Sampling of continuous-time signals

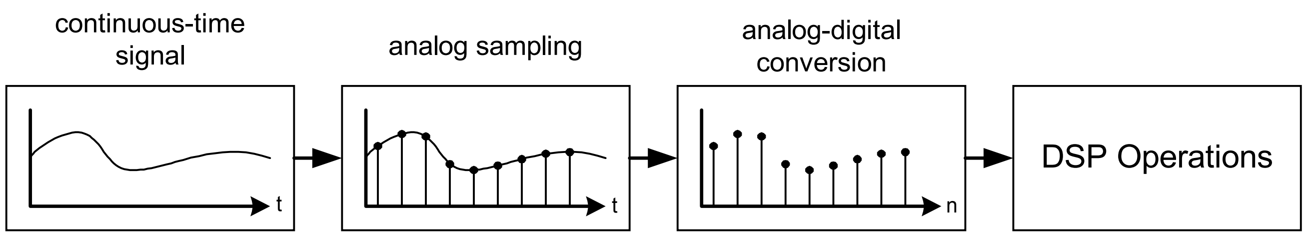
J. Carwardine

Outline

- Sampling theory.
- Anti-alias filters
- Sample-rate conversion (digital down sampling)
- Bandpass sampling.

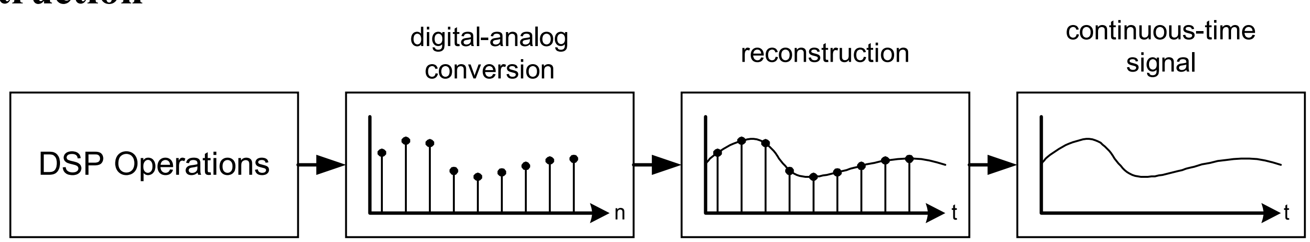
Key Elements of Sampling and Reconstruction

Sampling

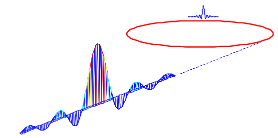


- A continuous-time signal is sampled at discrete time intervals and subsequently converted to a sequence of digital values for processing.

Reconstruction

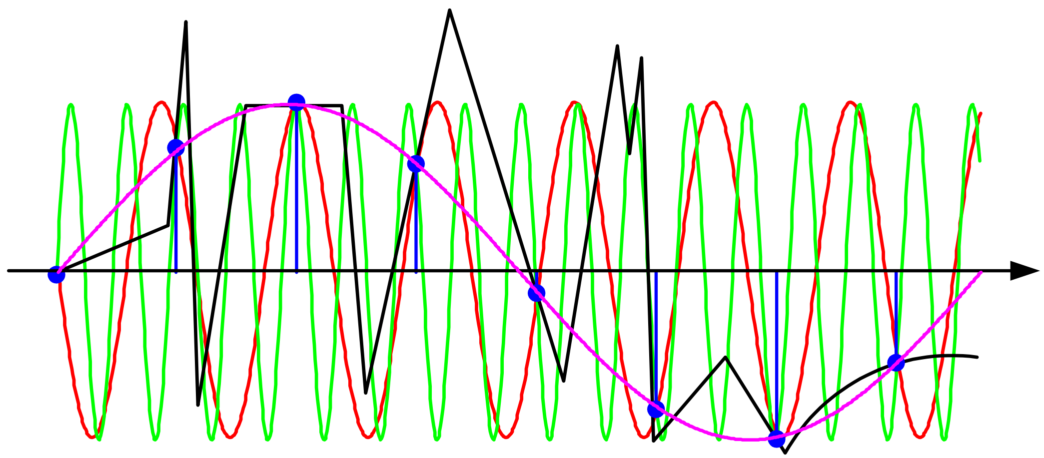


- The sequence of digital values is converted into a series of impulses at discrete time intervals before being reconstructed into a continuous-time signal.
- Sampling and Reconstruction are mathematical duals.

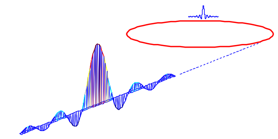


Ambiguity of Sampled-Data Signals

- Which continuous-time signal does this discrete-time sequence represent?

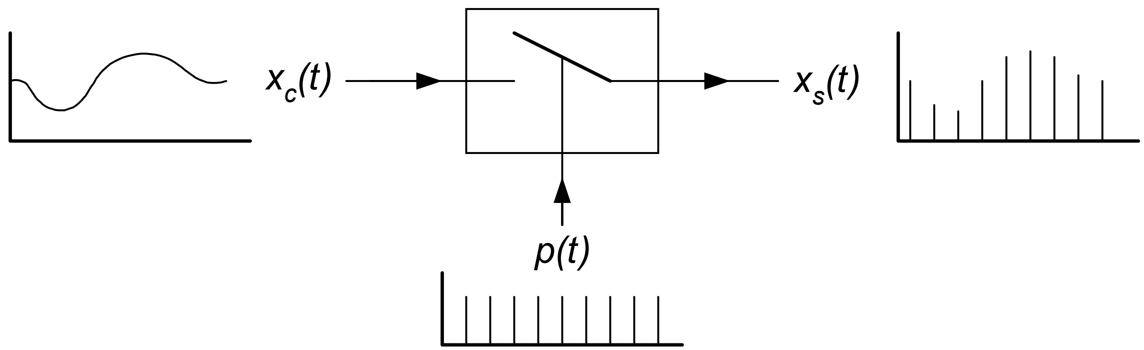


- Knowing the sampling rate, is not enough to uniquely reconstruct a continuous-time signal from a discrete-time sequence.
- The uncertainty is a result of *aliasing*.



Frequency-Domain View of Sampling

- Consider the sampling process as a time-domain multiplication of the continuous-time signal $x_c(t)$ with a sampling function $p(t)$, which is a periodic impulse function

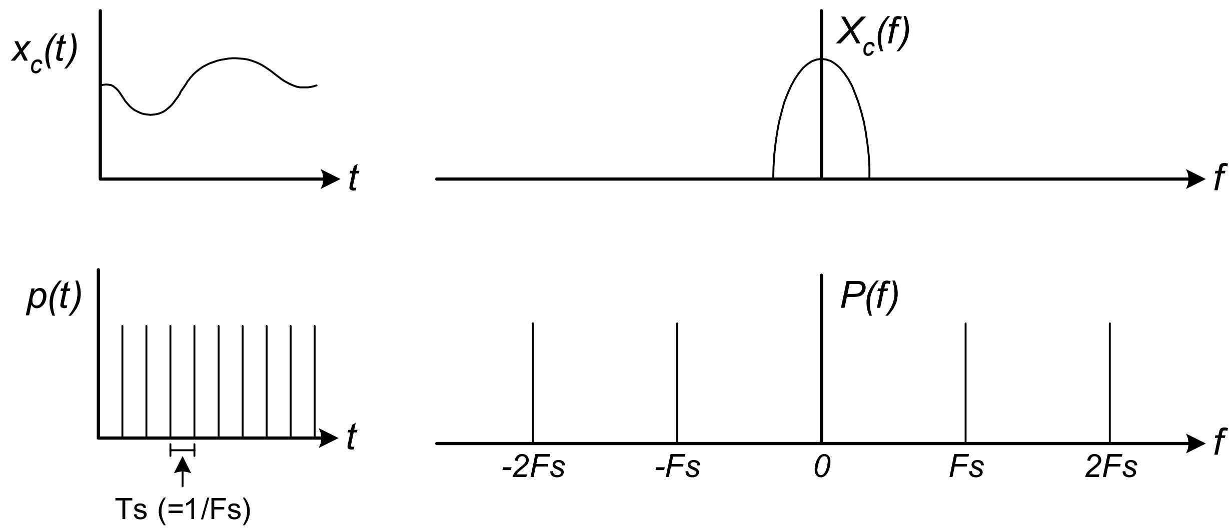


$$x_s(t) = x_c(t) \cdot p(t)$$

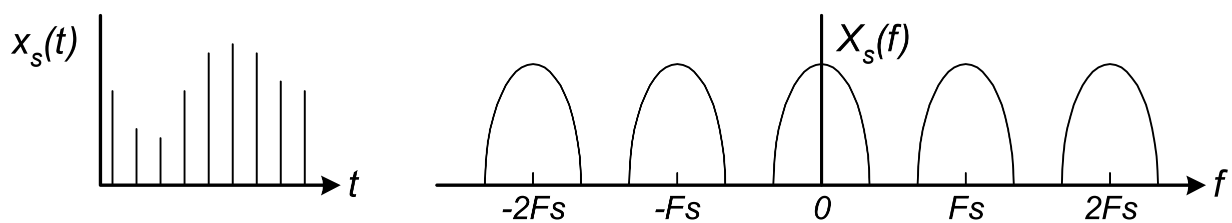
$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

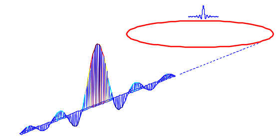
Frequency-Domain View of Sampling (cont)

- The time-domain and frequency-domain representation of the two signal is shown below

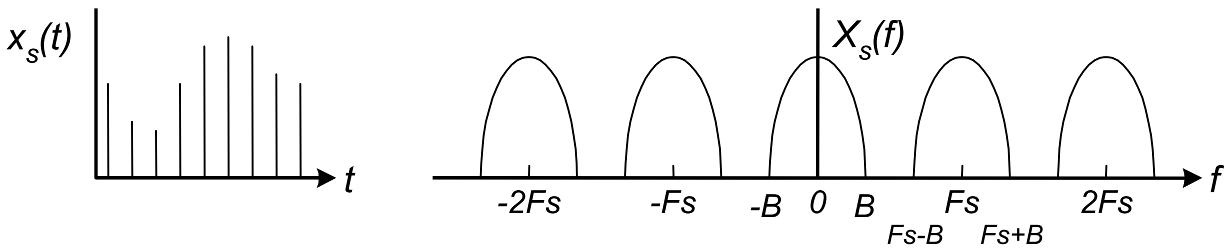


- The frequency-domain representation of the sampled-data signal is the convolution of the frequency domain representation of the two signals, resulting in





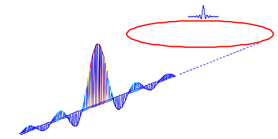
Shannon's Sampling Theorum



- Provided the sampling rate is more than twice the signal bandwidth, the image spectra do not overlap in frequency space.

