-ax}$$

$$= Cx^{n-1}e^{-ax}\left(-n + ax\right) \tag{iii}$$

At the equilibrium, $Cx_0^{n-1}e^{-ax_0}\left(-n + ax_0\right) = 0$

or $$x_0 = \frac{n}{a} \tag{iv}$$

Now, the second derivative of the potential is

$$V''(x) = \frac{d}{dx}\left[Ce^{-ax}\left(-x^{n-1}n + ax^n\right)\right]$$

$$= \left[anx^{n-1} - a^2x^n\right]Ce^{-ax} + \left[-n(n-1)x^{n-2} + anx^{n-1}\right]Ce^{-ax}$$

$$= \left[-n(n-1)x^{n-2} + 2anx^{n-1} - a^2x^n\right]Ce^{-ax} \tag{v}$$

Now, at the equilibrium position, we have

$$V''(x_0) = \left[-n(n-1)x_0^{n-2} + 2anx_0^{n-1} - a^2x_0^n\right]Ce^{-ax_0}$$

$$= \left[-n(n-1)\left(\frac{n}{a}\right)^{n-2} + 2an\left(\frac{n}{a}\right)^{n-1} - a^2\left(\frac{n}{a}\right)^n\right]Ce^{-a\left(\frac{n}{a}\right)}$$

This can be simplified to get

$$V''(x_0) = Ce^{-n}a^{2-n}n^{n-1} \quad \text{(vi)}$$

Then, the frequency of oscillation is

$$\omega = \sqrt{\frac{1}{m}V''(x_0)} = \sqrt{\frac{1}{m}Ce^{-n}a^{2-n}n^{n-1}} \quad \text{(vii)}$$

This is the required result.

Note: In this problem we made a simple approximation. The Lagrange's equation of motion, in general has the form

$$m\ddot{x} = -V'(x)$$

Expanding the potential about the equilibrium point, up to the second order, we get

$$V(x) \approx V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2$$

Then, $$V'(x) \approx V'(x_0) + V''(x_0)(x - x_0) + \frac{1}{2}V'''(x_0)(x - x_0)^2$$

But $V'(x_0) = 0$ and neglect the third order derivative and the other constant terms, and we get

$$V'(x) \approx V''(x_0)x$$

Then the equation of motion becomes

$$\ddot{x} + \frac{V''(x_0)}{m}x = 0 \text{ and the frequency is } \omega = \sqrt{\frac{V''(x_0)}{m}}$$

EXAMPLE 6.21 A particle in an isotropic three-dimensional harmonic oscillator potential has a natural frequency ω_0. Find its vibration frequencies if it is charged and is simultaneously acted on by uniform electric and magnetic fields.

Solution: We assume that the electric field E is along the x–direction and the magnetic field B is along the z–direction. Then we may write

$$E\hat{i} = -\nabla\Phi \text{ and } B\hat{k} = \nabla \times A \quad \text{(i)}$$

The scalar and vector potentials are

$$\Phi = eEx \text{ and } A = \frac{1}{2}\left(-By\hat{i} + Bx\hat{j}\right) \quad \text{(ii)}$$

Since the particle is placed in a harmonic oscillator potential and is acted upon by the uniform electromagnetic field. The potential energy is

$$V = \frac{1}{2}m\omega_0^2 r^2 + e\Phi - e\dot{r}.A$$

$$= \frac{1}{2}m\omega_0^2\left(x^2 + y^2 + z^2\right) - e\Phi + e\dot{r}.A \quad \text{(iii)}$$

The kinetic energy of the system is

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{iv}$$

Now we write the Lagrange's equation of motion for the three coordinates. They are

$$\ddot{x} + \omega_0^2 x - \frac{eB}{m}\dot{y} - \frac{eE}{m} = 0$$

$$\ddot{y} + \omega_0^2 y + \frac{eB}{m}\dot{x} = 0$$

and $$\ddot{z} + \omega_0^2 z = 0$$

The equation for the coordinate z shows that the vibration in the z–direction is at the natural frequency. In the first two equations we make a substitution, $x = x' + \frac{eE}{m\omega_0^2}$. Then we get

$$\ddot{x}' + \omega_0^2 x' - \frac{eB}{m}\dot{y} = 0 \tag{v}$$

and $$\ddot{y} + \omega_0^2 y + \frac{eB}{m}\dot{x}' = 0 \tag{vi}$$

Now, we try a solution of the form

$$x' = A'e^{i\omega t} \text{ and } y = B'e^{i\omega t} \tag{vii}$$

so that $\dot{x}' = i\omega A'e^{i\omega t}$ and $\ddot{x}' = -\omega^2 A'e^{i\omega t}$

Also, $\dot{y} = i\omega B'e^{i\omega t}$ and $\ddot{y} = -\omega^2 B'e^{i\omega t}$

Then, from (v) and (vi), we get

$$-\omega^2 A'e^{i\omega t} + \omega_0^2 A'e^{i\omega t} - \frac{eB}{m} i\omega B'e^{i\omega t} = 0$$

or $$\left(\omega_0^2 - \omega^2\right)A' - i\omega\frac{eB}{m}B' = 0 \tag{viii}$$

Similarly, $$-\omega^2 B'e^{i\omega t} + \omega_0^2 B'e^{i\omega t} + \frac{eB}{m} i\omega A'e^{i\omega t} = 0$$

$$\frac{eB}{m} i\omega A' + \left(\omega_0^2 - \omega^2\right)B' = 0 \tag{ix}$$

Now, we get the matrix equation

$$\begin{bmatrix} \omega_0^2 - \omega^2 & -i\frac{eB}{m}\omega \\ i\frac{eB}{m}\omega & \omega_0^2 - \omega^2 \end{bmatrix}\begin{bmatrix} A' \\ B' \end{bmatrix} = 0$$

To get the eigenfrequencies, we solve the secular equation

$$\begin{vmatrix} \omega_0^2-\omega^2 & -i\dfrac{eB}{m}\omega \\ i\dfrac{eB}{m}\omega & \omega_0^2-\omega^2 \end{vmatrix}=0$$

or $$\left(\omega_0^2-\omega^2\right)^2-\left(\frac{eB}{m}\omega\right)^2=0 \text{ or, } \left(\omega_0^2-\omega^2\right)=\pm\frac{eB}{m}\omega$$

or $$\omega^2\pm\frac{eB}{m}\omega-\omega_0^2=0 \qquad \text{(x)}$$

Then the two roots of this equation give the eigenfrequencies. They are

$$\omega_1=\frac{1}{2}\left(\frac{eB}{m}+\sqrt{\left(\frac{eB}{m}\right)^2+4\omega_0^2}\right) \text{ and;}$$

$$\omega_2=\frac{1}{2}\left(-\frac{eB}{m}+\sqrt{\left(\frac{eB}{m}\right)^2+4\omega_0^2}\right)$$

Then the three eigenfrequencies are ω_0, ω_1 and ω_2.

EXAMPLE 6.22 A block of mass m is attached to a wedge of mass M by a spring with spring constant k. The angle of the wedge is θ; with the horizontal. The wedge is free to slide on a horizontal surface. If the length of the spring is l_0 in its unstretched form find l_0', its length, when the block is at rest. Obtain the equations of motion and the eigenfrequencies of oscillation.

Solution: A schematic representation of the problem is shown in Figure 6.13.

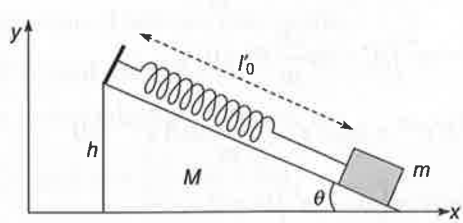

Fig. 6.13

When the block is in equilibrium, the net force along the direction of the inclined plane is zero. Therefore,

$$mg\sin\theta-k\left(l_0'-l_0\right)=0$$

or $$l_0'=\frac{mg\sin\theta}{k}+l_0 \qquad \text{(i)}$$

Now, let the left edge of the wedge be at x and the height of the wedge be h so that the coordinates of the block are

$$\left[(x+l\cos\theta),(h-l\sin\theta)\right]$$

where l is the instantaneous length of the spring.

The kinetic energy is the sum of the kinetic energies of the wedge and the block

$$T=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left[\left(\dot{x}+\dot{l}\cos\theta\right)^2+\left(\dot{l}\sin\theta\right)^2\right]$$

$$=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left[\dot{x}^2+2\dot{x}\dot{l}\cos\theta+\dot{l}^2\cos^2\theta+\dot{l}^2\sin^2\theta\right]$$

$$=\frac{1}{2}(M+m)\dot{x}^2+\frac{1}{2}m\dot{l}^2+m\dot{x}\dot{l}\cos\theta \tag{ii}$$

The potential energy is

$$V=\frac{1}{2}k(l-l_0)^2+mg(h-l\sin\theta) \tag{iii}$$

Then, the Lagrangian of the system is

$$L=T-V=\frac{1}{2}(M+m)\dot{x}^2+\frac{1}{2}m\dot{l}^2+m\dot{x}\dot{l}\cos\theta-\frac{1}{2}k(l-l_0)^2-mg(h-l\sin\theta) \tag{iv}$$

The Lagrange's equation of motion for the coordinate x is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$, which in the present case becomes

$$(M+m)\ddot{x}+m\ddot{l}\cos\theta=0 \tag{v}$$

Similarly, for the coordinate l, the Lagrange's equation of motion becomes

$$m\ddot{x}\cos\theta+m\ddot{l}-kl-(kl_0+mg\sin\theta)=0 \tag{vi}$$

Now, we put, $l=l'+\dfrac{kl_0+mg\sin\theta}{k}$ in equations (v) and (vi) to get

$$(M+m)\ddot{x}+m\ddot{l}'\cos\theta=0 \tag{vii}$$

and $$m\ddot{x}\cos\theta+m\ddot{l}'+kl'=0 \tag{viii}$$

A trial solution of these equations can be taken as

$$x=Ae^{i\omega t}\text{ and }l'=Be^{i\omega t} \tag{ix}$$

Then, equations (vii) and (viii) become

$$\omega^2(M+m)A+\left(\omega^2 m\cos\theta\right)B=0 \tag{x}$$

and $$\left(m\omega^2\cos\theta\right)A+\left(m\omega^2-k\right)B=0 \tag{xi}$$

From (x) and (xi), we form the secular equation

$$\begin{vmatrix} (M+m)\omega^2 & \omega^2\cos\theta \\ \omega^2\cos\theta & \left(m\omega^2-k\right) \end{vmatrix} = 0$$

or $$(M+m)\left(m\omega^2-k\right)\omega^2-\omega^4\cos^2\theta=0$$

or $$\left(Mm+m^2-\cos^2\theta\right)\omega^4-k(M+m)\omega^2=0 \tag{xii}$$

Evidently, $\omega=0$ is a root of the above equation. Also,

$$\left(Mm+m^2-\cos^2\theta\right)\omega^2-k(M+m)=0$$

or $$\omega^2=\frac{k(M+m)}{\left(Mm+m^2-\cos^2\theta\right)}=\frac{k(M+m)}{m\left(M+\sin^2\theta\right)}$$

This will give the other two roots of the equation (xii) as

$$\omega=\pm\sqrt{\frac{k(M+m)}{m\left(M+\sin^2\theta\right)}} \tag{xiii}$$

EXAMPLE 6.23 A ring of mass M and radius R is supported by a pivot at one point on the ring about which the ring is free to rotate in a vertical plane. A bead of mass m slides without friction about the ring. Obtain the equation of motion and the eigenfrequencies of small oscillations.

Solution: Referring to Figure 6.14, the coordinates of the centre of mass of the ring and the mass m are $(R\sin\theta, R\cos\theta)$ and $(R\sin\theta+R\sin\phi, R\cos\theta+R\cos\phi)$ respectively. Then, the kinetic energy can be written as

$$T=\frac{1}{2}\left\{\begin{array}{l} MR^2\left(\dot\theta^2\cos^2\theta+\dot\theta^2\sin^2\theta\right) \\ +mR^2\left[\left(\dot\theta\sin\theta+\dot\phi\sin\phi\right)^2+\left(\dot\theta\cos\theta+\dot\phi\cos\phi\right)^2\right] \end{array}\right\}$$

This can be simplified to get

$$T=MR^2\dot\theta^2+\frac{1}{2}mR^2\left[\dot\theta^2+\dot\phi^2+2\dot\theta\dot\phi\cos(\theta-\phi)\right] \tag{i}$$

The potential energy is

$$\begin{aligned} V&=-MgR\cos\theta-mg(R\cos\theta+R\cos\phi) \\ &=-(M+m)gR\cos\theta-mgR\cos\phi \end{aligned} \tag{ii}$$

Then, the Lagrangian of the system is

$$L = T - V = \begin{Bmatrix} MR^2\dot{\theta}^2 + \frac{1}{2}mR^2\left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right] \\ +(M+m)gR\cos\theta + mgR\cos\phi \end{Bmatrix} \quad \text{(iii)}$$

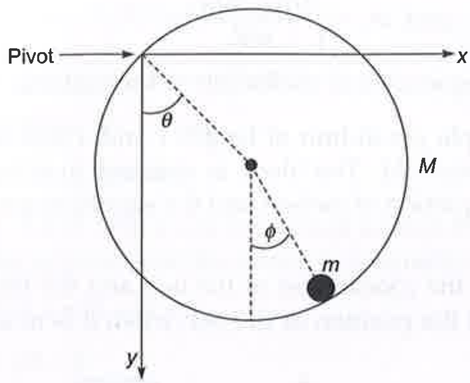

Fig. 6.14

The Lagrange's equation of motion for the coordinates θ and ϕ will be

$$(2M+m)R\ddot{\theta} + mR\ddot{\phi}\cos(\theta-\phi) + mR\dot{\phi}^2\sin(\theta-\phi) + (M+m)\sin\theta = 0 \quad \text{(iv)}$$

and $$R\ddot{\phi} + R\ddot{\theta}\cos(\theta-\phi) - R\dot{\theta}^2\sin(\theta-\phi) + g\sin\phi = 0 \quad \text{(v)}$$

For small oscillations, $\theta, \phi, \dot{\theta}$ and $\dot{\phi}$ are very small so that the higher order terms can be neglected and we can approximate the above two equations as

$$(2M+m)R\ddot{\theta} + mR\ddot{\phi} + (M+m)\theta = 0 \quad \text{(vi)}$$

$$R\ddot{\phi} + R\ddot{\theta} + g\phi = 0 \quad \text{(vii)}$$

Now, we try a solution of the form, $\theta = Ae^{i\omega t}$ and $\phi = Be^{i\omega t}$. Then the equations (vi) and (vii) become

$$\left[(M+m) - R(2M+m)\omega^2\right]A - \omega^2 mRB = 0$$

and $$-\omega^2 RA + \left(g - \omega^2 R\right)B = 0$$

or $$\begin{bmatrix} (M+m) - R(2M+m)\omega^2 & -mR\omega^2 \\ -R\omega^2 & g - R\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = 0 \quad \text{(viii)}$$

We write the secular equations as

$$\begin{vmatrix} (M+m) - R(2M+m)\omega^2 & -mR\omega^2 \\ -R\omega^2 & g - R\omega^2 \end{vmatrix} = 0 \quad \text{(ix)}$$

or $$\left[(M+m)-R(2M+m)\omega^2\right]\left(g-R\omega^2\right)-mR^2\omega^4=0$$

or $$\left(2R\omega^2-g\right)\left[MR\omega^2-(M+m)g\right]=0 \tag{x}$$

The roots of this equation are

$$\omega_1=\sqrt{\frac{g}{2R}} \text{ and } \omega_2=\sqrt{\frac{(M+m)g}{MR}} \tag{xi}$$

These are the eigenfrequencies of oscillation of the system.

EXAMPLE 6.24 A simple pendulum of length l and a bob of mass m is attached to a rectangular block of mass M. The block is attached to a horizontal spring of force constant k. Obtain the equation of motion and the eigenfrequencies of small oscillations of the system.

Solution: First we write the coordinates of the bob and the block. We set the origin of the coordinate system at the position of the bob when it is in equilibrium.

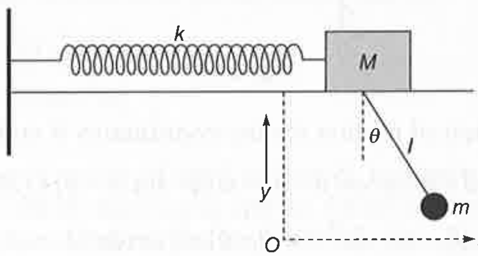

Fig. 6.15

Then the coordinates of the block are (x, l) and that of the bob are $(x+l\sin\theta, l-l\sin\theta)$. Then, the kinetic energy of the system is

$$T=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left[\left(\dot{x}+l\dot{\theta}\cos\theta\right)^2+l^2\dot{\theta}^2\sin^2\theta\right]$$

$$=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left(\dot{x}^2+l^2\dot{\theta}^2+2l\dot{x}\dot{\theta}\cos\theta\right) \tag{i}$$

The potential energy is

$$V=\frac{1}{2}kx^2+Mgl+mgl(1-\cos\theta) \tag{ii}$$

Then, the Lagrangian of the system is

$$L=T-V=\begin{bmatrix}\frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left(\dot{x}^2+l^2\dot{\theta}^2+2l\dot{x}\dot{\theta}\cos\theta\right)\\ -\frac{1}{2}kx^2-Mgl-mgl(1-\cos\theta)\end{bmatrix} \tag{iii}$$

Now, we write the Lagrange's equations of motion for the variables x and θ. For the coordinate x it is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

or $$\frac{d}{dt}\left(M\dot{x}+m\dot{x}+ml\dot{\theta}\cos\theta\right)-(-kx)=0$$

or $$(M+m)\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta+kx=0 \tag{iv}$$

Similarly, for the coordinate θ, we get

$$l\ddot{\theta}+\ddot{x}\cos\theta+g\sin\theta=0 \tag{v}$$

For small oscillations, $\theta, \phi, \dot{\theta}$ and $\dot{\phi}$ are very small so that the higher order terms can be neglected and we can approximate the above two equations as

$$(M+m)\ddot{x}+ml\ddot{\theta}+kx=0 \tag{vi}$$

$$l\ddot{\theta}+\ddot{x}+g\theta=0 \tag{vii}$$

Now, we try a solution, $x=Ae^{i\omega t}$ and $\theta=Be^{i\omega t}$. Using this in (vi) and (vii), we get

$$\left[k-\omega^2(M+m)\right]A-ml\omega^2B=0 \tag{viii}$$

and $$-\omega^2A+\left(g-\omega^2 l\right)B=0 \tag{ix}$$

Equations (viii) and (ix) can be written in a matrix form as

$$\begin{bmatrix} k-\omega^2(M+m) & -ml\omega^2 \\ -\omega^2 & g-l\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}=0$$

Then the secular equation is

$$\begin{vmatrix} k-\omega^2(M+m) & -ml\omega^2 \\ -\omega^2 & g-l\omega^2 \end{vmatrix}=0$$

or $$\left[k-\omega^2(M+m)\right]\left(g-l\omega^2\right)-ml\omega^4=0$$

or $$Ml\omega^4-\left[(M+m)g+kl\right]\omega^2+gk=0 \tag{x}$$

Roots of this equation are

$$\omega_1=\left[\frac{(M+m)g+kl+\sqrt{\left[(M+m)g+kl\right]^2-4Mlgk}}{2Ml}\right]^{1/2} \tag{xi}$$

and $$\omega_1 = \left[\frac{(M+m)g + kl - \sqrt{\left[(M+m)g + kl\right]^2 - 4Mlgk}}{2Ml}\right]^{1/2} \tag{xii}$$

These two roots give the eigenfrequencies of small oscillations.

EXAMPLE 6.25 Two identical pendula of length l and mass m are suspended side by side as shown in Figure 6.16. They are coupled together with a spring of force constant k. The spring is connected half way up the pendulum. Obtain the equations of motion and the eigenfrequencies of small oscillations.

Solution: We take θ_1 and θ_2 as the generalized coordinates. For small oscillations, these angles will be very small. The coordinates of the bobs are

$$(x_1, y_1) = (l\sin\theta_1, -l\cos\theta_1) \text{ and } (x_2, y_2) = (l\sin\theta_2, -l\cos\theta_2)$$

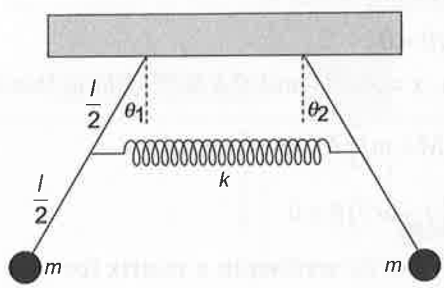

Fig. 6.16

Then the kinetic potential energies can be written as

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right)$$

$$= \frac{1}{2}ml^2\left(\dot{\theta}_1^2 \sin^2\theta_1 + \dot{\theta}_1^2\cos^2\theta_1 + \dot{\theta}_2^2\sin^2\theta_2 + \dot{\theta}_2^2\cos^2\theta_2\right)$$

$$= \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) \tag{i}$$

For small oscillations, θ_1 and θ_2 are very small, the extensions of the spring can be taken as $\left(-\frac{l}{2}\sin\theta_1 + \frac{l}{2}\sin\theta_2\right) \approx \frac{l}{2}(\theta_2 - \theta_1)$ and the corresponding potential energy is

$$V_1 = \frac{1}{2}k\left(\frac{l}{2}\right)^2(\theta_2 - \theta_1)^2$$

The gravitational potential energy is

$$V_2 = -mgl(\cos\theta_1 + \cos\theta_2)$$

$$= -mgl\left[1 - \frac{\theta_1^2}{2} + 1 - \frac{\theta_2^2}{2}\right] = -2mgl + mg\frac{l}{2}\left(\theta_1^2 + \theta_2^2\right)$$

Then the total potential energy is

$$V = \frac{1}{2}k\left(\frac{l}{2}\right)^2 (\theta_2 - \theta_1)^2 + \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2\right) \tag{ii}$$

Here, we have neglected the constant term in the potential energy.
Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{1}{2}k\left(\frac{l}{2}\right)^2 (\theta_2 - \theta_1)^2 - \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2\right) \tag{iii}$$

The Lagrange's equation of motion for the coordinates θ_1 and θ_2 can be obtained as

$$\ddot{\theta}_1 + \frac{g}{l}\theta_1 + \frac{k}{4m}(\theta_1 - \theta_2) = 0 \tag{iv}$$

$$\ddot{\theta}_2 + \frac{g}{l}\theta_2 + \frac{k}{4m}(\theta_2 - \theta_1) = 0 \tag{v}$$

Now, we put, $\omega_0^2 = \sqrt{\frac{g}{l}}$ and $\eta = \frac{kl}{4mg}$, then the above equations can be rewritten as

$$\ddot{\theta}_1 + \omega_0^2\theta_1 + \eta\omega_0^2(\theta_1 - \theta_2) = 0 \quad \text{or} \quad \ddot{\theta}_1 + \omega_0^2(1+\eta)\theta_1 - \eta\omega_0^2\theta_2 = 0 \tag{vi}$$

and
$$\ddot{\theta}_2 + \omega_0^2(1+\eta)\theta_2 - \eta\omega_0^2\theta_1 = 0 \tag{vii}$$

Now, make the substitution, $\theta_1 = \Theta_1 e^{i\omega t}$ and $\theta_2 = \Theta_2 e^{i\omega t}$ in the equations (vi) and (vii), to get

$$-\omega^2\Theta_1 + \omega_0^2(1+\eta)\Theta_1 - \eta\omega_0^2\Theta_2 = 0$$

or
$$\left[\omega_0^2(1+\eta) - \omega^2\right]\Theta_1 - \eta\omega_0^2\Theta_2 = 0 \tag{viii}$$

and
$$-\omega^2\Theta_2 + \omega_0^2(1+\eta)\Theta_2 - \eta\omega_0^2\Theta_1 = 0$$

or
$$-\eta\omega_0^2\Theta_1 + \left[\omega_0^2(1+\eta) - \omega^2\right]\Theta_2 = 0 \tag{ix}$$

Now, equations (viii) and (ix) can be written in the form of a matrix equation

$$\begin{bmatrix} \omega_0^2(1+\eta) - \omega^2 & -\eta\omega_0^2 \\ -\eta\omega_0^2 & \omega_0^2(1+\eta) - \omega^2 \end{bmatrix}\begin{bmatrix} \Theta_1 \\ \Theta_1 \end{bmatrix} = 0$$

Then the secular equation is

$$\begin{vmatrix} \omega_0^2(1+\eta)-\omega^2 & -\eta\omega_0^2 \\ -\eta\omega_0^2 & \omega_0^2(1+\eta)-\omega^2 \end{vmatrix} = 0$$

This can be rewritten by putting $\dfrac{\omega^2}{\omega_0^2} = \lambda$ as

$$\begin{vmatrix} 1+\eta-\lambda & -\eta \\ -\eta & 1+\eta-\lambda \end{vmatrix} = 0$$

or $$(1+\eta-\lambda)^2 - \eta^2 = 0$$

or $$\lambda^2 - 2(1+\eta)\lambda + (1+2\eta) = 0 \qquad \text{(x)}$$

This equation has the roots

$$\lambda_1 = 1 \text{ and } \lambda_2 = 1+2\eta$$

that is, $$\frac{\omega_1^2}{\omega_0^2} = \lambda_1 = 1 \text{ or } \omega_1 = \omega_0 \qquad \text{(xi)}$$

and $$\frac{\omega_2^2}{\omega_0^2} = \lambda_2 = 1+2\eta \text{ or, } \omega_2 = \omega_0\sqrt{1+2\eta} \qquad \text{(xii)}$$

Equations (xi) and (xii) are the eigenfrequencies with $\eta = \dfrac{kl}{4mg}$.

