Volume 3

Feynman Hughes Lectures

Matter-Wave Interacting Continued
Introduction to Quantum Mechanics
Scattering Theory
Perturbation Theory
Methods & Problems in QED

July 1968-June 1969

Notes taken & Transcribed by
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CONTENTS - VOLUME III

CHAPTER 4 (CONTINUED) PAGES

MATTER AND WAVES INTERACTING 1 - 28

CAVITY RESONATORS WITH ARBITRARY CHARGES AND CURRENTS
MORE ON CHARGES AND CURRENTS IN A CAN
WAVES IN FREE SPACE
ANALOGY BETWEEN WAVEGUIDES AND QUANTUM MECHANICS
ELECTROMAGNETIC MASS
MAGNETOSTATICS

CHAPTER 5

INTRODUCTION TO QUANTUM MECHANICS 29 - 100

PROBABILITIES AND AMPLITUDES
WATCHING ELECTRONS
DIRAC NOTATION
INTRODUCTION TO PATH INTEGRALS
FREE PARTICLE MOTION
SCHRÖDINGER EQUATION
DIRAC NOTATION
PENETRATION OF A COULOMB BARRIER
HYDROGEN ATOM

CHAPTER 6

SCATTERING THEORY 105 - 120

SCATTERING FROM A COULOMB BARRIER
QUANTUM THEORY OF SCATTERING

CHAPTER 7

QUANTUM THEORY OF ANGULAR MOMENTUM 120 - ~~166~~ 192

ROTATION OPERATORS
COMMUTATION RULES
CLEBSCH-GORDON COEFFICIENTS
SPIN ANGULAR MOMENTUM
ROTATION MATRICES
RESONANT SCATTERING OF PION AND PROTON
INTRODUCTION TO LIE GROUPS, SU2 AND SU3
PION AND NUCLEON SCATTERING
SU2 AND SU3 THEORY - QUARK THEORY
SOLVING THE DIRAC EQUATION FOR HYDROGEN ATOM
SOLVING THE HARMONIC OSCILLATOR BY OPERATORS

DATE	LECTURE	TOPIC	PAGE
1 JULY 1968	1.	SOLVING DIFFERENTIAL EQUATIONS IN TERMS OF ORTHOGONAL FUNCTIONS - FOURIER ANALYSIS	1
		• CAVITY RESONATOR WITH ARBITRARY CHARGE AND CURRENT DENSITIES INSIDE	3
8 JULY	2.	MORE ON THE CHARGES AND CURRENTS IN A CAN	4
		• ELECTRODYNAMICS AS A SERIES OF DRIVEN HARMONIC OSCILLATORS	4
		• VECTOR AND SCALAR POTENTIAL NOTATION	6
		• A PATH TO QUANTUM ELECTRODYNAMICS	8
15 JULY	3.	WAVES IN FREE SPACE	9
		• POLARIZATION AND PROPAGATION VECTORS	9
		• ELECTROMAGNETICS REDUCED TO THE COULOMB POTENTIAL AND A BUNCH OF DRIVEN OSCILLATORS	11
		• NORMALIZATION IN FOURIER THEORY	12
		• WAVEGUIDES	14
		• THE RECTANGULAR WAVEGUIDE AND CUTOFF FREQUENCY	15
22 JULY	4.	ANALOGY BETWEEN WAVE GUIDES AND QUANTUM MECHANICS	17
		• A WORLD OF 5 DIMENSIONS	18
		• ANGULAR MOMENTUM IN CIRCULARLY POLARIZED GUIDED WAVES AND AN EXTENSION TO QUANTUM MECHANICS	19
		• ON THE PREDICTION OF MAGNETIC MONOPOLES	21.
29 JULY	5.	ELECTROMAGNETIC MASS	22
5 AUG.	6.	MAGNETOSTATICS	25
		• MAGNETIC FIELD FROM A WIRE	26
		• FIELD FROM A SMALL LOOP OF WIRE	27
		• MAGNETIC MOMENT OF AN ATOM	28
30 SEPT	7.	INTRODUCTION TO QUANTUM MECHANICS	29
		• IDEA OF ENERGY STATES	30
		• QUANTIZED ANGULAR MOMENTUM	32
7 OCT	8.	MAGNETIC MOMENTS AND ANGULAR MOMENTUM	34
		• TWO HOLE EXPERIMENT - PROBABILITIES & AMPLITUDES	35
		• WATCHING THE ELECTRONS	39
23 OCT	9.	MORE ON WATCHING THE ELECTRONS	41
		• MEASUREMENT OF POSITION & MOMENTUM	43

LECTURE	TOPIC	PAGE
10	AMPLITUDE FOR EVENTS INVOLVING IDENTICAL PARTICLES	46
11	INTRODUCTION TO DIRAC NOTATION	47
	• NON-RELATIVISTIC ELECTRON AMPLITUDES	48
	• DEFINITION OF THE KERNEL	50
12	AMPLITUDE INVARIANCE FROM GALILEAN TRANSFORMATION	52
	• FREE PARTICLE AMPLITUDE	53, 55
13	CONSERVATION OF PROBABILITY	56
	INTRODUCTION TO PATH INTEGRALS	57
	• PARTICLE MOVING IN AN EXTERNAL POTENTIAL	59
	• INTERPERTATION OF PATH INTEGRALS	60
	• The CLASSICAL LIMIT	60
14	FREE PARTICLE RELEASED FROM A GAUSSIAN DISTRIBUTION	62
	• DERIVATION OF The UNCERTAINTY PRINCIPLE	63
	• FREE PARTICLE RELEASED FROM A MODULATED GAUSSIAN DISTRIBUTION	64
	• MOMENTUM REPRESENTATION	66
15.	THE SCHRÖDINGER EQUATION FROM PATH INTEGRALS	68
	• PERTURBING POTENTIALS	70
	• SPIN-1/2, SUPERPOSITIONS OF STATES	71
16.	THE DIRAC EQUATION	73
	• RELATIVISTIC DIRAC EQUATION	76
	• PAULI SPIN MATRICES	77
	• RELATIVISTIC PARTICLE WITH SPIN	
17	The PARTICLES IN THE PROTON (A SPECIAL LECTURE)	79
18	SOLUTIONS TO The Schrödinger EQUATION	81
	• PROPERTIES OF LINEAR OPERATORS	81-83
	• QUANTIZED ENERGY LEVELS	83
19.	PENETRATION OF A COULOMB BARRIER	84
	• RADIOACTIVE DECAY ANALOGY	85
	• SPherical WAVE SOLUTIONS	87

LECTURE	TOPIC	PAGE
20	TIME VARYING WAVE FUNCTION (COULOMB BARRIER PROBLEM CONTINUED)	90
	· CONTOUR INTEGRATION	92
	· PARTICLE AMPLITUDE OUTSIDE THE BARRIER	93
21	SCHRÖDINGER'S EQUATION FOR THE HYDROGEN ATOM	95
	· SPHERICALLY SYMMETRIC SOLUTION	96
	· SOLUTIONS WITH ANGULAR DEPENDENCE	97
	· ANGULAR MOMENTUM ANALOG	98
	· GUESSING THE SOLUTIONS TO THE DIFFERENTIAL EQUATION	99
	· ORBITAL ANGULAR MOMENTUM	101
22.	INTRODUCTION TO SCATTERING THEORY	103
	· USING THE SPHERICAL HARMONICS IN PROBLEM SOLVING	103
	· PARTICLE IN A COULOMB POTENTIAL AND NEAR THE ORIGIN	104
23	SCATTERING THEORY	105
	· PLANE WAVE SCATTERING OFF A COULOMB BARRIER	
	· ASYMPTOTIC SOLUTIONS	106
	· SCATTERING AMPLITUDE WITH ANGULAR DEPENDENCE	109
24.	QUANTUM THEORY OF SCATTERING	110
	· SCATTERING CROSS SECTION	
	· ELASTIC SCATTERING CROSS SECTION	111
	· THE PHYSICS OF SCATTERING	112
	· OPTICAL THEOREM, CONSERVATION OF PARTICLES DURING SCATTERING	114
	· RESONANCE SCATTERING, S-WAVES	118
25	ROTATION AND ANGULAR MOMENTUM	121
	· INTRODUCTION TO THE $\mathcal{D}_{ji}(R)$ OPERATOR	
	· COMMUTATION RULES	122
	· ROTATION - ANGULAR MOMENTUM ANALOGY	124
26	GENERATORS OF THE ROTATION OPERATOR	126
	· RAISING AND LOWER OPERATORS	
	· DERIVATION OF THE CLEBSCH-GORDON COEFFICIENTS	127

LECTURE	TOPIC	PAGE
27	SPIN ANGULAR MOMENTUM	130
	· SPIN 1/2 AND ROTATION MATRICES	131
	· COMPOSITION OF ANGULAR MOMENTUM for A 2 PARTICLE, SPIN 1/2 SYSTEM	133
28	PROBLEM SOLVING WITH ANGULAR MOMENTUM	138
	· RESONANT SCATTERING of The Pi-MESON AND The PROTON	138
	· ROTATION MATRICES for SPIN 1/2	142
	· ROTATION MATRICES for SPIN 3/2	143
	· AN EASY WAY TO FIND The ROTATION MATRIX ELEMENTS	145
	· RETURN TO PROTON SCATTERING	147
29	INTRODUCTION TO LIE GROUPS, SU2 AND SO3	148
	· The PHYSICS of LIE GROUPS	148, 149
	· GENERATORS of The GROUP	150
	· IRREDUCIBLE REPRESENTATIONS of The ROTATION GROUP	150
	· APPLICATIONS of The SU2 GROUP TO PHYSICS, SPIN 1/2 SYSTEM	151
	· STRONG INTERACTIONS, INTRODUCTION	152
30	PION - NUCLEON SCATTERING, AN APPLICATION of The SU2 GROUP	154
	INTRODUCTION TO The SU3 GROUP	159
31	MORE ON THE SU3 STUFF AND QUARKS	163
32.	SOLVING THE DIRAC EQUATION FOR THE HYDROGEN-LIKE ATOM	166
	· GUESSING The SOLUTIONS of The DIRAC EQ.	168
33	SOLVING THE DIRAC EQUATION FOR THE HYDROGEN ATOM	170
	· MAKING THE SOLUTION WORK	177
	· FINE LINE SPLITTING	178
	· DIFFICULTIES WITH QUANTUM ELECTRODYNAMICS	179
34	SOLVING THE HARMONIC OSCILLATOR PROBLEM with differential EQUATION TECHNIQUES	180
	· SOLVING THE HARMONIC OSCILLATOR BY OPERATORS	185
35	· A LITTLE THERMODYNAMICS	189
	· MORE THERMODYNAMICS	192

LECTURE	TOPIC	PAGE
36	METHODS AND PROBLEMS of QUANTUM ELECTRODYNAMICS	195
	· ZERO POINT ENERGY	197
	· ELECTRON-POSITRON ANNIHILATION	199
	· ON THE SIMPLICITY of NATURE	201
37	INTRODUCTION TO PERTURBATION THEORY	202
	· SEPARABLE SYSTEMS	203
	· TIME DEPENDENT PERTURBATION THEORY	205
38	INTERACTION of AN ATOM WITH AN ELECTROMAGNETIC FIELD	213
	· TRANSITION AMPLITUDE	217
	· RESONANT TRANSITIONS	219
39	COULOMB EXCITATION	222
	· ATOMIC IONIZATION	222
	· CALCULATING MATRIX ELEMENTS	223
40	TRANSITION RATES AND SCATTERING THEORY	227
	· SEPARABILITY of INTERNAL AND CENTER of MASS MOTIONS	228
	· ELASTIC SCATTERING TRANSITION RATES	230
	· SCATTERING CROSS-SECTION VIA TRANSITION RATE APPROACH	233

QED
CHAPTER 9 - SUPPLEMENT (DATE UNK).
(~MID-1969)

LECTURE	TOPIC	PAGE (Renumbered with Lecture 41.)
41.	INTERACTION of A RADIATION FIELD & MATTER	1
	• SPONTANEOUS EMISSION from AN ATOM IN A QUANTIZED FIELD	
	• SEPARATION of THE HAMILTONIAN	2
	• CALCULATING THE INTERACTION MATRIX ELEMENTS	6
	• RATE of SPONTANEOUS EMISSION	7
42.	REMARKS ON 1st order NON-RELATIVISTIC PERTURBATION THEORY	8
	• USE of CREATION AND ANNIHILLATION OPERATORS	9
	• RADIATION EQUILIBRIUM	10
	• MANY ELECTRON SYSTEM	10
	• DIRAC EQUATION INVARIANT FORM of PERTURBATION THEORY	11
	• SCATTERING of LIGHT FROM AN ATOM by 2nd order PERTURBATION THEORY	12.
	• TWO PHOTON LIGHT SCATTERING	14
43.	LIGHT SCATTERING from A RELATIVISTIC ELECTRON COMPTON SCATTERING	16
	• CALCULATING THE SCATTERING MATRIX ELEMENTS	17
	• SUM OVER NEGATIVE ENERGY STATES	21
	• PAIR PRODUCTION - NEGATIVE SEA AND SPACE TIME EXPLANATION	23
44.	MANIFESTLY INVARIANT THEORY of PAIR PRODUCTION	26
	• RULE for VIRTUAL ELECTRON PROPAGATION	26
	• RULE for PHOTON EMISSION AND ABSORPTION	27
	• COMPTON SCATTERING	27
	• INVARIANT MATRIX ELEMENT	29
	• MASS, ENERGY NORMALIZATION	29
	• GENERAL EQUATION for COMPUTING TRANSITION RATES	31
45.	RULES of QUANTUM ELECTRODYNAMICS	32
	• SUMMARY of 3-RULES	32
	• TWO PHOTO EMISSION - EXAMPLE of RULES	32
	• ELECTRON - ELECTRON SCATTERING, RULE for VIRTUAL PHOTON PROPAGATION	34

LECTURE	TOPIC	PAGE
46	HIGHER ORDER CORRECTION TERMS	39
	• FIRST ORDER CORRECTIONS	39
	• PAIR PRODUCTION FROM ELECTRON SCATTERING IN A NUCLEAR FIELD	40
	• SELF ENERGY OF THE ELECTRON	41
47.	MASS AND CHARGE RENORMALIZATION	45
	• SECOND ORDER CORRECTION TERMS	45
	• INFINITE SELF MASS	48
	• LAMBDA THEORY OF MASS RENORMALIZATION	49
	• CHARGE RENORMALIZATION (MORE ON)	50
48	AN INTRODUCTION TO WEAK INTERACTION	56

The lectures that comprise Volume 3 are a continuation of Volume 2. Feynman started this series earlier than normal, July versus October, since he felt he had not completed the prior year's lectures where he wanted to be. So this Volume 3 picks up where he left off and runs all the way into QED and weak interactions. I am sure the reader will find in it, as I did, the magic and genius of Feynman at his best.-jtn 12/20/2012.

1 July 68

1. SOLVING DIFFERENTIAL EQUATIONS IN TERMS OF ORTHOGOMAL FUNCTIONS

IN THE COURSE OF DISCUSSING CONFINED WAVES WE CAME TO THE SUBJECT OF EXPANDING FUNCTIONS IN TERMS OF ORTHOGONAL FUNCTION. THE EQUATION THAT LED US TO THIS SUBJECT CAME FROM SOLVING MAXWELL'S EQUATIONS INSIDE A CAVITY,

$$\nabla \cdot E = 0 \qquad \nabla \cdot B = 0 \qquad \nabla \times E = -\partial B/\partial t \qquad \nabla \times B = \partial E/\partial t$$

FROM THE LAST TWO EQUATIONS WE WERE LEFT WITH THE EQUATION

$$\nabla \times (\nabla \times E) = -\frac{\partial^2 E}{\partial t^2}$$

WE SAID IF E WAS PERIODIC WITH FREQUENCY ω, THEN WE HAD TO SOLVE

$$\nabla \times (\nabla \times E) = -\omega^2 E$$

SUBJECT TO THE CONDITION THAT $\nabla \cdot E = 0$ INSIDE AND $E_{TANGENTIAL} = 0$ AT THE WALLS. IN SOLVING THIS EQUATION THE PROBLEM WAS REDUCED TO A SET OF CHARACTERISTICS MODES OF VIBRATIONS E_n EACH WITH ITS OWN FREQUENCY, ω_n. THE FUNCTIONS E_n OR THE EIGENFUNCTIONS SATISFY THE CONDITION OF ORTHOGONALITY

$$\int \bar{E}_n \cdot \bar{E}_m \, d^3\bar{R} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

THERE ARE MANY PROBLEMS WHICH WE CAN SOLVE WHEN THE FUNCTION IS EXPANDED IN A SERIES OF THESE EIGENFUNCTIONS. SUPPOSE, FOR EXAMPLE, WE WERE GIVEN THE ELECTRIC FIELD AT SOME INITIAL TIME $t=0$, $\bar{E}(x,y,z,0) = \bar{E}_0(x,y,z)$. WHAT WOULD BE THE ELECTRIC FIELD AT SOME LATER TIME t. THE ELECTRIC FIELD IS EXPRESSED AS A SOPERPOSITION OF ALL THE INDIVIDUAL MODES EACH CONTRIBUTING A CERTAIN AMOUNT $a_n(t)$, I.E.,

$$E(x,y,z,t) = \sum_n a_n(t) \, E_n(x,y,z)$$

THE SUM OVER ALL n TAKES IN ALL POSSIBLE MODES. WE NOTE IF THE EQUATION IS FIRST MULTIPLIED BY E_m AND INTEGRATED OVER ALL SPACE, THAT

$$\int \bar{E} \cdot \bar{E}_m \, d^3\bar{R} = \sum_n a_n(t) \int \bar{E}_m \cdot \bar{E}_n \, d^3\bar{R}$$

AS WE JUST SAY THE RIGHT HAND INTEGRAL IS ZERO FOR ALL $n \neq m$ SO ONLY ONE TERMS REMAIN AND THAT IS WHEN $n = m$. THUS WE CAN FIND EACH COEFFICIENT $a_m(T)$,

$$a_m(t) = \int \bar{E}_m \cdot \bar{E} \, d^3\bar{R}$$

THE INTEGRAL TELLS US THE DEGREE TO WHICH E CONTAINS THE PARTICULAR MODE SHAPE E_m. THE ABOVE EQUATION IS THE INVERSE EQUATION OF $E = \Sigma a_n E_n$ AND TELLS US HOW TO FIND ALL THE a_n'S.

RETURN NOW TO THE PROBLEM OF FINDING THE FIELD AT SOME LATER TIME AFTER THE INITIAL CONDITION $\bar{E} = \bar{E}_0$, WE WRITE

$$\bar{E}(x,y,z,t) = \sum a_n(t)\, \bar{E}_n(R)$$

THIS FUNCTION MUST SATISFY

$$\nabla \times (\nabla \times E) = \frac{\partial^2 E}{\partial t^2}$$

OR UPON SUBSTITUTING,

$$\sum a_n(t)\, \nabla \times (\nabla \times \vec{E}_n(R)) = \sum_n \frac{\partial^2 a_n(t)}{\partial t^2} E_n(R)$$

WE HAVE SEEN IN A PREVIOUS ANALYSIS THE EIGENFUNCTIONS SATISFY THE THE EQUATION,

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}_n) = -\omega_n^2 \bar{E}(\bar{R})$$

SUBJECT TO THE CONDITION $\bar{\nabla} \cdot \bar{E}_n = 0$. NOW IF THIS RESULT IS PUT INTO THE ABOVE EQUATION THE COEFFICIENTS MUST SATISFY THE EQUATION,

$$\ddot{a}_n(t) = -\omega_n^2 a_n(t)$$

THIS IS JUST THE EQUATION FOR A HARMONIC OSCILLATOR SO WE'RE REALLY GOING AROUND IN A CIRCLE BECAUSE WE FIRST ASSUMED OUR SOLUTION WAS SINUSOIDAL.

SINCE WE KNOW THE INITIAL ELECTRIC FIELD $\bar{E}_0(R)$ WE CAN FIND $a_n(0)$ from

$$E_0(R) = \sum_n a_n(0)\, E_n(R)$$

or

$$a_m(0) = \int E_m \cdot E_0\, d^3R$$

BUT TO SOLVE $\ddot{a}_n = -\omega_n^2 a_n$ WE NEED MORE INFORMATION THAN JUST $a_n(0)$; WE ALSO NEED THE VELOCITY OF a_n INITIALLY, I.E., $\dot{a}_n(0)$. SO NEED TO KNOW dE_0/dt WHICH IS SOME FUNCTION OF POSITION $\bar{R}$ SAY $\bar{F}_0(R)$. THEN

$$\bar{F}_0(R) = \sum_n \dot{a}_n(0)\, E_n(\bar{R})$$

where

$$\dot{a}_m(0) = \int \bar{E}_n(R) \cdot \bar{F}_0(R)\, d^3R$$

THIS PROBLEM COULD HAVE BEEN SOLVED IF B HAD BEEN GIVEN INITIALLY, SINCE $-(\nabla \times E_n) = i\omega_n B_n$, B_n IS GIVEN AS

$$B_n = \frac{i}{\omega_n} (\nabla \times E_n)$$

IF E_0, AND B_0 ARE GIVEN, THE PROBLEM CAN BE SOLVED.

A MORE DIFFICULT PROBLEM TO SOLVE IS WHEN THE CAVITY HAS SOME CHARGES AND CURRENTS INSIDE. THIS CASE IS LIKE A DRIVEN OSCILLATOR where THE FIELDS SATISFY THE SET OF DIFFERENTIAL EQUATIONS,

$$\bar{\nabla}\cdot\bar{E} = \rho \qquad \bar{\nabla}\cdot\bar{B} = 0 \qquad \bar{\nabla}\times\bar{E} = -\partial\bar{B}/\partial t \qquad \bar{\nabla}\times\bar{B} = \partial\bar{E}/\partial t + \bar{j}(R,t)$$

AGAIN FROM THE LAST TWO EQUATIONS WE GET

$$\nabla\times\nabla\times E = \frac{\partial^2 E}{\partial t^2} + \frac{\partial \dot{j}}{\partial t}$$

WE ARE GOING TO ASSUME THAT SOMEHOW I KNOW THE CURRENTS AND CHARGE DISTRIBUTIONS INSIDE THE CAVITY. IN REALITY I'LL NEVER KNOW WHAT ~~the~~ ρ's AND j's ARE PRODUCED INSIDE BECAUSE OF THE BACK ACTION OF THE INDUCED WALL CURRENTS AND CHARGES ON EXTERNALLY APPLIED ρ's AND j's.

NOW I'LL WRITE THE ELECTRIC FIELD AS

$$E(R,t) = \Sigma_n\, a_n(t)\,\bar{E}_n(R) + \bar{X}(t,\bar{R})$$

THE EXTRA TERM IS NEEDED BECAUSE THE EQUATION $\nabla\times\nabla\times E$ IS SUBJECT TO THE CONDITION THAT $\nabla\cdot E = \rho$. THAT IS WHEN WE EXPAND $\nabla\times\nabla\times E$ AS $\nabla(\nabla\cdot E) - \nabla^2 E$, THE FIRST TERM IS $\nabla\rho$ NOT 0 AS BEFORE. BUT BY WRITING THE FIELD IN THE ABOVE FORM WE SEE THAT $\bar{\nabla}\cdot\bar{X} = \rho$. LET ME IGNORE THIS TERM FOR THE MOMENT AND WRITE

$$\Sigma_n\, a_n(t)\,(-\omega_n^2)\,E_n(R) = \Sigma\,\ddot{a}_n(t)\,E_n(R) + \frac{\partial \dot{j}}{\partial t}$$

I'd LIKE TO WRITE $j(R,t)$ IN TERMS OF $E_n(R)$ IN THE FOLLOWING WAY,

$$j(R,t) = \Sigma_n\, J_n(t)\, E_n(R)$$

where THE COEFFICIENTS ARE

$$J_n(t) = \int j(R,t)\cdot E_n(R)\, d^3R$$

PUTTING $j(R,t)$ INTO THE ABOVE EQUATION AND AGAIN EQUATING COEFFICIENTS WE FIND THAT

$$-\omega_n^2\, a_n(t) = \ddot{a}_n(t) + \dot{J}_n(t)$$

THIS differENTIAL EQUATION IS LIKE A forCED OSCILLATOR, I.E., $F - \omega_0^2 X = \ddot{X}$ If $J_n(t)$ HAPPENED TO BE $K\cos\omega t$ THEN IF $a_n(t) = \alpha\cos\omega t$ WE FIND AS A STEADY STATE SOLUTION,

$$\alpha = \frac{K\omega}{\omega^2 - \omega_n^2}$$

If THE CAVITY HAS LOSES ASSOCIATED WITH IT, A TERM $i\omega\gamma$ WOULD APPEAR AND REPRESENT THE DAMPING.

2. More on the charges and currents in a can

I'd like to fix up a few things that I messed up last time. You all tried to tell me what I was doing wrong but I couldn't see it. We had written down Maxwell's equations for some arbitrary charges and currents inside a metal can; they were:

$$\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0 \qquad \bar{\nabla}\cdot\bar{B}=0 \qquad \bar{\nabla}\times\bar{B} = \frac{\partial\bar{E}}{\partial t} + \bar{j} \qquad \bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

Note that ρ and $\bar{j}$ are the charges and currents <u>inside</u> the can — not on the wall. The above set of equations could be combined into an equation which was written as

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}) = \frac{\partial^2\bar{E}}{\partial t^2} + \dot{j}$$

In solving this equation we subjected it to the condition that $E_{TANG.}=0$ on the surface. When the electric field varied sinusoidally, the solution to the above equation was expressible in terms of an infinite set of characteristic modes of vibrations E_n and each having frequency ω_n. The eigenfunctions as we called them were solutions under the charge free condition that $\bar{\nabla}\cdot\bar{E}_n=0$. But when I had $\bar{\nabla}\cdot\bar{E}_n = \rho/\epsilon_0$ I did not expressed the electric field properly.

When I wrote for the solution

$$\bar{E} = \sum a_n(t)\, E_n$$

and found the set of harmonic oscillator equations

$$\ddot{a}_n(t) - \omega_n^2 a_n(t) = \frac{d\dot{j}_n}{dt}$$

I did not consider all the components of the electric field. In other words, I have to add something to $\sum a_n(t)\, E_n$ to straighten everything out.

One way to correct the problem is to take advantage of a characteristic of all vector fields, namely, any field can be decomposed into a transverse and longitudinal component subject to the condition that if

$$\bar{E} = \bar{E}_T + \bar{E}_L$$

Then, $\bar{\nabla}\times\bar{E}_L = 0$ and $\bar{\nabla}\cdot\bar{E}_T = 0$

If we put this new notation into Maxwell's equations, we have that

$$\bar{\nabla}\cdot\bar{E}_L = \rho/\epsilon_0 \qquad \bar{\nabla}\times\bar{B}_T = \frac{\partial\bar{E}}{\partial t} + \dot{j} \qquad \bar{\nabla}\times\bar{E}_T = -\frac{\partial\bar{B}}{\partial t} \qquad \bar{\nabla}\cdot\bar{B}_L = 0$$

First we note that $\bar{B}$ is entirely transverse since it comes from the transverse component of the electric field and $\bar{\nabla}\cdot\bar{B}_L = 0$.

If I combine the equation $\bar{\nabla}\times\bar{E}_L = 0$ with $\bar{\nabla}\cdot\bar{E}_L = \rho/\epsilon_0$, I find I must solve an electrostatic equation. That is, for $\bar{\nabla}\times\bar{E}_L = 0$, $\bar{E}_L = -\nabla\varphi$ so that $\nabla\cdot E_L = -\nabla^2\varphi = \rho/\epsilon_0$. At each moment I must solve an electrostatic problem subject to the condition that $E_{L\,TANG} = 0$ at the surface. It is this additional problem that has to be solved that I forgot.

Let me go on by writing $\bar{\nabla}\times\bar{B}_T = \frac{\partial\bar{E}}{\partial t} + \bar{j}$ in terms of the two $\bar{E}$-field components,

$$\bar{\nabla}\times\bar{B}_T = \frac{\partial\bar{E}_T}{\partial t} + \bar{j} + \frac{\partial\bar{E}_L}{\partial t}$$

Now I am going to write $\bar{j} + \frac{\partial\bar{E}_L}{\partial t}$ as $\bar{j}_T$ because $\bar{j}$ is a perfectly good vector and can be broken up just like any other field. Since $\bar{\nabla}\cdot\bar{j}_T = 0$ by our theorem we require that

$$\bar{\nabla}\cdot\bar{j} + \frac{\partial\,\bar{\nabla}\cdot\bar{E}_L}{\partial t} = 0$$

or

$$\bar{\nabla}\cdot\bar{j} + \frac{1}{\epsilon_0}\frac{\partial\rho}{\partial t} = 0$$

which is just the conservation of charge law. Since $\bar{j} = j_L + j_T$, j_L must equal $-\frac{\partial E_L}{\partial t}$.

Now to clear up the trouble. Last time when I worked this problem what I really solved was the equation

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}_{Tn}) = \omega_n^2 E_{Tn}$$

where

$$\bar{E}_T = \sum a_n(t) E_n \qquad \text{and} \qquad \bar{\nabla}\cdot\bar{E}_T = 0$$

In the solution when

$$\ddot{a}_n(t) - \omega_n^2 a_n(t) = \frac{d\bar{j}_n}{dt}$$

j_n was written as $\int j\cdot E_n\, dVOL$. Since $j = j_T + j_L$, this integral can be written as

$$j_n = \int j_T\cdot E_n\, dVOL + \int j_L\cdot E_n\, dVOL$$

The last term can be integrated by parts since $j_L = \nabla\chi$ in order to satisfy the requirement that $\nabla\times j_L = 0$. Therefore,

$$\int j_L\cdot E_n\, dVOL = \int \nabla\chi\cdot E_n\, dVOL = \int \chi(\bar{n}\cdot\bar{E}_n)\, dsurf + \int \chi\,\nabla\cdot E_n\, dVOL.$$

The first term on the right I'll throw out because $j_L = \frac{\partial E_L}{\partial t} = \nabla\frac{\partial\varphi}{\partial t}$ and φ is instantaneously constant on the surface. That is

$$\int \chi\,(n\cdot E_n)\, dsurf = \text{constant}\int n\cdot E_n\, dsurf = \text{const}\int \nabla\cdot E_n\, dVOL = 0$$

and $\nabla\cdot E_n = 0$ on the surface

WHAT WE HAVE SHOWN IS THAT TO SOLVE THE PROBLEM OF ARBITRARY CHARGES AND CURRENTS IN A CAN, YOU HAVE TO SOLVE TWO EQUATIONS. ONE PROBLEM IS THE CHARGE AND CURRENT FREE PROBLEM WHERE $\bar{\nabla} \cdot E_T = 0$ AND THE OTHER PROBLEM IS THE ELECTROSTATIC ONE WHERE $\nabla \cdot E_L = \rho/\epsilon_0$. WE SHOWED HOW TO FIND $\bar{E}_T$ AND $\bar{E}_L$ WAS FOUND THROUGH A SERIES OF ELECTROSTATIC PROBLEMS.

ONE FURTHER POINT WE COULD HAVE WORKED THIS PROBLEM OUT IN TERMS OF $\bar{B}$ AND FOUND THAT j_T ALONE DRIVES THE $\bar{B}$ EQUATION. THAT IS, FROM MAXWELL'S EQUATIONS WE GE

$$\nabla \times (\nabla \times B) = \frac{\partial^2 B}{\partial t^2} + \bar{\nabla} \times \bar{j} = \frac{\partial^2 B}{\partial t^2} + \bar{\nabla} \times \bar{j}_T$$

SINCE $\nabla \cdot B = 0$, THIS BECOMES

$$\nabla^2 B - \frac{\partial^2 B}{\partial t^2} = \bar{\nabla} \times \bar{j}_T$$

VECTOR AND SCALAR POTENTIAL NOTATION

I COULD HAVE WORK THIS PROBLEM OUT IN TERMS OF THE VECTOR AND SCALAR POTENTIAL. THIS IS USUALLY DONE IN ALL OF THE LITERATURE. IF WE RECALL $B = \nabla \times A$ AND $E = -\nabla\varphi + \frac{\partial A}{\partial t}$

NOTE IF THE ELECTRIC FIELD IS BROKEN INTO A TRANSVERSE AND LONGITUDINAL PART, THEN

$$E_T = \frac{\partial A}{\partial t} \quad \text{AND} \quad E_L = -\nabla\varphi$$

SINCE $\nabla \times E_L = 0$ AND $\nabla \cdot E_T = 0$

AS YOU MAY RECALL FROM PREVIOUS LECTURES THE SCALAR AND VECTOR POTENTIALS DO NOT UNIQUELY DEFINE THE ELECTRIC FIELD. WE ARE, THEREFORE, FREE TO CHOSE THE GAUGE TRANSFORMATION THAT IS THE MOST USEFUL. IF WE CHOSE THE LORENTZ GAUGE, I.E.,

$$\nabla \cdot A = \frac{\partial \varphi}{\partial t}$$

THE POTENTIAL EQUATIONS ARE UNCOUPLED,

$$\nabla^2 \bar{A} - \frac{\partial^2 \bar{A}}{\partial t^2} = \bar{j} \quad \text{AND} \quad \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial t^2} = \rho$$

THIS SHOWS THAT THE CHARGE DISTRIBUTION ρ DRIVES THE φ EQUATION AND THE CURRENT DISTRIBUTION j DRIVES THE $\bar{A}$ EQUATION.

IF WE CHOOSE A DIFFERENT GAUGE, THE SOLUTIONS FOR $\bar{E}$ ARE THE SAME BUT THE POTENTIALS ARE DIFFERENT. ONE GAUGE WE MIGHT TRY IS THE COULOMB GAUGE, $\bar{\nabla} \cdot \bar{A} = 0$. WITH THIS CHOICE MAXWELL'S EQUATIONS GIVES US

$$\bar{\nabla} \cdot \bar{E} = \nabla \cdot \left(-\nabla\varphi + \frac{\partial A}{\partial t}\right) = \rho/\epsilon_0$$

OR

$$\nabla^2 \varphi = -\rho/\epsilon_0$$

The solution of this time independent equation has been worked out before as

$$\varphi(1) = \int \frac{\rho(2,t)}{r_{12}} \, dVol_2$$

This is the coulomb potential without a retarded time.

From the other Maxwell equation $\bar{\nabla} \times \bar{B} = \frac{\partial \bar{E}}{\partial t} + \bar{j}$ we have

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = \frac{\partial^2 \bar{A}}{\partial t^2} - \frac{\partial}{\partial t} \bar{\nabla} \varphi + \bar{j}$$

Since $\nabla \times (\nabla \times A) = \nabla(\bar{\nabla} \cdot \bar{A}) - \nabla^2 A = -\nabla^2 A$ with $\nabla \cdot A = 0$, this equation becomes,

$$\nabla^2 A - \frac{\partial^2 A}{\partial t^2} = \bar{j} + \frac{\partial E_L}{\partial t}$$

The term on the right is just j_T as we defined it earlier. So we see it is just the transverse part of j that drives $\bar{A}$.

One further point regarding vector fields being expressed as the sum of two part; you might want to find explicit formulas for the two parts in terms of the field itself. To do this let $\bar{C}$ be any vector field such that

$$\bar{C} = \bar{C}_T + \bar{C}_L$$

subject to the condition that

$$\bar{\nabla} \cdot \bar{C}_T = 0 \quad \text{and} \quad \bar{\nabla} \times \bar{C}_L = 0$$

This implies that $\bar{C}_L = \nabla \chi$ where χ is some scalar potential. If we compute $\bar{\nabla} \cdot \bar{C}$ we find,

$$\bar{\nabla} \cdot \bar{C} = \nabla \cdot C_L$$

making the substitution,

$$\nabla \cdot C = \nabla^2 \chi$$

Since $\nabla^2 X =$ constant has a solution which we just found

$$\chi(1) = \int \frac{(\nabla \cdot C)_2}{r_{12}} \, dV_2$$

we have that

$$C_L = \nabla \left[\int \frac{(\nabla \cdot C)_2}{r_{12}} \, dV_2 \right]$$

The use of the coulomb gauge in solving problems of this type shows the nature of electromagnetism is two-fold: the fields from charges and currents are nothing more than a bunch of harmonic oscillators driven by the amount of charge or current in the particular mode you are studying. And secondly the potentials come from electrostatics and behave the coulomb action at a distance law.

A PATH TO QUANTUM ELECTRODYNAMICS

Read this carefully: jtn note

The COULOMB GAUGE has historical importance in that it provided a pedagogically easy way into QED. Working from classical physics the student learns electrodynamics, then quantum mechanics, relativity, and finally emerges into QED. The path is like a trail across two mountains by way of the saddle. There is, however, a more elegant way into QED using the Lorentz gauge which takes you around the mountain rather than the hard way over it. This new approach was first present by Feynman in 1948.

The new approach discovered in 1948 points out that very often that physical theories can be written in many ways - each different and each with its own features. Some formulations let you see a lot; others are very limited in the visibility they provide. It is a very powerful tool to possess a knowledge of all formulations to a given problem. The person who has this knowledge can guess at what is not so transparent without becoming bogged down in terrible mathematics problem concerned with proving existence, convergence of a series, or some other property of the problem. The point I want to make is that there is another way to learn how to solve problems than the traditional Euclidean ordering of assumptions and hypotheses. Very often the non-Euclidean approach will prove to be the easiest.

The Reader should pay close attention here to what Feynman is saying; he is pointing out his fundamental approach to understanding the world. He is always looking for back doors, some that are initially locked, to "get into the house". With multiple ways into a theory each offers its own unique explanation or interpretation; each offers a new window into our world and understanding. This is the Feynman "essence". jtn note

3. WAVES IN FREE SPACE

IN THE LAST LECTURE WE TALKED ABOUT THE SOLUTION OF THE ELECTRIC FIELD INSIDE A CAN. I'd LIKE TO FINISH THE DISCUSSION WITH SOLVING for THE MODES IN VACUUM. THE SIMPLEST SOLUTION TO THIS PROBLEM IS A SET OF PLANE WAVES, EACH REPRESENTING A PARTICULAR MODE. TO SOLVE THE PROBLEM WE WILL APPLY THE COULOMB GAUGE TO MAXWELL'S EQUATIONS IN free SPACE.

AS YOU RECALL $\bar{B} = \bar{\nabla} \times \bar{A}$ AND $\bar{E} = -\nabla\varphi + \partial\bar{A}/\partial t$, WITH THESE RELATIONS APPLIED TO MAXWELL'S EQUATIONS,

$$\bar{\nabla} \times \bar{E} = -\rho/\epsilon_0 \qquad \bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad \bar{\nabla} \times \bar{B} = \frac{\partial \bar{E}}{\partial t} + \bar{j} \qquad \bar{\nabla} \cdot \bar{B} = 0$$

SUBJECT TO THE COULOMB GAUGE $\nabla \cdot A = 0$, WE FOUND THAT THE SCALAR POTENTIAL HAD THE FORM

$$\nabla^2 \varphi = -\rho/\epsilon_0$$

WE HAVE PREVIOUSLY SOLVED THIS EQUATION IN ELECTROSTATICS WHEN $\rho = 0$. THE EQUATION WHICH $\bar{A}$ MUST SATISFY IS A LITTLE MORE COMPLICATED,

$$\nabla^2 \bar{A} - \frac{\partial^2 \bar{A}}{\partial t^2} = -\bar{\nabla}\frac{\partial \varphi}{\partial t} + \bar{j}$$

WHEN THE RIGHT SIDE EQUALS ZERO, WE HAVE THE SIMPLE WAVE EQUATION,

$$\bar{A} = \bar{a}\, e^{i(\omega t - K_z Z - K_x X - K_Y Y)}$$

where $\bar{a}$ IS SOME VECTOR AND $\bar{K} = K_x \bar{e}_x + K_Y \bar{e}_Y + K_Z \bar{e}_Z$ IS THE PROPAGATION VECTOR AND THE K'S ARE THE WAVE NUMBER ASSOCIATED WITH PROPAGATION IN THEIR RESPECTIVE DIRECTION.

WE WANT TO SOLVE THE WAVE EQUATION

$$\nabla^2 A - \frac{\partial^2 A}{\partial t^2} = 0$$

WITH THE ABOVE SOLUTION. IF WE NOTE THAT THE OPERATORS $\partial/\partial t$ AND $\partial/\partial x$, $\partial/\partial Y$, $\partial/\partial X$, ON $\bar{A}$ HAVE THE SAME EFFECT AS MULTIPLYING $\bar{A}$ by $i\omega$ AND $-iKX$, $-iK_Y Y$, $-iK_Z Z$ RESPECTIVELY, WE HAVE THAT,

$$(-K_X^2 - K_Y^2 - K_Z^2 + \omega^2)\,\bar{a} = 0$$

THIS EQUATION IS SATISFIED IF

$$\omega_K^2 = \bar{K} \cdot \bar{K}$$

INSTEAD OF DESCRIBING EACH MODE BY A PARTICULAR INDEX n AS WE DID IN THE CASE OF THE CAN, WE NOW DESCRIBE EACH MODE BY A K VALUE CONSISTING OF THREE NUMBERS.

Now so far we haven't used the fact that $\bar{\nabla}\cdot\bar{A} = 0$ so let's write it out,

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

and we find that

$$K_x a_x + K_y a_y + K_z a_z = 0 = \bar{K}\cdot\bar{a}$$

This says that the propagation vector is perpendicular $\bar{a}$. There are two directions which we can choose for $\bar{a}$; they correspond to directions of polarization and we can denote these as

$$\bar{a} = \bar{e}_1 \quad \text{or} \quad \bar{a} = \bar{e}_2$$

Note that $\bar{e}_1$ and $\bar{e}_2$ are orthogonal so that $\bar{e}_1 \circ \bar{e}_2 = 0$ and consequently $\bar{e}\cdot\bar{K} = 0$.

This has all been for the free wave or oscillator equation. Now we would liked to find the solution of the forced oscillator, i.e., when $-\bar{\nabla}\partial\varphi/\partial t + \bar{j} \neq 0$. Oh, one other thing. We can write the vector potential in the following notation

$$\bar{A}_{K,i} = \bar{e}_i\, e^{-i(\bar{K}\cdot\bar{R})}$$

The orthogonality of this set of functions is verified if we multiply $\bar{A}_{K,i}$ by the complex conjugate (for convenience) of a different mode say $A_{K',i'}$ then integrate over all space,

$$\int A_{K,i}\, \cancel{e^{-i(\bar{K}\cdot\bar{R})}} \cdot A^*_{K',i'}\, dVol = \int \bar{e}_i\, e^{-i(\bar{K}\cdot\bar{R})} \cdot e_{i'}\, e^{+i(\bar{K}'\cdot\bar{R})}\, dvol$$

$$\int A_{K,i}\cdot A^*_{K',i'}\, dV = \int \bar{e}_i \circ \bar{e}_{i'}\, (dV)\, e^{+i(\bar{K}'+\bar{K}')\cdot\bar{R}}$$

The right side is zero unless $i = i'$ and $K' = K$. Therefore, the $\bar{A}_{K,i}$'s form an orthogonal set of equations which enable us to write any other function in terms of.

Now then if we want to solve the equation

$$\bar{\nabla}^2\bar{A} - \frac{\partial^2\bar{A}}{\partial t^2} = -\bar{\nabla}\frac{\partial\varphi}{\partial t} + \bar{j}$$

we can try the solution

$$\bar{A}(\bar{R},t) = \sum_{K,i} \alpha_{K,i}(t)\, \bar{A}_{K,i}$$

or

$$\bar{A}(\bar{R},t) = \sum_{K,i} \alpha_{K,i}(t)\, \bar{e}_i\, e^{-i\bar{K}\cdot\bar{R}}$$

This solution we put into the above equation and find that,

$$\sum_{K,i}\left[K^2\alpha_{K,i}-\ddot{\alpha}_{K,i}(t)\right]\bar{e}_i\, e^{-i\bar{K}\cdot\bar{R}} = -\bar{\nabla}\frac{\partial\varphi}{\partial t}+\dot{\bar{j}}$$

To isolate one particular mode shape we can multiply both sides by a different mode, say $\bar{A}_{L,j}$ and integrate. In fact let's take the complex conjugate of $\bar{A}_{L,j}$ and get

$$\int\sum_{K,i}\left[K^2\alpha_{K,i}-\ddot{\alpha}_{K,i}(t)\right]\bar{e}_j\cdot\bar{e}_i\, e^{i(\bar{L}-\bar{K})\cdot\bar{R}}\,dVOL = \int\left(-\nabla\frac{\partial\varphi}{\partial t}\right)\cdot\bar{e}_j\, e^{i\bar{L}\bar{R}}\,dVOL + \int\dot{\bar{j}}(\bar{R},t)\cdot\bar{e}_j\, e^{i\bar{L}\cdot\bar{R}}\,d^3R$$

The left side is just $L^2\alpha_{L,j}(t)-\ddot{\alpha}_{L,j}(t)$. The first term on the right is zero so we are left with

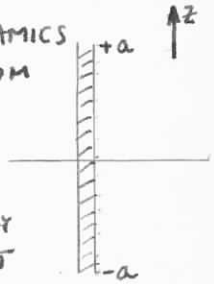
$$L^2\alpha_{L,j}(t)-\ddot{\alpha}_{L,j}(t) = \int\dot{\bar{j}}(\bar{R},t)\cdot\bar{e}_j\, e^{+i\bar{L}\cdot R}\,d^3R$$

This equation says that you take the component of the current density $\dot{j}$ projected in the direction of polarization and multiply by the sinusoidal distribution of modes in space and upon integrating you'll find the driving force to each mode L,j. A short-hand of the above equation is

$$L^2\alpha_{L,j}-\ddot{\alpha} = \bar{e}_j\cdot\dot{\bar{j}}_L(t)$$

The quantity L^2 is actually $\omega^2_{L,j}$ and what we really have here is all of electrodynamics in one equation – just a set of driven harmonic oscillators. Since we have already solved the electrostatics problems we have all of electromagnetics. While this equation obscures several apparent consequences of wave motion namely that waves travel at the speed of light and the electromagnetic energy is in the field, it does provide an easy way into quantum electrodynamics and we'll use it later.

As an example of how to apply this new approach to electrodynamics suppose we had a wire lying in the z direction and running from $+a$ to $-a$. If the current density is only in the z direction, then,



$$\dot{j}_z(\bar{R},t) = \bar{I}(z,t)\,\delta(x)\,\delta(y)$$

where $\bar{I}(z,t)$ is the final distribution of currents which is usually never know. It is the value thought to be applied to the wire but which has ignored all the back actions produced by the associated fields. The actual field I_A is given more accurately as

$$I_A = I + \Delta I + \Delta^2 I \approx I$$

This back reaction is due to the finite size of the wire and the delta function no longer is valid. But assuming we can get

AWAY WITH THE ROUGH APPROXIMATION, THEN TO FIND THE PROJECTION of $\dot{j}$ IN MODE L WE USE THE FORMULA

$$j_L(R,t) = \bar{e}_z \int I(z,t)\,\delta(x)\,\delta(y)\,e^{i(L_z z + L_x X + L_y Y)}\,dx\,dy\,dz$$
$$= \bar{e}_z \int_{-a}^{a} I(z,t)\,e^{iL_z z}\,dz$$

SINCE $I(z,t)$ IS KNOWN, WE CAN FIND THE WAVE IN THE MODE L by COMPUTING $\bar{e}\cdot j_L(t)$. THEN YOU CAN FIND THE SOLUTION TO THE FORCED EQUATION.

NORMALIZATION IN FOURIER THEORY

THE MATHEMATICAL TOOL of PROJECTING A FUNCTION INTO A SERIES of SINES WAVES IS QUITE POWERFUL IN SOLVING DIFFICULT PROBLEMS. I'd LIKE TO DISCUSS SOMETHING WHICH I HAVEN'T DONE YET IN REGARDS TO FOURIER ANALYSIS.

IF I COMPUTE THE INTEGRAL $\int (\bar{A}^*_{K',i'}\cdot\bar{A}_{K,i})\,dVOL$ I WILL FIND IT EQUALLING $\int e^{-i\bar{K}\cdot\bar{R}}\,e^{+i\bar{K}'\cdot\bar{R}}\,\bar{e}_i\cdot\bar{e}_{i'}\,dVOL$. I SAID THIS ONLY HAS A VALUE WHEN $K=K'$ AND $i=i'$ AND THAT VALUE IS ONE. WE SAY THE ANSWER IS NORMALIZED TO ONE. BUT THAT IS NOT A TRUE STATEMENT BECAUSE IT DOESN'T EQUAL ONE; IT EQUALS A VOLUME of SPACE WHICH CAN HAVE LOT of INFINITIES IN IT. FURTHER I WANT TO POINT THAT THE SUM $\Sigma_{K,i}$ IS LIKE AN INTEGRAL WHICH ALSO HAS A NORMALIZATION of A VOLUME NOT UNITY. I WANT TO SHOW THAT WHEN GOING FROM THE SUM TO THE INTEGRAL

$$\Sigma_K \longrightarrow \int \frac{d^3K}{(2\pi)^3}$$

THE VALUE IN THE INTEGRAL GIVES A NORMALIZATION PER UNIT VOLUME. THE NOTATION UNDER THE INTEGRAL ACTUALLY MEANS $\frac{dK_x}{2\pi}\frac{dK_y}{2\pi}\frac{dK_z}{2\pi}$

LET ME SHOW THIS CROSSOVER IS VALID if I TAKE AN EXAMPLE of A FINITE BUT ENORMOUSLY LARGE RECTANGULAR WORLD WITH SIDE DIMENSIONS a AND b. IN ORDER for THE WAVE TO FIT IN SIDE THE BOX WE MUST HAVE THE CONDITION ON $\cos K_x X$ AND $\cos K_y Y$ of $K_x a = n_x\pi$ AND $K_y b = m_y\pi$. TO BE MORE EXACT I SHOULD TAKE THE SINE BUT I'M TOO SLOPPY. THE VECTOR POTENTIAL WILL HAVE A SOLUTION

$$\bar{A} = \gamma\,\bar{e}\,e^{-i\bar{K}\cdot\bar{R}}\cos(K\cdot R)$$

AND THE NORMALIZATION CONDITION $\int A^*\cdot A\,dVOL = 1$ WOULD REQUIRE THAT $\gamma^2 = 2/VOL$ SINCE

$$1 = \gamma^2 \int \bar{e}_i\cdot\bar{e}_j\,\cos^2(K\cdot R)\,dVOL = \frac{\gamma^2}{2}\,VOL.$$

Therefore,

$$\bar{A} = \sqrt{\frac{2}{VOL}}\,\bar{e}\,\cos(\bar{K}\cdot\bar{R})$$

This is the right way to write the mode. This difference in $\bar{A}$ has a reflection in the equation for $\dot{j}_L$

$$\dot{j}_L = \sqrt{\frac{2}{VOL}} \int \bar{j}(R,t) \cdot \bar{e}_j \cos(K \cdot R)\, d\,VOL$$

so $\dot{j}_L$ is weaker by an amount $\sqrt{2/VOL}$.

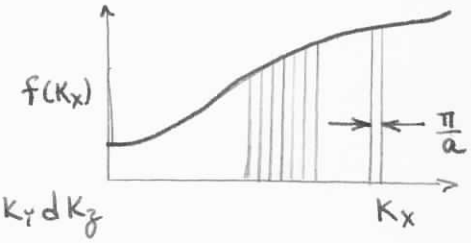
To find the true value of $\bar{A}(\bar{R},t)$ we must include this correction into the series. This time we'll denote $\alpha_{L,i}(t)$ with a superscript v to designate the new normalization, i.e.,

$$\bar{A}(\bar{R},t) = \sum_{L,i} \check{\alpha}_{L,i}(t) \left(\frac{2}{VOL}\right) \bar{e} \cos(\bar{L} \cdot \bar{R})$$

where $\check{\alpha}_{L,i} = \alpha_{L,i} \sqrt{2/VOL}$

The sum over L is actually a sum over n_x, n_y, and n_z. But

$$\sum_{n_x} f(K_x) = \frac{a}{\pi} \int f(K_x)\, dK_x$$



where π/a is a very small number interval along the K_x axis. Now the triple sum would mean that

$$\sum_{n_x} \sum_{n_y} \sum_{n_z} = \frac{abc}{\pi\pi\pi} \iiint f(K_x, K_y, K_z)\, dK_x\, dK_y\, dK_z$$

If we multiply both sides by $2/VOL$ and notice that $abc = VOL$ we get

$$\sum_L \frac{2}{VOL} = \int \frac{2\, d^3K}{\pi^3}$$

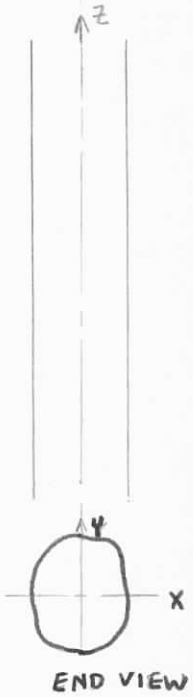
This whole exercise has been to get the volumes to cancel out.

The non-skilled might notice the extra 2 and the lack of $(2\pi)^3$ in the denominator. The 2 in the numerator can be explained by my persistent use of $\cos(K \cdot R)$. If I had use the exponential ~~notion~~ notation it wouldn't have appeared. Also when averaging $\cos^2$ we get a 1/2 so the twos divide out. The missing twos in the denominator has to do with the fact that when fitting waves inside a big box the solutions are that of a set of standing waves made up of two running waves going in the opposite direction. When summing $\sum_{N_x}$ to $\frac{a}{\pi}\sum_{K_x}$ we only considered the forward wave so we must multiply through by 2 to correct things. Thus we get

$$\sum_L \left(\frac{2}{VOL}\right) = \int \frac{d^3K}{2 \cdot 2 \cdot 2} \frac{2}{VOL} \cdot \frac{VOL}{\pi^3}$$

$$\sum_L = \int \frac{d^3K}{(2\pi)^3}$$

I WOULD LIKE TO DEAL WITH A LONG PIPE-SHAPED WAVEGUIDE which HAS ITS LONG DIMENSION IN The Z-DIRECTION. TO WORK OUT The CHARACTERISTIC MODES of PROPAGATION I MUST SOLVE The FOLLOWING EQUATION,



$$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}) = \frac{\partial^2\bar{E}}{\partial t^2}$$

SUBJECT To The CONDITION THAT $\bar{\nabla}\cdot\bar{E}=0$ INSIDE AND $\bar{E}_{TANG}=0$ ON The SURFACE. I HAVE ALREADY SHOWN This EQUATION WILL HAVE The SOLUTION

$$\bar{E}(\bar{R},t) = \bar{E}(x,y)\, e^{i(\omega t - Kz)}$$

The Above EQUATION CAN be SIMPLIFIED TO,

$$\nabla^2\bar{E} = -\frac{\partial^2 E}{\partial t^2} = -\omega^2\bar{E}$$

SINCE $\bar{\nabla}\cdot\bar{E}=0$. NOW WE HAVE THREE EQUATIONS TO SOLVE For The TWO DIMENSIONAL CASE

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)E_{x,y,z} = (\omega^2-K^2)\,E_{x,y,z}$$

BECAUSE $\bar{\nabla}\cdot\bar{E}=0$, I.E.,

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0$$

The Above THREE EQUATIONS ARE NOT INDEPENDENT of ONE ANOTHER, I.E.,

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y} = iK\,E_z$$

WE DO HAVE A CHOICE of E IN The X,Y PLANE because WE CAN break E_z INTO A TRANSVERSE AND LONGITUDINAL COMPONENT, I.E., $\bar{\nabla}\times\bar{E}_z=0$ or $\bar{\nabla}\cdot\bar{E}_z=0$. The first CONDITION IMPLIES THAT $E_x = \frac{\partial\varphi(x,y)}{\partial x}$ AND $E_y = \frac{\partial\varphi(x,y)}{\partial y}$. This CAN be PUT INTO The Above DIFFERENTIAL EQUATION AND for The X-COMPONENT WE CAN INTEGRATE ONCE TO FIND,

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi = \lambda^2\varphi$$

This EQUATION CAN be SOLVED SUBJECT TO The boundary CONDITION THAT φ = CONSTANT ON The surface of The PIPE AND for CONVENIENCE I'LL choose IT TO be ZERO. When E HAS both A TRANSVERSE AND LONGITUDINAL PART The MAGNETIC FIELD IS PERPENDICULAR TO The Z-DIRECTION. BECAUSE The B-FIELD IS TRANSVERSE WE CALL IT The TM, TRANSVERSE MAGNETIC, MODE. If E_z SATISFIES The CONDITION $\nabla\cdot E_z=0$ Then The ELECTRIC FIELD ONLY HAS COMPONENTS IN The X,Y PLANE

AND $E_z = 0$. THIS IS CALLED THE TE, TRANSVERSE ELECTRIC, MODE. BECAUSE $\nabla \cdot E_z = 0$ WE HAVE THE RELATIONS THAT

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

THIS DIFFERENTIAL EQUATION IS ONLY SATISFIED IF $E_x = \frac{\partial \psi}{\partial y}$ AND $E_y = -\frac{\partial \psi}{\partial x}$. THAT IS TO SAY THE DIFFERENTIAL RELATIONSHIP BETWEEN E_x AND E_y PERMITS ON 1 free FUNCTION AS A SOLUTION, NAMELY ψ. THUS WE FIND A THIRD DIFFERENTIAL EQUATION

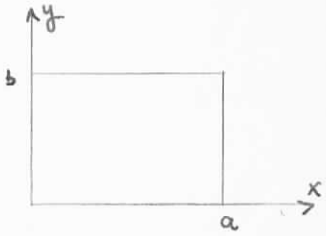
$$-\left(\frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y}\right)\psi = \lambda^2 \psi$$

SUBJECT TO THE BOUNDARY CONDITION THAT THE NORMAL TO THE WALL IS ZERO AT THE SURFACE, I.E.,

$$\left.\frac{d\psi}{dn}\right|_{surf} = 0$$

The RECTANGULAR WAVEGUIDE

LET'S CONSIDER AN EXAMPLE OF A RECTANGULAR WAVEGUIDE WHICH HAS THE DIMENSIONS a AND b IN THE x, y, PLANE. IN order TO fit A BOUNDARY CONDITION OF $\varphi = 0$ ON THE SURFACE WE CHOOSE THE POTENTIAL TO BE of THE form



$$\varphi = \sin \frac{n\pi}{a} x \ \sin \frac{m\pi}{b} y$$

If THIS IS PUT INTO THE DIFFERENTIAL EQUATION, WE GET

$$\left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)\varphi = \lambda^2 \varphi = (\omega^2 - K^2)\varphi$$

OR

$$\omega^2 - K^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$$

SOLVING for K,

$$K = \pm\left[\omega^2 - \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\right]^{1/2}$$

The TWO VALUES of K, + AND −, GIVE THE TWO DIRECTIONS of PROPAGATION of THE WAVE. AS ω GET LARGE, SO DOES K; IMPLIES λ GETS SMALLER. BUT for LOW FREQUENCIES, λ STARTS TO GET BIGGER. If ω GETS SMALL ENOUGH THAT THE BRACKETED QUANTITY GOES NEGATIVE, K becomes IMAGINARY. If K IS IMAGINARY THE ELECTRIC FIELD IS NOT TRANSMITTED but rather EXPONENTIALLY ATTENUATED AS IT GOES INTO PIPE. FROM OUR SOLUTION

$$\bar{E} = \bar{E}_0 e^{i(\omega t - Kz)}$$

If $K = iK'$, THEN

$$\bar{E} = E_0 e^{K'z} e^{i\omega t}$$

Thus when

$$\omega_c^2 = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right)$$

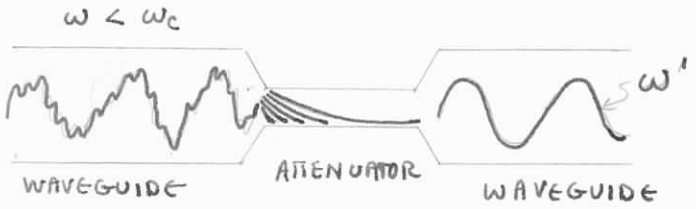
The wave propagation is cutoff so we call this the cut off frequency. Suppose for simplicity $a = b$ and $n = m = 1$ then

$$\omega_c = \frac{\pi c}{a}$$

The oscillating field will penetrate into the guide only a distance of the order of $1/K'$ where

$$K' = \sqrt{\omega_c^2 - \omega^2}$$

When ω is just below ω_c, K' is small and the field can penetrate a long distance into the guide. When $\omega \ll \omega_c$, $K' \sim \pi/a$ and the field dies off very fast. One way to isolate a particular frequency ω' is to build an attenuator which has the dimension a which makes it close to the cutoff frequency. ω' will be the last exponentially decaying wave far out in the guide and it can be amplified and then transmitted on.



4. ANALOGY BETWEEN WAVE GUIDES AND QUANTUM MECHANICS

I WANT TO POINT OUT AN INTERESTING THING WITH REGARDS TO WAVE-GUIDES. WE HAVE SEEN THAT THE WAVE NUMBER CAN BE WRITTEN AS

$$K = \sqrt{\omega^2 - \lambda_m^2}$$

where λ_m ARE THE CHARACTERISTIC WAVELENGTHS for EACH MODE m. IN ANOTHER LECTURE WE DISCUSSED THE PHASE VELOCITY of A WAVE or THE SPEED AT which NODES TRAVEL AND found THAT

$$v_{phase} = \frac{\omega}{K} = \frac{\omega}{\sqrt{\omega^2 - \lambda_m^2}}$$

THIS CAN be REARRANGED INTO THE FOLLOWING form

$$v_{phase} = \frac{c}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}$$

where ω_c IS THE CUTOFF FREQUENCY of the WAVEGUIDE.

Now THE GROUP VELOCITY TELLS US how FAST SIGNALS or ENERGY IS being PROPAGATED down THE GUIDE. WE KNOW THAT

$$v_{group} = \frac{d\omega}{dK}$$

UPON DIFFERENTIATING $\omega = \sqrt{K^2 + \lambda_m^2}$ we find

$$v_{group} = \frac{K}{\omega} = \frac{1}{v_{phase}}$$

OR EXPRESSED DIFFERENTLY

$$v_{phase}\, v_{group} = c^2$$

THIS IS AN INTERESTING RESULT because IN QUANTUM MECHANICS A PARTICLE WITH SOME VELOCITY v HAS AN ENERGY,

$$E = \sqrt{p^2 + m^2}$$

Where THE ENERGY E IS GIVEN by $\hbar\omega$ AND THE MOMENTUM p by $\hbar K$ THE VELOCITY of THE PARTICLE IS GIVEN ANALOGOUSLY AS

$$v = \frac{d\omega}{dK} = \frac{dE}{dP}$$

USING for E THE ABOVE VALUE WE JUST HAVE THAT

$$Ev = p$$

AND E IS LIKENED TO A MOVING MASS. USING THE SUBSTITUTIONS JUST WRITTEN WE HAVE THAT

$$K = \sqrt{\frac{\omega^2}{c^2} - \frac{m^2c^2}{\hbar^2}}$$

DRAWING THE ANALOGY THEN THE EIGENVALUES λ_m^2 ARE RELATED TO THE MASS SQUARED OF A PARTICLE BY $c^2/\hbar^2$

A WORLD OF 5 DIMENSIONS

WE COULD AMUSE OURSELVES AT THIS POINT BY NOTING THAT WE HAVE BEEN SOLVING A MORE GENERAL DIFFERENTIAL EQUATION GIVEN AS

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \varphi = m^2 \varphi$$

WHERE THE m^2 ARE THE EIGENVALUES OF SOLUTION. NOW IF WE REWRITE THIS EQUATION TO INCORPORATE AN ADDITIONAL DIMENSION, THE W-SPACE, WE HAVE THAT

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial w^2} \right) \varphi = 0$$

WHERE THE W SPACE IS CONFINED TO A REGION BOUNDED BY TWO PLANES. IF THE VARIATION OF φ IN THE W SPACE IS GIVEN BY $\sin n\pi w/a$, THEN SECOND DERIVATIVE OF φ WITH RESPECT TO W JUST GIVES $n^2\pi/a^2$. SO IF

$$\varphi \propto \sin \frac{n\pi w}{a} \, e^{-i(\omega t - K_x X - K_Y Y - K_z Z)}$$

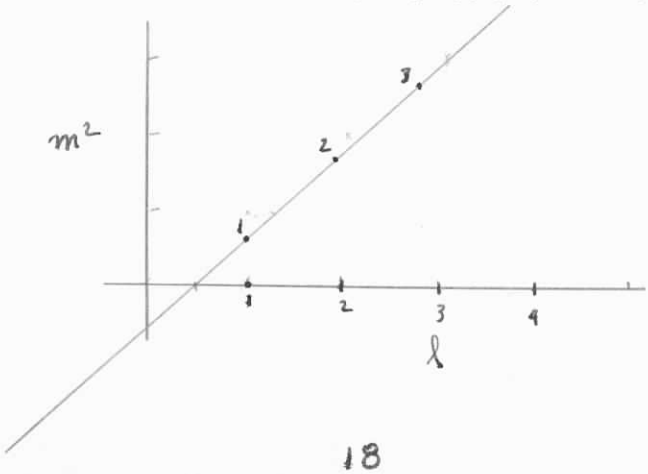
THE PARTICULAR SOLUTION TO THE DIFFERENTIAL EQUATION IS THEN

$$\omega^2 - K^2 = \frac{n^2\pi^2}{a^2} = m^2$$

THIS IS THE SAME FORM AS

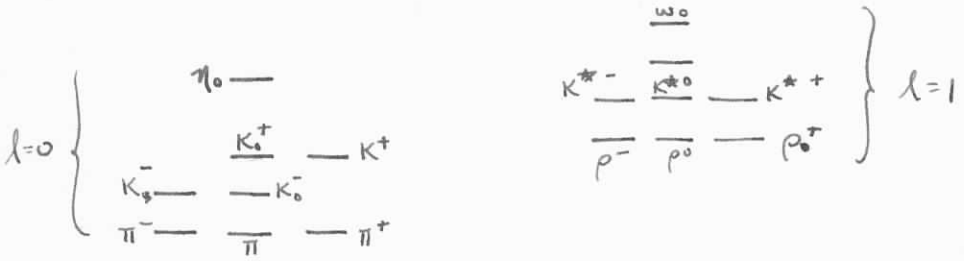
$$E^2 - P^2 = h^2 \frac{n^2\pi^2}{a^2}$$

IF WE MULTIPLY THROUGH BY $\hbar^2$. IF WE DRAW THE ANALOGY THEN THE MASS OF THE PARTICLE IS RELATED TO DISCRETE INTEGER VALUES OF $\hbar$. A FURTHER DEPENDS ON THE DIMENSIONS OF THE W-SPACE IN WHICH THE MASS IS PERMITTED TO WIGGLE ABOUT IN. SO IT WOULD NOT BE UNUSUAL TO PLOT M^2 VERSUS SOME INTEGER AND FIND A ONE TO ONE CORRESPONDENCE. AS A MATTER OF FACT THAT VERY THING HAS BEEN DONE IN THE REALM OF HIGH ENERGY PHYSICS. THE PLOT IS CALLED A REGGI TRAJECTORY AND LOOKS LIKE THE FOLLOWING:



$m^2 = -0.46 + 1.01\,l$

The interpertation of this function is a little difficult at this time. It is believed that the higher ~~the~~ values of m^2 are nothing more than excited states of some ground state. There is some experimental evidence that this is true. In higher energy physics there are curious octets of particles that appear to be related. Two octets in particular are mesons with angular momentum, $l=0$, and $l=1$. Pictorially the two sets can be represented as



Expetimentally the curve has been determined for $l = 1/2, 1, 2, 3$ and successively higher resonances for which $l = 4$ and 5. So while there is an interesting coincidence in the differential equations, maybe the analogy should be pursued in light of the recent advancements in high energy physics. At least it should be pursued to the point where the analogy breaks down. Sometimes it is easier to see the generality in a certain mathematical expression than it is to find a specific physical problem which obeys that generality. This is often the difficult point where mathematicians fail to make any sense out of physics.

Angular Momentum in Circularly Polarized Guided Waves and an Extension to Quantum Mechanics

We have seen how the characteristic modes of a guided wave can somehow be related to particle mass. We might wonder what happens then to a wave that has its electric field vector rotating as it propagates down the pipe. Certainly a particle undergoing rotation has a certain angular momentum associated with it. Then should a guided have have a certain angular momentum? The answer is indeed it does. However for the case of a wave in a square guide, the can absorbs the angular momentum. But in the case of a circularly guided wave, the angular momentum will be carried down the wave pipe.

A circularly polarized wave can be expressed as a real and imaginary part

$$E_x = e^{i(\omega t - Kz)} = \cos(\omega t - Kz)$$
$$E_y = i\, e^{i(\omega t - Kz)} = -\sin(\omega t - Kz)$$

The STRENGTH of The Ex AND Ey THUS VARY SINUSOIDLY TO GIVE The NET RESULT of A ROTATION of The $\bar{E}$ IN The XY PLANE NORMAL TO The DIRECTION of PROPAGATION. The SENSE of The ROTATION IS DETERMINED by The RIGHT AND LEFT HAND RULE WITH The THUMB POINTING IN The DIRECTION of PROPAGATION AND The ~~curl~~ FINGERS CURLING IN The DIRECTION of ROTATION.

I'd LIKE TO SHOW YOU THAT The ANGULAR MOMENTUM CARRIED by The WAVE IS PROPORTIONAL TO The ENERGY IN The WAVE by $1/\omega$. THAT IS TO SAY The RATE AT which ENERGY IS BEING Absorbed IS EQUIVALENT TO The TORQUE ON The ELECTRIC FIELD.

SUPPOSE AN ATOM RESPOND TO AN EXTERNAL FIELD LIKE AN OSCILLATOR TO A DRIVING force. AS The CHARGE IS DISPLACED IT TENDS TO RETURN TO ITS EQUILIBRIUM CONDITION but The EXTERNAL force IS ROTATING SO IT TRIES TO follow by TURNING but ALWAYS LAGS by SOME ANGLE, θ, The VELOCITY AT which The OSCILLATOR TURNS IS $v = r\omega$.

Now The WORK DONE PER SECOND ON The ATOM IS JUST

$$\frac{W}{\text{sec}} = F \sin\theta \cdot v = qE \sin\theta \, r\omega$$

BUT RECALL ALSO THAT The RATE of CHANGE of The ANGULAR MOMENTUM IS JUST The TORQUE, L SO THAT

$$L = F \cdot r \sin\theta = qEr \sin\theta$$

SO WE HAVE

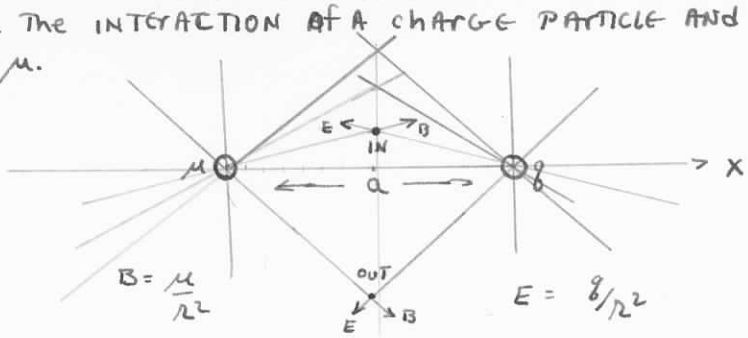
$$\text{Power Absorbed} = \overset{\text{TORQUE}}{\cancel{\text{ANGULAR MOMENTUM}}} \times \omega = \cancel{\text{TORQUE}}$$

WE CAN RELATE THIS IDEA TO The Theory of The PHOTON which IS JUST LIGHT AND behaves IN A WAVELIKE MANNER. EACH PHOTON CARRIES AN ENERGY E EQUIVALENT TO $\hbar\omega$. BY ANALOGY Then $\hbar$ MUST be IN UNITS of ANGULAR MOMENTUM, IN FACT ONLY INTEGRAL MULTIPLES of $\hbar$ ARE PERMITTED ACCORDING TO QUANTUM Theory. TO SAY THAT A PHOTON IS RIGHT or LEFT CIRCULARLY POLARIZED IS TO SAY IT HAS A SPIN $+\hbar$ or $-\hbar$, RESPECTIVELY. These TWO STATES of SPIN ARE The ONLY ONES PERMITTED for A PHOTON.

AT This POINT I GUESS I HAVE finished WHAT I WANT TO SAY ABOUT ELECTRICITY. I HAVEN'T SAID MUCH ABOUT MAGNETISM SO far AND IT MIGHT be GOOD TO DO SO before I GO INTO QUANTUM Theory. SO UNLESS There ARE ANY MORE QUESTIONS —

On the prediction of magnetic monopoles

The possibility of magnetic monopoles existing was first postulated by Dirac. To see how the reasoning for the existence of such a thing consider the interaction of a charge particle and monopole of strength μ.



The lines of magnetic and electric field are directed radially from the charge centers. If we calculate the poynting vector $\bar{S} = \bar{E} \times \bar{B}$ in the region where the lines cross and integrate over all the space through the moment arm about the x axis, you find a net angular momentum about x which is proportional to $q\mu$. This is just a number and surprisingly independent of the separation distance, a. Since Dirac thought that the angular momentum must be expressed in units of $\hbar$ for consistency with quantum mechanics, he found the strength of the monopole to be

$$\mu = \frac{\hbar}{q}$$

A way to prove this same result is to consider the electric charge to be slowly rotated about the monopole at radius a and with a velocity, $v = a\omega$. Since the $\bar{B}$-field is directed radially outward in the x direction the force on the charge is

$$F = q\,\bar{v} \times \bar{B} = q v \frac{\mu}{a^2}$$

Since the torque about μ is just Fa, i.e.

$$\tau = q v \frac{\mu}{a} = q \mu \omega$$

Since torque is simply angular momentum times ω we have that

$$\mu q = L$$

again independent of the spacing a.

Recalling that $\hbar c/e = 137$, we see that a magnetic monopole is 137 times stronger than e. Clearly such a particle would leave visible tracks in cloud chambers, etc. Since no such reactions have been observed, the existence of monopoles have been ruled out.

ELECTROMAGNETIC MASS

Ref. VOL. II CHAPT 28

WE HAVE TWO MORE LECTURES before we break for VACATIONS so I'd LIKE TO fool AROUND WITH ANYTHING YOU WANT TO DISCUSS before I START INTO MAGNETISM. LAST TIME I WAS ASKED TO DISCUSS The MOMENTUM AND FIELD ASSOCIATED WITH AN ELECTRON so I'LL DO THAT.

LET'S first CALCULATE The ENERGY IN The ELECTROMAGNETIC field of A CHARGED PARTICLE. SUPPOSE THAT A CHARGE q IS DISTRIBUTED ON The surface of A BALL WITH RADIUS a. The TOTAL ENERGY IS GIVEN by

$$U = \frac{\epsilon_0}{2}\int E^2 dVOL + \frac{\epsilon_0 c^2}{2}\int B^2 dVOL$$

first LET'S CONSIDER A PARTICLE AT REST SO THAT $\bar{B} = \bar{v}\times\bar{E}$ IS ZERO. SINCE The ELECTRIC field IS GIVEN AS

$$E = \frac{q}{4\pi\epsilon_0 r^2}$$

for A CHARGED BALL WE HAVE

$$U = \frac{\epsilon_0}{2}\frac{q^2}{(4\pi\epsilon_0)^2}\int_a^\infty \frac{4\pi r^2 dr}{r^4} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{2a}$$

If WE USE The SYMBOL e^2 TO DENOTE $q^2/4\pi\epsilon_0$, Then WE CAN WRITE

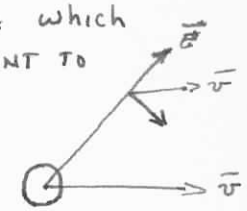
$$U = \frac{e^2}{2a}$$

FROM RELATIVITY WE KNOW THAT ASSOCIATED WITH ENERGY IS A MASS GIVEN by

$$m = \frac{U}{c^2} = \frac{e^2}{2ac^2}$$

If WE SOLVED for The RADIUS of The ELECTRON WE WOULD FIND THAT $a \sim 1.4\times10^{-13}$ CM

I'd NOW LIKE TO COMPUTE The ENERGY for A MOVING CHARGE which has IN ADDITION TO ITS ELECTRIC FIELD A MAGNETIC FIELD. WE WANT TO SEE how The MASS INCREASES WITH VELOCITY. The MAGNETIC FIELD IS GIVEN BY $\bar{B} = \frac{\bar{v}}{c^2}\times\bar{E}$. The MAGNITUDE of B IS



$$|B| = |E|\frac{v}{c^2}\sin\theta$$

SO THAT $|B|^2 = |E|^2\frac{v^2}{c^4}\sin^2\theta = \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2\frac{v^2}{c^4}\sin^2\theta$. WE SEE THAT SINCE $|B|^2$ IS PROPORTIONAL TO v^2 WE CAN THINK of The ADDED ENERGY TERM BEING DUE TO The KINETIC ENERGY ASSOCIATED WITH The MOVING FIELD.

If we now carry out the integral

$$U = \frac{\epsilon_0}{2} \int (E^2 + c^2 B^2)\, dvol$$

we get from the second term

$$I = \frac{\epsilon_0}{2} \int c^2 \frac{v^2}{c^4} \sin^2\theta \frac{q^2}{(4\pi\epsilon_0) r^4} 2\pi \sin\theta\, d\theta\, r^2 dr =$$

$$= \frac{v^2}{c^4} E^2 \frac{1}{2} \int_{-1}^{1} \sin^2\theta\, d(\cos\theta) = \frac{v^2}{c^4} \frac{E^2}{2} \left(\cos\theta - \frac{\cos^3\theta}{3}\right)\Big|_{-1}^{1}$$

$$B^2 = \frac{2}{3} \frac{v^2}{c^2} E^2$$

In other words the magnetic to electrostatic energy is

$$\frac{B^2}{E^2} = \frac{2}{3} \frac{v^2}{c^2}$$

The total energy U is just $E^2 + B^2$ or

$$U = E^2 + \frac{2}{3} \frac{v^2}{c^2} E^2$$

The total mass associated with this energy is

$$\frac{U}{c^2} = m_{elec} + \frac{2}{3} \frac{v^2}{c^2} m_{elec}$$

where $\frac{E^2}{c^2}$ is the electromagnetic mass, m_{elec}.

But a puzzling thing has happened. Apparently the only mass measurable is due to ~~m~~ electromagnetic energy. Isn't there any convential mass like that associated with inertia and proportional to the kinetic energy like $m_{mech} = 2E/v^2$?

Let's look at the problem from yet another point of view. We know that momentum is related to mass by the objects motion. Let's then find the momentum density in the field around a moving charge. where,

$$\bar{g} = \epsilon_0 \bar{E} \times \bar{B}$$

is the momentum density. If we integrate the symmetric field over all space, the resultant momentum parallel to $\bar{v}$ is

$$\bar{P} = \int \bar{g}\, dvol = \epsilon_0 \int |E|^2 \frac{\bar{v}}{c^2} \sin^2\theta\, 2\pi r^2 d\theta \sin\theta\, dr.$$

Since E is independent of θ for $v \ll c$ we have

$$\bar{P} = \frac{4}{3 \cdot 2} \frac{e^2}{a c^2} \bar{v} = \frac{2}{3} \frac{e^2}{a c^2} \bar{v}$$

Now we have another form for the electromagnetic mass and a total of three altogether:

BY RELATIVITY $m = \frac{e^2}{2ac^2}$

BY TOTAL ENERGY $m = \frac{2}{3}\frac{e^2}{ac^2}$

BY MOMENTUM $m = \frac{2}{3}\frac{e^2}{ac^2}$

For three different approaches we found three different answers. Which is right? Well, from a historical point of view we ought to discard the relativistic result because in 1890 relativity was not known.

When attempts we made to determine the mass by stopping the objects the answers were inconsistent. Lorentz realized that when going to higher velocities the spherical ball becomes squashed into an ellipsoid and the momentum is altered by a factor $(1-v^2/c^2)^{-1/2}$, i.e.,

$$\bar{p} = \frac{2}{3}\frac{e^2}{ac^2}\frac{\bar{v}}{(1-v^2/c^2)^{1/2}}$$

This was all before Einstien by the way. What Einstein found was the generalities in the Lorentz transformations.

The difficulty in the mass derived using relativity to begin with assumed no unbalance forces present to change the momentum density. But there are forces around which hold the charge on the ball. To be consistent we must include these forces, called Poincaré stresses, which hold the ball together. These extra nonelectrical forces have an energy associated with them that when added to the electromagnetic energy give the right mass. To summarize then

AT REST $U_{ELEC} + U_{STRESS} = U_{TOTAL}$

IN MOTION $U_{ELEC}(v) + U_{STRESS}(v) = U_{TOT}(v)/c^2 = m/\sqrt{1-v^2/c^2}$

$\bar{P}_{MOM}(v) + \bar{P}_{STRESS}(v) = m\bar{v}/\sqrt{1-v^2/c^2}$

For a discussion of whether there is any electromagnetic mass see the reference page 28+10

6. MAGNETOSTATICS

So far we have only talked about electricity and now I'd like to start discussing magnetism and the magnetic property of matter. This will lead us into Quantum Theory because may phenomena cannot be analyzed using classical physics.

Beginning with Maxwell's equations,

$$\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0 \qquad \bar{\nabla}\cdot\bar{B}=0 \qquad \bar{\nabla}\times\bar{E} = -\partial\bar{B}/\partial t \qquad \bar{\nabla}\times\bar{B} = \frac{\partial\bar{E}}{\partial t} + \bar{j}/\epsilon_0 c^2$$

We will first simplify the discussion by assuming the currents are steady, i.e., moving at a constant velocity. $\bar{j}$ and ρ are, therefore, independent of time and the law of conservation of charge shows that

$$\bar{\nabla}\cdot\bar{j} = \partial\rho/\partial t = 0$$

which indicates that there isn't any charge piling up anywhere. We call this special case Magnetostatics. The above equations become separable into two pairs: one for the magnetic field and one for the electric field:

$$\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0 \qquad \nabla\times E = 0 \qquad \text{and} \qquad \bar{\nabla}\cdot\bar{B} = 0 \;,\; \bar{\nabla}\times\bar{B} = \bar{j}/\epsilon_0 c^2$$

In electrostatics we saw that $\bar{\nabla}\times\bar{E}=0$ we could write E as a gradient, $\bar{E} = -\nabla\bar{\varphi}$ and the pair of equations reduced to $\nabla^2\varphi = -\rho/\epsilon_0$. Since $\nabla\times B \neq 0$ we can't do the same and write $\bar{B}$ as a gradient. But we can write it as a curl since $\bar{\nabla}\cdot\bar{B}=0$, i.e., let $\bar{B} = \bar{\nabla}\times\bar{A}$. Then

$$\bar{\nabla}\times\bar{\nabla}\times\bar{A} = -\nabla^2\bar{A} + \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) = \bar{j}/\epsilon_0 c^2$$

We call $\bar{A}$ the vector potential and have discussed it before.

It would be helpful to choose a gauge such that $\bar{\nabla}\cdot\bar{A}=0$ and the above equation becomes analagous to electrostatics. This is called the Coulomb gauge. It leaves us with

$$\bar{\nabla}^2\bar{A} = -\bar{j}/\epsilon_0 c^2$$

In rectangular coordinates this equations means

$$\nabla^2 A_x = j_x/\epsilon_0 c^2 \qquad \nabla^2 A_y = j_y/\epsilon_0 c^2 \qquad \nabla^2 A_z = j_z/\epsilon_0 c^2$$

In cylindrical or spherical coordinates we can't make the separation so easily since radial components can be projected onto the different axes. Again the general form of these equations is just like $\nabla^2\varphi = -\rho/\epsilon_0$ which we solved previous

$$\varphi(1) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(2)\,dV_2}{r_{12}}$$

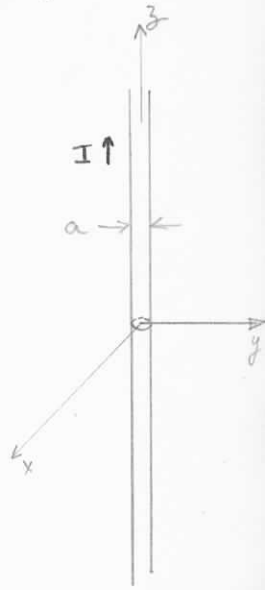
And, therefore,

$$A_x(1) = \frac{1}{4\pi\epsilon_0 c^2}\int \frac{j_x(2)\,dV_2}{r_{12}}$$

And the same for A_y & A_z.

MAGNETIC FIELD FROM A WIRE

I'd LIKE TO WORK A PROBLEM NOW TO SHOW YOU HOW THIS WORKS. LET'S TAKE A STRAIGHT WIRE OF RADIUS a AND LENGTH INFINITELY LONG. THE WIRE CARRIES A STEADY CURRENT I.



SINCE THE CURRENT RUNS ALONG THE z AXIS WE GUESS RIGHT AWAY THAT $A_x = A_y = 0$. BY ANALOGY TO THE ELECTRICAL PROBLEM OF A LONG THIN CHARGE DISTRIBUTION ρ WHICH IS THE CHARGE PER UNIT LENGTH THE ELECTRIC FIELD AT A RADIUS r IS

$$E = \frac{q}{4\pi\epsilon_0 r}$$

where $q = 4\pi a^2 \rho$

SINCE THE POTENTIAL φ IS DIFFERENTIATED TO GET E, WE HAVE THAT

$$E = -\frac{d}{dr}\varphi$$

OR

$$\varphi = -\frac{q}{2\pi\epsilon_0} \ln r$$

WE CAN THEREFORE CONCLUDE THAT FOR THE MAGNETIC CASE

$$A_z = -\frac{I}{2\pi\epsilon_0 c^2} \ln r = -\frac{\pi a^2 j_z}{2\pi\epsilon_0 c^2} \ln (x^2+y^2)^{1/2}$$

NOW WE CAN COMPUTE $\bar{B}$,

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0$$

$$B_x = \frac{\partial A_z}{\partial x} = -\frac{I}{2\pi\epsilon_0 x^2}\frac{y}{r^2}$$

$$B_y = -\frac{\partial A_z}{\partial x} = +\frac{I}{2\pi\epsilon_0 c^2}\frac{x}{r^2}$$

THE MAGNITUDE OF B IS GIVEN AS

$$|B|^2 = B_x^2 + B_y^2 = \frac{1}{2\pi\epsilon_0 c^2}\frac{I}{r}$$

IF WE LOOK DOWN ALONG THE WIRE WE SEE $\bar{B}$ IN THE x, y PLANE. AT A GIVEN RADIUS r THE B IS OF THE SAME INTENSITY SO IT CIRCULATES IN A CIRCLE. SINCE $\bar{B}$ IS SYMMETRICAL THERE CAN'T BE ANY FLUX RADIALLY OUTWARD LIKE $\bar{E}$. WE CAN CHANGE TO A LINE INTEGRAL AND GET

$$\oint \bar{B}\cdot d\bar{l} = \frac{1}{\epsilon_0 c^2}\int \bar{j}\cdot n\, dA$$

THIS TELLS US HOW MUCH CURRENT IS COMING THROUGH THE ~~AREA~~ OF THE CIRCLE OF CIRCUMFERENCE $2\pi r$, I.E.,

$$B\cdot 2\pi r = \frac{I}{\epsilon_0 c^2}$$

$$B = \frac{I}{2\pi\epsilon_0 c^2}$$

USING THIS SAME INTEGRAL INSIDE THE WIRE WE FIND

$$B 2\pi r = \frac{I r^2}{a^2 \epsilon_0 c^2}$$

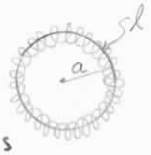
THEN

$$B = \frac{I r}{2\pi \epsilon_0 a^2 c^2}$$

THE $\bar{B}$ IS LINEAR IN DISTANCE TILL IT REACHES THE SURFACE THEN IT FALLS AS $1/r$

THE INTEGRAL CAN BE USED TO CALCULATE THE FIELD OF A SOLENOID WITH N TURNS. IF THE SOLENOID IS WOUND AROUND A TORUS THE FIELD IS GIVEN EXACTLY BY

$$B \cdot l = \frac{1}{\epsilon_0 c^2} N I$$

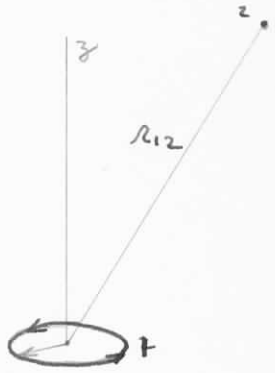


OR

$$B = \frac{1}{\epsilon_0 c^2}\left(\frac{N}{l}\right) I$$

IF THE RADIUS a GET VERY BIG THE SOLENOID APPROXIMATES A STRAIGHT SOLENOID SO THE FORMULA IS GOOD THERE ALSO.

THE FIELD FROM A SMALL LOOP



SUPPOSE WE HAVE ONE SMALL LOOP OF WIRE CARRYING A CURRENT I AND WE WANT TO KNOW WHAT B IS FAR AWAY. WE COULD FIND IT BY CALCULATING $\bar{A}$, I.E.,

$$\bar{A}(\bar{r}_2) = \int \frac{I\, d\bar{l}_1}{r_{12}}$$

SINCE $\bar{B} = \bar{\nabla}_2 \times \bar{A}$ AND $\nabla \frac{1}{r} = \frac{\bar{R}}{r^3}$

$$\bar{B}(\bar{r}_2) = \int \frac{I\, \bar{r}_{21} \times d\bar{l}}{r_{12}^3}$$

NOW $\bar{r}_{21}$ IS $\bar{r}_2 - \bar{r}_1$. IF WE ARE VERY FAR AWAY FOR THE LOOP, THEN $\bar{r}_1$ IS ESSENTIALLY FIXED. THE INTEGRAL $\int \bar{r}_1 \times d\bar{l}_1$ IS ZERO. THE INTEGRAL OVER $\int I\, \bar{r}_2 \times d l_1$ YIELDS THE AREA OF THE LOOP TIMES THE CURRENT.

THE ANALOGY IN ELECTRICITY IS THE ELECTRIC DIPOLE WHICH IS GIVEN BY $\varphi = \frac{\bar{P} \cdot \bar{R}}{r^3}$ AND $\bar{E} = -\bar{\nabla}\varphi = \left(\frac{\bar{P}}{r^3} - \frac{3\bar{R}(\bar{R}\cdot\bar{P})}{r^3}\right)$. THE FIRST TERM IS THE ON AXIS TERM AND THE SECOND INCLUDE ELEVATION ANGLE VARIATIONS. $\bar{P}$ HERE IS THE ELECTRIC DIPOLE AND IS $q\bar{D}$. NOW WE HAVE A MAGNETIC DIPOLE μ OF STRENGTH I (area) of Loop)

$$\mu = I \pi a^2$$

THEREFORE, WE CAN WRITE

$$\bar{A} = \frac{1}{4\pi\epsilon_0 c^2} \frac{\bar{\mu} \times \bar{R}}{R^3}$$

The INTEGRAL EQUATION for $\bar{B}$, i.e.,

$$\bar{B}(2) = \frac{1}{4\pi\epsilon_0 c^2}\int \frac{I\,\bar{r}_{21}}{r_{12}^3} \times d\bar{l} = \frac{1}{4\pi\epsilon_0}\int \frac{j(1)\times \bar{e}_{12}}{r_{12}^2}\, dV_1$$

HAS A NAME AND IS CALLED THE BIOT-SAVART LAW. IT IS FAIRLY USEFUL WHEN FINDING $\bar{B}$ for AN I odd SHAPE CURRENT DISTRIBUTION.

The IDEA of A MAGNETIC MOMENT IS IMPORTANT IN ATOMIC Theory SINCE ATOMS behave AS SMALL MAGNETS WITH A MOMENT GIVEN by

$$\mu = \frac{I\,\text{area}}{\epsilon_0 c^2}$$

If WE HAVE A MODEL of AN ELECTRON GOING AROUND IN A CIRCLE of RADIUS a WITH VELOCITY v, The CURRENT PASSING A POINT IN ONE SECOND IS $\quad I = \frac{q v}{2\pi a}$

Then

$$\mu = \frac{\pi a^2 I}{\epsilon_0 c^2} = \frac{q v}{2\epsilon_0 c^2}$$

~~W~~ WE MULTIPLY by The MASS m AND RECALL THAT The ANGULAR MOMENTUM IS defined AS $\bar{J} = m a \bar{v}$, Then WE CAN WRITE

$$\bar{\mu} = \frac{q_{el}}{m_{el}\, 2\epsilon_0 c^2}\,\bar{J}$$

Therefore WE find The MAGNETIC MOMENT of A CIRCULATING ELECTRON IS DIRECTLY RELATED TO ITS ANGULAR MOMENTUM THROUGH FUNDAMENTAL PHYSICAL CONSTANTS

If The ELECTRON IS PUT INTO A MAGNETIC FIELD $\bar{B}$ IT WILL be SEEN TO PRECESS ABOUT $\bar{B}$ AT A CERTAIN RATE ω_p. THAT IS, The ELECTRON ITSELF IS SPINNING ABOUT ITS AXIS AND RESISTS The TORQUE ON IT, i.e.,

$$\text{Torque} = B\mu \sin\theta = j\,\omega_p \sin\theta$$

OR $\quad \omega_p = \frac{\mu}{j}\bar{B}$

SINCE $\bar{B}$ IS KNOW AND ω_p determinable TO high ACCURACIES, μ AND j CAN be PRECISELY KNOWN.

7. INTRODUCTION TO QUANTUM MECHANICS

The LAST TIME WE MET WE WERE DISCUSSING MAGNETOSTATICS AND I HAD PLAN TO CONTINUE ON WITH MAGNETISM UNTIL WE CAME TO QUANTUM THEORY. BUT I HAVE decided I'd rather begin with QUANTUM Theory so THAT IS WHAT I'LL DO. SINCE THIS IS suppose TO be AN ELEMENTARY course IN PHYSICS I'LL begin WITH SOME ELEMENTARY QUANTUM Theory. TODAY I WON'T GIVE A LECTURE ON The deepest ASPECTS of The SUBJECT

As YOU ALL probably KNOW The behavior of THINGS IS very Different ON The ATOMIC SCALE THAN IT IS ON The LARGE or MACROSCOPIC SCALE. The World does NOT behave LIKE billiard balls, WATER WAVES, or other MACROSCOPIC ENTITIES when VIEWED ON AN ATOMIC LEVEL. The SUBJECT IS CONFUSED BY OUR OWN INTUITION which CAN ONLY ATTEMPT TO GIVE MEANING TO THIS ATOMIC World BASED ON ~~REAL~~ BIG world ANALOGS. So WE TRY TO MAKE MODELS; WE SAY The ATOM IS LIKE A MASS ON A SPRING free TO VIBRATE. BUT THIS MODEL DOES NOT help US GET CLOSER TO UNDERSTANDING The REAL behavior of AN ATOM. There IS NO WAY TO GET PERPETUALLY CLOSER TO REALITY BY SUCH IDEALIZATIONS. IT IS hard for The HUMAN brAIN - BUT IT IS The ONLY brAIN WE HAVE SO WE'LL HAVE TO MAKE do. So The first THING TO UNDERSTAND ~~STAND~~ ABOUT QUANTUM MECHANICS IS THAT There AIN'T NO WAY TO EASILY ~~UNSTA~~ UNDERSTAND IT - SOMETHING LIKE YOUR WIFE! The ONE WAY TO ACCURATELY describe The QUANTUM MECHANICAL ATOM IS IN ITS OWN WAY.

I'LL TALK ABOUT The ATOM IN A SEMI-SLOPPY WAY, i.e., NOTHING WILL BE RIGHT OR WRONG. If YOU GET CONFUSED, YOU ARE PROBABLY ON The RIGHT TRACK. NEXT TIME I GIVE MOST QUALITIES of A QUANTUM MECHANICAL (Q.M.) SYSTEM. This MEANS I'LL TELL The STORY NOT AS PARTICLES and NOT AS WAVES - I'LL TELL IT AS IT IS.

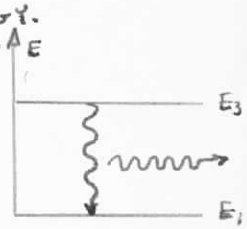
ALTHOUGH The ATOM IS NOT LIKE A PARTICLE or WAVE, IN EXPLAINING CERTAIN PHENOMENA WE CAN GET A GOOD IDEA how it behaves if WE COMPARE IT TO A PARTICLE or WAVE. IT IS GOOD TO HAVE A NUMBER of SLOPPY MODELS if YOU UNDERSTAND EACH MODEL'S LIMITATIONS.

Our CLASSICAL MODEL of The ATOM HAS AN ELECTRON MOVING AROUND A NUCLEUS. The ELECTRON CREATES A CLOUD of NEGATIVE CHARGE which CAN SHAKE BACK AND FORTH - IN SO DOING IT CAN EMIT LIGHT. AS ENERGY IS RADIATED AWAY from The ATOM, The ELECTRON MUST START TO SPIRAL IN TOWARDS The NUCLEUS. BUT when EXAMINING The SPECTRUM of HYDROGEN WE DON'T SEE ALL SORTS of frequencies CHARACTERISTIC of DIFFERENT ORBIT. The HYDROGEN SPECTRUM IS CHARACTERIZED BY ONLY A few frequencies.

Our model does not work. While it does give the right polarization sense to the radiated electromagnetic waves from radiating atom, it is not accurate enough. So we change our model; To get the right frequency we assume the electron is somehow connected to the nucleus by a spring. The coupled spring-mass system now oscillates at a characteristic frequency. As you ~~Rember~~ Remember in our discussions of scattering theory, we modeled the atom as a spring-mass oscillators and worked out a lot of interesting things. But all is not right. We still can't explain how the atom is ionized, i.e., and electron removed from the atom. Do we have to break the spring? If we do that, the spring is non-linear and we get in all sorts of trouble. Again, you must know the limitations of your model.

So far we have only talked about one frequency characteristic of the atom but could an atom have many modes of vibration and therefore frequencies which are somehow related. The answer is, of course, yes. So we must look closer at these frequencies. If a system vibrates at two frequencies ω_1 and ω_2, we can find a third frequencies ω_3 which equals $\omega_1+\omega_2$. Another possibility is that $\omega_1+\omega_2=\omega_3+\omega_4$; a classical system behaving like this is said to be non-linear. That is, some vibration has an amplitude $x=\cos\omega_1 t+\cos\omega_2 t$ where we now have a radiation term proportional to the cross product term, $\cos\omega_1 t+\cos\omega_2 t$. Which we can write as $\cos\omega_1 t\cos\omega_2 t=\cos(\omega_1-\omega_2)t+\cos(\omega_1+\omega_2)t$. This is how we get beat frequencies. One type of frequency combination that we don't get is $2\omega_1+\omega_2$ characteristic of an x^3 term.

Our examination of different models has now led us to the point where we find the atom sitting in states of different energy, call them E_1, E_2, E_3, etc. Here I have assumed the state of the internal motion of an atom can be defined by its energy. Further we must permit some mechanism for changing energy states. We are led to the sensational formula



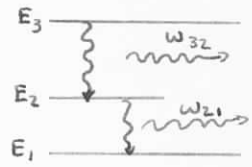
$$E_3-E_1=\hbar\omega_{31}$$

Where ω_{31} is the characteristic frequency of the emitted energy. ω_{31} is the difference in frequencies ω_3 and ω_1 characteristic of the two energy states E_3, and E_1, i.e., $E_3=\hbar\omega_3$ and $E_1=\hbar\omega_1$ so that

$$E_3-E_1=\hbar(\omega_3-\omega_1)$$

WE CAN NOW BEGIN TO UNDERSTAND THE CASE where $\omega_1 + \omega_2 = \omega_3 + \omega_4$. MAYBE THE ATOM DIDN'T GET ALL THE WAY TO STATE 1 FROM 3 BUT STOPPED AT AN INTERMEDIATE STATE E_2. THEN WE WOULD HAVE

$$E_3 - E_2 = \hbar\,\omega_{32} = \hbar(\omega_3 - \omega_2)$$
$$E_2 - E_1 = \hbar\,\omega_{21} = \hbar(\omega_2 - \omega_1)$$



If WE ADD THESE EQUATIONS WE GET THE SAME AS before, i.e.,

$$E_3 - E_1 = \hbar(\omega_3 - \omega_1) = \hbar\,\omega_{31}$$

NOTICE EACH PHOTON BEING EMITTED COMES OUT WITH A UNIQUE FREQUENCY AND THIS EXPLAINS WHY WE DON'T GET ANY DOUBLE FREQUENCY COMBINATIONS.

THE FACT THAT THE EMITTED PHOTON WAS CHARACTERISTIC of FREQUENCY WAS NOT PROVEN UNTIL THE EXPERIMENTS WITH PHOTOCELLS. IN THIS EXPERIMENT ~~ELECTRONS~~ PHOTONS WERE BOMBARDED AGAINTS AL CONDUCTOR AND THE RELEASED ELECTRONS HAD A FREQUENCY GIVEN BY $(E_{START} - E_{END})/\hbar = \omega$. THIS IS A VERY UN-CLASSICAL RESULT SINCE IT WAS BELIEVED THAT A HIGHER INTENSITY BOMBARDING BEAM WOULD RELEASE MORE ELECTRONS.

BY COMPARING EXPERIMENTS WHICH USED BOMBARDING BEAMS OF ELECTRONS AND PHOTONS ATOMS WERE EXCITED TO DIFFERENT STATES CHARACTERIZED BY UNIQUELY DEFINED ENERGIES. THUS ASSOCIATED WITH THE ELECTRON AND PHOTON WERE CHARACTERISTICS FREQUENCIES AND ENERGIES RELATED BY THE FORMULA

$$\hbar \times (\text{frequency of thing}) = \text{ENERGY of THING}$$

NOW WE HAVE A PROBLEM. IN THE PHOTO ELECTRIC EFFECT MENTIONED ABOVE THE LIGHT OF PHOTON ACTED LIKE RAIN, i.e., PARTICLES HAVING AN ENERGY $E = \hbar\omega$. THE RAINDROP ANALOGY COULD BE EXPLAINED IN TERMS OF ENERGY BUT NOT IN TERMS OF FREQUENCY. THE FREQUENCY OF THE PHOTON IS ASSOCIATED WITH THE WAVE NATURE OF THE PHOTON. THE WAVE VARIES SINUSOIDALLY IN TIME AS $e^{-i\omega t}$. BUT THE CONNECTION BETWEEN ENERGY AND FREQUENCY IS NOT ALL. THIS IS NOT A RELATIVISTICALLY INVARIANT CONCEPT AS WE HAVE DEVELOPED. ONE MAN'S ENERGY IS NOT ANOTHER MANS ENERGY if THE OTHER MAN IS OBSERVING THING FROM A MOVING COORDINATE SYSTEM. THE MORE ACCURATE WAVE EQUATION FOR A PHOTON IS PROPORTIONAL TO $e^{-i(\omega t - kz)}$ ω BY ITSELF IS NOT RELATIVISTICALLY INVARIANT BUT IT IS THE TIME COMPONENT OF A FOUR VECTOR WHICH IS INVARIANT. THE SPACE COMPONENTS OF THE FOUR VECTOR IS GIVEN BY THE WAVE NUMBERS K_x, K_y, AND K_z. THE TRANSFORMATION EQUATIONS FOR THE FREQUENCY - WAVE NUMBER FOUR-VECTOR IS

$$\omega' = \frac{\omega - v k_z}{\sqrt{1 - v^2/c^2}} \qquad K_z' = \frac{K_z - \omega v/c}{\sqrt{1 - v^2/c^2}} \qquad K_x' = K_x \qquad K_y' = K_y$$

This describes how THE WAVE WOULD LOOK TO A PASSING OBSERVER.

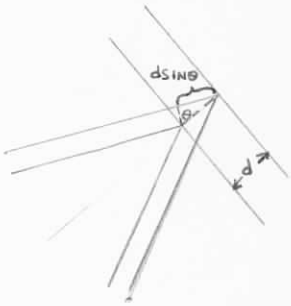
FORTUNATELY for us ENERGY IS The TIME COMPONENT of The MOMENTUM four-VECTOR. This is Good. OTherwise relativity would NOT be CONSISTENT. WE NOW have developed A LOGICAL AND SOUND REASON for EQUATING WAVE MOMENTUM TO WAVE NUMber by The RULE

$$\vec{p} = \hbar \vec{K}$$

This follows DIRECTLY from The prior EQUATION of frequency AND ENERGY. The ELECTRON, which IS PERHAPS MORE EASY TO DESCRIBE IN TERMS of ENERGY AND MOMENTUM, DOES HAVE A WAVE NUMber, $\bar{K}$, AND frequency, ω, ASSOCIATED WITH IT THAT YIELDS CERTAIN WAVE CHARACTERISTICS. AN EXPERIMENT TO verify This UNUSUAL PROPERTY of PARTICLES AND WAVES IS TO SCATTER ELECTRONS AND X-RAYS of A CRYSTAL. The diffraction PATTERNS ARE VIRTUALLY IDENTICAL. BOTH EXHIBIT A COHERENT REFLECTION obeying The CONDITION THAT

$$2d \sin\theta = n\lambda$$

where d is The SEPARATION DISTANCE BETWEEN PLANES.



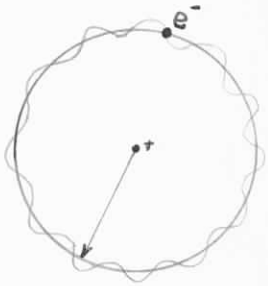
WE ARE LED TO A VERY INTERESTING CONCLUSION AS A RESULT of This LINE of REASONING - ALL OBJECTS, i.e., PHOTONS, ELECTRONS, NEUTRONS, PROTON, ATOM, etc, UNDER CERTAIN CIRCUMSTANCES ~~be~~ ARE best described by WAVES AND AT OTher TIMES by PARTICLES.

I WANT TO CONTINUE ON AND DISCUSS SOME MORE CURIOUS THINGS ABOUT ATOMS. WE HAVE LEARNED THAT ENERGY LEVELS ASSOCIATED WITH SPECTRAL LINES Show A high degree of REGULARITIES. For The hydrogen These ENERGY LEVELS ARE rather REMARKABLY AND SIMPLY defined AS

$$E = -\frac{\text{Rydberg}}{n^2}$$

Where The NUMERATOR IS SOME CONSTANT TERM WHILE n IS A WHOLE INTEGER. NO OTher ATOM IS LIKE This but WHY Should This be TRUE for hydrogen? This SET The CLUE which WAS The beginning of The end for NATURE.

Rutherford SET OUT TO SOLVE This MYSTERY by MODELLING his ATOM with The ELECTRON CIRCULATING The NUCLEUS. The DYNAMIC LAWS which BALANCE The forces WERE WRITTEN, i.e.,



$$F_{\text{ELECTROSTATIC}} = F_{\text{DYNAMIC}}$$

$$\frac{q^2}{4\pi\epsilon_0 R^2} = \frac{mv^2}{R} \quad \text{(CENTRIFUGAL force)}$$

or

$$\frac{e^2}{R^2} = \frac{mv^2}{R}$$

The ENERGY ASSOCIATED WITH This MOTION IS JUST

$$E = \frac{1}{2}mv^2 - \frac{e^2}{R} = -\frac{mv^2}{R}$$

NOW SUPPOSE THAT THE MOMENTUM HAS A WAVE NUMBER ASSOCIATED WITH IT - WE JUST PROVED THIS IDEA WAS CONSISTENT WITH RELATIVITY. THEN WE MUST HAVE THAT

$$mv = \frac{\hbar}{\lambda} = \frac{2\pi\hbar}{\lambda} = \hbar k$$

WE'RE ONE ARE WAY. IF WE CAN NOW CRAZILY IMAGINE THAT ONLY DEFINITE DISCRETE VALUES FOR THE ANGULAR MOMENTUM ARE POSSIBLE, THEN THE WAVE MUST SOME-HOW FIT NICELY INTO THE CIRCLE. THIS WILL ONLY HAPPEN IF

$$n\lambda = 2\pi R$$

PUTTING THESE RESULTS TOGETHER WE SEE THAT INDEED THE ANGULAR MOMENTUM MUST BE CONSTRAINED TO CERTAIN VALUES,

$$mvR = n\hbar$$

THIS IS THE SO-CALLED QUANTIZATION RULE OF ANGULAR MOMENTUM.