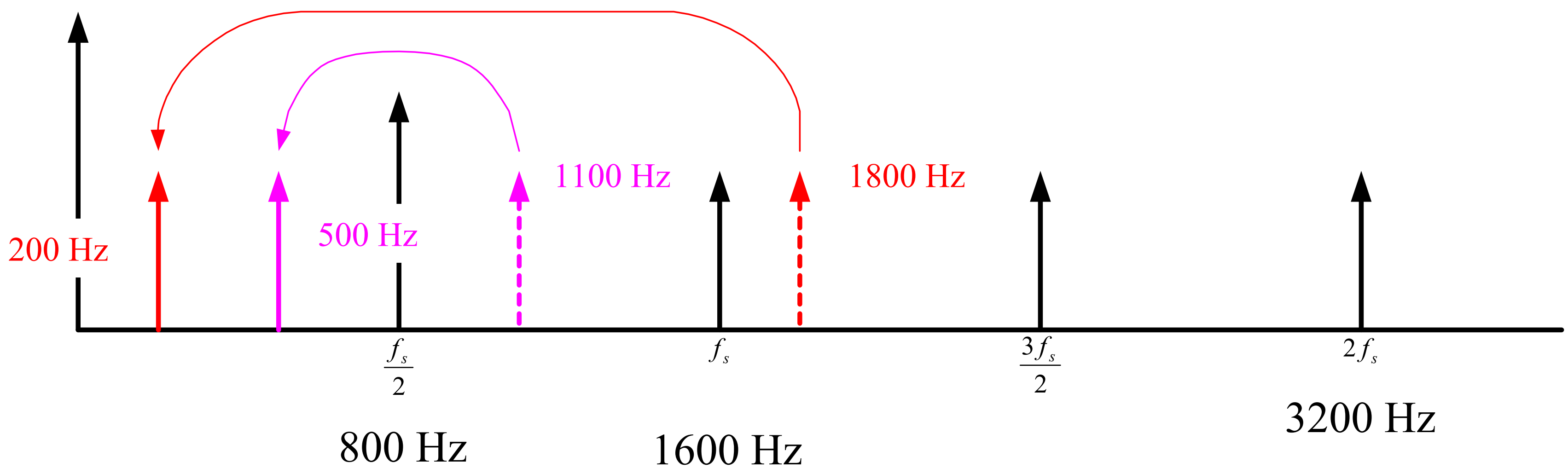
- This leads to Shannon's Sampling Theorem, which states

  A band limited continuous-time signal, with highest frequency B*Hz can be uniquely recovered from its samples provided that the sampling rate* F_s *is greater than 2*B *samples per second.*



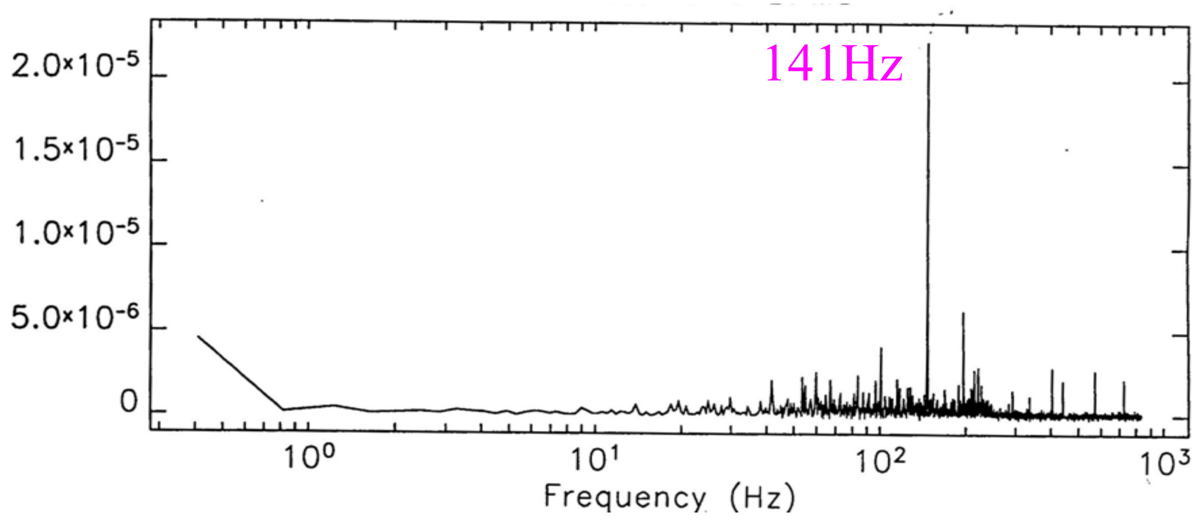
Aliasing of Tones

- Single Frequency Tones greater than $f_s/2$ appear as aliases.
- Consider the following spectrum that is sampled at 1600Hz.

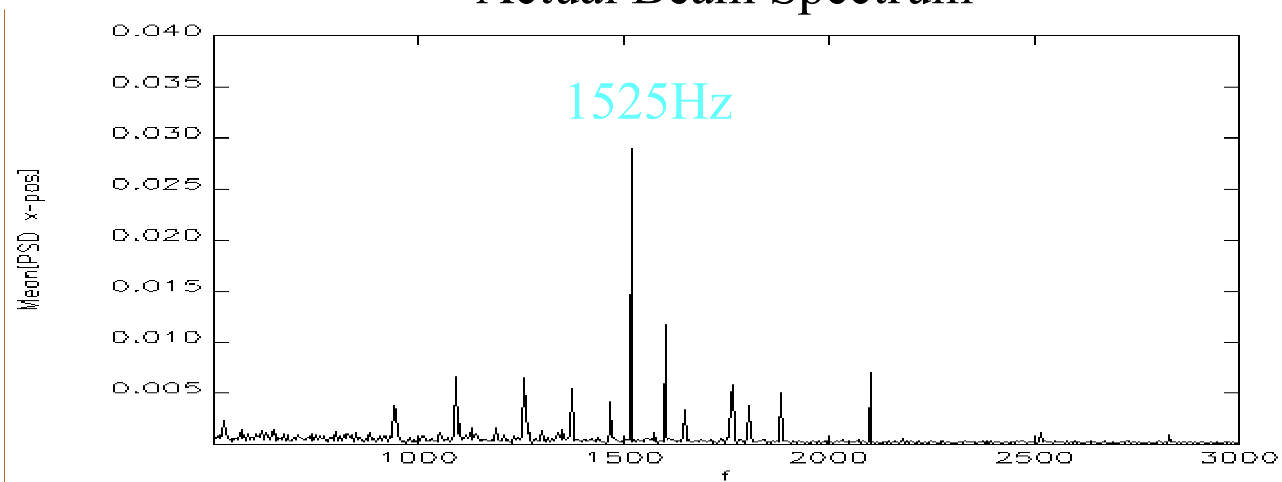


Corrector PS Oscillation Aliased by RT Feedback

Spectrum sampled at 1666Hz (orbit feedback)



Actual Beam Spectrum



Mathematical Explanation of Aliasing

- Consider the continuous-time sinusoid described by the expression

$$x(t) = \sin(2\pi f t + \phi)$$

- Sampling this at intervals T results in the discrete-time sequence

$$x[n] = \sin(2\pi f T n + \phi) = \sin(\omega_d n + \phi)$$

- Since the sequence is unaffected by the addition of any integer multiple of 2π, we can write ***x[n]*** as

$$x[n] = \sin(2\pi f T n \pm 2\pi m + \phi)$$
$$= \sin[2\pi T\left(f \pm \frac{m}{Tn}\right)n + \phi]$$

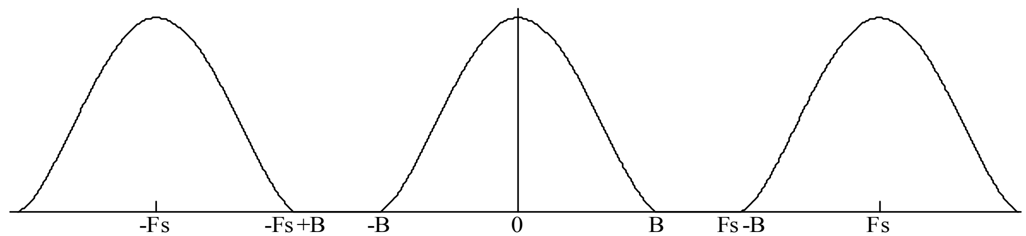
- This must hold for any integer ***m***, so let's pick integer values of ***m/n*** and replace the ratio by another integer ***k***. We'll also replace ***1/T*** by the sampling rate ***Fs**,* giving

$$x[n] = \sin[2\pi T\left(f \pm kF_S\right)n + \phi]$$

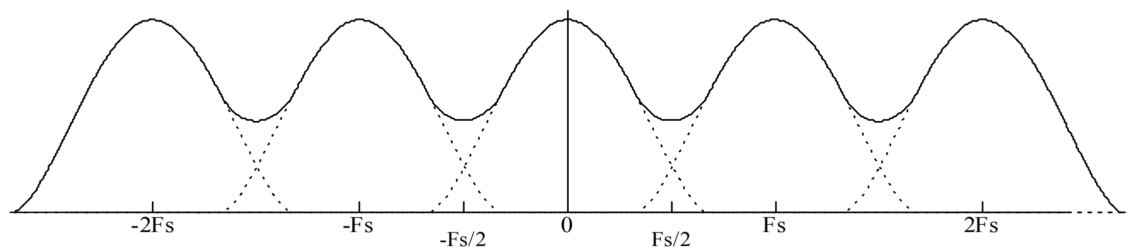
- The implication is that when sampling at a frequency ***Fs***, we cannot distinguish between ***f***, and a frequency ***f* ±*kFs*** where ***k*** is any integer.

Three Cases of Sampling

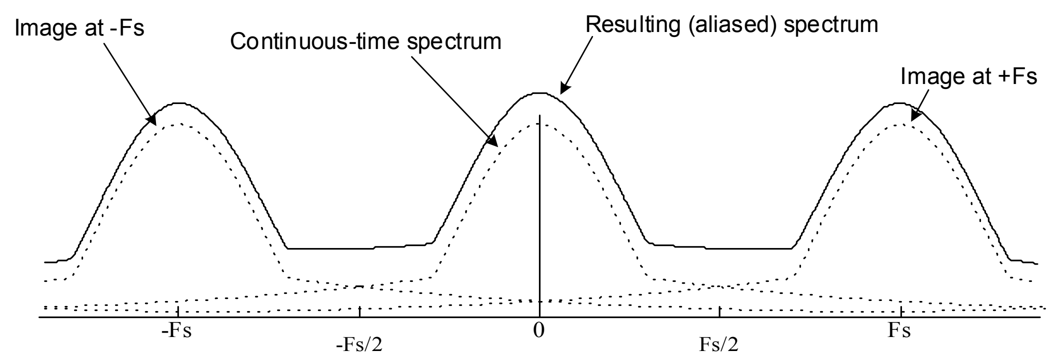
- Signal is band-limited
- *Fs>2B*



- Signal is band-limited
- *Fs<2B*



- Signal is not band-limited

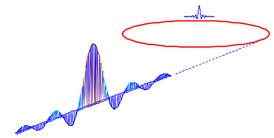


How to avoid signal contamination by aliasing

- Digitize the analog signal at least 2x the highest frequency component of interest (Shannon’s Sampling Theory).
- Use an analog anti-aliasing filter to get rid of unwanted higher frequency components <u>before</u> the digitizer (it’s too late otherwise).

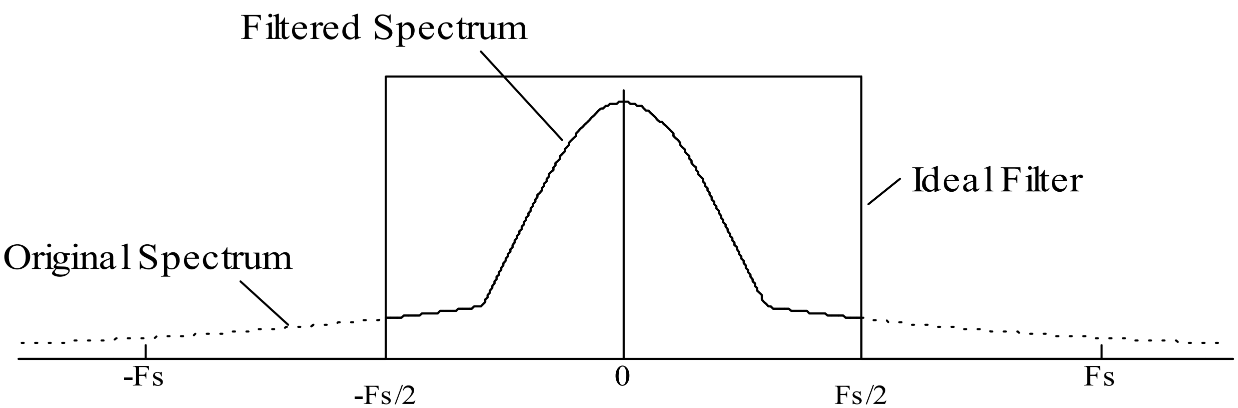
- Realize there will *always* be aliasing to some degree, the question is how much can be tolerated...