EXAMPLE: 6.26 A triple pendulum is arranged as shown in Figure 6.17. Each pendulum has a length l and bob of mass m. The bobs are connected with identical springs of force constant k. Obtain the eigenfrequencies and corresponding eigenvectors of small oscillations.

Solution: In this problem, we take θ_1, θ_2 and θ_3 as the generalized coordinates. Since the pendula are identical, we write the total kinetic energy of the system as

$$T = \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right) \qquad \text{(i)}$$

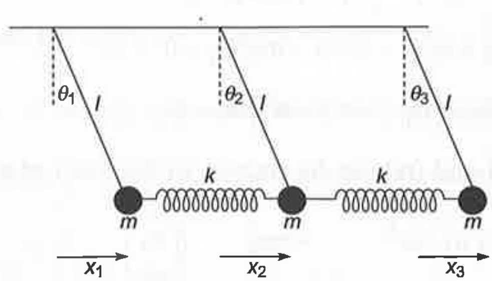

Fig. 6.17

The potential energy has two parts, that is, potential energy of the spring and the gravitational potential energy.

The potential energy of the spring can be written as

$$V_1 = \frac{1}{2}k\left[x_3^2 + (x_3 - x_2 - x_1)^2 + (x_2 - x_1)^2\right]$$

$$= \frac{1}{2}k\left[x_1^2 + x_2^2 + x_2^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1\right]$$

For small oscillations, θ_1, θ_2 and θ_3 are very small and we may write $x_i = l\theta_i$ so that the above expression becomes

$$V_1 = \frac{1}{2}kl^2\left[\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3 - 2\theta_3\theta_1\right] \tag{ii}$$

The gravitational potential energy is

$$V_2 = -mgl(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)$$

For small oscillations, we may write; $\cos\theta_i = 1 - \frac{\theta_i^2}{2}$, then; the above expression after dropping the constant term becomes

$$V_2 = \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2 + \theta_3^2\right) \tag{iii}$$

Then, the total potential energy of the system is

$$V = \frac{1}{2}\left[\left(mgl + kl^2\right)\left(\theta_1^2 + \theta_2^2 + \theta_3^2\right) - 2kl^2\left(\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1\right)\right] \tag{iv}$$

The matrices representing the kinetic and potential energies are

$$T = \begin{bmatrix} ml^2 & 0 & 0 \\ 0 & ml^2 & 0 \\ 0 & 0 & ml^2 \end{bmatrix} \text{ and;}$$

$$V = \begin{bmatrix} mgl + kl^2 & -kl^2 & -kl^2 \\ -kl^2 & mgl + kl^2 & -kl^2 \\ -kl^2 & -kl^2 & mgl + kl^2 \end{bmatrix}$$

The eigenfrequencies can be obtained by solving, $\left|V - \omega^2 T\right| = 0$

That is,

$$\begin{vmatrix} mgl + kl^2 - ml^2\omega^2 & -kl^2 & -kl^2 \\ -kl^2 & mgl + kl^2 - ml^2\omega^2 & -kl^2 \\ -kl^2 & -kl^2 & mgl + kl^2 - ml^2\omega^2 \end{vmatrix} = 0$$

or

$$\left(mgl + kl^2 - ml^2\omega^2\right)^3 - 3k^2\left(mgl + kl^2 - ml^2\omega^2\right) - 2k^3 = 0 \tag{v}$$

This is a cubic equation in $\left(mgl + kl^2 - ml^2\omega^2\right)$ and can be solved to get the eigenfrequencies as

$$\omega_1 = \left(\frac{g}{l} + \frac{2k}{m}\right)^{1/2}, \quad \omega_2 = \left(\frac{g}{l} + \frac{2k}{m}\right)^{1/2} \text{ and } \omega_3 = \left(\frac{g}{l} + \frac{k}{m}\right)^{1/2} \tag{vi}$$

Now, to determine the eigenvectors, we assume that $m = 1, g = 1$ and $l = 1$ units so that the eigenfrequencies become

$$\omega_1^2 = 1 + 2k \ , \ \omega_2^2 = 1 + 2k \text{ and } \omega_3^2 = 1 - k \tag{vii}$$

Similarly, the matrix representing $\left[V - \omega^2 T\right]$ becomes

$$\left[V - \omega^2 T\right] = \begin{bmatrix} 1 + k - \omega^2 & -k & -k \\ -k & 1 + k - \omega^2 & -k \\ -k & -k & 1 + k - \omega^2 \end{bmatrix}$$

Then, the eigenvectors can be determined by solving the matrix equation

$$\begin{bmatrix} 1 + k - \omega^2 & -k & -k \\ -k & 1 + k - \omega^2 & -k \\ -k & -k & 1 + k - \omega^2 \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix} = 0 \tag{viii}$$

Case 1: $\omega_1^2 = 1 + 2k$

Let the eigenvectors be (a_{11}, a_{21}, a_{31}). Then (viii) becomes

$$\begin{bmatrix} -k & -k & -k \\ -k & -k & -k \\ -k & -k & -k \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = 0$$

or $\quad -ka_{11} - ka_{21} - ka_{31} = 0$ or, $a_{11} + a_{21} + a_{31} = 0$

or $\quad a_{21} = -(a_{11} + a_{31}) = -2\alpha$ (say) $\qquad$ (ix)

Case 2: $\quad \omega_1^2 = 1 + 2k$

Let the eigenvectors be (a_{12}, a_{22}, a_{32}). Then (viii) becomes

$$\begin{bmatrix} -k & -k & -k \\ -k & -k & -k \\ -k & -k & -k \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = 0$$

or $\quad -ka_{12} - ka_{22} - ka_{32} = 0$ or, $a_{12} + a_{22} + a_{32} = 0$

Now, we take $a_{22} = 0$ and $a_{12} = -a_{32} = \beta$ (say) $\qquad$ (x)

Case 3: $\omega_3^2 = 1 - k$

Let the eigenvectors are; (a_{13}, a_{23}, a_{33}). The (viii) becomes

$$\begin{bmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = 0$$

Thus, we get the following three equations:

$$2ka_{13} - ka_{23} - ka_{33} = 0$$
$$-ka_{13} + 2ka_{23} - ka_{33} = 0$$
$$-ka_{13} - ka_{23} + 2ka_{33} = 0$$

These equations can be solved to get

$$a_{13} = a_{23} = a_{33} = \gamma \text{ (say)} \qquad \text{(xi)}$$

From the equations (ix), (x) and (xi), we write the matrix representing the eigenvectors as

$$A = \begin{bmatrix} \alpha & \beta & \gamma \\ -2\alpha & 0 & \gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \qquad \text{(xii)}$$

The values of α, β and γ can be determined by solving the equation $A^T TA = I$.

That is,
$$\begin{bmatrix} \alpha & -2\alpha & \alpha \\ \beta & 0 & \beta \\ \gamma & \gamma & \gamma \end{bmatrix} \begin{bmatrix} ml^2 & 0 & 0 \\ 0 & ml^2 & 0 \\ 0 & 0 & ml^2 \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma \\ -2\alpha & 0 & \gamma \\ \alpha & -\beta & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On multiplication, this becomes

$$ml^2 \begin{bmatrix} 6\alpha^2 & 0 & 0 \\ 0 & 2\beta^2 & 0 \\ 0 & 0 & 3\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or
$$\alpha = \frac{1}{\sqrt{6ml^2}}, \; \beta = \frac{1}{\sqrt{2ml^2}} \text{ and } \gamma = \frac{1}{\sqrt{3ml^2}} \qquad \text{(xiii)}$$

Equation (xiii) can be substituted in (xii) to get the eigenvectors.

EXERCISES

6.1 A uniform rod of length l and mass m is supported at the ends by identical springs of force constant k. The system can be set into vibration by pressing one end of the rod and then releasing. Determine the normal frequencies.

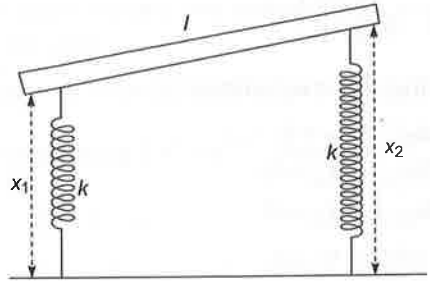

Fig. 6.18

6.2. A molecule consists of three identical atoms located at the vertices of a 45° right triangle. Each pair of atoms interacts by an effective spring potential, with all spring constants equal to k. Considering only the planar motion of the molecule obtain the eigenfrequencies and eigenvectors.

6.3 A triple pendulum consists of three masses m_1, m_2 and m_3 with lengths of the pendulum l_1, l_2 and l_3 as shown in Figure 6.19. Obtain the general equations of motion. Determine the eigenfrequencies and eigenvectors when $m_1 = m_2 = m_3$ and $l_1 = l_2 = l_3$.

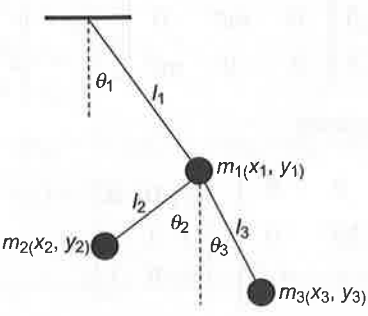

Fig. 6.19

6.4 Three equal masses are placed on the vertices of an equilateral triangle. The masses are connected together using identical springs of force constant k. Determine the eigenfrequencies and the corresponding eigenvectors of small oscillations.

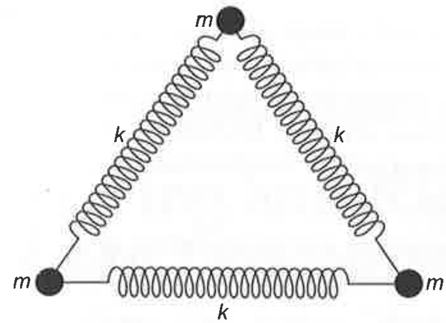

Fig. 6.20

6.5 A spring of length l, mass m and force constant k is hanging vertically from a rigid support. A block of mass M is attached to its free end. If the block is pulled down a little and then released, obtain the frequency of small oscillations.

6.6 A particle of mass m is moving under the action of a central field potential given by the expression, $-Ar^{-\alpha}$ where A and α are constants. Obtain the equation of motion and the eigenfrequencies of small oscillations.

6.7 Two blocks of masses m_1 and m_2 sliding over each other are connected to rigid walls using springs of force constants k_1 and k_2 respectively. Assuming that there is no friction between the blocks, deduce the equations of motion and then obtain the eigenfrequencies of oscillation.

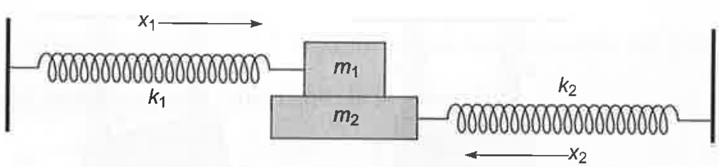

Fig. 6.21

6.8 A rectangular plate of mass m supports four identical springs of force constant k at its four corners. If only vertical motion is possible, obtain the eigenfrequencies and eigenvectors of small vibrations of the plate.

6.9 A fly wheel of moment of inertia I rotates in a horizontal plane about an axis passing through its centre and perpendicular to its plane. A mass m is attached to its centre using a spring of length l and force constant k so that it can slide freely along one of the spokes. Obtain the eigenfrequencies of small oscillations.

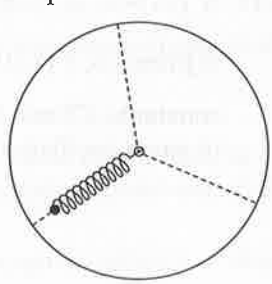

Fig. 6.22

6.10 A smooth uniform circular hoop of mass M and radius R swings in a vertical plane about a point O about which it is hinged freely from a rigid support. A bead of mass m slides without friction on the hoop. Obtain the characteristic frequencies about the stable equilibrium.

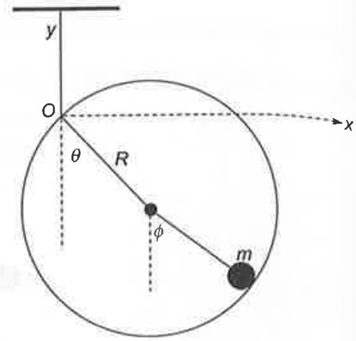

Fig. 6.23

6.11 Obtain the eigenfrequencies and eigenvectors of a coupled oscillator whose Lagrangian is given by,

$$L = ma^2\left(2\dot{x}^2 - 10\dot{x}\dot{y} + 13\dot{y}^2\right) - mga\left(5x^2 - 22xy + 25y^2\right).$$

6.12 A particle of mass m moves along the x–direction under the influence of a potential energy $V(x) = -Kxe^{-ax}$, where K and a are positive constants. Obtain the frequencies of small oscillations.

6.13 A small sphere of mass m and radius r hangs like a pendulum between the plates of a parallel plate capacitor from an insulating rod of length l. The sphere is charged to a potential V and the plates are grounded. Obtain the frequency of small oscillations of the pendulum and find the voltage at which such an oscillation occurs.

6.14 A thin square sheet of metal of mass m hangs from two identical springs of force constant k. The springs can move only in a vertical plane. Obtain the frequency of small vibrations.

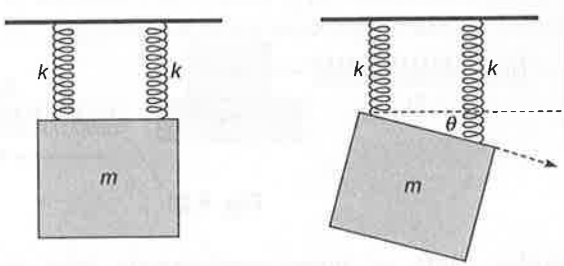

Fig. 6.24

6.15 A particle of mass m moving in two dimensions in a potential given by the expression, $V(x,y) = -\frac{1}{2}kx^2 + \frac{1}{2}\lambda_0 x^2y^2 + \frac{1}{4}\lambda_1 x^4$, where k, λ_0 and λ_1 are positive constants. Obtain the Lagrange's equations of motion and the normal frequencies of small oscillations.

7

CHAPTER

Scattering and Rigid Body Dynamics

CONCEPTS AND FORMULAE

7.1 LABORATORY AND CENTRE OF MASS FRAMES

A laboratory frame is the one in which the target particle is assumed to be at rest and the incident particle is moving.

A coordinate system attached to the centre of mass which is at rest is known as a centre of mass frame.

7.2 SCATTERING CROSS SECTION IN LAB AND CENTRE OF MASS FRAME

The scattering cross section $\left(\dfrac{d\sigma}{d\Omega}\right)$ is defined as the number of particles scattered through the unit solid angle in unit time. It is given by

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{7.1}$$

The relation between the scattering cross section in the lab and the centre of mass frames is given by

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{\sin\theta_l}{\sin\theta_c}\frac{d\theta_l}{d\theta_c}\left.\frac{d\sigma}{d\Omega}\right|_{lab} \tag{7.2}$$

where, θ_l and θ_c are the scattering angle in the lab and centre of mass frames respectively.

7.3 ROTATING FRAME OF REFERENCE

To describe the general motion of a body, a stationary or fixed frame of reference is not sufficient. We need to consider a frame of reference which is rotating about one of the axes.

If a vector $\vec{A}$ of constant magnitude rotates with a constant angular velocity ω about an axis, then

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A} \tag{7.3}$$

7.4 EQUATION OF MOTION OF A PARTICLE IN A ROTATING FRAME OF REFERENCE

The general motion of a particle of mass m in a rotating frame of reference can be described by the equation

$$m\vec{a}' = m\vec{a} - 2m(\vec{\omega} \times \vec{v}') - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - m\frac{d\vec{\omega}}{dt} \times \vec{r} \tag{7.4}$$

In this expression, the first term on the RHS is the external force; $\vec{F}_{ext} = m\vec{a}$. The second term on the RHS $2m(\vec{\omega} \times \vec{v}')$, is the Corioli's force. The third term on the RHS $m\vec{\omega} \times (\vec{\omega} \times \vec{r})$, is the centripetal force. The last term is valid only if there is change in angular velocity.

7.5 RIGID BODY

A body in which the distance between the particles remains unchanged during the motion of the body is known as a rigid body.

The number of degrees of freedom of a rigid body is 6.

7.6 EULER'S AND CHASLE'S THEOREM

Euler's theorem states that a general displacement of a rigid body whose one point is fixed is a rotation about some axis.

Chasle provided a more general theorem which states that the most general displacement of a rigid body is the translation of the rigid body and a rotation of the rigid body.

7.7 EULERIAN ANGLES

Eulerian angles are the angles of rotation about the three independent axes executed in a specific order. As a result of the three successive rotations a transformation from the space fixed coordinate system to the body fixed coordinate system can be achieved.

7.8 MOMENT OF INERTIA TENSOR

Since the moment of inertia of a rigid body which is in motion depends on the various components of its velocity, it can be represented as a tensor. That is,

$$I = \begin{bmatrix} \sum_i m_i\left(y_i^2 + z_i^2\right) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i x_i y_i & \sum_i m_i\left(x_i^2 + z_i^2\right) & -\sum_i m_i z_i y_i \\ -\sum_i m_i x_i z_i & -\sum_i m_i z_i y_i & \sum_i m_i\left(x_i^2 + y_i^2\right) \end{bmatrix} \tag{7.5}$$

The diagonal elements of the matrix are called the moment of inertia coefficients and the off diagonal elements are called the products of inertia.

7.9 ANGULAR MOMENTUM

The angular momentum can be determined as

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{7.6}$$

7.10 PRINCIPAL AXIS OF INERTIA

We can select the body set of axes such that the off diagonal elements of the moment of inertia tensor are zero. Such an axis is known as the principal axes on inertia and the corresponding moments of inertia are known as the principal moments of inertia.

$$I' = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{7.7}$$

where, I_1 is the principal moment of inertia.

Then,
$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{7.8}$$

7.11 ROTATIONAL KINETIC ENERGY OF A RIGID BODY

The rotational kinetic energy of a body is given by;

$$T = \frac{1}{2}\vec{\omega}.\vec{L} = \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right) \tag{7.9}$$

7.12 EULER'S EQUATIONS OF MOTION OF A RIGID BODY

Euler's equations of motion are;

$$\begin{aligned} \tau_x &= I_x\dot{\omega}_x + \left(I_z - I_y\right)\omega_y\omega_z \\ \tau_y &= I_y\dot{\omega}_y + \left(I_x - I_z\right)\omega_z\omega_x \\ \tau_z &= I_z\dot{\omega}_z + \left(I_y - I_x\right)\omega_x\omega_y \end{aligned} \tag{7.10}$$

SOLVED PROBLEMS

EXAMPLE 7.1 Obtain an expression for the scattering cross section (Rutherford scattering formula) in the laboratory frame of reference.

Solution: We first define a differential scattering cross section through the equation

$$d\sigma = \frac{dN}{I} \tag{i}$$

where, dN is the number of particles scattered through an annular region with radius b and thickness db, that is, through an annular region of area $2\pi ada$ and I is the intensity of incident particles. Since the intensity is equal to the number of particles that are incident on unit area in unit time, we have;

$$dN = 2\pi b.I.db$$

Then, the differential scattering cross section becomes

$$d\sigma = 2\pi b.db \tag{ii}$$

This expression, we may rewrite as

$$d\sigma = 2\pi b.\left|\frac{db}{d\theta}\right| d\theta \tag{iii}$$

where θ is the scattering angle. In this expression we have taken the modulus of the derivative since it is a negative quantity.

In general, scattering is a three-dimensional process and we usually write the scattering cross section in terms of a solid angle rather than the plane angle. The solid angle between the cones of semivertical angles θ and $\theta + d\theta$ is $d\Omega = 2\pi \sin\theta d\theta$. Therefore, we may write

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{iv}$$

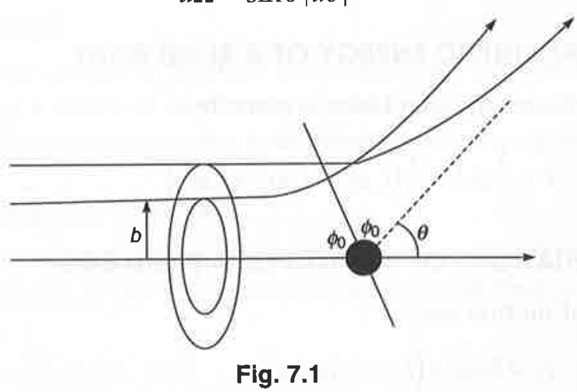

Fig. 7.1

To obtain the Rutherford scattering formula, we make use of the Lenz vector $\vec{A}$ which is defined through

$$\vec{A} = \vec{p} \times \vec{L} + \frac{\mu k \vec{r}}{r} \tag{v}$$

Note that the Lenz vector is a constant of motion for the potential that varies as $\frac{1}{r}$. In (v) $\vec{p}$ and $\vec{L}$ are the linear and angular momenta. Before scattering, we take; $\vec{p} = p\hat{i}$ and $\hat{i}$ is the direction of the incident particles and take $\hat{j}$ perpendicular to it; both are in the plane of the paper. The direction of the angular momentum $\vec{L}$ is $\hat{k}$ and is perpendicular to the plane of the paper; $\vec{L} = (\mu v)a\hat{k}$.