LET'S GO ~~BACK~~ ON - DAMN THE TORPEDOES - AND PUT THE DYNAMIC LAWS TOGETHER WITH THIS QUANTIZATION RULE,

$$mv^2 = \frac{e^2}{R} = \frac{mve^2}{n\hbar}$$

THAT IS

$$v = \frac{e^2}{n\hbar}$$

THE ENERGY LEVELS ARE FINALLY DERIVED,

$$E = -\frac{mv^2}{2} = -\left(\frac{me^4}{2\hbar^2}\right)\frac{1}{n^2}$$

THE QUESTION WE FINALLY HAVE TO ANSWER IS $\left(\frac{me^4}{2\hbar^2}\right)$ EQUAL TO THE RYDBERG CONSTANT? THE ANSWER IS YES. $\frac{me^4}{2\hbar^2} = 13.6\,eV$; THIS IS THE ENERGY NEEDED TO IONIZE THE HYDROGEN ATOM.

SO WHAT DID WE LEARN? WELL, WE SAW THAT ATOMS MUST EXIST IN CERTAIN STATES ; THAT WAVES CAN ONLY SIT STILL IF THEY FIT INTO WHATEVER CONFINES THEM EVENLY ; AND THAT TO DO SO GIVES RISE TO QUANTIZATION RULES. IN ESSENCE WHAT WE HAVE DESCRIBED A TWO DIMENSIONAL STANDING WAVE WHERE WE HAVE BEEN SLOPPY AND IGNORED ANY RADIAL COMPONENTS. AFTER RUTHERFORD MADE HIS DISCOVERY SCHROEDINGER FINALLY CAME AROUND TWO YEARS LATER AND WROTE THE THREE DIMENSIONAL WAVE EQUATION TO DESCRIBE THESE STANDING WAVES IN DIFFERENTIAL FORM. THAT WAS A GREAT MOMENT.

I HAVE BEEN TOYING WITH THE IDEA THAT IT WOULD BE FUN TO DESCRIBE QUANTUM MECHANICS FROM A SET OF AXIOMS. THIS WOULD BE A BACKHANDED WAY TO DESCRIBE QUANTUM PHENOMENA BUT IT MIGHT BE INTERESTING. I MIGHT TRY TO DEVELOPE THIS FURTHER AND TRY IT OUT ON YOU BUT NOT THIS WEEK.

8. MAGNETIC MOMENTS AND ANGULAR MOMENTUM ref. VOL III CHAPT 34 & 35

LAST TIME WE DEVELOPED THE IDEA THAT A CONFINED SYSTEM HAS A SET OF DEFINITE ENERGY LEVELS ASSOCIATED WITH IT. TO THIS SYSTEM WE THEN IDENTIFIED A STANDING WAVE, REPRESENTING THE LOCATION OF THE ELECTRON, AND SOMEHOW FITTING NEATLY INTO THE CIRCUMFERENCE IN WHOLE UNITS OF ANGULAR MOMENTUMS, $\hbar$. THE NUMBER OF NODES, n, IN A CIRCLE WAYS

$$n = \frac{2\pi a}{\lambda}$$

WHERE a IS THE CIRCLE RADIUS, AND λ THE WAVELENGTH. THE ANGULAR MOMENTUM OF THE PARTICLE WAS $J = mva = pa$ AND THE TWO IDEAS WERE RELATED BY THE EQUATION

$$n = \frac{J}{\hbar} = \frac{pa}{\hbar}$$

NOW THE IDEA OF ANGULAR MOMENTUM IS DEFINED OR ASSOCIATED WITH A UNIQUE AXIS. IF ANOTHER AXIS IS ROTATED THROUGH SOME ANGLE θ RELATIVE TO THE STARTING AXES, THEN THE NEW ANGULAR MOMENTUM IS DIFFERENT. THEN THE WHOLE INTEGER IDEA WOULD BE DESTROYED. BUT NATURE IS INGENIOUS AND REQUIRES THAT J HAVE ONLY ONE VALUE ABOUT ANY AXIS. AN EXPERIMENT TO PROVE THIS PHENOMENA IS REALLY THE WAY NATURE BEHAVES IS THE STERN-GERLACH EXPERIMENT. IN THIS EXPERIMENT THE ATOM HAS A MAGNETIC MOMENT, μ, WHICH IS PROPORTIONAL TO ITS ANGULAR MOMENTUM, I.E.,

$$\mu = g\left(\frac{qe}{2m}\right) J$$

WHERE g IS A FACTOR CHARACTERISTIC OF THE STATE OF THE ATOM. WHEN THE ATOM IS IN A MAGNETIC ~~FO~~ FIELD $\bar{B}$, THERE IS A MAGNETIC ENERGY ASSOCIATED WITH THE SYSTEM,

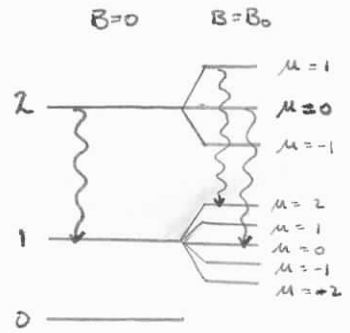
$$U_{MAG} = -\bar{\mu}\cdot\bar{B} = -g\,\bar{J}\cdot\bar{B}$$

IF $\bar{B}$ WAS IN THE z DIRECTION THE ENERGY WOULD BE DISTRIBUTED AS

$$-\mu B\cos\theta$$

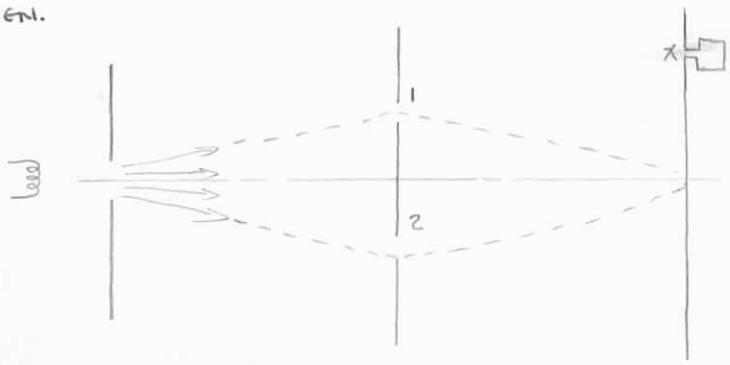
AND ALL ANGLES θ ALL POSSIBLE IN CLASSICAL PHYSICS. BUT IT WAS FOUND THAT $\bar{\mu}$ HAD ONLY CERTAIN ALLOWED VALUES GIVING RISE TO TWO DISTINCT POINTS ON THE PHOTOGRAPHIC PLATE. THUS THE ANGULAR MOMENTUM HAS ONLY A FINITE SET OF VALUES.

THERE IS YET ANOTHER EXPERIMENT THAT INDICATES THAT THE ANGULAR MOMENTUM CAN ASSUME DEFINITE VALUE. WHEN THE ELECTRON IN A HYDROGEN ATOM MAKES A TRANSITION IT EMITS A CHARACTERISTIC FREQUENCY - WE HAVE TALKED ABOUT THIS. NOW IF THE TRANSITION OCCURS IN A MAGNETIC FIELD THE CHARACTERISTIC EMISSION LINE ACTUALLY CONSISTS OF A NUMBER OF DIFFERENT FREQUENCIES EACH REPRESENTING TRANSITIONS FROM SLIGHTLY DIFFERENT ENERGY STATES. IN EXAMINING EACH ENERGY STATE OF THE ELECTRON WE SEE IT SPLITTING INTO SLIGHTLY DIFFERENT VALUES; THE DIFFERENCE BEING ASSOCIATED WITH THE MAGNETIC ENERGY DUE TO THE INTERACTING MAGNETIC MOMENT AND THE MAGNETIC FIELD.

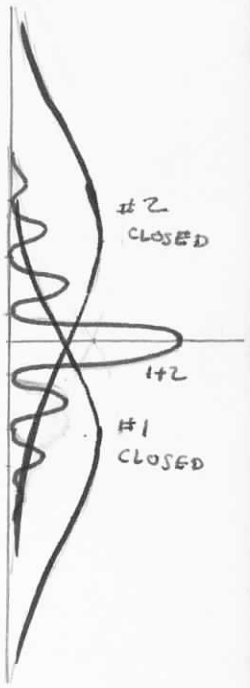


THE TRANSITION IN THE ABSENCE OF A MAGNETIC FIELD WOULD BE FROM STATE 2_0 TO 1_0. WHEN A FIELD B IS PRESENT AN ADDITIONAL 10 TRANSITIONS ARE POSSIBLE, I.E., $2_1 \to 1_2$, $2_1 \to 1_1$, $2_1 \to 2_0$, $2_1 \to 1_{-1}$, $2_1 \to +1_{-2}$, AND $2_{-1} \to 1_2$, $2_{-1} \to 1_1$, $2_{-1} \to 1_0$, $2_{-1} \to 1_{-1}$, $2_{-1} \to 1_{-2}$. THIS IS CALLED THE ZEEMAN EFFECT AND FURTHER INDICATES THAT J CAN HAVE ONLY DISCRETE VALUES

NOW I'M GOING TO DESCRIBE AN EXPERIMENT WHICH IS SO DESIGNED THAT EVERYTHING HAPPENS AT ONCE, I.E., THE PARTICLE AND WAVE PICTURES CANNOT BE USED TO EXPLAIN WHAT HAPPENS. NEITHER VIEW BY ITSELF WILL EXPLAIN THE PHENOMENA. I'LL TAKE A SOURCE, S, OF ELECTRONS OR PHOTONS AT A UNIQUE FREQUENCY AND A DEFINITE POLARIZATION. BEYOND THE SOURCE IS A BARRIER WITH TWO HOLES IN IT; EACH HOLE HAS A DIAMETER WHICH IS VERY BIG COMPARED TO THE BARRIER WIDTH - THIS IS AN IDEALIZED CASE WHICH PERMITS ME TO IGNORE THE BARRIER AS A SECONDARY SOURCE. FINALLY AT ~~THE~~ A DISTANCE BEYOND THE BARRIER I HAVE A DETECTOR AND SCREEN.



When both slits are open we get a photographic record on the screen that looks just like a diffraction pattern which could be associated with interferring waves. The intensity distribution would look like the side sketch. The intensity is twice what would come from one slit. If slit one or two is shut the other intensity distributions are recorded



Now if a photomultiplier records the number of photons striking the point x per second, we might ask how the counting varies with x. Remember that it counts or it doesn't count, i.e., the photons are discrete particles striking the plate. Surely we would expect the counts per second as a function of x to reproduce the summed curve on the side. And, in fact, we do read this. So the intensity is associated with the number of clicks, bangs, or counts per second. We are thus faced with a phenomena which clearly gives rise to a wave phenomena, i.e., interference, and at the same time implying that the cause of the interference is raindrop like in nature - or to say it another way the wave-particle duality of nature is simultaneously demonstrated here. For that reason the human brain is boggled by the idea.

Suppose we cooled down the light and made it weaker so that most of the time not much is happening. Only a few clicks per second will be recorded at the multiplier. The counts really are an indication of the photon arriving at a certain point x each second. This implies that you can't predict where the photon will land. This further suggests that may be there ~~are~~ is some mechanism back in the source which tells the photon where to go. We really need some random motion or mechanism which permits the photon to smell out which hole it should go through so as to produce the right curve.

The first obvious statement that can be made is that the photon goes through either hole 1 or hole 2. This hypothesis implies that the probability of arrival of the photon is just the sum through each hole, i.e.,

$$P = P_1 + P_2$$

In quantum mechanics the idea of a probability of an event occurring can be represented quantitatively by the absolute square of a complete number called a probability amplitude. Call it φ. The complex amplitude of an event happening by either paths 1 or 2 is

$$\varphi = \varphi_1 + \varphi_2$$

WE JUST SAID THE PROBABILITY OF AN EVENT HAPPENING WAS PROPORTIONAL TO $|\varphi|^2$ SO THAT

$$P = |\varphi|^2 = |\varphi_1 + \varphi_2|^2 = |\varphi_1|^2 + |\varphi_2|^2 + 2|\varphi_1\varphi_2|$$

where THE INDIVIDUAL PROBABILITIES ARE $P_1 = |\varphi_1|^2$ AND $P_2 = |\varphi_2|^2$. FROM WAVE THEORY WE CAN REPRESENT φ IN COMPLEX NOTATION AS $\rho e^{i\theta}$ where θ IS SOME PHASE ANGLE INDICATING when THE WAVE STARTED OUT AND from which PART OF THE SOURCE. EXPANDING THIS IDEA WE SEE THAT

$$\varphi_1 + \varphi_2 = \rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}$$

or $$P = |\varphi_1 + \varphi_2|^2 = |\rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}|^2 = \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)$$

FINALLY

$$P = P_1 + P_2 + \text{A CORRECTION TERM}$$

WE SEE THAT THE QUANTUM MECHANICAL CALCULATION NOW GIVES US A CORRECTION TO THE PROBABILITY, $P_1 + P_2$, which WE WOULD PREDICT USING CLASSICAL PHYSICS. IN TYPICAL EXPERIMENTS WITH LIGHT λ IS SO LARGE COMPARED TO THE HOLE SIZE THAT θ VARIES EXTREMELY RAPIDLY AND THE COSINE TERM AVERAGES OUT; THAT IS WHY WE GET THE CLASSICAL RESULT. BUT WHEN THE EXPERIMENT IS RUN ON THE TINY SCALE, THE RESULTS ARE VERY DIFFERENT.

TO SUMMARIZE THE TWO IMPORTANT IDEAS developed here:

1. THE TOTAL PROBABILITY THAT SOMETHING CAN HAPPEN = | AMPLITUDE THAT IT CAN HAPPEN |^2

2. IF AN EVENT CAN HAPPEN IN A NUMBER OF WAYS, i, THEN THE AMPLITUDE THAT IT CAN HAPPEN AT ALL = $\sum_i$ AMPLITUDE THAT IT HAPPENS IN THE MANNER i

THESE POINTS ARE VERY SUBTLE AND VERY IMPORTANT TO THE UNDERSTANDING OF QUANTUM MECHANICS. IT IS TOO BAD THAT WE DON'T HAVE A DIRECT WAY TO MEASURE THE COMPLEX AMPLITUDE - IT WOULD BE EASIER TO UNDERSTAND IF WE COULD. PERHAPS TO EXPAND ON THE SECOND POSTULATE, THE AMPLITUDE TO GO FROM THE SOURCE TO X VIA PATH 1 AND 2 CAN BE WRITTEN IN THE FOLLOWING MANNER

$$\varphi_{1x} = [\text{AMP}(x \leftarrow 1)][\text{AMP}(1 \leftarrow S)]$$

$$\varphi_{2x} = [\text{AMP}(x \leftarrow 2)][\text{AMP}(2 \leftarrow S)]$$

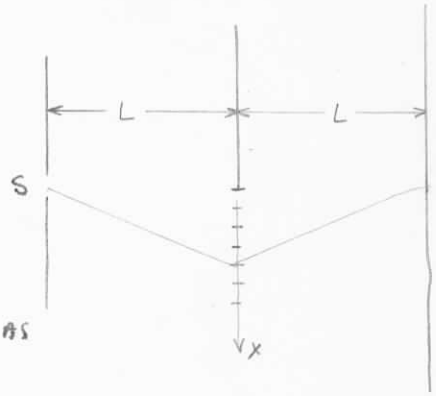
AND

$$\varphi_x = \varphi_{1x} + \varphi_{2x}$$

IN THE CASE of definite ENERGY the AMPLITUDE IS PROPORTIONAL TO exp (i DISTANCE APART $/\lambda$). Where the DISTANCE IS the LENGTH from S TO X VIA the PARTICULAR SLIT. SUPPOSE WE HAD A LOT of SLITS then

$$\varphi \propto \sum_{\text{SLITS}} e^{i\,\frac{\text{DISTANCE}}{\lambda}}$$

AS AN EXAMPLE CONSIDER the CASE where the BARRIER IS GONE IN the LOWER HALF of the ARRANGEMENT ON the SIDE, BY CONTINUIN TO ADD MORE AND MORE HOLES the MAXIMUM INTENSITY IS NOT INCREASED INDEFINITELY AS WE SEE IF the HOLES BECOME A CONTINUUM then



$$\sum e^{i\,\frac{\text{DISTANCE}}{\lambda}} \rightarrow \int_{x=0}^{\infty} e^{i2\frac{\sqrt{L^2+x^2}}{\lambda}}\,dx$$

TO A GOOD APPROXIMATION THIS INTEGRAL CAN be WRITTEN AS

$$e^{2i\frac{L}{\lambda}} \int_0^{\infty} e^{i\frac{2x^2}{\lambda L}}\,dx$$

EVALUATING

$$\int_0^{\infty} e^{i\frac{2x^2}{\lambda L}}\,dx = \int_0^{\infty} \cos\frac{2x^2}{\lambda L}\,dx + i\int_0^{\infty} \sin\frac{2x^2}{\lambda L}\,dx$$

$$= \frac{\lambda L}{2}\left[\frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}}\right] = \frac{\sqrt{\pi}}{2}\frac{\lambda L}{4}\left[1+i\right]$$

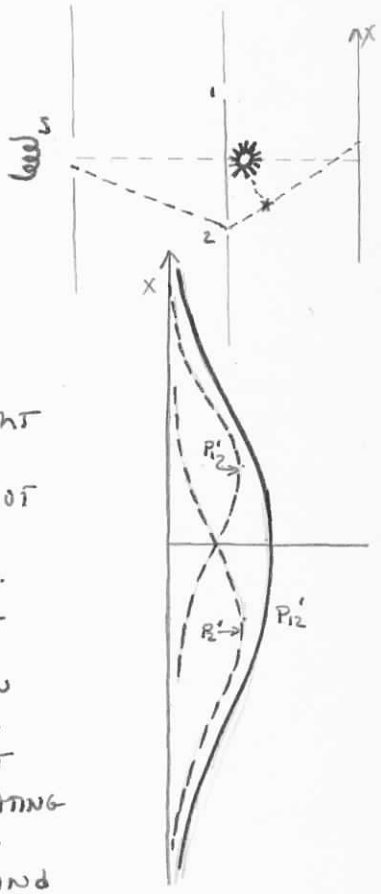
SINCE the INTEGRAL IS FINITE the INTENSITY DOES NOT CONTINUE TO INCREASE UNBOUNDED. SINCE the PHASE CHANGE IS PROPORTIONAL TO x^2 WE GET A feel for how the ~~FESU~~ RESULTANT INTENSITY CURVE CAN be PLOTTED,

YOU CAN SEE THAT ADDING ALL these EQUAL PHASES AFTER A CERTAIN POINT DOES NOT CHANGE the TOTAL RESULTANT INTENSITY VERY MUCH. THIS IS A CORNU SPIRAL; MANY of YOU HAVE SEEN IT before.

I'd LIKE TO NOW DISCUSS SOME of the PITFALLS IN OUR ATOMIC MODEL which RESULT from the PROPOSITIONS WE MADE ON the PREVIOUS PAGE.

WE HAVE SAID THAT THERE IS AN AMPLITUDE ASSOCIATED WITH THE PROBABILITY THAT AN ELECTRON OR PARTICLE WILL GO THROUGH HOLE 1 or 2. BUT MAYBE THE ELECTRON SOMEHOW BREAKS UP AND HALF GOES THROUGH ONE HOLE AND HALF THROUGH THE OTHER. SINCE WE IMAGINE THE ELECTRON IS A DISCRETE INDIVISIBLE ENTITY, THIS IDEA DOESN'T LOOK GOOD. LET'S ADD TO THE EXPERIMENT A LIGHT behind THE BARRIER SO AS THE ELECTRON COMES THROUGH A HOLE THE LIGHT IS SCATTERED AND WE CAN TELL WHEN IT COMES THROUGH. If THE FLASHES OCCUR behind EITHER HOLE 1 or 2 ~~ore~~ PROPOSITION IS VERIFIED.

LET'S MAKE A LIST AND MARK OFF TWO COLUMNS, ONE for ELECTRONS THRU 1 AND ONE for HOLE 2. THE TOTAL NUMBER of ELECTRONS COMING THROUGH AND HITTING THE SCREEN IS JUST $N_T = N_1 + N_2$. If WE PLOT THE NUMBERS OVER AND OVER for different LOCATIONS, x, WE GET THE CURVES P_1' AND P_2' – THE SUM BEING P_{12}'. THAT IS, THE NUMBER of ELECTRONS IN COLUMN 1, N_1, GIVES US THE CURVE P_1' WHILE N_2 GIVES US CURVE P_2'. THIS IS NOT TOO SURPRISING; IT IS THE SAME RESULT AS WHEN THE other HOLE IS CLOSED. HOWEVER, THE TOTAL PROBABILITY CURVE P_{12}' IS different from THE INTERFERENCE CURVE P_{12} THAT WE GOT LAST TIME WE DID THIS EXPERIMENT – WITHOUT THE LIGHT.



SO YOU SAY, ALRIGHT FEYNMAN GET THE GOD DAMN LIGHT OUT OF THERE because IT IS SCREWING UP THE RESULTS. THE RESULTS ARE AFFECTED by THE LIGHT, AH-HA! THIS IS NOT UNUSUAL THAT THE LIGHT WOULD KNOCK THE ELECTRONS OUT of THEIR NORMAL PATHS AND SO AFFECT THE CURVES. SO TURN DOWN THE LIGHT; NOW IT WON'T KNOCK THE HELL OUT of ELECTRON. BUT NOW WE WON'T ALWAYS SEE THE ELECTRON WHEN THE COUNTER MAKES A recording. THE LOW INTENSITY LIGHT MEANS THERE ARE fewer PHOTONS COMING OUT per SECOND so THERE ARE fewer TO SCATTER. WE MUST NOW ADD A THIRD COLUMN TO OUR LIST; THIS TIME INDICATING WHEN AN ELECTRON WAS COUNTED but NOT ~~SEEEN~~. SEEN. If WE LOOK AT THE CURVES from N_1, N_2, AND N_3 WE find THAT N_1 YIELDS P_1', N_2 YIELDS P_2' AND their SUM IS P_{12}'; COUNT N_3, THE "NOT SEEN" ELECTRONS GIVES US A "WAVY" DISTRIBUTION LIKE P_{12}. If THE ELECTRONS ARE NOT SEEN, WE GET INTERFERENCE. IN other words WHEN A PHOTON from OUR LIGHT SOURCE HITS THE ELECTRON, IT destroys ITS INTERFERENCE properties AND MAKES IT APPEAR PARTICLE-LIKE. TAKE THE photon SOURCE AWAY AND THE ELECTRON ACTS LIKE WAVES.

MAYBE WE SHOULD HAVE CONDUCTED THIS EXPERIMENT WITH GENTLER PHOTONS SO THAT THEY WOULDN'T DISTURB THE ELECTRON SO MUCH. THAT IS WE COULD REDUCE THE AFFECT INFINITESIMALLY BY RECALLED THE PHOTON HAS A MOMENTUM, $P = h/\lambda$. IF WE LOWER THE PHOTON'S FREQUENCY, I.E., INCREASE IT WAVELENGTH λ, IT WON'T JOLT THE ELECTRON SO MUCH. NOW THE PHOTON IS SO WEAK THAT THE INTERACTION IS INCOMPREHENSIBLE. BUT TO ACHIEVE THIS λ IS SO LONG, I.E., INTO THE RADIO WAVES AND LOWER THAT YOU CAN'T "SEE" THE ELECTRON. REMEMBER THAT IN RESOLVING TWO POINTS USING WAVES THAT THE MINIMUM RESOLUTION IS PROPORTIONAL TO λ. SO NOW WE CAN'T TELL WHICH HOLE THE ELECTRON WENT THROUGH. WE ARE FORCED TO MARK THE THIRD COLUMN EVERYTIME THE COUNTER CLICKS AND YOU OBSERVE THAT THE COUNT IN COLUMN 3 GETS QUITE LARGE SO WE HAVE THE EXPECTED WAVE-LIKE INTERFERENCE.

9. MORE ON WATCHING THE ELECTRON Ref. VOL III CHAPT. I

LAST TIME WE WERE TALKING ABOUT QUANTUM THEORY IN SEVERAL EXPERIMENTS. THE PRINCIPLES THAT WE DEVELOPED WERE:

(1). The PROBABILITY of AN EVENT IN AN IDEAL EXPERIMENT IS GIVEN by The SQUARE of The ABSOLUTE VALUE of A COMPLEX NUMBER φ which is CALLED The PROBABILITY AMPLITUDE.

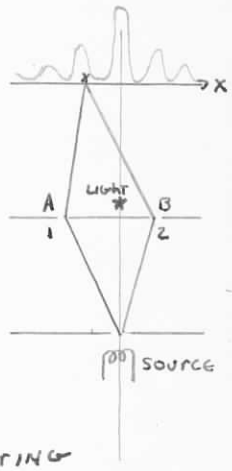
$$P = |\varphi|^2$$

(2) When AN EVENT CAN OCCUR IN SEVERAL ALTERNATIVE WAYS, The PROBABILITY AMPLITUDE FOR The EVENT IS The SUM of The PROBABILITY AMPLITUDES FOR EACH WAY considered SEPARATELY - INTERFERENCE OCCURS,

$$\varphi = \varphi_1(x) + \varphi_2(x)$$
$$P = |\varphi_1 + \varphi_2|^2$$

3. If AN EXPERIMENT IS PREFORMED which IS CAPABLE of DETERMINING WHETHER ONE or ANOTHER ALTERNATIVES IS ACTUALLY TAKEN, The PROBABILITY of The EVENT IS The SUM of The PROBABILITIES for EACH ALTERNATIVE. The INTERFERENCE IS LOST

$$P = P_1 + P_2$$



In The TWO HOLE EXPERIMENT WE HAD TWO DISTINGUISHABLE EVENTS:

1. The ELECTRON ARRIVED AT X AND A PHOTON APPEARED AT "A"
2. The ELECTRON ARRIVED AT X AND A PHOTON APPEARED AT "B"

EACH EVENT HAS AN AMPLITUDE AND PROBABILITY. SINCE The EVENTS ARE DISTINCT, The PROBABILITIES ARE ADDED by RULE 3 ABOVE. To find The AMPLITUDE of EVENT 1 WE MUST COMPUTED The AMPLITUDE BASED ON The PROBABILITY THAT A PHOTON WAS SCATTERED AT "A" AND, Therefore WE COULD "SEE" The ELECTRON. If WE CALL The SCATTERING AMPLITUDE IS SEEN Through hole 1 "a", The AMPLITUDE of EVENT 1 IS GIVEN by $a\varphi_1$. BECAUSE The EXPERIMENT IS NEVER perfect, There WILL be SOME PROBABILITY of A PHOTON BEING SCATTERED AT HOLE 2. If WE CALL THIS SCATTERING AMPLITUDE, b, IT MEANS WE WRONGLY SAW The ELECTRON. The AMPLITUDE THAT IT WENT Through HOLE 2 IS The $b\varphi_2$. Now The TOTAL AMPLITUDE TO find AN ELECTRON AT X AND A photon AT A IS

$$\Phi_A = a\varphi_1 + b\varphi_2 \qquad a+b=1$$

The PROBABILITY of This EVENT OCCURRING IS found by RULE 2 ABOVE

$$P_A = |a\varphi_1 + b\varphi_2|^2$$

IN A NORMALLY WELL designed EXPERIMENT $b \ll a$ SO $P_A \approx |a\varphi_1|^2$; The a^2 here IS JUST A NORMALIZATION FACTOR.

If you recall our tabulations last time we had two columns one marked N_1 and the other N_2. We have just found out what goes into column N_1. What goes into column N_2? Well, going through the same time of argument,

- The amplitude that an electron is found at x and a photon is seen at B is $a\varphi_2$
- The amplitude that we saw a photon a B when the electron went thru A is $b\varphi_1$

The probability to find an electron at x and a photon at B is

$$P_B = |b\varphi_1 + a\varphi_2|^2$$

This then is the N_2 in our tabulation.

To carry this experiment further let's assume that $\varphi_1(x)$ and $\varphi_2(x)$ are about the same but differ by some small phase angle. This supposes that the path lengths from the source to x via the two routes is very nearly the same. We can write,

$$\varphi_1(x) = \sqrt{P_1}\, e^{i\delta_1(x)} + \sqrt{P_2}\, e^{i\delta_2(x)}$$

where $\sqrt{P_1}$ and $\sqrt{P_2}$ are weighting factors such that $|\varphi_1|^2 = P_1$ and $|\varphi_2|^2 = P_2$. The phase angles δ_1 and δ_2 are functions of x. Further for simplicity assume a and b are real. The probability of event 1 occuring is then

$$P_{1_A} = \left| a\sqrt{P_1}\, e^{i\delta_1(x)} + b\sqrt{P_2}\, e^{i\delta_2(x)} \right|^2$$
$$= a^2 P_1 + b^2 P_2 + ab\sqrt{P_1P_2}\cos(\delta_1 - \delta_2)$$

For event 2

$$P_{2_B} = \left| b\sqrt{P_1}\, e^{i\delta_1} + a\sqrt{P_2}\, e^{i\delta_2} \right|^2$$
$$= b^2 P_1 + a^2 P_2 + ab\sqrt{P_1P_2}\cos(\delta_1 - \delta_2)$$

The total probability of the event occurring is the sum of the two individual probabilities just calculated.

$$P = (a^2 + b^2)(P_1 + P_2) + 2ab\sqrt{P_1P_2}\cos(\delta_1 - \delta_2)$$

Notice if you could absolutely tell which hole the electron went thru b would be zero and we have

$$P = a^2(P_1 + P_2) \qquad \text{for} \quad a = 1,\ b = 0$$

This just the result we expected. Let's look at another extreme where we chose a wavelength for our light so large compared to the hole spacing that we can't tell which hole the electron went through. This corresponds to $a = b$ and

$$P = 2a^2\left[P_1 + P_2 + 2\sqrt{P_1P_2}\cos(\delta_1 - \delta_2)\right]$$

or

$$P = 2a^2\,|\varphi_1 + \varphi_2|^2$$

So we get back the expected wave result.

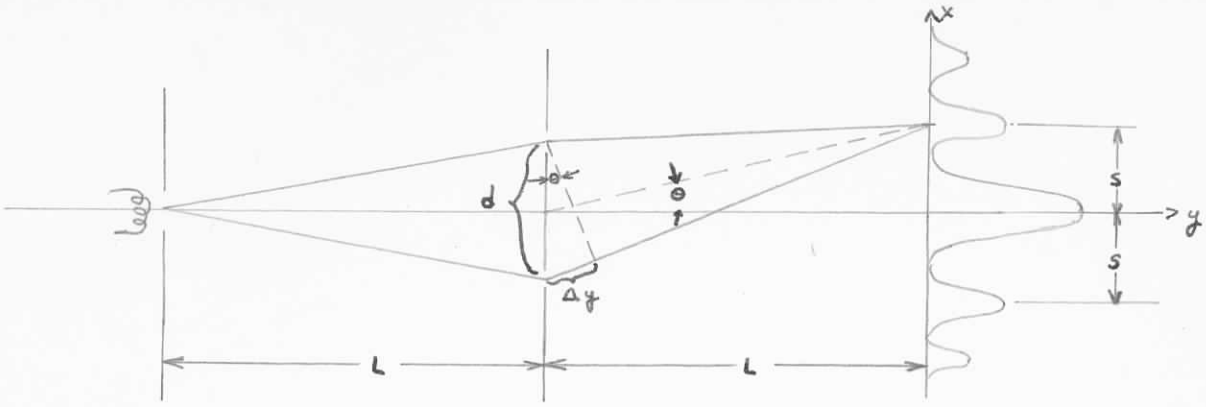
BETWEEN THE TWO EXTREMES of λ where IN ONE CASE YOU GOT PERFECT RESOLUTION AND IN THE OTHER where YOU GOT NO RESOLUTION, LIES THE RANGE where YOU EXPERIENCE THIS WAVE-PARTICLE NATURE.

THE MATHEMATICAL EXPLANATION GIVEN here IS MUCH BETTER IN EXPLAINING WHAT REALLY GOES ON THAT THE SO-CALLED WAVE PACKET REDUCTION BUSINESS. THE IDEA THAT THE ELECTRONS CAN be ANY where behind THE BARRIER AND SUDDENLY, UPON GOING THROUGH A HOLE, SQUEEZED DOWN TO A VERY NARROW REGION IS A VERY UNREAL REPRESENTATION OF WHAT GOES ON. THE EXPLANATION OF AN EVENT IS MUCH MORE COMPLICATED THAN THIS. THE POSITION of THE PHOTON AND ELECTRON MUST be SPECIFIED ALONG WITH THE VALUES for a AND b. WHEN MORE PARTICLES ARE INVOLVED IT IS NECESSARY TO SPECIFY ALL THEIR LOCATIONS AND THE NUMBER of VARIABLES GOES UP RAPIDLY. IT IS NECESSARY TO forget THE IDEA of A WAVE PROPAGATING IN SPACE AND REPLACE IT BY A MATHEMATICAL REPRESENTATION of A PROBABILITY AMPLITUDE which ITSELF IS TIME AND SPACE VARYING. IT IS hard for THE HUMAN MIND TO DO THIS BUT ITS THE ONLY WAY TO UNDERSTAND WHAT IS GOING ON.

MEASUREMENT of POSITION AND MOMENTUM Ref. VOL. III chapt 2

I'd LIKE TO build A FRAMEWORK for BETTER UNDERSTANDING THE IDEA of A PROBABILITY AMPLITUDE AND how IT IS EFFECTED by THE PRESENCE of SOME APPARATUS THAT PERMITS LOCATION of THE ELECTRON. If THE ELECTRON IS EJECTED from THE SOURCE WITH A PARTICULAR ENERGY E GIVEN by $P^2/2M$, THEN over THE PATH LENGTH TO THE DETECTOR THE AMPLITUDE IS PROPORTIONAL TO $e^{iP(\text{LENGTH})}$. I HAVEN'T PROVEN TO YOU why THIS RELATIONSHIP IS TRUE AND I HAVEN'T BOTHERED WITH SOME INESSENTIAL COMPLICATIONS LIKE POLARIZATION - SO ACCEPT THAT I CAN MAKE THIS REPRESENTATION. NOW THE QUESTION of CHANGING THE AMPLITUDE REDUCES TO ASKING how MY OBSERVATION CHANGES THE MOMENTUM of THE PARTICLE AS WELL AS ITS PATH LENGTH.

SINCE WE KNOW THAT LIGHT of A GIVEN WAVELENGTH λ HAS A MOMENTUM ASSOCIATED WITH IT GIVEN by $P = h/\lambda$, WE CAN ASK for A GIVEN λ, SHORT ENOUGH TO "SEE" THE ELECTRON, CAN WE CHANGE ITS MOMENTUM ENOUGH TO SHIFT ITS POSITION from A MAXIMUM TO THE ADJOINING MAXIMUM IN THE INTERFERENCE PATTERN. IN OTHER ~~other~~ words WILL WE SMEAR THE PICTURE. TO WORK THIS OUT, LET'S DRAW A BETTER PICTURE.



FIRST, SINCE THE HOLE SPACING IS d, WE NEED A WAVELENGTH $\lambda < d$. THE MOMENTUM OF THE LIGHT IS $P_L = \frac{\hbar}{\lambda} > \frac{h}{d}$.
THE MOMENTUM IN THE ELECTRON IS P_e SO THE AMPLITUDE CAN BE WRITTEN AS $\propto e^{iP_eY/h}$. BY GEOMETRY WE CAN COMPUTE THE PATH DIFFERENCE Δy. THE PATH DIFFERENCE CORRESPONDS TO A PHASE SHIFT $\delta = \frac{\Delta Y}{\lambda}$. SINCE $\lambda = \hbar/P_L$ WE HAVE

$$\delta = \frac{P}{\hbar}\Delta Y$$

Δy IS GIVEN APPROXIMATELY BY θd. θ CAN BE REFERRED TO THE DISPLACEMENT ALONG THE SCREEN AS

$$\theta = \frac{x}{L}$$

SO WE HAVE $\Delta Y = \frac{xd}{L}$. THE PHASE DIFFERENCE δ IS THEN

$$\delta = \delta_1 - \delta_2 = \frac{P}{\hbar}\frac{xd}{L}$$

IF THE BUMPS ARE A DISTANCE S APART AND THE ELECTRON IS SHIFTED THROUGH THIS DISTANCE δ HAS UNDERGONE A 2π SHIFT SO THAT

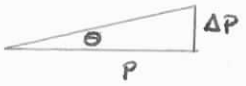
$$\frac{Pds}{\hbar L} = 2\pi$$

OR

$$\frac{Pds}{hL} = 1 \quad \rightarrow \quad \frac{s}{L} = \frac{h}{Pd}$$

THE MOMENTUM NEEDED TO DEFLECT THE ELECTRON THROUGH AN ANGLE θ IS JUST

$$\Delta P = P\theta = \frac{Ps}{L}$$



BUT WE JUST SAW THAT THE LIGHT OF WAVELENGTH $\lambda = d$ CAN IMPART A SIDEWISE MOMENTUM OF $PS/L = h/d$. SO, INDEED, THE OBSERVING APPARTUS CAN SMEAR THE ELECTRON.

IF WE SUBSTITUTE FOR PS/L h/d IN THE ABOVE RESULT WE FIND THAT

$$(\Delta P)d \approx h$$

IN ORDER TO KNOW IF THE ELECTRON IS GOING UP OR DOWN AFTER IT IS HIT $\Delta P < \frac{h}{d}$ BUT IN ORDER TO SEE WHERE THE ELECTRON

IS $\lambda > \frac{h}{d}$. THUS THERE IS AN UNCERTAINTY IN MOMENTUM RELATED TO THE UNCERTAINTY IN POSITION BY THE RULE

$$\Delta P \Delta X \approx h$$

THIS SAYS YOU CANNOT PREPARE AN EXPERIMENT SUCH THAT YOU KNOW PRECISELY WHERE THE PARTICLE IS AND IN WHICH DIRECTION IT IS MOVING. HEISENBERG FIRST RECOGNIZED THIS FUNDAMENTAL LIMITATION OF QUANTUM THEORY AND REALIZED THAT FOR THE THEORY TO BE CONSISTENT THIS "UNCERTAINTY' PRINCIPLE" MUST HOLD. THE UNCERTAINTY PRINCIPLE REALLY REFERS TO THE PREDICTABILITY OF AN EVENT AND HAS NO DEPENDENCE ON THE PAST OR HOW THE EVENT WAS INITIATED OR PREPARED.

10. AMPLITUDES FOR EVENTS INVOLVING IDENTICAL PARTICLES

Ref: VOL III CHAPTER 3

The LECTURE INVOLVED A NUMBER of QUESTION WITH The DISCUSSION ENDING UP CONSIDERING AMPLITUDES of EVENTS INVOLVING INDENTICAL PARTICLES. The LECTURE followed CHAPTER 3 IN VOLUME III.

II. INTRODUCTION TO DIRAC NOTATION

Ref. CHAPT III VOLUME III

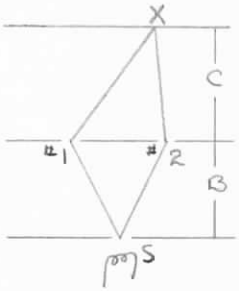
So far we have been talking about amplitudes and how to compute them for different events. I like to consider the case where the particle is non-relativistic; further I'll disregard any spin or internal mechanism which could lead to an internal breakup. I only want to be concerned with position. In order to define the position I must assign an amplitude to finding the particle at any one spot. The amplitude is a function of x, y, z and time, in general, so that $\text{Amp} = \psi(\bar{x}, t)$.

In order to work at problems in quantum mechanics a shorthand used to denote amplitudes is Dirac notation. To show you how it works recall the two hole experiment where I had an electron starting at ~~x~~ some source S and going to X via one or the other of two holes. We would write the above sentence like

$$\langle X \mid \text{2 hole apparatus} \mid \text{at } S \rangle$$

The bracket on the right denotes the initial state while the one on the left denotes the final state. Between the two lines the apparatus or mechanism separating the two states is noted. This is purely a short hand notation and anything can go in the bracket but the critical thing to remember is that the complete bracket enclosed by < > is equivalent to the phrase "The amplitude that" and, therefore, it is just a number - but a complex number. Sometimes the apparatus is left out and you'll see a notation like $\langle X | S \rangle$. This still means the same thing.

Now the question might come up what kind of initial and final states are there and what happens when there are alternative ways the apparatus affects the measurement. Again recall the two hole experiment as shown on the right, the amplitude that an electron gets from S to X is written as,



$$\langle X | A | S \rangle = \langle X | C | \#1 \rangle \langle \#1 | B | S \rangle + \langle X | C | \#2 \rangle \langle \#2 | B | S \rangle$$

In general when a number of alternative paths are present we have the sum of each one,

$$\langle X | A | S \rangle = \sum_i \langle X | C | i \rangle \langle i | B | S \rangle$$

We have used here the idea of multiplying the amplitude to get part way to X by the amplitude to get the rest of the way. That is there is a certain amplitude for the electron to get to hole #1 and then to X over region C. I should point out that I have ignored the finite size of the hole and, therefore, haven't bothered computed amplitudes to get to different parts of the hole.

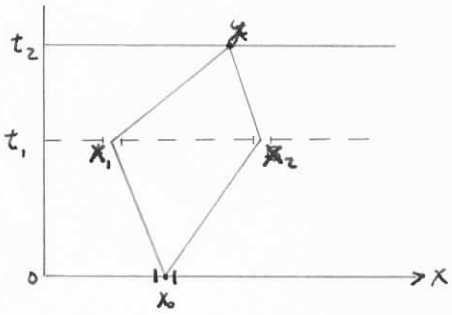
This notation is sometimes baffling and even magical at times. For instance if I move the source to different places, the formula is still right. I can generalize the expression for the apparatus as,

$$A = C|\#1\rangle\langle\#1|B + C|\#2\rangle\langle\#2|B$$

This expression is hungry to work on a state $|\rangle$ or $\langle|$ so it is the same thing as an operator. Whenever the bracket $\langle\ \rangle$ is not closed it denotes a complex algebra. The above expression can be half complete in which case it is still considered an operator, i.e.,

$$A|S\rangle = C|1\rangle\langle 1|B|S\rangle + C|2\rangle\langle 2|B|S\rangle$$

Let's now return to the original problem of a non-relativistic electron which we'll imagine has no internal motion and no spin - thus we will ignore how the electron comes through the slot and only be concerned with whether or not it came through. at all. The problem is analogous to the two slit problem only we'll imagine the particle travels only in one dimension x and time is measured vertically like the sketch to the right,



The electron will be released from some point x_0 at time = 0 and at time equal t_1 there is an amplitude to find it at x_1 or x_2. There will then be an amplitude to get to point y at time t_2. By imagining a series of infinitesimal holes at t_1 and considering all but two, x_1 & x_2, closed the problem is just like the two hole problem discussed before.

I am going to define a function $U(t', t)$ as the apparatus waiting time. By this I mean to describe transition from some time t to a latter time t'; in essence it is the same thing as "just hang around." If I write this function more explicitly for the case in question, $U(t_2, t_1)$ is the same thing as C in the previous discussion and $U(t_1, 0)$ is just like B. In the Dirac notation we can write

$$\langle y|A|x_0\rangle = \langle y|U(t_2,t_1)|x_1\rangle\langle x_1|U(t_1,0)|x_0\rangle + \langle y|U(t_2,t_1)|x_2\rangle\langle x_2|U(t_1,0)|x_0\rangle$$

If there were now i of the infinitesimal holes and we didn't look at t_1 so that $U(t_2, t_0) = U(t_2, 0)$ and

$$\langle Y|U(t_2,0)|x_0\rangle = \sum_i \langle y|U(t_2,t_1)|x_i\rangle\langle x_i|U(t_1,0)|x_0\rangle$$

To be more precise we have computed the amplitude of finding the particle in some incremental distance dx_i so we should have written,

$$\langle y | U(t_2, 0) | x_0 \rangle = \sum_i \langle y | U(t_2, t_1) | x_i \rangle \langle x_i | U(t_1, 0) | x_0 \rangle dx_i$$

The probability to find the particle in dx_i is given by

$$P(x_i)\, dx_i = |\langle y | U(t_2, t_1) | x_i \rangle \langle x_i | U(t_1, 0) | x_0 \rangle|^2 dx_i$$

The operator here which is

$$U(t_2, 0) = \sum_i U(t_2, t_1) |x_i\rangle \langle x_i| U(t_1, 0)\, dx_i$$

can be written as an integral in the limit,

$$U(t_2, 0) = \int U(t_2, t_1) |x\rangle dx \langle x | U(t_1, 0)$$

As a short commentary, in order for the above equations to have meaning and be right we must express every possible way the damn thing can get through to y. The only way to completely describe the system is to say where it is at every instant after some initial condition, x_0, where it was started. Each position of the particle corresponds to a different state i.

Let's go back to the 2 hole case above and only look at the apparatus from t_1 on. In otherwords we are not concerned about the initial state $|x_0\rangle$ or how the electron was prepared. We know that

$$\langle y | U(t_2, 0) | x_0 \rangle = \langle y | U(t_2, t_1) | x_2 \rangle \langle x_2 | U(t_1, 0) | x_0 \rangle + \langle y | U(t_2, t_1) | x_1 \rangle \langle x_1 | U(t_1, 0) | x_0 \rangle$$

We could write the ~~write~~ right-hand side in different notation as,

$$\langle y | U(t_2, t_1 | ? \rangle = \langle y | U(t_2, t_1) | x_1 \rangle a_1 + \langle y | U(t_2, t_1) | x_2 \rangle a_2$$

Where the state $|?\rangle$ has a question mark standing for, the state which has an amplitude to be at x_1 and x_2. Specifically those amplitudes are a_1 and a_2,

$$|?\rangle = |x_1\rangle a_1 + |x_2\rangle a_2 = \left| \begin{matrix} \text{amplitude at } x_1 \\ \text{and} \\ \text{amplitude at } x_2 \end{matrix} \right\rangle$$

When I write it this way instead of using the terms $\langle x_2 | U(t_2, 0) | x_0 \rangle$ for a_2 and $\langle x_1 | U(t_1, 0) | x_0 \rangle$ for a_1, I generalize the results greatly. I could have arrived at the same a_1 and a_2 by some other arrangement of initial state and $U(t_1, 0)$ such that from time t_1 on I couldn't tell the difference in the results. But as soon as I specify a_1 and a_2, I can henceforth predict the location of the particle.

The future is thus seen to be independent of the past in a certain sense. As soon as the amplitudes are written, i.e., a_1 and a_2, the past has been be summarized, the present described and sufficient detail given to predict the future. The guts of quantum mechanics is in describing the presence so you can predict the future.

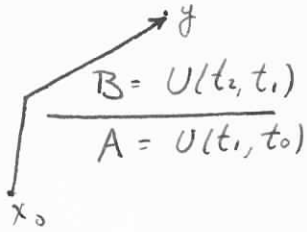
Because we have added the two states $|x_1\rangle a_1$ and $|x_2\rangle a_2$ to get a new state we call $|x_1\rangle$ and $|x_2\rangle$ base states in that a superposition of base states leads to a new state. Sometimes you may see this written as

$$|\psi\rangle = |x\rangle\, \psi(x)$$

where $\psi(x)$ is the amplitude to get at x_1 and x_2

In the way of a summary then we have

$$U(t_2, t_0) = \sum_x U(t_2, t_1)\,|x\rangle\langle x|\, U(t_1, t_0)$$



which is equivalent to

$$BA = \sum_x \langle y|B|x\rangle\langle x|A|x_0\rangle$$
$$= \sum_x B|x\rangle\langle x|A$$

A constraint on this abstraction representation is that $\sum_x |x\rangle\langle x| = 1$. This is the ultimate of abstraction and it says to add through all x's and you get unity. It is essentially a statement of completeness and assures us of preserving probability of all events adding to one.

Definition of a Kernel

To help understand this amplitude idea better I'd like to introduce another labeling scheme, I'll define the amplitude $\langle y|U(t_2, t_0)|x_0\rangle$ to be the following complex function

$$\langle y|U(t_2, t_0)|x_0\rangle = K(y, t_2; x_0, t_0)$$

Let me for convenience sake change y to x_2 so that the form of the equation is more consistent. The amplitude given in the above form must satisfy the rule for combining amplitudes of events occurring in succession in time, i.e.,

$$\psi(x) = K(x_2, t_2; x_0, t_0) = \int K(x_2, t_2; x_1, t_1)\, K(x_1, t_1; x_0, t_0)\, dx_1$$

In quantum mechanical notation $K(x_2, t_2; x_0, t_0)$ is called a kernel and represents the amplitude to go from point x_0, t_0 to x_2, t_2.

As a special case of a particle moving between two points, consider the case of motion in a vacuum. We have to solve the previous equations where the conditions are constant, i.e., there is no scattering between points. Subject to this constraint and assuming the particle was released at point $x_0 = t_0 = 0$, we can can write

$$K(x_2, t_2; x_1, t_1) = k(x_2 - x_1, t_2 - t_1) = k(x_2, t_2)$$

And,

$$k(x_2, t_2) = \int k(x_2 - x_1, t_2 - t_1)\, k(x_1, t_1)\, dx_1$$

The form of this equation is like a convolution, i.e.,

$$h(t) = \int f(\tau)\, g(t - \tau)\, d\tau$$

So if we take the fourier transform of $k(x_2, t_2)$ we have

$$\mathcal{K}(p; t) = \int e^{ipx}\, k(x, t)\, dx = \mathcal{K}(p; t_2 - t_1)\, \mathcal{K}(p, t_1)$$

The parameter p is constant throughout this calculation. A function which has the above property is

$$K(p, t) = e^{-i E(p) t}$$

The function $E(p)$ can be real or complex but for the sum of the probabilities for each event to be unity $E(p)$ must be real.

In order to pursue this example it is necessary to assume the Galilean transformation for the observer since we are considering low velocities. If we use this idea we find that $E(p) \propto p^2$ and the proportionality constant is $1/2m$. To show this is true the Galilean invariance implies that $x' = x - vt$ and $t' = t$. Therefore,

$$K(x_2', t_2'; x_1', t_1') = K(x_2 - vt_2, t_2'; x_1 - vt_1, t_1')$$
$$= K(x_2, t_2; x_1, t_1)$$

An important fact here is the K is not uniquely determined defined. It is always possible for every K to find another, say, $\tilde{K}$ which gives the same probability and differs in phase, i.e.,

$$\tilde{K}(x_2, t_2; x_1, t_1) = e^{i[\chi(x_2, t_2), \chi(x_1, t_1)]}\, K(x_2, t_2; x_1, t_1)$$

Or written another way

$$\tilde{\psi}(x, t) = e^{i\chi(x,t)}\, \psi(x, t)$$

where χ is some real number.

12. AMPLITUDE INVARIANCE TO GALILEAN TRANSFORMATIONS

I'd LIKE TO RETURN TO A PROBLEM STARTED LAST TIME where WE DISCUSSED The INVARIANCE of The AMPLITUDE or WAVE FUNCTION TO A GALILEAN TRANSFORMATION. ASSUME WE HAVE SOME MAN READING A BOOK ON QUANTUM MECHANICS AS he IS FLYING ALONG IN A PLANE AT SOME VELOCITY v AND he READS The MYSTIC FORMULA for COMPUTING AMPLITUDES SO he WRITES IT DOWN where he USES $x' = x - vt$, $t' = t$

$$K(x_2' - x_1', t_2' - t_1')$$

To The observer DO The SAME THING ON The GROUND he WOULD WRITE $K(x_2 - x_1, t_2 - t_1)$. If The TWO ARE TO GET The SAME RESULTS,

$$K(x_2 - x_1, t_2 - t_1) = K(x_2 - x_1 + vt_1 - vt_2, t_2 - t_1)$$

BUT THIS EQUALITY IS PHYSICALLY NOT CORRECT BECAUSE The GROUND observer COULD HAVE CHOSEN A DIFFERENT K WHICH DIFFERS FROM The ABOVE by A PHASE difference, I.E.,

$$\tilde{K}(x_2 - x_1, t_2 - t_1) = e^{i[\chi(x_2, t_2) - \chi(x_1, t_1)]} K(x_2 - x_1, t_2 - t_1)$$

This NEW K GIVES The SAME PHYSICAL RESULT SINCE The PROBABILITY GOES AS K^2 AND The EXPONENTIAL GOES OUT. WE HAVE for The AMPLITUDE TO ARRIVE AT x_2 VIA THIS NEW $\tilde{K}$

$$\tilde{\psi}(x_2) = \int e^{i(\chi_2 - \chi_1)} K(x_2 - x_1, t_2 - t_1)\, e^{ix_1} f(x_1)\, dx_1$$

$$\tilde{\psi}(x_2) = e^{i\chi_2} \psi(x_2)$$

To GO BACK AND CORRECT The ABOVE EQUALITY RELATING The TWO AMPLITUDE WE HAVE

$$K(x_2 - x_1, t_2 - t_1) = e^{i[\chi_v(x_2, t_2) - \chi_v(x_1, t_1)]} K(x_2 - x_1 - vt_2 + vt_1, t_2 - t_1)$$

where $\chi(x_2, t_2)$ AND $\chi(x_1, t_1)$ ARE FUNCTIONS of VELOCITY v of The MOVING observer. NOW The PHYSICS WILL be The SAME AND WE CAN PROCEED.

FIRST, The EXPONENTIAL MUST be A FUNCTION of The difference BETWEEN x_2 AND x_1; MORE SPECIFICALLY IT IS A LINEAR RELATIONSHIP. LET ME WRITE THIS AS

$$K(x_2 - x_1, t_2 - t_1) = e^{i[a(v)(x_2 - x_1) + b(v)(t_2 - t_1)]} K(x_2 - x_1 - vt_2 + vt_1, t_2 - t_1)$$

or SIMPLIFYING by LETTING $x = x_2 - x_1$ & $t = t_2 - t_1$

$$K(x, t) = e^{i[a(v)x + b(v)t]} K(x - vt, t)$$

NOW INSIDE The PLANE WE HAVE A LITTLE GIRL RIDING A KIDDY CAR UP AND DOWN The AISLE WITH VELOCITY U DOING QUANTUM MECHANICS AS she GOES. To GUY ON The PLANE WITH her READING his BOOK AND DOING QUANTUM MECHANICS he WOULD RELATE her RESULTS TO his by The RELATIONSHIP,

$$K(x-ut,t) = e^{i[a(v)(x-ut)+b(v)t]} K[x-(u+v)t,t]$$

Here I just used the transformation laws with a velocity u. To the fellow on the ground he must see an amplitude,

$$K(x,t) = e^{i[a(u)x+b(u)t]} K(x-ut,t)$$

Using the above relationship for $K(x-ut,t)$, the girl's amplitude appears to be

$$K(x,t) = e^{i[a(u)x+b(u)t]} e^{i[a(v)(x-ut)+b(v)t]} K[x-(u+v)t,t]$$

Using the law of compound velocities we can write this as

$$K(x,t) = e^{i[a(v+u)x+b(v+u)t]} K[x-(u+v)t,t]$$

For the equality to hold, the exponentials must be equal which requires that,

$$a(v+u) = a(u) + a(v)$$

A function which is the sum of two different functions must be linear so that $a(v) = \frac{mv}{\hbar}$ $\quad a(u) = \frac{m}{\hbar}u$

Here m is characteristic of the particle and $\hbar$ is used for dimensional completeness but I'll leave it out since I know where to stick it back in. The second coefficient is a little tougher on the brain,

$$b(v+u) = b(u) - u\,a(v) + b(v)$$

This can be written as

$$b(v+u) = b(v) + b(u) - m(uv)$$

To solve this differentiate with respect to u and set $u=0$,

$$b'(v) = b'(0) - mv = \beta - mv$$

where β is some constant number. Now integrating back,

$$b(v) = \beta v - \frac{mv^2}{2}$$

Going back to $K(x,t)$ we have found that for $u=0$

$$K(x,t) = e^{i[mvx - \frac{mv^2}{2}t + \beta vt]} K(x-vt,t)$$

We have thus solved for the phase relationship between the two amplitudes and we now can determine the exact formula for K.

Suppose I let $v = x/t$ which I can do then if I ignore the β term for the moment, I have

$$K(x,t) = e^{i[m\frac{x^2}{t} - \frac{m}{2}\frac{x^2}{t^2}t]} K(0,t)$$

$K(0,t)$ is just some function of time, $f(t)$, say while the exponential becomes just $\frac{mx^2}{2t}$. Therefore, we have

$$K(x,t) = e^{i\frac{mx^2}{2t\hbar}} F(t)$$

OR WRITTEN OUT

$$K(x_2-x_1, t_2-t_1) = e^{i\frac{m}{2}\frac{(x_2-x_1)^2}{(t_2-t_1)\hbar}} F(t_2-t_1)$$

WE NOW MUST FIND $F(t)$. TO DO THIS RECALL LAST TIME WE FOUND USING FOURIER TRANSFORM THEORY THAT

$$k(p,t) = \int e^{-ipx} K(x,t)\,dx = e^{i\epsilon(p)t}$$

SUBSTITUTING for $K(x,t)$

$$k(p,t) = \int e^{-ipx} e^{i\frac{mx^2}{2t}} F(t)\,dx$$

I'LL WRITE THIS AS

$$k(p,t) = \int_{-\infty}^{\infty} e^{i\frac{m}{2t}\left[x-\frac{tp}{m}\right]^2} e^{-\frac{im}{2t}\frac{t^2p^2}{m^2}} F(t)\,dx$$

THIS INTEGRAL IS of the form of A GAUSSIAN INTEGRAL. To preform the INTEGRATION I shift ORIGINS by LETTING $x + \frac{-tp}{m} = y$

$$k(p,t) = e^{-\frac{ip^2}{2mt}} \int_{-\infty}^{\infty} e^{-\frac{im}{2t}y^2}\,dy$$

upon INTEGRATING USING TABLES

$$I = \int e^{-\alpha y^2}\,dy = \sqrt{\frac{\pi}{\alpha}}$$

$$k(p,t) = \sqrt{\frac{2\pi\hbar t i}{m}}\, e^{-i\frac{p^2}{2m}t} F(t) \qquad \alpha = \frac{m}{2ti}$$

Therefore WE HAVE THAT

$$\sqrt{\frac{2\pi\hbar t i}{m}}\, e^{-i\frac{p^2}{2m}t} F(t) = e^{-i\epsilon(p)t}$$

For THIS EQUALITY TO hold TWO THINGS MUST be VALID,

(1) $\epsilon(p) = \frac{p^2}{2m}$

(2) $F(t) = \sqrt{\frac{m}{2\pi\hbar t i}}$

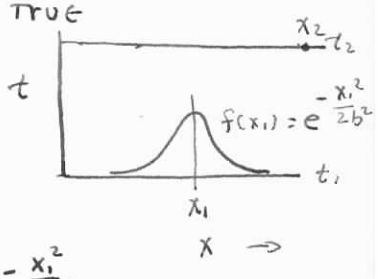
WE HAVE FINALLY COMPLETED the AMPLITUDE for A free PARTICLE

$$K(x_2-x_1, t_2-t_1) = e^{i\frac{m}{2}\frac{(x_2-x_1)^2}{(t_2-t_1)\hbar}} \sqrt{\frac{m}{2\pi\hbar i(t_2-t_1)}}$$

THIS DESCRIBES the COMPLETE GUTS of NON-RELATIVISTIC QUANTUM MECHANIC AND, Therefore, IS A VERY POWERFUL TOOL TO USE IN WORKING A NUMBER of PROBLEMS.

FREE PARTICLE AMPLITUDE

ONE THING MIGHT be worth MENTIONING before WE GO ON. You WILL NOTICE THAT AS K IS WRITTEN IT APPEARS THAT IT BLOWS UP AS $t_2 - t_1$ Approaches 0. BUT THIS IS NOT The CASE. If X_2 IS FAR from X_1, YOU CAN'T EXPECT The PARTICLE To be AT X_2 when IT HAS ONLY GONE A SHORT TIME, ϵ, from t_1. You WILL NOTICE THAT If $t_2 - t_1$ IS ABOUT 0, The phASE VARIES TREMENDOUSLY. The rapidly OSCILLATING PHASE SMOOTHS OUT The DISTRIBUTION FUNCTION $f(X_1)$ which describes The PARTICLE ABOUT X_1. To show THIS IS TRUE WE KNOW THAT



$$\psi(X_2) = \int_{-\infty}^{\infty} K(X_2 - X_1, t_2 - t_1)\, f(X_1)\, dX_1$$

$$= \int_{-\infty}^{\infty} e^{i \frac{m}{2} \frac{(X_2 - X_1)^2}{\epsilon \hbar}} \sqrt{\frac{m}{2\pi i \epsilon \hbar}}\, f(X_1)\, dX_1$$

If WE choose A LITTLE GAUSSIAN FUNCTION for $f(X_1)$ SAY $e^{-\frac{X_1^2}{2b^2}}$ WE CAN find $\psi(X_2)$. Upon SUBSTITUTION AND REORGANIZING

$$\psi(X_2) = \sqrt{\frac{m}{2\pi i \epsilon \hbar}}\, e^{i \frac{m}{2\epsilon\hbar} X_2^2} \int_{-\infty}^{\infty} e^{-\alpha X_1^2 - \beta X_1}$$

where $\alpha = \frac{1}{2b^2} - \frac{im}{2\epsilon}$ AND $\beta = \frac{i m X_2}{\epsilon \hbar}$

The SOLUTION TO THIS INTEGRAL IS

$$I = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/4\alpha} = \sqrt{\frac{2\pi b^2 \epsilon \hbar}{\epsilon\hbar - i m b^2}}\, e^{-\frac{m^2 X_2^2 b^2}{2\epsilon\hbar(\epsilon\hbar - i m b^2)}}$$

· COMBING

$$\psi(X_2) = \sqrt{\frac{m}{2\pi i \epsilon\hbar} \times \frac{2\pi b^2 \epsilon\hbar}{(\epsilon\hbar - i m b^2)}}\, e^{i \frac{m X_2^2}{2\epsilon\hbar}}\, e^{-\frac{m^2 X_2^2 b^2}{2\epsilon\hbar(\epsilon\hbar - i m b^2)}}$$

$$= \sqrt{\frac{m b^2}{m b^2 + i\epsilon\hbar}}\, e^{-\frac{m^2 X_2^2}{2\epsilon\hbar}\left[\frac{b^2}{(\epsilon\hbar - i m b^2)} + \frac{i}{m}\right]}$$

$$\psi(X_2) \doteq 1\, e^{-\frac{i m X_2^2}{\hbar \epsilon}} \simeq 0$$

The rapidly varying exponential is on The AVERAGE zero so you would Expect TO find The PARTICLE AT X_2.

ONE WAY TO SOLVE The dèLEMMA of K blowing up is TO LET $\hbar$ be SLIGHTLY IMAGINARY, I.E, LET $\hbar \rightarrow \hbar - i\delta$. When you MAKE The SUBSTITUTION AND LET $\delta \rightarrow 0$ The ANSWER CONVERGES TO 0 rapidly. IN The Above EXAMPLE

$$\psi(X_2) = e^{-\frac{i m X_2^2}{\epsilon\hbar(1 - i\delta/\hbar)}} = \exp\left[-\frac{i m X_2^2}{\epsilon\hbar}(1 - i\delta/\hbar)\right]$$

$$= \exp\left[-\frac{i m X_2^2}{\epsilon\hbar} - \frac{m X_2^2 \delta}{\epsilon \hbar^2}\right]$$

if $\delta \simeq \frac{\epsilon}{m X_2^2}$ Then $\psi(X_2)$ damps rapidly TO zero AS $e^{-\frac{1}{\hbar^2}}$

13. CONSERVATION of PROBABILITY

Ref. FEYNMAN & HIBBS, Chapter 4 PAGE 83

LAST TIME WE WERE TALKING ABOUT THE MOTION OF A FREE PARTICLE IN EMPTY SPACE AND WE FOUND THAT THE AMPLITUDE TO GET BETWEEN TWO POINTS x_2 AND x_1, SAY, WAS GIVEN AS $K(x_2-x_1, t_2-t_1)$, i.e., IT DEPENDED ON THE INITIAL AND FINAL POINTS AND THEIR DIFFERENCE. FURTHER WE SAY THAT THE AMPLITUDE TO GET TO x_2 COULD be derived FROM AN INTEGRAL EXPRESSION where THE INTEGRAND WAS GIVEN AS

$$K(x_2-x_1, t_2-t_1)\, f(x_1)$$

The FUNCTION $f(x_1)$ IS THE AMPLITUDE of THE PARTICLE ARTICLE ARRIVING AT x_1. THUS UP TO x_1 ALL THE PAST history of THE PARTICLE IS DESCRIBED BY GIVING $f(x_1)$. Therefore, WE HAVE

$$\psi(x_2) = \int_{-\infty}^{\infty} K(x_2-x_1, t_2-t_1)\, f(x_1)\, dx_1$$

WE COULD GO ON TO A NEW LOCATION x_3 AND FIND THE AMPLITUDE of ARRIVING THERE, AS

$$\psi(x_3) = \int K(x_3-x_2, t_3-t_2)\, \psi(x_2)\, dx_2$$

This AMPLITUDE IS THE SAME AS

$$K(x_3-x_1, t_3-t_1) = \int K(x_3-x_1, t_3-t_1)\, f(x_1)\, dx_1$$
$$= \int K(x_3-x_2, t_3-t_2)\, K(x_2-x_1, t_2-t_1)\, dx_2$$

If WE ASK WHAT THE PROBABILITY IS To find THE PARTICLE AT x_2, WE HAVE THAT

$$P(x_2) = |\psi(x_2)|^2$$

WHILE THE PROBABILITY THAT THE PARTICLE <u>IS</u> AT x_2 AT t_2 IS

$$P(x_2)\,dx = \int |\psi(x_2)|^2\, dx_2$$

The probability THAT There IS A PARTICLE AT t_1 IS JUST $\int |f(x_1)|^2 dx_1$. IN order TO PRESERVE THE CONSERVATION of MATTER, These TWO probabilities MUST be EQUAL. IN GENERAL WE CAN WRITE THIS CONSERVATION of PROBABILITY AS,

$$\int f(x_1) f^*(x_1)\, dx_1 = \iiint K(x_2-x_1, t_2-t_1)\, f(x_1)\, K^*(x_2-x_1', t_2-t_1)\, f^*(x_1')\, dx_1'\, dx_2\, dx_1$$

IN order for THIS EQUALITY TO hold for ARBITRARY VALUES of $f(x_1)$ WE MUST HAVE THAT

$$\int K^*(x_2-x_1', t_2-t_1)\, K(x_2-x_1, t_2-t_1)\, dx_2 = \delta(x_1'-x_1)$$

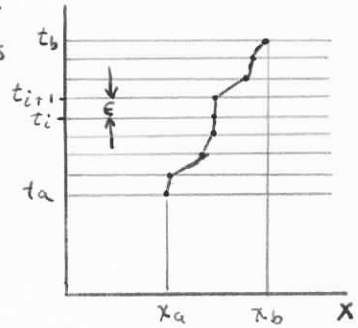
where $\delta(x_1'-x_1)$ IS A DELTA FUNCTION SUCH THAT $\delta(x_1'-x_1) = 0$ if $x_1' \neq x_1$ AND $= 1$ if $x_1' = x_1$

INTRODUCTION TO PATH INTEGRALS

ref. Feynman & Hibbs
Chapter 2-9

There are directions which I would like to go now. I want to use our equation of a free particle amplitude to work some problem so you will better understand it. The other topic is to respond to some of your requests for discussing the idea of a path integral. I'll do the latter first and then return to problem solving.

You recall I talked about an amplitude for a particle to leave some point x_1, t_1, and arrive at a point x_2, t_2. We have just calculated the value for the empty space case. We also talked about all the different ways to get from point x_1, t_1 to x_2, t_2 and each way had its own amplitude



Let's imagine that we get to x_b by a particular path as shown on the right. The time interval $t_b - t_a$ is divided into N intervals each ϵ (seconds) apart. The amplitude for the trajectory shown will be denoted as $x(t)$. The amplitude to arrive via a specific $x(t)$ trajectory is $\Phi[x(t)]$. This function is really a functional since it depends on each value $x(t)$. We can write the amplitude to get from a to b in the following form

$$K(b,a) = \sum_{\substack{\text{over all} \\ \text{paths from a to b}}} \Phi[x(t)]$$

In integral form this becomes

$$K(b,a) = \int \Phi[x(t)]\, \mathcal{D}x(t)$$

Now I'd like to describe what I mean when I write $\Phi[x(t)]$. If we divide the time interval $t_b - t_a$ up into these little units of ϵ, we have

$$N\epsilon = t_b - t_a$$

where $\epsilon = t_{i+1} - t_i$ and $t_0 = t_a$, $x_0 = x_a$, $t_N = t_b$, and $x_N = x_b$. At each time t_i we now write or assign a value x_i to the particle's position. At each point t_i we connect to t_{i-1} by a straight line until we have the jagged line between x_b and x_a as shown. The amplitude to get to x_b, t_b from x_a, t_a is given by the product of the amplitude to get between x_{i+1}, t_{i+1}, and x_i, t_i,

$$[K(x_b, x_{N-1}, \epsilon)] \ldots [K(x_{i+1}, x_i; \epsilon)] \ldots [K[x_2, x_1; \epsilon][K(x_1, x_a; \epsilon] = f(x_i)$$

This long product of amplitudes is what I mean by $\Phi x(t)$, i.e.,

$$f(x_i) = \Phi[x_1, x_2, \ldots, x_i] = \Phi[x(t)]$$

The resulting integral relationship is

$$K(b,a) = \underbrace{\int\int \cdots \int}_{N-1 \text{ fold integration}} \Phi[x(t)]\, dx_1\, dx_2 \cdots dx_{N-1}$$

In computing this amplitude we must chose a normalizing factor which depends on ϵ such that the value of $K(b,a)$ does not blow up in the limit as $\epsilon \to 0$. But this is the idea and the above formula completely describes quantum mechanics.

Typically when the particle is moving in regions of potentials the values for K get very complicated. But for short enough time intervals the particle acts like a free particle, i.e., it is unresponsive to the external forces acting on it. Thus to first order approximation

$$\Phi[x(t)] = \cdots e^{i\frac{m}{2\epsilon}(x_{i+1}-x_i)^2} \cdots e^{i\frac{m}{2\epsilon}(x_2-x_1)^2} e^{i\frac{m}{2\epsilon}(x_1-x_0)^2}$$

or

$$\Phi[x(t)] = \exp \frac{im}{2\hbar} \sum_{i=0}^{N-1} \left(\frac{x_{i+1}-x_i}{\epsilon}\right)^2 \epsilon$$

Here I put back in $\hbar$ and rewrote the exponential a little differently. In this form observe that as $\epsilon \to 0$ the squared term gets more and more like a velocity in the classical sense of the definition, i.e.,

$$\Phi = \exp \frac{im}{2\hbar} \sum \left(\frac{x_{i+1}-x_i}{t_{i+1}-t_i}\right)^2 (t_{i+1}-t_i)$$

This functional is, therefore, replaceable by the integral

$$\Phi = \exp \frac{i}{\hbar} \int_a^b \frac{m\dot{x}^2}{2}\, dt$$

The integral has dimensions of action, i.e., energy times time and again has classical connotation therefore we can replace it by $S[x(t)]$ where

$$S[x(t)] = \int_a^b \frac{m\dot{x}^2}{2}\, dt$$

Our final results gives

$$\Phi = (\text{constant})\, e^{\frac{i}{\hbar} S[x(t)]}$$

I want to go back to the trouble we eluded to with regards to chosing a normalizing factor which didn't let the amplitude blow up in the limit as $\epsilon \to 0$. From the free particle case we found the factor to be,

$$A^{-1} = \sqrt{\frac{m}{2\pi\hbar i\epsilon}}$$

I prefer to handle this factor by incorporating it into the differentials as

$$\frac{dx_1}{A}\ \frac{dx_2}{A} \cdots \frac{dx_{N-1}}{A}$$

The script Differential notation $\mathcal{D}[x(t)]$ has the meaning

$$\int\int\cdots\int \mathcal{D}x(t) = \int\int\cdots\int A^N \, dx_1 \, dx_2 \cdots dx_{N-1}$$

Using this notation the Amplitude to go from a to b is

$$K(b,a) = \int_a^b e^{\frac{i}{\hbar} S[b,a]} \, \mathcal{D}x(t)$$

This expression is called a path integral.

We saw that the exponential contained the action of the particle moving between two points. Further we assumed first order approximation to a free particle, i.e., there were no potentials acting. But in general the action is the time integral of the Lagrangian of the system, i.e.,

$$S = \int_{t_a}^{t_b} L(\dot{x}, x, t)\, dt$$

Where the Lagrangian $L(\dot{x}, x, t)$ is defined to be

$$L(\dot{x}, x, t) = (\text{kinetic} - \text{potential}) \text{ energy}$$
$$= \frac{m}{2}\dot{x}^2 - V(x,t)$$

Thus we have to add the additional term to the exponential to consider the effect of forces acting on the particle,

$$\Phi = \exp \frac{i}{\hbar}\left[\frac{m}{2}\sum_i \left(\frac{x_{i+1} - x_i}{\epsilon}\right)^2 \epsilon - \sum_i V(x_i)\,\epsilon\right]$$

For short time intervals we can average the potential between the two points and write

$$K(x_b, x_a; t_b, t_a) = \sqrt{\frac{m}{2\pi\hbar i(t_b - t_a)}}\; e^{\frac{i}{\hbar}\left[\frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a} - V\left(\frac{x_b + x_a}{2}\right)(t_b - t_a)\right]}$$

Continuing, when a vector potential is present we must add another term, $-x\frac{q}{c}A_0(x,t)$ to the above exponential. Again averaging for short intervals

$$K(b,a) = \sqrt{\frac{m}{2\pi\hbar i(t_b - t_a)}}\; e^{\frac{i}{\hbar}\left[\frac{m}{2}\frac{(x_b - x_a)^2}{(t_b - t_a)} - q(t_b - t_a)V\left(\frac{x_b + x_a}{2}\right) - \frac{q}{c}(x_b - x_a)A\left(\frac{x_b + x_a}{2}\right)\right]}$$

where q is the charge on the particle. Note that the exponential is the action of a particle moving in an electromagnetic field. The Lagrangian of the particle is given by

$$L = \frac{m}{2}\dot{x}^2 + qV(x,t) - \frac{q}{c}\dot{x}A(x,t)$$

If we let the amplitude of the free particle motion be described by $\Phi_0[X(t)]$ and have the particle moving in an external electromagnetic field described by the four-vector, A_μ, then the amplitude for every path is

$$\Phi_A[X(t)] = \Phi_0[X(t)] \exp \frac{ie}{c}\int_a^b A_\mu(X,t)\,dX_\mu$$

where $$\int_a^b A_\mu(X,t)\,dX_\mu = \int_a^b V\,dt - \int_a^b A\dot{X}\,dt$$

INTERPERTATION of PATH INTEGRALS

We have just written down all of non-relativistic and relativistic quantum mechanics and it would be good to understand specifically what the notation means. First observe that the contribution of a certain path to the total amplitude of an event has a phase proportional to the action S, i.e.,

$$\Phi[X(t)] = (\text{constant})\, e^{\frac{i}{\hbar}S[X(t)]}$$

This is, in fact, a phase contribution since $\hbar$ has dimensions of erg-seconds just like the action. Observe the changes in the action produce a phase shift in each path's contribution. Further by factoring out the free particle amplitude as $\Phi_0[X(t)]$, we see that the only effect of electromagnetic phenomena on particle motion is to change the phase of the path integral. This is very important to understand and makes electrical forces seem cute in how they affect a particle.

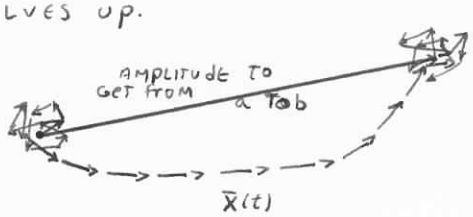
For the case where the vector potential is given by a pure gradient then the path does not depend on which way the particle moves because the phase never changes. This explains why pure gradient never produce any effects. This explains why it was okay to forget the β term when I was working the free particle problem. You recall the specific term was $\beta X t = \beta X$ so β is just $V\left(\frac{X_2+X_1}{2}\right)$, the constant potential acting on the particle.

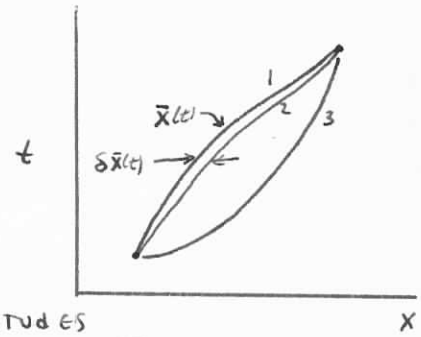
The CLASSICAL LIMIT

Let's see how we explain classical phenomena from the path integral concept. At first glance it appears that all paths contribute equally to the amplitude and it is not clear why one path is unique and corresponds to classical path. We know that the action is a minimum for the actual dynamical path - but why?

First observe that in the classical scale the action is enormous compared to $\hbar$ ($= 10^{-27}$ erg-sec). The phase contribution $S/\hbar$ is likewise enormous, i.e., rapidly oscillatory. Even small variations of S corresponds to enormous phase changes such that the total contribution will add to ~~together a~~ zero as the oscillations average to zero.

Thus for differing paths with differring actions the contributions cancel each other out. But for the special path where the action is a minimum, first order changes in S do not change S so the contributions of nearby paths begin to add in phase and do not cancel out. We can imagine the situation shown below where ~~near~~ the different phase interfer with eachother and just knot themselves up. But for the classical path the interference is constructive and don't eat themselves up.





All the squiggly short lines come from amplitudes computed for path beyond first order changes in S, $\delta\bar{x}(t)$.

14. FREE PARTICLE RELEASED WITH A GAUSSIAN DISTRIBUTION

I WOULD LIKE TO DISCUSS further The CASE of free PARTICLE MOTION where The INITIAL POSITION of The PARTICLE WAS GIVEN AS A GAUSSIAN ABOUT SOME ORIGIN. THAT IS To SAY The PARTICLE LOCATION IS GIVEN by

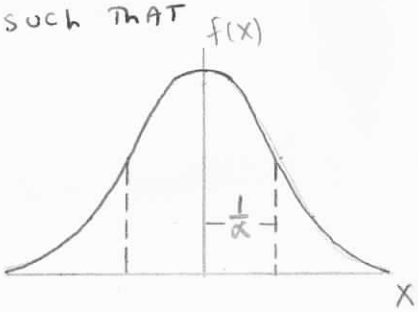
$$f(x) = e^{-\frac{\alpha}{2}x^2}$$

The Probability of finding The PARTICLE AT POINT x is Therefore GIVEN by,

$$P(x) = e^{-\alpha x^2} = |f(x)|^2$$

USUALLY This is NORMALIZED by The FACTOR $\sqrt{\alpha/\pi}$ such THAT

$$f(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha}{2}x^2}$$



if WE START The PARTICLE off AT $t_0 = 0$ AND ASK WHAT IS The AMPLITUDE TO find IT LATTER AT POINT x' AT time T, WE hAVE THAT

$$\psi(x') = \sqrt[4]{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{m}{2\pi i \hbar T}}\, e^{\frac{i m (x'-x)^2}{2T}}\, e^{-\frac{\alpha}{2}x^2}\, dx$$

This INTEGRAL IS of The form

$$\int_{-\infty}^{\infty} \left[\exp\left(-Ax^2 - Bx\right)\right] dx = \sqrt{\frac{\pi}{A}}\; e^{B^2/4A}$$

where $\quad A = \frac{\alpha}{2} - \frac{im}{2T} \quad$ AND $\quad B = \frac{i m x'}{T}$

Therefore

$$\psi(x') = \sqrt[4]{\frac{\alpha}{\pi}} \sqrt{\frac{\pi}{\frac{\alpha}{2} - \frac{im}{2T}} \cdot \frac{m}{2\pi i T}}\; e^{\frac{i m x'^2}{2T}}\; e^{-\frac{m^2 x'^2}{T^2} \Big/ 4\left(\frac{\alpha}{2} - \frac{im}{2T}\right)}$$

$$= \sqrt[4]{\frac{\alpha}{\pi}} \sqrt{\frac{\pi}{\left(\frac{\alpha}{2} - \frac{im}{2T}\right)} \cdot \frac{m}{2\pi i T}}\; e^{\frac{\alpha}{2}(x')^2 \frac{im}{2T} \Big/ \left(\frac{\alpha}{2} - \frac{im}{2T}\right)}$$

Or

$$\psi(x') = \sqrt[4]{\frac{\alpha}{\pi}}\; e^{-\frac{\alpha}{2}\frac{(x')^2}{\left(1 + \frac{iT\alpha}{m}\right)}}\; \frac{1}{\sqrt{1 + \frac{iT\alpha}{m}}}$$

TO UNDERSTAND THIS RESULT FIRST NOTICE WHEN $T \simeq 0$, I.E., WHEN The PARTICLE hASN'T GONE VERY FAR, WE GET The ORIGINAL PROBABILITY DISTRIBUTION. AS T GETS VERY LARGE The PARTICLE SPREADS LIKE A WAVE ORIGINATING AT A POINT.

TO COMPUTE THE PROBABILITY TO FIND THE PARTICLE A X',

$$P(X') = |g(x')|^2$$

where

$$g(x') = \psi(x') = \sqrt[4]{\frac{\alpha}{\pi}} \sqrt{\frac{1}{1 + \frac{iT\alpha}{m}}} \; e^{-\frac{\alpha}{2} \frac{\left(1 - \frac{iT\alpha}{m}\right)}{(1 + T^2\alpha^2/m^2)} X'^2}$$

THEN

$$P(X') = \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{1}{1 + \frac{T^2\alpha^2}{m^2}}} \; e^{-\frac{\alpha \, |X'|^2}{\left(1 + \frac{T^2\alpha^2}{m^2}\right)}}$$

NOTICE IF WE LET

$$\beta = \frac{\alpha}{\left(1 + \frac{T^2\alpha^2}{m^2}\right)}$$

THEN WE HAVE THAT

$$P(X') = \sqrt{\frac{\beta}{\pi}} \; e^{-\beta X'^2}$$

WHICH SAYS THE PROBABILITY HAS BROADEN BY AN AMOUNT EQUAL TO $\frac{1}{1 + \frac{T^2\alpha^2}{m^2}}$.

THIS RESULT IS QUITE IMPORTANT FOR IT TELLS US HOW THE PROBABILITY DISTRIBUTION SPREADS WITH TIME. IF WE CALL THE WIDTH OF THE DISTRIBUTION, ΔX, THE ROOT MEAN SQUARE DEVIATION IS

$$(\Delta X)^2 = \frac{1}{\beta} = \frac{1}{\alpha}\left(1 + \frac{T^2\alpha^2}{m^2}\right)$$

$$= \frac{1}{\alpha} + \frac{T^2\alpha}{m^2}$$

SINCE $\frac{1}{\alpha} = (\Delta X)_0^2$ IS THE INITIAL RMS DEVIATION WE HAVE

$$(\Delta X)^2 = (\Delta X)_0^2 + \left(\frac{T}{\Delta X_0 m}\right)^2$$

NOW LET'S RETURN TO THE QUESTION OF HOW THE PROBABILITY DISTRIBUTION SPREADS WITH TIME. CLASSICALLY THIS RESULTS CORRESPONDS TO AN INITIAL UNCERTAINTY IN POSITION WHICH GROWS WITH TIME. IF I CALL (Δv_0) THE CLASSICAL UNCERTAINTY IN VELOCITY, I SEE THAT

$$\Delta v_0 = \frac{\Delta X_0}{T}$$

SUCH THAT

$$(\Delta v_0)^2 = V^2$$

THEN

$$(\Delta X)^2 = (\Delta X)_0^2 + \left(\frac{1}{\Delta v_0 m}\right)^2 = (\Delta X)_0^2 + \frac{1}{V^2 m^2}$$

I should be careful with the $\hbar$ term which I have left out so far,

$$(\Delta x)^2 = (\Delta x)_0^2 + \left(\frac{\hbar}{\Delta v_0 m}\right)^2$$

The second term on the right hand side represents an extra deviation

$$\Delta x_1 = \frac{\hbar}{\Delta v_0 m}$$

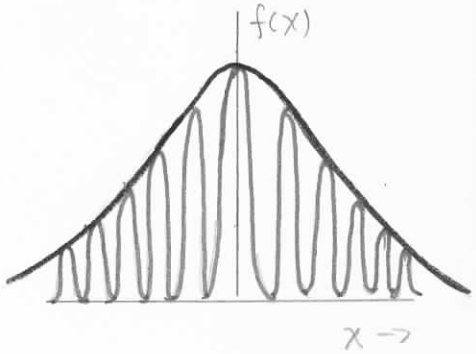
This term has physical significance since $\Delta v_0 m$ is just the uncertainty in the particles momentum, Δp. The fact that $\hbar$ appears in our result convinces us that the spreading in probability is ~~Gaussi~~ quantum mechanical in nature, i.e., unique to small particle and small regions of space. The important result is, therefore, the uncertainty in position at some time T is Δx_1 and that the relationship with the uncertainty in momentum is

$$\Delta x_1 \, \Delta p = \hbar$$

This is the uncertainty principle which you are familiar with.

As another example consider a gaussian distribution which is now modulated by e^{ikx} such that

$$f(x) = \sqrt[4]{\frac{\alpha}{\pi}}\, e^{-\frac{\alpha}{2}x^2}\, e^{ikx}$$



Now the amplitude the particle at x' is given by

$$\psi(x') = \sqrt[4]{\frac{\alpha}{\pi}}\sqrt{\frac{m}{2\pi i\hbar T}}\int_{-\infty}^{\infty} e^{\frac{im}{2T}(x'-x)^2}\, e^{-\frac{\alpha}{2}x^2}\, e^{ikx}\, dx$$

This integral is of the form

$$I = \int_{-\infty}^{\infty} e^{-Ax^2 - Bx}\, dx = \sqrt{\frac{\pi}{A}}\, e^{B^2/4A}$$

where we have to find A and B. If we expand the exponential

$$\frac{im}{2T}(x'^2 - 2x'x + x^2) - \frac{\alpha x^2}{2} + ikx = \frac{imx'^2}{2T} - \left(\frac{\alpha}{2} - \frac{im}{2T}\right)x^2 - \left(\frac{im}{T}x' - ik\right)x$$

Therefore

$$A = \frac{\alpha}{2} - \frac{im}{2T} \qquad B = i\left(\frac{mx'}{T} - k\right) = \frac{imx'}{T}\left(1 - \frac{kT}{mx'}\right)$$

Then

$$\psi(x') = \sqrt[4]{\frac{\alpha}{\pi}}\sqrt{\frac{m}{2\pi i\hbar T}}\sqrt{\frac{\pi}{\frac{\alpha}{2} - \frac{im}{2T}}}\; e^{\frac{imx'^2}{2T}}\; e^{+\left(\frac{imx'}{T}\right)^2\left(1 - \frac{kT}{mx'}\right)^2 \Big/ 2\left(\alpha - \frac{im}{T}\right)}$$

The coefficient can be written as

$$\sqrt[4]{\frac{\alpha}{\pi}}\;\frac{1}{\sqrt{1 + \frac{iT\alpha}{m}}}$$

The EXPONENTIAL CAN BE MANIPULATED AROUND TO A BETTER FORM

$$\frac{\frac{imx'^2}{2T} - \frac{m^2x'^2}{T^2}\left(1 - \frac{2kT}{mx'} + \frac{k^2T^2}{m^2x'^2}\right)}{2(\alpha - \frac{im}{T})} = \frac{\frac{imx'^2}{T}(\alpha - \frac{im}{T}) - \frac{m^2x'^2}{T^2} + \frac{2kTmx'}{T} - k^2}{2(\alpha - \frac{im}{T})}$$

EXPANDING

$$\frac{\frac{i\alpha mx'^2}{T}}{2(\alpha - \frac{im}{T})} + \cancel{\frac{m^2x'^2}{T^2}} - \cancel{\frac{m^2x'^2}{T^2}} + \frac{kx'\left(\frac{2m}{T} - \frac{k}{x'}\right)}{2(\alpha - \frac{im}{T})}$$

The first term ~~can~~ can be written as

$$\frac{i\alpha mx'^2}{T2(\alpha - \frac{im}{T})} = \frac{i\alpha mx'^2}{-2\frac{im}{T}T(1 - \frac{\alpha T}{im})} = -\frac{\alpha}{2}\frac{x'^2}{(1 + \frac{iT\alpha}{m})}$$

The SECOND TERM CAN BE WRITTEN

$$\frac{kx'}{2(\alpha - \frac{im}{T})}\left(\frac{2m}{T} - \frac{k}{x'}\right) = \frac{ikx'\left(1 - \frac{kT}{2mx'}\right)}{(1 + \frac{iT\alpha}{m})}$$

Therefore

$$\psi(x') = \sqrt[4]{\frac{\alpha}{\pi}}\sqrt{\frac{1}{1 + \frac{iT\alpha}{m}}}\; e^{\left[-\frac{\alpha}{2}x'^2 + ikx'\left(1 - \frac{kT}{2mx'}\right)\right]/(1 + \frac{iT\alpha}{m})}$$

NOTICE WHEN $T \simeq 0$ WE GET THE ORIGINAL DISTRIBUTION

$$\psi(x') = \sqrt[4]{\frac{\alpha}{\pi}}\; e^{-\frac{\alpha}{2}x'^2 + ikx'} = f(x')$$

THIS WE EXPECTED.

MORE IMPORTANTLY if $k = \frac{2mx'}{T}$ WE GET THE FREE PARTICLE AMPLITUDE AS COMPUTED ON PAGE 62. IF WE RESTORE THE $\hbar$ INTO THE FORMULA FOR k

$$k = \frac{m}{\hbar}\frac{x'}{T}$$

BUT x'/T IS THE ANALOG OF VELOCITY AND mv IS MOMENTUM p SUCH THAT

$$p = \hbar k$$

The QUANTITY k IS THE PHASE CHANGE PER UNIT DISTANCE OF A WAVE. SINCE THE WAVELENGTH IS THE DISTANCE OVER WHICH THE PHASE CHANGES BY 2π WE HAVE THAT

$$k = 2\pi/\lambda$$

CORRESPONDINGLY THEN $p = h/\lambda$. WE HAVE THUS TIED TOGETHER The idea of A MOVING AMPLITUDE WITH WAVE behavior CHARACTERIZED BY A WAVELENGTH. IT IS THIS RESULT THAT IS FUNDAMENTAL TO THE UNDERSTANDING OF QUANTUM MECHANICS.

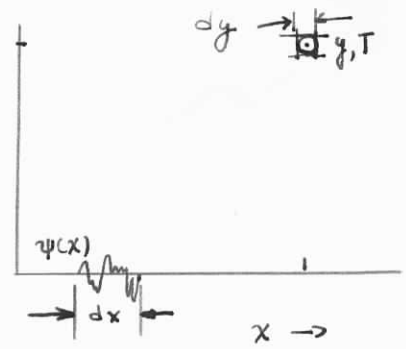
18. MOMENTUM REPRESENTATION

Ref FEYNMAN & HIBBS CHAPTER 5-1

SO FAR WE HAVE BEEN CONCERNED WITH EXPRESSING THE PROBABILITY OF AN EVENT OCCURRING IN TERMS OF THE PARTICLE'S POSITION, x, WE WROTE A WAVE FUNCTION, $\psi(x)$, EXPRESSING THE AMPLITUDE OF FINDING A PARTICLE SUCH THAT THE PROBABILITY OF THE PARTICLE TO BE AT x WITHIN A RANGE dx IS $P(x)dx = |\psi(x)|^2 dx$. WE SHOULD ASK WHAT IS THE ANALOG FOR THE PARTICLE TO HAVE A CERTAIN MOMENTUM p IN RANGE dp.

CONSIDER THE SITUATION WHERE WE KNOW THE PARTICLE IS INITIALLY DESCRIBED BY A FUNCTION $\psi(x)$ AND WE ASK WHAT IS THE PROBABILITY TO FIND IT A POSITION y AT SOME LATER TIME T. IF THE PARTICLE IS INITIALLY LOCALIZED TO WITHIN dx AT $t=0$, THEN FOR THE PARTICLE TO BE FOUND AT y IT MUST OF HAD A PARTICULAR VELOCITY WHICH WE CAN THINK OF IN THE CLASSICAL SENSE OF VELOCITY, I.E.,

$$V = \frac{y}{T}$$



THAT IS ALSO THE PARTICLE HAD A MOMENTUM GIVEN BY

$$p = mV = m\frac{y}{T}$$

SINCE THE INITIAL UNCERTAINTY IN THE POSITION WAS dx, THE ERROR IN DETERMINENG p CAN BE MADE SUFFICENTLY SMALL BY TAKING T SUFFICIENTLY LARGE, I.E., $\Delta p = m\,dx/T$.

THUS THE QUESTION OF ASKING WHAT IS THE PROBABILITY OF FINDING THE PARTICLE AT y WITHIN dy IS THE SAME AS ASKING WHAT IS ITS PROBABILITY OF HAVING VELOCITY V IN dV,

$$P(y)\,dy = P(VT)\,d(VT) = P(VT)\,T\,dV$$

WE KNOW THE AMPLITUDE TO FIND THE PARTICLE AT y AT T IS GIVEN BY

$$\psi(y) = \sqrt{\frac{m}{2\pi i\hbar T}} \int_{-\infty}^{\infty} e^{\frac{im}{2T}(x-y)^2}\,\psi(x)\,dx$$

IF WE SUBSTITUTE $y = VT$ INTO THIS AMPLITUDE

$$\psi(VT) = \sqrt{\frac{m}{2\pi i\hbar T}} \int_{-\infty}^{\infty} e^{\frac{imV^2T}{2}}\, e^{\frac{imx^2}{2T}}\, e^{-\frac{imxV}{2}}\,\psi(x)\,dx$$

WE CAN NOT INTERPERT V AS VELOCITY UNLESS T IS VERY LARGE IN WHICH CASE $e^{\frac{imV^2T}{2}}$ IS RAPIDLY OSCILLATORY AND DOES NOT CONTRIBUTE TO THE INTEGRAL. FURTHER $e^{imx^2/2T}$ IS NEGLIGIBLY SMALL AND $e^{imx^2/2T}$ IS ABOUT EQUAL TO 1. THEREFORE,

$$P(V) = P(VT)\,T = |\psi(VT)|^2 = \frac{m}{2\pi\hbar}\left|\int e^{-imxV}\,\psi(x)\,dx\right|^2$$

NOTICE THAT THIS PROBABILITY IS INDEPENDENT OF T WHICH IS A GOOD THING. THIS SAYS YOU WILL MEASURE THE SAME VELOCITY IN AN HOUR AND A HALF THAT YOU DID AT AN HOUR. FINALLY SINCE

$$\mathcal{P}(v)\,dv = \mathcal{P}(p)\,dp = \mathcal{P}(p)\,m\,dv$$

WE HAVE

$$\mathcal{P}(p)\,dp = \frac{dp}{2\pi\hbar}\left|\int_{-\infty}^{\infty} e^{-\frac{imvx}{\hbar}}\,\psi(x)\,dx\right|^2$$

$$\mathcal{P}(p)\,dp = \frac{dp}{2\pi\hbar}\left|\int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}}\,\psi(x)\,dx\right|^2$$

OBSERVE THAT THE MOMENTUM PROBABILITY IS INDEPENDENT OF THE PARTICLE MASS AND, THEREFORE, IT APPLIES UNIVERSALLY.

IT IS COMMON TO WRITE THE MOMENTUM AMPLITUDE AS

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}}\,\psi(x)\,dx$$

THUS GIVEN THE WAVE FUNCTION $\psi(x)$ WE CAN FIND A CORRESPONDING $\phi(p)$ BY TAKING ITS FOURIER TRANSFORM. THE QUESTION OF WHERE THE FACTOR $1/2\pi\hbar$ APPEARS IS SOMEWHAT ARBITRARY AND IS DIFFERENT FOR EACH BOOK YOU READ. NOTE THAT IF YOU ARE GIVEN $\phi(p)$, YOU CAN FOURIER TRANSFORM BACK TO FIND $\psi(x)$, I.E.,

$$\psi(x) = \int_{-\infty}^{\infty} \phi(p)\,e^{\frac{ipx}{\hbar}}\,dp$$

SINCE WE CAN COMPUTE $\psi(x)$ BY THE INTEGRAL $\int_{-\infty}^{\infty} K(x,x';T)\,f(x')\,dx'$ WHERE $K(x,x';T)$ IS THE PARTICLE'S KERNEL, WE SHOULD NOT BE SURPRISED TO FIND AN ANALOG

$$\chi(p) = \int_{-\infty}^{\infty} k(p,p';T)\,\varphi(p')\,\frac{dp'}{2\pi}$$

SINCE WE KNOW WHAT $K(x,x';T)$ IS, WE HAVE FOUND THE SPECIFIC FREE PARTICLE FORM. WHAT IS $k(p,p';T)$? IF WE SUBSTITUTE FOR $f(x')$, $\int \varphi(p')\,e^{ip'x'/\hbar}\,dp'$ WE HAVE

$$\psi(x) = \int_{-\infty}^{\infty} K(x,x';T)\int_{-\infty}^{\infty} \varphi(p')\,e^{\frac{ip'x'}{\hbar}}\,dp'\,dx'$$

WORKING THROUGH AFTER SUBSTITUTING

$$k(p,p';T) = \delta(p-p')\,e^{\frac{ip^2}{2m}T}$$

WHERE $\delta(p-p')$ IS THE DIRAC DELTA FUNCTION

15. THE SCHRODINGER EQUATION

Ref. FEYNMAN AND HIBBS CHAPTER 9-1

WE HAVE BEEN QUANTUM MECHANIZING THINGS BY WRITING A WAVE FUNCTION

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x,x';t,t')\,\psi(x',t')\,dx'$$

WHERE $K(x,x';t,t')$ IS THE KEY CALCULATING MACHINE IN QUANTUM MECHANICS FOR IT CARRIES THE INITIAL AMPLITUDE $\psi(x',t')$ INTO THE FUTURE AND LET'S US PREDICT EVENTS. FOR THE CASE OF THE FREE PARTICLE WE FOUND THAT

$$K(x,x';t,t') = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}\; e^{\frac{im}{2\hbar}\frac{(x-x')^2}{(t-t')}}$$

IF WE NOW STUDY THE BEHAVIOR OF $\psi(x,t)$ FOR VERY SHORT TIME INTERVALS SUCH THAT $t-t' = \epsilon$ WE CAN WRITE

$$\psi(x,t+\epsilon) = \int_{-\infty}^{\infty} \sqrt{\frac{m}{2\pi i\hbar\epsilon}}\; e^{\frac{im}{2\hbar\epsilon}(x-x')^2}\,\psi(x',t)\,dx'$$

NOTICE IF x' IS APPRECIABLY DIFFERENT FROM x, THEN AS ϵ APPROACHES ZERO THE EXPONENTIAL OSCILLATES VERY RAPIDLY. THUS THE INTEGRATION GIVES US BACK $\psi(x',t')$ WHICH IS GOOD; I.E. IF WE DON'T GO ANYWHERE WE STILL SHOULD OBSERVE THE SAME PROBABILITY.

NOW AS x' NEARS x, SAY $x' = x+\eta$, THE EXPONENTIAL BEGINS TO VARY SLOWLY SO WE DO GET SOME SIGNIFICANT CONTRIBUTIONS. WE CAN REPLACE $\psi(x',t')$ BY $\psi(x+\eta,t)$ AND EXPAND IN A TAYLOR SERIES SINCE WE ARE ONLY INTERESTED IN LOW ORDER η, I.E.,

$$\psi(x+\eta,t) = \psi(x,t) + \eta\frac{\partial\psi}{\partial x} + \frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}$$

NOW WE MUST INTEGRATE

$$\psi(x,t+\epsilon) = \int_{-\infty}^{\infty} \sqrt{\frac{m}{2\pi i\hbar\epsilon}}\; e^{\frac{im\eta^2}{2\hbar\epsilon}}\left[\psi(x,t) + \eta\frac{\partial\psi}{\partial x} + \frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}\right]d\eta$$

WERE WE SUBSTITUTED $x' = x+\eta$ AND $dx' = d\eta$.

THE FIRST INTEGRAL TURNS OUT TO BE JUST $\psi(x,t)$,

$$\psi(x,t)\int_{-\infty}^{\infty}\sqrt{\frac{m}{2\pi i\hbar\epsilon}}\; e^{\frac{im\eta^2}{2\hbar\epsilon}}\,d\eta = \psi(x,t)\sqrt{\frac{m}{2\pi i\hbar\epsilon}}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} = \psi(x,t)$$

THIS IS JUST A GAUSSIAN INTEGRAL WHICH WE HAVE EVALUATED BEFORE. THE SECOND INTEGRAL, $\int \eta\, e^{\frac{im\eta^2}{2\epsilon\hbar}}\,d\eta$ INTEGRATES TO 0. THE THIRD TERM CAN BE OBTAINED BY DIFFERENTIATING THE FIRST INTEGRAL WITH RESPECT TO m, I.E.,

$$\cancel{\sqrt{\frac{m}{2\pi i\hbar\epsilon}}\,\frac{\eta^2}{2}} \qquad I = \frac{i}{\epsilon}\int\frac{\eta^2}{2}\,e^{\frac{im\eta^2}{2\epsilon\hbar}}\,d\eta = \frac{d}{dm}\int e^{\frac{im\eta^2}{2\epsilon\hbar}}\,d\eta$$

$$= \frac{d}{dm}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} = -\frac{1}{2}\frac{(2\pi i\hbar\epsilon)^{1/2}}{m^{3/2}}$$

THEN

$$\frac{\partial^2\psi}{\partial x^2}\int\sqrt{\frac{m}{2\pi i\hbar\epsilon}}\;\frac{\eta^2}{2}\,e^{\frac{im\eta^2}{2\hbar\epsilon}}\,d\eta = \frac{\partial^2\psi}{\partial x^2}\,\frac{i\hbar\epsilon}{2m}$$

WE HAVE SOFAR FOUND THAT

$$\psi(x, t+\epsilon) = \psi(x,t) + 0 \cdot \frac{\partial \psi}{\partial x} + \frac{i\hbar\epsilon}{2m}\frac{\partial^2 \psi}{\partial x^2}$$

SINCE ϵ IS SMALL WE CAN EXPAND THE LEFT SIDE IN A TAYLOR SERIES ALSO,

$$\psi(x,t) + \epsilon\frac{\partial \psi}{\partial t} = \psi(x,t) + \cancel{\frac{\partial \psi}{\partial x}} + \frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial x^2}$$

THIS REQUIRES THAT $\psi(x,t)$ MUST SATISFY THE DIFFERENTIAL,

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial x^2}$$

FOR THE ABOVE EQUALITY TO HOLD TO FIRST ORDER. THIS DIFFERENTIAL EQUATION (DIFF.EQ) IS OFTEN SEEN WRITTEN AS

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

OR

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

THIS DIFFERENTIAL EQUATION IS SCHRÖDINGER'S EQUATION. SOLVING THIS EQUATION IS THE USUAL WAY OF DOING QUANTUM MECHANICS.

TO GENERALIZE THIS RESULT TO THREE DIMENSIONS WE WOULD START WITH THE THREE DIMENSIONAL KERNAL,

$$K(\bar{x},t) = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}\; e^{\frac{im}{2\hbar}\frac{[(x-x')^2+(y-y')^2+(z-z')^2]}{(t-t')}}$$

THE RESULTING EQUATION IS

$$-\frac{\hbar}{i}\frac{\partial\psi(\bar{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi(\vec{x},t)$$

I MIGHT POINT OUT THAT MANY PEOPLE LIKE TO WORK IN THE MOMENTUM REPRESENTATION WHERE IN THREE DIMENSIONS

$$\varphi(\vec{p},t) = e^{i\frac{p^2}{2m}(t-t')}\,\varphi(\bar{p},t')$$

WHERE THE MOMENTUM AMPLITUDE IS GIVEN BY

$$\varphi(\bar{p},t) = \int e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}}\,\psi(\bar{x},t)\,d^3\vec{x}$$

SINCE $\psi(\bar{x},t)$ IS GIVEN BY THE FOURIER TRANSFORM OF $\varphi(\bar{p},t)$,

$$\psi(\bar{x},t) = \int e^{\frac{i}{\hbar}\bar{p}\cdot\bar{x}}\,\varphi(\bar{p},t)\frac{d^3\bar{p}}{(2\pi\hbar)^3}$$

IT IS EASY TO FIND THE ANALOGOUS DIFFERENTIAL DIFFERENTIAL FOR $\varphi(p)$,

$$-\frac{\hbar}{i}\frac{d}{dt}\int e^{i/\hbar\,\bar{p}\cdot x}\,\varphi(\bar{p},t)\frac{d^3\bar{p}}{(2\pi\hbar)^3} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\int e^{\frac{i}{\hbar}\bar{p}\cdot\bar{x}}\,\varphi(\bar{p},t)\frac{d^3\bar{p}}{(2\pi\hbar)^3}$$

THIS SIMPLY BECOMES

$$-\frac{\hbar}{i}\frac{\partial\,\varphi(\bar{p},t)}{\partial t} = +\frac{p^2}{2m}\,\varphi(\bar{p},t)$$

THIS DIFFERENTIAL EQUATION IS OFTEN EASIER TO SOLVE SINCE IT IS ONLY A FIRST ORDER DERIVATIVE. YOU SHOULD NOTICE THAT DIFFERENTIATING BY $\frac{\hbar}{i}\frac{\partial}{\partial x}$ IS THE SAME THING AS MULTIPLYING BY P. WE CALL THIS THEN THE MOMENTUM OPERATOR SINCE ITS ACTION ON THE WAVE FUNCTION IS THE SAME AS MULTIPLICATION BY P. THE INVERSION OF THIS OPERATION IS DIFFERENTIATION BY $-i\hbar\frac{\partial}{\partial p}$ WHICH LIKE A MULTIPLICATION BY X. WE HAVE, THEREFORE, TWO REPRESENTATIONS, POSITION AND MOMENTUM, WHICH CAN BE USED IN SOLVING SCHRÖDINGERS EQUATION OR FOR COMPUTING THE AMPLITUDE VIA THE PATH INTEGRAL APPROACH. BUT BOTH REPRESENTATIONS ARE RELATED BY TAKING FOURIER TRANSFORMS.

ONE INTERESTING SOLUTION TO THE DIFF. EQ JUST DERIVED IS FOR THE FREE PARTICLE WHICH HAS A WAVE FUNCTION GIVEN BY

$$\psi = e^{-i/\hbar(Et - Px)}$$

INSERTING INTO

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

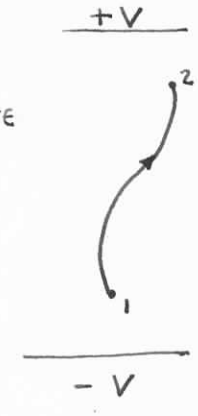
WE FIND THAT

$$E = -\frac{\hbar^2}{2m}\left(+\frac{iP}{\hbar}\right)^2 = \frac{P^2}{2m}$$

THIS IS JUST THE ENERGY THE FREE PARTICLE HAS.

PERTURBING POTENTIALS

NEXT I WANT TO DISCUSS WHAT HAPPENS WHEN THE FREE PARTICLE IS NOW DISTURBED BY POTENTIALS AND FORCES LIKE THOSE RESULTING FROM ELECTRICALLY CHARGED PLATES. HERE WE ARE ONLY GOING TO BE CONCERNED WITH EXTERNAL POTENTIALS ACTING ON A CHARGED PARTICLE. THIS IS JUST ELECTROSTATICS.