Anti-alias filters



Anti-Alias Filters

- Since aliasing will occur in all real systems, anti-alias filters are used to reduce the effect to acceptable levels.
- An ideal anti-alias filter would pass, unaffected, all frequencies below the folding frequency, but attenuate to zero all frequencies above the folding frequency.



- To compute the impulse response of this idealized anti-alias filter, we can take the inverse Fourier transform of its frequency response.

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi ft} df$$

Impulse Response of the Ideal Anti-Alias Filter

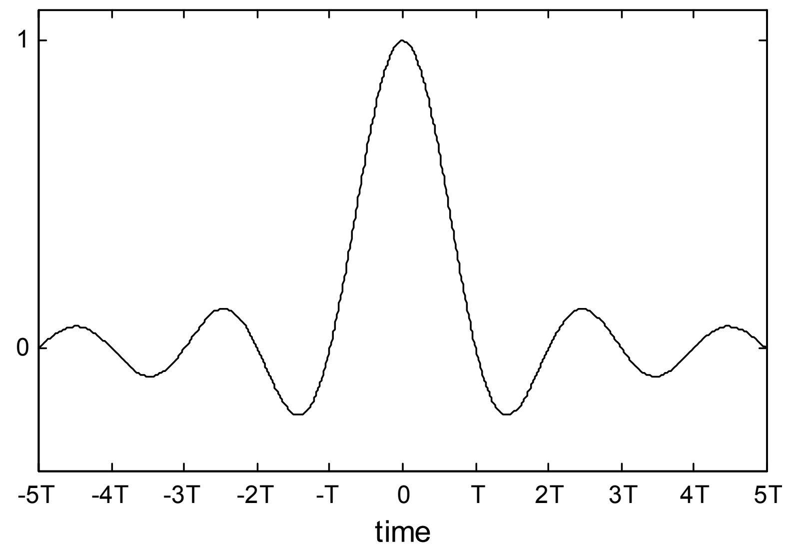
- Computing the inverse Fourier transform...

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi ft} df$$
$$= \int_{-F_s/2}^{F_s/2} e^{j2\pi ft} df$$
$$= \frac{1}{j2\pi t}\left[e^{j2\pi ft} \right]_{-F_s/2}^{F_s/2}$$
$$= \frac{1}{j2\pi t}\left[e^{j2\pi \frac{F_s}{2} t} - e^{-j2\pi \frac{F_s}{2} t} \right]$$
$$= \frac{1}{2\pi t}\sin(\pi F_s t)$$

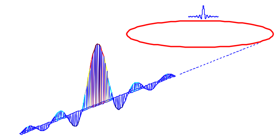
- Using the substitution $Fs=1/T$, we get

$$h(t) = \frac{1}{2T}\frac{\sin(\pi \cdot t/T)}{\pi \cdot t/T}$$

This is a doubly-infinite sinc function

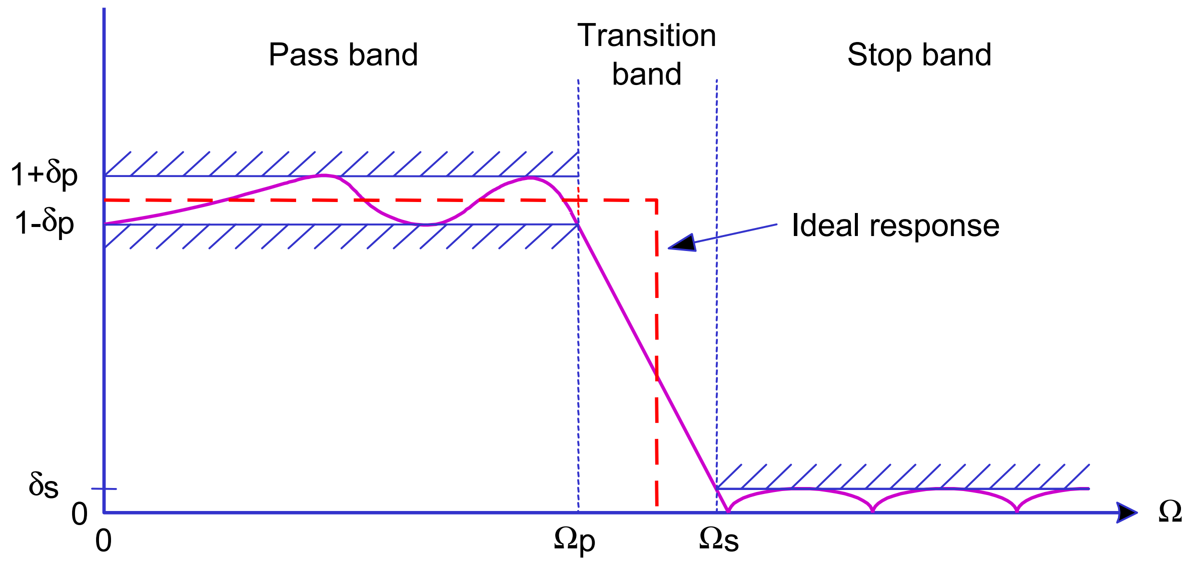


- Since realizable filters cannot be non-causal, practical anti-alias filter involve some compromises in system performance.



Frequency Response of Practical Filters

- When a realizable impulse response is generated, the frequency response of the resulting filter is compromised from the ideal response
  - The passband may not be flat
  - There is a finite width to the transition from passband to stopband
  - The stopband will not have infinite attenuation
  - The phase response will not be zero for all frequencies.



Anti-Alias Filter Considerations

- Must maintain accuracy commensurate with ADC resolution
  - Reduce alias contamination below quantization noise of ADC
  - Keep filter pass-band attenuation within ADC resolution
- Parameters to adjust
  - Sample Frequency
  - Filter Type
  - Filter cutoff frequency
  - Filter Order

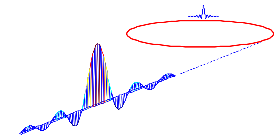
Anti-Alias Filter Requirements vs digitizer resolution

- ADC quantization noise and resolution (1 LSB) in percent and dB

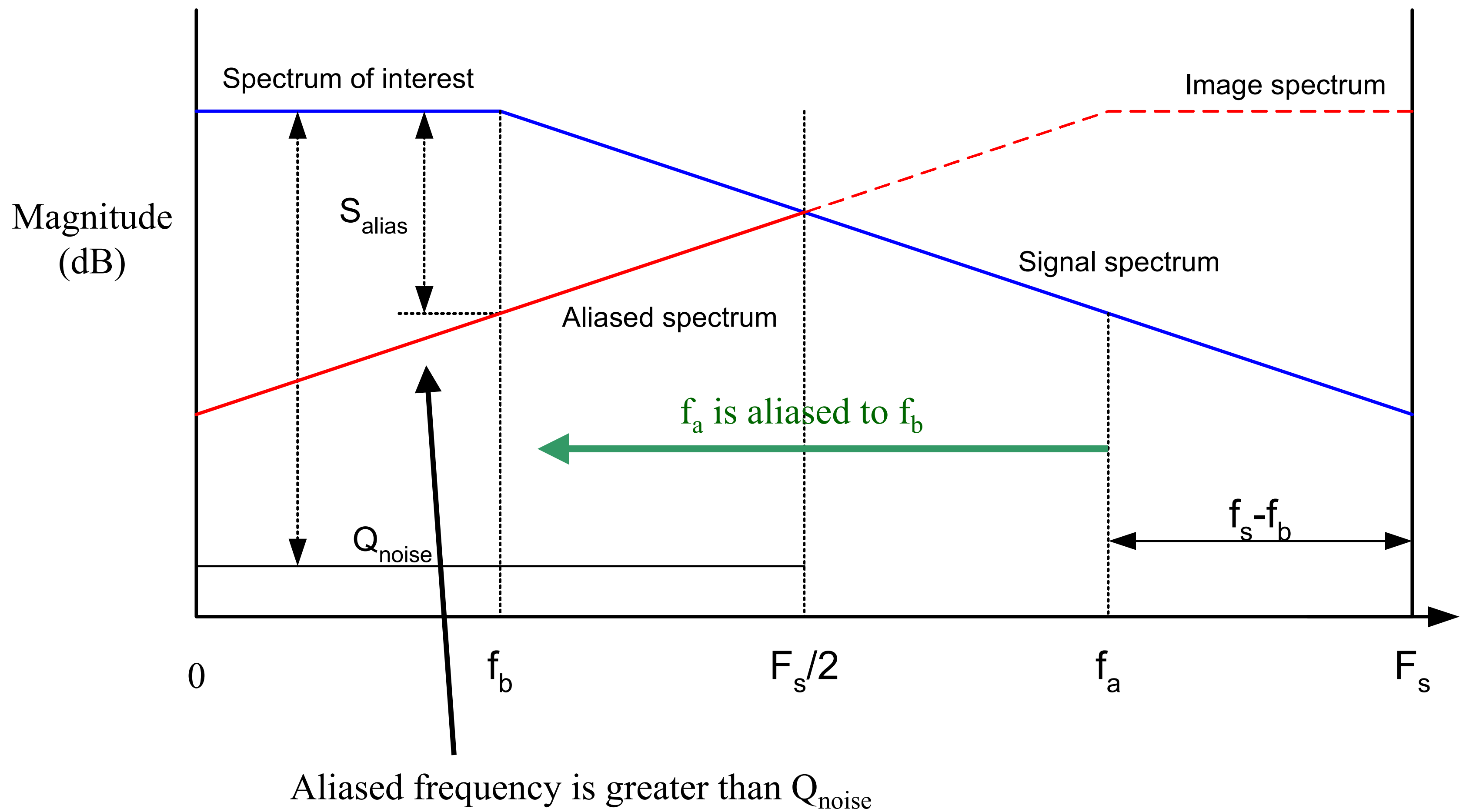
		Resolution	
N	Q-noise(dB)	Percent	dB
08	-49.9	0.3906	-0.03400
10	-62.0	0.0977	-0.00849
12	-74.0	0.0244	-0.00212
14	-86.0	0.0061	-0.00053
16	-98.1	0.0015	-0.00013
18	-110.1	0.0004	-0.00003