The Lenz vector of the initial state can be written as

$$\vec{A} = (\mu v)(\mu v b)\left(\hat{i} \times \hat{k}\right) + \mu k\frac{\vec{r}}{r} = -(\mu v)^2 b\hat{j} - \mu k\hat{i} \tag{vi}$$

The Lenz vector of the final state is

$$\vec{A} = (\mu v)(\mu v b)\left(\hat{i}' \times \hat{k}\right) + \mu k\frac{\vec{r}}{r} = (\mu v)^2 b\hat{j}' + \mu k\hat{i}' \tag{vii}$$

Note that the Lenz vector is same for both the initial and final states since it is a constant.

Now, take the dot product of the equations (vi) and (vii) with $\hat{i}$. From (vi), we get

$$\vec{A}.\hat{i} = -(\mu v)^2 b\hat{i}.\hat{j} - \mu k\hat{i}.\hat{i} = -\mu k \tag{viii}$$

From (vii), we get

$$\vec{A}.\hat{i} = (\mu v)^2 b\hat{i}.\hat{j}' + \mu k\, i.\hat{i}' = 2T\mu b(-\sin\theta) + \mu k(\cos\theta) \tag{ix}$$

Here we have used $T = \frac{1}{2}\mu v^2$; $\hat{i}.\hat{j}' = -\sin\theta$ and $\hat{i}.\hat{i}' = \cos\theta$

Equating the RHS of (viii) and (ix), we get

$$-\mu k = 2T\mu b(-\sin\theta) + \mu k(\cos\theta)$$

or $$k(1+\cos\theta) = 2Tb\sin\theta$$

or $$b = \frac{k(1+\cos\theta)}{2T\sin\theta} = \frac{k}{2T}\cot\left(\frac{\theta}{2}\right) \tag{x}$$

Differentiating this expression with respect to θ, we get

$$\frac{db}{d\theta} = -\frac{k}{4T}cosec^2\frac{\theta}{2} \tag{xi}$$

Using (x) and (xi) in equation (iv), it becomes

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\theta}\left(\frac{k}{2T}\cot\left(\frac{\theta}{2}\right)\right)\frac{k}{4T}cosec^2\frac{\theta}{2}$$

$$= \frac{k^2}{16T^2}\frac{1}{\sin^4(\theta/2)} \tag{xii}$$

This is Rutherford scattering formula. If the charge fixed at the centre of force is Ze and the charge of the incident particle is $Z'e$, then $k = ZZ'e^2$.

EXAMPLE 7.2 A stream of particles are scattered by a heavy rigid sphere of radius R. Find the effective cross section for scattering.

Solution: For a rigid sphere, we write

$$V(r) = 0 \text{ for } r > R \text{ and } V(r) = \infty \text{ for } r < R \tag{i}$$

From Figure 7.2, we see that

Impact parameter, $\quad b = R\sin\theta_0 \quad$ (ii)

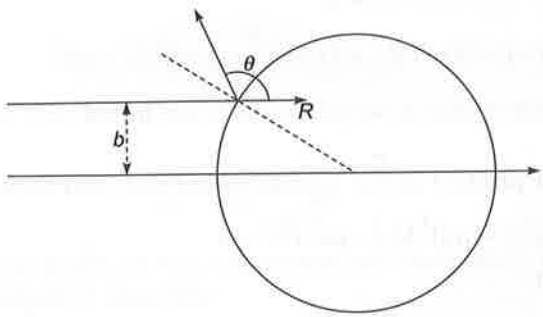

Fig. 7.2

and $\quad \theta_0 = \dfrac{\pi - \theta}{2}$

Then, $$\frac{b}{R} = \sin\left(\frac{\pi - \theta}{2}\right) = \cos\left(\frac{\theta}{2}\right) \tag{iii}$$

Differentiating this expression with respect to θ, we get

$$\frac{db}{d\theta} = -\frac{R}{2}\sin\left(\frac{\theta}{2}\right) \tag{iv}$$

Now, we have the differential scattering cross section, $\dfrac{d\sigma}{d\Omega} = \dfrac{b}{\sin\theta}\left|\dfrac{db}{d\theta}\right|$. Using (iv), this expression becomes

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{R}{2}\sin\left(\frac{\theta}{2}\right)$$

Using (iii), we write

$$\frac{d\sigma}{d\Omega} = \frac{R\cos\left(\frac{\theta}{2}\right)}{\sin\theta}\frac{R}{2}\sin\left(\frac{\theta}{2}\right) = \frac{R^2}{4} \tag{v}$$

Now, the total scattering cross section is

$$\sigma = \int_0^{4\pi} \frac{d\sigma}{d\Omega} d\Omega = \int_0^{4\pi} \frac{R^2}{4} d\Omega = \pi R^2 \tag{vi}$$

EXAMPLE 7.3 Obtain a relation between the scattering angles in the lab and centre of mass system.

Solution: Let, $\vec{u}_1$ and $\vec{v}_1$ be the initial and final velocities of the mass m_1 in the lab frame; $\vec{u}_2$ and $\vec{v}_2$ be the initial and final velocities of the mass m_2 in the lab frame; and $\vec{V}_{cm}$ be the velocity of the centre of mass in the lab frame. Similarly, $\vec{u}_1'$ and $\vec{v}_1'$ be the initial and final velocities of the mass m_1 in the centre of mass frame, $\vec{u}_2'$ and $\vec{v}_2'$ be the initial and final velocities of the mass m_2 in the centre of mass frame. A schematic diagram of the scattering process in the lab and the centre of mass frames is given in Figure 7.3. Let θ_l be the scattering angle in the lab frame and is measured as the angle between the final and initial directions of the scattered particle; θ_c, the scattering angle in the centre of mass frame, which is the angle between the initial and final directions of the relative vector connecting the two particles.

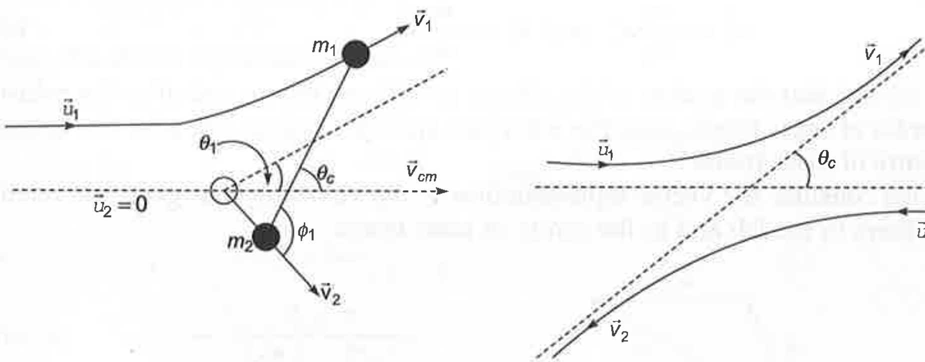

Fig. 7.3

From the law of conservation of momentum, in the centre of mass frame, we write

$$m_1 u_1' + m_2 u_2' = m_1 v_1' + m_2 v_2' = 0 \tag{i}$$

The velocity of the centre of mass is;

$$\vec{V}_{cm} = \frac{m_1 \vec{u}_1}{m_1 + m_2} + \frac{m_2 \vec{u}_2}{m_1 + m_2} \tag{ii}$$

In the lab frame, we consider the second mass is at rest, i.e., $\vec{u}_2 = 0$. Therefore,

$$\vec{V}_{cm} = \frac{m_1 \vec{u}_1}{m_1 + m_2} \tag{iii}$$

The initial velocity of the masses in the centre of mass system can be obtained as

$$\vec{u}_1' = \vec{u}_1 - \vec{V}_{cm} = \vec{u}_1 - \frac{m_1\vec{u}_1}{m_1+m_2} = \frac{m_2\vec{u}_1}{m_1+m_2} \tag{iv}$$

and
$$\vec{u}_2' = \vec{u}_2 - \vec{V}_{cm} = -\frac{m_1\vec{u}_1}{m_1+m_2} \tag{v}$$

If we assume only elastic scattering, law of conservation of energy in the centre of mass frame may be expressed as

$$\frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \tag{vi}$$

Now we substitute (iv) and (v) in equations (i) and (vi), then in the centre of mass frame we may have

$$\vec{v}_1' = \vec{u}_1' = \frac{m_2\vec{u}_1}{m_1+m_2} \quad \text{and} \quad \vec{v}_2' = -\vec{u}_2' = \frac{m_1\vec{u}_1}{m_1+m_2} \tag{vii}$$

From (vii), by dividing one equation by the other, we get

$$u_2' = -\frac{m_1}{m_2}u_1' \quad \text{and} \quad v_2' = -\frac{m_1}{m_2}v_1' \tag{viii}$$

Thus, we see that the motion of the masses is collinear before and after the collision in the centre of mass frame. Also the velocities are same before and after the collision in the centre of mass frame.

Now, we consider the vector representation of the velocities that gives the relation between them in the lab and in the centre of mass frame.

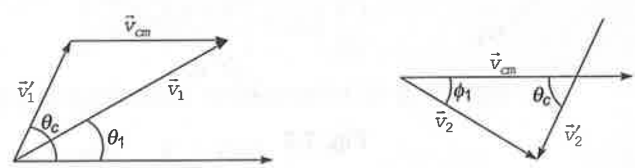

Fig. 7.4

From the first figure, we get

$$v_1'\sin\theta_c = v_1\sin\theta_l \tag{ix}$$

and
$$v_1'\cos\theta_c + V_{cm} = v_1\cos\theta_l \tag{x}$$

Dividing (x) by (ix), we get

$$\tan\theta_l = \frac{\sin\theta_c}{\cos\theta_c + \dfrac{V_{cm}}{v_1'}}$$

From equations (iii) and (vii), we have

$$\frac{V_{cm}}{v_1'} = \frac{m_1}{m_2}$$

Then, $$\tan\theta_l = \frac{\sin\theta_c}{\cos\theta_c + \dfrac{m_1}{m_2}} \tag{xi}$$

From the second figure, we get

$$v_2 \sin\phi_l = v_2' \sin\theta_c \tag{xii}$$

and $$v_2 \cos\phi_l + v_2' \cos\theta_c = V_{cm} \tag{xiii}$$

Dividing the first equation by the second equation, we get

$$\tan\phi_l = \frac{\sin\theta_c}{\dfrac{V_{cm}}{v_2'} - \cos\theta_c}$$

Again, from equations (iii) and (vii), we have

$$\frac{V_{cm}}{v_2'} = 1$$

Then, the above expression becomes

$$\tan\phi_l = \frac{\sin\theta_c}{1-\cos\theta_c} = \frac{2\sin\dfrac{\theta_c}{2}\cos\dfrac{\theta_c}{2}}{2\sin^2\dfrac{\theta_c}{2}} = \cot\frac{\theta_c}{2}$$

or $$\tan\phi_l = \tan\left(\frac{\pi}{2} - \frac{\theta_c}{2}\right)$$

that is, $$\phi_l + \frac{\theta_c}{2} = \frac{\pi}{2} \tag{xiv}$$

Equations (xi) and (xiv) give the relation between the scattering angle and the recoil angle in the lab frame and the scattering angle in the centre of mass frame.

Note: From equations (xi) and (xiv) we see that

if $m_2 >> m_1$; $\theta_c = \theta_l$ and $\phi_l = \dfrac{\pi}{2} - \dfrac{\theta_l}{2}$

if $m_2 = m_1$; $\theta_c = 2\theta_l$ and $\phi_l = \dfrac{\pi}{2} - \theta_l$

EXAMPLE 7.4 Show that the total scattering cross section is infinity for scattering from a potential $V(r) = \dfrac{\alpha}{r^2}$, where α is a positive quantity.

Solution: For the central motion, we have

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V \tag{i}$$

This can be rearranged to get

$$\dot{r} = \left[\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)\right]^{1/2} \tag{ii}$$

We have $\quad \dot{r} = \frac{dr}{dt} = \frac{dr}{d\phi_0}\frac{d\phi_0}{dt} = \frac{dr}{d\phi_0}\frac{l}{mr^2}$

or $\quad \frac{dr}{d\phi_0} = \frac{\dot{r}}{l/mr^2} = \frac{mr^2}{l}\left[\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)\right]^{1/2}$

or

$$d\phi_0 = \frac{l\,dr}{mr^2\left[\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)\right]^{1/2}} = \frac{l\,dr}{r^2\left[2m\left(E - V - \frac{l^2}{2mr^2}\right)\right]^{1/2}}$$

This can be integrated to get

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{l\,dr}{r^2\left[2m(E-V) - \left(l^2/r^2\right)\right]^{1/2}} \tag{iii}$$

Here we have dropped the constant term representing the initial position. Now, we put; $l = mv_\infty b$ and $E = \frac{1}{2}mv_\infty^2$ so that equation (i) becomes

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{mv_\infty b\,dr}{r^2\sqrt{2m}\left[(E-V) - \left(m^2v_\infty^2b^2/2mr^2\right)\right]^{1/2}}$$

$$= \int_{r_{\min}}^{\infty} \frac{mv_\infty b\,dr}{r^2\sqrt{2mE}\left[1 - \frac{V(r)}{E} - \frac{b^2}{r^2}\right]^{1/2}}$$

$$= \int_{r_{\min}}^{\infty} \frac{b\,dr}{r^2\left[1 - \frac{b^2}{r^2} - \frac{V(r)}{E}\right]^{1/2}} \tag{iv}$$

From Figure 7.1 we see that the scattering angle; $\theta = \pi - 2\phi_0$. Therefore,

$$\theta = \pi - 2b \int_{r_{\min}}^{\infty} \frac{dr}{r^2\left[1 - \frac{b^2}{r^2} - \frac{V(r)}{E}\right]^{1/2}} \tag{v}$$

Now we substitute $V(r) = \frac{\alpha}{r^2}$ in this expression to get

$$\theta = \pi - 2b \int_{r_{\min}}^{\infty} \frac{dr}{r^2\left[1 - \frac{b^2}{r^2} - \frac{\alpha}{Er^2}\right]^{1/2}} = \pi - 2b \int_{r_{\min}}^{\infty} \frac{dr}{r^2\left[1 - \frac{1}{r^2}\left(b^2 + \frac{\alpha}{E}\right)\right]^{1/2}} \tag{vi}$$

In equation (iv), we put, $r = \frac{1}{u}$ and $dr = -\frac{du}{u^2}$

$$\theta = \pi + 2b \int_{1/r_{\min}}^{0} \frac{du}{\left[1 - u^2 r_{\min}^2\right]^{1/2}}$$

$$= \pi + 2\frac{b}{r_{\min}} \sin^{-1}\left(\frac{u}{r_{\min}}\right)\Bigg|_{1/r_{\min}}^{0}$$

$$= \pi - 2\frac{b}{r_{\min}} \sin^{-1} 1 = \pi - \frac{\pi b}{r_{\min}} \tag{vii}$$

Now, we find $r_{\min}$ using the expression for the total energy. That is,

$$E = \frac{l^2}{2mr^2} + V(r) = \frac{m^2 v_\infty^2 b^2}{2mr^2} + \frac{\alpha}{r^2} = \frac{1}{2} m v_\infty^2 \frac{b^2}{r^2} + \frac{\alpha}{r^2}$$

$$= E\frac{b^2}{r^2} + \frac{\alpha}{r^2} = \frac{E}{r^2}\left(b^2 + \frac{\alpha}{E}\right)$$

that is,
$$r^2 = \left(b^2 + \frac{\alpha}{E}\right) \quad \text{or} \quad r_{\min} = \left(b^2 + \frac{\alpha}{E}\right)^{1/2} \tag{viii}$$

Then,
$$\theta = \pi - \frac{\pi b}{\sqrt{b^2 + (\alpha/E)}} = \pi\left(1 - \frac{b}{\sqrt{b^2 + (\alpha/E)}}\right) \tag{ix}$$

This can be substituted in the expression, $\frac{d\sigma}{d\Omega}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$ and after simplification, we get

$$\frac{d\sigma}{d\Omega}=\frac{\pi^2\alpha}{E}\frac{(\pi-\theta)}{\theta^2(2\pi-\theta)^2\sin\theta} \tag{x}$$

The total scattering cross section is

$$\begin{aligned}\sigma_{total} &= \int\frac{d\sigma}{d\Omega}d\Omega \\ &= \frac{\pi^2\alpha}{E}\int\frac{(\pi-\theta)}{\theta^2(2\pi-\theta)^2\sin\theta}(2\pi\sin\theta d\theta) \\ &= \frac{2\pi^3\alpha}{E}\int_0^{\pi}\frac{(\pi-\theta)}{\theta^2(2\pi-\theta)^2}d\theta=\infty \end{aligned} \tag{xi}$$

Hence, proved.

EXAMPLE 7.5 Show that the differential scattering cross section for the scattering produced by a repulsive central force $f=kr^{-3}$ is given by, $d\sigma(\theta)d\theta=\frac{k}{4E}\frac{(1-x)}{(2-x)^2\sin\pi x}dx$, where $x=\theta/\pi$ and E is the energy.

Solution: Given that the force, $f=kr^{-3}$ so the potential energy is

$$U=\frac{k}{2r^2}=\frac{ku^2}{2} \tag{i}$$

The differential equation for the orbit is

$$\frac{d^2u}{d\theta^2}+u=-\frac{m}{l^2}\frac{dU}{du}=-\frac{mk}{l^2}u$$

or
$$\frac{d^2u}{d\theta^2}+\left(1+\frac{mk}{l^2}\right)u=0 \tag{ii}$$

This equation has a solution

$$u=A\cos\omega\theta+B\sin\omega\theta \tag{iii}$$

where,
$$\omega=\sqrt{1+\frac{mk}{l^2}} \tag{iv}$$

Initially, when the particle is at a large distance $(r=-\infty)$ from the scattering centre, the angle $\theta=\pi$ and after scattering and when $r\to\infty$, let $\theta=\theta_s$. Then we can determine the value of the constants A and B. From the first condition, we write

$$A\cos\pi\omega+B\sin\pi\omega=0 \quad \text{or,} \quad A=-B\tan\pi\omega \tag{v}$$

From the second condition, we get

$$A\cos\omega\theta_s + B\sin\omega\theta_s = 0$$

Using (v) in this expression

$$-B\tan\pi\omega\cos\omega\theta_s + B\sin\omega\theta_s = 0$$

or $$\sin\omega\theta_s\cos\pi\omega - \cos\omega\theta_s\sin\pi\omega = 0$$

or $$\sin\omega(\theta_s - \pi) = 0$$

or $$\omega(\theta_s - \pi) = \pi \tag{vi}$$

Using $x = \theta/\pi$ this expression can be written as

$$\omega = \frac{1}{x-1} \quad \text{or,}\ \omega^2 = 1 + \frac{mk}{l^2} = \frac{1}{(x-1)^2} \tag{vii}$$

Now, we have; $l = mv_\infty b = (2mE)^{1/2} b$, where b is the impact parameter. Then, (vii) becomes

$$1 + \frac{k}{2Eb^2} = \frac{1}{(x-1)^2}$$

or $$b^2 = -\frac{k}{2E}\left[\frac{(1-x)^2}{x(x-2)}\right] \tag{viii}$$

Differentiating this expression, we get

$$2bdb = -\frac{k}{2E}\left[\frac{2(x-1)}{x(x-2)} - \frac{(x-1)^2}{x^2(x-2)} - \frac{(x-1)^2}{x(x-2)^2}\right]dx$$

$$= -\frac{k}{2E}\left[\frac{2x(x-1)(x-2) - (x-1)^2(x-2) - x(x-1)^2}{x^2(x-2)^2}\right]$$

$$= -\frac{k}{2E}\left[\frac{2(1-x)}{x^2(x-2)^2}\right] \tag{ix}$$

The differential scattering cross section is

$$d\sigma(\theta)d\theta = \frac{b}{\sin\theta}|db| = \frac{k}{4E}\frac{(1-x)}{(2-x)^2\sin\pi x}dx \tag{x}$$

Hence, proved.

EXAMPLE 7.6 The potential corresponding to a central force is given by; $V = 0$ for $r > a$ and $V = -V_0$ for $r \le a$. Show that scattering produced by such potential in classical approach is identical to the refraction of light rays produced by the sphere of radius a and relative refractive index $\mu = \sqrt{1+(V_0/E)}$. Also show that the differential scattering cross section is $\dfrac{d\sigma}{d\Omega} = \dfrac{\mu^2 a^2}{4\cos(\theta/2)} \dfrac{[\mu\cos(\theta/2)-1]}{[\mu^2+1-2\mu\cos(\theta/2)]^2}[\mu-\cos(\theta/2)]$ and the total scattering cross section is $\sigma = \pi a^2$.