AS YOU RECALL NOT LONG AGO WE MENTIONED THAT GIVEN THE FREE PARTICLE AMPLITUDE TO GO ACROSS A PATH IN A POTENTIAL HAS ITS PHASE CHANGED BY THE AMOUNT $-i/\hbar\, e\int V(x)\,dt$. WE CAN EVALUATE THIS INTEGRAL IF WE TAKE A VERY SHORT TIME, I.E.,

$$e\int V(x)\,dt = \frac{e}{2}\left[V(x_2,t_2) + V(x_1,t_1)\right]$$

HERE I HAVE JUST AVERAGED THE POTENTIAL BETWEEN THE TO POINTS. IF A VECTOR POTENTIAL IS ALSO ACTING WE MUST ADD STILL ANOTHER PHASE TERM

$$\frac{i}{\hbar}\frac{e}{2}\left[A(x_2,t_2) + A(x_1,t_1)\right]\cdot(\bar{x}_2 - \bar{x}_1)$$

WHERE I AVERAGED BETWEEN THE TWO INTEGRATION POINTS AGAIN. THE TOTAL MODIFICATION TO THE PARTICLE'S KERNEL IS

$$K(x_2, x_1; t_2, t_1) = \sqrt{\frac{m}{2\pi i\hbar(t_2-t_1)}}\; e^{\frac{im}{2\hbar}\frac{(x_2-x_1)^2}{(t_2-t_1)}}\; e^{-\frac{ie}{\hbar}\frac{1}{2}\left[V(x_2,t_2)+V(x_1,t_1)\right](t_2-t_1)}$$

$$\cdot\; e^{i/\hbar\,\frac{e}{2}\left[A(x_2,t_2)+A(x_1,t_1)\right](x_2-x_1)}$$

If we just have a scalar potential V acting then the phase change is just $e^{-i/\hbar\, eVt}$. Since we have that $\psi(x,t) = e^{-i/\hbar\,(Et-px)}$, the effect of a potential V is to add some energy to the system - a very understandable thing to happen.

Going back to Schrödinger's equation to see how it is modified, notice we have the particle's kernel

$$K = e^{\frac{im}{2\hbar}\frac{(x-x')}{(t'-t)}}\; e^{-\frac{ie}{\hbar}V(x)\,(t'-t)}$$

For small time periods, ϵ,

$$\psi(x,t+\epsilon) = \sqrt{\frac{m}{2\pi i\hbar\epsilon}}\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\epsilon}}\; e^{-\frac{i\epsilon eV(x)}{\hbar}}\;\psi(x+\eta,t)\,d\eta$$

We have to expand the exponential $e^{-\frac{ieV(x)}{\hbar}\epsilon}$ for small ϵ,

$$e^{-\frac{ieV(x)}{\hbar}\epsilon} = 1 - \frac{ieV(x)}{\hbar}\epsilon$$

When this is multiplied by the expansion of $\psi(x+\eta,t)$ there is one extra term hanging on at the end which when carried through gives

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + eV(x,t)\,\psi$$

Again we see a little extra energy coming in due to the scalar potential V such that

$$E = \frac{p^2}{2m} + eV$$

Where eV is the potential energy. Remembering this for of E is often a heuristic rule to plug into the above equation if you know classical mechanics. But it is better to know Schrödinger's equation and find that $E = \frac{p^2}{2m} + V(x)$

Spin one-half, superposition of states

Vol. III Chapter 6

As an aside to today's lecture let me show you how rotational invariance lets us predict the state of a system. Consider an electron with a definite momentum. There are two states in which it can be found - it can have spin-up or spin-down. That is to say the particle has positive angular momentum about some z axis or along the -z axis. There is no amplitude to find the electron to be in one state if it is in the other.

We have to ask what another observer would see if he was using a different coordinate system, call it the prime system, rotated through some angle θ, to the unprimed system. Since the electron has not gone into a new state, we have to express the $+z'$ state as combination of the $+z$ and $-z$ states, i.e.,

$$|+z'\rangle = C\left(\begin{smallmatrix}+z'\\-z'\end{smallmatrix}\right)|+z\rangle + C\left(\begin{smallmatrix}+z'\\-z'\end{smallmatrix}\right)|-z\rangle$$

THE COEFFICIENTS $C\binom{+z'}{+z}$ AND $C\binom{+z'}{-z}$ AND AMPLITUDES TO FIND THE PARTICLE IN $|+z'\rangle$ STATE IF IT IS IN THE $+z$ STATE AND TO FIND IT IN $|+z'\rangle$ IF IT IS IN $|-z\rangle$ INITIALLY.

$$C\binom{+z'}{+z} = \langle +z|+z'\rangle$$

$$C\binom{+z'}{-z} = \langle -z|+z'\rangle$$

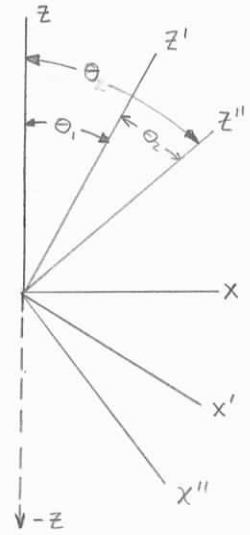
Therefore WE CAN WRITE

$$|+z'\rangle = \langle +z|+z'\rangle\,|+z\rangle + \langle -z|+z'\rangle\,|-z\rangle$$

THERE IS A CORRESPONDING EXPRESSION FOR THE $|-z'\rangle$ STATE,

$$|-z'\rangle = \langle +z|-z'\rangle\,|+z\rangle + \langle -z|-z'\rangle\,|-z\rangle$$

TO FIND OUT WHAT THE AMPLITUDES ARE CONSIDER THE DRAWING ARGUING FROM THE PICTURE WE HAVE



$$|+z'\rangle = \cos\frac{\theta_1}{2}\,|+z\rangle + \sin\frac{\theta_1}{2}\,|-z\rangle$$

THE REASON FOR HALF ANGLES IS TO PRESERVE THE PHYSICAL STATE OF THE SYSTEM. IF $\theta_1 = 0$ $|+z'\rangle = |+z\rangle$ AS EXPECTED. BUT IF $\theta_1 = 180°$ THEN $|+z'\rangle = |-z\rangle$ WHICH SAYS WE NOW SEE THE ELECTRON SPINNING DOWN. THE CORRESPONDING $|-z'\rangle$ STATE IS

$$|-z'\rangle = -\sin\frac{\theta_1}{2}\,|+z\rangle + \cos\frac{\theta_1}{2}\,|-z\rangle$$

IF STILL ANOTHER COORDINATE SYSTEM HAD BE USED SAY THE DOUBLE PRIME ONE WE COULD RELATE IT TO THE PRIME SYSTEM, I.E.,

$$|+z''\rangle = \cos\frac{\theta_2}{2}\,|+z'\rangle + \sin\frac{\theta_2}{2}\,|-z'\rangle$$

IF WE SUBSTITUTE FOR $|+z'\rangle$ AND $|-z'\rangle$

$$|+z''\rangle = \cos\frac{\theta_2}{2}\left[\cos\frac{\theta_1}{2}\,|+z\rangle + \sin\frac{\theta_1}{2}\,|-z\rangle\right] + \sin\frac{\theta_2}{2}\left[-\sin\frac{\theta_1}{2}\,|+z\rangle + \cos\frac{\theta_1}{2}\,|-z\rangle\right]$$

THIS CAN BE EXPANDED AND SIMPLIFIED TO

$$|+z''\rangle = \cos\frac{(\theta_1+\theta_2)}{2}\,|+z\rangle + \sin\left(\frac{\theta_1+\theta_2}{2}\right)|-z\rangle$$

SINCE THE TOTAL ROTATION θ IS JUST $\theta_1+\theta_2$, THE RESULTS ARE CONSISTENT. WE CAN EXTEND THIS ANALYSIS TO THREE DIMENSIONS AND DETERMINE ALL THE COEFFICIENTS IN TERM OF ONLY TWO STATES - THAT IS AN AMAZING RESULT.

16. THE DIRAC EQUATION

SO FAR WE HAVE DEVELOPE QUANTUM MECHANICS TO THE POINT WHERE WE HAD AN EXPRESSION FOR SCHROEDINGER'S EQUATION,

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$$

THIS IS THE PROPER FORM OF THE DIFFERENTIAL EQUATION IN THE ABSENSE OF ANY EXTERNAL POTENTIAL. WHEN AN EXTERNAL FIELD IS TURNED ON, THE MODIFICATION TO THE ABOVE EQUATION IS

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)\cdot\left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)\psi + eV\psi$$

THIS IS THE GENERALIZED NON-RELATIVISTIC SCHROEDINGER EQUATION.

NOW IF WE ARE AIMING FOR A RELATIVISTICALLY INVARIANT EXPRESSION WE WILL HAVE TO MAKE AN IDENTIFICATION AS TO WHAT REPRESENTS THE RELATIVISTIC ENERGY AND MOMENTUM. IF WE REWRITE THE ABOVE EXPRESSION AS

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} + eV\right)\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)^2\psi$$

WE CAN READILY IDENTIFY THE ENERGY AND MOMENTUM IN THE CLASSICAL SENSE AS

$$E = -\frac{\hbar}{i}\frac{\partial}{\partial t} + eV$$

WHILE

$$p = \left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)$$

THAT IS,

$$E\psi = \frac{p^2}{2m}\psi$$

WE HAVE TO BE CAREFUL HERE BECAUSE THE ENERGY IS ACTUALLY SHIFTED BY THE REST ENERGY, mc^2 SO THAT

$$E = \frac{p^2}{2m} + mc^2$$

THIS RESULT FOLLOWS FROM THE RELATIVISTIC EXPRESSION FOR THE ENERGY WHICH WE HAVE PREVIOUSLY DEVELOPED, I.E.,

$$E = \sqrt{m^2c^4 + p^2c^2}$$

THE EFFECT OF mc^2 IS TO SHIFT THE ZERO ENERGY VALUE. IF WE WRITE $W = p^2/2m$ THEN $\psi_{\text{NON-RELATIVISTIC}} \sim e^{-iWt}$ AND

$$\psi_{\text{RELATIVISTIC}} = e^{-imc^2}\psi_{NR}$$

The problem we have in developing the relativistic form of the Schroedinger equation is this; it really is a proportionality

$$\text{If} \quad E = mc^2 + \frac{p^2}{2m} \quad \text{is to} \quad \left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\psi_{NR} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{eA}{c}\right)^2\psi_{NR} + mc^2\psi_{NR}$$

$$\text{The} \quad E = \sqrt{m^2c^4 + p^2c^2} \quad \text{is to} \quad \underline{\text{WHAT EQUATION ??}}$$

So we have cooked up ~~and~~ analogy now let's see what we can do with it. First we might make an immediate substitution into the relativistic expression for E, i.e.,

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\psi = \sqrt{m^2c^4 + c^2\left(\frac{\hbar}{i}\bar{\nabla} - \bar{A}\right)^2}\ \psi$$

Now the right side turns out to be a real mess because of the operation under the radical; the operation expanded out becomes

$$\left(\frac{\hbar}{i}\nabla - A\right)^2 = -\hbar^2\nabla^2 + \bar{A}\cdot\bar{A} - \frac{\hbar}{i}\nabla\cdot A - \frac{\hbar}{i}\bar{A}\cdot\bar{\nabla}$$

The solution to this problem is easy when there isn't any external field $\vec{A}$ because the fourier transform can be taken to express the equation in momentum space and perform the operation; later the inverse transform can be employed to get the answer in coordinate space. But when a vector potential is present, taking fourier transforms gives complicated convolution integrals which paralyze the mathematics. So another approach is sought.

A second try would be to square the relativistic energy expression such that

$$E^2 = m^2c^4 + p^2c^2$$

and now make the identification as

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)^2\psi = m^2c^4\psi + \left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)^2\psi$$

You have to be careful here. This equation is not the same as the above expression even though it looks like it. Further this is not derived by writing

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)^2\psi = \left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\sqrt{m^2c^4 + c^2\left(\frac{\hbar}{i}\bar{\nabla} - \bar{A}\right)^2}\ \psi$$

and then rearranging the right side as

$$\sqrt{m^2c^4 + c^2\left(\frac{\hbar}{i}\nabla - A\right)^2}\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\psi = \sqrt{m^2c^4 + c^2\left(\frac{\hbar}{i}\nabla - A\right)^2}\ \sqrt{m^2c^4 + c^2\left(\frac{\hbar}{i}\nabla - A\right)^2}\ \psi$$

where the substitution was made. The reason this is not legitimate is that in general $\bar{A}$ depends on time and therefore the radical ~~radial~~ is not commutative with the time differentiation in the energy expression $\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)$.

THE EQUATION WE HAVE GUESSED AT,

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)^2 \psi = m^2c^4\psi + \left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)^2 \psi$$

IS CALLED THE KLEIN-GORDON EQUATION BECAUSE SCHROEDINGER derived IT before his MORE FAMOUS EQUATION which WE WROTE AT THE START OF THIS LECTURE. SCHROEDINGER HAD TRIED SOLUTIONS of The FORM

$$\psi(x,t) = e^{-imc^2t}\,\chi(x,t)$$

And found THAT

$$-\frac{1}{i}\frac{\partial}{\partial t}\psi = e^{-imc^2t}\left[m\chi - \frac{1}{i}\frac{\partial}{\partial t}\chi\right]$$

SO The TIME OPERATION SHOULD BE REPLACED BY $mc^2 - \frac{1}{i}\frac{\partial}{\partial t}$.

HOWEVER IT WAS NOT UNTIL he REALIZED THAT $-\frac{1}{i}\frac{\partial}{\partial t}$ WAS SMALL COMPARED TO mc^2 And IGNORABLE THAT he WAS ABLE TO SOLVE A LARGE NUMBER OF PROBLEM. THAT IS, FROM The ABOVE EQUATION WE CAN SUBSTITUTE TO GET

$$\left[\left(m - \frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\left(m - \frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right) - \left(\frac{\hbar}{i}\nabla - A\right)^2\right]\psi = m^2\psi$$

MULTIPLYING OUT

$$\left[m^2 - 2m\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right) + \left(\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)^2 - \left(\frac{\hbar}{i}\nabla - A\right)^2\right]\psi = m^2\psi$$

And SUBTRACTING OUT The m^2 TERM And REARRANGING,

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - A\right)^2\psi - \frac{\left(\frac{\hbar}{i}\frac{\partial}{\partial t} - V\right)^2\psi}{2m}$$

IT WAS The DROPPAGE of The LAST TERM for The NON RELATIVISTIC CASE THAT PERMITTED SCHROEDINGER TO WRITE HIS NOW FAMOUS EQUATION And SOLVED A LOT OF PROBLEMS.

TO verify The ABOVE EQUATION IS A LEGITIMATE RELATIVISTIC EQUATION WE COULD TRY TO FIND The HYDROGEN ENERGY LEVELS WITH IT. If WE APPLY A SOLUTION of The FORM

$$\psi(x,y,z,t) = e^{-iEt}\,\phi(x,y,z)$$

TO The differential EQUATION

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - \frac{Ze^2}{r}\right)^2\psi = m^2c^4\psi - \frac{\hbar^2}{2m}\nabla^2\psi$$

ENERGY VALUES CAN BE CALCULATED, i.e.,

$$\left(E - \frac{Ze^2}{r}\right)\phi = m^2c^4\phi - \hbar^2\nabla^2\phi$$

Here WE ASSUME The hydrogen ATOM WAS AT REST And The proton had NO SELF-REACTION. The ENERGY VALUES CAN BE SOLVED AS IN VOL III CHAPTER 19 TO YIELD

$$E_n = -\frac{me^4}{2\hbar^2}\frac{1}{n^2} = -\frac{13.6}{n^2}\ \text{VOLTS}$$

WE HAVE ANOTHER STEP TO GO AND THAT IS TO PUT THIS EXPRESSION INTO THE CORRECT RELATIVISTIC FORM. IT IS POSSIBLE TO WRITE THIS SAME EQUATION AS

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\underline{\psi} = \frac{1}{2m}\left[\bar{\sigma}\cdot\left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)\right]\left[\bar{\sigma}\cdot\left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)\right]\underline{\psi}$$

TO SHOW THAT THIS IS EQUIVALENT TO THE PREVIOUSLY DEVELOPED EQUATION WE NEED TO MAKE USE OF THE PROPERTIES OF THE SPIN MATRICES, I.E.,

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

AND

$$\sigma_x\sigma_y = i\sigma_z = -\sigma_y\sigma_x \quad \text{ETC. CYCLICALLY ANTI-COMMUTE}$$

TO SHOW THIS MORE SPECIFICALLY EXPANDING OUT THE RIGHT SIDE

$$\left[\sigma_x(\nabla_x - A_x) + \sigma_y(\nabla_y - A_y) + \sigma_z(\nabla_z - A_z)\right]\left[\sigma_x(\nabla_x - A_x) + \sigma_y(\nabla_y - A_y) + \sigma_z(\nabla_z - A_z)\right]$$

MULTIPLYING

$$= \sigma_x^2(\nabla_x - A_x)(\nabla_x - A_x) + \sigma_y^2(\nabla_y - A_y)(\nabla_y - A_y) + \sigma_z^2(\nabla_z - A_z)(\nabla_z - A_z) = (\bar{\nabla} - \bar{A})^2$$

THE CROSS-TERMS ARE OF THE FORM

$$\sigma_x\sigma_y(\nabla_x - A_x)(\nabla_y - A_y) + \sigma_y\sigma_x(\nabla_y - A_y)(\nabla_x - A_x) =$$

$$i\sigma_z\left\{(\nabla_x - A_x)(\nabla_y - A_y)\psi - (\nabla_y - A_y)(\nabla_x - A_x)\psi\right\} =$$

$$i\sigma_z\left\{\cancel{\frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi} - \cancel{\frac{\partial}{\partial y}\frac{\partial}{\partial x}\psi} - \frac{\partial}{\partial x}A_y\psi - A_x\frac{\partial}{\partial y}\psi + \frac{\partial}{\partial y}A_x\psi + A_y\frac{\partial}{\partial x}\psi + \cancel{A_xA_y\psi} - \cancel{A_yA_x\psi}\right\} =$$

$$i\sigma_z\left\{-\cancel{A_y\frac{\partial}{\partial x}\psi} - \psi\frac{\partial}{\partial x}A_y - \cancel{A_x\frac{\partial\psi}{\partial y}} + \cancel{A_x\frac{\partial}{\partial y}\psi} + \psi\frac{\partial A_x}{\partial y} + \cancel{A_y\frac{\partial\psi}{\partial x}}\right\} =$$

$$i\sigma_z\psi\left\{\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right\} = -i\sigma_z\psi(\nabla\times A)_z$$

THE BRACKETED TERM IS JUST THE z COMPONENT OF THE CURL OF $\bar{A}$ OR SINCE $\bar{\nabla}\times\bar{A} = B$ IT IS JUST THE z COMPONENT OF $\bar{B}$. AND WE VERY THE SENSATIONAL FORMULA GIVEN ABOVE.

WE STILL DON'T QUITE HAVE THE FINAL RELATIVISTIC FORM BECAUSE WE NEED THE REST ENERGY TERM mc^2 SO LET'S WRITE

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)^2\psi - \left[\bar{\sigma}\cdot\left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)\right]^2\psi = m^2c^4\psi$$

BUT WAIT! WE DIDN'T SQUARE THE EXPRESSION RIGHT AND IT IS COCKEYED. TO BE INVARIANT WE MUST HAVE ALL THE SPATIAL COMPONENTS GROUPED WITH THE TIME EXPRESSION IN A NICE LINE LIKE $(E - P)(E + P)$. SO LET'S TRY THE FOLLOWING

$$\left[-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV - \bar{\sigma}\cdot\left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)\right]\left[-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV + \sigma\cdot\left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)\right]\psi = m^2c^4\psi$$

THIS LOOKS BETTER AND IS IN AN INVARIANT FORM. IF WE COMPUTE THE HYDROGEN ENERGY LEVELS NOW WE GET THE RIGHT ANSWERS. BUT WE STILL LIMIT THE RESULT TO EXTERNAL POTENTIALS ONLY. IT DOES, HOWEVER, SATISFACTORILY PREDICT THE SPIN ORBIT COUPLING. THIS EQUATION IS ACTUALLY THE DIRAC EQUATION IN SLIGHTLY DIFFERENT NOTATION. IT CAN BE WRITTEN IN A MORE COMMON FORM BY DEFINING THE FOLLOWING

$$\left[-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV + \bar{\sigma}\cdot\left(\frac{\hbar}{i}\bar{\nabla} - \bar{A}\right)\right]\psi = m\chi$$

SUCH THAT

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV - \bar{\sigma}\cdot\left(\frac{\hbar}{i}\bar{\nabla} - \bar{A}\right)\right]\chi = m\psi$$

AND THIS IS THE MORE COMMON FORM OF THE DIRAC EQUATION.

WE CAN STREAMLINE THE MATHEMATICAL FORM BY RETURNING TO MATRICES AND DEFINING

$$\underline{\underline{\phi}} = \begin{pmatrix} \psi_+ \\ \psi_- \\ \chi_+ \\ \chi_- \end{pmatrix}$$

AND ESTABLISHING A SET OF 4 COMPONENT OPERATORS SIMILAR TO THE PAULI SPIN MATRICES.

THE GENERALIZED FORM OF THE DIRAC EQUATION IS WRITTEN

$$\left\{\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\gamma_t - \bar{\gamma}\cdot\left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)\right\}\underline{\underline{\varphi}} = m\underline{\underline{\varphi}}$$

where

γ_T OPERATING ON $\underline{\underline{\varphi}}$ PRODUCED χ_+ AND χ_-. THEREFORE

$$\gamma_t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

AND SIMILARLY

$$\gamma_{x,y,z} = \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \sigma_{x,y,z} \\ & & 0 & 0 \\ -\sigma_{x,y,z} & & 0 & 0 \end{pmatrix}$$

THE PROPERTIES OF THE 4 DIMENSIONAL INVARIANT OPERATOR IS

$$\gamma_t^2 = 1 \qquad \gamma_x^2 = \gamma_y^2 = \gamma_z^2 = -1$$

ALL γ'S ARE ANTI-COMMUTATIVE, I.E., $\gamma_x\gamma_t = -\gamma_t\gamma_x$

The PREDICTED ENERGY VALUES DO AGREE QUITE WELL WITH EXPERIMENT. HOWEVER The fINE LINE STRUCTURE of The VARIOUS LEVELS IS NOT IN AGREEMENT And WORSE The WRONG NUMBER of fINE STRUCTURE STATES ARE PREDICTED. SO WHAT AT fIRST SEEMED NICE QUICKLY TURNED OUT TO BE ALL WRONG. WHAT WE HAVE OMITTED SO fAR IS CONSIDERATION of The ELECTRON SPIN.

TO DESCRIBE A RELATIVISTIC PARTICLE WITH SPIN WE MUST INCLUDE A DESCRIPTION of The INTRINSIC Spin of The PARTICLE. TO DO This WE NEED TWO FUNCTIONS, CALL Them $\psi_+(\bar{x},t)$ And $\psi_-(\bar{x},t)$ where ψ_+ IS The AMPLITUDE TO BE AT $\bar{x}$ WITH SPIN UP ALONG The z AXIS WHILE ψ_- IS The AMPLITUDE TO BE AT $\bar{x}$ WITH SPIN DOWN. WE HAVE TO, therefore, LEARN how TO HANDLE The PAIR of differential EQUATIONS,

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \frac{1}{2m}\left(+\frac{\hbar}{i}\bar{\nabla} - \bar{A}\right)\cdot\left(\frac{\hbar}{i}\bar{\nabla} - \bar{A}\right)\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}$$

BUT WAIT! This IS WRONG. IT WOULD BE RIGHT if There WASN'T A MAGNETIC field AROUND. BUT IN The PRESENCE of A field DIRECTED ALONG The z AXIS THERE IS AN ENERGY ASSOCIATED WITH The ELECTRON SPINNING UP which IS different from The ENERGY IT HAS WHEN SPINNING DOWN. The CHANGES BEING

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\psi_+ = \frac{1}{2m}\left(\frac{\hbar}{i}\bar{\nabla} - \bar{A}\right)^2\psi_+ + \frac{-e\hbar}{2mc}B_z\psi_+$$

And

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\psi_- = \frac{1}{2m}\left(\frac{\hbar}{i}\bar{\nabla} - A\right)^2\psi_- + \frac{e\hbar}{2mc}B_z\psi_-$$

These EQUATIONS ARE COMPLETE FOR A MAGNETIC field And The z AXIS BUT LACK GENERALIZATION TO ANY field ORIENTATION, SAY B_x And B_y. EXTENDING TO ANY field DIRECTION WE GET

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\psi_+ = \frac{1}{2m}\left(\frac{\hbar}{i}\bar{\nabla} - \bar{A}\right)^2\psi_+ - \frac{e\hbar}{2mc}B_z\psi_+ + \frac{e\hbar}{2mc}B_x\psi_- + \frac{e\hbar}{2mc}iB_y\psi_-$$

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\psi_- = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - A\right)^2\psi_- + \frac{e\hbar}{2mc}B_z\psi_- + \frac{e\hbar}{2mc}B_x\psi_+ - \frac{e\hbar}{2mc}B_y i\psi_+$$

IT MORE CONVENIENT MATHEMATICALLY TO WRITE THIS ALL IN MATRIX NOTATION by fIRST defining

$$\underline{\psi} = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}$$

And Then The SET of TWO x TWO MATRIX OPERATORS

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

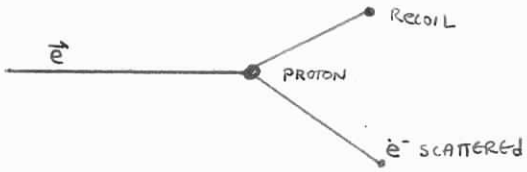
These ARE CALLED The PAULI SPIN EXCHANGE OPERATORS. THUS IN The MORE GENERALIZED FORM WE HAVE

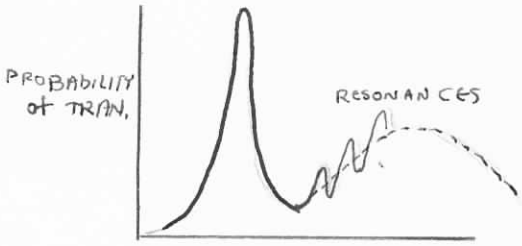
$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - eV\right)\underline{\psi} = \frac{1}{2m}\left(\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)\cdot\left(\frac{\hbar}{i}\bar{\nabla} - e\bar{A}\right)\psi - \frac{e\hbar}{2mc}\bar{B}\cdot\bar{\sigma}\,\underline{\psi}$$

17. THE PARTICLES IN THE PROTON

OVER THE HOLIDAYS DR. FEYNMAN GAVE A PRESENTATION AT AE MEETING IN SAN DIEGO. THE INQUISITIVE NATURE OF MALIBU AUDIENCE FINALLY CONVINCED DR. FEYNMAN TO DISCUSS HIS TALK.

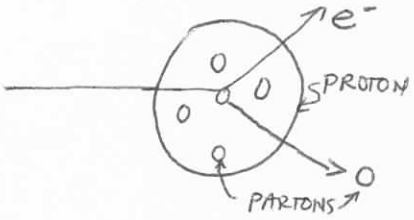
RECENT STRONG INTERACTION EXPERIMENTS AT THE STANFORD LINEAR ACCELERATOR HAVE UNDER COVERED A SUBPARTICLE NATURE OF THE PROTON. THIS IS THE FIRST TIME THAT THE PROTON WAS SHOWN TO HAVE A UNIQUE QUALITY ABOUT IT. IN THE EXPERIMENT 20 BeV COLLISIONS WERE INITIATED WITH HYDROGEN GAS BEING BOMBARDED BY RELATIVISTIC ELECTRONS. THE SPECTRUM OF THE SCATTERED ELECTRONS WERE EXAMINED AT DIFFERENT ANGLES. THUS BY STANDARD SCATTERING THEORY SIMILIAR TO BILLARD BALL COLLISION THE MOMENTUM LOSS TO THE PROTON COULD BE MEASURED. IN THE COLLISION PROCESS THE PROTON COULD BE SMASHED INTO PIECES JUST AS A BILLARD BALL HIT TO HARD WOULD CRACK. THE EXCITED PROTON WOULD CONTINUE TO GIVE OFF MORE ENERGY AS IT DISINTEGRATES. A PLOT OF THE PROBABILITY OF MOMENTUM TRANSFER VERSUS THE ENERGY LOST BY THE ELECTRON. THE BUMPS GET MORE IMPORTANT AS THE ENERGY OF ELECTRON GETS BIGGER





ENERGY LOSS by $\bar{e}$

IF IT IS ASSUMED THAT THERE ARE PARTS OR PARTONS OF THE PROTON WHICH HAVE A PROBABILITY DISTRIBUTION FOR THEIR MOMENTA. WHAT MAY HAPPEN IS THAT THE ELECTRON STRIKES ONE PARTON AND THAT PARTICLES GETS ALL THE ENERGY OF THE COLLISION, ~~CALL IT Q~~. THEN THE ENERGY LOSS DURING THE COLLISION IS



$$\Delta E = \left(\frac{P+Q}{2m}\right)^2 - \frac{P^2}{2m} - \epsilon$$

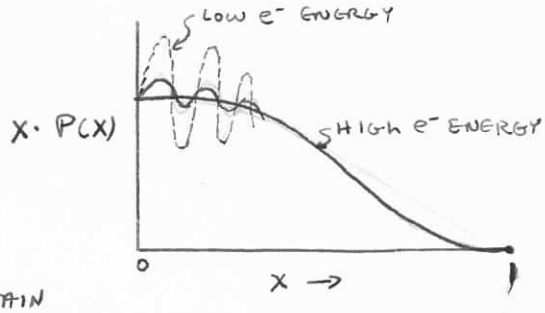
where ϵ is the PARTON BINDING ENERGY ; Q IS IS THE ELECTRON MOMENTUM AND P IS THE PARTON MOMENTUM.

THE ABOVE EQUATION CAN BE WRITTEN AS

$$\Delta E = \frac{Q^2}{2m} + \frac{PQ}{2m} - \epsilon$$

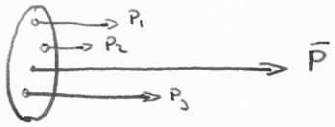
NOTE AS $Q \rightarrow \infty$ THE TERM PQ GROWS WITH Q^3 BUT ϵ WILL NOT. SINCE $Q^2/2m$ IS KNOW, I.E., 20 BeV, WE WILL DROP ϵ. IT THEREFORE REMAINS TO FIND P, THE MOMENTUM DISTRIBUTION OF THE PARTONS

ONE WAY TO TEST THIS THEORY IS TO REALIZE THAT AS Q APPROACHES ∞ THE MOMENTUM CURVES MUST APPROACH A UNIVERSAL DISTRIBUTION. IF I PLOT THE FRACTION OF PARTONS X HAVING A GIVEN MOMENTA I WILL GET A CURVE LIKE THE FOLLOWING



THE CURVE HAS BEEN EXPERIMENTALLY VERIFIED AND IT HAS AN IMPORTANT ASPECT; IT INDICATES THAT THERE IS 0 PROBABILITY THAT ALL THE MOMENTUM IS IN ONE PARTICLE. IT MAY BE IN TWO OR MORE BUT NOT ONE.

CONSIDERING A RELATIVISTIC PROTON FOR A MINUTE. THE PROTON IS LIKE A DISK WITH A CERTAIN MOMENTUM $\vec{P}$. IT SEEM LOGICAL TO THINK OF $\vec{P}$ AS BEING MADE UP OF A SUM OF MOMENTA, ONE FOR EACH PARTON.



$$P = \sum_i \mu_i p_i$$

18. SOLUTIONS TO SCHRÖDINGER'S EQUATION

WE HAVE BEEN DISCUSSING THE SCHRÖDINGER AND DIRAC EQUATIONS BUT NOW I'D LIKE TO DETERMINE THE SPECIFIC VALUES OF THE DISCRETE ENERGY LEVELS PREDICTED BY THE EQUATION.

FIRST, CONSIDER THE SCHRÖDINGER EQUATION

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = H\psi$$

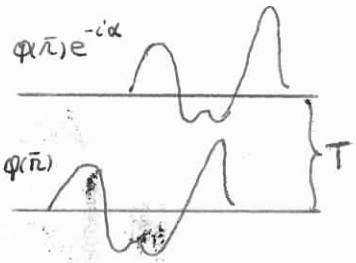
Where H here IS NOT A NUMBER BUT REALLY AN OPERATOR. WE HAVE CALLED IT THE HAMILTONIAN OPERATOR. FOR A PARTICLE MOVING IN A POTENTIAL IT IS GIVEN BY

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})$$

USING THIS OPERATOR FORM SCHRÖDINGER'S EQUATION BECOMES

$$-\frac{\hbar}{i}\frac{\partial \psi(\bar{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\bar{r},t) + V(\bar{r})\psi(\bar{r},t)$$

TO TRY TO UNDERSTAND THIS DIFFERENTIAL EQUATION CONSIDER THE CASE WHEN H IS NOT A FUNCTION OF TIME. THIS TURNS OUT TO BE A VERY INTERESTING CASE BECAUSE IT IS POSSIBLE TO FIND A PARTICULAR FUNCTION DESCRIBING THE PARTICLE'S AMPLITUDE WHICH PROPAGATES TO A NEW POSITION IN TIME T BUT RETAINS ITS INITIAL SHAPE. IT IS OFTEN TIMES HARD TO FIND SUCH A FUNCTION BUT IT IS POSSIBLE. IF WE WAIT FOR A TIME 2T, WE WILL FIND THE INITIAL DISTRIBUTION $\varphi(\bar{r})$ REPEATING ITSELF. AT EACH TIME T THE FUNCTION IS PERIODIC, I.E.,



$$\alpha = ET/\hbar$$

where E IS A NUMBER CHARACTERISTIC OF THE POINT where $\varphi(\bar{r})$ REPEATS ITSELF.

WE THUS HAVE THE INTUITIVE SOLUTION WHICH HAS TO BE SOLVED ANALYTICALLY

$$\psi(\bar{r},t) = \varphi(\bar{r})\, e^{-iET/\hbar}$$

PLUGGING THIS FUNCTION INTO SCHRÖDINGERS EQUATION WE GET THAT

$$E\varphi(\bar{r})\, e^{-iET/\hbar} = e^{-iET/\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\varphi + V\varphi\right]$$

OR MORE SIMPLY

$$H\varphi(\bar{r}) = E\varphi(\bar{r})$$

WE HAVE TO SOLVE THE EQUATION WHERE AN OPERATOR WORKING ON $\varphi(\bar{r})$ GIVES THE SAME RESULT AS MULTIPLYING IT BY A NUMBER E. IN THE MATHEMATICAL CIRCLES $\varphi(\bar{r})$ IS CALLED THE EIGENFUNCTION WHILE E describes THE EIGENVALUES, I.E., THE ENERGY OF EACH STATE CAN BE described BY A NUMBER WHICH IS RELATED TO A CERTAIN FREQUENCY. AS AN OPERATOR H IS WHAT IS CALLED A LINEAR OPERATOR HAVING THE PROPERTY THAT

$$H(\varphi_1 + \varphi_2) = H\varphi_1 + H\varphi_2$$

SUPPOSE WE HAD THE CASE WHERE

$$H\varphi_1 = E_1\varphi_1 \quad \text{AND} \quad H\varphi_2 = E_2\varphi_2$$

THEN THE COMPLETE SOLUTION FOR ψ IS GIVEN AS

$$\psi(\bar{r},t) = C_1 e^{-iE_1t}\varphi_1(\bar{r}) + C_2 e^{-iE_2t}\varphi_2(R)$$

WHILE THIS IS A SOLUTION TO SHRÖDINGER'S EQUATION IS IS NOT CHARACTERIZED BY A DEFINITE ENERGY.

SO FAR I HAVE SAID E IS A NUMBER BUT IS IT REAL? IF YOU WILL PERMIT ME TO WRITE DOWN A PARTICULAR PROPERTY THAT H HAS, I CAN SHOW YOU THAT E MUST BE REAL. THE PROPERTY IS

$$\int f^*(Hg)\, d^3\bar{r} = \int (Hf)^* g\, d^3\bar{r}$$

WHERE f AND g ARE ANY TWO FUNCTIONS. UPON SUBSTITUTING FOR H

$$\int f^*\left(-\frac{\hbar^2}{2m}\nabla^2 g\right) + \int f^* V g \overset{?}{=} \int \left(-\frac{\hbar^2}{2m}\nabla^2 f\right)^* g + \int (Vf)^* g$$

WHERE V HERE IS REAL.

THE FUNCTIONS f AND g ARE ONE IN THE SAME AND EQUAL $\psi(\bar{r})$. IF WE DEFINE

$$\psi^*\psi = P(\text{ROBABILITY}),\ P(\bar{r},t)$$

AS WE HAVE BEFORE, IT IS TRUE THAT

$$\int \psi^*\psi\, d^3\bar{R} = \int P\, d^3\bar{R} = \text{CONSTANT}$$

THAT IS TO SAY THE PROBABILITY OF FINDING THE PARTICLE REMAINS CONSTANT. FOR THIS TO HOLD

$$\frac{d}{dt}\int \psi^*\psi\, d^3\bar{R} = 0$$

$$\int \frac{d\psi^*}{dt}\psi + \int \psi^*\frac{d\psi}{dt} = \int i(H\psi^*)\psi - i\int \psi^*(H\psi)$$

SINCE H HAS THE ABOVE PROPERTY OF BEING HERMITEAN, I.E., SATISFY

$$\int f^*(Hg)\, d^3\bar{r} = \int (Hf)^* g\, d^3\bar{r}$$

THE CONSTANCY DOES HOLD. THIS PROPERTY EVOLVES FROM TWICE INTEGRATING THE ABOVE EQUALITY AND EVALUATING; IN ONE DIMENSION

$$-\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} f\frac{d^2g^*}{dx^2}\,dx + \int_{-\infty}^{\infty} Vg^* f\,dx = -\frac{\hbar^2}{2m}\left(\frac{dg^*}{dx}f - g^*\frac{df}{dx}\right)\Bigg|_{-\infty}^{+\infty} -$$

$$-\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} g^*\frac{d^2f}{dx^2}\,dx + \int_{-\infty}^{\infty} Vg^* f\,dx$$

IF f, g FALL OFF TO ZERO AT INFINITY THE INTEGRATED PARTS VANISH AND THE IDENTITY ESTABLISHED.

Thus with H hermitean

$$\int \varphi^* (H\varphi)\, dvol = E \int \varphi^* \varphi \, dvol$$

can be written as

$$\int (H\varphi)^* \varphi \, dvol = E^* \int \varphi^* \varphi^* \, dvol$$

but $\int \varphi^* \varphi \, dvol = \int \varphi \varphi^* \, dvol$. Subtracting the two

$$E = E^*$$

Therefore E must be real. If it were imaginary the solution

$$\psi(\vec{r}, t) = \varphi(\vec{r})\, e^{-iEt}$$

would exponentially expand or contract which is not physically what happens.

I said that $\int \psi^* \psi \, dvol$ was constant independent of time; let me show you this is true for

$$\psi = C_1 e^{-iE_1 t} \varphi_1 + C_2 e^{-iE_2 t} \varphi_2$$

$$\int \psi^* \psi = C_1 C_1^* \int \varphi_1^* \varphi_1 + C_2^* C_2 \int \varphi_2^* \varphi_2 + C_1^* C_2\, e^{i(E_1 - E_2)t} \int \varphi_1^* \varphi_2 + C_2^* C_1\, e^{-i(E_1 - E_2)t} \int \varphi_2^* \varphi_1 \, dvol.$$

If E_1 does not equal E_2 then $\int \varphi_1^* \varphi_2 = \int \varphi_2^* \varphi_1 = 0$ for $\int \psi^* \psi$ to be constant independent of time. To show this is true consider 2 states which give different energies for the same hamiltonian, $H\varphi_1 = E_1 \varphi_1$, $H\varphi_2 = E_2 \varphi_2$

Then

$$\int (H\varphi_1)^* \varphi_2 \, dvol = E_1 \int \varphi_1^* \varphi_2 \, dvol$$

but this is the same as

$$\int \varphi_1^* (H\varphi_2)\, dvol = E_2 \int \varphi_1^* \varphi_2 \, dvol$$

subtracting

$$(E_1 - E_2) \int \varphi_1^* \varphi_2 \, dvol = 0$$

since $E_1 \neq E_2$ $\quad \int \varphi_1^* \varphi_2 \, dvol = 0$

This is called the orthogonality relationship of two eigenfunctions of different eigenvalues.

Quantized Energy Level

I'd like to now explain the existence of the discrete energy levels associated with the eigenvalues of the Schrödinger equation. Specifically, I'll take the simple case of a one dimensional motion and consider the effect of a potential "well" on the motion.

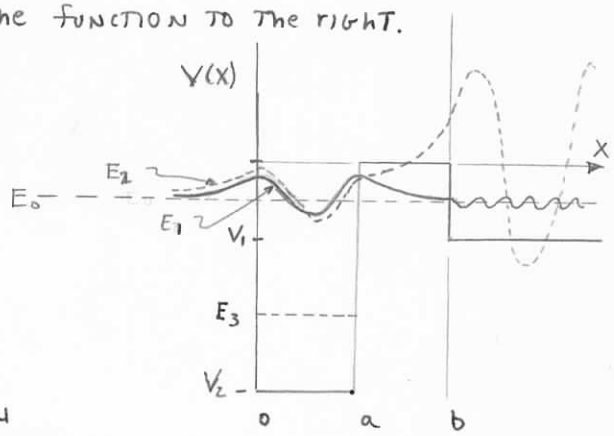
The lecture proceeded to cover the material in Vol III chapter 16-6 and, therefore, is not presented here.

19. PENETRATION of A COULOMb POTENTIAL

LAST TIME WE TALKED ABOUT DISCRETE ENERGY LEVELS BEING ASSOCIATED WITH A PARTICLE TRAPPED IN A POTENTIAL WELL AND THAT for PARTICLE MOTION WITH ENERGIES GREATER THAN The WELL THAT There IS A CONTINUOUS ENERGY DISTRIBUTION. IN THAT EXAMPLE I CHOSE TO LEAVE OUT The INTERACTION of The PARTICLE WITH The field AND The ENERGY LEVELS WERE of ZERO WIDTH. YOU CAN TAKE OTHER TYPES of POTENTIAL WELLS, E.G., A PARABOLIC FUNCTION LIKE $V(x) = ax^2$ where $a = 1/2\, m\omega^2$ AND WORK OUT ALL The ASSOCIATED ENERGY LEVELS BY SOLVING The differential EQUATION. FOR The CASE JUST describe The LEVELS TURN OUT TO BE JUST $E_n = \hbar\omega(n+1/2)$ where ω IS The NATURAL frequency of The SYSTEM IN A CLASSICAL SENSE.

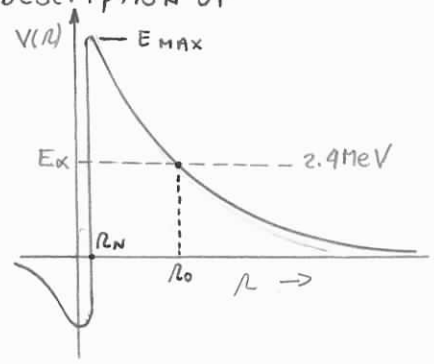
I WOULD LIKE TO MAKE A bAd PEDAGOGICAL MOVE NOW AND proceed TO DISCUSS A MUCH MORE COMPLICATED problem which ordinarily would ONLY be ATTEMPTED AFTER SOME development of EASIER PROBLEMS. BUT SINCE YOU'RE NOT foolin' ME AND I KNOW YOU'VE ALL HAD THIS STUFF before I'LL CONTINUE bECAUSE WE CAN LEARN A LOT of PHYSICS from The EXERCISE.

SUPPOSE I hAd A POTENTIAL GIVE bY The FUNCTION TO The RIGHT. When The PARTICLE HAS ENERGY E_3 WHICH IS BETWEEN V_1 AND V_2, I.E., $V_2 < E_3 < V_1$, ~~The~~ IT IS bound TO A SET of FINITE ENERGY LEVELS GIVEN bY The SOLUTION WE HAD LAST TIME.



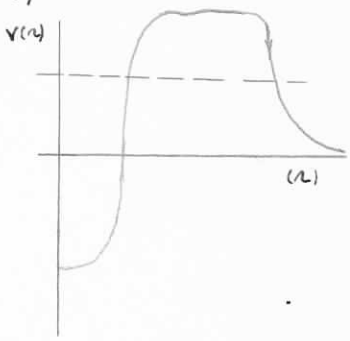
NOW OBSERVE The REGION of INTEREST When $V_1 < E < 0$. IN THIS REGION The PARTICLE IS IN bETWEEN A free STATE AND A bound STATE. THAT IS TO SAY, There IS ALWAYS A FINITE POSSIBILITY THAT The PARTICLE CAN be OUTSIDE The BOUNDARY b. The WAVE FUNCTION bETWEEN a AND b IS EXPONENTIALLY decaying bUT AT The boundary UNDERGOES A TRANSITION TO A SINUSOIDAL VARIATION. THE behavior of The PARTICLE IN REGION $x > b$ IS of INTEREST bECAUSE VERY SMALL CHANGES IN The WAVE FUNCTION IN REGION a-b CAN SIGNIFICANTLY CHANGE The behavior of The FUNCTION IN REGION b>a. The AMPLITUDE of The SINUSOIDAL VARIATIONS ARE SENSITVE TO The SLOPE of The WAVE FUNCTION AT a AND b. WE WANT TO DESCRIBE The WAVE FUNCTION AND PROBABILITY DISTRIBUTION AS The PARTICLE "LEAKS" THROUGH The bARRIER.

THIS PROBLEM HAS PHYSICAL SIGNIFICANCE IN THAT IT DESCRIBES THE PROCESS OF RADIOACTIVE DECAY. TO MAKE THIS POINT CLEARER RADIUM EMITS α PARTICLES FROM TIME TO TIME SUCH THAT OVER 1600 YEARS HALF THE ATOMS HAVE FIRED OFF AN α PARTICLE. THIS α PARTICLE HAS AN ENERGY OF 2.4 MeV. THE BEHAVIOR OF THE PARTICLE IN THE VICINITY OF THE NUCLEUS IS THAT OF A CHARGED PARTICLE IN A COULOMB POTENTIAL FIELD, I.E., IT IS REPELLED INVERSELY AS THE ~~SQUAR~~ DISTANCE FROM THE NUCLEUS. THE RADIUS OF THE NUCLEUS IS GIVEN AS $r_N = 1.2 \times 10^{-13}$ cm $A^{1/3}$ where for RADIUM THE $A = 86$. THE DESCRIPTION OF THE POTENTIAL FUNCTION LOOKS SOMETHING LIKE,



SOMEWHAT TO SCALE THE PARTICLE COMES OUT WITH ENERGY E_α. THIS ENERGY IS CONSIDERABLY LESS THAN E_{MAX}. THAT IS TO SAY ONCE THE PARTICLE GETS OUT IT HAS TO ACQUIRE AN ENERGY $E_{MAX} - E_\alpha$ TO GET BACK IN. BUT THEN HOW DID IT GET IN THE FIRST PLACE?

TO SOLVE THIS PROBLEM I'M GOING TO MAKE SOME ASSUMPTIONS TO SIMPLY THE WORK. FIRST I'LL CONSIDER TIME = 0 TO MEAN THAT NO PARTICLE IS BEYOND r_0. SAID ANOTHER WAY THE WAVE FUNCTION HAS NO EXPONENTIAL PORTIONS IN THE REGION $r > r_0$. THIS REPRESENTS AN EXPERIMENTAL PROBLEM RATHER THAN AN ANALYTIC ONE SO LET ME GET BY WITH IT. I AM ALSO GOING TO CONSIDER THAT I AM SO FAR AWAY FROM THE BARRIER THAT ITS THICKNESS AND QUALITY ARE NOT IMPORTANT. IN OTHER WORDS I COULD HAVE A FUNCTION LIKE,



WE'LL FIND IN WORKING THIS PROBLEM THAT IN order TO describe THE PARTICLE behavior QUANTUM MECHANICALLY THAT WE WILL NEED THE THEOREMS WE developed LAST TIME. IN PARTICULAR WE WILL BE LOOKING FOR THE TIME DEPENDENT SOLUTION

$$\psi(x,t) = \sum_n C_n e^{-iE_n t} \varphi_n(x)$$

THE COEFFICIENTS C_n ARE TIME DEPENDENT QUANTITIES. FOR THE CASE where ALL THE C_n's ARE 0 EXCEPT ONE THEN THE SOLUTION SIMPLE REDUCES TO $\psi = e^{-iE_n T} \varphi(x)$. When There is ONE DEFINITE ENERGY STATE DESCRIBING THE WAVE FUNCTION THE PARTICLE IS IN A STEADY-STATE CONDITION; ANY OTHER TIME IT IS MADE UP of A SUPERPOSITION of MANY Different STATES or FREQUENCIES. IN ORDER TO COMPUTE THE PARTICLE'S PROBABILITY DISTRIBUTION FROM THE ABOVE WAVE FUNCTION WE MUST find $P(x,t) = \psi^* \psi^*$ OR

$$P(x,t) = \sum_n \sum_m C_m^* C_n e^{i(E_m - E_n)t} \varphi_m^* \varphi_n$$

RECALL LAST TIME WE SAID THAT $\int_{-\infty}^{\infty} P(x,t)\,dx$ WAS A CONSTANT INDEPENDENT OF TIME. WELL, THE SAME MUST HOLD WHICH IMPLIES THAT

$$\int \varphi_m^* \varphi_n \, dx = \delta_{nm}$$

WHEN THIS HOLDS WE HAVE

$$\int \psi^* \psi \, dx = \sum_n C_n^2 = 1$$

WHERE WE HAVE NORMALIZED THE PROBABILITY TO BE ONE. THE C_n^2's DESCRIBE THE PROBABILITY OF THE PARTICLE TO BE IN STATE n.

THE PROBLEM WE WILL BE SOLVING IS AN INITIAL VALUE PROBLEM WHEREIN THE WAVE FUNCTION A TIME $t=0$ IS PRECISELY KNOWN, I.E.,

$$\psi(x,0) = f(x) = \sum_n C_n \varphi_n(x)$$

WHAT WE WANT TO ESTABLISH IS THE WAVE FUNCTION AT SOME LATER TIME. TYPICALLY IT IS QUITE HARD TO EXPRESS $f(x)$ AS A FOURIER SERIES IN $\varphi(x)$ BUT IT CAN BE EASY IF WE FIND THE COEFFICIENTS IN THE FOLLOWING CUTE WAY,

$$\int \varphi_m^* f(x)\, dx = \sum_n C_n \int \varphi_m^* \varphi_n \, dx$$

SINCE $\int \varphi_m^* \varphi_n \, dx = \delta_{nm}$ WE IMMEDIATELY FIND THE COEFFICIENT C_m

$$C_m = \int_{-\infty}^{\infty} \varphi_m^* f(x)\, dx$$

THIS IS A REMARKABLY SIMPLE RESULT BUT IS ONE OF THE MOST IMPORTANT OBSERVATIONS OF QUANTUM MECHANICS. IT PROVIDES A REAL POWER TO SOLVE SOME VERY DIFFICULT PROBLEMS.

FOR THOSE WHO LIKE TO SEE MORE MATHEMATICAL WORK I'LL SHOW YOU HOW TO GUESS AT $f(x)$ BASED ON SOMETHING YOU ALREADY LEARNED. IF YOU INSERT THE COEFFICIENTS C_n THAT WAS JUST FOUND INTO $\psi(x,t)$

$$\psi(x,t) = \sum_n \varphi_n(x)\, e^{-iE_n t} \int \varphi_n^*(x')\, \psi(x',0)\, dx'$$

WHERE I HAVE USED A PRIME UNDER THE INTEGRAL TO DENOTE THAT THE INTEGRATION IS OVER A DIFFERENT RANGE OF x. I CAN REARRANGE THIS EXPRESSION TO READ

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x,t;x',0)\, \psi(x',0)\, dx'$$

WHERE UPON EXAMINATION THE KERNEL IS RELATED TO THE WAVE FUNCTION AS

$$K(x,t;x',0) = \sum_n \varphi_n(x)\, \varphi_n^*(x')\, e^{-iE_n t}$$

THIS IS NOT THE GENERAL EXPRESSION FOR THE KERNEL BECAUSE WE HAVE ASSUMED THE HAMILTONIAN TO BE TIME INDEPENDENT. THAT MEANS THAT NO ONE IS TURNING POTENTIALS ON OR OFF AND THUS CREATING TIME VARYING INTERACTIONS.

ANOTHER PROBLEM WHICH WE MUST SOLVE IS HOW TO GO FROM A SET OF DISCRETE ENERGY LEVELS TO A CONTINUUM OF LEVELS WHICH ARE NOT SEPARATED BY FINITE ENERGY LEVELS. THE TASK IS TO TURN A SUM OF DISCRETE STATES INTO AN INTEGRAL. THE ONLY WAY I KNOW GRACEFULLY TO DO THIS IS TO ASSUME THE CONTINUUM IS REALLY A SET OF DISCRETE ENERGY LEVELS INFINITESIMALLY CLOSE AND IN THE LIMIT LET THE SPACING GO TO ZERO.

TO MAKE THIS APPROXIMATION IT IS NECESSARY TO PUT THE PARTICLE AND ITS "WELL" INTO A GIGANTIC BOX. A BOX SO BIG THAT WE WON'T WORRY ABOUT THE PARTICLE EVER HITTING THE WALLS. WE COULD LEAVE THE REFLECTIONS IN THE ANALYSIS BUT IT ONLY SERVES TO COMPLICATE THE COMPUTATIONS. THE BOX CAN BE THOUGHT OF AS A POTENTIAL OF INFINITE SIDES SEPARATED BY A DISTANCE L WHICH IN THE LIMIT GOES TO INFINITY.

I WOULD LIKE TO WORK WITH SPHERICAL, THREE-DIMENSIONAL WAVES FOR THIS PROBLEM. THE HAMILTONIAN IN SPHERICAL COORDINATES PRODUCES THE FOLLOWING SCHRÖDINGER EQUATION,

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r) - V(r)\psi = E\psi$$

I AM GOING TO ASSUME THAT THE WAVE FUNCTION HAS A RADIAL DEPENDENCE DUE TO COULOMB ATTRACTION TO A CENTRAL FORCE. THUS THE SOLUTIONS WILL NOT DEPEND ON ANY ANGLES BUT LATER WELL COME BACK AND DO IT WILL THE ANGULAR DEPENDENCE.

BEFORE WE GO ON I WANT TO WRITE THE LAPLACIAN IN A DIFFERENT FORM. SINCE THERE IS ONLY A RADIAL DEPENDENCE

$$\nabla^2\psi = \frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\psi = \frac{1}{r}\frac{d^2}{dr^2}(r\psi)$$

THIS IDENTITY CAN EASILY BE PROVEN AND WILL TURN OUT TO BE QUITE HELPFUL. IF WE SUBSTITUTE THIS INTO SCHRÖDINGER'S EQUATION WE HAVE THAT

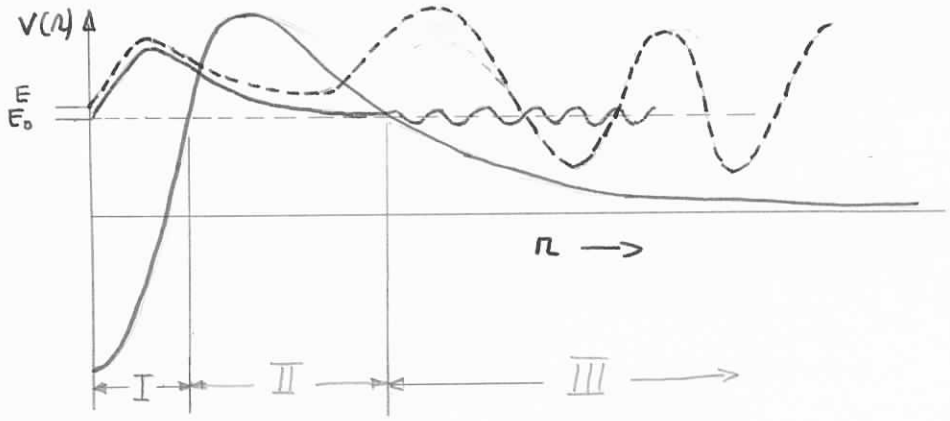
$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}(r\psi) - V(r)(r\psi) = E(r\psi)$$

NOTICE THAT THIS DIFFERENTIAL EQUATION IS IN THE FORM OF A ONE DIMENSIONAL RELATIONSHIP IF I REPLACE $r\psi$ BY Ψ AND LET $r \to x$. I MUST REQUIRE, HOWEVER, THAT Ψ APPROACH ZERO AS $r \to 0$ SO THAT THE WAVE FUNCTION REMAINS FINITE. THE NORMALIZATION INTEGRATION IS WRITTEN AS

$$\int \psi^2 4\pi r^2\, dr = \int (r\psi)^2 dr = \frac{1}{4\pi}$$

THUS THE SPHERICAL CASE CAN BE SOLVED JUST LIKE THE ONE DIMENSIONAL CASE.

NOW WE RETURN TO THE PROBLEM OF RADIOACTIVE DISINTEGRATION OR PENETRATION OF A BARRIER.



THE WAVE FUNCTIONS ARE DESCRIBED IN THE THREE REGIONS SHOWN ABOVE. FOR THE SPECIAL FUNCTION CORRESPONDING TO ENERGY E_0 THE STEADY STATE BEHAVIOR IN REGION III IS A VERY LOW AMPLITUDE SINUSOIDAL VARIATION. IN SMALL VARIATION IN ENERGY WILL SIGNIFICANTLY CHANGE THE BEHAVIOR IN REGION III WHILE IN REGION I THE VARIATIONS IN WAVE FUNCTION ARE VERY SMALL. I WILL CHARACTERIZE THE BEHAVIOR OF THE WAVE FUNCTION IN REGION I AND III AS FOLLOWS,

$$\text{REGION I} \qquad \varphi_E^{I}(r) = f(r)$$

$$\text{REGION III} \qquad \varphi_E^{III} = a\left[\cos(kr+\delta_0) + \frac{2(E-E_0)}{\Gamma}\sin(kr+\delta_0)\right]$$

THE FUNCTION $f(r)$ IS NEARLY THE SAME FOR ALL VALUES OF E SUFFICIENTLY CLOSE TO E_0. IN REGION III THE SPATIAL WAVE FUNCTION φ_E DEPENDS ON a WHICH IS A FACTOR WHICH WILL LET ME TIE THE TWO WAVE FUNCTIONS TOGETHER. Γ IS A SMALL ENERGY VALUE. THE FIRST IN φ^{III}, I.E., $a\cos(kr+\delta_0)$ IS THE STEADY STATE OSCILLATION ASSOCIATED WITH ENERGY E_0. THE SECOND TERM $\frac{E-E_0}{\Gamma}\sin(kr+\delta_0)$ IS THE ADDITIONAL LARGE VARIATION DUE TO THE EXPONENTIAL BEHAVIOR IN THE BARRIER.

IN THE ABOVE EXPRESSIONS THE FUNCTIONS φ^{I} AND φ^{III} ARE NOT NORMALIZED. SO TO DO THAT WE WILL DEFINE A NORMALIZING FACTOR $\alpha(E)$ SUCH THAT

$$\varphi^{I} = \alpha(E)\, f(r)$$

$$\varphi^{III} = \alpha(E)\, a\left[\cos(kr+\delta_0) + \frac{2(E-E_0)}{\Gamma}\sin(kr+\delta_0)\right]$$

ONE ADDITIONAL FACT IS THAT AT TIME $t=0$ THE WAVE FUNCTION IS GIVEN BY

$$\psi(r,0) = F(r)$$

FURTHER I AM GOING TO REQUIRE THAT THE NORMALIZATION BE SUCH THAT

$$\int |\varphi_E^I(r)|^2 \, dr = 1 = \int F(r)\, dr$$

SO WHAT IS $\alpha(E)$ FOR THIS TO OCCUR? WELL INSIDE THIS GIGANTIC BOX THAT I HAVE I MUST REQUIRE THAT

$$\int_0^L |\varphi_E(r)|^2 \, dr = 1 = |\alpha(E)|^2 \int_0^L F(r)^2 \, dr +$$

$$\cancel{|\alpha(E)|^2} \int_0^L a^2 |\alpha(E)|^2 \left\{ \cos^2(kr+\delta_0) + \frac{(E-E_0)^2}{\Gamma^2} 4 \sin^2(kr+\delta_0) + \frac{4(E-E_0)}{\Gamma} \cos(kr+\delta_0) \sin(kr+\delta_0) \right\} dr$$

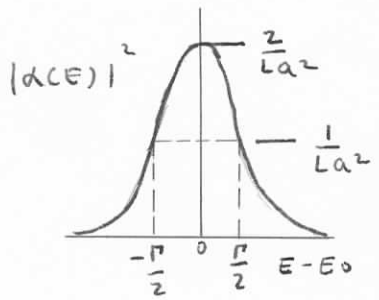
THE FIRST INTEGRAL IS SOME FINITE TERM. THE SECOND INTEGRAL CONSISTS OF 3 PARTS. THE INTEGRALS OF $\sin^2 x$ AND $\cos^2 x$ WILL GIVE AN AVERAGE VALUE OF $1/2$ WHILE THE $\sin x \cos x$ TERM IS ON THE AVERAGE ZERO. THEREFORE WE CAN WRITE

$$1 = \frac{L}{2} a^2 |\alpha(E)|^2 \left[1 + \frac{4(E-E_0)^2}{\Gamma^2} \right] + \text{FINITE TERMS AS } L \to \infty$$

SO AS $L \to \infty$ WE CAN FORGET THE FINITE TERMS. SOLVING FOR $|\alpha(E)|^2$ THEN

$$|\alpha(E)|^2 = \frac{2}{La^2} \frac{\Gamma^2/4}{\left[(E-E_0)^2 + \Gamma^2/4\right]}$$

THIS IS AN INTERESTING RELATIONSHIP. THE PROBABILITY FOR THE PARTICLE TO BE FOUND IN A NORMALIZED ENERGY STATE IS GIVEN BY THE CURVE. $\Gamma/2$ IS INTERPERTED AS THE WIDTH OF THE ENERGY STATE E_0. AS Γ BECOME MORE NARROWER, THE ENERGY BECOMES MORE DEFINITE.



20. TIME VARYING WAVE FUNCTION.

NOW WE MUST EVALUATE ψ AS A FUNCTION of TIME. TO DO THIS LET'S REVIEW the MATH. If $\varphi_E(r)$ IS The DEFINITE SPACE WAVE FUNCTION, THEN

$$\psi(r,t) = \sum_E C(E)\, \varphi_E(r)\, e^{-iEt}$$

Where WE HAVE SUMMED OVER ALL POSSIBLE ENERGYE VALUES. At TIME $t=0$ WE KNOW THAT $\psi(0) = F(r) = \sum_E C(E)\varphi_E$. BY MULTIPLY THIS EQUATION BY A DIFFERENT ENERGY STATE AND INTEGRATE The COEFFICIENTS ARE FOUND, I.E.,

$$\int \varphi_{E'}^*(r)\, F(r)\, dr = \sum_E C(E) \int \varphi_{E'}^* \varphi_E\, dr$$

BY The ORTHOGONALITY CONDITION $\int \varphi_{E'}^* \varphi_E\, dr = \delta(E'-E)$ And Therefore

$$\int \varphi_{E'}^*(r)\, F(r)\, dr = C(E')$$

Thus The expression for $\psi(r,t)$ CAN be EXPRESSED IN TERMS of $\int \varphi_E^*(r)\, F(r)\, dr$, I.E., IN TERMS of The INITIAL CONDITIONS. LET'S EVALUATE The COEFFICIENTS,

$$\int \varphi_E^* F(r)\, dr = \int_{\text{INSIDE}} \alpha^*(E)\, f(r)\, F(r)\, dr + \int_{\text{OUT}} \{\varphi^{III}\}^* F(r)\, dr$$

BUT SINCE $F(r) = 0$ OUTSIDE AN $\int |F(r)|^2 dr = 1$ INSIDE IT TURNS OUT THAT The COEFFICIENTS ARE JUST

$$C(E) = \alpha^*(E)$$

SO WE CAN WRITE

$$\psi(t) = \sum \alpha^*(E)\, \varphi_E(r)\, e^{-iEt} = \psi^I + \psi^{III}$$

And

IN REGION I $\quad \psi^I(r,t) = \sum_E \alpha^*(E)\, \alpha(E)\, f(r)\, e^{-iEt}$

IN REGION III $\quad \psi^{III}(r,t) = \sum_E \alpha^*\alpha\, a \left\{ \cos(kr+\delta_0) + \frac{2(E-E_0)}{\Gamma} \sin(kr+\delta_0) \right\} e^{-iEt}$

I MAY HAVE TROUBLE WITH The + SIGN IN ψ^{III} BECAUSE I HAVE defined Γ TO be POSITIVE AND IT MAY TURN OUT THAT I SHOULD HAVE USED The NEGATIVE SIGN; WE'LL SEE.

INSIDE THE BARRIER, REGION I

Now I'd LIKE TO DISCUSS WHAT The TIME DEPENDENT WAVE FUNCTION LOOKS LIKE INSIDE THE BARRIER. The AMPLITUDE for The PARTICLE TO be INSIDE IS

$$\psi^I(r,t) = \sum_E |\alpha(E)|^2 e^{-iEt} f(r)$$

SUBSTITUTING FOR $|\alpha(E)|^2$ AND REARRANGING THE EXPONENTIAL

$$\psi^I(r,t) = \sum_E \frac{2}{La^2} \frac{\Gamma^2/4}{\left[(E-E_0)+\Gamma^2/4\right]} e^{-i(E-E_0)t} \cdot e^{-iE_0 t} f(r)$$

NOW WHAT THE HELL IS THE SUM OVER ALL ENERGY STATES? WELL, WE ARE GOING TO CHANGE TO AN INTEGRAL RATHER THAN A SUM; THIS IS A TRICK PHYSICIST PULL TO GET AWAY FROM WORKING OUT THE SUMMATION IN ORDER TO SWITCH TO AN INTEGRAL WE MUST SATISFY THE BOUNDARY CONDITIONS AT THE EDGE OF THE BOX, I.E., AT $r=L$. THEREFORE IN REGION III WE MUST REQUIRE THAT

$$\sin(kL + \delta(E)) = 0$$

WHICH GIVES THE CONDITION THAT

$$k_n L + \delta(E_n) = n\pi$$

AND THE k_n'S ARE RELATED TO THE ENERGY STATES BY

$$\frac{k_n^2}{2m} = E$$

WHEN THE ENERGY IS CHANGED FROM ONE STATE TO THE ADJACENT ONE, n IS INCREASED BY 1 WHILE δ SHIFTS BY AN AMOUNT $1/L$. THEREFORE, FOR LARGE L δ IS ALMOST CONSTANT. AN OTHER WAY TO SAY THAT IS FOR LARGE L THE STATES HAVE k VALUE DIFFERING BY π/L.

THUS THE SUMMATION CAN BE CHANGED TO AN INTEGRATION BY THE TRANSFORMATION

$$\Sigma_n = \int dk \frac{L}{\pi}$$

SINCE $dE = \frac{k\,dk}{m} = v\,dk$ WE CAN WRITE THIS AS

$$\Sigma_E = \int \left(\frac{L}{\pi v}\right) dE$$

WE CAN RETURN TO THE EXPRESSION FOR ψ^I AND WRITE

$$\psi^I(r,t) = \frac{2}{\pi a^2} \int \frac{dE}{v} \frac{\Gamma^2/4}{\left[(E-E_0)^2+\Gamma^2/4\right]} e^{-i(E-E_0)t} \cdot e^{-iE_0 t} f(r)$$

NOTICE THAT THE L DIMENSION DROPPED OUT AS WE EXPECTED. TO EVALUATE THIS INTEGRAL IT ONLY HAS A VALUE WHEN E IS ABOUT EQUAL TO E_0 SINCE Γ IS SO LARGE. FURTHER THE VELOCITY IS NEARLY CONSTANT SO THAT WE CAN TAKE IT OUT FRONT; WE ARE LEFT WITH THE FOLLOWING INTEGRAL TO EVALUATE,

$$I_1 = \int \frac{\Gamma^2/4\, d(E-E_0)\; e^{-i(E-E_0)t}}{(E-E_0)^2 + \Gamma^2/4}$$

I AM GOING TO EVALUATE THIS INTEGRAL BY USING CONTOUR INTEGRATION TECHNIQUES AS FOLLOWS. REWRITE THE INTEGRAL AS

$$I_1 = \int d(E-E_0) \frac{\Gamma/2}{\left(E-E_0+\frac{i\Gamma}{2}\right)} \cdot \frac{\Gamma/2}{\left(E-E_0+\frac{i\Gamma}{2}\right)} e^{-i(E-E_0)t}$$

SUBSTITUTE $\nu = E - E_0$ AND REWRITE

$$I_1 = \int d\nu \frac{\Gamma/2}{\left(\nu+\frac{i\Gamma}{2}\right)} \cdot \frac{\Gamma/2}{\left(\nu-\frac{i\Gamma}{2}\right)} e^{-i\nu t}$$

NOW BREAK THE INTEGRAL INTO TWO PARTS

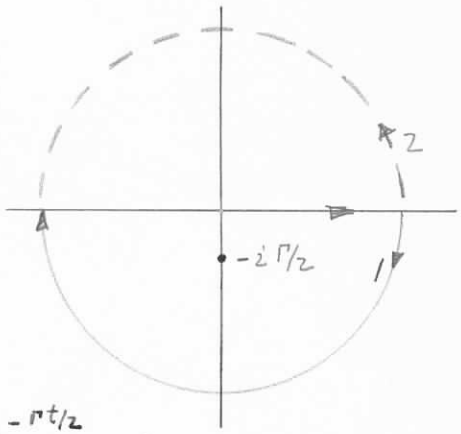
$$I_1 = I_{1a} + I_{1b} = i \int d\nu \left[\frac{\Gamma/2}{\nu+i\Gamma/2} - \frac{\Gamma/2}{\nu-i\Gamma/2}\right] e^{-i\nu t}$$

where

$$I_{1a} = i \int d\nu \frac{\Gamma/2\, e^{-i\nu t}}{\nu+i\Gamma/2}$$

AND

$$I_{1b} = -i \int d\nu \frac{\Gamma/2\, e^{-i\nu t}}{\nu-i\Gamma/2}$$



TO EVALUATE I_{1a} EXAMINE THE CONTOUR. THE INTEGRAND HAS A POLE AT $\nu = -i\Gamma/2$. THE INTEGRATION ABOUT PATH 2 GIVES A ZERO VALUE SINCE IT DOES NOT ENCLOSE A POLE. WHEN INTEGRATING OVER THE BOTTOM CONTOUR FOR TIMES GREATER THAN 0 THE VALUE OF THE INTEGRAL IS GIVEN BY

$$I_{1a} = \cancel{i}\left[2\pi i \cdot \text{SUM of the residues}\right]$$

$$= \cancel{i}\, 2\pi i \frac{\Gamma^2/4\, e^{-it(-i\Gamma/2)}}{-i\Gamma/2 - i\Gamma/2} = -\cancel{2}\pi e^{-\Gamma t/2} \quad \text{for } t > 0$$

THE SECOND INTEGRAL I_{1b} IS 0 FOR $t > 0$ WHICH IS THE PHYSICAL CONSTRAINT ON THE MATHEMATICS.

SO FINALLY WE FIND THAT

$$\psi^I(r,t) = \frac{\Gamma}{a^2 v_0} e^{-\Gamma t/2} e^{-iE_0 t} f(r)$$

THE PROBABILITY OF FINDING THE PARTICLE ANYWHERE INSIDE IS

$$P^I = |\psi^I|^2 = \left|\frac{\Gamma}{a^2 v_0}\right|^2 e^{-\Gamma t}$$

where THE INTEGRAL $\int f^2(r)\, dr = 1$. If WE REQUIRE THAT AT $t=0$ $P^I = 1$ THEN THE CONDITION REQUIRES

$$\frac{\Gamma}{a^2 v_0} = 1$$

where a defines THE DIMENSIONS OF THE OUTGOING WAVE.

WE CAN NOW INTERPERT Γ AS The MEAN LIFETIME of DISINTEGRATION OR MORE CORRECTLY

$$\frac{\Gamma}{\hbar} = \frac{1}{\tau}$$

where τ = The half LIFE = $\hbar/\Gamma$. When Γ IS SMALL THIS IMPLIES There is WEAK COUPLING BETWEEN The PARTICLE AND THE WELL. Thus The probability THAT IT GETS OUT OF The WELL IS high.

PARTICLE AMPLITUDE OUTSIDE THE BARRIER

WE NOW WANT TO ESTABLISH The EXPRESSION FOR The AMPLITUDE of FINDING The PARTICLE IN REGION III, i.e.,

$$\psi^{III}(r,t) = \sum_E |\alpha(E)|^2 a \left\{ \cos(kr+\delta_0) + 2\frac{(E-E_0)}{\Gamma}\sin(kr+\delta_0)\right\} e^{-iEt}$$

CONVERTING TO The INTEGRAL

$$\psi^{III}(r,t) = \int \frac{L\,dE}{\pi L a^2 v_0}\frac{2\,\Gamma^2/4}{\left[(E-E_0)^2+\Gamma^2/4\right]}\, a\left\{\cos(kr+\delta_0)+2\frac{(E-E_0)}{\Gamma}\sin(kr+\delta_0)\right\}e^{-iEt}$$

which CAN be REARRANGED AS

$$\psi^{III}(r,t) = \frac{2}{2a\pi v_0}\int \frac{dE\;\Gamma^2/4\cdot e^{-iEt}}{\left[(E-E_0)^2+\Gamma^2/4\right]}\;\frac{2}{\Gamma}\left\{\frac{\Gamma}{2}\cos(kr+\delta_0)+(E-E_0)\sin(kr+\delta_0)\right\}$$

NOW I'LL MAKE The SUBSTITUTIONS

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \qquad \sin x = -\frac{i}{2}\left(e^{+ix}-e^{-ix}\right)$$

Then The BRACKETED TERM becomes

$$\frac{\Gamma}{2}\left(\frac{e^{ix}+e^{-ix}}{2}\right) - \frac{(E-E_0)i}{2}\left(e^{+ix}-e^{-ix}\right) =$$

$$\frac{1}{2}\left\{ e^{ix}\left[\frac{\Gamma}{2} - i(E-E_0)\right] + e^{-ix}\left[\Gamma/2 + i(E-E_0)\right]\right\} =$$

$$\frac{1}{2}\left\{ -i\left[(E-E_0)+i\Gamma/2\right]e^{+i(kr+\delta_0)} + i\left[(E-E_0)-i\Gamma/2\right]e^{-i(kr+\delta_0)}\right\}$$

If I divide The BRACKET by $(E-E_0)^2+\Gamma^2/4 = (E-E_0+i\Gamma/2)(E-E_0-i\Gamma/2)$ I will END UP WITH

$$\psi^{III}(r,t) = \frac{\Gamma}{2a\pi v_0}\int dE\, e^{-iEt}\left[\frac{-i\;e^{i(kr+\delta_0)}}{(E-E_0-i\Gamma/2)} + \frac{i\;e^{-i(kr+\delta_0)}}{(E-E_0+i\Gamma/2)}\right]$$

This IS The SUM of TWO INTEGRALS WHICH I MUST EVALUATE, i.e.,

$$\psi^{III}(r,t) = \frac{\Gamma i}{2a\pi v_0}\left(-I_{IIIa} + I_{IIIb}\right)$$

I'LL STUDY THESE ONE AT A TIME SO FIRST LETS TAKE $I_{III\,A}$

$$I_{III\,A} = \int dE\, e^{-iEt} \left[\frac{-i\, e^{+i(kr+\delta_0)}}{(E-E_0 - i\Gamma/2)} \right]$$

THE ONLY TIME THIS INTEGRAL HAS A VALUE IS WHEN E IS ABOUT EQUAL TO E_0. OR SAID ANOTHER WAY $k \approx k_0$, I.E.,

$$E_0 = \frac{k_0^2}{2m} \quad \text{AND} \quad (k-k_0)\frac{k_0}{m} = E-E_0 \quad \text{OR} \quad (k-k_0) = \frac{E-E_0}{v}$$

SO ASSUMING $E \sim E_0$ THEN

$$I_{III\,A} = e^{-iE_0t}\, e^{i(k_0r+\delta_0)} \int \frac{d(E-E_0)\, e^{-i(E-E_0)t}\, e^{i(k-k_0)r}}{(E-E_0 - i\Gamma/2)}$$

REPLACE $E-E_0$ BY ν AND $k-k_0$ BY $E-E_0/v_0$ WE GET

$$I_{III\,A} = e^{-iE_0t}\, e^{i(k_0r+\delta_0)} \int \frac{d\nu}{\nu - i\Gamma/2}\, e^{-i\nu(t - r/v_0)}$$

NOW THE INTEGRAL HAS A POLE IN THE UPPER HEMISPHERE SO THE VALUE OF THE INTEGRAL FOR $t > 0$ IS

$$I_{III\,A} = e^{-iE_0t}\, e^{i(k_0r+\delta_0)}\, 2\pi i\, e^{\Gamma/2\,(t - r/v_0)}$$

FOR $t - r/v > 0$. IF $t - r/v$ IS < 0 THEN $I_{III\,A} = 0$. THE SECOND INTEGRAL CAN BE INTEGRATED THE SAME WAY BUT $I_{III\,b} = 0$ UNLESS $t + r/v < 0$. BUT SINCE t AND r ARE BOTH POSITIVE THIS CONDITION HAS NO WAY OF BEING SATISFIED. THE CONDITION THAT $t - r/v > 0$ IS A SIMPLE WAY OF SAYING YOU HAVE TO WAIT FOR A TIME t BEFORE THE PARTICLE CAN REACH YOU; UNTIL THEN YOU WON'T FIND IT.

THUS WE CAN WRITE

$$\psi^{III}(r,t) = \frac{-i}{2\pi a v_0}\, \Gamma \cdot 2\pi i\, e^{-iE_0t}\, e^{i(k_0r+\delta_0)}\, e^{-\Gamma/2\,(r/v_0 - t)}$$

$$\psi^{III}(r,t) = \frac{\Gamma}{a v_0}\, e^{-iE_0t}\, e^{i(k_0r+\delta_0)}\, e^{-\Gamma/2\,(r/v_0 - t)}$$

THE PROBABILITY OF FINDING THE PARTICLE IN RANGE dR IS

$$P(r)\,d(r) = (\psi^{III})^*(\psi^{III})\,dr = dr\, \frac{\Gamma^2}{a^2 v_0^2}\, e^{-\Gamma(r/v - t)}$$

SINCE $a^2 = \Gamma/v_0$

$$P(r)\,dr = \frac{\Gamma\, dr}{v_0}\, e^{-\Gamma(r/v - t)}$$

TO FIND THE PROBABILITY OF DETECTING THE PARTICLE OUTSIDE WE INTEGRATE TO FIND

$$P = \int_0^{tv} \frac{\Gamma}{v}\, dr\, e^{-\Gamma(r/v - t)} = e^{-\Gamma t}(e^{\Gamma t} - 1) = 1 - e^{-\Gamma t}$$

THUS THE PROBABILITY OF DETECTING THE PARTICLE FALLS EXPONENTIALLY WITH TIME.

21. SCHRÖDINGER'S EQUATION FOR THE HYDROGEN ATOM

Ref. VOL III
CHAPTER 19

LAST TIME WE ENDED UP DISCUSSING BRIEFLY how WE COULD EXTEND The PENETRATION of A BRRRIER TO The PROBLEM of SCATTERING from A barrier or POTENTIAL. Before I GET TOO far ALONG ON This TOPIC I WANT TO DISCUSS The Three dIMENSIONAL SOLUTION TO The Schrodinger EQUATION.

I WILL CONSIDER FIRST THE COULOMB POTENTIAL CREATED BY A HYDROGEN NUCLEUS. The STRENGTh of The POTENTIAL VARIES AS

$$V(r) = -\frac{e^2}{r}$$

The PROBLEM IS EASILY EXTENDABLE TO OTher ATOMS WHICH hAVE COULOMb FIELDS VARYING AS $-Ze^2/r$. WE WILL CONSIDER The CASE where The PARTICLES ARE ALL BOUND TO This SPherICALLY SYMMETRIC POTENTIAL.

THE SCHRÖDINGER EQUATION for This PROBLEM WOULD be EXPRESSED AS

$$H\Psi = E\Psi$$

Where WE ARE LOOKING for The definite ENERGY STATES which ARE The SOLUTION

$$\Psi = e^{-i\frac{Et}{\hbar}}\,\psi(\vec{r})$$

The HAMILTONIAN for This PROBEM IS GIVEN by

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r} \qquad \text{where } e^2 = \frac{q^2}{4\pi\epsilon_0}$$

Therefore, WE MUST SOLVE The 3-DIMENSIONAL DIfferENTIAL EQUATION

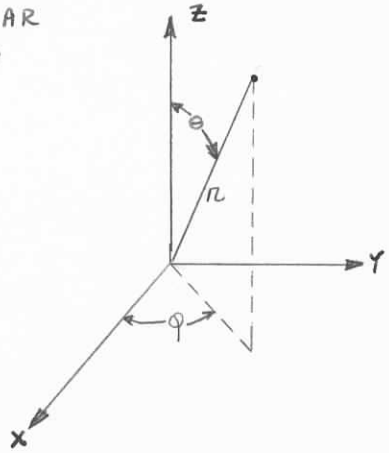
$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{-e^2}{r}\right)\psi(\vec{r}) = E\,\psi(\vec{r})$$

When YOU first LOOK AT This EQUATION YOU MIGHT LOGICALLY GUESS THAT The ANSWER depends JUST ON r but This IS NOT TRUE. IN OTher words JUST BECAUSE The POTENTIAL IS SPherICALLY SYMMETTIC DOESN'T MEAN The ANSWER MUST be LIKEWISE. WE MUST WRITE The differENTIAL EQUATION OUT IN TERMS of SpherICAL COORDINATES.

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right] = E\,\psi(\vec{r})$$

To GET THIS EQUATION from The LAPLACIAN IN RECTANGULAR COORDINATES The following TRANSFORMATION IS USED

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$



WE would LIKE TO STUDY The SOLUTIONS TO The The ABOVE EQUATION for SEVERAL DIfferENT CASES.

SPHERICALLY SYMMETRIC SOLUTION

NOW WE SAID IN GENERAL THAT A SPHERICALLY SYMMETRIC POTENTIAL DID NOT HAVE A SPHERICALLY SYMMETRIC SOLUTION. HOWEVER, IT IS POSSIBLE THAT THE ANSWER COULD HAVE THIS PROPERTY IF IT DID NOT DEPEND ON θ OR φ. IF THIS WERE THE CASE THEN WE HAVE TO SOLVE THE EQUATION

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(r\psi) + \frac{e^2}{r}\psi(r) = E\psi(r)$$

TO SOLVE THIS EQUATION WE MIGHT TRY THE SOLUTION

$$\psi = e^{-\alpha r}$$

DIFFERENTIATING THIS TWICE AS REQUIRED WE HAVE THAT

$$\frac{d^2\psi}{dr^2} = -2\alpha e^{-\alpha r} + \alpha^2 r e^{-\alpha r}$$

Therefore, $$\frac{1}{r}\frac{d^2\psi}{dr^2} = \left(-\frac{2\alpha}{r} + \alpha^2\right)e^{-\alpha r}$$

THE SCHRÖDINGER EQUATION BECOMES

$$-\frac{\hbar^2}{2m}\left(-\frac{2\alpha}{r} + \alpha^2\right)\psi - \frac{e^2}{r}\varphi = E\varphi$$

IN ORDER FOR THIS EQUALITY TO HOLD THE $1/r$ TERMS MUST CANCEL, I.E.,

$$\frac{\hbar^2\alpha}{m} = e^2 \qquad \text{OR} \qquad \alpha = \frac{me^2}{\hbar^2}$$

THE ENERGY STATES ARE GIVEN BY

$$E = -\frac{\hbar^2}{2m}\alpha^2 = -\frac{me^4}{2\hbar^2}$$

FOR ATOMS LARGER THAN HYDROGEN THE ENERGY IS GIVEN AS

$$E = -Z^2\frac{me^4}{2\hbar^2}$$

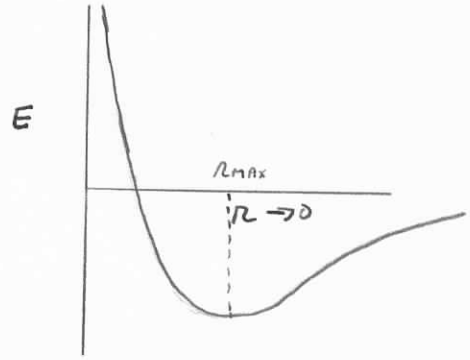
WE HAVE THUS FOUND THE LOWEST ENERGY STATE OF HYDROGEN. IF IN THE SOLUTION $\psi = e^{-\alpha r}$ WE IDENTIFY α AS $1/r_0$ SOME INVERSE LENGTH THE $r_0 = \frac{\hbar^2}{me^2}$. THIS IS COMMONLY CALLED THE BOHR RADIUS. IT IS THE RADIUS OF THE HYDROGEN, I.E., THE DISTANCE FROM THE NUCLEUS THAT THE COULOMB FIELD DROPS $1/e$. THE NUMERICAL VALUE OF $r_0 = .528$ Å (ANGSTROMS). THE ENERGY ASSOCIATED WITH THIS LOWEST STATE IS

$$E = -\frac{me^4}{2\hbar^2} = -13.6\,\text{eV}$$

WHICH IS CALLED ONE RYDBERG. THE FACT THAT IT IS NEGATIVE IMPLIES THAT THE PARTICLE IS BOUND TO THE NUCLEUS. THEREFORE ONE RYDBERG IS THE AMOUNT OF ENERGY NEEDED TO LIFT THE ELECTRON FROM THE BOUND STATE TO THE FREE STATE. FOR ATOMS WITH BIGGER ATOMIC NUMBERS Z^2 TIMES THIS ENERGY IS REQUIRED TO FREE THE ELECTRON.

I'd LIKE TO EXPLAIN WHAT I MEAN BY THE "SIZE" OF THE ATOM. WHY CAN'T THE ELECTRON GET ANY CLOSER TO THE NUCLEUS THAN A DISTANCE r_0? WE SUPPOSE SOMEHOW IT DID AND ACQUIRED A POTENTIAL ENERGY GIVEN BY $-e^2/b$. THAT IS, YOU NOW KNOW THE ELECTRON IS DEFINITELY WITHIN A RADIUS b OF THE NUCLEUS. IN ADDITION TO POTENTIAL ENERGY IT HAS A KINETIC ENERGY GIVEN BY $K.E = P^2/2M$. SINCE THE MOMENTUM P IS RELATED TO THE DISTANCE b THROUGH $\hbar$, I.E., $P = \hbar/b$ WE HAVE A TOTAL ENERGY STATE OF THE ELECTRON GIVEN BY

$$E = \frac{\hbar^2}{2mb} - \frac{e^2}{b}$$



THE FUNCTION E LOOKS LIKE THE PICTURE ON THE RIGHT. AS SEEN BY THE CURVE THERE IS SOME OPTIMAL RADIUS WHERE E IS A MINIMUM. TO FIND THAT POINT LET'S DIFFERENTIATE E WITH RESPECT TO b. AND SET EQUAL TO ZERO

$$\frac{dE}{db} = -\frac{\hbar^2}{2mb_0} + \frac{e^2}{b_0^2} = 0$$

SOLVING FOR b_0 WE FIND THAT

$$b_0 = \frac{\hbar^2}{me^2} = \text{bohr radius}$$

THUS THE UNCERTAINTY PRINCIPLE PREDICTS THE SAME SIZE FOR THE ATOM THAT THE SCHRODINGER EQUATION YIELDS. IT BETTER OR SOMETHINGS WRONG WITH QUANTUM MECHANICS.

SOLUTIONS WITH ANGULAR DEPENDENCE

NOW I'd LIKE TO COME BACK TO THE GENERAL ANALYSIS AND CONSIDER THE CASE WHERE THE SOLUTION TO SCHRODINGER'S EQUATION IS IN GENERAL A FUNCTION OF θ, AND φ., I.E., $\psi = \psi(r, \theta, \varphi)$. THINK FOR THE MOMENT WHAT WOULD HAPPEN IF ψ HAD A PARTICULAR DEPENDENCE SO THAT WHEN IT WAS TWICE DIFFERENTIATED THE SAME FUNCTIONAL DEPENDENCE WAS PRODUCED. IN OTHER WORDS WE CAN WRITE ψ AS THE PRODUCT OF TWO FUNCTIONS

$$\psi = Y(\theta, \varphi)\, f(r)$$

IF WE SUBSTITUTE THIS SOLUTION INTO THE SCHRODINGER EQUATION WE HAVE THAT

$$-\frac{\hbar^2}{2m}\left[Y(\theta,\varphi)\frac{1}{r}\frac{\partial^2}{\partial r^2} r f(r) + \frac{f(r)}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right\}\right] = \left[E - V(r)\right] f(r)\, Y(\theta,\varphi)$$

NOW WOULDN'T IT BE MARVELOUS IF BY SOME FREAKISH ACCIDENT OF LIFE THAT WE WERE SUCCESSFUL IN SEARCHING DILIGENTLY TO FIND A FUNCTION $Y(\theta,\varphi)$ THAT WHEN OPERATED ON BY OPERATOR $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$ WOULD ALWAYS PRODUCE A CONSTANT? IN OTHER WORDS WE HAVE THAT

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial Y}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} = -K\, Y(\theta,\varphi)$$

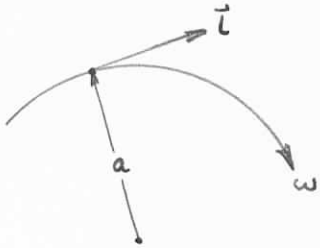
ASSUMING for The MOMENT WE found such A FUNCTION THEN WE HAVE A SOLUTION TO The EQUATION

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}(rF) + \left[V(r) + \frac{\hbar^2 K}{2mr^2}\right]F = EF(r)$$

which WE MULTIPLY by $Y(\theta,\varphi)$ from The previous EQUATION TO GET TO ANSWER $\psi(r,\theta,\varphi)$. If YOU LET ME GET BY WITHOUT QUESTIONING The MATHEMATICAL INTEGRITY of The ABOVE REARRANGEMENTS of TERMS, I'd LIKE TO EXPLAIN WHY The ABOVE EQUATION IS JUST LIKE The SPHERICALLY SYMMETRIC CASE EXCEPT for The Added POTENTIAL. IT REALLY DOESN'T BOTher ME ANY MORE TO PULL These MATHEMATICAL SHENANIGANS because I CAN'T Remember how TO PROVE WHAT I did. I JUST KNOW THAT deep INSIDE of ME IT IS OKAY SO LET ME off The hook.

TO EXPLAIN This Added POTENTIAL ENERGY I'LL APPEAL TO A CLASSICAL ANALOG SO The TERM WON'T APPEAR SO MYSTERIOUS. IMAGINE A PARTICLE MOVING AROUND SOME CENTER of FORCE. The PARICLE HAS SOME ANGULAR MOMENTUM L GIVEN AS

$$L = mva = ma^2\omega$$



The ENERGY The PARTICLE HAS DUE TO ITS ANGULAR MOMENTUM IS

$$\frac{1}{2}mv^2 = \frac{m}{2}\left(\frac{L}{ma}\right)^2 = \frac{L^2}{2ma^2}$$

SINCE The ANGULAR MOMENTUM MUST be CONSERVED L IS A CONSTANT of The MOTION SO There IS INdeed AN EXTRA ~~E~~ force ACTING DUE TO The CENTRIFUGAL POTENTIAL. The TOTAL POTENTIAL IS Therefore GIVEN by

$$V(a) + \frac{L^2}{2ma^2}$$

The POSITIVE SIGN INdicates A REPULSIVE force. If WE HAVE NO ANGULAR MOMENTUM, WE GET The $1/a$ TYPE POTENTIAL BACK.

NOW LOOK AT THIS POTENTIAL AND COMPARE IT TO The bracketed QUANTITY ABOVE, i.e.,

$$V(a) + \frac{L^2}{2ma^2} \;\vdots\; V(r) + \frac{\hbar^2 K}{2mr^2}$$

SINCE a AND r ARE The SAME The POTENTIALS ARE IDENTICAL if WE EQUATE The ANGULAR MOMENTUM, L, WITH $\hbar K$. WE'LL COME BACK TO further AMPLIFY THIS EQUALITY AND INTERPRET K AS ANGULAR MOMENTUM IN UNITS of $\hbar$.

NOW WE OUGHT TO RETURN TO The FUNCTION $Y(\theta,\varphi)$ AND See if WE CAN UNDERSTAND how SUCH A FUNCTION COULD SATISFY The COMPLICATED differential EQUATION,

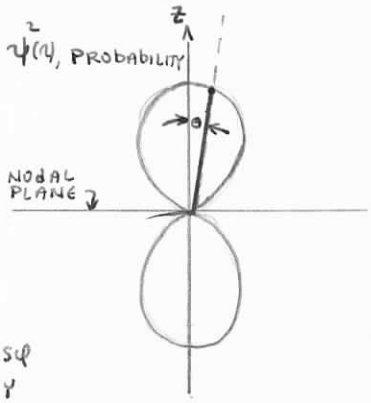
$$\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} = -K\,Y(\theta,\varphi)$$

WE SHOULD BE SMART ENOUGH TO BE ABLE TO GUESS AT A CORRECT SOLUTION TO THIS EQUATION. SUPPOSE WE CONSIDER FIRST A θ DEPENDENCE AND TRY $Y(\theta,\varphi) = \cos\theta$. WELL PUT IT IN THE EQUATION -

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(-\sin\theta\sin\theta\right) = -K\cos\theta$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta} = -K\cos\theta$$

OR $K=2$. SEE, THAT'S NOT SO HARD TO DO! IF WE PLOT THE SOLUTION $\psi(r,\theta) = f(r)\cos\theta$ OR BETTER LET'S PLOT THE PROBABILITY $\psi^2(r)$ WE GET A CURVE LIKE THE ONE HERE.



THE CURVE INDICATES THE STRENGTH OF EACH SOLUTION OR A RELATIVE PROBABILITY OF FINDING THE PARTICLE AT SOME ANGLE FROM THE Z.

NOW IF YOU USE YOUR BRAIN RATHER THAN YOUR KNOWLEDGE YOU CAN PROBABLY DEDUCE SOME OTHER SOLUTIONS. THE FIRST OBVIOUS ONES WOULD COME FROM A ROTATION OF THE NODAL PLANE SO THAT IT LIES ON THE X OR Y AXIS. IN OTHER WORDS $\sin\theta\cos\varphi$ AND $\sin\theta\sin\varphi$ ARE ALSO SOLUTIONS. THIS IMMEDIATELY FOLLOWS FROM THE INITIAL ARBITRARY CHOICE OF THE Z AXIS, I.E., $Y = \cos\theta$. FOR THIS CASE WE HAVE THE THREE SOLUTIONS

$$\psi(r,\theta,\varphi) = \frac{f(r)}{r}\underbrace{(r\cos\theta)}_{z}, \quad \frac{f(r)}{r}\underbrace{(r\sin\theta\cos\varphi)}_{x}, \quad \frac{f(r)}{r}\underbrace{(r\sin\theta\sin\varphi)}_{Y}$$

EACH OF THESE SOLUTIONS HAVE A K-VALUE OF 2.

IF WE NOW CONTINUED TO USE OUR BRAIN TO PREDICT MORE SOLUTIONS, WE WOULD CONCLUDE THERE MUST BE AN INFINITE NUMBER OF SOLUTIONS TO THE DIFFERENTIAL EQUATION BECAUSE THE NODAL PLANE CAN BE ROTATED THROUGH 360°. HOWEVER, WE MUST REMEMBER THAT WHEN EVER 2 SOLUTIONS GIVE DIFFERENT ENERGIES THE MUST SATISFY THE ORTHOGONALITY CONDITION

$$\int \psi_n^* \psi_m \, d\vec{R} = 0$$

WHEN WE APPLY THIS CONDITION WE FIND THAT THERE ARE ONLY THREE MUTUALLY ORTHOGONAL SOLUTIONS.

IN THE EVENT THAT THERE ARE MORE THAT ONE SOLUTION FOR THE SAME ENERGY, I.E.,

$$H\psi_a = E_0\psi_a \quad \text{AND} \quad H\psi_b = E_0\psi_b$$

THE ENERGY STATE IS DEGENERATE AND THERE IS ALWAYS ANOTHER SOLUTION GIVEN BY THE SUM OF ψ_a AND ψ_b, I.E.,

$$H(\psi_a+\psi_b) = E_0(\psi_a+\psi_b.$$

NOW THAT WE ARE SO DAMN SMART IN GUESSING SOLUTIONS WE MIGHT CONTINUE ON A LITTLE BIT. WE HAVE CHOSEN AN ORTHOGONAL SET OF "STATES" TO REPRESENT THE SOLUTIONS IN, I.E., X, Y, AND Z SO LET'S SEE IF THE PRODUCTS OF THESE STATES ARE ALSO SOLUTION. THUS FOR THE SECOND CASE WE HAVE A POSSIBILITY OF NINE MORE SOLUTIONS,

$$X^2, Y^2, Z^2, XY, XZ, YZ, YX, ZX, ZY \rightarrow X^2, Y^2, Z^2, XY, XZ, YZ$$

BUT THE LATER SIX ARE REALLY JUST THREE DUE TO THE COMMUTATIVE NATURE OF X, Y, AND Z. THE SIX POSSIBILITIES ARE THEREFORE

$$\begin{aligned}
XY &= R^2 \sin^2\theta \sin\varphi \cos\varphi \\
XZ &= R^2 \sin\theta \cos\theta \cos\varphi \\
YZ &= R^2 \sin\theta \cos\theta \sin\varphi \\
X^2 &= R^2 \sin^2\theta \cos^2\varphi \\
Y^2 &= R^2 \sin^2\theta \sin^2\varphi \\
Z^2 &= R^2 \cos^2\theta
\end{aligned}$$

SOLVING FOR K WE FIND THAT IT EQUALS 6 FOR ALL THE ABOVE BUT Z^2. THE REASON WE DON'T GET A SOLUTION FOR Z^2 IS THAT IS DOES NOT HAVE A ZERO AVERAGE VALUE. INSTEAD YOU CAN SHOW THAT THE AVERAGE VALUE OF $\overline{Z^2}$ IS JUS $1/3\, r^2$ BECAUSE OF THE SYMMETRIC OF X^2; Y^2, AND Z^2. THUS WE COULD USE THE SOLUTION $3Z^2 - r^2 = r^2(3\cos^2\theta - 1)$. WE ALSO FIND THAT $X^2 - Y^2$ IS A SOLUTION WHILE X^2 AND Y^2 ARE NOT INDEPENDENT SOLUTIONS FOR THE REASONS JUST DESCRIBED. THE COMPLETE LIST OF K, 6 SOLUTIONS CAN BE WRITTEN AS

$$\begin{aligned}
\psi &= \frac{f(r)}{2r^2} \underbrace{r^2 \sin^2\theta \sin 2\varphi}_{XY} \\
&= \frac{f(r)}{2r^2} \underbrace{r^2 \cos\varphi \sin 2\theta}_{XZ} \\
&= \frac{f(r)}{2r^2} \underbrace{r^2 \sin\varphi \sin 2\theta}_{YZ} \\
&= \frac{f(r)}{r^2} \underbrace{r^2 \sin^2\theta \cos 2\varphi}_{X^2 - Y^2} \\
&= \frac{f(r)}{r^2} \underbrace{r^2 (3\cos^2\theta - 1)}_{2Z^2 - X^2 - Y^2}
\end{aligned}$$

IF I WERE TO KEEP GOING COMPUTING TRIPLE PRODUCTS, ELIMINATING THE NON-REDUNDANT CASES, AND THE NON-ZERO AVERAGE VALUE PRODUCT TERMS, I WOULD FIND 7 $Y(\theta, \varphi)$'S AND THEIR K VALUE WOULD BE 12. IF l IS THE CASE NUMBER THEN I WILL FIND $2l+1$ FUNCTIONS HAVING A K VALUE OF $l(l+1)$.

SOMETIMES IT IS MORE CONVENIENT TO COMBINE THE $Y(\theta,\varphi)$'s IN EXPONENTIAL NOTATION AS FOLLOWS

$$\sin^2\theta\, e^{2i\varphi}, \quad \sin 2\theta\, e^{i\varphi}, \quad (3\cos^2\theta - 1)\, e^{0i\varphi}, \quad \sin 2\theta\, e^{-i\varphi}, \quad \sin^2\theta\, e^{-2i\varphi}$$

I CAN THEREFORE IDENTIFY AN EXPONENTIAL NUMBER m TO GENERALIZE THIS REPRESENTATION WHERE $m = 2, 1, 0, -1, -2$ FOR THIS CASE. THUS m TAKES ON THE VALUES $-\ell \le m \le \ell$

WHAT WE REALLY HAVE CONSTRUCTED IN A REALLY SLOPPY WAY ARE SPHERICAL HARMONICS $Y_{\ell,m}(\theta,\varphi)$ WHICH ARE GIVEN AS

$$Y_{\ell,m}(\theta,\varphi) = A\, P_\ell^{|m|}(\cos\theta)\, e^{im\theta} \qquad A = \frac{2}{\sqrt{2\ell+1}}$$

WHERE A IS SOME NORMALIZING FACTOR. THE FUNCTIONS $P_\ell^{|m|}(\cos\theta)$ ARE CALLED THE ASSOCIATED LEGENDRE POLYNOMINALS.

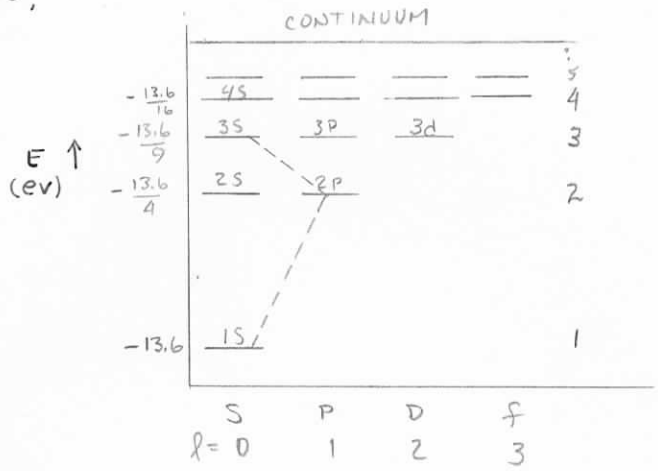
WE NOW SHOULD ATTACH PHYSICAL MEANING TO ℓ AND m BECAUSE THEY DO HAVE A SIGNIFICANT INTERPERTATION. ℓ IS THE ORBITAL ANGULAR MOMENTUM WHILE m IS THE z-COMPONENT OF THE ORBITAL ANGULAR MOMENTUM. THE VARIOUS VALUES OF m AND ℓ HAVE BEEN GIVEN NAMES SO WE CAN ESTABLISH THE DICTIONARY OF TERMS HERE

ℓ	m	ANGULAR DEPENDENCE OF AMPLITUDE	NUMBER OF STATES	K	NAME
0	0	1	1	0	S-WAVE {SPHERICALLY SYMMETRIC
1	+1	$-\frac{1}{\sqrt{2}}\sin\theta\, e^{i\varphi}$	3	2	P-WAVE {1 NODAL PLANE
	0	$\cos\theta$			
	-1	$\frac{1}{\sqrt{2}}\sin\theta\, e^{-i\varphi}$			
2	2	$\frac{\sqrt{6}}{4}\sin^2\theta\, e^{2i\varphi}$	5	6	d-WAVE {2 NODAL PLANE
	1	$\frac{\sqrt{6}}{2}\sin\theta\cos\theta\, e^{i\varphi}$			
	0	$\frac{1}{2}(3\cos^2\theta - 1)$			
	-1	$-\frac{\sqrt{6}}{2}\sin\theta\cos\theta\, e^{-i\varphi}$			
	2	$\frac{\sqrt{6}}{4}\sin^2\theta\, e^{-2i\varphi}$			
3			7	12	f
4			9	20	G
5			11	30	H
			⋮	⋮	⋮
			$(2\ell+1)$	$\ell(\ell+1)$	

SO FAR WE HAVE JUST DESCRIBED THE WAVE FUNCTION OF HYDROGEN for THE CASE $\ell = 0$, $m = 0$, I.E., THE SPHERICALLY SYMMETRIC CASE. THIS IS THE STATE where THE ELECTRON HAS NO ANGULAR MOMENTUM. BUT WE HAVE JUST SHOW THAT There EXISTS A WHOLE SET of SOLUTIONS DEPENDING ON THE VALUES of ℓ AND m. THE COMPLETE SOLUTION TO THE PROBLEM IS EXPRESSIBLE AS

$$\psi_{n,\ell,m} = Y_{\ell,m}(\theta,\phi)\, F_{n,\ell}(r)$$

where n IS THE PRINCIPLE QUANTUM NUMBER. The reference DEVELOPES THE MATHEMATICS IN MORE DETAL BUT A PICTURE of THE COMPLETE hydrogen ENERGY LEVEL,



TRANSITIONS bETWEEN THE VARIOUS LEVEL CAN OCCUR AND when THEY DO A CHARACTERISTIC FREQUENCY IS EMITTED which REPESENTS THE ENERGY LOST IN THE JUMP. OF COURSE ENERGY COULD BE ABSORBED AND MOVE TO A HIGH STATE.

22. INTRODUCTION TO SCATTERING THEORY

LAST TIME WE SAW how The SOLUTION TO The schrödinger EQUATION for The HYDROGEN CAN be EXPRESSED AS The PRODUCT of TWO FUNCTIONS $Y_{\ell,m}(\theta,\varphi)\, f_\ell(r)$, i.e.,

$$\psi_{\ell m}(r,\theta,\varphi) = Y_{\ell,m}(\theta,\varphi)\, f_\ell(r)$$

SCHRÖDINGER'S EQUATION for which THIS IS A SOLUTION IS GIVEN by

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\,\psi(r,\theta,\varphi) = E\psi$$

AND The $Y_{\ell,m}$'s SATISFIED The DIFFERENTIAL EQUATION

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial Y_{\ell,m}}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{\ell m}}{\partial\theta^2} = \ell(\ell+1)\,Y_{\ell m}$$

AND $f_\ell(r)$ SATISFIES The EQUATION

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(r f_\ell) + \left[V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right] f_\ell(r) = E f_\ell(r)$$

I WANT TO POINT OUT SOME HANDY THINGS ABOUT These FUNCTIONS AND SOLUTIONS. The $Y_{\ell,m}$'s ARE GIVEN BY

$$Y_{\ell,m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^{\,m}(\cos\theta)\, e^{im\varphi}$$

The $m=0$ $Y_{\ell,m}$ IS A SPECIAL SOLUTION,

$$Y_{\ell,0}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\; P_\ell(\cos\theta)$$

A LOT of TIMES WE WILL EXPAND FUNCTIONS IN TERMS of The $Y_{\ell,m}$'s LIKE The FUNCTION

$$\frac{1}{\sqrt{1-2r\cos\theta+r^2}} = \sum_{\ell=0}^{\infty} r^\ell P_\ell(\cos\theta)$$

SEVERAL PROPERTIES of The $Y_{\ell,m}$'s which WE WILL ALSO USE IS The NORMALIZATION AND ORTHOGONALITY RELATIONSHIP

$$\int_0^\pi\int_0^{2\pi} Y^*_{\ell'm'}(\theta,\varphi)\, Y_{\ell,m}(\theta,\varphi)\, d\varphi\, \sin\theta\, d\theta = \delta_{mm'}\,\delta_{\ell\ell'}$$

when The SOLUTION IS SYMMETRIC ABOUT The z AXIS

$$\int_0^\pi Y^*_{\ell'm}\, Y_{\ell m}(\theta,\varphi)\, \sin\theta\, d\theta = 2\pi\,\delta_{mm'}$$

IT IS POSSIBLE TO EXPAND A FUNCTION ON The SURFACE of A UNIT Sphere IN TERMS of The $Y_{\ell m}$'s, i.e.,

$$f(\theta,\varphi) = \sum a_{\ell m}\, Y_{\ell,m}(\theta,\varphi)$$

WE HAVE ALREADY MENTIONED how TO SOLVE for The $a_{\ell,m}$'s by The following RULE

$$a_{\ell,m} = \iint Y^*_{\ell,m} \; F(\theta,\varphi) \sin\theta \, d\theta \, d\varphi$$

This is ANALOGOUS TO EXPANDING A function IN TERMS of A fourier SERIES ON A LINE BETWEEN 0 AND 1. FOR The spherical NATURE of The EXPANSIONS These ARE CALLED spherical HARMONICS.