For a ***perfect*** ADC of N bits

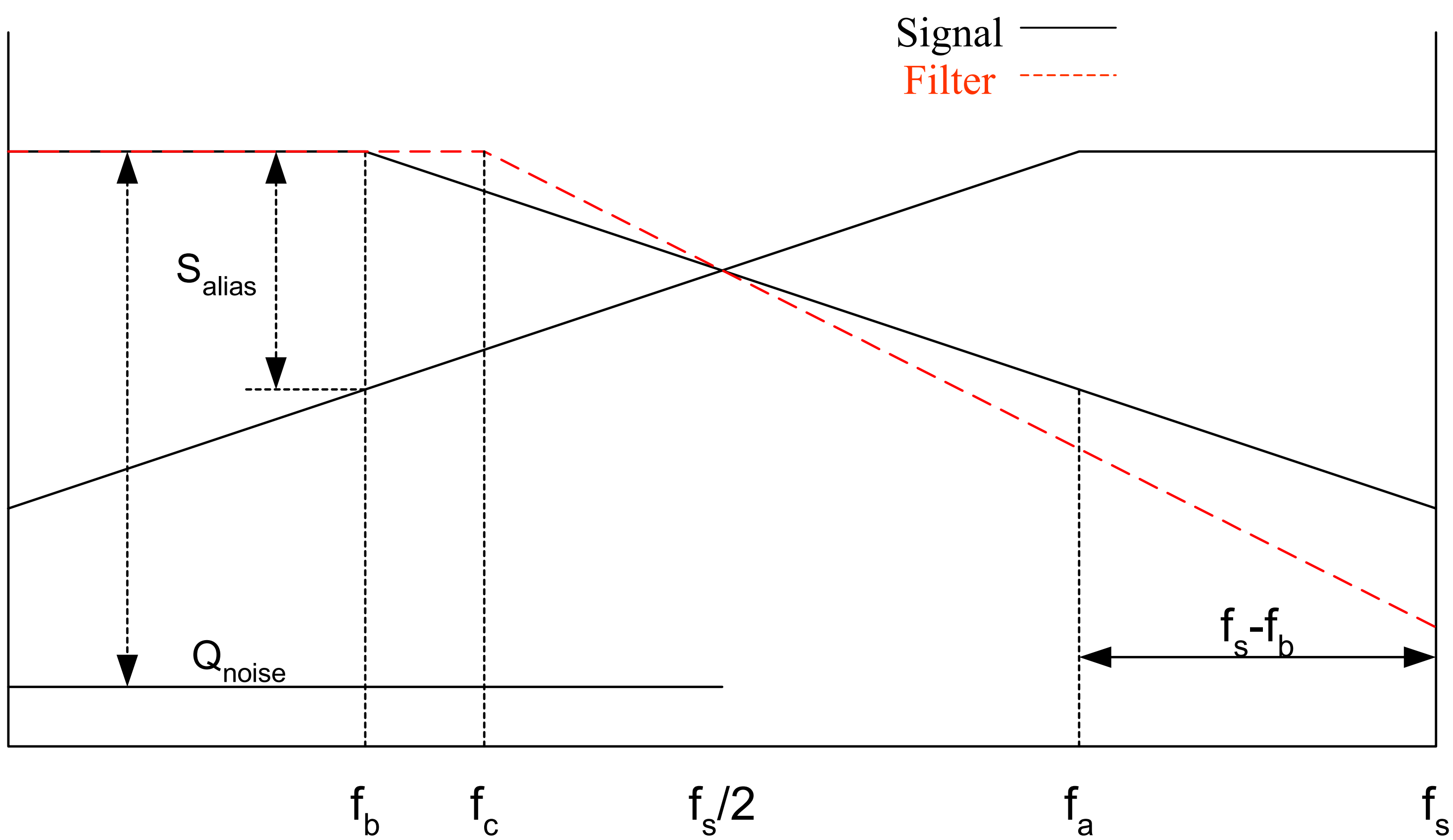
$$SNR_{dB} = 6.02N + 1.76$$



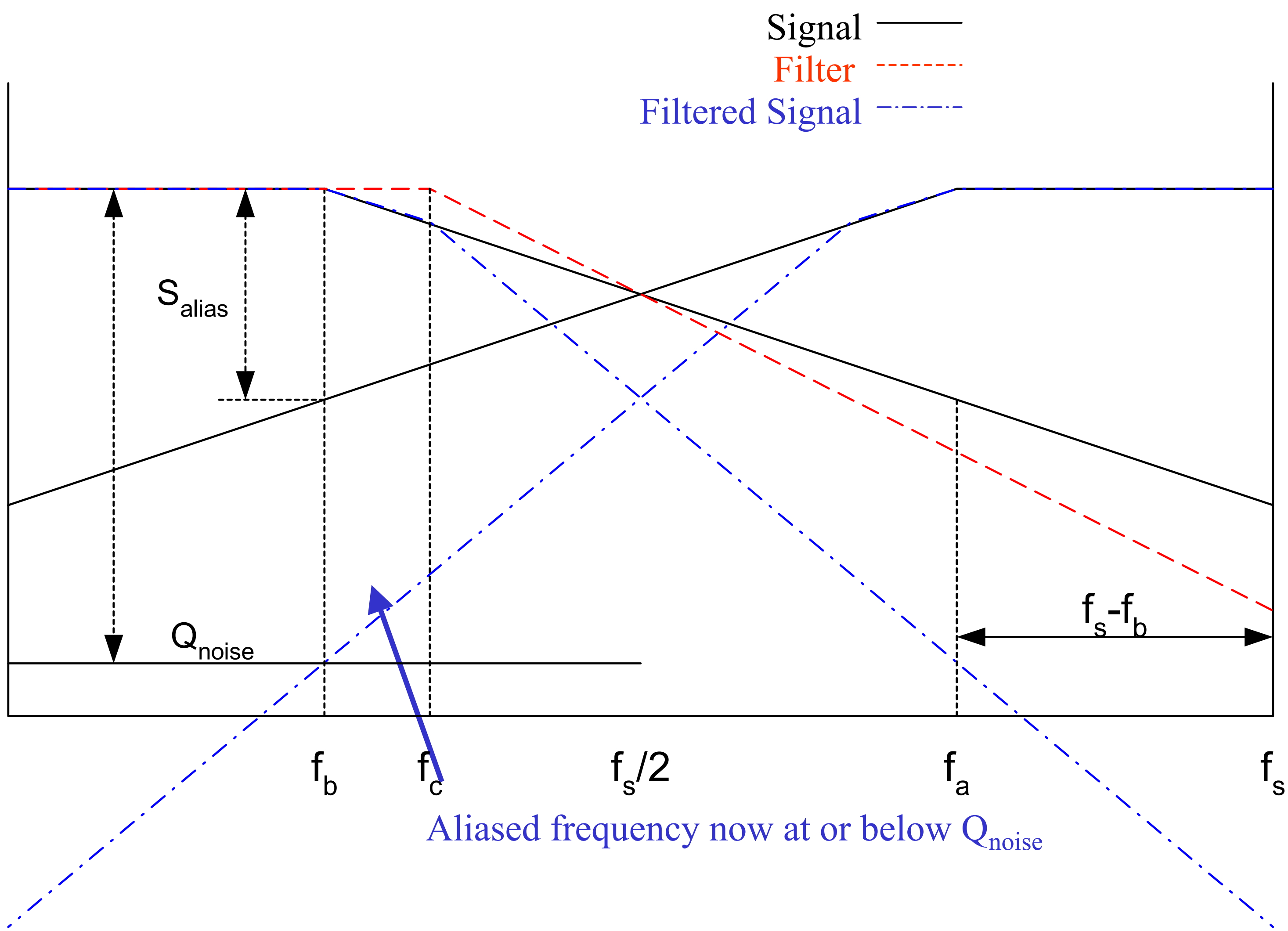
Anti-alias filter criteria in frequency space



Anti-Alias Filter design considerations



Anti-Alias Filter design considerations (2)



Other Anti-Alias Filter Considerations

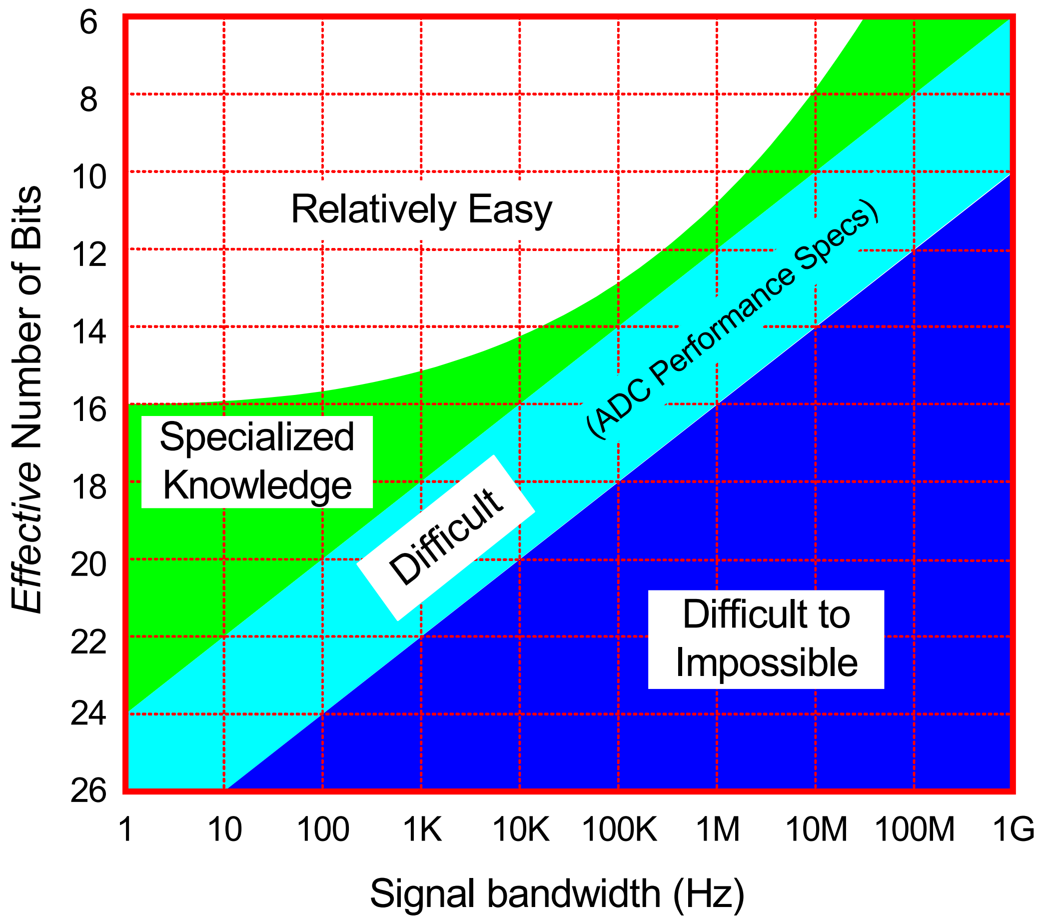
- Filter phase shift may be important consideration in stability of feedback applications
- Filter passband undulations may be undesirable in high resolution measurement applications
  - No passband undulations - Butterworth, Bessel, Chebychev I
  - Passband undulations - Chebychev II, Eliptical
- Filter roll-off affects amplitude of frequencies near cutoff.

- Anti-alias filter design is usually a compromise, because in control applications, the filter characteristics can significantly impact system closed-loop performance.

- To achieve sufficient signal quality for high-performance beam position measurements, anti-alias filters are absolutely necessary and can be very challenging to implement.

Digitizer performance trade-offs

- Getting even 16-bit performance is not as simple as just using a 16-bit digitizer!