Solution: From Figure 7.5, applying the laws of conservation of momentum and energy, we write

$$p\sin\alpha = p'\sin\beta$$

and $$\frac{p^2}{2m} = \frac{p'^2}{2m} - V_0 \quad \text{or} \quad E = \frac{p'^2}{2m} - V_0$$

or $$\frac{p'^2}{2m} = E + V_0 \quad \text{or,} \quad p'^2 = 2mE\left(1+\frac{V_0}{E}\right) = p^2\left(1+\frac{V_0}{E}\right)$$

that is, $$p' = p\sqrt{1+(V_0/E)} \tag{i}$$

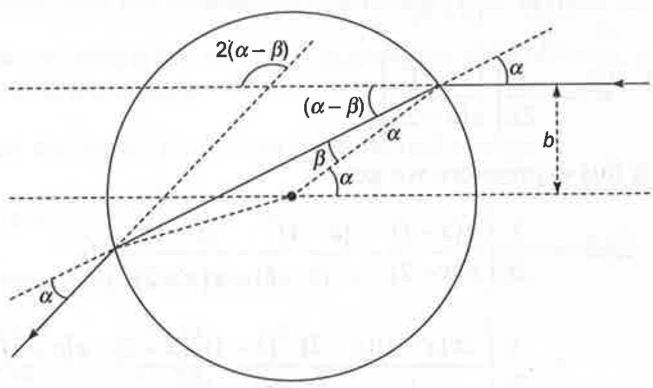

Fig. 7.5

From Snell's law, we have

$$\mu = \frac{\sin\alpha}{\sin\beta}$$

and given that $\mu = \sqrt{1+(V_0/E)}$

From the figure, we write the scattering angle as

$$\theta = 2(\alpha-\beta) \quad \text{or} \quad \beta = \alpha - \frac{\theta}{2}$$

Then, $$\sin\beta = \sin\left(\alpha - \frac{\theta}{2}\right) = \sin\alpha\cos\frac{\theta}{2} - \cos\alpha\sin\frac{\theta}{2}$$

$$\frac{\sin\alpha}{\mu} = \sin\alpha\cos\frac{\theta}{2} - \cos\alpha\sin\frac{\theta}{2} = \sin\alpha\left(\cos\frac{\theta}{2} - \cot\alpha\sin\frac{\theta}{2}\right)$$

or $$\left(\cos\frac{\theta}{2} - \cot\alpha\sin\frac{\theta}{2}\right) = \frac{1}{\mu} \qquad \text{(ii)}$$

Now, from the figure, we have

$$b = a\sin\alpha \ \text{ or } \ \sin\alpha = \frac{b}{a}$$

or $$1+\cot^2\alpha = \frac{1}{\sin^2\alpha} \ \text{ or, } \ \cot^2\alpha = \frac{1}{\sin^2\alpha} - 1 = \frac{a^2}{b^2} - 1$$

or $$\cot\alpha = \left(\frac{a^2}{b^2} - 1\right)^{1/2} \qquad \text{(iii)}$$

Now, from (ii), $$\cot\alpha = \frac{\cos(\theta/2) - (1/\mu)}{\sin(\theta/2)} \qquad \text{(iv)}$$

From equations (iii) and (iv)

$$\left(\frac{a^2}{b^2} - 1\right)^{1/2} = \frac{\cos(\theta/2) - (1/\mu)}{\sin(\theta/2)}$$

or $$\frac{a^2}{b^2} = \left[1 + \frac{\cos(\theta/2) - (1/\mu)}{\sin(\theta/2)}\right]^2$$

$$= \frac{\sin^2(\theta/2) + \cos^2(\theta/2) - (2/\mu)\cos(\theta/2) + \left(1/\mu^2\right)}{\sin^2(\theta/2)}$$

or $$b^2 = a^2\frac{\mu^2\sin^2(\theta/2)}{\mu^2 + 1 - 2\mu\cos(\theta/2)} \qquad \text{(v)}$$

Differentiating this expression with respect to θ, we get

$$2b\frac{db}{d\theta} = \mu^2 a^2 \frac{d}{d\theta}\left[\frac{\sin^2(\theta/2)}{\mu^2 + 1 - 2\mu\cos(\theta/2)}\right]$$

$$= \mu^2 a^2\left[\frac{2\sin(\theta/2)\cos(\theta/2)}{\mu^2 + 1 - 2\mu\cos(\theta/2)} - \frac{2\mu\sin^3(\theta/2)}{\left[\mu^2 + 1 - 2\mu\cos(\theta/2)\right]^2}\right]$$

This can be simplified to get

$$b\frac{db}{d\theta}=\frac{\mu^2a^2}{2}\frac{\sin(\theta/2)\left[\mu\cos(\theta/2)-1\right]}{\left[\mu^2+1-2\mu\cos(\theta/2)\right]^2}\left[\cos(\theta/2)-\mu\right] \quad \text{(vi)}$$

Now, the differential scattering cross section is given by; $\frac{d\sigma}{d\Omega}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$. Using (vi) in this expression, we get the differential scattering cross section as

$$\frac{d\sigma}{d\Omega}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|=\frac{\mu^2a^2}{2\sin\theta}\frac{\sin(\theta/2)\left[\mu\cos(\theta/2)-1\right]}{\left[\mu^2+1-2\mu\cos(\theta/2)\right]^2}\left[\mu-\cos(\theta/2)\right]$$

$$=\frac{\mu^2a^2}{4\cos(\theta/2)}\frac{\left[\mu\cos(\theta/2)-1\right]}{\left[\mu^2+1-2\mu\cos(\theta/2)\right]^2}\left[\mu-\cos(\theta/2)\right] \quad \text{(vii)}$$

The scattering angle θ varies from zero to a maximum value corresponding to $b=a$ so that $\sin\alpha=1$ and from equation (ii)

$$\cos\left(\frac{\theta_{\max}}{2}\right)=\frac{1}{\mu} \quad \text{(viii)}$$

The total scattering cross section is

$$\sigma=\int\frac{d\sigma}{d\Omega}d\Omega=2\pi\int_0^{\theta_{\max}}\frac{d\sigma}{d\Omega}\sin\theta d\theta$$

$$=2\pi\int_0^{\theta_{\max}}\frac{\mu^2a^2}{4\cos(\theta/2)}\frac{\left[\mu\cos(\theta/2)-1\right]}{\left[\mu^2+1-2\mu\cos(\theta/2)\right]^2}\left[\mu-\cos(\theta/2)\right]\sin\theta d\theta$$

$$=\pi\int_0^{\theta_{\max}}\mu^2a^2\sin(\theta/2)\frac{\left[\mu\cos(\theta/2)-1\right]}{\left[\mu^2+1-2\mu\cos(\theta/2)\right]^2}\left[\mu-\cos(\theta/2)\right]d\theta$$

We now put $x=\cos(\theta/2)$ and $dx=-\frac{1}{2}\sin(\theta/2)d\theta$, then

$$\sigma=2\pi\mu^2a^2\int_{1/\mu}^{1}\frac{(\mu x-1)(\mu-x)}{\left(\mu^2+1-2\mu x\right)^2}dx \quad \text{(ix)}$$

After the integration and applying the limits, we get the total scattering cross section as

$$\sigma=\pi a^2 \quad \text{(x)}$$

Thus, the total scattering cross section is equal to the geometrical cross section of the sphere. (To get the upper and lower limits of the integration, we have used equation viii.)

EXAMPLE 7.7 Two masses m_1 and m_2 are connected by a massless rod of length a. If the rod is placed along the z-axis with its centre of mass at the origin, obtain the principal moments of inertia.

Solution: The rod is placed along the z axis with its centre of mass at the origin. Therefore, for both masses $x = y = o$. As a result, the off diagonal elements of the moment of inertia tensor are zero, i.e. $I_{xy} = I_{yz} = I_{zx} = 0$.

Now, let the mass m_1 be at $z = z_1$ and the mass m_2 is at $z = -z_2$ so that

$$z_1 + z_2 = a \tag{i}$$

Since the centre of mass of the system is at the origin, we have

$$m_1 z_1 = m_2 z_2 \tag{ii}$$

Then, using (i) in (ii), we can write

$$\frac{m_1 + m_2}{m_2} = \frac{a}{z_1}, \quad \text{or, } z_1 = \frac{m_2 a}{m_1 + m_2} \tag{iii}$$

Similarly, $$z_2 = \frac{m_1 a}{m_1 + m_2} \tag{iv}$$

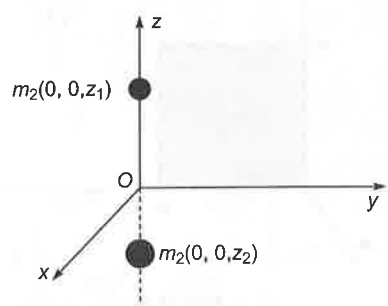

Fig. 7.6

Now, the principal moment of inertia I_{xx} is given by

$$I_1 = \sum_i m_i \left(y_i^2 + z_i^2\right) = \sum_i m_i z_i^2 = m_1 z_1^2 + m_2 z_2^2, \text{ since } y = 0.$$

Therefore, $$I_1 = m_1 \left(\frac{m_2 a}{m_1 + m_2}\right)^2 + m_2 \left(\frac{m_1 a}{m_1 + m_2}\right)^2$$

$$= \frac{(m_1 + m_2) m_1 m_2 a^2}{(m_1 + m_2)^2} = \frac{m_1 m_2 a^2}{m_1 + m_2} \tag{v}$$

Similarly, we have

$$I_2 = \sum_i m_i \left(x_i^2 + z_i^2\right) = \sum_i m_i z_i^2 = m_1 z_1^2 + m_2 z_2^2 = \frac{m_1 m_2 a^2}{m_1 + m_2} \tag{vi}$$

Finally, $$I_3 = \sum_i m_i \left(x_i^2 + y_i^2\right) = 0 \qquad \text{(vii)}$$

EXAMPLE 7.8 A square plate of side a and mass M is shown in Figure 7.7. Assume the plate has uniform density. Find the moment of inertia tensor. Also find the principal moments on inertia and the directions of the principal axes.

Solution: From the figure we see that $x = y = a$ and $z = 0$. The principal moment of inertia I_{xx} can be determined as

$$I_{xx} = \int \sigma\left(y^2 + z^2\right)dxdy = \sigma \int_0^a y^2 dy \int_0^a dx$$

where $\sigma = \dfrac{M}{a^2}$, the mass per unit area of the plate. On integration, the above expression gives

$$I_{xx} = \sigma \frac{a^4}{3} = \frac{Ma^2}{3} \qquad \text{(i)}$$

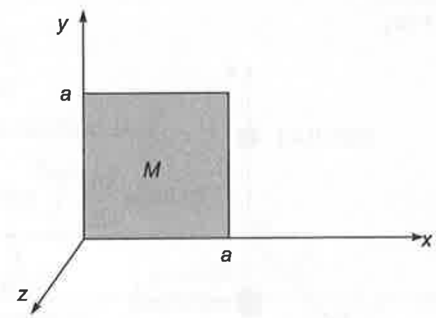

Fig. 7.7

Similarly, we get

$$I_{yy} = \sigma \frac{a^4}{3} = \frac{Ma^2}{3} \qquad \text{(ii)}$$

Also, $$I_{zz} = \sigma \int \left(x^2 + y^2\right)dxdy = \sigma \left[\int_0^a x^2 dx + \int_0^a y^2 dy\right]$$

$$= \sigma \frac{a^4}{3} + \sigma \frac{a^4}{3} = 2\sigma \frac{a^4}{3}$$

$$= \frac{2}{3} Ma^2 \qquad \text{(iii)}$$

Finally, $$I_{xy} = -\sigma \int_0^a x dx \int_0^a y dy$$

$$= -\sigma\frac{a^4}{4} = -\frac{Ma^4}{4} \tag{iv}$$

Equations (i) through (iv) give the components of the moment of inertia tensor. Therefore, the moment of inertia tensor is given by

$$I = \begin{bmatrix} \frac{Ma^2}{3} & -\frac{Ma^2}{4} & 0 \\ -\frac{Ma^2}{4} & \frac{Ma^2}{3} & 0 \\ 0 & 0 & \frac{2Ma^2}{3} \end{bmatrix} \tag{v}$$

Now, we determine the principal moments of inertia by diagonalising the moment of inertia tensor by evaluating the determinant $|I - \lambda| = 0$. That is,

$$\begin{vmatrix} \frac{Ma^2}{3} - \lambda & -\frac{Ma^2}{4} & 0 \\ -\frac{Ma^2}{4} & \frac{Ma^2}{3} - \lambda & 0 \\ 0 & 0 & \frac{2Ma^2}{3} - \lambda \end{vmatrix} = 0 \tag{vi}$$

or $$\left(\frac{2Ma^2}{3} - \lambda\right)\begin{vmatrix} \frac{Ma^2}{3} - \lambda & -\frac{Ma^2}{4} \\ -\frac{Ma^2}{4} & \frac{Ma^2}{3} - \lambda \end{vmatrix} = 0$$

Adding the second row to the first row, and replacing the first row with the sum, we get

$$\left(\frac{2Ma^2}{3} - \lambda\right)\begin{vmatrix} \frac{Ma^2}{12} - \lambda & \frac{Ma^2}{12} - \lambda \\ -\frac{Ma^2}{4} & \frac{Ma^2}{3} - \lambda \end{vmatrix} = 0$$

that is, $$\left(\frac{Ma^2}{12} - \lambda\right)\left(\frac{2Ma^2}{3} - \lambda\right)\begin{vmatrix} 1 & 1 \\ -\frac{Ma^2}{4} & \frac{Ma^2}{3} - \lambda \end{vmatrix} = 0$$

or $$\left(\frac{Ma^2}{12} - \lambda\right)\left(\frac{2Ma^2}{3} - \lambda\right)\left(\frac{7Ma^2}{12} - \lambda\right) = 0 \tag{vii}$$

Therefore, the principal moments of inertia are

$$I_1 = \frac{1}{12}Ma^2\,,\ I_2 = \frac{2}{3}Ma^2 \text{ and } I_3 = \frac{7}{12}Ma^2 \tag{viii}$$

Now, for $I_1 = \frac{1}{12}Ma^2$, the characteristic equations are

$$\left(\frac{1}{3}Ma^2 - \frac{1}{12}Ma^2\right)x - \frac{1}{4}Ma^2 y = 0$$

$$-\frac{1}{4}Ma^2 x + \left(\frac{1}{3}Ma^2 - \frac{1}{12}Ma^2\right)y = 0$$

and $$\left(\frac{2}{3}Ma^2 - \frac{1}{2}Ma^2\right)z = 0$$

Solving these three equations, we see that; $x = y$ and $x = 0$. Therefore, $(\hat{i} + \hat{j})$ gives the direction of one of the principal axes.

Then for $I_2 = \frac{2}{3}Ma^2$, the characteristic equations are

$$\left(\frac{1}{3}Ma^2 - \frac{2}{3}Ma^2\right)x - \frac{1}{4}Ma^2 y = 0$$

and $$-\frac{1}{4}Ma^2 x + \left(\frac{1}{3}Ma^2 - \frac{2}{3}Ma^2\right)y = 0$$

These two equations have only a single possible solution $x = y = 0$ and x_3 is arbitrary. Thus, the direction of the other principal axis is $\hat{k}$.

Then for $I_3 = \frac{7}{12}Ma^2$, the characteristic equations are

$$\left(\frac{1}{3}Ma^2 - \frac{7}{12}Ma^2\right)x - \frac{1}{4}Ma^2 y = 0$$

$$-\frac{1}{4}Ma^2 x + \left(\frac{1}{3}Ma^2 - \frac{7}{12}Ma^2\right)y = 0$$

and $$\left(\frac{2}{3}Ma^2 - \frac{7}{2}Ma^2\right)z = 0$$

Solving these three equations, we get

$$y = -x \text{ and } z = 0$$

Therefore, the direction of the third principal axis is $(\hat{i} - \hat{j})$.

EXAMPLE 7.9 Find the moment of inertia tensor and the principal moments of inertia of a solid cube of side a, mass M and uniform density ρ about a vertex.

Solution: First we determine the elements of the moment of inertia tensor. We have the general expression for the elements of the moment of inertia tensor as

$$I_{ij} = \int_V \rho\left(r^2\delta_{ij} - r_i r_j\right)dV \tag{i}$$

Then,
$$I_{xx} = \rho\int\left[\left(x^2+y^2+z^2\right)-x^2\right]dxdydz$$

$$= \rho\int\left(y^2+z^2\right)dxdydz = \rho\int_0^a y^2dy\int_0^a dx\int_0^a dz + \rho\int_0^a z^2dz\int_0^a dx\int_0^a dy$$

$$= \rho\frac{a^3}{3}a^2 + \rho\frac{a^3}{3}a^2 = 2\rho\frac{a^5}{3}$$

$$= \frac{2}{3}Ma^2 \tag{ii}$$

Similarly,
$$I_{yy} = I_{zz} = \frac{2}{3}Ma^2 \tag{iii}$$

Then,
$$I_{xy} = -\int \rho xydxdydz = -\rho\int_0^a xdx\int_0^a ydy\int_0^a dz$$

$$= -\rho.\frac{a^2}{2}.\frac{a^2}{2}.a = -\rho\frac{a^5}{4} = -\frac{1}{4}Ma^2 \tag{iv}$$

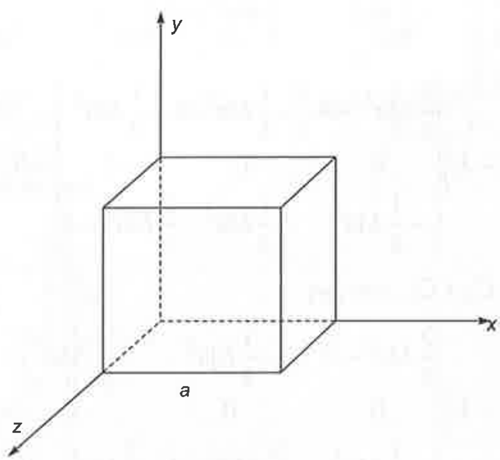

Fig. 7.8

In a similar way we can show that the other off diagonal elements are also equal to $-\frac{1}{4}Ma^2$. Therefore, the moment of inertia tensor is

$$I=\begin{bmatrix} \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 \end{bmatrix} \tag{v}$$

To find the principal moments of inertia we diagonalise the above matrix using the determinant equation

$$\begin{vmatrix} \frac{2}{3}Ma^2-\lambda & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2-\lambda & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2-\lambda \end{vmatrix}=0 \tag{vi}$$

Now rewriting the second row $R_2 \to R_2 + R_3$, we get

$$\begin{vmatrix} \frac{2}{3}Ma^2-\lambda & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ 0 & \frac{11}{12}Ma^2-\lambda & -\frac{11}{12}Ma^2+\lambda \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2-\lambda \end{vmatrix}=0$$

or
$$\left(\frac{11}{12}Ma^2-\lambda\right)\begin{vmatrix} \frac{2}{3}Ma^2-\lambda & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ 0 & 1 & -1 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2-\lambda \end{vmatrix}=0$$

Now, writing $C_2 \to C_2 + C_3$, we get

$$\left(\frac{11}{12}Ma^2-\lambda\right)\begin{vmatrix} \frac{2}{3}Ma^2-\lambda & -\frac{1}{2}Ma^2 & -\frac{1}{4}Ma^2 \\ 0 & 0 & -1 \\ -\frac{1}{4}Ma^2 & \frac{5}{12}Ma^2-\lambda & \frac{2}{3}Ma^2-\lambda \end{vmatrix}=0$$

or
$$\left(\frac{11}{12}Ma^2-\lambda\right)\begin{vmatrix} \frac{2}{3}Ma^2-\lambda & -\frac{1}{2}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{5}{12}Ma^2-\lambda \end{vmatrix}=0$$

Again, writing $C_1 \to C_1 + C_2$, we get;

$$\left(\frac{11}{12}Ma^2 - \lambda\right)\begin{vmatrix} \frac{1}{6}Ma^2 - \lambda & -\frac{1}{2}Ma^2 \\ \frac{1}{6}Ma^2 - \lambda & \frac{5}{12}Ma^2 - \lambda \end{vmatrix} = 0$$

or
$$\left(\frac{1}{6}Ma^2 - \lambda\right)\left(\frac{11}{12}Ma^2 - \lambda\right)\begin{vmatrix} 1 & -\frac{1}{2}Ma^2 \\ 1 & \frac{5}{12}Ma^2 - \lambda \end{vmatrix} = 0$$

that is,
$$\left(\frac{1}{6}Ma^2 - \lambda\right)\left(\frac{11}{12}Ma^2 - \lambda\right)\left(\frac{11}{12}Ma^2 - \lambda\right) = 0 \qquad \text{(vii)}$$

Therefore, the principal moments of inertia are

$$I_1 = \frac{1}{6}Ma^2 \;,\; I_2 = \frac{11}{12}Ma^2 \;\text{ and }\; I_3 = \frac{11}{12}Ma^2 \qquad \text{(viii)}$$

To find the direction of the principal axes, we find the eigenvectors of the characteristic equation given in equation (vi). With $\lambda = I_1 = \frac{1}{6}Ma^2$

$$\begin{bmatrix} \frac{2}{3}Ma^2 - \frac{1}{6}Ma^2 & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 - \frac{1}{6}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 - \frac{1}{6}Ma^2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

This can be simplified to get

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

Multiplying by 4, we get

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

Then, the characteristic equations are

$$2\alpha-\beta-\gamma=0,\ -\alpha+2\beta-\gamma=0 \ \text{ and } -\alpha-\beta+2\gamma=0 \tag{ix}$$

Solving these equations, we see that $\alpha=\beta=\gamma=1$. Therefore, the eigen vector is

$$\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix} \tag{x}$$

Clearly, this represents a principal axis corresponding to one of the diagonals of the cube. Since $I_2=I_3$, the other two principal axes are any two mutually perpendicular axes and also perpendicular to the first principal axis.

EXAMPLE 7.10 A thin disc of mass M and radius R is placed in the $x-y$ plane. A point mass $m=\frac{5}{4}M$ is attached to its edge as shown in Figure 7.9. Find the moment of inertia tensor of the combination of the disc and the point mass about the point A. Also determine the principal moments of inertia and the principal axes about the point A. Given that the moment of inertia of the disc about its centre of mass is

$$I_d=\frac{MR^2}{4}\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$$

Solution: We have the components of the moment of inertia tensor given by the equation

$$I_{ij}=\int_V \rho\left(r^2\delta_{ij}-r_ir_j\right)dV \tag{i}$$

Then, the moment of inertia of the point mass about the point A can be obtained as

$$I_m=\frac{5MR^2}{4}\begin{bmatrix}1&-1&0\\-1&1&0\\0&0&2\end{bmatrix} \tag{ii}$$

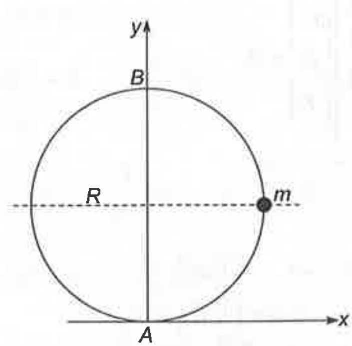

Fig. 7.9

The moment of inertia of the disc about the centre of mass is given. Therefore, we can obtain its moment of inertia about a parallel axis passing through the point A as

$$I'_d = \frac{MR^2}{4}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{iii}$$

Since the moment of inertia of a system of particles is equal to the sum of the moments of inertia of the particles constituting the system, we can find the moment of inertia of the disc and the point mass is equal to the sum of the corresponding moments of inertia. Therefore,

$$I = \frac{5MR^2}{4}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \frac{MR^2}{4}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \frac{MR^2}{4}\begin{bmatrix} 10 & -5 & 0 \\ -5 & 6 & 0 \\ 0 & 0 & 16 \end{bmatrix} \tag{iv}$$

Now, we find the principal moments of inertia by solving the secular equation;

$$\frac{MR^2}{4}\begin{vmatrix} 10-\lambda & -5 & 0 \\ -5 & 6-\lambda & 0 \\ 0 & 0 & 16-\lambda \end{vmatrix} = 0$$

or $$(16-\lambda)\left(\lambda^2 - 16\lambda + 35\right) = 0 \tag{v}$$

The roots of this equation are

$$\lambda_1 = 16,\ \lambda_2 = 8-\sqrt{29} \text{ and } \lambda_3 = 8+\sqrt{29} \tag{vi}$$

Therefore, the three principal moments of inertia are

$$I_1 = 4MR^2,\ I_2 = \left(2-\frac{\sqrt{29}}{4}\right)MR^2 \text{ and } I_3 = \left(2+\frac{\sqrt{29}}{4}\right)MR^2 \tag{vii}$$

Now, the eigenvectors or the direction cosines corresponding to $I_1 = 4MR^2$, (α,β,γ) can be obtained by solving the matrix equation

$$\frac{MR^2}{4}\begin{bmatrix} 10-16 & -5 & 0 \\ -5 & 6-16 & 0 \\ 0 & 0 & 16-16 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

or
$$\begin{bmatrix} -6 & -5 & 0 \\ -5 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

that is, $-6\alpha - 5\beta = 0$

and $-5\alpha - 10\beta = 0$

Note that all the coefficients of the third equation are zero.
This shows that the only possible solutions are

$$\alpha = \beta = 0 \text{ and } \gamma \text{ is arbitrary.} \tag{viii}$$

Now, from the definition of the direction cosines, we have

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{ix}$$

Therefore, from (viii) we see that; $\gamma = 1$.

For the principal moments of inertia I_2 and I_3, the eigenvectors can be determined from

$$\frac{MR^2}{4}\begin{bmatrix} 2 \pm \sqrt{29} & -5 & 0 \\ -5 & -2 \pm \sqrt{29} & 0 \\ 0 & 0 & 8 \pm \sqrt{29} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

Then the characteristic equations are

$$\left(2 \pm \sqrt{29}\right)\alpha - 5\beta = 0$$

$$-5\alpha - \left(2 \pm \sqrt{29}\right)\beta = 0$$

and $\left(8 \pm \sqrt{29}\right)\gamma = 0$

This immediately follows that $\gamma = 0$

From, $-5\alpha - \left(2 \pm \sqrt{29}\right)\beta = 0$, we get

$$\frac{\alpha}{\beta} = -\frac{\left(2 \pm \sqrt{29}\right)}{5} = 0.677 \text{ or } -1.477 \tag{x}$$

Now, from (ix), since $\gamma = 0$, we have $\alpha^2 + \beta^2 = 1$

or
$$1 + \frac{\beta^2}{\alpha^2} = \frac{1}{\alpha^2}, \text{ or } \alpha = \left(1 + \frac{\beta^2}{\alpha^2}\right)^{-\frac{1}{2}} = 0.828 \text{ or } 0.561 \tag{xi}$$

$$\beta = \left(1 - \alpha^2\right)^{\frac{1}{2}} = 0.561 \text{ and } 0.828 \tag{xii}$$

Since the principal axes for I_2 and I_3 are orthogonal, the condition

$$\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 = 0 \qquad \text{(xiii)}$$

must be satisfied. Therefore, the principal axes are given by; $(0,0,1)$, $(0.561, 0.828, 0)$ and $(-0.828, 0.561, 0)$ (xiv)

These are the required results.