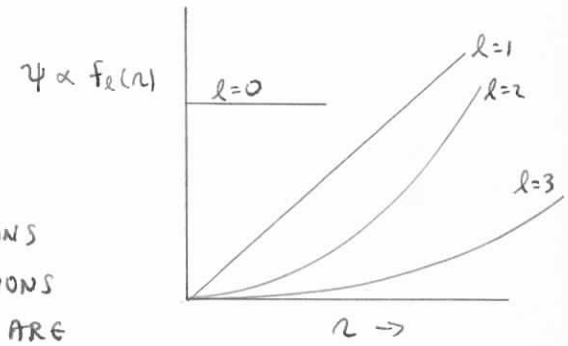
ONE MORE INTERESTING FACT which I'd LIKE TO POINT OUT About These SOLUTIONS IS The behavior of The f_ℓ's NEAR The ORIGIN. SINCE The POTENTIAL ENERGY TERMED IS GOVERNED by The CENTRIFUGAL POTENTIAL when The radius GETS CLOSE TO ZERO, i.e., $V(r) \propto 1/r$ VERSUS $1/r^2$. Thus $f_\ell(r)$ MUST SATISFY The ~~SOLUT~~ EQUATION

$$-\frac{1}{r}\frac{d^2}{dr^2} f_\ell(r) + \frac{\ell(\ell+1)}{r^2} f_\ell(r) = 0$$

The RIGHTHAND SIDE IS ZERO BECAUSE The E'S ARE CONSTANT AND BECOME NEGLIGIBLE IN The LIMIT. I'LL TRY The SOLUTION $f_\ell(r) = r^\sigma$ AND UPON DIFFERENTIATING

$$-\sigma(\sigma+1) r^{\sigma-2} + \ell(\ell+1) \frac{r^\sigma}{r^2} = 0$$

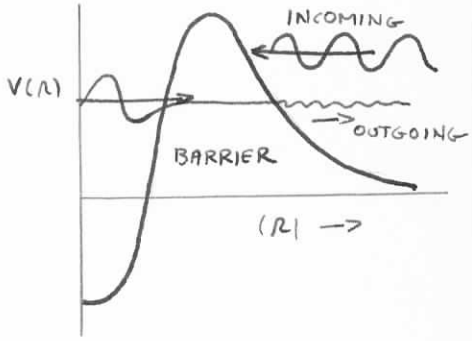
OR The EQUALITY $\ell(\ell+1) = \sigma(\sigma+1)$ which GIVES The RESULTS $\sigma = \ell$ AND $\sigma = -(\ell+1)$. The SOLUTION $r^{-(\ell+1)}$ DOES NOT TURN OUT TO BE PHYSICALLY ALLOWED IN QUANTUM MECHANICS BECAUSE IT CANNOT BE NORMALIZED OVER ALL SPACE, AND STILL BE FINITE. Thus The behavior of $f_\ell(r)$ NEAR The ORIGIN IS GIVEN by r^ℓ. The AMPLITUDES for The first few VALUES of ANGULAR MOMENTUM ARE,



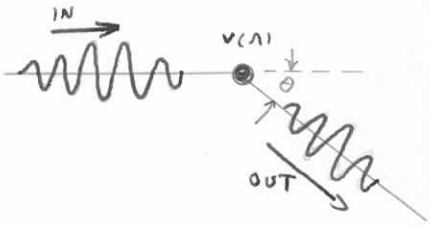
NOTICE AS The ANGULAR MOMENTUM GETS higher WITH CORRESPONDINGLY higher VALUES of ℓ The LIKELYhood of find The PARTICLE NEAR The ORIGIN DECREASES RAPIDLY. The reason The functions r^ℓ ARE COMBINED WITH The ANGULAR FUNCTIONS IS TO Produce NICE SMOOTH FUNCTIONS THAT ARE WELL behaved LIKE, $r\cos\theta = z$.

23. SCATTERING THEORY

WE ARE NOW READY TO START THE PROBLEM WE INTENDED TO DO SOME TIME BACK. WE WORKED ON PENETRATION OF A COULOMB BARRIER WHERE THE WAVE WAS INCIDENT ON THE BARRIER FROM THE "INSIDE", I.E., THE PARTICLE WAS TRAPPED IN THE POTENTIAL. THE INVERSE PROBLEM IS TO STUDY THE BEHAVIOR OF A WAVE INSIDENT ON THE OUTSIDE OF THE BARRIER AND TO SEE WHAT HAPPENS TO IT.



THE SIMPLEST EXAMPLE TO WORK OUT IN SCATTERING PROBLEM IS THE CASE OF SCATTERING FROM A SPHERICALLY SYMMETRIC COULOMB POTENTIAL BY A WAVE PACKET AND CALCULATE THE PROBABILITY THAT THE SCATTERED WAVE COMES OUT AT SOME ANGLE. THE SIDE DRAWING DEPICTS THIS CASE.

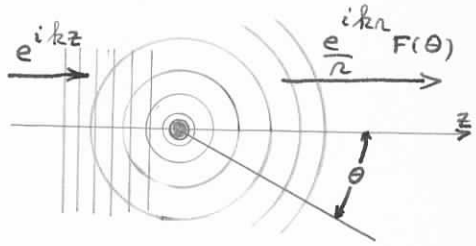


IN GENERAL THIS PROBLEM WOULD REQUIRE WORKING WITH TIME DEPENDENT WAVE FUNCTIONS AND THEREFORE GET QUITE COMPLICATED. HOWEVER, IT IS POSSIBLE TO SOLVED THE PROBLEM FOR A SPECIAL CASE. THE SPECIAL CASE DEPENDS ON THE NATURE OF THE INCOMING WAVE PACKET AND TO SOLVE THE PROBLEM IT IS NECESSARY THE WAVE PACKET IN VERY LONG AND HAS A LARGE AMPLITUDE. IN OTHER WORDS IN THE LIMIT WE ARE CONSIDERING A PLANE WAVE INCIDENT ON A SPHERICAL POTENTIAL. BECAUSE THE WAVELENGTH OF THE PLANE WAVE IS LARGE THE ENERGY OF THE PARTICLE-WAVE IS NEARLY DEFINITE AND GIVEN BY $E = k^2/2m$. BUT BE CAREFUL, WE ARE NOT TALKING ABOUT A "BEAM" OF INCOMING PARTICLE.

NOW WHEN THIS PLANE-WAVE OR WAVE PACKET PASSES OVER, THROUGH, OR AROUND THE SPHERICAL POTENTIAL OF THE HYDROGEN ATOM OR WHATEVER, THE INTERACTION SETS UP OUTGOING SPHERICALLY SYMMETRIC WAVES. THE PROBLEM WE WILL WORK OUT WILL CONSIDER THE SCATTERING CENTER AT REST SO THE INCIDENT WAVE WILL NOT TRANSFER ANY ENERGY TO THE ATOM, I.E., IT WILL NOT JIGGLE IT OR SET UP INTERNAL MOTIONS IN THE NUCLEUS.

WE HAVE SET UP THE PROBLEM SO THAT WE CAN WRITE DOWN THE ASYMPTOTIC SOLUTION of THE WAVE FUNCTION AS THE WAVE GETS VERY far AWAY from THE SCATTERING CENTER. THAT IS THE WAVE QUITES INTERACTING AND SETTLES DOWN INTO A STEADY STATE SOLUTION AS THE RADIAL DISTANCE from THE ATOM GETS VERY LARGE. THUS WE WRITE

$$\psi(r,\theta) \underset{r\to\infty}{=} e^{ikz} + \frac{e^{ikr}}{r} F(\theta)$$



The first TERM REPRESENTS THE INCOMING PLANE WAVE. The SECOND TERM IS THE OUTGOING SPHERICAL WAVE which IS SYMMETRIC ABOUT THE Z AXIS SO There IS NO φ dependence. The MOMENTUM or k IS THE SAME for both WAVES SINCE WE HAVE ASSUMED NO RECOIL or ENERGY TRANSfer TO THE ATOM. THUS WE WILL CONSIDER ONLY ELASTIC SCATTERING WITH AN AMMOVABLE SCATTERER. SINCE THE POTENTIAL $V(r)$ IS LIMITED IN RANGE THE ABOVE SOLUTION IS VALID IN LIMIT for LARGE r.

NOW WE MUST EXPLOIT OUR NEW KNOWLEDGE IN HOW TO EXPRESS FUNCTIONS IN TERMS of THE spherical HARMONICS. BECAUSE WE HAVE CONSIDERED THE CASE where ψ IS SYMMETRIC ABOUT THE Z AXIS WE CAN WRITE,

$$\psi(r,\theta) = \sum_{l=0}^{\infty} C_l f_l(r) P_l(\cos\theta)$$

where THE FUNCTIONS $f_l(r)$ SATISFY THE differENTIAL EQUATION

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(r f_l) + \left[V(r) + \frac{\hbar l(l+1)}{2mr^2}\right] f_l(r) = E f_l(r)$$

IN order TO SOLVE for THE C_l's IN THE ABOVE EXPANSION IT IS NECESSARY TO COMPUTE $f_l(r)$. IN PARTICULAR WE ARE INTERESTED IN $f_l(r)$ far OUT. TAKING A CLUE from THE PREVIOUS PROBLEM of BARRIER PENETRATION WE WILL GUESS THE form of $f_l(r)$ TO be.

$$f_l(r) = \frac{1}{r}\sin\left(kr + \frac{l\pi}{2} + \delta_l\right)$$

HOW DO I KNOW THIS IS A SOLUTION? WELL LOOK AT THE DIFFERENTIAL EQUATION FOR $f_\ell(r)$ IN THE LIMITS AS $r \to \infty$. THE POTENTIAL TERM IS NEGLIGIBLE AND WE HAVE TO SATISFY

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}(r f_\ell(r)) = E\, r f_\ell(r)$$

SO IT FOLLOWS IMMEDIATELY THAT $r f_\ell(r)$ MUST BE A SINE OR COSINE FUNCTION. THE δ_ℓ REPRESENTS A PHASE SHIFT TERM CAUSED BY THE ACTION OF THE POTENTIAL ON THE WAVE FUNCTION. WHEN $\delta_\ell = 0$ THERE IS NO DEVIATION IN PHASE WHICH IMPLIES NO POTENTIAL IS ACTING. RECALL WHEN WE TALKED ABOUT THE PATH INTEGRAL APPROACH TO QUANTUM MECHANICS THIS IS THE VERY RESULT WE PREDICTED (SEE PAGE 60). THE OTHER PHASE SHIFT TERM $\ell\pi/2$ IS RELATED TO THE ANGULAR MOMENTUM THE WAVE HAS AS IT PASSES THE ATOM.

OKAY, NOW LET'S USE THE EXPRESSION FOR $\psi(r,\theta)$ ~~FAR~~ OUT TO COMPUTE THE C_ℓ's. LET'S WRITE OUT WHAT WE KNOW SO ~~FAR~~

$$e^{ikz} + \frac{e^{ikr}}{r}F(\theta) = \sum_{\ell=0}^{\infty} C_\ell \frac{\sin(kr + \ell\frac{\pi}{2} + \delta_\ell)}{r} P_\ell(\cos\theta)$$

BUT SINCE $z = r\cos\theta$ THIS BECOMES

$$e^{-ikr\cos\theta} + \frac{e^{ikr}}{r}F(\theta) = \sum C_\ell \frac{\sin(kr + \ell\frac{\pi}{2} + \delta_\ell)}{r} P_\ell(\cos\theta)$$

LET'S ALSO EXPAND $\sin(kr + \ell\pi/2 + \delta_\ell)$ AS AN EXPONENTIAL, I.E.,

$$\sin(kr + \tfrac{\ell\pi}{2} + \delta_\ell) = \frac{1}{2i}\left[e^{i(kr + \frac{\ell\pi}{2} + \delta_\ell)} - e^{-i(kr + \frac{\ell\pi}{2} + \delta_\ell)}\right]$$

THEREFORE

$$e^{-ikr\cos\theta} + \frac{e^{ikr}}{r}F(\theta) = \sum \frac{C_\ell}{2i}\left[e^{i(kr + \frac{\ell\pi}{2} + \delta_\ell)} - e^{-i(kr + \frac{\ell\pi}{2} + \delta_\ell)}\right]\frac{P_\ell(\cos\theta)}{r}$$

IN THIS FORM WE HAVE TURNED THE INCIDENT PLANE WAVE - OUTGOING SPHERICAL WAVE PROBLEM INTO PURE SPHERICAL INCOMING AND OUTGOING WAVES AS OBSERVED BY A GUY AT THE NUCLEUS. BUT, BE CAREFUL, IN DOING THIS WE HAVE NOT NECESSARILY PRESERVED THE SPHERICALLY SYMMETRIC FORM OF THE SOLUTION.

TO CONTINUE ON IT IS NECESSARY TO EXPAND THE INCOMING WAVE IN TERMS OF SPHERICAL WAVES, I.E.,

$$e^{ikr\cos\theta} = \sum_\ell j_\ell(kr)\, P_\ell(\cos\theta)$$

THE PROBLEM IS REALLY GETTING TO BE A MESS WHEN I MAKE THIS EXPANSION

$$\sum_\ell j_\ell(kr) P_\ell(\cos\theta) + \frac{e^{ikr}}{r} F(\theta) = \sum \frac{C_\ell}{2i}\left[e^{i(kr+\frac{\ell\pi}{2}+\delta_\ell)} - e^{-i(kr+\frac{\ell\pi}{2}+\delta_\ell)} \right] \frac{P_\ell(\cos\theta)}{r}$$

I DON'T KNOW THE j_ℓ's, THE C_ℓ's, OR THE FUNCTION $F(\theta)$. BUT I CAN INDEED FIND THEM. SO LETS TEAR THIS EXPRESSION APART SOLVE FOR THE UNKNOWNS AND PUT IT ALL BACK TOGETHER AGAIN. FIRST, LET'S WORK ON THE j_ℓ's BECAUSE WE CAN FIND THEM THROUGH THE ORTHOGONALITY RELATIONSHIP PREVIOUSLY DISCUSSED. SO LET'S DO IT,

$$\int_{-1}^{1} e^{ikr\cos\theta} P_{\ell'}(\cos\theta)\, d(\cos\theta) = \sum_\ell \int_{-1}^{1} j_\ell(kr) P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, d(\cos\theta)$$

WHICH PRODUCES THE ℓ' VALUE OF j_ℓ,

$$\frac{2}{2\ell+1} j_{\ell'}(kr) = \int_{-1}^{1} e^{ikr\cos\theta} P_{\ell'}(\cos\theta)\, d\cos\theta$$

DROPPING THE PRIME NOTATION BECAUSE OF THE GENERALITY OF THIS RESULT, I CAN SHOW YOU HOW THIS EQUALITY WORK; LET'S TRY $\ell = 0$ FOR WHICH $P_0(\cos\theta) = 1$ AND WE HAVE TO EVALUATE

$$2 j_\ell(kr) = \int_{-1}^{1} e^{ikr\cos\theta}\, d\cos\theta$$

UPON INTEGRATION,

$$2 j_0(kr) = \frac{1}{ikr}\left(e^{ikr} - e^{-ikr} \right) = 2\frac{\sin kr}{kr}$$

THE BRACKETED QUANTITY REPRESENTS AN INCOMING AND OUTGOING WAVE. FOR $\ell = 1$

$$j_1(kr) = \int_{-1}^{1} e^{ikr\cos\theta} \cos\theta\, d(\cos\theta) \propto \frac{\sin kr - kr\cos kr}{(kr)^2}$$

THE SPECIFIC NAME GIVEN THESE FUNCTIONS IS SPHERICAL BESSEL FUNCTIONS OF HALF INTEGERS.

THE FUNCTION $F(\theta)$ IS TREATED IN A SIMILAR FASHION AND EXPANDED AS

$$F(\theta) = \sum_\ell a_\ell P_\ell(\cos\theta)$$

The SECRET NOW IS TO PUT THIS MESS BACK TOGETHER NOW AND MATCH THE TWO PARTS, I.E., INCOMING AND OUTGOING WAVES. I WILL BE LOOKING FOR A PARTICULAR SET OF j_ℓ'S WHICH HAS AS ITS ASYMPTOTIC LIMIT THE SOLUTION $\frac{\sin(kr+\ell\pi/2)}{kr}$ SO HERE WE GO

$$\sum_\ell \frac{1}{2ikr}\left[e^{i(kr+\frac{\ell\pi}{2})} - e^{-i(kr+\frac{\ell\pi}{2})}\right]P_\ell(\cos\theta) + \sum_\ell \frac{e^{ikr}}{r} a_\ell P_\ell(\cos\theta) =$$

$$\sum_\ell \frac{C_\ell}{2i}\left[e^{i(kr+\frac{\ell\pi}{2}+\delta_\ell)} - e^{-i(kr+\frac{\ell\pi}{2}+\delta_\ell)}\right]\frac{P_\ell(\cos\theta)}{r}$$

NOW THE $e^{-i(kr+\ell\pi/2)}$ TERMS MUST MATCH AND THEY WILL IF

$$C_\ell = \frac{e^{i\delta_\ell}}{kr}$$

NOW WE HAVE TO MATCH $e^{+i(kr+\ell\pi/2)}$,

$$\frac{e^{i(kr+\frac{\ell\pi}{2})}}{2ikr} + \frac{a_\ell}{r}e^{ikr} = \frac{C_\ell}{2i\cancel{kr}}e^{i(kr+\frac{\ell\pi}{2}+\delta_\ell)} = \frac{e^{2i\delta_\ell}}{2ikr}e^{i(kr+\frac{\ell\pi}{2})}$$

SOLVING FOR a_ℓ

$$a_\ell = \frac{1}{2ik}\left[e^{2i\delta_\ell} - 1\right]$$

SO THAT WE FINALLY HAVE THE FUNCTION $F(\theta)$,

$$F(\theta) = \frac{1}{2ik}\sum_\ell (2\ell+1)P_\ell(\cos\theta)\left[e^{2i\delta_\ell} - 1\right]$$

THE δ_ℓ'S MUST BE COMPUTED BEFOREHAND FOR THEY DEPEND ON THE NATURE OF THE EXPERIMENT. A COMPUTER IS TYPICALLY USED TO TABULATE THEIR VALUES SO THE AMPLITUDE OF THE SCATTERED WAVE IS DEFINED EXPLICITLY.

24. Quantum Theory of Scattering

As a review from last time we had come to discuss scattering theory. We assumed a steady incident wave which was essentially infinite in extent as viewed by the scattering atom. Part of this incoming wave is deflected by some scatterer and from the intensity of the scattered wave, the probability of scattering is computed.

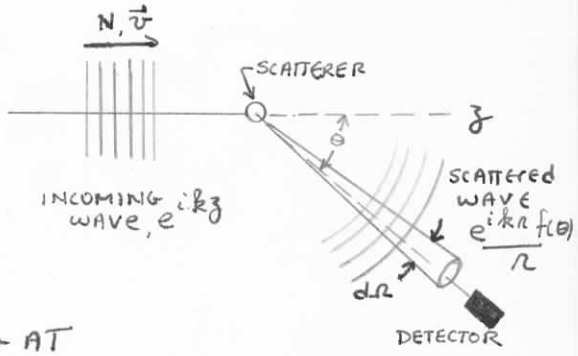
The asymptotic form of the time-independent wave function was given as

$$\psi(r;\theta) = e^{ikz} + \frac{e^{ikr}}{r} f(\theta)$$

Here we have ignored the finite time interval in which the incoming beam collides with the scatterer. Since the incident beam and the scattered beam remain steady, all probabilities become independent of time so that the complete time dependent wave function can be written as

$$\Psi = \psi(r,\theta)\, e^{-\frac{iEt}{\hbar}} = \left[e^{ikz} + \frac{e^{ikr}}{r} f(\theta) \right] e^{-iEt/\hbar}$$

We saw the scattered amplitude varied as a function of θ. I'd like to discuss in more detail the nature of this function, $f(\theta)$, and first off I'd like to talk about how it is recorded.



If a detector is set up at some and θ to the z axis and it subtends a solit angle $d\Omega$, then we would like to compute how many particles arive at the detector per second. The strength of the incoming particles is given by N particles per cubic centimeter traveling at a velocity $\vec{v}$. The number of particles scattered per second in the direction of θ through a solid angle $d\Omega$ is

$$N v_{IN}\, \sigma\, d\Omega$$

where v_{IN} - equal the velocity of the incoming wave.

σ WILL TURN OUT TO HAVE THE DIMENSION OF AREA AND IS ANALOGOUS TO THE CLASSICAL ~~AN~~ DEFINITION OF SCATTERING CROSS SECTION. HOWEVER IT IS NOT PROPER TO THINK OF σ HERE AS THE "AREA" OF THE SCATTERING CENTER BUT RATHER AN EFFECTIVE AREA WHICH THE INCOMING WAVE MUST HIT TO COME OUT AT THE ANGLE θ. IN GENERAL σ IS A FUNCTION OF BOTH θ AND φ BUT FOR OUR CASE IT IS JUST A FUNCTION OF θ. SINCE Nv IS ANALOGOUS TO A CURRENT, WE CAN THINK OF THE INCIDENT CURRENT $I = Nv$ AND THE DEFLECTED PARTICLE CURRENT AS $I\sigma d\Omega$. THE CROSS SECTION, $\sigma d\Omega$ IS JUST THE RATIO OF THE CURRENTS.

I NOW WANT TO COMPARE THE PROBABILITY~~ES~~ OF FINDING THE PARTICLE IN THE INCIDENT WAVE TO THE PROBALITY OF FINDING THE PARTICLE IN SOME AREA $R^2 d\Omega$ AT THE DETECTOR. IN THE INCIDENT BEAM, THE PROBABILITY OF FINDING THE PARTICLE IS

$$P_{IN} = |e^{ikz}|^2 = 1$$

IN THE SCATTERED BEAM THE PROBABILITY IS

$$P_{OUT} = \frac{|f(\theta)|^2}{R^2}$$

IF THE INCOMING BEAM VELOCITY IS DIFFERENT FROM THE OUTGOING VELOCITY THEN

$$P_{OUT} = \frac{1}{R^2}|f(\theta)|^2 = N v_{IN}\,\sigma(\theta)\,\frac{1}{R^2}\frac{1}{v_{OUT}}$$

WHILE $P_{IN} = N$. THEREFORE THE RATIO OF P_{OUT} TO P_{IN} IS

$$\frac{P_{OUT}}{P_{IN}} = \frac{N v_{IN}\,\sigma(\theta)\,1/R^2\,\frac{1}{v_{OUT}}}{N} = \frac{1/R^2\,|f(\theta)|^2}{1}$$

SO THAT

$$\sigma(\theta) = \frac{v_{OUT}}{v_{IN}}|f(\theta)|^2$$

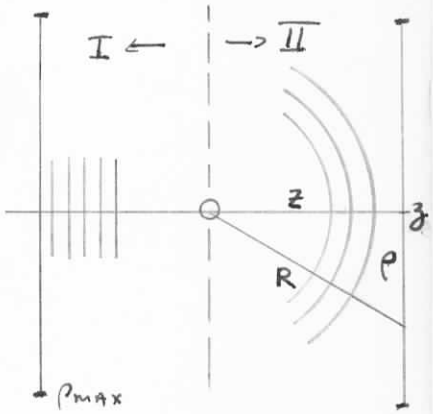
IF SCATTERING IS ELASTIC THEN $v_{OUT} = v_{IN}$ AN $\sigma(\theta) = |f(\theta)|^2$

BEFORE I CONTINUE ON I WOULD LIKE TO DISCUSS PHYSICALLY WHAT HAPPENS TO THE ORIGINAL WAVE WHEN A SCATTERED WAVE IS GENERATED BY THE INTERACTION. WE WILL LEARN THAT THE INCOMING BEAM IS REDUCED IN INTENSITY AS THE SECONDARY WAVE GENERATED BY THE SCATTERER - BEAM INTERACTION INTERFER. THE NET EFFECT OF THE INTERFERENCE IS REDUCE THE STRENGTH OF THE DEFLECTED BEAM BELOW THE STRENGTH THE WAVE WOULD HAVE IF NO SCATTERING CENTER WAS PRESENT. THIS IS JUST ANOTHER WAY OF SAYING THE SCATTERING CENTER CASTS A SHADOW IN THE FORWARD DIRECTION. HOW THEN DO WE KNOW THAT THE WAVE FUNCTION

$$e^{ikz} + \frac{e^{ikr}}{r} f(\theta)$$

PREDICTS SUCH A BEHAVIOR. OR SAID ANOTHER WAY HOW DOES IT TELL US THAT PARTICLES ARE CONSERVED AFTER THE SCATTERING PROCESS?

TO ANSWER THE QUESTION LET'S CALCULATE HOW MANY PARTICLES THERE ARE IN REGION I AND REGION II. BY SUBTRACTING THE TWO QUANTITIES WE CAN SHOW THAT THERE ARE FEWER PARTICLES IN II THAN IN I. TO DO THIS WE'LL TAKE TWO VERY LARGE AREAS, EQUALE IN SIZE AND CALCULATE THE NUMBER OF PARTICLES GOING THROUGH.



THE INTENSITY OF PARTICLES INTO A_I IS

$$\text{INTENSITY AT } A_I = \int_0^{\rho_{MAX}} |e^{ikz}|^2 \, 2\pi\rho \, d\rho = \int_0^{\rho_{MAX}} 2\pi\rho \, d\rho$$

WHERE ρ_{MAX} IS THE RADIUS OF THE AREA A_I WHICH IN THE LIMIT GOES TO INFINITY. THE INTENSITY OF PARTICLES OUT THROUGH A_{II} IS

$$\text{INTENSITY AT } A_{II} = \int_0^{\rho_{MAX}} \left| e^{ikz} + \frac{e^{ikR}}{R} f(\theta) \right|^2 2\pi\rho \, d\rho$$

THIS CAN BE EXPANDED TO THE FORM

$$\int_0^{\rho_{MAX}} \left[1 + \frac{e^{ik(R-z)} f(\theta) - e^{-ik(R-z)} f^*(\theta)}{R} + \frac{1}{R^2} \right] 2\pi\rho \, d\rho$$

FOR THIS CALCULATION WE CAN NEGLECT THE $1/R^2$ TERM SINCE WE WILL LET $\rho_{MAX} \to \infty$ IN THE LIMIT.

The PARTICLES LOST would be The difference of These TWO INTEGRALS or JUST

$$\Delta \text{ PARTICLES LOST} = -\int_0^{\rho_{MAX}} \left[f(\theta)\, e^{ik(R-z)} + e^{-ik(R-z)} f^*(\theta) \right] \frac{2\pi\rho\, d\rho}{R}$$

IN order TO EVALUATE The INTEGRAL I WILL MAKE SOME UNNECCESSARY APPROXIMATIONS. FOR LARGE R WE CAN WRITE

$$R = \sqrt{z^2+\rho^2} \simeq z + \rho^2/2z$$

So The TERM $R - z$ IS JUST $\rho^2/2z$. WHILE IN The DENOMINATOR WE CAN by WITH JUST LETTING $R = z$ SINCE WE DON'T HAVE TO WORRY About ANY PHASE effect AS IN The EXPONENTIAL. WE CAN NOW DO The INTEGRATION rather QUICKLY. For The first TERM WE GET

$$\int_0^{\rho_{MAX}} 2\pi\left(\frac{\rho\, d\rho}{z}\right) f(\theta)\, e^{ik\rho^2/2z} = \frac{2\pi}{ik}\left[e^{\frac{ik}{2z}\rho_{MAX}^2} - 1 \right] f(0)$$

Where I HAVE EVALUATED $f(\theta)$ ALONG The z AXIS SO THAT $\theta = 0$. This INTEGRAL YIELD The INFINITELY OSCILLATORY WAVE which WE HAVE ENCOUNTERED before. And IN The LIMIT AS $\rho_{MAX} \to \infty$ The ONLY PHYSICAL SOLUTION CAN be A WAVE which SLOWLY PETERS OUT. So WE ARE LEFT WITH The ANSWER TO The first INTEGRAL AS JUST

$$\int_0^{\rho_{MAX}} 2\pi\left(\frac{\rho\, d\rho}{z}\right) f(\theta)\, e^{ik\rho^2/2z} = -\frac{2\pi f(0)}{ik}$$

BY SIMILAR EVALUATION WE find THAT

$$\int_0^{\rho_{MA}} 2\pi\left(\frac{\rho\, d\rho}{z}\right) f(\theta)\, e^{-ik\rho^2/2z} = +\frac{2\pi f^*(0)}{ik}$$

So The LOST PARTICLES ARE GIVEN by

$$\Delta \text{ PART} = -\left[\frac{2\pi f(0)}{ik} + \frac{2\pi f^*(0)}{ik} \right] = \frac{2\pi i}{k}\left[f^*(0) - f(0) \right]$$

This ANSWER LOOKS PRETTY USELESS but IT DOES TURN OUT TO be The correct ONE WE WANT. TO PUT IT IN A MORE AMENABLE form WE WILL WRITE $f(0)$ IN ITS GENERAL COMPLEX form,

$$f(0) = f_{REAL} + i f_{IMAG}.$$

SO THAT

$$f^*(0) = f_{REAL} - i f_{IMAG}.$$

Subtracting $f^*(0)$ from $f^*(0)$ we have that

$$f^*(0) - f(0) = -2i\, f_{IMAG}$$

Thus the number of particles lost is given by the important result,

$$\Delta \text{PARTICLES} = \frac{4\pi}{k} f(0)_{\text{IMAGINARY PART}}$$

If we ask the same question for the classical case, i.e., how many particles are lost per second due to the total scattering we have that

$$\text{Number lost per second} = N v\, \sigma_{TOTAL}$$

where $\sigma_{TOTAL} = \int \sigma(\theta)\, d\Omega$ which is the total elastic scattering occurring. Thus we can simply compute the loss by

$$\frac{\text{NO. of PARTICLES LOST / SEC}}{\text{No of PARTICLES IN / SES}} = \frac{N v\, \sigma_{TOT}}{N v} = \sigma_{TOTAL}$$

So we can conclude that for conservation of particles to hold during the collision process it ~~mus~~ the following rule must be satisfied,

$$\sigma_{TOT}(\theta) = \frac{4\pi}{k} \operatorname{Im} f(0)$$

This is usually called the OPTICAL Theorem. The Theorem tells us how to relate the total cross section and the imaginary part of the forward scattering amplitude. It just tells us how the beam is reduced in the forward direction in proportion to everything that is scattered away through all angles.

For the case of elastic scattering only the above result tells us that

$$\frac{4\pi}{k} \operatorname{Im} f(0) = \int |f(0)|^2\, d\Omega$$

which is the requirement to conserved particles during the scattering. It says you cannot have an object there sucking

UP PARTICLES. QUANTUM MECHANICALLY THEN THERE IS NO SUCH THING AS A BLACK BOX ABSORBING INCOMING PARTICLES; THERE MUST BE SOME DIFFRACTION, I.E., INTERFERENCE OF WAVES DUE TO THE ELASTIC SCATTERING.

WE NOW WANT TO FIND AN EXPLICIT FORM FOR $f(\theta)$. ACTUALLY WE ARE NOW INTERESTED SPECIFICALLY IN THE IMAGINARY PART OF $f(\theta)$. AND WE BETTER FIND OUT THAT $\mathrm{Im}\, f(0)$ IS A POSITIVE NUMBER. WE DID THIS LAST TIME AND FOUND THAT

$$f(\theta) = \frac{1}{2ik} \sum_{\ell} (2\ell+1) P_\ell(\cos\theta) \left(e^{2i\delta_\ell} - 1 \right)$$

THE IMPORTANT TERM IS $(e^{2i\delta_\ell} - 1)$ WHICH IS THE DIFFRACTED WAVE OR INTERFERENCE TERM. IF WE WRITE THIS IN A LITTLE DIFFERENT FORM, I.E.,

$$e^{2i\delta_\ell} - 1 = \tfrac{1}{2}(\cos 2\delta_\ell - 1) + \tfrac{1}{2} i \sin 2\delta_\ell$$

SO THAT

$$f(\theta) = \frac{1}{2k} \sum_{\ell} (2\ell+1) \sin 2\delta_\ell \, P_\ell(\cos\theta) + \frac{1}{k} \sum_{\ell} (2\ell+1) \left(\frac{1-\cos 2\delta_\ell}{2} \right) P_\ell \cos\theta$$

SINCE $1 - \cos 2\delta_\ell = 2\sin^2\delta_\ell$, WE HAVE FOR THE IMAGINARY PART OF $f(0)$ THE VALUE

$$\mathrm{Im}\, f(0) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_\ell$$

BY OUR THEOREM THEN

$$\sigma_{TOT}(\theta) = \frac{4\pi}{k} \mathrm{Im}\, f(0) = \frac{4\pi}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_\ell$$

IT MIGHT BE INTERESTING TO SEE IF WE CAN VERIFY THIS ANSWER BY RECALLING THAT

$$\sigma_{TOTAL} = |f(\theta)|^2 = f(\theta) f^*(\theta)$$

USING THE ABOVE EXPRESSION FOR $f(\theta)$ WE CAN WRITE,

$$\sigma_{TOTAL} = \frac{1}{(2ik)}\frac{1}{(2ik)} \sum_{\ell}^{\infty}\sum_{\ell'}^{\infty} (2\ell+1)(2\ell'+1)(e^{2i\delta_\ell}-1)(e^{-2i\delta_{\ell'}}-1)\, P_\ell \cos\theta\, P_{\ell'} \cos\theta$$

BY CONVERTING TO INTEGRALS, ~~AND~~ INTEGRATING OVER ALL ANGLES, AND USING THE FACTS THAT

$$\int_{-1}^{1} P_{\ell'}(\cos\theta)\, P_\ell(\cos\theta)\, 2\pi\, d(\cos\theta) = \frac{4\pi}{2\ell+1}\,\delta_\ell\,\delta_{\ell'}$$

$$\sigma_{TOTAL} = \frac{4\pi}{4k^2}\sum_{\ell} \frac{(2\ell+1)(2\ell+1)}{2\ell+1}(e^{2i\delta_\ell}-1)(e^{-2i\delta_\ell}-1)$$

SIMPLIFYING THE EXPONENTIAL

$$(e^{2i\delta_\ell}-1)(e^{-2i\delta_\ell}-1) = 1 - e^{2i\delta_\ell} - e^{-2i\delta_\ell} + 1 = 2 - 2\cos 2\delta_\ell = 4\sin^2\delta_\ell$$

SO WE HAVE THE SAME RESULT BY SOME MIRACLE,

$$\sigma_{TOTAL} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

THE USUAL WAY TO PURSUE THIS DISCUSSION IS TO SPEND VIRTUALLY ALL YOUR EFFORT IN WORKING OUT THE δ_ℓ's FOR A SPECIFIC POTENTIAL FUNCTION AND MAYBE 10 PERCENT OF THE TIME IS SPENT SHOWING HOW VARIOUS δ_ℓ's WORK INTO THE EQUATIONS. I'd LIKE SOMEDAY TO REDO THIS APPROACH AND EMPHASIZE THE IMPORTANCE OF THE δ_ℓ's.

SO WHAT I'd LIKE TO DO here IS TO ILLUSTRATE THE δ_ℓ TERM for CERTAIN CIRCUMSTANCES. FIRST WE SHOULD ASK TO WHAT DEGREE IS THE RESULT OBTAINED GENERAL? WELL, THEY ARE NOT GENERAL for SEVERAL REASONS. FIRST THE INCOMING BEAM NEED NOT BE SPHERICALLY SYMMETRIC. HOWEVER for VERY LONG WAVES spherical WAVES WILL COME OUT. WHEN THE ANGULAR MOMENTUM IN THE BEAM IS large, i.e., big ℓ QUANTITIES, THE WAVE ACTS NEARLY INDEPENDENT of THE SCATTERING POTENTIAL AS IT PASSES by. IN order words THE LARGER THE ℓ THE SMALLER THE CHANCE THE WAVE WILL FIND OUT WHAT'S INSIDE THE ATOM. For LOW VALUES of ℓ THE WAVE HAS A higher probability of BEING found NEAR THE CENTER of THE POTENTIAL SO IT IS deflected MORE.

To clarify what I mean a little more the asymmetric form of the incoming wave implies that quantum mechanically there is angular momentum being carried. When this occurs the solutions get more complicated. One example is when the incoming beam has an intrinsic spin, then the wave function will have to be expanded in terms of the spherical harmonics. When three particles are involved in the scattering process additional complications arise. Also when the incoming and outgoing waves have different momentums the results are more complicated than presented here.

All I have proven is that a solution for the ℓth wave is possible but I have not determined the relative size of the resulting phase shift $e^{2i\delta_\ell}$. In order to get everything to fit together it is usually necessary to write the phase shift as $\eta_\ell e^{2i\delta_\ell}$. So what is η_ℓ? Well, for our case η_ℓ was equal to 1. because we conserved particles, i.e., the angular momentum was the same in the two waves. If the incoming wave can turn into something else or knock apart the scattering center or some other crazy thing then $\eta_\ell < 1$ which says I had to use up some particles (energy) to excite the atom or something else. This usually makes the problem hard to solve.

Resonance Scattering

Going way back to the problem that ~~propte~~ prompted this discussion, we wanted to know what happened to a wave as it was beamed at a coulomb potential and the wave had a frequency ~~given by~~ close to the resonance frequency for leaking out of the well. The energy was then varied about this resonance energy E_0 according to our equation $\frac{1}{(E-E_0)^2+\Gamma^2/4}$. For the sake of definiteness I will consider an s-wave ($\ell=0$) resonance only and see what happens.

Remember we expressed the wave function outside the barrier in the form

$$\psi_{out} \propto \cos(kr + \Delta) + \frac{2(E-E_0)}{\Gamma} \sin(kr + \Delta)$$

By expanding this in exponential form we can write

$$\psi_{out} \propto e^{i(kr+\Delta)}\left[1 - \frac{2i(E-E_0)}{\Gamma}\right] + e^{-i(kr+\Delta)}\left[1 + \frac{2i(E-E_0)}{\Gamma}\right]$$

This is the some of an incoming and outgoing wave. Since $e^{2i\delta_l}$ is a measure of the strength of the two wave functions we write for $l=0$

$$e^{2i\delta_0} = e^{2i\Delta} \frac{\left[1 - \frac{2i(E-E_0)}{\Gamma}\right]}{\left[1 + \frac{2i(E-E_0)}{\Gamma}\right]}$$

The term involving Δ is critical but for the moment we assume nothing else is scattering but the scatterer, i.e., there isn't any background scattering. For that case Δ is 0. But before using that approximation we can rewrite the above expression in a more useful form, i.e.,

$$e^{2i\delta_0} = e^{2i\Delta} e^{2i\delta_{res}}$$

where the second phase shift is due to the resonance scattering,

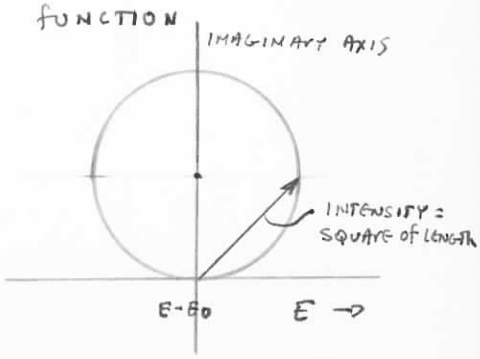
$$e^{2i\delta_{res}} = \frac{E - E_0 + i\Gamma/2}{E - E_0 - i\Gamma/2}$$

The general expressions for the lth resonance would be dependent upon

$$\delta_l = \Delta_l + \delta_{res,l}$$

Now far away from the resonance Γ is negligible and $\delta_{res} \sim 0$. Near resonance, $E \simeq E_0$, $\delta_{res} \simeq 1$ and the scattering reaches a maximum. I have screwed up the math along the way because I wanted to plot the function $e^{2i\delta_{res}} - 1$ as a function $E - E_0$. If I do

$$e^{2i\delta_{res}} - 1 = \frac{i\Gamma}{E - E_0 - i\Gamma/2}$$



The GRAPHICAL REPRESENTATION of The SCATTERING INTENSITY IS A GOOD WAY TO VISUAL These KINDS of PROBLEMS. The AMPLITUDE of SCATTERING AROUND RESONANCE IS THEN PROPORTIONAL TO

$$\frac{1}{2ik}\frac{i\Gamma}{(E-E_0-i\Gamma/2)}$$

The RESONANCE CROSS-SECTION IS GIVEN BY

$$\sigma_{RES} = \frac{4\pi}{k^2}\frac{\Gamma^2}{[(E-E_0)^2+\Gamma^2/4]}$$

IT IS INTERESTING TO INTEGRATE σ_{RES} AROUND The RESONANT ENERGY TO FIND OUT WHAT ~~σ_{MAX}~~ IS I.E., WHEN $E=E_0$.

$$\cancel{\sigma_{MAX}} = \int \sigma_{RES}\, dE = \int \frac{\Gamma^2\, dE}{(E-E_0)^2+\Gamma^2/4} = \frac{\Gamma^2\pi}{\Gamma/2} = 2\pi\Gamma$$

Then $\sigma_{MAX} = 4\pi/k^2$. THAT IS

$$\sigma_{TOTAL} = \sum_{l=0}^{\infty}\frac{4\pi}{k^2}(2l+1)\sin^2\delta_l$$

for $l=0$, $\delta_l=0$

$$\sigma_{TOTAL} = \sigma_{MAX} = \frac{4\pi}{k^2}$$

Suppose THAT ONLY The S AND p WAVE RESONANCES ARE IMPORTANT THEN SINCE $l=0$ AND $l=1$ THEN $f(\theta) = a + b\cos\theta$ AND

$$\sigma^2 = |f(\theta)|^2 = |a|^2 + 2|a||b|\cos\theta + |b|^2\cos^2\theta$$

Here The $|B|^2$ TERM IS The RESONANCE TERM; $|a|^2$ IS NEARLY CONSTANT AND REPRESENTS The BACKGROUND IN SCATTERING AND $|a||b| = (a b^* + a^* b)$ REPRESENTS The INTERFERENCE EFFECT.

25. ROTATION AND ANGULAR MOMENTUM

A WHILE BACK WE TALKED ABOUT THE IDEA OF AN N-STATE SYSTEM where THE STATE VECTORS $|\psi_j\rangle$ COULD BE REPRESENTED IN SOME OTHER COORDINATE SYSTEM, SAY THE PRIMED SYSTEM. TO GET FROM ONE STATE TO THE OTHER WE INTRODUCED THE IDEA OF ROTATION OF THE STATE VECTORS. THE IDEA OF ROTATION SOMEHOW SEEMS TO SUGGEST THE CLASSICAL CONCEPT OF ANGULAR MOMENTUM; SO PERHAPS WE CAN FIND A CORRELATION BETWEEN ROTATIONS AND ANGULAR MOMENTUM.

LET'S DEFINE THE OPERATOR $\mathcal{D}_{ji}(R)$ WHICH WORKS ON THE STATES $|\psi_i\rangle$ IN THE S COORDINATE SYSTEM TO PRODUCE THE STATES $|\psi_j\rangle$, I.E.,

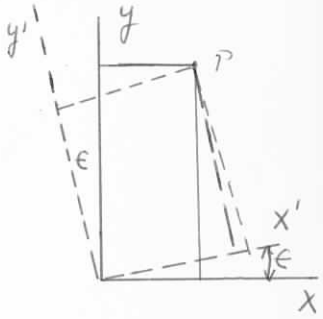
$$|\psi_j\rangle = \sum_i \mathcal{D}_{ji}(R)\,|\psi_i\rangle$$

Here R DENOTES THE ROTATION OF S INTO S' or SAID MATHEMATICALLY $S' = RS$. BY THE SAME TOKEN WE COULD ROTATE AGAIN THROUGH $\mathcal{R}$ TO GET $S'' = \mathcal{R}S' = \cancel{TS}$. BY THESE SUCCESSIVE ROTATIONS THE OPERATOR $\mathcal{D}_{ji}(R'')$ MUST SATISFY THE RELATIONSHIP

$$\mathcal{D}_{ji}(R'') = \sum_k \mathcal{D}_{jk}(T)\,\mathcal{D}_{ki}(R)$$

IN WORDS, A R'' ROTATION CAN FIRST BE EFFECTED BY ROTATING THROUGH R THEN THROUGH T. THIS IS A MATRIX EQUATION AND WE MUST FIND THE PARTICULAR MATRICES THAT SATISFIES THE RELATIONSHIP. TO DO THIS WE WILL ~~WE~~ CONSIDER TINY ROTATIONS, I.E., INFINITESIMAL ROTATIONS, ABOUT ALL AXIS AND SEE WHAT HAPPENS.

FIRST, WE WILL CONSIDER A SMALL ROTATION ϵ ABOUT THE Z-AXIS, I.E. for $\theta \simeq \epsilon \simeq 0$ THE ROTATION WOULD BE



$$x' = x\cos\theta + y\sin\theta \quad = x + y\epsilon$$

$$y' = -x\sin\theta + y\cos\theta \quad = -x\epsilon + y$$

WE WILL DENOTE $\mathcal{D}_{ji}(R)$ AS $\mathcal{D}_{ji}(1)$ BECAUSE THE COORDINATES ARE ESSENTIALLY UNCHANGED. HERE $\mathcal{D}_{ji}(1)$ IS GIVEN BY A first order VARIATION FROM UNITY

$$\mathcal{D}_{ji}(1) = \delta_{ji} - i\epsilon\,(J_z)_{ji}$$

where $i = \sqrt{-1}$ IN THE RIGHT SIDE TERM IN ϵ. NOW WE HAVE INTRODUCED ANOTHER SYMBOL $(J_z)_{ji}$ which for THE MOMENT IS ASSOCIATED WITH THE FACT THAT WE ROTATED ABOUT THE z AXIS. HAD WE ROTATED ABOUT X, OR Y WE WOULD HAVE USED J_x, J_y RESPECTIVELY

LATER WE WILL DISCUSS IN MORE DETAIL THE PHYSICAL SIGNIFICANCE OF J_z, J_x, AND J_y. FOR NOW WE WILL ASSUME IT TO BE A ROTATION OPERATOR.

THE J'S HAVE A VERY IMPORTANT AN IMPORTANT MATHEMATICAL PROPERTY WHICH THEY SATISFY. - THEY ARE HERMETIAN OPERATORS. THAT MEANS THAT THE MATRIX ELEMENTS SATISFY THE MANIPULATION

$$(J_z)^*_{ji} = (J_z)^\dagger_{ij}$$

THIS MEANS TO GET THE i,j ELEMENT OF $J_z^\dagger$ (J_z-DAGGER) YOU GO TO THE j,i ELEMENT OF J_z AND TAKE ITS COMPLEX CONJUGATE. THE REASON WHY J_z MUST BE HERMETIAN EVOLVES FROM A MORE BASIC CONSTRAINT HAVING TO DO WITH ~~THE~~ CAUSALITY AND THE CONSERVATION OF PROBABILITY. THUS THE ROTATION OPERATOR $R_z(\epsilon)$ IS THE FOLLOWING

$$R_z(\epsilon) = 1 - i\epsilon J_z$$

AND SIMILARLY FOR x, AND y. THEN FOR ANY ARBITRARY ROTATION ABOUT SOME AXIS COCKED TO THE ORIGINAL x, y, z COORDINATES, SAY IN THE DIRECTION $\vec{N}$ WE CAN DEDUCE THAT

$$R = 1 - i\epsilon \vec{J}\cdot\vec{N}$$

BY THIS NOTATION WE MEAN THE SET OF THREE ROTATIONS ϵN_z, ϵN_y, ϵN_x IN THE ORDER GIVEN SUCH THAT

$$1 - \epsilon \vec{J}\cdot\vec{N} = (1 - i\epsilon N_z J_z)(1 - i\epsilon N_y J_y)(1 - i\epsilon N_x J_x)$$

NOW WHAT PROPERTIES MUST J_x, J_y, AND J_z HAVE FOR ALL THIS TO WORK OUT? WE WILL FIND OUT THAT FOR THE J'S TO SATISFY THE ABOVE ROTATIONAL PROPERTIES THAT THEY MUST COMMUTE ACCORDING TO THE RULE

$$J_x J_y - J_y J_x = i J_z$$

AND TWO OTHER SUCH RULES OBTAINED BY CYCLIC ROTATIONS OF x, y AND z

$$J_y J_z - J_z J_y = i J_x$$

$$J_z J_x - J_x J_z = i J_y$$

THIS IS A VERY IMPORTANT PROPERTY OF REAL SPACE; IT IS A NECESSARY RESULT WHICH MUST HOLD FOR SPATIAL SYMMETRY TO BE PERSERVED.

To verify that such a rule naturally evolves from a rotation like $1-i\epsilon\vec{J}\cdot\vec{N}$ we will show that the equation is correct in the second order. If it is correct in the second order, it is correct to 3rd order, etc. We want to prove then that,

rotation ϵ about x + rotation η about y − rot. ϵ about x − η about y =

rotation of $\epsilon\eta$ about z

By a second order rotation we mean the operator $\mathcal{D}$ is expressable in exponential form, i.e., $\mathcal{D} = e^{-i\theta J_z}$ or expanded out

$$\mathcal{D} = 1 - i\theta J_z - \frac{\theta^2}{2} J_z^2 + \cdots$$

Thus we have to work out

$$\left(1 + i\eta J_y - \frac{\eta^2}{2} J_y^2\right)\left(1 + i\epsilon J_x - \frac{\epsilon^2 J_x^2}{2}\right)\left(1 - i\eta J_y - \frac{\eta^2 J_y^2}{2}\right)\left(1 - i\epsilon J_x - \frac{\epsilon^2}{2} J_x^2\right) = 1 - i\epsilon\eta J_z$$

Note the particular order in which the rotations are expressed; its sort of like reading Chinese - you have to read from right to left. This is important however when expanding out that we express all the products just as they occur. To expand this out the 0 order terms is easy its just 1. The first order terms are

$$i\epsilon J_x + i\eta J_y - i\epsilon J_x - i\eta J_y = 0$$

Now we need the second order terms,

$$-\eta^2 J_y^2 - \epsilon^2 J_x^2 - \epsilon\eta J_y J_x + \eta^2 J_y^2 + \epsilon\eta J_y J_x + \epsilon\eta J_x J_y + \epsilon^2 J_x^2 - \epsilon\eta J_y J_x$$

Subtracting out we are left with $\epsilon\eta J_x J_y - \epsilon\eta J_y J_x$. Therefore equating the right and left sides we find that

$$-J_x J_y + J_y J_x = -i J_z$$

After correcting for a sign mistake. Thus

$$J_x J_y - J_y J_x = i J_z$$

or written alternately as

$$[J_x, J_y] = i J_z$$

This result is one of the basic principals of Quantum Mechanics; I ask you to hold it dear to your heart.

So far we have only been considering the mathematics of rotation matrices and showed the important correlation to a fundamental property of space. The implication being that the results of an experiment is independent of the orientation of the axes. This means the solutions to the Schrodinger equation are the same if we rotate the coordinates say through an angle θ about z, i.e.,

$$\psi(x,y,z) \rightarrow \psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, z)$$

This means for a small rotation ϵ we have that

$$\psi' = \mathcal{D}\psi(x,y,z) = \psi(x+\epsilon y, y - x\epsilon, z)$$

If I change labels for the moment and let $L_z = J_z$ we can write to first order

$$(1 - i\epsilon L_z)\psi = \psi(x,y,z) + \epsilon y \frac{\partial \psi}{\partial x} - \epsilon x \frac{\partial \psi}{\partial y}$$

This means that the operation L_z is the same thing as

$$L_z = \frac{1}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

If I put in $\hbar$ where it belong; over i, then

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Now a while back we represented momentum as the operation $\frac{\hbar}{i}\frac{\partial}{\partial x} = P_x$ Thus L_z is equivalent to

$$L_z = x P_y - y P_x$$

Now we can give some physical significance to the quantity L_z because it looks like the z component of the classical mechanics angular momentum,

$$\vec{L} = \vec{R} \times \vec{P}$$

Classically, however, the quantity $P_x X - X P_x = 0$ since P and X commute. In quantum mechanics there is a distinctly different result since

$$X P_x - P_x X = x\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x} x$$

When this operator works on the function ψ we find that

$$x \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x) - \frac{\hbar}{i}\frac{\partial}{\partial x} x\psi(x) = x\frac{\hbar}{i}\frac{\partial}{\partial x}\psi - \frac{\hbar}{i}\psi(x) - \frac{\hbar}{i} x\frac{\partial\psi}{\partial x} = -\frac{\hbar}{i}\psi(x)$$

WE SEE THAT THE ANSWER IS NOT ZERO AND THAT THE OPERATOR IS IDENTICAL TO MULTIPLICATION BY $i\hbar$, I.E.,

$$x p_x - p_x x = i\hbar$$

IF $\hbar$ WERE 0, THEN CLASSICAL AND QUANTUM MECHANICS WOULD BE THE SAME; SINCE $\hbar \neq 0$, WE HAVE TO LEARN QUANTUM MECHANICS.

SINCE WE SUBSTITUTED L_z FOR J_z IT MUST BE TRUE THAT THE L'S SATISFY THE COMMUTATION RULES PREVIOUSLY ESTABLISHED, E.G.

$$L_x L_y - L_y L_x = i L_z$$

TO CHECK THIS $L_x = z p_y - y p_z$ AND $L_y = x p_z - z p_x$. THEN WE HAVE TO WORK OUT $L_y(L_x\psi)$, I.E.,

$$-\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\psi$$

WHAT THE HELL DOES ALL THAT OPERATION MEAN. WELL, LET'S PLOW AHEAD AND WRITE IT OUT CAREFULLY AND COMPARE IT TO $L_x(L_y\psi)$.

$$L_y(L_x\psi) = -x\frac{\partial}{\partial z}\left(z\frac{\partial\psi}{\partial y}\right) + xy\frac{\partial^2}{\partial z^2}\psi - z^2\frac{\partial^2\psi}{\partial x\partial y} + zy\frac{\partial^2\psi}{\partial x\partial z}$$

$$= -xz\frac{\partial^2\psi}{\partial z\partial y} - x\frac{\partial\psi}{\partial y} + \quad '' \quad - \quad '' \quad + \quad ''$$

NOW $L_x(L_y\psi)$ CAN BE FOUND BY CHANGING X AND Y IN THE ABOVE RESULTS SO THAT THE FIRST 4 TERMS ARE THE SAME BUT THE LAST TERM IS $-y\,\partial\psi/\partial x$. THEREFORE

$$L_x(L_y\psi) - L_y(L_x\psi) = x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x}$$

IF WE DIVIDE THIS BY i WE HAVE THAT

$$L_x L_y - L_y L_x = \frac{1}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial\psi}{\partial x}\right) = L_z$$

THEREFORE INDEED A RESIDUAL IS LEFT AND THE TERMS L_x AND L_y DO NOT COMMUTE. THEY BETTER HADN'T!

$$x \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x) - \frac{\hbar}{i} \frac{\partial}{\partial x} x \psi(x) = x \frac{\hbar}{i} \frac{\partial}{\partial x} \psi - \frac{\hbar}{i} \psi(x) - \frac{\hbar}{i} x \frac{\partial \psi}{\partial x} = -\frac{\hbar}{i} \psi(x)$$

WE SEE THAT THE ANSWER IS NOT ZERO AND THAT THE OPERATOR IS IDENTICAL TO MULTIPLICATION BY $i\hbar$, I.E.,

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$$L_y(L_x \psi) = -x \frac{\partial}{\partial z}\left(z \frac{\partial \psi}{\partial y}\right) + xy \frac{\partial^2}{\partial z^2} \psi - z^2 \frac{\partial^2 \psi}{\partial x \partial y} + zy \frac{\partial^2 \psi}{\partial x \partial z}$$

$$= -xz \frac{\partial^2 \psi}{\partial z \partial y} - x \frac{\partial \psi}{\partial y} + \quad '' \quad - \quad '' \quad + \quad ''$$

NOW $L_x(L_y \psi)$ CAN BE FOUND BY CHANGING X AND Y IN THE ABOVE RESULTS SO THAT THE FIRST 4 TERMS ARE THE SAME BUT THE LAST TERM IS $-y\, \partial\psi/\partial x$. THEREFORE

$$L_x(L_y \psi) - L_y(L_x \psi) = x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x}$$

IF WE DIVIDE THIS BY i WE HAVE THAT

$$L_x L_y - L_y L_x = \frac{1}{i}\left(x \frac{\partial}{\partial y} - y \frac{\partial \psi}{\partial x}\right) = L_z$$

THEREFORE INDEED A RESIDUAL IS LEFT AND THE TERMS L_x AND L_y DO NOT COMMUTE. THEY BETTER HADN'T!

26. GENERATORS of The ROTATION OperATOR

I'd LIKE TO STUDY The PROPERTIES of J MORE CLOSELY AND LOOK AT The CHARACTERTICS of The STATES THAT The OperATOR PRODUCES. WE WILL SOON DISCOVER THAT The Schrodinger EQUATION IS INCOMPLETE IN ITS PREDICTION of how NATURE behAVES. The STATES Left OUT which The J's PRODUCE ARE The ½ INTEGER SPIN STATES. SINCE NATURE TAKES ADVANTAGE of ALL The POSSIBILITIES AVAILABLE TO her, AND Then SOME, Schrodingers EQUATION IS INCOMPLETE. IN order TO APPRECIATE how The PROPERTIES of SPACE ARE reflected IN TO our RESULTS WE'LL HAVE TO GET A LITTLE MORE ABSTRACT IN our REPRESENTATION. BECAUSE WE CAN'T PHYSICALLY describe or INTERPERT WHAT $J_x J_y - J_y J_x$ MEANS THINGS WILL GET MORE COMPLICATED - BUT ONLY TO The HUMAN brAIN.

WE MUST START TO develope our ANALYSIS WITH The FACT THAT The J's ARE HERMETIAN, I.E., $J_{ij} = (J_{ji})^*$. WE CAN define The NON-HERMETIAN OPERATORS

$$J_+ = J_x + i J_y$$
$$J_- = J_x - i J_y$$

WE CAN USE These operators AND WRITE

$$J_z J_+ = J_+ (J_z + 1)$$
$$J_z J_- = J_- (J_z - 1)$$

AND

$$J_+ J_- - J_- J_+ = 2 J_z$$

IT IS ALWAYS POSSIBLE TO find A UNITARY TRANSFORMATION TO DIAGONALIZE A hermetIAN MATRIX SUCH THAT $U^{-1} J U = J'$. When THIS IS REALIZED Then WE CAN WRITE

$$J_z |m\rangle = m |m\rangle$$

GIVEN The STATE $|m\rangle$ WE CAN FIND ANOther STATE BY OPERATING ON $|m\rangle$ WITH J_+. Therefore WE WANT TO ASK WHAT IS The RESULT of $J_z (J_+ |m\rangle)$? USING The formULA JUST WRITTEN down

$$J_z (J_+ |m\rangle) = J_+ (J_z + 1) |m\rangle$$

BUT The RIGHT SIDE IS JUST $J_+ (m+1) |m\rangle$. SINCE $(m+1)$ IS JUST A NUMBER IT CAN COME Through The OPERATOR J_+

Thus we have the answer

$$J_z(J_+|m\rangle) = (m+1)(J_+|m\rangle)$$

which indicates that the eigenvalue has been increased by 1 with the operation of J_z. on $J_+|m\rangle$. Since $J_z|m+1\rangle = (m+1)|m+1\rangle$ we conclude that $J_+|m\rangle$ raises the eigenfunction to the next higher state. We can write this as

$$J_+|m\rangle = C_{m+\frac{1}{2}}|m+1\rangle$$

Where $C_{m+\frac{1}{2}}$ is some constant which I have evaluated between m and m+1.

If I pursue a similar argument for J_- operating on the state $|m\rangle$, I'll find it lowers the state by one and we can write

$$J_-|m\rangle = C_{m-\frac{1}{2}}|m-1\rangle$$

I can find $C_{m+\frac{1}{2}}$ by the mathematical manipulation

$$\langle m+1|J_+|m\rangle = C_{m+\frac{1}{2}}$$

since $(J_+)^\dagger = J_-$. Likewise

$$\langle m|J_-|m+1\rangle = C^*_{m+\frac{1}{2}}$$

We can check what we have done by evaluating $J_+J_- - J_-J_+$ and see if it equals $2J_z$.

$$J_+(J_-|m\rangle) = C^*_{m-\frac{1}{2}} J_+|m-1\rangle$$
$$= C^*_{m-\frac{1}{2}} C_{m-\frac{1}{2}}|m\rangle$$

and

$$J_-(J_+|m\rangle) = C_{m+\frac{1}{2}} J_-|m+1\rangle$$

substituting m+1 for m

$$J_-(J_+|m\rangle) = C_{m+\frac{1}{2}} C^*_{m+\frac{1}{2}}|m\rangle$$

And $2J_z|m\rangle = 2m|m\rangle$ by definition.

It must be true then that

$$|C_{m-\frac{1}{2}}|^2 - |C_{m+\frac{1}{2}}|^2 = 2m$$

To continue on let me substitute the function f_m for the polynomials $|C_{m+\frac{1}{2}}|^2$ such that

$$f_m - f_{m-1} = -2m$$

Now for such a relationship to yield a first order expression in m on the right side f_m must have the form that it is one order higher in m. Thus I can write

$$f_m = am^2 + bm + c$$

and then $f_m - f_{m-1} = -2m$ becomes

$$am^2 + bm + c - a(m-1)^2 - b(m-1) - c = -2m$$

simplifying $\quad 2ma - a + b = -2m$

which means $\quad a = -1$ and $b = -1$. Therefore

$$f_m = -m^2 - m + c = -m(m+1) + c$$

which is mathematically equivalent to

$$f_m = -(m+\tfrac{1}{2})^2 + \gamma$$

where γ is a constant.

Our grand result comes out to be

$$|C_{m+\frac{1}{2}}|^2 = \gamma - (m+\tfrac{1}{2})^2$$

Now γ is a positive number but it appears that after m gets big enough $|C_{m+\frac{1}{2}}|^2$ goes negative and we get no answer. What happens is m cannot continue to increase indefinitely. As J_+ operates on $|m\rangle$ there will a value for m which when raised again by J_+ gives zero. This is the only way for this mess to work out.

Suppose j is the value which m reaches and cannot exceed. This means that $J_+|j\rangle = 0$. Also we have that

$$(J_+J_- - J_-J_+)|j\rangle = 2j$$

or

$$|C_{j-\frac{1}{2}}|^2 = 2j$$

Substituting into our grand formula we find a relationship which γ must satisfy,

$$2j = \gamma - (j - \tfrac{1}{2})^2$$

or

$$\gamma = (j + \tfrac{1}{2})^{1/2}$$

and finally

$$|C_{m+\frac{1}{2}}|^2 = (j+\tfrac{1}{2})^2 - (m+\tfrac{1}{2})^2$$
$$= (j+m)(j+m+1)$$

WERE NOT QUITE DONE BECAUSE WE HAVE TO COUNT DOWNWARD AND WE RUN INTO TROUBLE BECAUSE WE WON'T IN GENERAL REACH $-j$. FOR INSTANCE IF $j = 3.3$ THEN COUNTING DOWN BY ONE WE HAVE $3.3, 2.3, 1.3, +0.3, -0.7, -1.7, -2.7$. THEREFORE WE HAVE TO REQUIRE THAT THE NUMBER OF SPACINGS BETWEEN j AND $-j$ BE INTEGRAL, I.E., $2j$ IS A POSITIVE INTEGER E.G. $2.5, 1.5, 0.5, -0.5, -1.5, -2.5$. THIS TURNS OUT TO BE ALL WE CAN DEDUCE; THERE ISN'T ANY MORE. ARE CONCLUSIONS ARE THAT m CAN RUN OVER THE VALUES $-j$ TO j AND THERE WILL BE $2j+1$ OF THEM. (IF $j = 2.5$, $2j+1 = 6$ AND $-2.5 \le m \le +2.5$).

27. ANGULAR MOMENTUM, SPIN

LAST TIME WE STARTED TO FIND MATRICES WHICH SATISFIED THE COMMUTATOR RELATIONSHIPS LIKE $J_x J_y - J_y J_x = i J_z$. WE IDENTIFIED WITH EACH SYSTEM A TOTAL ANGULAR MOMENTUM j WHICH CAN TAKE ON $2j+1$ VALUE. IN ADDITION WE DENOTED THE z-COMPONENT OF ANGULAR MOMENTUM AS m AND SAW THAT ITS VALUE RANGED FROM $+j$ TO $-j$. THE OPERATOR J_z ON THE STATE $|m\rangle$ PRODUCES THE EIGENVALUES, m, I.E.,

$$J_z |m\rangle = m |m\rangle$$

FURTHER WE FOUND THAT THE SPECIAL OPERATOR J_- HAD THE PECULIAR PROPERTY OF LOWERING THE STATE OF THE SYSTEM,

$$J_- |m\rangle = \sqrt{j(j+1) - m(m-1)}\ |m-1\rangle$$

AND

$$J_+ |m\rangle = \sqrt{j(j+1) - m(m+1)}\ |m+1\rangle$$

SEVERAL INTERESTING AND USEFUL EXAMPLES OF THE ABOVE RELATIONSHIPS ARE THE FOLLOWING:

(1) $j = 0$, $m = 0$, THIS IS THE TRIVIAL CASE OF NO ANGULAR MOMENTUM

(2) $j = 1/2$, $m = \begin{matrix} +1/2 \\ -1/2 \end{matrix} \rangle 1$, $J_- |+\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}(\tfrac{3}{2}) - \tfrac{1}{2}(-\tfrac{1}{2})}\ |-\tfrac{1}{2}\rangle = 1\ |-\tfrac{1}{2}\rangle$

IN THIS CASE OF SPIN 1/2 THE OPERATOR J_- ACTS ON THE STATE TO INVERT IT.

(3) $j = 1$,

$m = 1 \rangle \sqrt{2}$
$m = 0 \rangle \sqrt{2}$
$m = -1$

$J_- |1\rangle = \sqrt{2}\ |0\rangle$
$J_- |0\rangle = \sqrt{2}\ |-1\rangle$
$J_- |-1\rangle = 0$

(4) $j = 3/2$

$m = 3/2 \rangle \sqrt{3}$
$m = 1/2 \rangle \sqrt{4}$
$\cancel{m = 0}$
$m = -1/2 \rangle \sqrt{3}$
$m = +3/2$

$J_- |3/2\rangle = \sqrt{3}\ |1/2\rangle$
$J_- |1/2\rangle = \sqrt{4}\ |-1/2\rangle$
$J_- |-1/2\rangle = \sqrt{3}\ |-3/2\rangle$
$J_- |-3/2\rangle = 0$

(5) $j = 2$

$m = 2 \rangle \sqrt{4}$
$m = 1 \rangle \sqrt{6}$
$m = 0 \rangle \sqrt{6}$
$m = -1 \rangle \sqrt{4}$
$m = -2$

SPIN 1/2 AND ROTATION MATRICES.

I'd LIKE TO NOW STUDY THE SPECIFIC CASE where $j = 1/2$. To DO THIS I WANT TO define The OPERATOR J_z AS

$$J_z = \tfrac{1}{2}\sigma_z$$

Where IN TURN The OPERATOR σ_z hAS The PROPERTY ThAT $\sigma_z |+1/2\rangle = |+1/2\rangle$ AND $\sigma_z |-1/2\rangle = -|-1/2\rangle$. AS AN OPERATOR MATRIX ThEN

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

IN A SIMILAR MANNER I CAN INVENT σ_x AND σ_y. TO DO THIS I REALIZE ThAT $J_+ |+1/2\rangle = 0$ AND $J_- |+1/2\rangle = |-1/2\rangle$. If I ADD These TWO EQUATIONS AND USE The FACT ThAT $J_+ = J_x + iJ_y$ ThEN I hAVE ThAT $2J_x |+1/2\rangle = |-1/2\rangle$ such ThAT

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

BY A SIMILAR MANIPULATION of SUBTRACTING THE TWO EQUATIONS I GET ThAT

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

WE WANT TO FIND OUT WhAT THE ROTATION OPERATOR $\mathcal{D}$ MEANS for The CASE $j = 1/2$. LATER YOU CAN EXTEND IT TO HIGher VALUES of j but The MATh GETS TOO COMPLICATED. FOR AN ARBITRARY ROTATION ABOUT A UNIT VECTOR $\vec{N}$ The OPERATOR $\mathcal{D} = \exp i\theta(\vec{J}\cdot\vec{N})$.

If WE first CONSIDER a ROTATION φ ABOUT The z AXIS ThEN $\mathcal{D}$ OPERATING ON The STATE PRODUCES A PHASE shift, I.E.

$$\mathcal{D}|m\rangle = e^{im\varphi}|m\rangle$$

This IS ANOTher WAY of THE STATE $|m\rangle$ IS of DEFINITE ANGULAR MOMENTUM, AND IT MUST BE CONSERVED UNDER A ROTATION. NOW LET'S TALK ABOUT A ROTATION ABOUT The y AXIS Through AN ANGLE θ; The $\mathcal{D}$ IS GIVEN BY $e^{iJ_y\theta}$ OR ALTERNATELY $\mathcal{D} = \exp \frac{i\sigma_y}{2}\theta$. TO UNDERSTAND WhAT This EXPONENTIAL OPERATOR MEANS WE MUST EXPAND IT IN A POWER SERIES,

$$\mathcal{D} = 1 + \frac{i}{2}\sigma_y\theta + \frac{1}{2!}\left(\frac{i}{2}\sigma_y\theta\right)^2 + \cdots$$

If YOU REALIZE ThAT $(\sigma_y)^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 1$ The UNITY MATRIX, ThEN The SERIES CAN be SEPARATED IN TO TWO

SERIES WHICH ARE $\cos \theta/2$ AND $\sin \theta/2$. THE RESULT BEING

$$\mathcal{D} = 1 \cos \frac{\theta}{2} + i \sigma_y \sin \theta/2$$

EXPANDING THIS OUT

$$\mathcal{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \theta/2 + i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin \theta/2$$

$$= \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix}$$

ALRIGHT NOW, HOW DO YOU USE THIS MATRIX OPERATOR? SUPPOSE YOU HAD A STATE $|+\rangle$ WHICH DESCRIBED A PARTICLE WITH SPIN UP ALONG AN AXIS WHICH IS AT AN ANGLE OF θ TO ANOTHER COORDINATE SYSTEM. WHAT DOES THE STATE $|+\rangle$ LOOK LIKE TO THE OTHER GUY? WORKING OUT THE MATH,

$$\mathcal{D}|+\rangle = \mathcal{D}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \cos \theta/2 \, |+\rangle + \sin \theta/2 \, |-\rangle$$

SIMILARLY for THE SPIN DOWN CASE $|-\rangle$,

$$\mathcal{D}|-\rangle = \cos \theta/2 \, |-\rangle - \sin \theta/2 \, |+\rangle$$

TO CHECK THESE RESULT OBSERVE WHAT HAPPENS IF $\theta = 180°$ AND $90°$. NOTE ALSO THAT THE PROBABILITY IS CONSERVED IN BOTH OF THESE TRANSFORMATION SINCE $\cos^2 \theta/2 + \sin^2 \theta/2 = 1$

I WOULD LIKE TO POINT OUT A VERY REMARKABLE AND CURIOUS THING ABOUT NATURE. IF TWO FULL ROTATIONS ARE MADE ABOUT THE z AXIS IT IS THE SAME AS DOING NOTHING WHILE ROTATION THROUGH $360°$ IS LIKE CHANGING THE SIGN OF THE WAVE FUNCTION. IF THIS DIDN'T HAPPEN WE WOULDN'T GET THE RIGHT PHYSICS. THE EXPLANATION FOR THIS CAN BE SEEN IN THE EQUATION $\mathcal{D}|m\rangle = e^{im\varphi}|m\rangle$

WE HAVE SHOWN THAT m CAN BE ONLY WHOLE OR HALF INTEGERS. FOR INTEGRAL ~~MULT~~ VALUES OF m ROTATION THROUGH 2π PRODUCES A $180°$ PHASE SHIFT WHILE FOR $1/2$ INTEGER VALUES OF m A 4π ROTATION PRODUCES 0 PHASE SHIFT SO THE ORIGINAL WAVE FUNCTION IS RESTORED.

IT IS ALSO INTERESTING TO NOTE THE EXCHANGE of 2 PARTICLES PRODUCES THE SAME AFFECT ON THE STATE of THE SYSTEM AS A ROTATION of ONE PARTICLE RELATIVE TO THE OTHER. BOTH EVENTS CHANGE THE SIGN of THE WAVE FUNCTION. IT IS ALSO TRUE THAT SUCH A PHENOMENA WOULD ~~NO~~ BE IMPOSSIBLE if THE WORLD WERE NON RELATIVISTIC, I.E., ~~THAT IS~~ WITH GALILEAN TRANSFORMATIONS IT IS IMPOSSIBLE TO PROVE THAT SPIN AND STATISTICS ARE RELATED. IT IS ONLY THE CONCEPT of RELATIVISTIC INVARIANCE THAT MAKES IT POSSIBLE TO TIE THE TWO TOGETHER AND DESCRIBE THINGS THE WAY THEY ARE.

COMPOSITION of ANGULAR MOMENTUM FOR 2 SPIN ½ PARTICLES.

SUPPOSE WE NOW COMPLICATE THE PROBLEM BY CONSIDERING A TWO PARTICLE SYSTEM EACH WITH THEIR OWN ANGULAR MOMENTA. EACH PARTICLE (for INSTANCE AN ELECTRON AND PROTON) HAS ITS OWN ANGULAR MOMENTUM j_a AND j_b, SAY, AND ITS OWN z-COMPONENT of ANGULAR MOMENTUM m_a AND m_b RESPECTIVELY. WE WANT TO ASK WHAT ARE THE POSSIBLE STATES for THIS SYSTEM. IT WILL TURN OUT THERE ARE QUITE A few, $(2j_a+1)(2j_b+1)$ IN FACT.

SUPPOSE for THE EXAMPLE WE CONSIDER A SIMPLE ATOM WITH NO NUCLEAR SPIN AND CONSIDER IT IN THE p-STATE, I.E., IT HAS orbital ANGULAR MOMENTUM $L=1$. ALSO WE CONSIDER THE ELECTRON TO HAVE SPIN ½. THEN IN OUR NOTATION $j_a = 1$ AND $j_b = 1/2$. THE GRAND TOTAL ANGULAR MOMENT J HAS A VALUE of 3/2 AND ½. BY our RULE for THE NUMBER of STATES, I.E., $(2j_a+1)(2j_b+1)$ THERE MUST BE A TOTAL of $(2\cdot 1+1)(2\cdot 1/2+1) = 6$ STATES ALTOGETHER. Where ARE THEY?

WELL for THE p-STATE WITH $L=1$ THERE ARE 3 POSSIBLE orbital STATES,

$j_a = L = 1$ IMPLIES $|+1\rangle_a$, $|0\rangle_a$, $|-1\rangle_a$

WHILE for THE ELECTRON THERE ARE TWO POSSIBLE STATES,

$j_b = 1/2$ IMPLIES $|+1/2\rangle_b$ AND $|-1/2\rangle_s$.

The MEANING of These SYMBOLS IS TO IMPLIE The SENSE AND MAGNITUDE of The orbITAL ANGULAr MOMENTUM WHILE AT THE SAME TIME TO CHARACTERIZE The SPIN of The ELECTRON AS EITHER UP OR down. THUS our SIX STATES ARE

$$|+1\rangle_a |+\tfrac{1}{2}\rangle_b, \quad |+1\rangle_a |-\tfrac{1}{2}\rangle_b, \quad |0\rangle_a |+\tfrac{1}{2}\rangle_b, \quad |0\rangle_a |-\tfrac{1}{2}\rangle_b,$$
$$|-1\rangle_a |+\tfrac{1}{2}\rangle_b, \quad |-1\rangle_a |-\tfrac{1}{2}\rangle_b$$

The SYMBOL $|+\rangle_a$ MEANS The ELECTRON IS orbITING IS SOME CLOCKWISE SENSE WITH ONE UNIT OF ANGULAr MOMENTUM, $\hbar$; $|0\rangle_a$ IMPLIES NO ANGULAr MOMENTUM, AND $|-1\rangle_a$ IMPLIES OPPOSITE SENSE rotATION. The SYMBOL $|+\tfrac{1}{2}\rangle_b$ SAYS The ELECTRON hAS SPIN UP; WHILE $|-\tfrac{1}{2}\rangle_b$ dENOTES SPIN down.

FOR NOW WE ARE GOING TO DISREGARd ANY SPIN-orbIT INTERACTION but ~~to~~ RATHER JUST TRY TO COMPOSE from The STATES AVAILABLE THE STATES of dEfINITE ANGULAR MOMENTUM. HOW DO WE DO The ALGEBrA?

FIRST WE SHOULd ASK WHAT IS THE GrAND J_z of The SYSTEM If WE WANT TO CONSTRUCT A STATE of dEfINITE ANGULAR MOMENTUM. BUT WHAT The HELL DOES J_z MEAN? IT IS AN OPERATOR SURE ENOUGH but NOW TWO QUANTITIES ARE INVOLVED. SOMEHOW THOUGH WE MUST INTERPERT IT TO MEAN THAT WE WILL UNdERGO SOME rotATION ϵ ABOUT The z AXIS. IN OTHER WORDS The WAVE FUNCTION CHANGES TO first order by $\psi \to \psi - i\epsilon J_z \psi$. Our QUESTION IS WHAT DOES J_z DO TO ψ.

CONSIDER THE STATE where $\psi = |+1\rangle_a |+\tfrac{1}{2}\rangle$ for which WE rOTATE ABOUT z. Then J_z ON $|+1\rangle_a |+\tfrac{1}{2}\rangle_b$ IS EQUIVALENT TO CHANGING The PHASE by $e^{i\epsilon} e^{i\epsilon/2}$. i.e.,

$$J_z |+1\rangle_a |+\tfrac{1}{2}\rangle_b = e^{i\epsilon} e^{i\epsilon/2} |+1\rangle_a |+\tfrac{1}{2}\rangle_b$$

This IS EQUIVALENT TO A COMBINED $\mathcal{D}$ of $\mathcal{D}^a \mathcal{D}^b$ OR

$$\mathcal{D} = \mathcal{D}^a \mathcal{D}^b = (1 - i\epsilon J_a)(1 - i\epsilon J_b)$$

which TO first order IS JUST

$$\mathcal{D} = 1 + i\epsilon (J_a + J_b)$$

WE CAN CONCLUDE THAT for TWO OBJECTS THE GRAND J_z IS The SUM Of The TWO COMPONENTS $J_{az} + J_{bz}$ AND ALSO $J = J_a + J_b$. THUS THE TWO ANGULAR MOMENTA Add.

LET'S POSTULATE SOME STATE M WHICH IS A COMPLETE STATE Of DEFINITE ANGULAR MOMENTUM; IT MAY be THE PRODUCT of SEVERAL STATES. GIVEN THIS STATE Then THE GRAND J_z ACTING ON IT WILL PRODUCE M, i.e.

$$J_z |M\rangle = M|M\rangle$$

BUT WE HAVE JUST SHOWN THAT $J_z = J_{az} + J_{bz}$ SO THAT

$$(J_{az} + J_{bz})|M\rangle = M|M\rangle$$

Now WHAT THE HELL DOES THIS MEAN? WELL WE have TO WORK OUT THE ALGEBRA for EACH STATE TO SEE WHAT HAPPENS.

WE WILL START OUT WITH THE biggest M VALUE which WE CAN CONSTRUCT. LET ME CALL THIS M-VALUE m_c for COMBINED M AND SIMILARLY j_c for The COMBINE j_a AND j_b. Then LETS START WITH $j_c = 3/2$ $(1 + 1/2)$ AND $m_c = 3/2$. THE STATE WE ARE AFTER IS THEREfore $|+1\rangle_a |+1/2\rangle_b$, There ARE NO STATES WITH higher TOTAL ANGULAR MOMENTUM. WE MUST work OUT

$$(j_{za} + j_{zb})|+1\rangle_a |+1/2\rangle_b$$

The SUBSCRIPTS ARE IMPORTANT IN WORKING OUT THIS EQUATION BECAUSE j_{za} CAN ONLY ACT ON The STATE $|+1\rangle_a$ while SIMILARLY j_{zb} CAN ONLY ACT ON $|+1/2\rangle_b$. Therefore

$$(j_{za} + j_{zb})|+1\rangle_a |+1/2\rangle_b = j_{za}|+1\rangle_a |+1/2\rangle_b + j_{zb}|+1\rangle_a |+1/2\rangle_b$$

Now $j_{za}|+1\rangle_a$ JUST PRODUCES The EIGEN VALUE +1 AND SIMILARLY for $j_{zb}|+1/2\rangle_b = +1/2|+1/2\rangle_b$ SO THAT

$$(j_{za} + j_{zb})|+1\rangle_a |+1/2\rangle_b = +1|+1\rangle_a |+1/2\rangle_b + 1/2|+1\rangle_a |+1/2\rangle_b = 3/2|+1\rangle_a |+1/2\rangle_b$$

WE HAVE JUST SHOWN THAT m_c does IN FACT EQUAL 3/2, i.e.,

$$j_c |+1\rangle_a |+1/2\rangle_b = m_c |m_c\rangle = 1/2 |m_c\rangle$$

Now that we have computed the state $j_c = 3/2$ and $m_c = 3/2$ what about the state $j_c = 3/2$ and $m_c = 1/2$ since we know that m_c must take on the values $3/2, 1/2, -1/2, -3/2$ for $j_c = 3/2$? How do we compute the angular momentum for state $m_c = 1/2$? Well, we use the lowering operator on the state $m_c = 3/2$ which we just determined. Let's do it — Remember our rule

$$j_{-c} \, | m_c = 3/2 \rangle = \sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{3}{2}(\tfrac{3}{2}-1)} \; | \tfrac{3}{2} - 1 \rangle = \sqrt{3} \, | m_c = 1/2 \rangle$$

So ~~we~~ we know what the state $|m_c = 3/2\rangle$ is and all we have to do is operate on it with j_{-c} and divide by $\sqrt{3}$.

$$| m_c = 3/2 \rangle = \frac{1}{\sqrt{3}} \left(j_{-a} + j_{-b} \right) | +1 \rangle_a \, | +1/2 \rangle_b$$

Recalling that j_{-a} will only work on $|+1\rangle_a$ to lower it and similarly for j_{-b} on $|+1/2\rangle_b$ we have that

$$| 1/2 \rangle_c = \frac{1}{\sqrt{3}} \left[\sqrt{2} \, | 0 \rangle_a \, | +1/2 \rangle_b + 1 \, | +1 \rangle_a \, | -1/2 \rangle_b \right]$$

Where again we used the general formula $J_- | m \rangle = \sqrt{j(j+1) - m(m-1)} \; | m-1 \rangle$ but we already worked it out for the case $j = 1$ and $1/2$

$$J_- \, | 1 \rangle = \sqrt{2} \, | 0 \rangle, \quad J_- \, | 1/2 \rangle = | -1/2 \rangle$$

Therefore we see that j_{-c} operating on $|m_c = 3/2\rangle$ state turns the orbit down by one quantum of angular momentum while leaving the spin the same but also leaves the angular momentum the same but turns the electron upside down. Summarizing then

$$\text{for } j_c = 3/2, \; m_c = 1/2, \quad \psi = \sqrt{2/3} \, | 0 \rangle_a \, | +1/2 \rangle_b + \sqrt{1/3} \, | +1 \rangle_a \, | -\tfrac{1}{2} \rangle_a$$

Notice that the total intensity adds to 1 $[(\sqrt{2/3})^2 + (\sqrt{1/3})^2]^{1/2}$. This is always a good check at the end to make sure you didn't screw anything up.

We have another state to work out for $j_c = 3/2$ and it is for $m_c = -1/2$. The $m_c = 1/2$ state can be determine in a similar manner as above since

$$j_{-c} \, | m = 1/2 \rangle_c = \sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - (\tfrac{1}{2})(\tfrac{1}{2}-1)} = \sqrt{4} \; | m_c = -1/2 \rangle$$

THE MATH IS GETTING MORE MESSY BUT LET'S GO AHEAD AND WORK IT OUT,

$$|m_c = -1/2\rangle = \frac{1}{\sqrt{9}}\left(j_{-a} + j_{-b}\right)\left[\sqrt{2/3}\,|0\rangle_a|+\tfrac{1}{2}\rangle_b + \sqrt{1/3}\,|+1\rangle_a|-1/2\rangle_b\right] =$$

$$= \frac{1}{\sqrt{9}}\left[\sqrt{2}\sqrt{2/3}\,|-1\rangle_a|+1/2\rangle_b + \sqrt{2}\sqrt{1/3}\,|0\rangle_a|-1/2\rangle_b + \sqrt{2/3}\,|0\rangle_a|-1/2\rangle_b\right]$$

$$|-1/2\rangle_c = \sqrt{\tfrac{1}{3}}\,|-1\rangle_a|+1/2\rangle_b + \sqrt{2/3}\,|0\rangle_a|-1/2\rangle_b$$

TO GET THE LAST STATE, $|-3/2\rangle_c$ WE WILL LOWER $|-1/2\rangle_c$ AND WE CAN DO IT DIRECTLY FROM THE ABOVE RESULT REALIZING THAT $j_{-a}|-1\rangle_a = 0$ SINCE WE ARE AT THE LAST STATE. THEREFORE WE HAVE

for $j_c = 3/2$, $m_c = -3/2$, $\psi = \sqrt{2/3}\,|-1\rangle_a|-1/2\rangle_b$

TABULATING OUR RESULTS

$$|j_c = 3/2, m_c = 3/2\rangle = 3/2\,|+1\rangle_a|+\tfrac{1}{2}\rangle_b$$

$$|j_c = 3/2, m_c = 1/2\rangle = \sqrt{\tfrac{2}{3}}\,|0\rangle_a|+1/2\rangle_b + \sqrt{1/3}\,|+1\rangle_a|-1/2\rangle_b$$

$$|j_c = 3/2, m_c = -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|-0\rangle_a|+1/2\rangle_b + \sqrt{2/3}\,|0\rangle_a|-1/2\rangle_b$$

$$|j_c = 3/2, m_c = -3/2\rangle = \sqrt{2/3}\,|-1\rangle_a|-1/2\rangle_b$$

ONCE WE GET THE HANG OF THE CALCULATE ~~YO~~ WE CAN GO WILD AND FIND WHAT THE STATES ARE FOR $j_c = 1/2$, I.E., for $m_c = 1/2$ AND $-1/2$. WITH A LITTLE THOUGHT

$$|j_c = 1/2, m_c = 1/2\rangle = \sqrt{1/3}\,|0\rangle_a|+1/2\rangle_b - \sqrt{2/3}\,|+1\rangle_a|-1/2\rangle_b$$

$$|j_c = 1/2, m_c = -1/2\rangle = \sqrt{1/3}\,|0\rangle_a|-1/2\rangle_b - \sqrt{2/3}\,|-1\rangle_a|+1/2\rangle_b$$

ITS NOT MUCH VALUE TO KEEP THIS STUFF UP BECAUSE ALL THESE COEFFICIENTS AND STATES HAVE BEEN WORKED OUT AND TABULATED. THEY ARE CALLED THE CLEBSCH-GORDON COEFFICIENTS AND YOU CAN FIND THEM IN A NUMBER OF BOOK. BUT IF YOU ARE EVER ON A DESERT ISLAND YOU CAN COMPUTE THEM BY STARTING WITH THE GENERAL LOWERING RULE AND THE HIGHEST VALUE for m_c.

28. PROBLEM SOLVING WITH ANGULAR MOMENTUM

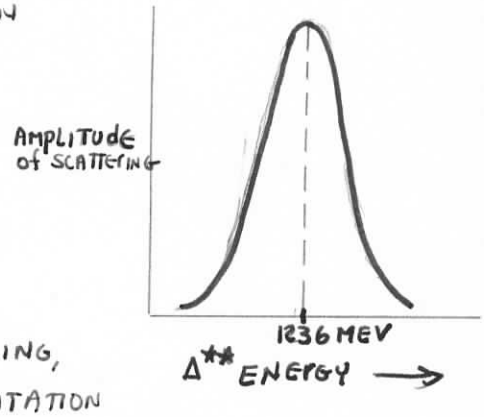
Ref. VOL III
Chapter 17

I'd NOW LIKE TO show YOU how TO USE The CONCEPT of ANGULAR MOMENTUM THAT WE HAVE developed TO SOLVE SOME INTERESTING PROBLEMS. WHAT I HAVE TO DO IN PARTICULAR IS TO DEVELOPE specific EXPRESSION for The $\mathcal{D}$ MATRICES THAT WE HAVE DISCUSSED

RESONANT SCATTERING Of THE Pi-MESON And The PROTON

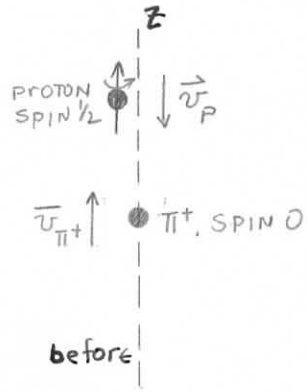
AN INTERESTING EXAMPLE TO START OUT WITH IS SCATTERING of Pi-MESONS, π^+, FROM A PROTON. IT TURNS OUT THAT THERE IS A VERY STRONG RESONANCE SCATTERING for THIS INTERACTION when The PARTICLE form by The INTERACTION, CALLIT Δ^{**}, HAS AN ENERGY of AROUND 1236 MeV. THE INTERACTION IS The FOLLOWING

$$\pi^+ + p \longrightarrow \Delta^{**} \longrightarrow p + \pi^+$$

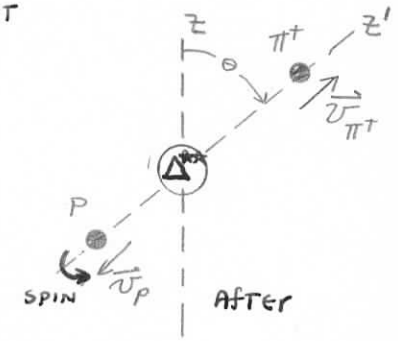


THE EXCITED ENERGY STATE of The Pi MESON And PROTON QUICKLY DISINTEGRATES ~~TO~~ BACK INTO The CONSTITUENT PARTS. The RESONANCE IS QUITE STRONG NEAR 1236 MeV And ALL The SCATTERING, i.e., 99 % of IT, IS DUE TO The RESONANT EXCITATION of The Δ^{**} PARTICLE. WHAT WE WOULD LIKE TO DO IS TO VERIFY THAT The Δ^{**} HAS A TOTAL ANGULAR MOMENTUM of 3/2. The REASON WE ARE INTERESTED IN DOING THAT IS BECAUSE The π^+ HAS A SPIN ZERO And The PROTON HAS AN INTRINSIC SPIN of 1/2. So SOMEHOW DURING The "COLLISION" ANGULAR MOMENTUM WAS IMPARTED TO The SYSTEM. IN order TO UNDERSTAND how The Δ^{**} CAN PICK UP ANGULAR MOMENTUM WE WILL LOOK AT The INTERACTION ~~IN~~ FROM A coordinate SYSTEM IN which The Δ^{**} IS AT REST.

To SIMPLIFY The PROBLEM INITIALLY WE WILL ASSUME The PROTON GAS, or SCATTERING CENTER, HAS BEEN POLARIZED SO THAT ALL The SPIN VECTORS ARE IN AN "UP" direction ALONG SOME Z AXIS which defines The VELOCITY VECTOR of The APPROACHING PION (Pi MESON).



When the particles combine to form the intermediate resonant particle, what state will the Δ^{**} particle be in? Remember that the π^+ has no intrinsic spin while the proton has spin 1/2. This initial condition requires that in order to conserve momentum when the Δ^{**} breaks apart the spin angular momentum along the z' axis must be 1/2. This is required since a particle, the π^+, moving along the z axis cannot contribute ~~to~~ any angular momentum about that axis by virtue of its motion. Since the π^+ has no spin, we know the proton must have spin 1/2 or spin "up" after disintegration. Later we will consider the case where the proton comes in with spin down. After we do that then we can consider the proton gas to be a mixture of spin up or down protons because we can average our two results. Therefore, we know that $j_z = 1/2 = m_{maximum}$. Early we said the total angular momentum of the resonant particle was 3/2. The way the total angular momentum is greater is that in the rest mass coordinate system the π^+ has orbital angular momentum which, upon reaching collision distance with the proton, causes the proton to tumble. The idea is sort of like apply a torque to a little precessing gyro which adds angular momentum. The important difference being here that there are fixed quanta of angular momentum which can be transferred in the interaction.



To summarize then the system is initially in a pure state with a total angular momentum $j = 3/2$ and a z component of angular momentum of $m = 1/2$.

Again our question is to figure out what happens when Δ^{**} disintegrates. We want to see what the scattered proton looks like relative to its initial condition on the z-axis. We have suggested that if we examine

THE INTERACTION from The CENTER of MASS SYSTEM THEN The TWO PARTICLES WILL EXPLODE MOVING AWAY ALONG SOME AXIS z' which IS ROTATED Through AN ANGLE θ RELATIVE TO THE INITIAL z-AXIS. SINCE ANGULAR MOMENTUM ALONG The z' AXIS MUST be $1/2$, WE KNOW THAT $m' = 1/2$. THE ANSWER WE ARE LOOKING for IS The RESULT of ROTATING The $j = 3/2$, $m = 1/2$ STATE Through AN ANGLE θ. IN order TO PROJECT The $m = 1/2$ ONTO The ROTATED AXIS WE MUST COMPUTE The MATRIX ELEMENTS

$$\langle \tfrac{1}{2} | D^{j=3/2}_{R(\theta)} | \tfrac{1}{2} \rangle$$

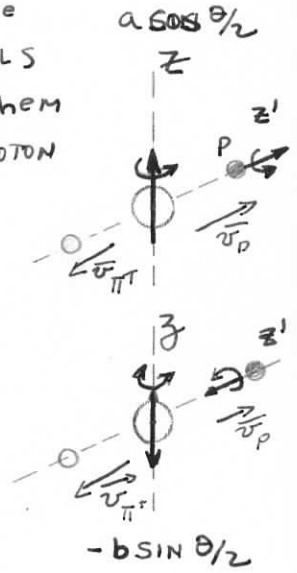
WHAT This MATRIX ELEMENT TELLS US IS how MUCH of The OLD STATE, m, IS IN The NEW STATE m'. WE WILL find THAT The final STATE DOES NOT depend ON ANY ANGULAR RELATIONSHIP-which IS GOOD. The SCATTERING CAN OCCUR AT ANY ANGLE.

NOW ONE IMPORTANT ROTATION RELATIONSHIP which WE WILL USE, AND which WE HAVE ALREADY ESTABLISHED, IS THAT TO find The PROJECTION of The PROTON SPIN (SPIN = $1/2$) AFTERWARD THAT IS ON The ORIGINAL z AXIS. TO DO THIS The AMPLITUDE TO be SPIN "UP" IS $\cos\theta/2$ AND The AMPLITUDE TO be IN A SPIN DOWN STATE IS $-\sin\theta/2$. BY UP AND DOWN WE MEAN AS SEEN ALONG The z' AXIS. If The Δ^{**} HAS ITS SPIN INITIAL "UP", I.E., IN The $+z'$ direction, IT WILL EMIT A PROTON with SPIN, LIKEWISE, IN The $+z'$ direction with AN AMPLITUDE, SAY, a. If Δ^{**} HAS A SPIN DOWN INITIALLY Then The PROTON would DISINTEGRATE with SPIN DOWN ALSO BUT WITH AN AMPLITUDE b. (The AMPLITUDES a AND b ARE UNIQUELY DETERMINED by The INTERNAL GUTS of The DISINTEGRATION PROCESS; The DETAILS of The PROCESS BEING so obscure THAT WE WON'T PURSUE Them NOW). Thus The AMPLITUDE TO find A DISINTEGRATED PROTON SPINNING ALONG The ORIGINAL z AXIS IS

$$\psi_z(\theta) = a\cos\theta/2 - b\sin\theta/2$$

~~Thus~~ The PROBABILITY of SUCH AN EVENT OCCURRING IS

$$f_z(\theta) = |\psi_z(\theta)|^2 = |a|^2\cos^2\theta/2 + |b|\sin^2\theta/2$$



This function can be simplified by using the trigonometric identities $\sin^2\theta/2 = 1/2(1-\cos\theta)$ and $\cos^2\theta/2 = 1/2(1+\cos\theta)$ into the form

$$f(\theta) = \beta(1+\alpha\cos\theta)$$

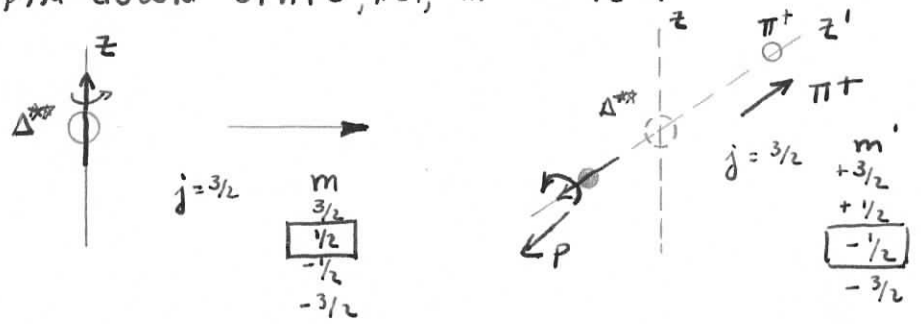
where α and β are functions of a and b.

$$\beta = \left(\frac{|a|^2+|b|^2}{2}\right)$$

$$\alpha = \left(\frac{|a|^2-|b|^2}{|a|^2+|b|^2}\right)$$

If the interaction has the property that parity is conserved then the amplitudes b is equal to a or $-a$. This implies that the two disintegrations are reflections of one another. In this case $\alpha = 0$ and $\beta = |a|^2$. Thus the disintegration is likely to occur in any direction. Usually in these disintegrations there is an angular distribution for the probability of scattering so that the process is asymmetric. That is, it lacks symmetry under a reflection.

Things are starting to get complicated because the are a lot of combinations of final states of the system possible as a result of the disintegration. For instance let's consider the proton initial spinning up so that the initial state is $j = 3/2$, $m = +1/2$. For the $j = 3/2$ state the z component of angular momentum, m can be $3/2, 1/2, -1/2$, or $-3/2$. What would be the amplitude for the disintegration into the spin down state, i.e., $m' = -1/2$?



For disintegration into the same state but with proton spin "down" along the z' axis, the amplitude would be

$$\underbrace{a\left(\langle \tfrac{1}{2}|\mathcal{D}^{3/2}|+\tfrac{1}{2}\rangle a\right)}_{\text{Amp. for + helicity proton}} + \underbrace{a\left(\langle -\tfrac{1}{2}|\mathcal{D}|+\tfrac{1}{2}\rangle b\right)}_{\text{Amp. for - helicity proton}}$$

The general result of this disintegration is to end up not with a "pure" state in the z' coordinate system but with a linear superposition of the four possible states $m' = 3/2, 1/2, -1/2, -3/2$. We must work out the matrix elements in more detail.

Rotation Matrices for Spin 1/2

Let's take the easiest case to begin computing these matrix element. Let's suppose the system has an initial total angular momentum $j = 1/2$ so that $m = 1/2$ or $-1/2$. This is the case we have already worked out. If $m = 1/2$ initially then the two possible final states of $m' = 1/2$ and $-1/2$ have the following amplitudes

$$\langle r\tfrac{1}{2} | \mathcal{D}^{j=1/2} | +1/2 \rangle = \cos \theta/2$$

$$\langle -\tfrac{1}{2} | \mathcal{D}^{j=1/2} | +1/2 \rangle = +\sin \theta/2$$

If $m = -1/2$ initially then disintegration into $m' = +1/2$ or $-1/2$ is given by the amplitudes

$$\langle +1/2 | \mathcal{D}^{j=1/2} | -1/2 \rangle = -\sin \theta/2$$

$$\langle -1/2 | \mathcal{D}^{j=1/2} | -1/2 \rangle = \cos \theta/2$$

These are very useful transformations to remember when working these kinds of problems. Lets put them in a little table

Initial state	×	=	Final state
$\lvert + \rangle$	c	=	$\lvert + \rangle$
$\lvert + \rangle$	s	=	$\lvert - \rangle$
$\lvert - \rangle$	$-s$	=	$\lvert + \rangle$
$\lvert - \rangle$	$+c$	=	$\lvert - \rangle$

where c stands for $\cos \theta/2$ and s stands for $\sin \theta/2$.

To illustrate this rotation business let's assume the disintegration of a $j = 1/2$ particle with $m = +1/2$ into an $m' = -1/2$ state. That is we want the amplitude

$$a\left(\langle 1/2 | \mathcal{D}^{1/2} | +1/2 \rangle a\right) + a\left(\langle -1/2 | \mathcal{D}^{1/2} | +\tfrac{1}{2} \rangle b\right)$$

NOW ITS JUST A MATTER of LOOKING AT OUR TABLE BECAUSE WE HAVE The MATRIX ELEMENTS ALREADY WORKED OUT. THAT IS

$$\text{AMPLITUDE PROTON UP} = a^2 \cos \theta/2 + ab \sin \theta/2$$

The PROBABILITY of SCATTERING INTO THIS STATE IS Then

$$a^4 \cos^2 \theta/2 + a^2 b^2 \sin^2 \theta/2 \longrightarrow \alpha + \beta \cos\theta$$

This is The CASE WE DISCUSSED EARLIER. WE COULD ASK ALSO AT THIS TIME WHAT IS The AMPLITUDE TO SCATTER WITH The PROTON INITIAL DOWN, WE would HAVE THAT

$$\text{amp for proton down} = b(\langle +\tfrac{1}{2} | \mathcal{D}^{j=1/2} | -\tfrac{1}{2} \rangle a) + b(\langle -\tfrac{1}{2} | \mathcal{D}^{j=1/2} | -\tfrac{1}{2} \rangle b)$$

$$\text{probability of scattering with protons down} = a^2 b^2 \sin^2 \theta/2 + b^4 \cos^2 \theta/2$$

Therefore, if WE HAD SCATTERING IN A ~~DIRTY~~ BARREL of dirty PROTONS, I.E. TOTALLY UNPOLARIZED Then WE CAN JUST AVERAGE The UP AND DOWN CASES TO GET The UNPOLARIZED PROBABILITY distribution of

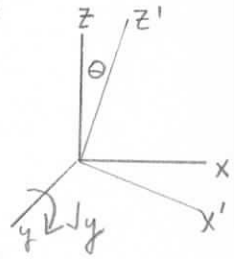
$$\frac{a^4 + b^4}{2} \cos^2 \theta/2 + a^2 b^2 \sin^2 \theta/2$$

NOTE if PARITY IS CONSERVED, $|b| = |a|$, Then The SCATTERING IS EQUALLY PROBABLE OVER ALL ANGLES. This IMPLIES THAT YOU STAY IN The SAME STATE.

ALL This STUFF IS VERY PRETTY BECAUSE WE CAN PREDICT A LOT FROM VERY LITTLE. ALL WE NEED TO ASSUME IS The superposition RULE AND WE CAN WORK OUT A hellova LOT. IT IS THIS Approach WE PURSUE IN TRYING TO WORK OUT The PHYSICS of STRONG INTERACTIONS BECAUSE WE DON'T UNDERSTAND INTERNAL MECHANISM. WE DO UNDERSTAND The PROPERTIES of SPACE AND THAT ALONE IS SUFFICIENT TO MAKE A LOT of PREDICTIONS.

ROTATION MATRICES for SPIN 3/2 PARTICLE

NOW LET'S RETURN TO The INITIAL PROBLEM of COMPUTE The $\mathcal{D}$ for $j = 3/2$. WE KNOW The PROBLEM IS reducible TO A ROTATION ABOUT The Z AXIS by AN ANGLE θ. Therefore



$$\mathcal{D}^{j=3/2}_{R(\theta)} = e^{i J_y \theta}$$

Where J_y is The ROTATION GENERATOR ABOUT The Y AXIS.

WE NOW MUST WORK OUT $\langle -1/2 | \mathcal{D}^{j=3/2} | +1/2 \rangle$. TO DO THAT WE HAVE TO EXPAND OUT THE EXPONENTIAL FORM OF $\mathcal{D}$,

$$\langle -1/2 | \; 1 + i J_y \theta - J_y^2 \frac{\theta^2}{2} + \cdots | 1/2 \rangle$$

SO WE HAVE A SERIES OF TERMS TO COMPUTE. EACH TERM GIVES US AN ELEMENT OF THE ROTATION MATRIX. RECALL NOW THAT WE HAVE WORKED OUT WHAT THE RAISING AND LOWER OPERATORS J_+ AND J_- DO ON THE STATES $3/2, 1/2, -1/2, -3/2$, I.E.,

$$J_- | 3/2 \rangle = \sqrt{3} \, | 1/2 \rangle \quad \rightarrow \quad \langle 1/2 | J_- | 3/2 \rangle = \sqrt{3}$$
$$J_- | 1/2 \rangle = \sqrt{4} \, | -1/2 \rangle \quad \rightarrow \quad \langle -1/2 | J_- | 1/2 \rangle = \sqrt{4}$$
$$J_- | -1/2 \rangle = \sqrt{3} \, | -3/2 \rangle \quad \rightarrow \quad \langle -3/2 | J_- | -1/2 \rangle = \sqrt{3}$$
$$J_- | -3/2 \rangle = 0$$

AND

$$J_+ | 3/2 \rangle = 0 \quad \rightarrow$$
$$J_+ | 1/2 \rangle = \sqrt{3} \, | 3/2 \rangle \quad \rightarrow \quad \langle 3/2 | J_+ | +1/2 \rangle = \sqrt{3}$$
$$J_+ | -1/2 \rangle = \sqrt{4} \, | +1/2 \rangle \quad \rightarrow \quad \langle 1/2 | J_+ | -1/2 \rangle = \sqrt{4}$$
$$J_+ | -3/2 \rangle = \sqrt{3} \, | -1/2 \rangle \quad \rightarrow \quad \langle -1/2 | J_+ | -3/2 \rangle = \sqrt{3}$$

WE DON'T WANT J_+ OR J_- BUT RATHER J_y. RECALLING THE DEFINITION OF $J_\pm = J_x \pm i J_y$ WE CAN OBTAIN J_y BY SUBTRACTING J_+ FROM J_-, I.E.,

$$J_y = \frac{1}{2i} \left(J_+ - J_- \right)$$

SO WHAT IS THE MATRIX ELEMENT $\langle -1/2 | J_y | +1/2 \rangle$? WELL IT IS

$$\frac{1}{2i} \langle -1/2 | J_+ - J_- | +1/2 \rangle = \frac{1}{2i} \left(\sqrt{0} - \sqrt{4} \right) = -\frac{1}{i}$$

YOU CAN KEEP GOING FOR ALL THE POSSIBLE 16 MATRIX ELEMENTS BUT YOU CAN SEE THINGS ARE GETTING QUITE MESSY BECAUSE YOU HAVE TO SQUARE THE J_y, ETC. IT IS A HELLUVA WAY TO RUN A RAILROAD AND IF IT KEPT GOING WE'D ALL BE LOST. WOULDN'T IT BE NICE IF THE SERIES $e^{iJ_y\theta}$ WAS CLOSED AND AFTER SOME POWER THE SERIES STARTED A RECURSION, I.E., THAT YOU CAN EXPRESS THE FINAL TERM IN SOME LINEAR COMBINATION OF THE PREVIOUS TERMS? WITHOUT PURSUING THE POINT THE J'S DO HAVE THIS PROPERTY AND FOR $j = 3/2$ THE J_y^4 TERM IS THE LAST ONE. IT CAN BE EXPRESSED IN TERMS OF J_y^3, J_y^2, J_y, AND A CONSTANT.