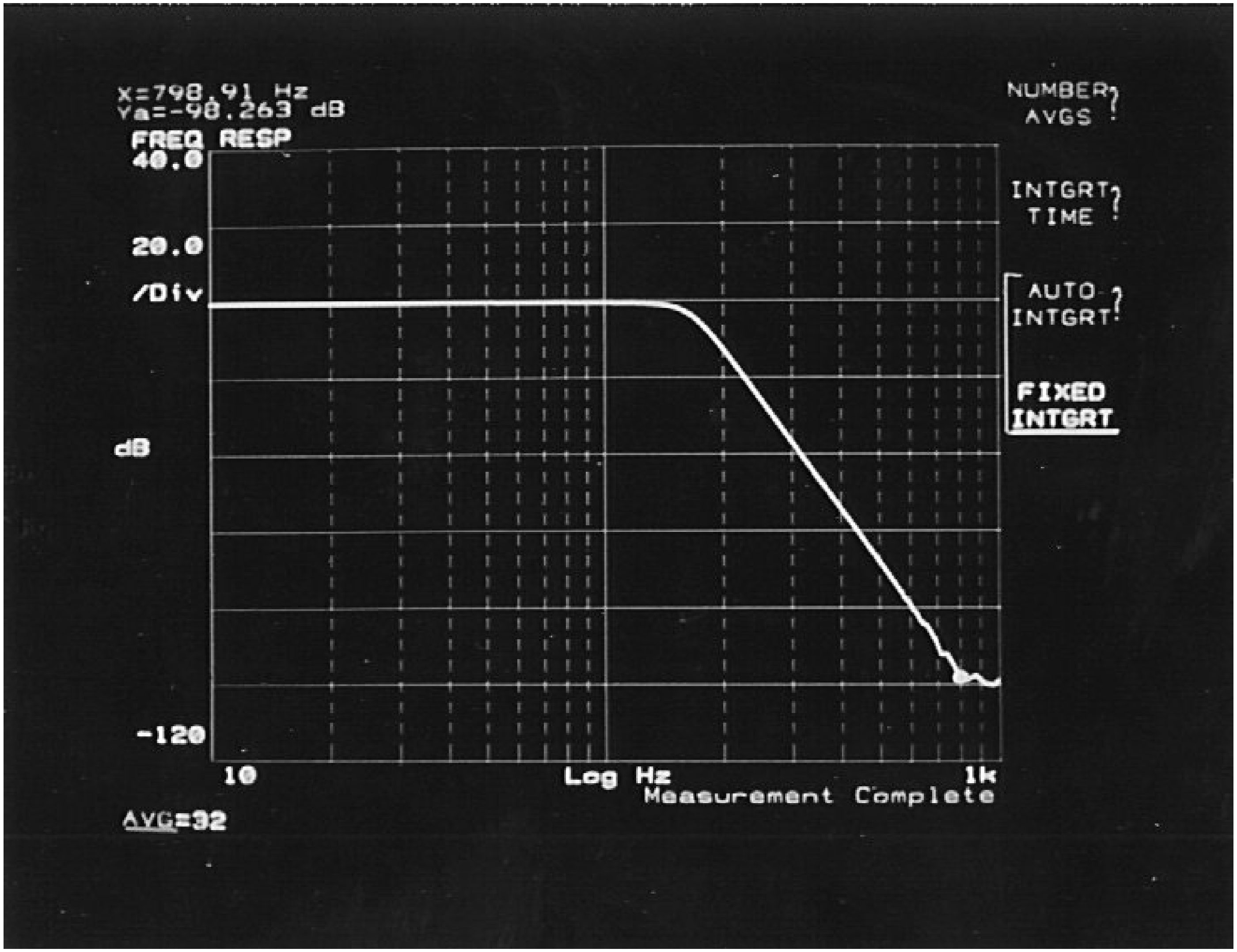


Ref: "Practical Limits of Analog-to-Digital Conversion" (Jerry Horn)

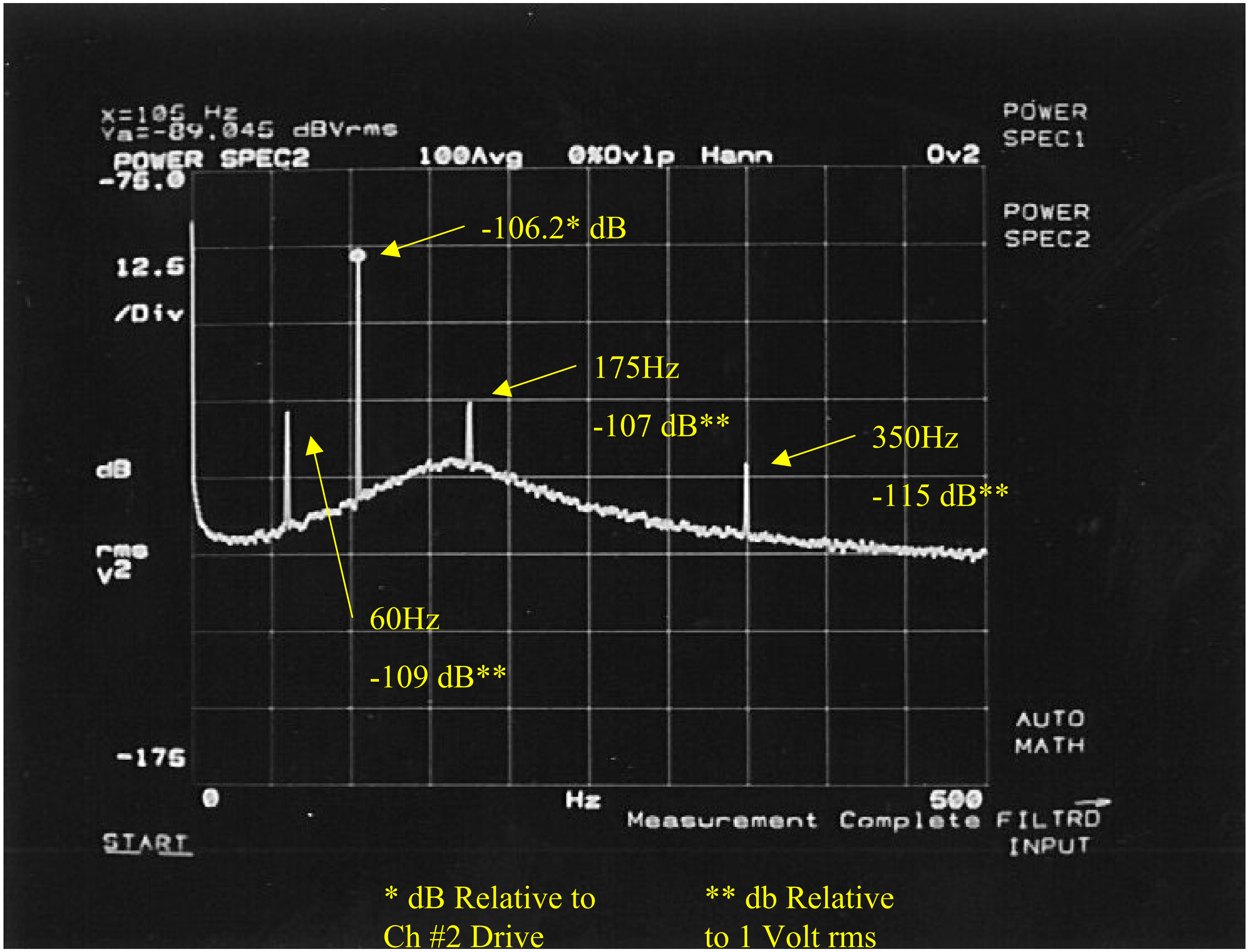
Measured Filter Performance

Bandwidth (3 dB)	165 Hz
Attenuation (at 800 Hz)	98 dB
Spurious Free Dynamic Range (45 Hz Full-Scale Input)	90 dB
Noise and Pickup	-115 dB
Adjacent Channel Crosstalk (At 105 Hz)	-116 dB

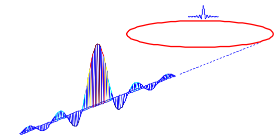
Filter Frequency Response (Average = 32)



Cross Talk on Channel #1

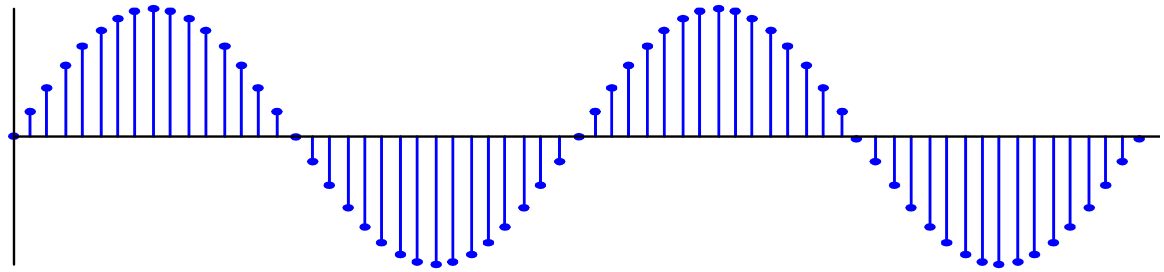


Sample-rate conversion
(digital down-sampling)



Sample-Rate Conversion

- Consider the following sampled-data discrete-time sinusoid



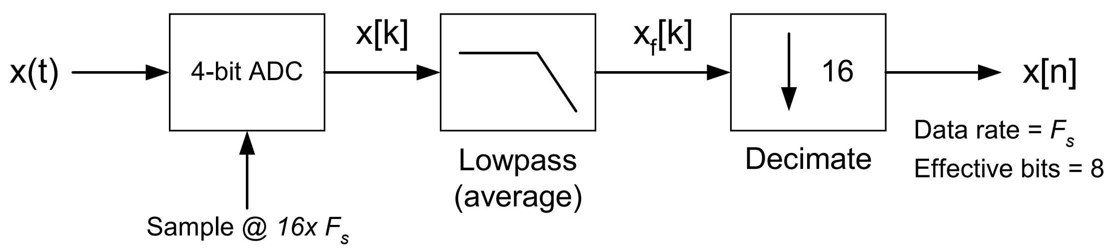
- This has 64 samples per period, so the discrete-time frequency is $2\pi/64$.
- If the sample-rate had been 128kS/s, this would represent a 2kHz sinusoid.
- How would we represent this signal at a sample-rate of 16kS/s or 256kS/s?

- The signal can be *decimated* to produce fewer samples per period (ie reduce the sample-rate) or *interpolated* to produce more samples per period (ie increase the sample rate).

Motivation for Decimation of Discrete-Time Signals

- If the signal is to be archived, then reduction of data storage might be an objective.
- In real-time applications, unnecessarily high data rates requires additional processing power with consequential impact on cost or performance.
- When implementing filters, it is sometimes impossible to achieve the required performance at high sample rates because of wordlength effects.

- Almost always, something is done to the signal before it is decimated.
  - Example: 16-times over-sampling ADC, with 4-bit quantizer.



Decimation and Filter Implementation

- Consider a signal that is sampled at 1MS/s, where we need to implement a lowpass filter with cutoff at 1kHz.
- A simple one-pole lowpass filter that meets this requirement would have the transfer function

$$H(z) = \frac{0.001}{1 - 0.999z^{-1}}$$

- We would need a dynamic range of 2000 to represent these filter coefficients in a DSP chip, implying at least 11 bits.

- However, if we reduced the sampling rate from 1MS/s to 10kS/s, the same 1kHz lowpass filter would be implemented with the transfer function

$$H(z) = \frac{0.1}{1 - 0.9z^{-1}}$$

- We only need a dynamic range of 20 to represent the coefficients of this filter, so we could use a DSP chip with fewer bits.
- We could also use a much slower DSP to implement the same filter since data only arrives every 100μS rather than every 1μS.