EXAMPLE 7.11 Four equal masses m each lie in the $xy-$plane at positions $(a,0),(-a,0),(0,2a)$ and $(0,-2a)$. Find the moment of inertia tensor. Also find the moment of inertia of rotation about an axis in the direction of $\hat{n}$ that lies *equally* between the axes in the first quadrant.

Solution: We start with the general expression for the elements of moment of inertia tensor of a system of particles. It is given by

$$I_{ij} = \sum_n m_n \left(r_n^2 \delta_{ij} - x_{n_i} x_{n_j}\right) \qquad \text{(i)}$$

where $r_n^2 = x_{n_1}^2 + x_{n_2}^2 + x_{n_3}^2$. In the present problem, there are four masses. For all the four masses, one of the coordinates is zero and hence $I_{ij} = 0$ for all $i \neq j$, that is, all the off diagonal elements are zero. The diagonal elements can be determined as follows:

$$I_{11} = \left(m_1 r_1^2 - x_1^2\right) + \left(m_2 r_2^2 - x_2^2\right) + \left(m_3 r_3^2 - x_3^2\right) + \left(m_4 r_4^2 - x_4^2\right)$$

$$= m\left(a^2 - a^2\right) + m\left(a^2 - a^2\right) + m\left(4a^2 - 0\right) + m\left(4a^2 - 0\right)$$

$$= 8ma^2$$

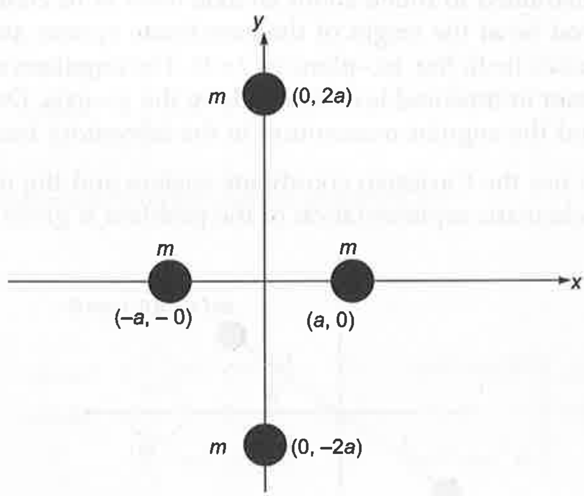

Fig. 7.10

Similarly, we can have

$$I_{22} = 2ma^2 \text{ and } I_{33} = 10ma^2$$

Therefore, the moment of inertia tensor is given by

$$I = \begin{bmatrix} 8ma^2 & 0 & 0 \\ 0 & 2ma^2 & 0 \\ 0 & 0 & 10ma^2 \end{bmatrix} \tag{ii}$$

For the second part of the problem, the given direction makes equal angle with the positive axes. Therefore, the direction cosines (α, β, γ) are equal. Then, the moment of inertia about this axis is given by

$$I = I_{11}\alpha^2 + I_{22}\beta^2 + I_{33}\gamma^2 - 2I_{12}\alpha\beta - 2I_{23}\beta\gamma - 2I_{31}\gamma\alpha$$

$$= 8ma^2\alpha^2 + 2ma^2\beta^2 + 10ma^2\gamma^2 = 20ma^2\alpha^2 \tag{iii}$$

Also, the direction cosines satisfy the equation

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{iv}$$

In the present problem, it becomes

$$3\alpha^2 = 1 \text{ or, } \alpha^2 = \frac{1}{3} \tag{v}$$

Using this in (iii), we get

$$I = \frac{20}{3}ma^2 \tag{vi}$$

EXAMPLE 7.12 Two equal point masses m are connected by a rigid, massless rod of length $2l$. It is constrained to rotate about an axle fixed at its centre at an angle θ. Let the centre of the rod be at the origin of the coordinate system and the axle along the z-axis and the masses lie in the xz-plane at $t = 0$. The angular velocity of the rotation of the axle is constant in time and is directed along the z-axis. Determine the moment of inertia tensor and the angular momentum in the laboratory frame of reference.

Solution: Here we use the Cartesian coordinate system and the masses are initially in the xz-plane. A schematic representation of the problem is given in Figure 7.12.

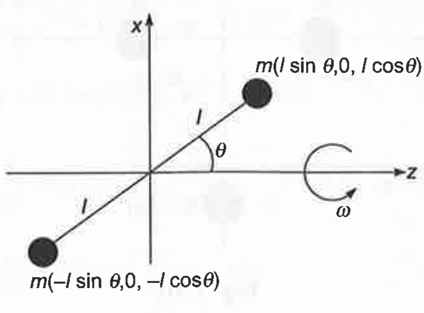

Fig. 711

The coordinates of the masses are given in the figure. We can determine the components of the moment of inertia tensor about the origin using the general expression

$$I_{ij} = \sum_n m_n \left(r_n^2 \delta_{ij} - x_{n_i} x_{n_j} \right) \tag{i}$$

Proceeding as in the case of the previous problem, we get

$$I_{xx} = 2ml^2 \cos^2\theta,\ I_{xy} = 0,\ I_{xz} = -2ml^2 \cos\theta \sin\theta$$

$$I_{yx} = 0,\ I_{yy} = 2ml^2,\ I_{yz} = 0$$

and $I_{zx} = -2ml^2 \cos\theta \sin\theta,\ I_{zy} = 0,\ I_{zz} = -ml^2 \sin^2\theta$

Then the moment of inertia tensor can be written as

$$I = \begin{bmatrix} 2ml^2\cos^2\theta & 0 & -ml^2 \sin 2\theta \\ 0 & 2ml^2 & 0 \\ -ml^2 \sin 2\theta & 0 & 2ml^2 \sin^2\theta \end{bmatrix} \tag{ii}$$

In the second part of the problem, we assume a laboratory frame $x'y'z'$ such that the z'-axis coincides with the z-axis of the rotating coordinate system that we considered in the first part at $t = 0$. Then, the transformation can be written as

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} \tag{iii}$$

where, $C = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the transformation matrix. Now, the moment of inertia tensor in the laboratory frame can be obtained as

$$I' = C^T I C \tag{iv}$$

Therefore,

$$I' = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2ml^2\cos^2\theta & 0 & -ml^2 \sin 2\theta \\ 0 & 2ml^2 & 0 \\ -ml^2 \sin 2\theta & 0 & 2ml^2 \sin^2\theta \end{bmatrix} \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This expression can be simplified to get

$$I' = \begin{bmatrix} 2ml^2\left(\cos^2\theta\cos^2\omega t + \sin^2\omega t\right) & ml^2\sin^2\theta\sin 2\omega t & -ml^2\sin 2\theta\cos\omega t \\ ml^2\sin^2\theta\sin 2\omega t & 2ml^2\left(\cos^2\theta\sin^2\omega t + \cos^2\omega t\right) & -ml^2\sin 2\theta\sin\omega t \\ ml^2\sin 2\theta\cos\omega t & ml^2\sin 2\theta\sin\omega t & 2ml^2\sin^2\theta \end{bmatrix}$$

Now, the angular momentum in the laboratory frame is

$$L = I'\omega = I'\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \tag{v}$$

Substituting for I' and simplifying, we get

$$L = ml^2\omega\begin{bmatrix} -\sin 2\theta\cos\omega t \\ \sin 2\theta\sin\omega t \\ 2\sin^2\theta \end{bmatrix} \tag{vi}$$

In writing the matrix elements of the angular velocity, we considered the fact that the angular velocity is directed in the z-direction and has a magnitude ω. The other two components of the angular velocity are zero.

EXAMPLE 7.13 Obtain the equation of motion of a particle in a rotating coordinate system.

Solution: We consider two coordinate systems: xyz which is fixed in space and $x'y'z'$ fixed in the body. Initially, at $t = 0$, we assume that their origins and base vectors coincide with each other. The body fixed coordinate system can rotate with respect to the space fixed coordinate system.

Now, consider a point P inside the body. Let (x, y, z) be the coordinates of the point in the unprimed coordinate system and (x', y', z'), the coordinates in the primed coordinate system. Initially, at $t = 0$ since the origins are coinciding, we have the position vector of the point as

$$r = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{(in the space fixed coordinate system)}$$

and

$$r = x'\hat{i}' + y'\hat{j}' + z'\hat{k}' \quad \text{(in the body fixed coordinate system)}$$

Note that since the unprimed coordinates are fixed in space the corresponding unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ are also fixed in space; whereas the unit vectors in the primed coordinate system $\hat{i}', \hat{j}'$ and $\hat{k}'$ will change their direction with time as the body rotates.

To obtain the equation of motion in the rotating coordinate system, we first find the rate of change of the position vector as the axis rotates, then relation between the acceleration in the two coordinate systems and finally the equation of motion.

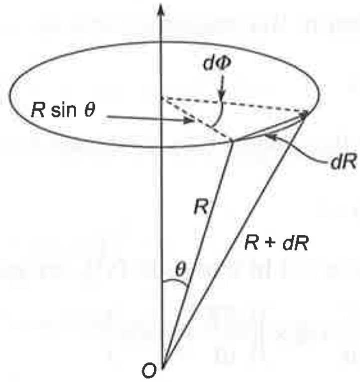

Fig. 7.12

From the geometry of Figure 7.12, we have

$$dR = R\sin\theta d\Phi = d\vec{\Phi}\times\vec{R}$$

Then,
$$\frac{dR}{dt} = \frac{d\vec{\Phi}}{dt}\times\vec{R} = \vec{\omega}\times\vec{R} \tag{i}$$

where, ω is the angular velocity. This expression gives the rate of change of a vector due to the rotation of the coordinate system. This can be expressed in the form of an operator $\frac{d}{dt} = \vec{\omega}\times$, and can be operated on any vector. For example, we can write

$$\frac{d\hat{i}'}{dt} = \vec{\omega}\times\hat{i}' \tag{ii}$$

Then we can write the rate of change of a vector $\vec{r}$ when the primed coordinate system is rotating with an angular velocity ω, with respect to the unprimed coordinate system. That is,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x'\hat{i} + y'\hat{j} + z'\hat{k}\right)$$

$$= \dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}' + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

$$= \dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}' + x'\left(\vec{\omega}\times\hat{i}'\right) + y'\left(\vec{\omega}\times\hat{j}'\right) + z'\left(\vec{\omega}\times\hat{k}'\right)$$

$$= \dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}' + (\vec{\omega}\times\vec{r}) = \frac{d\vec{r}'}{dr} + (\vec{\omega}\times\vec{r}) \tag{iii}$$

This can be written as

$$\left(\frac{dr}{dt}\right)_{\text{unprimed}} = \left(\frac{dr}{dt}\right)_{\text{primed}} + (\vec{\omega}\times\vec{r}) \tag{iv}$$

Equation (iv) can be written in the operator form as

$$\frac{d}{dt}=\frac{d'}{dt}+\vec{\omega}\times \tag{v}$$

Applying this operator to the vector $\vec{r}$ in the space fixed coordinate system. That is,

$$\frac{d\vec{r}}{dt}=\frac{d'\vec{r}}{dt}+\vec{\omega}\times\vec{r} \tag{vi}$$

Again applying the operator (v) to equation (vi), we get

$$\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)=\left(\frac{d'}{dt}+\vec{\omega}\times\right)\left(\frac{d'\vec{r}}{dt}+\vec{\omega}\times\vec{r}\right)$$

or
$$\frac{d^2r}{dt^2}=\frac{d'^2r}{dt}+\frac{d'}{dt}(\vec{\omega}\times\vec{r})+\vec{\omega}\times\frac{d'r}{dt}+\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$=\frac{d'^2r}{dt}+\frac{d'\vec{\omega}}{dt}\times\vec{r}+\vec{\omega}\times\frac{d'r}{dt}+\vec{\omega}\times\frac{d'r}{dt}+\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$=\frac{d'^2r}{dt}+\frac{d'\vec{\omega}}{dt}\times\vec{r}+2\vec{\omega}\times\frac{d'r}{dt}+\vec{\omega}\times(\vec{\omega}\times\vec{r}) \tag{vii}$$

This expression can be written as

$$\vec{a}=\vec{a}'+\frac{d'\vec{\omega}}{dt}\times\vec{r}+2\vec{\omega}\times\frac{d'r}{dt}+\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

or
$$\vec{a}'=\vec{a}-2\vec{\omega}\times\frac{d'r}{dt}-\vec{\omega}\times(\vec{\omega}\times\vec{r})-\frac{d'\vec{\omega}}{dt}\times\vec{r} \tag{viii}$$

This expression gives the relation between the acceleration in the space fixed (unprimed) and body fixed (primed) coordinate systems.

If the particle or body has a mass m, then the equation of motion is

$$m\vec{a}'=m\vec{a}-2m\vec{\omega}\times\frac{d'r}{dt}-m\vec{\omega}\times(\vec{\omega}\times\vec{r})-m\frac{d'\vec{\omega}}{dt}\times\vec{r} \tag{ix}$$

In this expression, $m\vec{a}'$ is the force in the primed system, $m\vec{a}$ is the force in the unprimed system, $2m\vec{\omega}\times\frac{d'r}{dt}$ is known as the Corioli's force, $m\vec{\omega}\times(\vec{\omega}\times\vec{r})$ is the centripetal force and $m\frac{d'\vec{\omega}}{dt}\times\vec{r}$ is a term which is significant only if there is change in the angular velocity.

EXAMPLE 7.14 Obtain the equation of motion in the rotating frame of reference by using the Lagrangian formulation.

Solution: The Lagrangian of a particle of mass m in an inertial frame of reference is given by

$$L=T-V=\frac{1}{2}mv^2-V \tag{i}$$

The velocities in the inertial and a rotating frame of reference are related through

$$\frac{dr}{dt}=\frac{d'r}{dt}+(\omega\times r) \quad \text{or,} \quad v=v'+(\omega\times r) \tag{ii}$$

where, ω is the angular velocity of the rotating frame.
Then equation (i) becomes

$$L=\frac{1}{2}m\left[v'+(\omega\times r)\right]^2-V$$

$$=\frac{1}{2}mv'^2+mv'(\omega\times r)+\frac{1}{2}m(\omega\times r).(\omega\times r)-V \tag{iii}$$

Now, $$\frac{\partial L}{\partial \dot{r}}=\frac{\partial L}{\partial v'}=m\left[v'+(\omega\times r)\right] \tag{iv}$$

Further, $$\frac{\partial L}{\partial r}=m(v'\times\omega)+m\omega\times(\omega\times r)-\nabla V \tag{v}$$

The Lagrange's equation of motion is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0$. Using (iv) and (v) this can be written as

$$m\frac{d}{dt}\left[v'+(\omega\times r)\right]-\left[m(v'\times\omega)+m\omega\times(\omega\times r)-\nabla V\right]=0$$

or $$m\frac{dv'}{dt}+2m(\omega\times v')+m(\omega\times\omega\times r)+m(\dot{\omega}\times r)+\nabla V=0$$

or $$m\frac{dv'}{dt}=-\nabla V-2m(\omega\times v')-m(\omega\times\omega\times r)-m(\dot{\omega}\times r)$$

This can be written as

$$m\frac{dv'}{dt}=m\frac{dv}{dt}-2m(\omega\times v')-m(\omega\times\omega\times r)-m(\dot{\omega}\times r) \tag{vi}$$

where, $m\frac{dv}{dt}=-\nabla V$ the external force acting on the particle. Equation (vi) is the equation of motion in a rotating frame of reference.

EXAMPLE 7.15 A particle of mass m is thrown vertically downwards from a height H with a velocity u. Obtain an expression for the shift from the vertical due to Corioli's force as a function of time.

Solution: First we will obtain the equations of motion of a particle of mass m near the surface of the earth. Let (x',y',z') be the coordinates of the particle at any instant of time with respect to a rotating frame of reference. Also let $(\hat{i}',\hat{j}',\hat{k}')$ be the unit vectors along these axes taken in the same order. The equation of the motion of the particle in the presence of gravitational and Corioli's force can be written as

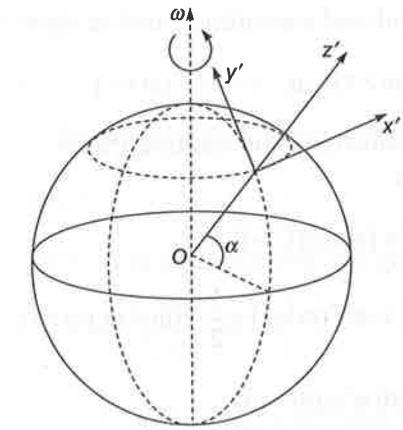

Fig. 7.13

$$m\ddot{r} = -mg\hat{k}' - 2m(\omega' \times \dot{r}) \tag{i}$$

with $\quad r = x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$. Then equation (i) can be written as

$$m\left(\ddot{x}'\hat{i}' + \ddot{y}'\hat{j}' + \ddot{z}'\hat{k}'\right) = -mg\hat{k}' - 2m\left(\omega_x'\hat{i}' + \omega_y'\hat{j}' + \omega_z'\hat{k}'\right) \times \left(\dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}'\right)$$

This can be simplified to get

$$m\left(\ddot{x}'\hat{i}' + \ddot{y}'\hat{j}' + \ddot{z}'\hat{k}'\right) = -mg\hat{k}' - 2m\left[\hat{i}'\left(\omega_y'\dot{z}' - \omega_z'\dot{y}'\right) - \hat{j}'\left(\omega_x'\dot{z}' - \omega_z'\dot{x}'\right) + \hat{k}'\left(\omega_x'\dot{y}' - \omega_y'\dot{x}'\right)\right]$$

Now we put, $\omega_x' = 0 \quad \omega_y' = \omega\cos\alpha$ and $\omega_z' = \omega\sin\alpha$ in the above expression to get

$$m\left(\frac{d^2x'}{dt^2}\hat{i}' + \frac{d^2y'}{dt^2}\hat{j}' + \frac{d^2z'}{dt^2}\hat{k}'\right) = -mg\hat{k}' - 2m\omega\begin{bmatrix}\hat{i}'\left(\cos\alpha\dfrac{dz'}{dt} - \sin\alpha\dfrac{dy'}{dt}\right) \\ +\hat{j}'\sin\alpha\dfrac{dx'}{dt} - \hat{k}'\cos\alpha\dfrac{dx'}{dt}\end{bmatrix} \tag{ii}$$

Then, the equations of motion are

$$m\frac{d^2x'}{dt^2} = -2m\omega\left(\cos\alpha\frac{dz'}{dt} - \sin\alpha\frac{dy'}{dt}\right) \tag{iii}$$

$$m\frac{d^2y'}{dt^2} = -2m\omega\sin\alpha\frac{dx'}{dt} \tag{iv}$$

$$m\frac{d^2z'}{dt^2} = -mg + 2m\omega\cos\alpha\frac{dx'}{dt} \tag{v}$$

Equations (iii) to (v) are the equations of motion.

In the present problem, the motion of the particle is along the vertical, that is, in the z-direction. So we take the Corioli's force along the other directions as zero. That is, $\frac{dx'}{dt} = \frac{dy'}{dt} = 0$. Then the equations of motion (iii), (iv) and (v) become

$$\frac{d^2x'}{dt^2} = -2\omega\cos\alpha\frac{dz'}{dt} \tag{vi}$$

$$\frac{d^2y'}{dt^2} = 0 \tag{vii}$$

$$\frac{d^2z'}{dt^2} = -g \tag{viii}$$

Integrating equation (viii), we get

$$\frac{dz'}{dt} = -gt + C$$

where, C is the constant of integration and can be determined from the initial conditions: at $t = 0$, $x' = y' = 0$ and $\frac{dx'}{dt} = \frac{dy'}{dt} = 0$. Also $z = H$ and, $\frac{dz'}{dt} = -u$. Using these conditions, we get $C = -u$. Therefore,

$$\frac{dz'}{dt} = -gt - u \tag{ix}$$

Substituting this in (vi), we get

$$\frac{d^2x'}{dt^2} = 2\omega(u + gt)\cos\alpha \tag{x}$$

On integration, the above equation gives

$$\frac{dx'}{dt} = 2\omega\left(ut + \frac{1}{2}gt^2\right)\cos\alpha \tag{xi}$$

From the initial condition we see that the integration constant is zero. Again integrating equation (xi), we get

$$x' = \frac{1}{3}\omega g t^3 \cos\alpha + \omega u t^2 \cos\alpha \tag{xii}$$

Equation (xii) is the required result.

EXAMPLE 7.16 A particle of mass m is thrown vertically upwards with a velocity u in the northern hemisphere at a latitude α. Determine the deviation of the particle from the vertical when it returns. The variation in the gravity may be neglected.

Solution: We start with the equations of motion of the particle that we have obtained in the previous problem. They are

$$\frac{d^2x'}{dt^2} = -2\omega\cos\alpha\frac{dz'}{dt} \tag{i}$$

$$\frac{d^2y'}{dt^2} = 0 \tag{ii}$$

$$\frac{d^2z'}{dt^2} = -g \tag{iii}$$

Further, the initial conditions are; $t = 0,\ x' = y' = 0,\ \frac{dx'}{dt} = \frac{dy'}{dt} = 0$ and $z' = 0,\ \frac{dz'}{dt} = 0$. Integrating (iii), we get

$$\frac{dz'}{dt} = -gt + C$$

where, C is the constant of integration. From the initial conditions we see that $C = u$. Then,

$$\frac{dz'}{dt} = -gt + u \tag{iv}$$

This can be substituted in equation (i) to get

$$\frac{d^2x'}{dt^2} = 2\omega\cos\alpha(gt - u)$$

On integration, we get

$$\frac{dx'}{dt} = 2\omega\cos\alpha\left(\frac{1}{2}gt^2 - ut\right) \tag{v}$$

Here the constant of integration is zero. Integrating one more time and using the initial conditions, we get

$$x' = 2\omega\cos\alpha\left(\frac{1}{6}gt^3 - \frac{1}{2}ut^2\right) \tag{vi}$$

This expression gives the deviation from the vertical at a given time. Now we have to determine the deviation when the particle returns to the ground. This can be obtained by substituting with the total time taken by the particle.

When the particle reaches the maximum height its velocity becomes zero. Then from (iv), the time for the particle to reach the maximum height can be obtained as $t = \frac{u}{g}$, so that the total time taken by the particle to return back to the ground is $T = \frac{2u}{g}$. Using this in (vi), we get

$$x' = 2\omega\cos\alpha\left[\frac{1}{6}g\left(\frac{2u}{g}\right)^3 - \frac{1}{2}u\left(\frac{2u}{g}\right)^2\right]$$

$$= -\frac{4}{3}\frac{u^3}{g^2}\omega\cos\alpha \qquad \text{(vii)}$$

This expression gives the deviation from the vertical when the particle returns to the ground.