The WAY WE find The recursion relationship is, ~~To~~ AS AN EXPAMPLE TO form The operator $(J_z - 3/2)(J_z - 1/2)(J_z + 1/2)(J_z + 3/2)$ for The CASE $j = 3/2$ AND LET IT SEPARATELY OPERATE ON The four STATE $|3/2\rangle$, $|1/2\rangle$, $|-1/2\rangle$, $|-3/2\rangle$. BECAUSE The OPERATOR produces 0 for EACH TERM WE CAN WRITE

$$(J_z - 3/2)(J_z - 1/2)(J_z + 1/2)(J_z + 3/2) = 0$$

This CAN be EXPANDED OUT TO GET

$$J_z^4 - 5/2\, J_z^2 + 9/16 = 0 \quad \rightarrow \quad J_z^4 = 5/2\, J_z^2 - 9/16$$

The idea for J_Y is JUST The SAME BECAUSE SOME GUYS Z AXIS is SOME other GUY'S Y AXIS. The RESULT IS GENERAL. For SPIN 1/2 The SERIES STOPS AFTER The 3RD TERM SINCE $(J_z + 1/2)(J_z - 1/2) = 0$ which IMPLIES $J_z^2 = 1/4$

EASY WAY TO FIND THE ROTATION MATRICES

LET'S TRY TO find The $\mathcal{D}$'s IN ANOther WAY which ISN'T SO MESSY. This IS PARTICULARLY APPEALING TO DO BECAUSE The $\mathcal{D}$'s don'T depend ON WHAT The hell The problem or MODEL IS THAT YOU ARE WORKING ON - IT COULD be A RESONANCE, IT COULD be AN ATOM, IT COULD BE ANYTHING.

LET'S START WITH AN OBJECT OF SPIN 1/2. IT HAS TWO STATES $|+\rangle$ AND $|-\rangle$; They correspond TO SPIN "UP" AND "down". For A SYSTEM WITH TWO SPIN 1/2 OBJECTS WE CAN MAKE A SYSTEM WITH TOTAL $j = 1$ or 0. LET'S DENOTE The POSSIBLE STATES for $j = 1$ AS

$$j = 1 \qquad |++\rangle \;,\; \frac{1}{\sqrt{2}}|+-\rangle + \frac{1}{\sqrt{2}}|-+\rangle \;,\; |--\rangle$$

WE NEED TO CONTINUE TO MODEL SYSTEMS WITH different ANGULAR MOMENTUM.

For A 3 OBJECT SYSTEM EACH WITH SPIN 1/2 WE CAN HAVE A MAXIMUM J_z of 3/2 which WE WILL denote by $|+++\rangle$, I.E., ALL SPINS UP. NOW WE CAN WORK ON This STATE WITH The LOWERING OPERATOR TO GET IT DOWN TO The STATE $J_z = 1/2$.

If we do that we find that there are 3 particles to flip over individually so we can have the state

$$j_z = 1/2 \quad \rightarrow \quad \frac{1}{\sqrt{3}}\left(|++-\rangle + |+-+\rangle + |-++\rangle \right)$$

We can keep going for the states $j_z = -1/2$ and $j_z = -3/2$

$$j_z = -1/2 \qquad 1/\sqrt{3}\left(|--+\rangle + |-+-\rangle + |+--\rangle \right)$$

And

$$j_z = -3/2 \qquad |---\rangle$$

For more objects the extension is straightforward.

Now let's see how this will help us work out the matrices like $\langle -1/2 | \mathcal{D}^{j=3/2} | +1/2 \rangle$. If we don't worry about the order of the states we can write this as

$$\langle --+ | \mathcal{D}^{j=3/2} | ++- \rangle$$

We want all the different ways that we can turn the state $|++-\rangle$ into the state $|--+\rangle$. Here's where our little transformation rule comes in because we want to flip the spin sense several different time. Given that our ~~tha~~ table

+	→ +	c
+	→ −	s
−	→ +	−s
−	→ −	s

already computes the amplitudes for us we can go right ahead a figure out the various combinations. There are three ways to do it

Initial	+	+	−		+	+	−		+	+	−				
	↓	↓	↓	+	↓	↓	↓	+	↓	↓	↓				
Final	+	−	−		−	+	−		−	−	+				
Amp	c	s	c	+	s	c	c	+	s	s	(−s)	$= 2c^2s - s^3$			

Thus the total amplitude is just $2\cos^2\theta/2 \sin\theta/2 - \sin^3\theta/2$. It's magic! Further it's a helluva easier. To exercise this cute idea further what about $\langle +1/2 | \mathcal{D}^{3/2} | +1/2 \rangle$?

Initial	+	+	−		+	+	−		+	+	−				
	↓	↓	↓	+	↓	↓	↓	+	↓	↓	↓				
Final	+	+	−		−	+	+		+	−	+				
Amplitude	c	c	c		s	c	(−s)		c	s	(−s)	$= c^3 - 2s^2c$			

or total amplitude is $\cos^3\theta/2 - 2\sin^2\theta/2 \cos\theta/2$

YOU CAN SIMPLIFY THESE RESULTS FURTHER USING THE VARIOUS TRIGONOMETRIC IDENTITIES IF YOU WANT BUT THE IMPORTANT THING IS HOW EASY IT CAN BE FIGURED OUT.

IF WE GO BACK TO OUR PROTON SCATTERING PROBLEM where WE WERE FIGURING OUT THE AMPLITUDE TO START OUT WITH AN "UP" PROTON AND TO END WITH AN "UP" PROTON ON THE Z' AXIS WE HAD THE AMPLITUDE GIVEN BY

$$a\left(\langle +1/2 | D^{j=3/2} | +1/2 \rangle a\right) + a\left(\langle -1/2 | D^{j=3/2} | +1/2 \rangle b\right)$$

WHICH CAN EASILY BE WORKED OUT NOW AS

$$a^2 (c^3 - 2s^2 c) + ab(2c^2 s - s^3)$$

THE PROBABILITY OF SCATTERING THE PROTON UP IS THEN

$$a^4 (c^3 - 2s^2 c) + a^2 b^2 (s^3 - 2c^2 s)^2$$

IF THE PROCESS CONSERVES PARITY THEN $|b| = |a|$ AND WE CAN EXPAND OUT THE FUNCTIONS AS

$$a^4 (c^6 + s^6)$$

IF WE WRITE $\cos^2\theta/2 = \frac{1+\cos\theta}{2} = \frac{1+x}{2}$ AND $\sin^2\theta/2 = \frac{1-\cos\theta}{2} = \frac{1-x}{2}$

THEN THE PROBABILITY OF SCATTERING BECOMES

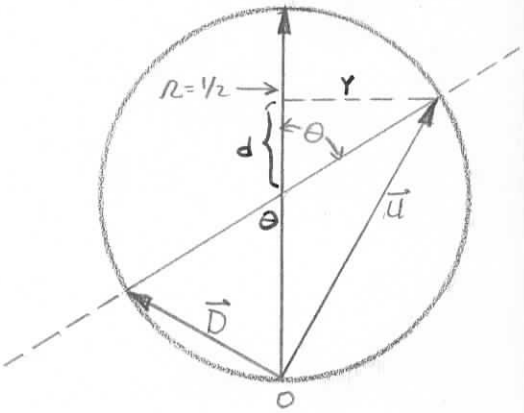
$$\frac{a^4}{4}(1+3x^2) = \frac{a^4}{4}(1+3\cos^2\theta)$$

THUS WE HAVE WORKED OUT THE COMPLETE ANGULAR DEPENDENCE OF THE SCATTERING OF A SPIN 1/2 PARTICLE.

A CUTE GEOMETRICAL WAY OF COMPUTING THE AMPLITUDE OF SCATTERING A SPIN 1/2 PARTICLE INTO A DIRECTION θ IS BY DRAWING A UNIT CIRCLE. THE LENGTH OF THE VECTOR $\vec{U}$ GIVES THE ~~AMPLITUDE~~ PROBABILITY TO SCATTER UP AND $|\vec{D}|$ GIVES THE PROBABILITY ~~AMPLITUDE~~ TO SCATTER DOWN.

$$|\vec{u}|^2 = (1/2 + d)^2 + Y^2$$
$$= (1/2 + 1/2\cos\theta)^2 + 1/4 \sin^2\theta$$
$$= \frac{1}{4}(1+\cos\theta)^2 + \frac{1}{4}\sin^2\theta = \underline{\cos^2\theta/2}$$

$$|\vec{D}|^2 = 1/4 + 1/4 - 2\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\cos 2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$
$$= \sin^2\theta/2$$



SOMEDAY CHILDREN IN ELEMETRY SCHOOL WILL LEARN QUANTUM MECHANICS THIS WAY!

29. LIE GROUPS, SU(2) AND SU(3)

There EXISTS IN The REALM of MATHEMATICS A SUBJECT CALLED LIE GROUPS AND LIE ALGEBRA. SOME of YOU MAY be FAMILIAR WITH The SU(2) AND SU(3) GROUPS BECAUSE They ARE USEFUL IN INTERPERTING The RESULTS of STRONG INTERACTIONS. LET'S TAKE A LITTLE TIME TO UNDERSTAND PHYSICALLY WHAT The hell They ARE AND how WE USE The MATHEMATICS IMPLIED by Their forms.

IN A MATHEMATICAL SENSE The Theory of GROUPS deals with A SET, USUALLY AN INFINITE SET, of ELEMENTS which ARE RELATED To EACH other by CERTAIN COMBINATION RULES. ONE SUCH RULE IS The COMMUTATOR RELATIONSHIP which WE HAVE DISCUSSED before. GROUP MULTIPLICATIONS NEED NOT be COMMUTATIVE but when They ARE The GROUP IS CALLED ABELIAN. When WE SPEAK of A GROUP REPRESENTATION WE ARE TALKING ABOUT A SET of SQUARE (AND NONSINGULAR) MATRICES EACH MATRIX "REPRESENTING" EACH GROUP ELEMENT. If The MATRICES ARE $n \times n$ Then WE HAVE A N-DIMENSIONAL REPRESENTATION. EVERY ~~DOES~~ CONTINUOUS GROUP HAS A MATRIX REPRESENTATION.

So much for The PRELIMINARIES - WHAT KIND of CONTINUOUS GROUPS ARE POSSIBLE? ONE CONTINUOUS GROUP IS The SET of ROTATIONS IN 3 DIMENSIONS; or IT COULD be A 4-DIMENSIONAL ROTATION, ETC. BECAUSE ROTATIONS ARE MADE UP of A INFINITE SET of INFINITESIMAL ROTATIONS They ARE INdeed Lee GROUPS. THIS R3 GROUP describes ALL ROTATIONS IN EUCLIDEAN Three-SPACE.

While SPATIAL ROTATIONS ARE ONE CLASSIFICATION of A GROUP WE CAN REPRESENT The PHYSICAL EQUIVALENT by POLARIZATION of LIGHT, AS AN EXAMPLE. As YOU PROBABLY KNOW, if I START AN ELLIPTICALLY POLARIZED LIGHT Through A SERIES of QUARTER WAVE PLATES, birefringent MATERIAL, AND other JUNK, The Thing COMING OUT WILL be of DIFFERENT SHAPE THAN WHAT WENT IN. IN PARTICULAR if There WASN'T ANY ABSORBTION IN The PATH Then I COULD HAVE JUST HAVE EASILY REPLACED ALL The POLARIZING STUFF WITH ONE PIECE which ~~wf~~ PERFORMED The EQUALENT POLARIZATION ON The WAVE.

WE CAN CONCLUDE THAT THE OPERATION ON LIGHT AT EACH "THING" PRODUCES A TOTAL EFFECT AFTER A SERIES OF THESE "THINGS" WHICH IS EQUIVALENT TO ANOTHER OBJECT OF THE SAME CLASS. THAT IS TO SAY WITH ANOTHER MEMBER OF THE POLARIZER SET WE CAN UNDO WHAT WE DID. THE GROUP IS THEN THE EFFECT CREATED BY PUTING LIGHT THROUGH THE BIREFRINGENT GUNK. THE OPERATION IS TO DISTURB THE POLARIZATION.

ANOTHER EQUIVALENT GROUP TO THE PHYSICAL ONE JUST DESCRIBED IS THE ONE COMPOSED OF A 2X2 MATRIX REPRESENTATION $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ WHERE THE ELEMENT CAN BE COMPLEX. WE WILL CONSIDER THE MATRIX TO BE RESTRICTED TO THE CLASS WITH ITS DETERMINANT 1 AND FURTHERMORE IT MUST BE UNITARY. THAT MEANS ITS INVERSE MUST EQUAL ITS HERMETIAN CONJUGATE, $M^{-1} = M^{+}$. GROUPS OF THIS GENERIC CLASS HAVE $n^2 - 1$ INDEPENDENT PARAMETERS, I.E., FOR A 2X2 REPRESENTATION THERE ARE 3 REAL PARAMETERS. THE SIMPLEST EXAMPLE IS ~~E~~ THE EULER ANGLE REPRESENTATION OF THREE DIMENSIONAL ROTATIONS, I.E., THE 3 ROTATION ANGLES ARE α, β, γ. THE SET OF 2X2 UNITARY MATRICES IS CALLED SU(2).

YOU MIGHT BE WONDERING WHY THE 2X2 COMPLEX MATRIX JUST DESCRIBED IS IDENTICAL TO A 3X3 REAL ROTATION MATRIX. WELL, ITS AN UNFORTUNATE COINCIDENCE THAT THE 3X3 REAL MATRIX, R3, AND SU(2) ARE THE SAME. IT IS JUST THE WAY THINGS TURNED OUT.

IN ADDITION TO THE LIE GROUPS JUST DESCRIBED THERE ARE ALL SORTS OF OTHER POSSIBILITIES. THE IS A WHOLE SET OF REAL ORTHOGONAL GROUPS R3, 4, 5, ..., N. THERE ARE A WHOLE SET OF THE SPECIAL UNITARYE MATRICES: SU(2), SU(3), ---SU(N). IN ADDIATION TO THESE GROUPS THERE IS A SPECIAL GROUP OF A FINITE ORDER WHICH CORRESPONDS TO THE LORENTZ TRANSFORMATIONS. THEN THERE ARE A COUPLE OF NUTBALL MATTICES WHICH DON'T FALL INTO ANY OF THE ABOVE CLASSES; THESE ARE CALLED E AND F ~~AND~~ GROUPS AND ARE 8X8 OR 16X16 - I DON'T REMEMBER. BEYOND THE 16TH ORDER OR SO THE GROUP CLASSIFICATION, FORTUNATELY, GETS SIMPLER.

WHEN LIE WAS WORKING ON THIS MATERIAL he NOTICED The IMPORTANT FACT THAT by STUDING The PROPERTIES of INFINITESIMALLY SMALL ROTATIONS, I.E., Those of The form $1 + i\epsilon G_j$, THAT he could GENERATE ALL THE FINITE OPERATIONS. FurTher he observed THAT The GENERATORS G_j COULD be WRITTEN AS A LINEAR COMBINATION of A COEFFICIENT, A REAL CONSTANT CALLED The GROUP STRUCTURE CONSTANT, TIMES A CERTAIN SUBSET of The G'S, I.E.,

$$G = \sum_j C_j G_j$$

The TOTAL NUMBER of INDEPENDENT G'S is A FUNDAMENTAL PROPERTY of The GROUP. The OTHER CRITCAL ~~ALGGRAD~~ ALGEBRAIC RELATIONSHIP ESTABLISHED by LIE IS THE COMMUTATOR RELATIONSHIP

$$G_i G_j - G_j G_i = \sum_k C_{ij}^{k} G_k$$

WE MENTIONED THAT FOR The SU2 GROUP There ARE 3 INDEPENDENT VARIABLES. THIS RESULTS FROM INITIALLY 8 PARAMETERS (2 PARAMETERS for EACH COMPLEX ELEMENT) BUT BECAUSE The MATRIX MUST be UNITARY WE CUT OUT 4 PARAMETERS; The fifth PARAMETER IS LOST DUE TO The ~~DE~~ RESTRICTION ON The DETERMINENT TO be 1. The GENERAL RULE of $n^2 - 1$ INDEPENDENT PARAMETERS INDICATES for The SU3 MATRIX, WE MUST HAVE 8 PARAMETERS. FOR The REAL ORTHOGONAL MATRICES The NUMBER of INDEPENDENT PARAMETERS IS $\frac{1}{2} n(n-1)$. FOR The R(3) MATRIX There ARE AGAIN 3 INDEPENDENT PARAMETERS. for ANY OTHER order of SU AND R The NUMBER of PARAMETERS ARE NOT EQUAL.

MOST of The TIME WE WILL WORK WITH IRREDUCIBLE REPRESENTATIONS of THE ROTATION GROUP. The IRREDUCIBILITY of A MATRIX IMPLIES THAT YOU HAVE FOUND The FUNDAMENTAL MATRIX REPRESENTATION which CANNOT be further SIMPLIFIED by CHANGING The BASE. Of INTEREST ARE The D^j MATRICES which CANNOT be REDUCED TO DIAGONAL BLOCKS WITH DIMENSIONS SMALLER THAN $(2j+1)$. AS AN EXAMPLE TAKE A 6X6 MATRIX IF THIS MATRIX OPERATES ON A 6 COMPONENT VECTOR, IT IS The SAME AS MULTIPLYING The first 4 COMPONENTS by The 4X4 MATRIX AND The LAST TWO by The 2X2 MATRIX.

$$\left(\begin{array}{cccc|cc} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 \\ a_{21} & \cdot & \cdot & \cdot & 0 & 0 \\ a_{31} & \cdot & \cdot & \cdot & 0 & 0 \\ a_{41} & \cdot & \cdot & a_{44} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{array}\right)$$

The 0 -COMPONENTS DO NOT COUPLE The 4X4 WITH The 2X2 MATRIX.

APPLICATIONS of The SU2 GROUP TO PHYSICS

LET'S SEE WHAT THE PREVIOUS MATHEMATIC REPRESENTATIONS MEAN WHEN WE APPLY The CONCEPTS TO A PHYSICAL SYSTEM. WE WILL BEGIN WITH The SU2 GROUP which IMMEDIATELY LIMITS US TO A 2 STATE SYSTEM. WE WILL DENOTE The TWO STATES AS $|1\rangle$ AND $|2\rangle$. ASSOCIATED WITH EACH STATE IS AN AMPLITUDE TO BE IN EITHER STATE (These AMPLITUDE COEFFICIENTS correspond TO The STRUCTURE CONSTANTS while The STATES $|1\rangle$ AND $|2\rangle$ ARE The Group MEMBERS; WE WILL CONSIDER here 2 INDEPENDENT SETS). The SYSTEM Then HAS AN AMPLITUDE describable by

$$|\psi\rangle = C_1 |1\rangle + C_2 |2\rangle$$

NOW The first IMPORTANT POINT TO ESTABLISH IS The STATE REPRESENTATION IN SOME OTHER COORDINATE SYSTEM - CALL IT The PRIME SYSTEM. The NEW STATES $|1'\rangle$ AND $|2'\rangle$ WILL, IN GENERAL, BE LINEAR COMBINATIONS of $|1\rangle$ AND $|2\rangle$, i.e.,

$$|1'\rangle = \alpha_{11} |1\rangle + \alpha_{12} |2\rangle$$
$$|2'\rangle = \alpha_{21} |1\rangle + \alpha_{22} |2\rangle$$

WHILE

$$|\psi\rangle = C_1' |1'\rangle + C_2' |2'\rangle$$

Thus The AMPLITUDES TRANSFORM ACCORDING TO The OPERATION

$$\begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

The α MATRIX Produces The SU2 Group AND IS ITS REPRESENTATION.

AS AN EXAMPLE, The ELECTRON IS A TWO STATE SYSTEM WITH SPIN EITHER $+\frac{1}{2}$ or $-\frac{1}{2}$. Thus WE HAVE The CASE where $|+\frac{1}{2}\rangle = |1\rangle$ AND $|-\frac{1}{2}\rangle = |2\rangle$. The ABOVE REPRESENTATION JUST CONFIRMS WHAT WE HAVE BEEN DOING ALL ALONG; WE HAVE BEEN PROJECTING A SPIN STATE ALONG ANOTHER SET of AXES TO find The ROTATED AMPLITUDE TO BE IN The STATE. WE SAW THAT The REPRESENTATION WAS JUST A LINEAR ONE.

STRONG INTERACTIONS

LET'S CONSIDER ANOTHER TWO STATE SYSTEM THIS TIME IN A LITTLE MORE DETAIL. CONSIDER WHAT HAPPENS WHEN A NEUTRON AND PROTON INTERACT THROUGH SOME COMPLICATED WAY - A NUCLEUS IS FORMED. THE COMPLEX INTERACTION FORCES DEPEND ON EVERYTHING; THE WAY THE PROTON IS SPINNING WHEN IT COMES TOGETHER WITH THE NEUTRON, AND ALL SORTS OF OTHER MESSY THINGS. IF WE, FOR THE MOMENT, CONSIDER THE TWO MASSES ARE EQUAL, THEN IN THE MESS LIES A MIRACULOUS SIMPLICITY WHICH SAYS THAT A NEUTRON - PROTON INTERACTION IS DESCRIBABLE BY THE INTERACTIONS OF A PROTON - PROTON OR A NEUTRON - NEUTRON. THAT IS, THE INTERACTION FORCE JUST BECOMES A FUNCTION OF DISTANCE.

NOW, OBVIOUSLY, SUCH AN EQUAL MASS ASSUMPTION LEADS TO SOME FALLACIOUS PHYSICAL RESULTS BECAUSE WE ALL KNOW THAT A PROTON REPELS ANOTHER PROTON WHEN THE TWO ARE BROUGHT CLOSE TOGETHER. ALTERNATELY THE NEUTRONS BEING NEUTRAL DO NOT EXPERIENCE THE ELECTRICAL REPULSION FORCE. THEREFORE, IN A PROTON - NEUTRON INTERACTION THERE ARE TERMS INVOLVING THE NEUCLEAR AND ELECTRICAL FORCES. FOR A PROTON AND NEUTRON THE SLIGHT MASS DIFFERENTIAL LEADS TO THE ELECTRICAL DISSIMILARITY WHILE THE NUCLEAR FORCES ARE IDENTICAL FOR BOTH PARTICLES. THE ELECTRICAL FORCE INVOLVED HERE IS A FACTOR OF 137 TIMES SMALLER THAN THE NUCLEAR FORCES SO WE ARE IGNORING A FRACTION OF A PERCENT CORRECTION TO THE NUCLEAR FORCES. QUANTUM ELECTRODYNAMICS LET'S US APPROPRIATE CALCULATE ALL THE CORRECTION FORCES.

WHAT WE WANT TO CONSIDER THEN IS THE STRONG INTERACTIONS, I.E., THOSE INTERACTIONS WHICH IGNORE THE ELECTRICAL FORCES. BY DOING THAT THE P-P, P-N, AND N-N ARE ALL ONE IN THE SAME AND CAN BE EASILY STUDIED SINCE THE DYNAMIC FORCES INVOLVED ARE ALL THE SAME. TO SIMPLIFY THINGS LET'S CALL A NEW PARTICLE THE NUCLEON. THE NUCLEON CAN BE IN ONE OF TWO STATES: EITHER IT CAN BE A NEUTRON OR IT CAN BE A PROTON. SOMETIMES WE WON'T KNOW WHETHER WE ARE DEALING WITH A

PROTON OR A NEUTRON BECAUSE THE MATHEMATICS BLINDS US. TO CHECK AND SEE WHAT WE HAVE PHYSICALLY WE CAN TURN ON SOME ELECTRICITY.

BY MAKING THE ABOVE ASSUMPTIONS THE THEORY OF 2 STATE SYSTEMS LET'S US WORK OUT THE AMPLITUDE TO CREATE NEUCLEONS, N' BY COMPUTING $C_1 |N\rangle + C_2 |P\rangle$.

LET'S CONSIDER AN OBJECT MADE UP OF 2 NUCLEONS LIKE A DEUTRON. SINCE THE PROTON AND NEUTRON LEAD TO DEGENERATE STATES WHAT KINDS OF STATES CAN WE MAKE? IT TURNS OUT THAT WE CAN MAKE THE FOLLOWING STATES,

$$|N\rangle|N\rangle, \quad \frac{1}{\sqrt{2}}\left(|N\rangle|P\rangle + |P\rangle|N\rangle\right), \quad \frac{1}{\sqrt{2}}\left(|N\rangle|P\rangle - |P\rangle|N\rangle\right), |P\rangle|P\rangle$$

THE $|P\rangle|P\rangle$ STATE TURNS OUT NOT TO BE A BOUND STATE; HOWEVER, IT DOES HAVE A SCATTERING RESONANCE JUST ABOVE THE 0 BINDING CONDITION. THE SAME HOLDS TRUE FOR THE $|N\rangle|N\rangle$ AND $|N\rangle|P\rangle + |P\rangle|N\rangle$ STATES; THEY ARE NOT BOUND. THE $|N\rangle|P\rangle - |P\rangle|N\rangle$ STATE IS A BOUND STATE AND NEVER CHANGES.

TO ILLUSTRATE THE ABOVE THERE ARE THREE ELEMENTS, BERYLLIUM 10, BORON 10 AND CARBON 10 WHICH HAVE THE FOLLOWING STATE DESCRIPTION

Be^{10}	B^{10}	C^{10}
6N 4P	5N 5P	6P 4N

IF WE SUBTRACT A NUCLEAR BALL OF 4N AND 4P WE GET A STATE CORRESPONDING TO

$$|N\rangle|N\rangle \qquad |N\rangle|P\rangle \qquad |P\rangle|P\rangle$$

THIS MANIPULATION IS JUST LIKE WHAT WE DID FOR THE ELECTRON WITH TWO SPINS $|+\tfrac{1}{2}\rangle$ AND $|-\tfrac{1}{2}\rangle$. WE FOUND THE FOLLOWING STATES POSSIBLE

$$|+\tfrac{1}{2}\rangle|+\tfrac{1}{2}\rangle, \quad \frac{1}{\sqrt{2}}\left(|+\tfrac{1}{2},-\tfrac{1}{2}\rangle + |-\tfrac{1}{2},+\tfrac{1}{2}\rangle\right), \quad \frac{1}{\sqrt{2}}\left(|+\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},+\tfrac{1}{2}\rangle\right)$$

$$|-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

THE MATH IS THE SAME FOR BOTH PROBLEMS SO THE SU2 GROUP CAN BE APPLIED.

30 PION - NUCLEON SCATTERING

I'd LIKE TO WORK OUT A PROBLEM IN The AREA of STRONG INTERACTIONS WHICH Should GIVE YOU A BETTER GUT feel for how TO MANIPULATE This SU2 STUFF. LET'S TAKE The CASE of A Pi MESON SCATTERING off of A NUCLEON.

The Pi MESON CAN BE IN ONE of Three STATES: π^+, π^0, or π^-. THE NUCLEON IS IN TWO POSSIBLE STATES: P OR N., I.E, ITS A PROTON or A NEUTRON. IN ORDER TO Keep OUR SENSES Through The PROBLEM WE'LL CONSIDER CONSIDER The PROTON TO BE CONTINUOUSLY POLARIZED Through The SCATTERING PROCESS; Thus we don'T have TO worry About SPIN FLIP. SOME of The CASE POSSIBLE ARE

CASE 1 $\quad \pi^+ + P \longrightarrow \pi^+ + P$

CASE 2 $\quad \pi^- + P \longrightarrow \pi^- + P$

$\xrightarrow{\text{OR}} \pi^0 + N$

CASE 3 $\quad \pi^+ + N \longrightarrow \pi^+ + N$

$\xrightarrow{\text{OR}} \pi^0 + P$

CASE 4 $\quad \pi^- + N \longrightarrow \pi^- + N$

There ISN'T A REACTION shown for The π^0 PARTICLE BECAUSE IT DISINTEGRATES ABOUT 10^8 TIMES AS FAST AS The π^- And π^+. FOR EACH of The ABOVE CASES There IS A different AMPLITUDE for INTERACTION. IN TOTAL There ARE 10 POSSIBLE AMPLITUDES for SCATTERING. BUT BECAUSE of The MARVELOUS SYMMETRY which NUCLEAR forces ~~SPACE~~ EXhibits WE ARE ABLE TO REDUCE The NUMBER of INdependENT AMPLITUDES DOWN TO JUST 2. LET'S SEE how IT hAPPENS.

THE PROPERTY of STRONG NUCLER forces which PRODUCES The MARVELOUS SYMMETRY IS NOT UNDERSTOOD. WE WILL, however, REFER TO IT AS ISOTOPIC SPIN SYMMETRY. BECAUSE The NUCLEON EXISTS IN ONE of TWO POSSIBLE STATES IT IS EASIER

FOR THE MIND TO THINK IN TERMS OF THE TWO STATE SYSTEM OF AN OBJECT OF SPIN 1/2 IN A THREE DIMENSIONAL SPACE. THIS FAKE SPACE THE PHYSICISTS CALL ISOTOPIC SPIN SPACE. IT IS JUST A TRICK THEY PULL TO SIMPLIFY THEIR WORK BECAUSE THE PROBLEM OF STRONG INTERACTION REDUCES TO ONE OF MANIPULATING A SU2 GROUP. SINCE ALL THE SCATTERING AMPLITUDE COEFFICIENTS HAVE BEEN TABULATED FOR THE ANGULAR MOMENTUM ANALOG, THE PHYSICIST CAN GO TO A BOOK AND LOOK UP HIS ANSWER. ITS SNEAKY WHAT THEORETICAL PHYSICIST DO BUT THE INCREDIBLE THING IS ITS WORKS AS FAR AS WE KNOW. EVEN THOUGH WE DON'T UNDERSTAND THE DYNAMICAL LAWS GOVERNING THE NATURE OF THE TWO INDEPENDENT SCATTERING AMPLITUDES, WE CAN DEDUCE A LOT BECAUSE OF THIS AMAZING SYMMETRY LAW.

BY CONSTRUCTING THIS ISOTOPIC SPIN SPACE WE ARE ABLE TO CLASSIFY THE NUCLEON AS AN OBJECT WITH ISOTOPIC SPIN 1/2 WHILE THE PI MESON IS AN OBJECT WITH ISOTOPIC SPIN 1. TO WORK OUT THE AMPLITUDES FOR COMBINING THE VARIOUS CASES ENUMERATED ABOVE WE WILL BE GUIDED BY THE CONSERVATION RULE OF PRESERVING THE SYSTEM'S ISOTOPIC SPIN SYMMETRY. THIS IS ANALOGOUS TO CONSERVING ANGULAR MOMENTUM.

THE PROBLEM IS NOW REDUCED TO A 2 OBJECT SYSTEM; THE PION, π, AND THE NUCLEON η, WITH ISOTOPIC SPIN $I=1$ AND $I=1/2$ RESPECTIVELY. THE TOTAL SYSTEM WILL HAVE AN ISOTOPIC SPIN OF $I=3/2$ OR $I=1/2$. FOR THE STATE $I=3/2$ THERE ARE 4 POSSIBLE SUBSTATES OF THE SYSTEM; THEY ARE ANALOGOUS TO THOSE ON PAGE 137:

$$|+1\rangle|+1/2\rangle$$

$$\sqrt{2/3}\,|0\rangle|+1/2\rangle + \sqrt{1/2}\,|+1\rangle|-1/2\rangle$$

$$\sqrt{2/3}\,|0\rangle|-1/2\rangle + \sqrt{1/3}\,|-1\rangle|+1/2\rangle$$

$$|-1\rangle|-1/2\rangle$$

FOR THE STATE OF TOTAL ISOTOPIC SPIN 1/2 THERE ARE TWO POSSIBLE STATES, AGAIN ANALOGOUS TO PAGE 137:

$$\sqrt{1/3}\,|0\rangle\,|+1/2\rangle - \sqrt{2/3}\,|+1\rangle\,|-1/2\rangle$$

$$\sqrt{1/3}\,|0\rangle\,|-1/2\rangle - \sqrt{2/3}\,|-1\rangle\,|+1/2\rangle$$

WE SEE THAT FOR $I = 3/2, 1/2$ THERE EXIST A QUARTET AND DOUBLET STATE DESCRIPTION OF THE PION AND NUCLEON. SINCE THE PION HAS 3 POSSIBLE STATES π^+, π^0, π^- AND THE NUCLEON, n, HAS 2 POSSIBLE STATES $P\,(+1/2)$ AND $N\,(-1/2)$ WE CAN REWRITE THE DOUBLET AND QUARTET IN A MORE SIMPLIFIED NOTATION:

$$|I = 3/2,\ I_z = 3/2\rangle \longleftrightarrow (\pi^+, P) \qquad (a)$$

$$|I = 3/2,\ I_z = 1/2\rangle \longleftrightarrow \sqrt{2/3}\,(\pi^0 P) + \sqrt{1/3}\,(\pi^+ N) \qquad (b)$$

$$|I = 3/2,\ I_z = -1/2\rangle \longleftrightarrow \sqrt{2/3}\,(\pi^0 N) + \sqrt{1/3}\,(\pi^- P) \qquad (c)$$

$$|I = 3/2,\ I_z = -3/2\rangle \longleftrightarrow (\pi^-, N) \qquad (d)$$

$$|I = 1/2,\ I_z = +1/2\rangle \longleftrightarrow \sqrt{1/3}\,(\pi^0, P) - \sqrt{2/3}\,(\pi^+, N) \qquad (e)$$

$$|I = 1/2,\ I_z = -1/2\rangle \longleftrightarrow \sqrt{1/3}\,(\pi^0, N) - \sqrt{2/3}\,(\pi^-, P) \qquad (f)$$

IN ORDER TO USE THIS CRAZY STUFF TO WORK OUT SCATTERING AMPLITUDES FOR DIFFERENT REACTIONS IT WOULD BE WISE TO TURN SEVERAL OF THE ABOVE REPRESENTATIONS AROUND. SOLVING FOR (π^0, N) AND (π^-, P) FROM c AND f WE HAVE THAT

$$(\pi^0, N) = \sqrt{2/3}\,|3/2\rangle|-1/2\rangle + \sqrt{1/3}\,|+1/2\rangle|-1/2\rangle \qquad (G)$$

$$(\pi^-, P) = +\sqrt{1/3}\,|+3/2\rangle|-1/2\rangle - \sqrt{2/3}\,|+1/2\rangle|-1/2\rangle \qquad (h)$$

SOLVING FOR (π^0, P) AND (π^+, N) FROM b AND e WE HAVE THAT

$$(\pi^0, P) = \sqrt{2/3}\,|3/2\rangle|+1/2\rangle + \sqrt{1/3}\,|+1/2\rangle|+1/2\rangle \qquad (i)$$

$$(\pi^+, N) = \sqrt{1/3}\,|+3/2\rangle|+1/2\rangle - \sqrt{2/3}\,|+1/2\rangle|+1/2\rangle \qquad (j)$$

FINALLY FROM a AND d WE HAVE THE TWO PURE STATE

$$(\pi^+, P) = |3/2\rangle|3/2\rangle$$

$$(\pi^-, N) = |3/2\rangle|-3/2\rangle$$

WE NOW HAVE THE MACHINERY WE NEED TO CALCULATE THE VARIOUS SCATTERING AMPLITUDES. THE RULE OF THE GAME IS TO CONSERVE THE TOTAL ISOTOPIC SPIN OF THE SYSTEM. IF $I = 3/2$ INITIALLY, I MUST EQUAL $3/2$ AFTER THE SCATTERING. THE SAME IS TRUE IF $I = 1/2$ INITIALLY. WE WILL DEFINE THE AMPLITUDE TO SCATTER IF $I = 3/2$ a_3 AND THE AMPLITUDE TO SCATTER IF $I = 1/2$ WILL BE DENOTED BY a_1. THE EXACT EQUATIONS DETERMINING a_3 AND a_1 ARE THE MYSTERIES OF NATURE WHICH WE DON'T UNDERSTAND. BUT WITH a_3 AND a_1 WE CAN EXPRESS ALL THE OTHER POSSIBLE INTERACTIONS AS LINEAR COMBINATIONS OF THESE TWO ~~SET~~ AMPLITUDES.

THE AMPLITUDE a_3 IS THE DESCRIPTION OF THE SCATTERING,

$$\pi^+ + P \longrightarrow \pi^+ + P$$

THAT IS, THE AMPLITUDE TO START WITH A PURE STATE OF $I = 3/2$ AND END UP IN THE SAME STATE IS JUST a_3.

LET'S SEE WHAT THE AMPLITUDE FOR THE FOLLOWING REACTION TO OCCUR,

$$\pi^- + p \longrightarrow \pi^0 + N$$

THIS IS MORE COMPLICATED SO LET'S ~~A~~ WRITE IT OUT:

THE AMPLITUDE TO START WITH π^- AND p AND SCATTER INTO π^0 AND N IS THE SUM OF TWO TERMS WHERE THE FIRST TERM ESTABLISHES THE AMPLITUDE THAT (π^-, P) IS IN THE $I = 3/2$ WITH $I_z = -1/2$ $(+\sqrt{1/3})$ ~~FROM~~ FROM EQ. h TIMES a_3 WHICH IS THE AMPLITUDE TO SCATTER IF IT IS IN $I = 3/2$ STATE TIMES THE AMPLITUDE TO BE $A(\pi^0, N)$ IF $I = 3/2$, $I_z = -1/2$ $= \sqrt{2/3}$ FROM ~~C~~. THE SECOND AMPLITUDE IS THE PRODUCT OF 3 SIMILAR TERMS: FIRST THE AMPLITUDE THAT $(\pi^- P)$ IS IN THE STATE $I = 1/2$ $(-\sqrt{2/3}$ FROM h) TIMES THE AMPLITUDE TO SCATTER IF $I = 1/2 = a_1$ TIMES THE AMPLITUDE TO BE $A(\pi^0 N)$ IF $I = 1/2$ ($\sqrt{1/3}$ FROM f). TOGETHER THE SCATTERING AMPLITUDES LOOKS LIKE

$$\sqrt{1/3} \cdot a_3 \cdot \sqrt{2/3} + \sqrt{2/3} \cdot a_1 \cdot \sqrt{1/3} = \frac{\sqrt{2}}{3}(a_3 - a_1)$$

AS A SECOND EXAMPLE THE SCATTERING AMPLITUDE ~~TO~~ DESCRIBING THE FOLLOWING REACTION IS

$$(\pi^- P) \longrightarrow (\pi^-, P)$$

$$\text{AMPLITUDE} = \underset{\text{EQ.h} \cdot a_3 \cdot \text{EQ.C.}}{\sqrt{1/3} \cdot a_3 \cdot \sqrt{1/3}} + \underset{\text{EQ.h} \cdot a_1 \cdot (\text{EQ.f})}{(-\sqrt{2/3}) \cdot a_1 \cdot (-\sqrt{2/3})} = \frac{1}{3}(a_3 + 2a_1)$$

= AMP TO be IN The $|3/2, -1/2\rangle$ • AMP TO SCATTER AS $I = 3/2$ • AMP TO SCATTER from $|3/2, -1/2\rangle$ INTO $(\pi^- P)$ + AMPLITUDE TO be IN The $|1/2, -1/2\rangle$ STATE • AMP TO SCATTER AS $I = 1/2$ • AMP. TO SCATTER FROM $|1/2, -1/2\rangle$ INTO (π^-, P).

IN The world of Theoretical Physics IT IS COMMON TO RENAME The Above AMPLITUDE IN The following WAY

	AMPLITUDE	REACTION
f_+	$= a_3$	$\pi^+ P \to \pi^+ P$
f_-	$= 1/3(a_3 + 2a_1)$	$\pi^- P \to \pi^- P$
f_{ex}	$= \sqrt{2}/3\,(a_3 - a_1)$	$\pi^- P \to \pi^0 N$

where $f_{exchange} = \dfrac{f_+ - f_-}{\sqrt{2}}$. This IS ONLY FOR CONVENIENCE TO REWRITE The AMPLITUDES IN This FORM.

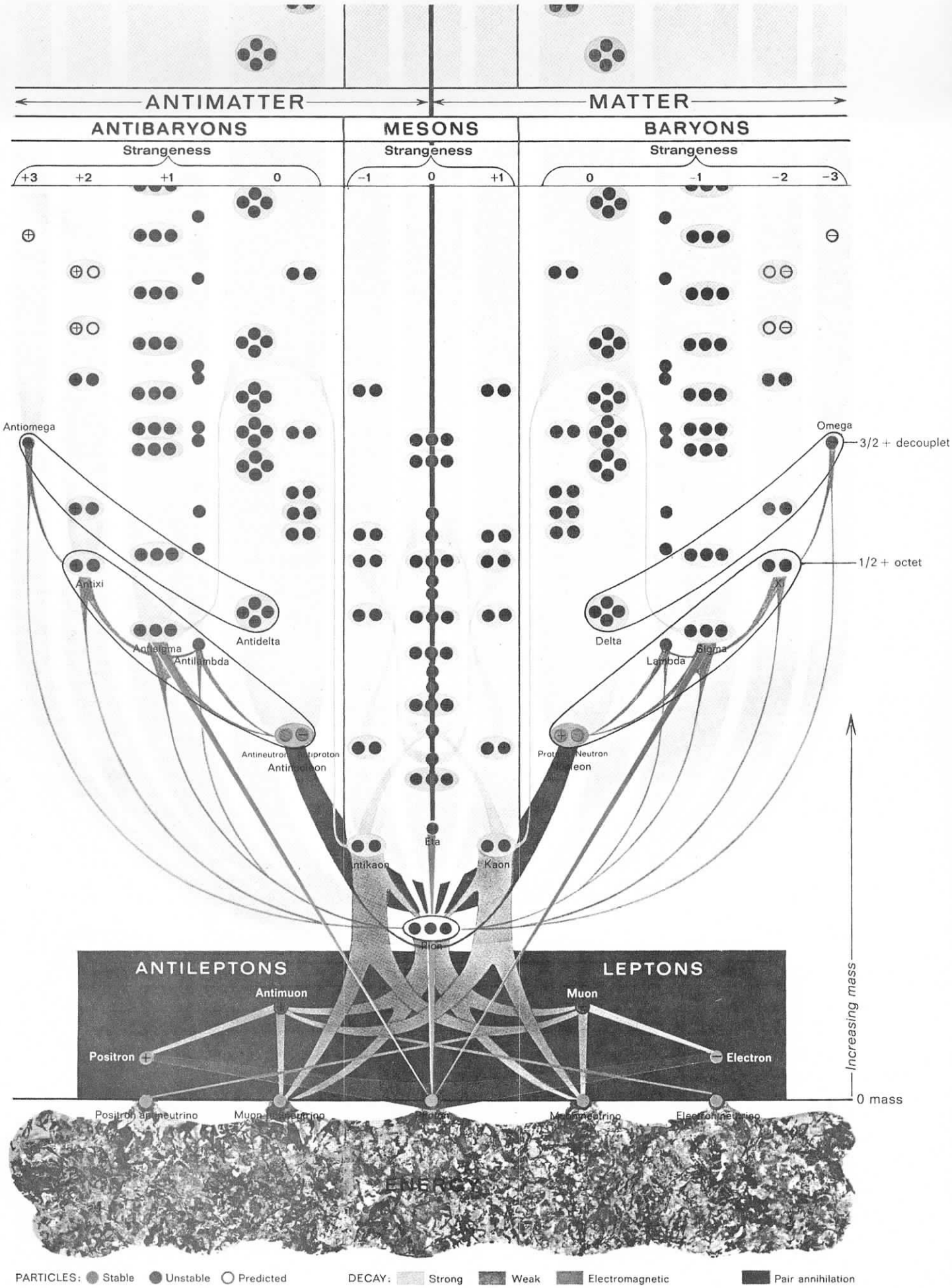
The other Three SCATTERING AMPLITUDES ARE FOUND IN A SIMILAR WAY TO The Above AMPLITUDES

$$\pi^- N \to \pi^- N = a_1$$
$$\pi^+ N \to \pi^+ N = 1/3\,(a_3 + 2a_1)$$
$$\pi^+ N \to \pi^0 P = \sqrt{2}/3\,(a_3 - a_1)$$

ONCE YOU GET The HANG OF ~~ho~~ IT CALCULATING The VARIOUS AMPLITUDES IS EASY SINCE WE HAVE ALREADY WORKED Them OUT. THE Theory of The SU2 GROUP IS ONE of The GUIDING PRINCIPALS OF STRONG INTERACTIONS. IT IS VERY WORTHWHILE GETTING FAMILIAR WITH IT.

SU3 GROUP

THE ADVANTAGE of WORKING WITH The SU2 Group IN The PREVIOUS CASE of STRONGLY INTERACTING PARTICLES WAS The VERY CLOSE ANALOG THAT WE found WITH ANGULAR MOMENTUM. IN effect OUR WORK WAS DONE ONCE WE HAD DRAWN The ANALOGY. There ARE OTHER PARTICLE WHICH INTERACT STRONGLY for WHICH WE ARE NOT SO FORTUNATE IN BEING ABLE TO CONTINUE ON WITH The SU2 ANALOGY. IN ITS PLACE The NEW PARTICLES FORM higher order MULTIPLETS LIKE OCTETS AND DECIMETS. BECAUSE The SU3 GROUP IS reducible DOWN TO 8 INdependent PARAMETERS The Group REPRESENTATION CONCEPT IS APPLIED TO These other PARTICLES. THINGS TEND TO GET VERY CONFUSING VERY FAST BECAUSE There IS A LOTE of PIddLING AROUNd TO WORK OUT ALL The COEFFICIENTS. IN ESSENSE WHAT WE WOULd LIKE TO DO IS TO EXTEND OUR IDEA of The PION-NUCLEON INTERACTIONS TO NEW PARTICLE REACTIONS LIKE The K-MESONS, The LAMBdA PARTICLES, SIGMA AND OMEGA PARTICLES, ETC. A VERY GRAPHICAL REPRESENTATION of The FAMILY of ELEMENTARY PARTICLES IS SHOWN ON The NEXT PAGE. This DIAGRAM APPEARED IN The JUNE 1968 ISSUE of FORTUNE MAGAZINE. The OCCURRENCE of The OCTET AND DECIMET GROUPINGS AS ENCLOSED BY The BLACK LINE have LED TO A formulation of The SYMMETRY LAWS GOVERNING MEMBERSHIP IN EACH Group. The DIAGRAM IN EFFECT IS ANALOGOUS TO The MENdeleyev TABLE of The ATOMIC ELEMENTS. THUS IS IS POSSIBLE TO CONSTRUCT THESE MULTIPLETS AND if There ARE ANY MISSING PARTICLES IS POSSIBLE PREDICT ITS EXISTENCE IN ADDITION A PRIORI ESTABLISH ITS ISOTOPIC SPIN, CHARGE, MASS, AND STRANGENESS. ONE SUCH PARTICLE PREDICTED THIS WAY WAS The Ω MINUS, The heaviest member of The decimet Group. ONE CAUTIONING NOTE ALL The IMPORTANT PARAMETERS EXCEPT for The ANGULAR MOMENTUM HAVE BEEN VERIFIED EXPERIMENTALLY for The Ω^-. If IT TURNS OUT TO BE different from The VALUE $I = 3/2$ EXPECTED, MAN WILL GET The FICKLE FINGER of FATE AWARd from GOd.



160

Diagram by Max Gschwind

WHAT WE HOPE TO BE ABLE TO DO WITH THE SU3 GROUP IN UNDERSTANDING THE GUIDING PRINCIPLES BEHIND THE APPARENT PERFECT SYMMETRY OF THESE PARTICLES IS TO REPRESENT IN THE FEWEST NUMBER OF CONSTANTS POSSIBLE ALL THE MULTITUDE OF INTERACTIONS. SO FAR, HOWEVER, THE SU3 GROUP IS NOT THE PERFECT MACHINERY THE SU2 GROUP IS. THE DISPARITIES IN MASS OF THE INTERACTING PARTICLES DISTURBS THE INHERENT SYMMETRY REPRESENTED BY THE SU3 GROUP. AT BEST THE SU3 GROUP ALLOWS US TO MAKE PREDICTIONS WHICH MIGHT BE IN ERROR BY AS MUCH AS 30%. BY ASSUMING THAT THE SU3 GROUP IS INDEED PERFECT WE CAN GET A SEMI-QUALITATIVE APPRECIATION FOR A PARTICULAR REACTION, I.E, IF ITS WEAK WE ONLY GET IN THE BALL PARK; OR IF ITS A BIG ~~FES~~ INTERACTION, WE WILL KNOW THAT. WE ARE IN SORT OF A SOUP BECAUSE WE CAN'T TELL HOW GOOD THE THEORY IS.

THE IDEA OF THE SU3 GROUP IS TO START WITH A SYSTEM WITH 3 POSSIBLE STATES, SAY $|A\rangle$, $|B\rangle$, AND $|C\rangle$ WHICH MIGHT CORRESPOND TO STRANGENESS 0, -1, AND -2. WE CAN PERFORM A TRANSFORM ON THESE STATES BY TAKING THE COMPLEX CONJUGATE OF THE MATRIX ELEMENT, I.E, WE CAN CONSTRUCT THE STATES $|\bar{A}\rangle$, $|\bar{B}\rangle$, AND $|\bar{C}\rangle$. PHYSICALLY ITS LIKE CONSIDERING THE ANTIPARTICLES. THERE ARE NINE POSSIBLE STATES OF THE PARTICLE-ANTIPARTICLE ENSEMBLE CONSIDERED. THERE IS A SINGLET REPRESENTATION WHICH REMAINS INVARIANT UNDER ROTATION; IT IS THE STATE

$$\frac{1}{\sqrt{3}}\left(|A\rangle|A\rangle + |B\rangle|B\rangle + |C\rangle|C\rangle\right)$$

THE OTHER EIGHT POSSIBILITIES CAN BE REPRESENTED DIAGRAMMATICALLY AS

ISOTOPIC SPIN				STRANGENESS
1/2	$\bar{B}C$	$C\bar{A}$		-1
1	$\bar{B}A$	$\frac{1}{\sqrt{2}}(\bar{A}A-B\bar{B})$	$\bar{A}B$	0
0		$\frac{1}{\sqrt{6}}(A\bar{A}+B\bar{B}-2C\bar{C})$		
1/2		$\bar{C}A$	$\bar{C}B$	1
	-	0	+	← CHARGE

THE FORM of The OCTET JUST DIAGRAM IS EXEMPLIED BY THE OCTET of PARTICLES ON The DIAGRAM of PAGE .

	ISOTOPIC SPIN	STRANGENESS
Ξ^- Ξ^0	1/2	−1
Σ^- Σ^0 Σ^+	1	0
Λ	0	0
N P	1/2	+1

The DECIMET IS A LITTLE MORE COMPLICATED; The EXAMPLE BEING

					I	S
			Ω^-	1674 Mev		−2
		Ξ^0	Ξ^-	1528 Mev	1/2	−1
	Y^+	Y^0	Y^-	1382 Mev		0
Δ^{++}	Δ^+	Δ^0	Δ^-	1236 Mev		1

31. MORE ON THE SU3 STUFF AND QUARKS

LAST TIME WE WERE DISCUSSING HOW THE SU3 GROUP CAN BE USED TO DESCRIBE THE APPARENT SYMMETRIES AND PATTERNS EXISTING BETWEEN THE VARIOUS ELEMENTARY PARTICLES. IN STUDYING THE NATURE OF THE SU3 GROUP IT IS CONVENIENT TO LABEL THE STATES IN A NEW WAY BY THINKING IN TERMS OF "QUARKS." A QUARK IS AN IMAGINARY OBJECT CONJURED UP TO MAKE THIS STUFF EASIER FOR THE HUMAN MIND TO UNDERSTAND. QUARKS DO NOT EXIST; OR I BETTER SAY THEY HAVEN'T BEEN FOUND YET.

THERE ARE SOME PHYSICISTS WHO HAVE CONVINCED THE GOVERNMENT THAT IT IS IN THE INTEREST OF THE ~~GOVERNMENT~~ A SCIENCE THAT THEY BE GRANTED THOUSANDS OF DOLLARS TO GO HUNT FOR QUARKS IN THE GODDAMNEST PLACES. - IT COULDN'T BE IN A NICE HANDY PLACE LIKE THE LA BREA TAR PITS. ~~TA~~ NO, THEY HAVE TO GO GET A TON OF CRAP OUT OF THE DEEPEST PACIFIC TRENCH, HAUL IT BACK TO A LABORATORY, ELECTROLYZE THE HELL OUT OF IT FOR 10 YEARS, GO OUT WITH THEIR GIRLFRIEND, AND ULTIMATELY PROCLAIM THERE AREN'T ANY QUARKS AROUND. THIS IS WHAT I PAY TAXES FOR! SO MUCH FOR THE EDITORIALIZING.

THE QUARKS ARE DESCRIBED TO HAVE ~~AN ISOTOPIC~~ SPIN 1/2, A CHARGE OF 1/3 AND ARE CAPABLE OF HAVING THREE DISTINCTS QUALITIES OR CHARACTERS CALL THEM A, B, AND C. SOME PEOPLE CONTEND THAT THE THREE QUARKS CAN BE BOUND TOGETHER SOMEHOW. SO FAR WE CAN'T EXPLAIN ~~BY~~ BY ANY KNOWN DYNAMICAL LAWS HOW SUCH PARTICLES WOULD BE BOUND SINCE THE REST MASS IS GREATER THAN 4 BeV.

LET'S SEE HOW THE QUARKERIANS PLAY THEIR GAME. SUPPOSE WE HAVE THE QUARKS EXISTING IN THE FOLLOWING STATES:

$$C^{+1/3} \qquad Y = -2/3 \qquad I = 0$$

$$A^{-1/3}, B^{2/3} \qquad Y = +1/3 \qquad I = 1/2$$

THE SUPERSCRIPTS DENOTE THE CHARGE NUMBER; Y = THE HYPER~~ONIC~~ CHARGE WHICH IS THE STRANGENESS + THE BARYON CHARGE.

FROM THESE STATES WE CAN ASSEMBLE The following SET of 4 STATES for The case $Y = 1$ AND $I = 3/2$

-1	0	$+1$	$+2$	ELECTRIC Charge
(AAA)	(AAB)	(ABB)	(BBB)	

The NOTATION here A IS A ShorthAND for A MORE EXPLICIT REPRESENTATION

$$(AAB) = \frac{1}{\sqrt{3}}\left(|A\rangle|A\rangle|B\rangle + |A\rangle|B\rangle|A\rangle + |B\rangle|A\rangle|A\rangle\right)$$

The OTher STATES ARE ANALOGOUS. These Then ARE The 4 STATES for $y = +1$ or STRANGENESS $= 0$. They ARE MORE COMMONLY CALLED Δ BARYONS : Δ^-, Δ^0, Δ^+, Δ^{++} - This WE NOTED LAST TIME but didN'T EXPLAIN where WE GOT The PARTICLES from.

If WE NOW bring IN The C PARTICLE, WE CAN form The TRIPLET STATES for $Y = 0$ AND $I = 1$:

-1	0	$+1$	charge
CAA	CAB	CBB	
(Y^-)	(Y^0)	Y^+	"Y-PARTICLES"

The NEXT STATE IS A DOUPLET STATE for $Y = -1$, $I = 1/2$; IT IS

$-$	0
CCA	CCB
Ξ^{*-}	Ξ^{*0}

FIND The SINGLET IS formed for $Y = -2$, $I = 0$;

$$CCC, \ \Omega^-$$

This IS The GREAT DECIMET. EACH STATE Differs by About 146 MeV from The NEXT. IT IS Theorized THAT The C QUARK CARRIES ALL The MASS, i.e., $m_A = m_B = 0$. If WE PURSUE This IDEA further AND RECALL THAT WE defined The QUARK TO HAVE SPIN 1/2 AND if The STATES ARE SYMMETRICAL, Then IT MUST be TRUE THAT for 3 PARTICLE WITH SPIN 1/2 The TOTAL SPIN MUST be 3/2 AND INdeed IT checks THAT $J = 3/2$. IT'S KIND of CUTE SO far.

WE HAVEN'T WRITTEN ALL The POSSIBLE STATES YET because for SYMMETRICAL QUARKS WE MUST INCLUDE STATES WITH different z SPIN COMPONENTS, i.e., LIKE $A_+(A_+A_+)$ where The SUPSCRIPTS denote SPIN "up". This SECOND SET IS for The CASE of TOTAL SPIN 1/2, i.e., $J = 1/2$.

As another short hand notation let me write $A[AB]$. This stands for a state in which the total symmetry is preserved but that it is anti-symmetric in quality and spin. I mean by the notation the following

$$A[AB] = A_+(A_+B_-) - A_+(A_-B_+) + A_+(B_-A_+) - A_+(B_+A_-)$$

In this new set the two symmetrical states AAA and CCC are not permitted. We then have the following octet,

	$A[AB]$ neutron	$B[AB]$ proton	$I = 1/2$	$Y = +1$
$A[CA]$ Σ^-	$\underbrace{A[CB] + B[C,A]}_{\Sigma^0}$	$\underbrace{B[CB]}_{\Sigma^+}$	$I = 1$	$Y = 0$
	$A[C,B] - B[C,A]$ Λ^0		$I = 0$	
$C[C,A]$ Ξ^-	$C[C,B]$ Ξ^0		$I = 0$	

This is called the nucleon octet and we wrote it down last time. There are 8 states with two possible spins so we have a total of 16 possible states altogether. For the case $J = 3/2$ and the 10 state system there were 4 possible spins for $J = 3/2$ so that there were 40 possible states. The total states then is $16 + 40 = 56$. This is what we would expect for a thing which has 6 possible permutations (i.e., 3 qualities, and 2 spins): $6 \cdot 7 \cdot 8 / 6 = 56$.

This is the game physicists play to describe the symmetry they see in the particles. The theory is not perfect and one of its weak points is in attributing the full quark mass to the C quality particle. This is not true because it does not hold for degenerate states. The quark idea doesn't work well for for higher angular momentum. Adding more quarks does not work. However, we can sidestep the difficulties if we assume that 2 quarks do in fact spin around each other ~~have~~ and have therefore orbital angular momentum. The system no longer stays in the S-state. This idea makes the quakerians happy but as yet we can't explain why two quarks would stick together. Now you know all there is to know about quarks - that's it!

32. SOLVING THE DIRAC EQUATION FOR THE HYDROGEN ATOM

Ref: FEYNMAN QUANTUM ELECTRODYNAMICS by BENJAMIN.

WE SHOULD MOVE ON WITH THE COURSE AND THE NEXT THING I WANT TO DO IS WORK OUT THE SOLUTIONS TO THE DIRAC EQUATION FOR THE HYDROGEN OR HYDROGEN-LIKE ATOMS. I HAVE PREPARED THIS LECTURE THREE TIMES AND LOST MY NOTES EACH TIME SO LET ME SEE how FAR I CAN GO BY REMEMBERING WHAT I did.

YOU MIGHT RECALL IN LECTURE 16 which STARTS ON PAGE 73 WE developed THE DIRAC EQUATION BY WORKING UP from The SCHRÖDINGER EQUATION. ON PAGE 77 WE FINALLY ARRIVED AT The DIRAC EQUATION which WE WILL USE HERE. I Should ALSO POINT OUT The properties of The σ AND γ MATRICES which WE INTRODUCED THEN. THE DIRAC EQUATION CAN BE EXPRESSED IN The COMPACT form,

$$\gamma_\mu (i \nabla_\mu - e A_\mu) \psi = m \psi$$

THIS IS THE 4-VECTOR FORM OF THE EQUATION. HERE $\nabla_\mu = \partial/\partial t, \partial/\partial x, \partial/\partial y$ AND $\partial/\partial z$; $A_\mu = A_t$ or V, A_x, A_y, AND A_z ; AND $\gamma_\mu = \gamma_t, \gamma_x, \gamma_y, \gamma_z$. If WE WRITE THIS EQUATION OUT IN MORE DETAIL WE HAVE THAT

$$\gamma_t \left(i \frac{\partial}{\partial t} - eV\right)\psi - \gamma_{x,y,z}\left(-i \frac{\partial}{\partial x,y,z} - A_{x,y,z}\right)\psi = m\psi$$

THE PROBLEM WE ARE GOING TO SOLVE IS A SPECIFIC CASE for The hydrogen LIKE ATOMS. WE ARE GOING TO idealize The PROBLEM BY ASSUMING The NUCLEUS TO be INFINETLY MASSIVE AND further THAT The NUCLEUS DOES NOT HAVE ANY INTRINSIC. LATER WE WILL find THAT NUCLEAR SPIN ACCOUNT for The hyperfine STRUCTURE of The hydrogen SPECTRUM. Thus WE ARE INTERESTED IN The SPECIAL CASE where

$$A_{x,y,z} = 0 \qquad \text{AND} \qquad V = \frac{eZ}{r}$$

The COULOMB POTENTIAL USED here HAS further ASSUMED NO SELF-ACTION of The ELECTRON ; such AN ASSUMPTION IGNORES The LAMB SHIFT CORRECTION which QUANTUM ELECTRODYNAMICS PREDICTS. LET'S WRITE OUT WHAT WE HAVE SO FAR ("I'M REMINDED of SOMETHING Prof. Wheeler SAID TO WHEN I WAS WORKING OUT A PROBLEM; he SAID, 'LITTLE STEPS for LITTLE feet'") ; SO LET'S TAKE LITTLE STEPS,

$$\gamma_t \left(i \frac{\partial}{\partial t} - \frac{Ze^2}{r}\right)\psi - \gamma_x\left(-i\frac{\partial}{\partial x}\right)\psi - \gamma_y\left(-i\frac{\partial}{\partial y}\right)\psi - \gamma_z\left(-i\frac{\partial}{\partial z}\right)\psi = m\psi$$

BEFORE WE GO ON FROM here WE BETTER UNDERSTAND WHAT THE EQUATION MEANS. FIRST WE WONT TO RECALL WHAT The γ's ARE. RECALL PAGE 77 AND

$$\gamma_t = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) \qquad \gamma_{x,y,z} = \left(\begin{array}{c|c} 0 & \sigma_{x,y,z} \\ \hline -\sigma_{x,y,z} & 0 \end{array}\right)$$

EXPANDED OUT WE hAVE THAT

$$\gamma_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}$$

Some of The PROPERTIES which The γ's obey ARE The FOLLOWING

$$\gamma_t^2 = 1, \quad \gamma_x^2 = \gamma_y^2 = \gamma_z^2 = -1, \quad \gamma_t \gamma_{x,y,z} + \gamma_{x,y,z} \gamma_t = 0$$

$$\gamma_x \gamma_y + \gamma_y \gamma_x = 0, \quad \gamma_x \gamma_z + \gamma_z \gamma_x = 0, \quad \gamma_y \gamma_z + \gamma_z \gamma_y = 0$$

ALSO

$$\gamma_t \gamma_{x,y,z} = \begin{pmatrix} 0 & \sigma_{x,y,z} \\ \sigma_{x,y,z} & 0 \end{pmatrix}, \quad \gamma_x \gamma_y \gamma_z \gamma_t = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

NOW ALL THESE 4 X 4 MATRICES OPERATE ON 4 COMPONENT doodledac ψ which CAN be REWRITTEN AS $\begin{pmatrix} u \\ v \end{pmatrix}$ where u AND v ARE 2 COMPONENT doodle DACS. NOTICE from The form of γ'S THAT The PRESENCE of The 2X2 0 MATRICES IN The upper LEFT AND LOWER RIGHT PORTS of The 4X4 IMPLY THAT The $\begin{pmatrix} u \\ v \end{pmatrix}$ WILL NOT BECOME MIXED. IN ORDER WORDS The DIRAC EQUATION DISGUISES ACTUALLY 2 EQUATIONS.

WE ARE OUT TO find The ENERGY LEVELS of The ATOM SO WE ARE LOOKING for A SOLUTION which hAS The form,

$$\psi(x,y,z,t) = \varphi(x,y,z)\, e^{-iEt}$$

Then SINCE $i \frac{\partial}{\partial t} \psi = E \psi$ WE CAN WRITE

$$\left(E - \frac{Ze^2}{r}\right)\begin{pmatrix} u \\ v \end{pmatrix} + \left(i \gamma_t \gamma_{x,y,z}\right)\left(\partial / \partial_{x,y,z}\right)\psi - m \gamma_t \psi = 0$$

HERE I MULTIPLIED Through by γ_t AND USED The FACT THAT $\gamma_t^2 = 1$. SINCE WE KNOW WHAT $\gamma_t \gamma_{x,y,z}$ IS from ABOVE WE hAVE THAT,

$$\left(E - \frac{Ze^2}{r}\right)\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \cdot i \vec{\nabla} \begin{pmatrix} u \\ v \end{pmatrix} - m \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)\begin{pmatrix} u \\ v \end{pmatrix} = 0$$

WE CAN WRITE THIS EQUATION OUT TO GET The 2 EQUATIONS:

$$\left(E + m - \frac{Ze^2}{r}\right) u + i(\bar{\sigma} \cdot \bar{\nabla}) v = 0$$

$$\left(E + m - \frac{Ze^2}{r}\right) v + i(\bar{\sigma} \cdot \bar{\nabla}) u = 0$$

if WE REDEFINE u AND v WE CAN GET rid of The "i", i.e., LET $u \to u$ AND $v \to iv'$ THEN WE find WE hAVE ACTUALLY 4 differential EQUATIONS TO SOLVE

$$\left(E - m - Ze^2/r\right) u - (\bar{\sigma} \cdot \bar{\nabla}) v' = 0$$

$$\left(E + m - Ze^2/r\right) v' + (\bar{\sigma} \cdot \bar{\nabla}) u = 0$$

GUESSING THE SOLUTIONS TO THE DIRAC EQUATION.

SO NOW WE HAVE 4 ~~FUN~~ EQUATIONS WHICH WE HAVE TO SOLVE. THINGS ARE GETTING COMPLICATED. WE CAN PROCEED 2 WAYS FROM HERE: WE CAN GO AHEAD AND SOLVE THE EQUATIONS BY THE BRUTE-FORCE TECHNIQUE OF APPLYING THE RIGORS OF MATHEMATICS OR WE CAN PURSUE A MORE INTELLECTUALLY STIMULATING PATH BY APPLYING OUR POWERS OF REASON TO GUESS THE FORM OF THE ANGULAR DEPENDENCE WHICH THE SOLUTIONS WILL TAKE. ONCE WE GUESS AT A SOLUTION, WE CAN SEE HOW IT WORKS OUT. FOR THIS PRESENTATION WE'LL TRY THE LATTER PATH AND SUBSTITUTE YAK-YAK FOR RIGOR AND SEE WHAT HAPPENS.

WHERE DO WE BEGIN? WELL, LET'S TAKE A QUE FROM THE WAY WE SOLVED THE SCHRODINGER EQUATION. WE KNOW ALREADY THAT AN OBJECT HAVING A z-ANGULAR MOMENTUM DESCRIBED BY l CAN HAVE AN ANGULAR DISTRIBUTION CHARACTERIZEDY BY THE FUNCTION

$$\sin^{l}\theta \, e^{il\phi}$$

MULTIPLYING THIS BY A RADIAL FUNCTION r^{l} WE HAVE ALREADY A LOGICAL TRIAL SOLUTION

$$\psi(r,\theta,\phi) = r^{l} \sin^{l}\theta \, e^{il\phi}$$

WE CAN SIMPLIFY THE MATHEMATICAL FORM OF THE SOLUTION BY WRITING

$$(r \sin\theta \, e^{i\phi})^{l} = (x+iy)^{l}$$

WHERE WE USED THE FACT THAT,

$$x = r\sin\theta\cos\phi \quad \text{AND} \quad y = r\sin\theta\sin\phi$$

IT IS HANDY TO KNOW THE NEXT SOLUTION IN THE SERIES IF l IS THE HIGHEST z-COMPONENT ANGULAR MOMENTUM, THEN FOR $m = l-1$ WE HAVE A SOLUTION OF THE FORM $z(x+iy)^{l-1}$. SO FAR WE HAVEN'T SAID ANYTHING ~~ANY~~ ABOUT SOLUTIONS TO DIFFERENTIAL EQUATIONS; WE HAVE ONLY DISCUSSED THE PROPERTIES OF ANGULAR MOMENTUM. WE KNOW THAT BECAUSE THE ELECTRON(S) ARE IN A CENTRAL POTENTIAL THAT THE EXIST IN A STATE OF DEFINITE ANGULAR MOMENTUM — CALL IT j. WE ARE LOOKING FOR A z-COMPONENT OF j WHICH IS $m_j = 1/2$. THAT IS, THE ELECTRON WILL BE IN TWO STATES OF SPIN; EITHER ~~OR~~ UP OR DOWN.

Our problem is to understand the behavior of the wave functions. The orbital angular momentum l can be $j \pm 1/2$ depending on the electron spin. It is a little more convenient to redefine $j \pm 1/2$ in terms of k such that

$$k = j - 1/2 \quad \text{and} \quad k+1 = j + 1/2$$

The $l = k+1$ and $l-1 = k$.

Now ~~we~~ we can write down the form of u and v and they are, with explanation

$$u = \begin{pmatrix} \text{electron is } +1/2; \text{ orbit ang. mom. } m_l = k & (x+iy)^k \left[r\, a(r) + z\, b(r) \right] \\ \text{electron is } -1/2 \text{ orbit ang. mom} = k+1 & (x+iy)^{k+1}\, c(r) \end{pmatrix}$$

v has an equivalent form with different radial functions, i.e.,

$$v = \begin{pmatrix} (x+iy)^k \left[r\, A(r) + z\, B(r) \right] \\ (x+iy)^{k+1} \left[C(r) \right] \end{pmatrix}$$

These are the mathematical forms which we wanted to deduce. As you see it is a great short cut to working out the differential equations. But we're not done yet.

33. SOLVING THE DIRAC EQUATION FOR THE HYDROGEN ATOM

LAST TIME WE STARTED TO SOLVE THE DIRAC EQUATION,

$$\gamma_\mu (i\nabla_\mu - eA_\mu)\psi = m\psi$$

FOR THE HYDROGEN OR HYDROGEN LIKE ATOM. WE FIND THAT THE PROBLEM WAS REALLY REDUCIBLE TO TWO EQUATIONS OF THE FOLLOWING FORM,

$$(E - V - m)u = (\bar{\sigma}\cdot\bar{\nabla})v$$
$$(E - V + m)v = -(\bar{\sigma}\cdot\bar{\nabla})u$$

WHERE WE USED THE SUBSTITUTION $\binom{u}{iv}$ FOR ψ AND USED THE POTENTIAL V AS A STRAIGHT COULOMB POTENTIAL, I.E.,

$$V = -\frac{Ze^2}{r}$$

WE SHALL DEFINE FOR CONVENIENCE SAKE THE PARAMETER η WHICH STANDS FOR Ze^2 BECAUSE I DON'T WANT TO KEEP WRITING IT ALL THE TIME.

NOW BY SOME HOCKUS-POCKUS AND FAST TALKIN' I DERIVED THE FUNCTIONAL FORM OF u AND V WHICH ARE THEMSELVES TWO COMPONENT THINGS. BY MAGIC THEN FOR A STATE WITH ANGULAR MOMENTUM $l = j - 1/2$ WE HAVE THAT

$$u = \begin{pmatrix} ar + bz \\ c(x+iy) \end{pmatrix}(x+iy)^l$$

$$v = \begin{pmatrix} Ar + Bz \\ C(x+iy) \end{pmatrix}(x+iy)^l$$

THE FACTORS a, b, c, A, B, C ARE ALL FUNCTIONS OF r. THUS, AWAY WE GO —

WE FIRST NEED TO CALCULATE $(\sigma\cdot\nabla)u$ AND $(\bar{\sigma}\cdot\bar{\nabla})v$. FOR THE DEFINITION OF $\bar{\sigma}$ AND $\bar{\nabla}$, IT CAN BE SHOWN THAT

$$(\bar{\sigma}\cdot\bar{\nabla}) = \begin{pmatrix} \partial/\partial z & \partial/\partial x - i\partial/\partial y \\ \partial/\partial x + i\partial/\partial y & -\partial/\partial z \end{pmatrix}$$

WHAT WE NEED TO CALCULATE THEN IS $(\bar{\sigma}\cdot\bar{\nabla})\binom{u_1}{u_2}$ WHICH IS

$$(\bar{\sigma}\cdot\bar{\nabla})\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{matrix} \partial/\partial z\, u_1 + (\partial/\partial x - i\partial/\partial y)u_2 \\ (\partial/\partial x + i\partial/\partial y)u_1 - (\partial/\partial z)u_2 \end{matrix}$$

FIRST LET'S TAKE $(\bar{\sigma}\cdot\bar{\nabla})V$,

$$(\bar{\sigma}\cdot\bar{\nabla})V = \begin{pmatrix} \partial/\partial z & \partial/\partial x - i\partial/\partial y \\ \partial/\partial x + i\partial/\partial y & -\partial/\partial z \end{pmatrix}\begin{pmatrix} A r + Bz \\ C(x+iy) \end{pmatrix}(x+iy)^{l}$$

TERM by TERM WE GET

$$\frac{\partial}{\partial z} A r = r\frac{\partial}{\partial z} A\left(\sqrt{x^2+y^2+z^2}\right) + A\frac{\partial r}{\partial z} = A'\frac{z}{r} + \frac{Az}{r}$$

NOTICE THAT The FACTOR $(x+iy)^{l}$ IS NOT bothered by $\partial/\partial z$. NEXT,

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) C(x+iy)^{l+1} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy)^{l+1} C\left(\sqrt{x^2+y^2+z^2}\right)$$

NOTICE PLEASE THAT I CAN MAKE THINGS A LITTLE EASIER FOR MYSELF IF I NOTICE THAT,

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy)^{l} = 2l(x+iy)^{l-1}$$

AND ALSO $\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x+iy)^{l} = 0$

The Above differentiation RESULTS IN The following,

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy)^{l} C = 2(l+1)C + C'\left(\frac{x}{r} - \frac{iy}{r}\right)$$

For The TERM $\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) A r (x+iy)^{l}$ I GET SIMILARLY

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) A(r)(x+iy)^{l} r = A' + \frac{A}{r} + B$$

ALSO $\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) Bz = \frac{B'z}{r}$

Then $-\frac{\partial}{\partial z} C(x+iy)^{l+1} = -C'\frac{z}{r}(x+iy)^{l+1}$

I forgot The TERM $\partial/\partial z\, Bz$ which produces $B'z^2/r + C$.

The GrAND RESULT of ALL THIS TEDIOUS EXERCISE IS The following,

$$(E-V-m)\begin{pmatrix} ar + bz \\ c(x+iy) \end{pmatrix}(x+iy)^{l} = \begin{pmatrix} A'z + \frac{A}{r}z + \frac{B'z^2}{r} + B + C'\frac{(x^2+y^2)}{r} + 2(l+1)C \\ \left(A' + \frac{A}{r} + \frac{B'z}{r} - C'\frac{z}{r}\right)(x+iy) \end{pmatrix}(x+iy)^{l}$$

This IS GOOD BECAUSE The $(x+iy)^{l}$ FACTORS GO OUT.

BUT WE NEED A SIMILAR EXPRESSION for The OTher SET of EQUATIONS

$$(E-V+m)\begin{pmatrix} Ar + Bz \\ C(x+iy) \end{pmatrix}(x+iy)^{l} = -(\bar{\sigma}\cdot\bar{\nabla})\begin{pmatrix} ar + bz \\ c(x+iy) \end{pmatrix}(x+iy)^{l}$$

SINCE The OPERATOR IS The SAME AND The form of a, b, c AND A, B, C BEING SIMILAR WE HAVE The RIGHT SIDE IS JUST

$$-\begin{pmatrix} a'z + \frac{a}{r}z + \frac{b'z^2}{r} + b + c'\frac{(x^2+y^2)}{r} + 2(l+1)c \\ \left(a' + \frac{a}{r} + \frac{b'z}{r} - c'\frac{z}{r}\right)(x+iy) \end{pmatrix}(x+iy)^{l}$$

Now what functions can possibly satisfy these terrible equalities? In order to succeed at all I must proceed to compare coefficients. I will leave it as an exercise to show that $B' = C'$, $B = C$, $b' = c'$ and $b = C$. This is a consequence of no z term in the $C(x+iy)$ factor, thus it must be true that $(B'-C')z/r = 0$; likewise for b' and c'. The constant of integration is next proven to be zero. Using these facts we then can simplify the terms,

$$B'\frac{z^2}{r} + C'\frac{(x^2+y^2)}{r} = B'\frac{(x^2+y^2+z^2)}{r} = B'r$$

Therefore we have that

$$\begin{pmatrix} ar + bz \\ b(x+iy) \end{pmatrix} = \begin{pmatrix} A'z + \frac{Az}{r} + B'r + B + 2(l+1)B \\ (A' + \frac{A}{r})(x+iy) \end{pmatrix}$$

From which we conclude that

$$b = (A' + \frac{A}{r})$$

and that

$$ar + z(A' + \frac{A}{r}) = A'z + \frac{Az}{r} + B'r + B(2l+3)$$

subtracting out $A'z + Az/r$ and substituting $l = j - 1/2$ we have that

$$a = 2(j+1)\frac{B}{r} + B'$$

Our first pair of equations can be expressed as

$$(E-V-m)a = 2(j+1)\frac{B}{r} + B' \qquad (1)$$

$$(E-V-m)b = A' + \frac{A}{r} \qquad (2)$$

In addition to these equations we have two more which by symmetry we can deduce to be the following,

$$-(E-V+m)A = 2(j+1)\frac{b}{r} + b' \qquad (3)$$

$$-(E-V+m)B = a' + a/r \qquad (4)$$

I now have 4 equations in 4 unknowns which have to be simultaneously solved for certain values of energy, E. I have now a problem of solving linear first order differential equations. I should however notice right away that the 4 variables are uncoupled in the 4 equations and the problem is simplified by have a, and B coupled in equations 1 and 4 and A, and b coupled in (2) and (3).

LET'S PICK EQUATION (1) AND (4) TO SOLVE FOR a AND B BY SETTING $b = A = 0$ FOR THE MOMENT. LATER YOU CAN GO BACK AND SOLVE (2) AND (3) SIMULTANEOUSLY BY SETTING $a = B = 0$. THUS YOU CAN GET ALL THE EIGENVALUES. SO WE WANT TO SOLVE THE SET OF EQUATIONS

$$\left(E + \frac{n}{r} - m\right) a = 2(j+1)\frac{B}{r} + B'$$

$$-\left(E + \frac{n}{r} + m\right) B = a' + a/r$$

THESE TWO COUPLED FIRST ORDER EQUATIONS ARE MATHEMATICALLY EQUIVALENT TO SCHRÖDINGER'S 2ND ORDER DIFFERENTIAL EQUATION, I.E., THE SOLUTIONS ARE ESSENTIALLY IN THE SAME FORM. BECAUSE THE SOLUTION TO THE GENERAL DIFFERENTIAL EQUATION IS IMPORTANT THEY GET THEIR OWN NAME - THEY'RE CALLED THE CONFLUENT HYPERGEOMETRIC FUNCTIONS. IT TURNS OUT THE HARMONIC OSCILLATOR EQUATION HAS A SOLUTION WHICH IS A SUBCLASS OF THE HYPER GEOMETRIC FUNCTIONS; THEY ARE THE HERMITE POLYNOMIALS. IT'S TYPICAL WHEN TEACHING THE YOUNG STUDENT HOW TO SOLVE SUCH PROBLEM THAT HE IS INSTRUCTED IN THE WONDERFULNESS OF THE FUNCTIONS AND THUS HE IS TAUGHT TO BE PROFICIENT IN SOLVING ABOUT THREE PROBLEMS IN QUANTUM MECHANICS: THE FREE PARTICLE, THE PARTICLE IN A COULOMB POTENTIAL, AND THE HARMONIC OSCILLATOR. THE HELLUVA IT IS THE POOR KID GETS DISILLUSIONED AS SOON AS HE REALIZES THAT HE SOLVED VERY SPECIAL PROBLEMS WHICH HAVE PARALYZED HIS THINKING IN TO HOW TO HANDLE WEIRD POTENTIALS. IT'S ANOTHER EXAMPLE OF OUR STUPID PEDAGOGICAL TECHNIQUES.

WELL, AT ANY RATE I'LL PROCEED ON WITH THE VARIOUS STEPS WHICH I WON'T BE ABLE TO EXPLAIN WHY I'M USING THEM ONLY THAT I KNOW I'LL BE GUIDED TO THE RIGHT ANSWER IN END. WHAT WE START OUT TO DO IS TO ESTABLISH THE ASYMPTOTIC FORM OF THE SOLUTION; THEN TAKING THAT SOLUTION AS A TRIAL SOLUTION WE PLUG IT BACK INTO THE EQUATIONS, TURN THE CRANK AND SEE WHAT HAPPENS.

SINCE $r \to \infty$ IN THE LIMIT WE HAVE THE EQUATIONS

$$(E-m)\,a = B'$$
$$(E+m)\,B = a'$$

FROM WHICH WE CAN OBTAIN BY DIFFERENTIATING

$$B'' = -(E^2 - m^2)\,B$$

IF WE DEFINE $w^2 = m^2 - E^2$ THEN THE WAVE FUNCTION HAS THE ASYMPTOTIC FORM OF $e^{\pm wr}$. IF $E < m$ THEN THE ELECTRON IS IN A BOUND STATE AND THE SOLUTION IS ~~SINUSOIDAL~~ EXPONENTIAL. IF $E > m$ THEN THE PARTICLE IS FREE AND THE SOLUTION IS SINUSOIDAL IN FORM. SINCE IN THE CASE $E < m$ MUST NOT BE EXPONENTIALLY DIVERGING WE MUST PHYSICALLY REQUIRE THE SOLUTION TO BE OF THE FORM e^{-wR}.

WITH THE ASYMPTOTIC FORM OF THE SOLUTION ESTABLISHED WE WILL GUESS AT THE FORM OF THE SOLUTIONS FOR THE GENERAL SOLUTION; I'D WOULD ADD THAT IT IS NOT CLEAR WHAT JUSTIFICATION I CAN OFFER FOR DOING WHAT I'M DOING BUT "DAMN THE TORPEDOES ...." I'LL TRY THE SOLUTIONS

$$a = \alpha\, e^{-wr} \qquad B = \beta\, e^{-wr}$$

PUTTING THESE FUNCTIONS INTO THE PAIR OF EQUATIONS WE FIND THAT CONVENIENTLY THE e^{-wR} DEPENDENCE DROPS OUT AND WE ARE LEFT WITH THE EQUATIONS

$$\left(E + \frac{n}{r} - m\right)\alpha = 2(j+1)\frac{\beta}{r} + \beta' - w\beta$$
$$-\left(E + \frac{n}{r} + m\right)\beta = \alpha' + \frac{\alpha}{r} - w\alpha$$

NOW WHY THESE EQUATIONS ARE EASIER TO SOLVE THAT THE OTHERS IS NOT A IMMEDIATELY APPARENT TO THE HUMAN EYE. WHAT WE HAVE TO DO IS PROPOSE THAT α AND β HAVE A SERIES EXPANSION FORM WHICH WE CAN WRITE AS (BECAUSE WE'RE SMART IN KNOWING THE ANSWER),

$$\alpha = \sum_{n=0}^{\infty} p_n\, r^{n+\gamma}$$
$$\beta = \sum_{n=0}^{\infty} q_n\, r^{n+\gamma}$$

WE HAVE ADDED THE 0-ORDER TERM r^{γ} TO ACCOUNT for THE FACT THAT WE MAY NOW NECESSARILY START THE SERIES WITH AN INTEGER; LATER I'LL DETERMINE THE EQUATION THAT γ MUST OBEY; I'LL do THAT by EXAMINING WHAT HAPPENS NEAR THE ORIGIN. TO SIMPLIFY WHAT I WANT TO NEXT I WILL REARRANGE THE EQUATIONS TO PUT THE ~~RIGHT~~ SAME order TERMS TOGETHER, I.E.,

$$(E-m)\alpha + W\beta = (2j+1)\frac{\beta}{r} + \beta' - \frac{\eta}{r}\alpha$$

$$-(E+m)\beta + W\alpha = \frac{\eta}{r}\beta + \alpha' + \frac{\alpha}{r}$$

PUTTING IN THE POLYNOMIAL FUNCTIONS

$$(E-m)\sum p_n r^{n+\gamma} + W\sum q_n r^{n+\gamma} = (2j+1)\sum q_n r^{n+\gamma-1} + \sum q_n (n+\gamma) r^{n+\gamma-1} - \eta \sum p_n r^{n+\gamma-1}$$

$$-(E+m)\sum q_n r^{n+\gamma} + W\sum q_n r^{n+\gamma} = \eta \sum q_n r^{n+\gamma-1} + \sum p_n (n+\gamma) r^{n+\gamma-1} + \sum p_n r^{n+\gamma-1}$$

NOW IF WE RAISE n by 1 INTEGER BY JUST redefining $n \to n+1$ ON THE RIGHT WE CAN REDUCE THE PAIR of EQUATIONS TO THE form,

$$(E-m)p_n + W q_n = -\eta\, p_{n+1} + (2j+2+n+\gamma+1)\, q_{n+1}$$

$$-(E+m)q_n + W p_n = \eta\, q_{n+1} + (n+\gamma+2)\, p_{n+1}$$

These EQUATIONS AREN'T QUITE AS hard TO SOLVE AS THEY LOOK. I WANT TO BE CARE THOUGH THAT I GET THE first TERM IN THE SERIES RIGHT SO LET ME SEE WHAT HAPPENS when $n=0$. For $n=0$ THE LEFT SIDE of THE EQUATIONS MUST MATCH THE RIGHT WHICH IMPLIES THAT THE COEFFICIENT of THE $r^{\gamma-1}$ TERM MUST BE 0, I.E.,

$$-\eta\, p_0 + (2j+2+\gamma)\, q_0 = 0$$

$$\eta\, q_0 + (\gamma+1)\, p_0 = 0$$

COMBINING These EQUATIONS WE GET THAT

$$\eta^2 + (\gamma+1)(2j+2+\gamma) = 0$$

AND SOLVING THE QUADRATIC PRODUCES THE CONSTAINT ON γ

$$\gamma = -j - 3/2 + \sqrt{(j+\tfrac{3}{2})^2 - \eta^2}$$

The result we have just obtained is interesting in that it gives us the correction to the Schrödinger equation near the origin, i.e.,

$$\psi \propto r \cdot r^{\gamma-1} = r^{\gamma}$$

For the electron with $j = 1/2$, $\gamma = -1 + (1-\eta^2)^{1/2}$. Numerically $\eta = \frac{Ze^2}{\hbar c} = \frac{Z}{137.04}$ where $\frac{e^2}{\hbar c} = \alpha$ the fine structure constant. Since η^2 is $\ll 1$ we can approximate $(1-\eta^2)^{1/2}$ by $1 - \frac{1}{2}\eta^2$. Thus for the hydrogen atom $\psi \propto \approx r^{-1/40{,}000}$.

We now need the general relationship between the p's and q's. To return to the equations

$$(E-m)\,p_n + W q_n = -\eta\, p_{n+1} + (2j+3+n+\gamma)\, q_{n+1}$$

$$-(E+m)\, q_n + W p_n = n\, q_{n+1} + (\eta+\gamma+2)\, p_{n+1}$$

Multiply the first by $(E+m)$ and the bottom by $(E-m)$ and add, and after some manipulation

$$[-\eta(E+m) + W(\eta+\gamma+2)]\, p_{n+1} = -[(n+\gamma+3+2j)(E+m) + \eta W]\, q_{n+1}$$

If this horrible mess is correct in representing the correct correlation between the p's and q's we should be able to check it for $n=0$, i.e., let's find p_0/q_0 which earlier we found it to be $-\eta/\gamma+1$. Substitute in $n-1$ and we get

$$-\eta(E+m) + W(\gamma+1)\, p_0 = -[(\gamma+2+2j)(E+m) + \eta W]\, q_0$$

Since we earlier proved that $(2j+2+\gamma) = -\frac{\eta^2}{\gamma+1}$, you can juggle the terms around to show in fact the recursion relationship holds.

Since we talked about the asymptotic limit of the solution, we should see what happens for large n. If we do that we find that

$$W p_n = -(E+m)\, q_n \qquad \text{for large } n$$

such that $\quad q_n = -\frac{W}{E+m}\, p_n$

Taking this relationship and plugging it back in the

GENERAL FORMULA WE GET THAT

$$(E-m)\,p_n + W q_n = -\eta\, p_{n+1} + (2j+3+n+\gamma)\, q_{n+1}$$

BECOMES

$$(E-m)\,p_n - \frac{W^2}{E+m}\, p_n = -\eta\, p_{n+1}\left(\frac{W}{E+m}\right)$$

MULTIPLYING THROUGH BY $E+m$ AND SUBSTITUTING $W^2 = m^2 - E^2$ WE FIND THAT

$$p_{n+1} = \frac{2W}{n}\, p_n$$

WHAT KIND OF SERIES HAS THIS PROPERTY? THAT IS EACH TERM DIFFERS FROM ITS NEIGHBOR BY $2W/n$. IT TURNS OUT THAT THIS CORRESPONDS TO A GEOMETRIC SERIES

$$r^{\gamma} e^{2WR} = \sum \frac{(2W)^n}{n!}\, r^{n+\gamma}$$

THUS WE FIND THAT THE SOLUTION EXPONENTIALLY DIVERGES FOR LARGE n WHICH SAYS THAT THE ELECTRONS CAN'T BE BOUND TO THE NUCLEUS AND THERE ARE NO ATOMS! WELL, EVIDENTLY SOMETHING HAS TO BE STRAIGHTENED OUT BECAUSE THE FUNCTIONS DOESN'T BLOW UP AND WE MUST FIND OUT HOW TO EXPLAIN THIS APPARENT DISCREPANCY.

MAKING THE SOLUTION WORK

OUR SOLUTION DOESN'T WORK BECAUSE IT GOES OFF TO INFINITY AND CLEARLY THIS IS NOT RIGHT. WE HAVE TO FIND A SOLUTION WHICH WORKS. THE ONLY WAY THAT EVERYTHING WILL WORK OUT IS IF THE SERIES SUDDENLY STOPS AND BEYOND SOME TERM THE COEFFICIENTS ARE ZERO. THEN AS WE GO FURTHER OUT IN THE SERIES THE SOLUTION e^{+WR} FALLS OFF FASTER THAN THE r^n TERM. IN ORDER TO ESTABLISH THE CRITERIA THERE MUST THEREFORE BE SOME n FOR WHICH $p_{n+1} = q_{n+1} = 0$. FOR OUR RECURSION FORMULA THEN WE MUST REQUIRE THAT

$$\frac{p_n}{q_n} = \frac{E+m}{W}$$

OR THAT

$$\frac{E+m}{W} = -\frac{[(n+\gamma+2j+3)(E+m) + \eta W]}{-\eta(E+m) + W(n+1+\gamma)}$$

Now it would be some small miracle if this relationship ~~does~~ works. In general, it will not so that the solution explodes. But for a particular set of energy values it will work and it is those that we are after. There will clearly be more than one value since n appears on the right side and it can be a number of discrete integers.

If we multiply the above relationship out and rearrange things,

$$-\eta(E+m)^2 - (E+m)W(2n+2\gamma+3+2j) - \eta W^2 = 0$$

If I substitute for W^2, m^2-E^2 then

$$\eta(2E^2+2Em) - (E+m)W(2n+2\gamma+3+2j) = 0$$

or

$$\frac{\eta E}{W} = n+\gamma+j+3/2$$

Now if we substitute for γ, $= -j-3/2+\sqrt{(j+\frac{1}{2})^2-\eta^2}$ the $+j+3/2$ eat each other up and we are left with

$$\frac{\eta E}{W} = n+\sqrt{(j+\tfrac{1}{2})^2-\eta^2}$$

By squaring both sides, substituting for W^2, m^2-E^2 and rearrange you can show that

$$E = m\left\{1+\frac{\eta^2}{\left[n+\sqrt{(j+\frac{1}{2})^2-\eta^2}\right]^2}\right\}^{-1/2}$$

This is the grand result we were after, it gives the position of the energy levels for the hydrogen atom. If we substitute back in the expression for η i.e.,

$$\eta = \frac{Ze^2}{\hbar c} = Z\alpha$$

where α is the fine structure constant and equal to $1/137.04$. For small values of η, i.e., for hydrogen we can write to a good approximation

$$E = mc^2\left[1-\frac{1}{2}\frac{\eta^2}{n+|j+\frac{1}{2}|}\right]$$

$$= mc^2 - \frac{Z^2e^4m}{2(n')^2\hbar^2}$$

where I have defined

$$n' = n+|j+\tfrac{1}{2}|$$

The ANSWER WE HAVE OBTAINED GIVES US The ENERGY EIGEN-VALUES for ALL The bound STATES. IT IS PRECISELY The Bohr formula where The binding ENERGY $\frac{Z^2 e^4 m}{2\hbar^2}$ IS CALLED A RYdberg AND EQUALS -13.60 EV.

CONTAINED IN OUR RESULT IS The FACT THAT A degeneracy IN The ENERGY LEVELS EXIST. This IS BECAUSE The TERM $n + j + 1/2$ CAN GIVE The SAME RESULT for 2 different VALUES of n AND j. For INSTANCE if $n = 0$ AND $j = 3/2$ $n' = 2$ AND if $n = 1$ AND $j = 1/2$, n' ALSO $= 2$. However, WE ALSO SEE THAT The LEVELS ARE NOT EXACTLY DEGENERATE because WE HAVE LEFT OUT The n^2 TERM IN The denominator. Therefore The LEVELS ARE SPREAD A LITTLE BIT. The spreading IS referred TO A The fine LINE STRUCTURE of hydrogen.

I Should POINT OUT THAT WE HAVE IGNORED The OTHER SOLUTION when $a = 0$ AND $B = 0$ BUT IT TURNS OUT YOU GET A SIMILAR SOLUTION AND The EXTRA SOLUTION LETS YOU ACCOUNT for EVEN AND odd PARITY WAVE FUNCTIONS.

Further The dirac EQUATION IS INCOMPLETE IN THAT IT IGNORES The SELF-ENERGY of The ELECTRON. When The dynamical field of The ELECTRON IS CORRECTED for, IT IS FOUND THAT The $n = 0$, $j = 3/2$ AND $n = 1$, $j = 1/2$ ($2S_{1/2}$ AND $2P_{1/2}$) STATES ARE STILL DEGENERATE. The SLIGHT DISCREPANCY IS CALLED The LAMB Shift AND IT HAS BEEN SHOWN Theoretically THAT The Shift corresponds TO About 1057.70 MEGACYCLE. EXPERIMENTALLY The RESULT differs by About .2 MEGACYCLE AND IT IS CLAIMED THAT both ANSWERS HAVE AN ACCURACY GOOD TO 0.1 mc. Perhaps, QUANTUM ELECTRODYNAMICS IS NOT EXACT IN ITS form. BUT MORE LIKELY IT IS A PROBLEM IN referencing The shift of The TWO LINE CENTERS. WE Should Theoretically compute The CENTER of The TWO LINES AND Then SEE how IT OVERLAYS WITH The EXPERIMENTAL RESULTS. Then we'd KNOW if WE WERE IN TROUBLE.

34. SOLVING THE HARMONIC OSCILLATOR PROBLEM WITH DIFFERENTIAL EQUATION TECHNIQUES.

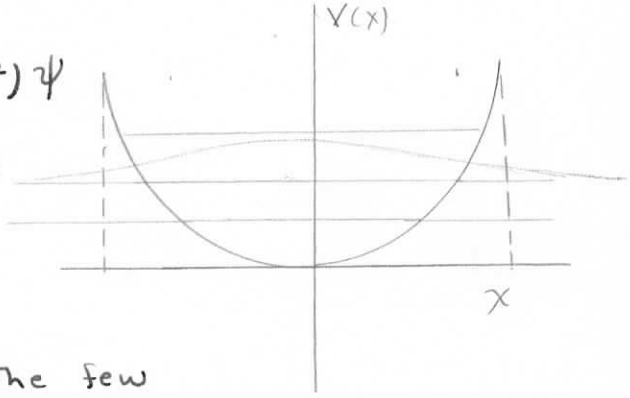
WE JUST WORKED OUT A RATHER HARD PROBLEM BY FINDING THE SOLUTIONS, I.E., THE ENERGY LEVELS OF HYDROGEN BY SOLVING THE DIRAC EQUATION. WE SHOULD, PERHAPS, BACK-UP FOR A FEW MINUTES AND CONSIDER A MORE EASY PROBLEM TO SOLVE BUT ONE THAT HAS A GOOD DEAL OF UTILITY IN HELPING US SOLVE OTHER MORE DIFFICULT PROBLEM. I AM REFERRING TO THE QUANTUM ANALOG OF THE HARMONIC OSCILLATOR. IN THIS PROBLEM WE WILL BE DEALING WITH A PARABOLIC SHAPED POTENTIAL INSTEAD OF A COULOMB POTENTIAL. THE PARABOLIC POTENTIAL PROBLEM IS AN IMPORTANT ONE BECAUSE VERY OFTEN WE CAN USE IT TO GET AN APPROXIMATION TO A MORE DIFFICULT PROBLEM.

IN CLASSICAL PHYSICS THE HARMONIC OSCILLATOR IS A VERY USEFUL TOOL IN UNDERSTANDING THE BEHAVIOR OF A COMPLEX SYSTEM. SIMILARLY IN THE REALM OF QUANTUM PHYSICS WE CAN LEARN A GREAT DEAL OF Q.E.D. BY STUDYING THE OSCILLATORY BEHAVIOR OF THE MODAL BEHAVIOR OF AN ELECTROMAGNETIC FIELD CONTAINED WITHIN A LARGE (PERHAPS, INFINITELY LARGE) BOX. THE KEY POINT BEING THAT EACH MODE CAN BE REPRESENTED AS A CLASSICAL HARMONICS OSCILLATOR. OTHER QUANTUM PHENOMENA CAN BE UNDERSTOOD IN AN ANALOGOUS; FOR INSTANCE THE BEHAVIOR OF SOUND AND HEAT IN A MATERIAL CAN BE MORE ACCURATELY UNDERSTOOD. LATER ON WE'LL SHOW YOU SOME SPECIFIC EXAMPLES. NOW WE WANT TO SOLVE THE HARMONIC OSCILLATOR PROBLEM IN GENERAL.

THERE MUST BE AT LEAST A 100 DIFFERENT WAYS TO SOLVE THE PROBLEM BUT I'M ONLY GOING TO DISCUSS 2 PARTICULAR WAYS HERE – ONE WAY WILL SOLVE THE DIFFERENTIAL EQUATION IN A STRAIGHTFORWARD MANNER; THE SECOND APPROACH WILL USE THE OPERATOR TECHNIQUE WHICH WE HAVE AT OUR DISPOSALE.

WE SHALL BE CONSIDERING A POTENTIAL OF THE GENERAL FORM $V = \frac{k}{2}x^2$. THE PROPORTIONALITY CONSTANT k CAN BE EXPRESSED IN TERMS OF A NATURAL FREQUENCY, ω_0, AND SYSTEM MASS, m, JUST AS IN THE CLASSICAL CASE. THUS WE COULD

SUBSTITUTE for k, $m\omega_0^2$. The WAVE EQUATION THAT WE ARE INTERESTED IN SOLVING THEN IS



$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \left(\frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2\right)\psi$$

SINCE THE PARTICLE'S MOMENTUM CAN be EXPRESSED AS $\frac{\hbar}{i}\frac{\partial}{\partial x}$ WE CAN WRITE

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}m\omega_0^2 x^2\psi$$

The HARMONIC OSCILLATOR IS ONE of The few TIME dependent EQUATIONS which CAN be SOLVED IN QUANTUM MECHANICS. SINCE our SPECIFIC GOAL IS TO find The ENERGY levels of A HARMONIC OSCILLATOR WE WANT TO TRY The SOLUTION

$$\psi(x,t) = e^{-iEt/\hbar}\,\varphi(x)$$

If WE SUBSTITUTE THIS INTO The ABOVE EQUATION WE HAVE

$$E\varphi = -\frac{\hbar^2}{2m}\frac{d^2\varphi}{d^2x^2} + \frac{1}{2}m\omega_0^2 x^2\varphi$$

NOW WE WANT TO SOLVE THIS EQUATION. TO do THAT WE'LL first MAKE A SUBSTITUTION TO GET rid of The CONSTANTS, DEFINE,

$$\xi = \sqrt{\frac{m\omega_0}{\hbar}}\,x \qquad \text{AND} \qquad \epsilon = E/\hbar\omega_0$$

Thus WE HAVE TO SOLVE

$$\frac{1}{2}\frac{d^2\varphi}{d\xi^2} + \left(\epsilon - \frac{1}{2}\xi^2\right)\varphi = 0$$

This differential EQUATION IS of The CONFLUENT hypergeometric form which MEANS WE CAN TRY finding AN ASYMPTOTIC SOLUTION AND Then PLUGGING IT IN correct for for SMALL x or ξ. This IS The SAME APPROACH WE JUST USED IN SOLVING The dirac EQUATION. Therefore, TAKING A hINT FROM our previous EXAMPLE LET'S TRY The SOLUTION

$$\varphi = e^{-\frac{1}{2}\xi^2}\,w(\xi)$$

WE WANT TO TWICE differentiate This with respect to ξ

$$\frac{d\varphi}{d\xi} = \varphi' = -\xi e^{-\frac{1}{2}\xi^2}w(\xi) + e^{-\frac{1}{2}\xi^2}w'(\xi)$$

$$\varphi'' = \xi^2 e^{-\frac{1}{2}\xi^2}w(\xi) - e^{-\frac{1}{2}\xi^2}w(\xi) - \xi e^{-\frac{1}{2}\xi^2}w'(\xi) - \xi e^{-\frac{1}{2}\xi^2}w'(\xi) + e^{-\frac{1}{2}\xi^2}w''(\xi)$$

$$= e^{-\frac{1}{2}\xi^2}\left[w''(\xi) - 2\xi w'(\xi) + (\xi^2-1)w(\xi)\right]$$

SUBSTITUTING INTO THE EQUATION AND SIMPLIFYING WE HAVE

$$(\epsilon - \tfrac{1}{2})\, W(\xi) = \xi W' - \tfrac{1}{2} W''$$

SO FAR WE HAVE FOUND A SOLUTION FOR LARGE ξ SUCH THAT THE WAVE FUNCTION DROPS OFF EXPONENTIALLY AS ξ^2. WHETHER OR NOT THIS IS IN FACT THE RIGHT SOLUTION REMAINS TO BE SEEN. BUT FOLLOWING BLINDLY IN THE STEPS OF OUR FOREFATHERS WE PROCEED TO FIND A SOLUTION TO THE ABOVE EQUATION BY SUBSTITUTING FOR $W(\xi)$ A SERIES IN ξ, I.E.,

$$W(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$$

THUS WE KNOW THAT

$$W'(\xi) = \sum_{n=0}^{\infty} n a_n \xi^{n-1} = \sum_{n=1}^{\infty} (n+1) a_{n+1} \xi^n$$

AND

$$W''(\xi) = \sum (n)(n-1) a_n \xi^{n-1} = \sum_{n=2}^{\infty} (n+1)(n+2) a_{n+2} \xi^n$$

BY PUTTING THESE TERMS IN THE EQUATION

$$(\epsilon - \tfrac{1}{2}) \sum_{n=0}^{\infty} a_n \xi^n = \sum_{n=0}^{\infty} n a_n \xi - \tfrac{1}{2} \sum_{n=2}^{\infty} (n+1)(n+2) a_{n+2} \xi^n$$

WE THEREFORE OBTAIN A RECURSION RELATIONSHIP OF THE FORM

$$(\epsilon - \tfrac{1}{2})\, a_n = n a_n - (n+1)(n+2) \frac{a_{n+2}}{2}$$

OR

$$a_{n+2} = - \frac{2(\epsilon - \tfrac{1}{2} - n)}{(n+1)(n+2)} a_n$$

THUS WE HAVE SOLVED THE HARMONIC OSCILLATOR - GRAND, MARVELOUS, WASN'T IT EASY? LET'S SEE HOW WELL IT AGREES WITH OUR ASYMPTOTIC SOLUTION THAT WE GUESSED ABOVE. TAKE n VERY LARGE; THEN ~~TOO~~ A GOOD APPROXIMATION

$$a_{n+2} = \frac{2}{n} a_n$$

IF WE LOOK LONG ENOUGH TO FIND A SERIES WITH COEFFICIENTS OF THE nth AND $n+2$ TERM IN THE RATIO OF $2/N$ WE FIND IT TO BE

$$\sum \frac{\xi^{2m}}{m!}$$

WHICH TURNS OUT TO BE THE SERIES FOR e^{ξ^2}. SUDDENLY OUR SOLUTION IS EXPLODING EXPONENTIALLY; IT IS ENDLESS; IT IS AN INFINITE SERIES. SOMETHING'S WRONG AND WITH OUR LAST EXAMPLE WE PROBABLY KNOW WHAT'S WRONG.

If the solution is not to continue on indefinitely, there must be a particular coefficient which goes to 0. In order to satisfy the constraint of the vanishing coefficient, the energy, ϵ, must be an integer plus $+\frac{1}{2}$, i.e.,

$$\epsilon = k + \tfrac{1}{2}$$

With this constraint we can find the polynomial form of the wave function; some guy got his name on these polynomials so we call them Hermite polynomials. Let's see if we can find a few.

Consider $k = 0$ or $\epsilon = \frac{1}{2}$ then $E = \frac{1}{2}\hbar\omega_0$ while

$$a_{n+2} = \frac{2n}{(n+1)(n+2)} a_n$$

From this formula we see that for $n = 0$, $a_2 = 0$ so the only term in the polynomial is a constant a_0

$$\varphi_0(x) = e^{-\frac{1}{2}\xi^2} W(\xi) = e^{-\frac{1}{2}\xi^2}(a_0 + 0 \ldots)$$

This wave function is therefore called the ground state. It represents the lowest possible energy state that the particle can be in. Let's go on. Notice though the ground state has a gaussian distribution.

Consider $k = 1$ or $\epsilon = 3/2$ then $E = 3/2\hbar\omega_0$. This time $(\epsilon - \frac{1}{2} - m)$ gives $n = 1$ as the last term in the series. But notice since $a_1 \neq 0$, a_0 must $= 0$ since $a_2 = 0$ is beyond the end of series. The wave function for the "first excited state" is

$$\varphi_1(x) = \xi e^{-\frac{1}{2}\xi^2}$$

The shape of this curve is just like $x \cdot$ a gaussian.

Consider now $k = 2$ for which $\epsilon = 5/2$ and the energy $E = 5/2\hbar\omega_0$. Things now get a little more complicated. The last term in the series is a_4 therefore there is no odd integer terms. The terms that we have are

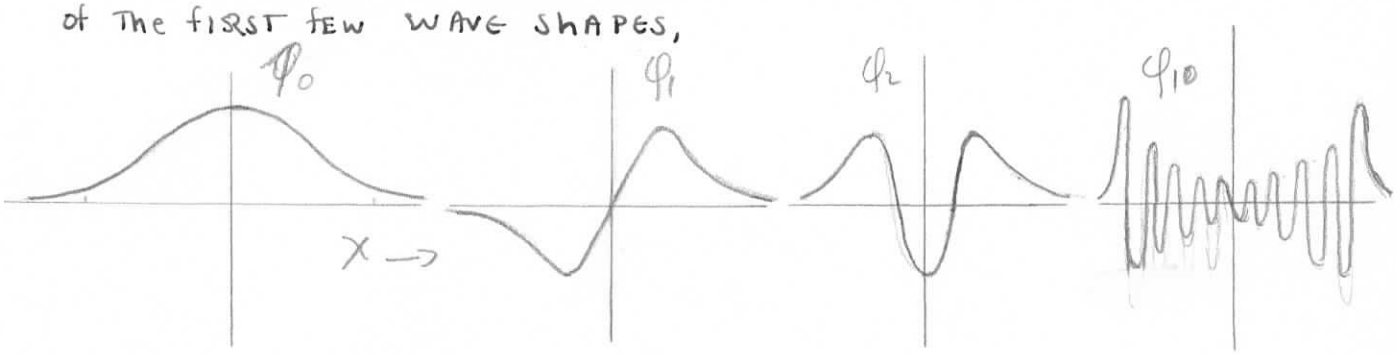
$$a_0, \quad a_2 = -2\frac{(5/2 - \frac{1}{2} - 0)}{2} a_0 = -2a_0, \quad a_4 = -2\frac{(5/2 - \frac{1}{2} - 2)}{12} a_2 = 0$$

Therefore $\varphi_2(\xi) = a_0(1 - 2\xi^2) e^{-\frac{1}{2}\xi^2}$

The higher energy levels continue to be evenly spaced and incremental integer values given by

$$E = (k + \tfrac{1}{2})\hbar\omega_0 \qquad k = 1, 2, 3, \ldots, N$$

The higher order wave functions can be found by extending the example presented here further or by looking up the specific Hermite polynomials, $H_k(\xi)$. As a visual representation of the first few wave shapes,



The probability of find the particle in the potential well is given by the square of $\varphi(x)$. Notice as k gets large the probability aproaches the classical limit, i.e., the particle swingings back and forth like with a ~~rendu~~ lower velocity near the "edges" of the well so it is more likely to be found there than near the center where it has a fast velocity.