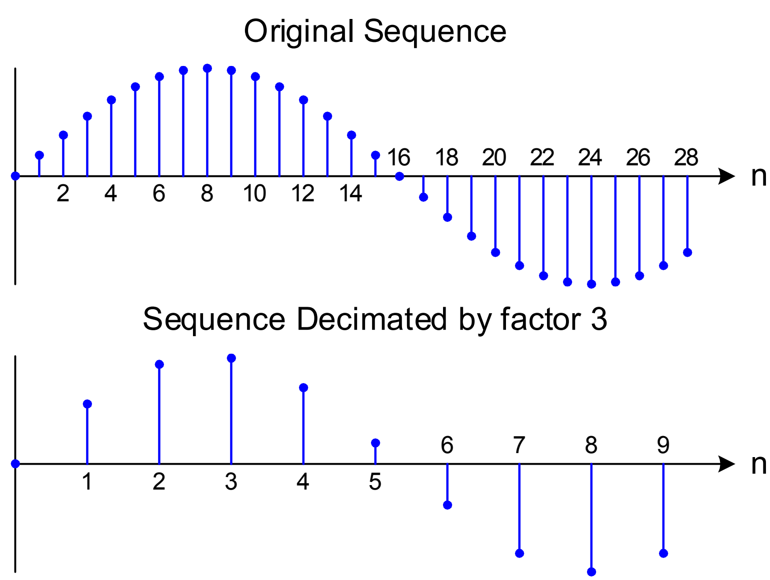
The Decimation Process

- Decimating a signal by a factor M can be represented by the expression

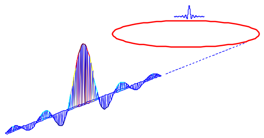
$$y[n] = x[M \cdot n]$$

For example, if $M = 3$ then $y[0] = x[0],\quad y[1] = x[3],\quad y[2] = x[6],$ etc

- An example of decimating a sinusoidal signal by a factor 3 is shown below

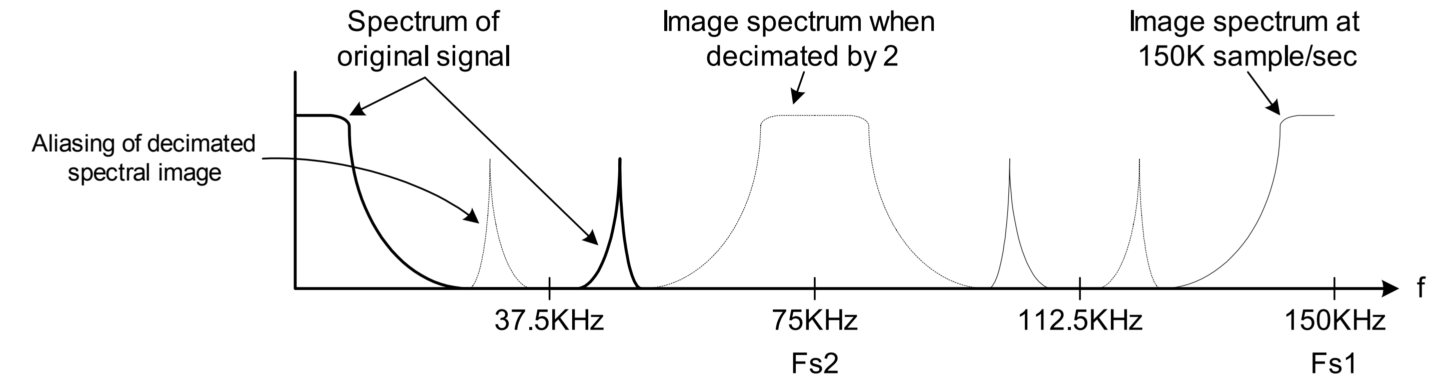


- Note that to avoid aliasing, we must obey Shannon's sampling theorem after decimation.

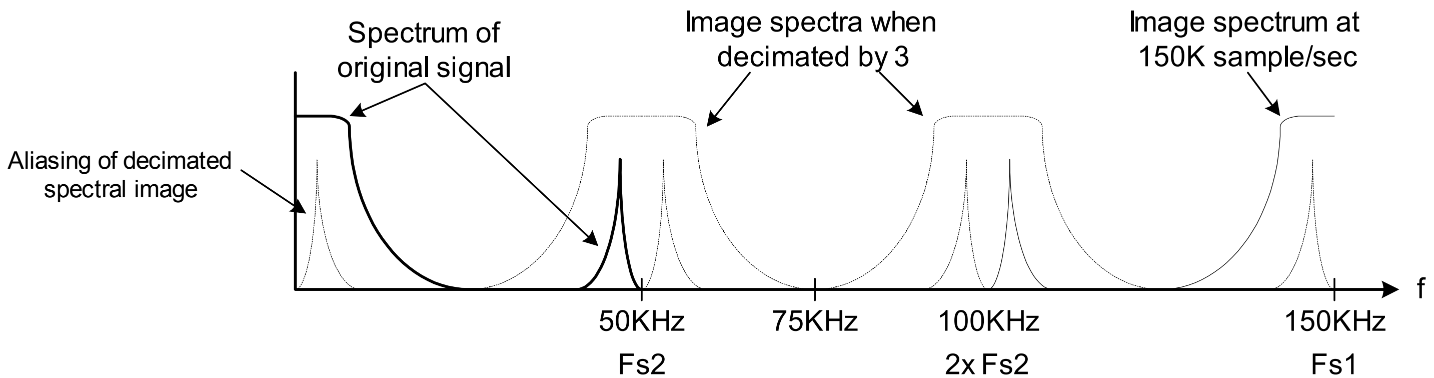


Decimation in Frequency Space

- Decimating by a factor two splits the frequency space from DC to the original sampling rate into two, with additional image spectra appearing about the new sampling rate. Note that there is an implied reduction in sampling rate along with the decimation process.

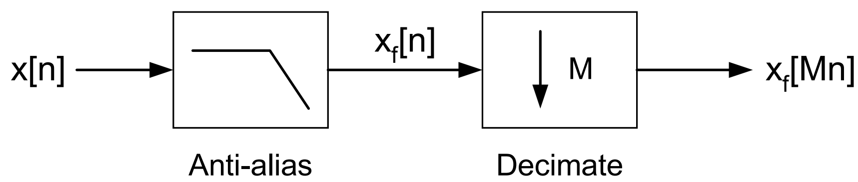


- If the same signal were decimated by a factor three, the new spectrum would be

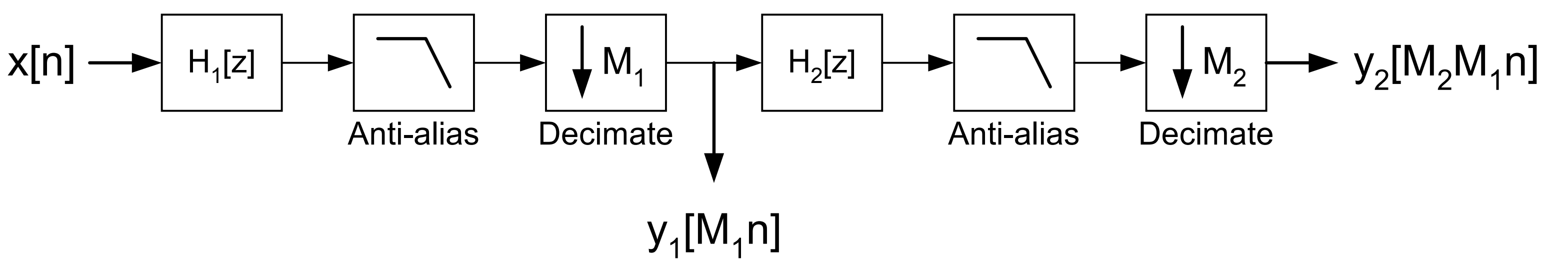


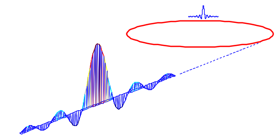
Anti-Alias Filters for Decimation

- The decimation process can be thought of as the sampling of a discrete-time sequence.
- Anti-alias filters are required prior to decimation in the same way they are required in the continuous-time domain before sampling.



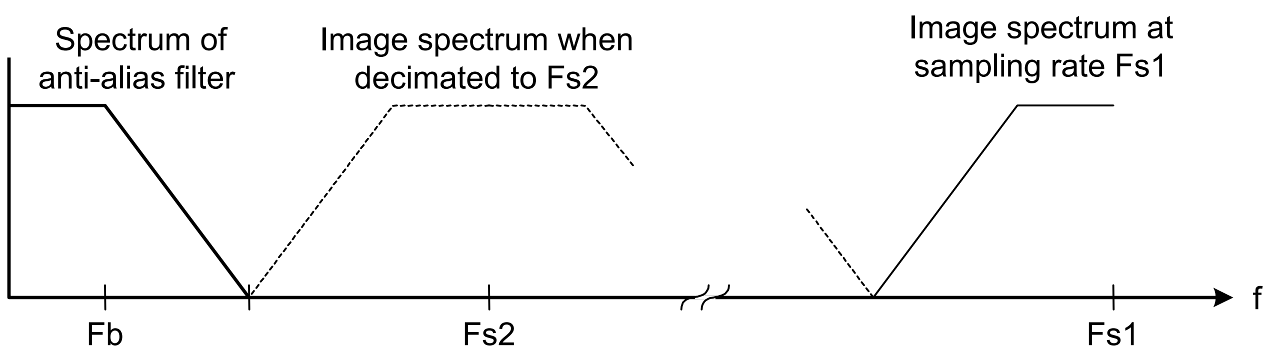
- In a multi-stage decimation system, anti-alias filters are required before every decimation stage, regardless of what else is done to the signal.



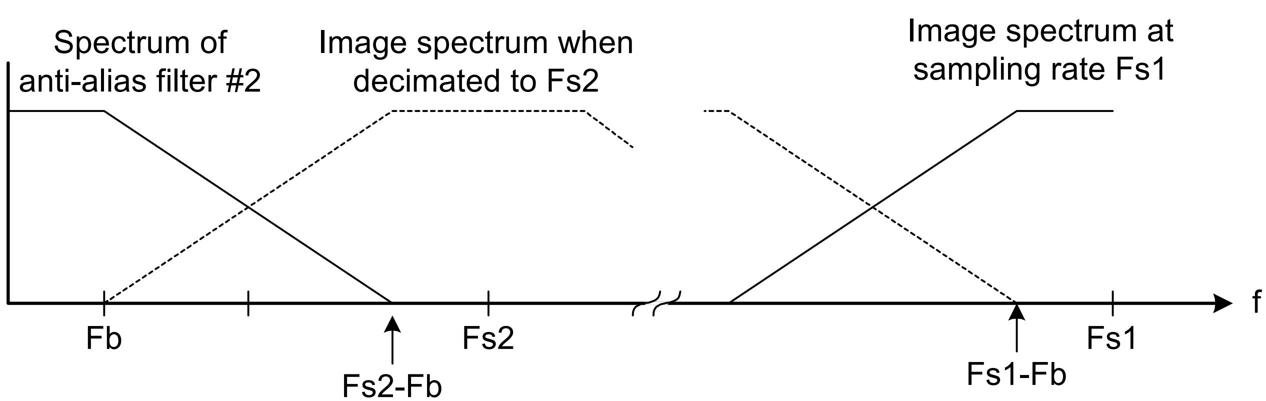


Anti-Alias Filter Generic Requirements

- The anti-alias filter is required to prevent aliasing when the original spectrum at sample-rate #1 is down-sampled to sample-rate #2.



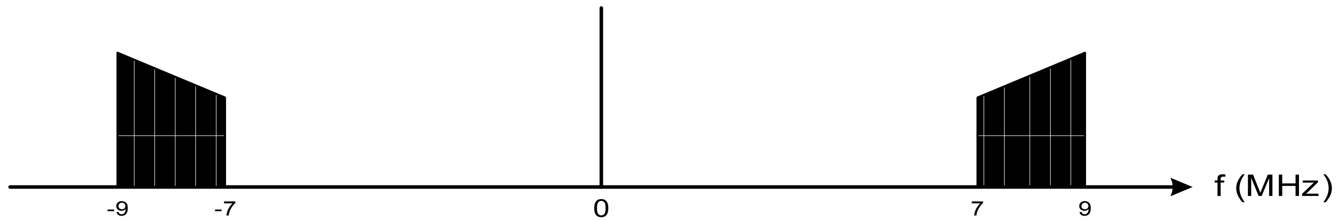
- More relaxed requirements on the anti-alias filter slope that still avoid aliasing



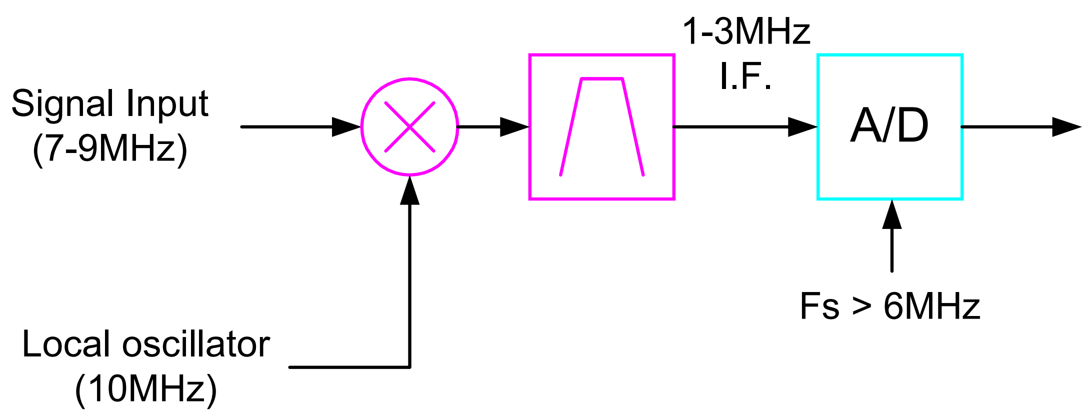
Bandpass Sampling
where aliasing is a good thing

Sampling Band-limited Signals

- Consider a 2MHz band-limited signal riding on an 8MHz carrier.