Note: If the maximum height travelled by the particle is h, then, we have, from energy conservation principle, $u = \sqrt{2gh}$. Then,

$$x' = -\frac{4}{3g^2}(2gh)^{3/2}\omega\cos\alpha = -\frac{8}{3}\omega\sqrt{\frac{2h^3}{g}}\cos\alpha \qquad \text{(viii)}$$

In the northern hemisphere, the shift will be towards east and in the southern hemisphere, the shift will be towards west.

EXAMPLE 7.17 A pendulum with a heavy bob suspended by using a long string from a rigid support is known as Foucault's pendulum. Due to the effect of Corioli's force, the plane of oscillation of the pendulum will rotate from east to west in the northern hemisphere and from west to east in the southern hemisphere. Obtain the equations of motion of such a pendulum.

Solution: Let m be the mass of the bob and l be the length of the pendulum. The coordinates of the bob are given by (x, y, z). From Figure 7.14 we write the components of the tension as

$$T_x = -T\frac{x}{l},\ T_y = -T\frac{y}{l} \text{ and } T_z = T\frac{l-z}{l} \qquad \text{(i)}$$

The general equation of motion of the bob is given by

$$m\ddot{r} = T + mg - 2m(\omega\times\dot{r}) - m\omega\times(\omega\times r) \qquad \text{(ii)}$$

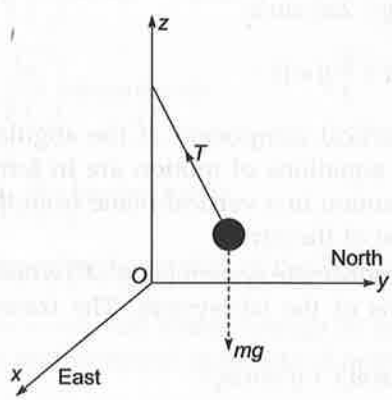

Fig. 7.14

Then the components of the force are

$$m\ddot{x} = -T\frac{x}{l} + 2m\omega\dot{y}\sin\alpha - 2m\omega\dot{z}\cos\alpha \tag{iii}$$

$$m\ddot{y} = -T\frac{y}{l} - 2m\omega\dot{x}\sin\alpha \tag{iv}$$

and $$m\ddot{z} = T\frac{l-z}{l} - mg + 2m\omega\dot{x}\cos\alpha \tag{v}$$

In writing equation (v), we have taken $g = -g\hat{k}$ and the higher order term $m\omega \times (\omega \times r)$ is neglected. Further, we assume small amplitude oscillations so that the vertical motion of the bob is also neglected. Then,

$$\dot{z} = 0\text{, and } \ddot{z} = 0\text{. Also } \frac{l-z}{l} \approx 1$$

With this assumption, equation (v) becomes

$$T = mg - 2m\omega\dot{x}\cos\alpha \tag{vi}$$

Substituting this in (iii) and (iv) for T, we get

$$\ddot{x} = -\frac{g}{l}x + \frac{2\omega}{l}x\dot{x}\cos\alpha + 2\omega\dot{y}\sin\alpha \tag{vii}$$

and $$\ddot{y} = -\frac{g}{l}y + \frac{2\omega}{l}y\dot{x}\cos\alpha - 2\omega\dot{x}\sin\alpha \tag{viii}$$

These expressions can be simplified by neglecting the terms containing $x\dot{x}$ and $y\dot{x}$ since the oscillations are of small amplitudes. Therefore,

$$\ddot{x} = -\frac{g}{l}x + 2\omega\dot{y}\sin\alpha$$

$$\ddot{x} - 2\omega_z\dot{y} + \frac{g}{l} = 0 \tag{ix}$$

and $$\ddot{y} = -\frac{g}{l}y - 2\omega\dot{x}\sin\alpha$$

$$\ddot{y} + 2\omega_z\dot{x} + \frac{g}{l}y = 0 \tag{x}$$

where $\omega_z = \omega\sin\alpha$, the vertical component of the angular velocity of rotation of the earth. In (ix) and (x), the equations of motion are in terms of ω_z. Then, we need to decouple the oscillatory motion in a vertical plane from the vertical component of the angular velocity of rotation of the earth.

For this we introduce a coordinate system (x', y', z') whose origin and z'-axis coincides with the origin and z-axis of the lab system. The transformation equations can be written as

$$x = x'\cos\omega_z t + y'\sin\omega_z t \tag{xi}$$

and $$y = -x'\sin\omega_z t + y'\cos\omega_z t \tag{xii}$$

These expressions can be substituted either in (ix) or (x) and simplified to get

$$\left(\ddot{x}'+\frac{g}{l}x'\right)\cos\omega_z t+\left(\ddot{y}'+\frac{g}{l}y'\right)\sin\omega_z t=0 \tag{xiii}$$

This equation is valid for all values of ω_z and therefore the coefficients must vanish separately. That is,

$$\ddot{x}'+\frac{g}{l}x'=0\,;\text{ and }\;\ddot{y}'+\frac{g}{l}y'=0 \tag{xiv}$$

The effect of decoupling is a precession of the plane of oscillation of the pendulum at a frequency ω_z. Note that this precession is in the clockwise direction in the northern hemisphere and in the anticlockwise direction in the southern hemisphere. Now, we may write solution to the above equations as

$$x'=A\cos\left(\sqrt{\frac{g}{l}}t\right)\quad\text{and}\quad y'=B\cos\left(\sqrt{\frac{g}{l}}t\right) \tag{xv}$$

These expressions can be used in (xi) and (xii) and simplified to get

$$x=A\cos\left(\sqrt{\frac{g}{l}}t\right)\cos\omega_z t+B\cos\left(\sqrt{\frac{g}{l}}t\right)\sin\omega_z t \tag{xvi}$$

and

$$y=-B\cos\left(\sqrt{\frac{g}{l}}t\right)\sin\omega_z t+B\cos\left(\sqrt{\frac{g}{l}}t\right)\cos\omega_z t \tag{xvii}$$

Now we need to find the values of A and B from the initial conditions. We assume that the bob is released from $y=a$ along y–direction. Then, at $t=0$; $x=0$, $\dot{x}=0$, $y=a$ and $\dot{y}=0$. Then from the equations (xvi) and (xvii), we get; $A=0$ and $B=a$. Then,

$$x=a\cos\left(\sqrt{\frac{g}{l}}t\right)\sin\omega_z t\qquad\text{and; } y=-a\cos\left(\sqrt{\frac{g}{l}}t\right)\sin\omega_z t \tag{xviii}$$

or

$$x=a\cos\left(\sqrt{\frac{g}{l}}t\right)\sin(\omega\sin\alpha)t \tag{xix}$$

and

$$y=-a\cos\left(\sqrt{\frac{g}{l}}t\right)\sin(\omega\sin\alpha)t \tag{xx}$$

Equations (xix) and (xx) describe the motion of the Foucault's pendulum.

EXAMPLE 7.18 Express the rotational kinetic energy in terms of the components of the moment of inertia tensor and principal moments of inertia.

Solution: We consider a rigid body one point of which is fixed. The body is capable of rotation about an axis passing through this fixed point.

To start with, we write the expression for the kinetic energy of a system of particles. It is given by

$$T = \frac{1}{2}\sum_j m_j v_j^2 \tag{i}$$

where $\vec{v}_j$ is the velocity of the j^{th} particle. If $\vec{\omega}$ is the angular velocity with which the body is rotating, then we have $\vec{v}_j = \vec{\omega} \times \vec{r}_j$. Using this equation (i) can be written as

$$T = \frac{1}{2}\sum_j m_j \vec{v}_j . \left(\vec{\omega} \times \vec{r}_j\right) = \frac{1}{2}\sum_j m_j \vec{\omega} . \left(\vec{r}_j \times \vec{v}_j\right) \tag{ii}$$

Now we rewrite the above expression as

$$T = \frac{1}{2}\vec{\omega} . \sum_j m_j \left(\vec{r}_j \times \vec{v}_j\right) = \frac{1}{2}\vec{\omega} . \sum_j \left(\vec{r}_j \times m_j \vec{v}_j\right)$$

or $$T = \frac{1}{2}\vec{\omega} . \sum_j \left(\vec{r}_j \times \vec{p}_j\right) = \frac{1}{2}\vec{\omega} . \vec{L} \tag{iii}$$

where $\vec{L}$ is the total angular momentum of the body. Again, we have the angular momentum given by the expression: $\vec{L} = I\vec{\omega}$, then

$$T = \frac{1}{2} I \omega^2 \tag{iv}$$

where I is the moment of inertia of the body about the axis of rotation. Now, let us write the rotational kinetic energy in terms of the components of the moment of inertia and the angular velocity. That is,

$$T = \frac{1}{2}\vec{\omega} . \vec{L} = \frac{1}{2}\left(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}\right) . \left(L_x \hat{i} + L_y \hat{j} + L_z \hat{k}\right)$$

$$= \frac{1}{2}\left(\omega_x L_x + \omega_y L_y + \omega_z L_z\right)$$

$$= \frac{1}{2}\begin{bmatrix} \omega_x \left(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\right) + \omega_y \left(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\right) \\ \omega_z \left(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\right) \end{bmatrix}$$

Using the symmetry property of the moment of inertia tensor: $I_{ij} = I_{ji}$, we rewrite the above expression as

$$T = \frac{1}{2}\left[I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y + 2I_{yz}\omega_y\omega_z + 2I_{zx}\omega_z\omega_x\right] \tag{v}$$

Equation (v) gives the rotational kinetic energy in terms of the components of the moment of inertia tensor.

Now, we write the rotational kinetic energy in terms of the principal moments of inertia I_1, I_2 and I_3 using $L = I_1\omega_x\hat{i} + I_2\omega_y\hat{j} + I_3\omega_z\hat{k}$ in the expression $T = \frac{1}{2}\vec{\omega}.\vec{L}$. Then,

$$T = \frac{1}{2}\left(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}\right).\left(I_1\omega_x\hat{i} + I_2\omega_y\hat{j} + I_3\omega_z\hat{k}\right)$$

$$= \frac{1}{2}\left(I_1\omega_x^2 + I_2\omega_y^2 + I_3\omega_z^2\right) \qquad \text{(vi)}$$

This gives the rotational kinetic energy in terms of the principal moments of inertia.

EXAMPLE 7.19 Obtain the Euler's equations of motion of a rigid body: (i) by Newton's method, and (ii) by Lagrange's method.

Solution:

(i) We consider a rigid body whose one point is fixed and rotates with an angular velocity ω about an axis passing through the fixed point. In the space fixed frame of reference, we have the torque acting on the body

$$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{space} \qquad \text{(i)}$$

We use the transformation equation between the space and body fixed coordinate systems

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + \omega \times$$

to rewrite the above expression as

$$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{space} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} \qquad \text{(ii)}$$

From the previous problem, we have

$$L = I_1\omega_x\hat{i} + I_2\omega_y\hat{j} + I_3\omega_z\hat{k}$$

so that $\left(\dfrac{dL}{dt}\right)_{body} = I_1\dot{\omega}_x\hat{i} + I_2\dot{\omega}_y\hat{j} + I_3\dot{\omega}_z\hat{k}$ (iii)

Also, we have $\vec{\omega} = \left(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}\right)$ (iv)

Using (iii) and (iv) in (ii), we get

$$\tau = \left(I_1\dot{\omega}_x\hat{i} + I_2\dot{\omega}_y\hat{j} + I_3\dot{\omega}_z\hat{k}\right) + \left[\left(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}\right) \times \left(I_1\dot{\omega}_x\hat{i} + I_2\dot{\omega}_y\hat{j} + I_3\dot{\omega}_z\hat{k}\right)\right]$$

This expression can be expanded to get:

$$\tau_x\hat{i}+\tau_y\hat{j}+\tau_z\hat{k}=\left\{\begin{matrix}\left[I_1\dot{\omega}_x+(I_3-I_2)\omega_y\omega_z\right]\hat{i}\\ +\left[I_2\dot{\omega}_y+(I_1-I_3)\omega_x\omega_z\right]\hat{j}+\left[I_3\dot{\omega}_z+(I_2-I_1)\omega_y\omega_x\right]\hat{k}\end{matrix}\right\}$$

Then the components of the torque are

$$\left.\begin{aligned}\tau_x&=I_1\dot{\omega}_x+(I_3-I_2)\omega_y\omega_z\\ \tau_y&=I_2\dot{\omega}_y+(I_1-I_3)\omega_x\omega_z\\ \tau_z&=I_3\dot{\omega}_z+(I_2-I_1)\omega_y\omega_x\end{aligned}\right\}\tag{v}$$

These are Euler's equations of motion of a rigid body one point of which is fixed and acted upon by a torque.

(ii) Now we use the Lagrange's method to obtain the above equations of motion. Here we use the Eulerian angles (ϕ,θ,ψ) as the generalized coordinates for the motion of the rigid body. The rotational kinetic energy of the body is given by

$$T=\frac{1}{2}\left(I_1\omega_x^2+I_2\omega_y^2+I_3\omega_z^2\right)$$

and the Lagrangian is

$$L=T-V=\frac{1}{2}\left(I_1\omega_x^2+I_2\omega_y^2+I_3\omega_z^2\right)-V(\phi,\theta,\psi)\tag{vi}$$

We have to express the kinetic energy in terms of the derivatives of the generalized coordinates. This can be done using the following relations

$$\left.\begin{aligned}\omega_x&=\dot{\phi}\sin\theta\sin\psi-\dot{\theta}\cos\psi\\ \omega_y&=\dot{\phi}\sin\theta\cos\psi-\dot{\theta}\sin\psi \quad\text{and}\\ \omega_z&=\dot{\phi}\cos\theta+\dot{\psi}\end{aligned}\right\}\tag{vii}$$

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0\tag{viii}$$

For the variable, ψ this equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}}\right)-\frac{\partial L}{\partial\psi}=0\ \text{ or, }\ \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\psi}}\right)-\frac{\partial T}{\partial\psi}=-\frac{\partial V}{\partial\psi}\tag{ix}$$

Now, $T \equiv T(\omega_x, \omega_y, \omega_z)$ and hence,

$$\frac{\partial T}{\partial \dot{\psi}} = \frac{\partial T}{\partial \omega_x}\frac{\partial \omega_x}{\partial \dot{\psi}} + \frac{\partial T}{\partial \omega_y}\frac{\partial \omega_y}{\partial \dot{\psi}} + \frac{\partial T}{\partial \omega_z}\frac{\partial \omega_z}{\partial \dot{\psi}} \tag{x}$$

From (vii) we see that; $\frac{\partial \omega_x}{\partial \dot{\psi}} = 0$, $\frac{\partial \omega_y}{\partial \dot{\psi}} = 0$ and $\frac{\partial \omega_z}{\partial \dot{\psi}} = 1$. Therefore,

$$\frac{\partial T}{\partial \dot{\psi}} = \frac{\partial T}{\partial \omega_z} = I_3 \omega_z \tag{xi}$$

Similarly, $$\frac{\partial T}{\partial \psi} = \frac{\partial T}{\partial \omega_x}\frac{\partial \omega_x}{\partial \psi} + \frac{\partial T}{\partial \omega_y}\frac{\partial \omega_y}{\partial \psi} + \frac{\partial T}{\partial \omega_z}\frac{\partial \omega_z}{\partial \psi} \tag{xii}$$

Again from (vii), we get

$$\frac{\partial \omega_x}{\partial \psi} = \omega_y, \quad \frac{\partial \omega_y}{\partial \psi} = -\omega_x \text{ and } \frac{\partial \omega_z}{\partial \dot{\psi}} = 0.$$

Then, $$\frac{\partial T}{\partial \psi} = \frac{\partial T}{\partial \omega_x}\omega_y - \frac{\partial T}{\partial \omega_y}\omega_x = (I_1 - I_2)\omega_x \omega_y \tag{xiii}$$

Using (xi) and (xiii) in (ix), we get

$$\frac{d}{dt}(I_3 \omega_z) - (I_1 - I_2)\omega_x \omega_y = -\frac{\partial V}{\partial \psi}$$

or, $$I_3 \ddot{\omega}_z - (I_1 - I_2)\omega_x \omega_y = -\frac{\partial V}{\partial \psi} \tag{xiv}$$

Now, $\left(-\frac{\partial V}{\partial \psi}\right)$, the vertical component of the generalized force which is the z–component of the torque (τ_z). Therefore,

$$\tau_z = I_3 \ddot{\omega}_z - (I_1 - I_2)\omega_x \omega_y \tag{xv}$$

This is one of the Euler's equations of motion. The other two equations can be obtained by cyclic permutation of the indices in the above equation. It is to be noted that the equations thus obtained do not correspond to the generalized coordinates ϕ and θ.

EXAMPLE 7.20 Show that for the torque free rotation of a rigid body, the rotational kinetic energy and the angular momentum are conserved.

Solution: We start with the Euler's equations of motion. They are

$$\tau_x = I_1\dot{\omega}_x + (I_3 - I_2)\omega_y\omega_z$$

$$\tau_y = I_2\dot{\omega}_y + (I_1 - I_3)\omega_x\omega_z$$

$$\tau_z = I_3\dot{\omega}_z + (I_2 - I_1)\omega_y\omega_x$$

When there is no external torque acting on the body, these equations become

$$I_1\dot{\omega}_x = (I_2 - I_3)\omega_y\omega_z \quad \text{(i)}$$

$$I_2\dot{\omega}_y = (I_3 - I_1)\omega_x\omega_z \quad \text{(ii)}$$

$$I_3\dot{\omega}_z = (I_1 - I_2)\omega_x\omega_y \quad \text{(iii)}$$

Multiplying equation (i) with ω_x, equation (ii) with ω_y and equation (iii) with ω_z, we get

$$I_1\omega_x\dot{\omega}_x = (I_2 - I_3)\omega_x\omega_y\omega_z \quad \text{(iv)}$$

$$I_2\omega_y\dot{\omega}_y = (I_3 - I_1)\omega_x\omega_y\omega_z \quad \text{(v)}$$

$$I_3\omega_z\dot{\omega}_z = (I_1 - I_2)\omega_x\omega_y\omega_z \quad \text{(vi)}$$

Adding the equations (iv), (v) and (vi), we get

$$I_1\omega_x\dot{\omega}_x + I_2\omega_y\dot{\omega}_y + I_3\omega_z\dot{\omega}_z = 0$$

or
$$\frac{d}{dt}\left(\frac{1}{2}I_1\omega_x^2 + \frac{1}{2}I_2\omega_y^2 + \frac{1}{2}I_3\omega_z^2\right) = 0$$

or
$$\left(\frac{1}{2}I_1\omega_x^2 + \frac{1}{2}I_2\omega_y^2 + \frac{1}{2}I_3\omega_z^2\right) = \frac{1}{2}\vec{L}.\vec{\omega} = \text{constant} \quad \text{(vii)}$$

Therefore, the total rotational kinetic energy is conserved in the absence of any external torque.

Then, by definition; $\vec{\tau} = \dfrac{d\vec{L}}{dt}$ and since the external torque is zero, $\dfrac{d\vec{L}}{dt} = 0$ or the angular momentum $\vec{L}$ is conserved.

EXAMPLE 7.21 Obtain the expression for the angular velocity of precession of a symmetrical top which is under motion in a force free field.

Solution: We consider a symmetrical top with z–axis as the symmetry axis, such that $I_1 = I_2 \neq I_3$. The fixed point of the top is at the origin. Then the Euler's equations of motion in the force free field becomes

$$I_1\dot{\omega}_x = (I_1 - I_3)\omega_y\omega_z \quad \text{(i)}$$

$$I_1\dot{\omega}_y = (I_3 - I_1)\omega_x\omega_z \quad \text{(ii)}$$

$$I_3\dot{\omega}_z = 0 \quad \text{(iii)}$$

Note that in the RHS of the first equation and in the LHS of the second equation we replaced I_2 by I_1. From equation (iii) it immediately follows that; $\omega_z = constant$. This means that the angular velocity of motion of the top about the z-axis is a constant.

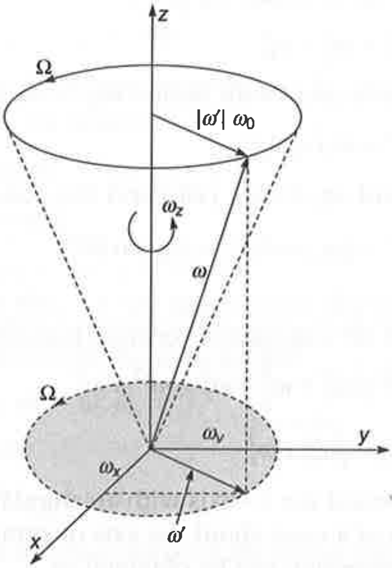

Fig. 7.15

Now, let us make a substitution as

$$\Omega = \frac{(I_3 - I_1)}{I_1}\omega_z \tag{iv}$$

Then the equations (i) and (ii) take the form

$$\dot{\omega}_x = -\Omega\omega_y \text{ and } \dot{\omega}_y = \Omega\omega_x \tag{v}$$

Differentiating the first expression in (v), and then using the second expression in it, we get

$$\ddot{\omega}_x = -\Omega\dot{\omega}_y = -\Omega^2\omega_x$$

or
$$\ddot{\omega}_x + \Omega^2\omega_x = 0 \tag{vi}$$

This equation representing a simple harmonic motion can have a solution of the form

$$\omega_x = \omega_0 \cos\Omega t \tag{vii}$$

where, ω_0 is a constant. Also in the present case at $t = 0$, $\omega_x = 0$ and hence we have chosen the phase constant in the argument of the sine function as zero.

Now, differentiating (vii), we get

$$\dot{\omega}_x = -\omega_0\Omega\sin\Omega t \tag{viii}$$

This can be used in (v) to get

$$-\omega_0\Omega\sin\Omega t = -\Omega\omega_y \text{ or; } \omega_y = \omega_0 \sin\Omega t \tag{ix}$$

Squaring and adding (viii) and (ix), we get

$$\omega_x^2 + \omega_y^2 = \omega_0^2 \tag{x}$$

This equation represents a circle with radius ω_0. Now, let us write

$$\omega' = \omega_x\hat{i} + \omega_y\hat{j}$$

as the resultant of ω_x and ω_y. Using (viii) and (ix), this can be written as

$$\omega' = (\omega_0 \cos\Omega t)\hat{i} + (\omega_0 \sin\Omega t)\hat{j} \tag{xi}$$

Evidently, $|\omega'| = \omega_0$. This will rotate about the $z-$ axis with an angular velocity Ω. The resultant of ω_z and ω' will have a constant magnitude as follows:

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2 = \omega'^2 + \omega_z^2$$

or

$$|\omega| = \sqrt{\omega_0^2 + \omega_z^2} \tag{xii}$$

This vector precesses around the z–axis with an angular velocity Ω. Thus the vector $\vec{\omega}$ moves over the surface of a cone about the axis of symmetry, here the z-axis.

Finally, period of the precession can be obtained as

$$T = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega_z}\left(\frac{I_3 - I_1}{I_1}\right) \tag{xiii}$$

and the frequency as the reciprocal of the time period.

EXAMPLE 7.22 A body moves about a point O under no force. The principal moments of inertia of the body about the fixed point are in the ratio 3:5:6. Initially, the angular velocity has the components given by $\omega_{x'} = \omega_{z'} = \omega_0$ and $\omega_{y'} = 0$ about the corresponding principal axes. Show that at any time t the angular velocity about the y' axis is given by; $\omega_{y'} = \frac{3\omega_0}{\sqrt{5}}\tanh\left(\frac{\omega_0 t}{\sqrt{5}}\right)$.