Another interesting fact about the solution which is pecular to the harmonic oscillator - if I take the fourier transform of $\varphi(x)$ into momentum space, i.e.,

$$\Phi(p) = \int \varphi(x)\, e^{ipx}\, dx$$

we find the momentum amplitude is of the same form as position amplitude $\varphi(x)$. In momentum space we have the differential equation

$$E\Phi = \frac{p^2}{2m}\Phi(p) + \frac{1}{2}m\omega_0^2\left(i\frac{d}{dp}\right)^2\Phi(p)$$

where we substituted $i\,d/dp$ for x. Since the equations are of the same form except for the constants, we would expect to get the same form of solutions for $\varphi(x)$ and $\Phi(p)$. We find the interesting symmetry between x and p produces equivalent ground state amplitudes,

$$\varphi(x) = e^{-\frac{1}{2}m\omega_0 x^2} = e^{-\text{Potential Energy}/\hbar\omega_0}$$

And

$$\Phi(p) = e^{-\frac{1}{2}\frac{p^2}{m\omega_0}} = e^{-\text{Kinetic Energy}/\hbar\omega_0}$$

SOLVING THE HARMONIC OSCILLATOR BY OPERATORS

WE WILL NOW ATTEMPT TO APPLY THE MAGIC OF COMMUTING OPERATORS TO SOLVING THE HARMONIC OSCILLATOR. WE WON'T HAVE TO CONCERN OURSELVES WITH SOLVING DIFFERENTIAL EQUATIONS BUT WE WILL HAVE TO USE THE ABSTRACT PROPERTIES OF OPERATORS TO WORK OUR WAY THROUGH. I DON'T WANT TO IMPLY THIS TECHNIQUE IS ANY BETTER THAT THE OTHER WAY THAT I JUST SHOWED; IT IS EQUIVALENT.

WE START BY WRITING THE HAMILTONIAN FOR THE SYSTEM. THIS TIME, THOUGH, FOR REASONS OF HIGH CLASS CONVENTION WE SWITCH FROM x TO THE VARIABLE q TO DENOTE POSITION, THUS

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2$$

WE WILL NEED THE COMMUTATION RELATION BETWEEN p AND q BUT WE'VE CALCULATED IT BEFORE,

$$pq - qp = \hbar/i$$

I WILL INVENT AN OPERATOR a WHICH IS GIVEN AS

$$a = \sqrt{\frac{m\omega_0}{2}}\, q + i\sqrt{\frac{1}{2m\omega_0}}\, p$$

BECAUSE OF THE $\sqrt{-1} = i$ IN THE RIGHT SIDE, THE OPERATOR IS NON-HERMITEAN SO WE WILL DEFINE THE HERMITEAN ADJOINT TO BE

$$a^* = \sqrt{\frac{m\omega_0}{2}}\, q - i\sqrt{\frac{1}{2m\omega_0}}\, p$$

USING a AND a^* WE CAN SOLVE FOR q AND p IN TERMS OF THEM SO,

$$q = \sqrt{\frac{2}{m\omega_0}}\,(a + a^*)$$
$$p = \sqrt{2m\omega_0}\, i(a^* - a)$$

IN ORDER TO FIGURE OUT WHAT ALL THESE SYMBOLS MEAN WE HAVE TO CONTINUE A LITTLE FURTHER AND CALCULATE THE PRODUCT a^*a

$$a^*a = \frac{m\omega_0}{2} + \frac{i}{2}(qp - pq) + \frac{p^2}{2m\omega_0} = \frac{m\omega_0}{2} q^2 + \frac{p^2}{2m\omega_0} - \frac{1}{2}$$
$$= \frac{H}{\hbar\omega_0} - \frac{1}{2}$$

SIMILARLY WE CAN FIND THE PRODUCT $aa^* = \frac{H}{\hbar\omega_0} + \frac{1}{2}$ BY SYMMETRY WITHOUT WORKING IT OUT. SO WHAT!

WELL, hold ON; LOOK WHAT WE HAVE. a and a^* COMMUTE ACCORDING TO The RULE

$$a a^* - a^* a = 1$$

AND

$$H = \tfrac{1}{2}\hbar\omega_0 + \hbar\omega_0 a^* a$$

WE ALMOST HAVE The ENERGY LEVELS if WE define The OPERATOR $N = a^* a$ Then $H = (N + 1/2)\hbar\omega_0$ which LOOKS AWFULLY FAMILAR BUT NOW WHAT The hell DOES N MEAN? As AN OPERATOR N HAS EIGENVALUES AND IT IS Them THAT WE ARE AFTER. TO find The EIGENVALUES, SUBJECT TO The COMMUTATION RELATION ON $a^* a$ WE'LL SUPPOSE THAT ONE STATE $|n\rangle$ HAS The ~~PAR~~ PARTICULAR EIGENVALUE n when WORKED OVER BY N, I.E.,

$$N|n\rangle = n|n\rangle$$

OKAY, NOW LET'S USE A LITTLE of OUR MAGIC AND SEE if WE CAN GET SOME MORE EIGENVALUES. LET'S ASK WHAT HAPPENS if WE LET N WORK OVER The STATE $a^*|n\rangle$, I.E.,

$$N a^* |n\rangle = a^* a a^* |n\rangle$$

USING The COMMUTATION RULE ABOVE $a a^* = 1 + a^* a$

$$N a^* |n\rangle = a^* |n\rangle + a^* a^* a |n\rangle$$

BUT! $a^* a |n\rangle = n|n\rangle$ SINCE $N = a^* a$ SO WE HAVE

$$N a^* |n\rangle = a^* |n\rangle + n a^* |n\rangle$$

where WE PULLED n Through a^* SINCE IT IS JUST A NUMBER. Therefore WE HAVE found THAT

$$N(a^*|n\rangle) = (n+1)(a^*|n\rangle)$$

This EQUATION TELLS US THAT a^* RAISES The EIGENVALUE of STATE by 1 INTEGER. WE COULD KEEP GOING AND EVENTUALLY GET $n+2$, $n+3$, ETC.

THAT'S ALL VERY NICE but how ABOUT GOING down from n. WELL, IT TURNS OUT WE CAN GO DOWN if WE CONSIDER The STATE

$$N(a|n\rangle) = (n-1)|n\rangle$$

The ANALOGOUS MANIPULATION IS STRAIGHT forward.

ANOTHER WAY TO EXPRESS $a^* |n\rangle$ IS TO RELATE IT TO THE STATE $|n+1\rangle$ SINCE IT PRODUCES THE $n+1$ EIGENVALUE. LET'S MAKE THE EQUALITY

$$a^* |n\rangle = C_n |n+1\rangle$$

SIMILARLY for $a|n\rangle$ IT IS PROPORTIONAL TO THE $|n-1\rangle$ STATE SO THAT

$$a|n\rangle = d_n |n-1\rangle$$

Now The d_n AND C_n COEFFICIENTS ARE NOT UNCORRELATED

$$\langle n+1| a^* |n\rangle = C_n$$

$$\langle n-1| a |n\rangle = d_n \rightarrow \langle n|a|n+1\rangle = d_{n-1} = C_n^*$$

TO DETERMINED THE VALUES of THE C'S WE'LL ASSUME THE STATES TO BE NORMALIZED. THEN

$$a^* a |n\rangle = n |n\rangle$$

but

$$a^* a |n\rangle = a^* C^*_{n-1} |n-1\rangle$$

$$= C^*_{n-1} a^* |n-1\rangle$$

$$= C^*_{n-1} C_{n-1} |n\rangle$$

SO THAT

$$n |n\rangle = C^*_{n-1} C_{n-1} |n\rangle$$

AND

$$C^*_{n-1} C_{n-1} = n$$

OR

$$C_{n-1} = \sqrt{n}$$

WE CAN THEN WRITE

$$a^* |n\rangle = \sqrt{n+1}\ |n+1\rangle$$

$$a |n\rangle = \sqrt{n}\ |n-1\rangle$$

I ALWAYS REMEMBER THE SQUARE ROOT TERM BY BLINDLY PUTTING IN THE HIGHEST STATE of THE ONE I'M GOING TO or COMING FROM. IT helps TO Keep THINGS STRAIGHT.

NOW WE HAVE TO STRAIGHTEN OUT SOMETHING. LOOKS WHAT HAPPENS if $n = -1$, $C_{n-1} = i$ which IS IMPOSSIBLE SO n CANNOT BE LESS THAN 0. WELL, WHAT if IT IS π (3.1416)? WON'T WE Keep LOWERING π BY -1 UNTIL WE GO BELOW 0; for THAT MATTER WE CAN ALWAYS Keep COUNTING down UNTIL THE EIGENVALUES GO IMAGINARY. SO SOMETHING'S WRONG. THERE MUST BE A "LOWEST" EIGENVALUE BELOW which YOU ONLY PRODUCE 0.

CONSIDER k TO BE THE LOWEST EIGENVALUE THEN IT MUST BE TRUE THAT

$$a|k\rangle = 0$$

THAT IS, a TRYES TO LOWER THE LOWEST EIGENSTATE. SINCE $N|k\rangle = a^* a|k\rangle = k|k\rangle = 0$, IT MUST BE TRUE THAT $k=0$ OR $N=0$. THIS CORRESPONDS TO THE LOWEST STATE OF THE HARMONIC OSCILLATOR. FOR THIS STATE $H = \frac{1}{2}\hbar\omega_0$. BEYOND THIS GROUND STATE WE RAISE THE ENERGY CONTENT OF THE SYSTEM BY ONE UNIT OF $\hbar\omega_0$.

WE'RE NOT QUITE DONE SINCE WE DON'T HAVE THE WAVE FUNCTIONS FOR EACH EIGENSTATE. TO FIND THEM WE START WITH THE LOWEST STATE

$$a|0\rangle = 0$$

AND SUBSTITUTE FOR a ITS PARTICULAR VALUE

$$\left(\sqrt{\frac{m\omega_0}{2}}\, q + \sqrt{\frac{1}{2m\omega_0}}\frac{d}{dq}\right)\varphi_0(q) = 0$$

WE CAN REARRANGE ALL THIS STUFF AND GET

$$m\omega_0 q = \frac{d\,\varphi_0(q)/dq}{\varphi(q)} = \frac{d}{dq}\ln\varphi$$

INTEGRATING

$$\ln\varphi = -m\omega_0 q^2/2 + C$$

$$\varphi = C e^{-\frac{m\omega_0}{2}q^2}.$$

THIS IS THE SAME GROUND STATE WAVE FUNCTION THAT WE FOUND WITH THE OTHER SOLUTION. BY SUCCESS INTEGRATIONS ALL THE HERMITE POLYNOMIALS CAN BE OBTAINED.

35. ~~34~~. A LITTLE THERMODYNAMICS

Ref. VOL I FEYNMAN chapter 40, 41

I SAID EARLIER I'd GIVE A few EXAMPLES of The USEFULNESS of The HARMONIC OSCILLATOR IN SOLVING A NUMBER of PROBLEMS. ONE INTERESTING APPLICATION IS IN The AREA of Thermodynamics. What WE WANT TO do IS RELATE OUR QUANTUM MECHANICAL CONCEPT of A QUANTIZED HARMONIC OSCILLATOR TO The Thermodynamics of A CRYSTAL, SAY. Our ANALOGY WILL BE TO RELATE EACH VIBRATING MODE of The CRYSTAL TO A corresponding MODE of The HARMONIC OSCILLATOR. Then AS The ENERGY CONTENT of The CRYSTAL CHANGES, i.e., AS IT TEMPERATURE CHANGES, The CHANGE MUST BE IN UNITS of $\hbar\omega$.

CONSIDER The CRYSTAL TO BE MADE UP off A LOT of VIBRATORY MODES EACH denoted BY AN ENERGY; $E_1, E_2, E_3, \ldots E_n$. WE WILL TALK ABOUT EACH STATE i defining A CERTAIN ENERGY CONTENT E_i. This LET'S US AVOID ANY DEGENERACY PROBLEMS BY Specifying The ENERGY E_i AND SAYING THAT THAT PARAMETER defines The STATE of The SYSTEM. Now IT IS IMPORTANT TO POINT OUT THAT SPECIFYING The TEMPERATURE of The SYSTEM DOESN'T MEAN ANYTHING - IT IS NOT A PRECISE definition of The STATE of The SYSTEM. If WE disturb The SYSTEM BY A heat bath for A LONG PERIOD of TIME, our KNOWLEDGE of The SYSTEM HAS NOT CHANGED. IT'S SORT of LIKE HAVING A 6-sided DIE; EACH SIDE or NUMBER HAVING A CERTAIN PROBABILITY of BEING TURNED UP. IT DOESN'T MATTER TO OUTCOME of ROLL WHETHER The die IS Thrown BY HAND or SPUN AROUND IN A wheel AND Then STOPPED; The OUTCOME IS STILL The SAME.

JUST AS WE MIGHT ASK WHAT IS The PROBABILITY of The die BEING IN A PARTICULAR STATE (i.e., EXPOSING AN A priori GUESSED NUMBER) WE CAN ASK WHAT IS The PROBABILITY of The CRYSTAL BEING IN A PARTICULAR STATE i. The WHOLE of Thermodynamics IS CONTAINED IN This rather SIMPLE formulation of The PROBABILITY JUST described; IT IS GIVEN by

$$P_i = \text{CONSTANT} \times e^{-\frac{E_i}{kT}}$$

where k IS CALLED The BOLTZMAN CONSTANT. The formula IS CALLED The BOLTZMANN LAW. The PROPORTIONALITY CONSTANT, CALL IT $1/Q$, IS determined from The NORMALIZATION CONDITION.

WHAT WE'd LIKE TO find OUT IS for A SPECIFIC TEMPERATURE, T, WHAT WOULD The INTERNAL ENERGY of The SYSTEM be. LET'S CALL The TOTAL ENERGY U The ENERGY DUE TO ALL The OSCILLATORS; IT IS AN AVERAGE ENERGY SO TO EVALUATE IT for The MULTI-STATE SYSTEM WE HAVE TO SUM The PROBABITIES OVER ALL POSSIBLE STATES, i.e.,

$$U = \sum_{i=1}^{N} p_i E_i = \sum_{i=1}^{N} \frac{1}{Q} E_i \, e^{-E_i/kT}$$

The FACTOR Q IS JUST $\Sigma e^{-E_i/kT}$ The SUM OVER ALL STATES which MUST SATISFY The CONDITION $\Sigma p_i = 1$. LET'S PUT The NUMBERS IN for The HARMONIC OSCILLATOR,

$$U = \frac{\sum \hbar\omega_0 (n+1/2)\, e^{-(n+1/2)\hbar\omega_0/kT}}{\sum e^{-(n+1/2)\hbar\omega_0/kT}}$$

To SIMPLIFY THIS MATHEMATICALLY LET $x = e^{-\hbar\omega_0/kT}$, Then

$$\frac{U}{\hbar\omega_0} = \frac{\sum (n+1/2)\, x^n}{\sum x^N} = \frac{1}{2} + \frac{\sum n x^n}{\sum x^n}$$

You SHOULD ALL KNOW The GEOMETRIC SERIES

$$\sum x^N = \frac{1}{1-x}$$

which CAN be diffentiated WITH RESPECT TO n TO GIVE

$$\frac{d}{dx} \sum x^n \Rightarrow \sum n x^n = \frac{x}{(1-x)^2}$$

Therefore,

$$\frac{\sum n x^n}{\sum x^n} = \frac{x}{(1-x)^2} \cdot (1-x) = \frac{x}{1-x}$$

So The ENERGY of The HARMONIC OSCILLATOR IS

$$U = \frac{\hbar\omega_0}{2} + \hbar\omega_0 \frac{e^{-\hbar\omega_0/kT}}{1 - e^{-\hbar\omega/kT}}$$

OR REARRANGING WE HAVE

$$U = \frac{\hbar\omega_0}{2} + \frac{\hbar\omega_0}{e^{\hbar\omega_0/kT} - 1}$$

LET'S SEE WHAT THIS EQUATION TELLS US,

$$U = \frac{\hbar\omega_0}{2} + \frac{\hbar\omega_0}{e^{+\hbar\omega_0/kT} - 1}$$

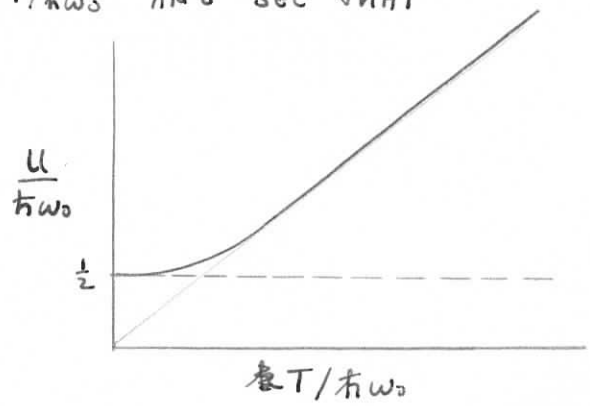
FIRST AT $T \to 0$ THE DENOMINATOR BLOWS UP INFINITELY LARGE SO THE SECOND TERM IS 0. WE ARE LEFT WITH THE ZERO POINT ENERGY OF THE SYSTEM $\hbar\omega_0/2$. THIS ENERGY IS DUE TO ATOMIC MOTION, I.E., ELECTRON MOVEMENT BUT THE ATOM ITSELF IS STATIC AND NOT MOVING. NOW AS $T \to \infty$ AND $kT \gg \hbar\omega_0$ THEN TO A GOOD APPROXIMATION

$$e^{\hbar\omega_0/kT} - 1 = 1 + \frac{\hbar\omega_0}{kT} - 1 = \frac{\hbar\omega_0}{kT}$$

AND NEGLECTING $\hbar\omega_0/2$ WE FIND THAT

$$U = kT$$

THUS WE CAN PLOT $U/\hbar\omega_0$ VESUS $T/\hbar\omega_0$ AND SEE THAT IN THE LIMIT AS T GETS LARGE THE ~~SOLI~~ RESULTS APPROACH THE CLASSICAL LIMIT kT WHILE A LOW VALUES OF T THE QUANTUM BEHAVIOR OF THE SYSTEM IS MORE ACCURATELY PREDICTED BY THE QUANTUM HARMONIC OSCILLATOR.



MORE ON THERMODYNAMICS

I would like to discuss further what I mean by the term temperature. On the quantum level it doesn't appear meaningful to talk about one atom being at a different "temperature" than another because I then I have difficulty talking about the temperature of billions of particles. It is not good to say that the temperature depends on the size of the collection taken.

The concept of a definite temperature for a system is ~~a~~ just a neat statement of your knowledge of the thing. It in no way describes what the thing is. You may be examining a piece of cheese in a box or a collection of atoms in a gas; the temperature of the system is simply an extra piece of information about the system. Our general knowledge of the system which may be at some temperature T is described by the probability relationship of being found in one energy level or another, say E_1 and E_0. The simplest way to understand what this means is to bring the system in question (an atom, a molecule, a crumb, a marble, a boulder), into contact with a very large heat bath which is at temperature T. You then ask what is the probability that after a long period of time, i.e., time for the system under observation to get in equilibrium with the heat bath, the system will be found in the ith energy state. Well, we already know that the probability is given by

$$P_n = \psi_n^2 \propto e^{-E_n/kt}$$

where the proportionality constant is

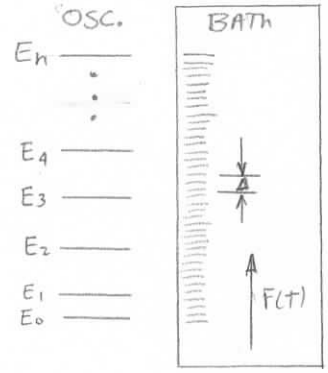
$$\frac{1}{Q} = \frac{1}{\sum_{n=1}^{N} e^{-E_n/kt}}$$

Sometimes we write the mean energy $Q = e^{-F(t)/kt}$ where $F(t)$ is the free energy of the system.

The important point I want to make is that it is perfectly legitimate to talk about an atom at temperature T if it starts out from this imaginary heat bath which is at temperature T.

IT IS LEGITIMATE BECAUSE YOU CAN USE THAT KNOWLEDGE ABOUT THE ATOM TO MAKE PREDICTIONS. OF COURSE ONCE THE ATOM STARTS INTERACTING WITH OTHER SYSTEMS YOU RAPIDLY LOSE YOUR KNOWLEDGE ABOUT IT.

IT IS POSSIBLE TO PROVE THE ABOVE RELATIONSHIP FOR P_n IN FACT IT IS EASIER TO PROVE QUANTUM MECHANICALLY THAN IT IS CLASSICALLY. HOWEVER THE PROVE IS SLIGHTLY SUBTLE, I.E. IT REQUIRES SOME HAND WAVING SO I SKETCH WHAT IS DONE. CONSIDER A SYSTEM LIKE A QUANTIZED HARMONIC OSCILLATOR BROUGHT INTO CONTACT WITH A GIGANTIC HEAT BATH. THE HEAT BATH IS AT THIS FREE ENERGY VALUE $F(t)$ WITH MANY MANY ENERGY LEVELS CLOSELY PACKED TOGETHER.



THE BILLIONS OF STATES IN THE BATH, ALL NEAR THE SAME ENERGY LEVEL, HAVE A DENSITY OF $N(F)$ IN A RANGE Δ. THE DENSITY $N(F)$ TIMES Δ THEN GIVES US THE NUMBER OF STATES IN RANGE Δ. THE DEPENDENCE OF N ON F IS A TREMENDOUS RELATIONSHIP,

$$N \propto F^{K}$$

WHERE K IS THE NUMBER OF PARTS IN THE BATH. THE EASIEST WAY TO APPRECIATE THIS IS TO CONSIDER HOW MANY WAYS THE ENERGY F IN UNITS OF $\hbar\omega_0$ CAN BE DISTRIBUTED AMONG THE CONSTITUTE PARTS. AS A SIMPLE EXAMPLE IF $F = 3\hbar\omega_0$ AND THE BATH ONLY HAD $K=2$ ATOM THEN THERE ARE 6 DIFFERENT WAYS TO DISTRIBUTE 3 UNITS OF $\hbar\omega_0$ AMONG THE 2 ATOM. NOW IF WE HAD LIKE AVOGADROS NUMBER OF ATOM (I.E., A MOLE OF STUFF IN THE BATH) $K = 10^{23}$ SO $N \propto F^{10^{23}}$ WHICH IS A SENSATIONAL NUMBER.

ANOTHER POINT WE NEED TO CONSIDER IS THE POSSIBILITY OF DEGENERATE STATES, I.E., STATES WITH THE SAME ENERGY LEVELS - CALL THEM E_a AND E_b. WHEN $E_a = E_b$ THERE IS AN AMPLITUDE TO BE EITHER IN STATE a OR STATE b. IF THE SYSTEM STARTS IN a AND IS LATER PERTURBED, I.E., BROUGHT INTO CONTACT WITH A BATH FOR INSTANCE, THERE IS SOME PROBABILITY PER SECOND THAT IT MAKES A TRANSITION TO THE OTHER STATE. THE FUNDAMENTAL FACT THAT

WE MUST CONSIDER IS THAT WHEN EQUILIBRIUM HAS BEEN REACHED THE PROBABILITIES ARE EQUAL. IT IS THIS KEY POINT THAT WE USE TO COMPUTE THE PROBABILITY P_N.

WHAT WE WANT TO SHOW IS FOR WHAT PROBABILITY RELATIONSHIP WITH DOES THE TRANSITION STOP. CONSIDER FOR SIMPLICITY A TWO STATE SYSTEM E_1 AND E_2 AND RESPECTIVE PROBABILITIES P_1 AND P_2 TO BE IN EACH STATE. ASSUME THE ENTIRE SYSTEM STARTS AT ENERGY G. WHAT ARE THE NUMBER OF STATES AVAILABLE TO THE SYSTEM? IF THE SYSTEM STARTS OUT IN E_1 THEN THE POSSIBLE STATES AVAILABLE IN RANGE Δ IS $N(G-E_1)\Delta$. SIMILARLY FOR THE INITIAL STATE OF E_2 THERE ARE $N(G-E_2)\Delta$ STATES AVAILABLE. TO BE IN EQUALIBRIUM THEN WE MUST REQUIRE THAT

$$\frac{P_1}{P_2} = \frac{N(G-E_1)\,\Delta}{N(G-E_2)\,\Delta}$$

SINCE $N \propto F^{k}$ OR MORE EXPLICITY $N \propto F^{kR}$ WHERE R IS A CONSTANT DEPENDING UPON THE PARTICLES OF THE BATH (I.E., A MONOTOMIC GAS ETC). THEN

$$\frac{P_1}{P_2} = \frac{(G-E_1)^{k}}{(G-E_2)^{k}}$$

IN GENERAL $G >>> E_2$ SO THAT

$$\frac{P_1}{P_2} = \frac{(1-E_1/G)^{k}}{(1-E_2/G)^{k}}$$

USING THE LIMIT RELATIONSHIP OF $\left(1-\frac{x}{N}\right)^{n} = e^{-x}$ FOR LARGE N WE HAVE THAT

$$\frac{P_1}{P_2} = \frac{e^{-E_1 k/G}}{e^{-E_2 k/G}}$$

BY DEFINING $kT = \frac{G}{K}$ AS THE ENERGY PER DEGREE OF FREEDOM WE ARRIVE AT THE ANSWER

$$\frac{P_1}{P_2} = \frac{e^{-E_1/kT}}{e^{-E_2/kT}}$$

36. METHODS AND PROBLEMS of QUANTUM ELECTRODYNAMICS

Now that we have an understanding of the quantum properties of a harmonic oscillator we are now ready to move on to more difficult problems in quantum electrodynamics, Q.E.D. But before we move into the problems I want to make sure you understand the methods of our analysis.

We are going to be discussing the properties of electromagnetic radiation in a box and its interaction with atoms also in the box. The dimensions of the box are not critical since we will eventually let the dimensions go to infinity. Now the behavior of the radiation in the box is best analyzed in terms of a summation of oscillatory modes which behave classically as harmonic oscillators each with frequency ω_n. The energy in each mode is just $\hbar\omega_n$. When there is one unit of energy, $\hbar\omega_n$, in the box we say that there is one phonon inside. When the energy content is $2\hbar\omega_n$ then there are 2 phonons inside, etc. The EM field thus becomes just a lot of classical oscillators. But we have to be careful since quantum mechanically ~~ther~~ each mode has a zero point energy of $\frac{1}{2}\hbar\omega_0$. Since there are an infinite number of possible modes in the box, it looks like there is an infinite amount of energy inside - even if the system is in its ground state. Something must be wrong! The rest energy cannot be infinite. In order to continue our analysis we will fix the problem with infinite rest energy by subtracting it away. In otherwords we magically shift the zero point energy by the amount $\frac{1}{2}\hbar\omega_0$.

The real problem we are omitting by this redefinition of the zero point energy is the problem of an inconsistency with gravity theory. So far we don't ~~have~~ ~~hav~~ a quantum theory of gravity. Thus we don't know really what we are subtracting away when we take out $\frac{1}{2}\hbar\omega$ in the energy level. Since there isn't a good quantum theory of gravity we just subtract away the problem by not worrying about it. I don't mean to imply that this is valid

TRICK but rather the INFINITE REST ENERGY PROBLEM IS SYMPTOMATIC of our GENERAL ~~IN~~ IGNORANCE of WHAT HAPPENS IN the REALM of VERY SMALL DISTANCES, I.E. LESS THAN 10^{-14} CM AND for VERY high frequencies GREATER THAN GAMMA RAYS. QED COULD BE IN TROUBLE AS the frequency APPROACHES ~~∞~~ INFINITY but SO FAR the theory WORKS WELL UP INTO the GAMMA RAY frequencies SO WE keep USING IT. ONE REASON WE CONTINUE TO USE the theory IS THAT AS YET WE HAVE NOT FOUND A PHYSICAL PROCESS which IS SENSISTIVE TO A high frequency CUTOFF POINT. IN other words TO PREVENT the INFINITY from OCCURRING WE ARBITRARY CUT off the frequency AT SOME LARGE but finite VALUE. THE PIECE WE THROW AWAY TAKES WITH IT OUR IGNORANCE. However THROWING AWAY THAT LIMITING PIECE CAUSES difficulties IN THAT the PROBABILITIES NO LONGER Add UP TO ONE. The REAL QUESTION IS if IN the LIMIT ALL the INCONSISTENCIES AND INFINITIES disolve AWAY IN the EXACTNESS of A COMPLETE THEORY. IN SUMMARY THEN QED ~~SUFFERS~~ EXPERIENCES the FOLLOWING UNIQUE PROPERTIES,

(1). There ARE INFINITIES dotted Through the ANALYSES
(2). WE don'T KNOW the SIGNIFICANCE of the INFINITIES
(3) NOR do WE APPRECIATE the LACK of SENSITIVETY of ALL PHYSICAL PROCESSES SO far CONSIDERED TO these high FREQUENCY CUTOFFS.

WHAT WE END UP WITH IS SORT of LIKE A SHELL GAME where WE keep SHUFFLING the PROBLEM AROUND UNTIL IT BECOMES TOTALLY CONFUSING TO the OBSERVER AND he IS UNABLE TO SELECT the CORRECT SHELL TO LOOK UNDER. The THEORETICAL PHYSICIST, PLAYING the ROLE AS the TOWN CHARLATAN, MIGHT CHAGRINNED SOME DAY TO find OUT THAT SOMEONE STOLE his PEA (PROBLEM) AND ALL the TIME he WAS IN FACT FOOLING HIMSELF INTO THINKING he KNEW WHAT he WAS DOING.

ZERO POINT ENERGY

WHAT THEN of THESE INFINITIES THAT bother us? HOW CAN WE deal WITH THEM AND STILL REMAIN KNOWLEDGABLE ABOUT WHAT WE ARE doING. THE PROBLEM IS TO UNDERSTAND THE SUMMATION $\sum^{\infty} \frac{1}{2}\hbar\omega_0$ which is ADDING UP ALL THE ZERO POINT ENERGYES for EACH of THE MODES IN THE box. BECAUSE THERE ARE AN INFINITE NUMBER of THESE LEVELS WE CAN TRANSform TO AN INTEGRATION OVER ALL frequencies

$$\sum \tfrac{1}{2}\hbar\omega \rightarrow 2\int_0^{\infty} V \frac{d^3\mathbb{K}}{(2\pi)^3}\, \tfrac{1}{2}\hbar\omega$$

The 2 OUT front of THE INTEGRAL TAKES IN THE TWO POSSIBLE POLARIZATIONS of THE GROUND STATE. WHAT I AM GOING TO DO NEXT IS NOW SHIFT THE ZERO POINT ENERGY by AN AMOUNT EQUAL TO THE ZERO POINT ENERGY of THE VACUUM, I.E.,

$$\tfrac{1}{2}\hbar\omega_{0_{VAC}} = \tfrac{1}{2}\hbar|K|$$

SINCE $K = \frac{n\omega}{c}$, $n = 1$ for A VACUUM AND $c = 1$ by definition. n is THE INDEX of REFRACTION of THE VACUUM.
The INTEGRAL WE WANT TO EVALUATE NOW BECOMES,

$$2\int_0^{\infty} V \frac{d^3\mathbb{K}}{(2\pi)^3}\, \tfrac{1}{2}\hbar(\omega - |K|)$$

WHAT WE HAVE DONE IN ESSENCE IS TO INJECT INTO THE CAVITY, which INITIAL WAS A PURE VACUUM, A VERY diffuse hydrogen GAS (I.E., A bunch of WEAKLY COUPLED HARMONIC OSCILLATORS). WE ARE NOW GOING TO find OUT how THE ATOMS INTERACT WITH THE ELECTROMAGNETIC field IN SUCH A WAY AS TO SHIFT THE frequency SPECTRUMS of THE ATOMS IN SUCH A WAY AS TO RESULT IN A redefinition of THE ZERO POINT ENERGY, I.E., frequency. THE hydrogen GAS HAS AN INDEX of refraction n which IS PROPORTIONAL TO THE NUMBER of ATOMS IN THE VOLUME, I.E., ITS DENSITY AS WELL AS depending UPON THE frequency of EM-field, ω, I.E.

$$n^2 - 1 = \frac{N}{V} g(\omega)$$

RECALL THIS formula from VOL II PAGE 179 where

$$g(\omega) = -\frac{4\pi e^2}{m\omega^2}$$

WE SHOWED IN THE PREVIOUS VOLUME (P. 169) THAT for A VERY THIN GAS THAT

$$n \simeq 1 + \frac{N g(K)}{V}$$

where for A THIN GAS $\omega \simeq |K|$ WHILE $k_{gas} = n\omega = n|K|_{vacuum}$. Therefore

$$\omega = n|K| = \left(1 - \frac{N g(K)}{V}\right)|K|$$

WE SEE THEN THAT THE frequency ω does IN FACT shift SLIGHTLY AS IT "INTERACTS" WITH THE VACUUM. RETURNING NOW TO OUR INTEGRAL WE HAVE THAT

$$2\int_0^\infty \frac{d^3\mathbb{K}}{(2\pi)^3} V \frac{1}{2}\hbar(\omega - |K|) = 2\int_0^\infty V \frac{d^3\mathbb{K}}{(2\pi)^3} \frac{1}{2}\hbar\left[|K| - \frac{N|K|g(K)}{V} - |K|\right]$$

$$= -N\hbar \int_0^\infty |K|\, g(|K|) \frac{d^3\mathbb{K}}{(2\pi)^3}$$

This RESULT IS IMPORTANT SINCE IT TELLS US THE AMOUNT OF EXTRA ENERGY IN THE BOX AFTER THE ATOMS ARE INJECTED THE EXTRA ENERGY COMES IN FROM THE ENERGY of INTERACTION of THE ATOM AND THE ELECTROMAGNETIC field of THE VACUUM. REWRITING THIS IN A LITTLE different form,

$$\frac{\text{EXTRA ENERGY}}{\text{ATOM}} \propto -\int_0^\infty K^3 g(K)\, dK$$

For A THIN GAS WE NOTED THAT $g(K) \propto -\frac{1}{K^2}$ SO THE INTEGRAL diverges AS A QUADRATIC. This finite energy per ATOM IS Thus THE difference BETWEEN THE INFINITE ENERGY IN THE BOX DUE TO THE ATOMS' REST ENERGY MINUS THE REST ENERGY DUE TO THE VACUUM IN THE BOX.

THIS SLIGHT ENERGY SHIFT IN EACH ENERGY LEVEL of THE ATOM IS VERY difficult TO OBSERVE. THE first TIME IT WAS observed WAS THE SLIGHT ANOMALY IN THE hydrogen spectrum BETWEEN THE 2S AND 2P STATES. The 2P STATE IS 2 fold degenerate WITH THE ENERGY shift CALLED THE LAMB shift. IT IS THIS correction due TO THE INTERACTION WITH THE ELECTROMAGNETIC field. IT IS THE correction TO THE dirac EQUATION AND comes from THE ZERO POINT MOTION of THE SYSTEM. The difference IN ENERGY LEVELS IS GIVEN by

$$\Delta(E_{2p} - E_{2s}) = \int K^3\, dK\, [g_{2p}(K) - g_{2s}(K)]$$

IT IS AMAZING HOW ACCURATE THE NON-RELATIVISTIC CALCULATION IS IN PREDICTING THIS ENERGY DIFFERENCE. THE THEORY OF OSCILLATORS WORKS WELL UP TO GAMMA RAY FREQUENCIES BEFORE IT IS NECESSARY TO DETERMINE THE g'S BY PAIR PRODUCTION METHODS. IN THIS EXAMPLE THE DEPENDENCE OF g ON K IS PROPORTIONAL TO $1/K^4$ SO INSTEAD OF A QUADRATIC DIVERENGENCE WE GET A LOGARITHMIC ONE. IT IS ALSO INTERESTING TO NOTE THAT THE ANALYIS JUST OUTLINED IS A PRECISE IN DETERMINING THE LAMB SHIFT AS IS THE DIFFICULT APPROACH OF DETERMINING THE SELF-ENERGY OF THE ELECTRON AND WORKING OUT THE DIRAC EQUATION WITH A DIFFERENT HAMILTONIAN. THE RESULTS ARE THE SAME.

THERE IS ONE IMPORTANT POINT THAT MUST BE MADE REGARDING THE OBSERVABLE NATURE OF THIS ENERGY DIFFERENCE. IF INITIALLY THE ATOMS ARE BROUGHT FROM OUTSIDE OF THE BOX FROM SOME STORAGE PILE THEN THE ATOMS HAVE INFINITE ENERGY SINCE $\int K^3 g(K) dK$ IS INFINITE. THEREFORE THE g'S WILL BE THE SAME AND THE ~~EN~~ ENERGY SHIFT IS NOT OBSERVABLE. IF THE ATOMS ARE IN THE BOX INITIALLY AND ARE AT DIFFERENT LEVELS, THEN THE g'S WILL BE DIFFERENT SO ΔE IS MEASURABLE JUST BECAUSE YOU MOVE THE ATOM AROUND. IT SOUNDS STRANGE BUT IT'S THE WAY THINGS WORK OUT. ANOTHER THING THAT MAY SEEM STRANGE IS THAT EVEN IF THE BOX IS "DARK" THERE IS STILL A PROBABILITY THAT AN INTERACTION WILL TAKE PLACE. JUST BECAUSE OUR EYEBALL IS NOT SEEING AND ELECTROMAGNETIC FIELD INSIDE THE BOX DOESN'T PRECLUDE A FIELD FROM BEING IN THERE.

ELECTRON - POSITRON ANNIHILATION

CONSIDER WHAT HAPPENS WHEN AN ELECTRON AND POSITRON COLLIDE. SINCE EACH PARTICLE HAS MASS mc^2 WE EXPECT THE TOTAL ENERGY RELEASED TO BE $2mc^2$. THE MASS m IS THE TOTAL PARTICLE MASS AND IS THE SUM OF THE REST MASS PLUS THE ELECTROMAGNETIC MASS ASSOCIATED WITH FIELD INTERACTION, I.E.,

$$m = m_0 + 4\pi e^2 \int_0^\infty K dK$$

where we substituted for $g(K) - \frac{4\pi e^2}{K^2}$.

UNFORTUNATELY THE MASS IS INFINITE WHEN THE INTEGRAL IS EVALUATED. BUT WE WILL CUT OFF K AT SOME MAXIMUM VALUE, K_{MAX} SUCH THAT

$$m = m_0 + 4\pi e^2 \int_0^{K_{MAX}} K\,dK$$

m_0 IS CALLED THE NORMAL MASS OF THE PARTICLE WILL THE SECOND TERM IS THE ELECTROMAGNETIC MASS. BUT WAS IS m_0? WE CAN'T MEASURE m_0 ANYWAY WE CAN ONLY DETERMINE THE PARTICLE'S INERTIA WHICH GIVES US THE TOTAL MASS m. NOTICE HOWEVER AS WE CHANGE K_{MAX} IT IS NECESSARY TO CHANGE THE VALUE OF m_0 SO THAT THE TOTAL REMAINS THE SAME. YOU MIGHT BE WONDERING THEN IF m_0 CAN GO NEGATIVE. THE ANSWER IS YES IF $K_{MAX} \to \infty$. THIS ARGUMENT SUGGESTS THE POSSIBILITY THAT THE QED THEORY EVENTUALLY LEADS TO INCONSISTENCIES WHEN THE LIMIT IS TAKEN. MAYBE THE CONCEPT OF NEGATIVE MASS WILL FORCE US TO MODIFY OUR THEORY.

THE THEORY RUNS INTO TROUBLE AT THE VERY HIGH FREQUENCIES ARE AT VERY SMALL DIMENSIONS. IN ORDER FOR THE WORLD TO BE CONSISTENT WITH THE CUTOFF IDEA IT MUST BE MADE UP OF TINY DOTS WHOSE DIMENSIONS ARE CONSISTENT WITH THE CUTOFF FREQUENCY. BUT ALSO ALLOWING FOR SUCH FINITE STRUCTURE PROPERTIES OF SPACE-TIME MUST AT THE SAME TIME BE CONSISTENT WITH RELATIVITY, PROBABILITY, CAUSALITY, ETC. THE THEORY WILL HAVE TO BE VERY CLEVERLY CONTRIVED. IT WILL NEVER BE DEVELOPED BY ANY HIT OR MISS TECHNIQUE AS IS OFTEN THE CASE WITH EAGER YOUNG THEORETICAL PHYSICISTS. OFTEN TIMES THESE YOUNSTERS HEAR THAT QED IS IN TROUBLE IN SOME EXPERIMENTAL ASPECT SO THEY SET OUT TO RECONSTRUCT A NEW THEORY WHICH WILL EXPLAIN THE DIFFICULTIES. BUT THE WAY THEY DO IT IS SO OFTEN TASTELESS THAT I CRINGE WHEN I SEE WHAT THEY ARE DOING. AS AN EXAMPLE NOTHING IS MORE SIMPLY BEAUTIFUL THAN THE FORM OF THE HAMILTONIAN

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 - \frac{1}{2}\hbar\omega$$

WHEN A STUDENT BRASHLY SCREWS THIS UP BY ADDING A q^3 TERM OR SOMETHING LIKE THAT I REALLY GET UPSET. THIS IS JUST TASTELESS.

ON THE SIMPLICITY of NATURE

NATURE IS VERY CLEVER AND VERY SUBTLE. She SEEKS TO MINIMIZE The COMPLICATIONS AND AS A RESULT PHYSICAL LAWS CAN ~~TAKE The~~ very oftEN be reduced TO SIMPLE MATHEMATICAL EQUATIONS. Modifying These SIMPLE EQUATIONS TO OBTAIN A different physical RESULT which IS OBSERVABLE AND A MORE ACCURATE REPRESENTATION of The REAL WORLD ALMOST REQUIRES The WORK of AN ARTISAN. MAN IS ALWAYS CONFRONTED WITH ~~fear~~ frightening RESTRICTIONS when he ATTEMPTS TO CONSTRUCT A NEW PHYSICAL LAW. The obvious difficulties PRESENTED by The LIMITATIONS of his OWN BRAIN IS foremost IN The picture. BUT IN The LAB he SEE A WORSE MESS. DIRT GETS IN his APPARATUS, he HAS TO PUMP AIR OUT of his TEST CHAMBERS, he fights AN' EVERY ENDING BATTLE WITH ENVIRONMENTAL FACTORS which change his RESULTS AND Thus create disagreement with his Theory. TRY TO VISUALIZE SOMEONE PREDICTING The behavior of A Thunderstorm if he NEVER SAW. JUST IMAGINE WHAT GOES ON: The SUN'S INCIDENT heat CAUSES CURRENTS IN The ATMOSphere which CAUSES EVAPORATION of WATER AND Then changes IN PRESSURE AND TEMPERATURE LEAD TO CONDENSATION INTO CLOUDS AND Then ELECTROSTATIC charge reach A breakdown POINT suddenly RELEASING TREMENDOUS QUANTITIES of ENERGY ACCOMPLIED by The down pouring of RAIN OR, perhaps, The beauty of A SNOW FALL ~~which~~ WITH The INDIVIDUALITY of each SNOWFLAKE. No MAN COULD have EVER have dreamed A dream LIKE THAT if he had NOT EXPERIENCED IT first. AS WE CONTINUE TO PURSUE ALL ASPECTS of PURE SCIENCE WE SHOULD be MOTIVATED by THIS SIMPLICITY AND BEAUTY of NATURE which IS ALL AROUND US. ~~When~~ WITH SOME IMAGINATION AND CLEVER THINKING WE WILL ONE ~~DA~~ DAY UNDERSTAND WHY WE ARE here.

37. A PHILOSOPHICAL NOTE

NIELS Bohr ONCE SAID There ARE TWO KINDS of TRUTHS. There ARE ORDINARY TRUTHS AND Then There ARE deep TRUTHS. AN ORDINARY TRUTH IS ONE Which The CONTRARY IS FALSE. EXAMPLE: The HOUSE IS GREEN AND The house IS NOT GREEN. CLEARLY ONE STATEMENT IS FALSE. Now A deep TRUTH IS ONE IN which The CONTRARY IS ALSO A deep TRUTH but NOT NECESSARILY FALSE. EXAMPLE: MAN LIVES TO SERVE GOD; OR MAN does NOT LIVE TO SERVE God. OR PERHAPS, MAN IS NOT JUST A MACHINE AND MAN IS JUST A MACHINE. There ARE NUMEROUS SUCH EXAMPLES of SUCH deep TRUTHS. THINK ABOUT IT —

INTRODUCTION TO PERTURBATION THEORY

WE HAVE BEEN LEARNING The BASIC MACHINERY of QUANTUM MECHANICS. WE HAVE TALKED ABOUT The free PARTICLE behavior AND how WE USE AMPLITUDE WAVE FUNCTIONS TO PREDICT ITS behavior. WE STUDIED The PROPERTIES of SCATTERING AND QUANTUM ANGULAR MOMENTUM THEORY. RECENTLY WE discussed The QUANTUM HARMONIC OSCILLATOR. WE WOULD NOW LIKE TO APPLY ALL THIS POWERFUL MACHINERY TO SOLVING MORE COMPLICATED AND MORE REALISTIC PROBLEM. Our BASIC PROBLEM WILL BE INVOLVING INTERACTIONS BETWEEN OSCILLATORS AND ELECTROMAGNETIC fields. WE WANT TO WORK OUT SOME of The RESULTING ENERGY LEVELS for SUCH A SYSTEMS. To BEGIN WE NEED ONLY A few SIMPLE TOOLS; ONE BEING The Schrödinger EQUATION for The TIME INDEPENDENT HAMILTONIAN, i.e.,

$$H|\varphi_n\rangle = E_n|\varphi_n\rangle$$

Where The WAVE FUNCTIONS φ_n ARE ORTHOGONAL SUBJECT TO The CONDITION THAT $\langle\varphi_m|\varphi_n\rangle = 0$

THESE TWO SIMPLE POINTS ARE VERY IMPORTANT TO REMEMBER BECAUSE THEY ARE VERY USEFUL. NOW WE WON'T CONCERN OURSELVES here WITH ANY of The COMPLICATIONS ARISING from A NON-LINEAR OSCILLATOR. IN GENERAL The NON-LINEARITIES ARE NOT VERY IMPORTANT but MORE honestly very few PEOPLE have SUCCEEDED IN WORKING ANY NON-LINEAR PROBLEMS OUT EXACTLY.

SINCE THE MAJORITY OF QUANTUM MECHANICAL PROBLEMS CANNOT BE SOLVED EXACTLY - EVEN LINEAR PROBLEMS; IT IS OFTEN ADVANTAGEOUS TO APPLY PHYSICAL AND MATHEMATICAL APPROXIMATIONS IN ORDER TO OBTAIN A NEARLY CORRECT ANSWER. THE ABILITY TO WISELY APPLY APPROXIMATIONS AT THE APPROPRIATE TIME IN SOLVING A PROBLEM IS A GREAT GIFT AND ONE WHICH EVERYONE SHOULD ~~LEARN~~ SEEK TO ACQUIRE. THE GENERAL METHOD OF ATTACKING A PROBLEM IN QUANTUM MECHANIC IS TO APPLY PERTURBATION THEORY.

THERE ARE TWO TYPES OF PERTURBATIONS THEORIES WHICH CAN BE USED. THERE IS THE TIME-INDEPENDENT AND TIME-DEPENDENT PROBLEMS. THE TIME-INDEPENDENT PROBLEMS SEEK TO FIND THE STATIONARY EIGENSTATES AND EIGENVALUES OF THE HAMILTONIAN H. BY A TIME-INDEPENDENT HAMILTONIAN H WE NECESSARILY ARE DEALING WITH AN ISOLATED SYSTEM, ONE WHICH IS NOT INTERACTING WITH AN ADJACENT SYSTEM. SUCH AN ISOLATED SYSTEM CAN UNDERGO A TRANSITION BETWEEN STATIONARY STATES IF THE SYSTEM INITIALLY STARTED OUT IN A STATE WHICH WAS NOT AN EIGENSTATE OF H. MOST SYSTEMS ARE NOT ISOLATED FROM THE EXTERNAL WORLD SO THAT TRANSITIONS BETWEEN DIFFERENT STATIONARY STATES CAN OCCUR WHEN SUBJECT TO EXTERNALLY APPLIED DISTURBANCE. SUCH PROBLEMS AS PARTICLE COLLISION AND DECAY ARE REPRESENTED BY EXTERNAL INTERACTION FORCES. SOLVING THESE PROBLEMS IS OF MORE INTEREST BUT INVOLVES TIME-DEPENDENT ANALYSES WHICH IS INHERENTLY MORE COMPLICATED THAN THE TIME-INDEPENDENT PROBLEMS. WE THEREFORE WILL BE DEALING PRIMARILY WITH THE TIME-DEPENDENT PERTURBATION THEORY.

SEPARABLE SYSTEMS

COMMON TO BOTH TYPES OF PERTURBATION THEORY IS THE IDEA OF SEPARABLE HAMILTONIANS. WHEN WE WRITE THE TOTAL HAMILTONIAN H IN SCHRÖDINGER'S EQUATION

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = H\psi$$

IT IS COMPOSED OF TWO PARTS. THE WAVE FUNCTION ψ IS GIVEN BY $e^{-iE_nt}|\varphi_n\rangle$ FOR THE TIME-INDEPENDENT CASE, AND $\sum_n C_n e^{-iE_nt}|\varphi_n\rangle$ FOR THE TIME DEPENDENT CASE.

THE HAMILTONIAN H IS EXPRESSED AS

$$H = H_0 + H_1$$

where H_0 IS THE UNPERTURBED HAMILTONIAN WHICH WE POSTULATE WE KNOW HOW TO ACCURATELY DEFINE. H_1 IS THE WEAK INTERACTION HAMILTONIAN WHICH HAS ITS OWN ENERGY SPECTRUM, CALL IT E_{1m}. Therefore There ARE TWO SETS of WAVE FUNCTIONS ψ_0 AND ψ_1 EACH WITH ENERGY LEVELS E_{0n} AND E_{1m} RESPECTIVELY. THE HAMILTONIAN H_0 WILL ONLY OPERATE ON ψ_0 WHILE H_1 WILL ONLY OPERATE ON ψ_1. The TWO PIECES of The HAMILTONIAN OPERATE ON different PARTS of The WAVE FUNCTION. If WE NOW define SOME SUPER WAVE FUNCTION Φ Which IS A FUNCTION of The VARIABLES of The SYSTEM Then

$$H|\Phi\rangle = (H_a + H_b)|\Phi\rangle = W|\Phi\rangle$$

where W IS The COMPOSITE ENERGY SPECTRUM of The SYSTEM.

JUST AS AN EXAMPLE TO SHOW YOU how COMPLICATED H CAN GET LET'S CONSIDER A HELIUM ATOM WITH TWO ELECTRONS AND NUCLEUS. The COORDINATES of The ELECTRONS ARE DENOTED BY (x_1, y_1, z_1) AND The SECOND ELECTRON by (x_2, y_2, z_2). If WE CONSIDER The NUCLEUS TO BE of FINITE MASS Then IT WILL MOVE AND HAVE COORDINATES (x_N, y_N, z_N). Therefore The HAMILTONIAN IS

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + \frac{P_N^2}{2m_N} + \frac{e^2}{r_{12}} - \frac{Ze^2}{r_{1N}} - \frac{Ze^2}{r_{2N}}$$

Schrodinger's EQUATION BECOMES

$$-\frac{\hbar^2}{2m}\nabla_1^2\varphi - \frac{\hbar^2}{2m}\nabla_2^2\varphi - \frac{\hbar^2}{2M}\nabla_N + V\varphi = E\varphi(x_1, y_1, z_1, x_2, y_2, z_2, x_N, y_N, z_N)$$

IN THIS HAMILTONIAN The EXTRA TERM H_1 IS ASSOCIATED WITH The DYNAMICAL RESPONSE of The NUCLEUS TO The MOTION of The ELECTRONS AROUND IT. Therefore

$$H_0 = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + \frac{e^2}{r_{12}} -$$

AND

$$H_1 = \frac{P_N^2}{2m_N} - \frac{Ze^2}{r_{1N}} - \frac{Ze^2}{r_{2N}}$$

IN The LIMIT AS The NUCLEUS MASS IS TAKEN TO BE INFINITE Then $H_1 = 0$ AND WE ARE LEFT WITH JUST H_0.

The point of writing out H for the helium atom is to show when I write $\Phi(x_0, x_1)$ I am making a shorthand notation for the independent variables contained in the system.

In order to see how $H|\Phi\rangle$ produces $W|\Phi\rangle$ as expressed earlier let's try a composite wave function of the form

$$\Phi(x_0, x_1) = \varphi_n^0(x_0)\,\varphi_m^1(x_1)$$

We guessed at the product form because of our earlier work in amplitudes. Plugging Φ into the above equation

$$H|\Phi\rangle = (H_0 + H_1)\left(|\varphi_n^0\rangle|\varphi_m^1\rangle\right) = H_0\,\varphi_n^0\,\varphi_m^1 + H\,\varphi_n^0\,\varphi_m^1 = W\varphi_n^0\,\varphi_m^1$$

Since H_0 only operates on φ_n^0 and H_1 only operates on φ_m^1, we have

$$\varphi_m^1 H_0 \varphi_n^0 + \varphi_n^0 H_1 \varphi_m^1 = \varphi_m^1 E_n^0 \varphi_n^0 + \varphi_n^0 E_m^1 \varphi_m^1$$

E_n^0 and E_m^1 are just numbers so that they can come out as

$$(E_n^0 + E_m^1)\,\varphi_m^0\,\varphi_m^1 = W\,\varphi_n^0\,\varphi_m^1$$

So that

$$W = E_n^0 + E_m^1$$

Therefore the energies add while the wave functions multiply. This is a very important properties of separable systems which we will use in more complicated problem. The key point is that the hamiltonian and energy spectrums are additive.

TIME-DEPENDENT PERTURBATION THEORY

I'd like to talk about the time-dependent perturbation theory because it has enormous practical utility in quantum mechanic. It constitutes the underpinning of the analysis of 2 systems far apart or weakly coupled through a slight interacting field. A typical example is one system initially in the ground state while another is in some excited state; at some later time the two systems can interact and transfer energy. The problem is to figure out the probability that the transition will occur. Such an example would be a cavity resonator consisting of millions and millions of harmonic oscillators which can interact with some atoms in the cavity to undergo an absorption of a photon or perhaps, emit a photon.

AGAIN WE WILL BE CONCERNED ONLY WITH TIME-INDEPENDENT HAMILTONIANS INITIALLY, I.E., $H_0 \neq H_0(t)$. THIS MEANS THE STATE IS INITIALLY PREPARED BY THE EXPERIMENTER. THE WILL HOWEVER BE A TIME DEPENDENT PIECE OF THE HAMILTONIAN, CALL IT H', WHICH ARISES FROM THE TIME VARYING INTERACTING FIELD. THE TOTAL HAMILTONIAN IS THEN

$$H = H_0 + H_1(t)$$

AND THE SCHRODINGER EQUATION IS

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = H\psi = [H_0 + H_1(t)]\,\psi(t)$$

WE WILL ASSUME AT TIME $t=0$ WE KNOW PRECISELY THE STATE OF THE SYSTEM TO BE $\psi(t) = \psi(0)$ AT $t=0$. WE WANT TO KNOW WHAT STATE THE SYSTEM WILL BE IN AT SOME LATER TIME t. NOW H_0 DESCRIBES TWO NONINTERACTING SYSTEMS AND $H_1(t)$ DESCRIBES THEIR MUTUAL INTERACTION.

WE WILL SOLVE THIS ~~Th~~ PROBLEM BY USING AN UNPEDISTRIAN APPROACH. FIRST WE WILL SOLVE THE EASY PROBLEM, I.E.,

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = H_0\,\psi(t)$$

BUT TO SOLVE THIS WE NEED TO KNOW WHAT $\psi(t)$ IS. WELL, LET'S TRY THE SOLUTION

$$\psi(t) = e^{-i/\hbar\, t H_0}\,\psi(0)$$

IT'S NOT TOO DIFFICULT TO SHOW THIS IS A SOLUTION BUT WHAT THE HELL DOES IT MEAN $e^{-i/\hbar\, t H_0}$? WHAT IS AN OPERATOR DOING IN THE EXPONENT? WELL IF WE EXPAND THE EXPONENT IN A POWER SERIES THE OPERATION BECOMES CLEAR SINCE WE KNOW WHAT IT MEANS TO OPERATE WITH H_0 ONCE, TWICE, THREE-TIMES, ETC.

NOW IF $\psi(0)$ IS JUST φ_n THEN WOULDN'T YOU EXPECT THAT $e^{-i/\hbar\, t H_0}\varphi_n$ WOULD PRODUCE $e^{-i/\hbar\, t E_n}\varphi_n$? SURE YOU WOULD BECAUSE $H_0\varphi_n = E_n\varphi_n$. LET'S SEE IF WE CAN PROVE IT, FIRST WE KNOW THAT

$$e^{-i/\hbar\, t H_0} = 1 - \frac{i}{\hbar}tH_0 + \frac{1}{2}\left(-\frac{i}{\hbar}t\right)^2 H_0 H_0 + \frac{1}{3!}\left(-\frac{i}{\hbar}t\right)^3 H_0 H_0 H_0 + \cdots$$

THEN $e^{-i/\hbar\, t H_0}\varphi_n$ PRODUCES

$$\varphi_n - i/\hbar\, t\,\underbrace{H_0\varphi_n}_{E_n\varphi_n} + (\cdots)\,H_0\underbrace{(H_0\varphi_n)}_{\underbrace{H_0\,E_n\varphi_n}_{E_n\cdot E_n\varphi_n}} + (\cdots)\,H_0 H_0\underbrace{(\underbrace{H_0\varphi_n}_{E_n\varphi_n})}_{\underbrace{E_n^2\varphi_n}_{E_n^3\varphi_n}} + \cdots = e^{-i/\hbar\, t E_n}\varphi_n$$

NOW THAT WE UNDERSTAND WHAT THE SYMBOLS MEAN WE SHOULD PROVE THAT $\psi(t) = e^{-i/\hbar\, t H_0}\psi(0)$ IS INDEED A SOLUTION TO SHRÖDINGER'S EQUATION - SO LET'S SEE:

$$-\frac{\hbar}{i}\frac{\partial \psi(t)}{\partial t} = -\frac{\hbar}{i}\frac{\partial}{\partial t}\left[e^{-i/\hbar\, t H_0}\psi(0)\right] = H_0\psi(t)$$

PRETTY EASY huh? BUT YOU STILL MIGHT BE WONDERING IF WHAT I did WAS MATHEMATICALLY LEGITIMATE SO I'LL PROVE IT. WE HAVE TO differentiate THE POWER SERIES EXPANSION FOR $e^{-i/\hbar\, t H_0}$

$$-\frac{\hbar}{i}\frac{\partial \psi(t)}{\partial t} = -\frac{\hbar}{i}\left[-\frac{i}{\hbar}H_0 + \left(-\frac{i}{\hbar}\right)^2 t\, H_0 H_0 + \frac{1}{2!}\left(-\frac{i}{\hbar}\right)^3 t^2 H_0 H_0 H_0 + \cdots\right]\psi(0)$$

$$= H_0\left[1 - i/\hbar\, t H_0 + \frac{1}{2!}\left(-\frac{i}{\hbar}t\right)^2 H_0 H_0 + \cdots\right]\psi(0)$$

$$= H_0\, e^{-i/\hbar\, t H_0}\psi(0) = H_0\psi(t) \qquad \text{Q.E.D.}$$

THE NOTATION HERE IS FAIRLY SIMPLE BUT IT IS hard TO figure OUT WHAT IS GOING ON. NOW IT IS NECESSARY TO SPECIFY THE form of $\psi(0)$ which CAN BE DONE MOST SIMPLY AS A POWER SERIES,

$$\psi(0) = \sum_n C_n \varphi_n$$

where THE C_n'S CAN BE found BY PROJECTING $\psi(0)$ INTO EACH φ_n AS

$$C_n = \langle \varphi_n | \psi(0) \rangle$$

This THEN SPECIFIES THE INITIAL STATE of THE SYSTEM BUT IN GENERAL $\psi(t)$ IS GIVEN BY

$$\psi(t) = \sum_n C_n e^{-i/\hbar\, t H_0}\varphi_n = \sum_n C_n e^{-i/\hbar\, t E_n}\varphi_n$$

So far WE HAVE JUST CONCERNED OURSELVES WITH THE first PART of THE HAMILTONIAN, H_0, WE MUST NOW CONSIDER THE TIME DEPENDENT PART, $H_1(t)$.

Our PROBLEM IS TO SOLVE THE SCHRÖDINGER EQUATION WITH THE TIME dependent PART of THE HAMILTON. THE SOLUTION WILL be of THE GENERAL form

$$\psi(t) = \left\{e^{-i/\hbar\int_0^t [H_0 + H_1(t')]\,dt'}\right\}\psi(0)$$

BUT WAIT WE CAN'T PROVE THIS SOLUTION IS CORRECT AS WE did LAST TIME. THIS HAPPENS BECAUSE IN GENERAL if A AND B ARE TWO OPERATORS THEN

$$e^{(A+B)} \neq e^A e^B$$

If A AND B ARE NON-COMMUTATIVE, I.E., $(AB - BA) \neq 0$. SO WE HAVE TO GO ABOUT THE PROOF ANOTHER WAY. WE WON'T do IT TOO ELEGANTLY SO NO ONE GET LOST IN THE MATH.

I AM GOING TO TRY AND SOLVE FOR H_1 SUBJECT TO THE CONDITION THAT IT IS TO BE SMALL. I WILL TRY THE SOLUTION OF THE FORM

$$\psi(t) = e^{-i/\hbar\, t H_0}\, \chi(t)$$

Where $\chi(t)$ IS A SLIGHTLY PERTURBED WAVE FUNCTION from $\psi(0)$. SUBSTITUTING THIS DIRECTLY INTO SCHRÖDINGER'S EQUATION

$$i\hbar \frac{\partial}{\partial t}\psi(t) = i\hbar \frac{\partial}{\partial t}\left[e^{-i/\hbar\, t H_0}\chi(t)\right] = H_0\chi(t)e^{-i/\hbar\, t H_0} + i\hbar\, e^{-i/\hbar\, t H_0}\frac{\partial \chi(t)}{\partial t}$$

$$= H_0\psi(t) + i\hbar\, e^{-i/\hbar\, t H_0}\frac{\partial \chi(t)}{\partial t}$$

Now THIS MUST EQUAL $(H_0 + H_1(t))\psi(t) = (H_0 + H_1(t))\, e^{-i/\hbar\, t H_0}\chi(t)$

Therefore WE HAVE AN EQUATION for $H_1(t)$

$$H_1(t) = \frac{i\hbar}{\chi(t)}\frac{\partial \chi(t)}{\partial t}$$

OR

$$\chi(t) = e^{-i/\hbar \int_0^t H_1(t')\,dt'}$$

The ALTERNATE WAY TO EXPRESS THIS differential EQUATION IS

$$i\hbar \frac{\partial \chi(t)}{\partial t} = e^{i/\hbar\, t H_0} H_1(t)\, e^{-i/\hbar\, t H_0}\chi(t)$$

This IS ALMOST A SCHRÖDINGER EQUATION fo $\chi(t)$. If WE define A NEW HAMILTONIAN

$$H_1'(t) = e^{i/\hbar\, t H_0} H_1(t)\, e^{-i/\hbar\, t H_0}$$

WE HAVE The SCHRÖDINGER EQUATION

$$-\frac{\hbar}{i}\frac{\partial \chi(t)}{\partial t} = H_1'(t)\,\chi(t)$$

This IS A VERY USEFUL EQUATION BECAUSE IT TELLS how The disturbance PROPAGATES WITH TIME. FROM The ABOVE RESULTS WE CAN CONCLUDE THAT if ψ SATISFIES

$$i\hbar \frac{\partial \psi}{\partial t} = [H_0 + H_1(t)]\,\psi(t)$$

THEN

$$\psi'(t) = e^{+i/\hbar\, t H_0}\psi \quad \text{WILL SATISFY,}$$

$$i\hbar \frac{\partial \psi'}{\partial t} = H_1'(t)\,\psi'$$

TO SEE THIS SUBSTITUTE for ψ'

$$i\hbar \frac{\partial}{\partial t}\left[e^{i/\hbar\, t H_0}\psi\right] = e^{i/\hbar\, t H_0} H_1(t)\, e^{-i/\hbar\, t H_0}\, e^{i/\hbar\, t H_0}\psi$$

$$i\hbar\left[H_0\, i/\hbar\, e^{i/\hbar\, t H_0}\psi + e^{i/\hbar\, t H_0}\frac{\partial \psi}{\partial t}\right] = e^{i/\hbar\, t H_0} H_1(t)\,\psi$$

$$-e^{i/\hbar\, t H_0} H_0\psi + i\hbar\, e^{i/\hbar\, t H_0}\frac{\partial \psi}{\partial t} = e^{i/\hbar\, t H_0} H_1(t)\,\psi$$

DIVIDE THROUGH BY $e^{i/\hbar\, t H_0}$ AND REARRANGE

$$i\hbar \frac{\partial \psi}{\partial t} = [H_0 + H_1(t)]\,\psi \qquad \text{Q.E.D.}$$

In order to figure out what $H_1'(t)$ represents we need to work out matrix elements; the matrix element giving us us the transition probabilities. In general

$$\int \varphi_m^* A \varphi_n \, dVol = \langle \varphi_m^* | A | \varphi_n \rangle = A_{nm}$$

In our case $A = H_1'(t)$ so that

$$\int \varphi_m^* H_1'(t) \varphi_n \, dVol = \int \varphi_m^* e^{i/\hbar t H_0} H_1(t) e^{-i/\hbar t H_0} \varphi_n \, dVol$$

since $e^{-i/\hbar t H_0} \varphi_n = e^{-i/\hbar t E_n} \varphi_n$ and $\varphi_m^* e^{i/\hbar t H_0} = e^{i/\hbar E_m t} \varphi_m^*$

we have that

$$\int = e^{i/\hbar (E_m - E_n) t} \int \varphi_m^* H_1(t) \varphi_n \, dVol = e^{i/\hbar (E_m - E_n) t} \langle \varphi_m | H_1(t) | \varphi_n \rangle$$

Now physically ψ' is the time development of ψ but the propagation is modulated by component $e^{i/\hbar (E_m - E_n) t}$ due to the presence of $H_1(t)$ or a small interaction force. Although $H_1(t)$ is in general weak the phase shift could be quite rapid.

Obviously is $H_1(t)$ were negligible then the two systems would never interact and there wouldn't be anything of interest occurring. When $H_1(t)$ is small but not negligible we have some interesting physics. Suppose at time $t=0$ we specify $\psi(t)$ precisely, i.e, we know $\psi(0)$ and $\psi'(0)$. What our formula says that if we want to know what to expect of the wave function at some latter time $t = T$ then all we need to calculate is

$$\psi(T) = e^{-i/\hbar T H_0} \psi'(T)$$

$\psi'(t)$ is in general a power series expansion such as

$$\psi'(t) = \psi_0'(t) + \psi_1'(t) + \psi_2'(t) + \psi_3'(t) + \cdots \psi_n'(t)$$

Where the zeroth order term $\psi_0'(t)$ we know equals $\psi'(0)$. We are usually interested in the first order approximation which tells us to what order in H_0 we need, i.e., first also. Our Schrödinger equation becomes

$$i\hbar \frac{\partial}{\partial t} [\psi_0'(t) + \psi_1'(t)] = H_1'(t) [\psi_0'(t) + \psi_1'(t)]$$

or

$$i\hbar \frac{\partial}{\partial t} [\psi_0'(0) + \psi_1'(t)] = H_1'(t) [\psi'(0) + \psi_1'(t)]$$

This equation reduces to

$$i\hbar \frac{\partial}{\partial t} \psi_1'(t) = H_1'(t)\, \psi(0)$$

which can be integrated for $\psi_1'(t)$ to get

$$\psi_1'(t) = -i/\hbar \int_0^t H_1'(t')\, dt'\, \psi(0)$$

Therefore we have found $\psi'(t)$ to first order,

$$\psi'(t) = \psi(0) - i/\hbar \int_0^t H_1'(t')\, dt'\, \psi(0)$$

If we want $\psi'(t)$ at time $t = T$ we simply integrate dt' between 0 and T. It is a little more messy to go to the second order terms and higher but they are obtained in a straightforward manner. For the second order term

$$\psi_2'(T) = \left(-\frac{i}{\hbar}\right)^2 \int_0^T dt \int_0^t dt'\, H'(t)\, H'(t')\, \psi'(0)$$

The third order term is

$$\psi_3'(T) = -\left(-\frac{i}{\hbar}\right)^3 \int_0^T dt \int_0^t dt' \int_0^{t'} dt''\, H'(t)\, H'(t')\, H'(t'')\, \psi'(0)$$

As we said earlier for the questions we want to answer the first order term will be sufficient.

So now we want to find what the amplitude will be for the system to end up in some final state, ψ_f, if it is initially prepared in ~~state~~ some initial state ψ_i. How do we calculate this. Well if we have the system describable by a wave function $\psi(t)$ we evaluate it at time $t = T$ at the time of interest and then project this wave function into the final state ψ_f' that we'd like it to be in. Mathemathically this is said

$$\langle \psi_f' | \psi'(T) \rangle$$

I should point out this is mathematically equivolent to $\langle \psi_f | \psi(T) \rangle$. To prove that it is

$$\langle \psi_f' | \psi'(T) \rangle = \langle e^{\frac{i}{\hbar} T H_0} \psi_f | e^{i/\hbar T H_0} \psi \rangle = \langle \psi_f | e^{-i/\hbar t H_0} e^{i/\hbar t H_0} | \psi \rangle$$
$$= \langle \psi_f | \psi(T) \rangle$$

That's our answer. So let's do the work.

LET'S PROJECT $\psi'(T)$ INTO $\psi_f'(T)$ AND SEE WHAT WE GET

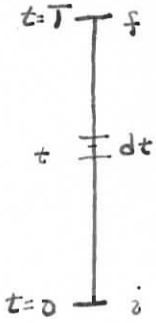
$$\langle \psi_f' | \psi'(T) \rangle = \langle \psi_f' | \underset{\text{INITIAL}}{\psi_i'} \rangle - \frac{i}{\hbar} \langle \psi_f' | \int_0^T H_1'(t)\,dt \, | \psi_i' \rangle$$

which can be rewritten as

$$\langle \psi_f' | \psi'(T) \rangle = \langle \psi_f' | \psi_i' \rangle - \frac{i}{\hbar} \int_0^T dt \, \langle \psi_f' | H_1'(t) | \psi_i' \rangle$$

The first term on the right is just the amplitude to end up in the final state if there were no disturbance around. Usually it is taken that the initial an final states are orthogonal such that $\langle \psi_f' | \psi_i' \rangle = 0$ and we are interested solely in the second term on the right. Since it is a very important concept that we have developed here we will continue to describe in more detail physically what is contained in the integral. You really don't know anything when you see abstract formulas like the above.

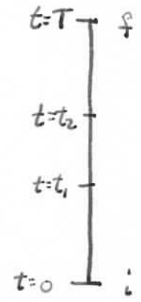
To further understand what is happening we will use some of our path integral concepts which we have developed. Let's draw a little diagram like on the right where we start at some initial state and ask it to end up in some final state. Starting with initial amplitude ψ_i the system developes in such a manner as to end up in some final state ψ_f. However at some time t along the way there appears an interacting force which produces an amplitude for transition from one state to another. The transition I like to think of as a scattering process. The amplitude to scatter then has a phase proportional to $-i\,H_1(t)\,dt$ where the dt scattering occurs over a range of time dt. Since the scattering can take place at anytime between 0 and T, we integrate over this range. It is quite possible that the scattering may occur not once but twice, three-time, etc. The integral is all inclusive of all possible interaction. The amplitude to start in ψ_i and end in ψ_f is read from right to left in the integral, i.e., right to left is the time developement of the system



$$\langle \psi_f | \psi_i \rangle = \int_0^T \langle \psi_f | \, e^{-iH_0(T-t)} (-iH_1(t)\,dt)\, e^{-i/\hbar\, t H_0} \, | \psi_i \rangle$$

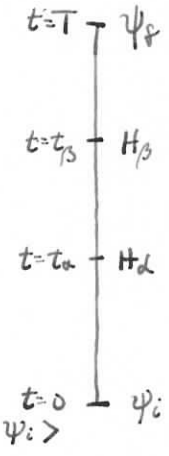
If two scatterings occur then we have the situation where

$$\langle \psi_f | \psi_i \rangle = \int_0^T \int_0^{t_2} (-i)^2 dt_2\, dt_1 \; \langle \psi_f | e^{-i(T-t_2)H_0} H_1(t_2) e^{-i(t_2-t_1)H_0} H_1(t_1) e^{-it_1 H_0} | \psi_i \rangle$$



In this case we start in state ψ_i and propagate as a free particle until acted upon by the disturbance at $t=t_1$ which produces a phase shift given by $H_1(t_1) e^{-it_1 H_0}$. Then the particle continues on until acted upon by the second scattering disturbance which produces another phase shift given by $H_1(t_2) e^{-i(t_2) H_0}$. Finally the particle propagates on proportional to $e^{-i(T) H_0}$ as a free particle. The total amplitude to start at i and end at f is given by the product of the individual amplitudes. The diagram, commonly called a Feynman diagram after the guy who first developed their use, is just a simple way to save you all the time grinding through these messy integrals. There isn't anything intuitively mysterious about their form its just another useful tool in describing these kinds of processes.

You have to be careful though in applying the integral expression because time order is critical in describing which scattering occurs when. As example consider a process described by the diagram on the right where there are now two different types of disturbances occurring, types H_α and H_β, say. H_β must occur after H_α chronologically as depicted. Therefore, the integral must reflect this time ordering, i.e.,



$$\langle \psi_f | \psi_i \rangle = (-i)^2 \int_0^T \int_0^{t_\beta} dt_\alpha\, dt_\beta \; \langle \psi_f | e^{-iH_0(T-t_\beta)} H_\beta(t_\beta) \underbrace{e^{-iH_0(t_\beta - t_\alpha)} H_\alpha(t_\alpha)}_{\hookrightarrow\, e^{-iH_0 t_\alpha} |\psi_i\rangle}$$

This is the amplitude if H_α occurs before H_β but there is also the possibility that H_β occurs before H_α. That case is given by

$$(-i)^2 \int_0^T \int_{t_\beta}^T dt_\alpha\, dt_\beta \; \langle \psi_f | e^{-iH_0(T-t_\alpha)} H_\alpha(t_\alpha) e^{-iH_0(t_\alpha - t_\beta)} H_\beta(t_\beta) e^{-it_\beta H_0} | \psi_i \rangle$$

The total amplitude is the sum of these two amplitudes if it is not known whether H_α will occur before H_β.

38. INTERACTION OF AN ATOM WITH AN ELECTROMAGNETIC FIELD

LAST TIME WE LEARNED HOW THE HAMILTONIAN OF A SYSTEM COULD BE BROKEN INTO TWO PARTS - ONE EASY PIECE, H_0, AND ANOTHER PIECE, H_1, WHICH IS A MORE COMPLICATED FUNCTION OF TIME DESCRIBING THE INTERACTION BETWEEN THE PARTS OF THE SYSTEM. THE RULE WHICH WE DEVELOPED WAS THAT THE AMPLITUDE TO SCATTER IN TIME dt IS PROPORTIONAL TO $-iH_1(t)\,dt$. THIS REPRESENTS AN EXTRA PIECE OF THE ANSWER WHICH IS A CORRECTION TO THE H_0 PIECE WHICH IS FOR NON-INTERACTING PARTS. THE TRANSITION OF A SYSTEM FROM SOME INITIAL STATE i TO SOME FINAL STATE f WAS DESCRIBABLE BY THE APPROXIMATIONS:

1st order $\quad \langle \psi_f | e^{-iH_0T} | \psi_i \rangle$

2nd order $\quad \langle \psi_f | e^{-iH_0(T-t)} \left(-\frac{i}{\hbar} H_1(t)\,dt\right) e^{-iH_0t} | \psi_i \rangle$

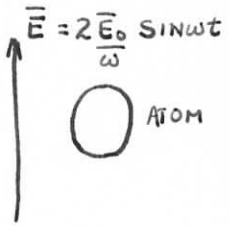
THE 1st ORDER TERM IS THE AMPLITUDE TO GO FROM i TO f WITHOUT ANY INTERACTION IN TIME T. THE SECOND ORDER TERM CORRECTS THE RESULT FOR A SCATTERING WHICH OCCURS AT TIME t IN TRANSIT FROM i TO f. THE COMPLETE ANSWER MUST INCLUDE THE POSSIBILITY THAT THE SCATTERING CAN OCCUR ANYWHERE BETWEEN i AND f OR BETWEEN $0 \le t \le T$. THE ANSWER IS THEN

$$\langle \psi_f | \psi_i \rangle = \langle \psi_f | e^{-iH_0t} | \psi_i \rangle + \int_0^T \langle \psi_f | e^{-iH_0(T-t)} \left(-\frac{i}{\hbar} H_1(t)\,dt\right) e^{-iH_0t} | \psi_i \rangle$$

WE WOULD LIKE TO APPLY THIS RESULT TO SOLVE SOME PROBLEMS WHICH ONLY REQUIRE THE FIRST ORDER CORRECTION TERM. LATTER WE'LL GET MORE COMPLICATED AND CORRECT TO SECOND ORDER SCATTERING.

THE PROBLEM WE WILL CONSIDER IS AN ATOM IN AN ELECTRIC FIELD $\bar{E}$. $\bar{E}$ IS SINUSOIDALLY VARYING AT FREQUENCY ω AND IS THEREFORE DESCRIBABLE BY A VECTOR POTENTIAL $\bar{A}$, WHERE

$$\frac{\partial \bar{A}}{\partial t} = \bar{E} \rightarrow \bar{A} = 2A_0 \cos\omega t, \quad A_0 = \frac{E_0}{\omega}$$



WE WILL FIRST CONSIDER A FREQUENCY SUFFICIENTLY LOW THAT THE WAVELENGTH IS LONG RELATIVE TO THE ATOMIC DIMENSION; THUS WE DON'T HAVE TO WORRY ABOUT VARIATIONS OF $\bar{A}$ ACROSS THE ATOM.

To begin the problem it is assumed that somehow we know the energy states of the atom in the absence of the external field. In principal we can in fact calculate the energy states of an unperturbed atom but it is quite difficult as we have pointed out earlier. Consider chromium, for instance, with 24 electrons. This is a very complicated structure which requires many independent variables: 3 for each electron and 3 for the nucleus or 75 total. But nevertheless we assume H_0 is solved and all the eigenfunctions φ_n have been worked out so that

$$H_0 \varphi_n = E_n \varphi_n$$

is known. The φ_n's then describe the behavior of all the electrons within the atom itself. Now we want to know what happens when the $\bar{E}$-field is present and how H_0 changes. For the level of accuracy here we will consider the change to H_0 is proportional to $\bar{E}$. There is no point in going to second order E terms since we are only concerned with first order variations in H_0. This is a key point in simplifying the work tremendously so I suggest you get use to making these approximations yourselves.

We now must write the hamiltonian for the system. We know that $H = P^2/2m$ is not the complete hamiltonian when an electric field is present. The correct hamiltonian for a particle in a vector potential $\bar{A}$ is

$$H = \frac{(\bar{P} - \frac{e}{c}\bar{A})^2}{2m} = \frac{1}{2m}\left(-\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)\cdot\left(-\frac{\hbar}{i}\bar{\nabla} - \frac{e}{c}\bar{A}\right)$$

This hamiltonian then works on ψ to produce the energy states of the system. Expanding out $H\psi$ we have that

$$H\psi = \frac{1}{2m}(-\hbar^2\nabla^2)\psi + \frac{1}{2m}\left[\frac{\hbar e}{ic}\bar{\nabla}\cdot(\bar{A}\psi)\right] +$$
$$\frac{1}{2m}\left[\frac{\hbar e}{ic}\bar{A}\cdot(\bar{\nabla}\psi)\right] + \frac{1}{2m}\left(\frac{e^2}{c^2}\right)(\bar{A}\cdot\bar{A})\psi$$

I should point out here, before I continue, if the particle (atom, molecule, etc) is made up of a lot of electrons, we must keep track of them all in the hamiltonian, i.e.,

$$H_0 = \sum_j \frac{p_j^2}{2m_j} + \sum_j \sum_k V(\bar{r}_{jk})$$

where $V(\bar{r}_{jk})$ is the interaction potential between the j and k particles. The total hamiltonian is

$$H = \sum_j \frac{\left[\bar{P}_j - \frac{e}{c}\bar{A}(x_j, y_j, z_j)\right]^2}{2m_j} + \sum_j \sum_k V(r_{jk})$$

The vector potential must be evaluated at each of the jth particle.

Now we notice in the expanded form of H three terms comining in which involve $\vec{A}$. These are the extra pieces which change our results from the case of no external field, i.e., the H_0 piece which is $-\frac{\hbar^2}{2m}\nabla^2\psi$. Since $H_1(t)$ lacks physical interpertation we will simply compute it by taking the difference between H and H_0. In other words

$$H_1(t) = H - H_0$$

or

$$H_1(t) = \sum_j \frac{e_j}{2m_j c}\left[\bar{P}_j \cdot \bar{A}(x_j) + \bar{A}(x_j)\cdot\bar{P}_j\right] + \sum \frac{e_j^2}{2m_j}\bar{A}(x_j)\cdot\bar{A}(x_j)$$

If we take the freedom to select the gage transformation $\bar{\nabla}\cdot\bar{A} = 0$ (i.e., the plane wave is assumed transverse) then

Proof:

$$\bar{P}\cdot\bar{A} = \bar{A}\cdot\bar{P}$$

$$(\nabla\cdot(A\psi)) = (A\cdot\nabla)\psi$$

$$\underbrace{(\nabla\cdot A)}_{0}\psi + \bar{A}\cdot\nabla\psi = A\cdot\nabla\psi$$

If we choose a different gage we must include a scalar potential χ in the hamiltonian.

One other point we agree that A_0 is a small potential and that terms in A_0^2 can be ignored since we are not computing terms in H_1^2. Therefore we have that

$$H_1(t) = \sum_j \frac{e_j}{m_j c}\bar{P}\cdot\bar{A}$$

SINCE WE HAVE CHOSEN A GAGE WHICH MAKES $\bar{A}$ TRANSVERSE WE WILL DEFINE THE DIRECTION OF A_0 TO BE THE x DIRECTION SUCH THAT

$$\vec{A} = A\,\bar{e}_x = 2A_{0x}\cos\omega t$$

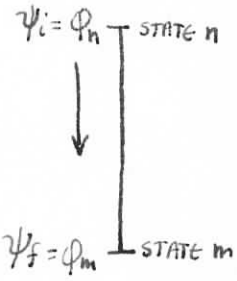
THEN THE OPERATOR

$$\vec{P} = \frac{\hbar}{i}\frac{\partial}{\partial x_j}$$

I WOULD LIKE TO DEFINE A NEW OPERATOR $j_x = \frac{e_j P x_j}{m_j c}$ WHICH IS NOT TO BE CONSIDERED A CURRENT IN THE CLASSICAL SENSE BUT RATHER A TRUE OPERATOR WORKING ON ψ.

NOW WE ARE READY TO GO TO WORK. FIRST WE WILL TAKE THE ZERO ORDER TERM WHICH DESCRIBES THE BEHAVIOR OF THE SYSTEM WITH NO EXTERNAL FIELD. OF COURSE, WE KNOW THE ANSWER ALREADY,

$$\langle \psi_f | e^{-i H_0(t)} | \psi_i \rangle$$



CONSIDER THE SYSTEM INITIALLY IN STATE n AND SUPPOSE WE WANT TO KNOW WITH WHAT AMPLITUDE IT WILL BE FOUND IN STATE m AT TIME T. THEN

$$\langle \varphi_m | \varphi_n \rangle = \langle \varphi_m | e^{-i H_0(t)} | \varphi_n \rangle$$

BUT WE KNOW WHAT $e^{-i H_0 T}$ ON $|\varphi_n\rangle$ IS IT IS JUST $e^{-i E_n T}|\varphi_n\rangle$ AND THEREFORE OUR ANSWER IS

$$e^{-i E_n T}\langle \varphi_m | \varphi_n \rangle$$

GENERALLY WE WANT THE TRANSITION TO OCCUR BETWEEN TWO EIGENSTATES WHICH IMPLIES BY ORTHOGONALITY THAT

$$\langle \varphi_m | \varphi_n \rangle = 0$$

IF THE STATES ARE NOT DEGENERATE. THE WHOLE PHYSICAL PROBLEM THUS REDUCES TO EVALUATING THE 1ST ORDER TERM WHICH WE WILL NOW DO.

TRANSITION AMPLITUDE

The AMPLITUDE TO GO from STATE n TO STATE m IN The presence of A SCATTERING POTENTIAL IS

$$S_{mn} = \int_0^T \langle \varphi_m | e^{-iH_0(T-t)} \frac{-i}{\hbar} H_1(t) e^{-iH_0 t} | \varphi_n \rangle dt$$

WE CAN START TO EVALUATE THE INTEGRAL by LETTING $e^{-iH_0 t}$ work ON The STATES $|\varphi_m\rangle$ AND $|\varphi_n\rangle$.

$$e^{-iH_0 t} | \varphi_n \rangle = e^{-iE_n t} | \varphi_n \rangle$$

$$\langle \varphi_m | e^{iH_0(T-t)} = \langle \varphi_m | e^{iE_m(T-t)}$$

The LAST EXPRESSION COMING from The HERMITEAN PROPERTY of φ_m. ALSO $e^{-iE_m T}$ CAN be brought OUTSIDE The INTEGRAL SINCE IT IS CONSTANT with TIME. Therefore we have simplified The INTEGRAL to,

$$S_{mn} = -\frac{i}{\hbar} e^{-iE_m T} \int_0^T dt \, e^{i/\hbar (E_m - E_n) t} \langle \varphi_m | H_1 | \varphi_n \rangle$$

The probability TO MAKE A TRANSITION between n and m IS Thus proportional TO The SQUARE of The MATRIX ELEMENT $\langle \varphi_m | H_1 | \varphi_n \rangle$ which MUST be INTEGRATED. The CONSTANT phase factor $e^{-i\frac{E_m t}{\hbar}}$ GOES OUT IN The SQUARING.

$$P(n \to m) \propto |S_{mn}|^2 = \left| \int_0^T dt \, e^{i/\hbar (E_m - E_n) t} \langle \varphi_m | H_1 | \varphi_n \rangle \right|^2$$

What This RESULT SAYS PHYSICALLY IS THAT PROBABILITY TO MAKE A TRANSITION BETWEEN TWO STATES depends ON The AMOUNT of $H_1(t)$ fourier COMPONENT CONTAINED IN The frequency RANGE As defined by The difference BETWEEN The ENERGY LEVELS,

$$\omega_{mn} = \frac{E_m - E_n}{\hbar}$$

If $H_1(t)$ HAS A ~~SM~~ LARGE frequency COMPONENT corresponding TO TRANSITION frequency Then The perturbation or SCATTER IS LIKEWISE STRONG. ANOTHER WAY TO THINK of THIS PHENOMENA IS TO DRAW AN ANALOG with The power density of H_1 IN The APPROPRIATE frequency RANGE. THAT IS The whole STORY of TRANSITION AMPLITUDES AND IT IS QUITE PRETTY AND SIMPLE; IT should be LEARNED by The STUDENTS EARLIER THAN IT IS TAUGHT SINCE ALL of CLASSICAL PHYSICS STEMS from The CONCEPT.

WE SHOULD NOW SPECIALIZE THE ANALYSIS TO THE PROBLEM WE ~~WHICH~~ WISH TO SOLVE BY PUTTING IN OUR EXPRESSION FOR H_1

$$H_1 = 2 j_x A_0 \cos \omega t$$

IF WE PUT THIS INTO S_{mn} WE CAN BRING OUT OF THE INTEGRAL THE TIME INDEPENDENT MATRIX ELEMENT $\langle \varphi_m | j_x | \varphi_n \rangle$ AND GET

$$S_{mn} = -\frac{i}{\hbar} e^{-i\frac{E_m T}{\hbar}} \langle \varphi_m | j_x | \varphi_n \rangle \int_0^T dt \, e^{\frac{i}{\hbar}(E_m - E_n)t} \, 2 A_0 \cos \omega t$$

IN ORDER TO GET AN ANSWER WE WILL SUBSTITUTE FOR $\cos \omega t$

$$\cos \omega t = \tfrac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

AND STREAMLINE THE NOTATION BY LETTING $(j_x)_{mn} = \langle \varphi_m | j_x | \varphi_n \rangle$.
THE EXPRESSION FOR S_{mn} THEN BECOMES INTEGRABLE

$$S_{mn} = -\frac{i}{\hbar} e^{-i\frac{E_m T}{\hbar}} (j_x)_{mn} \int_0^T A_0 \left[e^{i\left(\frac{E_m - E_n}{\hbar} + \omega\right)t} + e^{i\left(\frac{E_m - E_n}{\hbar} - \omega\right)t} \right] dt$$

IF WE INTEGRATE AND CONSIDER A_0 CONSTANT

$$S_{mn} = -\frac{i}{\hbar} e^{-i\frac{E_m T}{\hbar}} (j_x)_{mn} \left[\frac{A_0 \, e^{i\left(\frac{E_m - E_n}{\hbar} + \omega\right)t} - 1}{\left(\frac{E_m - E_n}{\hbar} + \omega\right)} + \frac{A_0 \, e^{i\left(\frac{E_m - E_n}{\hbar} - \omega\right)T}}{\left(\frac{E_m - E_n}{\hbar} - \omega\right)} \right]$$

THE MATRIX ELEMENT $(j_x)_{mn}$ IS A COMPLEX NUMBER WHICH HAS TO BE WORKED OUT FOR EACH PROBLEM TO BE SOLVE. FOR NOW WE WILL LEAVE IT JUST AS A MATRIX ELEMENT AND ASSUME IT CAN BE CALCULATED.

BY EXAMINING S_{mn} YOU SEE THAT IF A_0 IS WEAK YOU WONT GET MUCH AMPLITUDE TO SCATTER. UNLESS, HOWEVER, THE DENOMINATORS GET QUITE SMALL. FOR SUCH A CASE S_{mn} CAN GET BIG EVEN IF A_0 IS WEAK BY HAVING THE DENOMINATOR GET CLOSE TO ZERO. IF THE EXTERNAL FIELD IS FLUCTUATING AT FREQUENCY ω AND IT TURNS OUT THAT

$$\omega = \pm\left(\frac{E_m - E_n}{\hbar}\right) = \pm \omega_{mn}$$

THEN THE ATOM IS EXCITED AT ITS RESONANT TRANSITION FREQUENCY BETWEEN STATE n AND m, AND THE SYSTEM REALLY RUNS UP OR DOWN DEPENDING UPON THE SIGN OF THE ENERGY DIFFERENCE. IF $E_m > E_n$ AND WE ASK WHAT IS THE TRANSITION AMPLITUDE TO GO FROM n TO m, WE ARE CONSIDERING RESONANT ABSORBTION. WHILE FOR $E_m < E_n$ WE DRIVE THE SYSTEM DOWN AND HAVE RESONANT EMISSION.

NOTICE ALSO from the form of the TERMS IN THE BRACKET Either ONE or the other TERM IS IMPORTANT. If $\omega = -\omega_{mn}$ the first TERM IS IMPORTANT WHILE the SECOND TERM JUST SITS there rapidly OSCILLATING NEVER CONTRIBUTING MUCH TO the EMISSION PROCESS. FOR THAT REASON WE WILL forget the SECOND TERM by LIMITING OURSELVES TO the CASE FOR EMISSION. It SIMPLIFIES THE CALCULATION SO YOU don'T GET LOST.

The ANSWER IS NOW AT hand for WE WANT the SQUARE of S_{mn} TO GET the PROBABILITY THAT the SYSTEM WILL MAKE the TRANSITION of INTEREST.

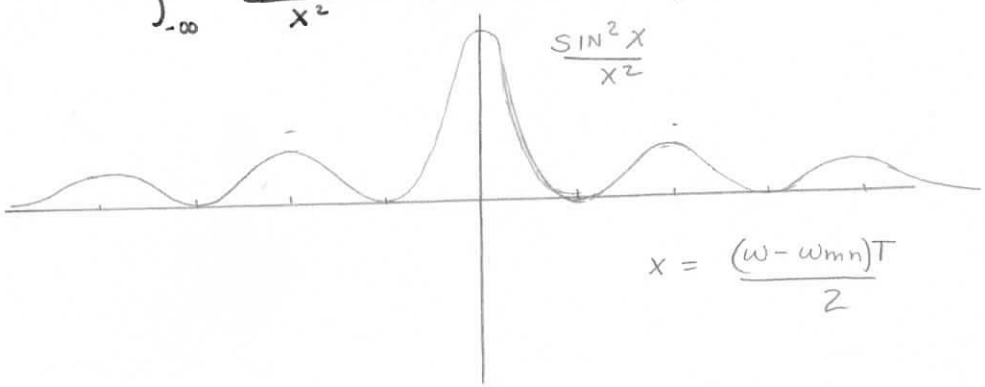
$$P(n \to m) = |S_{mn}|^2 = \frac{1}{\hbar^2} |(\mathcal{J}_x)_{mn}|^2 |A_0|^2 \left[\frac{e^{i(\omega-\omega_{mn})t} - 1}{\omega - \omega_{mn}} \right]^2$$

The SQUARED bracket CAN be EXPANDED OUT AND SIMPLIFIED TO GIVE the ANSWER

$$P(n \to m) = \frac{1}{\hbar^2} |(\mathcal{J}_x)_{mn}|^2 |A_0|^2 \, 4 \frac{\sin^2 (\omega-\omega_{mn}) \frac{T}{2}}{(\omega-\omega_{mn})^2}$$

THE EXPRESSION $\frac{4 \sin^2 (\omega-\omega_{mn}) T/2}{(\omega-\omega_{mn})^2}$ IS VERY LARGE WHEN RESONANCE IS APPROACHED, I.E., NEAR $\omega = \omega_{mn}$ the PROBABILITY IS PROPORTIONAL TO T^2. The SHARP RESONANCE which DEMONSTRATES this T^2 dependence IS IN PRACTICE IS VERY difficult TO duplicate IN the LAB. PROBLEMS INVOLVING TURNING ON A_0 AND TURNING IT off INTRODUCE other frequency COMPONENTS which DEGENERATE the SPIKE of the RESONANCE. The PROBABILITY DISTRIBUTION depends ON the FUNCTION $\frac{\sin^2 x}{x^2}$ A LITTLE LESS STRONGLY THAN INDICATED when AN AVERAGE over ~~the~~ A LIMITED frequency RANGE IS performed. The ACTUAL AVERAGED DEPENDENCE IS PROPORTIONAL TO $\delta(\omega-\omega_{mn}) T$

$$\int_{-\infty}^{\infty} 4 \frac{\sin^2 x}{x^2} dx = 2\pi \delta(\omega-\omega_{mn}) T$$



The bumpy character of the transition probability is quite similar to the diffraction pattern from a slit. The bumpy character can be observed if all the aspects of the experiment are tightly controlled. Practically the bumpiness is lost and the curve dies off as the average of the sinusoidally damped curve.

Since A_0 in practice has ~~en~~ frequency components other than the resonant frequency it is necessary to integrate over the expected frequency range or band width. If we think in terms of the mean value of A_0 to be analogous to the power contained in the field then

$$\overline{A_0^2} = \text{Power } (\omega)$$

And,

$$P(n \to m) = \int \cancel{\hbar} 2 |(j_x)_{mn}|^2 P(\omega) \pi \, \delta(\omega - \omega_{nm}) T \, d\omega$$

Which is quickly integrated to get

$$P(n \to m) = \frac{2\pi}{\hbar^2} |(j_x)_{mn}|^2 P(\omega_{mn}) T$$

The probability per unit time becomes proportional to the intensity or power of the field which is concentrated at frequency ω_{mn}. The proportionality constant depends on the complex matrix elements which in turn depends on the internal structure of the atom. $|(j_x)_{mn}|$ has ~~prop~~ the property of a hermitean operator such that

$$(j_x)_{mn} = (j_x)_{nm}$$

which tells us that the odds ~~of~~ driving the system up are just as good as driving us down.

If the transition occurs into a continuum of energy state or at least extremely closely spaced states, then we must sum over the density of states $\rho(E)$ or else integrate.

$$\sum_{m=1}^{\infty} P(n \to m) = \sum \frac{2\pi}{\hbar} |(j_x)_{mn}|^2 \delta(\omega - \omega_{mn}) |A_0|^2$$

If we have the atom being illuminated with a light which is not monochromatic then the power or intensity $I(\omega)$ at frequency ω_{mn} produces the probability per seconds of

$$\frac{P(n \to m)}{\text{sec}} = \frac{2\pi}{\hbar} \left| (\mathcal{J}_x)_{mn} \right|^2 \frac{I(\omega_{mn})}{(\omega_{mn})^2}$$

$$= \frac{2\pi}{\hbar} \left| \frac{(\mathcal{J}_x)_{mn}}{\omega_{mn}} \right|^2 I(\omega_{mn})$$

The strength of the transition, ~~is~~ given by the square matrix element, is sometimes called the dipole matrix element. Since the transition probability is proportional to the time the result becomes large only for a narrow bandwidth of frequencies centered at ω_{mn}. The bandwidth $\Delta\omega$ is on the order of $1/T$. Thus the longer you wait the closer you must excite the atom at resonance. One additional comment the total proportality constant $\frac{2\pi}{\hbar} \left| \frac{(\mathcal{J}_x)_{mn}}{\omega_{mn}} \right|^2$ is sometime called the Einstein transition element.

39. COULOMB EXCITATION

LAST TIME WE ENDED UP TALKING ABOUT MAKING TRANSITIONS IN A CONTINUUM OF ENERGY STATES. WE SAW THAT IT WAS NECESSARY TO SUM OVER ALL POSSIBLE STATES IN THE CONTINUUM SINCE IT IS MEANINGLESS TO TALK ABOUT A TRANSITION TO ONE STATE IN THE CONTINUUM. THE RESULT WE OBTAINED THEN WAS

$$\sum_m \frac{P(n \to m)}{\text{SEC}} = \sum_m \frac{2\pi}{\hbar} |A_{mn}|^2 \delta(E_m - E_n - \hbar\omega)$$

where A_{mn} IS THE MATRIX ELEMENT OF THE PERTURBATION, $H_1 = 2A\cos\omega t$; IT IS THE VECTOR POTENTIAL. IF WE CAN ASSUME THE CONTINUUM IS DESCRIBABLE BY A DENSITY FUNCTION $\rho(E_m)$ THEN WE CAN INTEGRATE OVER ALL STATES E_m

$$\sum_m \frac{P(n \to m)}{\text{SEC}} \to \int^{E_m} \frac{2\pi}{\hbar} |A_{mn}|^2 \delta(E_m - E_n - \hbar\omega) \rho(E_m)\, dE_m$$

$$= \frac{2\pi}{\hbar} |A_{mn}|^2 \rho(E)$$

where $E = E_n + \hbar\omega$. THIS THEN IS THE PROBABILITY PER UNIT TIME TO MAKE A TRANSITION TO SOME ARBITRARY STATE IN THE CONTINUUM.

ATOMIC IONIZATION

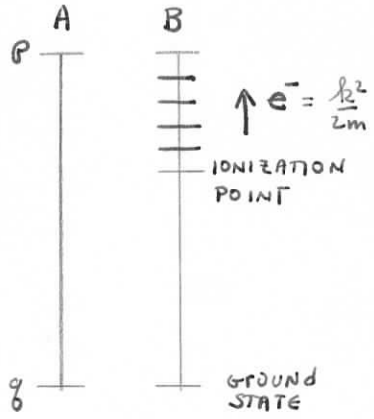
I'd LIKE TO WORK OUT A PROBLEM IN WHICH THE PERTURBING POTENTIAL H_1 IS NOT A FUNCTION OF TIME. IN THIS SPECIAL CASE THE PROBABILITY TO MAKE A TRANSITION PER SECOND IS JUST

$$\frac{P(n \to m)}{\text{SEC}} = \frac{2\pi}{\hbar} |(H_1)_{mn}|^2 \delta(E_m - E_n)$$

THE PROBLEM INVOLVES A HIGHLY EXCITED ATOM A AND ANOTHER ATOM B WHICH IS NEARBY AND WE WANT TO KNOW WITH WHAT PROBABILITY THE EXCITED ATOM INDUCES IONIZATION IN THE OTHER. THE HAMILTONIAN OF THE SYSTEM IS JUST

$$H = H_0 + \frac{e^2}{R_{ab}}$$

WHERE $H_0 = H_A + H_B$ THE SUM OF THE HAMILTONIANS OF THE NON-INTERACTING ATOM. THE COULOMB POTENTIAL IS A FUNCTION OF THE ELECTRON SEPARATION DISTANCE ONLY AND DOES NOT DEPEND ON TIME.



INITIALLY ATOM B IS IN THE GROUND STATE describable by the wave function φ_0^B. ATOM A IS IN THE EXCITED STATE p, ϕ_p^A. Therefore the INITIAL WAVE FUNCTION of the SYSTEM φ_m IS

$$\varphi_m = \varphi_0^B \varphi_p^A$$

The FINAL STATE finds ATOM B IONIZED with ITS ELECTRON MOVING WITH MOMENTUM k while ATOM A HAS FALLEN TO ITS GROUND STATE q. Therefore the FINAL STATE φ_n of the SYSTEM IS

$$\varphi_n = \varphi_k^B \varphi_q^A$$

The RESPECTIVE ENERGY STATES INITIALLY AND FINALLY ARE

$$E_m = E_0^B + E_p^A$$

And

$$E_n = E_k^B + E_q^A$$

ALL WE HAVE TO DO TO FINISH the problem IS TO COMPUTE the INTEGRAL

$$\int \langle \varphi_m | (H_1)_{mn} | \varphi_n \rangle \, dVOL = \int \varphi_q^A \varphi_k^B \frac{e^2}{r_{ab}} \varphi_0^B \varphi_p^A \, d^3r_a \, d^3r_b$$

The PROBABILITY TO MAKE the TRANSITION IS JUST

$$\frac{P(m \to n)}{sec} = \int^{E_k} 2\pi \, |H_1|^2 \, \delta\left(E_k^B + E_q^A - E_0^B - E_p^B \right) \rho(E_k) \, dE_k$$

Where WE INTEGRATE OVER ALL POSSIBLE MOMENTUM VALUES.

TO WORK OUT the INTEGRAL WE KNOW THAT the ELECTRON IN A CONTINUUM HAS A WAVE FUNCTION proportional TO e^{ikx} Thus $\varphi_k^B \propto e^{ikx}$, I.E. the ELECTRON IS free. The INTEGRATION IS TAKEN OVER ALL POSSIBLE MOMENTUMS SO THAT

$$\rho(E_k) \, dE_k \longrightarrow \frac{d^3k}{(2\pi)^3}$$

And the INTEGRATION IS NOW IN k-SPACE which IS EASIER TO work with. For the MOMENT WE ARE GOING TO AVERAGE H_1 over ALL directions of k AND TAKE IT OUT of the INTEGRAL.

WE NOW HAVE,

$$\frac{P(m \to n)}{SEC} = \frac{2\pi}{\hbar} \overline{|(H_1)_{mn}|^2} \int \delta\left(\frac{k^2}{2m} + E_g^A - E_0^B - E_p^A\right) \frac{d^3k}{(2\pi)^3}$$

LETS DEFINE $E_g^A - E_0^A - E_p^A$ AS $-W$ ASINCE IT IS THE EXCESS ENERGY REQUIRED OVER THE IONIZATION LEVEL ; E_0^B IS defined here from THE IONIZATION POINT SO IT IS A NEGATIVE ENERGY. If WE CONSIDER THE k-SPACE TO BE SPHERICALLY SYMMETRIC THEN

$$\frac{P(m \to n)}{SEC} = \frac{2\pi}{\hbar} \overline{|H_1|^2} \int \frac{4\pi k^2}{(2\pi)^3} \delta\left(\frac{k^2}{2m} - W\right) dk$$

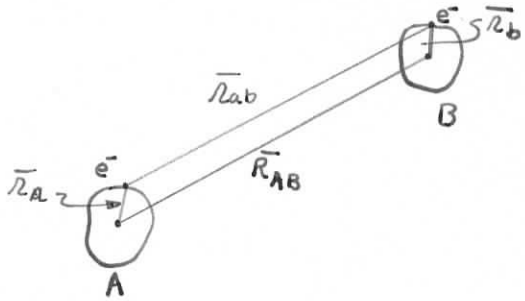
TO EXPLICITLY EVALUATE THE INTEGRAL WE HAVE TO USE THE PROPERTY of A delta FUNCTION THAT

$$\delta(f(x))\, dx = \frac{1}{|f'(x)|}$$

PERFORMING THE differentiation AND SIMPLIFYING WE HAVE

$$\frac{P(m \to n)}{SEC} = \frac{2}{\hbar\pi} \overline{H_1^2}\, m k$$

where $k = \sqrt{2mW}$. THAT'S THE FINAL RESULT EXCEPT for THE FACT THAT WE NEVER EVALUATED H_1. USUALLY IT IS TOO DIFFICULT TO MAKE AN EXPLICIT EVALUATION BUT SOMETIME IT IS POSSIBLE TO MAKE AN APPROXIMATION.