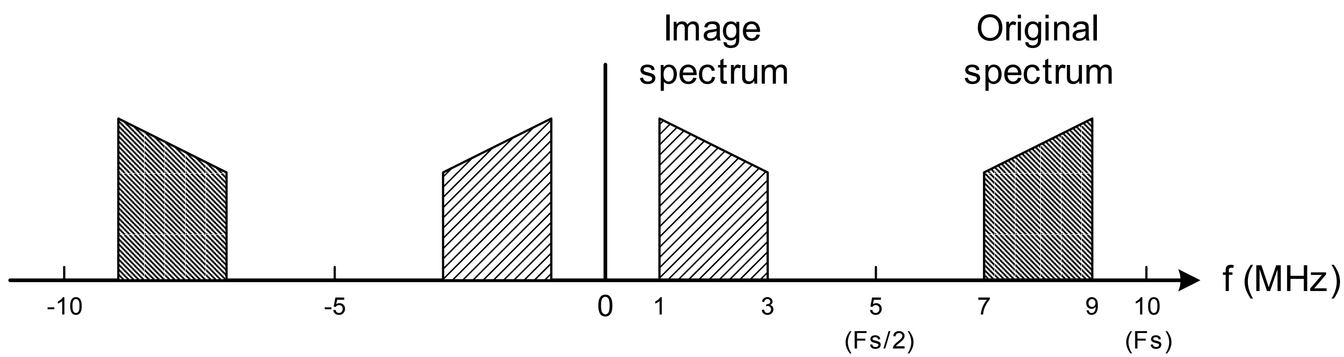


- The IF could be extracted by mixing with a local oscillator at 10MHz and sampled at 6MHz, or could be directly sampled at > 18MHz.

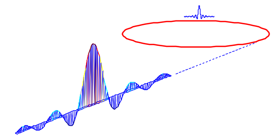


Bandpass Sampling Example

- Instead, let’s directly sample the signal at only 10M samples/second.

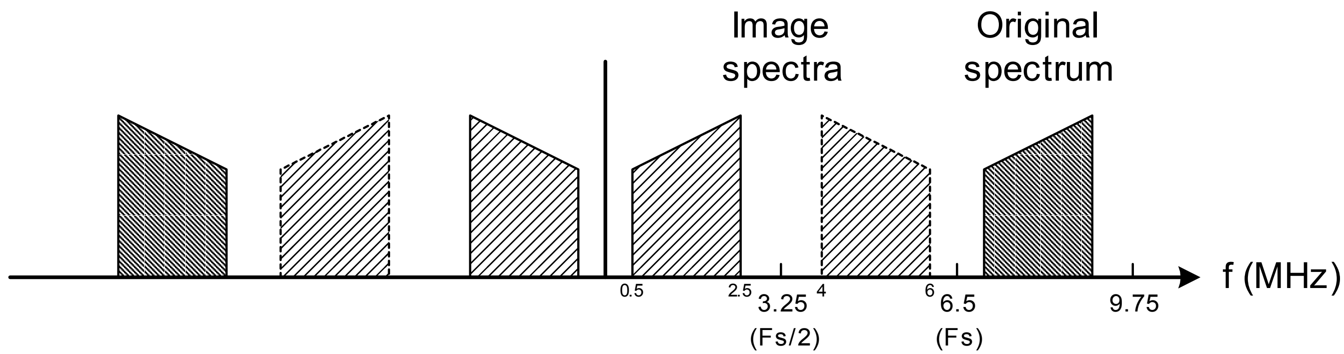


- In this case the Nyquist frequency would be 5MHz, and the original spectrum is in the range of *Fs/2* to *Fs*, instead of the range *DC-Fs/2* (as we are used to seeing).
- The original spectrum is aliased into the lower half of the frequency band, reflected about the Nyquist rate of 5MHz, appearing in the frequency range 3Mhz - 1MHz.
- So, we have successfully sampled the signal using a sampling rate almost half the ‘officially’ required rate



Bandpass Sampling Example (cont)

- What if we sample at only 6.5M samples/second??

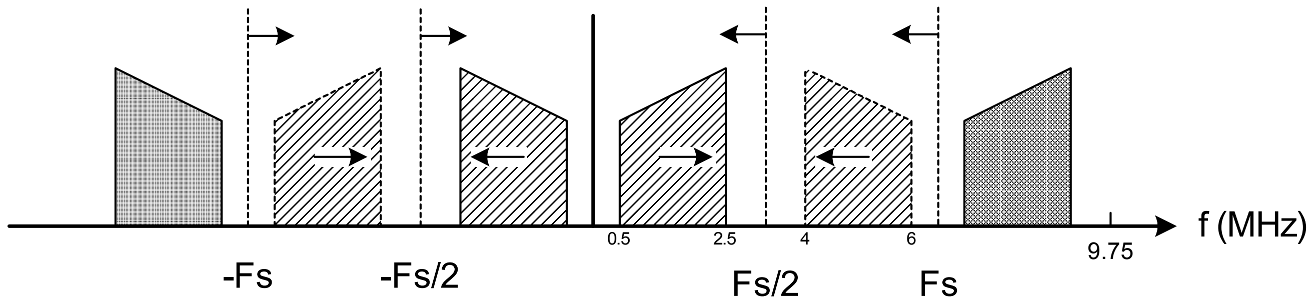


- This time the original spectrum lies between *Fs* and 1.5*Fs*.
- Here, the spectrum is reflected about the sampling rate, to appear in the range from *Fs/2* to *Fs*, spanning 6MHz - 4MHz.
- It is then reflected a second time about *Fs/2*, finally appearing in the lower half of the sampled frequency range between 0.5MHz and 2.5MHz.

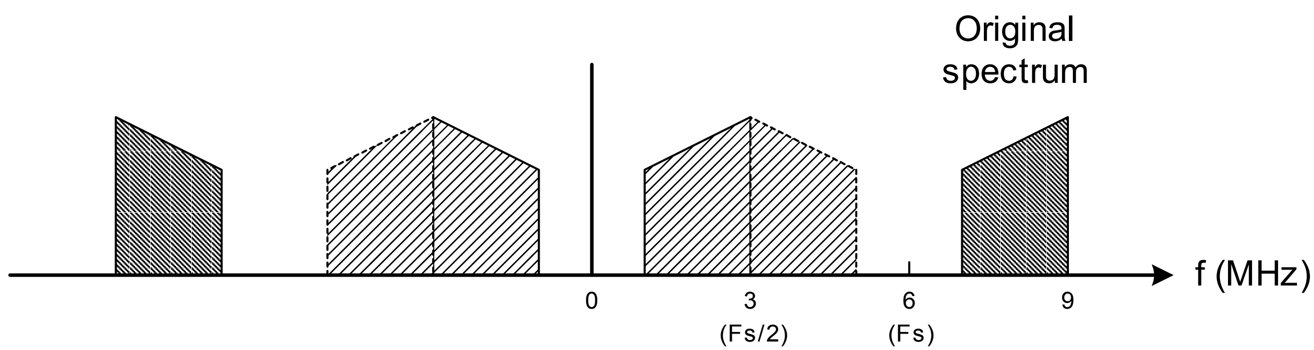
Can we sample at an even lower rate and still get a unique spectrum??

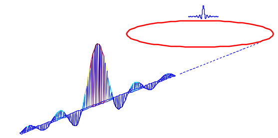
Lower Limit of Sampling Rate for Bandpass Example

- We can indeed sample at less than 6.5M samples/second, but not by much.
- To consider what happens, let's revisit the case from the last slide and examine how the spectral images behave when we reduce the sampling rate below 6.5M samples/sec



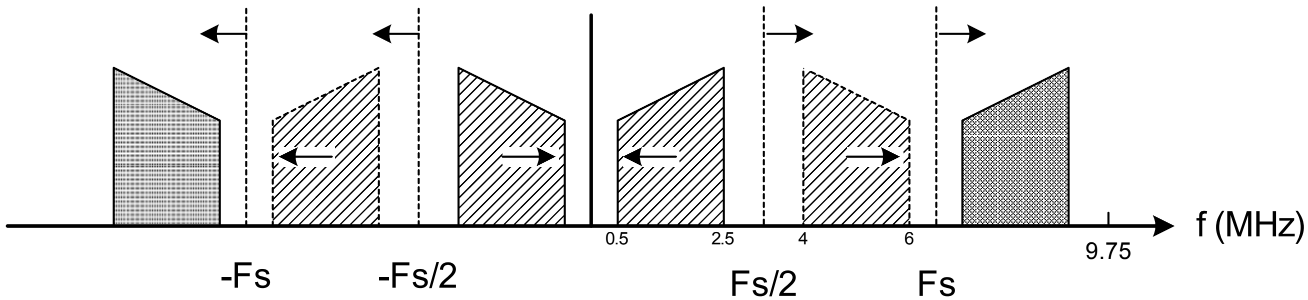
- As the sample rate is reduced, the image spectra move closer together, until eventually they collide when the sample rate gets to 6M samples/second.



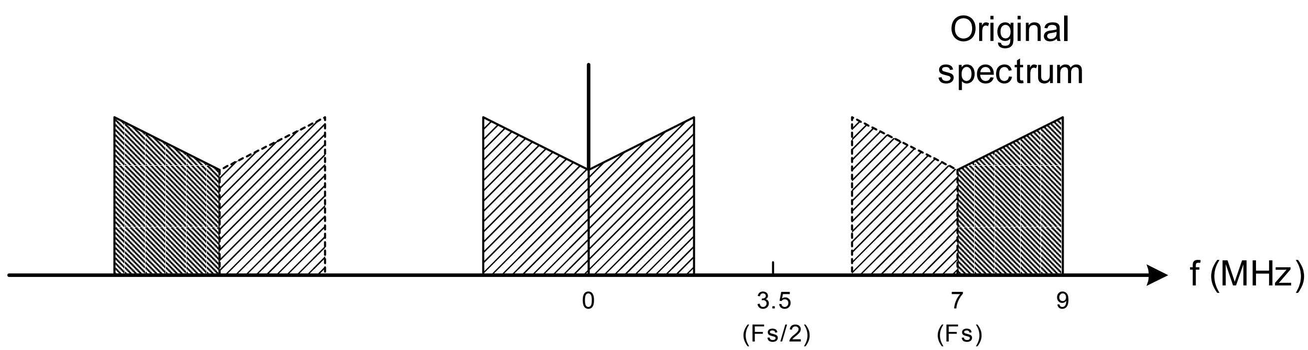


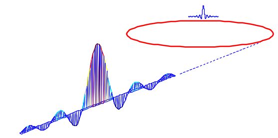
Upper Limit of Sampling Rate for Bandpass Example

- Now let's consider what happens if we increase the sampling rate from 6.5M samples/second.



- This time, the image spectra move further apart until eventually the first image spectrum collides with the original spectrum when the sampling rate reaches 7M samples/second.