Solution: Given that the principal moments of inertia are in the ratio 3:5:6. Then, we write

$$I_1 \equiv 3I,\ I_2 \equiv 5I \text{ and } I_3 \equiv 6I \tag{i}$$

Also given that

$$\omega_{x'} = \omega_{z'} = \omega_0 \text{ and } \omega_{y'} = 0 \tag{ii}$$

We have the Euler equations in the absence of external torque given by

$$I_1\dot{\omega}_{x'} + (I_3 - I_2)\omega_{y'}\omega_{z'} = 0 \tag{i}$$

$$I_2\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{z'} = 0 \tag{ii}$$

$$I_3\dot{\omega}_{z'} + (I_2 - I_1)\omega_{y'}\omega_{x'} = 0 \tag{iii}$$

Using the given conditions, the Euler's equations can be written as

$$3\dot{\omega}_{x'} + \omega_{y'}\omega_{z'} = 0 \tag{iv}$$

$$5\dot{\omega}_{y'} - 3\omega_{x'}\omega_{z'} = 0 \tag{v}$$

$$3\dot{\omega}_{z'} + \omega_{y'}\omega_{x'} = 0 \tag{vi}$$

From the equations (iv) and (v), by multiplying (iv) by $3\omega_{x'}$ and (v) by $\omega_{y'}$ and then adding together, we can have the relation

$$9\omega_{x'}\dot{\omega}_{x'} + 5\omega_{y'}\dot{\omega}_{y'} = 0$$

On integration this would yield

$$9\omega_{x'}^2 + 5\omega_{y'}^2 = \alpha \text{ (say)} \tag{vii}$$

Similarly, from the equations (iv) and (vi), by multiplying (iv) with $\omega_{x'}$ and (vi) with $\omega_{z'}$ and subtracting the second from the first, we get

$$\omega_{x'}\dot{\omega}_{x'} - \omega_{z'}\dot{\omega}_{z'} = 0$$

On integration, we will have

$$\omega_{x'}^2 - \omega_{z'}^2 = \beta \text{ (say)} \tag{viii}$$

The values of α and β can be obtained from the initial conditions $\omega_{x'} = \omega_{z'} = \omega_0$ and $\omega_{y'} = 0$. Therefore, we see that; $\alpha = 9\omega_0^2$ and $\beta = 0$.

Then from (viii), we get; $\omega_{x'}^2 = \omega_{z'}^2$ or $\omega_{x'} = \omega_{z'}$. Then equation (v) becomes

$$5\dot{\omega}_{y'} - 3\omega_{x'}^2 = 0 \tag{ix}$$

From (vii) we have; $\omega_{x'}^2 = \frac{1}{9}\left(9\omega_0^2 - 5\omega_{y'}^2\right)$, so that the above expression can be written as

$$5\frac{d\omega_{y'}}{dt} = \frac{1}{3}\left(9\omega_0^2 - 5\omega_{y'}^2\right)$$

or $$dt = 15\frac{d\omega_{y'}}{9\omega_0^2 - 5\omega_{y'}^2}$$

or $$\int dt = 15\int \frac{d\omega_{y'}}{9\omega_0^2 - 5\omega_{y'}^2} = 3\int \frac{d\omega_{y'}}{\frac{9}{5}\omega_0^2 - \omega_{y'}^2}$$

On integration, this would yield

$$t = \frac{\sqrt{5}}{\omega_0}\tanh^{-1}\left(\frac{\sqrt{5}}{3\omega_0}\omega_{y'}\right)$$

or $$\omega_{y'} = \frac{3\omega_0}{\sqrt{5}} \tanh\left(\frac{\omega_0 t}{\sqrt{5}}\right) \tag{x}$$

Hence, proved.

EXAMPLE 7.23 The principal moments of inertia of a rigid body rotating about a fixed point are such that $I_1 = I_2 \neq I_3$. The components of the torque along the corresponding principal axes are $-\xi\omega_{x'}, -\xi\omega_{y'}$ and $-\xi\omega_{z'}$. Obtain the expressions of the components of the angular velocity along the principal axes as a function of time.

Solution: Given that; $I_1 = I_2 \neq I_3$, also $\tau_{x'} = -\xi\omega_{x'}, \tau_{y'} = -\xi\omega_{y'}$ and, $\tau_{z'} = -\xi\omega_{z'}$. Then the Euler's equations of motion become

$$I_1\dot{\omega}_{x'} + (I_3 - I_1)\omega_{y'}\omega_{z'} = -\xi\omega_{x'} \tag{i}$$

$$I_1\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{z'} = -\xi\omega_{y'} \tag{ii}$$

$$I_3\dot{\omega}_{z'} = -\xi\omega_{z'} \tag{iii}$$

From (iii), we get

$$\frac{\dot{\omega}_{z'}}{\omega_{z'}} = -\frac{\xi}{I_3}$$

On integration, this would yield

$$\ln\omega_{z'} = -\frac{\xi}{I_3}t + \ln\alpha \tag{iv}$$

where α is a constant.

Then, $$\omega_{z'} = \alpha e^{-\frac{\xi}{I_3}t} \tag{v}$$

Multiplying equation (i) by $\omega_{x'}$ and equation (2) by $\omega_{y'}$, then adding together, we get

$$\omega_{x'}\dot{\omega}_{x'} + \omega_{y'}\dot{\omega}_{y'} = -\frac{\xi}{I_1}\left(\omega_{x'}^2 + \omega_{y'}^2\right)$$

or $$\frac{1}{2}\frac{d}{dt}\left(\omega_{x'}^2 + \omega_{y'}^2\right) = -\frac{\xi}{I_1}\left(\omega_{x'}^2 + \omega_{y'}^2\right)$$

that is, $$\frac{d\left(\omega_{x'}^2 + \omega_{y'}^2\right)}{\left(\omega_{x'}^2 + \omega_{y'}^2\right)} = -2\frac{\xi}{I_1}dt$$

On integration, we get

$$\ln\left(\omega_{x'}^2 + \omega_{y'}^2\right) = -2\frac{\xi}{I_1}t + \ln\beta^2$$

or $$\omega_{x'}^2 + \omega_{y'}^2 = \beta^2 e^{-2\frac{\xi}{I_1}t} \tag{vi}$$

A possible solution to this equation can be taken as

$$\omega_{x'} = \beta e^{-\frac{\xi}{I_1}t} \sin\delta \text{ and } \omega_{y'} = \beta e^{-\frac{\xi}{I_1}t} \cos\delta \tag{vii}$$

Substituting this in equation (i), we get

$$I_1\beta\left(-\frac{\xi}{I_1}e^{-\frac{\xi}{I_1}t}\sin\delta + e^{-\frac{\xi}{I_1}t}\cos\delta\frac{d\delta}{dt}\right) + \alpha\beta\left(I_3 - I_1\right)e^{-\frac{\xi}{I_1}t}e^{-\frac{\xi}{I_3}t}\cos\delta = -\xi\beta e^{-\frac{\xi}{I_1}t}\sin\delta$$

This can be simplified to get

$$I_1\frac{d\delta}{dt} + \alpha\left(I_3 - I_1\right)e^{-\frac{\xi}{I_3}t} = 0$$

or

$$\frac{d\delta}{dt} = \alpha\frac{\left(I_1 - I_3\right)}{I_1}e^{-\frac{\xi}{I_3}t} = -\sigma e^{-\frac{\xi}{I_3}t} \tag{viii}$$

where, $\sigma = \alpha\dfrac{\left(I_3 - I_1\right)}{I_1}$. On integration, equation (viii) gives

$$\delta = \sigma\left(\frac{I_3}{\xi}\right)e^{-\frac{\xi}{I_3}t} + \varepsilon \tag{ix}$$

where, ε is a constant of integration. Substituting this in equation (vii), we get

$$\omega_{x'} = \beta e^{-\frac{\xi}{I_1}t}\sin\left(\frac{\sigma I_3}{\xi}e^{-\frac{\xi}{I_3}t} + \varepsilon\right) \tag{x}$$

$$\omega_{y'} = \beta e^{-\frac{\xi}{I_1}t}\cos\left(\frac{\sigma I_3}{\xi}e^{-\frac{\xi}{I_3}t} + \varepsilon\right) \tag{xi}$$

and

$$\omega_{z'} = \alpha e^{-\frac{\xi}{I_3}t} \tag{xii}$$

Equations (x), (xi) and (xii) are the required results.

EXAMPLE 7.24 Discuss the motion of a symmetric top in a force field. Obtain the expression for the kinetic energy of rotation.

Solution: We consider a symmetrical top one point of which is fixed at the origin. We use the unprimed coordinate system as the inertial frame and the primed coordinate system as the rotating coordinate system whose axes represent the principal axes of the spinning top. Also both coordinate systems have the same origin. At $t = 0$ the corresponding axes of both coordinate systems coincide each other.

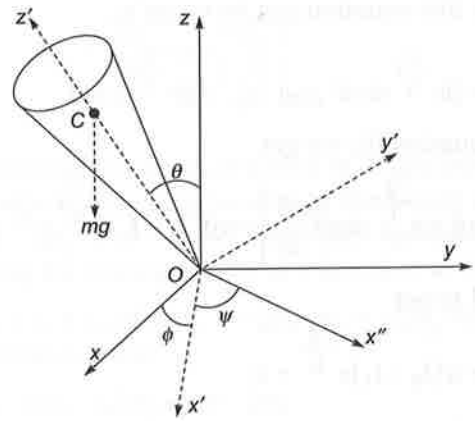

Fig. 7.16

In Figure 7.16, ϕ, θ and ψ are the Eulerian angles. Let the unit vectors along (x, y, x) be $\hat{i}, \hat{j}, \hat{k}$ and that along (x', y', z') be $(\hat{i}', \hat{j}', \hat{k}')$. Then the angular velocity of rotation of the primed coordinate system with respect to the unprimed coordinate system can be written as

$$\omega = \omega_{x'} i' + \omega_{y'} j' + \omega_{z'} k' \tag{i}$$

Since the top is spinning about the z' − axis, then in the angular velocity of the top, the corresponding component of the angular velocity will have an additional term; say $\xi k'$. Therefore, the angular velocity of the top is

$$\omega = \omega_{x'} i' + \omega_{y'} j' + (\omega_{z'} + \xi) k' \tag{ii}$$

If I_1, I_2 and I_3 are the principal moments of inertia about the body set of axes, then the angular momentum of the spinning top can be written as

$$L = I_1 \omega_{x'} i' + I_2 \omega_{y'} j' + I_3 (\omega_{z'} + \xi) k' \tag{iii}$$

Using Euler's equations of motion, we can rewrite the above equation as

$$L = \begin{Bmatrix} \left[I_1 \dot{\omega}_{x'} + (I_3 - I_2) \omega_{y'} \omega_{z'} \right] i' + \left[I_2 \dot{\omega}_{y'} + (I_1 - I_3) \omega_{x'} \omega_{z'} \right] j' \\ + \left[I_3 \dot{\omega}_{z'} + (I_2 - I_1) \omega_{y'} \omega_{x'} + I_3 \xi \right] k' \end{Bmatrix} \tag{iv}$$

The transformation equation that gives the torque in the space fixed coordinate system is

$$\left(\frac{dL}{dt}\right)_{space} = \left(\frac{dL}{dt}\right)_{body} + \omega \times L \tag{v}$$

Using equation (ii) and (iv), we can evaluate the terms in the RHS of equation (v) and then substituting and simplifying, we get

$$\tau = \begin{Bmatrix} \left[I_1\dot{\omega}_{x'} + (I_3 - I_2)\omega_{y'}\omega_{z'} + I_3\omega_{y'}\xi\right]i' \\ +\left[I_2\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{z'} - I_3\omega_{x'}\xi\right]j' \\ +\left[I_3\left(\dot{\omega}_{z'} + \dot{\xi}\right) + (I_2 - I_1)\omega_{x'}\omega_{y'}\right]k' \end{Bmatrix} \tag{vi}$$

Now, the torque about the centre of mass can be written as

$$\tau = lk' \times F = (lk') \times (-mgk) \tag{vii}$$

where, l is the distance of the centre of mass from the fixed point; here the origin. Now, we have

$$k = (k.i')i' + (k.j')j' + (k.k')k'$$

$$= \cos\left(\frac{\pi}{2} - \theta\right)j' + \cos\theta k' = \sin\theta j' + \cos\theta k' \tag{viii}$$

Then, $$\tau = -mgl(k' \times k) = -mgl\left[k' \times (\sin\theta j' + \cos\theta k')\right]$$

or $$= mgl\sin\theta i' \tag{ix}$$

From equations (vi) and (ix), we write

$$mgl\sin\theta i' = \begin{Bmatrix} \left[I_1\dot{\omega}_{x'} + (I_3 - I_2)\omega_{y'}\omega_{z'} + I_3\omega_{y'}\xi\right]i' \\ +\left[I_2\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{z'} - I_3\omega_{x'}\xi\right]j' \\ +\left[I_3\left(\dot{\omega}_{z'} + \dot{\xi}\right) + (I_2 - I_1)\omega_{x'}\omega_{y'}\right]k' \end{Bmatrix} \tag{x}$$

Comparing the coefficients, we get

$$I_1\dot{\omega}_{x'} + (I_3 - I_2)\omega_{y'}\omega_{z'} + I_3\omega_{y'}\xi = mgl\sin\theta$$

$$I_2\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{z'} - I_3\omega_{x'}\xi = 0$$

and $$I_3\left(\dot{\omega}_{z'} + \dot{\xi}\right) + (I_2 - I_1)\omega_{x'}\omega_{y'} = 0 \tag{xi}$$

For a symmetric top, $I_1 = I_2$, then from the last expression of (xi), we get

$$\dot{\omega}_{z'} + \dot{\xi} = 0 \text{ or } \omega_{z'} + \xi = C, \text{ a constant.}$$

Therefore; $\xi = C - \omega_{z'}$. This can be substituted in the first two expressions of (xi) to get

$$I_1\dot{\omega}_{x'} - I_1\omega_{y'}\omega_{z'} + I_3\omega_{y'}C = mgl\sin\theta \tag{xii}$$

and $$I_1\dot{\omega}_{y'} + I_1\omega_{x'}\omega_{z'} - I_3\omega_{x'}C = 0 \tag{xiii}$$

Now, for $\psi = 0$, we have; $\omega_{x'} = \dot{\theta}$, $\omega_{y'} = \dot{\phi}\sin\theta$ and $\omega_{z'} = \dot{\phi}\cos\theta$. Then, (xii) and (xiii) becomes

$$I_1\ddot{\theta} - I_1\dot{\phi}^2\sin\theta\cos\theta + I_3\dot{\phi}\sin\theta C = mgl\sin\theta \tag{xiv}$$

$$I_1\left(\ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta\right) + I_1\dot{\phi}^2\sin\theta\cos\theta - I_3\dot{\theta}C = 0 \tag{xv}$$

If the precession is with a constant angular velocity, then $\dot{\theta} = constant$ or $\ddot{\theta} = 0$. Then from (xiv), we have

$$-I_1\dot{\phi}^2 \sin\theta\cos\theta + I_3\dot{\phi}\sin\theta C = mgl\sin\theta$$

or
$$I_1\dot{\phi}^2 \cos\theta - I_3\dot{\phi}C + mgl = 0 \tag{xvi}$$

This is quadratic equation in $\dot{\phi}$ which has the solutions given by

$$\dot{\phi} = \frac{I_3 C \pm \sqrt{I_3^3 C^2 - 4mglI_1 \cos\theta}}{2I_1 \cos\theta} \tag{xvii}$$

The roots are real if $I_3^3 C^2 - 4mglI_1 \cos\theta \geq 0$. Now, multiplying the three expressions in equation (xi) by $\omega_{x'}, \omega_{y'}$ and $(\omega_{z'} + \xi)$ respectively, and putting $I_1 = I_2$, we get

$$I_1\omega_{x'}\dot{\omega}_{x'} + (I_3 - I_1)\omega_{x'}\omega_{y'}\omega_{z'} + I_3\omega_{x'}\omega_{y'}\xi = mgl\omega_{x'}\sin\theta$$

$$I_1\omega_{y'}\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{y'}\omega_{z'} - I_3\omega_{x'}\omega_{y'}\xi = 0$$

and
$$I_3\left(\dot{\omega}_{z'} + \dot{\xi}\right)(\omega_{z'} + \xi) = 0$$

Adding these expressions together, and putting $\omega_{x'} = \dot{\theta}$, we obtain

$$I_1\left(\omega_{x'}\dot{\omega}_{x'} + \omega_{y'}\dot{\omega}_{y'}\right) + I_3\left(\dot{\omega}_{z'} + \dot{\xi}\right)(\omega_{z'} + \xi) = mgl\dot{\theta}\sin\theta$$

that is,
$$\frac{1}{2}\frac{d}{dt}\left[I_1\left(\omega_{x'}^2 + \omega_{y'}^2\right) + I_3\left(\omega_{z'} + \xi\right)^2\right] = \frac{d}{dt}(-mgl\cos\theta)$$

or
$$\frac{1}{2}I_1\left(\omega_{x'}^2 + \omega_{y'}^2\right) + \frac{1}{2}I_3\left(\omega_{z'} + \xi\right)^2 + mgl\cos\theta = E \tag{xviii}$$

where E is a constant. Putting $\omega_{z'} + \xi = C$, $\omega_{x'} = \dot{\theta}$ and $\omega_{y'} = \dot{\phi}\sin\theta$ the above equation becomes

$$\frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3C^2 + mgl\cos\theta = E \tag{xix}$$

This expression is equivalent to the energy conservation principle with kinetic energy $T = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3C^2$ and potential energy equal to $V = mgl\cos\theta$.

EXAMPLE 7.25 A rigid wheel has principal moments of inertia $I_1 = I_2 \neq I_3$ about the body fixed principal axes. At the centre of the wheel a bearing is attached about which the wheel can rotate about a space fixed axis without any friction. Obtain the constant angular velocity at which the wheel can rotate without exerting a torque on the bearing.

Solution: We start with the Euler's equations of motion of a rigid body. They are

$$\tau_x = I_1\dot{\omega}_{x'} + (I_3 - I_2)\omega_{y'}\omega_{z'}$$

$$\tau_y = I_2\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{z'}$$

$$\tau_z = I_3\dot{\omega}_{z'} + (I_2 - I_1)\omega_{y'}\omega_{x'}$$

In the given problem, we have $I_1 = I_2 \neq I_3$ and the torque is zero; so that the above equations become

$$I_1\dot{\omega}_{x'} + (I_3 - I_1)\omega_{y'}\omega_{z'} = 0 \text{ or, } I_1\dot{\omega}_{x'} - I_1\omega_{y'}\omega_{z'} + I_3\omega_{y'}\omega_{z'} = 0 \quad \text{(i)}$$

$$I_1\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{z'} = 0 \text{ or, } I_1\dot{\omega}_{y'} + I_1\omega_{x'}\omega_{z'} - I_3\omega_{x'}\omega_{z'} = 0 \quad \text{(ii)}$$

$$I_3\dot{\omega}_{z'} + (I_1 - I_1)\omega_{y'}\omega_{x'} = 0 \text{ or, } I_3\dot{\omega}_{z'} = 0 \quad \text{(iii)}$$

Equation (iii) can be integrated to get

$$\omega_{z'} \equiv \Omega, \text{ a constant.} \quad \text{(iv)}$$

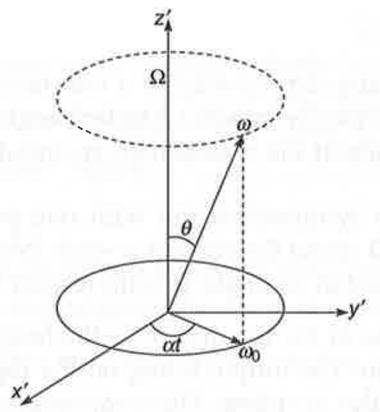

Fig. 7.17

With this equations (i) and (ii) become

$$I_1\dot{\omega}_{x'} - I_1\omega_{y'}\Omega + I_3\omega_{y'}\Omega = 0 \text{ or, } \dot{\omega}_{x'} = \frac{(I_1 - I_3)}{I_1}\omega_{y'}\Omega \quad \text{(v)}$$

and $$I_1\dot{\omega}_{y'} + I_1\omega_{x'}\Omega - I_3\omega_{x'}\Omega = 0 \text{ or } \dot{\omega}_{y'} = -\frac{(I_1 - I_3)}{I_1}\omega_{x'}\Omega \quad \text{(vi)}$$

Differentiating these equations, we get

$$\ddot{\omega}_{x'} = \frac{(I_1 - I_3)}{I_1}\dot{\omega}_{y'}\Omega \quad \text{and } \ddot{\omega}_{y'} = -\frac{(I_1 - I_3)}{I_1}\dot{\omega}_{x'}\Omega \quad \text{(vii)}$$

These expressions can be written as

$$\ddot{\omega}_{x'} = -\left[\frac{(I_1 - I_3)}{I_1}\Omega\right]^2 \omega_{x'} = -\alpha^2\omega_{x'} \quad \text{(viii)}$$

and $$\ddot{\omega}_{y'} = -\left[\frac{(I_1 - I_3)}{I_1}\Omega\right]^2 \omega_{y'} = -\alpha^2\omega_{y'} \quad \text{(ix)}$$

where, $\alpha = \dfrac{(I_1 - I_3)}{I_1}\Omega$. The general solution to the above equations can be written as

$$\omega_{x'} = \omega_0 \cos(\alpha t + \phi) \text{ and } \omega_{y'} = \omega_0 \sin(\alpha t + \phi) \tag{x}$$

Now, the resultant of the angular velocity can be obtained as

$$\begin{aligned}\omega &= \sqrt{\omega_{x'}^2 + \omega_{y'}^2 + \omega_{z'}^2} \\ &= \sqrt{\omega_0^2 \cos^2(\alpha t + \phi) + \omega_0^2 \sin(\alpha t + \phi) + \Omega^2} \\ &= \sqrt{\omega_0^2 + \Omega^2}\end{aligned} \tag{xi}$$

Thus, we see that the angular velocity is a constant. Further, since $\omega_{z'} = \Omega$ is a constant, the total angular velocity makes a constant angle θ with the z'-axis. Also, the plane of ω and z' rotates about the z'-axis with an angular velocity α.

EXAMPLE 7.26 A heavy symmetrical top with one point fixed is precessing at a constant angular velocity Ω about the vertical z–axis. What is the minimum spin about the z'–axis which is inclined at an angle θ with respect to the z–axis?