If THE ATOMS ARE FAR APART WE CAN APPROXIMATE THE ELECTRON SEPARATION DISTANCE r_{ab} by

$$r_{ab} = R_{AB} - \vec{r}_a \cdot \vec{R}_{AB} + \vec{r}_b \cdot \vec{R}_{AB}$$

where $\vec{R}_{AB}$ IS THE DISTANCE BETWEEN THE TWO NUCLEI AND $\vec{r}_a$ AND $\vec{r}_b$ ARE THE ELECTRON'S LOCATION RELATIVE TO THEIR RESPECTIVE NUCLEUS. THUS WE HAVE

$$\frac{1}{r_{ab}} = \frac{1}{R_{AB}} + \frac{1}{R_{AB}^3}\left(\vec{r}_a \cdot \vec{R}_{AB} + \vec{r}_b \cdot \vec{R}_{AB}\right)$$

Therefore

$$\langle \varphi_n | H_1 | \varphi_m \rangle = \int \varphi_g^A \varphi_k^B \; e^2 \left[\frac{1}{R_{AB}} + \frac{1}{R_{AB}^3}(\vec{r}_a \cdot \vec{R}_{AB} - \vec{r}_b \cdot \vec{R}_{AB})\right] \varphi_0^B \varphi_p^A \, d^3r_a d^3r_b$$

If we integrate out the first term, we find that it is zero. This is a consequence of the integration over $d^3 r_a$, i.e.,

$$\frac{1}{R_{AB}} \int \varphi_q^A(a) \varphi_p^A(a)\, d^3 r_a = 0$$

because φ_q^A and φ_p^A are orthogonal eigenfunction as we constructed the problem. The second term

$$\int\int \varphi_q^A(a) \varphi_k^B(b) \left(\frac{\bar{r}_a \cdot \bar{R}_{AB}}{R_{AB}^3} \right) \varphi_o^B(b)\, \varphi_p^A(a)\, d^3 r_b\, d^3 r_a$$

again produces 0 when integrating over $d^3 r_b$ because the function $\bar{r}_a \cdot \bar{R}_{AB} / R_{AB}^3$ does not depend on r_b. By a similar approach the third term $\propto (-\bar{r}_b \cdot \bar{R}_{AB}/R_{AB}^3)$ when integrated over $d^3 r_a$ gives zero. So the answer is nothing! No not quite. Obviously our approximation is not good enough so we have to go to second order expansions of $1/r_{ab}$. Expanding to third order we have

$$\frac{1}{r_{ab}} = \frac{1}{R_{AB}} + \frac{1}{R_{AB}^2} \left(\frac{\bar{r}_a \cdot \bar{R}_{AB}}{R_{AB}} - \frac{\bar{r}_b \cdot \bar{R}_{AB}}{R_{AB}} \right) + \frac{X_a X_b + Y_a Y_b - 2 Z_a Z_b}{R_{AB}^3}$$

For the first new term

$$\int_{r_b}\int_{r_a} \varphi_q^A(a)\, \varphi_k^B(b)\, e^2 \frac{X_a X_b}{R_{AB}^3}\, \varphi_o^B(b)\, \varphi_p^A(a)\, d^3 r_a\, d^3 r_b$$

To evaluate this integral we'll separate it in the following way

$$(H_1)_{mn} = \frac{1}{R_{AB}^3} \left[\int \varphi_q^A(a)\, e\, X_a\, \varphi_p^A(a)\, d^3 r_a \right] \left[\int \varphi_k^B(b)\, e X_b\, \varphi_o^B(b)\, d^3 r_b \right]$$

The bracketed quantities are known as the electric dipole matrix elements

$$\int \varphi_q^A(a)\, e X_a\, \varphi_p^A(a)\, d^3 r_a = (e X_{pq})^A$$

$$\int \varphi_k^B(b)\, e X_b\, \varphi_o^B(b)\, d^3 r_b = (e X_{ok})^B$$

The other directions Y and Z produce similiar dipole terms which can be averaged over all directions to produce an average dipole matrix element. So our final result is

$$\frac{P(m \to n)}{\text{sec}} \propto \frac{1}{R_{AB}^6} |X^A|^2 |X^B|^2\, m k$$

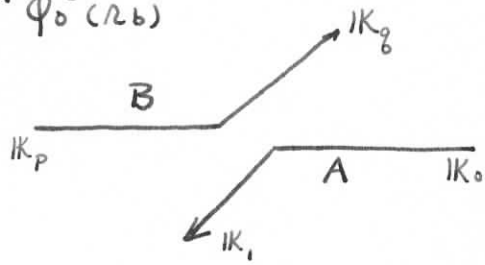
If we continued this problem a little further by complicating the interaction we can consider what happens if the two atoms are in motion relative to each other. We want the probability to make a transition as the excited atom flies by. The hamiltonian of the system as viewed from the center of mass is

$$H = \underbrace{H_A + H_B + \frac{P_A^2}{2M_A} + \frac{P_B^2}{2M_B}}_{H_0} + \underbrace{\frac{e^2}{|r_a + R_A - (r_b + R_b)|}}_{H_1}$$

The initial state of the system has a wave function and energy of

$$\varphi_m = e^{i\mathbb{K}_p \cdot \mathbb{R}_B} \varphi_p^A(r_a)\, e^{i\mathbb{K}_0 \cdot \mathbb{R}_A} \varphi_0^B(r_b)$$

$$E_m = \frac{P_B^2}{2m_B} + \frac{P_0^2}{2M_A} + E_p^B + E_0^A$$



$\mathbb{K}_p$ and $\mathbb{K}_0$ denote the momentum of the atom A and B while $P_B^2 = \mathbb{K}_p \cdot \mathbb{K}_p$ and $P_0^2 = \mathbb{K}_0 \cdot \mathbb{K}_0$. In the final state

$$\varphi_n = e^{i\mathbb{K}_q \cdot \mathbb{R}_B} \varphi_q^B(r_b)\, e^{i\mathbb{K}_1 \cdot R_A} \varphi_1^A(r_a)$$

$$E_n = \frac{\mathbb{K}_q^2}{2M_B} + \frac{\mathbb{K}_1^2}{2M_A} + E_q^B + E_1^A$$

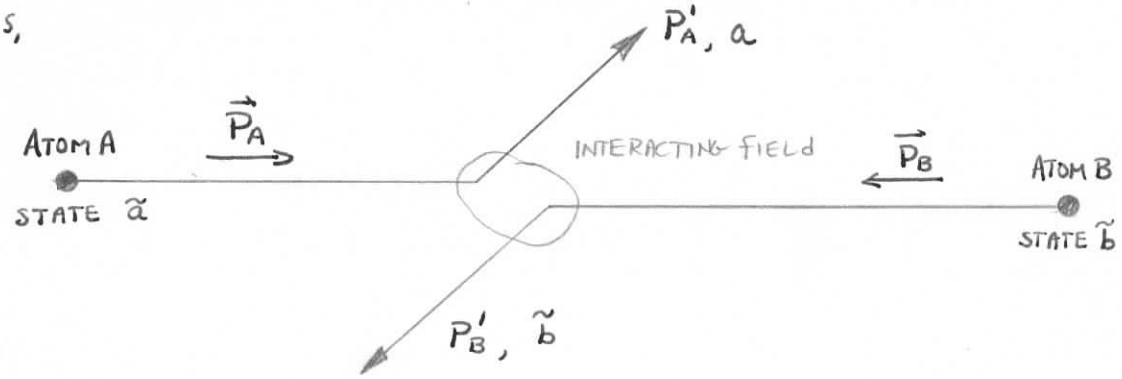
In this interaction it is possible to separate the translational motion of the center of mass from the internal motions of the atoms so that

$$(H_1)_{mn} = \int e^{i\mathbb{K}_q \cdot \mathbb{R}_B} e^{i\mathbb{K}_1 \cdot R_A} \varphi_q^B(r_b)\, \varphi_1^A(r_a) \frac{e^2}{|\bar{r}_a + \bar{R}_A - \bar{r}_b - \bar{R}_b|} e^{i\mathbb{K}_p \cdot \mathbb{R}_B} e^{i\mathbb{K}_0 \cdot R_A} \varphi_p^A \varphi_0^B\, d^3r_a\, d^3r_b\, d^3R_A \cdot d^3R_B$$

40. TRANSITION RATES AND SCATTERING THEORY

I'd LIKE TO RETURN TO THE PROBLEM THAT I STARTED LAST TIME SINCE I didN'T GET IT finished. I WAS discussing, WHAT TURNS OUT TO be, A COMPLICATED INTERACTION. I had TWO PARTICLES COLLIDING AND IN The PROCESS of SCATTERING They EXCHANGE ENERGY SO ONE ATOM IS NO LONGER ~~IONIZED WH~~ EXCITED WHILE The other ATOM BECOMES IONIZED. BECAUSE I didN'T GET FAR I'LL START ALL OVER AGAIN.

I HAVE TWO ATOMS A AND B COMING TOWARDS EACHOTHER. ATOM A HAS MOMENTUM P_A WHILE BEING IN AN EXCITED STATE $\tilde{a}$. After The COLLISION IT HAS MOMENTUM P'_A WHILE MAKING A TRANSITION TO The GROUND STATE. ATOM B, COMMING TOWARDS A AT MOMENTUM P_B, COLLIDES WITH A and SCATTERS off WITH MOMENTUM P'_B AND IN AN IONIZED STATE. DIAGRAMATICALLY WE HAVE The following PROCESS,



TO SUMMARIZES THEN The CONDITIONS of The PROBLEM

	ATOM	STATE	MOMENTUM	ENERGY
INITIAL CONDITIONS	A	$\tilde{a}$	P_A	E_A
	B	b	P_B	E_B
FINAL CONDITIONS	A	a	P'_A	E'_A
	B	$\tilde{b}$	P'_B	E'_B

IN ORDER TO SIMPLIFY THE PROBLEM INITIALLY I WILL NOT CONSIDER ANY INTERNAL EXCITATIONS OF THE NUCLEI. THEREFORE, I WILL LIMIT MY DISCUSSION HERE TO THE PHENOMENA OF ELASTIC SCATTERING. THE INELASTIC CASE IS STRAIGHT FORWARD BUT IT GETS MESSY DRAGGING ALONG ALL THE TERMS. IT'LL BE HARD ENOUGH KEEPING THE ELASTIC CASE STRAIGHT.

THE ANSWER TO THE PROBLEM IS CONTAINED IN THE GENERAL FORMULA,

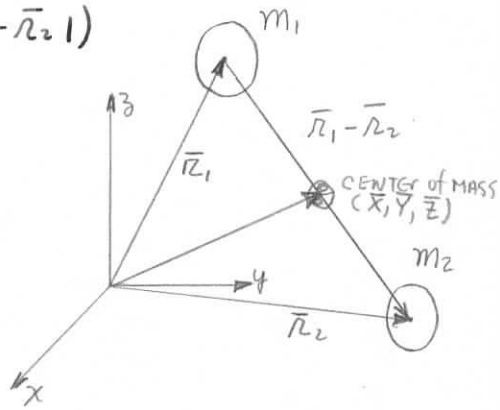
$$\frac{P(i\to f)}{\Delta t} = \text{TRANSITION RATE} = \frac{2\pi}{\hbar} \left| H'_{if} \right|^2 \delta(E_f - E_i)$$

ALL WE HAVE TO DO IS EVALUATE THIS SIMPLE EXPRESSION! IN ORDER TO EVALUATE THE EQUATION WE NEED THE INITIAL WAVE FUNCTION OF THE SYSTEM. TO WRITE THE INITIAL WAVE FUNCTION I WILL MAKE USE OF THE THEOREM THAT THE INTERNAL MOTION OF THE ATOM CAN BE SEPARATED FROM IT'S CENTER OF MASS MOTION. THIS IS SUCH A USEFUL CONCEPT TO KNOW THAT I'd LIKE TO PROVE IT TO YOU.

SEPARABILITY OF INTERNAL AND CENTER OF MASS MOTIONS

CONSIDER A TWO PARTICLE SYSTEM WITH A HAMILTONIAN GIVEN BY

$$H = \frac{1}{2m_1}\nabla^2_{r_1} + \frac{1}{2m_2}\nabla^2_{r_2} + V(|\bar{r}_1 - \bar{r}_2|)$$



BECAUSE I'M LAZY AND BECAUSE OF THE SYMMETRY OF THE PROBLEM I'M ONLY GOING TO WORK WITH THE x-COMPONENTS. AND LATER ADD IN THE y AND z-TERMS.

I CAN FIRST DEFINE, IN THE CLASSICAL SENSE, THE X LOCATION OF THE CENTER OF MASS OF THE SYSTEM, I.E.,

$$\bar{X} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

I WILL CALL THE DISPLACEMENT, $x_1 - x_2$, ξ AND USE THE FOLLOWING RELATIONSHIP FROM THE THEORY OF DIFFERENTIATION.

If $x_1 - x_2 = \xi$ THEN

$$\frac{\partial}{\partial x_1} = \frac{\partial \xi}{\partial x_1}\frac{\partial}{\partial \xi} + \frac{\partial \bar{X}}{\partial x_1}\frac{\partial}{\partial \bar{X}}$$

where $\bar{X} = \frac{m_1}{m_1+m_2} x_1 + \frac{m_2}{m_1+m_2} x_2 = \frac{m_1}{M} x_1 + \frac{m_2}{M} x_2$

AND $M = m_1 + m_2$. THEN

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \xi} + \frac{m_1}{M}\frac{\partial}{\partial \bar{X}}$$

SIMILARLY

$$\frac{\partial}{\partial x_2} = -\frac{\partial}{\partial \xi} + \frac{m_2}{M}\frac{\partial}{\partial \bar{X}}$$

IN THE HAMILTONIAN THEN

$$\frac{1}{2m_1}\nabla_{r_1}^2 \to \frac{1}{2m_1}\left(\frac{\partial}{\partial x_1}\right)^2 = \frac{1}{2m_1}\left(\frac{\partial^2}{\partial \xi^2} + \frac{2m_1}{M}\frac{\partial}{\partial \xi}\frac{\partial}{\partial \bar{X}} + \frac{m_1^2}{M^2}\frac{\partial^2}{\partial \bar{X}^2}\right)$$

$$\frac{1}{2m_2}\nabla_{r_2}^2 \to \frac{1}{2m_2}\left(\frac{\partial}{\partial x_2}\right)^2 = \frac{1}{2m_2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{-2m_2}{M}\frac{\partial}{\partial \xi}\frac{\partial}{\partial \bar{X}} + \frac{m_2^2}{M^2}\frac{\partial^2}{\partial \bar{X}^2}\right)$$

AND $V(\bar{r}_1 - \bar{r}_2) \to V(\xi)$

ADDING UP THE TERMS AND USING

$$Y_1 - Y_2 = \eta \quad \text{AND} \quad z_1 - z_2 = \gamma$$

$$H = \left[\frac{1}{2m_1}\left(\frac{m_1^2}{M^2}\right) + \frac{1}{2m_2}\left(\frac{m_2^2}{M^2}\right)\right]\left[\frac{\partial^2}{\partial \bar{X}^2} + \frac{\partial^2}{\partial \bar{Y}^2} + \frac{\partial^2}{\partial \bar{Z}^2}\right] + \left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \gamma^2}\right] + V(\xi,\eta,\gamma)$$

THIS CAN BE REDUCED TO THE SIMPLER FORM

$$H = \frac{1}{2M}\nabla_{\bar{R}}^2 + \frac{1}{2\mu}\nabla_{\rho}^2 + V(\bar{\rho})$$

where $\vec{\rho} = \xi\,\bar{i} + \eta\,\bar{j} + \gamma\,\bar{k}$ IS THE RELATIVE COORDINATE BETWEEN ATOMS AND μ IS THE REDUCED MASS OF THE SYSTEM,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$\bar{R}$ DENOTES THE CENTER OF MASS LOCATION AND WE SEE THE HAMILTONIAN SPLITS INTO TWO DISTINCT PARTS

$$H = H_{\bar{R}} + H_{\bar{\rho}}$$

where

$$H_{\bar{R}} = \frac{1}{2M}\nabla_{\bar{R}}^2 = \text{CENTER OF MASS MOTION}$$

$$H_{\bar{\rho}} = \frac{1}{2\mu}\nabla_{\rho}^2 + V(\bar{\rho}) = \text{INTERNAL INTERACTION.}$$

The reason we were successful in splitting up the motion into two parts came from adding up the two gradient terms $\nabla_{r_1}^2$ and $\nabla_{r_2}^2$ which contained cross terms which were equal but opposite in sign. Therefore the motions fortunately were uncoupled.

I should point out whenever one mass, say m_1, is much greater than m_2, then the reduced mass μ just becomes equivalent to the light mass m_2

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{m_1\left(1+\frac{m_2}{m_1}\right)} = m_2\left(1 - \frac{m_2}{m_1} + \cdots\right) \simeq m_2 \quad \text{if } m_2 \ll m_1$$

Thus in the case of hydrogen where m_1 = the neutron and m_2 = the electron $\mu \simeq$ mass of the electron since $\frac{m_N}{m_{e^-}} \simeq 1800$. But the fact that the reduced mass is the exact formulation of the Hamiltonian was recognized by Rutherford in the early days of quantum mechanics so we cannot ignore it in any precise calculations. End of side note!

ELASTIC SCATTERING TRANSITION RATES

Alright then, what is the wave function for the system initially and finally? We have established the fact that the Hamiltonian is separable into two parts: one corresponding to the center of mass motion, $H_{\bar{R}}$, and the other part describes the internal motion, $H_{\bar{\rho}}$. We know then

$$H = H_{\bar{R}} + H_{\bar{\rho}} = \frac{P^2}{2M} + \mathcal{E}_i$$

where $P^2/2M$ is the energy due to the moving system and $\mathcal{E}_i$ are the energy levels of the system. The wave function is easily established by the product of the two wave functions

$$\psi \propto e^{i\bar{P}\cdot\bar{R}}\,\varphi_i(\rho)$$

Here $e^{i\bar{P}\cdot\bar{R}}$ is the wave function describing the center of mass motion and $\varphi_i(\rho)$ is the wave function describing the internal motion of the system.

SLOWLY The PROBLEM IS TURNING INTO A REAL MORASS because The PROBLEM IS BECOMING obscured by The COMPLEX NOTATION NEEDED TO KEEP EVERYTHING STRAIGHT. SO WE CONTINUE by EXPRESSING EVERYTHING IN TERMS of difference of DISTANCES ρ_a, ρ_b, AND $\bar{r}_a - \bar{r}_b$. WE WILL hold off INTEGRATING OVER ρ_a AND ρ_b SINCE WE DON'T KNOW HOW TO NOR WILL WE BE ABLE TO by MAKING APPROXIMATIONS; AS YET WE CAN EXPLAIN The MACHINERY of ATOM ENOUGH TO WRITE DOWN The DYNAMICAL LAWS. IT IS SORT of INTERESTING; QUANTUM MECHANICS IS NOTHING MORE THAN The APPLICATION of ONEUPSMANSHIP TO PHYSICS. WE USE The CONCEPT WE KNOW but IN The END WE NEVER REALLY CALCULATE ANY THING - IT'S ALWAYS beyond our KNOWLEDGE.

WHAT WE WANT TO DO NOW IS TO REARRANGE The INTEGRAL INTO FUNCTIONS of The SEPARATION DISTANCES. TO DO THAT WE REWORK The EXPONENTIAL TERMS

$$e^{-i P_a' \cdot \bar{r}_a} e^{-i P_b' \cdot \bar{r}_b} e^{i \bar{P}_a \cdot \bar{r}_a} e^{i \bar{P}_b \cdot \bar{r}_b} =$$

$$e^{-i P_a' \cdot \bar{r}_a} e^{i P_a' \cdot \bar{r}_b} e^{-i P_a' \cdot \bar{r}_b} e^{-i P_b' \cdot \bar{r}_b} e^{i \bar{P}_a \cdot \bar{r}_a} e^{-i \bar{P}_a \cdot \bar{r}_b} e^{+i \bar{P}_a \cdot \bar{r}_b} e^{i \bar{P}_b \cdot \bar{r}_b}$$

WE CAN REARRANGE THIS NOW

$$e^{-i P_a' (\bar{r}_a - \bar{r}_b)} e^{-i (P_a' + P_b') \cdot \bar{r}_b} e^{i \bar{P}_a \cdot (\bar{r}_a - \bar{r}_b)} e^{i (\bar{P}_a + \bar{P}_b) \cdot \bar{r}_b}$$

If WE INTEGRATE THIS FUNCTION OVER r_b WE GET The INTERESTING RESULT THAT

$$\int e^{+i [(\bar{P}_a + \bar{P}_b) - (P_a' + P_b')] \cdot \bar{r}_b} \, d r_b = \begin{cases} 0 & \text{if } P_a + P_b \neq P_a' + P_b' \\ 1 & \text{if } P_a + P_b = P_a' + P_b' \end{cases}$$

WHICH TELLS US THAT The PROBLEM WILL ONLY WORK OUT if YOU CONSERVE MOMENTUM DURING The INTERACTION. SO if YOU DIDN'T MAKE The INITIAL STATEMENT ABOUT CONSERVING MOMENTUM, The EQUATIONS MAKE IT FOR YOU. IT'S INTERESTING; YOU CAN be dumb AND NATURE TAKES CARE of YOU!

WE CAN NOW WRITE THE INITIAL AND FINAL WAVE FUNCTIONS

INITIALLY $\psi_i = e^{i\bar{P}_a\cdot\bar{R}_a}\,\varphi_{\tilde{a}}(\rho_a)\,e^{i\bar{P}_b\cdot\bar{R}_b}\,\varphi_b(\rho_b)$

FINALLY $\psi_f = e^{i\bar{P}_a'\cdot\bar{R}_a}\,\varphi_a(\rho_a)\,e^{i\bar{P}_b'\cdot\bar{R}_b}\,\varphi_{\tilde{b}}(\rho_b)$

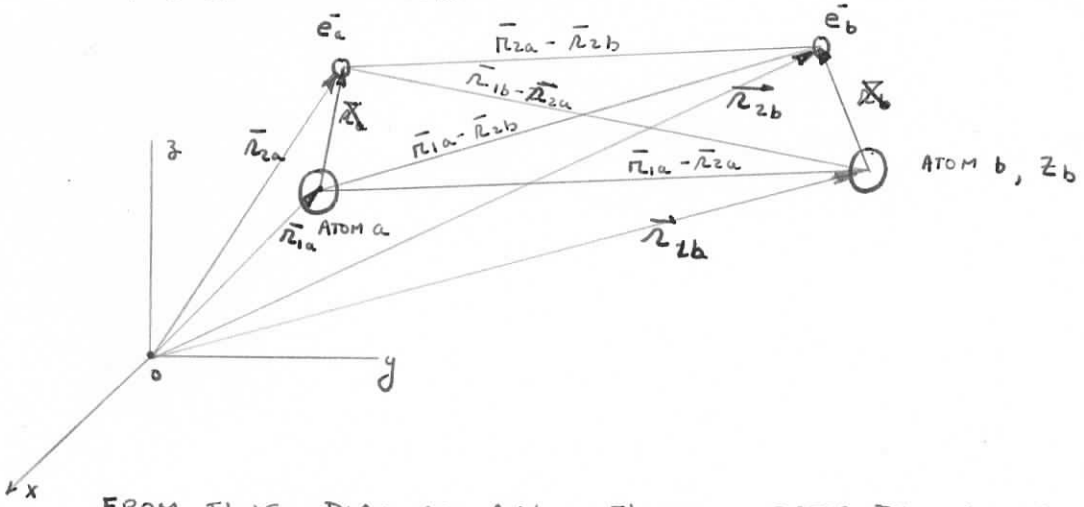
WE NOW MUST EVALUATE THE INTEGRAL,

$$\langle\psi_f|H_{if}|\psi_i\rangle = \int \psi_f^* V \psi_i \, dVOL.$$

SO LET'S WRITE IT OUT,

$$\int = \int d^3\bar{R}_a\, d^3\bar{R}_b\, d^3\bar{\rho}_a\, d^3\bar{\rho}_b \Big[e^{-i\bar{P}_a'\cdot\bar{R}_a}\,\varphi_a^*(\rho_a)\, e^{-i\bar{P}_b'\cdot\bar{R}_b}\,\varphi_{\tilde{b}}^*(\rho_b)\, V(\bar{\rho}_a,\bar{\rho}_b,\bar{R}_a-\bar{R}_b)\, e^{i\bar{P}_a\cdot\bar{R}_a}\,\varphi_{\tilde{a}}(\rho_a)\, e^{i\bar{P}_b\cdot\bar{R}_b}\,\varphi_b(\rho_b)\Big]$$

THIS LOOKS TERRIBLY CONFUSING BUT IF YOU FOLLOW THROUGH IT YOU MIGHT UNDERSTAND IT. TO CONTINUE WE'LL HAVE TO MAKE SOME APPROXIMATIONS ON THE POTENTIAL FUNCTION V. THE CO-ORDINATES WHICH ARE SIGNIFICANT ARE:



FROM THIS DIAGRAM AND WITH SOME PATIENTS WE CAN WRITE DOWN THE APPROPRIATE POTENTIAL FUNCTION

$$V = \frac{e^2}{|\bar{R}_{2a}-\bar{R}_{2b}|} - \frac{Z_a e^2}{|\bar{R}_{1a}-\bar{R}_{2b}|} - \frac{Z_b e^2}{|\bar{R}_{1b}-\bar{R}_{2a}|} + \frac{Z^2 e^2}{|\bar{R}_{1a}-\bar{R}_{2a}|}$$

NOW THAT WE HAVE AN EXPRESSION FOR V WE SEE THAT IT IS NOT IN THE FORM WE WANT IT SO THAT WE CAN PERFORM THE INTEGRATION; WE NEED $V(\rho_a,\rho_b,\bar{R}_a-\bar{R}_b)$. SO WE HAVE TO RE-EXPRESS V IN TERMS OF THE RIGHT VARIABLES.

TRANSITION RATE, SCATTERING CROSS-SECTION, AND ELASTIC SCATTERING

The PROBLEM HAS NOW BEEN PUT INTO The GENERAL FORM which IS READY TO be INTEGRATED

$$\langle \psi_f | H_{if} | \psi_f \rangle = \int e^{-i P_a' \cdot (\bar{r}_a - \bar{r}_b)} \varphi_a^* \varphi_b^* V e^{-i \bar{P}_a \cdot (\bar{r}_a - \bar{r}_b)} \varphi_a \varphi_b \, d r_a \, d p_a \, d p_b$$

SUBJECT TO The CONSTRAINT THAT $P_a + P_b = P_a' + P_b'$. IN ORDER TO GO ON from here IT IS NECESSARY TO MAKE SOME APPROXIMATIONS.

Before I GO ON I WANT TO MAKE USE of The SCATTERING CROSS SECTION which WE INTRODUCED A WHILE BACK. IT CAN BE SHOWN THAT THE CROSS SECTION IS RELATED TO The TRANSITION RATES BY The RELATIVE COLLISION VELOCITY, i.e.,

$$\frac{dP}{dt} = v \sigma$$

Thus

$$\sigma = \frac{2\pi}{\hbar v} | H_{if} |^2 \, \delta (E_f - E_i)$$

LET'S TRY TO WORK OUT AN EASY PROBLEM WITH ELASTIC SCATTERING. BY ASSUMING UNEXCITED INTERNAL MOTION WE CAN Perform The INTEGRATION.

ASSUME Then LOW ENERGY ELASTIC SCATTERING AND The ATOM REMAIN IN The GROUND STATE. WE HAVE NOW CHANGED The PROBLEM AND ARE NO LONGER WORKING WITH TRANSITIONS of ONE EXCITED STATE TO A GROUND STATE. Thus The MATRIX ELEMENT BECOMES

$$\langle \psi_f | H_{if} | \psi_i \rangle = \left[\iint V(|\bar{r}_a - \bar{r}_b|, p_a, p_b) \, |\varphi_a^*(p_a)|^2 |\varphi_b(p_b)|^2 \, d^3 p_a \, d^3 p_b \right] \times$$
$$\left[\int e^{i (P_a - P_a') \cdot (\bar{r}_a - \bar{r}_b)} \, d^3 (|\bar{r}_a - \bar{r}_b|) \right]$$

BECAUSE WE CAN'T DO The DOUBLE INTEGRAL IN The first BRACKETS WE'LL JUST CALL IT $U(\bar{r}_a - \bar{r}_b)$, The AVERAGE ELECTRICAL POTENTIAL. IT'S AN AVERAGE of The POSITION of The CHARGES WITHIN The ATOMS. If YOU ASSUME THAT The CHARGE IS SPHERICALLY DISTRIBUTED OR ASSUME SOME OTHER INTEGRABLE MODEL, Then YOU CAN EXPLICITLY EVALUATE IT. OTHERWISE YOU DON'T WORRY ABOUT IT AND ESTABLISH IT Through EXPERIMENTS.

BY THE PREVIOUS ASSUMPTIONS WE NOW HAVE REDUCED THE PROBLEM CONSIDERABLY SINCE NOW WE MUST INTEGRATE,

$$\langle \psi_f | H_{if} | \psi_i \rangle = \int d^3(\bar{r}_a - \bar{r}_b)\, U(r_a - r_b)\, e^{i(P_a - P_a')\cdot(\bar{r}_a - \bar{r}_b)}$$

BY TRANSFORMING TO A NEW COORDINATE $t = \bar{r}_a - r_b$ WE CAN USE FOURIER TRANSFORM THEORY TO INTEGRATE

$$H_{if} = \int d^3 t \quad U(t)\, e^{iQt}$$

where $Q = P_a - P_a'$; NOTICE THAN THE TRANSFORMED MOMENTUM Q ALSO EQUALS $P_b' - P_b$ DUE TO CONSERVATION OF MOMENTUM. TO SHOW YOU HOW THIS INTEGRATION WORKS – LET'S ASSUME THAT

$$U = \frac{e^2}{z}$$

THEN

$$H_{if} = \int_0^\infty e^2 d^3 z \; \frac{e^{iQz}}{z} = \frac{4\pi e^2}{Q^2}$$

THE SCATTERING CROSS-SECTION IS THEN

$$\sigma = \frac{2\pi}{\hbar v} \left(\frac{4\pi e^2}{Q^2} \right)^2 \delta(E_f - E_i)$$

NOW WE HAVE TO FIGURE OUT WHAT TO DO WITH THE DELTA FUNCTION. WHAT WE ARE INTERESTED IN DOING IS SUMMING OVER THE DENSITY OF STATES IN A GIVEN MOMENTUM RANGE; WE CAN LOOK FOR EITHER P_a' or P_b'. WE MUST SUM OVER THE DENSITY OF STATES IN RANGE dP_a', I.E.,

$$\sigma = \frac{2\pi}{\hbar v} \left(\frac{16\pi^2 e^4}{Q^4} \right) \delta(E_f - E_i) \; \frac{d^3 P_a'}{(2\pi)^3}$$

THE ENERGY DIFFERENCE $E_f - E_i$ CAN BE EVALUATED

$$E_f - E_i = \frac{(P_a')^2}{2m_a} + \frac{(P_b')^2}{2m_b} - \frac{(P_a)^2}{2m_a} - \frac{P_b^2}{2m_b}$$

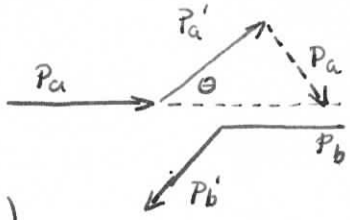
SINCE WE ARE SUMMING OVER P_a' WE MUST GET RID OF P_b' BY USING THE CONSERVATION OF MOMENTUM

$$(P_b')^2 = (P_a + P_b - P_a')^2$$

IF WE ASSUME THE INTEGRATION OVER $d^3 P_a'$ IS SPHERICALLY SYMMETRICAL THE $d^3 P_a' \rightarrow 4\pi (P_a')^2 \, d P_a'$ AND THE INTEGRATION WITH THE DELTA FUNCTION IS STRAIGHT FORWARD

If we have the special case where $P_b = P_b'$; i.e. the B atom comes in and leaves with the same momentum while scattering, then

$$Q = P_a - P_a' = 2 P_a \sin\frac{\theta}{2}$$



and

$$\sigma = \frac{2\pi}{\hbar v} \times \frac{16\pi^2 e^4}{Q^4} \int \frac{(P_a')^2 dP_a' d\Omega_a'}{(2\pi)^3} \delta\left(\frac{P_a'^2}{2m_a} - \frac{P_a^2}{2m_a}\right)$$

To integrate we need to use the delta function property:

$$\delta f(x)\, dx = \frac{1}{|f'(x)|}$$

$$\sigma = \frac{4 e^4 P_a' d\Omega_a'}{2\hbar v m_a Q^4}$$

where $d\Omega_a'$ is the element of solid angle for the scattered a particle. Since $Q = 2 P_a \sin\theta/2$

$$\sigma = \frac{1}{2} \frac{e^2 P_a'}{\hbar v m_a P_a^4} \frac{d\Omega_a'}{\sin^4\theta/2}$$

Here then is the familiar scattering law which depends on $1/\sin^4\theta/2$ as the scattering angular dependence.

QED. Supplement

41. INTERACTION OF A RADIATION FIELD AND MATTER

- SPONTANEOUS EMISSION FROM AN ATOM IN A QUANTIZED FIELD -

I'd NOW LIKE TO TALK ABOUT The first REAL PROBLEM IN QUANTUM ELECTRODYNAMICS. The problem I WANT TO WORK OUT IS TO PREDICT WITH WHAT PROBABILITY AN ATOM WILL EMIT A photon, EXCT The radiation FIELD, AND EMIT "LIGHT." To work This problem we have to go beyond The CLASSICAL Theory of LIGHT SCATTERING from ATOM AND QUANTIZE The radiation field. WE SHALL CONSIDER A SINGLE ELECTRON SYSTEM, I.E., AN hydrogen ATOM because There AREN'T TOO MANY THINGS TO KEEP TRACK Of. This is ONE of The REAL PROBLEMS of QED; KEEPING STRAIGHT The NOTATION IS A MAJOR TASK. I hope TO MAKE The PRESENTATION IS AS CLEAR AS POSSIBLE.

WE WILL CONSIDER THE ATOM TO INITIALLY BE IN A STATE of ENERGY E_1 AND SEE WHAT HAPPENS AS IT MAKES A TRANSITION TO STATE E_0. THE TRANSITION WILL TAKE PLACE IN The "DARK". BY "dark" I MEAN THAT The BACKGROUND radiation IS IN ITS GROUND STATE; I.E. There ARE NO PHOTONS IN ANY MODES. The ENERGY IN EACH MODE of The FIELD IS The GROUND STATE ENERGY $\hbar\omega/2$. WE HAVE PULLED The first SLIGHT of hand WHICH CHARACTERIZES Q.E.D by redefining The VACUUM STATE TO be The ZERO ENERGY STATE. WE DO This BECAUSE if WE SUM OVER ALL MODES, The ZERO POINT ENERGY IS INFINITE. TO MAKE The ENERGY FINITE IT IS NECESSARY TO CUTOFF of The frequency RANGE IN AN ARTIFICIAL WAY. This IS The first SYMPTOM of INTERNAL DIFFICULTIES WITH QED.

WE WILL TAKE The ATOM TO BE IN AN ENORMOUS box which CONTAINS MANY MODES of The radiation FIELD. The box IS LARGE SO WE DON'T HAVE TO worry ABOUT REFLECTIONS from The SIDES. EACH MODE IN The FIELD WILL BE MODELED AS A CLASSICAL HARMONIC OSCILLATOR. EACH MODE WILL BE CHARACTERIZED BY TWO INDICES, $\mathbb{K}$ AND i. $\mathbb{K}$ denotes The WAVE NUMBER of The mode while i represents The STATE of POLARIZATION which IS EITHER up or down so i HAS 2 VALUES ONLY. The displacement of The $\mathbb{K}^{Th}$ MODE IS GIVEN by coordinate $q_{\mathbb{K},i}$. WE WILL frequently BE CONCERNED

WILL THE NUMBER OF MODES PER UNIT VOLUME WHICH IS $\frac{d^3 \mathbb{K}}{(2\pi)^3}$. The COORDINATE $q_{\mathbb{K},i}$ IS IMPORTANT BECAUSE IT CHARACTERIZES The STRENGTH of The VECTOR POTENTIAL IN THAT MODE. The VECTOR POTENTIAL of THE FIELD IS DESCRIBABLE IN The following form

$$\vec{A} = \sum_{\mathbb{K},i=1,2} \frac{\vec{e}_i}{\sqrt{2\omega_{\mathbb{K}}}} q_{\mathbb{K},i} \; e^{i\mathbb{K}\cdot\mathbb{X}}$$

$\vec{e}_i$ IS A UNIT VECTOR IN The direction of The POLARIZATION of The field. The denominator $\sqrt{2\omega_{\mathbb{K}}}$ IS A NORMALIZATION FACTOR WHICH IS RELATED TO $\mathbb{K}$ SINCE $\omega_{\mathbb{K}}^2 = c^2\mathbb{K}^2$. $\vec{e}$ IS ORTOGONAL TO $\vec{\mathbb{K}}$, i.e. $\vec{\mathbb{K}}\cdot\vec{e}_{\mathbb{K},i} = 0$.

NOW WE HAVE TO WRITE DOWN THE HAMILTONIAN of THE SYSTEM. There will be three PARTS ONE DUE TO The ATOM' BY ITSELF, The SECOND PART DUE TO THE FIELD OSCILLATORS AND The Third PIECE IS THE INTERACTION PART, i.e.,

$$H = H_{ATOM} + H_{OSCIL.} + H_{INTER.}$$

THE OSCILLATOR TERM CAN BE WRITTEN OUT IN MORE DETAIL AS

$$H_{OSC} = \sum_{\mathbb{K},i} \frac{1}{2}\left(P_{\mathbb{K},i}^2 + \omega_{\mathbb{K},i}^2 q_{\mathbb{K},i}^2\right)$$

IT IS BEST TO CHANGE OVER TO OPERATOR NOTATION for WORKING WITH THIS HAMILTONIAN. RECALL OUR SOLUTION TO The HARMONIC OSCILLATOR BACK ON PAGE 185. WE defined The OPERATORS,

$$q_{\mathbb{K},i} = \sqrt{\frac{2}{m\omega_{\mathbb{K},i}}}\left(a_{\mathbb{K},i} + a^*_{\mathbb{K},i}\right)$$

AND

$$P_{\mathbb{K},i} = i\sqrt{\frac{\hbar m \omega_{\mathbb{K},i}}{2}}\left(a^*_{\mathbb{K},i} - a_{\mathbb{K},i}\right)$$

a and a^* WERE Referred TO CREATION AND ANNIHILATION OPERATORS IN THAT They worked ON The WAVE FUNCTION of The OSCILLATOR TO EITher RAISE ITS ENERGY BY ONE UNIT of $\hbar\omega$ (ONE PHOTON) OR REDUCE The ENERGY BY ONE $\hbar\omega$ (EMIT ONE PHOTON). The OPERATORS ARE ALSO SUBJECT TO The COMMUTATOR RELATIONSHIP

$$aa^* - a^*a = 1$$

USING THE OPERATOR NOTATION JUST DESCRIBED, WE CAN WRITE THE HAMILTONIAN OF THE OSCILLATOR AS

$$H_{osc} = \sum \hbar \omega_{\mathbb{k},i} \, a^*_{\mathbb{k},i} \, a_{\mathbb{k},i}$$

the VECTOR POTENTIAL MUST ALSO be EXPRESSED IN TERMS a AND a^*. TO DO THAT JUST SUBSTITUTE $q_{\mathbb{k},i}$ INTO $\mathbb{A}$,

$$\mathbb{A} = \sum_{\mathbb{k},i} \left[\frac{\vec{e}^{\,*}_{\mathbb{k},i}}{\sqrt{2\omega_{\mathbb{k},i}}} a^*_{\mathbb{k},i} \, e^{-i\mathbb{k}\cdot\mathbb{x}} + \frac{\vec{e}_{\mathbb{k},i}}{\sqrt{2\omega_{\mathbb{k},i}}} a_{\mathbb{k},i} \, e^{i\mathbb{k}\cdot\mathbb{x}} \right]$$

The first TERM IS The CREATION TERM WHILE The SECOND IS The ANNIHILATION TERM. BY WE MEAN The $a^*_{\mathbb{k},i}$ WILL CAUSE The field TO INCREASE from The GROUND STATE by ONE QUANTUM $\hbar\omega$ or ONE PHOTON.

NOW WE HAVE TO THINK about how AN ELECTRON behaves IN AN EXTERNAL field. WE HAVE ALREADY SEEN THAT WITH AN INFINITELY HEAVY NUCLEUS WE CAN WRITE The HAMILTONIAN for AN ATOM IN The FIELD AS

$$H_0 = \frac{(\bar{P} - \frac{e}{c}\bar{A})^2}{2m} + V(r)$$

The MASS IS The ELECTRON MASS WHILE $\bar{P}$ IS The MOMENTUM of The ATOM. IF WE EXPAND THIS HAMILTONIAN OUT, WE WILL GET

$$H_0 = \frac{P^2}{2m} - \frac{e}{2mc}\left[\bar{P}\cdot\bar{A} + \bar{A}\cdot\bar{P}\right] + \frac{e^2}{2mc^2}\bar{A}\cdot\bar{A} + V(r)$$

NOW WE CAN SINGLE OUT The THREE TERMS IN The TOTAL HAMILTONIAN $H = H_{osc} + H_{ATOM} + H_{INT}$

$H_{osc} = \sum \hbar \omega_{\mathbb{k},i} \, a^* a$, behavior of field without AN ATOM TO INTERACT WITH

$H_{ATOM} = \frac{P^2}{2m} + V(r)$, HYDROGEN HAMILTONIAN WITHOUT A field TO INTERACT WITH

$H_{INT.} = -\frac{e}{2mc}\left[\bar{P}\cdot\bar{A} + \bar{A}\cdot\bar{P}\right] + \frac{e^2}{2mc^2}\bar{A}\cdot\bar{A}$, INTERACTION TERMS between field AND ATOM

The SUM of The PIECES GIVES The TOTAL COMPLETE DESCRIPTION of The SYSTEM.

WHAT HAS SO FAR APPEARED TO BE QUITE SIMPLE AND STRAIGHTFORWARD IS, ON THE CONTRARY, QUITE SUBTLE. WHEN WE EXPRESSED $\bar{A}(x)$ IN TERMS OF THE FIELD OPERATORS a AND a^*, WE HAVE INTRODUCED A LOT OF TECHNICAL DIFFICULTIES. SUCH PROBLEMS AS INFINITIES CROP UP WHICH HAVE TO BE DELT WITH BY TRICKS AND MAGIC AND A LOT OF HAND WAVING. AS WE UNDERSTAND Q.E.D TODAY, THERE IS ONLY ONE FORMULATION WHICH RESULTS FROM MODELING THE FIELD AS A SERIES OF OSCILLATORS. IT IS A CONSEQUENCE OF THIS FORMULATION THAT LEADS TO THE TECHNICAL DIFFICULTIES WHICH CAUSE US TO DOUBT THE VALIDITY OF Q.E.D. WE SHALL DISCUSS THESE PROBLEMS AS THEY COME UP. SO LET'S PROCEED AND SEE WHAT IS IN STORE FOR US.

WE WILL BEGIN BY EXAMINING THE INTERACTION PIECE OF THE HAMILTONIAN WHICH CAUSES ALL THE DIFFICULTY. SINCE WE ARE GOING TO RESTRICT OUR PROBLEM TO FIRST ORDER PERTURBATION EFFECTS IT IS POSSIBLE TO DISCARD THE SECOND ORDER TERM IN $\bar{A}$, I.E. THE A^2 TERM. ALSO IF WE CONSIDER THE GAGE $\bar{\nabla}\cdot\bar{A}=0$ THEN $\bar{P}\cdot\bar{A} = \bar{A}\cdot\bar{P}$ AND,

$$H_{INT} = -\frac{e}{mc}\,\bar{P}\cdot\bar{A}(x)$$

$\bar{P}$ IS THE MOMENTUM OF ELECTRON AND THE VECTOR POTENTIAL IS DEFINED AT THE POSITION OF THE ELECTRON. NOW SUBSTITUTING FOR $\bar{A}$ WE HAVE

$$H_{INT} = -\frac{e}{mc}\sum_{\mathbb{K},i}\left[\left(\frac{P\cdot\bar{e}^*_{\mathbb{K},i}}{\sqrt{2\omega_{\mathbb{K},i}}}\right)e^{-i\mathbb{K}\cdot x}\,a^*_{\mathbb{K},i} + \left(\frac{\bar{P}\cdot\bar{e}_{\mathbb{K}i}}{\sqrt{2\omega_{\mathbb{K},i}}}\right)e^{i\mathbb{K}\cdot x}\,a_{\mathbb{K},i}\right]$$

NOW WE ARE READY TO COMPUTE THE TRANSITION RATE,

$$\frac{\text{Prob}(i\to f)}{\text{SEC}} = \frac{2\pi}{\hbar}\,|H_{INT\,(fi)}|^2\,\delta(E_f - E_i)$$

BUT TO EVALUATE THIS EXPRESSION WE MUST COMPUTE THE MATRIX ELEMENT $\langle\psi_f|H_{INT}|\psi_i\rangle$ AND THEN SQUARE IT TO GET THE PROBABILITY. OKAY, WE NEED TO DESCRIBE THE INITIAL AND FINAL WAVE FUNCTION OF THE SYSTEM.

THE INITIAL WAVE FUNCTION IS GIVEN AS

$$\psi_i = |A_1; 0,0,0,\ldots,0,0,0\rangle$$

IN WORDS THIS SAYS THAT THE ATOM IS IN STATE 1 (AN EXCITED STATE) WHILE THE FIELD IS IN THE GROUND STATE THUS ALL THE ZEROES (AN INFINITE NUMBER OF THEM) DESCRIBE THE ENERGY CONTENT OF EACH MODE. THE FINAL STATE OF THE SYSTEM IS

$$\psi_f = |A_0; 0,0,0,1,\ldots,0,0,0\rangle$$

THIS SAYS THE ATOM IS NOW IN ITS GROUND STATE WHILE THE FIELD HAS ONE MODE WHICH HAS ABSORBED ONE PHOTON OF ENERGY, THE ENERGY THE ATOM LOST. WE'LL CHARACTERIZE THE EXCITED MODE AS $k = l$ AND $i = j$. THE ENERGY STATES OF THE SYSTEM ARE GIVEN AS

CONDITION	ENERGY	ATOM	FIELD
INITIALLY	$E_i = \epsilon_1 + 0$	$= \epsilon_1$	$+ 0$
FINALLY	$E_f = \epsilon_0 + \hbar\omega_l$	$= \epsilon_0$	$+ \hbar c l$

LET'S WRITE OUT ψ_i AND ψ_f IN TERMS OF THE PRODUCT OF WAVE FUNCTION TO HELP CLARIFY THE MEANING OF THE DIRAC NOTATION ~~NS~~ USED ABOVE.

$$\psi_i = |A_1; 0,00,\ldots 0,0,0\rangle = \Phi_1(x_0)\,\varphi_0(q_1)\,\varphi_0(q_2)\,\varphi_0(q_3)\ldots\varphi_0(q_n)$$

$\Phi_1(x_0)$ IS THE WAVE FUNCTION OF THE ATOM IN THE FIRST EXCITED STATE. THE MATHEMATICAL EXPRESSION OF $\Phi_1(x_0)$ IS x TIMES A GAUSSIAN DISTRIBUTION. THE WAVE FUNCTION OF THE Nth MODE OF THE FIELD IS GIVEN BY A GAUSSIAN DISTRIBUTION SINCE IT IS IN THE GROUND STATE (ALL MODES HAVE GAUSSIAN DISTRIBUTIONS INITIALLY THEREFORE $\varphi_0(q_{k,i,n}) \propto e^{-\frac{1}{2}\omega_{ki}^2 q_N^2}$). IN THE FINAL STATE THE SYSTEM WAVE FUNCTION IS GIVEN BY

$$\psi_f = |A_0; 0,0,0,1,\ldots,0,0,0\rangle = \Phi_0(x_0)\,\varphi_0(q_1)\,\varphi_0(q_2)\,\varphi_0(q_3)\,\varphi_1(q_4)\ldots\varphi_0(q_n)$$

HERE THE ATOM HAS A WAVE FUNCTION $\Phi_0(x_0)$ WHICH IS A GAUSSIAN SINCE IT IS IN THE GROUND STATE. THE FIELD NOW HAS ONE MODE WHICH IS EXCITED, CALL IT THE l,j MODE WHICH HAS A WAVE FUNCTION $\varphi_1(q_{k,j,4})$. ~~whi~~ NOW WE CAN TRY TO COMPUTE THE RATE OF TRANSITION WITH THE INFORMATION AVAILABLE TO US.

The MATRIX ELEMENT MUST BE WORKED OUT, i.e.,

$$(H_{INT})_{fi} = \langle A_0;\, 0,0,0,1,\ldots 0,0,0 \,|\, H_{INT} \,|\, A_1;\, 0,0,0,\cdots 0,0,0 \rangle$$

NOW SUBSTITUTE H_{INT} INTO THIS EXPRESSION,

$$(H_{INT})_{fi} = -\frac{e}{mc}\sum_{\mathbb{K},i} \langle A_0;\, 0,0,0,1,\ldots 0,0 \,|\left\{\frac{(\bar{p}\cdot\bar{e}^*_{\mathbb{K},i})}{\sqrt{2\omega_{\mathbb{K},i}}}e^{-i\mathbb{K}\cdot x}a^*_{\mathbb{K},i} + \frac{(\bar{p}\cdot\bar{e}_{\mathbb{K},i})}{\sqrt{2\omega_{\mathbb{K},i}}}a_{\mathbb{K},i}e^{i\mathbb{K}\cdot x}\right\}|\, A_1;\, 000,\cdots 0,00\rangle$$

WE HAVE TO FIGURE OUT WHAT THIS MESS REDUCES DOWN TO. IN ORDER TO PROCEED WE HAVE TO RECALL WHAT $a_{\mathbb{K},i}$ AND $a^*_{\mathbb{K},i}$ DO TO THE WAVE FUNCTIONS OF THE FIELD. FORTUNATELY WE ALREADY KNOW THE ANSWER; WE WORKED IT OUT BACK ON PAGE 187. a AND a^* HAVE THE PROPERTY THAT

$$a^* \varphi_n = \sqrt{\frac{n+1}{\cancel{2\omega}}}\,\varphi_{n+1}$$

$$a\,\varphi_n = \sqrt{\frac{n}{\cancel{2\omega}}}\,\varphi_{n-1}$$

THAT IS a^* CREATES A QUANTUM OF ENERGY AND THUS RAISE THE STATE OF THE SYSTEM ONE UNIT. a, ON THE OTHER HAND, LOWERS THE EIGENSTATE ONE STEP THUS CORRESPONDS TO EMISSION OF ONE PHOTON. WE SHOULD NOTE THAT THE OPERATION $a\,\varphi_0$ PRODUCES ZERO SINCE φ_0 IS THE GROUND YOU CAN'T GO LOWER THAN THAT ONE. SIMILARLY THE PRODUCT $\varphi_0 a^* = 0$ BY COMPLEX CONJUGATION RULES. THEREFORE WE DON'T HAVE TO WORRY ABOUT THE TERM $a|A_1;0,0,0\cdots 00\rangle = 0$; THIS IS JUST A RESULT OF THE ABSENCE OF ANY PHOTONS INITIALLY WHICH CAN BE ANNIHILATED.

SO NOW WE SEE WHAT HAPPENS WHEN WE CALCULATE THE EFFECT OF $a^*_{\mathbb{K},i}$ ON $|A_0;\, 0,0,0,0,\ldots 0,0\rangle$. SINCE $a^*_{\mathbb{K},i}\,\varphi_0(q_n) = \sqrt{\frac{n+1}{\cancel{2\omega_{\mathbb{K},i}}}}\,\varphi_{n+1}$ IT APPEARS THAT EACH MODE OF THE FIELD GETS EXCITED BY ONE PHOTON. BUT THAT'S NOT WHAT HAPPENS BECAUSE EACH PAIR OF MODES MUST BE ORTHOGONAL BETWEEN THE INITIAL AND FINAL STATES. SINCE IN THE FINAL STATE THERE IS ONE PHOTON IN THE $\mathbb{L},j$ MODE, ALL THE OTHER $\mathbb{K},i$ MODE MUST REMAIN ZERO OTHERWISE WE ARE PRODUCING MORE PHOTONS (i.e., ENERGY) THAN THE ATOM CAN SUPPLY. THE SUMMATION OVER ALL $\mathbb{K},i$ IS ZERO UNLESS $\mathbb{K},i = \mathbb{L},j$. THEREFORE $a^*_{\mathbb{L},j}\,\varphi_0(q^n_{\mathbb{L},j}) = \sqrt{\frac{1}{2\omega_{\mathbb{L}}}}\,\varphi_1(q_0)$ AND

$$H_{fi} = -\frac{e}{\cancel{2}mc\sqrt{\omega_{\mathbb{L}}\cdot 2}}\langle A_0 \,|\, \bar{p}\cdot\bar{e}^*_{\mathbb{L},j}\, e^{-i\mathbb{L}\cdot x} \,|\, A_1 \rangle$$

WE CAN NOW SQUARE THIS EXPRESSION TO COMPUTE THE TRANSITION RATE,

$$\frac{P(i\to f)}{SEC} = \frac{e^2}{4m^2c^2\omega_{\mathbb{L}}^2}\left|\langle A_0|\,\bar{p}\cdot\bar{e}^{*}_{\mathbb{L},j}\,e^{-i\mathbb{L}\cdot x}\,|A_1\rangle\right|^2 \times \frac{2\pi}{\hbar}\delta(E_0+\hbar\omega_{\mathbb{L}}-E_1)$$

HERE WE PUT IN THE INITIAL AND FINAL ENERGY VALUES IN THE DELTA FUNCTION. SO THIS IS THE RESULT BUT IN A FORM WHICH IS NOT TOO USEFUL. USUALLY WE ARE INTERESTED IN THE PROBABILITY OF FINDING THE EMITTED PHOTON IN SOME DIRECTION IN A SOLID ANGLE $d\Omega$. WE WANT TO INTEGRATE THE ABOVE EXPRESSION OVER THE VOLUME $\frac{d^3\mathbb{L}}{(2\pi)^3}$ OR IN TERMS OF SOLID ANGLES OVER THE DIFFERENTIAL VOLUME,

$$\frac{d^3\mathbb{L}}{(2\pi)^3} \longrightarrow \frac{\mathbb{L}^2\,d\mathbb{L}}{(2\pi)^3}\,d\Omega$$

THUS

$$\frac{P(i\to f)}{SEC} = \frac{e^2}{4m^2c^2\omega_{\mathbb{L}}^2}\left|\langle A_0|\,\bar{p}\cdot\bar{e}^{*}_{\mathbb{L}j}\,e^{-i\mathbb{L}\cdot x}\,|A_1\rangle\right|^2\cdot\frac{2\pi}{\hbar}\int\delta(E_0-E_1+\hbar c\mathbb{L})\,\frac{\mathbb{L}^2\,d\mathbb{L}}{(2\pi)^3}\,d\Omega$$

HERE I SUBSTITUTED $\omega_{\mathbb{L}} = \mathbb{L}c$ INTO THE DELTA ~~FO~~ FUNCTION TO GET THE RIGHT FORM FOR INTEGRATION. IT IS NOW STRAIGHT FORWARD FROM HERE SINCE $\delta(f(x))\,dx = \frac{1}{f'(x)} = \frac{1}{c\hbar}$

$$\frac{P(i\to f)}{SEC} = \frac{e^2}{16m^2c^2\omega_{\mathbb{L}}^2\pi^2}\,\frac{\mathbb{L}^2}{c\hbar^2}\,d\Omega\,\left|\langle A_0|\,\bar{p}\cdot\bar{e}^{*}_{\mathbb{L},j}\,e^{-i\mathbb{L}\cdot x}\,|A_1\rangle\right|^2$$

WE CAN SIMPLIFY THIS SOMEWHAT SINCE $\frac{\mathbb{L}^2}{\omega_{\mathbb{L}}^2} = \frac{1}{c^2}$

$$\frac{P(i\to f)}{SEC}\ \text{INTO}\ d\Omega = \frac{e^2}{16\pi^2m^2c^5\hbar^2}\left|\langle A_0|\,\bar{p}\cdot\bar{e}^{*}_{\mathbb{L},j}\,e^{-i\mathbb{L}\cdot x}\,|A_1\rangle\right|^2 d\Omega$$

AND THIS IS THE ANSWER FOR CALCULATING THE RATE OF SPONTANEOUS EMISSION OF LIGHT AT FREQUENCY $\omega_{\mathbb{L}} = \frac{E_1-E_0}{\hbar}$ INTO A DIRECTION $d\Omega$. THE ANSWER IS ALSO VALID FOR INDUCED ABSORPTION SINCE THE MATRIX ELEMENTS ARE EQUIVALENT.

42. REMARKS ON 1ST ORDER NON-RELATIVISTIC PERTURBATION THEORY

Before I GO ON TO DISCUSS The 2ND order PERTURBATION Theory AND EVENTUALLY DISCUSS The RELATIVISTICALLY INVARIANT forms of QED, I'd LIKE TO DISCUSS The LIMITATIONS of The Theory AS WE CURRENTLY KNOW IT.

WE HAVE been dealing WITH PROBLEMS where WE WANT TO CALCULATE The PROBABILITY PER SECOND THAT A SYSTEM WILL MAKE A TRANSITION FROM SOME INITIAL STATE i TO SOME FINAL STATE f if The SYSTEM IS ACTED UPON BY AN EXTERNAL POTENTIAL. WE LEARNED The FORMULATION of The TRANSITION RULE (SOMETIMES CALLED Fermi's GOLDEN RULE) AS

$$\text{TRANS. } \frac{\text{Prob}}{\text{SEC}} = \frac{2\pi}{\hbar} \left| {}_f (\text{POTENTIAL})_i \right|^2 \cdot (\text{DENSITY of STATES})$$

where The DENSITY of STATES IS GIVEN BY

$$\frac{K^2 \, dK \, d\Omega}{(2\pi c)^3 \, dE} = \frac{\omega^2 \, d\Omega}{(2\pi c)^3 \hbar \, d\omega}$$

The POTENTIAL COULD be A VECTOR POTENTIAL of The FORM

$$\vec{A} = \sum_{\mathbb{K},i} \sqrt{\frac{4\pi\hbar c^2}{2\omega_{\mathbb{K}}}} \left[a^*_{\mathbb{K},i} \, e^{-i\mathbb{K}\cdot\mathbb{X}} \vec{\varepsilon}_i + a_{\mathbb{K},i} \, e^{i\mathbb{K}\cdot\mathbb{X}} \vec{\varepsilon}_i \right]$$

FROM This POTENTIAL WE developed The INTERACTION HAMILTONIAN AND WROTE IT AS

$$H_{INT} = \frac{e}{mc} \vec{P}_0 \cdot \vec{A}(X_0)$$

$\vec{P}_0$ being The ELECTRON MOMENTUM AND $\vec{A}(X_0)$ IS EVALUATED AT The POSITION of The ELECTRON. This INTERACTION TERM WAS A CONSEQUENCE of EXPANDING The ~~HA~~ TOTAL HAMILTONIAN of The SYSTEM

$$H = \frac{1}{2m} \left(\vec{P} - \frac{e}{c} \vec{A} \right)^2 + eV$$

NOW There IS A PIECE of The HAMILTONIAN, i.e., $\frac{e^2}{2mc^2} \vec{A} \cdot \vec{A}$ which WE HAVE NEGLECTED. The REASON WE NEGLECTED IT

WAS DUE TO THE APPROXIMATION MADE IN CALCULATING THE AMPLITUDE TO GO FROM i TO f, RECALL THAT

$$\langle \psi_f | \psi(T) \rangle = \langle \psi_f | \psi_i \rangle - \frac{i}{\hbar} \langle \psi_f | \int_0^T H_{INT}(t)\, dt | \psi_i \rangle + \cdots$$

BY EXPANDING TO FIRST ORDER WE ARE PHYSICALLY RESTRICTING THE POTENTIAL TO ACT ONLY ONCE BETWEEN i AND f. AS A CONSEQUENCE OF THIS FIRST ORDER APPROXIMATION, WE HAVE NO USE FOR THE $\bar{A}\cdot\bar{A}$ TERM. ~~which~~ THIS TERMS DESCRIBES A TWO-FOLD ACTION OF $\bar{A}$ ON THE TRANSITION. SINCE WE ARE DEALING WITH FIRST ORDER APPROXIMATIONS THE OPERATORS a AND a^* ONLY WORK ON THE PHOTON STATE ONE TIME. THIS MEANS THAT THE FINAL AND INITIAL STATE CAN DIFFER BY ONLY ONE PHOTON. WHAT THEN OF A SYSTEM WHERE OUR CAVITY IS FILLED WITH A LOT OF PHOTONS IN SAY MODE k, j (CALL THEM $n_{k,j}$ PHOTONS) AND A TRANSITION OCCURS BETWEEN i AND f?

USING THE CREATION AND ANNIHILATION OPERATORS a^* AND a, RESPECTIVELY, WE SHOULD FIND IN GENERAL THAT THE PROBABILITY TO MAKE A TRANSITION IN THE DARK IS PROPORTIONAL TO ~~THE DARK~~ $n+1$. THIS FOLLOWS FROM THE QUANTUM THEORY OF OSCILLATORS WHICH PREDICTS

$$\langle n+1 | a^* | n \rangle = \sqrt{n+1}$$

AND THE PROBABILITY TO ABSORB A PHOTON IS PROPORTIONAL TO n. THIS FOLLOWS FROM

$$\langle n-1 | a | n \rangle = \sqrt{n}$$

THE PROBABILITIES ARE THE SQUARE OF THESE MATRIX ELEMENTS. THE LAST EXAMPLE WE WORK OUT WAS FOR THE CASE $n=0$. THIS ASSUMED ALL THE MODES OF THE FIELD WERE IN THE GROUND STATE. THUS ONLY EMISSION COULD TAKE PLACE WHEREIN THE ATOM EXCITED THE FIELD WITH ONE PHOTON. WE CALLED THIS EMISSION SPONTANEOUS IN THE SENSE THAT IT DIDN'T NEED TO BE FORCED EXTERNALLY. THUS WITHOUT DOING TOO MUCH WORK WE HAVE LEARNED HOW TO EXTEND OUR RESULTS TO A SYSTEM WHERE BOTH ABSORPTION AND EMISSION CAN TAKE PLACE. FURTHER WE CAN CONCLUDE THAT:

$$\text{Probability of Emission}_{i \to f} = C_{if}\,(1 + n_k)$$

AND

$$\text{Probability of Adsorption}_{f \to i} = B_{if}\,(n_\ell)$$

RADIATION EQUILIBRIUM

The coefficients C & B are equal due to the fact that the complex conjugate of the vector potential equals the vector potential itself. From the results thus obtained we can ~~conclude~~ deduce something about thermal equilibrium. Suppose I have two states of a system, m and n, each containing N_m and N_n atoms per unit volume. From thermodynamics we saw that for equilibrium the two states must be in the ratio of

$$\frac{N_m}{N_n} = e^{-(E_m - E_n)/kT}$$

This system will be in equilibrium with the radiation field if the number of atoms going from m to n per unit time by absorption of photons ($\hbar\omega_{mn}$) equals the number going from n to m by emission. We must require then

$$N_n n_{mn} = N_m (1 + n_{mn})$$

or using the above relationship

$$n_{mn} = e^{-(E_m - E_n)/kT} (1 + n_{mn})$$

solving for n_{mn},

$$n_{mn} = \frac{1}{e^{(E_m - E_n)/kT} + 1} = \frac{1}{e^{\hbar\omega_{mn}/kT} + 1}$$

This is the Planck black-body distribution law for n_{mn} photons of frequency, ω_{mn}.

I might add one further comment that the emission process described here differs from the classical results, i.e., when the field is modeled as a classical wave, by the extra piece due to spontaneous emission. The results for absorption are consistent.

MANY ELECTRON SYSTEM

So far we have restricted our analysis to a single electron system, i.e, the hydrogen atom. In general we will have a interaction Hamiltonian which is the sum of all the individual electrons, i.e.,

$$H_{INT.} = \sum_{\mu} \frac{e_\mu}{m_\mu c} \vec{P}_\mu \cdot \vec{\mathcal{E}}_{\mathbb{K},i} \, e^{-i\mathbb{K}\cdot\mathbb{X}}$$

When dealing with many electrons at a time it is more convenient to define a new operator $J(\mathbb{x})$, the current density,

$$\mathbb{J}(\mathbb{x}) = \sum_{n} \frac{e_n}{m_n c} P_n \, \delta^3(\mathbb{x} - \mathbb{x}_n)$$

The interaction term H_{INT} can be converted to an integral of the form

$$H_{INT} = \int \mathbb{J}(\mathbb{x}) \cdot \mathbb{A}(\mathbb{x}) \, d^3\mathbb{x}$$

Substituting for $\mathbb{A}(\mathbb{x})$ we have

$$H_{INT} = \frac{1}{\sqrt{2\omega_{\mathbb{k}}}} \int \left[\mathbb{J}(\mathbb{x}) \cdot \mathbb{e}_i^* \, a_{\mathbb{k}i}^* \, e^{-i\mathbb{k}\cdot\mathbb{x}} + \mathbb{J}(\mathbb{x}) \cdot \mathbb{e}_i \, a_{\mathbb{k},i} \, e^{i\mathbb{k}\cdot\mathbb{x}} \right] d^3\mathbb{x}$$

Since $\mathbb{j}(\mathbb{k}) = \int \mathbb{J}(\mathbb{x}) \, e^{+i\mathbb{k}\cdot\mathbb{x}} \, d^3\mathbb{x}$ = fourier transform of $\mathbb{J}(\mathbb{x})$ we observe that the perturbation gets strong if there is a lot of juice ~~of~~ in the $\mathbb{k}^{Th}$ mode. The amplitude to make emissions and absorptions is then:

$$\text{Amp. to emit} \propto \sqrt{n+1} \; |\mathbb{j}(\mathbb{k})|_{mn} \, \bar{e}_{\mathbb{k},i}$$

$$\text{Amp. to absorb} \propto \sqrt{n} \; |\mathbb{j}(\mathbb{k})|_{mn} \, \bar{e}_{\mathbb{k},i}$$

So far we have only considered time dependent perturbation theory as it applies to the Schrödinger equation. We had solved the wave equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

where H was expanded in a power series, $H_0 + H' + H'' + \cdots$. A more accurate representation of the system is to make the Hamiltonian invariant under translation. This is done by using the dirac equation, ~~equation~~

$$H_{D,0} = \boldsymbol{\alpha} \cdot \mathbb{P} + \beta m$$

where no external potentials are acting. If potentials are acting then

$$H_{D,\bar{A}} = \boldsymbol{\alpha} \cdot \left(\mathbb{P} - \frac{e}{c} \mathbb{A} \right) + \beta m + V$$

$\boldsymbol{\alpha}$ and β are operators which work on the spin variables of the system. In the dirac equation the interaction term becomes

$$H_{INT} = -\frac{e}{c} \boldsymbol{\alpha} \cdot \mathbb{A}$$

WE CAN WRITE THE HAMILTONIAN OF THE SYSTEM THEN AS

$$H = H_{0,A} + H_{osc}$$

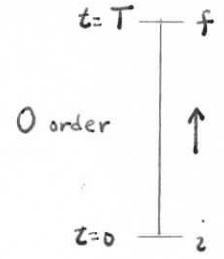
AND SINCE $H_{0,\bar{A}} = H_{0,0} + H_{INT}$ WE SEE THAT AGAIN THE HAMILTONIAN BREAKS IN 3 PARTS. FROM THIS EQUATION WE CAN GO ON TO DEVELOPE A MANIFESTLY RELATIVISTIC FORM FOR THE INTERACTION PROBLEM WITHOUT DOING A LOT OF TEDIOUS WORK. I'd LIKE FIRST TO WORK OUT SOME MORE PROBLEMS.

SCATTERING OF LIGHT FROM AN ATOM

SECOND ORDER TIME DEPENDENT PERTURBATION THEORY

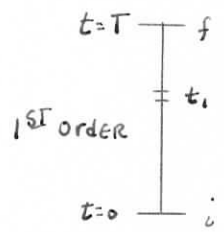
WE ARE NOW READY TO GET MORE COMPLICATED BY CONSIDERING WHAT HAPPENS IN THE NEXT ORDER APPROXIMATION. LET'S REVIEW BRIEFLY WHERE WE ARE AND WHAT A 2nd PERTURBATION MEANS PHYSICALLY DURING A TRANSITION PROCESS. IMAGINE A TRANSITION TAKING PLACE IN THE TIME INTERVAL $0 \le t \le T$. WE WANT TO KNOW WITH WHAT PROBABILITY A SYSTEM, STARTING IN STATE i, WILL BE FOUND IN A PARTICULAR STATE f AFTER A TIME t. ONE WAY TO GET FROM $i \to f$ IS TO GO DIRECTLY WITHOUT ANY DISTURBANCE; WE CALL THIS THE ZERO ORDER TERM. THE AMPLITUDE TO GO FROM $i \to f$ WITHOUT AN INTERACTION IS JUST

$$\langle \psi_f | \psi_i \rangle = \delta_{fi}\, e^{-i(E_f - E_i)T} = \begin{cases} 1 & \text{if } E_f = E_i \\ 0 & \text{if } E_f \neq E_i \end{cases}$$



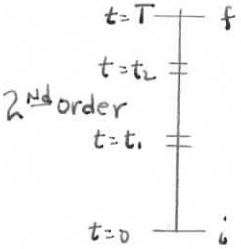
IN THE FIRST ORDER TERM THE INTERACTION OCCURS ONCE AT TIME $t = t_1$ AND THE AMPLITUDE TO ARRIVE AT f IS,

$$-\frac{i}{\hbar}\int_0^T dt \,\langle \psi_f |\, e^{+iE_f(T-t)} (H_{INT})_{fi}\, e^{+iE_i t} \,| \psi_i \rangle$$



THE SECOND ORDER TERM ALLOWS FOR TWO INTERACTIONS ONE AT t_1 AND THE OTHER AT t_2. THE AMPLITUDE TO MAKE A TRANSITION IN SECOND ORDER IS

$$+\sum_k \frac{(-i)^2}{\hbar^2}\int_0^T dt_2 \int_0^{t_2} dt_1 \,\langle \psi_f |\, e^{iE_f(T-t_2)} (H_{INT})_{fk}\, e^{+iE_k(t_2-t_1)} (H_{INT})_{ki}\, e^{iE_i t_1} \,| \psi_i \rangle$$



THE DIAGRAMS SHOW THE TIME ORDERING OF THE EVENTS. IN THE SECOND ORDER TERM $t_2 > t_1$ AND WE MUST SUM OVER ALL POSSIBLE INTERMEDIATE STATES k.

The INTERMEDIATE STATE IS OFTEN CALLED A VIRTUAL STATE SINCE IT ONLY LASTS A SHORT TIME. IN ORDER TO EVALUATE THE INTEGRAL WE HAVE TO AVERAGE OVER A LONG TIME T AS WE did IN THE first ORDER PROBLEM. TO PUT THE INTEGRAND INTO A MORE INTEGRABLE form CONSIDER THE FOLLOWING EXPONENTIAL RELATIONSHIP

$$e^{iE_f(T-t_2)}\, e^{iE_k(t_2-t_1)}\, e^{iE_i t_1} = e^{iE_i T}\, e^{i(E_f-E_i)(T-t_2)}\, e^{i(E_k-E_i)(t_2-t_1)}$$

When T IS VERY LARGE e^{iE_iT} AVERAGES OUT TO 1. The MIDDLE TERM $e^{i(E_f-E_i)(T-t_2)}$ WILL PRODUCE THE DELTA FUNCTION, $\delta(E_f-E_i)$ SIMILAR TO LAST TIME. The LAST TERM MUST BE INTEGRATED AND IS of THE form

$$\int_0^\infty e^{i(E_k-E_i)t}\, dt$$

HERE WE HAVE AN INTEGRAL which IS divergent AND OSCILLATES About ZERO UNLESS t_2-t_1 IS SMALL. WE FIRST INSERT A SMALL CONVERGING FACTOR, $e^{-\epsilon t}$, INTO THE INTEGRAL AND INTEGRATE.

IN GENERAL

$$\int_0^\infty e^{ix-\epsilon x}\, dx = \frac{1}{x+i\epsilon} = P.V\,\frac{1}{x} - i\pi\,\delta(x)$$

Therefore

$$\int_0^\infty e^{i(E_k-E_i)t-\epsilon t}\, dt = \frac{i}{E_k-E_i+i\epsilon}$$

AND FINALLY SUMMING OVER ALL POSSIBLE STATES k, WE HAVE AS THE 2ND order TRANSITION AMPLITUDE,

$$M_{fi}^{(2)} = -\sum_k \frac{i}{\hbar^2}\,\frac{(H_{INT})_{fk}\,(H_{INT})_{ki}}{E_k-E_i+i\epsilon}$$

The MATRIX ELEMENTS $(H)_{fk}$ AND $(H)_{ki}$ ARE NEVER REALLY CALCULATED SO WE LEAVE The ANSWER IN This form. WE CAN COMPARE This RESULT TO The ZERO AND first Order AMPLITUDE

$$M_{fi}^{(0)} = 1 \quad , \quad M_{fi}^{(1)} = -\frac{i}{\hbar}\,(H_{INT})_{fi}$$

The formula for CALCULATING THE TRANSITION PROBABILITY CAN BE GENERALIZED IN THE FOLLOWING SIMPLE FORM:

$$\frac{\text{TRANS. PROB}}{\text{SEC}} = \frac{2\pi}{\hbar} |M_{fi}|^2 \, \delta(E_f - E_i)$$

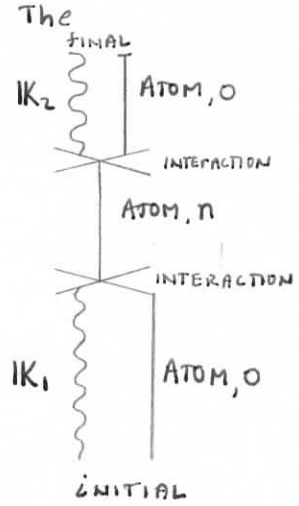
This is a beautifully simple result because it is now possible to go on to higher order terms without doing a lot of work. For example, the third order term is given by

$$M_{fi}^{(3)} = \sum_k \sum_\ell \frac{(H_{INT})_{f\ell} (H_{INT})_{\ell k} (H_{INT})_{ki}}{(E_\ell - E_i + i\epsilon)(E_k - E_i + i\epsilon)}$$

At first glance you may think the denominator is screwy and unsymmetric and should be $(E_\ell - E_k + i\epsilon)(E_k - E_i + i\epsilon)$. The important point to remember in writing down these expression is the strength of the interaction (or resonance, if you like) depends on how far you are away from the initial state - always! If all the expansion terms are fighting eachother at the same time, it is necessary to add up all the M_{fi}'s and then square the sum to insert in the transition rate formula. One final comment here: the factor $1/(E_k - E_i + i\epsilon)$ is just the fourier transform of the free particle propagator $e^{i(E_k - E_i)t}$. So by transforming into momentum space our rules for computing amplitudes stays quite simple.

LIGHT SCATTERING FROM AN ATOM

Now let's return to the problem of light scattering from an atom. If the incident radiation scatters off the atom without exciting atom, how can this process be described? Assume the atom starts in the ground state and ends up in the ground state. The incoming photon has a momentum and polarization $\mathbb{K}_1, \vec{\epsilon}_1$ and the outgoing photon has momentum and polarization $\mathbb{K}_2, \vec{\epsilon}_2$. The diagram on the right describes the process described. The interactions occur twice but only 1 photon is exchanged at a time. First a photon is absorbed and then a second remitted. Since we know the formula for single photon absorption and emission, we can write down M_{fi}



$$(H_{INT})_{fk} \rightarrow \left(\frac{e}{mc\sqrt{2\omega_{\mathbb{K}_2}}} \bar{P}_0 \cdot \bar{\epsilon}_2^* e^{-i\mathbb{K}_2 \cdot \mathbb{X}_0} \right)_{on}$$

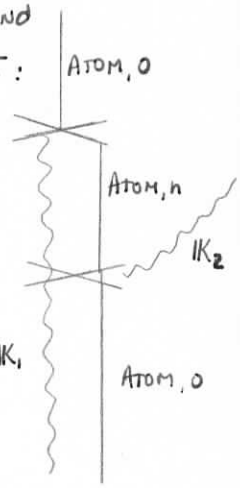
And

$$(H_{INT})_{ki} \rightarrow \left(\frac{e}{mc\sqrt{2\omega_{\mathbb{K}_1}}} \bar{P}_0 \cdot \bar{\epsilon}_1 e^{+i\mathbb{K}_1 \cdot \mathbb{X}_0} \right)_{no}$$

The intermediate state of the system, n, must be summed over to include all possible interactions. The matrix elements $(\ )_{on}$ and $(\ )_{no}$ denote transitions between the excited state n and the ~~ground~~ initial state 0 and the ~~ground~~ initial state 0 to excited state n respectively. We have the according to our rule for determining M_{fi}

$$M_{fi}^{\mathbb{K}_1,\mathbb{K}_2} = \sum_n \left(\frac{e}{mc\sqrt{2\omega_{\mathbb{K}_2}}} \bar{P}_0 \cdot \bar{\epsilon}_2^* e^{-i\mathbb{K}_2 \cdot \mathbb{X}_0} \right)_{on} \frac{1}{(\epsilon_n - \epsilon_0 - \hbar\omega_{\mathbb{K}_1})} \left(\frac{e}{mc\sqrt{2\omega_{\mathbb{K}_1}}} \bar{P}_0 \cdot \bar{\epsilon}_1 e^{i\mathbb{K}_1 \cdot \mathbb{X}_0} \right)_{no}$$

Have we summed over all possible intermediate states? No! There is another way that the same process can occur. This time the photon $\mathbb{K}_1$ does not actually interact with the atom but rather the atom emits a photon $\mathbb{K}_2$ and then turns around and absorbs it back. The diagram for the process is on the right:



The amplitude for this process is

$$M_{fi}^{\mathbb{K}_1,\mathbb{K}_1} = \sum_n \left(\frac{e}{mc\sqrt{2\omega_{\mathbb{K}_2}}} \bar{P} \cdot \bar{\epsilon}_1 e^{+i\mathbb{K}_2 \cdot \mathbb{X}_0} \right)_{on} \frac{1}{(\epsilon_n + \hbar\omega_1 + \hbar\omega_2 - \epsilon_0 - \hbar\omega_1)} \left(\frac{e}{c} \frac{P_0 \cdot \bar{\epsilon}_2^*}{\sqrt{2\omega_{\mathbb{K}_2}}} e^{-\mathbb{K}_2 \cdot \mathbb{X}_0} \right)_{no}$$

The denominator tells us that initially

$$E_i = \epsilon_0 + \hbar\omega_1$$

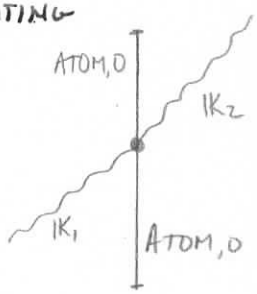
i.e., the atom is in energy state ϵ_0 and the photon exists. And the intermediate state

$$E_k = \epsilon_n + \hbar\omega_1 + \hbar\omega_2$$

i.e., the atom is in energy state ϵ_n and both photons exist. We must add the two amplitudes together to get the total amplitude. ~~Do we~~

Now do we have everything computed right? No!! What about the $\bar{A} \cdot \bar{A}$ term? What do this term do? In this process photon $\mathbb{K}_1$ is absorbed and $\mathbb{K}_2$ is emitted without exciting the atom. The amplitude for this transition is

$$\frac{e^2}{mc^2} \bar{\epsilon}_2^* \cdot \bar{\epsilon}_1 \left(e^{-i(\mathbb{K}_2 - \mathbb{K}_1) \cdot \mathbb{X}_0} \right)_{00}$$

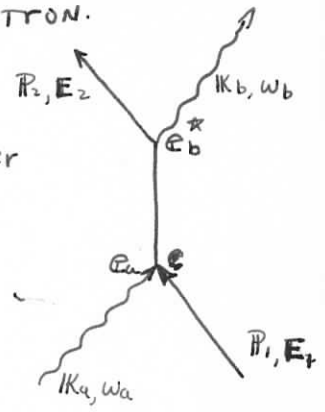


Thus we have three terms to add together to get the complete and accurate scattering amplitudes.

43. LIGHT SCATTERING FROM A RELATIVISTIC ELECTRON

I'd NOW LIKE TO WORK OUT The PROBLEM of LIGHT SCATTERING from AN ATOM where The INCIDENT PHOTON HAS AN ENERGY COMPARABLE TO The REST MASS of AN ELECTRON. For The ENERGY RANGE of INTEREST The GAMMA RAYS HAVE SUCH A high MOMENTUM THAT The POSSIBILITY EXISTS TO PRODUCE A PAIR, i.e. AN ELECTRON AND POSITRON. WE ARE INTERESTED HERE IN The CASE of The ENERGETIC PHOTON HITTING AN ELECTRON which IS FREE (ITS ENERGY IS GREATER THAN The BINDING ENERGY of BeV). So The PROBLEM IS TO SEE WHAT HAPPENS when A SLUGGISH ELECTRON GETS CLOBBERED by A POWERFUL PHOTON.

WE SHALL ASSUME The ELECTRON HAS MOMENTUM $\mathbb{P}_1$ AND The ~~PHOTON~~ HAS REST ENERGY EQUAL TO m_1c^2 where m_1 IS The MASS of The ELECTRON. The RELATIVISTIC ENERGY of The ELECTRON IS THEN $E = \sqrt{\mathbb{P}_1^2 + m_1^2}$ where $c=1$. The PHOTON IS described by A FREQUENCY, ω_a, AND WAVE NUMBER, $\mathbb{K}_a$. The PHOTON IS further described by A PARTICULAR POLARIZATION VECTOR $\mathbb{e}_a$. For EACH VALUE of $\mathbb{K}_a$ THERE ARE TWO POSSIBLE VALUES for $\mathbb{e}_a$, EACH BEING AT RIGHT ANGLES TO EACH OTHER. The PROBLEM IS TO FIGURE OUT WHAT HAPPENS when The PHOTON INTERACTS WITH The ELECTRON. The DIAGRAM for THIS INTERACTION IS TO The RIGHT:



#1

The DIAGRAMS TELLS US THAT AN ELECTRON ~~of MOMENTUM~~ $\mathbb{P}_1$, ENERGY E_1 Absorbs A PHOTON of FREQUENCY ω_A, WAVE NUMBER $\mathbb{K}_A$ (i.e., IT IS SCATTERED by The VECTOR POTENTIAL of The QUANTUM field); Then The ELECTRON PROPAGATES AS A FREE PARTICLE WITH ENERGY $E_1 + \hbar\omega_a$ UNTIL FINALLY The ELECTRON EMITS A PHOTON of ENERGY $\hbar\omega_b$, WAVE NUMBER $\mathbb{K}_b$ AND POLARIZATION $\mathbb{e}_b^*$.

I WANT TO CLARIFY, before I GO ON, WHAT I MEAN when I refer TO A RELATIVISTIC or NON-RELATIVISTIC PROBLEM. The NON RELATIVISTIC PROBLEMS RELATE TO SYSTEMS where ALL The VELOCITIES ARE LOW AS COMPARED TO The SPEED of LIGHT. When I AM

dealing relativistic problems I will primarily be working in relationships which can best be described as manifestly relativistically invariant. Here I am stealing Schwinger's terminology because it correctly ~~defines~~ defines the situation. By this exotic nomenclature we refer to a method of analysis which in itself is not relativistically invariant, only because it doesn't appear to be; but ultimately the solution, and the analysis, is truly relativistically invariant. This is a tremendous help in work out problems in Q.E.d. because relativistic terminology gets very confusing and hard to keep straight unless you know what you are doing. I shall be working with the manifestly relativistic notation and show you how to demonstrate that the answer is actually invariant.

We can now start the problem by asking with what probability the system has to start in its initial state i and end in the final state as described by the diagram on the previous page. From our rule for calculating the transition rate,

$$\text{Prob.} \frac{\text{transition}}{\text{sec}} = \frac{2\pi}{\hbar} |M_{fi}|^2 \, \delta(E_2 - E_1 - \omega_a + \omega_b)$$

The delta function contains the conservation of energy criteria for the reaction to go. In addition momentum must be satisfied therefore

$$\mathbb{P}_2 + \mathbb{K}_b = \mathbb{P}_1 + \mathbb{K}_a$$

where I loosely interchange $\mathbb{K}_b$ and ω_b.

To calculate the transition matrix element M_{fi} we have just learned from our last lecture that

$$M_{fi} = \sum_k H'_{fk} \frac{1}{E_i - E_k + i\epsilon} H'_{ki}$$

Here we sum over all possible intermediate states k where the electron has energy $\mathbb{P}_1 + \omega_a$. We must calculate the interaction terms for the process. The process, by the way, is commonly referred to as the Compton effect. This effect consists of the scattering of energetic photons by a free electron.

SINCE WE ARE DEALING WITH THE RELATIVISTIC TREATMENT OF THE ELECTRON, WE MUST DERIVE THE INTERACTION TERM FROM THE DIRAC HAMILTONIAN. IN GENERAL THE DIRAC EQUATION IS GIVEN BY

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \left[\bar{\alpha}\cdot\left(-i\bar{\nabla} + \frac{e}{c}\bar{A}\right) + \beta m + eA_t\right]\psi$$

where $A_t = V$ the SCALAR POTENTIAL. This EQUATION doesnt LOOK INVARIANT IN form BUT IT IS GOOD for WORKING OUT QUANTUM MECHANICAL PROBLEMS. IT IS NOT IN AN OBVIOUSLY INVARIANT FORM. LET'S SEE WHAT CAN BE DONE TO MAKE IT LOOK INVARIANT. The DIRAC HAMILTONIAN IS GIVEN BY

$$H_{D,A} = \alpha\cdot\left(\bar{P} - \frac{e}{c}\bar{A}\right) + \beta m + A_t(x,t)$$

which SOLVES The ABOVE EQUATION, VIZ

$$\frac{\hbar}{i}\frac{\partial}{\partial t}\psi = H_{D,A}\psi$$

If WE MULTIPLY BOTH SIDES of THIS EQUATION AND REARRANGE The TERMS WE GET

$$\left[\beta\left(\frac{\hbar}{i}\frac{\partial}{\partial t} - eA_t\right) - \beta\bar{\alpha}\cdot\left(i\nabla - \frac{e}{c}\bar{A}\right)\right]\psi = \beta^2 m\psi = m\psi$$

where $\beta^2 = \gamma_t^2 = 1$ AND $\beta\alpha_x = \gamma_x$, $\beta\alpha_y = \gamma_y$, $\beta\alpha_z = \gamma_z$. AND $\alpha_{x,y,z} = \gamma_t\gamma_{x,y,z}$. γ_μ IS A four VECTOR which WE HAS discussed PREVIOUSLY AND whose PROPERTIES AND COMMUTATION RULES ARE KNOWN. WE CAN SUBSEQUENTLY reduce The DIRAC EQUATION TO The RELATIVISTICALLY APPEARING form

$$\left[\gamma_\mu\left(p_\mu - \frac{e}{c}A_\mu\right)\right]\psi = m\psi$$

WE Therefore have demonstrated The INVARIANCE of The ORIGINAL Expression for The DIRAC EQUATION.

RETURNING TO The INTERACTION PIECE THAT WE WANT, IN The ABSENCE of AN EXTERNAL POTENTIAL The DIRAC HAMILTONIAN, $H_{D,0}$, IS JUST

$$H_{D,0} = \bar{\alpha}\cdot\bar{P} + \beta m$$

SUBTRACTING $H_{D,0}$ from $H_{D,A}$ WE GET The INTERACTION TERM

$$H' = -\frac{e}{c}\bar{\alpha}\cdot\bar{A}(x) + A_t(x,t)$$

For NOW WE WILL ONLY CONSIDER A VECTOR POTENTIAL $\bar{A}(x)$, i.e., $A(x)$ IMPLIES THAT The INTERACTION HAMILTONIAN IS INFLUENED by The QUANTUM field VARIABLES WHILE AND A_t IMPLIES The PRESENCE of ATOMIC VARIABLES CREATING The POTENTIAL.

Now the solution to the Dirac equation for a free ~~equ~~ electron in the absence of a potential is given by the solution

$$\psi = u\, e^{-i\bar{p}\cdot\bar{x}}$$

which produces

$$(\bar{\alpha}\cdot\bar{p} + \beta m)\, u = E_p u$$

when the plane wave piece $e^{-i\bar{p}\cdot\bar{x}}$ is taken out. The function u is four ~~vect~~ by four matrix describing the photon energy and momentum. We are interested in computing the matrix element between states 1 and 2 with state 3 being the intermediate state. This intermediate state, or virtual state, has momentum $P_3 = P_1 + \hbar k_a$. Now we can write the matrix element M_{fi}

$$M_{fi} = \sum_3 \frac{e^2}{c^2} \frac{\overbrace{(u_2^* (\alpha\cdot\epsilon_b^*)\, u_3)}^{H'_{fk}} \cdot 1 \cdot \overbrace{(u_3^* (\alpha\cdot\epsilon_a)\, u_1)}^{H'_{ki}}}{\sqrt{2\omega_b}\,(E_{p_1} + \hbar k_a - E_{p_3} + i\epsilon)\,\sqrt{2\omega_a}}$$

Reading right to left the matrix element says that electron in state u_1 absorbs the photon of polarization in direction ϵ_a; then it propagates as a free electron of energy $E_{p_1} + \hbar k_a - E_{p_3}$ whereupon it emits a photon of polarization ϵ_b^* (the star denotes emission) and finally we ask to find the electron in one of its eigenstates u_2^*.

How do we calculate H'_{fi}? Well, we know what the the vector potential is, $\vec{A} = \sum_L \frac{1}{\sqrt{2\omega_L}} \left[\epsilon_L^* a_L^* e^{-iL\cdot x} + \epsilon_L a_L e^{iL\cdot x}\right]$ and,

$$H_{ki} = \int \psi_k \left\{ \alpha\cdot \sum_l \frac{1}{\sqrt{2\omega_L}} \left[\epsilon_L^* a_L^* e^{-iL\cdot x} + \epsilon_L a_L e^{iL\cdot x}\right] \right\} d\,vol$$

In the expression for the vector potential a_L and a_L^* are, respectively, the operators for the emission and absorption by the Lth oscillator of a photon. But to evaluate the integral we must know what ψ_i and ψ_k are. Well ψ_i denote the composite wave function for the system where the electron is in state 1 and all the photons are in the field. ψ_k denotes the state where the electron is excited to state 3 and the photons are in the kth mode.

$$\psi_i = \psi_1 \varphi_L$$
$$\psi_f = \psi_3 \varphi_k$$

$$H'_{ki} = \int \psi_3^* \varphi_k^* H_{INT} \varphi_l \psi_1 \, d^3x$$

SUBSTITUTING IN FOR H_{INT}

$$H'_{ki} = \int \psi_3^* \varphi_k^* \left\{ \sum_L \frac{e}{c} \frac{\bar{\alpha} \cdot a_L^* \bar{e}_L^*}{\sqrt{2\omega_L}} e^{iL\cdot x} + \frac{e}{c} \frac{\bar{\alpha} \cdot a_L \bar{e}_L}{\sqrt{2\omega_L}} e^{-iL\cdot x} \right\} \varphi_l \psi_1$$

SINCE a_L WORKS DIRECTLY ON φ_L TO KILL THE L^{Th} PHOTON TO IKa

$$\int \varphi_k^* a_L \varphi_L \, dvol = \sqrt{\frac{n+1}{2\omega_{Ka}}}$$

n = NUMBER OF PHOTONS WE HAVE

IF WE ONLY HAVE ONE PHOTON PRESENT IN THE PROCESS

$$H'_{ki} = \sum_L \int \varphi_k^* a_L \varphi_L \int \psi_3^* (\bar{\alpha} \cdot \bar{e}_L) e^{-iL\cdot x} \frac{d^3x}{\sqrt{2\omega_L}}$$

SINCE a_L^* HAS NO IMMEDIATE SIGNIFICANCE.

THUS

$$H'_{ki} = \frac{1}{\sqrt{2\omega_{Ka}}} \int \psi_3^* (\bar{\alpha} \cdot \bar{e}_a) \psi_1 \frac{d^3x}{\sqrt{2\omega_L}}$$

WE NOW MUST SUBSTITUTE IN THE INTEGRAL THE FREE PARTICLE WAVE FUNCTIONS, I.E.,

$$H'_{ki} = \frac{1}{\sqrt{2\omega_{Ka}}} \int U_3^* e^{-iP_3\cdot x} (\bar{\alpha} \cdot \bar{e}_a) e^{+iKa\cdot x} U_1 e^{iP_1\cdot x} \frac{d^3x}{\sqrt{2\omega_L}}$$

OR

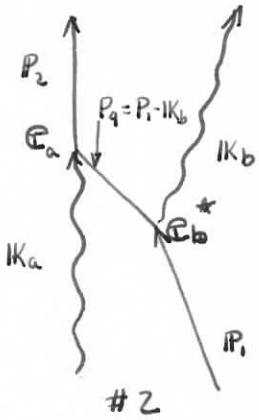
$$= \frac{1}{\sqrt{2\omega_{Ka}}} \int U_3^* (\bar{\alpha} \cdot \bar{e}_a) U_1 e^{-i[P_3 + Ka - P_1]\cdot x} \frac{d^3x}{\sqrt{2\omega_L}}$$

THE INTEGRATION OVER d^3x PRODUCES THE DELTA FUNCTION $\delta(P_3 - Ka - P_1)$ WHICH IS NOTHING MORE THAN A STATEMENT OF THE CONSERVATION OF MOMENTUM. IF YOU DIDN'T REMEMBER TO RIGHT IT DOWN INITIALLY, THE RULE SAYS YOU DON'T GO ANYWHERE UNLESS YOU CONSERVE MOMENTUM. THE INTEGRATION OVER THE VOLUME HAS BEEN NORMALIZED TO $V = 1$ FOR SIMPLICITY. THE SECOND PART OF THE MATRIX ELEMENT, I.E, THE H'_{fk} IS FOUND IN A SIMILAR WAY AS H'_{ki}. THIS TIME HOWEVER WE WILL USE THE a_L^* OPERATOR TO EMIT THE PHOTON AGAIN. THUS WE GET THE FIRST TERM ON THE RIGHT HAND SIDE OF M_{fi}, IE

$$H'_{fk} \quad \frac{1}{\sqrt{2\omega_{Kb}}} U_2^* (\alpha \cdot e_b^*) U_3$$

THIS COMPLETES M_{fi} WHEN THE SUMMATION IS TAKEN OVER ALL THE VIRTUAL STATES 3.

IT IS NOT OBVIOUS FROM THE SUMMATION OVER ALL THE INTERMEDIATE STATES THAT WE FORGOT SOMETHING - BUT WE DID. THERE IS ANOTHER WAY THE SCATTERING CAN HAPPEN AND THAT PROCESS HAS THE ELECTRON EMITTING A PHOTON AND SUBSEQUENTLY ABSORBING THE INCIDENT PHOTON. THE DIAGRAM FOR THIS PROCESS IS TO THE RIGHT. THE MATRIX ELEMENT FOR THIS PROCESS CAN BE WRITTEN DOWN RIGHT AWAY FROM OUR KNOWLEDGE OF THE LAST EXERCISE, I.



$$M_{fi}^{(2)} = \sum_q \frac{e^2}{c^2} \frac{[u_2^*(\alpha\cdot\epsilon_a)u_q]\,[u_q^*(\alpha\cdot\epsilon_b^*)u_1]}{\sqrt{2\omega_a}\,[(E_{p_1}+\omega_a)-(E_{pq}+\omega_a+\omega_b)]\sqrt{2\omega_b}}$$

THE DENOMINATOR CAN BE PUT IN A DIFFERENT FORM SINCE ENERGY MUST BE CONSERVED, I.E. $E_{p_1}+\omega_a = E_2+\omega_b$ SO THAT

$$M_{fi}^{(2)} = \sum_q \frac{e^2}{c^2} \frac{[u_2^*(\alpha\cdot\epsilon_a)u_q]\,[u_q^*(\alpha\cdot\epsilon_b^*)u_1]}{\sqrt{2\omega_a}\,[E_{p_2}-E_{pq}-\hbar\omega_a]\sqrt{2\omega_b}}$$

THE OTHER MATRIX ELEMENT FOR PROCESS 1 IS

$$M_{fi}^{(1)} = \sum_3 \frac{e^2}{c^2} \frac{[u_2^*(\alpha\cdot\epsilon_b^*)u_3]\,[u_3^*(\alpha\cdot\epsilon_a)u_1]}{\sqrt{2\omega_b}\,[E_{p_1}-E_{p_3}+\hbar\omega_a]}$$

THE TOTAL ~~SO~~ MATRIX ~~for~~ ELEMENT for THE SCATTERING IS THE SUM OF THE TWO PIECES: $M_{fi}^{(1)} + M_{fi}^{(2)}$.

NOW SOMETHING USUAL OCCURS WHEN SUMMING OVER ALL POSSIBLE INTERMEDIATE STATES - THE DIRAC EQUATION ALLOWS FOR POSITIVE AND NEGATIVE ENERGY VALUES. WHAT THAT MEANS IS THERE IS A PROBABILITY TO MAKE A TRANSITION INTO A NEGATIVE ENERGY STATE. WHILE THIS SEEMS LIKE A PHYSICALLY STUPID IDEA, WE CAN PURSUE THE IDEA TO SEE WHAT HAPPENS. WHAT WE WILL FIND OUT IS THAT THE NEGATIVE ENERGY STATES CORRESPOND TO PAIR PRODUCTION.

LET'S BEGIN BY SUMMING OVER THE INTERMEDIATE STATE FOR PROCESS 1 AND BREAK THE SUMMATION INTO TWO PARTS: ONE OVER POSITIVE ENERGIES OR MOMENTA AND THE OTHER PART SUMS OVER NEGATIVE ENERGIES.

$$M_{fi}^{(1)} = \sum_{P_3>0} \frac{[U_2^*(\alpha\cdot e_b^*)U_3][U_3^*(\alpha\cdot e_a)U_1}{\sqrt{4\omega_a\omega_b}\,[(E_{p_1}+K_a)-E_{pa}]} + \sum_{P_3<0} \frac{[U_2^*(\alpha\cdot e_b^*)U_3][U_3^*(\alpha\cdot e_a)U_1]}{\sqrt{2\omega_a\omega_b}\,[(E_{p_1}+K_a)+E_{p3}]}$$

WHERE $E_{P3} = \pm\sqrt{m^2c^4+c^2P_3^2}$

BY PUTTING THE TWO PIECES OVER A COMMON DENOMINATOR THE SUMMATION CAN BE TAKE OVER ALL VIRTUAL STATES OF ENERGY E_{P3}, I.E., OVER ALL STATES OF U_3 WHICH INCLUDES 4 (2 for EACH SPIN STATE AND 2 for EACH ENERGY STATE).

$$M_{fi}^{(1)} = \sum \frac{[U_2^*(\alpha\cdot e_b^*)U_3](E_{p_1}+K_a+E_{P3})[U_3^*(\alpha\cdot e_a)U_1]}{\sqrt{4\omega_a\omega_b}\,[(E_{p_1}+K_a)^2-E_{P3}^2]} + \sum \frac{[U_2^*(\alpha\cdot e_b^*)U_3](E_{p_1}+K_a-E_{P3})[U_3^*(\alpha\cdot e_a)U_1]}{\sqrt{4\omega_a\omega_b}\,[(E_{p_1}+K_a)^2-E_{P3}^2]}$$

The DENOMINATOR NO LONGER DEPENDS ON The SIGN of The ENERGY.

NOW WE USE THE FACT THAT

$$(\alpha\cdot P_3+\beta m)U_3 = \pm E_{P3}U_3$$

IS A SOLUTION TO THE DIRAC EQUATION TO PERMIT PUTTING BOTH PIECES OVER The COMMON DENOMINATOR

$$M_{fi}^{(1)} = \sum_{ALL\,U_3} \frac{[U_2^*(\alpha\cdot e_b^*)(E_{p_1}+W_a+\alpha\cdot P_3+\beta m)U_3U_3^*(\alpha\cdot e_a)U_1]}{\sqrt{4\omega_a\omega_b}\,[(E_{p_1}+K_a)^2)-E_{P3}^2]}$$

SINCE U_3 forms A COMPLETE SET of WAVE FUNCTIONS The SUM OVER ALL U_3 MUST BE 1, I.E., $\sum_{U_3} U_3U_3^* = 1$

$$M_{fi}^{(1)} = U_2^*(\alpha\cdot e_b^*)\frac{(E_{p_1}+\hbar\omega_a+\alpha\cdot P_3+\beta m)(\alpha\cdot e_a)U_1}{\sqrt{4\omega_a\omega_b}\,[(E_{p_1}+\omega_a)^2-E_{P3}^2]}$$

PAIR PRODUCTION Theory

PAIR PRODUCTION, NEGATIVE SEA AND SPACE-TIME EXPLANATION

THERE IS ANOTHER WAY TO LOOK AT COMPTON SCATTERING WHICH IS A LITTLE MORE SUBTLE IN IT PHYSICAL CONNOTATION BUT IS MATHEMATICALLY, AND PHYSICALLY, ANALOGOUS TO COMPTON SCATTERING; THAT IS THE IDEA OF PAIR PRODUCTION. SINCE PAIR PRODUCTION IS INTERESTING I WOULD LIKE TO GIVE IT A LITTLE MORE ATTENTION AT THIS TIME.

PAIR PRODUCTION CAN RESULT IN SEVERAL WAYS: A SINGLE PHOTON OF ENERGY GREATER THAN $2m$ CAN INTERACT WITH A NUCLEAR FIELD EITHER BY THE PHOTON CREATING A PAIR AND THE ELECTRON INTERACTING WITH THE FIELD OR BY THE PHOTON CREATING THE PAIR WITH THE POSITRON INTERACTING WITH THE FIELD. ALTERNATELY TWO PHOTONS CAN ANNIHILATE CREATE AN ELECTRON AND POSITRON. IN THE LATTER CASE ONE PHOTON CAN BE SUPPLIED BY THE COULOMB FIELD OF THE NUCLEUS, I.E.,

$$\gamma + \gamma \rightarrow e^- + e^+$$

The opposite reaction here is pair annihilation where the electron and positron produce two gamma rays.

THE INTERESTING THING ABOUT PAIR PRODUCTION WAS THAT THE EXISTENCE OF THE POSITRON WAS PREDICTED BY THE DIRAC EQUATION. IT IS A CONSEQUENCE OF THE DIRAC WAVE EQUATION THAT PARTICLES CAN HAVE ENERGIES GIVEN BY

$$E = \pm \sqrt{m^2c^4 + p^2c^2}$$

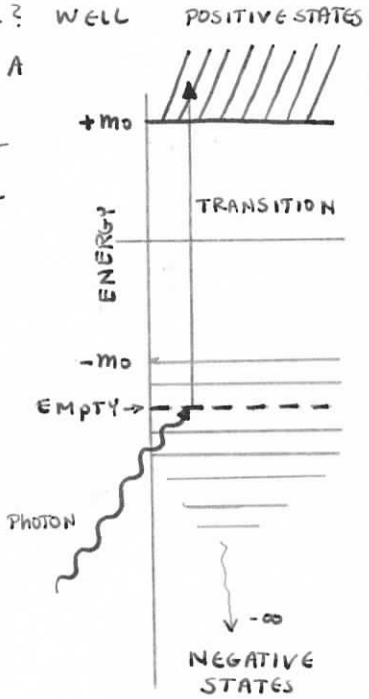
where for A PHOTON of ZERO REST MASS $\omega = K$. The POSITIVE ENERGY describes THE ELECTRON WITH ORDINARY CHARACTER WHILE the NEGATIVE ENERGY SOLUTION describes NEW PARTICLE.

The IDEA, first proposed by DIRAC, TO EXPLAIN THE NEGATIVE ENERGY ROOT CONSIDER A NEGATIVE SEA OF STATES which THE POSITIVE ELECTRON or PARTICLE COULD MAKE TRANSITIONS INTO AND FROM. For AN ELECTRON TO GET KNOCKED OUT OF THE NEGATIVE SEA IT had TO RECEIVE $2m_0c^2$ from THE ENERGETIC PHOTON. IN ESSENCE AN ELECTRON AND A HOLE ARE CREATED; THE HOLE BEING THE POSITRON. DIRAC IMPROPERLY IDENTIFIED THE HOLE AS

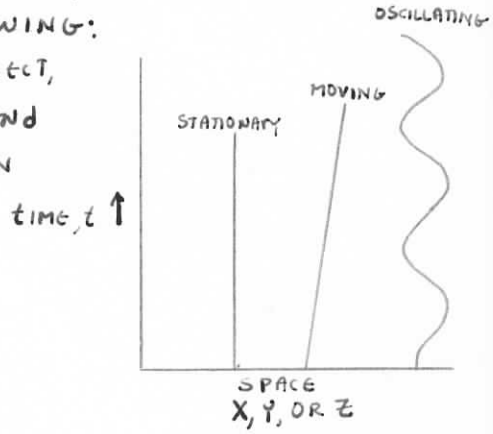
A PROTON. IN 1932 ANDERSON EXPERIMENTALLY DETECTED THE POSITRON.

ONE DIFFICULTLY WITH THE NEGATIVE SEA IDEA IS THAT THE PROCESS NECESSARY TO PRODUCE THE PAIR IN ADDITION IS A SOURCE OF ENERGY. THIS MAKES IT SEEM LIKE AN IMPOSSIBLE THEORY. BUT THE DIRAC EQUATION WAS SO SUCCESSFUL IN PREDICTING THE ATOMIC SPECTRUMS INCLUDING SPIN ~~AND~~ IN A RELATIVISTIC CONDITION THAT THE THEORY MUST BE RIGHT. A WAY OUT OF THE DILEMMA IS TO PROPOSE THAT THE NEGATIVE STATES ARE NEARLY FULL. A VACUUM STATE CAN BE DEFINED AS ONE WHERE ALL THE NEGATIVE STATES ARE FULL (DON'T WORRY ABOUT THE INFINITE AMOUNT OF ENERGY NEEDED TO FILL THE BOTTOMLESS PIT) AND ALL THE POSITIVE STATES ARE EMPTY. WHAT STOPS TRANSITIONS BETWEEN POSITIVE AND NEGATIVE STATES IS THE EXCLUSION PRINCIPLE. AN ELECTRON, OF EITHER SIGN, CANNOT GO INTO ITS ANTI-STATE IF IT IS OCCUPIED.

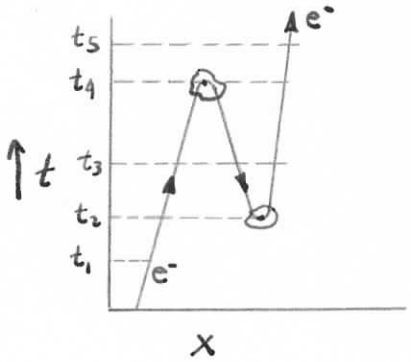
NOW DIRAC ARGUED THAT IF THE ELECTROMAGNETIC FIELD ~~WIT~~ HAD ENOUGH ENERGY IT COULD LIFT THE NEGATIVE ELECTRON, SOMETIMES CALLED THE NEGATON, INTO A POSITIVE ENERGY STATE. FOR THIS TO HAPPEN TWO PHOTONS ARE NEEDED TO CONSERVE ENERGY AND MOMENTUM. HOW DOES THIS PROCESS LOOK? WELL AN ELECTRON OF POSITIVE ENERGY IS CREATED IN ADDITION A NEGATIVE STATE IS LEFT EMPTY IN THE SEA. THE EMPTY STATE OR HOLE IS LIKE A WATER BUBBLE. IT CAN FLOAT AROUND CARRYING CHARGE. IT IS NECESSARY TO CANCEL OUT THIS CHARGED STATE BY SOME TRICK. DIRAC DIDN'T KNOW HOW TO FILL THE HOLE BACK UP BY THE DECAY OF AN ELECTRON INTO THE HOLE. IT WAS LEFT TO OPPENHEIMER TO DEMONSTRATED THE SYMMETRY OF THE TRANSITION AND THE IDEA OF PAIR ANNIHILATION WAS LATER VERIFIED. EXPERIMENTALLY BY ANDERSON. UNFORTUNATELY THIS HOLE-BUSINESS IS PHYSICALLY DIFFICULT TO WORK WITH AND FOR SOME TIME, BETWEEN 1929 AND 1949, QED DIDN'T MAKE MUCH PROGRESS DUE TO THE DIFFICULTIES ENCOUNTERED.



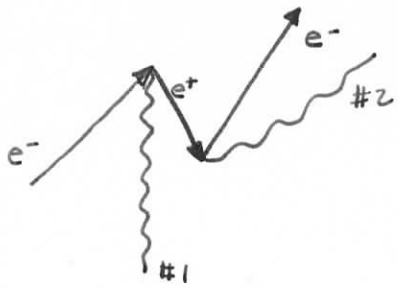
IN '48 - '49 FEYNMAN PROPOSED HIS SPACE-TIME ARGUMENTS for WORKING OUT THESE QED PROBLEM. SINCE THEN THE MACHINERY of SPACE-TIME PHYSICS HAS PROVEN TO BE A GREAT TOOL IN MANIPULATING AND SOLVING SUCH PROBLEMS AS PAIR PRODUCTION. THE SIMPLE CONCEPT of A PARTICLE MOVING IN SPACE-TIME CAN be REDUCED TO A 2-DIMENSIONAL DRAWING:



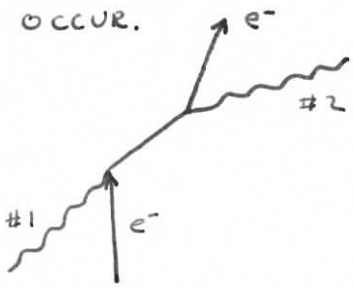
THE 3 MOTIONS SHOWNS INDICATE A STATIONARY OBJECT, ONE THAT IS MOVING UNDER AN APPLIED FORCE, AND ANOTHER ONE WHICH IS OSCILLATORY DUE TO AN EXTERNAL FIELD. A PARTICLE FOLLOWING A SPACE TIME TRAJECTORY CAN be FOUND TO BE MOVING forward AS WELL AS BACKWARD IN TIME AS SEEN IN THE OTHER DIAGRAM: HERE THE ELECTRON IS MOVING forward IN TIME AT t_1. BUT A t_2 SUDDENLY TWO NEW PARTICLE APPEAR UNTIL AT t_4 SUDDENLY TWO PARTICLES CONVERGE, ANNIHILATE, AND THE ELECTRON A NEW ELECTRON CONTINUES ON AT t_5. THE PARTICLE MOVING BACKWARD IN TIME CORRESPONDS TO THE POSITRON. IN order TO TURN THE ELECTRON AROUND IN TIME IT IS NECESSARY TO HAVE A FIELD PRESENT TO SCATTER off of.