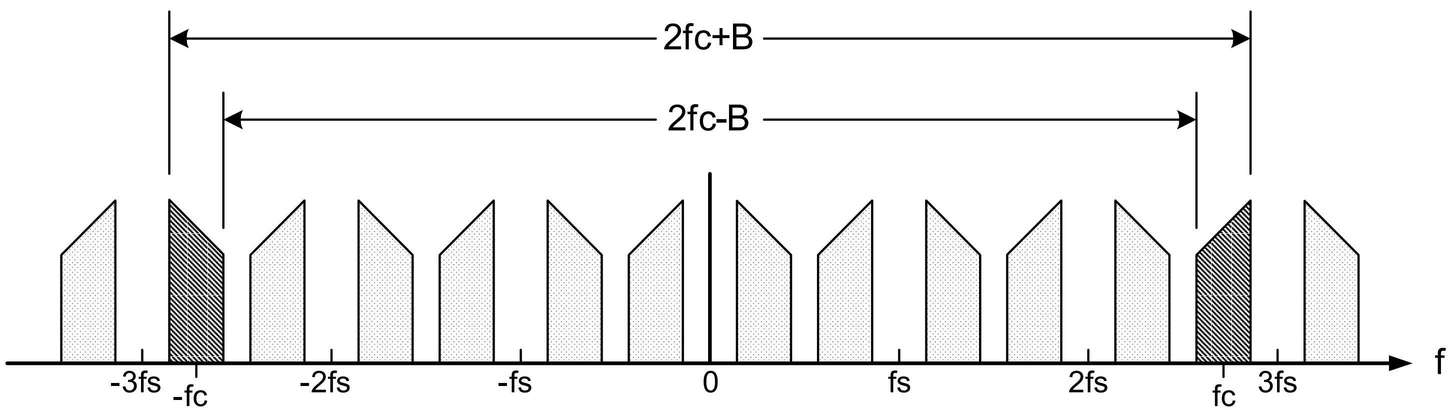


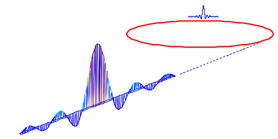
General Case of Bandpass Sampling

- In general, it can be shown that if there are m image spectra between the original and its negative image, the range of possible sampling frequencies is given by the expression

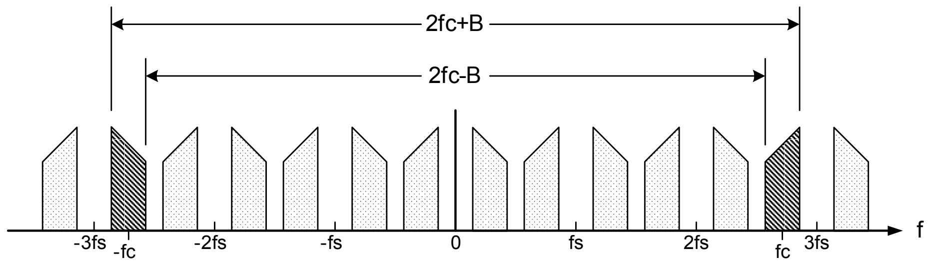
$$\frac{2fc - B}{m} \geq fs \geq \frac{2fc + B}{m+1}$$

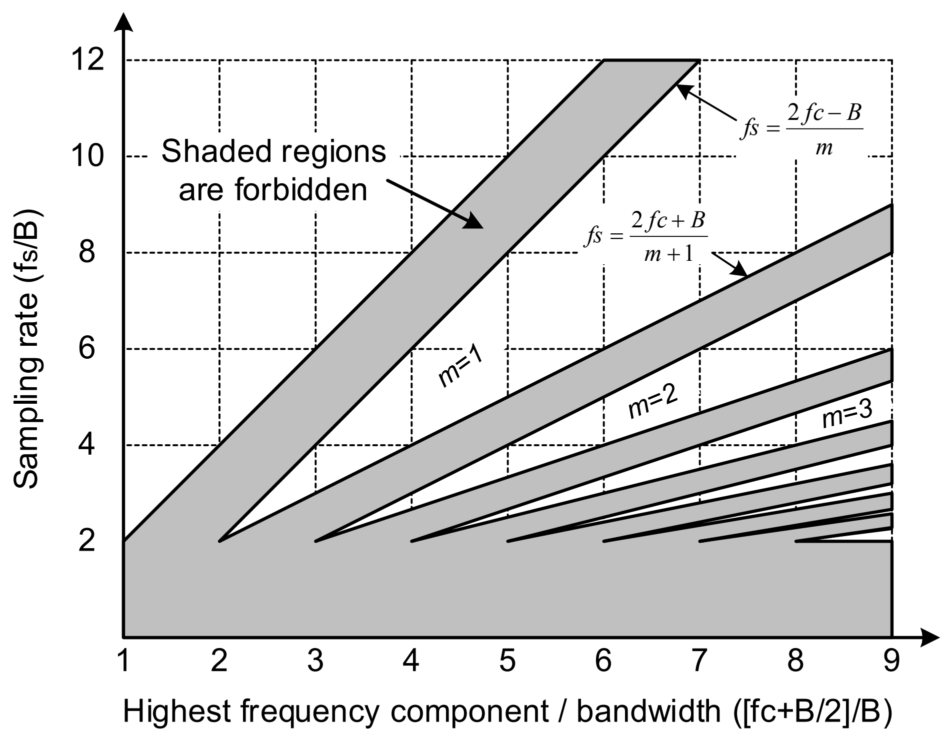
- Example with $m = 5$





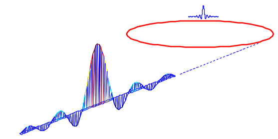
Graphical Representation of Possible Sampling Rates



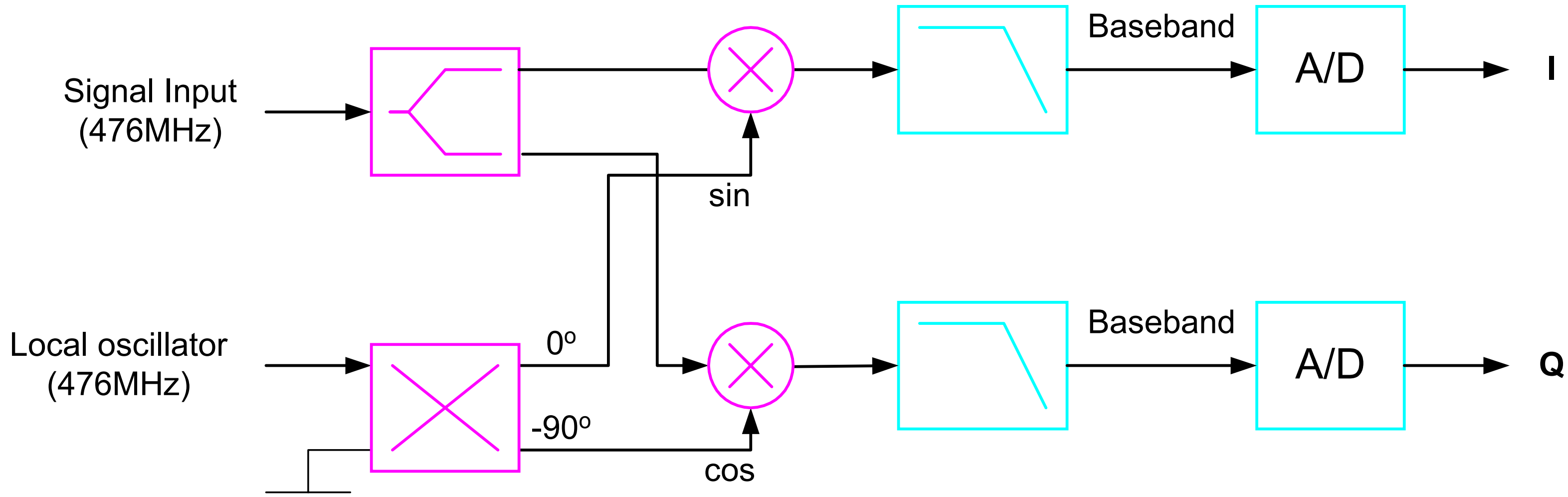


Front-End Requirements for Bandpass Sampling

- When designing the front-end for a bandpass application, there are a couple of points to bear in mind.
- Firstly, the analog front-end circuits and sample/hold must be designed for the maximum signal bandwidth (ie $fc+B/2$), which can be several times the sampling rate.
- Secondly, the anti-alias filter must be a <u>bandpass</u> filter since noise both above and below the band of interest can be aliased into the baseband.



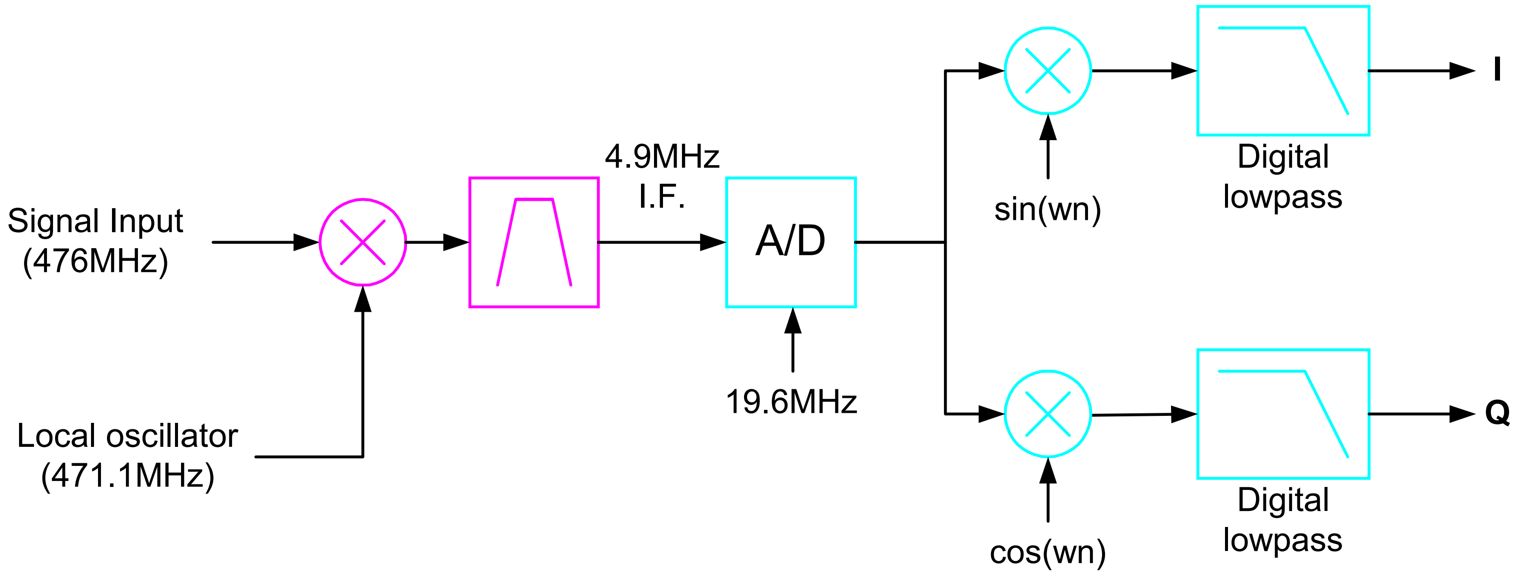
Analog I/Q Detector



- Issues: DC offsets in mixer, quadrature phase errors, impedance matching, ...

Quadrature Sampling with Digital Mixing

- In practice, it can be very difficult to implement the I/Q mixing without error so that the two channels match each other exactly.
- Digital technology now offers a completely digital approach to this problem.



- The continuous-time signal is sampled at exactly 4 times the IF frequency.
- Digital sine and cosine signals are multiplied with the incoming discrete-time sequence to generate the real and imaginary part of the signal.