Solution: Let m be the mass of the top and h be the height of the centre of mass from the fixed point at the origin. The torque acting on the top due to the gravity is in the direction perpendicular to the xz–plane. The components of this torque along the body fixed axes are

$$\tau_{x'} = mgh\sin\theta\cos\psi,\ \tau_{y'} = mgh\sin\theta\sin\psi \text{ and } \tau_{z'} = 0 \tag{i}$$

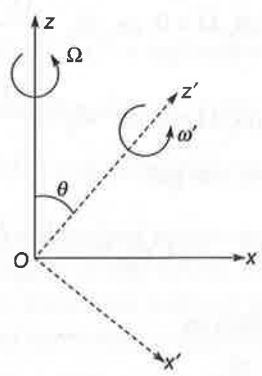

Fig. 7.18

The Euler's equations of motion for a symmetrical top with $I_1 = I_2$ are

$$\left.\begin{aligned} &I_1\dot{\omega}_{x'} + (I_3 - I_1)\omega_{y'}\omega_{z'} = \tau_{x'} \\ &I_1\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{z'} = \tau_{y'} \\ &I_3\dot{\omega}_{z'} = \tau_{z'} \end{aligned}\right\} \tag{ii}$$

From (i) and (ii), we get

$$I_1\dot{\omega}_{x'} + (I_3 - I_1)\omega_{y'}\omega_{z'} = mgh\sin\theta\cos\psi \tag{iii}$$

$$I_1\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{z'} = mgh\sin\theta\sin\psi \tag{iv}$$

and $$I_3\dot{\omega}_{z'} = 0 \tag{v}$$

Now, we have the components of the angular velocity in terms of the Euler angles as

$$\omega_{x'} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_{y'} = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

and $$\omega_{z'} = \dot{\phi}\cos\theta + \dot{\psi}$$

In the present problem it is given that θ is a constant and hence, $\dot{\theta} = 0$. Also $\dot{\phi} = \Omega$, constant and $\ddot{\phi} = 0$. Then the components of the angular velocities are

$$\omega_{x'} = \Omega\sin\theta\sin\psi;\ \ \omega_{y'} = \Omega\sin\theta\cos\psi \text{ and } \omega_{z'} = \Omega\cos\theta + \dot{\psi} \tag{vi}$$

With this equation (iii) becomes

$$I_1\Omega\left(\dot{\theta}\cos\theta\sin\psi + \dot{\psi}\sin\theta\cos\psi\right) + (I_3 - I_1)(\Omega\sin\theta\cos\psi)(\Omega\cos\theta + \dot{\psi}) = mgh\sin\theta\cos\psi$$

Putting $\dot{\theta} = 0$ and dividing throughout by $\sin\theta\cos\psi$ the above expression reduces to

$$I_1\Omega\dot{\psi} + \Omega(I_3 - I_1)(\Omega\cos\theta + \dot{\psi}) = mgh$$

This can be further simplified to get

$$\Omega^2(I_1 - I_3)\cos\theta - \Omega I_3\dot{\psi} + mgh = 0 \tag{vii}$$

This expression can be rearranged to get

$$\omega' = \dot{\psi} = \frac{\Omega^2(I_1 - I_3)\cos\theta + mgh = 0}{\Omega I_3} \tag{viii}$$

Further, for from (vii) we see that for Ω to be real, we must have

$$I_3^2\omega'^2 - 4(I_1 - I_3)mgh\cos\theta \geq 0$$

or $$\omega' \geq \frac{1}{I_3}\sqrt{4(I_1 - I_3)mgh\cos\theta} \tag{ix}$$

where the equality gives the minimum value of the spin about the z'–axis.

EXAMPLE 7.27 A uniform disc of radius r rolls on a perfectly rough surface such that the point of contact describes a circle of radius R. The normal to the disc makes a fixed angle θ with the vertical. Obtain the angular velocity of motion $\dot{\phi}$ around the $z-$axis so that the disc continues to move around the vertical axis.

Solution: Let the body fixed axes be attached to the body such that the $z'-$axis and hence $\hat{k}'$ is normal to the plane of the disc as shown in Figure 7.19. The unit vector $\hat{j}'$ is always pointing from the point of contact to the centre of the disc and $\hat{i}'$ is into the plane of the paper. Since the angle θ is fixed, the components of the angular velocity along the body fixed axes becomes;

$$\omega_{x'} = \dot{\phi}\sin\theta\sin\psi\ ;\ \omega_{y'} = \dot{\phi}\sin\theta\cos\psi \text{ and } \omega_{z'} = \dot{\phi}\cos\theta + \dot{\psi} \tag{i}$$

With this the Euler equations of motion are;

$$\tau_{x'} = I_1\dot{\omega}_{x'} + (I_3 - I_1)\omega_{y'}\omega_{z'}$$

$$\tau_{y'} = I_1\dot{\omega}_{y'} + (I_1 - I_3)\omega_{x'}\omega_{z'}$$

$$\tau_{z'} = I_3\dot{\omega}_{z'}$$

Here, due to the symmetry of the disc we have taken $I_1 = I_2$. Using the expressions for the components of angular velocity given in (i), the Euler equations become;

$$\tau_{x'} = I_1\left(\dot{\phi}\dot{\psi}\sin\theta\cos\psi\right) + (I_3 - I_1)\left(\dot{\phi}\sin\theta\cos\psi\right)\left(\dot{\phi}\cos\theta + \dot{\psi}\right) \tag{ii}$$

$$\tau_{y'} = -I_1\left(\dot{\phi}\dot{\psi}\sin\theta\sin\psi\right) + (I_1 - I_3)\left(\dot{\phi}\sin\theta\sin\psi\right)\left(\dot{\phi}\cos\theta + \dot{\psi}\right) \tag{iii}$$

The third equation will be satisfied identically since $\tau_{z'} = 0$.

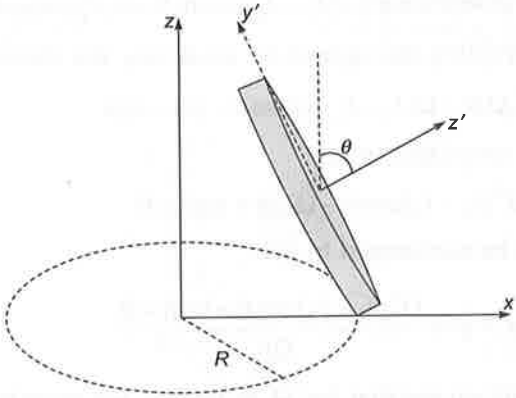

Fig. 7.19

Initially, when $t = 0$, the total force acting at the point of contact is the weight of the disc which is acting vertically and the force due to friction is horizontal. It is given by;

$$F = mg\hat{k} - (R - r\cos\theta)m\dot{\phi}^2\hat{i} \tag{iv}$$

The total torque due to this force on the disc is

$$\tau = -r\hat{j}' \times \left[mg\hat{k} - (R - r\cos\theta) m\dot{\phi}^2 \hat{i} \right]$$

This expression can be evaluated by making use of the relations between the unit vectors in the unprimed and primed coordinate systems. Then we get

$$\tau = -rm\left[g\cos\theta - (R - r\cos\theta)\dot{\phi}^2 \sin\theta \right]\hat{i}' \tag{v}$$

Now, the initial conditions are; at $t=0$, $\phi = \frac{\pi}{2}$ and $\psi = 0$. Then at $t=0$, $\tau_{y'} = 0$ as we have seen through equation (v) and is satisfied via the initial conditions. From (ii) and (v), we see that

$$-rm\left[g\cos\theta - (R - r\cos\theta)\dot{\phi}^2 \sin\theta \right] = \begin{bmatrix} I_1\left(\dot{\phi}\dot{\psi}\sin\theta\cos\psi\right) + \\ (I_3 - I_1)\left(\dot{\phi}\sin\theta\cos\psi\right)\left(\dot{\phi}\cos\theta + \dot{\psi}\right) \end{bmatrix}$$

Now, we apply the rolling condition, i.e., $R\dot{\phi} = -r\dot{\psi}$ in the above expression along with the initial conditions and rearrange to get

$$\dot{\phi}^2 = \frac{mgr\cos\theta}{\left[I_3\dfrac{R}{r} + (I_1 - I_3)\cos\theta + mr(R - r\cos\theta) \right]\sin\theta} \tag{vi}$$

For a uniform disc, we have

$$I_3 = \frac{1}{2}mr^2 \text{ and } I_1 = \frac{1}{4}mr^2$$

Substituting these in (vi), we get

$$\dot{\phi}^2 = \frac{mgr\cos\theta}{\left[\dfrac{1}{2}mr^2\dfrac{R}{r} + \left(\dfrac{1}{4}mr^2 - \dfrac{1}{2}mr^2 \right)\cos\theta + mr(R - r\cos\theta) \right]\sin\theta}$$

$$= \frac{mgr\cos\theta}{\left[\dfrac{1}{2}mr^2\dfrac{R}{r} - \dfrac{1}{4}mr^2\cos\theta + mr(R - r\cos\theta) \right]\sin\theta}$$

$$= \frac{4g\cos\theta}{(6R - 5r\cos\theta)\sin\theta} \tag{vii}$$

Equation (vii) gives the required result.

EXAMPLE 7.28 A uniform right circular cone of semi-vertical angle α moves under no forces except at its vertex which is fixed. It is set rotating about a generator. Show that its axis describes a right cone of semi-vertical angle β given by; $\tan\beta = \frac{1}{2}\tan\alpha + 2\cot\alpha$.

Solution: Let the cone rotate about the vertex O and the z–axis as the axis of the cone. Let h be the height and r the radius of the base of the cone. Let ω be the initial angular velocity with which the cone is rotating. Then,

$$\omega_{x'} = \omega \sin\alpha,\ \omega_{y'} = 0 \text{ and } \omega_{z'} = \omega\cos\alpha \tag{i}$$

For the cone with height h and radius r, has the principal moments of inertia

$$I_1 = I_2 = \frac{3}{20}m\left(r^2 + 4h^2\right) \text{ and } I_3 = \frac{3}{10}mr^2 \tag{ii}$$

In the present problem, the Euler equations can be written as

$$I_1\dot{\omega}_{x'} + \left(I_3 - I_1\right)\omega_{y'}\omega_{z'} = 0 \quad \text{or,} \quad \dot{\omega}_{x'} = \frac{\left(I_1 - I_3\right)}{I_1}\omega_{y'}\omega_{z'} \tag{iii}$$

$$I_1\dot{\omega}_{y'} + \left(I_1 - I_3\right)\omega_{x'}\omega_{z'} = 0 \quad \text{or,} \quad \dot{\omega}_{y'} = -\frac{\left(I_1 - I_3\right)}{I_1}\omega_{x'}\omega_{z'} \tag{iv}$$

$$I_3\dot{\omega}_{z'} = 0 \quad \text{or;} \quad \dot{\omega}_{z'} = 0 \tag{v}$$

From (v), on integration, we get

$$\omega_{z'} = a\text{, constant} \tag{vi}$$

From the initial condition, we have; $\omega_{z'} = \omega\cos\alpha$ or $a = \omega\cos\alpha$ (vii)

Dividing (iii) by (iv), we get

$$\frac{\dot{\omega}_{x'}}{\dot{\omega}_{y'}} = -\frac{\omega_{y'}}{\omega_{x'}} \quad \text{or;} \quad \omega_{x'}\dot{\omega}_{x'} + \omega_{y'}\dot{\omega}_{y'} = 0 \tag{viii}$$

On integration, this expression gives

$$\omega_{x'}^2 + \omega_{y'}^2 = b\text{, constant} \tag{ix}$$

Using the initial conditions, $\omega_{x'} = \omega\sin\alpha$ and $\omega_{y'} = 0$, the above equation gives

$$b = \omega^2\sin^2\alpha \tag{x}$$

Now, the direction cosines are proportional to, $I_1\omega_{x'}, I_2\omega_{y'}$ and $I_3\omega_{z'}$. If θ is the angle between the space fixed $z-$axis and the z' axis, then

$$\cos\theta = \frac{I_3\omega_{z'}}{\sqrt{I_1^2\omega_{x'}^2 + I_2^2\omega_{y'}^2 + I_3^2\omega_{z'}^2}} = \frac{I_3\omega_{z'}}{\sqrt{I_1^2\left(\omega_{x'}^2 + \omega_{y'}^2\right) + I_3^2\omega_{z'}^2}}$$

Substituting the initial values of the components of the angular velocity, we get

$$\cos\theta = \frac{I_3\omega\cos\alpha}{\sqrt{I_1^2\omega^2\sin^2\alpha + I_3^2\omega^2\cos^2\alpha}} = \frac{I_3}{\sqrt{I_1^2\tan^2\alpha + I_3^2}} \tag{xi}$$

On rearranging, we get

$$\cos\theta = \frac{1}{\left[\left(\frac{I_1}{I_3}\tan\alpha\right)^2 + 1\right]^{1/2}} \quad \text{or,} \quad \sec\theta = \left[\left(\frac{I_1}{I_3}\tan\alpha\right)^2 + 1\right]^{1/2}$$

From this expression, we see that

$$\tan\theta = \frac{I_1}{I_3}\tan\alpha = \frac{1}{2}\left(\frac{r^2+4h^2}{r^2}\right)\tan\alpha$$

Since $\frac{h}{r} = \cot\alpha$ the above expression becomes

$$\tan\theta = \frac{1}{2}\left(1+4\cot^2\alpha\right)\tan\alpha = \frac{1}{2}\tan\alpha + 2\cot\alpha \tag{xii}$$

The RHS of the above expression is a constant since α is a constant. Also $\theta = \beta$ and hence,

$$\tan\beta = \frac{1}{2}\tan\alpha + 2\cot\alpha \tag{xiii}$$

Hence proved.

EXERCISES

7.1 Determine the impact parameter and the differential scattering cross section for α particles of energy $10^{-12}\,J$ scattered by $Pb(Z=82)$ nucleus for a scattering angle of 30°.

7.2 Find the differential scattering cross section for the scattering of particles by the potential $V(r) = \alpha\left(\frac{1}{r} - \frac{1}{R}\right)$ for $r < R$ and $V(r) = 0$ for $r > R$.

7.3 Find the horizontal component of the Corioli's force acting on a body of mass 1 kg moving northward with a horizontal velocity 100 m/s at a latitude of 45° north.

7.4 If a body in the northern hemisphere falls freely under the effect of gravity from a height h, show that its deviation from the vertical when it reaches the ground is $\frac{2}{3}\omega h\left(\frac{2h}{g}\right)^{1/2}\cos\alpha$, where ω is the angular velocity of earth's rotation and α is the latitude.

7.5 A tidal current is running due north in the northern latitude λ with velocity v in a channel of width b. Prove that the level of water on the east coast is raised above that on the western coast by $(2\,bv\,\omega\sin\lambda)g$ where ω is the earth's angular velocity.

7.6 A train of mass m is travelling with a uniform velocity v along a parallel latitude. Show that the difference between the lateral force on the rails when it travels towards east and when it travels towards west is $4\ mv\ \omega \cos \lambda$, where λ is latitude and ω is the angular velocity of the earth.

7.7 Foucault's pendulum is a simple pendulum suspended by a long string from a high ceiling. The effect of Corioli's force on the motion of the pendulum is to produce a precession or rotation of the plane of oscillation with time. Find the time for one rotation for the plane of oscillation of the Foucault pendulum at 30° latitudes.

7.8 Find the principal moments of inertia about the centre of mass of a flat rigid body in the shape of a 45° right triangle with uniform mass density. Also obtain the principal axes.

7.9 Find the moment of inertia tensor, principal axes and principal moments of inertia of cylinder of radius R and height h.

7.10 Three equal mass points are located at $(a, 0, 0)$, $(0, a, 2a)$, and $(0, 2a, a)$. Find the moment of inertia tensor, the principal axes, and the principal moments of inertia.

7.11 Find the principal axis of rotation and the principal moments of inertia for a thin uniform rectangular plate of mass m and dimensions $2a$ by a for rotation about axes passing through (a) the centre of mass, and (b) a corner.

7.12 Determine the moment of inertia tensor about the origin of a uniform solid cone of mass m whose base circle is described by the equation $(x-a)^2 + z^2 = a^2$ and $y = 0$, whose vertex is at the point $(a, 5a, 0)$.

7.13 Show that a principal moments of inertia cannot exceed the sum of the other two.

7.14 A pendulum of length l with two bobs, one of mass m_1 at the end of the rigid support and one of mass m_2 halfway down. Let the system be at rest so the masses lie along the z axis. Find the moment of inertia tensor.

7.15 A particle of mass m is located at $x = 2$, $y = 0$, $z = 3$. (a) Find its moments and products of inertia relative to the origin; and (b) the particle undergoes pure rotation about the z-axis through a small angle α. Show that its moments of inertia are unchanged to first order in α if $\alpha \ll 1$.

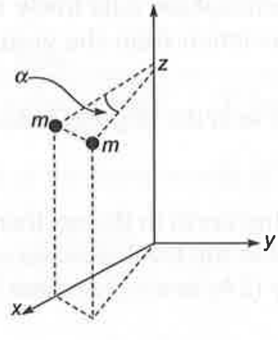

Fig. 7.20

7.16 A gyroscope wheel is at one end of an axle of length l. The other end of the axle is suspended from a string of length L (Fig. 7.21). The wheel is set into motion so that it executes uniform precession in the horizontal plane. The wheel has mass M and moment of inertia about its centre of mass I_0. Its spin angular velocity is ω_s. Neglect the mass of the shaft and of the string. Find the angle β that the string makes with the vertical assuming β as very small.

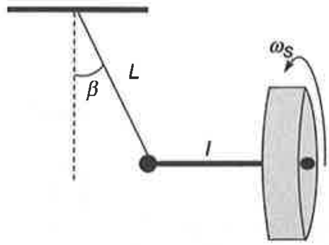

Fig. 7.21

7.17 A top of mass m spins with angular speed ω about its axis. The moment of inertia of the top about the spin axis is I_0, and the centre of mass of the top is a distance l from the point. The axis is inclined at angle θ with respect to the vertical, and the top is undergoing uniform precession. If the top is in an elevator, with its tip held to the elevator floor by a frictionless pivot. Find the rate of precession, (i) when the elevator is at rest; and (ii) when the elevator is moving down with an acceleration $2g$.

7.18 Show that a body whose principal moments of inertia I_1, I_2 and I_3 are all different, can rotate uniformly around one of them, say z'.

7.19 Study the motion of a "skating top", a spinning symmetrical top constrained to remain in contact with a smooth horizontal surface.

7.20 A cone of height h and semi-angle α rolls without slipping inside a fixed cone of semi-angle β with $\beta > \alpha$. The axis of the inner cone rotates about the axis of the outer cone with a constant angular velocity ω. Find the angular velocity, angular momentum and kinetic energy of the cone.

7.21 Using Euler's equations find the torque required to rotate a rectangular plate about a diagonal with a constant angular velocity ω.

Appendix

List of Integrals Used in this Book

1. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right)$

2. $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{a^2+x^2}\right)$

3. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) = -\cos^{-1}\left(\frac{x}{a}\right)$

4. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos^{-1}\left(\frac{x}{a}\right)$

5. $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b)$

6. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) = \frac{1}{a}\sin^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$

7. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$

8. $\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\left(\frac{x\sqrt{ab}}{a}\right)$

9. $\int \frac{dx}{x\left(a^2+x^2\right)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$

10. $\int \frac{x\,dx}{\left(a^2+x^2\right)} = \frac{1}{2}\ln\left(a^2+x^2\right)$

11. $\int \frac{x^2\,dx}{\left(a^2+x^2\right)} = x - a\tan^{-1}\left(\frac{x}{a}\right)$

12. $\int \frac{x^3\,dx}{\left(a^2+x^2\right)} = \frac{1}{2}\left[x^2 - a^2\ln\left(a^2+x^2\right)\right]$

13. $\int x\left(\sqrt{x-a}\right)dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}a(x-a)^{5/2}$

14. $\int \left(\sqrt{ax+b}\right)dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b}$

15. $\int \frac{x}{\sqrt{x\pm a}}\,dx = \frac{2}{3}(x\pm 2a)\sqrt{x\pm a}$

16. $\int \left(\sqrt{x^2\pm a^2}\right)dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left(x\pm\sqrt{x^2\pm a^2}\right)$

17. $\int \left(\sqrt{a^2-x^2}\right)dx = \frac{1}{2}x\sqrt{a^2-x^2} \pm \frac{1}{2}a^2\tan^{-1}\left(\frac{x\sqrt{a^2-x^2}}{a^2-x^2}\right)$

18. $\int \left(x\sqrt{x^2\pm a^2}\right)dx = \frac{1}{3}\left(x^2\pm a^2\right)^{3/2}$

19. $\int \frac{x}{\sqrt{x^2\pm a^2}}\,dx = \sqrt{x^2\pm a^2}$

20. $\int \frac{x}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2}$

21. $\int xe^x\,dx = (x-1)e^x$

22. $\int xe^{ax}\,dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$

23. $\int x^2e^x\,dx = \left(x^2-2x+2\right)e^x$

24. $\int \ln x\,dx = x\ln x - x$

25. $\int \frac{\ln(ax)}{x} dx = \frac{1}{2}\left[\ln(ax)\right]^2$

26. $e^x \int \ln(ax+b)\, dx = \frac{ax+b}{a}\ln(ax+b) - x$

27. $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4}\sin 2x$

28. $\int \sin^3 x dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x$

29. $\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4}\sin 2x$

30. $\int \cos^3 x dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x$

31. $\int \sin x \cos x dx = -\frac{1}{2}\cos^2 x$

32. $\int \tan^2 x dx = -x + \tan x$

33. $\int \tan^3 x dx = \ln(\cos x) + \frac{1}{2}\sec^2 x$

34. $\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln(\sec x \tan x)$

35. $\int e^x \sin x dx = \frac{1}{2}e^x(\sin x - \cos x)$

36. $\int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x)$

37. $\int e^{ax} \sin bx dx = \frac{1}{\left(a^2+b^2\right)} e^{ax}(a \sin bx - b \cos bx)$

38. $\int e^{ax} \cos bx dx = \frac{1}{\left(a^2+b^2\right)} e^{ax}(b \sin bx - a \cos bx)$

39. $\int x \cos x dx = \cos x + x \sin x$

40. $\int x \cos ax dx = \frac{1}{a^2}\cos ax + \frac{1}{a} x \sin ax$

Bibliography

Alexei Deriglazov, *Classical Mechanics, Hamiltonian and Lagrangian Formalism*, Springer, New York, (2010).

ChinmoyTaraphdar, *The Classical Mechanics*, Asian Books Private Limited, New Delhi, (2007).

D Ter Haar, *Elements of Hamiltonian Mechanics*, Pergamon Press, Oxford, (1971).

David Morin, *Introductory Classical Mechanics: With Problems and Solutions*, Cambridge University Press, Cambridge, (2004).

Dieter Strauch, *Classical Mechanics: An Introduction*, Springer, New York, (2009).

Donald T Greenwood, *Classical Dynamics*, Dover Publications Inc., New York, (1977).

Douglas Gregory, *Classical Mechanics*, Cambridge University Press, Cambridge, (2006).

Edward A Desloge, *Classical Mechanics: Vol. 1*, John Wiley and Sons, New York, (1982).

H. Goldstein, C. Poole & J. Safko, *Classical Mechanics*, Addison Wesley, San Francisco, (2000).

John R Tylor, *Classical Mechanics*, University Science Books, California, (2005).

Keith R Symon, *Mechanics*, Addison Wesley Publishing Company, California, (1971).

L A Pars, *A Treatise on Analytical Dynamics*, Heinemann Educational Books Ltd., London, (1965).

M G Calkin, *Lagrangian and Hamiltonian Mechanics*, World Scientific Publishing Pvt. Ltd., Singapore, (1998).

Mark D Andrema, *Analytical Dynamics: Theory and Applications*, Plenum Publishers, New York, (2006).

Walter Greiner, *Classical Mechanics: Point Particles and Relativity*, Springer, New York, (2004).

Index

Analytical Problems in Classical Mechanics

Complete Solutions

The book is meant for the students, both graduate and postgraduate, to understand the application of various formulations of classical mechanics and equip them for solving mechanical problems. Theory notes are given at the beginning of each chapter.

This book contains problems of various difficulty levels. Most of the problems are discussed in detail for understanding of the applications of various formulations. Certain number of problems are discussed in all formulations which will help the students to make a comparison between different formulations. A large number of problems are selected from various university examinations as well as from competitive examinations like NET, JEST, GATE and Civil Services. Practice problems are also given at the end of each chapter. Consequently, this book will also be helpful to the students preparing for such examinations.

Salient Features:

- Application of various formulations of classical mechanics.
- Theory notes.
- Problems in all formulations.
- Classical problems with applications of various formulations.

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