IN THE CASE of COMPTON SCATTERING THERE ARE TWO WAYS IN which THE INTERACTION CAN OCCUR.



1.



2.

44. MANIFESTLY INVARIANT THEORY of PAIR PRODUCTION

I'd NOW LIKE TO GO INTO THE QUANTITATIVE THEORY of PAIR PRODUCTION. IN THE PROCESS ~~of~~ WE WILL COME ACROSS THE THREE MAJOR RULES MAKING UP ALL of Q.E.D. IN ORDER TO DEVELOPE THE THEORY IN A RELATIVISTICALLY SIMPLE FORM WHICH IS EASY TO COMPARE WITH THE OLD THEORY of SCATTERING I'd LIKE TO INTRODUCE SOME NEW NOTATION.

THE 4-VECTOR γ_μ HAS PREVIOUSLY BEEN defined SO THAT IT SATISFIES THE COMMUTATION RULE

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\,\delta_{\mu\nu}$$

If a_μ IS ANOTHER 4-VECTOR SATISFYING THIS SAME RULE, THEN define THE MATRIX $\not{a}$ TO BE

$$\not{a} = a_\mu \gamma_\mu = a_t \gamma_t - a_x \gamma_x - a_y \gamma_y - a_z \gamma_z$$

THIS MATRIX IS RELATIVISTICALLY INVARIANT AND Therefore WILL BE QUITE USEFUL. IN THE SAME WAY WE CAN define THE MOMENTUM 4-VECTOR p_μ IN A RELATIVISTIC FORM, I.E., $\not{p} = p_\mu \gamma_\mu$. ACCORDINGLY THE DIRAC EQUATION CAN BE EXPRESSED AS,

$$i\gamma_\mu(-ip_\mu)u = p_\mu \gamma_\mu u = m u \quad \longrightarrow \quad \not{p} u = m u$$

ONE FINAL NOTATION RULE FOR TWO INVARIANT MATRICES $\not{a}$ AND $\not{b}$ THE FOLLOWING IS TRUE:

$$\not{a}\not{b} = -\not{b}\not{a} + 2(a \cdot b) \qquad \text{where } a \cdot b = a_\mu b_\mu$$

AND

$$\not{a}\not{a} = a \cdot a$$

NOW WE WRITE DOWN ONE of OUR IMPORTANT PROPAGATION RULES

RULE: AN ELECTRON PROPAGATES FROM ONE INTERACTION TO ANOTHER VIRTUALLY WITH AN AMPLITUDE PROPORTIONAL TO $\dfrac{1}{\not{p} - m + i\epsilon}$

THIS FREE PARTICLE PROPAGATOR WAS OBTAINED PREVIOUSLY BY TAKING THE ~~MO~~ FOURIER TRANSFORM of THE POSITION PROPAGATOR.

THUS WE WILL BE WORKING IN THE MOMENTUM-ENERGY REPRESENTATION RATHER THAN IN SPACE-TIME; IT MAKES THINGS A LOT EASIER. SINCE THERE MAY BE SOME CONCEPTUAL DIFFICULTY IN UNDERSTANDING THE OPERATION OF THE MATRIX $\not{p}$ IN THE DENOMINATOR, THE FACTOR CAN BE REWRITTEN AS

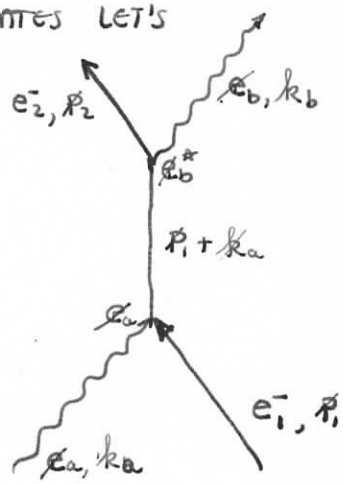
$$\frac{1}{\not{p} - m + i\epsilon} = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}$$

NOW THE DENOMINATOR IS A PURELY SCALAR QUANTITY WITH A SMALL IMAGINARY PIECE WHICH WILL EVENTUALLY BE OMITTED. THE STRENGTH OF THE INTERACTION DEPENDS ON THE DEVIATION OF THE ENERGY OF THE PHOTON, $\not{p}$, AND ELECTRON, m, ENERGY.

IF THE INCIDENT PHOTON IS TAKEN TO BE A VECTOR POTENTIAL WITH POLARIZATION ϵ_μ THEN THE COUPLING OF THE ELECTRON WITH THE PHOTON IS GIVEN BY $\not{\epsilon}$.

RULE: (i). THE AMPLITUDE TO ABSORB A PHOTON IS $\not{\epsilon}$
(ii) THE AMPLITUDE TO EMIT A PHOTON IS $\not{\epsilon}^*$

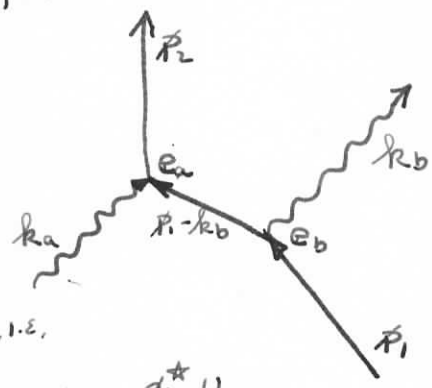
WITH THE RULE FOR COUPLING THE ELECTRON WITH THE FIELD AND THE RULE FOR VIRTUAL PROPAGATION IN THE EXCITED STATES LET'S LOOK AGAIN A THE COMPTON SCATTERING.



CONSIDER THE DIAGRAM TO THE RIGHT. IN THE MOMENTUM REPRESENTATION, THE MATRIX ELEMENT FOR THE PROCESS CAN BE COMPUTED DIRECTLY FROM THE SECOND ORDER PERTURBATION TERM (THIS IS REQUIRED BECAUSE THE ELECTRON INTERACTS TWICE). IF THE ELECTRON STARTS IN STATE U_1, ABSORBS PHOTON k_a, PROPAGATES WITH ENERGY $\not{p}_1 + \not{k}_a$, EMITS PHOTON k_b, AND PROPAGATES WITH MOMENTUM $\not{p}_2$ INTO STATE $\bar{U}_2$, THE AMPLITUDE TO DO ALL THIS IS FOUND TO BE PROPORTIONAL TO

$$\bar{U}_2 \not{\epsilon}_b^* \frac{1}{\not{p}_1 + \not{k}_a - m + i\epsilon} \not{\epsilon}_a U_1$$

To this diagram we must add the amplitude to emit photon b first since the timing of the event is not important. Later we will see that this is just a statement of the relativistic invariance of the process. The amplitude for this second process to occur is



$$\bar{u}_2 \not{e}_a \frac{1}{\not{p}_1 - \not{k}_b - m + i\epsilon} \not{e}_b^* u_1$$

In order to conserve momentum and energy in the process $\not{p}_2 = \not{p}_1 - \not{k}_b + \not{k}_a$. The total matrix element is the sum of these two terms, i.e.

$$M_{12} = \bar{u}_2 \not{e}_b^* \frac{1}{\not{p}_1 + \not{k}_a - m + i\epsilon} \not{e}_a u_1 + \bar{u}_2 \not{e}_a \frac{1}{\not{p}_1 - \not{k}_b - m + i\epsilon} \not{e}_b^* u_1$$

Here I have slipped in the 4-vector $\not{k}$ (ω, k_x, k_y, k_z) in preparation of converting this result to a manifestly invariant form. Right now it is a real mess; it's a mixture of our old technique of computing matrix elements together with some mysterious 4-vector notation. Somehow this all ends up to be invariant and I'd now like to see if I can prove it is.

Now I forgot some factors in writing the above matrix element which I now must straighten out. First the strength of the coupling of the electron is proportional to $\sqrt{\frac{4\pi e^2}{2\omega_a}}$ for the absorption of the photon and proportional to $\sqrt{\frac{4\pi e^2}{2\omega_b}}$. Thus M_{12} is proportional to $\sqrt{\frac{4\pi e^2}{2\omega_a} \cdot \frac{4\pi e^2}{2\omega_b}}$. Let me define the polarization 4-vector e_μ as

$$e_\mu \gamma_\mu = \not{e} \sqrt{4\pi e^2}$$

Therefore, $\not{e}_a^* \not{e}_b^* 4\pi e^2$ is the coupling strength of the photon and electron. We still have to worry about the factor $\left(\frac{1}{2\omega_a} \cdot \frac{1}{2\omega_b}\right)^{1/2}$ because it is not in an invariant form. So let's redefine M_{12}.

LET'S DEFINE AN INVARIANT MATRIX ELEMENT $\mathcal{M}^R$ which IS RELATED TO THE OLD MATRIX ELEMENT M COMPOTED BY THE NON-RELATIVISTIC PERTURBATION THEORY. THE TWO M'S ARE RELATED BY A SCALE FACTOR $\left(\frac{1}{2\omega_a 2\omega_b}\right)^{1/2}$, I.E,

$$\mathcal{M}^{\text{RELATIVISTIC}} = M^{\text{old Theory}} \sqrt{2\omega_a 2\omega_b}$$

WE HAVE ANOTHER MISTAKE BECAUSE THE WAVE FUNCTIONS u_1 WERE ASSUMED TO OBEY THE NORMALIZATION CRITERIA $u_1^* u_1 = 1$. BUT IN THE RELATIVISTIC CASE THE PLANE WAVE DENSITY IS NO LONGER INVARIANT AS THE BOX SHRINKS. THAT IS TO SAY AS THE DENSITY OF THE BOX INCREASES DUE TO SHRINKING BOX DIMENSIONS THE ENERGY CONTENT STARTS TO INCREASE PROPORTIONALLY AS THE SPEED INCREASES. THUS WE HAVE AN UNRELATIVISTIC CHOICE OF NORMALIZATION CONSTANTS WHICH CAN BE CORRECTED FOR BY CHOOSING THE NORMALIZATION TO BE

$$u_1^* u_1 = 2E_1$$

IN THIS EQUATION WE HAVE SET THE 4TH COMPONENT OF THE 4-VECTOR CURRENT EQUAL TO THE 4TH COMPONENT OF THE MOMENTUM 4-VECTOR P_μ. FROM OUR EARLIER WORK $\bar{u} = u^* \beta$ where $\beta = \gamma_t$ AND $\bar{u}\beta = u^* \beta^2 = u^*$. THUS $u_1^* u_1 = \bar{u}\beta u_1 = \bar{u}\gamma_t u = 2E$ THE SPACE COMPONENTS OF THE CURRENT FOUR VECTOR ARE NORMALIZED TO $u_1^* u_1 = 2m$ SUCH THAT

$$\bar{u}\gamma_x u = 2P_x \qquad \bar{u}\gamma_y u = 2P_y \qquad \bar{u}\gamma_z u = 2P_z$$

WE NOW HAVE A RELATIVISTICALLY INVARIANT NORMALIZATION OF THE WAVE FUNCTIONS. WE DO THIS BY SCALING ALL THE u'S UP BY A FACTOR $\sqrt{E_1 E_2}$ which MAKE THE MATRIX ELEMENTS BIGGER PROPORTIONATELY. TO TAKE THIS SCALE CHANGE INTO ACCOUNT WE MUST DIVIDE IT OUT. THUS WHILE THIS FACTOR IS NOT IN ITSELF INVARIANT IT IS JUST THE RIGHT SIZE TO MAKE THE ANSWER COME OUT RELATIVISTICALLY INVARIANT.

FROM THE OLD SCATTERING THEORY (1.2, THE NONRELATIVISTIC CASE) THE TRANSITION RATE COULD BE EXPRESSED IN TERMS OF THE SCATTERING CROSS SECTION AS

$$\frac{\text{TRANSITION RATE}}{\text{SEC}} = \sigma v_{\text{RELATIVE}} = \left[2\pi\,\delta(E_2+\omega_b-\omega_a-E_1)\,(2\pi)^3\delta^3(\mathbf{P}_2+\mathbf{K}_b-\mathbf{K}_a-\mathbf{P}_1)\right]\cdot \left[|M_{fi}|^2\,\frac{d^3\mathbf{K}_b}{2\omega_b(2\pi)^3}\cdot\frac{d^3\mathbf{P}_2}{2E_2(2\pi)^3}\right]\cdot\left[\frac{1}{2E_1\cdot 2\omega_a}\right]$$

I HAVE ADDED A FEW EXTRA TERMS IN THE EXPRESSION PREVIOUSLY USED. THE DELTA FUNCTION IS NOW A 4 DIMENSIONAL ONE IN TERMS OF THE 4-MOMENTUM; IT DESCRIBES THE CONSERVATION OF ENERGY AND MOMENTUM. I DID THIS TO KEEP THE ANSWER MANIFESTLY INVARIANT. THE INTEGRATION OVER $d^3\mathbf{K}_b$ IS OVER THE FINAL MOMENTUM STATES OF THE PHOTON. THE d^3P_2 TERM WAS PUT IN SO THE 4-DIMENSIONAL DELTA FUNCTION IS COMPLETE. NOW THE RESULT HAS A CHANCE OF BEING INVARIANT. THE VELOCITY IS THE RELATIVE VELOCITY BETWEEN THE PARTICLES. IN WORDS THE TRANSITION RULE IS

$$\text{RATE} = \text{CROSS SECTION} \times \text{RELATIVE VELOCITY} = \left\{\begin{array}{l}2\pi \times \text{4-DIMENSIONAL DELTA FUNCTION} \times \\ \text{MATRIX ELEMENT SQUARED DIVIDED BY} \\ \text{TWICE THE ENERGY OF EVERY PARTICLE IN \& OUT}\end{array}\right.$$

$$\text{rate} = (2\pi)^4\,\delta^4(P_{2\mu}+k_{b\mu}-P_{1\mu}-k_{a\mu})\times(M_{fi})^2 \times \frac{1}{2E_1\cdot 2\omega_a}\times\frac{d^3\mathbf{K}_a}{2\omega_a(2\pi)^3}\,\frac{d^3\mathbf{P}_2}{2E_1(2\pi)^3}$$

SINCE THE FORM OF THIS EQUATION CONTAINS THE RELATIVE VELOCITY OF THE PARTICLES INVOLVED IN THE COLLISION IT DOES NOT APPEAR TO BE RELATIVISTICALLY INVARIANT. BUT WE CAN DEMONSTRATE THE MANIFESTLY INVARIANT FORM OF THE RATE EQUATION BY SHOWING THAT

$$v_{rel}\,(2E_1\,2\omega_a) = (v_1-v_2)(2E_1\cdot 2\omega_a) = \left[-\mathbf{P}_1^2\,\mathbf{k}_a^2 + (\vec{P}_1\cdot\vec{k}_a)^2\right]^{\frac{1}{2}}$$

IS AN INVARIANT EXPRESSION. TO PROVE THE ABOVE EQUALITY I'LL WORK IN ONE DIMENSION TO SAVE SOME WORK. THUS THE 4-VECTOR $\mathbf{P}_1$ HAS THE COMPONENTS $\mathbf{P}_{1x} = E_1, P_{1x}$ BUT $P_{1x} = mV_1 = E_1V_1$; THEREFORE $\mathbf{P}_1 = E_1, E_1V_1$. SIMILARLY THE 4-VECTOR $\mathbf{k}_a$ HAS THE COMPONENTS $\omega_a, \omega_a v_a$. HERE I INCLUDE v_a FOR COMPLETENESS; FOR A PHOTON $v_a = c = 1$.

INSERTING $\bar{P}_1$ AND k_a IN TO THE EQUATION

$$-\bar{P}_1^2 k_a^2 + (\bar{P}_1 \cdot \bar{k}_a)^2 = -(E_1^2 - E_1^2 V_1^2)(\omega_a^2 - \omega_a^2 V_a^2) + (E_1 \omega_a - E_1 \omega_a \vec{v}_1 \cdot \vec{v}_a)^2$$

EXPANDING AND CONSIDERING $\bar{v}_1 \cdot \bar{v}_a = 0$, I.E., THE PARTICLES COLLIDE HEADON WE CAN COLLECT TERMS AND HAVE LEFT

$$-\bar{P}_1 k_a^2 + (\bar{P}_1 \cdot \bar{k}_a)^2 = [E_1 \omega_a (v_a - v_1)]^{\frac{3}{2}}$$

THE SQUARE ROOT THEN GIVES JUST THE RIGHT ANSWER WITH THE FACTOR OF 2 ADDED TO SCALE THE ENERGIES RIGHT; THUS

$$4E_1 \omega_a (v_a - v_1) = [-\bar{P}_1^2 k_a^2 + (P_1 \cdot k_a)^2]^{1/2}$$

I'D NOW LIKE TO CONSIDER THE FACTOR $d^3 k_a \, d^3 P_f$ AND SHOW IT CAN BE EXPRESSED IN THE MANIFESTLY INVARIANT FORM OF $2\pi \, \delta(k_a^2 - 0) \frac{d^4 k_a}{(2\pi)^4} \, \delta(P_f^2 - m^2) \frac{d^4 P_f}{(2\pi)^4}$. CONSIDER NOW A SINGLE PARTICLE INTERACTION TO MAKE THINGS SIMPLER. WHEN WE SUBSTITUTE THIS INTO THE RATE FORMULA WE HAVE THE COMPLETE MANIFESTLY INVARIANT FORMULA WE DESIRE!

$$\sigma v_{rel} = 2\pi \frac{1}{2E_1 2\omega_a} |\mathcal{M}_{fi}^R|^2 \times (2\pi)^4 \delta^4(P_{2\mu} + k_{b\mu} - P_{1\mu} - k_{a\mu}) \times 2\pi \, \delta(P_2^2 - m^2) \frac{d^4 P_2}{(2\pi)^4}$$

IN GENERAL TERMINOLOGY

$$\text{TRANSITION RATE} = 2\pi \left[\prod \left(\frac{1}{2 \cdot \text{INCOMING PARTICLE ENERGY}} \right) \right] \left(\text{RELATIVISTIC MATRIX ELEMENT} \right)^2 (2\pi)^4 \, \delta^4\left(\sum_{\text{ALL IN}} P_{IN} - \sum_{\text{ALL OUT}} P_{OUT} \right) \cdot$$

$$\left[\prod \left(\frac{1}{2 \text{ OUTGOING PARTICLE ENERGY}} \right) \right] \delta(P_{OUT}^2 - m^2) \frac{d^4 P_{OUT}}{(2\pi)^4}$$

IN SUMMARY WE HAVE CONSIDERED ALMOST ALL OF THE RULES OF QED IN DERIVING THIS MANIFESTLY INVARIANT SCATTERING THEORY. I'D LIKE TO DISCUSS NOW THE ONE RULE WE HAVEN'T TALKED ABOUT.

45. RULES of QUANTUM ELECTRODYNAMICS

SO FAR WE HAVE DISCUSSED TWO of THE THREE RULES of Q.E.D.

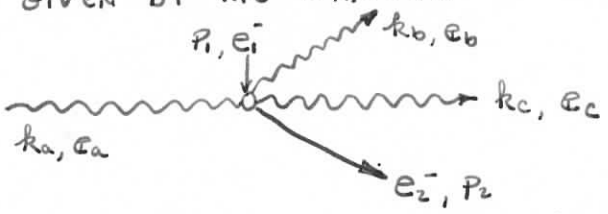
RULES.

1. THE AMPLITUDE for A VIRTUAL e^- TO PROPAGATE IS $\propto \frac{1}{\not{p} - m + i\epsilon}$
2. THE INTERACTION of THE PHOTON AND ELECTRON IS COUPLED BY THE TERM: $\sqrt{4\pi e^2}\,\not{\epsilon}$
3. THE AMPLITUDE for A VIRTUAL PHOTON TO PROPAGATE IS PROPORTIONAL TO, $\gamma_\mu \frac{1}{q^2} \gamma_\mu 4\pi e^2$

IT IS THE THIRD RULE which WE HAVE NOT DISCUSSED AND I'd LIKE TO DO THAT NOW. THEN WE WILL HAVE ALL THE CONCEPTUAL TOOLS NEEDED TO WORK OUT ALL KINDS of PROBLEMS. I WILL EMPHASIZE AS WE GO ON THE USE of SPACE-TIME (FEYNMAN) DIAGRAMS TO WORK OUT THE VARIOUS PROBLEMS. THE diAGRAMS SERVE AS A VERY USEFUL TOOL BECAUSE ONCE YOU ARE FAMILIAR WITH THEM AND CAN WRITE DOWN ALL THE TOPOLOGICALLY DIFFERENT diAGRAMS for A GIVEN PROBLEM, THEN IT IS NOLONGER NECESSARY TO WORK OUT THE MATHEMATICS; I.E., WRITE OUT ALL THE MATRIX ELEMENTS. BY STUDYING THE VARIOUS diAGRAMS IT WILL be POSSIBLE TO SEE which diAGRAM IS "STRONGER or WEAKER" THAN THE OTHERS AND THUS APPROPRIATELY SIMPLY THE SITUATION. If IT BECOMES NECESSARY TO COMPUTE THE EXPECTATION VALUES for THE PROCESS IN QUESTION, THEN THE DIAGRAM CAN be CONVERTED DIRECTLY INTO THE NECESSARY form by CAREFUL LABELING of THE diAGRAM.

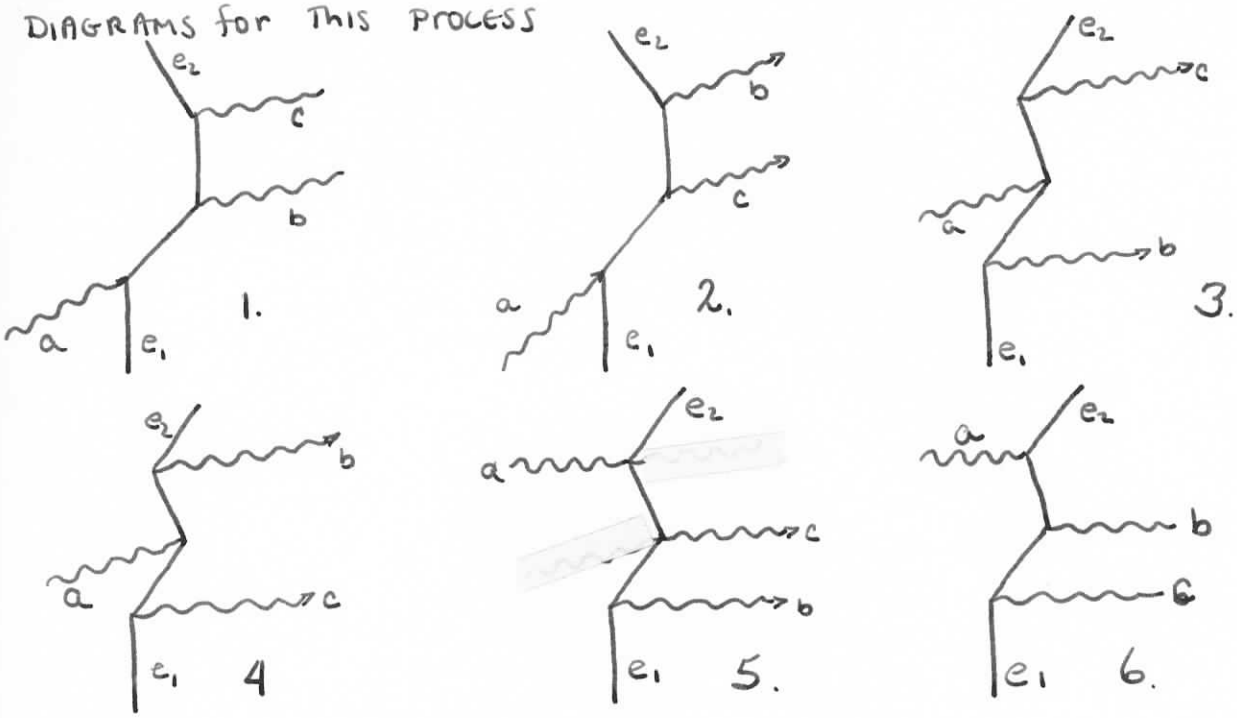
~ TWO PHOTON EMISSION ~

AS AN EXAMPLE of WHAT I MEAN, before I develope RULE 3, LET'S CONSIDER A PROCESS which MAY NOT HAVE BEEN OBSERVED. ASSUME WE HAVE A PHOTON SCATTERING OFF AN ELECTRON; WHAT IS THE PROBABILITY THAT TWO PHOTONS ARE SCATTERED OUT. THE PROCESS IS GIVEN BY THE diAGRAM

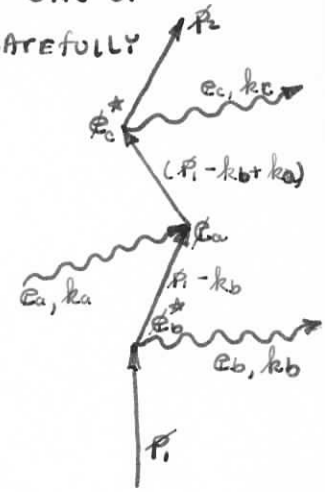


LET'S ASSUME THE ELECTRON IS INITIALLY AT REST, I.E. $\not{p}_1 = (m, 0, 0, 0)$. IN order TO CONSERVE MOMENTUM WE MUST HAVE THAT $\not{p}_1 + \not{k}_a = \not{p}_2 + \not{k}_b + \not{k}_c$.

NOW WE DRAW ALL THE TOPOLOGICALLY DIFFERENT FEYNMAN DIAGRAMS for THIS PROCESS



WE HAVE THEN 6 DIAGRAMS WHICH ALL MUST BE ADDED TOGETHER. I SHOULD POINT THAT EACH OF THESE DIAGRAMS CAN BE DECOMPOSED INTO EIGHT MORE DIAGRAMS if WE REVERSE THE TIME OCCURRANCE of EACH EVENT. BUT SINCE OUR THEORY IS RELATIVISTICALLY INVARIANT WE DON'T NEED TO DO THAT. FOR EXERCISE LET'S WRITE ONE of THESE DIAGRAMS OUT, SAY #3. LET'S DRAW IT AGAIN CAREFULLY LABELING EVERY THING.



FIRST $\not{p}_2 = \not{p}_1 + \not{k}_a - \not{k}_b - \not{k}_c$ OR $\not{p}_2 + \not{k}_c = \not{p}_1 - \not{k}_a + \not{k}_b$

WRITING OUT M_3

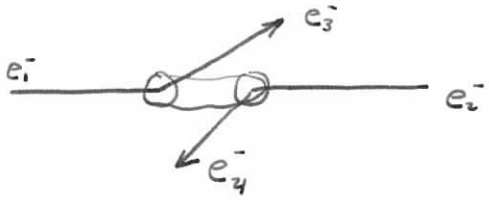
$$M_3 \propto \bar{u}_2 \, \not{\epsilon}_c^* \frac{1}{\not{p}_1 - \not{k}_b + \not{k}_a - m + i\epsilon} \not{\epsilon}_a \frac{1}{\not{p}_1 - \not{k}_b - m + i\epsilon} \not{\epsilon}_b^* u_1$$

Reading from right to left (the way we write the matrix element out), the electron of momentum $\not{p}_1$ in state u_1 emits photon k_b, propagates with energy $\not{p}_1 - \not{k}_b$, absorbs photon $\not{k}_a$, propagates to with momentum $\not{p}_1 - \not{k}_b + k_a$, finally it emits photon k_c and propagates on with momentum $\not{p}_2 + \not{k}_c$ and we ask the probability to be in state $\bar{u}_2$. Similarly for the other 5 processes M's are computed such that the total rate is

$$\text{RATE} = \sigma v = 2\pi \left(\frac{1}{\underbrace{2E_1 2\omega_a}_{\text{IN}} \underbrace{2\omega_b 2\omega_c 2E_2}_{\text{OUT}}} \right) \cdot \left| \sum_{n=1}^{5} M_N \right|^2 \times \text{density of state}$$

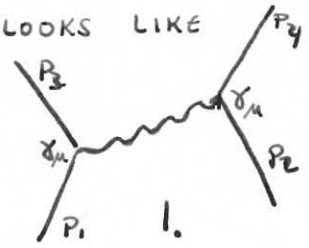
ELECTRON-ELECTRON SCATTERING,
THE RULE FOR VIRTUAL PHOTON PROPAGATION.

IN ORDER TO DEVELOPE THE RULE FOR VIRTUAL PHOTON PROPAGATION I WILL RETURN TO THE OLD WAY OF COMPUTING MATRIX ELEMENTS SO YOU CAN SEE BETTER WHERE IT COMES FROM. THE PROBLEM WE WANT TO CONSIDER IS TWO ELECTRON SCATTERING. IN THE REAL WORLD THE INTERACTION PICTURE LOOKS LIKE

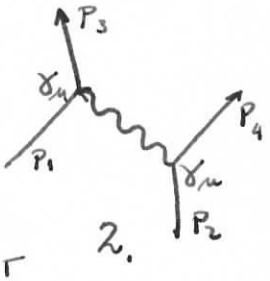


THE SCATTERING POTENTIAL IS DUE TO THE COULOMB INTERACTION OF THE TWO CHARGED PARTICLES. WE WILL CONSIDER THE CASE WHERE THE ELECTRONS MOVE FAST ENOUGH THAT THEIR MOTION IS AFFECTED BY RETARDATION EFFECTS. THUS WE MUST WORRY ABOUT Q.E.D WHICH ALLOWS FOR PHOTON EMISSION DURING THE EMISSION. OTHERWISE FOR SLOW MOVING ELECTRONS WE WOULD SUPPRESS THE RETARDATION EFFECTS AND CONSIDER THE STATIC INTERACTION OF THE COULOMB FIELD.

IN THE SPACE-TIME PICTURE THE INTERACTION LOOKS LIKE THE FOLLOWING:



HERE, THE ELECTRON 1 INTERACTS WITH THE COULOMB FIELD, RELEASING A PHOTON WHICH IS SUBSEQUENTLY ABSORBED BY THE OTHER INCOMING ELECTRON 2. IN THIS PROCESS THE VACUUM STATE IS EXCITED BY THE VIRTUAL PHOTON BUT THE VACUUM STATE IS RESTORED AS THE PHOTON IS ABSORBED. IT SHOULD BE CLEAR THAT THIS PROCESS COULD HAPPEN ANOTHER WAY IF THE SECOND ELECTRON EMITS THE PHOTON FIRST AND THEN ELECTRON 1 ABSORBS IT.



THE γ_μ'S WILL BE EXPLAINED IN A MINUTE. FOR PURPOSES OF SIMPLICITY WE WILL CONSIDER THE ELECTRONS AS DISTINGUISHABLE. WE COULD THINK OF ONE AS A MU MESON WHICH IS AN ELECTRON WITH A DIFFERENT MASS. THUS WE WON'T WORRY ABOUT THE EXCHANGE OF PARTICLES.

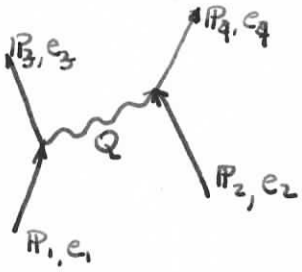
SINCE THIS IS A TWO INTERACTION PROCESS WE MUST HAVE THE INTERACTION TERM ACTING TWICE WHICH TELLS US WE WANT TO COMPUTE THE SECOND ORDER PERTURBATION MATRIX ELEMENT; I.E,

$$\mathcal{M}_{fi} = 4\pi e^2 \sum_k H'_{fk} \frac{1}{E_i - E_k} H'_{ki}$$

NOW WE ARE GOING TO WORK WITH THE OLD NON-RELATIVISTIC THEORY AND THEN SHOW IT IS MANIFESTLY INVARIANT. THE DESCRIPTION OF THE VARIOUS STATES OF THE SYSTEM IS AS FOLLOWS:

i,	INITIAL	e_1^-, e_2^-,	NO PHOTON
k,	INTERMEDIATE	e_3^-, e_2^-,	1 PHOTON
f,	FINAL	e_3^-, e_4^-,	NO PHOTON

1.



THE INTERMEDIATE MOMENTUM OF THE PHOTON IS $Q = P_1 - P_3$.
WE CAN WRITE DOWN THE MATRIX ELEMENT FOR THIS INTERACTION USING OUR RULES

$$M^1_{fi} = \frac{4\pi e^2}{\sqrt{2\omega_Q}} \frac{1}{\sqrt{2\omega_Q}} (\bar{u}_4^* \alpha \cdot \epsilon^* u_2) \frac{1}{E_1 + E_2 - (E_3 + E_2 + \omega_Q)} (\bar{u}_3^* \alpha \cdot \epsilon u_1)$$

HERE $\omega_Q = |Q|$ SINCE $c = 1$.
TO THIS TERM WE MUST ADD THE SECOND POSSIBLE AMPLITUDE,

$$M^2_{fi} = \frac{4\pi e^2}{\sqrt{2\omega_Q}} \frac{1}{\sqrt{2\omega_Q}} (u_3^* \alpha \cdot \epsilon u_1) \frac{1}{E_1 + E_2 - (E_1 + E_4 + \omega_Q)} (u_4^* \alpha \cdot \epsilon^* u_2)$$

2.

TO PUT THESE TERMS TOGETHER WE MUST PUT THEM OVER THE SAME DENOMINATOR. THIS CAN BE DONE USING THE CONSERVATION OF ENERGY RULE, I.E,

$$E_1 + E_2 = E_3 + E_4 \quad \text{OR} \quad E_1 - E_3 = E_4 - E_2$$

FROM M^1_{fi}

$$\frac{1}{E_1 + E_2 - (E_3 + E_2 + \omega_Q)} = \frac{1}{E_1 - E_3 - \omega_Q}$$

FROM M^2_{fi}

$$\frac{1}{E_1 + E_2 - (E_1 + E_4 + \omega_Q)} = \frac{1}{E_2 - E_4 - \omega_Q} = \frac{1}{-(E_1 - E_3) - \omega_Q}$$

THE COMMON DENOMINATOR HERE IS $(E_1 - E_3)^2 - \omega_Q^2$.

$$M = M' + M^2 = \frac{4\pi e^2}{2\omega_Q} \frac{\left[(E_1 - E_3) + \omega_Q \right] - \left[(E_1 - E_3) - \omega_Q \right]}{\left[(E_1 - E_3)^2 - \omega_Q^2 \right]} \left(u_4^* (\alpha \cdot \epsilon^*) u_2 \right) \left(u_3^* (\alpha \cdot \epsilon) u_1 \right)$$

THIS CAN BE CONSIDERABLY SIMPLIFIED BY SUBTRACTING AND DIVIDING OUT THE NUMERATOR AND DENOMINATOR $2\omega_Q$ AND LEAVING

$$M = \frac{4\pi e^2}{(E_1 - E_3)^2 - \omega_Q^2} \left(u_4^* (\alpha \cdot \epsilon^*) u_2 \right) \left(u_3^* (\alpha \cdot \epsilon) u_1 \right)$$

NOW RECALL THAT $\omega_Q = |Q|$ SO $\omega_Q^2 = Q \cdot Q = (\mathbb{P}_1 - \mathbb{P}_3)^2$
IF WE NOW IMAGINE THE PHOTON IS DESCRIBED BY A 4 MOMENTUM WITH $q_4 = E_1 - E_3$ AND SPACE PART $\bar{Q} = \mathbb{P}_1 - \mathbb{P}_3$
THEN $q_4^2 - \bar{Q} \cdot \bar{Q} = q^2$ SO THAT

$$M = \frac{4\pi e^2}{q^2} \left(u_4^* (\alpha \cdot \epsilon^*) u_2 \right) \left(u_3^* (\alpha \cdot \epsilon) u_1 \right)$$

THE STRENGTH OF THE INTERACTION DEPENDS ON THE DEVIATION OF q^2 FROM 0 MOMENTUM.

OUR ANSWER IS NOT COMPLETE IN THE ABOVE FORM BECAUSE WE MUST SUM OVER ALL POSSIBLE POLARIZATIONS γ_μ OF THE PHOTON

$$M = \sum_{\text{ALL POL}} \frac{4\pi e^2}{q^2} \left(u_4^* (\alpha \cdot \epsilon^*) u_2 \right) \left(u_3 (\alpha \cdot \epsilon) u_1 \right)$$

THIS IS EQUIVALENT TO SUM OF THE FOUR POLARIZATIONS,

$$-\frac{1}{q^2} \left(u_4^* \vec{\alpha}_{x,y,z} u_2 \right) \cdot \left(u_3^* \vec{\alpha}_{x,y,z} u_1 \right) + \frac{1}{q^2} \left(u_4^* \frac{\bar{\alpha} \cdot \bar{Q}}{Q^2} u_2 \right) \left(u_3^* \frac{\alpha \cdot Q}{Q^2} u_1 \right)$$

THIS DOESN'T LOOK RIGHT SINCE IT DOES NOT EQUAL THE ABOVE MATRIX ELEMENT; THERE IS AN EXTRA PIECE IN THE EXPRESSION INVOLVING $1/Q^2$. LET'S SEE IF WE CAN UNDERSTAND WHERE THIS EXTRA TERM CAME FROM BY FORCING THE EQUATION TO THE RELATIVISTICALLY INVARIANT. FIRST WE NEED THE FACT THAT $\bar{u}\beta = u^*$
THEN WE HAVE

$$-\frac{1}{q^2} \left(\bar{u}_4 \beta \vec{\alpha} u_2 \right) \cdot \left(\bar{u}_3 \beta \vec{\alpha} u_1 \right) + \frac{1}{q^2} \left(u_4^* \frac{\bar{\alpha} \cdot \bar{Q}}{Q^2} u_2 \right) \left(u_3^* \frac{\alpha \cdot Q}{Q^2} u_1 \right)$$

NOW $\beta\vec{\alpha} = \vec{\gamma}$ AND BY ADDING AND SUBTRACTING $(\bar{u}_4 \gamma_t u_2)$ & $(\bar{u}_3 \gamma_t u_1)$ WE GET THAT

$$M = \frac{4\pi e^2}{q^2}(\bar{u}_4 \gamma_\mu u_2)(\bar{u}_3 \gamma_\mu u_1) + \frac{1}{q^2}\underbrace{(u_4^* \frac{\alpha\cdot Q}{Q^2} u_2)(u_3^* \frac{\alpha\cdot Q}{Q^2} u_1) - (\bar{u}_4 \gamma_t u_2)(\bar{u}_3 \gamma_t u_1)}_{\text{debris}}$$

WE WOULD HAVE A NICE RELATIVISTICALLY INVARIANT RESULT IF SOMEHOW THE debris EQUALS ZERO. LET'S WORK ON THE TERM $u_3^* \frac{\alpha\cdot Q}{Q^2} u_1$; THIS EQUALS

$$u_3^* \frac{\alpha\cdot Q}{Q^2} u_2 = u_3^* (\alpha\cdot P_1 - \alpha\cdot P_3) u_1$$

BUT THE DIRAC EQUATION PREDICTS THAT $\alpha\cdot P_1 u_1 = E_1 u_1 - \beta m u_1$ AND $\alpha\cdot P_3 u_3^* = E_3 u_3^* - \beta m u_3^*$ SO

$$u_3^*(\alpha\cdot P_1 - \alpha\cdot P_3) u_1 = u_3^*[(E_1 - \beta m) - (E_3 - \beta m)] u_1$$
$$= u_3^* u_1 (E_1 - E_3)$$

SIMILARLY WE CAN SHOW THAT

$$u_4^* \alpha\cdot Q u_2 = u_4^* u_2 (E_4 - E_2)$$

SINCE $E_1 - E_3 = E_4 - E_2$

$$u_4^* u_2 (E_4 - E_2) = u_4^* u_2 (E_1 - E_3)$$

FINALLY THEN

$$(u_4^* \frac{\alpha\cdot Q}{Q^2} u_2)(u_3^* \frac{\alpha\cdot Q}{Q^2} u_1) = u_3^* u_1 u_4^* u_2 \frac{(E_1 - E_3)^2}{Q^2}$$

NOW THE TERM $(\bar{u}_4 \gamma_t u_2)(\bar{u}_3 \gamma_t u_1)$ CAN BE REWRITTEN SINCE $\beta = \gamma_t$ AND $\bar{u}\beta = u^*$, I.E.,

$$(\bar{u}_4 \gamma_t u_2)(\bar{u}_3 \gamma_t u_1) = (u_4^* u_2)(u_3^* u_1)$$

WE HAVE THEN

$$\text{debris} = u_4^* u_2 u_3^* u_1 \frac{(E_1 - E_3)^2}{Q^2} - u_4^* u_2 u_3^* u_1$$

$$= \frac{4\pi e^2}{q^2}\frac{1}{Q^2} u_4^* u_2 u_3^* u_1 [(E_1 - E_3)^2 - Q^2]$$

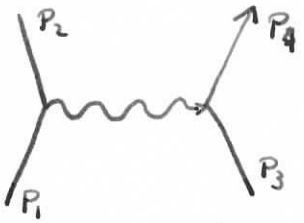
RECALL THAT $q^2 = (E_1 - E_3)^2 - Q^2$ SO WE HAVE THAT

DEBRIS : $\dfrac{U_2^* U_2 U_3^* U_1}{Q^2}$

AND THIS MATRIX ELEMENT IS NON ZERO. DAMN! TOO BAD. WE MUST FIND AN EXPLANATION OF WHY IT APPEARS IN OUR FORMULA.

IF WE THINK BACK TO WHAT WE ASSUMED WHEN DEVELOPING THE RULE FOR COMPUTING THE INTERACTION ~~AEA~~ POTENTIAL WE EXPRESSED THE VECTOR POTENTIAL IN TERMS OF PLANE TRANSVERSE WAVES. WHEN WE DID THAT WE ASSUMED THE COULOMB GAUGE WHICH MEANS WE HAVE A SECOND TERM, THE STATIC, INSTANTANEOUS COULOMB INTERACTION BETWEEN THE PARTICLES. THE DEBRIS WHICH WE CALCULATED IS THE FOURIER TRANSFORM OF THIS STATIC INTERACTION TERM, e^2/r_{ij}, IN MOMENTUM SPACE, I.E, $\int \frac{e^{iQ\cdot R}}{R} d^3R = \frac{4\pi}{Q^2}$

THIS TERM CORRESPONDS TO INSTANTANEOUS PROPAGATION OF THE VIRTUAL PHOTON,



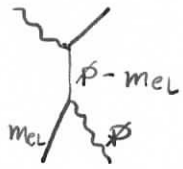
WE HAVE THEN A TOTAL OF THREE DIAGRAMS WHICH MUST BE SUMMED TOGETHER TO OBTAIN THE FINAL RESULT. THE SUMMATION OVER THE FOUR POLARIZATIONS IS SEEN TO BE EQUIVALENT TO TRANSVERSE WAVES PLUS AN INSTANTANEOUS COULOMB INTERACTION. HOWEVER SINCE OUR ANSWER IS INVARIANT UNDER A LORENTZ TRANSFORMATION, IT IS MEANINGFUL TO REDUCE THE FORMULA TO THE TWO DIFFERENT DYNAMICAL INTERACTIONS INVOLVING THE TRANSVERSE FIELD. THE DIAGRAM WE JUST COMPUTED IS, THEREFORE, ~~IN~~ IGNORED ~~WH~~ AND WE REPLACE IT BY THE GENERAL PHOTON COUPLING DIAGRAM

46. HIGHER ORDER CORRECTION TERMS

WE HAVE DEVELOPED A SET OF RULES for COMPUTING THE MATRIX ELEMENTS BETWEEN SOME INITIAL AND FINAL STATES. THE ANALYSIS of THE INTERACTION HAS BEEN CONSIDERED IN A SEQUENCE of PERTURBATION EACH WITH A STRENGTH of $e^2/\hbar c \simeq 1/137$. THE STUDY LED TO THE DEVELOPMENT OF THE RULE FOR COMPUTING THE SCATTERING RATES, I.E.,

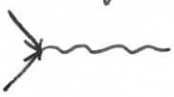
$$\text{RATES} = \left(\Pi_{IN} \frac{1}{2E_k}\right) |M_{fi}|^2 (2\pi)^4 \delta^4\left(\sum \mathcal{P}_{IN} - \sum \mathcal{P}_{OUT}\right) \cdot \Pi_{OUT} \frac{d^4 P_{OUT}}{(2\pi)^4} \delta(\mathcal{P}_{OUT}^2 - m_{OUT}^2)$$

THE RULES for CALCULATING M_{fi} WERE DERIVED AND WERE GIVEN IN THE form of PROPAGATOR



VIRTUAL ELECTRON PROPAGATOR $\dfrac{1}{\not{p} - m_{eL}}$

VIRTUAL PHOTON PROPAGATOR $4\pi e^2 \gamma_\mu \dfrac{1}{q^2} \gamma_\mu$



REAL PHOTON EMISSION, $\not{e}\sqrt{4\pi e^2}$

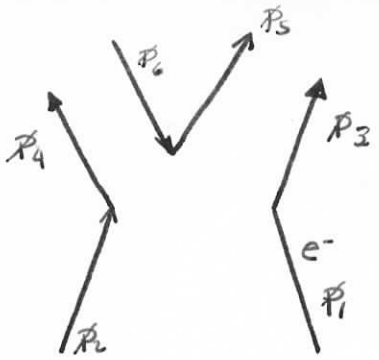
THE RULE for SUMMING OVER ALL MOMENTA REQUIRES INTEGRATION OVER $d^4k/(2\pi)^4$

FIRST ORDER PERTURBATION

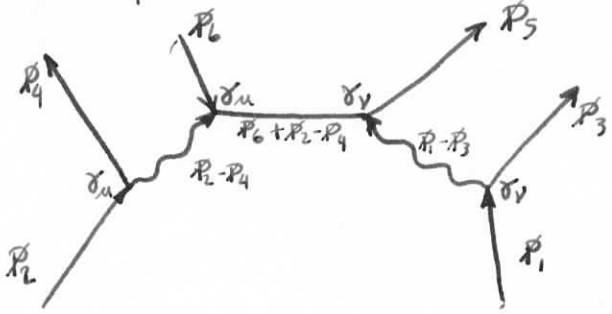
IF THE DESIRED ACCURACY of A PROCESS IS ABOUT 1%, THEN THE FIRST order PERTURBATION THEORY IS ADEQUATE, I.E, $(e^2/\hbar c)^1 = 1/137 \simeq 1\%$ THIS LOWEST order PERTURBATION IMPLIES THE INTERACTION ACTS ONLY ONCE. Therefore OUR DIAGRAMS CAN ONLY be COUPLED by diagrams ON THE RIGHT:

AS AN EXAMPLE of HOW THE RULES WORK LET'S CONSIDER THE POSSIBILITY of PAIR PRODUCTION FROM ELECTRON SCATTERING by A NUCLEAR FIELD. THE NUCLEAR field IS ESSENTIALLY A STATIONARY ELECTRON field AS CONSTRUCTED IN THE LAb. THUS THE PATHS IN THE CLOUD CHAMBER WOULD HAVE A FAST ELECTRON COMING IN WITH A PAIR BEING PRODUCED. THE ESSENTIALLY LINES of THE diagram ARE drawn NEXT.

PAIR PRODUCTION FROM ELECTRON SCATTERING FROM A NUCLEAR field.



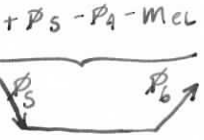
The problem is how to interconnect the kinks in the diagrams because for every kink we need a "wire" to hook onto. Since our interest is only to first order, we only have to worry about single line interconnects. Since there are 3 kinks, there should be 6 diagrams. One such diagram is



In this diagram P_1 and P_2 are the physically incoming particle, while $P_1 = (m_1, 0, 0, 0)$ and P_4, P_3 are the out going electrons. P_5 is the newly created electron while P_6 is it's created pair and represents a physically outgoing positron. Mathematically P_6 is considered an incoming particle. The pair production wing of the diagram is coupled to the collision by the polarized photons of 4 vector propagator γ_μ and γ_ν. We can not write down M for the process. Above each term in the expression for M I have expressed ~~for~~ the part of the diagram which is written out mathematically,

$$M = (\sqrt{4\pi e^2})^4 \underbrace{\frac{(\bar{u}_5 \gamma_\nu \cdot 1 \cdot \gamma_\mu u_6)}{P_6 + P_5 - P_4 - m_{el}}} (\bar{u}_3 \gamma_\nu u_1)(\bar{u}_4 \gamma_\mu u_2) \frac{1}{(P_2 - P_4)^2} \frac{1}{(P_1 - P_3)^2}$$

4 for each photon exchange



Virtual electron propagation

ν photon emission

μ photon emission

Virtual μ photon propagation $\frac{1}{q^2} = \frac{1}{(P_2 - P_4)^2}$

Virtual ν photon propagation $\frac{1}{q^2} = \frac{1}{(P_1 - P_3)^2}$

IN WRITING THIS SCATTERING MATRIX ELEMENT WE MUST DECIDE WHAT TO DO WITH u_6, I.E., WHAT DO WE USE FOR IT. IF WE CALL THE MOMENTUM OF A POSITRON, e^+, $\not{P}$, THEN $\not{P}_6$ EQUALS THE NEGATIVE OF $\not{P}$. SINCE $\not{P} u = -m_{eL} u$, $\not{P}_6 u_6 = +m_{eL} u_6$. CLEARLY WE ARE GETTING INTO DIFFICULTLY DISTINGUISHING BETWEEN PARTICLES WHICH ARE INCOMING AND OUTGOING BOTH PHYSICALLY AND MATHEMATICALLY.

LET'S DEFINE THE INITIAL AND FINAL STATE IN A TIME SENSE SUCH THAT THE INITIAL STATE OCCURS BEFORE THE FINAL STATE. ENTRANCE AND EXIT STATES WILL BE DEFINED IN A MATHEMATICAL SENSE. THE DIRECTION ARROWS WILL REFER TO ENTRANCE AND EXIT STATES. THUS THE INITIAL POSITRON IS IN THE EXIT STATE WHILE THE FINAL ~~POS~~ ELECTRON IS IN THE ENTRANCE STATE. WITH THIS DEFINITION WE CAN EXPRESS M_{fi} AS THE PRODUCT OF M_{EXIT} $M_{ENTRANCE}$.

I SHOULD POINT OUT IN THE PROCESS JUST DESCRIBED IF THE DENOMINATORS ARE SMALL, I.E., $\not{P}_2 \approx \not{P}_4$ AND $\not{P}_1 \approx \not{P}_3$ THEN THE SCATTERING IS STRONG AND QED IS A GOOD PREDICTOR OF THE COUPLING SUCH THAT THE RATE CAN BE ACCURACY, PRECISELY, DETERMINED. THIS IS NOT VERY INTERESTING QED UNLESS YOU LIKE PRECISION. ON THE OTHER HAND WHEN $\not{P}_2 - \not{P}_4$ AND $\not{P}_1 - \not{P}_3$ ARE LARGE, IMPLYING VERY SHORT DISTANCE COUPLING, THEN Q.E.D GETS INTERESTING AND THE ACCURACY STARTS TO FALL APART. ERRORS ON THE ORDER OF 5 TO 10 PERCENT ARE MORE COMMON THAN ERRORS OF A FEW PART IN A MILLION.

SELF-ENERGY OF THE ELECTRON

I'D LIKE TO DISCUSS HIGHER ORDER CORRECTION TO THE FIRST ORDER PERTURBATION THEORY JUST DESCRIBED. IN PARTICULAR I'D LIKE TO DISCUSS THE COMPLICATION OF M_{fi} DUE TO THE DIAGONAL ELEMENTS OF H. IT IS THE NON-DIAGONAL ELEMENTS OF H, THE H_{fi}, ELEMENTS THAT WE HAVE USED TO COMPUTE THE TRANSITION RATES. THE H_{ii} IS NOW TO BE THE INTERACTION OF THE STATE WITH ITSELF. AS A RESULT M_{ii} IS NOT A RATE OF TRANSITION BUT RATHER A SHIFT IN ENERGY. BUT WE MUST BE CAREFUL BECAUSE THE NORMALIZATION IS OFF.

Suppose I want to calculate the energy shift of the electron due to its action back on itself. I want then to compute $\Delta E = u^* H_{ii} u$. We know that $u^* u = 2E$ so let's divide through by $u^* u$ and get

$$\frac{\cancel{2}\,\Delta E}{\cancel{2E}} = \frac{u^* H_{ii} u}{u^* u} \quad u^* u$$

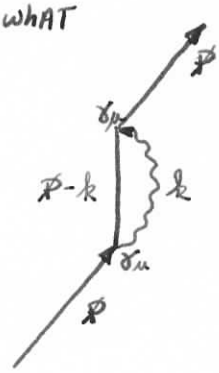
or

$$2E\,\Delta E = M_{ii}$$

Since $E^2 = P^2 + m^2$ and P is the same before and after the perturbation, the change in energy is equivalent to a change in mass. We can interpert the diagonal elements, M_{ii}, as a correction to the mass, actually it is the correction to the mass squared.

The physical interpertation of the self-energy correction is that an electron at rest has a finite probability of making a virtual transition to some new momentum state $\not{p}$ plus simultaneously existing with a photon. Ultimately the photon is reabsorbed back. Since the perturbation ~~is~~ acts twice, the correction is second order. The diagram for the process is to the right; unfortunately when the correction is computed, it comes out infinite. Wait, the one electron could have been at rest and that is an uninvariant calculation so let the electron be moving. Now we correct for the energy difference but we find that the Lorentz transformation does not give the right correction. We're still in trouble. What about the coulomb field? Did we forget it? Well e^2/r_{11} is likewise infinite so now we have 2 infinite energy results. How can we put them together ~~so~~ just so they balance and give a relativistically invariant result? One way, which we have already discussed, is to cut off the frequency at some arbitrary large value. But this approach is not manifestly invariant. This infinite energy problem plagued QED up to 1949, and even today it still brothers us. But we keep the shell game going by a lot of tricks to get around the problem but it is still with us.

LET'S TRY TO CALCULATE THE ENERGY SHIFT AND SEE WHAT HAPPENS, THE INTEGRAL OVER ALL PHOTONS IS



$$\Delta m^2 = M_{ii} = \int \left(\sqrt{4\pi e^2}\right)^2 \frac{1}{k^2} \frac{\bar{u}\,\gamma_\mu \cdot 1 \cdot \gamma_\mu u}{(\not{p}-\not{k})-m+i\epsilon} \frac{d^4k}{(2\pi)^4}$$

IT SHOULD BE EVIDENT HOW M_{ii} IS EXPRESSED. FROM PREVIOUS WORK. SINCE M_{ii} GETS SQUARED, I HAVE LEFT OUT THE i'S WHICH GO ALONG; THEY ARE JUST UNESSENTIAL COMPLICATIONS.

NOW WHAT VALUE OF k DO WE USE, I.E., WHAT MODE IS EXCITED? SINCE IN A CAVITY RESONATOR ANY MODE MAY BE EXCITED, WE MUST INTEGRATE OVER ALL POSSIBLE MOMENTA OF THE VIRTUAL PHOTON. I SHOULD POINT OUT WE ARE WORRIED ONLY ABOUT FIRST ORDER CORRECTIONS HERE EVEN THOUGH A TWO PHOTON EMISSION AND REABSORPTION IS POSSIBLE.

HOW DO WE DO THE INTEGRAL? IT LOOKS FRIGHTENING SINCE THERE ARE MATRICES IN THE DENOMINATOR. LET'S PUT THEM IN THE NUMERATOR AND COLLECT THE IMAGINARY PIECE ϵ IN THE DEFINITION OF THE MASS, I.E.,

$$\bar{u}\,\gamma_\mu \frac{1}{\not{p}-\not{k}-m} \gamma_\mu u = \bar{u}\,\gamma_\mu \frac{(\not{p}-\not{k}+m)}{(\not{p}-\not{k})^2-m^2} \gamma_\mu u$$

WE CAN WORK ON THE RIGHT SIDE SOME BY OBSERVING IF THE γ_μ'S WERE NEXT TO EACH OTHER THINGS WOULD BE COMPUTATIONALLY EASIER. THAT IS WE KNOW $\gamma_\mu\gamma_\mu = \gamma_t\gamma_t - \gamma_x\gamma_x - \gamma_y\gamma_y - \gamma_z\gamma_z = 1-(-1)-(-1)-(-1)$ OR $\gamma_\mu\gamma_\mu = 4$. CLEARLY $\gamma_\mu m \gamma_\mu = m\gamma_\mu\gamma_\mu = 4m$. BUT WHAT ABOUT $\gamma_\mu(\not{p}-\not{k})\gamma_\mu$. WE NEED THE RULE FOR $\gamma_\mu \not{a} \gamma_\mu$ WHICH YOU RECALL IS COMMUTATIVE BY THE 4 VECTOR COMMUTATION RULE.

$$\not{a}\not{b} = -\not{b}\not{a} + 2a\cdot b$$

THUS $\gamma_\mu \not{a} \gamma_\mu = \gamma_\mu(-\gamma_\mu \not{a}) + \gamma_\mu(2a) = -4\not{a} + 2\not{a} = -2\not{a}$

SO THAT

$$\bar{u}\,\gamma_\mu \frac{(\not{p}-\not{k}+m)}{(\not{p}-\not{k})^2-m^2}\gamma_\mu u = \bar{u}\,\frac{[-2\not{p}-2\not{k}+4m]}{(\not{p}-\not{k})^2-m^2}\,u$$

NOW $\not{p}u = mu$ SO WE HAVE $\bar{u}\,\dfrac{(2m-2\not{k})}{(\not{p}-\not{k})^2-m^2}\,u$

SINCE u IS NORMALIZED SUCH THAT $\bar{u}u = 2m$ WE HAVE THAT

$$+2m\, u\bar{u} = +4m^2$$

ALSO WE CAN SIMPLY $\bar{u}(2\not{k})u$ AS $2(2\bar{p}\cdot\bar{k}) = 4\bar{p}\cdot\bar{k}$ SINCE THE PHOTON HAS $m_0 = 0$. THE INTEGRAL OF INTEREST IS NOW

$$M = 4\pi e^2 \int \frac{4m^2 + 4p\cdot k}{(p-k)^2 - m^2} \frac{d^4k}{k^2(2\pi)^4}$$

THE DENOMINATOR CAN BE SIMPLIFIED BY NOTING THAT $\not{p}u = mu$ OR $\not{p}^2 = m^2$ THEREFORE

$$M = 4\pi e^2 \int \frac{4m^2 + 4p\cdot k}{(k^2 - 2p\cdot k + i\epsilon)} \cdot \frac{d^4k}{k^2(2\pi)^4}$$

IF WE ASSUME THE ELECTRON IS AT REST THEN $\not{p} = (m,0,0,0)$ AND ASSUME $\not{k} = (\nu, 0, 0, 0)$ SUCH THAT

$$M = 4\pi e^2 \int \frac{4m^2 + 4m\nu}{\nu^2 - 2m\nu + i\epsilon} \frac{d\nu\, d^3k\, (2\pi)^{-4}}{k^2}$$

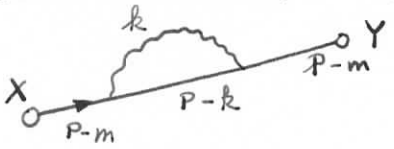
EVALUATING THIS INTEGRAL GIVES AN INFINITE VALUE.

47. MASS AND CHARGE RENORMALIZATION

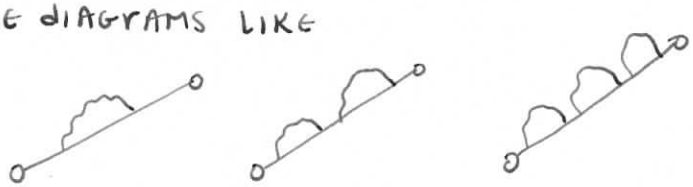
LAST TIME WE DISCUSSED THE INFINITE SELF-ENERGY OF THE ELECTRON BY WORKING OUT THE DIAGONAL ELEMENTS OF THE SCATTERING MATRIX M. THERE IS ANOTHER WAY TO CALCULATE THE SELF ENERGY USING THE DIAGRAM TECHNIQUE. AS AN EXAMPLE CONSIDER AGAIN THE COMPTON EFFECT AS THE BASIC REACTION AND NOW LET'S ASK HOW WE CAN CORRECT THE RESULTS BY GOING TO HIGHER ORDER IN THE PERTURBATION EXPANSION. THE SECOND ORDER CORRECTION LIMITS US TO ONE ADDITIONAL ELECTRON WHICH MUST BE EMITTED AND ABSORBED. THE POSSIBLE MODIFICATIONS TO THE FIRST ORDER PROCESS ARE MANY. IF THE VIRTUAL ELECTRON IS ALMOST FREE, I.E, THE PROPAGATOR IS CLOSE TO THE RIGHT MASS THEN THE INTERMEDIATE STATE CAN LAST A LONG TIME. IN THE INTERMEDIATE STATE MANY VIRTUAL PHOTONS CAN EXIST.

FIRST ORDER COMPTON SCATTERING

THE BASIC SECOND ORDER PROCESS IS THE FOLLOWING

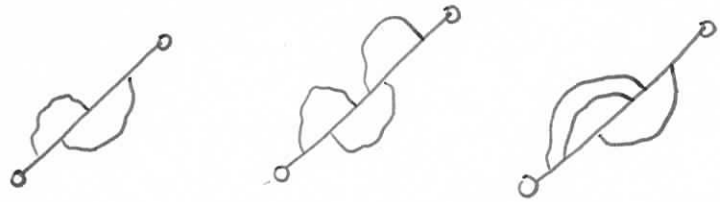


HERE THE ELECTRON STARTS IN SOME STATE X, AND WE REALLY AREN'T CONCERNED WHAT THE INITIAL STATE IS, AND PROPAGATES TO SOME FINAL STATE Y. THUS $X \to Y$ COULD BE ANY ONE OF THE THREE STRAIGHT LINES IN THE COMPTON SCATTERING DIAGRAM. IF THE ELECTRON IS NEARLY FREE, I.E., $P^2 = m^2$, THEN THERE MAY BE MORE PHOTONS EXISTING. IF WE CONCERN OURSELVES, FIRST, WITH ONLY ONE VIRTUAL PHOTON EXISTING AT A TIME, WE HAVE POSSIBLE DIAGRAMS LIKE



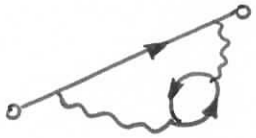
ETC.

LESS LIKELY, BUT POSSIBLY OCCURRING, TWO AND MORE VIRTUAL PHOTONS CAN EXIST SIMULTANEOUSLY. THEN WE GET DIAGRAMS LIKE,



ETC

EVEN WORSE THERE IS STILL ANOTHER POSSIBLE VIRTUAL STATE AND THAT IS WHERE THE VIRTUAL PHOTON, of sufficient energy, CREATES A VIRTUAL PAIR, which ANNIHILATE AND ULTIMATELY RETURN TO ~~TO~~ THE ELECTRON,



THIS LAST DIAGRAM DESCRIBES WHAT IS CALLED VACUUM POLARIZATION. I'LL COME BACK TO THIS PROCESS A LITTLE LATER.

RIGHT NOW I WANT TO CONSIDER RADIATIVE CORRECTIONS which ARE ONLY ACCURATE TO first order IN $e^2/\hbar c$, i.e. $\approx 1\%$. I HAVE THUS CHOSEN TO EXCLUDE ALL THE VIRTUAL STATES WHERE MORE THAN ONE PHOTON EXISTS. NONETHELESS I STILL HAVE A SERIES of DIAGRAMS of THE TYPE ON PAGE 45 which MUST BE DEALT WITH. SINCE THAT IS GOING TO CREATE ENOUGH PROBLEM IN ITSELF I CHOOSE TO MAKE IT AS EASY AS POSSIBLE BY SIMPLIFYING THE ANALYSIS.

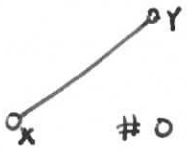
TO BEGIN THE ANALYSIS THE ELECTRON IS ASSUMED TO be NEARLY FREE, i.e. $p^2 \neq m^2$ PRECISELY, BUT INSTEAD

$$p^2 = m^2(1+\epsilon)^2$$

where $\epsilon = \frac{\hbar}{mT}$ from THE UNCERTAINTY PRINCIPLE. T IS THE INTERVAL BETWEEN SCATTERINGS. SINCE T IS LARGE, ϵ IS SMALL.

NOW CONSIDER THE ZERO order DIAGRAM for DETERMINING THE AMPLITUDE TO PROPAGATE from X TO Y, i.e.,

$$\text{AMP}(0) = Y \frac{1}{\not{p} - m} X$$



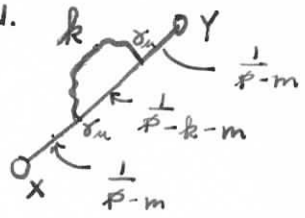
#0

REWRITING

$$\text{AMP}(0) = Y \frac{\not{p}+m}{\not{p}^2 - m^2} X = Y \frac{(P+m)X}{2m^2\epsilon}$$

TO order ϵ.

THE NEXT ORDER DIAGRAM IS INTERESTING BECAUSE IT ALLOWS FOR THE EMISSION AND REABSORPTION OF A VIRTUAL PHOTON. BECAUSE WE MUST SUM OVER ALL POSSIBLE MOMENTA OF THE VIRTUAL PHOTON THE AMPLITUDE IS



#1

$$AMP(1) = \int Y \frac{1}{\not p - m} \gamma_\mu \frac{1}{\not p - \not k - m} \gamma_\mu \frac{1}{\not p - m} X \frac{4\pi e^2}{k^2} \frac{d^4 k}{(2\pi)^4}$$

IF WE DEFINE THE VARIABLE $B(\not p)$ TO BE

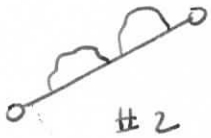
$$B(\not p) = \int \gamma_\mu \frac{1}{\not p - \not k - m} \gamma_\mu \frac{d^4 k}{k^2} \frac{(4\pi e^2)}{(2\pi)^4}$$

THEN WE CAN WRITE

$$AMP(1) = Y \frac{1}{\not p - m} B(\not p) \frac{1}{\not p - m} X$$

WE CAN NOW WRITE THE NEXT ORDER DIAGRAM AS

$$AMP(2) = Y \frac{1}{\not p - m} B(P) \frac{1}{\not p - m} B(P) \frac{1}{\not p - m} X$$



#2

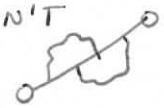
NOW WE HAVE A SERIES OF THESE AMPLITUDES WHICH MUST BE ADDED TO INCLUDE ALL POSSIBLE ~~ER~~ CORRECTIONS IN e^2. THUS TO SUM THE SERIES WE MAKE USE OF THE SENSATIONAL FACT THAT

$$\frac{1}{A+B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} - \cdots$$

WHERE A AND B ARE NONE COMMUTING OPERATORS. THE SUM OF ALL THESE "BUBBLE" AS SOME THINK OF THEM IS JUST

$$Y \frac{1}{\not p - m - B(\not p)} X = \sum_{\text{All } e^2 \text{ terms}} AMP$$

AGAIN OUR APPROXIMATION IS TO ORDER e^2 AND WE HAVEN'T INCLUDED THE DIAGRAMS TO ORDER e^4 WHICH ARE LIKE THE ONE ON THE RIGHT.



IT WOULD BE NICE IF $B(\not p)$ WERE JUST A NUMBER BECAUSE IT WOULD THEN CORRESPOND TO A CORRECTION TO PROPAGATOR, I.E., THE MASS IS CHANGED BY A SMALL AMOUNT. PHYSICALLY, THIS MEANS THAT THE EXPERIMENTALLY MEASURED MASS REALLY INCLUDES THE EFFECT OF THE VIRTUAL PROCESSES JUST MENTIONED.

Thus B represents the correction to the theoretical mass due to the self energy of the particle. If we assume that $B(\not{p})$ has the mathematical form of

$$B(\not{p}) = a(\not{p})\,\not{p} + b(\not{p})$$

Then the amplitude to get from X to Y is

$$Y \frac{1}{(1-a)\not{p} - (m+b)} X = Y \frac{[(1-A)\not{p} + (m+b)]}{(1-A^2)\not{p}^2 - (m+b)^2} X$$

or rewritten differently as

$$\frac{Y}{\sqrt{1-a}} \frac{1}{(\not{p} - m')} \frac{X}{\sqrt{1-a}}$$

where $$m' = \frac{m+b}{1-a} \approx m + b + ma$$

and $b + ma$ is the correction to the mass.

The extra term $\sqrt{1-a}$ under Y and X are curious terms because they are interpreted to mean that each time the photon interacts it has its charge slightly changed so the coupling strength is modified. The change in mass and charge of the electron are referred to as mass and charge renormalization. The idea then is with a somewhat different mass and somewhat different coupling the mass of the electron is really known because you can work backward from the experimentally determinable mass, m'.

Well, not quite! You see $B(\not{p})$ turns out to be an infinite integral. You can readily see that by observing that the integral is of the form $\int \frac{k^3 dk}{k^4} \propto \int \frac{dk}{k} \propto \ln k$ thus the integral is logarithmically divergent. What do we do? Certainly if $B = \infty$ the whole QED theory collapses. But since it is a good theory to explain the phenomena we see we would like to use it. We have to fix up the theory by modifying the rules of photon propagation.

LAMBDA THEORY for MASS RENORMALIZATION

SUPPOSE THAT THE PROPAGATOR for A VIRTUAL PHOTON IS $1/k^2$ for SMALL VALUES of k (I.E. CORRESPONDING TO WAVELENGTHS ON THE order of METERS, CENTIMETERS, MILLIMETERS & DONE TO WAVELENGTHS of 10^{-13} CM AND ENERGIES ON THE order of BEV'S). BEYOND THIS LIMIT WE WILL ASSUME THE PROPAGATOR FALLS off MORE RAPIDLY,

$$\frac{1}{k^2} \longrightarrow \frac{1}{k^2}\left[\frac{-\Lambda^2}{k^2-\Lambda^2}\right]$$

THIS CONVERGENCE FACTOR, CALL IT $C(k) = -\Lambda^2/k^2-\Lambda^2$, IS CHOSEN TO BE RELATIVISTICALLY INVARIANT. FOR SMALL VALUES of k^2 $C(k)$ IS ALMOST 1; for LARGE VALUES of k^2 $C(k)$ GOES AS $1/k^2$ SO THE WHOLE PROPAGATOR GOES AS $1/k^4$. THE Λ FACTOR IS CHOSEN TO MAKE ALL THIS APPROXIMATION STUFF WORK. WE SHOULD NOW TRY TO EVALUATE $B(p)$ USING THIS NEW PROPAGATOR.

Therefore WE WANT TO EVALUATE

$$B^{\Lambda} = 4\pi e^2 \int \gamma_\mu \frac{(\not p - \not k + m)}{(k^2 - 2p\cdot k + p^2 - m^2)} \gamma_\mu \frac{d^4k}{k^2}\left[\frac{-\Lambda^2}{k^2-\Lambda^2}\right]$$

USING THE RELATIONSHIPS THAT $\gamma_\mu\gamma_\mu = 4$ AND $\gamma_\mu \not a \gamma_\mu = -2\not a$ TO rid THE EQUATION of γ_μ'S AND ALSO NOTING TO REPLACE p^2 by m^2 SINCE WE WANT THE MASS CORRECTION THEN

$$B^{\Lambda} = 4\pi e^2 \int \frac{-2(\not p - \not k) + 4m}{-2p\cdot k + k^2} \frac{d^4k}{k^2}\left[\frac{-\Lambda^2}{k^2-\Lambda^2}\right]$$

THIS INTEGRAL CAN BE EVALUATED TO BE

$$B^{\Lambda} = b + ma = \frac{3}{2} m \frac{e^2}{\pi}\left[3\ln\left(\frac{\Lambda}{m}\right) + \frac{3}{4}\right] = \delta m$$

THE INTEGRAL NOW CONVERGES BUT IT DEPENDS ON Λ. AS Λ GETS bigger AND bigger, CORRESPONDING TO higher AND ~~freq~~ higher frequencies THE MASS CORRECTION GETS bigger LIKEWISE. THAT IS

$$m_{\text{EXPER.}} = m_{\text{THEORETICAL}}\left[1 + \frac{3e^2}{2\pi}\ln\left(\frac{\Lambda}{m}\right)\right]$$

THE PROBLEM IS YOU CAN'T TAKE THE LIMIT AS Λ GOES TO ∞ BECAUSE THE MASS CORRECTION BLOWS UP. If Λ WERE REAL AND had A PHYSICAL BASIS, WE COULD UNDERSTAND how TO USE IT. BUT WE WORK BLIND.

EVEN IF Λ WERE VERY LARGE SAY ~ 100 BeV THE MASS CORRECTION IS ON THE ORDER OF 10%. IT SEEMS LIKE THE MASS CORRECTION IS INSENSITIVE TO THE HIGH FREQUENCY CUT OFF.

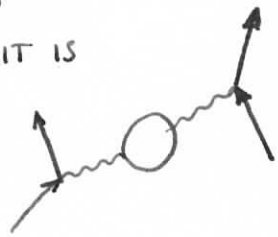
IN DEALING WITH HIGHER ORDER CORRECTIONS TO PROCESSES INVOLVING ELECTRON-PHOTON INTERACTION, LOGARITHMIC DIVERGENCES ARE THE WORSE KIND ENCOUNTERED, SO FAR THAT IS. HOWEVER, IF THE ANSWER TO EVERY PHYSICAL QUESTION IS EXPRESSED IN TERMS OF THE EXPERIMENTAL MASS, I.E., $m_{EXP.} = m_{THE.} + \delta m$ THEN EXPANDED TO FIRST ORDER IN δm, THE COEFFICIENT OF $\ln \Lambda/m$ BECOMES EQUAL TO ZERO AND THE RESULT IS INSENSITIVE TO THE CUTOFF FREQUENCY; THE LIMIT AS $\Lambda \to \infty$ CAN BE TAKEN AND THE RESULT IS FINITE.

WE THUS HAVE ANOTHER RULE FOR QED. WHEN SOLVING PROBLEMS OF THIS TYPE: (1) PUT IN AN ARBITRARY CUTOFF FACTOR $C(k^2) = -\Lambda^2/k^2-\Lambda^2$ AND OBSERVE THE SENSITIVITY TO Λ; (2) EXPRESS EVERYTHING IN TERMS OF THE EXPERIMENTAL MASS; (3) TAKE THE LIMIT AS $\Lambda \to \infty$ AND GET THE CORRECT ANSWER.

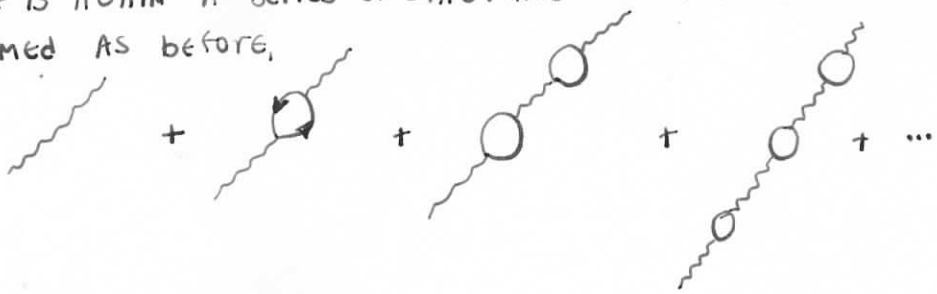
WHILE THIS PROCEDURE DOES SEEM TO AGREE QUITE ACCURATELY WITH EXPERIMENTS, IT IS NOT CLEAR THAT IT IS A VALID ONE. THE ANSWER STILL COULD BE DIVERGENT AS WE CONSIDER HIGHER ORDER CORRECTIONS, I.E., WHERE 2 OR MORE VIRTUAL PHOTONS EXIST SIMULTANEOUSLY. THE THEORY IS VALID TO CORRECTIONS IN ORDER e^2 BUT THE SUM OF THE SERIES GOES AS $n!\left(\frac{e^2}{\hbar c}\right)^n$. WHEN $n \approx 137$ THEN $n!$ STARTS TO DOMINATE THE SERIES AND IT MAY DIVERGE.

CHARGE RENORMALIZATION, VACUUM POLARIZATION

THERE IS ANOTHER INFINITY WHICH WE SHOULD DEAL WITH AND IT IS ASSOCIATED WITH THE PROPABILITY OF CREATING A VIRTUAL PAIR:



THERE IS AGAIN A SERIES OF DIAGRAMS WHICH MUST BE SUMMED AS BEFORE,



USING THE SAME TRICK AS before THE SERIES CAN be SUMMED TO PRODUCE, IN EFFECT, A MODIFIED CHARGE e' COUPLING THE PHOTON TO THE ELECTRON. e' IS THE EXPERIMENTAL CHARGE MEASURED AND IT differs FROM THE THEORETICAL VALUE AS

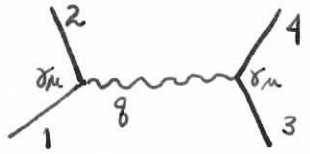
$$e_{exp} = \frac{e_{THEORETICAL}}{\sqrt{1-Y}} = e'$$

WHERE Y APPROACHES A CONSTANT. e' IS CALLED THE RENORMALIZED CHARGE. THE INTERPERTATION of THE CHARGE Shift IS THAT IT IS A SMALL CORRECTION TO THE COULOMB POTENTIAL. HOWEVER, A LOGARITHMIC DIVERGENCE STILL APPEARS SO WE EXPRESS THE ANSWER IN TERMS of e', TAKE THE LIMIT IN THE FIRST few orders of THE PERTURBATION. IT IS AMAZINGLY THAT WHILE ALL THESE TRICKS SEEM TO WORK AND GIVE THE RIGHT ANSWER WE DON'T UNDERSTAND WHY - THE PROBLEM IS TOO hard TOO SOLVE.

MORE ON CHARGE RENORMALIZATION

LAST TIME WE TALKED BRIEFLY ABOUT CHARGE RENORMALIZATION AND I'd LIKE TO DISCUSS IT NOW IN MORE DETAIL.

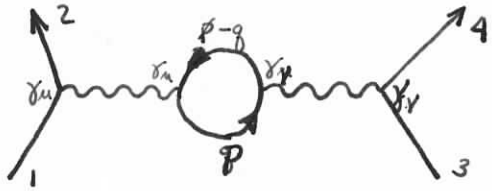
CONSIDER THE PROBLEM OF TWO ELECTRONS INTERACTING THROUGH THE COULOMB FIELD AND SCATTERING AT SMALL ANGLES



THE ELECTRONS ARE ~~COULOMB~~ COUPLED BY THE VIRTUAL PHOTON q WITH AN AMPLITUDE GIVEN BY:

$$4\pi e^2 (\bar{u}_4 \gamma_\mu u_3) \frac{1}{q^2} (\bar{u}_2 \gamma_\mu u_1)$$

THERE IS ANOTHER DIAGRAM, ONE OF MANY, FOR THE ABOVE INTERACTION WHEREIN A VIRTUAL PAIR IS PRODUCED, I.E.,



ON PAGE 50 ARE SOME OF THE OTHER POSSIBLE DIAGRAMS. TO COMPUTE THE CONTRIBUTION OF THIS DIAGRAM TO THE SCATTERING RATE WE HAVE TO FOLLOW THE ELECTRON AROUND THE CLOSED LOOP. WE START WITH $\sqrt{4\pi}\, e\, \gamma_\mu$ WHICH IS THE AMPLITUDE TO ANNIHILATE THE μ PHOTON; $1/\not{p}-m$ IS THE AMPLITUDE FOR THE ELECTRON TO PROPAGATE BETWEEN THE 2 PHOTON VERTICES; $\sqrt{4\pi}\, e\, \gamma_\nu$ IS THE AMPLITUDE TO EMIT A PHOTON OF POLARIZATION ν; AND $1/(\not{p}-\not{q}-m)$ IS THE AMPLITUDE FOR THE ELECTRON TO PROPAGATE BACK TO ITS STARTING PLACE. THE TOTAL AMPLITUDE IS THEN

$$4\pi e^2 (\bar{u}_2 \gamma_\mu u_1) \frac{1}{q^2} \left[\int \mathrm{Spur} \left(\gamma_\mu \frac{1}{\not{p}-m} \gamma_\nu \frac{1}{\not{p}-\not{q}-m} \right) \frac{d^4p}{(2\pi)^4} \right] \frac{1}{q^2} (\bar{u}_4 \gamma_\mu u_3)$$

THE INTEGRAL SUMS OVER ALL POSSIBLE MOMENTA P AND INITIAL STATES. A SIDE REMARK MAY BE IN ORDER TO EXPLAIN THIS POINT IN MORE DETAIL. THE INTEGRAL CONTAINS A SPUR WHICH MEANS WE SUM OVER ALL DIRECTIONS OF THE ELECTRON SPIN.

LET'S TAKE A LOOP WHERE WE START IN SOME STATE u_1 AND END UP IN THAT STATE AFTER CIRCULATING AROUND THE LOOP.
WE WANT TO EVALUATE

$$\sum_{\text{SPIN } u_1} (\bar{u}_1 M u_1)$$

WHERE M IS THE MATRIX ELEMENT TO GET AROUND THE LOOP,

$$M = \gamma_\mu \frac{1}{\not{p}-\not{q}-m} \gamma_\nu \frac{1}{\not{p}-m}$$

ORDINARILY WHEN A MATRIX IS PROJECTED INTO $\bar{u}u$ AND SUMMED OVER ALL VALUES OF u, THE RESULT IS THE TRACE OR SPUR OF THE MATRIX, I.E

$$\sum_{\text{ALL } u} (\bar{u} M u) = \text{Spur } M = \text{TRACE } M$$

WHERE TRACE M EQUALS THE SUM OF THE DIAGONAL ELEMENTS.
NOW SUPPOSE WE WANT THE SUM OVER 2 DIFFERENT SPINS. WHY NOT SUM OVER ALL u'S AND SOMEHOW RESTRICT THE RESULT TO THE FOUR SOLUTIONS TO TWO. SINCE u MUST SATISFY $\not{p}u = mu$ AN ELECTRON AT REST MUST SATISFY $m\gamma_t u = mu$ WHICH HAS THE SOLUTIONS

$$u = \underbrace{\underset{\text{SPIN}\uparrow}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}, \underset{\text{SPIN}\downarrow}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}}_{\text{ELECTRON STATES}}, \underbrace{\underset{\text{SPIN}\uparrow}{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}, \underset{\text{SPIN}\downarrow}{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}}_{\text{POSITRON STATES}}$$

WE WANT THE SOLUTIONS THAT SATISFY $\not{p}u = mu$ AND DON'T WANT THE SOLUTIONS THAT SATISFY $-\not{p}v = mv$. SO LET'S REWRITE THE Σ OVER SPINS OF u AS

$$\sum_{\text{SPIN } u} (\bar{u} M u) = \sum_{\text{ALL 4 } u\text{'S}} \frac{1}{2m} (\bar{u} M (\not{p}+m) u) = \text{Tr}\left[\frac{M(\not{p}+m)}{2m}\right]$$

THUS WHEN WE SUM OVER THE TWO NEGATIVE STATES WE GET ZERO.
SOME USEFUL PROPERTIES OF A SPUR ARE

$$\begin{aligned}
Sp\,[1] &= 4 \\
Sp\,[\gamma_\mu] &= 0 \\
Sp\,[\gamma_i \gamma_j] &= 0 \\
Sp\,[\gamma_5] &= 0 \\
Sp\,[\gamma_i \gamma_i] &= -4
\end{aligned}$$

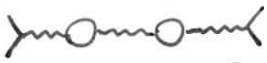
RETURNING TO THE INTEGRAL LET'S SEE WHAT HAPPENS WHEN WE TRY TO WORK IT OUT. CONSIDER THE INTEGRAL HAS A FORM OF THE FOLLOWING

$$\int sp\left[\gamma_\mu \frac{1}{\not p - m}\gamma_\nu \frac{1}{\not p - \not q - m}\right)\frac{d^4p}{(2\pi)^4} = a\,\delta_{\mu\nu} + bq^2\delta_{\mu\nu} + cq_\mu q_\nu + \cdots$$

WE SUPPOSE THAT q IS SMALL ENOUGH THAT WE CAN MAKE A POWER EXPANSION IN q OF THE TERM $1/(\not p - m - \not q)$. WHEN WE DO THAT AND EVALUATE THE INTEGRAL WE GET AN AMPLITUDE LIKE

$$-(4\pi e^2)^2 (u_2\gamma_\mu u_1)\frac{1}{q^2}\, a\, \frac{1}{q^2}(u_4\gamma_\mu u_3)$$

BUT AS q GETS SMALL THIS TERM MAY DOMINATE THE DIAGRAM WHICH IS TROUBLING SO WE MUST CONSIDER THE NEXT ORDER DIAGRAM,

 WHICH, WHEN EVALUATED PRODUCES

$$+(4\pi e^2)^3 (\bar u_2\gamma_\mu u_1)\frac{1}{q^2}\, a\, \frac{1}{q^2}\, a\, \frac{1}{q^2}(\bar u_4\gamma_\mu u_3)$$

NOW YOU KEEP DOING THIS AND FINALLY ADD UP ALL THE TERMS,

$$4\pi e^2(\bar u_4\gamma_\mu u_3)\left\{\frac{1}{q^2} - \frac{4\pi e^2 a}{q^4} + \frac{(4\pi e^2 a)^2}{q^6} + \cdots\right\}(\bar u_2\gamma_\mu u_1) = 4\pi e^2(u_4\gamma_\mu u_3)\frac{1}{q^2 - 4\pi e^2 a}(\bar u_2\gamma_\mu u_1)$$

THIS IS INTERESTING BECAUSE IT SAYS THE LAW FOR PHOTON PROPAGATION IS WRONG AND REALLY THE PHOTON ~~MA~~ HAS A REST MASS $4\pi e^2 a$. WELL EXPERIMENTALLY NO EVIDENCE SUPPORTS THAT CONCLUSION SO a MUST BE ZERO. IF $a \neq 0$ THE Q.E.D WOULD NOT DESCRIBE NATURE RIGHT.

TERMS INVOLVING q_μ ARE ZERO SINCE $u_2\gamma_\mu u_1$ = CURRENT = $j_\mu q_\mu$ AND THE DIVERGENCE OF THE CURRENT IS ZERO. SO WE WON'T WORRY ABOUT $q_\mu q_\nu$ TERMS. THAT LEAVES ONLY THE $bq^2\delta_{\mu\nu}$ TERM TO WORRY ABOUT.

BY A SIMILAR SERIES EXPANSION AS ABOVE FOR THE a TERM WE GET THAT THE LOOP AMPLITUDE IS GIVEN BY

$$4\pi e^2(\bar u_2\gamma_\mu u_1)\frac{1}{q^2 - q^2 b(4\pi e^2)}(\bar u_4\gamma_\mu \gamma_3)$$

I CAN REWRITE THIS AS

$$4\pi e^2\, j_\mu \frac{1}{q^2}\frac{1}{(1+4\pi e^2 b)}\, j_\mu$$

b HERE IS DEFINED AS A NEGATIVE QUANTITY. SO FINALLY WE CAN

WRITE $4\pi (e')^2 j_\mu \frac{1}{q^2} j_\mu$

WHERE THE NEW CHARGE $e' = \frac{e^2}{1 + 4\pi e^2 b}$

THE COUPLING LAW THUS CHANGES SO THE CURRENTS INTERACT ACCORDING TO MAXWELL'S COULOMB LAW BUT WITH A MODIFIED CHARGE. WHENEVER WE QUOTE "e" WE REALLY MEAN e' BECAUSE WE CAN'T SOLVE THE INTEGRAL for b. IN ESSENCE THE CHARGE IS RENORMALIZED AND THE LAW SAYS TO PLEASE EXPRESS THE RESULTS IN TERMS of e' BECAUSE e CAN'T be DETERMINED. THE INTEGRAL for b CONTAINS A LOGARITHMIC DIVERGENT TERM WHICH REQUIRES AN ARBITRARY CUTOFF TO BE MADE IN ORDER TO ~~EVA~~ OBTAIN A FINITE VALUE for b. THIS IS ANOTHE DILEMMA of Q.E.D.

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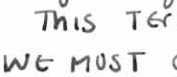
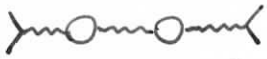
BUT AS q GETS SMALL THIS TERM MAY DOMINATE THE  DIAGRAM WHICH IS TROUBLING SO WE MUST CONSIDER THE NEXT ORDER DIAGRAM,

 WHICH, WHEN EVALUATED PRODUCES

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48. AN INTRODUCTION TO WEAK INTERACTION

THERE IS A CLASS OF INTERACTIONS WHICH DOES NOT OBEY THE KNOWN LAWS OF ELECTRODYNAMICS, GRAVITY, OR STRONG INTERACTIONS. THE INTERACTION WAS OBSERVED IN NUCLEI THAT UNDERGO A DECAY SUCH AS,

$$N \longrightarrow P + e^- + \bar{\nu} \quad \text{, ELECTRON EMISSION}$$

THIS IS CALLED β-DECAY. HERE A NEUTRON DECAYS INTO A PROTON BY EMITTING AN ELECTRON AND ANTINEUTRINO. THE NEUTRINO IS A MASSLESS PARTICLE WITH SPIN 1/2. OTHER TYPES OF β-DECAY ARE

$$P \rightarrow N + e^+ + \nu \quad \text{, POSITRON EMISSION}$$

$$P + e^- \rightarrow N + \nu \quad \text{, ELECTRON (K) CAPTURE}$$

$$P + \mu^- \rightarrow N + \nu_\mu \quad \text{MU-MESON CAPTURE}$$

THE CHARACTERISTICS OF THESE INTERACTIONS ARE THE FOLLOWING:

(1). REACTION TIMES ON THE ORDER OF 10^{-10} SEC

(2). RIGHT AND LEFT SYMMETRY IS ALWAYS VIOLATED

(3) STRANGENESS IS VIOLATED

IN ORDER TO EXPLAIN THESE INTERACTIONS FERMI (~1934) PROPOSED THAT THE OBEY THE SAME TYPE OF RULE AS ELECTRODYNAMICS WHICH MEANT THE TRANSITION RATE IS PROPORTIONAL TO $|M_{if}|^2$ WHERE M_{if} IS SOME MATRIX ELEMENT BETWEEN THE INITIAL AND FINAL STATE.

UNFINISHED TRANSCRIPTION FOLLOWS. THESE NOTES MAY NOT BE IN PROPER ORDER. THEY ARE INCLUDED HERE FOR TWO REASONS: 1) MAYBE THEY ARE IMPORTANT AND USEFUL TO SOMEONE AND 2) THESE ARE "RAW" LECTURE NOTES. SOME HAVE EXPRESS CURIOSITY OVER HOW I CAPTURED FEYNMAN WITHOUT A/V AID. TO GET THE TRANSCIPTION CLOSE TO REAL TIME I HAD TO RUSH HOME AND DO THE TRANSCIPTION THAT EVENING. (JJN 11/2012)

Weak Interaction

Weak interaction set of physics not explained by gravity - EM and Strong Interactions laws of physics.

Neutron decay in proton e^- and antineutrino

$$N \rightarrow P + e^- + \bar{\nu}$$ spin moment to left — electron emission

interact in ~~two~~ no known ways. ~~Also~~ ~~and~~ proton interacts as proton into neutron — spin moment to right

$$P \rightarrow N + e^+ + \nu$$ ~~electro~~ positron emission

also K-capture of e^- from Bohr orbit

$$P + e^- \rightarrow N + \nu \qquad \text{K-capture}$$

These called Beta Decay.

Not only examples of weak interaction

$$P + \mu^- \rightarrow N + \nu_\mu \qquad \text{mu-meson capture}$$

high energy collision

$$\mu^- \rightarrow e^- + \bar{\nu} + \nu_\mu$$

$$\pi \rightarrow \mu^+ + \nu_\mu$$

$$K^+ \rightarrow \mu^+ + \nu_\mu$$

All matrix elements give reaction times of 10^{-10} sec

ways to categorize these interactions

1. If involve ν is weak interaction
2. If involve strangeness and violate S-rules weak int.
3. General sig rate ~ constant
4. R & Left symmetry always violated

Not explained by gravity & electrodynamics

E-D satisfies gravity. Maybe ED is higher order correction to weak electron

2 kinds of neutrinos -

$\nu + (N) \rightarrow (P) + e^-$ reverse of other

reaction not make e^- makes μ's

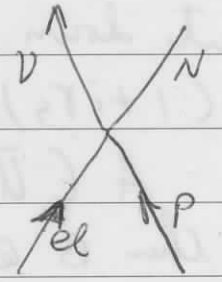
How to represent $N \rightarrow P + e^- + \bar{\nu}$ Fermi suggested from field theory that

$$\text{Rate} = \text{phase Space } |M_{if}|^2$$

said constant not depend on particle energy. He used perturbation idea but need way compute matrix element. He made simple guess, brilliant for the day $P + e^- \rightarrow N + \nu$

for photons $U_f \gamma_\mu U_i e_\mu$

guessed



$A(\bar{U}_\nu U_{el})(\bar{U}_N U_P)$ - rel. invariant matrix

or $B(\bar{U}_\nu \gamma_\mu U_{el})(\bar{U}_N \gamma_\mu U_P)$ + fermi guessed this only one

vector vector $C(U_\nu \gamma_\mu \gamma_\nu U_{ee})(U_N \gamma_\mu \gamma_\nu U_P)$ +

also $D(U_\nu \gamma_5 \gamma_\mu U_{ee})(U_N \gamma_5 \gamma_\mu U_P)$ +

and $E(U_\nu \gamma_5 U_{ee})(U_N \gamma_5 U_P)$

so add all. But one parity violated get more like

$+ B'(U_\nu \gamma_\mu \gamma_5 U_{ee})(U_N \gamma_\mu U_P')$

so get 10 constants. Hopeless now.

Landau suggested neutrino only spin one way so put factor in front ν wave functions to stop spin wrong way

so γ must satisfy $\gamma_\nu U_\nu = 0$

$$P_\nu = (E_\nu, 0, 0)$$

$$(\gamma_t - \gamma_z)\, u_\nu = 0 \rightarrow \gamma_t u_\nu = \gamma_z u_\nu$$

$$\gamma_x \gamma_y u_z = -u_z \qquad u = \gamma_t \gamma_z u$$

and $$-u = \gamma_x \gamma_y \gamma_z \gamma_t u$$

$$= i\gamma_5 u$$

$$(1 - i\gamma_5) u = 0$$

replace $u_\nu \rightarrow u_\nu' = (1 + i\gamma_5) u_\nu$ to get it spin right way.

Feynman propose all $u \rightarrow (1 + i\gamma_5) u$

ben changes sign

$$[\bar{u}_\nu (1 - i\gamma_5) \gamma_\mu (1 + i\gamma_5) u_{el}][\bar{u}_N (1 - i\gamma_5) \gamma_\mu (1 + i\gamma_5) u_p]$$

cuts down 1 possibility

$$(1 + i\gamma_5)(1 - i\gamma_5) = 0$$

$$A(\bar{u}_\nu u_{el})(\bar{u}_N u_p) = A(\bar{u}_\nu (1 - i\gamma_5)(1 + i\gamma_5) u_{el}) = 0$$

then B term

$$(1 + i\gamma_5)(1 + i\gamma_5) = 2(1 + i\gamma_5)$$

not in element for B decay

$$B[\bar{u}_\nu \gamma_\mu \frac{(1 + i\gamma_5)}{2} u_{el}][\bar{u}_N \gamma_\mu \frac{(1 + i\gamma_5)}{2} u_p]$$

get rid A - C and E terms but D term

$$\gamma_5 (1 + i\gamma_5) = (\gamma_5 - i) = -i(1 + i\gamma_5)$$

give for same B same as fermi guessed

$$B = \frac{G}{\sqrt{2}}$$

when theory invented several exp disagreed Feynman said theory so beautiful predicted experiment were wrong and they were

Beta decay combination B & D terms instead A & C

Do not need review on fundamentally important problem
I am solved. End summer supp such mess I understood

Examples of how works

Consider Neutron free - $N \to p + e^- + \bar{\nu}$

how compute Matrix element so get rate

$$M = [\, U_{el}\gamma(1+i\gamma_5)U_\nu)(\bar{U}_p\, \gamma_\mu(1+i\gamma_5)U_N)\,]$$

particles ~ at rest relative to energy of proton

big matrix element when large meet large:

$$\bar{U}_p\, \gamma_\mu U_N \sim U_p\, \gamma_t U_N = U_p^* \gamma_t \gamma_t U_N$$

NR theory $\sim U_p^* 1 U_N$

$$M = (U_{el}\, \gamma_t(1+i\gamma_5)U_\nu)(U_p^* U_N) +$$

fermi term [proton 1 not change spin. no change ang. mom]

want $\gamma_\mu \gamma_5 U_N$

$\mu = t$ $\quad \gamma_t \gamma_t \gamma_x \gamma_y \gamma_z$ $\quad$ need 2 γ's to get big 3 small

$\mu = x$ $\quad \gamma_x \gamma_t \gamma_x \gamma_y \gamma_z = \gamma_t \gamma_y \gamma_z = \gamma_t \sigma_x$

y

z

need heavy el & ν relativistic cause move fast

so must add to above $+ (U_{el}\, \gamma_x(1+i\gamma_5)U_\nu)(U_p^* \gamma_x U_N)$

$+ \quad y \quad y$

$+ \quad z \quad z$

Gamow teller

Selection Rules of Beta decay

fermi ~~tde~~ term allow for $J=2$ to $J=2$
or $J=0$ to $J=0$

~~rate depend~~ v^2

The Gamow-Teller terms allow for spin flip
$\Delta J = 1 \qquad J=3 \to 2$

Let's do example nucleus goes from $J=0$ to $J=0$
simplest case Matrix element would be

$$\mathcal{M} = (u_e \gamma_\mu (1+i\gamma_5) u_\nu) \cdot N$$ = nuclear matrix element

$$N = (\psi_N^* \gamma_5 \psi_N)$$ = regulated from nuclear physics
can't really compute it.

$$\text{Rate} = 2\pi\, \delta(P_\nu^2)\, 2\pi\, \delta(P_{el}^2 - m_{el}^2)\, \delta(\epsilon_\nu + \epsilon_{el} - W)\, N^2 G^2$$

disregard recoil of nucleus cause big momentum sink
W = total energy

$$\frac{d^4 P_\nu}{(2\pi)^4} \frac{d^4 P_{el}}{(2\pi)^4} \times \left[(u_{el} \gamma_\mu (1+i\gamma_5) u_\nu) \right]^2$$

$$\text{Rate} = (\text{const})\, \delta(\epsilon_\nu^2 - P_\nu^2)\, \delta(\epsilon_{el} - P_{el}^2 - m^2)\, d\epsilon_\nu\, d^3\bar{P}_\nu$$

$\epsilon_\nu = |P_\nu|$ from integral $\qquad \epsilon_{el} = \sqrt{m^2 + P_{el}^2}$

$$= (\text{const}) \frac{d^3 P_\nu}{(2\epsilon_\nu)} \frac{d^3 P_{el}}{2\epsilon_{el}} N^2\, \delta\left(P_\nu + \sqrt{m^2 + P_{el}^2} - W\right) [\;]^2$$

evaluate

$[\ ]^2$ - sum over both spins of neutrino & spin electron

$$\sum_{\substack{\text{spin}\\ el}} \sum_{\substack{\nu\\ \text{spin}}} [\ ]^2 = S_p\left[\not{P}_\nu (1+i\gamma_5)(\not{P}_\nu + 0)\gamma_t (1+i\gamma_5)(\not{P}_{el}+m)\right]$$

when $\sum_{u\,\text{spin}} (\bar{\lambda} A U)(\bar{U} B V) = \bar{\lambda} A (\not{P}+m) B \vec{V}$

in S_{pur} if have odd number γ's get 0. Get

$$2\, S_p\left[\underbrace{(1+i\gamma_5)\not{P}_\nu \gamma_t \not{P}_{el}\gamma_t}\right]$$

$\cancel{\gamma_5} = 0$ now $\gamma_t \not{P}_{el} = 2P_{el} - \not{P}_{el}\gamma_t$

$$= 2\left\{\left(2\,\epsilon_e\, S_p[\not{P}_\nu \gamma_t] - S_p[\not{P}_\nu \not{P}_{el}\gamma_t\gamma_t]\right\}\right.$$

$$2\left\{4\,\epsilon_e\,\epsilon_\nu - (\underbrace{\mathbf{P}_\nu \epsilon_{el} - \mathbf{P}_{el}\cdot\mathbf{P}_\nu}_{\to\ 0})\right\}$$

if integrate over all $\mathbf{P}_\nu$ $P_{el} \to$ "0

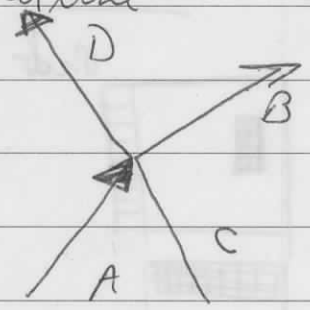
get $\epsilon_e\,\epsilon_\nu$ ~~gets~~ denominator terms

Rate $= P_\nu^2\, P_{el}^2\, dP_{el}$ Gamow Spectrum

substitute $P_\nu = W - \sqrt{m^2 + P_{el}^2}$ -

Fitch-Cronin experiment effect -

32 Topics of Large magnitude General outline -



Rule for 4 particle interaction

$(\bar{D}C)(\bar{B}A)$

$$[\bar{\psi}_D \gamma_\mu (1+i\gamma_5)\psi_C][\bar{\psi}_B \gamma_\mu (1+i\gamma_5)\psi_A]$$

$P \to N + e^+ + \nu$ RULE $\to (\bar{N}P)(\bar{\nu}e^-)$

or $P + e^- \to N + \nu$

Eg

$$[\bar{\psi}_\nu \gamma_\mu (1+i\gamma_5)\psi_{ee}][\bar{\psi}_N \gamma_\mu (1+i\gamma_5 c)\psi_S]G_{PN}$$

$c \simeq 1.23$

If consider $\bar{\psi}_P \gamma_\mu \psi_P$ like electric current operator in ordinary β decay. ~~too~~ Momentum is very small, 1-c slow moving potential goes to charge.

Other reactions $\mu^- + p \to N + \nu_\mu$

$\pi^- \to \mu^- + \bar{\nu}_\mu$

$\pi^- \to e^- + \bar{\nu}$ more momentum than so rate should be twice large due to mass difference

$m_\pi = 174$ MeV $m_\mu = 105$ MeV $m_{el} = 0.5$ MeV

$\tan\theta = \frac{m_\pi}{m_k} \simeq .24$

By Problems of physics

JTN NOTE (11/2012): I believe these FEYNMAN NOTES RELATED TO THE "FITCH-CRONIN EFFECT" EXPERIMENTALLY observed IN 1964 (ABOUT 5 yrs prior TO THIS LECTURE). The Experiment IN CP VIOLATION EARNED THEM THE Nobel PRIZE IN Physics IN 1980.

What couplings exist.

$$\frac{\text{Prob}}{\text{Prob}} \quad \frac{\pi \to \mu + \nu}{\pi \to e + \nu} = \frac{m_\mu^2}{m_e^2} \frac{\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2}{\left(1 - \frac{m_e^2}{m_\pi^2}\right)^2}$$

$$= \left(13.8 \times 10^{-4}\right)^{-1}$$

$$= \frac{107}{(.5)} \frac{\left[1 - \left(\frac{107}{174}\right)^2\right]^2}{\left[1 - \left(\frac{.5}{174}\right)^2\right]^2}$$

$$= 214 \left[1 - \left(\frac{5}{8}\right)^2\right]^2$$

$$= 50$$

$$\frac{3}{9} \quad 35 \qquad \left(\frac{1}{21}\right)^2 \qquad \begin{array}{r} 1.00 \\ .39 \\ \hline (.66)^2 \\ 66 \\ \hline 396 \\ 396 \\ \hline 4356 \end{array} \qquad \begin{array}{r} .020 \\ 50 \overline{)1.00} \end{array}$$

$$\frac{5}{8} \quad 40 \quad 5 \quad 16$$

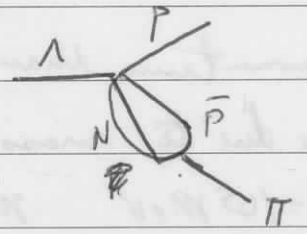
$$20 \times 10^{-3}$$

$K \to \mu + \nu$

$K \to \pi + \mu + \nu$

$\Lambda \to p + e^- + \bar{\nu}$

$\Lambda \to p + \pi$ changes strangeness doesn't emit neutrino.



6 basic reaction of weak interact.

$(\bar{N} P)(\bar{\nu} e^-)$

$(\bar{\mu}^- \nu_\mu)(\bar{\nu} e^-)$

$(\bar{\pi} p)(\bar{\nu}_\mu \mu^-)$

$(\bar{\Lambda} p)(\bar{\nu} e^-)$

$(\bar{\Lambda} p)(\bar{\nu} \mu^-)$

$(\bar{\Lambda} p)(\bar{p} N)$

$G J_\mu^{+} J_\mu$

$J_\mu = \bar{\nu} e + \bar{\nu}_\mu \mu + \bar{p}(\cos\theta N + \sin\theta P)$

NATIONAL
45-600
Made in U.S.A.

10 20 30 40 50

Separation

Booster squib driver circuit

delay separation

· t_{cutoff} rate

attitude accuracy

Yaw maneuver

Separation

NOTES TO MYSELF DURING HIS LECTURE. WORKING ON LAUNCH VEHICLE/SPACECRAFT SEPARATION AND I HAD TO ANALYZE THESE FACTORS —

$(\bar{\nu}e)(\bar{\nu}e)$ self interaction.

neutrino collapse $e^+ + e^- \rightarrow \nu + \bar{\nu}$ is it real

ν can carry out heat, center cools, collapse

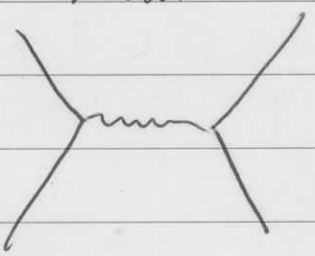
instead of $e^+ + e^- \rightarrow \gamma + \gamma$

source of super nova collapse.

$(\bar{N}P)(\bar{P}N)$ neutron proton interaction violates parity

$\therefore$ parity add mixture imperfect.

W meson



$M_W \approx 5$ Bev

$$4\pi f^2 \, J_\mu \frac{1}{q^2 - M_W^2} J_\mu^\dagger \approx \frac{4\pi f^2}{(M_W)^2} J_\mu J_\mu^\dagger$$

G

$$\frac{4\pi f^2}{m_W^2} = \frac{10^{-5}}{M_p^2} \rightarrow \frac{M_W}{M_p} \approx 10^3 f \approx 110$$

$$f \hat{\approx} k^2 \sim \left(\frac{1}{137}\right)^2 \sim \frac{1}{12}$$

$M_W \sim 100$ Bev

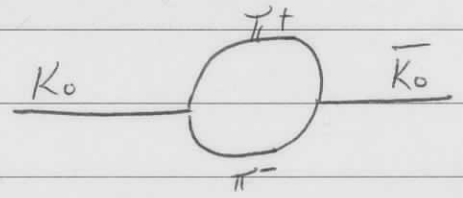
1B

1 more interaction not explained by any law - strong weak ED gravity.

K^0 and $\bar{K}_0$ produced by $P + N \xrightarrow{\text{Hi energy}} P + \Lambda + \bar{K}^0$ (Strange +1)

also make K_0 by $\bar{K}_0 = S+1$, $K_0 = S+1$

but same mass. Now suppose Here K^0 and it decays $K^0_0 \xrightarrow{a}$ into $\pi^0 + \pi^-$



symmetry. also $\bar{K}^0 \xrightarrow[a]{} \pi^- + \pi^+ \rightarrow K_0$

2 state system swings back and forth. Try find eigenstate that does not change

$\frac{1}{\sqrt{2}}(K_0 + \bar{K}_0)$ can decay into 2 π's — K^0 short state rate 10^{-10} sec

$\frac{1}{\sqrt{2}}(K_0 - \bar{K}_0)$ cannot decay into 2π — K^0 long state 10^{-8} sec

long state can go into 2π's 500 times slower

$K_0 \rightarrow \pi^+ + e^- + \nu$ such

maybe disintegrate into $\pi^- \neq \pi^+$ for anti particle decays
$a \neq a$

so $\bar{K}_0 - K_0 \neq 0$

<u>lack of CP symmetry.</u>